1. Jacek Brodzki, Graham A. Niblo, Nick Wright: Property A and exactness of the uniform Roe algebra

It has long been established that certain properties of groups can be described through properties of suitably chosen C^*-algebras associated with them. A model result in this direction is a theorem of Lance that a discrete group is amenable if and only if its reduced C^*-algebra $C^*_r(G)$ is nuclear.

Property A was introduced by Yu as a geometric analogue of the Følner criterion that describes amenability of a group. It implies many of the interesting consequences of amenability for a discrete group, for example, property A implies uniform embeddability in Hilbert space, which in turn gives the Coarse Baum-Connes conjecture and therefore the Novikov conjecture [1].

Property A and the uniform Roe algebra can be defined for arbitrary metric spaces. Let us recall the main definitions.

A uniformly discrete metric space (X, d) has *property A* if for all $R, \epsilon > 0$ there exists a family of finite non-empty subsets A_x of $X \times \mathbb{N}$, indexed by x in X, such that

- for all x, y with $d(x, y) < R$ we have $\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \epsilon$;
- there exists S such that for all x and $(y, n) \in A_x$ we have $d(x, y) \leq S$.

The *uniform Roe algebra*, $C^*_u(X)$, is the C^*-algebra completion of the algebra of bounded operators on $l^2(X)$ which have finite propagation. The details are as follows. A kernel $u\colon X \times X \to \mathbb{C}$ has *finite propagation* if there exists $R \geq 0$ such that $u(x, y) = 0$ for $d(x, y) > R$. If X is a proper discrete metric space, and $u\colon X \times X \to \mathbb{C}$ is a finite propagation kernel then for each x there are only finitely many y with $u(x, y) \neq 0$. Thus u defines a linear map from $l^2(X)$ to itself, $u * \xi(x) = \sum_{y \in X} u(x, y)\xi(y)$. Note that if additionally X has bounded geometry, then every bounded finite propagation kernel gives rise to a *bounded* operator on $l^2(X)$. The uniform Roe algebra is the completion of the algebra generated by bounded linear operators arising from bounded propagation kernels.

For a discrete group G, Yu's property A is equivalent both to the nuclearity of the uniform Roe algebra $C^*_u(G)$ and to the exactness of the reduced C^*-algebra $C^*_r(G)$. This follows from the results of

[1] G.L. Yu, *The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into a Hilbert space*, Inventiones Mathematicae 138 (2000), 201–240.

Anantharaman-Delaroche and Renault [2], Higson and Roe [3], Guentner and Kaminker [4], and Ozawa [5].

It is natural to state the following conjecture.

Conjecture 1.1. *A uniformly discrete bounded geometry metric space X has property A if and only if the uniform Roe algebra $C^*_u(X)$ is exact.*

In evidence for the conjecture we offer the following. The conjecture is true for any countable discrete group equipped with its natural coarse structure. It is then an easy exercise to show that the conjecture holds for any metric space which admits a proper co-compact action by a group of isometries.

We proved recently [6] that the conjecture also holds if the space is sufficiently group-like in the following sense.

One of the key ingredients in the proof of the conjecture for groups is the interplay between the left and the right action of a group on itself. By convention, the left action is by isometries while the right action has the curious property that each point is moved by the same distance by a given element of the group. By analogy with Euclidean geometry we call such transformations *translations* even though they are not in general isometries. It is often overlooked that it is the translation action, rather than the isometric action, that allows one to identify $C^*_r(G)$ with a subalgebra of the uniform Roe algebra $C^*_u(G)$. One may abstract from this the notion of a translation structure for a space. We can then say that a space is more or less group-like depending on how much this structure resembles the natural left-right multiplication structure on a group.

When the space X is sufficiently group-like in this sense, the conjecture holds for X. For example, this is the case when X embeds uniformly in a group.

[2] C. Anantharaman-Delaroche and J. Renault, Amenable groupoids. Monographs of L'Enseignement Mathématique, 36. Geneva, 2000.

[3] N. Higson and J. Roe, *Amenable group actions and the Novikov conjecture*, J. Reine Angew. Math. 519 (2000), 143–153.

[4] E. Guentner and J. Kaminker, *Exactness and the Novikov conjecture*, Topology 41 (2002), no. 2, 411–418.

[5] N. Ozawa, *Amenable actions and exactness for discrete groups*, C. R. Acad. Sci. Paris Sér. I Math. 330 (2000), no. 8, 691–695.

[6] J. Brodzki, G. A. Niblo, N. J. Wright, Property A, partial translation structures and uniform embeddings in groups, preprint, http://front.math.ucdavis.edu/math.OA/0603621.