1. Graham A. Niblo and Michah Sageev: The Kropholler conjecture

A finitely generated group G is said to split over subgroup H if and only if G may be decomposed as an amalgamated free product $G = A \underset{C}{*} B$ (with $A \neq C \neq B$) or as an HNN extension $G = A \underset{C}{*}$. The Kropholler conjecture is concerned with the existence of such splittings.

Given a subgroup H of a finitely generated group G the invariant $e(G, H)$ is defined to be the number of Freudenthal (topological) ends of the quotient of the Cayley graph of G under the action of the subgroup H. This number does not depend on the (finite) generating set chosen for G [3] so it is an invariant of the pair (G, H). For example, if G is a free abelian group and H is an infinite cyclic subgroup then $e(G, H) = 0$, if G has rank 1, $e(G, H) = 2$ if G has rank 2 and $e(G, H) = 1$ if G has rank greater than or equal to 3. This invariant generalises Stallings' definition of the number of ends of the group G since if $H = \{1\}$ then $e(G, H) = e(G)$.

In [4] Stallings showed that the group G splits over some finite subgroup C if and only if $e(G) \geq 2$. There are several important generalisations of this fact, the most wide ranging being the algebraic torus theorem, established by Dunwoody and Swenson [1]. This states that, under suitable additional hypotheses, if G contains a polycyclic-by-finite subgroup H of Hirsch length n with $e(G, H) \geq 2$ then either

(1) G is virtually polycyclic of Hirsch length $n + 1$,
(2) G splits over a virtually polycyclic subgroup of Hirsch length n,
(3) G is an extension of a virtually polycyclic group of Hirsch length $n - 1$ by a Fuchsian group.

This theorem generalises the classical torus theorem from low dimensional topology which asserts that a closed 3-manifold which admits an immersed incompressible torus either admits an embedded incompressible torus or has a Seifert fibration. These topological conclusions imply the algebraic conclusions for the fundamental group of the manifold. An important ingredient of the proof of the algebraic torus theorem is a special case of the so called Kropholler conjecture. Its original formulation relies on the following observation of Scott:

A subgroup H of a finitely generated group G satisfies $e(G, H) \geq 2$ if and only if G admits a subset A satisfying the following:

(1) $A = HA$,
(2) A is H-almost invariant, and
(3) A is H-proper, i. e. , neither A nor $G - A$ is H-finite.

We will refer to the subset A as a proper H-almost invariant subset. In his proof of the algebraic torus theorem for Poincaré duality groups Kropholler observed that, under certain additional hypotheses, if G admits a proper H-almost invariant set A such that $A = AH$ then G admits a splitting over some subgroup $C < G$ related to H (see [2] for an outline of the proof). He conjectured that the additional hypotheses were inessential. Specifically:

Conjecture 1.1 (The Kropholler conjecture)**.** *Let G be a finitely generated group and $H < G$. If G contains a proper H-almost invariant subset A such that $A = AH$ then G admits a non-trivial splitting over a subgroup C which is commensurable with a subgroup of H.*

The conjecture is known to hold when G is a Poincaré duality group or when G is word hyperbolic and H is a quasi-convex subgroup. In general it is known (for an arbitrary finitely generated group G) whenever H is a subgroup which satisfies the following descending chain condition:

Every descending chain of subgroups $H = H_0 \geq H_1 \geq H_2 \geq \ldots$ such that H_{i+1} has infinite index in H_i eventually terminates.

This condition holds for example for the class of finitely generated polycyclic groups, in which class the Hirsch length is the factor controlling the length of such a chain. This is a key ingredient in the proof of the full algebraic torus theorem.

An alternative, more geometric, point of view on the conjecture is provided by the following characterisation:

Theorem 1.2. *Given a finitely generated subgroup G and a subgroup $H < G$ the invariant $e(G, H)$ is greater than or equal to 2 if and only if G acts with no global fixed point on a CAT(0) cubical complex with one orbit of hyperplanes, and so that H is a hyperplane stabiliser. H admits a right invariant, proper H-almost invariant subset if and only if the action can be chosen so that H has a fixed point in the complex.*

References

[1] M. J. Dunwoody and E. L. Swenson, 'The algebraic torus theorem', *Invent. Math.* **140** (2000), no. 3, pp. 605–637.

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[3] G. P. Scott, 'Ends of pairs of groups', *Journal of Pure and Applied Algebra* **Vol. 11**, 1977, pp 179–198.

[4] J. R. Stallings, 'On torsion-free groups with infinitely many ends', *Ann. of Math.* **88**, (1968), pp. 312–334.