University of Southampton

Analysis and Control of Linear Repetitive Processes

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A thesis submitted for the degree of
Doctor of Philosophy
in the
Faculty of Engineering and Applied Science
Department of Electronics and Computer Science

July 2000

UNIVERSITY OF SOUTHAMPTON

ABSTRACT

FACULTY OF ENGINEERING AND APPLIED SCIENCE

ELECTRONICS AND COMPUTER SCIENCE

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Repetitive processes are a distinct class of 2D systems of both practical and algorithmic interest, with a growing list of application areas. Their main identifying characteristic is a series of sweeps, termed passes here, through a set of known dynamics with explicit interaction between successive outputs, or pass profiles, as the process evolves. As a result of the explicit dependence of the process dynamics on two independent variables (in the along the pass and pass to pass directions) existing theory cannot be applied. This fact, together with the growing list of applications areas, has prompted an ongoing research programme into the development of a 'mature' systems theory for these processes.

As part of this programme, this thesis gives new results on the analysis and control of the subclasses known as differential and discrete linear repetitive processes. Novel results are presented in three separate research areas. Firstly new stability results are presented, including the further development of a two-dimensional Lyapunov equation based approach. These results provide computable information of performance which is not available from alternative stability characterisations. An initial study of robustness analysis is provided, including a discussion of a potentially promising new approach to stability margin analysis. Preliminary results on the design of controller structures are given, including the use of simple structure control schemes and fast sampling considerations. Finally some areas for short to medium term future research are discussed.

Contents

Acknowledgements

This thesis would not have happened without the help and support of a number of people.

First and foremost, I am greatly indebted to my supervisor Professor Eric Rogers whose support, guidance and continual encouragement has been immeasurable. Couldn't have done it without you.

Appreciation must also go to Professor David Owens of the Department of Automatic Control and Systems Engineering of the University of Sheffield and Professor Krzysztof Galkowski of the Department of Robotics and Software Engineering of the Technical University of Zielona Gora, Poland, for help on our collaborations.

Well played to all of my friends, both at University and elsewhere, who have provided the sanity, good humour and beer to make my time at Southampton an enjoyable one. Special mention goes to a few : the ISIS boys, in particular my office mates, Manslow, Bailey and Dodgy, for providing such a stable and well balanced working environment(!); fellow team mates Alex Rogers the Cabin Boy, Jazzy Jeff and Gippa Zaris for their varying ability in answering questions at the Dolphin pub quiz; Izzy for the much appreciated away days and trips to see Cope; Jasmin and Pricey for their advice on my thesis contents and Tim for his immoral support.

Thanks to my family for their continual support and love over the past few years. I'll get a *proper* job soon - honest. And finally a big thank you to Neil for surviving life in the firing line.

Nomenclature

α Real constant pass length, $\alpha > 0$

$S(E_\alpha, W_\alpha, L_\alpha)$ Linear repetitive process of constant pass length α

E_α^M Banach product space, $= E_\alpha \times E_\alpha \times \cdots \times E_\alpha$ (M times)

$S(E_\alpha, W_\alpha, L_\alpha)_{\alpha \geq \alpha_0}$ Extended linear repetitive process

k Pass index

t, p Differential/discrete along the pass variable

$x_k(t)$ $n \times 1$ State vector on pass k, $0 \leq t \leq \alpha$, $k \geq 1$

$y_k(t)$ $m \times 1$ Output pass profile on pass k, $0 \leq t \leq \alpha$, $k \geq 1$

$u_k(t)$ $l \times 1$ Vector of control inputs on pass k, $0 \leq t \leq \alpha$, $k \geq 1$

$r_k(t)$ $m \times 1$ External reference vector on pass k, $0 \leq t \leq \alpha$, $k \geq 1$

$e_k(t)$ $m \times 1$ Error vector on pass k, $0 \leq t \leq \alpha$, $k \geq 1$

M Memory length of the process, $M \geq 1$

$L_D(A, B, C)$ Derived conventional linear system of $S(E_\alpha, W_\alpha, L_\alpha)$

$L_A^j(A, B_{j-1}, C, D_j)$ j^{th} Associated conventional linear system, $1 \leq j \leq M$, of $S(E_\alpha, W_\alpha, L_\alpha)$

Φ Augmented plant matrix of $S(E_\alpha, W_\alpha, L_\alpha)$

$\mathbb{Z}_+^2$ Set of ordered pairs of integers, $= \{(i, j) : i, j \geq 0,\ i, j \in \mathbb{Z}\}$

$\mathbb{Z}_+$ Set of nonnegative integers

$\mathbb{R}^n$ Euclidean n-space of real $n \times 1$ vectors

$\mathbb{R}_+$ Set of positive real numbers

$\mathbb{C}$ Set of complex numbers

X^d d^{th} Cartesian product of space X

X_e^d Extended space of X^d

$S(\mathbb{Z}_+, E)$ Linear space of all sequences on E, i.e. the functions $f : \mathbb{Z}_+ \longrightarrow E$

$B(\mathbb{Z}_+, E)$ Subspace of $S(\mathbb{Z}_+, E)$ of all bounded functions

$L_2^m[0, \alpha]$ Product space of square summable functions over $[0, \alpha]$

$\ell_2^m[0, \alpha]$ Sequence space of real square summable $m \times 1$ vectors of length α

U Open unit disc, $\{z \; : \; |z| < 1\}$

$\overline{U}$ Closed unit disc, $\{z \; : \; |z| \leq 1\}$

$\overline{U}^2$ Closed unit bidisc, $\{(z_1, z) \; : \; |z_1| \leq 1, \; |z| \leq 1\}$

$A \oplus B$ Direct sum of A and B

$A \otimes B$ Kronecker product of A and B

$A > 0, \; A \geq 0$ A is positive definite/positive semi-definite

A^* Complex conjugate transpose of A

PDH Positive Definite Hermitian matrix

$|| \cdot ||$ denotes both the norm on E_α and the induced operator norm

$||A||_p$ Nonnegative matrix associated with A

$r(\cdot)$ Spectral Radius of its argument

$\sigma(L_\alpha)$ Set of spectral values of L_α termed the spectrum of L_α

$\overline{\sigma}(\cdot)$ Largest singular value of its argument

$\underline{\sigma}(\cdot)$ Smallest singular value of its argument

$\mathcal{L}$ Laplace Transform operator

$S[\cdot]$ Stacking operator

V_0 Volterra operator $B(\mathbb{Z}_+, E) \longrightarrow B(\mathbb{Z}_+, E)$

$N_T(f)$ Total variation of f

$P_T L$ Natural projection of $L \in X_e^d$ into $X_{(0,T)}^d$

Chapter 1

Introduction

The concept of a repetitive process (at the time termed multipass processes) was introduced in the 70's as a result of research by the University of Sheffield into looking at the problems associated with long-wall coal cutting processes (Edwards, 1974; Boland and Owens, 1980; Edwards and Owens, 1982). Since the decline of the coal mining industry in the UK, attention in recent years has been focussed on application areas where analysis from a repetitive process perspective has advantages over available alternatives. These so-called algorithmic examples include the use of repetitive process theory in the algorithmic solution of nonlinear dynamic optimal control problems using the maximum principle (Roberts, 1994b; Roberts, 1996; Roberts, 2000). In addition, a recent important development has been the fact that the theory can be used within the algorithmic analysis of iterative learning control schemes - i.e. those where a procedure is repeatedly performed with a view to sequentially improving accuracy. Significant results on exploiting these links can be found in, for example, (Amann, 1996; Amann et al., 1996; Amann et al., 1998; Owens et al., 2000).

Repetitive processes have been defined (Edwards, 1974) as those involving the processing of a material or workpiece by a sequence of sweeps, termed passes, of the processing tool. On each pass, an output, or pass profile, is produced. One of the key characteristics of repetitive processes is that the output $y_k(t)$, $0 \leq t \leq \alpha$, (where t is the temporal (or spatial) independent variable and α is the pass length) generated during the kth pass acts as a forcing function on the next pass, and hence contributes to the dynamics of the new pass profile $y_{k+1}(t)$, $0 \leq t \leq \alpha$, $k \geq 0$. It is this interaction between successive pass profiles which leads to the unique control problem associated with these processes in that oscillations can occur in the output sequence of pass profiles which increase in amplitude from pass to pass. In the long-wall coal cutting example, where the main objective is to maximise coal extraction without penetrating the stone-coal interface, this can be seen via the presence of undulations in the newly cut coal floor, which means that cutting operations (i.e. productive work) must be suspended to enable their manual removal. This problem is one of the key factors behind the 'stop/start' cutting pattern of a typical working cycle in a coal mine. This behaviour can be easily generated in simulation studies and experiments on scaled models of industrial processes such as long-wall coal cutting - see (Smyth, 1992) for the details.

Attempts to control these processes using standard, termed 1D, techniques in general fail, since they ignore the inherent two-dimensional nature of the processes, i.e. information is propagated in two different directions - along a given pass (in the t direction) and from pass to pass (in the k direction). This has motivated the development of a rigorous stability theory for linear repetitive processes by Rogers and Owens (Owens, 1977; Rogers and Owens, 1992b).

In the most general case, a repetitive process has nonlinear dynamics and a variable pass length. Clearly to analyse such a process would be a formidable task. Hence research to date has been limited to processes with linear dynamics and a constant pass length α with the justification that the majority of practical examples fall into this category. A mathematical formulation of a linear repetitive process with constant pass length α has been proposed in (Owens, 1977) based on an abstract model in a Banach space setting, which includes all previously studied examples as special cases (but also allows for the consideration of those with a potentially more complex structure) and is the basis of the rigorous stability theory for these processes. In particular, this model admits analysis of differential and discrete subclasses of processes which are of direct industrial and algorithmic interest and which are the subject of this thesis.

The aim of this thesis is to make progress in the development of a 'mature' systems theory for linear repetitive processes with a constant pass length α. In particular, the areas of stability (including the further development of a Lyapunov equation based approach), robustness and controller structure design have been investigated, each of which are commented on below. The results presented form part of an ongoing research program by Rogers, Owens, Galkowski et al. and a summary of current progress can be found in (Owens and Rogers, 2000; Galkowski et al., 2000; Rogers et al., 2000a).

1.1 Stability

Chapter 3 introduces the rigorous stability theory for linear repetitive processes with a constant pass length α which is based on an abstract model in a Banach space setting. Here it is demonstrated how two distinct concepts exist, namely asymptotic stability and stability along the pass, which is not surprising since a repetitive process is governed by two independent variables. It is shown how asymptotic stability is a relatively weak definition of stability and, except in a few very special cases when it is all that is required, or in fact all that is achievable, it is the stronger condition of stability along the pass that is generally needed for acceptable systems performance.

Using techniques from functional analysis, the theory of Rogers and Owens is initially presented for the general abstract representation of a linear repetitive process with constant pass length α and is subsequently extended to the differential and discrete subclasses of processes which are the subject of this thesis. Here it is stressed how the accurate determination of process boundary conditions (termed 'simple' or 'dynamic' in chapter 2) is vital for correct stability classification. In fact, the misclassification of dynamic boundary conditions as simple could result in an unstable process being accepted as stable. This is a key distinguishing feature of linear repetitive processes and is a major reason why they cannot be analysed by direct application of standard Roesser/Fornasini-Marchesini based theory. A summary of the current situation in the research program into dynamic boundary condition analysis is given in (Galkowski et al., 2000).

As an immediate consequence of stability along the pass of differential and discrete processes, after a 'sufficiently large' number of passes the dynamics of the process can be replaced by those of a 1D stable system. Clearly strong computable information on the rate of approach of the output sequence of pass profiles to this so-called limit profile would be of interest, in addition to bounds on the 'error' $y_k - y_\infty$ on a given pass. Two possible routes are available for obtaining these performance predictions (which are not available from the standard Nyquist like stability tests), namely adopting a two-dimensional Lyapunov equation based approach (see chapter 4) or using time domain (or 'simulation-based') tests. It is shown how for the discrete subclass of processes the standard test for stability along the pass involves the computation of the eigenvalues of a potentially large dimensioned matrix for all points on the unit circle in the complex plane. Here, new tests are introduced for this subclass which replace these computationally intensive conditions with the one-off evaluation of the eigenvalues of a matrix with constant entries. The resulting conditions are sufficient in nature only, but this potential conservativeness is offset by the availability of performance measures at no extra computational cost. The theory presented in these sections is novel and provides the basis for the paper (Benton et al., 1998b).

Within chapter 2 it is illustrated how certain subclasses of linear repetitive processes can be written in the form of standard 2D state-space representations. Here links between the BIBO stability of these Roesser/Fornasini-Marchesini state-space models and the stability along the pass of linear repetitive processes are made, which allow the transfer of certain results and ideas between the two areas. Results obtained from exploiting these links can be found in section 3.8.

Finally, within this chapter on stability, a Volterra operator approach to stability analysis is introduced. These relatively new results indicate that the powerful theory of Volterra operators has a significant role to play in the analysis of discrete linear repetitive processes, and hence is an area where future research effort should be directed.

1.2 Lyapunov Equations

As a result of the 'equivalence' between the BIBO stability of 2D systems described by the Roesser model (and hence the Fornasini-Marchesini model) and the stability along the pass of discrete linear repetitive processes which is discussed in chapter 3, many well known tests available for the stability analysis of 2D linear systems may be applied here.

Chapter 4 considers the extent to which a Lyapunov equation based approach to the stability analysis of linear repetitive processes may be applicable. A review of the literature to date reveals that Lyapunov equations for systems with two independent time variables has been approached in essentially two different ways:

(i) the so-called 1D Lyapunov equation approach, which is termed 1D since the equation has an identical structure to that for discrete linear time-invariant systems, but with defining matrices which are functions of a complex variable; and

(ii) the so-called 2D Lyapunov equation approach, which is defined in terms of matrices with constant entries.

Initially, the 1D Lyapunov equation approach is introduced, here for the differential subclass of processes (the equivalent treatment for the discrete case can be found in any of the cited references within the main text) with simple boundary conditions. It is shown how the resulting condition based on this equation is found to be both necessary and sufficient for stability along the pass, and can be implemented via computations on matrices with constant entries. This test hence serves as an alternative to the stability along the pass tests of chapter 3, in particular the potentially computationally intensive Nyquist-like tests. In addition, it is shown how the tests provide computable information on the rate of approach of the output sequence of pass profiles to the limit profile on a given pass, and hence provide an alternative route to obtaining measures of performance prediction to the simulation-based stability tests of chapter 3.

Finally, in this part of the chapter, a 1D Lyapunov equation is developed for a subclass of differential processes with a particular type of dynamic boundary conditions (which are of special relevance to the area of delay-differential systems theory). Strict positive realness based tests to compute positivity are proposed which reduce the problem to a 1D problem by showing that the (necessary and sufficient) stability along the pass condition is equivalent to testing for positive realness of a certain 1D rational transfer-function matrix. The analysis presented here has been presented in (Benton et al., 2000c) and (Benton et al., 2000d).

In section 4.6 and onward, the so-called 2D Lyapunov equation approach to stability analysis is developed. The theory presented here provides the subject for the paper (Benton et al., 1999). Here it is shown how the existence of a positive definite solution pair to the 2D Lyapunov equation is, in general, only sufficient for stability along the pass. A counter-example is given which demonstrates that a stable along the pass process does not necessarily have the strictly bounded real property and hence doesn't satisfy the 2D Lyapunov equation. Two special cases are discussed, however, where the existence of a positive definite solution pair to the 2D Lyapunov equation is both necessary and sufficient for stability along the pass.

In section 4.7, a 2D Lyapunov equation is developed for a 2D Fornasini-Marchesini state-space model of the dynamics of a discrete linear repetitive process which involves the computation of generalised eigenvalues. The analysis presented here is the subject of (Benton et al., 2000a).

Despite the apparent conservativeness of the sufficient but not necessary nature of the 2D Lyapunov equation, this approach has a potentially major role to play in the analysis of discrete linear repetitive processes in terms of stability margins and robust stability theory, which is discussed in chapter 5. In addition, here it is shown how the equation provides measures on performance along a given pass which is not available from Roesser/Fornasini-Marchesini equivalent descriptions (for the discrete subclass of processes).

1.3 Robustness

When analysing a process, it is important to not only determine stability, but also to obtain some indication of as to how robust the system is to perturbations in the system. In chapter 5 the subject of robustness of linear repetitive processes is investigated. As a measure of 'how stable' a process is, or rather 'how far' from being unstable, the subjects of allowable parameter variation bounds and stability margins are considered.

Given a stable along the pass discrete process, the first of these areas considers how the stability of the process is affected by perturbations in the matrices which define the state-space model. Two types of perturbation are looked at within this thesis:

(i) structured, where the perturbation model structure and bounds on the individual elements of the perturbation matrices are known; and

(ii) unstructured, where at most a spectral norm bound on the perturbation is known.

The aim of the analysis presented has been to find methods of determining the minimum norm of the perturbation matrix $\Delta\Phi$ such that the perturbed process remains stable along the pass. A discussion of available methods is given. In section 5.4, it is indicated how, in many cases, a good lower bound for this often suffices. Here it is shown how the existence of a positive definite solution pair to the 2D Lyapunov equation of chapter 4 can be used as a starting point to obtaining such lower bounds. This application area of the 2D Lyapunov equation offsets some of its inherent conservativeness due to its sufficient only for stability along the pass nature, and this analysis can be found in (Benton et al., 1999).

To conclude the work performed on parameter variations, robustness analysis is presented using a Fornasini-Marchesini representation of the process dynamics to give various bounds on the minimum norm of the matrices of both structured and unstructured perturbations.

Stability margins give an indication as to the extent to which the 'singularities' of a stable along the pass process may be moved before the process becomes unstable. Given a stable along the pass process, the stability margin is defined as the shortest distance between the singularities of the process and the boundary of the stability region - in the case of discrete linear repetitive processes, the boundary of the unit bidisc. Different methods for evaluating stability margins are discussed within the chapter, and once again, it is indicated how a 2D Lyapunov equation based approach may be used to provide good lower bounds for the margins.

Finally, in section 5.9 some very recent results on the definition of a pole of a multidimensional system using the behavioural approach are interpreted in terms of discrete linear repetitive processes.

1.4 Controller Structures

Repetitive processes clearly introduce control problems which are outside the scope of existing 1D (feedback control) theory, and the question of when and under what conditions does a basic physically realisable stabilising controller exist is complicated by the fact that the process dynamics explicitly depend on two independent complex variables.

A general control problem can be formulated with the following aims:

(i) to define objectives;

(ii) to specify control structures; and

(iii) the development of design algorithms (ideally within a computer aided control system design environment).

The main focus within this chapter is (ii) above, a consideration of the controller structures available for these processes - further details of progress made in (i) and (iii) above can be found in, for example, (Smyth, 1992; Smyth et al., 1994).

A basic consideration of the sweeping action of information propagation, and hence the set of 'causal' information, indicates that repetitive process controller structures fall into two distinct categories:

(i) those which explicitly use information from the current pass only - so-called memoryless controllers; and

(ii) those which explicitly use information from the current pass and/or previous pass profiles, state vectors and input vectors - so-called controllers with memory.

Memoryless schemes (and, in particular, so-called point controllers which use data from the current time instant on the current pass only) clearly have the simpler structure in terms of data to be stored/logged, and hence should be fully investigated prior to the consideration of those with a potentially more complex structure (i.e. those in class (ii) above or alternatives).

Differential and discrete linear repetitive processes clearly have strong structural links with 1D linear systems (see chapter 2 for further details of these links), and hence the first attempt at controller design for these processes has been to exploit these links wherever possible and gauge to what extent 1D structures may be applied here. Section 6.2 introduces (current point) state feedback policies. These schemes are, in general, only implementable with an observer structure, hence output/error-actuated schemes are also given. In general these 1D control actions fail, since repetitive processes introduce control problems which are inherently two-dimensional in nature. In section 6.4 and onward, however, it is shown that for one subclass of practical interest - a class of so-called benchmark problems - a 1D control action is all that is required for acceptable systems performance, provided a high enough gain is applied. The general philosophy adopted in this work is in the spirit of (Sebek and Kraus, 1995) for other classes of 2D linear systems, i.e. the use of 'simple' structure controllers, and the novel analysis presented here provides the basis for the paper (Benton et al., 1998a). The work replaces the necessary and sufficient condition on gain for stability along the pass by a sufficient but not necessary alternative. This potential conservativeness is offset by the availability of strong information on performance along a given pass from this result at no extra computational cost, which is not available from Nyquist-like alternatives. Two refinements to the work are also presented, thus extending the range of application of the theory.

When one or more of the control objectives cannot be met by a current pass controller, one way forward is to look at controllers with memory. Within this chapter, an example of a memoryless linear state feedback law with proportional repetitive minor loop compensation is introduced. Here, it is demonstrated how the application of this type of structure to a class of benchmark problem can successfully give a solution to the so-called repetitive systems disturbance decoupling with stability problem.

In section 6.8, discrete processes are considered. It is shown how a discrete linear repetitive process can be regarded as being derived from a differential process under fast sampling conditions. It can be seen that these conditions give rise to 'high performance' control for one subclass of practical interest, and this analysis provides the basis for (Benton et al., 2000b).

Finally, an approach to controller design using the 2D Lyapunov equation of chapter 4 as a starting point is given in section 6.9. Here it is shown how the equation is used in the design of a current pass state feedback law with 'feedforward' of the previous pass output action (which is an example of a controller with memory), as a result of which the 2D Lyapunov equation is used as a sufficient condition for closed loop stability along the pass.

Chapter 2

Background

2.1 Introduction

In the most general case, a repetitive process has nonlinear dynamics over a finite, but variable, pass length. To analyse such a process would clearly be a formidable task. With this motivation, research in this area to date has been restricted to linear processes over a fixed finite pass length, with the justification that the vast majority of previously studied practical examples fall into this category.

Within this chapter, the models of linear repetitive processes used within the analysis presented in this thesis are formally introduced. Initially, a rigorous mathematical representation of linear repetitive processes with a constant pass length α is presented, which is then used as the basis for the stability theory introduced in chapter 3 and onward analysis. This theory applies to all examples of processes with linear dynamics and a constant finite pass length, including the subclasses of so-called differential and discrete linear repetitive processes which are the main subject of this thesis. Further industrial and algorithmic examples are briefly introduced to give an indication of possible future areas of application of the theory. It is shown how linear repetitive processes assume a two-dimensional nature and hence, in the stability analysis and the formulation of physically meaningful control policies, one possible way forward is to attempt to exploit structural links with 2D linear systems described by well known state-space models and with standard, termed 1D here, linear systems. With this motivation, links are drawn between linear repetitive process theory and, in particular, state-space model based approaches to these

2D systems. Finally, a 2D transfer-function based approach is presented which is used as the basis for some of the stability tests/controller design methods of the subsequent chapters.

Prior to the introduction of formal mathematical representations, a brief overview of 'how the processes actually work' is given, with the long-wall coal cutting process referred to by way of a physical example. Long-wall coal cutting is the most commonly encountered method of extracting coal from deep cast mines in Great Britain, and has the basic operation, as illustrated in figure 2.1, of a coal cutting machine being hauled along the entire length of the coal face (up to 300m in some mines) by resting on the so-called armoured face conveyer - a collection of loosely joined steel pans which rests on the newly cut floor profile.

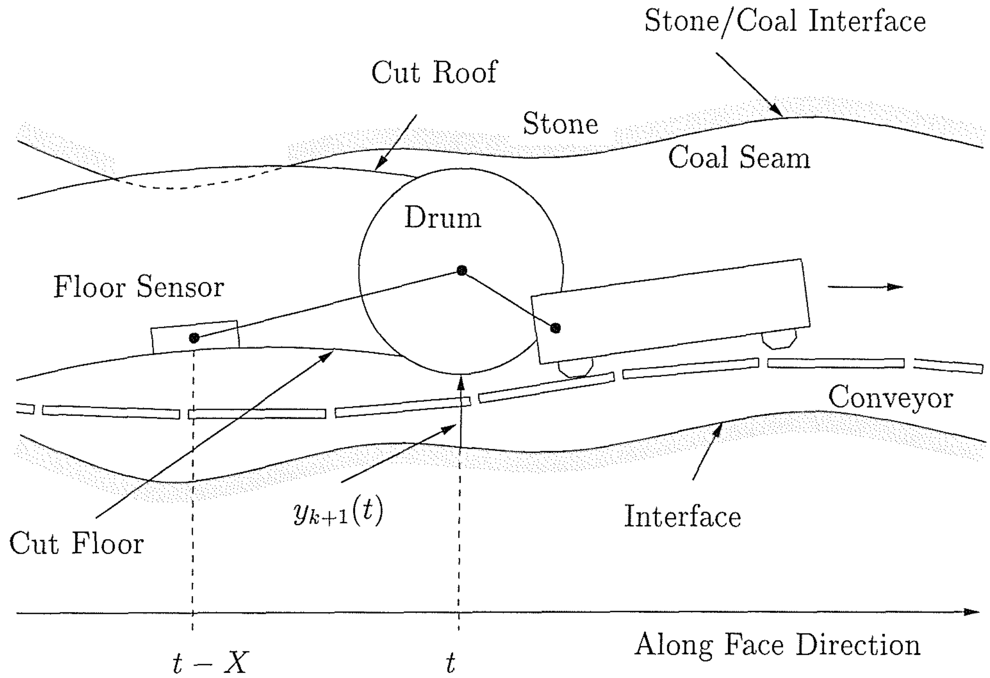

Figure 2.1: Side Elevation of a Long-wall Coal Cutting Installation

These machines generally cut in one direction only (left to right in figure 2.1) - more advanced bidirectional cutting is only really feasible in very rich seam mines - and are hauled back in reverse at high speed for the start of the next sweep, or pass, of the coal face. Between passes, the conveyer is snaked forward hydraulically so that it now rests of the floor of the profile produced during the previous pass. An idealised model of the process (see for example (Rogers and Owens, 1990a) for a detailed treatment) is based on the geometry shown in figure 2.2, which immediately confirms that this long-wall coal cutting example is indeed a repetitive process.

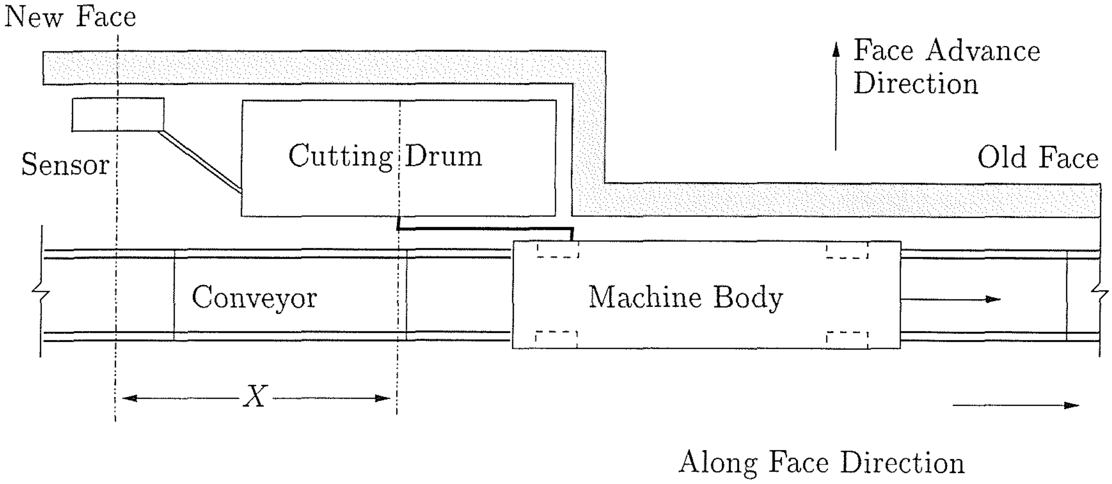

Figure 2.2: Plan of a Long-wall Coal Cutting Installation

Suppose now that α denotes the constant finite pass length (i.e. the total length of the coal face being mined) and $y_{k+1}(t)$, $0 \leq t \leq \alpha$, the height of the stone-coal interface above a fixed datum at 'point' t along pass $k+1$, $k \geq 0$. Then, with the further assumption that the conveyer moulds itself exactly on the newly cut floor profile (the so-called rubber conveyer assumption), a simple model of the process dynamics is

$$\begin{aligned} &y_{k+1}(t) = -k_1\, y_{k+1}(t-X) + k_2\, y_k(t) + k_1\, r_{k+1}(t) \\ &X > 0, \quad 0 \leq t \leq \alpha, \quad k \geq 0. \end{aligned} \tag{2.1}$$

Here on pass k, $r_k(t)$ is a new external variable taken to represent desired floor coal thickness, k_1 and k_2 are positive real scalars and X is the transport lag (time delay) by which the sensor lags the centre of the cutting drum in the along the pass direction. To complete the process description, the following initial conditions can be imposed without loss of generality,

$$y_{k+1}(t) = 0, \quad -X \leq t \leq 0, \quad k \geq 0. \tag{2.2}$$

Figure 2.3 gives a representation of the response of (2.1) (with $k_1 = 0.8$, $k_2 = 1$, $X = 1.25$ and $\alpha = 10$) over the first four passes to the conditions

$$r_{k+1}(t) = -1 \quad \text{and} \quad y_0(t) = 0, \qquad 0 \leq t \leq 10, \quad k \geq 0, \tag{2.3}$$

i.e. a downward unit step applied at $t = 0$ on each pass and a zero initial pass profile (figure based on simulation study given in (Rogers and Owens, 1992b)).

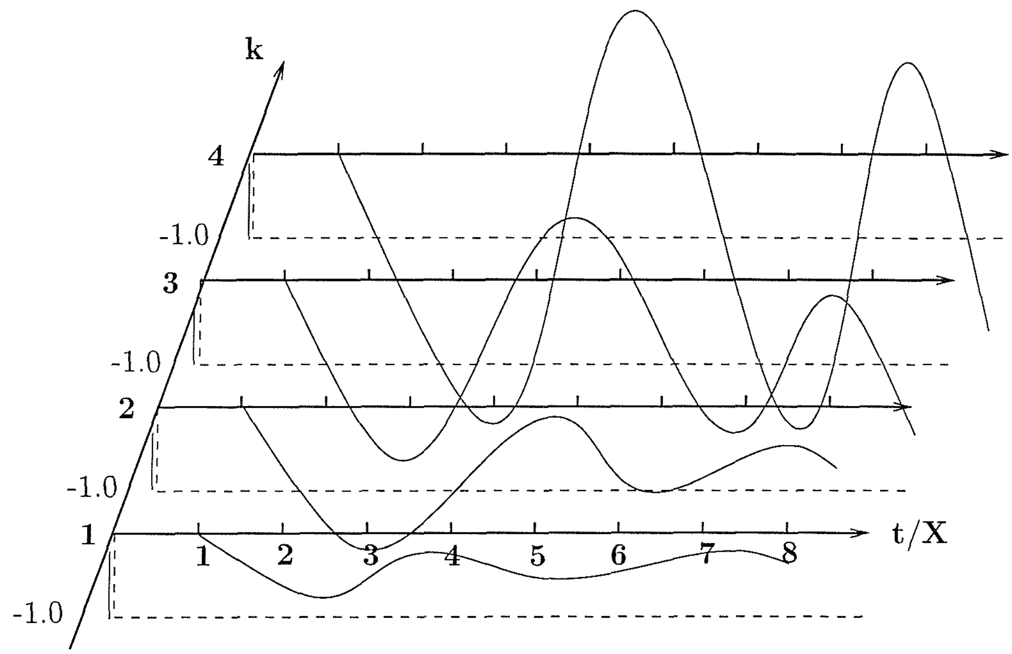

Figure 2.3: Representation of Dynamics of (2.1)-(2.2) under Conditions (2.3)

The point to note in figure 2.3 is that, although the first profile is an acceptable 'classical' response to a downward unit step input, oscillations are present in successive pass profiles which increase in amplitude in the pass to pass direction, and clearly a strong control action is required. This feature is caused by the interaction between successive pass profiles and illustrates the essential unique control problem associated with linear repetitive processes.

In abstract terms, a linear repetitive process can be visually represented, as shown in figure 2.4, by a set of two axes where the horizontal axis represents the time or distance along each pass and the vertical axis represents the pass number. The time axis can be measured in continuous or discrete variables, but the pass number is always a discrete measure. Now, since we are looking at processes with a constant pass length α, the time/distance axis is limited, as shown in figure 2.4, a repetitive process is a continuous-discrete or discrete-discrete system which is limited in one direction.

Associated with each point in the grid are the states, $x_k(t)$, the inputs $u_k(t)$ and the

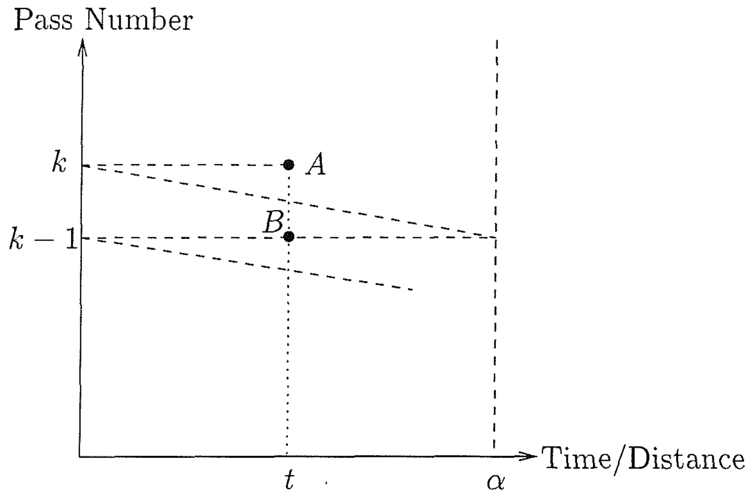

Figure 2.4: Two-Dimensional Nature of a Repetitive Process

outputs $y_k(t)$. Initial conditions, in the simplest possible situation (see later for a further discussion of this point) are specified at $t = 0$ on each pass and the initial pass profile is given (on the $k = 0$ axis).

So, on a given pass k, the process operates until $t = \alpha$ (i.e. the end of the pass is reached). The process then resets back to $t = 0$, all states are reset by the initial conditions, the pass number is iterated to $k + 1$ and the procedure repeats. Note that it is this passing movement through the positive quadrant which defines the causality of the process in the 'obvious intuitive' sense.

In terms of the long-wall coal cutting example, the finite length repeatable nature of the process is clearly seen when, at the end of a pass, the cutting machine is hauled back in reverse to the start of the pass, where it rests on the newly cut floor profile ready for the start of the next sweeping action.

The sweeping motion is termed 'unidirectional operation' in the sense that the relative motion between the tool and the material is processed in one direction only. (Edwards, 1974) discusses linear repetitive processes with a bidirectional sweeping action, where the material is processed in each direction alternatively. The difference here is that, within these so-called 'record and reverse' processes, the along the pass variable switches from t to $\alpha - t$ at the beginning of each pass. Since all research to date (and the vast majority of practical examples) has been into processes falling into the former category, processes with a bidirectional sweeping action are not considered within this thesis.

A unique feature of a repetitive process is that the output $y_k(t)$ on pass k is explicitly effected by a finite number of previous pass profiles. In the simplest situation, the output at point B in figure 2.4 acts as a forcing function on and hence contributes to the dynamics at point A. Processes with this feature have the so-called unit memory property. In the situation where it is the previous M pass profiles which contribute to the current one the process is called non-unit memory, where the integer M, $M \geq 1$, is termed the memory length of the process.

In the long-wall coal cutting example, the 'interaction between successive profiles' occurs in the form of oscillations caused by the machines weight as it comes to rest on the newly cut coal face ready for the start of the next pass along the coal wall, resulting in severe undulations in the newly cut floor profile (as illustrated in figure 2.3). This physical behaviour illustrates the unique control problem associated with these processes, namely the possible presence of oscillations in the output sequence of pass profiles, due to pass profile interaction, which increase in amplitude from pass to pass.

2.2 Original Approach to Stability Analysis

The first attempt at analysis of repetitive processes (then termed multipass processes) was by Edwards in the late 60's/early 70's as the result of research into the vertical steering of a long-wall coal cutting machine (Edwards, 1974). The original approach was to convert the output sequence of pass profiles $\{y_k(t)\}_{k\geq1}$, $0 \leq t \leq \alpha$, to a single pass, infinite length output $y(v)$, $0 \leq v < +\infty$, described by a differential/algebraic delay system in which the relationships between the variables are expressed only in terms of v. This technique expresses the process as a function of the single coordinate v, where, given the constant finite pass length α,

$$v = (k-1)\,\alpha + t \ = \text{total pass distance traversed up to the point } (k,t), \qquad (2.4)$$

and hence admits stability analysis by any of the well known classical techniques. In particular, the standard inverse Nyquist stability criterion was utilised in (Edwards, 1974) to assess stability of examples of repetitive processes such as long-wall coal cutting, ploughing and certain metal rolling operations.

It was observed in (Owens, 1977), however, that this modelling approach will almost always end in failure since it neglects the fact that the initial conditions are reset at the beginning of each pass (the $x_{k+1}(0) = d_{k+1}$ pass initial conditions) and the essential finite length repeatable nature of the processes, i.e. two of the inherent characteristics of repetitive processes.

2.3 A General Abstract Representation

To remove these deficiencies, a general abstract representation has been proposed by Edwards and Owens (Owens, 1977; Edwards and Owens, 1982) and subsequently developed by Rogers and Owens (Rogers and Owens, 1992b) with the following essential features,

(i) retention of initial conditions on each pass, and

(ii) treatment of all previously studied examples as special cases, with the provision for inclusion of those with a potentially more complex structure.

In the most general case, each pass k is characterised by a pass length α_k, which may vary from pass to pass, and nonlinear dynamics (see, for example, (Rogers and Owens, 1992b) for the details). To analyse such a process would clearly be a formidable task. This difficulty is avoided here by noting that the vast majority of processes of practical interest studied to date are of constant pass length with linear dynamics. Hence from this point onwards attention is restricted to linear processes with $\alpha_k \equiv \alpha, \ k \geq 0$.

It is clear that any representation of a linear repetitive process must include the following unique characteristics (which have been illustrated in figure 2.4),

(i) a number of passes through a known set of dynamics,

(ii) an initial pass profile $y_0(t)$ defined over $0 \leq t \leq \alpha$,

(iii) each pass subject to its own boundary conditions, disturbances and control inputs, and

(iv) explicit interaction between successive passes.

The general abstract representation of a linear repetitive process with constant pass length α can now be defined as follows,

Definition 2.1 (General Abstract Representation) *(Edwards and Owens, 1982; Rogers and Owens, 1992b) A linear repetitive process $S(E_\alpha, W_\alpha, L_\alpha)$ of constant pass length $\alpha > 0$ consists of a Banach space E_α, a linear subspace W_α of E_α, and a bounded linear operator L_α mapping E_α into itself. The system dynamics are described by linear recursion relations of the form*

$$y_{k+1} = L_\alpha\, y_k + b_{k+1}, \quad k \geq 0, \tag{2.5}$$

where $y_k \in E_\alpha$ is the pass profile on pass k and $b_{k+1} \in W_\alpha$, $k \geq 0$. Here the term $L_\alpha y_k$ represents the contribution from pass k to pass $k+1$ and b_{k+1} represents initial conditions, disturbances and control input effects on pass $k+1$.

In what follows, $||\cdot||$ is used to denote both the norm on E_α and the induced operator norm.

Processes described by (2.5) have the so-called unit memory property. In other words it is the previous pass only which explicitly contributes to the dynamics of the current pass.

Repetitive processes also exist where the current pass dynamics are a function of the independent inputs/disturbances to that pass and a finite number of previous pass profiles. A practical example of such a process occurs in certain classes of bench mining systems intended for use in more advanced 'relatively rich' mines where (typically) M lies in the range 20 to 50. A full description of the model of this example together with a complete stability characterisation for this class of processes is presented in (Rogers and Owens, 1992b). In this situation the process is termed non-unit memory and the dynamics can be represented by linear recursion relations of the form

$$y_{k+1} = \sum_{j=1}^{M} L_\alpha^j\, y_{k+1-j} + b_{k+1}, \tag{2.6}$$

where L_α^j, $1 \leq j \leq M$, is a bounded linear operator mapping E_α into itself, $y_k \in E_\alpha$, $k \geq 1 - M$, $b_{k+1} \in W_\alpha \subset E_\alpha$ and the integer M (as noted previously) is the memory length of the process.

Note that if $M = 1$, i.e. if the process is unit memory, then (2.6) reduces to (2.5) (with $L_\alpha := L_\alpha^1$), and hence (2.6) can be regarded as the natural non-unit memory generalisation of (2.5).

It should also be noted that the non-unit memory process (2.6) can be written in the 'unit memory' form (2.5) by considering the stacked vector

$$Y_{k+1} = \left[y_{k+2-M}^T, \cdots, y_{k+1}^T\right]^T \tag{2.7}$$

to be a pass profile in the Banach product space $E_\alpha^M = E_\alpha \times E_\alpha \times \cdots \times E_\alpha$ (M times). Then the non-unit memory process (2.6) can be written as

$$Y_{k+1} = \hat{L}_\alpha Y_k + \hat{b}_{k+1} \tag{2.8}$$

where

$$\hat{L}_\alpha := \begin{bmatrix} 0 & I & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & I \\ L_\alpha^M & L_\alpha^{M-1} & \cdots & L_\alpha^1 \end{bmatrix} \tag{2.9}$$

and

$$\hat{b}_{k+1} := \left[0, \cdots, b_{k+1}^T\right]^T. \tag{2.10}$$

Hence all results obtained for the unit memory abstract representation can be immediately generalised to the non-unit memory case.

2.4 Two Subclasses of Interest

The general abstract representation of section 2.3 admits all previously studied examples as special cases, but also allows those with a potentially more complex structure to be considered. To illustrate the generality of this representation, two special subclasses are introduced which are of both direct industrial and algorithmic relevance, and which form the subject of this thesis.

Section 2.4.1 introduces so-called differential linear repetitive processes, where the dynamics over a given pass evolve as a function of a continuous variable defined over the pass length, α. It should be stressed that this subclass of processes is distinct from the class of so-called continuous-discrete 2D systems reported in the literature (for example in (Kaczorek, 1996; Kaczorek, 1998)), in that, although the pass index is always an unlimited discrete variable, the continuous variable is finite in duration.

Discrete linear repetitive processes are introduced in section 2.4.2. Here the dynamics along a given pass evolve as a function of a discrete variable and, as such, can be thought of as the discrete analog to the differential processes presented in section 2.4.1.

2.4.1 Differential Processes

A differential non-unit memory linear repetitive process with constant pass length α and memory length M is described by the following state-space model over $0 \leq t \leq \alpha, \;\; k \geq 0$,

$$\begin{aligned} \dot{x}_{k+1}(t) &= A\, x_{k+1}(t) + B\, u_{k+1}(t) + \sum_{j=1}^{M} B_{j-1}\, y_{k+1-j}(t) \\ y_{k+1}(t) &= C\, x_{k+1}(t) + \sum_{j=1}^{M} D_j\, y_{k+1-j}(t). \end{aligned} \tag{2.11}$$

Here on pass k, $x_k(t)$ is the $n \times 1$ state vector, $y_k(t)$ is the $m \times 1$ output pass profile vector and $u_k(t)$ is the $l \times 1$ vector of control inputs.

To complete the process description, it is necessary to specify the 'boundary conditions', namely the state initial vector at $t = 0$ on each pass and the initial pass profiles $y_{1-j}(t)$, $1 \leq j \leq M$. The simplest possible form for these (see also below) is

$$\begin{aligned} x_{k+1}(0) &= d_{k+1}, \quad k \geq 0, \\ y_{1-j}(t) &= \hat{y}_j(t), \quad 0 \leq t \leq \alpha, \;\; 1 \leq j \leq M, \end{aligned} \tag{2.12}$$

where d_{k+1} is a constant $n \times 1$ vector, and the entries in the $m \times 1$ vector $\hat{y}_j(t)$, $1 \leq j \leq M$, are known functions of t. Note that there are M of them since the process explicitly uses information from the previous M passes.

Within this thesis, reference will often be made to the unit memory (i.e. $M = 1$) subclass of differential processes. To avoid any ambiguity later on, the unit memory model is explicitly stated here, as follows.

A differential unit memory linear repetitive process with constant pass length α is described by the following state-space model over $0 \leq t \leq \alpha,\ k \geq 0$,

$$\begin{aligned} \dot{x}_{k+1}(t) &= A\,x_{k+1}(t) + B\,u_{k+1}(t) + B_0\,y_k(t) \\ y_{k+1}(t) &= C\,x_{k+1}(t) + D_1\,y_k(t) \end{aligned} \tag{2.13}$$

with (simple) boundary conditions (see also later)

$$\begin{aligned} x_{k+1}(0) &= d_{k+1}, \quad k \geq 0, \\ y_0(t) &= \hat{y}_1(t), \quad 0 \leq t \leq \alpha. \end{aligned} \tag{2.14}$$

To write the non-unit memory differential process (2.11)-(2.12) in the form of the abstract representation $S(E_\alpha, W_\alpha, L_\alpha)$ of (2.6), first note that, over $0 \leq t \leq \alpha$, $k \geq 0$,

$$\begin{aligned} y_{k+1}(t) \;=\; & C \int_0^t e^{A(t-\tau)} \left\{ \sum_{j=1}^{M} B_{j-1}\,y_{k+1-j}(\tau) + B\,u_{k+1}(\tau) \right\} d\tau \\ & + Ce^{At}d_{k+1} + \sum_{j=1}^{M} D_j\,y_{k+1-j}(t). \end{aligned} \tag{2.15}$$

By taking the Banach space E_α to be the space $E_\alpha = L_2^m[0,\alpha] \cap L_\infty[0,\alpha]$ then L_α^j, $1 \leq j \leq M$, is defined over $0 \leq t \leq \alpha$ by

$$(L_\alpha^j\,y)(t) = C \int_0^t e^{A(t-\tau)} B_{j-1}\,y(\tau)\,d\tau + D_j\,y(t) \tag{2.16}$$

and

$$b_{k+1} = C \int_0^t e^{A(t-\tau)} B\,u_{k+1}(\tau)\,d\tau + Ce^{At}d_{k+1}. \tag{2.17}$$

Hence the differential process (2.11)-(2.12) is clearly a special case of the abstract model (2.6) already presented. Therefore all available results may be specifically interpreted for the differential subclass of processes.

In some cases, the boundary conditions of (2.14) are simply not strong enough to 'adequately' model the underlying dynamics of the process - even for preliminary simulation/control analysis. For example, the optimal control application (Roberts, 1996) requires the use of pass state initial vectors which are functions of the previous pass profile.

Other work (Owens and Rogers, 1999) has reported a general form of dynamic boundary conditions for differential unit memory linear repetitive processes. These conditions can be obtained by replacing the $x_{k+1}(0) = d_{k+1},\ k \geq 0$, term in (2.14) by

$$x_{k+1}(0) = d_{k+1} + \sum_{j=1}^{N} K_j\, y_k(t_j), \quad k \geq 0, \tag{2.18}$$

where $0 \leq t_1 < t_2 < \cdots < t_N \leq \alpha$ are N sample points along the previous pass profile and $K_j,\ 1 \leq j \leq N$, is an $n \times m$ matrix with constant entries.

Consider again the choice of $E_\alpha = L_2^m[0,\alpha] \cap L_\infty[0,\alpha]$. Then for a process defined by (2.13) and (2.18), with $D_1 \equiv 0$ for simplicity, it can be shown that, over $0 \leq t \leq \alpha$,

$$(L_\alpha\, y)(t) = C \int_0^t e^{A(t-\tau)} B_0\, y(\tau)\, d\tau + Ce^{At}\, \tilde{Y} \tag{2.19}$$

where

$$\tilde{Y} = \sum_{j=1}^{N} K_j\, y(t_j) \tag{2.20}$$

and

$$b_{k+1} = C \int_0^t e^{A(t-\tau)} B\, u_{k+1}(\tau)\, d\tau + Ce^{At} d_{k+1}. \tag{2.21}$$

The stability theory for linear repetitive processes has been developed (Owens, 1977; Rogers and Owens, 1992b) in terms of the abstract representation of definition 2.1 and necessary and sufficient conditions for the various stability properties expressed in terms of conditions on the bounded linear operator L_α. This theory is formally introduced in chapter 3 of this thesis (together with specific results for differential and discrete processes). For now it should be noted that the inclusion of the dynamic boundary condition term $\tilde{Y}$ in (2.18) effects L_α in (2.19) and hence has implications in terms of the stability of the process. This is discussed further in chapter 3.

2.4.2 Discrete Processes

The other main subclass of specific interest covered within this thesis is discrete linear repetitive processes. Such processes can be thought of as the natural discrete analog to the differential processes (2.11)-(2.12) presented in section 2.4.1 and take the following state-space form over $0 \le p \le \alpha,\ \ k \ge 0$,

$$x_{k+1}(p+1) = A\, x_{k+1}(p) + B\, u_{k+1}(p) + \sum_{j=1}^{M} B_{j-1}\, y_{k+1-j}(p)$$

$$y_{k+1}(p) = C\, x_{k+1}(p) + \sum_{j=1}^{M} D_j\, y_{k+1-j}(p). \tag{2.22}$$

Once again, on pass k, $x_k(p)$ is the $n \times 1$ state vector, $y_k(p)$ is the $m \times 1$ vector pass profile and $u_k(p)$ is the $l \times 1$ vector of control inputs, M is the memory length, and the initial conditions are taken to have the following form (see also later),

$$\begin{aligned} x_{k+1}(0) &= d_{k+1}, \quad k \ge 0, \\ y_{1-j}(p) &= \hat{y}_j(p), \quad 0 \le p \le \alpha,\ \ 1 \le j \le M, \end{aligned} \tag{2.23}$$

where d_{k+1} is a constant $n \times 1$ vector and the entries in the $m \times 1$ vector $\hat{y}_j(p)$, $1 \le j \le M$, are known functions of p.

As for the differential subclass of processes, we explicitly introduce the unit memory, i.e. $M = 1$, subclass of (2.22)-(2.23) as follows,

$$\begin{aligned} x_{k+1}(p+1) &= A\, x_{k+1}(p) + B\, u_{k+1}(p) + B_0\, y_k(p) \\ y_{k+1}(p) &= C\, x_{k+1}(p) + D_1\, y_k(p) \end{aligned} \tag{2.24}$$

with (simple) initial conditions (see also later)

$$\begin{aligned} x_{k+1}(0) &= d_{k+1}, \quad k \ge 0, \\ y_0(p) &= \hat{y}_1(p), \quad 0 \le p \le \alpha. \end{aligned} \tag{2.25}$$

In the same manner as for the differential case of the previous section, the discrete process (2.22)-(2.23) can be written in the abstract form (2.6) by considering the Banach product space $E_\alpha = \ell_2^m[0,\alpha]$ of sequences of real $m \times 1$ vectors of length α (corresponding to $p = 1,\ 2,\ \cdots,\ \alpha$ in (2.22)).

Then L_α^j, $1 \le j \le M$, in (2.6) is defined for $0 \le p \le \alpha$ by

$$(L_\alpha^j y)(p) = \begin{cases} D_j\, y(p), & p = 0 \\ \sum_{r=0}^{p-1} C\, A^{p-1-r}\, B_{j-1}\, y(r) + D_j\, y(p), & 1 \le p \le \alpha \end{cases} \tag{2.26}$$

and

$$b_{k+1} = \begin{cases} C\, A^p\, d_{k+1}, & p = 0 \\ \sum_{r=0}^{p-1} C\, A^{p-1-r}\, B\, u_{k+1}(r) + C\, A^p\, d_{k+1}, & 1 \le p \le \alpha. \end{cases} \tag{2.27}$$

As for the differential case in section 2.4.1, if the simple boundary conditions of (2.25) are not 'adequate' to model the dynamics of the process, dynamic boundary conditions may be employed. The most general set results from replacing $x_{k+1}(0) = d_{k+1}$, $k \geq 0$, in (2.25) by

$$x_{k+1}(0) = d_{k+1} + \sum_{j=1}^{N} K_j \, y_k(t_j), \quad k \geq 0, \tag{2.28}$$

where $0 \leq t_1 < t_2 < \cdots < t_N \leq \alpha$ are N sample points along the previous pass profile and K_j, $1 \leq j \leq N$, is an $n \times m$ matrix with constant entries. Note that this general form of boundary conditions are precisely those required in the nonlinear optimal control application of (Roberts, 1996).

The stability implications of the inclusion of these dynamic boundary conditions are discussed in chapter 3.

2.5 Further Examples of Repetitive Processes

The list of subclasses of linear repetitive processes with constant pass length α introduced in the previous section is by no means exhaustive. Here, other examples are presented which, although not specifically covered within the analysis in this thesis, serve to highlight the range of application of the theory developed to date, together with some areas for future development. For further details of these examples, see the cited references.

Example 2.1 (A Delay-Algebraic System) *(Rogers and Owens, 1992b) The scalar equation*

$$\begin{aligned} y_{k+1}(t) &= -k_1 \, y_{k+1}(t - X) + k_2 \, y_k(t) + k_1 \, r_{k+1}(t), \quad 0 \leq t \leq \alpha, \; k \geq 0, \\ y_{k+1}(t) &= 0, \; -X \leq t \leq 0, \; X > 0, \end{aligned} \tag{2.29}$$

where k_1 and k_2 are constants, has been shown to represent physical examples of repetitive processes such as long-wall coal cutting (see section 2.1 of this chapter) and certain metal rolling operations. Equation (2.29) has the structure of a unit memory linear repetitive process with constant pass length α, $E_\alpha = W_\alpha =$ the vector space of continuous functions on $[0, \alpha]$ satisfying $y(0) = 0$ and with norm

$$||y|| := \max_{0 \leq t \leq \alpha} |y(t)|. \tag{2.30}$$

The operator L_α in (2.5) can then be defined by expressing $y_1 = L_\alpha\, y_0$ in the form

$$\begin{aligned} y_1(t) &= -k_1\, y_1(t-X) + k_2\, y_0(t), \quad 0 \le t \le \alpha, \\ y_1(t) &= 0, \quad -X \le t \le 0. \end{aligned} \tag{2.31}$$

Example 2.2 (Matrix Recursion Relations) *(Rogers and Owens, 1992b) The discrete state vector model*

$$x_{k+1} = A\, x_k + B\, u_k, \quad x_k \in \mathbb{R}^n, \ u_k \in \mathbb{R}^l, \ k \ge 0, \tag{2.32}$$

can be regarded as a unit memory linear repetitive process with $E_\alpha = \mathbb{R}^n$, $W_\alpha =$ range of B and $b_{k+1} = B\, u_k$, $k \ge 0$.

Example 2.3 (Differential Processes with Interpass Smoothing) *(Rogers and Owens, 1992b) Interpass smoothing is a common feature of a number of industrial examples, such as long-wall coal cutting, and is, in effect, the dynamic interaction which occurs between passes and distorts the previous pass profile(s). (In the long-wall coal cutting example the source of this is the machine's weight (up to 5 tonnes) as it passes over the coal face).*

Consider, for simplicity, the unit memory differential process (2.13)-(2.14) with $D_1 \equiv 0$. Then one possible method of modelling the effects of interpass smoothing is to assume that the pass profile at point t on pass $k+1$ is a function of the state and inputs at this point on the current pass together with the complete pass profile on pass k.

For example, a candidate representation is

$$\begin{aligned} \dot{x}_{k+1}(t) &= A\, x_{k+1}(t) + B\, u_{k+1}(t) + B_0 \int_0^\alpha K(t,\tau)\, y_k(\tau) d\tau \\ y_{k+1}(t) &= C\, x_{k+1}(t), \quad 0 \le t \le \alpha, \ x_{k+1}(0) = d_{k+1}, \ k \ge 0, \end{aligned} \tag{2.33}$$

where the interpass term $B_0 \int_0^\alpha K(t,\tau)\, y_k(\tau) d\tau$ represents a 'smoothing out' of the previous pass profile in a manner governed by the properties of the kernel $K(t,\tau)$.

Note that the particular choice of

$$K(t,\tau) = \delta(t-\tau) I_m, \tag{2.34}$$

where δ denotes the Dirac delta function, reduces (2.33) to (2.13).

It is now simple to verify that (2.33) is a linear repetitive process in $E_\alpha = L_2^m[0,\alpha] \cap L_\infty[0,\alpha]$ *with*

$$(L_\alpha y)(t) = C\int_0^t e^{A(t-\tau)}B_0\int_0^\alpha K(\tau,t')\,y(t')dt'\,d\tau, \quad 0 \le t \le \alpha, \tag{2.35}$$

and

$$b_{k+1} = C\int_0^t e^{A(t-\tau)}B\,u_{k+1}(\tau)d\tau + Ce^{At}d_{k+1}, \quad 0 \le t \le \alpha,\ k \ge 0. \tag{2.36}$$

Example 2.4 (Delay-Differential Systems) *(Rogers and Owens, 1995b) A class of delay-differential system in* $\mathbb{R}^n$ *has the state-space form*

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B_0\,x(t-\alpha) + B\,u(t),\ t \ge 0, \\ x(t-\alpha) &:= x_0(t),\ 0 < t \le \alpha \end{aligned} \tag{2.37}$$

where A, B_0 *and* B *are constant* $n \times n$, $n \times n$ *and* $n \times l$ *matrices respectively.*

If the delay α *is interpreted as the pass length, then it is clear that these systems have strong structural similarities with linear repetitive processes described by a set of recursive differential equations. This can be seen by introducing the change of variables*

$$\begin{aligned} u_{k+1}(t) &:= u(k\alpha + t) \\ x_k(t) &:= x((k-1)\alpha + t),\ 0 \le t \le \alpha,\ k \ge 0, \end{aligned} \tag{2.38}$$

and denoting the pass profiles as

$$y_k(t) = x_k(t),\ k \ge 0. \tag{2.39}$$

Then (2.37) can be written as

$$\begin{aligned} \dot{x}_{k+1}(t) &= A\,x_{k+1}(t) + B\,u_{k+1}(t) + B_0\,y_k(t) \\ y_{k+1}(t) &= x_{k+1}(t),\ 0 \le t \le \alpha,\ k \ge 0, \end{aligned} \tag{2.40}$$

with boundary conditions

$$x_{k+1}(0) = x_k(\alpha), \quad k \ge 0, \tag{2.41}$$

i.e. the initial value of the state vector on pass $k+1$ *matches the final value of the state for the previous pass* k *- clearly a necessary condition for the continuity of the* *process. So the linear delay-differential system (2.37) can be interpreted as having the structure of a subclass of linear differential repetitive processes with interaction between pass boundary conditions.*

This fact clearly has implications regarding the interplay of concepts/theory between the areas of delay-differential systems and linear repetitive processes. Results obtained from the exploitation of these structural links can be found in the cited reference.

Example 2.5 (Iterative Learning Control) *eg. (Amann et al., 1996; Amann, 1996; Amann et al., 1998; Owens et al., 2000) The area of iterative learning control considers systems which repeatedly perform the same task with a view to sequentially improving accuracy. Original interest in this area arose as the result of robot operations on an assembly line where the robot is required to repeat the same task many times. The specified task can be taken as the requirement to track an external reference vector,* $r(t)$ *say, over a specified time interval* $0 \leq t \leq T$. *The objective is then to use the repetitive nature of the process to improve accuracy by changing the control input from trial to trial.*

One approach in the literature to this type of problem has been to view the system as having a 2D structure. Clearly, due to the finite pass length repeatable nature of the systems, iterative learning control has clear structural links with the area of linear repetitive processes. For further details of these links, see the cited references.

Example 2.6 (Solution of Nonlinear Dynamic Optimal Control Problems via the Maximum Principle) *(Roberts, 1994a; Roberts, 1996; Roberts, 2000) The cited references show that how, due to the existence of mixed boundary conditions, the solution of nonlinear dynamic optimal control problems via the maximum principle can often require an algorithm which iteratively updates a trial solution. In (Roberts, 1994a), it is shown that the structure of a discrete linear repetitive process arises in the analysis of the local convergence and stability properties of these iterative algorithms for solving (classes of) nonlinear dynamic optimal control problems. Since a trial solution is updated from iteration to iteration, the algorithm has information propagation in two independent directions, namely along the time horizon of the dynamic response and from iteration to iteration, and hence such algorithms have an inherent 2D/repetitive process structure. The resulting models have state initial vectors which are function of the previous pass profile, as in (2.28).*

Example 2.7 (Hybrid Systems) *(Franke, 1998) The cited reference concerns a 2D approach to so-called event driven hybrid systems. In simple terms, such a system has mixed characteristics in that the occurrence of some 'event' causes the process to switch to some other dynamics. As such, the system exhibits both continuous time and discrete event dynamics, and hence has some basic similarities with certain classes of iterative learning control schemes and therefore linear repetitive processes. The details of this (relatively) recent link with repetitive process theory can be found in the cited reference.*

2.6 A 2D Systems Approach

The dynamics of discrete linear repetitive processes clearly share some basic characteristics with 2D discrete linear systems recursive in the positive quadrant, i.e. systems which propagate information in two separate directions. Hence one possible approach to the stability analysis and development of meaningful control policies for these processes is to treat them as 2D discrete linear systems recursive over $\mathbb{Z}_+^2 = \{(i,j) : i,j \geq 0,\ i,j \in \mathbb{Z}_+\}$ and exploit links with the (relatively) well reported field of 2D linear systems theory. A key difference which should be stressed, however, is the fact that the pass length of a repetitive process (which corresponds to one direction of information propagation) is always finite by definition.

Within this section, Fornasini-Marchesini and Roesser state-space model interpretations of the dynamics of discrete linear repetitive processes described by (2.22)-(2.23) are presented. These models then form the basis for a discussion on how an equivalence can be developed between standard 2D systems stability concepts and the associated stability theory for this subclass of processes - see chapter 3 for further details of these concepts.

Motivated by research in the field of image enhancement and filtering, the following state-space model was introduced in (Roesser, 1975) for systems recursive in the positive quadrant (omitting the output equation which has no role in this work),

$$\begin{aligned} x_h(i+1,j) &= A_1\, x_h(i,j) + A_2\, x_v(i,j) + B_1\, u(i,j) \\ x_v(i,j+1) &= A_3\, x_h(i,j) + A_4\, x_v(i,j) + B_2\, u(i,j). \end{aligned} \tag{2.42}$$

Here i and j are positive integer valued horizontal and vertical coordinates, x_h is the $n \times 1$ vector of horizontally transmitted information, x_v is the $m \times 1$ vector of vertically transmitted information and u is the $l \times 1$ vector of control inputs. Note here that the local state $x(i, j)$ has been divided into horizontal and vertical components, each of which is propagated by a first order difference equation.

In Fornasini-Marchesini model structures (Fornasini and Marchesini, 1978), the state vector is not split into horizontal and vertical components and the output equation is once again not required. With $z(i, j)$ denoting the (appropriately dimensioned) state vector at (i, j), $i \geq 0$, $j \geq 0$, the general model of this type has the structure

$$\begin{aligned} z(i+1, j+1) &= A_5\, z(i+1, j) + A_6\, z(i, j+1) + A_7\, z(i, j) \\ &\quad + B_3\, u(i+1, j) + B_4\, u(i, j+1) \end{aligned} \tag{2.43}$$

where, as in (2.42), u is the appropriately dimensioned vector of control inputs.

In (Galkowski et al., 1995) and subsequently in (Galkowski et al., 1999b) it is shown that the dynamics of discrete linear repetitive processes can be represented by a dynamically equivalent singular Fornasini-Marchesini type model. The starting point for this model is the so-called augmented state vector for discrete linear repetitive processes, defined for the state-space model (2.24) as

$$Z_k(p) := \left[x_k^T(p),\; y_k^T(p)\right]^T. \tag{2.44}$$

It then follows immediately that the dynamics of (2.24) can be written in the form

$$E\, Z_{k+1}(p+1) = A_8\, Z_{k+1}(p) + A_9\, Z_k(p) + B_5\, u_{k+1}(p) \tag{2.45}$$

where

$$E = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix},\; A_8 = \begin{bmatrix} A & 0 \\ C & -I_m \end{bmatrix},\; A_9 = \begin{bmatrix} 0 & B_0 \\ 0 & D_1 \end{bmatrix} \text{ and } B_5 = \begin{bmatrix} B \\ 0 \end{bmatrix}. \tag{2.46}$$

This is a singular version of the Fornasini-Marchesini model of (2.43) with $A_6 = 0$, $B_4 = 0$. Other work (Galkowski et al., 1999b) has concluded that this singularity is not an intrinsic feature (in a well defined sense) of discrete linear repetitive processes, since a key property of the state-space model (2.24) is that it is nonsingular (also termed standard or regular). Further Fornasini-Marchesini type models of these processes are introduced in (Galkowski et al., 1999a) and (Galkowski et al., 1999b) which have been constructed via the development of a 'transformation theory' for nonsingular Fornasini-Marchesini and Roesser models from their singular counterparts. For a detailed treatment, refer to the papers cited.

In (Rocha et al., 1996) it has been argued that the discrete subclass of non-unit memory linear repetitive processes with state-space form (2.22) has a Roesser representation. In particular, the process has a Roesser-type structure where

(i) x_h - current pass state vector x_k (horizontally transmitted information),

(ii) x_v - current pass output vector y_k (vertically transmitted information).

To write (2.22)-(2.23) in Roesser form, introduce the following notation (as in (Rocha et al., 1996)),

$$z_k(p) := [y_{k-M}^T(p), \cdots, y_{k-1}^T(p)]^T \in \mathbb{R}^N, \quad N = mM. \tag{2.47}$$

Then it follows that (2.22)-(2.23) can be written

$$\begin{bmatrix} x_k(p+1) \\ z_{k+1}(p) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_k(p) \\ z_k(p) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u_k(p) \tag{2.48}$$

where

$$A_{11} = A, \quad B_1 = B, \quad A_{12} = [B_{M-1}, \cdots, B_0], \quad A_{21} = [0, \cdots, 0, C^T]^T, \quad \text{and}$$
$$A_{22} = \begin{bmatrix} 0 & I_m & & 0 \\ \vdots & & \ddots & \ddots \\ 0 & \cdots & 0 & I_m \\ D_M & \cdots & D_2 & D_1 \end{bmatrix}. \tag{2.49}$$

In particular, unit memory discrete linear repetitive processes (2.24)-(2.25) can be written

$$\begin{bmatrix} x_k(p+1) \\ z_{k+1}(p) \end{bmatrix} = \begin{bmatrix} A & B_0 \\ C & D_1 \end{bmatrix} \begin{bmatrix} x_k(p) \\ z_k(p) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k(p), \tag{2.50}$$

from where the so-called augmented plant matrix for this process can then be defined as

$$\Phi := \begin{bmatrix} A & B_0 \\ C & D_1 \end{bmatrix}. \tag{2.51}$$

In addition, if we define the augmented state vector as

$$X_k(p) := [x_k(p)^T, z_k(p)^T]^T \tag{2.52}$$

and define

$$X_k^{ij}(p) := [x_k(p+i)^T,\ z_{k+j}(p)^T]^T, \tag{2.53}$$

then the state-space equation (2.50) now reads

$$X_k^{11}(p) = \Phi\, X_k(p) + \overline{B}\, u_k(p), \quad k \geq 0,\ \ 0 \leq p \leq \alpha, \tag{2.54}$$

where

$$\overline{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}. \tag{2.55}$$

The 'equivalence' between Roesser / Fornasini-Marchesini models and certain classes of discrete linear repetitive processes has enabled the interchanging of stability tests between the two areas. This link, however, has not been useful in addressing currently open systems theoretic questions (such as what (if anything) is meant by reachability / controllability). Hence (based on the preliminary results in (Galkowski et al., 1995)) new nonsingular 2D linear systems representations of the dynamics of (2.24) have been introduced in (Galkowski et al., 1999b) which are then used with the singular Fornasini-Marchesini state-space models discussed previously in the characterisation of local reachability / controllability properties for these processes.

Introduce the following transformations into the discrete unit memory subclass of processes with state-space form (2.24),

$$\begin{aligned} \eta_k(p) &:= x_k(p+1) - A\, x_k(p) - B\, u_k(p) \\ \mu_k(p) &:= y_k(p) - C\, x_k(p). \end{aligned} \tag{2.56}$$

Then the following representation can be obtained

$$\begin{bmatrix} x_k(p+1) \\ \mu_{k+1}(p) \\ \eta_{k+1}(p) \end{bmatrix} = \begin{bmatrix} A & 0 & I \\ D_1C & D_1 & 0 \\ B_0C & B_0 & 0 \end{bmatrix} \begin{bmatrix} x_k(p) \\ \mu_k(p) \\ \eta_k(p) \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} u_k(p) \tag{2.57}$$

which is a standard (nonsingular) Roesser model whose state dimension is $2n+m$ as opposed to $2(n+m)$ for the singular Roesser model which can be developed from the singular Fornasini-Marchesini model of (2.45)-(2.46).

As a special case of this, consider now the case when D_1 is nonsingular and define the so-called restricted state vector for (2.24) as

$$\hat{z}_k(p) = [x_k^T(p), \mu_k^T(p)]^T. \tag{2.58}$$

Hence, here we have

$$\eta_k(p) = B_0 D_1^{-1} \mu_k(p) \tag{2.59}$$

and the following restricted 2D state-space model of Roesser type is obtained for the dynamics of (2.24),

$$\begin{bmatrix} x_k(p+1) \\ \mu_{k+1}(p) \end{bmatrix} = \begin{bmatrix} A & B_0 D_1^{-1} \\ D_1 C & D_1 \end{bmatrix} \begin{bmatrix} x_k(p) \\ \mu_k(p) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k(p). \tag{2.60}$$

This section has introduced some of the models from 'classical' 2D systems theory which are available for the subclass of discrete linear repetitive processes with constant pass length α which are employed throughout this thesis. The stability theory for these models is presented in chapter 3.

2.7 A 2D Transfer-Function Approach

A major basis for the analysis of 1D linear systems theory is the transfer-function representation. It is expected that such an approach may play a similar role for 2D systems, and in particular for linear repetitive processes.

(Rogers and Owens, 1989a) has developed a 2D transfer-function matrix description for differential processes using two separate transform parameters. Prior to the introduction of these transforms, some preliminary results and observations are required. Firstly the processes must be well posed in the sense that sequences of inputs are mapped to sequences of outputs. In addition, they must exhibit multipass causality. As an illustration of this last point, consider the differential process (2.11)-(2.12). Then, in this case, multipass causality means that the output $y_k(t)$ at any time t on pass k does not depend on information from any of the following sets,

$$\begin{aligned}
X &= \{x_k(\tau) \,:\, t < \tau \leq \alpha\} \cup \{x_l(t) \,:\, 0 \leq t \leq \alpha,\, l > k\} \\
D &= \{d_l \,:\, l > k\} \\
U &= \{u_k(\tau) \,:\, t < \tau \leq \alpha\} \cup \{u_l(t) \,:\, 0 \leq t \leq \alpha,\, l > k\} \\
Y &= \{y_k(\tau) \,:\, t < \tau \leq \alpha\} \cup \{y_l(t) \,:\, 0 \leq t \leq \alpha,\, l > k\}.
\end{aligned} \tag{2.61}$$

This set of causal information is illustrated in figure 2.5. (Note that this discussion of multipass causality extends in a natural manner to discrete processes described by (2.22)-(2.23)).

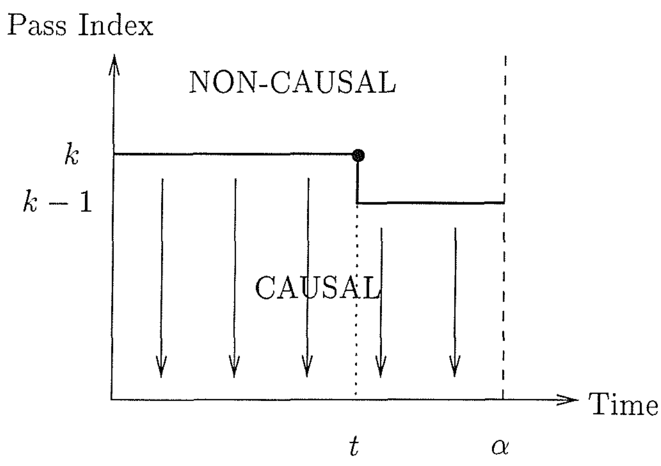

Figure 2.5: Set of Causal Information at Time t on Pass k

Under these conditions, a formal definition of a 2D transfer-function description for linear repetitive processes described by (in this case) a differential model can be given. In order to do this, some formal definitions of the transforms are first introduced. These have been included within the appendix section A.2.

Given these initial results, and proceeding as in (Rogers and Owens, 1989a), the 2D transfer-function of the non-unit memory differential process (2.11)-(2.12) can be written as

$$Y(s,z) = G(s,z)\,U(s,z) \tag{2.62}$$

where $G(s,z)$ is the $m \times l$ 2D transfer-function matrix given by

$$G(s,z) = (I_m - D(z))^{-1}C\{sI_n - A - B(z)(I_m - D(z))^{-1}C\}^{-1}B \tag{2.63}$$

and

$$B(z) = \sum_{j=1}^{M} B_{j-1}z^{-j}, \quad D(z) = \sum_{j=1}^{M} D_j z^{-j}. \tag{2.64}$$

Note that, in the differential model (2.11)-(2.12), two parameters are required to specify a variable (namely the pass index k and the time or distance along the pass t) and hence this is the basic reason why the transfer-function matrix for the process is 2D in nature. In (2.62)-(2.64) the Laplace transform variable s represents the along the pass dynamics whereas the second parameter z^{-1} is a 'backward' shift operator which takes account of the interaction between successive pass profiles.

After routine algebraic manipulation, (2.62)-(2.64) can be rearranged to give clearer insight into the physical structure of the processes, as follows,

$$Y(s,z) = G_0(s)\,U(s,z) + \sum_{j=1}^{M} G_j(s) z^{-j}\,Y(s,z) \tag{2.65}$$

where

$$\begin{aligned} G_0(s) &= C(sI_n - A)^{-1}B, \quad \text{and} \\ G_j(s) &= C(sI_n - A)^{-1}B_{j-1} + D_j, \quad 1 \le j \le M. \end{aligned} \tag{2.66}$$

Now, consider the two elements of this representation separately.

2.7.1 Derived Conventional Linear System

Firstly consider the subsystem described by

$$\begin{aligned} Y(s) &= G_0(s)\,U(s) \\ &= C(sI_n - A)^{-1}B\,U(s). \end{aligned} \tag{2.67}$$

This is just the transfer-function representation of a 1D linear system and represents the contribution of the current pass input vector acting alone to the current pass profile. To illustrate this, suppose that, in (2.11),

(i) the previous pass terms are deleted, i.e. $B_{j-1} \equiv 0,\ D_j \equiv 0,\ 1 \le j \le M$,

(ii) the pass subscript $k+1$ is dropped, and

(iii) the concept of a pass length is irrelevant.

Then (2.11) reduces to

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t), \quad x(0) = d, \end{aligned} \tag{2.68}$$

which is just the well known state-space model from conventional differential linear systems theory. With this in mind, (2.67) is termed the derived conventional linear system of (2.11)-(2.12), denoted $L_D(A,B,C)$. It then follows that, under the conditions (i) to (iii) above, $G(s,z)$ in (2.62) reduces to $G_0(s)$ which is just the transfer-function matrix of $L_D(A,B,C)$.

2.7.2 Associated Conventional Linear System

Now consider the set of subsystems described by $G_j(s)$, $1 \leq j \leq M$, of (2.65). It can be shown that $G_j(s)$ is the transfer-function of the j^{th} so-called associated conventional linear system, $L_A^j(A, B_{j-1}, C, D_j)$, described by the state-space model

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B_{j-1}\,y^{1-j}(t) \\ W^j(t) &= C\,x(t) + D_j\,y^{1-j}(t), \quad x(0) = 0. \end{aligned} \tag{2.69}$$

Now each subsystem (2.69) is a 1D linear system and in fact (2.69) can be regarded as describing the contribution of pass $k+1-j$ to the current one. This can be seen by restricting t to $[0, \alpha]$ and by setting $y^{1-j}(t)$ equal to the $(k+1-j)^{th}$ pass profile. Note that each of the j^{th} associated conventional linear systems can be written in the transfer-function matrix form

$$W^j(s) = G_j(s)\,Y^{1-j}(s) \tag{2.70}$$

with

$$G_j(s) = C(sI_n - A)^{-1}B_{j-1} + D_j. \tag{2.71}$$

2.7.3 Physical Interpretation

Returning now to the expansion (2.65) of $Y(s,z)$. Firstly note that $G(s,z)$ can be written

$$G(s,z) = \left(I_m - \sum_{j=1}^{M} G_j(s) z^{-j} \right)^{-1} G_0(s). \tag{2.72}$$

Then, what we have is a representation of a linear repetitive process which gives us physical insight into the structure of the system. Observation of figure 2.6 shows that $G_0(s)$ (the transfer-function matrix of the derived conventional linear system) has the effect of a dynamic pre-compensator with the $G_j(s)$ terms, $1 \leq j \leq M$, (the transfer-function matrices of the associated conventional linear systems) as feedback elements representing the crucial interaction terms.

It should be noted that this block diagram is not unique. The point is that it clearly highlights the fact that the process dynamics are constructed from the interconnection of subsystems whose dynamics are characterised by 1D linear systems

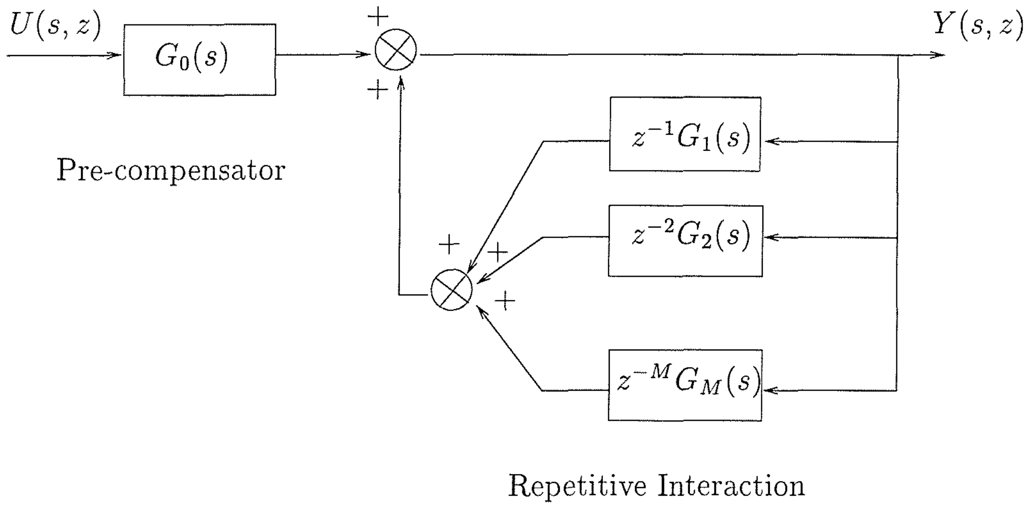

Figure 2.6: Block Diagram Interpretation of a Linear Repetitive Process

transfer-function matrices. With this motivation it appears that 1D linear systems methods may play a role in the design of control schemes for linear repetitive processes. This is discussed further in chapter 6.

One other point should be discussed prior to ending this section on 2D transfer-function matrices. In the unit memory, $M = 1$, case, $G_1(s)$ denotes the contribution of pass profile y_k to y_{k+1} and as such is termed the interpass transfer-function matrix. For the more general $M > 1$ non-unit memory case, let $Y(s)$ denote the combined effects of the previous M passes. Then

$$Y(s) := \sum_{j=1}^{M} W^j(s) = \sum_{j=1}^{M} G_j(s)\, Y^{1-j}(s). \tag{2.73}$$

This expression can be interpreted in 'unit memory' form by stacking up the $Y(s)$ terms and writing it as

$$\begin{bmatrix} Y^{2-M}(s) \\ \vdots \\ Y(s) \end{bmatrix} = G(s) \begin{bmatrix} Y^{1-M}(s) \\ \vdots \\ Y^0(s) \end{bmatrix}, \tag{2.74}$$

where

$$G(s) = \begin{bmatrix} 0 & I_m & & 0 \\ \vdots & & \ddots & \\ 0 & & & I_m \\ G_M(s) & & G_2(s) & G_1(s) \end{bmatrix} \tag{2.75}$$

is an $mM \times mM$ block companion matrix, and which, since the process has now been written in unit memory form, is termed the interpass transfer-function matrix.

This interpass transfer-function matrix will be referred to in subsequent chapters, together with the $mM \times mM$ constant coefficient block companion matrix defined by

$$D = \lim_{|s| \to \infty} G(s) \tag{2.76}$$

i.e.

$$D = \begin{bmatrix} 0 & I_m & & 0 \\ \vdots & \ddots & \ddots & \\ 0 & & 0 & I_m \\ D_M & & D_2 & D_1 \end{bmatrix}. \tag{2.77}$$

2.7.4 Discrete Processes

Equivalent discrete versions of the transfer-function matrix concepts introduced in this section are outlined in (Rogers and Owens, 1992b). Instead of the s/z transforms used for the differential processes, a z_1/z or 'double z' transform is used for discrete processes. Since the transfer-function matrices and results presented generalise in a natural manner, the details here are omitted.

2.8 Summary

This chapter has introduced some of the models available (and those which are used in the subsequent analysis within this thesis) for representing linear repetitive processes with a constant pass length α. The associated stability theory is presented in the following chapter.

A rigorous mathematical representation of linear repetitive processes with constant pass length α has been given. It has been illustrated how this abstract representation admits analysis of processes with certain special structures, with emphasis on the differential and discrete subclasses. These constitute the two main subclasses where research has been focussed to date, and are of both direct industrial and algorithmic interest. Further examples of areas where adopting a repetitive process approach has certain benefits over alternatives has been presented in section 2.5 which serves to illustrate areas where future research effort may be directed.

Section 2.6 has established links between certain subclasses of linear repetitive processes and well known models from 2D linear systems theory. Stability results obtained from exploiting the structural links between such processes and 2D linear systems can be found in (Rocha et al., 1996), amongst others, and are summarised in section 3.8. It should be stressed that not all linear repetitive processes have an associated 2D Roesser/Fornasini-Marchesini form. In particular, certain 'non-standard' forms, such as processes with interpass smoothing effects, have no 2D Roesser/Fornasini-Marchesini representation. Hence the application areas of this associated stability analysis is limited to those processes possessing certain special structures.

Finally, within section 2.7 a 2D transfer-function representation of a linear repetitive process has been presented. It is anticipated that such an approach, as in 1D linear systems theory, will play a significant role in the analysis and design of control schemes for these processes, in addition to providing physical insight into their structure.

Chapter 3

Stability

3.1 Introduction

A key property of any system, whether it represents a physical process or is of a purely algorithmic nature, is that of stability. Using techniques from functional analysis, a rigorous stability theory has been developed by Rogers and Owens for the abstract representation of linear repetitive processes introduced in chapter 2 (see, for example, (Rogers and Owens, 1992b)). This theory, presented here in section 3.2, demonstrates that two distinct concepts of stability exist, namely asymptotic stability and stability along the pass. This is not surprising since, as already illustrated, a linear repetitive process is governed by two independent variables, i.e. in the along the pass and the pass to pass directions. Within this chapter, it is highlighted that asymptotic stability is a relatively weak definition of stability and that in general (with a few notable exceptions) it is the stronger concept of stability along the pass which is required for acceptable systems performance.

In sections 2.4.1 and 2.4.2 of chapter 2, differential and discrete classes of linear repetitive processes were introduced which were shown to be special cases of the abstract representation of definition 2.1. Within sections 3.3 and 3.4 of this chapter, the stability theory for the abstract representation is specifically interpreted for these processes. Here it is shown how the determination of the boundary conditions (termed 'simple' or 'dynamic') is of vital importance. In fact, the misclassification of a process with dynamic boundary conditions as having simple boundary conditions could result in an unstable process being determined as stable.

Section 3.5 discusses simulation-based stability tests which assume that suitably well behaved plant step response data is available or can be obtained from simulation studies. In this and the subsequent sections it is demonstrated how, for the discrete subclass of processes, the standard test for stability along the pass involves calculating the eigenvalues of an $mM \times mM$ transfer-function matrix for all points on the unit circle in the complex plane, which can be computationally intensive even in the simplest of cases. With this motivation, new stability tests have been developed for this subclass which replace these complex computational conditions with a one-off computation of the eigenvalues of a matrix with constant entries. The resulting conditions are sufficient but not necessary, but serve to act as a simple 'acceptance criterion'. In addition, this conservativeness is offset by the availability of performance measures for a given pass, supplied by the new stability conditions at no extra computational cost. The theory in these sections is novel, and provides the basis of the paper (Benton et al., 1998b).

Within section 2.6 it was illustrated how certain classes of linear repetitive processes can assume a 'classical' 2D systems structure. In section 3.8 links are drawn between the stability along the pass of these discrete linear repetitive processes and the BIBO stability of systems described by the Roesser / Fornasini-Marchesini 2D state-space models.

The chapter concludes by introducing a Volterra operator based approach to the stability analysis of discrete linear repetitive processes.

3.2 Stability Theory for the General Abstract Representation

Within this section, the rigorous stability theory developed by Rogers and Owens for the abstract representation of a linear repetitive process with constant pass length α is presented, introducing the two separate concepts of asymptotic stability and stability along the pass.

3.2.1 Asymptotic Stability

In chapter 2 it was illustrated (via figure 2.3) that the unique control problem associated with linear repetitive processes is that the output sequence of pass profiles $\{y_k\}_{k\geq1}$ can contain oscillations which become unbounded from pass to pass. With this motivation, the natural intuitive definition of asymptotic stability is to demand that, given any initial profile y_0 and any disturbance sequence $\{b_k\}_{k\geq1}$ which 'settles down' to a steady disturbance b_∞ as $k \to +\infty$, after a 'sufficiently large' number of passes the output sequence of pass profiles 'settles down' to a steady profile y_∞ as $k \to +\infty$. The phrases in quotes are, of course, subject to interpretation and depend upon the application under consideration. This idea is illustrated in figure 3.1 (further discussion of the so-called limit profile y_∞ is given in section 3.2.2).

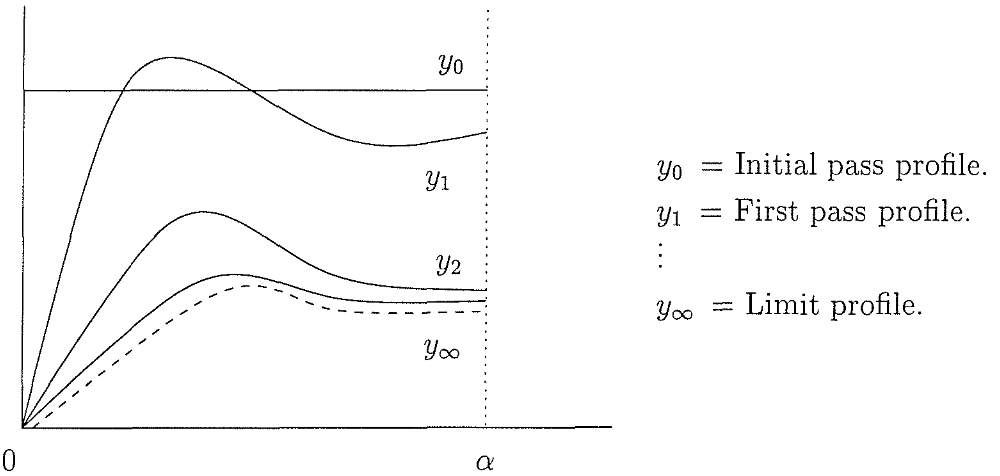

Figure 3.1: Asymptotic Stability of a Linear Repetitive Process

In practical applications, the effect of modelling errors and uncertainties will produce uncertainty in the structure of L_α in the abstract repetitive process model (2.5) and hence the following definition of asymptotic stability is used since this definition ensures that the 'set of all stable systems' is open (in a well defined sense) in the class of all linear repetitive processes. Note that here we only consider the unit memory case since all results obtained generalise in a natural manner to the case when $M > 1$. Also note that the results given here (i.e. in section 3.2) plus relevant proofs can be found in chapter 3 of (Rogers and Owens, 1992b).

Definition 3.1 (Abstract Representation - Asymptotic Stability) *A linear repetitive process $S(E_\alpha, W_\alpha, L_\alpha)$ of constant pass length $\alpha > 0$ is said to be asymp-*

totically stable if there exists a real scalar $\delta > 0$ such that, given any initial profile y_0 and any strongly convergent disturbance sequence $\{b_k\}_{k\geq 1} \subset W_\alpha$, the sequence $\{y_k\}_{k\geq 1}$ generated by the perturbed process

$$y_{k+1} = (L_\alpha + \gamma)y_k + b_{k+1}, \quad k \geq 0, \tag{3.1}$$

converges strongly to a so-called limit profile $y_\infty \in E_\alpha$ whenever $||\gamma|| \leq \delta$, where $||\cdot||$ denotes the norm on E_α.

Asymptotic stability is then the requirement that bounded disturbance (or forcing) sequences generate (in some well defined sense) bounded sequences of pass profiles. Note that this property has been augmented by the practically motivated requirement that asymptotic stability is retained in the presence of small modelling errors or simulation approximations.

Now consider the general abstract representation (2.5) of a linear repetitive process under asymptotic stability. In addition, consider the case of $b_k = b_\infty \equiv 0 \ \forall \ k \geq 0$ in (3.1), i.e. an absence of disturbances, which causes the set of pass profiles $\{y_k\}_{k\geq 1}$ to be strongly convergent to zero. Then taking

$$\gamma = \frac{L_\alpha \delta}{||L_\alpha||} \tag{3.2}$$

gives $||\gamma|| = \delta$. Since $y_k = (L_\alpha + \gamma)^k y_0$ is strongly convergent (by definition) it is bounded $\forall \ y_0 \in E_\alpha$.

Application of the Banach Steinhaus (Uniform Boundedness) Theorem A.1 now says $\exists$ real $M_\alpha > 0$ such that

$$||(L_\alpha + \gamma)^k|| \leq M_\alpha, \quad k \geq 0, \tag{3.3}$$

or equivalently

$$\left(1 + \frac{\delta}{||L_\alpha||}\right)^k ||L_\alpha^k|| \leq M_\alpha, \quad k \geq 0. \tag{3.4}$$

Defining

$$\lambda_\alpha = \left(1 + \frac{\delta}{||L_\alpha||}\right)^{-1} < 1, \tag{3.5}$$

we now consider the case of the 'real system', i.e. the system which is not subject to any small perturbations/modelling uncertainties, and $\therefore \ \gamma = 0$. Then

$$||y_k|| = ||L_\alpha^k y_0|| \leq ||L_\alpha^k|| \, ||y_0|| \leq M_\alpha \lambda^k ||y_0||. \tag{3.6}$$

Hence, in the absence of disturbances, since $0 < \lambda_\alpha < 1$, the output sequence of pass profiles $\{y_k\}_{k\geq 1}$ converges strongly to zero for all initial profiles, i.e. asymptotic stability is the requirement that the effects of the initial pass profile are rapidly attenuated.

In order to introduce a formal asymptotic stability result for the abstract representation some definitions are needed. Within the appendix section A.1 formal definitions of spectral values, spectrum and spectral radius of the bounded linear operator L_α are given.

Given these definitions, the following result now characterises asymptotic stability for the general abstract representation of (2.5),

Theorem 3.1 (Abstract Representation - Asymptotic Stability) *A linear repetitive process $S(E_\alpha, W_\alpha, L_\alpha)$ of constant finite pass length $\alpha > 0$ is asymptotically stable if, and only if,*

$$r(L_\alpha) < 1 \tag{3.7}$$

where $r(\cdot)$ denotes the spectral radius of its argument (throughout this thesis).

Note that if E_α is finite dimensional, this result is equivalent to the requirement that all eigenvalues of L_α lie in the open unit disc in the complex plane.

The condition of theorem 3.1 is not surprising since a superficial consideration of the abstract representation (2.5) indicates a similarity between the structure of $S(E_\alpha, W_\alpha, L_\alpha)$ and the well known linear time-invariant discrete time system, and hence in this sense it is to be expected that the stability of the process depends explicitly on the spectrum of L_α.

3.2.2 Limit Profile

Now this result gives little or no information regarding the transient behaviour of the process. For this type of information we look towards the so-called limit profile of the process. This is the 'steady state' profile, under asymptotic stability, which the output sequence of pass profiles tends towards after a sufficiently large number of passes, and is represented by y_∞ in figure 3.1. Formally, the limit profile can be defined as follows,

Definition 3.2 (Abstract Representation - Limit Profile) *Suppose that the linear repetitive process $S(E_\alpha, W_\alpha, L_\alpha)$ of definition 2.1 is asymptotically stable and that $\{b_k\}_{k\geq 1}$ is a disturbance sequence that converges strongly to a disturbance b_∞. Then the strong limit*

$$y_\infty := \lim_{k\to+\infty} y_k \tag{3.8}$$

is termed the limit profile corresponding to $\{b_k\}_{k\geq 1}$.

This definition implies that, under asymptotic stability, the output sequence of pass profiles $\{y_k\}_{k\geq 1}$ converges strongly to the limit profile y_∞ (in the sense of the norm on E_α), i.e.

$$\lim_{k\to+\infty} ||y_k - y_\infty|| = 0. \tag{3.9}$$

Corollary 3.1 *Suppose that the conditions of definition 3.2 hold. Then the corresponding limit profile is the unique solution of the linear equation*

$$y_\infty = L_\alpha\, y_\infty + b_\infty, \tag{3.10}$$

where

$$b_\infty := \lim_{k\to+\infty} b_k. \tag{3.11}$$

Clearly y_∞ is independent of the initial pass profile y_0 (as we would expect due to asymptotic stability) and independent of the direction of approach to b_∞, and by rearranging (3.10) as follows

$$y_\infty = (I_m - L_\alpha)^{-1} b_\infty, \tag{3.12}$$

we see that (3.10) has a unique solution due to the asymptotic stability condition (3.7). Note that equation (3.10) can be formally obtained from the asymptotic stability definition 3.1 be setting $\gamma = 0$ in (3.1) and replacing each term by its strong limit.

The following result shows that performance of an asymptotically stable process can be partially characterised by real scalars $M_\alpha > 0$ and $0 < \lambda_\alpha < 1$ describing the rate of approach of the output sequence of pass profiles $\{y_k\}_{k\geq 1}$ to the limit profile,

Theorem 3.2 (Abstract Representation - Asymptotic Stability) *Suppose that the linear repetitive process $S(E_\alpha, W_\alpha, L_\alpha)$ of definition 2.1 with constant pass length $\alpha > 0$ is asymptotically stable. Further, let this process be subjected to a constant disturbance sequence $b_{k+1} \equiv b_\infty$, $k \geq 0$, which generates the limit profile y_∞. Then there exists real scalars $M_\alpha > 0$ and $0 < \lambda_\alpha < 1$ such that*

$$||y_k - y_\infty|| \leq M_\alpha \lambda_\alpha^k \left\{ ||y_0|| + \frac{||b_\infty||}{1 - \lambda_\alpha} \right\}, \quad k \geq 0. \tag{3.13}$$

Note that in effect this result states that the output sequence $\{y_k\}_{k \geq 1}$ approaches the limit profile at a geometric rate governed by λ_α. For a further discussion/analysis of these so-called performance bounds see section 3.7.

3.2.3 Stability along the Pass

Asymptotic stability of $S(E_\alpha, W_\alpha, L_\alpha)$ guarantees the process has a limit profile. However it is not guaranteed that this limit profile has acceptable dynamic characteristics. A simple example which illustrates this key point for the differential subclass is given in section 3.3.2. Hence the natural definition of stability along the pass is to demand that the limit profile is stable in the standard, i.e. 1D, sense as the pass length becomes 'large', i.e. as $\alpha \to +\infty$. Now this intuitive definition of stability along the pass is not applicable if the limit profile is not a 1D linear system state-space model. Therefore the definition of stability along the pass is made in terms of the rate of approach of the output sequence of pass profiles to the limit profile.

This characterisation requires the introduction of the concept of a so-called extended linear repetitive process. This consists of a collection of models obtained by allowing the pass length α take values greater than some nominal value α_0, and can be formally defined as follows,

Definition 3.3 (Extended Linear Repetitive Process) *A collection of models of $S(E_\alpha, W_\alpha, L_\alpha)$ with pass lengths in the range $\alpha \geq \alpha_0$ is termed an extended linear repetitive process and is denoted $S(E_\alpha, W_\alpha, L_\alpha)_{\alpha \geq \alpha_0}$.*

Stability along the pass can then be defined by considering the rate of approach of the output sequence $\{y_k\}_{k \geq 1}$ to the limit profile as follows,

Definition 3.4 (Abstract Representation - Stability along the Pass) *The extended linear repetitive process $S(E_\alpha, W_\alpha, L_\alpha)_{\alpha\geq\alpha_0}$ is stable along the pass if there exists finite real scalars $M_\infty > 0$ and $0 < \lambda_\infty < 1$ (independent of α) such that, for each $\alpha \geq \alpha_0$ and for each constant disturbance sequence $b_{k+1} \equiv b_\infty$, $k \geq 0$, the output sequence from the model $S(E_\alpha, W_\alpha, L_\alpha)$ satisfies the inequality*

$$||y_k - y_\infty|| \leq M_\infty \lambda_\infty^k \left\{ ||y_0|| + \frac{||b_\infty||}{1 - \lambda_\infty} \right\}, \quad k \geq 0. \tag{3.14}$$

To be of use in a particular application, the abstract results must be convertible into a suitable computable form. Since this definition is not in an appropriate form for the derivation of a stability criterion, the following lemma is presented which implies a more useful definition of stability along the pass. Given theorem 3.2 (i.e. that the process is asymptotically stable - a necessary condition for stability along the pass), this result demands the existence of finite bounds M_∞ and λ_∞ for the scalars M_α and λ_α, as $\alpha \to +\infty$.

Lemma 3.1 *$S(E_\alpha, W_\alpha, L_\alpha)_{\alpha\geq\alpha_0}$ is said to be stable along the pass if, and only if, $\exists$ finite real scalars $M_\infty > 0$ and $0 < \lambda_\infty < 1$, independent of the pass length α, such that*

$$||L_\alpha^k|| \leq M_\infty \, \lambda_\infty^k \tag{3.15}$$

$\forall\, \alpha > 0,\ k \geq 0.$

Hence, in effect, stability along the pass of $S(E_\alpha, W_\alpha, L_\alpha)_{\alpha\geq\alpha_0}$ requires that the rate of convergence of the output sequence of pass profiles $\{y_k\}_{k\geq 1}$ to the limit profile y_∞ has a guaranteed geometric upper bound which is independent of the pass length α.

This result leads to the following which is one of several equivalent characterisations of stability along the pass for $S(E_\alpha, W_\alpha, L_\alpha)_{\alpha\geq\alpha_0}$,

Theorem 3.3 (Abstract Representation - Stability along the Pass) *The extended linear repetitive process $S(E_\alpha, W_\alpha, L_\alpha)_{\alpha\geq\alpha_0}$ is stable along the pass if, and only if,*

(a)

$$r_\infty := \sup_{\alpha\geq\alpha_0} r(L_\alpha) < 1 \tag{3.16}$$

and

(b)

$$M_0 := \sup_{\alpha \geq \alpha_0} \sup_{|z| \geq \lambda} ||(zI - L_\alpha)^{-1}|| < +\infty \tag{3.17}$$

for some real number $\lambda \in (r_\infty, 1)$.

Note that it can be shown that condition (b) can be relaxed to

$$M_0 := \sup_{\alpha \geq \alpha_0} \sup_{|z| = \lambda} ||(zI - L_\alpha)^{-1}|| < +\infty. \tag{3.18}$$

Part (a) of this result is equivalent to the requirement that all models with $\alpha \geq \alpha_0$ are asymptotically stable. This is the stronger requirement that asymptotic stability holds uniformly, i.e. that asymptotic stability is independent of pass length. Hence the reason for retaining the separate identities of (a) and (b) in theorem 3.3 despite the fact that (b) does imply (a).

It can be shown that the 'boundedness' condition (b) is equivalent to the requirement that $\exists\, \lambda \in (r_\infty, 1)$ such that

$$(zI - L_\alpha)y = \eta \tag{3.19}$$

has a uniformly bounded (with respect to α) solution $y \in E_\alpha\, \forall\, \eta \in E_\alpha$ satisfying $\sup_\alpha ||\eta|| < +\infty\, \forall\, |z| \geq \lambda$. In general this condition is very difficult to interpret. For the special cases of the differential and discrete processes introduced in sections 2.4.1 and 2.4.2, however, the stability results of sections 3.3 and 3.4 are obtained, respectively.

3.3 Stability Theory for Differential Processes

Within section 2.4.1 it was shown how differential processes with the state-space model (2.11) can be written in the form of the abstract representation (2.5), and hence the stability theory introduced in the previous section can be specifically interpreted for these processes. The theory is presented initially for differential processes with state-space model (2.11) and the simple boundary conditions (2.12), with the necessary amendments to the results to accommodate the dynamic boundary conditions of (2.18) given at the end of the section.

A discussion of the extension of these results to the discrete subclass of processes described by models of the form (2.22) is given in section 3.4.

3.3.1 Asymptotic Stability

The following result gives necessary and sufficient conditions for asymptotic stability of differential processes described by (2.11)-(2.12),

Theorem 3.4 (Asymptotic Stability - Differential Non-unit Memory) *(Rogers and Owens, 1992b) The differential non-unit memory linear repetitive process (2.11)-(2.12) is asymptotically stable if, and only if, all eigenvalues of the $mM \times mM$ block companion matrix D given by (2.77) have modulus strictly less than unity.*

The following corollary of theorem 3.4 can now be given for unit memory differential processes with state-space representation (2.13)-(2.14),

Corollary 3.2 (Differential Unit Memory Case) *(Rogers and Owens, 1992b) Setting $M = 1$ in theorem 3.4 gives the result that the unit memory differential linear repetitive process (2.13)-(2.14) is asymptotically stable if, and only if, all eigenvalues of the $m \times m$ matrix D_1 lie in the open unit circle in the complex plane, i.e. if, and only if,*

$$r(D_1) < 1. \tag{3.20}$$

Notice that the results of theorem 3.4 and corollary 3.2 are counter-intuitive, since what we have in effect is a stability condition which is independent of the system matrices A, B, B_0 and C. In particular, the result is independent of the eigenvalues of the matrix A which clearly govern the dynamics along a given pass. This is due entirely to the fact that the pass length α is finite and this changes drastically when the case of $\alpha \to +\infty$ is considered (see later).

As an illustration, consider the case of a differential unit memory single-input / single-output (SISO) process with zero control inputs, i.e. $u_{k+1}(t) \equiv 0,\ 0 \le t \le \alpha,\ k \ge 0$, and zero state initial conditions on each pass, i.e. $x_{k+1}(0) = 0,\ k \ge 0$. Then on the k^{th} pass the initial output is

$$y_k(0) = D_1^k y_0(0). \tag{3.21}$$

So, for the sequence $\{y_k(0)\}_{k\geq 1}$ not to become unbounded (in a well defined sense) as the pass index $k \to +\infty$, we require

$$r(D_1) = |D_1| < 1, \tag{3.22}$$

i.e. the condition of corollary 3.2, which can be tested by computing the eigenvalues of D_1 and displaying them relative to the unit circle in the complex plane (standard 1D test). Hence, in physical terms, asymptotic stability is the requirement that the initial output on each pass does not become unbounded as $k \to +\infty$, i.e. the effect of the initial profile is attenuated after a large number of passes.

3.3.2 Limit Profile

As for the abstract case, asymptotic stability guarantees the existence of a limit profile for the differential process with state-space model (2.11)-(2.12) as the following corollary shows,

Corollary 3.3 (Limit Profile - Differential Case) *(Rogers and Owens, 1992b) Suppose that the condition of theorem 3.4 holds and that a strongly convergent sequence $\{u_k\}_{k\geq 1}$ is applied. Then the limit profile for differential linear repetitive processes defined by (2.11)-(2.12) is described by the state-space model*

$$\begin{aligned} \dot{x}_\infty(t) &= A\, x_\infty(t) + B\, u_\infty(t) + \hat{B}\, y_\infty(t) \\ y_\infty(t) &= C\, x_\infty(t) + \hat{D}\, y_\infty(t), \quad 0 \leq t \leq \alpha, \ x_\infty(0) = d_\infty, \end{aligned} \tag{3.23}$$

where

$$\hat{B} = \sum_{j=1}^{M} B_{j-1}, \quad \hat{D} = \sum_{j=1}^{M} D_j, \quad \lim_{k\to\infty} u_k = u_\infty \quad \textit{and} \quad \lim_{k\to\infty} d_k = d_\infty, \tag{3.24}$$

or, since asymptotic stability ensures that $I_m - \hat{D}$ is nonsingular,

$$\begin{aligned} \dot{x}_\infty(t) &= (A + \hat{B}\,(I_m - \hat{D})^{-1}\, C)\, x_\infty(t) + B\, u_\infty(t) \\ y_\infty(t) &= (I_m - \hat{D})^{-1}\, C\, x_\infty(t), \quad 0 \leq t \leq \alpha, \ x_\infty(0) = d_\infty. \end{aligned} \tag{3.25}$$

This is obtained by replacing each term in the process description (2.11)-(2.12) by its strong limit. Note that the transfer-function matrix of this limit profile can also be obtained by setting $z = 1$ in the $m \times l$ 2D transfer-function matrix $G(s, z)$ given by (2.64).

Clearly (3.25) represents a standard state-space model from 1D linear differential systems theory. Hence, if the differential process (2.11)-(2.12) is asymptotically stable then, after a 'sufficiently large' number of passes, the process dynamics may be replaced by those of a 1D linear system. This fact has obvious implications from a feedback control point of view, which is discussed further in chapter 6 of this thesis on controller design.

Now asymptotic stability is a weak stability condition for the reason that, since the pass length α is fixed and finite, even an unstable 1D system can only produce a bounded output over such a duration. In this respect, asymptotic stability cannot guarantee that the resulting limit profile has acceptable dynamic characteristics and, in particular, that it is stable in the 1D sense.

The following simple example illustrates the point,

Example 3.1 (Asymptotic Stability $\not\Rightarrow$ Stability along the Pass) *Consider the following SISO differential unit memory process, where β is a real scalar,*

$$\begin{aligned} \dot{x}_{k+1}(t) &= -x_{k+1}(t) + u_{k+1}(t) + (1+\beta)\, y_k(t) \\ y_{k+1}(t) &= x_{k+1}(t) \\ x_{k+1}(0) &= 0, \;\; 0 \le t \le \alpha, \;\; k \ge 0. \end{aligned} \tag{3.26}$$

Then, since in terms of (2.13)-(2.14) $D_1 \equiv 0$, the process is asymptotically stable with limit profile

$$\begin{aligned} \dot{y}_\infty(t) &= \beta\, y_\infty(t) + u_\infty(t) \\ 0 \le t \le \alpha, &\;\; y_\infty(0) = 0. \end{aligned} \tag{3.27}$$

Also if $u_{k+1}(t) \equiv 1$ and $y_0(t) \equiv 0,\;\; 0 \le t \le \alpha,\;\; k \ge 0$, then it can be easily shown that the first pass profile is given by

$$y_1(t) = 1 - e^{-t}, \;\; 0 \le t \le \alpha. \tag{3.28}$$

But solving the limit profile differential equation (3.27) gives

$$y_\infty(t) = \beta^{-1}(e^{\beta t} - 1), \;\; 0 \le t \le \alpha. \tag{3.29}$$

So although the first pass profile (3.28) is clearly an acceptable dynamic characteristic response to the unit step command $u_1(t) \equiv 1$, the resulting limit profile has unacceptable dynamic characteristics. In particular, for $\beta > 0$, the dynamics of the limit profile increase exponentially and can be said to be 'unstable along the pass' in the obvious intuitive sense.

Despite the apparent weakness of asymptotic stability, cases do exist where this is all that is required (for example certain classes of iterative learning control schemes (Amann et al., 1996; Amann et al., 1998; Owens et al., 2000)) or, in fact, all that can be achieved (for example nonlinear optimal control using the maximum principle (Roberts, 1996; Roberts, 2000)). In this latter example, where a discrete unit memory linear repetitive process arises, the matrix corresponding to A never has all of its eigenvalues inside the unit circle in the complex plane which (as seen in the following section) is a necessary condition for stability along the pass. Hence asymptotic stability is all that is achievable here. In the majority of examples of repetitive processes, however, it is the stronger condition of stability along the pass which is required for acceptable systems performance.

3.3.3 Stability along the Pass

The stability along the pass result (theorem 3.3) for the abstract representation of definition 2.1 can be specifically interpreted for the differential subclass of processes as follows,

Theorem 3.5 (Stability along the Pass - Differential Case) *(Rogers and Owens, 1992b) Suppose that*

(i) the pair $\{C, A\}$ is observable;

(ii) the pair $\{A, \sum_{j=1}^{M} B_{j-1}\gamma^{j-1}\}$ is controllable at all but a finite number of points $\gamma_1, \gamma_2, \cdots, \gamma_q$ in the complex plane; and

(iii) $|sI_n - A - \sum_{j=1}^{M} B_{j-1}\gamma_i^{j-1}P(\gamma_i^{-1})^{-1}C|$ has no roots on the imaginary axis of the complex plane, $1 \le i \le q$, where

$$P(\gamma) = \gamma I_m - D_1 - \gamma^{-1}D_2 - \cdots - \gamma^{1-M}D_M. \qquad (3.30)$$

Then the extended linear repetitive process $S(E_\alpha, W_\alpha, L_\alpha)_{\alpha \geq \alpha_0}$ generated by differential models of the form (2.11)-(2.12) with $\alpha \geq \alpha_0$ is stable along the pass if, and only if,

(a)

$$r_\infty = \sup\{|z| : P(z) = 0\} < 1, \tag{3.31}$$

and

(b) there exists real numbers $\epsilon > 0$ and $r_\infty < \lambda < 1$ such that

$$|sI_n - A - \sum_{j=1}^{M} B_{j-1} z^{1-j} P(z)^{-1} C| \neq 0 \tag{3.32}$$

for all complex numbers z, s satisfying $|z| \geq \lambda$ and $Re\{s\} \geq -\epsilon$.

Note that condition (b) of this result is not computationally feasible.

At this stage, it is convenient to introduce the following definition,

Definition 3.5 (Asymptotic Stability Polynomial) *(Rogers and Owens, 1989b) The so-called asymptotic stability polynomial $P_a(z)$ for the differential process (2.11) is defined as*

$$P_a(z) := |Q(z)| \tag{3.33}$$

where

$$Q(z) = I_m - z^{-1} D_1 - \cdots - z^{-M} D_M, \tag{3.34}$$

and is to be regarded as a polynomial in z^{-1}.

Then it can easily be shown that condition (a) of theorem 3.5 for asymptotic stability can be replaced by

$$P_a(z) = |Q(z)| \neq 0 \quad \forall \quad |z| \geq 1. \tag{3.35}$$

It is also clear that, in this case, the spectral values of L_α $\forall$ $\alpha > 0$ are given by the solutions of $P_a(z) = 0$. Hence

$$r_\infty = \sup_{\alpha > 0} r(L_\alpha) < 1 \tag{3.36}$$

if condition (3.35) holds.

The stability along the pass polynomial for (2.11) can be defined as follows,

Definition 3.6 (Stability along the Pass Polynomial) *(Rogers and Owens, 1989b) The so-called stability along the pass polynomial $A_p(s,z)$ for the process (2.11) is defined as*

$$A_P(s,z) := |sI_n - A - \sum_{j=1}^{M} B_{j-1} z^{-j} Q(z)^{-1} C|, \tag{3.37}$$

with $Q(z)$ as in (3.34), and is regarded as a polynomial in s with coefficients which are rational functions in z^{-1}.

A simple argument now shows that (b) of theorem 3.5 is equivalent to the existence of real numbers $\epsilon > 0$ and $r_\infty < \lambda < 1$ such that

$$A_p(s,z) \neq 0 \tag{3.38}$$

for all complex z, s satisfying $|z| \geq \lambda$ and $\mathrm{Re}\{s\} \geq -\epsilon$.

The following alternative set of necessary and sufficient conditions now characterise stability along the pass of differential processes with state-space model (2.11)-(2.12). The result, in effect, replaces condition (b) of theorem 3.5 by two conditions which are both computationally feasible.

Theorem 3.6 (Stability along the Pass - Differential Case) *(Rogers and Owens, 1992b) Suppose that the assumptions of theorem 3.5 hold. Then the extended linear repetitive process $S(E_\alpha, W_\alpha, L_\alpha)_{\alpha \geq \alpha_0}$ generated by differential models of the form of (2.11)-(2.12) with $\alpha \geq \alpha_0$ is stable along the pass if, and only if,*

(a) all eigenvalues of the $mM \times mM$ block companion matrix D of (2.77) have modulus strictly less than unity;

(b) all eigenvalues of the matrix A have strictly negative real parts or, equivalently, the derived conventional linear system $L_D(A,B,C)$ described by the transfer-function matrix $G_0(s)$ of (2.65) is stable; and

(c) all eigenvalues of the $mM \times mM$ interpass transfer-function matrix $G(s)$ of (2.75) with $s = i\omega$ have modulus strictly less than unity for all real frequencies $\omega \geq 0$.

Now each of these conditions can be tested via well known 1D linear systems stability tests which are compatible with a computer aided analysis environment and has a well defined physical meaning.

Condition (a) is just the asymptotic stability condition already stated in theorem 3.4. (Note that this is as expected since asymptotic stability is a necessary condition for stability along the pass).

Condition (b) ensures that the derived conventional linear system is stable in the standard 1D sense. It is the requirement that the matrix A is Hurwitz and guarantees that the dynamics produced along any pass are uniformly bounded independent of the pass length. This condition is intuitively obvious since it prevents the presence of exponential growth terms within the along the pass dynamics.

For condition (c), we consider the special case of a SISO differential unit memory process (2.13) with zero state initial conditions and control inputs on each pass. Then the dynamics of the process along pass $k+1$ can be written

$$Y_{k+1}(s) = G_1(s)\,Y_k(s), \quad k \geq 0, \tag{3.39}$$

where

$$G_1(s) = C(sI_n - A)^{-1}B_0 + D_1. \tag{3.40}$$

Then, in this special case, the frequency component of the process can be written

$$Y_k(i\omega) = G_1^k(i\omega)\,Y_0(i\omega), \quad k \geq 0,\ \omega \geq 0. \tag{3.41}$$

Hence condition (c) is the requirement that the frequency component of the initial pass profile is attenuated from pass to pass, i.e. $|G_1(i\omega)| < 1 \ \forall\ \omega \geq 0$. In the general (multivariable) case, (c) can be tested by constructing the continuous curves (or characteristic loci) generated by the eigenvalues $g_j(s)$ of $G_1(s)$, $s = i\omega \ \forall\ \omega \geq 0$ and superimposing the unit circle on the resulting plots.

Returning to example 3.1, it is this third condition which is not satisfied here, since

$$G_1(s) = \frac{1+\beta}{s+1} \tag{3.42}$$

and hence, for $\beta > 0$, $|G_1(i\omega)| \not< 1 \, \forall \, \omega \geq 0$.

The three conditions of theorem 3.6 are tested in the order they are presented, i.e. with the most computationally intensive conditions computed last only if required. (Smyth, 1992) gives a comprehensive treatment of the testing of these three conditions for a given example.

3.3.4 Differential Processes with Dynamic Boundary Conditions

Within section 2.4.1 of chapter 2 so-called dynamic boundary conditions were introduced which cover cases where the simple boundary conditions of (2.14) are simply not strong enough to 'adequately' model the underlying dynamics of the process. The inclusion of dynamic boundary conditions affects the bounded linear operator L_α in the abstract formulation (2.5), and hence these conditions alone can destabilise the process, as outlined below.

It can easily be seen that asymptotic stability of the simple boundary condition case of (2.13) and (2.14), i.e. $r(D_1) < 1$, is a necessary condition for asymptotic stability of the dynamic boundary condition case of (2.13) and (2.18). Hence for simplicity, we set $D_1 \equiv 0$ in (2.13) for the remainder of this section.

It is shown in (Owens and Rogers, 1999) that, for differential unit memory processes with state-space model (2.13) and 'initial conditions' (2.18), asymptotic stability is determined by the following result,

Corollary 3.4 (Dynamic Boundary Conditions - Asymptotic Stability) *(Owens and Rogers, 1999) Suppose that the pair $\{A, B_0\}$ is controllable. Then the linear repetitive process $S(E_\alpha, W_\alpha, L_\alpha)$ generated by (2.13) and (2.18) (with $D_1 \equiv 0$) under a strongly convergent input sequence $\{u_k\}_{k\geq 1}$ is asymptotically stable if, and only if, all solutions of*

$$|zI_n - M(z)| = 0 \tag{3.43}$$

have modulus strictly less than unity, where

$$M(z) := \sum_{j=1}^{N} K_j C e^{\hat{A}(z)t_j} \tag{3.44}$$

with

$$\hat{A}(z) := A + z^{-1} B_0 C, \quad z \neq 0. \tag{3.45}$$

Note that we assume that N and t_j are fixed at the outset along the pass length α and, in particular, do not vary when developing stability along the pass criteria by letting $\alpha \to +\infty$.

Then, under the condition of corollary 3.4, the limit profile for the process is defined as follows,

Corollary 3.5 (Dynamic Boundary Conditions - Limit Profile) *(Owens and Rogers, 1999) The limit profile for an asymptotically stable differential linear repetitive processes defined by (2.13) and (2.18) (with $D_1 \equiv 0$) under a strongly convergent input sequence $\{u_k\}_{k\geq 1}$ is described by the state-space model*

$$\begin{aligned} \dot{x}_\infty(t) &= (A + B_0 C)\, x_\infty(t) + B\, u_\infty(t) \\ y_\infty(t) &= C\, x_\infty(t) \end{aligned} \tag{3.46}$$

with state initial vector $x_\infty(0)$ given by

$$x_\infty(0) = (I_n - M(1))^{-1} d_\infty \tag{3.47}$$

where u_∞ is the strong limit of $\{u_k\}_{k\geq 1}$, d_∞ is the strong limit of $\{d_k\}_{k\geq 1}$ and the invertability of the matrix $I_n - M(1)$ is guaranteed by asymptotic stability.

Once again, it is clear that, if the process is asymptotically stable, then its repetitive dynamics can, after a 'sufficiently large' number of passes, be replaced by those of a standard 1D linear state-space system.

Stability along the pass of processes with state-space model (2.13) and (2.18) can then be characterised as follows,

Theorem 3.7 (Dynamic Boundary Conditions - Stability along the Pass) *(Owens and Rogers, 1999) Suppose that $\{A, B_0\}$ is controllable and $\{C, A\}$ is observable. Then $S(E_\alpha, W_\alpha, L_\alpha)$ generated by (2.13) and (2.18) (with $D_1 \equiv 0$) is stable along the pass if, and only if,*

(a) the condition of corollary 3.4 holds,

(b) all eigenvalues of the matrix A have strictly negative real parts, and

(c)

$$\sup_{\omega \geq 0} r(G_1(i\omega)) < 1 \tag{3.48}$$

where $G_1(s) := C(sI_n - A)^{-1} B_0$.

Note that for the simple boundary condition case, the conditions of theorem 3.7 can be tested using standard 1D techniques - this is no longer true, however, when dynamic boundary conditions are employed.

It is clear from the above results that accurate determination of boundary conditions for a given example is vital for correct stability classification. In particular, with the wrong choice of boundary conditions, an unstable process may be accepted as asymptotically stable.

3.4 Stability Theory for Discrete Processes

Within this section, the abstract stability theory of section 3.2 is interpreted for the discrete subclass of processes. Here just the main results are quoted - in general, discussions of the results can be carried over from those given in the section on differential process stability given previously. For a detailed treatment see, for example, (Rogers and Owens, 1992b) for the simple boundary conditions case and (Galkowski et al., 1999a) for the case of dynamic boundary conditions and the relevant references cited within the text of the section.

An additional point should be noted here. Within section 2.6 of chapter 2 it was shown how certain classes of discrete processes can be written in 2D Roesser / Fornasini-Marchesini form. The extent to which so-called 'classical' 2D stability theory can be applied to these subclasses of linear repetitive processes is discussed within section 3.8 of this chapter.

Consider the subclass of discrete linear repetitive processes with state-space model (2.22)-(2.23). Then the following result characterises asymptotic stability for these processes,

Theorem 3.8 (Asymptotic Stability - Discrete Case) *(Rogers and Owens, 1992b) Discrete non-unit memory linear repetitive processes with state-space model (2.22)-(2.23) are asymptotically stable if, and only if, all eigenvalues of the $mM \times mM$ block companion matrix D given by the discrete form of (2.77) have modulus strictly less than unity.*

Asymptotic stability guarantees the existence of a limit profile for the process, which is defined in the following corollary,

Corollary 3.6 (Limit Profile - Discrete Case) *(Rogers and Owens, 1992b) Suppose that the condition of theorem 3.8 holds and that a strongly convergent sequence $\{u_k\}_{k\geq 1}$ is applied. Then the limit profile for discrete linear repetitive processes defined by (2.22)-(2.23) is described by the state-space model*

$$\begin{aligned} x_\infty(p+1) &= (A + \hat{B}\,(I_m - \hat{D})^{-1}\,C)\,x_\infty(p) + B\,u_\infty(p) \\ y_\infty(p) &= (I_m - \hat{D})^{-1}\,C\,x_\infty(p), \quad 0 \leq p \leq \alpha, \;\; x_\infty(0) = d_\infty, \end{aligned} \tag{3.49}$$

where

$$\hat{B} = \sum_{j=1}^{M} B_{j-1}, \quad \hat{D} = \sum_{j=1}^{M} D_j, \quad \lim_{k\to\infty} u_k = u_\infty \quad \text{and} \quad \lim_{k\to\infty} d_k = d_\infty. \tag{3.50}$$

The following is one of several equivalent sets of necessary and sufficient conditions for stability along the pass of discrete processes described by (2.22) and (2.23),

Theorem 3.9 (Stability along the Pass - Discrete Case) *(Rogers and Owens, 1992b) Suppose that*

(i) the pair $\{C, A\}$ is observable;

(ii) the pair $\{A, \sum_{j=1}^{M} B_{j-1}\gamma^{j-1}\}$ is controllable at all but a finite number of points $\gamma_1, \gamma_2, \cdots, \gamma_q$ in the complex plane; and

(iii) $|z_1 I_n - A - \sum_{j=1}^{M} B_{j-1}\gamma_i^{j-1} P(\gamma_i^{-1})^{-1} C|$ has no roots on the unit circle in the complex plane, $1 \leq i \leq q$, where $P(\gamma)$ is defined by (3.30).

Then the extended linear repetitive process $S(E_\alpha, W_\alpha, L_\alpha)_{\alpha\geq\alpha_0}$ generated by discrete models of the form of (2.22)-(2.23) with $\alpha \geq \alpha_0$ is stable along the pass if, and only if,

(a) all eigenvalues of the $mM \times mM$ block companion matrix D, constructed from the 2D transfer-function matrix $G(z_1, z)$ using the discrete form of (2.65) have modulus strictly less than unity;

(b) all eigenvalues of the matrix A have modulus strictly less than unity or, equivalently, the derived conventional linear system $L_D(A, B, C)$ is stable; and

(c) all eigenvalues of the $mM \times mM$ interpass transfer-function matrix $G(z_1)$, constructed from $G(z_1, z)$ using the discrete form of (2.65) have modulus strictly less than unity for all real frequencies z_1 satisfying $|z_1| = 1$.

Note that each of the conditions of theorem 3.9 can be tested using well known 1D linear system stability tests.

Within section 2.4.2 of chapter 2 dynamic boundary conditions were introduced for the discrete processes (2.24) which cover cases when the simple boundary conditions of (2.25) are not strong enough to model the process dynamics. Stability results for discrete processes with state-space structure (2.24) and (2.28) are now presented and these results are the discrete analog to the differential theory introduced in section 3.3.

As for the differential case, we see that asymptotic stability of processes described by (2.24) and (2.25) (i.e. $r(D_1) < 1$) is a necessary condition for processes with dynamic boundary conditions described by (2.24) and (2.28). Hence, for simplicity, we set $D_1 \equiv 0$ in (2.24) for the remainder of this section.

The following result introduced in (Rogers et al., 1998) characterises asymptotic stability of processes described by (2.24) and (2.28) and is the discrete counterpart to corollary 3.4,

Corollary 3.7 (Dynamic Boundary Conditions - Asymptotic Stability) *(Rogers et al., 1998) Suppose that the pair $\{A, B_0\}$ is controllable. Then the linear repetitive process $S(E_\alpha, W_\alpha, L_\alpha)$ generated by (2.24) and (2.28) (with $D_1 \equiv 0$) under a strongly convergent input sequence $\{u_k\}_{k\geq 1}$ is asymptotically stable if, and only if, all solutions of*

$$|zI_n - M(z)| = 0 \tag{3.51}$$

have modulus strictly less than unity, where

$$M(z) := \sum_{j=1}^{N} K_j C \hat{A}^{p_j}(z) \tag{3.52}$$

with

$$\hat{A}(z) := A + z^{-1} B_0 C, \quad z \neq 0. \tag{3.53}$$

The following corollary then defines the limit profile for the process,

Corollary 3.8 (Dynamic Boundary Conditions - Limit Profile) *(Rogers et al., 1998) The limit profile for discrete linear repetitive processes defined by (2.24) and (2.28) (with $D_1 \equiv 0$) under a strongly convergent input sequence $\{u_k\}_{k\geq 1}$ is described by the state-space model*

$$\begin{aligned} x_\infty(p+1) &= (A + B_0 C)\, x_\infty(p) + B\, u_\infty(p) \\ y_\infty(p) &= C\, x_\infty(p) \end{aligned} \tag{3.54}$$

with state initial vector $x_\infty(0)$ given by

$$x_\infty(0) = (I_n - M(1))^{-1} d_\infty \tag{3.55}$$

where u_∞ is the strong limit of $\{u_k\}_{k\geq 1}$, d_∞ is the strong limit of $\{d_k\}_{k\geq 1}$ and the invertability of the matrix $I_n - M(1)$ is guaranteed by asymptotic stability.

This corollary shows that, once again, under asymptotic stability the process dynamics may be replaced by those of a 1D discrete linear system, after a sufficiently large number of passes.

Stability along the pass can then be characterised by the following result,

Theorem 3.10 (Dynamic Boundary Conditions - Stability along the Pass) *(Rogers et al., 1998) Suppose that $\{A, B_0\}$ is controllable and $\{C, A\}$ is observable. Then $S(E_\alpha, W_\alpha, L_\alpha)$ generated by (2.24) and (2.28) (with $D_1 \equiv 0$) is stable along the pass if, and only if,*

(a) the condition of corollary 3.7 holds,

(b) all eigenvalues of the matrix A have modulus strictly less than unity, and

(c) all eigenvalues of the transfer-function matrix $G_1(z_1)$ have modulus strictly less than unity $\forall\, |z_1| = 1$, where $G_1(z_1) := C(z_1 I_n - A)^{-1} B_0$.

In contrast to the corresponding conditions for the differential subclass of processes, the conditions of theorem 3.10 can be tested via well known 1D linear systems tests. The starting point of this approach is to derive a 1D equivalent model of the dynamics of the process, as presented in (Galkowski et al., 2000). Much further work remains to be done on these promising initial results before the true potential of this approach can be realistically assessed, and thus this remains an open area for future research.

3.5 Simulation-Based Stability Tests

Within this section, time domain or 'simulation-based' tests for stability along the pass of the differential processes of (2.11)-(2.12) are presented based on the step response matrix of the associated conventional linear systems of (2.11). The results presented were first introduced in (Rogers and Owens, 1990b) (and subsequently extended in (Rogers and Owens, 1992a)) and provide an alternative route to performance prediction than the 1D Lyapunov approach to stability analysis presented in chapter 4. Extensive use is made of the well known results summarised in the appendix section A.3.

Now consider the subclass of differential processes with state-space form (2.11)-(2.12). The following analysis uses as a starting point the so-called associated conventional linear systems of (2.11), defined by (2.69), where it is assumed that each member of this set is controllable and observable. Further, the following assumptions concerning the step response matrix of each of these systems are made,

Assumption 3.1 *Write the j^{th} associated conventional linear system (2.69) in the convolution form $W^j = L^j\, y^{1-j}$ where*

$$(L^j\, y^{1-j})(t) = \int_0^t H^j(t')\, y^{1-j}(t-t')\, dt' + D_j\, y^{1-j}(t) \tag{3.56}$$

and $H^j(t)$, $1 \leq j \leq M$, is the $m \times m$ impulse response matrix

$$H^j(t) = Ce^{At}B_{j-1}. \tag{3.57}$$

Then it is assumed that the step response matrix

$$W^j(t) = \int_0^t H^j(t')\, dt' + D_j, \quad t \geq 0, \tag{3.58}$$

of this element is available and it is convenient to write this matrix in the form

$$W^j(t) = \begin{bmatrix} W^j_{11}(t) & \cdots & W^j_{1m}(t) \\ & \ddots & \\ W^j_{m1}(t) & \cdots & W^j_{mm}(t) \end{bmatrix}. \tag{3.59}$$

Here $W^j_{pv}(t)$ denotes the response of the p^{th} output channel to a unit step applied at $t = 0$ in the v^{th} input channel.

Assumption 3.2 *$W^j(t)$ is assumed to be a stable response. Formally it is required that*

$$\|W^j(t)\|_m \le \int_0^\infty \|H^j(t')\|_m \, dt' + \|D_j\|_m < +\infty, \tag{3.60}$$

where $\|\cdot\|_m = \max_i \sum_j |(\cdot)_{ij}|$ is the matrix norm induced by the vector norm $\|\cdot\|_m = \max_i |(\cdot)_i|$ in $\mathbb{R}^m$.

Note that under the standard controllability and observability assumptions, condition (3.60) holds if, and only if, all eigenvalues of the matrix A have strictly negative real parts (a necessary condition for stability along the pass). Further, it is assumed that $W^j(t)$ is available from appropriate simulation studies on the j^{th}, $1 \le j \le M$, element of (2.69) (see (Smyth, 1992) for further details of this point).

Suppose now that E_α in the abstract model $S(E_\alpha, W_\alpha, L_\alpha)$ is taken as $L^m_\infty(0, +\infty)$, where L_α has the block companion structure of

$$L_\alpha = \begin{bmatrix} 0 & I & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & I \\ L^M_\alpha & L^{M-1}_\alpha & & L^1_\alpha \end{bmatrix} \tag{3.61}$$

with L^j_α, $1 \le j \le M$, defined by

$$(L^j_\alpha \, y)(t) = C \int_0^t e^{A(t-\tau)} B_{j-1} \, y(\tau) d\tau + D_j \, y(t), \quad 0 \le t \le \alpha. \tag{3.62}$$

Further, define $L \in B(X^N, X^N)$, $X^N = L^N_\infty(0, +\infty)$, $N = mM$, as

$$L = \begin{bmatrix} 0 & I & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & I \\ L^M & & L^2 & L^1 \end{bmatrix}, \tag{3.63}$$

with L^j, $1 \leq j \leq M$, as in (3.56).

In which case it follows immediately that the natural projection (see definition A.11) of $L \in X_e^N$ into $X_{(0,\alpha)}^N = L_\infty^N(0,\alpha)$ is just L_α of (3.61), i.e.

$$P_\alpha L = L_\alpha, \quad 0 < \alpha < +\infty, \tag{3.64}$$

and

$$P_\infty L = L. \tag{3.65}$$

Now the result of lemma A.8 can be applied to each element in turn of the plant step response matrices $W^j(t)$, $1 \leq j \leq M$, to construct the matrix $||P_\infty L^j||_p$ of (A.36) and hence the $N \times N$ block companion matrix

$$||L||_p = \begin{bmatrix} 0 & I_m & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & I_m \\ ||L^M||_p & & ||L^2||_p & ||L^1||_p \end{bmatrix}. \tag{3.66}$$

It follows that the following application of the partial ordering of definition A.2 holds,

$$||L_\alpha||_p \leq ||L||_p, \quad 0 < \alpha < +\infty. \tag{3.67}$$

The following result then expresses stability along the pass of processes described by (2.11) in terms of the matrix $||L||_p$ of (3.66),

Theorem 3.11 (Simulation-based Stability along the Pass) *(Rogers and Owens, 1990b) Suppose that the matrix $||L||_p$ of (3.66) has been constructed for the differential non-unit memory linear repetitive process (2.11)-(2.12). Then the extended linear repetitive process $S(E_\alpha, W_\alpha, L_\alpha)_{\alpha \geq \alpha_0}$ generated by this model with $\alpha \geq \alpha_0$ is stable along the pass if*

$$r(||L||_p) < 1. \tag{3.68}$$

Note that this result is sufficient only, hence there exists examples which are stable along the pass but for which theorem 3.11 fails to produce a conclusive result. This potential conservativeness is offset by the fact that the result produces, at no extra computational cost, measures on performance along a given pass on key aspects of expected systems performance. For further details of these performance bounds see section 3.7.

At this stage, note that the initial entries in $W^j(t)$, $1 \le j \le M$, of (3.58) or (3.59) are simply the elements of the matrix D_j, $1 \le j \le M$, and hence the entries in

$$||D||_p = || \lim_{T \to 0+} (P_T L)||_p \tag{3.69}$$

are given by

$$||D||_p = \begin{bmatrix} 0 & I_m & & 0 \\ & \ddots & \ddots & \\ & & 0 & I_m \\ ||D_M||_p & & ||D_2||_p & ||D_1||_p \end{bmatrix}. \tag{3.70}$$

Further, by (a) of theorem 3.6, (2.11) is asymptotically stable if, and only if, the spectral radius of the matrix D is strictly less than unity. Application of the spectral radius inequality $r(D) \le r(||D||_p)$ of lemma A.1 now leads to the following result, which is clearly a simple preliminary test for the applicability of theorem 3.11 to a given example,

Lemma 3.2 *(Rogers and Owens, 1990b) Differential non-unit memory linear repetitive processes with state-space model (2.11)-(2.12) are asymptotically stable if*

$$r(||D||_p) < 1 \tag{3.71}$$

with $||D||_p$ given by (3.70).

The stability tests require the computation of the total variation of each element of the step response matrix of the associated conventional linear system. (Smyth, 1992) details the numerical and software aspects of implementing theorem 3.11 within a CAD environment. Note that there are a number of special cases where it is possible to obtain an explicit formula for $||L||_p$, with the consequent possibility of obtaining 'synthesis type' results for use in design studies. For a detailed treatment of these cases see, for example, (Rogers and Owens, 1992b). The advantage of this approach is that, unlike the stability tests of section 3.3, it can be extended to cases where it is necessary, for example, to include interpass smoothing effects in the basic model. Details of work undertaken on the use of these simulation-based tests in the specification and design of control schemes for these processes are given in (Smyth, 1992). In addition, in (Rogers and Owens, 1992a; Rogers and Owens, 1992b) it is demonstrated that the stability tests presented above provide computable information on performance along a given pass, which is not available from the Nyquist like tests of section 3.3. Further information on these so-called performance bounds is given in section 3.7. Also note that these results generalise to the discrete subclass of processes described by models of the form (2.22)-(2.23) as shown in (Rogers and Owens, 1992b).

Now recall the stability along the pass conditions of theorem 3.9 for the discrete subclass of processes, and note that condition (c) of this result involves calculating the eigenvalues of an $mM \times mM$ transfer-function matrix for all points on the unit circle. This can be computationally intensive even for the simplest of cases.

With this motivation, new stability tests are developed in the following section for the discrete subclass of processes which replace the potentially complex computational conditions mentioned above with sufficient but not necessary alternatives. This conservativeness is offset by the results supplying at no extra computational cost information on performance along a given pass (which are not available from theorem 3.9).

3.6 Simple Structure Stability Tests

In this section simple structure stability tests are developed for the discrete subclass of linear repetitive processes using some basic results from the theory of nonnegative matrices (included in the appendix section A.1). These tests replace the need to compute the eigenvalues of a transfer-function matrix for all points on the unit circle in the complex plane with a one-off computation of the eigenvalues of a matrix with constant entries, and are the subject of the paper (Benton et al., 1998b). In addition, it is shown how the tests produce information on performance along a given pass at no extra computational cost.

In order to develop the theory, some notation is needed. Consider the subclass of discrete non-unit memory linear repetitive processes with state-space representation (2.22)-(2.23). For this model, introduce the transfer-function matrix

$$G(z_1) := A_2 \left(z_1 I_n - A\right)^{-1} A_1 + A_3, \tag{3.72}$$

where

$$A_1 = [B_{M-1}, \cdots, B_0], \quad A_2 = \left[0, \cdots, C^T\right]^T \quad \text{and}$$

$$A_3 = \begin{bmatrix} 0 & I_m & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & I_m \\ D_M & \cdots & \cdots & D_1 \end{bmatrix}. \tag{3.73}$$

Then the following result (which is a restatement of theorem 3.9) gives necessary and sufficient conditions for stability along the pass of (2.22)-(2.23),

Theorem 3.12 (Stability along the Pass - Discrete Non-unit Memory) *Under the technical controllability and observability assumptions of theorem 3.9, processes described by (2.22)-(2.23) are stable along the pass if, and only if,*

(a)

$$r(A_3) < 1, \quad r(A) < 1, \tag{3.74}$$

and

(b) all eigenvalues of the transfer-function matrix $G(z_1)$ have modulus strictly less than unity $\forall\ |z_1| = 1$.

Note however that condition (b) requires the computation of the eigenvalues of the interpass transfer-function matrix $G(z_1)$ for all points on the unit circle - a task which involves working with an $mM \times mM$ transfer-function matrix. Hence a 'heavy' computational load could result for even the simplest cases of (2.22)-(2.23).

Here, alternative sufficient stability along the pass conditions are developed which involve the one-off computation of the eigenvalues of a matrix with constant entries. The new stability tests exploit some basic properties of the theory of nonnegative matrices which are reviewed in the appendix section A.1.

By considering the nonnegative matrix associated with each of the matrices in theorem 3.12, we obtain the following set of sufficient conditions for stability along the pass of discrete linear repetitive processes,

Theorem 3.13 *(Benton et al., 1998b) Under the assumptions of theorem 3.9, processes described by (2.22)-(2.23) are stable along the pass if*

(a)

$$r(||A_3||_p) < 1, \quad r(||A||_p) < 1, \qquad \textit{and} \tag{3.75}$$

(b)

$$r\left(||A_2||_p\,(I_n - ||A||_p)^{-1}\,||A_1||_p + ||A_3||_p\right) < 1. \tag{3.76}$$

Proof : *The proof of (a) is trivial on applying the spectral radius inequality of lemma A.1 to part (a) of theorem 3.12. To prove (b), first note that $(z_1 I_n - A)^{-1}$ can be represented by an absolutely convergent power series for $r(A) < |z_1|$, as follows,*

$$(z_1 I_n - A)^{-1} = \frac{1}{z_1}\sum_{r=0}^{\infty}\left(\frac{A}{z_1}\right)^r, \; z_1 \neq 0. \tag{3.77}$$

Hence by applying the properties of nonnegative matrices given in lemma A.1,

$$||(z_1 I_n - A)^{-1}||_p \leq \frac{1}{|z_1|}\sum_{r=0}^{\infty}\left(\frac{||A||_p}{|z_1|}\right)^r = (I_n - ||A||_p)^{-1}, \;\; \textit{on}\; |z_1| = 1. \tag{3.78}$$

Therefore

$$||G(z_1)||_p \leq ||A_2||_p\,(I_n - ||A||_p)^{-1}\,||A_1||_p + ||A_3||_p, \;\; \forall\, |z_1| = 1, \tag{3.79}$$

and the result follows immediately on using the partial ordering on matrices of definition A.2.

This result can be extended. Suppose that $||\cdot|| = \max_i \sum_j |(\cdot)_{ij}|$ is the matrix norm on $n_1 \times n_2$ matrices induced by the vector norm $\max_i |(\cdot)_i|$ in $\mathbb{R}^{n_2}$. Then the following corollary of theorem 3.13 gives an alternative sufficient condition for stability along the pass for the unit memory ($M = 1$) case. This condition follows immediately from the spectral radius inequality $r(||X||_p) \leq ||(||X||_p)|| = ||X||$ of lemma A.1.

Corollary 3.9 *(Benton et al., 1998b) Suppose that the pair $\{C, A\}$ is observable and the pair $\{A, B_0\}$ is controllable. Then, unit memory processes described by (2.24)-(2.25) are stable along the pass if*

(a)

$$||A_3|| \equiv ||D_1|| < 1, \quad ||A|| < 1, \qquad and \tag{3.80}$$

(b)

$$||A_2||(1 - ||A||)^{-1}\,||A_1|| + ||A_3|| < 1. \tag{3.81}$$

Proof : *Obtained trivially on setting $M = 1$ in the proof of theorem 3.13.*

Note that corollary 3.9 can only produce a conclusive result when applied to unit memory processes. This is because the block companion structure of $||A_3||_p$ means that $r(||A_3||) < 1$ can never hold in the non-unit memory case.

Similar results to those of corollary 3.9 have been reported by, for example, (Ahmed, 1980) for 2D linear systems described by the Roesser (Roesser, 1975) or Fornasini-Marchesini (Fornasini and Marchesini, 1978) models. There are however no Roesser / Fornasini-Marchesini alternatives to the performance measures which can be obtained from theorem 3.13 and which are presented in the next section.

3.7 Performance Bounds

An immediate conclusion of stability along the pass is that after a 'sufficiently large' number of passes, the dynamics of the process may be replaced by those of a stable 1D linear system. This fact is obviously of interest in terms of the specification and design of control schemes for these processes which is discussed in (Smyth, 1992) where it is argued that information on the following aspects of system performance would be of great use,

(i) the rate of approach of the output sequence of pass profiles to the resulting limit profile, and

(ii) the error $y_k - y_\infty$ on any pass k.

Within this section it is shown that the stability tests of the previous section produce at no extra cost computable information concerning the rate of approach to the limit profile together with bounds on performance along any given pass.

Return now to the abstract representation $S(E_\alpha, W_\alpha, L_\alpha)$ of definition 2.1 under stability along the pass, where no loss of generality arises from restricting our attention to the unit memory ($M = 1$) case. Suppose also that the disturbance sequence applied is constant from pass to pass (a relevant physical assumption in many cases (Smyth, 1992)), i.e. $b_k \equiv b_\infty,\ k \geq 0$.

Then the recursion relationship for the abstract model (2.5) can be rewritten as

$$\begin{aligned} y_k &= L_\alpha\, y_{k-1} + b_\infty \\ &= L_\alpha^k\, y_0 + \sum_{j=1}^{k} L_\alpha^{j-1}\, b_\infty. \end{aligned} \tag{3.82}$$

Similarly, the limit profile (3.10) (under stability along the pass and hence asymptotic stability, i.e. with $r(L_\alpha) < 1$) takes the form

$$y_\infty = \sum_{j=1}^{\infty} L_\alpha^{j-1}\, b_\infty. \tag{3.83}$$

Therefore the error term $y_k - y_\infty$ on a given pass k can be written

$$y_k - y_\infty = L_\alpha^k\, y_0 - \sum_{j=k+1}^{\infty} L_\alpha^{j-1}\, b_\infty, \tag{3.84}$$

and by looking at the nonnegative matrix associated with each side of (3.84) gives an estimate of convergence as

$$||y_k - y_\infty||_p \leq (\hat{L})^k \left\{ ||y_0||_p + \sum_{j=k+1}^{\infty} (\hat{L})^{j-1-k}\, ||b_\infty||_p \right\}, \tag{3.85}$$

where, for discrete processes with state-space representation (2.22)-(2.23) under the notation of (3.72)-(3.73),

$$\hat{L} = ||A_2||_p (I_n - ||A||_p)^{-1} ||A_1||_p + ||A_3||_p. \tag{3.86}$$

Further development of the last equation yields the following result,

Theorem 3.14 (Performance Bounds) *(Benton et al., 1998b) Suppose that $S(E_\alpha, W_\alpha, L_\alpha)$ generated by (2.22)-(2.23) is stable along the pass and that theorem 3.13 holds. Suppose also that the control input sequence applied is constant from pass to pass, i.e. $u_k \equiv u_\infty,\ k \geq 1$, and hence $b_k \equiv b_\infty,\ k \geq 1$, in the abstract* *model (2.5). Then for $\alpha \in (0,\infty)$ $\exists$ an $m \times m$ nonnegative matrix W and a real scalar $\gamma \in (r(\hat{L}), 1)$ such that the error $y_k - y_\infty$, $k \geq 0$, satisfies*

$$||y_k - y_\infty||_p \;\leq\; W\,\gamma^k\{||y_0||_p + (I_m - \hat{L})^{-1}\,||b_\infty||_p\}. \tag{3.87}$$

Proof : *Since $b_{k+1} \equiv b_\infty$, $k \geq 0$, the error term $y_k - y_\infty$ for the process can be expressed as (3.84) and therefore the inequality (3.85) holds as shown in the analysis above.*

To proceed, first note that, since theorem 3.13 holds, we have

$$r(\hat{L}) < 1. \tag{3.88}$$

Hence $(I_m - \hat{L})^{-1}$ exists and is nonnegative by lemma A.2, and it can easily be shown that

$$(I_m - \hat{L})^{-1} = \sum_{j=k+1}^{\infty} \hat{L}^{j-1-k}. \tag{3.89}$$

Therefore it remains to be shown that there exists a nonnegative matrix $W \geq 0$ and a real scalar $\gamma \in (r(\hat{L}), 1)$ such that

$$\hat{L}^k \leq W\gamma^k, \quad k \geq 0. \tag{3.90}$$

This follows on noting that $r(L_\alpha) \leq r(\hat{L}) < 1$ by lemma A.1, and hence it is possible to choose real numbers $\tilde{W} > 0$ and $\gamma \in (r(\hat{L}), 1)$ such that

$$||\hat{L}^k|| \leq \tilde{W}\gamma^k, \quad k \geq 0. \tag{3.91}$$

Further, it is clear that the partial ordering $\hat{L}^k \leq Q$ holds where Q is the $m \times m$ matrix with each element equal to $||\hat{L}||$. The result (3.87) now follows immediately on using (3.89) and defining W as the $m \times m$ nonnegative matrix with each element equal to $\tilde{W}$.

Suppose now that $y_k^i(p)$ and $y_\infty^i(p)$ denote the i^{th}, $1 \leq i \leq m$, output channels of $y_k(p)$ and $y_\infty(p)$ respectively. Suppose also that $||b_\infty||_p$ is available and assume, for simplicity, that the initial pass profile is zero, and introduce

$$\epsilon := (\epsilon_1, \cdots, \epsilon_m)^T = (\hat{L})^k (I_m - \hat{L})^{-1}||b_\infty||_p. \tag{3.92}$$

Then it follows immediately that

$$||y_k(p) - y_\infty(p)||_p \leq ||y_k - y_\infty||_p \leq \epsilon. \tag{3.93}$$

Hence $y_k^i(p)$ lies in the band defined by

$$\sqrt{\sum_{j=1}^{\alpha}(y_k^i(j) - y_\infty^i(j))^2} \leq \epsilon_i, \quad 1 \leq i \leq m. \tag{3.94}$$

Note that the width of these bands is, in effect, governed by $\hat{L}$.

Hence it has been shown that the output pass profile on pass k, $y_k(t)$, approaches the limit profile at a geometric rate governed by $\hat{L}$. This information is available at no extra computational cost from the sufficient stability tests of the previous section, and this offsets the conservative nature of the tests. A further discussion of performance bounds is given in section 6.4 of chapter 6.

3.8 Links between 2D Systems Stability and Repetitive Process Stability

In section 2.6 of chapter 2 it was shown that the dynamics of a large subclass of discrete linear repetitive processes can be represented by equivalent Roesser / Fornasini-Marchesini 2D state-space structures. Within this section, links are drawn between the stability of these discrete linear repetitive processes and the BIBO stability of 2D linear systems.

Consider again the Roesser state-space model (2.42) for 2D systems recursive in the positive quadrant. Then applying the 2D z-transform (where to follow 2D systems notation convention z and z_1 now represent 'backwards' shifts) yields the following 2D transfer-function matrix,

$$G(z_1, z) = \begin{bmatrix} z_1^{-1}I_n - A_1 & -A_2 \\ -A_3 & z^{-1}I_m - A_4 \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \tag{3.95}$$

Application of the BIBO stability results of Shanks (lemma A.11) or Huang (lemma A.12) then shows that this is dependent on the roots of the so-called characteristic polynomial of the system,

$$\rho_r(z_1, z) = \begin{vmatrix} I_n - z_1A_1 & -z_1A_2 \\ -zA_3 & I_m - zA_4 \end{vmatrix}. \tag{3.96}$$

Alternatively, use of Schur's formula yields

$$\rho_r(z_1, z) = |I_n - z_1 A_1|\,|I_m - zA_4 - z_1 z A_3 (I_n - z_1 A_1)^{-1} A_2|, \tag{3.97}$$

and hence the following result gives necessary and sufficient conditions for BIBO stability of the Roesser 2D state-space model (2.42),

Theorem 3.15 (BIBO Stability of Roesser Model) *(Boland and Owens, 1980) The Shanks (or equivalently, the Huang) stability test for the 2D Roesser model is equivalent to the following conditions*

(a) A_1 is a stability matrix (i.e. all eigenvalues lie in the open unit circle in the complex plane),

(b) A_4 is a stability matrix, and

(c) all eigenvalues of the transfer-function matrix

$$P(z_1^{-1}) := A_3(z_1^{-1} I_n - A_1)^{-1} A_2 + A_4 \tag{3.98}$$

with $|z_1| = 1$ lie in the open unit circle in the complex plane.

It can be shown that for the 2D systems described by the Roesser model (see eg. (Lu and Lee, 1985) for the details) that (a) and (b) are equivalent necessary conditions, hence either may be dispensed with.

At this stage it is convenient to introduce a formal definition of BIBO stability of linear repetitive processes, as follows,

Definition 3.7 (BIBO Stability) *(Rogers and Owens, 1992b) A linear repetitive process $S(E_\alpha, W_\alpha, L_\alpha)$ of constant pass length $\alpha > 0$ is said to be bounded input-bounded output (BIBO) stable if there exists a real scalar $\delta > 0$ such that, given any y_0 and $\{b_k\}_{k\geq 1} \subset W_\alpha$ bounded in norm (i.e. $\|b_k\| \leq c_1$ for some constant $c_1 \geq 0\ \forall\ k \geq 1$), the output sequence $\{y_k\}_{k\geq 1}$ generated by the perturbed process (3.1) is bounded in norm whenever $\|\gamma\| \leq \delta$.*

This definition demands that bounded disturbance sequences generate bounded sequences of pass profiles (i.e. the standard BIBO requirement) but also that this property is retained in the presence of small modelling errors.

In (Rocha et al., 1996) links between BIBO stability of 2D systems described by the Roesser state-space model and the concepts of asymptotic stability and stability along the pass of discrete linear repetitive processes were investigated. Taking each stability property in turn, the following conclusions were drawn. If a process is asymptotically stable, then the resulting limit profile is BIBO stable over the pass length α, which is finite, since over such a duration even an unstable 1D linear system can only produce a bounded output. Hence asymptotic stability is BIBO stability over the finite length only. Stability along the pass is then the stronger requirement that the process is BIBO stable uniformly (i.e. independent of the pass length). This equivalence is summarised in the following result,

Theorem 3.16 (BIBO Stability/Stability along the Pass Equivalence) *(Rocha et al., 1996) $S(E_\alpha, W_\alpha, L_\alpha)_{\alpha \geq \alpha_0}$ generated by (2.22)-(2.23) is stable along the pass if, and only if, it is BIBO stable in the sense of Shanks or Huang. (Note that the transfer-function matrix of (3.95) must have no nonessential singularities of the second kind).*

As a result of theorem 3.16, many tests available for checking BIBO stability of 2D linear systems described by the Roesser model can also be applied to testing for stability along the pass of discrete linear repetitive processes described by (2.22)-(2.23).

Within section 2.6, new Roesser-type representations of the dynamics of discrete unit memory linear repetitive processes were presented, which have proved useful in characterising local reachability / controllability properties for these processes (Galkowski et al., 1999b). Consider then the subclass of discrete linear repetitive processes with state-space model (2.57) (under transformation (2.56)) and introduce the characteristic polynomial for this process as

$$\rho_R(z_1, z) = \begin{vmatrix} I_n - z_1\hat{A} & -\hat{B} \\ -\hat{C} & I_{m+n} - z\hat{D} \end{vmatrix} \tag{3.99}$$

where

$$\hat{A} = A, \ \hat{B} = \begin{bmatrix} 0 & I \end{bmatrix}, \ \hat{C} = \begin{bmatrix} D_1C \\ B_0C \end{bmatrix}, \ \hat{D} = \begin{bmatrix} D_1 & 0 \\ B_0 & 0 \end{bmatrix}. \tag{3.100}$$

Then the Shanks test for stability (lemma A.11) says that the process (2.57) is BIBO stable if, and only if,

$$\rho_R(z_1, z) \neq 0, \quad |z_1| \leq 1, \ |z| \leq 1. \tag{3.101}$$

Note that, in this form, this test is not computationally feasible in all but the simplest cases. This problem can be overcome by using Huang's test (of lemma A.12) which states that (2.57) is BIBO stable if, and only if,

$$\begin{aligned} \rho_R(z_1, 0) &\neq 0, \quad |z_1| \leq 1, \quad \text{and} \\ \rho_R(z_1, z) &\neq 0, \quad |z_1| = 1, \ |z| \leq 1. \end{aligned} \tag{3.102}$$

Note that the stability conditions of (3.101) and (3.102) assume that the transfer-function matrix description of the underlying dynamics has no nonessential singularities of the second kind (Goodman, 1977).

It is also routine to show (Rogers and Owens, 1992b) that in the case of (2.57) these conditions can be reduced to the form of the following corollary,

Corollary 3.10 *(Galkowski et al., 1999b) The 2D BIBO linear systems stability test of Huang (Huang, 1972) applied to the dynamics of discrete linear repetitive processes written in the form (2.57) requires that*

(a) $r(D_1) < 1, \quad r(A) < 1,$ *and*

(b) all eigenvalues of the transfer-function matrix

$$G(z_1^{-1}) = C(z_1^{-1} I_n - A)^{-1} B_0 + D_1 \tag{3.103}$$

lie in the open unit circle in the complex plane $\forall \ |z_1^{-1}| = 1.$

The results presented above are based on a Roesser structure interpretation of the dynamics of the discrete process (2.24)-(2.25). Results obtained from interpreting the well known BIBO stability theory for Fornasini-Marchesini type structures can be found in (Galkowski et al., 1999b).

3.9 A Volterra Approach to Stability Analysis

Recent new results on the controllability of discrete linear repetitive processes (Dymkov et al., 2000) strongly suggest that the powerful theory of Volterra operators has a significant role to play in the onward development of a mature systems theory for linear (and nonlinear) repetitive processes. In this section the Volterra approach is used to study the stability properties of discrete linear repetitive processes.

Let E be a finite dimensional normed linear space over the complex field $\mathbb{C}$ with norm $\|\cdot\|_E$ and let $\mathbb{Z}_+$ be the set of nonnegative integers. Also let $S(\mathbb{Z}_+, E)$ be the linear space of all sequences on E, i.e. the functions $f : \mathbb{Z}_+ \longrightarrow E$. Then $S(\mathbb{Z}_+, E)$ is a locally convex Hausdorff topological space when equipped with the topology of uniform convergence on finite sets, i.e. the family of neighbourhoods is defined as

$$U_{M,\epsilon} = \{f : f \in S(\mathbb{Z}_+, E),\ \|f(k)\|_E < \epsilon,\ k \in N\} \tag{3.104}$$

where N is the set of all finite subsets from $\mathbb{Z}_+$, and ϵ ranges over the set $\mathbb{R}_+$ of all positive real numbers.

Suppose now that $B(\mathbb{Z}_+, E)$ denotes the subspace of $S(\mathbb{Z}_+, E)$ of all bounded functions, i.e. $f : \mathbb{Z}_+ \longrightarrow E$ such that $\sup_{k\in\mathbb{Z}_+} \|f(k)\|_E < +\infty$. Then it is a standard fact that $B(\mathbb{Z}_+, E)$ is dense in $S(\mathbb{Z}_+, E)$ with respect to the topology of uniform convergence over finite sets. Also $B(\mathbb{Z}_+, E)$ is a Banach space under a suitable norm definition, eg. $\|f\| = \sup_{k\in\mathbb{Z}_+} \|f(k)\|_E$.

Now let V and W be finite dimensional normed spaces over the complex field $\mathbb{C}$ and let $A : E \longrightarrow E$, $B : V \longrightarrow E$, $B_0 : W \longrightarrow E$, $C : E \longrightarrow W$, and $D_1 : W \longrightarrow W$ be linear operators. By letting $[0, \alpha]$ be the set of integers $\{0 \leq i \leq \alpha\}$ for given integer α, it is possible to describe the discrete unit memory linear repetitive process (2.24) as

$$\begin{aligned} x_{k+1}(p+1) &= A\, x_{k+1}(p) + B\, u_{k+1}(p) + B_0\, y_k(p) \\ y_{k+1}(p) &= C\, x_{k+1}(p) + D_1\, y_k(p) \end{aligned} \tag{3.105}$$

with respect to the unknown functions $x : \mathbb{Z}_+ \times [0, \alpha] \longrightarrow E$ and $y : \mathbb{Z}_+ \times [0, \alpha] \longrightarrow W$. The function x is the current pass state vector, y is the pass profile vector, and $u : \mathbb{Z}_+ \times [0, \alpha] \longrightarrow V$ is the control input vector.

The formal definition of a solution for (3.105) is as follows,

Definition 3.8 (Solution for (3.105)) *For a given control input vector $u_k(p)$, the pair of functions $\{x_k(p),\ y_k(p)\}$ defined on $\mathbb{Z}_+ \times [0, \alpha]$ with ranges in E and W respectively are said to be the solution of the equations of (3.105) if they satisfy them $\forall\, (k, p) \in \mathbb{Z}_+ \times [0, \alpha]$.*

It can easily be verified that, for any function $\gamma \in S(\mathbb{Z}_+, E)$ and any collection of elements $d_1,\ d_2,\ \cdots, d_\alpha$ from W, there is a unique solution to the equations of

(3.105) satisfying

$$x_k(0) = \gamma(k), \ \ k \in \mathbb{Z}_+, \ \ y_0(p) = d_p, \ \ p \in [0, \alpha]. \tag{3.106}$$

These are termed the initial conditions here.

The following analysis uses some properties of the Volterra operator which are reviewed in the appendix section A.5.

The structural properties of the process described by (3.105) and (3.106) can be studied by considering the space of bounded functions. Note that the closure of this set with respect to the topology of uniform convergence equals the space $S(\mathbb{Z}_+, E)$. Hence the question considered in the following analysis is of under what conditions the solution of (3.105)-(3.106) is bounded.

Suppose that the initial condition function and the control input satisfy the following,

$$\sup_{k\in\mathbb{Z}_+} \|\gamma(k)\|_E < +\infty, \quad \sup_{k\in\mathbb{Z}_+} \|u_{k+1}(p)\|_V < +\infty, \tag{3.107}$$

$\forall\, p \in [0, \alpha]$.

Then, without some additional assumptions, the solution to (3.105)-(3.106) may become unbounded, as illustrated in the following example,

Example 3.2 *Let $A = 0$, $C = 0$, $u_{k+1}(p) = 0$, $k \in \mathbb{Z}_+$, $p \in [0, \alpha]$. Then it can easily be seen that $y_k(p) = D_1^k\, d_p$ and $x_k(p) = B_0 D_1^k\, d_p$ is the solution of (3.105)-(3.106). Clearly $\sup_{k\in\mathbb{Z}_+} \|x_k(p)\| = +\infty$ and $\sup_{k\in\mathbb{Z}_+} \|y_k(p)\| = +\infty \; \forall\, p \in [0, \alpha]$, i.e. the solution is unbounded if the eigenvalues of D_1 lie outside the unit disc U.*

The following result then gives the condition for the existence of bounded solutions to (3.105)-(3.106),

Theorem 3.17 *If the spectrum $\sigma(D_1)$ of the operator D_1 lies within the unit disc, then, for any functions $\gamma(k)$ and $u_k(p)$ satisfying conditions (3.107), the solution*

$$x_k(p) = x(k, p, \gamma, d, u), \quad y_k(p) = y(k, p, \gamma, d, u) \tag{3.108}$$

of the system (3.105)-(3.106) satisfies the following conditions $\forall\, p \in [0, \alpha]$,

$$\sup_{k\in\mathbb{Z}_+} \|x_k(p)\| < +\infty, \quad \sup_{k\in\mathbb{Z}_+} \|y_k(p)\| < +\infty. \tag{3.109}$$

Proof : *Here we use the operator representation for the system (3.105)-(3.106) in the space* $B(\mathbb{Z}_+, E)$*. Introduce the operator* $T : B(\mathbb{Z}_+, E) \longrightarrow B(\mathbb{Z}_+, E)$ *defined as*

$$(Tf)(k+1) = f(k+1) - D_1 f(k), \quad k \in \mathbb{Z}_+. \tag{3.110}$$

Also define for each $p \in [0, \alpha]$ *the functions* $x_p : \mathbb{Z}_+ \longrightarrow E$, $y_p : \mathbb{Z}_+ \longrightarrow W$ *and* $u_p : \mathbb{Z}_+ \longrightarrow V$ *as*

$$(x_p)(k) := x_k(p), \quad (y_p)(k) := y_k(p), \quad (u_p)(k) := u_{k+1}(p), \quad k \in \mathbb{Z}_+. \tag{3.111}$$

Then (3.105) can be written as

$$(T\, y_p)(k) = (C\, x_p)(k), \tag{3.112}$$

which can be rewritten in the operator form

$$T\, y_p = C\, x_p, \quad p \in [0, \alpha]. \tag{3.113}$$

If the inverse operator T^{-1} *for* T *exists, then (3.113) yields*

$$y_p = T^{-1} C\, x_p, \tag{3.114}$$

and to establish the conditions for the existence of the T^{-1} *first note that (3.110) is a special case of the Volterra operator* V_0 *of definition A.49, and hence lemma A.13 may be applied. Further, it is easy to verify that the power series representation* $T(z)$ *of* T *is* $T(z) = I - zD_1$*. Since the spectral values of* $D_1 \in U$, $\det T(z) \neq 0$ *for* $|z| \leq 1$, $z \in \mathbb{C}$*. Hence* T *has the bounded inverse linear operator* T^{-1} *which has, as in (A.51), the form*

$$(T^{-1}\phi)(k) = \sum_{i=0}^{k} T_i \phi(k-i), \quad z \in \mathbb{C}, \tag{3.115}$$

where $T_i : E \longrightarrow E$, $i \in \mathbb{Z}_+$*, are linear operators satisfying* $\sum_{i=0}^{\infty} \|T_i\| < +\infty$.

At this stage, we have shown that (3.114) holds and it can be rewritten in the form

$$(y_p)(k) = (T^{-1} C\, x_p)(k), \quad k \in \mathbb{Z}_+. \tag{3.116}$$

Substituting (3.116) into (3.105) and using (3.115) gives

$$x_{k+1}(p+1) = A\, x_{k+1}(p) + B_0 \sum_{i=0}^{k} T_i C\, x_{k-i}(p) + B\, u_{k+1}(p). \tag{3.117}$$

Since the initial conditions $\gamma(k) = x_k(0)$ and control input function $u_k(p)$, $k \in \mathbb{Z}_+$, $p \in [0, \alpha]$, satisfy conditions (3.107), it follows immediately from (3.117) that

$$\sup_{k\in\mathbb{Z}_+} \|x_{k+1}(1)\|_E = \sup_{k\in\mathbb{Z}_+} \left\| A\, x_{k+1}(0) + B_0 \sum_{i=0}^{k} T_i C\, x_{k-i}(0) + B\, u_{k+1}(0) \right\|_E < +\infty. \tag{3.118}$$

Hence

$$\sup_{k\in\mathbb{Z}_+} \|x_{k+1}(2)\|_E = \sup_{k\in\mathbb{Z}_+} \left\| A\, x_{k+1}(1) + B_0 \sum_{i=0}^{k} T_i C\, x_{k-i}(1) + B\, u_{k+1}(1) \right\|_E < +\infty, \tag{3.119}$$

and continuing this procedure for all $p \in [0, \alpha]$ establishes the first conclusion of theorem 3.17 on $x_k(p)$.

Finally due to (3.116) we also have

$$\sup_{k\in\mathbb{Z}_+} \|y_k(p)\|_W \leq \|T^{-1}\| \sup_{k\in\mathbb{Z}_+} \|C\, x_p(k)\|_W < +\infty, \tag{3.120}$$

and the proof is complete.

Consider now the process (3.105) free of control inputs, i.e. with $u_k(p) \equiv 0$, $0 \leq p \leq \alpha$, $k \geq 0$. This homogeneous version of (3.105) can be represented as

$$\begin{aligned} x_{k+1}(p+1) &= A\, x_{k+1}(p) + B_0\, y_k(p) \\ y_{k+1}(p) &= C\, x_{k+1}(p) + D_1\, y_k(p). \end{aligned} \tag{3.121}$$

At this stage, we introduce the following definitions of stability,

Definition 3.9 (Exponential Stability) *The system (3.121) is said to be exponentially stable if there exists a real scalar q, $0 < q < 1$, such that the inequalities*

$$\|x(k, p, \gamma, d)\|_E \leq \lambda q^k, \quad \|y(k, p, \gamma, d)\|_W \leq \lambda q^k, \tag{3.122}$$

hold for all $\gamma \in B(\mathbb{Z}_+, E)$ and any collection of elements d_0, d_1, $\cdots$, d_α from W, where λ is some positive real scalar.

Definition 3.10 (Weak Exponential Stability) *The system (3.121) is said to be weakly exponentially stable if there exists a real scalar q, $0 < q < 1$, such that the inequalities*

$$\|x(k, p, \gamma, d)\|_E \leq \lambda q^k, \quad \|y(k, p, \gamma, d)\|_W \leq \lambda q^k, \tag{3.123}$$

hold for all $\gamma \in B(\mathbb{Z}_+, E)$ with $\|\gamma(k)\| \leq \omega\eta^k$, $\omega > 0$, $0 < \eta < 1$, and any collection of elements d_0, d_1, $\cdots$, d_α from W, where λ is some positive real scalar.

Note that the term 'exponential stability' arises from the fact that the decrease in the solution $\{x_k(p),\ y_k(p)\}$ with respect to the variable k is required to follow the exponential function $\exp\{k \ln q\}$.

It is now necessary to represent the solutions of (3.121) in the ring of power series. In order to do this, introduce the formal power series representation of $x_k(p)$ and $y_k(p)$ as

$$X(z,p) = \sum_{k=0}^{\infty} x_{k+1}(p)z^k, \quad Y(z,p) = \sum_{k=0}^{\infty} y_{k+1}(p)z^k, \quad p \in [0, \alpha]. \tag{3.124}$$

Substituting these power series expansions into (3.121) gives

$$\begin{aligned} X(z,p+1) &= A\,X(z,p) + zB_0\,Y(z,p) + B_0\,y(0,p) \\ Y(z,p) &= C\,X(z,p) + zD_1\,Y(z,p) + D_1\,y(0,p), \end{aligned} \tag{3.125}$$

and combining these two equations yields

$$\begin{aligned} X(z,p) &= \mathcal{A}^p(z)\,X(z,0) + \sum_{i=0}^{p-1} \mathcal{A}^{p-1-i}(z)\beta(z)d_i, \\ Y(z,p) &= (I - zD_1)^{-1}[C(\mathcal{A}^p(z)X(z,0) + \sum_{i=0}^{p-1} \mathcal{A}^{p-1-i}(z)\beta(z)d_i) + D_1 d_p], \end{aligned} \tag{3.126}$$

where

$$\mathcal{A}(z) = A + zB_0(I - zD_1)^{-1}C, \text{ and } \beta(z) = B_0 + zB_0(I - zD_1)^{-1}D_1. \tag{3.127}$$

The following result now characterises the property of exponential stability of processes described by (3.121), and hence of non-homogeneous processes described by (3.105),

Theorem 3.18 (Exponential Stability) *If the system (3.105) is exponentially stable then the following condition holds*

$$\det(\mathcal{A}(z) - \lambda I) \neq 0 \quad \forall\ |z| = 1,\ |\lambda| \geq 1. \tag{3.128}$$

Proof : *To obtain a contradiction, assume that (3.121) is exponentially stable but the condition of the result does not hold. Then we have* $\det(\mathcal{A}(z_0) - \lambda_0 I) = 0$ *for some* $|z_0| = 1$, $|\lambda_0| \geq 1$, *and there exists a nontrivial vector* $a \in E$ *such that* $\mathcal{A}(z_0)a = \lambda_0 a$.

Consider the solution pair $\{x_k(p), y_k(p)\}$ *of (3.121) generated by the following initial conditions*

$$x_k(0) = az_0^{-k}, \quad y_0(p) = 0, \quad z \in \mathbb{Z}_+, \quad k \in [0, \alpha]. \tag{3.129}$$

Then since the solution $x_k(p)$ *is stable,* $\exists$ *constants* $0 < q < 1$, $\lambda > 0$, *such that*

$$||x_k(p)||_E \leq \lambda q^k, \quad \forall\ p \in [0, \alpha]. \tag{3.130}$$

Since $q < 1$, $\exists$ *an integer* N *such that* $\lambda q^N \leq \frac{1}{2}||a||_E$. *Also,*

$$X(z,p) = \sum_{k=0}^{\infty} x_{k+1}(p) z^k \tag{3.131}$$

can be written as

$$X(z,p) = \mathcal{A}^p(z) X(z,0) = \mathcal{A}^p(z) a \sum_{k=0}^{\infty} (z/z_0)^k. \tag{3.132}$$

Suppose now that the analytic matrix function $\mathcal{A}^p(z)$ *can be expanded into the following convergent power series*

$$\mathcal{A}^p(z) = \sum_{s=0}^{\infty} A_s^{(p)} z^s. \tag{3.133}$$

Then $\exists$ *an integer* $s_0 \geq N$ *such that*

$$\left\| \sum_{i=s_0+1}^{\infty} A_i^{(p)} z_0^i \right\|_E \leq \frac{1}{4} \quad \forall\ p \in [0, \alpha]. \tag{3.134}$$

For the remainder of the proof, the norm of $x_k(p)$ *at* $k = s_0$ *is estimated. Note that* $x_{s_0}(p)$ *is the coefficient of the term* z^{s_0} *in the power series representation of* $X(z,p)$. *Hence (3.132)-(3.134) yields*

$$\begin{aligned} ||x_{s_0}(p)||_E &= \left\| \sum_{i=1}^{s_0} A_i^{(p)} z_0^{i-s_0} a \right\|_E = \left\| \sum_{i=0}^{s_0} A_i^{(p)} z_0^i a \right\|_E = \left\| \sum_{i=0}^{\infty} A_i^{(p)} z_0^i a - \sum_{i=s_0+1}^{\infty} A_i^{(p)} z_0^i a \right\|_E \\ &\geq ||A^{(p)}(z_0) a||_E - \left\| \sum_{i=s_0+1}^{\infty} A_i^{(p)} z_0^i a \right\|_E \geq |\lambda_0|^p ||a||_E - \frac{1}{4}||a||_E \geq \frac{3}{4}||a||_E, \end{aligned} \tag{3.135}$$

and due to the stability of $x_k(p)$, we have

$$||x_{s_0}(p)||_E \leq \lambda q^{s_0} \leq \lambda q^N \leq \frac{1}{2}||a||_E \tag{3.136}$$

which contradicts (3.135).

The following example demonstrates that (3.128) alone is not sufficient for exponential stability of processes described by (3.105),

Example 3.3 *Consider the discrete linear repetitive process with state-space representation*

$$\begin{aligned} x_{k+1}(p+1) &= \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} x_{k+1}(p) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} y_k(p) \\ y_{k+1}(p) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_{k+1}(p). \end{aligned} \tag{3.137}$$

In this case,

$$\mathcal{A}(z) = \begin{bmatrix} 1/2 & z \\ 0 & 1/2 \end{bmatrix} \tag{3.138}$$

and

$$\det[\mathcal{A}(z) - \lambda I] = (1/2 - \lambda)^2 \neq 0 \ \forall \, |z| = 1, \ \lambda \geq 1. \tag{3.139}$$

Consider now the solution of (3.137) under the bounded initial conditions $x_k(0) = \rho$, $k \in \mathbb{Z}_+$, $y_0(p) = 0$, $p \in [0, \alpha]$, where ρ is some constant. Also set $||x_k(p)|| = \max\{x_k^i(p)\}$ where $x_k^i(p)$, $i = 1, 2$, denotes the elements of the state vector on pass k. Then it can easily be shown that for $p \in [0, \alpha]$, $k \geq 0$,

$$\begin{aligned} x_{k+1}^1(p) &= y_{k+1}^2(p) = \left(\frac{2p+1}{2^p}\right)\rho \\ x_{k+1}^2(p) &= y_{k+1}^1(p) = \left(\frac{1}{2^p}\right)\rho. \end{aligned} \tag{3.140}$$

Hence

$$||x_k(p)|| \geq \frac{|\rho|}{2^p}, \quad ||y_k(p)|| = \frac{|\rho|}{2^p} \tag{3.141}$$

and this example is not exponentially stable in the space of bounded functions.

The following result relates to weak exponential stability,

Theorem 3.19 (Weak Exponential Stability) *The system (3.121) is weakly exponentially stable if, and only if, condition (3.128) holds.*

Necessity *: Follows from theorem 3.18*

Sufficiency *: Suppose that (3.128) holds, and note that this means that this condition also holds for $|z| \le 1$ and $|\lambda| \ge 1$. From (3.128) it follows that* $\det(I - \mu\mathcal{A}(z)) \neq 0$ *for $|z| \le 1$, $|\mu| \le 1$, and, since $\phi(z) = \det(I - \mu\mathcal{A}(z))$ is an analytic function, $\exists$ a real scalar $\hat{\rho} > 1$ such that* $\det(I - \mu\mathcal{A}(z)) \neq 0$ *for $|z| < \hat{\rho}$, $|\mu| < \hat{\rho}$. Therefore, (3.133) shows that the inverse matrix $(I - \mu\mathcal{A}(z))^{-1}$ can be written as*

$$(I - \mu\mathcal{A}(z))^{-1} = \sum_{k=0}^{\infty} \mu^k \sum_{i=0}^{\infty} A_i^{(k)} z^i, \quad |z| < \hat{\rho}, \ |\mu| < \hat{\rho}. \tag{3.142}$$

Let $\mu_0 = \frac{\hat{\rho}+1}{2}$ and $z_0 = \frac{\hat{\rho}+1}{2}$. Then clearly the series (3.142) converges at (μ_0, z_0). Hence $\exists$ a real constant $L > 0$ such that

$$|\mu_0^k A_i^{(k)} z_0^i| \le L \quad or \quad \left| \left(\frac{\hat{\rho}+1}{2}\right)^{i+k} A_i^{(k)} \right| \le L. \tag{3.143}$$

This inequality shows that $\|A_i^{(k)}\| \le L \left(\frac{2}{\hat{\rho}+1}\right)^{i+k}$.

It now follows from (3.126) that the power series representation $X(z,p)$ for the solution $x(k,p,\gamma,d)$ to (3.105) may be written as

$$X(z,p) = \mathcal{A}^p(z) \sum_{s=0}^{\infty} \gamma(s) z^s + \sum_{i=0}^{p-1} \mathcal{A}^{p-1-i}(z)\beta(z) d_i. \tag{3.144}$$

For the remainder of the proof, define for power series of the form $\Psi(z) = \sum_{i=0}^{\infty} a(i) z^i$, $a(i) \in E$, the mapping $\sigma_s(\Psi(z)) = a(s)$, $s \in \mathbb{Z}_+$, which is clearly linear. Then

$$\begin{aligned} x(s,p) = \sigma_s(X(z,p)) &= \sigma_s(\mathcal{A}^p(z) \sum_{i=0}^{\infty} \gamma(i) z^i) + \sum_{k=0}^{p-1} \sigma_s(\mathcal{A}^{p-1-k}(z)\beta(z) d_k) \\ &= \sigma_s \left(\sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} A_i^{(p)} \gamma(k-i) z^k \right) z^s \right) + \sum_{k=0}^{p-1} \sigma_s \left(\sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} A_j^{(k)} B_0 D_1^{i-j} \right) z^i \right) d_k \\ &= \left(\sum_{i=0}^{s} A_i^{(p)} \gamma(s-i) \right) + \sum_{k=0}^{p-1} \sum_{j=0}^{s} A_j^{(k)} B_0 D_1^{s-j} d_k. \end{aligned} \tag{3.145}$$

The condition $\det(\mathcal{A}(z)-\lambda)\neq 0$ *for* $|z|=1,\ |\lambda|\geq 1$ *means that* $\det(I-zD_1)\neq 0$ *for* $|z|=1$*. Hence the spectrum of the operator* D_1 *lies within the unit disc* U*, and there exists real scalars* $K>0,\ 0<\theta<1$*, such that* $|D_1^s|\leq K\theta^s\ \forall\ s\in\mathbb{Z}_+$*.*

The above bounds imply that

$$\begin{aligned}&\|x(s,p,\gamma,d)\|_E\\&\quad\leq L\omega\sum_{i=0}^{s}\left(\frac{2}{\hat{\rho}+1}\right)^{p+i}\eta^{s-i}+LK\|B_0\|\ \|d\|\sum_{k=0}^{p-1}\sum_{j=0}^{s}\left(\frac{2}{\hat{\rho}+1}\right)^{p+i}\theta^{s-j}.\end{aligned}\tag{3.146}$$

Define $r=\max\{\eta,\ \frac{2}{\hat{\rho}+1},\ \theta\}$*, and then clearly* $r<1$*. Then*

$$\begin{aligned}\|x(s,p,\gamma,d)\|_E&\leq L\omega\sum_{i=0}^{s}r^{p+s}+LK\|B_0\|\ \|d\|\sum_{k=0}^{p-1}\sum_{j=0}^{s}r^{k+s}\\&=L\omega(s+1)r^{s+1}r^{p-1}+LK\|B_0\|\ \|d\|(s+1)r^{s+1}r^{-1}(1-r^p)(1-r)^{-1},\end{aligned}\tag{3.147}$$

and the desired inequality

$$\|x(s,p,\gamma,d)\|_E\leq\frac{1}{2}\lambda q^s+\frac{1}{2}\lambda q^s=\lambda q^s\tag{3.148}$$

follows immediately where

$$q=\sqrt{r},\ \frac{1}{2}\lambda=\sup_s\{L\omega r^{p-1}(s+1)q^{s+1},\ LK\|B_0\|\ \|d\|r^{-1}(1-r^p)(s+1)q^{s+1}\}.\tag{3.149}$$

From the representation (3.126), it can be seen that the proof of stability for $y_k(p)$ *follows by analogy from the proof for* $x_k(p)$ *given above.*

3.10 Summary

Within this chapter the rigorous stability theory developed by Rogers and Owens for linear repetitive processes with a constant pass length α has been introduced. The theory is based on the abstract representation of the processes in a Banach space setting which was introduced in chapter 2 and covers the two separate concepts of asymptotic stability and stability along the pass. The existence of two separate types of stability for these processes is as expected since the processes depend explicitly on two independent variables. Asymptotic stability is the requirement that bounded sequences of inputs produce bounded sequences of outputs over the pass length, whereas stability along the pass is the requirement that this holds independently of the pass length (i.e. the case of letting $\alpha \longrightarrow +\infty$). Although examples do exist where asymptotic stability is all that is required (eg. (Owens et al., 2000)) or in fact all that is achievable (eg. (Roberts, 2000)), it is the stronger condition of stability along the pass which is of most interest here.

In sections 3.3 and 3.4 the theory developed initially for the general abstract representation of a linear repetitive process has been specifically interpreted for the differential and discrete subclasses of processes, firstly for the simple boundary condition case and then for dynamic boundary conditions.

In the simple boundary condition case, it has been shown that for both subclasses the resulting conditions for stability along the pass can be tested by applying well known tests from 1D linear systems theory.

In terms of $S(E_\alpha, W_\alpha, L_\alpha)$ generated by a differential process with dynamic boundary conditions, there are two cases to deal with:

(i) as $\alpha \longrightarrow +\infty$, allow $N \longrightarrow +\infty$ and $t_j \longrightarrow +\infty$; and

(ii) as $\alpha \longrightarrow +\infty$, keep N and t_j fixed.

In (Owens and Rogers, 2000) it is highlighted that the second case is of the most practical relevance, and hence is what has been considered in the literature to date. How to approach case (i) above remains an open research area.

For the second case above, for both the differential and discrete subclasses of processes, stability along the pass conditions have been given which clearly indicate that the accurate determination of boundary conditions for a given example is vital for correct stability characterisation. In terms of the tests, it is clear that the real problem arises with the condition for asymptotic stability (which is where the dynamic case differs from the simple boundary condition case).

In terms of the differential subclass of processes, in the general dynamic boundary condition case the resulting stability conditions can no longer be tested via standard 1D techniques. The problem of developing computationally efficient stability tests for this subclass is still open.

In terms of the discrete subclass of processes, however, for state initial vectors of the (most general) form

$$x_{k+1}(0) = d_{k+1} + \sum_{j=0}^{\alpha-1} k_j y_k(j) \tag{3.150}$$

the resulting stability along the pass conditions can be tested for via standard linear systems tests. For the details see the reference cited in the text.

For the differential and discrete subclasses of processes, as an immediate consequence of stability along the pass, after a 'sufficiently large' number of passes, the dynamics of the process under consideration may be replaced by those of a stable 1D linear system (or stable limit profile as it is termed here). Clearly strong measures on the following aspects of systems performance is of interest in terms of performance evaluation of a given example:

(i) the rate of approach of the output sequence of pass profiles to the limit profile; and

(ii) the error $y_k - y_\infty$ on a given pass k.

In terms of obtaining computable bounds on these aspects of performance prediction, two routes are available. One approach, the two-dimensional Lyapunov equation route, is fully detailed in chapter 4. Here, the time domain (also termed simulation-based) approach in the discrete case has been introduced. It has been shown that the standard test for stability along the pass involves the evaluation of a potentially large dimensioned matrix for all points on the unit circle in the complex plane. In section 3.6, for the discrete subclass of processes, this condition has been replaced by a one-off computation of a matrix with constant entries. Although the resulting condition for stability along the pass is sufficient in nature only, this potential conservativeness is offset by the availability of performance measures along a given pass from the new conditions at no extra computational cost. The theory presented here is novel and provides the subject of the paper (Benton et al., 1998b).

Within chapter 2 it was shown how certain subclasses of linear repetitive processes can be written in the form of 2D linear systems described by the Roesser or Fornasini-Marchesini state-space models. These 2D systems interpretations have led to the following advance in terms of systems theory for discrete linear repetitive processes. For the standard (i.e. nonsingular) model, a formal equivalence has been shown to exist between the stability along the pass of discrete linear repetitive processes and the BIBO stability of the corresponding Roesser (and hence Fornasini-Marchesini) state-space model interpretation of the process dynamics.

In addition, it has been shown in (Galkowski et al., 1999b) that consideration of the singular model has led to the development of a transition matrix (or fundamental matrix sequence) and hence a general response formula (which calculates the process response to a given input sequence and boundary conditions), which leads to a characterisation of certain reachability/controllability properties. See the cited reference for further details.

This chapter concludes by introducing a Volterra operator based approach to stability analysis of discrete linear repetitive processes and as such remains an area where future research effort should be directed. Although this route is very new, it appears that this approach may play a significant role in the stability analysis of such processes.

Chapter 4

1D and 2D Lyapunov Equations

4.1 Introduction

As a result of the 'equivalence' between the BIBO stability of 2D systems described by the Roesser model (and hence the Fornasini-Marchesini model) and the stability along the pass of discrete linear repetitive processes which has been presented in chapter 3, many well known tests available for the stability analysis of 2D linear systems may be applied to linear repetitive processes. Within this chapter, the question of to what extent a Lyapunov equation based approach to the stability analysis of these processes can be applied is considered.

The most basic aim of using these Lyapunov-type equations is to provide a suitable extension of conventional 1D theory. A review of the literature indicates that the problem of developing a Lyapunov-type equation for 2D linear systems described by, for example, the Roesser state-space model has been approached in essentially two different ways:

(i) the 1D Lyapunov equation approach, so-called because the equation has an identical structure to that for discrete linear time-invariant systems, but with defining matrices which are functions of a complex variable; and

(ii) the so-called 2D Lyapunov equation approach, defined in terms of matrices with constant entries.

Initially the 1D Lyapunov equation is introduced, firstly for the subclass of differential processes with simple boundary conditions - for the discrete case, see (Rogers and Owens, 1996). It is shown how the resulting necessary and sufficient conditions for stability along the pass can be implemented by computations on matrices with constant entries. In the discrete case, this serves as an alternative to the standard stability along the pass tests of chapter 3 which require the computation of the eigenvalues of a potentially large dimensional matrix for all points on the unit circle in the complex plane. It is highlighted how performance measures are available from the resulting condition for stability along the pass which provide computable information on the rate of approach of the output sequence of pass profiles to the limit profile on a given pass. The 1D equation does not, however, provide useful measures of relative stability, i.e. stability margins or robustness to, for example, uncertainties in the model description (unlike the 2D equation case - see chapter 5 for further details of these robustness measures). To conclude this analysis, a 1D Lyapunov equation characterisation of stability along the pass is introduced for a subclass of differential processes possessing dynamic boundary conditions of a special structure (which is of particular interest in terms of classes of delay-differential systems). New strict positive realness tests for the resulting condition are introduced, and the analysis presented here provides the basis for the papers (Benton et al., 2000c) and (Benton et al., 2000d).

In section 4.6 the so-called 2D Lyapunov equation is presented, which is defined in terms of matrices with constant entries. It is shown here how the existence of a positive definite solution pair to this equation, in general, provides a sufficient but not necessary condition for stability along the pass. In particular, a counter example is given which demonstrates that a stable along the pass process does not necessarily have the strictly bounded real property and hence doesn't satisfy the 2D Lyapunov equation. The analysis here can be found in (Benton et al., 1999). In section 4.7, a 2D Lyapunov equation is developed for a 2D Fornasini-Marchesini state-space model of the dynamics of a discrete linear repetitive process which involves the computation of generalised eigenvalues. The analysis of this section can be found in (Benton et al., 2000a). To offset this apparent conservativeness of the sufficient only nature of the 2D Lyapunov equation approach, it is shown in section 4.9 how the equation provides performance measures along a given pass.

4.2 1D Lyapunov Equation Approach

Within this and the following sections, the so-called 1D Lyapunov equation approach to the stability analysis of linear repetitive processes is introduced. Here the differential subclass of processes is considered (for a detailed treatment including relevant proofs see (Owens and Rogers, 1995)) - for the discrete case see, for example, (Rogers and Owens, 1996). As a starting point, consider the unit memory subclass of differential linear repetitive processes with state-space representation (2.13)-(2.14). Also, without loss of generality, set $d_{k+1} \equiv 0,\ k \geq 0$. Then the following result expresses stability along the pass in terms of a 1D Lyapunov equation. This result has been previously reported as theorem 3 in (Owens and Rogers, 1995).

Theorem 4.1 (1D Lyapunov Equation) *Suppose that the pair $\{C, A\}$ is observable and the pair $\{A, B_0\}$ is controllable. Then the extended linear repetitive process $S(E_\alpha, W_\alpha, L_\alpha)_{\alpha \geq \alpha_0}$ generated by (2.13) and (2.14) (with $x_{k+1}(0) \equiv 0,\ k \geq 0$) with $\alpha \geq \alpha_0$ is stable along the pass if, and only if,*

(a) $r(D_1) < 1$,

(b) $|sI_n - A| \neq 0,\ \ Re(s) \geq 0$, *and*

(c) $\exists$ *a rational polynomial matrix solution $P(s)$ of the Lyapunov equation*

$$G^T(-s)P(s)G(s) - P(s) = -I \tag{4.1}$$

bounded in an open neighbourhood of the imaginary axis of the complex plane with the properties that $P(s) \equiv P^T(-s)$ and

$$\beta_1^2 I \leq P(i\omega) = P^T(-i\omega) \leq \beta_2^2 I \quad \forall\ \omega \geq 0 \tag{4.2}$$

for some choices of real scalars $\beta_i \geq 1,\ i = 1,\ 2,$ where $G(s)$ is the interpass transfer-function matrix of the process, derived from the state-space quadruple $\{A, B_0, C, D_1\}$.

Note that the Lyapunov equation (4.1) in theorem 4.1 is identical in structure to that for 1D discrete linear systems, except for the fact that the coefficient matrices are functions of a complex variable. Hence it is termed 1D here to distinguish it from the alternative Lyapunov equation (termed 2D) for processes described by (2.24) and (2.25) which is developed in section 4.6.

Note that the scalars β_i, $i = 1, 2$, in this result play no role in the stability analysis but, as theorem 4.2 below shows, together with $P(s)$ they are the key to obtaining bounds on expected system performance.

As a starting point to the following analysis, assume that the differential process (2.13)-(2.14) is stable along the pass and consider the solution matrix $P(s)$ of the 1D Lyapunov equation of theorem 4.1. Then factorization techniques enable us to write

$$P(s) = F^T(-s)F(s) \tag{4.3}$$

where $F(s)$ is stable and minimum phase, and hence has a stable minimum phase inverse. Also, without loss of generality, let

$$\lim_{|s|\to+\infty} F(s) = P_\infty^{1/2} \tag{4.4}$$

where the matrix on the right hand side of this equation is the unique positive definite square root of $P_\infty > 0$ which solves

$$D_1^T P_\infty D_1 - P_\infty = -I, \tag{4.5}$$

where (4.5) can be obtained by defining $P_\infty := \lim_{|\omega|\to+\infty} P(i\omega)$ and observing that $\lim_{|s|\to+\infty} G(s) = D_1$.

Now consider the differential unit memory process which is free of control inputs, i.e. $u_{k+1}(t) \equiv 0$, $0 \le t \le \alpha$, $k \ge 0$. Then, in this situation, it follows that the process dynamics can be written in terms of the standard (1D) Laplace transform as

$$Y_{k+1}(s) = G(s)\, Y_k(s), \quad k \ge 0. \tag{4.6}$$

Also let

$$\hat{Y}_k(s) = F(s)\, Y_k(s), \quad k \ge 0, \tag{4.7}$$

denote 'filtered' (by the properties of $F(s)$) outputs. Then the following result (theorem 4 in (Owens and Rogers, 1995)) gives bounds on expected performance of the sequence of 'filtered' pass profiles,

Theorem 4.2 (1D Lyapunov Equation - Performance Bounds) *Suppose that $S(E_\alpha, W_\alpha, L_\alpha)$ generated by (2.13)-(2.14) (with zero control inputs and $x_{k+1}(0) = 0,\ k \geq 0$) is stable along the pass and set $X = L_2^m(0, +\infty)$. Then, $\forall\, k \geq 0$,*

$$||\hat{Y}_{k+1}||_X^2 = ||\hat{Y}_k||_X^2 - ||Y_k||_X^2 \tag{4.8}$$

and hence the 'filtered' sequence of pass profiles $\{||\hat{Y}_k||_X\}_{k\geq 0}$ is strictly monotonically decreasing to zero and satisfies, for $k \geq 0$, the inequality

$$||\hat{Y}_k||_X \leq \lambda^k ||\hat{Y}_0||_X \tag{4.9}$$

where

$$\lambda := (1 - \beta_2^{-2})^{1/2} < 1. \tag{4.10}$$

Also the actual output sequence of pass profiles $\{||Y_k||_X\}_{k\geq 0}$ is bounded by

$$||Y_k||_X \leq N\lambda^k ||Y_0||_X \tag{4.11}$$

where

$$N := \beta_2 \beta_1^{-1} \geq 1. \tag{4.12}$$

The 1D Lyapunov equation characterisation of stability along the pass provides the following information on the rate of approach of the output sequence of pass profiles produced by a stable example of the form (2.13) and (2.14) to the limit profile (a stable 1D differential linear system):

(i) the output sequence of 'filtered' pass profiles $\{||\hat{Y}_k||\}_{k\geq 0}$ consists of monotone signals converging to zero at a computable rate in $L_2^m(0, +\infty)$; and

(ii) the actual sequence of output pass profiles $\{||Y_k||\}_{k\geq 0}$ converges at the same geometric rate, but this is no longer necessarily monotonic. This deviation from monotonicity is described by the parameter N computed from the solution of the 1D Lyapunov equation of theorem 4.1 and (4.12).

It should be stressed that, for the discrete subclass of processes, there are no 2D Roesser/Fornasini-Marchesini alternatives to the performance information introduced here.

4.3 Solving the 1D Lyapunov Equation

To solve the 1D Lyapunov equation (and hence stability tests only involving computations on matrices with constant entries) in the general case requires the use of the Kronecker product, denoted $\otimes$, for matrices (as defined in definition A.6). In computational (or testing) terms, only the imaginary axis, i.e. $s = i\omega$, needs to be considered - the extension of this curve can be achieved (if required) by analytic continuation means. In particular, if conditions (a) and (b) of theorem 4.1 hold, the example under consideration is stable along the pass if, and only if, $\exists$ a positive definite Hermitian (denoted PDH) matrix $P(i\omega)$ which solves (4.1).

Suppose that a Hermitian matrix $P(i\omega)$ has been obtained. Then it follows immediately that the PDH requirement on $P(i\omega)$ is equivalent to it satisfying the so-called axis positivity property of Šiljak (Šiljak, 1971). In particular, the following result is an immediate consequence of Šijak's criterion for axis positivity of $P(i\omega)$,

Lemma 4.1 *Under the assumptions of theorem 4.1, differential linear repetitive processes described by (2.13) and (2.14) are stable along the pass if, and only if,*

(a) the conditions (a) and (b) of theorem 4.1 hold, and

(b) the solution matrix $P(i\omega)$ of the 1D Lyapunov equation (4.1) satisfies $P(0) > 0$ and $\det(P(i\omega)) > 0 \quad \forall\ \omega \geq 0$.

Further details on how these conditions can be applied to a particular example can be found in (Rogers et al., 1999).

Now note that the 1D Lyapunov equation (4.1) with $s = i\omega$ can be written as

$$(I_{m^2} - G^T(-i\omega) \otimes G^T(i\omega))S[P(i\omega)] = S[I] \tag{4.13}$$

where $S[\cdot]$ denotes the stacking operator. Also $\exists$ a unique solution matrix $P(i\omega)$ to this equation provided

$$\det(I_{m^2} - G^T(-i\omega) \otimes G^T(i\omega)) \neq 0 \quad \forall\omega. \tag{4.14}$$

Under the controllability and observability assumptions of theorem 4.1, the process is stable along the pass if, and only if, $\exists$ a PDH matrix $P(i\omega)$ which solves (4.1) $\forall\ \omega$. Then if condition (b) of lemma 4.1 holds for some arbitrary value of ω, say ω_o, we have $r(G(i\omega_o)) < 1$. Hence the following are an equivalent set of stability along the pass conditions to theorem 4.1,

Theorem 4.3 *The conditions of theorem 4.1 are equivalent to the following,*

(a) the conditions (a) and (b) of theorem 4.1 hold,

(b) for an arbitrary ω_o,

$$r(G(i\omega_o)) < 1, \quad \textit{and} \tag{4.15}$$

(c) (4.14) holds.

To apply the conditions of theorem 4.3, it is necessary to test these three constant matrices for stability in the 1D sense

(i) D_1 with respect to the unit circle in the complex plane,

(ii) $G(i\omega_o)$ with respect to the unit circle in the complex plane, and

(iii) A with respect to the imaginary axis in the complex plane,

and the determinant condition of (4.14), and hence this test is no more computationally efficient than alternatives.

The following result then gives alternative conditions for stability along the pass which are expressed in terms of the eigenvalues of constant matrices,

Theorem 4.4 *Suppose that the controllability and observability assumptions of theorem 4.1 hold. Then $S(E_\alpha, W_\alpha, L_\alpha)$ generated by (2.13) and (2.14) with $x_{k+1}(0) \equiv 0$, $k \geq 0$, is stable along the pass if, and only if,*

(a) conditions (a) and (b) of theorem 4.1 hold,

(b) (4.15) holds, and

(c)

$$\det(\eta^2 X_1 + \eta X_2 + X_3) \neq 0 \quad \forall\ \eta = i\omega,\ \forall\ \omega, \tag{4.16}$$

where

$$X_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & I \otimes A^T - A^T \otimes I \end{bmatrix} \quad \textit{and}$$

$$X_3 = \begin{bmatrix} I - D_1^T \otimes D_1^T & I \otimes B_0^T & B_0^T \otimes D_1^T & D_1^T \otimes D_1^T \\ D_1^T \otimes C^T & -I \otimes A^T & 0 & 0 \\ C^T \otimes I & 0 & -A^T \otimes I & 0 \\ C^T \otimes C^T & 0 & 0 & A^T \otimes A^T \end{bmatrix}. \tag{4.17}$$

Proof *: See (Rogers et al., 1999).*

In this last result, the matrices X_i, $1 \leq i \leq 3$, are composed of compatibly dimensioned Kronecker products of the matrices A, B_0, C and D_1 respectively. Also it can be seen that the matrix X_1 of (4.17) is singular and therefore the solutions cannot be obtained directly using existing software. Instead, extensive, but routine, algebraic manipulations must be performed to reformulate (4.16) as a condition involving a first order matrix polynomial which can easily be tested via existing software for computing generalised eigenvalues. See (Rogers et al., 1999) for a further discussion of this point.

4.4 Differential Processes with Dynamic Boundary Conditions Stability Tests

Within this section we develop a 1D Lyapunov equation characterisation of stability along the pass for a subclass of differential linear repetitive processes in the presence of so-called dynamic pass state initial conditions. The analysis presented here forms the subject for the papers (Benton et al., 2000c) and (Benton et al., 2000d).

Consider the subclass of unit memory differential linear repetitive processes of the form (2.13) with $m = n$, $C \equiv I_n$ and $D_1 \equiv 0$, i.e.

$$\dot{y}_{k+1}(t) = A\, y_{k+1}(t) + B\, u_{k+1}(t) + B_0\, y_k(t), \quad 0 \leq t \leq \alpha, \;\; k \geq 0. \tag{4.18}$$

Here we consider the process subject to a subclass of dynamic boundary conditions of the general form of (2.18), with $N = 1$, $K_1 = I_n$ and $t_1 = \alpha$, which are of particular interest in terms of links with delay-differential systems and also repetitive control schemes. In this case, the boundary conditions for (4.18) are

$$y_{k+1}(0) = d_{k+1} + y_k(\alpha), \quad k \geq 0. \tag{4.19}$$

Now note that in the testing for stability along the pass of processes with dynamic boundary conditions, it is the first part of theorem 3.7 which cannot be tested by direct application of 1D linear systems tests. The aim of the analysis in this section is to develop a 1D Lyapunov equation based interpretation of this condition for the special case of differential processes with the state-space representation (4.18) and dynamic boundary conditions of the form (4.19).

For asymptotic stability of differential processes described by (4.18) and (4.19), condition (a) of theorem 3.7 requires that all solutions of

$$|zI_n - e^{(A+z^{-1}B_0)\alpha}| = 0 \tag{4.20}$$

have modulus strictly less than unity $\forall\, \alpha \geq 0$. Now write $z = e^{s\alpha}$, and hence (4.20) reduces to the requirement that all solutions of

$$|sI_n - F(s)| = 0 \tag{4.21}$$

have strictly negative real parts where

$$F(s) = A + B_0 e^{-s\alpha}. \tag{4.22}$$

It can also be shown (using results in (Kamen, 1980)) that (4.21) reduces to the requirement that

$$|sI_n - F(e^{-i\omega\alpha})| \neq 0 \quad \forall\ \mathrm{Re}(s) \geq 0 \quad \forall\ \omega \in [0, 2\pi]. \tag{4.23}$$

The following result now expresses the condition of (4.23) in terms of a 1D Lyapunov equation (see (Brierley et al., 1982) for a similar approach for a class of differential linear systems with commensurate time delays),

Theorem 4.5 *Condition (4.21) holds if, and only if, for a given PDH matrix $Q(e^{i\omega})$, $\omega \in [0, 2\pi]$, the solution $P(e^{i\omega})$ of the matrix Lyapunov equation*

$$F^*(e^{i\omega})P(e^{i\omega}) + P(e^{i\omega})F(e^{i\omega}) = -Q(e^{i\omega}) \tag{4.24}$$

is PDH $\forall\, \omega \in [0, 2\pi]$, where $*$ denotes the complex conjugate transpose operation.

Proof : *To show sufficiency first note that, for any fixed $\omega_o \in [0, 2\pi]$, the matrix $F(e^{i\omega_o})$ is an $n \times n$ matrix with complex elements. Also let λ_0 be an eigenvalue of this matrix with corresponding eigenvector w_o. Then we have*

$$\begin{aligned} F(e^{i\omega_o})w_o &= \lambda_0 w_o \\ w_o^* F^*(e^{i\omega_o}) &= \bar{\lambda}_0 w_o^* \end{aligned} \tag{4.25}$$

where the bar denotes the complex conjugate operation. Now pre-multiplying (4.24) by w_o^* and post-multiplying by w_o yields

$$w_o^* Q(e^{i\omega_o})w_o = -w_o^*\{F^*(e^{i\omega_o})P(e^{i\omega_o}) + P(e^{i\omega_o})F(e^{i\omega_o})\}w_o. \tag{4.26}$$

Then, since both $P(e^{i\omega})$ and $Q(e^{i\omega})$ are PDH matrices $\forall \omega \in [0, 2\pi]$, and using (4.25) above, it follows that

$$w_o^* Q(e^{i\omega_o})w_o = -(\bar{\lambda}_0 + \lambda_0)w_o^* P(e^{i\omega_o})w_o. \tag{4.27}$$

Then we have

$$Re(\lambda) = \frac{1}{2}(\bar{\lambda} + \lambda) = -\frac{1}{2}\left(\frac{w^* Q(e^{i\omega})w}{w^* P(e^{i\omega})w}\right) < 0 \tag{4.28}$$

where now λ is any eigenvalue of $F(e^{i\omega})$ with corresponding eigenvector w. Hence sufficiency of theorem 4.5 holds.

To show necessity, consider (4.24) with an arbitrary PDH matrix $Q(e^{i\omega})$ on $[0, 2\pi]$. Then if (4.23) holds, it can be shown (Kamen, 1980) that all eigenvalues of the matrix $F(e^{i\omega})$ have strictly negative real parts $\forall\, \omega \in [0, 2\pi]$. Now define

$$P(e^{i\omega}) := \int_0^\infty e^{F^*(e^{i\omega})t} Q(e^{i\omega}) e^{F(e^{i\omega})t}\, dt \tag{4.29}$$

which is well defined since the eigenvalues of $F(e^{i\omega})$ (and $F^*(e^{i\omega})$) are in the left half of the complex plane. Also $P^*(e^{i\omega}) = P(e^{i\omega})$, $\forall\, \omega \in [0, 2\pi]$, and

$$\begin{aligned} F^*(e^{i\omega})P(e^{i\omega}) + P(e^{i\omega})F(e^{i\omega}) &= \int_0^\infty \Big(F^*(e^{i\omega})e^{F^*(e^{i\omega})t}Q(e^{i\omega})e^{F(e^{i\omega})t} \\ &\quad + e^{F^*(e^{i\omega})t}Q(e^{i\omega})e^{F(e^{i\omega})t}F(e^{i\omega})\Big)\, dt \\ &= \int_0^\infty \left(\frac{d}{dt}e^{F^*(e^{i\omega})t}Q(e^{i\omega})e^{F(e^{i\omega})t} \right. \\ &\quad \left. + e^{F^*(e^{i\omega})t}Q(e^{i\omega})\frac{d}{dt}e^{F(e^{i\omega})t}\right)\, dt \\ &= \int_0^\infty \frac{d}{dt}\left(e^{F^*(e^{i\omega})t}Q(e^{i\omega})e^{F(e^{i\omega})t}\right)\, dt \\ &= -Q(e^{i\omega}) \end{aligned} \tag{4.30}$$

where this last equality follows from the fact that $e^{F(e^{i\omega})t} \longrightarrow 0$, *and* $e^{F^*(e^{i\omega})t} \longrightarrow 0$, *as* $t \longrightarrow +\infty$. *Hence we have that*

$$F^*(e^{i\omega})P(e^{i\omega}) + P(e^{i\omega})F(e^{i\omega}) = -Q(e^{i\omega}),\ \forall\, \omega \in [0, 2\pi] \tag{4.31}$$

and $P^*(e^{i\omega}) = P(e^{i\omega})\ \forall\, \omega \in [0, 2\pi]$, *as required.*

Now define

$$F(z)|_{z=e^{i\omega}} := F_1(\omega) + iF_2(\omega) \tag{4.32}$$

where $F_1(\omega)$ and $F_2(\omega)$ are real $n \times n$ matrices. Also for a fixed $\omega_o \in [0, 2\pi]$, $F(e^{i\omega_o})$ is an $n \times n$ matrix with complex entries which can be written as

$$F(e^{i\omega_o}) = F_1(\omega_o) + iF_2(\omega_o). \tag{4.33}$$

Write the system $\dot{y} = F(e^{i\omega_o})y$ as

$$\dot{y}_r + i\dot{y}_i = (F_1(\omega_o) + iF_2(\omega_o))(y_r + iy_i) \tag{4.34}$$

where y_r and y_i denote the real and imaginary parts of y respectively. Then separating (4.34) into real and imaginary parts now yields

$$\begin{bmatrix} \dot{y}_r \\ \dot{y}_i \end{bmatrix} = \begin{bmatrix} F_1(\omega_o) & -F_2(\omega_o) \\ F_2(\omega_o) & F_1(\omega_o) \end{bmatrix} \begin{bmatrix} y_r \\ y_i \end{bmatrix}. \tag{4.35}$$

Introduce

$$\hat{F}(\omega) := \begin{bmatrix} F_1(\omega) & -F_2(\omega) \\ F_2(\omega) & F_1(\omega) \end{bmatrix}. \tag{4.36}$$

Then, in the SISO case, a necessary and sufficient condition for condition (a) of theorem 3.7 to hold is that $F_1(\omega) < 0\ \forall\, \omega \in [0, 2\pi]$, i.e. 1D stability of the real part of $F(e^{i\omega})$. Also in this case

$$\det(sI - \hat{F}(\omega)) = s^2 - 2f_1(\omega)s + f_1^2(\omega) + f_2^2(\omega) \tag{4.37}$$

where $f_j(\omega)$, $j = 1, 2$, are the SISO elements of $F_j(\omega)$, $j = 1, 2$, in (4.36).

Hence, in the SISO case, a necessary and sufficient condition for (a) of theorem 3.7 is that $f_1(\omega) < 0\ \forall\, \omega \in [0, 2\pi]$.

Example 4.1 *As an example, consider the following unforced differential process*

$$\dot{y}_{k+1}(t) = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix} y_{k+1}(t) + \begin{bmatrix} 0 & 0 \\ 0 & -c \end{bmatrix} y_k(t) \tag{4.38}$$

where a, b and c are positive real numbers, subject to the boundary conditions

$$y_{k+1}(0) = y_k(\alpha) \tag{4.39}$$

i.e. a special case of (4.19). Then in this case

$$F(z) = \begin{bmatrix} 0 & 1 \\ -a & -b - cz \end{bmatrix}, \; z = e^{i\omega}. \tag{4.40}$$

The solution of the Lyapunov equation (4.24) with $Q = I_2$ is

$$P(z) = \frac{1}{ay} \begin{bmatrix} |b + cz|^2 + a(a+1) & b + c\bar{z} \\ b + cz & a + 1 \end{bmatrix}, \; y = 2(b + c\cos\omega). \tag{4.41}$$

Then

$$det(P(z)) = \frac{|b + cz|^2 + (a+1)^2}{ay^2} \tag{4.42}$$

and $P(e^{i\omega})$ is PDH $\forall\, \omega \in [0, 2\pi]$ if, and only if, $y > 0$, i.e. if, and only if,

$$b + c\cos\omega > 0 \,\forall\, \omega \in [0, 2\pi]. \tag{4.43}$$

Hence (4.38) satisfies (a) of theorem 3.7 if, and only if, $b > c$.

4.5 Strict Positive Realness Based Tests

Within this section tests for condition (4.23) are developed using a strict positive realness approach. The analysis here forms the basis of the paper (Benton et al., 2000c).

First note that, on setting $z = e^{i\omega}$, (4.23) is equivalent (Kamen, 1980) to

$$\Delta(s, z) := \det(sI_n - F(z)) \neq 0, \; \mathrm{Re}(s) \geq 0, \; |z| = 1 \tag{4.44}$$

or

$$\Delta(s, e^{i\omega}) \neq 0, \; \mathrm{Re}(s) \geq 0, \; \omega \in [0, 2\pi]. \tag{4.45}$$

This is an equation with complex coefficients which are polynomial in $e^{i\omega}$ and it is required that all its roots should lie in the open left half of the s-plane. Using 'classical' root clustering theory, the condition for this (see, for example, (Jury, 1973)) is that the Hermite matrix obtained from the coefficients in $\Delta(s, e^{i\omega})$ is positive definite or, alternatively, the inner-wise matrix obtained from the coefficients must be positive inner-wise.

Consider the complex polynomial

$$B(s) = \sum_{i=0}^{n} b_i s^i. \tag{4.46}$$

Then the Hermite matrix, H, associated with $B(s)$ is obtained as follows,

$$H = \{h_{p,q}\} \tag{4.47}$$

where

$$\begin{aligned}
h_{p,q} &= 2(-1)^{\frac{p+q}{2}} \sum_{j=1}^{p} (-1)^j \operatorname{Re}(b_{n-j-1}\bar{b}_{n-p-q+j}), \quad p+q = \text{even}, \quad p \leq q, \\
h_{p,q} &= 2(-1)^{\frac{p+q-1}{2}} \sum_{j=1}^{p} (-1)^j \operatorname{Im}(b_{n-j-1}\bar{b}_{n-p-q+j}), \quad p+q = \text{odd}, \quad p \leq q \\
\text{and} \quad & h_{p,q} = h_{q,p}
\end{aligned} \tag{4.48}$$

(where Re and Im denote the real and imaginary parts of a complex number respectively). Also it can be shown (Kamen, 1980; Jury, 1973) that H positive definite $\forall\, \omega \in [0, 2\pi]$ (or $|e^{i\omega}| \in [-1, 1]$) is equivalent to the following conditions

$$H(e^{i0}) = H(1) > 0 \tag{4.49}$$

$$\det(H(e^{i\omega})) > 0 \,\forall\, \omega \in [0, 2\pi]. \tag{4.50}$$

The checking of (4.49) is straightforward and the more difficult condition of (4.50) can be checked using a positivity test. This is based on the fact that $\det(H(e^{i\omega}))$ is a function of $\cos\omega, \cos 2\omega, \cdots$ and, on setting $x = \cos\omega$, $\det(H(e^{i\omega}))$ becomes a function of x and its powers. Hence (4.50) becomes

$$\det(H(e^{i\omega})) = F(x) > 0, \; x \in [-1, 1]. \tag{4.51}$$

This last condition holds provided $F(x)$ has no real roots in the interval $[-1, 1]$. Also introduce the change of variable (a bilinear transform)

$$x = \frac{u-1}{u+1} \tag{4.52}$$

into (4.51) to yield the equivalent condition that

$$F_1(u) > 0, \ u \in [0, +\infty). \tag{4.53}$$

Then this condition can be checked using any of the computational positivity tests (Jury, 1973).

In the remainder of this section we develop a computationally more feasible alternative to the approach just presented. The starting point is to note that the condition to be tested here can be expressed as the requirement that a two variable polynomial of the general form

$$a(s,z) := s^p + \sum_{j=0}^{p-1} \sum_{i=0}^{q} a_{ij} s^j z^i \tag{4.54}$$

should satisfy

$$a(s,z) \neq 0, \ \mathrm{Re}(s) \geq 0, \ |z| \leq 1. \tag{4.55}$$

Firstly we show how (4.55) can be reduced to a one-dimensional problem by showing how it is equivalent to the positive realness of a certain 1D rational transfer-function matrix, which leads to a numerically efficient testing algorithm. The following analysis requires as background the results summarized next relating to the so-called strictly bounded real lemma (see for example (Anderson and Vongpanitlerd, 1973) for a detailed treatment).

Definition 4.1 (Strictly Bounded Real Matrices) *A real rational transfer-function matrix $G(s) = C_1(sI_n - A_1)^{-1}B_1$ is termed strictly bounded real if, and only if, the matrix A_1 is Hurwitz (i.e. all its eigenvalues have negative real parts) and*

$$I - G^T(-i\omega)G(i\omega) > 0 \, \forall \, \omega \in \mathbb{R}. \tag{4.56}$$

The well known strictly bounded real lemma (Anderson and Vongpanitlerd, 1973) takes the following form here,

Lemma 4.2 (Strictly Bounded Real Lemma) *Suppose that $G(s)$ is a proper rational transfer-function matrix and let $\{A_1, B_1, C_1, D_1\}$ be an associated minimal realization. Then this transfer-function matrix is strictly bounded real if, and only if, $\exists$ a real symmetric positive definite matrix P such that*

$$M = \begin{bmatrix} A_1^T P + PA_1 + C_1^T C_1 & PB_1 + C_1^T D_1 \\ (PB_1 + C_1^T D_1)^T & D_1^T D_1 - I \end{bmatrix} < 0. \tag{4.57}$$

One characterization of this strictly bounded real property (for the proof see, for example, (Gu and Lee, 1989)) is that $G(s)$ has this property if, and only if, for any given real symmetric matrix $Q > 0$, $\exists\, \epsilon > 0$ such that,

(i)

$$I - D_1^T D_1 > 0, \quad \text{and} \tag{4.58}$$

(ii) the algebraic Riccati equation

$$A_1^T P + PA_1 + (PB_1 + C_1^T D_1)(I - D_1^T D_1)^{-1}(B_1^T P + D_1^T C_1) + C_1^T C_1 + \epsilon Q = 0 \tag{4.59}$$

has a positive definite solution P.

Also the requirement for a minimal realization can be relaxed by the following result (also proved in (Gu and Lee, 1989)),

Lemma 4.3 *Suppose that $G(s)$ is strictly proper and let $\{A_1, B_1, C_1\}$ be a state-space realization with the pair $\{A_1, B_1\}$ controllable. Then $G(s)$ is strictly bounded real if, and only if, for any given real symmetric matrix $Q > 0$, $\exists$ a scalar $\epsilon > 0$ such that the algebraic Riccati equation*

$$A_1^T P + PA_1 + PB_1 B_1^T P + C_1^T C_1 + \epsilon Q = 0 \tag{4.60}$$

has a positive definite solution P.

Note that if (4.60) has a solution $P > 0$ for a given $\epsilon^* > 0$ then for any $\epsilon \in [0, \epsilon^*]$ this equation admits at least one positive definite solution.

If $G(s)$ is not strictly proper the following result (again from (Gu and Lee, 1989)) can be used,

Lemma 4.4 *Suppose that $\{A_1, B_1, C_1, D_1\}$ is a minimal realization of $G(s)$. Then $G(s)$ is strictly bounded real if, and only if, $G_m(s)$ is strictly bounded real where $G_m(s)$ is realized by $\{A_m, B_m, C_m\}$ where*

$$\begin{aligned} A_m &= A_1 + B_1(I - D_1^T D_1)^{-1} D_1^T C_1, \\ B_m &= B_1(I - D_1^T D_1)^{-\frac{1}{2}}, \\ C_m &= (I - D_1 D_1^T)^{-\frac{1}{2}} C_1. \end{aligned} \tag{4.61}$$

The key point here is that if A_1 is Hurwitz then this implies that A_m is Hurwitz and also the controllability of $\{A_1, B_1\}$ implies the controllability of $\{A_m, B_m\}$.

To apply these results, first note the following result (proved in (Gu and Lee, 1986)),

Lemma 4.5 *Consider the two variable polynomial $a(s,z)$ and suppose that $a(0,z) \neq 0 \,\forall\, |z| \leq 1$. Then (4.55) holds if, and only if,*

(a) $a(s,0)$ is Hurwitz, and

(b)

$$a(s,z) \neq 0, \; Re(s) = 0, \; |z| \leq 1. \tag{4.62}$$

Clearly it is the second of these conditions which is the most difficult to test. In what follows we develop a numerically efficient test based on treating $a(s,z)$ as a polynomial, denoted $a_s(z)$, in z with coefficients which are polynomials in s with s taking values on the extended imaginary axis of the complex plane.

The key point to note now is that (4.62) is true if, and only if, $a_s(z)$ has all its roots outside the unit circle for all s on the imaginary axis. Hence we can apply a 1D stability test to this condition using a point-wise approach, and here we use the Schur-Cohn test expressed in the following form (from (Ptak and Young, 1980)).

Lemma 4.6 (Schur-Cohn Test) *Let $a(z) = a_0 + a_1 z + \cdots + a_n z^n$, $a_0 \neq 0$, $a_n \neq 0$, be a polynomial with complex coefficients a_k, $k = 0, 1, \cdots, n$. Define also the triangular Toeplitz matrices*

$$D := \begin{bmatrix} \overline{a}_0 & \overline{a}_1 & \cdots & \overline{a}_{n-2} & \overline{a}_{n-1} \\ 0 & \overline{a}_0 & \overline{a}_1 & \cdots & \overline{a}_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \overline{a}_0 & \overline{a}_1 \\ 0 & 0 & \cdots & \cdots & \overline{a}_0 \end{bmatrix} \tag{4.63}$$

and

$$N := \begin{bmatrix} a_n & a_{n-1} & \cdots & a_2 & a_1 \\ 0 & a_n & a_{n-1} & \cdots & a_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_n & a_{n-1} \\ 0 & 0 & \cdots & \cdots & a_n \end{bmatrix}. \tag{4.64}$$

Then $a(z) \neq 0 \,\forall\, |z| \leq 1$ if, and only if, the Hermitian matrix

$$\Phi = D^*D - N^*N \tag{4.65}$$

is strictly positive (where again $*$ denotes the complex conjugate transpose operation).

Note also that if $a_0 \neq 0$ then if Φ is PDH $\Leftrightarrow$ the matrix $G = ND^{-1}$ is a strict contraction (see appendix definition A.5).

In the case under consideration here, the coefficient a_k is a polynomial in s, $s = i\omega$. Hence $\overline{a}_k = a_k(-s)$, $k = 0, 1, \cdots, n$. Also the triangular Toeplitz matrices D and N of (4.63) and (4.64) respectively can be constructed for this case. Similarly, define

$$\Phi(s) = D^T(-s)D(s) - N^T(-s)N(s) \tag{4.66}$$

and

$$G(s) = N(s)D^{-1}(s). \tag{4.67}$$

Then a simple controllable realization for $G(-s)$ is defined as follows

$$\hat{A} = \begin{bmatrix} -\hat{A}_1 & -\hat{A}_2 & -\hat{A}_3 & \cdots & -\hat{A}_p \\ I_q & 0 & 0 & \cdots & 0 \\ 0 & I_q & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \cdots & \cdots & 0 & I_q & 0 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} I_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \hat{C}^T = \begin{bmatrix} \hat{C}_1^T \\ \vdots \\ \hat{C}_p^T \end{bmatrix} \tag{4.68}$$

where

$$\hat{A}_{p-j} = \begin{bmatrix} a_{0j} & a_{1j} & a_{2j} & \cdots & a_{q-1j} \\ 0 & a_{0j} & a_{1j} & \cdots & a_{q-2j} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_{0j} & a_{1j} \\ 0 & 0 & \cdots & 0 & a_{0j} \end{bmatrix} \tag{4.69}$$

and

$$\hat{C}_{p-j} = (-1)^j \begin{bmatrix} a_{qj} & a_{q-1j} & a_{q-2j} & \cdots & a_{1j} \\ 0 & a_{qj} & a_{q-1j} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & a_{q-2j} \\ 0 & \cdots & \cdots & a_{qj} & a_{q-1j} \\ 0 & 0 & \cdots & 0 & a_{qj} \end{bmatrix} \tag{4.70}$$

are upper triangular Toeplitz matrices with real a_{kj} as defined in (4.54).

The next stage is to show that (4.55) is equivalent to $G(-s)$ being bounded real. To do this, first take $G(-s) = \hat{C}(sI - \hat{A})^{-1}\hat{B}$ as defined by (4.68). Then (a) of lemma 4.5 implies that $\det(sI - \hat{A})) = \det(D(-s)) = (a(s,0))^n$ is Hurwitz and hence $G(-s)$ is stable. Using (b) of lemma 4.5 we now have that $\Phi(i\omega)$ is PDH $\forall \omega \in \mathbb{R}$ and this, in turn, is equivalent to $G(-i\omega)$ being a strict contraction for each $\omega \in \mathbb{R}$. Hence $G(-s)$ is strictly bounded real.

Suppose now that $G(-s)$ is strictly bounded real. Then $\det(sI - \hat{A})$ is Hurwitz and hence (a) of lemma 4.5 holds. Also, since $G(-i\omega)$ is a strict contraction for each ω, this implies, by the Schur-Cohn test, that (b) of lemma 4.5 holds.

The arguments just given establish the following result,

Theorem 4.6 *Consider the two-variable polynomial $a(s,z)$ defined by (4.54) and $G(-s)$ defined by the state-space matrices of (4.68). Suppose also that $a(0,z) \neq 0 \; \forall \, |z| = 1$. Then this polynomial satisfies (4.55) if, and only if, $G(-s)$ is strictly bounded real.*

This leads immediately to the following algorithm for testing (4.55),

1. Input p, q and a_{ij} as defined in (4.54).
2. Test if $a(s,0)$ is Hurwitz and, if not, then stop since (4.55) does not hold (and hence the example under consideration is not stable along the pass).
3. Construct the matrices $\hat{A}$, $\hat{B}$, $\hat{C}$ and choose a positive definite matrix Q and a positive real scalar ϵ to solve the algebraic Riccati equation (4.60). If this equation has a solution then (4.55) holds. In which case proceed to test the other conditions for stability along the pass.

Note that the realisation defined by (4.68) may not be minimal and hence there could be numerical problems in solving the algebraic Riccati equation if the product pq is large. Hence an input normal realisation (Moore, 1981) should be used to obtain a minimal realisation prior to testing $G(-s)$ for the strict bounded realness property.

It is possible to avoid computing the solution of the algebraic Riccati equation here. This is based on the the fact that since $G(-s)$ is strictly proper, it is guaranteed to be strictly bounded real if $\det(I - G^T(-s)G(s)) \neq 0 \,\forall\, \mathrm{Re}(s) = 0$ or, equivalently, $\det(\Phi(s)) \neq 0 \,\forall\, \mathrm{Re}(s) = 0$. Also since we are using a minimal realization of $G(-s)$ it can be shown that this transfer-function matrix is strictly bounded real if, and only if, the Hamiltonian matrix

$$H_a := \begin{bmatrix} \hat{A} & \hat{B}\hat{B}^T \\ -\hat{C}^T\hat{C} & -\hat{A}^T \end{bmatrix} \tag{4.71}$$

has no purely imaginary eigenvalues. Note that the dimensions of this matrix are $2pq \times 2pq$ and hence if pq is 'large' then the eigenvalue computation cannot be expected to produce 'high accuracy' results.

Example 4.2 *As an example, suppose that*

$$a(s, z) = s + \gamma + (\beta + \lambda s)z \tag{4.72}$$

where $|\gamma| \neq |\beta|$ *and* $\gamma > 0$. *In this case, (4.55) clearly only holds if, and only if,*

$$G(-s) = \frac{\lambda s + \beta}{s + \gamma} \tag{4.73}$$

is strictly bounded real. Now set

$$A_m = \frac{\beta\lambda - \gamma}{1 - \lambda^2},\ B_m = 1,\ C_m = \frac{\beta - \gamma\lambda}{1 - \lambda^2} \tag{4.74}$$

(as per (4.61)) and hence strict bounded realness of $G(-s)$ *implies* $A_m < 0$. *Also let* P *be the solution of (4.60) with* $Q = 1$ *and then*

$$P^2 + 2A_m P + C_m^2 + \epsilon = 0. \tag{4.75}$$

Now we have that $P > 0$ *requires that* $A_m^2 > C_m^2 + \epsilon$ *which holds if, and only if,* $\gamma > |\beta|$ *(since* $A_m < 0$, $\gamma > 0$*). Hence we have stability when*

$$|\lambda| < 1,\ \gamma > |\beta| \geq 0. \tag{4.76}$$

4.6 The 2D Lyapunov Equation Approach

Within this section the so-called 2D Lyapunov equation approach to the stability analysis of linear repetitive processes is introduced. The analysis introduced here has been presented in (Benton et al., 1999) and (Benton et al., 2000a). Consider the subclass of unit memory discrete linear repetitive processes with state-space model (2.24) and simple boundary conditions (2.25). The starting point for the analysis presented in this section is the following set of necessary and sufficient conditions for stability along the pass which have been reported previously, but the proof here is more direct.

Theorem 4.7 (Stability along the Pass) *(Rogers and Owens, 1993) For the unit memory discrete process of (2.24)-(2.25), suppose that the pair $\{A, B_0\}$ is controllable and the pair $\{C, A\}$ is observable. Then the process is stable along the pass if, and only if,*

(a) all eigenvalues of the matrix D_1 have modulus strictly less than unity,

(b) all eigenvalues of the matrix A have modulus strictly less than unity, and

(c)

$$\rho(z_1, z) := det \begin{bmatrix} z_1 I_n - A & -B_0 \\ -C & zI_m - D_1 \end{bmatrix} \neq 0 \tag{4.77}$$

$\forall\ |z_1| \geq 1,\ \ |z| \geq 1.$

Proof : *In effect this consists of showing that (a), (b) and (c) here are equivalent to the conditions of theorem 3.3. To show necessity, first note that, since the spectrum of L_α, $\sigma(L_\alpha) = \sigma(D_1)$, the spectral radius of L_α in this case is independent of α, and hence, from theorem 3.3, we have $r_\infty = r(D_1)$, and hence condition (a) holds.*

Consider now the solution ζ of the equation

$$(zI - L_\alpha)\zeta = \zeta_0 \tag{4.78}$$

for some arbitrary $\zeta_0 \in E_\alpha$ and z such that $|z| \geq \lambda$ with $\lambda \in (r_\infty, 1)$. Equivalently, (4.78) can be written in the state-space form

$$\begin{bmatrix} z_1 I_n - A & -B_0 \\ -C & zI_m - D_1 \end{bmatrix} \begin{bmatrix} x(z_1) \\ \zeta(z) \end{bmatrix} = \begin{bmatrix} \zeta_0(z_1) \\ \zeta_0(z) \end{bmatrix} \tag{4.79}$$

with state vector $x(i)$, $i = 0,\ 1,\ 2,\ \cdots,\ x(0) = 0$, by applying the z-transform with variable denoted by z_1. It now follows from a routine argument that the existence of a uniform bound M_0 is equivalent to

$$\rho(z_1, z) \neq 0, \quad |z_1| \geq 1 - \epsilon,\ \ |z| \geq \lambda \tag{4.80}$$

where $\epsilon > 0$ is some real number. The observability assumption on $\{C, A\}$ guarantees that there are no 'hidden' unstable modes and the controllability assumption on $\{A, B_0\}$ ensures that all system modes are excited.

Now, use of Schur's formula yields

$$\rho(z_1, z) = \det(zI_m - D_1)\det(z_1 I_n - A - B_0(zI_m - D_1)^{-1}C) \tag{4.81}$$

and hence, given condition (a) of the result, (4.80) reduces to the requirement that $\exists$ a real number $\epsilon > 0$ such that

$$\det(z_1 I_n - A - B_0(zI_m - D_1)^{-1}C) \neq 0, \quad |z_1| \geq 1 - \epsilon, \ |z| \geq \lambda. \tag{4.82}$$

This requires that

$$|\det(z_1 I_n - A - B_0(zI_m - D_1)^{-1}C)| \geq (|z_1| - 1 + \epsilon)^n, \quad |z_1| \geq 1, \ |z| \geq \lambda, \tag{4.83}$$

and considering $|z| \longrightarrow +\infty$ now yields condition (b). The proof of necessity is completed by noting that

$$|\rho(z_1, z)| \geq (|z| - r_\infty)^m (|z_1| - 1 + \epsilon)^n > 0. \tag{4.84}$$

To prove sufficiency, first note that (a) trivially implies that $r_\infty < 1$. Consider also

$$\rho(z_1, z) \neq 0, \quad |z_1| \geq 1, \ |z| \geq 1 \tag{4.85}$$

which, since $r(D_1) < 1$, reduces to

$$\det(z_1 I_n - A - B_0(zI_m - D_1)^{-1}C) \neq 0, \quad |z_1| \geq 1, \ |z| \geq 1. \tag{4.86}$$

Also $(zI_m - D_1)^{-1}$ is strictly proper and this fact combined with $r(A) < 1$ (condition (b)) yields, for some $\epsilon > 0$, $r \gg 0$ and $\lambda \in (r_\infty, 1)$,

$$\det(z_1 I_n - A - B_0(zI_m - D_1)^{-1}C) \neq 0 \tag{4.87}$$

if either $|z_1| \geq 1 - \epsilon$ and $|z| \geq r$ and/or $|z_1| \geq r$ and $|z| \geq \lambda$. Consequently it only remains to consider (4.87) on the compact set $\{z_1 : 1 - \epsilon \leq |z_1| \leq r\} \times \{z : 1 \leq |z| \leq r\}$. A routine argument based on this fact leads, for some $\epsilon > 0$, and $\lambda \in (r_\infty, 1)$ to

$$\det(z_1 I_n - A - B_0(zI_m - D_1)^{-1}C) \neq 0, \quad |z_1| \geq 1 - \epsilon, \ |z| \geq \lambda. \tag{4.88}$$

At this stage, it remains to prove that the solution $\zeta(\cdot)$ of (4.78) generated by $\zeta_0(\cdot)$ is uniformly bounded in the sense that $\exists$ a constant M such that $||\zeta|| \leq M||\zeta_0||$, $\forall \zeta_0 \in$

E_α. To prove this first note that the set $\{A+B_0(zI_m - D_1)^{-1}C : |z| \geq \lambda\}$ is bounded in the sense of the norm. Also this set is relatively compact and therefore can be covered by a finite number of open balls B_j^a of centre A_j^a and radius δ_j^a such that

$$\sigma(A_j^a + \Gamma) \subset \{z_1 : |z_1| \leq 1 - \epsilon^a\} \,\forall\, \|\Gamma\| \leq \delta_j^a \tag{4.89}$$

where ϵ^a is a positive constant.

To choose appropriate A_j^a and δ_j^a note that if P_j is a Lyapunov matrix for A_j^a then it is also a Lyapunov matrix for $A_j^a + \Gamma \,\forall\, \|\Gamma\| \leq \delta_j^a$ in the sense that

$$(A_j^a + \Gamma)^T P_j (A_j^a + \Gamma) - P_j < -\hat{\epsilon} I_n \tag{4.90}$$

where $\hat{\epsilon}$ is a positive constant. Also, in B_j^a, (4.90) guarantees the existence of real scalars μ_j and ϵ_j such that $\|(A_j^a + \Gamma)^i\| \leq \mu_j \epsilon_j^i$, $i = 0,\, 1,\, \cdots$. Equivalently,

$$\|(A + B_0(zI_m - D_1)^{-1}C)^i\| \leq \mu_m \epsilon_m^i, \quad i = 0,\, 1,\, \cdots \tag{4.91}$$

where $\mu_a = \max_j \mu_i$ and $\epsilon_m = \max_j \epsilon_j$. This in turn means that the solution $\zeta(\cdot)$ of (4.78) is uniformly bounded $\forall\, \zeta_0 \in E_\alpha$ and the proof is complete.

The so-called augmented plant matrix for processes described by (2.24)-(2.25) has already been defined as

$$\Phi = \begin{bmatrix} A & B_0 \\ C & D_1 \end{bmatrix}. \tag{4.92}$$

Then, since $\rho(z_1, z) = \det(\text{diag}\{z_1 I_n, zI_m\} - \Phi)$, setting $z_1 = z = 1$ gives $r(\Phi) < 1$ as another necessary condition for stability along the pass. Clearly the three necessary conditions $r(D_1) < 1$, $r(A) < 1$ and $r(\Phi) < 1$ should be tested before proceeding with the analysis of a given example.

The so-called 2D Lyapunov equation (see (Lodge and Fahmy, 1981) for the case of 2D linear systems described by the Roesser state-space model) has the form

$$\Phi^T W \Phi - W = -Q \tag{4.93}$$

where $W = W_1 \oplus W_2$, and W_1, W_2 and Q are symmetric matrices of dimension $n \times n$, $m \times m$ and $(n+m) \times (n+m)$ respectively and $\oplus$ denotes the direct sum, i.e. $W = \text{diag}(W_1, W_2)$.

In nD linear systems theory, the so-called nD Lyapunov equation was first developed in (Piekarski, 1977) as a condition for the multivariate characteristic polynomial of an nD continuous linear system to be strictly Hurwitz, i.e. no zeros in the region $\mathrm{Re}(s_i)$, $1 \leq i \leq n$. This was then extended to the 2D discrete case using the double bilinear transform (Lodge and Fahmy, 1981). Here it was asserted that the existence of positive definite symmetric matrices Q and W satisfying (4.93) was a necessary and sufficient condition for BIBO stability. In (Anderson et al., 1986), however, it was subsequently shown that, in general, the 2D Lyapunov equation condition is sufficient but not necessary for the BIBO stability of such systems. The equation (4.93) is termed 2D to denote the fact that it is defined in terms of matrices which have constant entries (as opposed to the 1D Lyapunov equation of the previous sections of this chapter which has entries which are functions of a complex variable). The remainder of this section investigates the role of (4.93) in the stability analysis of discrete linear repetitive processes described by (2.24) and (2.25).

The following analysis makes use of the following results and definitions for so-called strictly bounded real matrices (see, for example, (Anderson and Vongpanitlerd, 1973) for a detailed treatment). These results form the discrete counterpart to the definitions for differential processes used in section 4.4.

Definition 4.2 (Strictly Bounded Real Matrices) *Let $S(\eta)$ be a square matrix of real rational functions in the complex variable η. Then $S(\eta)$ is termed strictly bounded real (SBR) provided*

(a) all poles of $S(\eta)$ lie in $|\eta| < 1$, and

(b) $I - S^T(e^{-i\omega})S(e^{i\omega}) > 0, \quad \forall\ \omega \in [0, 2\pi]$.

Conditions (a) and (b) can be reduced to conditions on the matrices of a minimal state-space realization of $S(\eta)$ using the following result, which is known as the bounded real lemma,

Lemma 4.7 (Bounded Real Lemma) *Suppose that the transfer-function matrix $S(\eta)$ has a minimal state-space realisation defined by the quadruple $\{F, G, H, J\}$ such that*

$$S(\eta) = H^T(\eta I - F)^{-1}G + J. \tag{4.94}$$

Then $S(\eta)$ is SBR if, and only if, $\exists$ a symmetric matrix $P > 0$ such that the matrix Q_1 given by

$$Q_1 = \begin{bmatrix} I - J^TJ - G^TPG & -(F^TPG + HJ)^T \\ -F^TPG - HJ & P - F^TPF - HH^T \end{bmatrix} \tag{4.95}$$

is positive definite.

Note that if $\{F, G, H, J\}$ is not a minimal realisation of $S(\eta)$ and $\exists$ a symmetric matrix $P > 0$ such that Q_1 of (4.95) satisfies $Q_1 > 0$, then $S(\eta)$ is still SBR, but the converse cannot be established. Also, if $Q_1 \geq 0$ then $S(\eta)$ is a bounded real matrix.

The answer to under what conditions is (4.93) solved by symmetric positive definite W and Q is based on the bounded real lemma 4.7 and is given by the following result,

Theorem 4.8 *Consider the case of discrete linear repetitive processes described by (2.24) and (2.25) and suppose that, for some nonsingular T, the transfer-function matrix*

$$G_1(z_1) := TG(z_1)T^{-1} = T[C(z_1I_n - A)^{-1}B_0 + D_1]T^{-1} \tag{4.96}$$

is SBR. Suppose also that $\{A, B_0\}$ is completely reachable and that $\{C^T, A\}$ is completely observable. Then $\exists$ symmetric matrices $Q > 0$ and $W = W_1 \oplus W_2 > 0$ such that the 2D Lyapunov equation (4.93) is satisfied.

Conversely, if (4.93) holds for symmetric $Q > 0$ and $W = W_1 \oplus W_2 > 0$, then $\exists$ a nonsingular matrix T such that $G_1(z_1)$ is SBR.

Note that the proof of this result is identical to that of theorem 1 in (Anderson et al., 1986) and hence is omitted here.

Consider now condition (c) of theorem 4.7. Then it follows immediately (by simple operations on the defining determinant) that this condition is equivalent to

$$\det \begin{bmatrix} z_1I_n - A & -B_0 \\ 0 & zI_m - G(z_1) \end{bmatrix} \neq 0 \quad \forall\ |z_1| \geq 1,\ |z| \geq 1. \tag{4.97}$$

Application of Huang's criterion for BIBO stability of 2D discrete linear systems (theorem A.12) now shows that (4.97) is equivalent to the following two conditions

(the first of which has already been established)

$$\begin{aligned} &\det(z_1 I_n - A) \neq 0 \quad \forall\ |z_1| \geq 1, \quad \text{and} \\ &\det(zI_m - G(z_1)) \neq 0 \quad \forall\ |z_1| = 1 \text{ and } |z| \geq 1. \end{aligned} \tag{4.98}$$

Now let $\lambda_i\{A\}$ and $\lambda_i\{G(e^{i\omega})\}$ denote the eigenvalues of A and $G(z_1)|_{z_1=e^{i\omega}}$ respectively. Then these two conditions become

$$\begin{aligned} &|\lambda_i\{A\}| < 1, \qquad 1 \leq i \leq n, \quad \text{and} \\ &|\lambda_i\{G(e^{i\omega})\}| < 1, \quad 1 \leq i \leq m, \quad \forall\ \omega \in [0, 2\pi], \end{aligned} \tag{4.99}$$

and, for any nonsingular T, the last condition of (4.99) is equivalent to

$$|\lambda_i\{G_1(e^{i\omega})\}| = |\lambda_i\{TG(e^{i\omega})T^{-1}\}| < 1, \quad 1 \leq i \leq m, \ \forall\ \omega \in [0, 2\pi]. \tag{4.100}$$

Suppose now that $G_1(z_1)$ is SBR. Then in the minimal realisation of this transfer-function matrix, we have immediately $|\lambda_i\{A\}| < 1, \ 1 \leq i \leq n$. Also by the SBR property,

$$I_m - G_1^T(e^{-i\omega})G_1(e^{i\omega}) > 0 \quad \forall\ \omega \in [0, 2\pi], \tag{4.101}$$

which implies that

$$\left|\lambda_i\{G(e^{i\omega})\}\right| < 1 \quad \forall\ \omega \in [0, 2\pi], \quad 1 \leq i \leq m. \tag{4.102}$$

As the counter-example given below demonstrates, however, the argument which establishes (4.102) cannot be reversed. Equivalently, (c) of theorem 4.7 does not imply that $G_1(z_1)$ is SBR. Hence there exists stable along the pass discrete linear repetitive processes with a $G_1(z_1)$ which are not SBR and therefore, by theorem 4.8, it follows that for such a process symmetric matrices $W = W_1 \oplus W_2 > 0$ and $Q > 0$ satisfying the corresponding 2D Lyapunov equation do not exist. This result can be illustrated as in figure 4.1 and is stated formally as follows,

Theorem 4.9 *Suppose that the pair $\{A, B_0\}$ is completely reachable and that the pair $\{C^T, A\}$ is completely observable. Then discrete linear repetitive processes described by (2.24) and (2.25) are stable along the pass if $G_1(z_1)$ of (4.96) is SBR.*

A counter-example to the converse of the result of theorem 4.9 is the 6-state, 2-input, 2-output process with the following state-space model describing the contribution

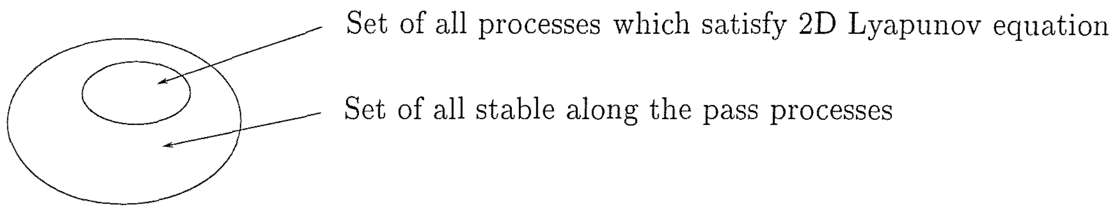

Figure 4.1: Illustration of sufficient but not necessary nature of the 2D Lyapunov equation for stability along the pass.

of the previous pass dynamics to those of the current pass over $0 \leq p \leq \alpha,\ \ k \geq 0$,

$$
\begin{aligned}
& x_{k+1}(p+1) \\
&= \begin{bmatrix} -2.81 & 1 & 0 & 0 & 0 & 0 \\ -2.657 & 0 & 1 & 0 & 0 & 0 \\ -0.845 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.81 & 1 & 0 \\ 0 & 0 & 0 & -2.657 & 0 & 1 \\ 0 & 0 & 0 & 0.845 & 0 & 0 \end{bmatrix} x_{k+1}(p) + \begin{bmatrix} 0.028 & 0 \\ 0.008 & 0 \\ 0.012 & 0 \\ 0 & 0.028 \\ 0 & -0.008 \\ 0 & 0.012 \end{bmatrix} y_k(p) \\
& y_{k+1}(p) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x_{k+1}(p) + \begin{bmatrix} 0.5 & 0.007 \\ -0.007 & 0.5 \end{bmatrix} y_k(p)
\end{aligned} \tag{4.103}
$$

First note that the necessary conditions of $r(D_1) < 1$, $r(A) < 1$ and $r(\Phi) < 1$, with Φ constructed from (4.92), are easily shown to hold in this case. Hence this example is stable along the pass if, and only if, the condition of (4.77) holds. The following analysis shows that this is the case using the equivalent formulation of (4.98).

Consider $G(z_1)$ which can be written in the form

$$
G(z_1) = \begin{bmatrix} 0.5 & G_a(z_1) \\ G_b(z_1) & 0.5 \end{bmatrix}, \tag{4.104}
$$

where

$$
\begin{aligned}
G_a(z_1) &= \frac{0.028z_1^2 - 0.008z_1 + 0.012}{z_1^3 - 2.81z_1^2 + 2.657z_1 - 0.845} + 0.007, \quad \text{and} \\
G_b(z_1) &= \frac{0.028z_1^2 + 0.008z_1 + 0.012}{z_1^3 + 2.81z_1^2 + 2.657z_1 + 0.845} - 0.007.
\end{aligned} \tag{4.105}
$$

Since $r(D_1) < 1$, to show that the conditions of (4.98) hold, and hence that the process is stable along the pass, it remains to be shown that the eigenvalues of (4.104) satisfy

$$
|\lambda_i\{G(e^{i\omega})\}| < 1, \qquad i = 1, 2,\ \ \omega \in [0, 2\pi]. \tag{4.106}
$$

First note that

$$\det(zI_2 - G(z_1)) = (z - 0.5)^2 - G_a(e^{i\omega})G_b(e^{i\omega}). \tag{4.107}$$

Also it is easily verified, by evaluating $|G_a(e^{i\omega})G_b(e^{i\omega})|\ \forall\, \omega \in [0, 2\pi]$ that

$$|G_a(e^{i\omega})G_b(e^{i\omega})| < 0.007. \tag{4.108}$$

This, in turn, implies that the values of z for which (4.107) is zero, i.e. the eigenvalues of $G(e^{i\omega})$, are all close to 0.5. Hence (4.106) holds and the process is stable along the pass.

Now we show that $G_1(z_1)$ is not SBR by showing that there is no nonsingular matrix T such that $G_1(z_1) = TG(z_1)T^{-1}$ satisfies

$$I - G_1^T(e^{-i\omega})G_1(e^{i\omega}) > 0, \quad \forall\ \omega \in [0, 2\pi]. \tag{4.109}$$

The approach used it to assume that a nonsingular matrix T does exist, and then to establish a contradiction. Suppose therefore that

$$P = T^T T. \tag{4.110}$$

Then (4.109) can be rewritten as

$$P - G^T(e^{-i\omega})PG(e^{i\omega}) > 0 \quad \forall\ \omega \in [0, 2\pi]. \tag{4.111}$$

Next we will show that $\exists$ no $P > 0$ such that the following two conditions hold,

$$P - G^T(1)PG(1) \ > \ 0, \quad \text{and} \tag{4.112}$$
$$P - G^T(-1)PG(-1) \ > \ 0. \tag{4.113}$$

From (4.104) and (4.105) we obtain

$$G(1) = \begin{bmatrix} 0.5 & \beta \\ -\alpha & 0.5 \end{bmatrix} \quad \text{and} \quad G(-1) = \begin{bmatrix} 0.5 & -\beta \\ \alpha & 0.5 \end{bmatrix} \tag{4.114}$$

where

$$\alpha = 0.000435 \quad \text{and} \quad \beta = 16.007. \tag{4.115}$$

Since P is symmetric, denote its elements by p_{11}, $p_{12} = p_{21}$ and p_{22}, and no loss of generality arises from setting $p_{11} = 1$. Hence, since P is positive definite, we have

$$|p_{12}| < \sqrt{p_{22}}. \tag{4.116}$$

Return now to (4.112), which can be rewritten as

$$P - G^T(1)PG(1)$$
$$= \begin{bmatrix} 0.75 + \alpha p_{12} - \alpha^2 p_{22} & -0.5\beta - (0.75 - \alpha\beta)p_{12} + 0.5\alpha p_{22} \\ -0.5\beta - (0.75 - \alpha\beta)p_{12} + 0.5\alpha p_{22} & -\beta^2 - \beta p_{12} + 0.75 p_{22} \end{bmatrix}. \tag{4.117}$$

Then for this matrix to be positive definite we require that

$$-\beta^2 - \beta p_{12} + 0.75 p_{22} > 0, \tag{4.118}$$

or, on rearranging and using (4.116),

$$0.75 p_{22} > \beta^2 - |\beta|\sqrt{p_{22}}. \tag{4.119}$$

On further rearranging we obtain

$$\sqrt{p_{22}} > \frac{2}{3}|\beta| \simeq 10. \tag{4.120}$$

Now rewrite (4.113) as

$$P - G^T(-1)PG(-1)$$
$$= \begin{bmatrix} 0.75 + \beta p_{12} - \beta^2 p_{22} & -0.5\alpha - (0.75 - \alpha\beta)p_{12} + 0.5\beta p_{22} \\ -0.5\alpha - (0.75 - \alpha\beta)p_{12} + 0.5\beta p_{22} & -\alpha^2 - \alpha p_{12} + 0.75 p_{22} \end{bmatrix}. \tag{4.121}$$

This matrix is positive definite provided

$$0.75 + \beta p_{12} - \beta^2 p_{22} > 0, \tag{4.122}$$

or, after similar analysis to the above,

$$\sqrt{p_{22}} < \frac{3}{2|\beta|} \simeq 0.1. \tag{4.123}$$

Condition (4.123) clearly contradicts (4.120), and hence our original assumption is invalid and there is no nonsingular matrix T such that (4.109) holds. Hence the immediate conclusion is that, for this example, symmetric matrices $W = W_1 \oplus W_2 > 0$ and $Q > 0$ which solve the corresponding 2D Lyapunov equation (4.93) do not exist. It has already been shown, however, that this process is stable along the pass, and hence this is a counter-example to the assertion that the existence of a positive definite solution pair $\{W, Q\}$ to the 2D Lyapunov equation (4.93) is equivalent to stability along the pass of discrete linear repetitive processes described by (2.24) and (2.25). Essentially the necessity part of the 2D Lyapunov equation result here is different from the 1D case.

Some special cases exist, however, where the 2D Lyapunov equation condition is both necessary and sufficient for stability along the pass, as the following sections show.

4.6.1 Special Case 1 - Φ is Normal

Suppose that the augmented plant matrix Φ (4.92) of the process under consideration is normal, i.e.

$$\Phi^T\Phi = \Phi\Phi^T. \tag{4.124}$$

Then, following the analysis in (Fadali and Gnanasekaran, 1989) for systems described by the Roesser 2D state-space model, necessity is immediate since (4.93) is equivalent to

$$\Phi^T W\Phi - W < 0, \tag{4.125}$$

i.e. $r(\Phi) < 1$, which is a necessary condition (see earlier in this section) for stability along the pass. Note that equation (4.125) is structurally similar to the Lyapunov equation for 1D discrete linear time-invariant systems with W constrained to be positive definite block diagonal matrix. In particular, under stability along the pass, it follows that the augmented plant matrix Φ is stable in the 1D sense. (Note that the converse is not generally true.) This necessary condition is expressed in terms of a matrix with constant entries and hence should be tested before proceeding further with the stability analysis of a given example.

To prove the converse, i.e. that if $r(\Phi) < 1$ then (4.125) holds, denote the eigenvalues of Φ by ω_i, $1 \leq i \leq n+m$, and the corresponding eigenvector matrix by R. Then, since Φ is normal,

$$\Phi = R\,\mathrm{diag}\{\omega_i\}_{1\leq i\leq n+m}\,R^* \tag{4.126}$$

where $*$ denotes the complex conjugate transpose operator.

Substituting (4.126) and $W = I_{n+m}$ into (4.125) then yields

$$\Phi^T W \Phi - W = R\left(|\Omega|^2 - I_{n+m}\right) R^* \tag{4.127}$$

where $\Omega = \text{diag}\{\omega_i^2\}_{1 \leq i \leq n+m}$, and this matrix is negative definite since $|\omega_i|^2 < 1$, $1 \leq i \leq n+m$, by the assumption that $r(\Phi) < 1$. Equivalently (4.125), the 1D discrete linear systems Lyapunov equation holds under the choice of $W = I_{n+m}$ and, since I_{n+m} is block diagonal under any partition, Φ also satisfies the 2D Lyapunov equation for stability along the pass.

Hence we have established the following corollary of theorem 4.9 (see (Fadali and Gnanasekaran, 1989) for the 2D Roesser model case).

Corollary 4.1 (Φ is Normal) *Suppose that the augmented plant matrix Φ for discrete linear repetitive processes described by (2.24)-(2.25) is normal. Then such processes are stable along the pass if, and only if, there exists symmetric positive definite matrices W and Q which satisfy the Lyapunov equation (4.93)*

This result can be extended slightly. A matrix Φ' is said to be 2D similar to Φ if $\Phi' = T^{-1}\Phi T$ where $T = T_1 \oplus T_2$ is a similarity transform and T_1 and T_2 are both invertible. Then it can easily be verified that the steps of the above analysis also hold if the augmented plant matrix Φ is 2D similar to normal, and the following corollary is obtained,

Corollary 4.2 (Φ is 2D Similar to Normal) *Suppose that the augmented plant matrix Φ' for discrete linear repetitive processes described by (2.24)-(2.25) can be written*

$$\Phi' = T^{-1}\Phi T \tag{4.128}$$

where $T = T_1 \oplus T_2$, T_1 and T_2 are invertible, and Φ is normal. Then such processes are stable along the pass if, and only if, there exists symmetric positive definite matrices W and Q which satisfy the 2D Lyapunov equation (4.93).

4.6.2 Special Case 2 - Process is SISO

Another special case is when the process is SISO. Then in this case it follows immediately that the two conditions for the SBR property are equivalent. In particular,

$$|\lambda_1\{G(e^{i\omega})\}| < 1 \quad \forall\ \omega \in [0, 2\pi], \tag{4.129}$$

and

$$1 - G(e^{-i\omega})G(e^{i\omega}) > 0 \quad \forall\ \omega \in [0, 2\pi], \tag{4.130}$$

are equivalent, and hence we have the following corollary of theorem 4.9.

Corollary 4.3 (SISO Processes) *SISO discrete linear repetitive processes described by (2.24)-(2.25) are stable along the pass if, and only if, there exists symmetric positive definite matrices W and Q which solve the 2D Lyapunov equation (4.93).*

4.7 2D Fornasini-Marchesini Model Based Lyapunov Equation

In 2D linear systems analysis there are (as noted previously in this thesis) two commonly used and extensively studied state-space models, namely those due to Roesser (Roesser, 1975) and Fornasini-Marchesini (Fornasini and Marchesini, 1978). Within chapter 2 both models have been presented and in chapter 3 it was shown that an 'equivalence' exists between the BIBO stability of systems described by the Roesser state-space model (and hence also those described by the Fornasini-Marchesini state-space model) and the stability along the pass of the discrete subclass of linear repetitive processes. This fact enables the interchange, to great effect, of stability tests between these two areas. Here, 2D Fornasini-Marchesini model based Lyapunov equations are developed - the analysis here is presented in (Benton et al., 2000a).

The subsequent analysis uses the following Roesser model of the discrete linear repetitive process of (2.24) and (2.25) as a starting point, which has already been introduced in section 2.6 of chapter 2,

$$\begin{bmatrix} x(k,p+1) \\ \mu(k+1,p) \\ \eta(k+1,p) \end{bmatrix} = \begin{bmatrix} A & 0 & I_n \\ D_1C & D_1 & 0 \\ B_0C & B_0 & 0 \end{bmatrix} \begin{bmatrix} x(k,p) \\ \mu(k,p) \\ \eta(k,p) \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} u(k,p). \tag{4.131}$$

An equivalent Fornasini-Marchesini model of (4.131), with the control input term deleted, can be obtained as follows

$$\begin{bmatrix} x(k,p+1) \\ \Lambda(k+1,p) \end{bmatrix} = \hat{A}_1 \begin{bmatrix} x(k,p) \\ \Lambda(k+1,p) \end{bmatrix} + \hat{A}_2 \begin{bmatrix} x(k,p+1) \\ \Lambda(k,p) \end{bmatrix} \tag{4.132}$$

where

$$\hat{A}_1 = \begin{bmatrix} A & 0 & I_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ D_1C & D_1 & 0 \\ B_0C & B_0 & 0 \end{bmatrix} \quad \text{and} \quad \Lambda(k,p) = \begin{bmatrix} \mu(k,p) \\ \eta(k,p) \end{bmatrix}. \tag{4.133}$$

Given the stability equivalence, we can now state the following result.

Theorem 4.10 *$S(E_\alpha, W_\alpha, L_\alpha)$ generated by (4.132) and (4.133) is stable along the pass if, and only if,*

$$\rho(z_1, z) := det(I_{2n+m} - z_1\hat{A}_1 - z\hat{A}_2) \neq 0 \text{ in } \overline{U}^2, \tag{4.134}$$

where $\overline{U}^2 = \{(z_1, z) : |z_1| \leq 1,\ |z| \leq 1\}$.

Note that the necessary conditions $r(D_1) < 1$ and $r(A) < 1$ should clearly be tested before recourse to the condition of theorem 4.10.

By Huang's criterion (lemma A.12), (4.134) is equivalent to the requirements that

(i)

$$\rho(z_1, 0) \neq 0, \quad |z_1| \leq 1, \quad \text{and} \tag{4.135}$$

(ii)

$$\rho(z_1, z) \neq 0,\ |z_1| = 1, \quad |z| \leq 1. \tag{4.136}$$

Suppose, therefore that the matrix H_{fm} defined by

$$H_{\text{fm}} := \hat{A}_1 + e^{i\omega}\hat{A}_2 \tag{4.137}$$

satisfies

$$r(H_{\text{fm}}) < 1 \quad \forall\, \omega \in [0, 2\pi]. \tag{4.138}$$

Then the images of the unit polydisc $\{(z_1, z) : |z_1| = 1, |z| = 1\}$ under the polynomial functions

$$\hat{q}_1(z_1, \hat{\eta}) = \det(I_{2n+m} - z_1\hat{A}_1 - z_1\hat{\eta}\hat{A}_2) \tag{4.139}$$

and

$$\hat{q}_2(\hat{\eta}, z) = \det(I_{2n+m} - z\hat{\eta}\hat{A}_1 - z\hat{A}_2) \tag{4.140}$$

coincide with the images of the polynomial function $\det(I_{2n+m} - z_1\hat{A}_1 - z\hat{A}_2)$ when acting on the sets

$$\overline{U}^2 \cap \{(z_1, z) : |z_1| \geq |z|\} \tag{4.141}$$

and

$$\overline{U}^2 \cap \{(z_1, z) : |z| \geq |z_1|\}. \tag{4.142}$$

Now, since $\hat{q}_1(0, \hat{\eta}) \neq 0$, $|\eta| \leq 1$ and $r(\hat{A}_1 + e^{i\omega}\hat{A}_2) < 1$ by assumption, $\hat{q}_1(z_1, e^{i\omega}) \neq 0$, $|z_1| \leq 1$, $\hat{q}_1(z_1, \hat{\eta}) \neq 0$ in $\overline{U}^2$ by Huang's criterion. The same property holds for $\hat{q}_2(\hat{\eta}, z)$ and hence (4.134) holds.

Conversely, stability along the pass implies that $\det(I_{2n+m} - z_1\hat{A}_1 - z\hat{A}_2) \neq 0$ in $\overline{U}^2$. Hence $\det(I_{2n+m} - z_1\hat{A}_1 - ze^{i\omega}\hat{A}_2) \neq 0$, $|z_1| \leq 1$, $|z| \leq 1$, which means that $r(\hat{A}_1 + e^{i\omega}\hat{A}_2) < 1$. Hence the following result has been established,

Theorem 4.11 *Suppose that $r(D_1) < 1$ and $r(A) < 1$. Then $S(E_\alpha, W_\alpha, L_\alpha)$ generated by (2.24) and (2.25) is stable along the pass if, and only if,*

$$r(\hat{A}_1 + e^{i\omega}\hat{A}_2) < 1 \quad \forall\ \omega \in [0, 2\pi]. \tag{4.143}$$

In particular, we have

Corollary 4.4 *Necessary conditions for stability along the pass of $S(E_\alpha, W_\alpha, L_\alpha)$ generated by (2.24) and (2.25) are that $r(\hat{A}_1 + \hat{A}_2) < 1$ and $r(\hat{A}_1 - \hat{A}_2) < 1$.*

Use of this last result now leads to the following 1D Lyapunov equation interpretation of stability along the pass. The proof of this result is omitted here since it is identical to that in (Rogers and Owens, 1996) for an essentially Roesser based 2D systems model interpretation of the dynamics of discrete linear repetitive processes.

Theorem 4.12 *Suppose that $r(D_1) < 1$ and $r(A) < 1$. Then $S(E_\alpha, W_\alpha, L_\alpha)$ generated by (2.24) and (2.25) is stable along the pass if, and only if, the 1D Lyapunov equation*

$$(\hat{A}_1 + e^{-i\omega}\hat{A}_2)^T P(e^{i\omega})(\hat{A}_1 + e^{i\omega}\hat{A}_2) - P(e^{i\omega}) = -I \tag{4.144}$$

has a positive definite Hermitian (PDH) solution $P(e^{i\omega})$.

Testing for stability using this last theorem involves obtaining a Hermitian solution matrix $P(e^{i\omega})$ to (4.144) and testing to see if this matrix is positive definite $\forall\, \omega \in [0, 2\pi]$. Applying standard positivity tests means that this is equivalent to the following two conditions,

(i)

$$P(e^{i\omega_o}) > 0 \text{ for any } \omega_o \in [0, 2\pi] \quad \text{and} \tag{4.145}$$

(ii)

$$\det(P(e^{i\omega})) > 0 \ \forall\, \omega \in [0, 2\pi]. \tag{4.146}$$

Hence it is not necessary to test all of the principal minors of $P(e^{i\omega})$ for positivity. Instead, it is enough to test at one point and then to ensure, using only the determinant, that none of the eigenvalues (or the principal minors) of $P(e^{i\omega})$ changes sign for $\omega \in [0, 2\pi]$. (See the proof of the next result for the arguments which establish this fact.)

The following result is the first step in obtaining a stability test for the 1D Lyapunov equation condition which only involves computations with matrices which have constant entries, where, for ease of notation, we write $\hat{G}(e^{i\omega}) = \hat{A}_1 + e^{i\omega}\hat{A}_2$.

Theorem 4.13 *$S(E_\alpha, W_\alpha, L_\alpha)$ generated by (2.24) and (2.25) is stable along the pass if, and only if,*

(a) $r(D_1) < 1$ and $r(A) < 1$,

(b) P the solution of

$$\hat{G}^T(e^{-i\omega_o})P\hat{G}(e^{i\omega_o}) - P(e^{i\omega_o}) = -I \tag{4.147}$$

is positive definite for any $\omega_o \in [0, 2\pi]$, and

(c)

$$det(I - \hat{G}^T(e^{-i\omega}) \otimes \hat{G}^T(e^{i\omega})) \neq 0\, \forall\, \omega \in [0, 2\pi]. \tag{4.148}$$

Proof: *Here it is only required to prove that (b) and (c) are equivalent to (4.144). To do this, consider (4.144) written as*

$$(I - \hat{G}^T(e^{-i\omega}) \otimes \hat{G}^T(e^{i\omega}))S[P(e^{i\omega})] = S[I] \tag{4.149}$$

where $S[\cdot]$ denotes the stacking operator. For the existence of a unique solution we require that

$$det(I - \hat{G}^T(e^{-i\omega}) \otimes \hat{G}^T(e^{i\omega})) \neq 0 \,\forall\, \omega \in [0, 2\pi]. \tag{4.150}$$

Also for $P(e^{i\omega})$ to be PDH, it is required that the eigenvalues of this matrix remain positive $\forall\ \omega \in [0, 2\pi]$. These eigenvalues are continuous functions of ω and hence they will always be positive if $P(e^{i\omega})$ is positive definite for an arbitrary value of ω and (4.149) holds. This proves the equivalence of (b) and (c) to (4.149).

Using this last result, it is possible to follow (Rogers and Owens, 1996) and obtain a stability test which involves the computation of generalised eigenvalues.

The following result now gives a sufficient condition for stability along the pass in terms of a 2D Lyapunov equation interpretation of (4.143) (termed the generalised 2D Lyapunov equation associated with state-space model (4.132)) (see (Hinamoto, 1993) for the case of 2D systems described by the Fornasini-Marchesini state-space model),

Theorem 4.14 *$S(E_\alpha, W_\alpha, L_\alpha)$ generated by (4.132) and (4.133) is stable along the pass if $\exists$ a $(2n+m) \times (2n+m)$ symmetric matrix $P > 0$ such that*

$$Q = \begin{bmatrix} \beta_1 P & 0 \\ 0 & \beta_2 P \end{bmatrix} - \hat{A}^T P \hat{A} > 0 \tag{4.151}$$

where β_1 and β_2 are positive real numbers which satisfy $\beta_1 + \beta_2 = 1$ and $\hat{A} = \begin{bmatrix} \hat{A}_1 & \hat{A}_2 \end{bmatrix}$.

Proof: *Suppose that the conditions of this theorem hold. Assume that*

$$\det(I_{2n+m} - z_1\hat{A}_1 - z\hat{A}_2) = 0 \tag{4.152}$$

for some $(z_1, z) \in \mathbb{C}^2$. Hence $\exists$ a $2n+m$ vector $\hat{x}$, say, $\hat{x} \neq 0$ such that

$$(I_{2n+m} - z_1\hat{A}_1 - z\hat{A}_2)\hat{x} = 0 \tag{4.153}$$

for some $(z_1, z) \in \mathbb{C}^2$. *Equivalently,*

$$\hat{x} = (z_1\hat{A}_1 + z\hat{A}_2)\hat{x}. \tag{4.154}$$

Hence, using (4.151) and (4.154) (where the superscript $*$ *denotes the complex conjugate or complex conjugate transpose as appropriate)*

$$\begin{aligned}\hat{x}^*P\hat{x} &= \hat{x}^*\begin{bmatrix} z_1^*I_{2n+m} & z^*I_{2n+m}\end{bmatrix}\hat{A}^TP\hat{A}\begin{bmatrix} z_1I_{2n+m} \\ zI_{2n+m}\end{bmatrix}\hat{x} \\ &= (\beta_1|z_1|^2 + \beta_2|z|^2)\hat{x}^*P\hat{x} - \hat{x}^*\begin{bmatrix} z_1^*I_{2n+m} & z^*I_{2n+m}\end{bmatrix}Q\begin{bmatrix} z_1I_{2n+m} \\ zI_{2n+m}\end{bmatrix}\hat{x}.\end{aligned} \tag{4.155}$$

Using this last equation, we now have that

$$(\beta_1|z_1|^2 + \beta_2|z|^2 - 1)\hat{x}^*P\hat{x} = \hat{x}^*\begin{bmatrix} z_1^*I_{2n+m} & z^*I_{2n+m}\end{bmatrix}Q\begin{bmatrix} z_1I_{2n+m} \\ zI_{2n+m}\end{bmatrix}\hat{x}. \tag{4.156}$$

Also, since $P > 0$ *and* $Q > 0$, *we have that* $\hat{x}^*P\hat{x} > 0$ *and*

$$\hat{x}^*\begin{bmatrix} z_1^*I_{2n+m} & z^*I_{2n+m}\end{bmatrix}Q\begin{bmatrix} z_1I_{2n+m} \\ zI_{2n+m}\end{bmatrix}\hat{x} > 0. \tag{4.157}$$

Hence, using (4.156) and (4.157), we have that

$$\beta_1|z_1|^2 + \beta_2|z|^2 - 1 > 0 \tag{4.158}$$

or

$$\frac{|z_1|^2}{\frac{1}{\beta_1}} + \frac{|z|^2}{\frac{1}{\beta_2}} > 1, \tag{4.159}$$

and hence the region which satisfies (4.152) in the $(|z_1|, |z|)$ *- plane is the region outside the ellipse given by*

$$\frac{|z_1|^2}{\frac{1}{\beta_1}} + \frac{|z|^2}{\frac{1}{\beta_2}} = 1. \tag{4.160}$$

Note again that if (4.152) holds then (4.159) is valid.

Now, to obtain a contradiction, suppose that

$$det(I_{2n+m} - z_1\hat{A}_1 - z\hat{A}_2) \neq 0 \tag{4.161}$$

for $(z_1, z) \in \mathbb{C}^2$ *satisfying*

$$\frac{|z_1|^2}{\frac{1}{\beta_1}} + \frac{|z|^2}{\frac{1}{\beta_2}} \leq 1. \tag{4.162}$$

Also introduce

$$\overline{U}_p^2 = \{(z_1, z) : \beta_1|z_1|^2 + \beta_2|z|^2 \leq 1\}. \tag{4.163}$$

Then if (4.161) holds, $\overline{U}^2 \subset \overline{U}_p^2$*. Hence if (4.161) is valid for* $(z_1, z) \in \overline{U}_p^2$ *then this condition is also valid for* $(z_1, z) \in \overline{U}^2$*, and hence this establishes that (4.151) is a sufficient condition for stability along the pass of (2.24). Figure 4.2 shows a schematic of this condition in the* $(|z_1|, |z|)$ *plane.*

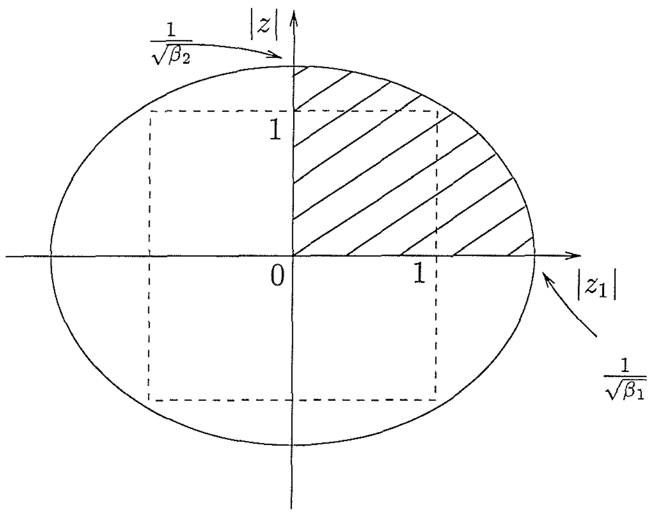

Figure 4.2: Illustration of the condition in theorem 4.14.

It follows from this result that stable along the pass examples with roots of $\det(I_{2n+m} - z_1\hat{A}_1 - z\hat{A}_2)$ in the shaded regions S_1 and S_2 of figure 4.3 do not satisfy the 2D Lyapunov equation (4.151), i.e. it is not possible to find admissible β_i, $i = 1, 2$, such that the roots of $\det(I_{2n+m} - z_1\hat{A}_1 - z\hat{A}_2)$ in both areas S_1 and S_2 are outside the corresponding ellipse. This means that there is a (potentially) large number of examples whose stability properties cannot be confirmed by use of (4.151).

If we impose $\beta_1 = \beta_2 = \frac{1}{2}$, then the following result is obtained (once again, see (Hinamoto, 1993) for the case of 2D linear systems described by the Fornasini-Marchesini state-space model),

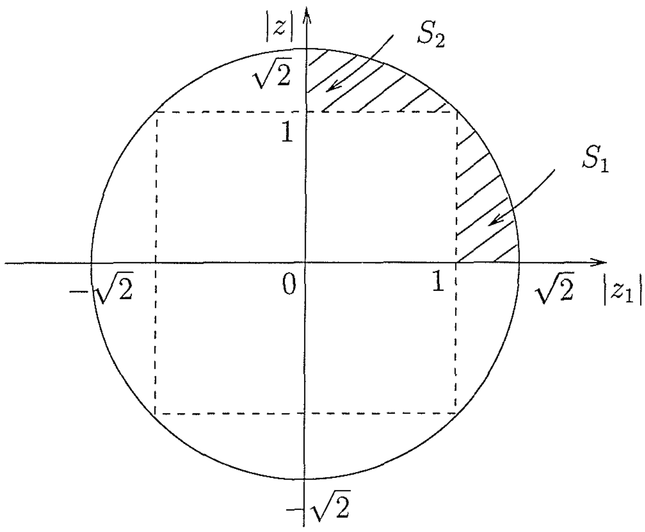

Figure 4.3: Illustration of the sufficiency of theorem 4.14.

Theorem 4.15 *The 2D Lyapunov equation for (4.132)-(4.133) in the case when $\beta_1 = \beta_2 = \frac{1}{2}$ holds if, and only if, the $(2n+m) \times (4n+2m)$ matrix $\hat{A}$ can be decomposed as*

$$\hat{A} = T^{-1}R \begin{bmatrix} \Gamma & 0 \end{bmatrix} S \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} \tag{4.164}$$

where R and S are compatibly dimensioned orthogonal matrices, T is a nonsingular matrix, and

$$\Gamma = diag\{r_1, \cdots, r_{2n+m}\} \tag{4.165}$$

where $|r_j| < \frac{1}{\sqrt{2}}$, $j = 1, 2, \cdots, 2n+m$.

Proof: *To show sufficiency, suppose that (4.164) holds and set $P = T^TT > 0$. Then in the case when $\beta_1 = \beta_2 = \frac{1}{2}$ we have that*

$$\begin{aligned} Q &= \begin{bmatrix} \beta_1 P & 0 \\ 0 & \beta_2 P \end{bmatrix} - \hat{A}^T P \hat{A} \\ &= \begin{bmatrix} \beta_1 T^T T & 0 \\ 0 & \beta_2 T^T T \end{bmatrix} - \begin{bmatrix} T^T & 0 \\ 0 & T^T \end{bmatrix} S^T \begin{bmatrix} \Gamma^2 & 0 \\ 0 & 0 \end{bmatrix} S \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} \\ &= \begin{bmatrix} T^T & 0 \\ 0 & T^T \end{bmatrix} S^T \begin{bmatrix} \beta_1 I_{2n+m} - \Gamma^2 & 0 \\ 0 & \beta_2 I_{2n+m} \end{bmatrix} S \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix}. \end{aligned} \tag{4.166}$$

Here we clearly have $Q > 0$ and hence theorem 4.14 holds for stability along the pass.

To show necessity, suppose that $\exists$ symmetric $P > 0$ such that (4.151) holds, i.e. $Q > 0$, when $\beta_1 = \beta_2 = \frac{1}{2}$. Then since $P > 0$, $\exists\, T$, $det(T) \neq 0$, such that $P = T^T T$. Hence using (4.151) we have that

$$\begin{bmatrix} \sqrt{\beta_1} T^T & 0 \\ 0 & \sqrt{\beta_2} T^T \end{bmatrix} \begin{bmatrix} \sqrt{\beta_1} T & 0 \\ 0 & \sqrt{\beta_2} T \end{bmatrix} - \hat{A}^T T^T T \hat{A} > 0 \tag{4.167}$$

or, equivalently,

$$\begin{bmatrix} \beta_1 I_{2n+m} & 0 \\ 0 & \beta_2 I_{2n+m} \end{bmatrix} - \tilde{A}^T \tilde{A} > 0 \tag{4.168}$$

where

$$\tilde{A} = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \end{bmatrix}, \quad \tilde{A}_1 = T\hat{A}_1 T^{-1} \quad \text{and} \quad \tilde{A}_2 = T\hat{A}_2 T^{-1}. \tag{4.169}$$

Now write the singular value decomposition of $\tilde{A}$ as

$$\tilde{A} = R \begin{bmatrix} \Gamma & 0 \end{bmatrix} S \tag{4.170}$$

where R, S and Γ are defined as in (4.164). Hence, on substituting (4.170) into (4.168), we have that

$$\begin{bmatrix} \beta_1 I_{2n+m} & 0 \\ 0 & \beta_2 I_{2n+m} \end{bmatrix} - S^T \begin{bmatrix} \Gamma^2 & 0 \\ 0 & 0 \end{bmatrix} S > 0 \tag{4.171}$$

or, since $\beta_1 = \beta_2 = \frac{1}{2}$,

$$\begin{bmatrix} \beta_1 I_{2n+m} - \Gamma^2 & 0 \\ 0 & \beta_2 I_{2n+m} \end{bmatrix} > 0. \tag{4.172}$$

This last equation implies that $|r_j| < \frac{1}{\sqrt{2}}$, $j = 1, 2, \cdots 2n + m$. Consequently $\hat{A}$ can be decomposed as in (4.164).

In what follows, a less conservative version of the sufficient condition of (4.151) is developed. For this, we need the standard fact that for a positive definite (or positive semi-definite) real matrix P it is always possible to write it as

$$P = U^T \Gamma U \tag{4.173}$$

where U is orthogonal and $\Gamma = \mathrm{diag}\{\sigma_1, \cdots, \sigma_{2n+m}\}$ with, in the positive definite case, $\sigma_j > 0$, $1 \leq j \leq 2n+m$. Also set $\Gamma^{\frac{1}{2}} = \mathrm{diag}\{\sigma_1^{\frac{1}{2}}, \cdots, \sigma_{2n+m}^{\frac{1}{2}}\}$. Then

$$P = (P^{\frac{1}{2}})^T P^{\frac{1}{2}} \tag{4.174}$$

where $P^{\frac{1}{2}} = \Gamma^{\frac{1}{2}} U$. For ease of notation, write $P^{\frac{T}{2}} = (P^{\frac{1}{2}})^T$ and then (4.151) can be written as

$$Q = \begin{bmatrix} P^{\frac{T}{2}}(\beta_1 I_{2n+m})P^{\frac{1}{2}} & 0 \\ 0 & P^{\frac{T}{2}}(\beta_2 I_{2n+m})P^{\frac{1}{2}} \end{bmatrix} - \hat{A}^T P^{\frac{T}{2}} I_{2n+m} P^{\frac{1}{2}} \hat{A}. \tag{4.175}$$

In what follows, we consider the following 2D Lyapunov equation defined by appropriately dimensioned positive definite matrices P, W_1, W_2 and R, and which clearly reduces to (4.151) if $W_1 = \beta_1 I_{2n+m}$, $W_2 = \beta_2 I_{2n+m}$, and $R = I_{2n+m}$,

$$Q = \begin{bmatrix} P^{\frac{T}{2}} W_1 P^{\frac{1}{2}} & 0 \\ 0 & P^{\frac{T}{2}} W_2 P^{\frac{1}{2}} \end{bmatrix} - \hat{A}^T P^{\frac{T}{2}} R P^{\frac{1}{2}} \hat{A}. \tag{4.176}$$

The following result can now be established (see (Lu, 1994a) for the case of 2D linear systems described by the Fornasini-Marchesini state-space model),

Theorem 4.16 *Discrete linear repetitive processes giving rise to the 2D Fornasini-Marchesini state-space model matrices $\hat{A}_1$ and $\hat{A}_2$ are stable along the pass if $\exists$ positive definite matrices P, W_1, W_2 and R such that Q defined by (4.176) is positive definite and that*

$$R - W_1 - W_2 \geq 0. \tag{4.177}$$

Proof: *Suppose that the conditions of the theorem hold but the process under consideration is unstable along the pass. Then this means that $\exists$ $(z_1, z) \in \overline{U}^2$ such that*

$$det(I_{2n+m} - z_1\hat{A}_1 - z\hat{A}_2) = 0. \tag{4.178}$$

This, in turn, means that $\exists$ $q \neq 0$ such that

$$q = \hat{A} \begin{bmatrix} z_1 I_{2n+m} \\ z I_{2n+m} \end{bmatrix} q, \tag{4.179}$$

where $\hat{A} = \begin{bmatrix} \hat{A}_1 & \hat{A}_2 \end{bmatrix}$.

Hence we have that

$$q^* P^{\frac{T}{2}} R P^{\frac{1}{2}} q = L - M, \tag{4.180}$$

where

$$L = q^* \begin{bmatrix} \bar{z}_1 I_{2n+m} & \bar{z} I_{2n+m} \end{bmatrix} \begin{bmatrix} P^{\frac{T}{2}} W_1 P^{\frac{1}{2}} & 0 \\ 0 & P^{\frac{T}{2}} W_2 P^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} z_1 I_{2n+m} \\ z I_{2n+m} \end{bmatrix} q, \quad \textit{and}$$

$$M = q^* \begin{bmatrix} \bar{z}_1 I_{2n+m} & \bar{z} I_{2n+m} \end{bmatrix} Q \begin{bmatrix} z_1 I_{2n+m} \\ z I_{2n+m} \end{bmatrix} q. \tag{4.181}$$

From this last equation it follows that

$$\begin{aligned} & q^* P^{\frac{T}{2}} (R - |z_1|^2 W_1 - |z|^2 W_2) P^{\frac{1}{2}} q \\ & \qquad = -q^* \begin{bmatrix} \bar{z}_1 I_{2n+m} & \bar{z} I_{2n+m} \end{bmatrix} Q \begin{bmatrix} z_1 I_{2n+m} \\ z I_{2n+m} \end{bmatrix} q. \end{aligned} \tag{4.182}$$

Also, since $(z_1, z) \neq 0$, $\begin{bmatrix} z_1 I_{2n+m} \\ z I_{2n+m} \end{bmatrix} q \neq 0$ *and* $Q > 0$, *means that the right hand side of (4.182) is negative. Conversely, however, the facts that* $|z_1| \leq 1, |z| \leq 1$ *and* $R - W_1 - W_2 \geq 0$ *imply that* $R - |z_1|^2 W_1 - |z|^2 W_2 \geq 0$, *i.e. the left hand side of (4.182) is nonnegative and we have a contradiction to our original assumption.*

This completes the proof.

The following corollary gives a special case of theorem 4.16.

Corollary 4.5 *Discrete linear repetitive processes giving rise to the 2D Fornasini-Marchesini state-space model matrices* $\hat{A}_1$ *and* $\hat{A}_2$ *are stable along the pass if* $\exists$ *a matrix* $P > 0$ *such that*

$$Q = \begin{bmatrix} P^{\frac{T}{2}} W_1 P^{\frac{1}{2}} & 0 \\ 0 & P^{\frac{T}{2}} W_2 P^{\frac{1}{2}} \end{bmatrix} - \hat{A}^T P \hat{A} > 0, \tag{4.183}$$

where $W_1 = U^T \Gamma_1 U$ *and* $W_2 = U^T \Gamma_2 U$, *with* U *orthogonal and*

$$\begin{aligned} \Gamma_1 &= diag\{\sigma_{1,1}, \cdots, \sigma_{1,2n+m}\} \quad \textit{and} \\ \Gamma_2 &= diag\{\sigma_{2,1}, \cdots, \sigma_{2,2n+m}\} \end{aligned} \tag{4.184}$$

with $\sigma_{1,j} > 0, \ \sigma_{2,j} > 0, \ \sigma_{1,j} + \sigma_{2,j} = 1, \ 1 \leq j \leq 2n + m.$

Proof : *Set* $R = I_{2n+m}$ *and note that* $I_{2n+m} - W_1 - W_2 = 0$. *Applying theorem 4.16 now gives the result.*

Suppose now that $U = I_{2n+m}$. Then we have the following corollary,

Corollary 4.6 *Discrete linear repetitive processes giving rise to 2D Fornasini-Marchesini state-space model matrices $\hat{A}_1$ and $\hat{A}_2$ are stable along the pass if $\exists P > 0$ such that*

$$Q = \begin{bmatrix} P^{\frac{T}{2}} W_1 P^{\frac{1}{2}} & 0 \\ 0 & P^{\frac{T}{2}} W_2 P^{\frac{1}{2}} \end{bmatrix} - \hat{A}^T P \hat{A} > 0 \tag{4.185}$$

where $W_j = diag\{\sigma_{j,1}, \cdots, \sigma_{j,2n+m}\}$, $j = 1, 2$, satisfies the conditions of the previous corollary.

In addition, suppose that $R = I_{2n+m}$, $W_1 = \beta_1 I_{2n+m}$, and $W_2 = \beta_2 I_{2n+m}$ where $\beta_1 > 0$, $\beta_2 > 0$ and $\beta_1 + \beta_2 = 1$. Then we obtain the condition of theorem 4.14 as a special case of theorem 4.16 and this is the essential reason that this theorem is a less conservative result. Finally, note that the choice of $R = I_{2n+m}$ in each of these corollaries incurs no loss of generality (this can be established by considering a transformation of the form $\hat{A}_j^t = T^{-1} \hat{A}_j T$, $j = 1, 2$) and hence the following result can be stated.

Theorem 4.17 *Discrete linear repetitive processes giving rise to 2D Fornasini-Marchesini state-space model matrices $\hat{A}_1$ and $\hat{A}_2$ are stable along the pass if $\exists$ matrices $P > 0$, $W_1 > 0$ and $W_2 > 0$ such that*

$$Q = \begin{bmatrix} P^{\frac{T}{2}} W_1 P^{\frac{1}{2}} & 0 \\ 0 & P^{\frac{T}{2}} W_2 P^{\frac{1}{2}} \end{bmatrix} - \hat{A}^T P \hat{A} > 0 \tag{4.186}$$

and

$$I_{2n+m} - \hat{W}_1 - \hat{W}_2 \geq 0. \tag{4.187}$$

Now we consider the numerical solution of the generalised 2D Lyapunov equation (4.186), i.e. given $\hat{A}$ find P, $W_1 > 0$ and $W_2 > 0$ such that Q defined by (4.186) is positive definite and (4.187) holds. In what follows, we establish a result that relates the existence of such positive definite P, W_1 and W_2 to a norm minimisation problem. After this, the numerical solution of the norm minimisation problem is discussed.

The analysis which follows requires consideration of $\tilde{A} = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \end{bmatrix}$ where $\tilde{A}_j = T^{-1} \hat{A}_j T$, $j = 1, 2$, for some nonsingular matrix T. Also let $|| \cdot ||$ denote the induced

2-norm of the matrix involved. Now suppose that $\exists\ P > 0$, $W_1 > 0$ and $W_2 > 0$ such that $Q > 0$ in (4.186) and (4.187) holds. Then we show below that this is equivalent to

$$\min \left\| \tilde{A} \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} \right\| < 1 \tag{4.188}$$

where the minimum is sought with T, V_1 and V_2 all nonsingular and subject to $I - V_1^{-T}V_1^{-1} - V_2^{-T}V_2^{-1} \geq 0$.

To establish (4.188) first suppose that $\exists\ P > 0$, $W_1 > 0$ and $W_2 > 0$ such that $Q > 0$ in (4.186) and (4.187) holds. Now write

$$W_1 = V_1^{-T}V_1^{-1},\ W_2 = V_2^{-T}V_2^{-1} \tag{4.189}$$

and set $T^{-1} = P^{\frac{1}{2}}$. In which case we have, from (4.186) and (4.187),

$$\begin{bmatrix} V_1^T T^T & 0 \\ 0 & V_2^T T^T \end{bmatrix} Q \begin{bmatrix} TV_1 & 0 \\ 0 & TV_2 \end{bmatrix} = I - \begin{bmatrix} V_1^T & 0 \\ 0 & V_2^T \end{bmatrix} \tilde{A}^T \tilde{A} \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} \tag{4.190}$$

and

$$I - V_1^{-T}V_1^{-1} - V_2^{-T}V_2^{-1} \geq 0. \tag{4.191}$$

Hence, since $Q > 0$, (4.190) implies that

$$\left\| \tilde{A} \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} \right\| < 1 \tag{4.192}$$

and hence (4.188) holds.

Suppose now that (4.188) holds. In which case, $\exists$ some nonsingular T, V_1 and V_2 satisfying (4.192) and hence

$$I - V_1^{-T}V_1^{-1} - V_2^{-T}V_2^{-1} \geq 0 \tag{4.193}$$

and also

$$\tilde{Q} \equiv I - \begin{bmatrix} V_1^T & 0 \\ 0 & V_2^T \end{bmatrix} \tilde{A}^T \tilde{A} \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} > 0. \tag{4.194}$$

Selecting $P = T^{-T}T^{-1}$, $W_1 = V_1^{-T}V_1^{-1}$, $W_2 = V_2^{-T}V_2^{-1}$ implies, by (4.194), that Q of (4.186) is positive definite where

$$Q = \begin{bmatrix} T^{-T}V_1^{-T} & 0 \\ 0 & T^{-T}V_2^{-T} \end{bmatrix} \tilde{Q} \begin{bmatrix} V_1^{-1}T^{-1} & 0 \\ 0 & V_2^{-1}T^{-1} \end{bmatrix} \tag{4.195}$$

and (4.193) gives (4.187).

The optimisation problem of (4.188) is quite well known in the literature. For example, efficient solution methods can be found in (Luenberger, 1984).

4.8 Solving the 2D Lyapunov Equation

Here, algorithms are given for solving the 2D Lyapunov equation of the previous sections. Consider again the 2D Lyapunov equation (4.93), i.e.

$$\Phi^T W \Phi - W = -Q. \tag{4.196}$$

Then no loss of generality arises from assuming that Q has the form

$$Q = \begin{bmatrix} K^T & 0 \\ L & N \end{bmatrix} \begin{bmatrix} K & L^T \\ 0 & N^T \end{bmatrix} = \begin{bmatrix} K^T K & K^T L^T \\ LK & LL^T + NN^T \end{bmatrix} \tag{4.197}$$

where K and N are $n \times n$ and $m \times m$ nonsingular matrices respectively, and L is an $m \times n$ matrix.

The 2D Lyapunov equation (4.196) can now be rewritten as the following three expressions,

$$\begin{aligned} W_1 - A^T W_1 A - C^T W_2 C &= K^T K, && (4.198) \\ W_2 - D_1^T W_2 D_1 - B_0^T W_1 B_0 &= LL^T + NN^T, && (4.199) \\ -B_0^T W_1 A - D_1^T W_2 C &= LK. && (4.200) \end{aligned}$$

Hence finding symmetric positive definite matrix solutions of (4.196) is equivalent to finding symmetric positive definite matrices W_1 and W_2, nonsingular matrices K and N, and a matrix L such that (4.198) - (4.200) hold. Here two algorithms for solving this problem are given, starting with one based on the use of spectral factorisation.

Algorithm 1

Step 1: Find an $n \times n$ symmetric positive definite matrix W_1 such that

$$W_1 - G_1^T(e^{-i\omega}) W_1 G_1(e^{i\omega}) > 0 \; \forall \, \omega \in [0, 2\pi] \tag{4.201}$$

where

$$G_1(e^{i\omega}) = B_0(e^{i\omega}I_m - D_1)^{-1}C + A. \tag{4.202}$$

Step 2: Find an $m \times m$ nonsingular matrix N such that

$$W_1 - G_1^T(e^{-i\omega})W_1G_1(e^{i\omega}) - C^T(e^{i\omega}I_m - D_1)^{-1}NN^T(e^{-i\omega}I_m - D_1)^{-1}C > 0,$$
$$\forall\, \omega \in [0, 2\pi]. \tag{4.203}$$

Step 3: Find an $n \times n$ rational matrix $W(z_1^{-1})$ such that

$$\begin{aligned} W_1 - G_1^*(z_1)W_1G_1(z_1^{-1}) \quad - \quad & C^T(z_1^*I_m - D_1^T)NN^T(z_1^{-1}I_m - D_1)^{-1}C \\ & = W^*(z_1)W(z_1^{-1}). \end{aligned} \tag{4.204}$$

Step 4: Find an $n \times n$ matrix K and an $m \times n$ matrix L such that

$$W(z_1^{-1}) = K + L^T(z_1I_m - D_1)^{-1}C. \tag{4.205}$$

Step 5: Find an $m \times m$ symmetric positive definite matrix W_2 as the solution of

$$W_2 - D_1^TW_2D_1 = B_0^TW_1B_0 + LL^T + NN^T. \tag{4.206}$$

It can be shown (Agathoklis et al., 1989) that if $G_1(z_1^{-1})$ is minimal and strictly positive real then it is always possible to obtain the positive definite solutions to the 2D Lyapunov equation using the above algorithm. Also well known 1D methods can be used at each step. The execution of steps 1 and 2 requires a method for testing the strictly bounded real property. Note, however, that although such algorithms exist, no algorithm has been developed yet which ensures that if a positive definite matrix W_1 satisfying (4.201) exists, this W_1 will be found. In step 2, a simple choice for N is $N = \epsilon I_m$ where ϵ is sufficiently small to ensure that (4.203) holds. Note, however, that a 'very small' ϵ could lead to a Q which is 'almost' positive semi-definite since $\det(Q) = (\epsilon^m K)^2$. In step 3, (4.204) can be rewritten as

$$I - G_r^*(z_1)G_r(z_1^{-1}) = W^*(z_1)W(z_1^{-1}) \tag{4.207}$$

where

$$G_r^*(z_1^{-1}) = \begin{bmatrix} TAT^{-1} \\ 0 \end{bmatrix} + \begin{bmatrix} TB_0 \\ N \end{bmatrix} (z_1^{-1}I_m - D_1)^{-1}CT^{-1} \tag{4.208}$$

and $W_1 = T^T T$. Finally, note that the spectral factorisation problem here, and the realisation of $W(z_1^{-1})$, have been well studied and numerous algorithms are available in the open literature (see, for example, (Agathoklis et al., 1989) and the relevant references therein). Also at step 5, a simple 1D Lyapunov equation has to be solved.

Algorithm 2

The following are the steps in the solution algorithm which uses a matrix Riccati equation.

Step 1: Find an $n \times n$ positive definite matrix W_1 such that

$$W_1 - G_1^T(e^{-i\omega})W_1 G_1(e^{i\omega}) > 0 \,\forall\, \omega \in [0, 2\pi], \tag{4.209}$$

where $G_1(e^{i\omega})$ is as in (4.202).

Step 2: Find and $m \times m$ nonsingular matrix N such that

$$W_1 - G_1^T(e^{-i\omega})W_1 G_1(e^{i\omega}) - C^T(e^{i\omega}I_m - D_1)^{-1}NN^T(e^{-i\omega}I_m - D_1)^{-1}C > 0,$$
$$\forall\, \omega \in [0, 2\pi]. \tag{4.210}$$

Step 3: Find a $m \times m$ symmetric positive definite matrix W_2 as the solution of the following matrix Riccati equation

$$D_1^T W_2 D_1 + HFG + NN^T + B_0^T W_1 B_0 = 0 \tag{4.211}$$

where

$$\begin{aligned} H &= B_0^T W_1 A + D_1^T W_2 C \\ F &= (W_1 - A^T W_1 A - C^T W_2 C)^{-1} \\ H &= (A^T W_1 B_0 + D_1^T W_2 C)^T. \end{aligned} \tag{4.212}$$

The first two steps of this algorithm are identical to the previous one but here the spectral factorization has been replaced by (4.211) to determine W_2. Also the matrix Q can be obtained by substituting $Q = W_1 \oplus W_2$ in the 2D Lyapunov equation and it can be shown (Agathoklis et al., 1989) that $Q > 0$.

4.9 Performance Bounds

Within this section computable bounds on performance along a given pass are introduced which use the positive definite solution pair $\{W, Q\}$ of the 2D Lyapunov equation (4.196) as a starting point. So, suppose that the 2D Lyapunov equation (4.196) holds and introduce

$$\begin{aligned}||x_{k+1}(p+1)||^2_{W_1} &:= x^T_{k+1}(p+1)W_1x_{k+1}(p+1), \quad \text{and}\\ ||y_{k+1}(p)||^2_{W_2} &:= y^T_{k+1}(p)W_2y_{k+1}(p).\end{aligned} \tag{4.213}$$

Then applying these definitions to (4.196) gives

$$||x_{k+1}(p+1)||^2_{W_1} + ||y_{k+1}(p)||^2_{W_2} - ||x_k(p)||^2_{W_1} - ||y_k(p)||^2_{W_2} = -||x_k(p)||^2_I - ||y_k(p)||^2_I. \tag{4.214}$$

Now suppose that $x_{k+1}(0) = 0$, $k \geq 0$, and introduce for $j \geq 0$,

$$\begin{aligned}||x_{j+1}||^2_{W_1} &:= \sum_{p=0}^{\infty} ||x_{j+1}(p+1)||^2_{W_1}\\ ||y_{j+1}||^2_{W_2} &:= \sum_{p=0}^{\infty} ||y_{j+1}(p+1)||^2_{W_2}.\end{aligned} \tag{4.215}$$

Then applying these summations to (4.214) gives

$$\begin{aligned}||x_{k+1}||^2_{W_1} + ||y_{k+1}||^2_{W_2} &= ||x_k||^2_{W_1-I} + ||y_k||^2_{W_2-I}\\ &\leq \lambda(||x_k||^2_{W_1} + ||y_k||^2_{W_2}).\end{aligned} \tag{4.216}$$

Now assume that

$$\begin{aligned}W_1 - I &\leq \lambda_1 W_1\\ W_2 - I &\leq \lambda_2 W_2,\end{aligned} \tag{4.217}$$

where λ_1 and λ_2 are real positive scalars.

Then

$$||(x_{k+1}, y_{k+1})||^2 := ||x_{k+1}||^2_{W_1} + ||y_{k+1}||^2_{W_2} \leq \lambda(||(x_k, y_k)||^2) \tag{4.218}$$

where $\lambda = \max(\lambda_1, \lambda_2)$. The process is stable along the pass if $\lambda < 1$ which is always guaranteed to be true if $W_1,\ W_2 \geq I$. Hence we have a geometric convergence to zero in the pass to pass direction.

It is possible to compute the rate λ using the fact that if a square matrix $\hat{W}$ satisfies $\hat{W} = \hat{W}^T \geq I$ then $\exists\, \hat{\lambda} \in [0, 1) : \hat{W} - I \leq \hat{\lambda}\hat{W}$. Then it is easily shown that

$$\hat{W} - I \leq \hat{W} - \frac{1}{\overline{\sigma}^2(\hat{W})}\hat{W} = (1 - \frac{1}{\overline{\sigma}(\hat{W})^2})\hat{W}. \tag{4.219}$$

Hence in the case of (4.218)

$$\lambda = \max\left(1 - \frac{1}{\overline{\sigma}(W_i)^2}\right), \ i = 1, 2. \tag{4.220}$$

4.10 Summary and Conclusions

Within this chapter the question of to what extent a Lyapunov based approach to the stability analysis of linear repetitive processes is available has been addressed. A study of the literature to date has revealed that the development of Lyapunov-type equations for 2D systems described by the Roesser/Fornasini-Marchesini state-space models has been approached in essentially two different ways:

(i) the so-called 1D Lyapunov equation approach, defined in terms of matrices which are functions of a complex variable; and

(ii) the so-called 2D Lyapunov equation, defined in terms of matrices with constant entries.

Initially, the 1D equation has been investigated, firstly for differential processes with simple boundary conditions. The term 1D refers to the fact that the equation has an identical structure to that for discrete linear time-invariant systems but with defining matrices which are functions of a complex variable. The resulting condition for stability along the pass based on this equation is both necessary and sufficient (as opposed to the sufficient only nature of the 2D Lyapunov equation condition - see later), and can be implemented by computations on matrices with constant entries. Hence this result serves as an alternative to previously presented/developed tests (see chapter 3 for the details) for stability along the pass which require the computation of the eigenvalues of a potentially large dimensioned matrix for all points on the unit circle. In addition, it has been shown how the 1D equation approach provides performance information on the rate of approach of the output sequence of pass profiles to the limit profile. It should be stressed, however, that the 1D equation does not provide any useful measures of relative stability, such as stability margins or robustness measures to, for example, uncertainties in the model description or parameter variations (unlike the 2D Lyapunov equation case - see below). Some comments on methods of solution of the 1D Lyapunov equation have been made. For full details of such techniques see the relevant references cited within the text.

To conclude the analysis of the 1D Lyapunov equation approach, a subclass of processes with dynamic boundary conditions have been considered, which have links with certain classes of delay-differential systems and area of repetitive control. A 1D Lyapunov equation characterisation of stability along the pass has been introduced for this subclass and shown to provide a stability condition which is both necessary and sufficient. Strict positive realness based tests to compute positivity have been developed which reduce the problem to a 1D problem by showing that the condition is equivalent to testing for positive realness of a certain 1D rational transfer-function matrix. The analysis presented in this section on dynamic boundary conditions provides the basis for the papers (Benton et al., 2000c) and (Benton et al., 2000d).

In section 4.6 and onward, the so-called 2D Lyapunov equation approach has been considered, which is defined in terms of matrices with constant entries. Here it has been shown that, in general, the existence of a positive definite solution pair to the 2D Lyapunov equation is a sufficient but not necessary condition for stability along the pass of discrete linear repetitive processes. A counter-example is given to show that a stable along the pass process does not necessarily imply that the process is strictly bounded real and hence satisfies the 2D Lyapunov equation. Two special cases have been presented, however, when the 2D Lyapunov condition provides necessary and sufficient conditions for stability along the pass - SISO systems and the case when the augmented plant matrix of the process is normal. In section 4.7 a 2D Lyapunov equation has been developed for a 2D Fornasini-Marchesini state-space model, which involves the computation of generalised eigenvalues. The analysis presented here on the 2D Lyapunov equation for discrete linear repetitive processes provides the basis for the paper (Benton et al., 2000a).

Despite its apparent conservativeness, the 2D Lyapunov equation approach has a potentially major role to play in the analysis of discrete linear repetitive processes in terms of stability margins and robust stability theory as discussed in the following chapter. In addition, the performance measures of section 4.9 are not available from Roesser/Fornasini-Marchesini alternatives (for the discrete subclass of processes).

Note that progress can be made in terms of the development of a 2D Lyapunov equation for the differential subclass of processes. This subject remains an area where future research effort should be focussed.

Chapter 5

Robustness

5.1 Introduction

In addition to determining whether or not a given process is stable along the pass, it is important to obtain measures of 'how stable' the process is or, more specifically, 'how far' it is from being unstable. Within this chapter, the subject of robustness of linear repetitive processes is considered. The first area looked at is how sensitive the property of stability along the pass is to system parameter variations. Secondly the subject of stability margins is introduced. For both areas of robustness analysis, discussions on the available methods of computation of the bounds/margins are given. In addition it is noted that, in many cases, evaluation of the exact bound or margin is not necessary (or possible) in which case good lower bounds may suffice. With this motivation, a Lyapunov equation based approach to robustness analysis for the two areas of parameter variation bounds and stability margins is given, using the 2D Lyapunov equation of chapter 4 as a starting point, and hence this work can be seen to be an application of the theory presented in the relevant sections therein. The analysis of these sections can be found in (Benton et al., 1999).

A valid criticism of the work to date on stability margins for 2D linear systems is the lack of a 'transparent' link to resulting systems performance. In particular, consider the 1D linear continuous time case and suppose that all the system poles lie to the left of the line $\text{Re}(s) = -\sigma$, $\sigma > 0$. Then this can be directly related to the system performance to, say, a step command.

To expand on this last point, consider, for simplicity, the unforced state-space model

$$\dot{x}(t) = A\,x(t), \quad x(t) \in \mathbb{R}^n, \; x(0) = x_0, \tag{5.1}$$

and let A have distinct eigenvalues λ_i, $1 \leq i \leq n$. Then the system performance is given by

$$x(t) = \sum_{i=1}^{n} T\,\mathrm{diag}\{e^{\lambda_i t}\}_{1 \leq i \leq n}\, T^{-1}\, x_0 \tag{5.2}$$

where T is the eigenvector matrix of A. Hence the stability margin here has a 'transparent' link to resulting system performance.

In the 2D/nD case, such a link is not present in previous work and it is clear that this is a problem which must be addressed before any further progress is possible. Here the basis of one highly promising approach is utilised by, in effect, specialising recent work on a pole theory for nD linear systems based on the behavioural approach (Wood et al., 2000). This is discussed further in section 5.9.

It should be noted that the analysis presented in this chapter provides only an introductory consideration of the subject area of robust stability theory for discrete linear repetitive processes - much further research effort is required before an objective appraisal of the techniques presented can be made.

5.2 Parameter Variations

The first stage in the analysis of a given linear repetitive process is to decide whether it is stable or not. If the process is stable along the pass, it is then important to consider how this property is affected in the presence of system parameter variations. Such variations can arise as the result of, for example, model inaccuracy or measurement noise, and the analysis presented here determines the degree to which the process will tolerate system parameter variations without becoming unstable along the pass.

In the case of discrete linear systems described by, for example, the Roesser 2D state-space model, this general area has been studied under two different types of perturbations in the matrices which define the state-space model as follows:

(i) structured, where the perturbation model structure and bounds on the individual elements of the perturbation matrix are known; and

(ii) unstructured, where at most a spectral norm bound on the perturbation is known.

5.2.1 Problem Statement

As a starting point, consider the subclass of unit memory discrete linear repetitive processes with the state-space model (2.24)-(2.25). Assume that the process is stable along the pass and is free of control inputs, i.e. $u_k(p) \equiv 0,\ 0 \le p \le \alpha,\ k \ge 0$. Then this so-called nominal system can be written in the following form over $0 \le p \le \alpha,\ k \ge 0$,

$$\begin{bmatrix} x_k(p+1) \\ z_{k+1}(p) \end{bmatrix} = \begin{bmatrix} A & B_0 \\ C & D_1 \end{bmatrix} \begin{bmatrix} x_k(p) \\ z_k(p) \end{bmatrix} \tag{5.3}$$

where $z_{k+1}(p) := y_k(p),\ 0 \le p \le \alpha,\ k \ge 0$. Using the augmented state matrix notation of (2.51)-(2.53), the nominal system can be rewritten over $0 \le p \le \alpha, k \ge 0$, as

$$X_k^{11}(p) = \Phi\, X_k(p). \tag{5.4}$$

Note that, since the process is stable along the pass (by assumption), the three conditions of theorem 4.7 hold, in addition to the necessary condition of $r(\Phi) < 1$.

Now consider the subclass of discrete processes with the following additive perturbation structure for the augmented plant matrix,

$$\Phi_{\text{per}} = \Phi + \Delta\Phi. \tag{5.5}$$

Here

$$\Delta\Phi = \begin{bmatrix} \Delta A & \Delta B_0 \\ \Delta C & \Delta D_1 \end{bmatrix} \tag{5.6}$$

represents the matrix of unstructured perturbations, with elements having the same dimensions as for Φ. Then the perturbed unforced system has the following state-space representation over $0 \le p \le \alpha,\ k \ge 0$,

$$\begin{bmatrix} x_k(p+1) \\ z_{k+1}(p) \end{bmatrix} = \Phi_{\text{per}} \begin{bmatrix} x_k(p) \\ z_k(p) \end{bmatrix} = \begin{bmatrix} A+\Delta A & B_0+\Delta B_0 \\ C+\Delta C & D_1+\Delta D_1 \end{bmatrix} \begin{bmatrix} x_k(p) \\ z_k(p) \end{bmatrix} \tag{5.7}$$

or

$$X_k^{11}(p) = \Phi_{\text{per}}\, X_k(p). \tag{5.8}$$

The question addressed in the following sections, then, is what conditions need to be imposed on the structure of the perturbation matrix (5.6) to ensure that the perturbed nominal system remains stable along the pass? (Note that, clearly, the three conditions $r(D_1 + \Delta D_1) < 1$, $r(A + \Delta A) < 1$ and $r(\Phi + \Delta\Phi) < 1$ are required to hold in addition to condition (c) of theorem 4.7 with Φ replaced by Φ_{per}.)

Now for the nominal stable along the pass discrete process (5.3), define the set of unstructured unstable along the pass perturbations of the form (5.6) as

$$S_u := \{\Delta\Phi : \Delta\Phi \in \mathbb{C}^{(n+m)\times(n+m)},\ \Phi + \Delta\Phi \text{ is unstable along the pass}\}. \tag{5.9}$$

The exact bound for stable along the pass perturbations can now be defined as

$$v := \inf_{\Delta\Phi \in S_u} ||\Delta\Phi||. \tag{5.10}$$

Then, given a nominal stable along the pass discrete process with augmented plant matrix Φ, an unforced discrete linear repetitive process with augmented plant matrix Φ_{per} will remain stable along the pass if

$$||\Delta\Phi|| < v. \tag{5.11}$$

Thus, the aim of the analysis presented here is to find methods of determining the minimum norm of the matrix $\Delta\Phi$ such that the perturbed system remains stable along the pass, or at least a good lower bound for it. In this latter case, a lower bound, v_b, for v provides a sufficient condition for stability along the pass, as follows

$$||\Delta\Phi|| < v_b \leq v. \tag{5.12}$$

A review of the literature indicates that, for systems described by the Roesser / Fornasini-Marchesini 2D state-space models, this problem has been approached in essentially two different ways :

(i) methods for evaluating the exact bound for stable along the pass perturbations, see, for example, (Lu, 1994b; Lu, 1989); and

(ii) methods for obtaining a lower bound for v, see, for example, (Lu, 1994b) for one such 2D Lyapunov equation based approach.

For the remainder of this chapter, the following notation conventions are used. Singular values of a matrix F are defined as the square root of the eigenvalues of F^*F, $\overline{\sigma}(F)$ and $\underline{\sigma}(F)$ denote the largest and smallest singular values of F respectively, the vector norm $||x||$ for $n \times 1$ vectors x is given by $(\sum_{i=1}^{n} x_i^2)^{1/2}$ and the matrix norm $||F||$ is the the induced 2-norm and is equal to $\overline{\sigma}(F)$.

5.3 The Exact Bound for Stable Perturbations

Consider the subclass of stable along the pass unit memory discrete linear repetitive processes of the form (2.24)-(2.25) with nominal system (5.3), and let Φ_{per} denote the augmented plant matrix of the process which has been subjected to unstructured perturbations with the additive perturbation structure of (5.5).

The so-called characteristic equation of the unperturbed stable process is defined (as in (4.77)) as

$$\begin{aligned} \rho(z_1, z) &= \begin{vmatrix} I_n - z_1 A & -z_1 B_0 \\ -zC & I_m - zD_1 \end{vmatrix} \\ &= \det[z_1^{-1} I_n \oplus z^{-1} I_m - \Phi] \end{aligned} \tag{5.13}$$

and since the process is stable along the pass, we have

$$\rho(z_1, z) \neq 0 \ \forall \ (z_1, z) \in \overline{U}^2. \tag{5.14}$$

Now introduce the characteristic equation of the perturbed system (5.7) as

$$\begin{aligned} \rho_{\text{per}}(z_1, z) &= \begin{vmatrix} I_n - z_1(A + \Delta A) & -z_1(B_0 + \Delta B_0) \\ -z(C + \Delta C) & I_m - z(D_1 + \Delta D_1) \end{vmatrix} \\ &= \det[z_1^{-1} I_n \oplus z^{-1} I_m - \Phi_{\text{per}}] \end{aligned} \tag{5.15}$$

and let

$$\begin{aligned} \tau(z_1, z) &= [z_1^{-1} I_n \oplus z^{-1} I_m - \Phi]^{-1} \\ \text{and} \quad \tau_{\text{per}}(z_1, z) &= [z_1^{-1} I_n \oplus z^{-1} I_m - \Phi_{\text{per}}]^{-1}. \end{aligned} \tag{5.16}$$

Then, following (Lu, 1989) for the case of discrete systems with dynamics described by the Roesser 2D state-space model, a sufficient condition for the perturbed system to remain stable along the pass is given by the following lemma,

Lemma 5.1 *Given the unit memory discrete linear repetitive process (2.24)-(2.25) subject to unstructured perturbations with the additive perturbation structure of (5.5), the perturbed process remains stable along the pass provided the perturbation matrix (5.6) satisfies*

$$||\Delta\Phi|| < ||\tau(e^{i\omega_1}, e^{i\omega_2})||^{-1} \quad \forall\ 0 \leq \omega_1, \omega_2 \leq 2\pi. \tag{5.17}$$

Now, from the definition of $\tau(z_1, z)$ we can write

$$||\tau(e^{i\omega_1}, e^{i\omega_2})||^{-1} = \underline{\sigma}[e^{i\omega_1} I_n \oplus e^{i\omega_2} I_m - \Phi], \tag{5.18}$$

and, since the singular values of a given matrix are continuous functions of the matrix entries, (5.18) achieves its minimum on Ω where

$$\Omega = \{(\omega_1, \omega_2) : 0 \leq \omega_1, \omega_2 \leq 2\pi\}. \tag{5.19}$$

Thus (5.17) may be restated as

$$||\Delta\Phi|| < q, \tag{5.20}$$

where

$$q := \min_{\Omega} \underline{\sigma}[e^{i\omega_1} I_n \oplus e^{i\omega_2} I_m - \Phi]. \tag{5.21}$$

Then we have the following result (which is proved in (Lu, 1989) for the case of 2D discrete systems described by the Roesser state-space model),

Theorem 5.1 *Given a stable along the pass discrete linear repetitive process (2.24)-(2.25), the exact bound, v, for stable along the pass perturbations of the form (5.6) is given by*

$$v = q \tag{5.22}$$

with q defined as in (5.21).

Hence, from this result, it follows that the tightest upper bound for unstructured complex perturbations that will not cause system instability is provided by q defined above. The question remaining is, for a given example, how can this q be evaluated?

The literature provides several approaches for evaluating this exact bound for stable along the pass perturbations. A computationally feasible method is given in (Lu, 1989), which involves computing the infimum of the minimum singular values of a two variable complex matrix of size $(m+n) \times (m+n)$ over Ω, which can be computationally intensive. Two alternative methods for computing the exact perturbation bound v are presented in (Lu, 1994b). The first method, in effect, reduces the calculation to a 1D minimisation problem where the objective function is the stable perturbation bound of a family of 1D discrete systems, which is then solved using the bisection method. The second approach uses a direct optimisation technique. Both methods are more numerically efficient, and so can be seen as an improvement on the original approach.

The following section implements the idea that it is not always necessary to know the exact bound for stable perturbations. Instead a good lower bound often suffices.

5.4 A Lyapunov Approach to Perturbation Bounds

Within this section, a Lyapunov equation based approach to finding good lower bounds for v is presented. Consider the unforced stable along the pass discrete linear repetitive process (5.3). The starting point for the following analysis, then, is to assume that this process satisfies the 2D Lyapunov equation of chapter 4, i.e. that $\exists$ a positive definite solution pair $\{W, Q\}$ to equation (4.93). It should be stressed that the assumption that the 2D Lyapunov equation is satisfied is stronger than assuming stability along the pass alone due to its sufficient but not necessary nature.

Following (Lu, 1994b) for the case of 2D discrete systems described by the Roesser state-space model, and given the positive definite matrices $W = W_1 \oplus W_2$ and Q as solution to the 2D Lyapunov equation (4.93), construct the Lyapunov function $\phi_k(p)$ as

$$\phi_k(p) := \phi_k^h(p) + \phi_k^v(p) \tag{5.23}$$

with

$$\phi_k^h(p) := x_k^T(p) W_1 x_k(p) \quad \text{and} \quad \phi_k^v(p) := z_k^T(p) W_2 z_k(p) \tag{5.24}$$

where, as before, $z_k(p) = y_{k-1}(p)$, $0 \le p \le \alpha$, $k \ge 1$. Hence, using the augmented state vector notation of (5.4), we have

$$\phi_k(p) = X_k^T(p) W X_k(p). \tag{5.25}$$

The function $\phi_k(p)$ represents the energy stored in the delays and, since W_1 and W_2 are positive definite, $\phi_k(p) > 0$ provided $X_k(p) \neq 0$.

Define

$$\begin{aligned} \phi_k^{11}(p) &:= \phi_k^h(p+1) + \phi_{k+1}^v(p) \\ &= X_k^{11T}(p) W X_k^{11}(p) \end{aligned} \tag{5.26}$$

using the notation of (5.8). Then equations (5.23)-(5.26) together with the perturbed process state-space representation (5.7) and the 2D Lyapunov equation (4.93) can be used to compute $\Delta\phi_k(p)$, defined as

$$\begin{aligned} \Delta\phi_k(p) &:= \phi_k^{11}(p) - \phi_k(p) \\ &= X_k^{11T}(p) W X_k^{11}(p) - X_k^T(p) W X_k(p) \\ &= -X_k^T(p)\{W - \Phi_{\text{per}}^T W \Phi_{\text{per}}\} X_k(p). \end{aligned} \tag{5.27}$$

Thus if $(W - \Phi_{\text{per}}^T W \Phi_{\text{per}})$ is positive definite, then we have $\Delta\phi_k(p) \leq 0$ and $\Delta\phi_k(p) = 0$ only when $X_k(p) = 0$. Then, in this situation, a routine argument (see, for example, (El-Agizi and Fahmy, 1979)) can be used to show that the energy stored in the delays is decreasing and hence that the perturbed process (5.7) satisfies the 2D Lyapunov equation. Therefore the perturbed process is stable along the pass.

Now (5.27) can be expanded to give

$$\Delta\phi_k(p) = -X_k^T(p) Q X_k(p) + 2X_k^T(p)\Phi^T W \Delta\Phi X_k(p) + X_k^T(p)(\Delta\Phi)^T W \Delta\Phi X_k(p). \tag{5.28}$$

Since W is positive definite, it can be factored to give $W = W^{T/2}W^{1/2}$, where $W^{T/2} := (W^{1/2})^T$ and $W^{1/2}$ some matrix known as a square root of W. In addition, since W is symmetric, so is its square root, i.e. $W^{1/2} \equiv W^{T/2}$. Hence

$$\Delta\phi_k(p) \leq -\left[\underline{\sigma}(Q) - 2\,\|\Phi^T W^{1/2}\|\,\overline{\sigma}^{1/2}(W)\,\|\Delta\Phi\| - \overline{\sigma}(W)\,\|\Delta\Phi\|^2\right]\|X_k(p)\|^2. \tag{5.29}$$

Clearly

$$\Phi^T W^{1/2} W^{1/2} \Phi = W - Q \tag{5.30}$$

which gives

$$\|\Phi^T W^{1/2}\| \le \overline{\sigma}^{1/2}(W - Q). \tag{5.31}$$

Then an upper bound for $\Delta\phi_k(p)$ is given by

$$\Delta\phi_k(p) \;\le\; -\left[\underline{\sigma}(Q) - 2\overline{\sigma}^{1/2}(W - Q)\,\overline{\sigma}^{1/2}(W)\,\|\Delta\Phi\| - \overline{\sigma}(W)\,\|\Delta\Phi\|^2\right]\|X_k(p)\|^2. \tag{5.32}$$

Since the right hand side of this inequality is a quadratic in $\|\Delta\Phi\|$, it follows that if

$$\|\Delta\Phi\| < \frac{[\overline{\sigma}(W - Q) + \underline{\sigma}(Q)]^{1/2} - \overline{\sigma}^{1/2}(W - Q)}{\overline{\sigma}^{1/2}(W)} \tag{5.33}$$

then $\Delta\phi_k(p) < 0 \,\forall\, 0 \le p \le \alpha,\ k \ge 0$, and hence the perturbed system remains stable along the pass. Therefore a lower bound, v_b, for v can be obtained by setting v_b equal to the right hand side of (5.33).

An alternative Lyapunov function may be defined as

$$\psi_k(p) := [\phi_k^h(p) + \phi_k^v(p)]^{1/2} \tag{5.34}$$

with $\phi_k^h(p)$ and $\phi_k^v(p)$ defined as in (5.24).

Similarly define

$$\psi_k^{11}(p) := [\phi_k^h(p+1) + \phi_{k+1}^v(p)]^{1/2}, \tag{5.35}$$

and, as before, compute

$$\begin{aligned}
\Delta\psi_k(p) :=&\; \psi_k^{11}(p) - \psi_k(p)\\
=&\; [X_k^T(p)\Phi_{\mathrm{per}}^T W \Phi_{\mathrm{per}} X_k(p)]^{1/2} - [X_k^T(p) W X_k(p)]^{1/2}\\
\le&\; [X_k^T(p)\Phi^T W \Phi X_k(p)]^{1/2} - [X_k^T(p) W X_k(p)]^{1/2}\\
&\quad + \left|[X_k^T(p)\Phi_{\mathrm{per}}^T W \Phi_{\mathrm{per}} X_k(p)]^{1/2} - [X_k^T(p)\Phi^T W \Phi X_k(p)]^{1/2}\right|\\
\le&\; \frac{-X_k^T(p) Q X_k(p)}{[X_k^T(p)(W - Q)X_k(p)]^{1/2} + [X_k^T(p) W X_k(p)]^{1/2}}\\
&\quad + \overline{\sigma}^{1/2}(W)\|\Delta\Phi\|\,\|X_k(p)\|\\
\le&\; -\left[\frac{\underline{\sigma}(Q)\|X_k(p)\|^2}{[\overline{\sigma}^{1/2}(W - Q) + \overline{\sigma}^{1/2}(W)]\,\|X_k(p)\|} - \overline{\sigma}^{1/2}(W)\|\Delta\Phi\|\,\|X_k(p)\|\right]\\
=&\; -\left[\frac{\underline{\sigma}(Q)}{\overline{\sigma}^{1/2}(W - Q) + \overline{\sigma}^{1/2}(W)} - \overline{\sigma}^{1/2}(W)\|\Delta\Phi\|\right]\|X_k(p)\|.
\end{aligned} \tag{5.36}$$

Hence, following a similar argument to that above, if

$$||\Delta\Phi|| < \frac{\underline{\sigma}(Q)}{\overline{\sigma}(W) + \overline{\sigma}^{1/2}(W)\overline{\sigma}^{1/2}(W-Q)} \tag{5.37}$$

then $\Delta\psi_k(p) < 0 \; \forall \; 0 \leq p \leq \alpha, \; k \geq 0$, and therefore the perturbed system (5.7) remains stable along the pass. Hence, setting v_b equal to the right hand side of (5.37) gives an alternative lower bound for v.

Note that other bounds based on a Lyapunov equation approach are possible - see, for example, (Tzafestas et al., 1992).

This section on a Lyapunov approach to parameter variation bounds concludes by considering the special case where $||\Phi|| < 1$. The two bounds for v presented earlier in this section require a solution to the 2D Lyapunov equation (4.93) which can be computationally intensive. However, if $||\Phi|| < 1$, the matrix Q defined by

$$Q = I_{n+m} - \Phi^T\Phi \tag{5.38}$$

is positive definite, which implies that the 2D Lyapunov equation with constant coefficients (4.93) has a positive definite solution with $W = I_{n+m}$ and Q given by (5.38).

Then since

$$\begin{aligned} \overline{\sigma}(\Phi^T\Phi) &= ||\Phi||^2 \quad \text{and} \\ \underline{\sigma}(I_{n+m} - \Phi^T\Phi) &= 1 - ||\Phi||^2, \end{aligned} \tag{5.39}$$

and by denoting the bound of (5.33) as v_b^1 and that of (5.37) as v_b^2 in this case, the two perturbation bounds v_b^1 and v_b^2 become

$$\begin{aligned} v_b^1 &= \frac{[\overline{\sigma}(\Phi^T\Phi) + \underline{\sigma}(I_{n+m} - \Phi^T\Phi)]^{1/2} - \overline{\sigma}^{1/2}(\Phi^T\Phi)}{\overline{\sigma}^{1/2}(I_{n+m})} \\ &= 1 - ||\Phi|| \end{aligned} \tag{5.40}$$

$$\begin{aligned} v_b^2 &= \frac{\underline{\sigma}(I_{n+m} - \Phi^T\Phi)}{\overline{\sigma}(I_{n+m}) + \overline{\sigma}^{1/2}(I_{n+m})\overline{\sigma}^{1/2}(\Phi^T\Phi)} = \frac{1 - ||\Phi||^2}{1 + ||\Phi||} \\ &= 1 - ||\Phi|| \end{aligned} \tag{5.41}$$

ie. for the special case of $||\Phi|| < 1$ the two perturbation bounds v_b^1 and v_b^2 are identical and equal to $1 - ||\Phi||$, which can be evaluated without the need to solve the 2D Lyapunov equation.

Note that v_b^1 and v_b^2 are lower bounds for the actual permissible parameter variation bound v, and hence the least conservative bounds are those which are as high as possible. Therefore, amongst all equivalent realisations of the system matrix $T\Phi T^{-1}$, it is best to seek a similarity transformation $T = T_1 \oplus T_2$ such that $||T\Phi T^{-1}||$ is minimised so as to achieve the largest possible stability robustness bounds v_b^1 and v_b^2. This point is discussed further in section 5.8.

5.5 Fornasini-Marchesini Model Based Analysis

Here robustness analysis is performed using a Fornasini-Marchesini model as a starting point. The analysis in this section uses some results from the theory of nonnegative matrices which are summarised in the appendix section A.1.

Consider the discrete unit memory linear repetitive process with state-space representation (2.24)-(2.25) and assume that the following necessary conditions for stability along the pass hold,

$$r(D_1) < 1 \quad \text{and} \quad r(A) < 1, \tag{5.42}$$

i.e. conditions (a) and (b) of theorem 4.7. Then the following result gives a condition for stability along the pass,

Theorem 5.2 *Discrete linear repetitive processes with unforced dynamics described by (5.3) are stable along the pass if, and only if,*

$$det(I_{n+m} - z_1 A_1 - z A_2) \neq 0 \quad \forall \ (z_1, z) \in \overline{U}^2 \tag{5.43}$$

where

$$A_1 = \begin{bmatrix} A & B_0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 0 \\ C & D_1 \end{bmatrix} \tag{5.44}$$

and $\overline{U}^2 = \{(z_1, z) \ : \ |z_1| \leq 1, \ |z| \leq 1\}$.

Proof: *Follows immediately on noting that, from theorem 4.7, stability along the pass holds if, and only if,*

$$\rho(z_1, z) \neq 0 \ \forall \ (z_1, z) \in \overline{U}^2. \tag{5.45}$$

To proceed, we need the following result,

Lemma 5.2 *Consider the transfer-function matrix*

$$\hat{G}_1(z_1) = (I_{n+m} - z_1 A_1)^{-1} \tag{5.46}$$

and write its Maclaurin series expansion as

$$\hat{G}_1(z_1) = (I_{n+m} - z_1 A_1)^{-1} = \sum_{j=0}^{\infty} A_1^j z_1^j. \tag{5.47}$$

Then

$$||\hat{G}_1(z_1) z A_2||_p \leq H \tag{5.48}$$

where

$$H := \sum_{j=0}^{\infty} ||A_1^j A_2||_p, \; \forall \, (z_1, z) \in \overline{U}^2. \tag{5.49}$$

Proof: *Follows immediately from applying the properties of nonnegative matrices given in section A.1 and hence the details are omitted.*

Now we have the following sufficient condition for stability along the pass,

Theorem 5.3 *Discrete linear repetitive processes with unforced dynamics described by (5.3) are stable along the pass if*

$$r(H) < 1. \tag{5.50}$$

Proof: *Since $r(A_1) < 1$, condition (5.43) for stability along the pass is equivalent to*

$$det(I_{n+m} - (I_{n+m} - z_1 A_1)^{-1} z A_2) \neq 0 \quad \forall \; (z_1, z) \in \overline{U}^2 \tag{5.51}$$

and this condition holds provided

$$r(\hat{G}_1(z_1) z A_2) < 1 \quad \forall \; (z_1, z) \in \overline{U}^2. \tag{5.52}$$

Hence, using the spectral radius inequality of lemma A.1, we have

$$r(\hat{G}_1(z_1) z A_2) \leq r(||\hat{G}_1(z_1) z A_2||_p) \leq r(H) \quad \forall \; (z_1, z) \in \overline{U}^2 \tag{5.53}$$

and the proof is complete.

An identical analysis, and hence the proof is omitted here, leads to the following result,

Theorem 5.4 *Discrete linear repetitive processes with unforced dynamics described by (5.3) are stable along the pass if*

$$r(H_1) < 1 \tag{5.54}$$

where

$$H_1 := \sum_{j=0}^{\infty} ||A_2^j A_1||_p. \tag{5.55}$$

As a special case, suppose that all elements in A_1 and A_2 are positive. Then in this case the matrices H and H_1 as defined above are given as follows,

$$\begin{aligned} H &= (I_{n+m} - A_1)^{-1} A_2 \quad \text{and} \\ H_1 &= (I_{n+m} - A_2)^{-1} A_1. \end{aligned} \tag{5.56}$$

Now consider the same discrete process, where the matrices A_1 and/or A_2 are subject to additive perturbations as follows,

$$\begin{aligned} A_1 &\longrightarrow A_1 + \Delta A_1 \\ A_2 &\longrightarrow A_2 + \Delta A_2 \end{aligned} \tag{5.57}$$

and the perturbation matrices ΔA_i, $i = 1, 2$, can have the following forms:

(i) $||\Delta A_i||_p \leq \hat{a}_i F_i$, $i = 1, 2$, where $\hat{a}_i > 0$ and F_i is a nonnegative matrix. This is the case when highly structured information is available on the perturbations of the entries in A_i; or

(ii) $||\Delta A_i|| \leq f_i$, $i = 1, 2$, where $f_i > 0$ and, for a matrix X, $||X|| = \overline{\sigma}(X) \equiv (\lambda_{\max}(X^T X))^{\frac{1}{2}}$ where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue. This corresponds to the case when the perturbations are unstructured and only a spectral norm bound on the perturbation is known.

As in the analysis of the previous section, the starting point of what follows is the assumption that the nominal (i.e. unforced unperturbed) process is stable along the pass. In which case the aim of the remainder of this section is to find bounds on

$||\Delta A_i||_p$, $i = 1, 2$, for perturbations of the type (i) above and on $||\Delta A_i||$, $i = 1, 2$, for type (ii).

The analysis of this section requires the following result,

Lemma 5.3 *Write the Maclaurin series for*

$$T(z_1, z) := (I_{n+m} - z_1 A_1 - z A_2)^{-1} \tag{5.58}$$

as

$$T(z_1, z) = \sum_{j=0}^{\infty} (z_1 A_1 + z A_2)^j. \tag{5.59}$$

Then

$$||T(z_1, z)||_p \leq \hat{H} := I_{n+m} + \hat{H}_1 + \hat{H}_2 \quad \forall \ (z_1, z) \in \overline{U}^2 \tag{5.60}$$

where

$$\begin{aligned} \hat{H}_1 &:= (I_{n+m} - L_1 L_2)^{-1}(L_1 + L_1 L_2) \\ \hat{H}_2 &:= (I_{n+m} - L_2 L_1)^{-1}(L_2 + L_2 L_1) \end{aligned} \tag{5.61}$$

and

$$L_1 := \sum_{j=1}^{\infty} ||A_1^j||_p, \quad L_2 := \sum_{j=1}^{\infty} ||A_2^j||_p. \tag{5.62}$$

Proof: *First note that*

$$T(z_1, z) = I_{n+m} + \tilde{H}_1 + \tilde{H}_2 \tag{5.63}$$

where

$$\begin{aligned} \tilde{H}_1 &:= A_1 f_1(A_1, A_2, z_1, z), \\ \tilde{H}_2 &:= A_2 f_2(A_1, A_2, z_1, z), \end{aligned} \tag{5.64}$$

where $f_i(\cdot)$, $i = 1, 2$, are nonlinear functions of their arguments. It now follows (by extensive, but routine, manipulations) that

$$\begin{aligned} \tilde{H}_1 &= \sum_{j=1}^{\infty} A_1^j z_1^j (I_{n+m} + \tilde{H}_2) \\ \tilde{H}_2 &= \sum_{j=1}^{\infty} A_2^j z^j (I_{n+m} + \tilde{H}_1) \end{aligned} \tag{5.65}$$

and hence, for $|z_1| \leq 1$ *and* $|z| \leq 1$,

$$\begin{aligned} ||\tilde{H}_1||_p &\leq L_1(||I_{n+m} + \tilde{H}_2||_p) \leq L_1 + L_1||\tilde{H}_2||_p \\ ||\tilde{H}_2||_p &\leq L_2(||I_{n+m} + \tilde{H}_2||_p) \leq L_2 + L_2||\tilde{H}_1||_p. \end{aligned} \tag{5.66}$$

Hence on solving the two inequalities of (5.66) we have that

$$\begin{aligned} ||\tilde{H}_1||_p &\leq \hat{H}_1 := (I_{n+m} - L_1L_2)^{-1}(L_1 + L_1L_2) \\ ||\tilde{H}_2||_p &\leq \hat{H}_2 := (I_{n+m} - L_2L_1)^{-1}(L_2 + L_2L_1) \end{aligned} \tag{5.67}$$

and the proof is complete.

Note that for $\hat{H}$ in lemma 5.3 to exist, we require

$$r(L_1L_2) = r(L_2L_1) < 1 \tag{5.68}$$

and hence this fact can be regarded as another sufficient condition for stability along the pass.

In the case when $||A_i||_p = A_i, i = 1, 2$, i.e. when all elements of A_1 and A_2 are positive and $r(A_1 + A_2) < 1$ then $\hat{H} = (I_{n+m} - A_1 - A_2)^{-1}$. Also if $||A_i||_p \neq A_i$, $i = 1, 2$, and in addition $r(||A_1||_p + ||A_2||_p) < 1$ then $\hat{H} \leq (I_{n+m} - ||A_1||_p - ||A_2||_p)^{-1}$.

The following result now gives sufficient conditions for stability along the pass under the structured perturbations of case (i) defined above.

Theorem 5.5 *The following are sufficient conditions for stability along the pass of discrete linear repetitive processes under the structured perturbations of case (i) defined above (i.e.*$||\Delta A_i||_p \leq \hat{a}_i F_i$, $i = 1, 2$),

$$r(H_1 + (I_{n+m} + L_1)\sum_{i=1}^{2} \hat{a}_i F_i) < 1 \tag{5.69}$$

or

$$r(H_2 + (I_{n+m} + L_2)\sum_{i=1}^{2} \hat{a}_i F_i) < 1. \tag{5.70}$$

Proof: *The proofs of these two conditions follow the same basic steps and hence only the proof of (5.69) is given here.*

The closed loop system for the perturbed process in this case is stable along the pass if, and only if,

$$det(I_{n+m} - z_1(A_1 + \Delta A_1) - z(A_2 + \Delta A_2)) \neq 0 \quad \forall\ (z_1, z) \in \overline{U}^2 \tag{5.71}$$

or, since $r(A_1) < 1$, this reduces to

$$det(I_{n+m} - (I_{n+m} - z_1A_1)^{-1}zA_2 - (I_{n+m} - z_1A_1)^{-1}(z_1\Delta A_1 + z\Delta A_2)) \neq 0 \tag{5.72}$$

for $(z_1, z) \in \overline{U}^2$. This condition holds if

$$r((I_{n+m} - z_1A_1)^{-1}zA_2 + (I_{n+m} - z_1A_1)^{-1}(z_1\Delta A_1 + z\Delta A_2)) < 1 \quad \forall\ (z_1, z) \in \overline{U}^2. \tag{5.73}$$

Using (5.47) and (5.49), this last equation leads to the following inequalities

$$\begin{aligned} r(\hat{G}_1(z_1)zA_2 + \hat{G}_1(z_1)(z_1\Delta A_1 + z\Delta A_2)) &\leq r(H_1 + \|\tilde{\Delta}\|_p) \\ &\leq r(H_1 + (I_{n+m} + L_1)\sum_{i=1}^{2} \hat{a}_iF_i) \end{aligned} \tag{5.74}$$

where

$$\|\tilde{\Delta}\|_p := \|\Delta A_1\|_p + \|\Delta A_2\|_p \tag{5.75}$$

and the proof is complete.

The following result gives sufficient conditions for the stability along the pass of discrete processes subject to the unstructured perturbations of case (ii) defined above.

Theorem 5.6 *The following are sufficient conditions for stability along the pass of discrete linear repetitive processes under the unstructured perturbations of case (ii) defined above,*

$$f_1 + f_2 < \frac{1 - \overline{\sigma}(H)}{\overline{\sigma}(I_{n+m} + L_1)} \tag{5.76}$$

or

$$f_1 + f_2 < \frac{1 - \overline{\sigma}(H_1)}{\overline{\sigma}(I_{n+m} + L_2)}. \tag{5.77}$$

Proof: *Again, the proofs of these two conditions follow identical steps and hence only (5.76) is proved here. The proof relies on the fact that if X is an $l \times l$ matrix then $r(X) \leq \overline{\sigma}(X)$. Hence we have the inequality*

$$r(H + (I_{n+m} + L_1)(||\Delta A_1||_p + ||\Delta A_2||_p)) \leq f_1 + f_2\,\overline{\sigma}(I_{n+m} + L_1) + \overline{\sigma}(H) \quad (5.78)$$

and the result follows immediately.

In addition, the following stability along the pass conditions apply to both types of perturbations,

Theorem 5.7 *Discrete linear repetitive processes are stable along the pass under both types of perturbations introduced under (i) and (ii) above if the following conditions hold for the structured and unstructured perturbations respectively,*

$$r(\hat{H}\sum_{i=1}^{2}\hat{a}_i F_i) < 1, \quad i = 1, 2, \quad (5.79)$$

and

$$f_1 + f_2 < \frac{1}{\overline{\sigma}(\hat{H})} \quad (5.80)$$

where $\hat{H}$ is defined by (5.60).

Proof: *Since the nominal system is assumed to be stable along the pass, it follows that the perturbed system will also be stable along the pass if, and only if,*

$$det(I_{n+m} - (I_{n+m} - z_1 A_1 - zA_2)^{-1}(z_1\Delta A_1 + z\Delta A_2)) \neq 0 \quad (5.81)$$

for $(z_1, z) \in \overline{U}^2$. The result now follows immediately on using (5.59) and (5.60).

Note now that the conditions of (5.76) and (5.77) require that $\overline{\sigma}(H) < 1$ and $\overline{\sigma}(H_1) < 1$ respectively. This restriction is not required by (5.80) and hence it is a less conservative alternative to these two conditions.

In addition, upper bounds for $\hat{a}_i$, $i = 1, 2$, can be computed as

$$\hat{a}_1\,\overline{\sigma}(HF_1) + \hat{a}_2\,\overline{\sigma}(HF_2) < 1. \quad (5.82)$$

It is possible to provide an alternative upper bound for $T(z_1, z)$ to that of (5.60), as can be seen in the following result,

Lemma 5.4 *$T(z_1, z)$ of (5.58) satisfies the following inequality for $(z_1, z) \in \overline{U}^2$,*

$$\|T(z_1, z)\|_p \leq \hat{L} := I_{n+m} + L_1 + L_2 + \hat{L}_1 + \hat{L}_2 \tag{5.83}$$

where

$$\begin{aligned} \hat{L}_1 &:= \hat{B}^{-1}\hat{C} \\ \hat{L}_2 &:= \hat{D} + \hat{E}\hat{B}^{-1}\hat{C} \end{aligned} \tag{5.84}$$

and

$$\begin{aligned} \hat{B} &:= I_{n+m} - \overline{H}_1 - \overline{H}_1(I_{n+m} - \overline{H}_2)^{-1}\overline{H}_2, \\ \hat{C} &:= \overline{H}_1(I_{n+m} + L_1 + L_2) + \overline{H}_1(I_{n+m} - \overline{H}_2)^{-1}\overline{H}_2(I_{n+m} + L_1 + L_2), \\ \hat{D} &:= (I_{n+m} - \overline{H}_2)^{-1}\overline{H}_2(I_{n+m} + L_1 + L_2), \\ \hat{E} &:= (I_{n+m} - \overline{H}_2)^{-1}\overline{H}_2, \\ \overline{H}_1 &:= \sum_{j=1}^{\infty} \|A_1^j A_2\|_p, \\ \overline{H}_2 &:= \sum_{j=1}^{\infty} \|A_2^j A_1\|_p. \end{aligned} \tag{5.85}$$

Proof: *First write the Maclaurin series for $T(z_1, z)$ in the form*

$$T(z_1, z) = I_{n+m} + \sum_{j=1}^{\infty} A_1^j z_1^j + \sum_{j=1}^{\infty} A_2^j z + K_1 + K_2 \tag{5.86}$$

where

$$\begin{aligned} K_1 &:= \sum_{j=1}^{\infty} A_1^j z_1^j z(I_{n+m} + \tilde{H}_1 + \tilde{H}_2) \\ K_2 &:= \sum_{j=1}^{\infty} A_2^j A_1 z_1 z^j (I_{n+m} + \tilde{H}_1 + \tilde{H}_2). \end{aligned} \tag{5.87}$$

Hence we have that

$$\begin{aligned} \tilde{H}_1 &= \sum_{j=1}^{\infty} A_1^j z_1^j + K_1 \\ \tilde{H}_2 &= \sum_{j=1}^{\infty} A_2^j z + K_2. \end{aligned} \tag{5.88}$$

It now follows that

$$\begin{aligned} \|K_1\|_p &\leq \overline{H}_1(I_{n+m} + L_1 + L_2 + \|K_1\|_p + \|K_2\|_p) \\ \|K_2\|_p &\leq \overline{H}_2(I_{n+m} + L_1 + L_2 + \|K_1\|_p + \|K_2\|_p) \end{aligned} \tag{5.89}$$

and hence

$$\begin{aligned} \|K_1\|_p &\leq \hat{L}_1 := \hat{B}^{-1}\hat{C} \\ \|K_2\|_p &\leq \hat{L}_2 := \hat{D} + \hat{E}\hat{B}^{-1}\hat{C}. \end{aligned} \tag{5.90}$$

The matrix $\hat{L}$ in (5.83) exists provided

$$r(\overline{H}_1 + \overline{H}_1(I_{n+m} - \overline{H}_2)^{-1}\overline{H}_2) < 1 \tag{5.91}$$

and this is another sufficient condition for stability along the pass. This condition can also be used to derive sufficient conditions for stability along the pass as alternative to (5.79) and (5.80).

In summary, this section has presented robustness analysis based on a Fornasini-Marchesini model of the dynamics of a discrete linear repetitive process. Given a stable along the pass unforced system, bounds on the permissible parameter variations have been derived for both cases of structured and unstructured perturbations. Note that to fully exploit these results the least conservative set for a particular example should obviously be used.

5.6 Stability Margins

The second type of relative stability analysis to be discussed within this thesis is that of stability margins. In 1D systems theory, the stability margin is defined as a measure of the distance between the dominant poles (or eigenvalues) of the system and the stability limit (for discrete systems this is just the boundary of the unit circle). Then a necessary and sufficient condition for 1D stability is that this measure, σ say, is strictly greater than zero. (Note that if $\sigma = 0$, then a root lies on the unit circle, whereas $\sigma < 0$ means that at least one root lies outside the stability region and the process is unstable).

Within this section, candidate definitions of stability margins for discrete linear repetitive processes are given. Note here that, since a fundamental difference between systems in one dimension and those in n, $n \geq 2$, is that the singularities are no longer isolated poles, but multidimensional manifolds, the stability margin for a repetitive process is given in terms of analytic regions of functions in two variables.

5.6.1 Problem Statement

In characterising systems behaviour, besides stability, it is extremely useful (or even essential) to have an indication of to what extent the poles of the system may be moved before it becomes unstable. In 1D systems theory, the distance of the dominant eigenvalues from the stability limit (the so-called stability margin) is used as a measure for this.

Consider the subclass of discrete linear unit memory processes with the state-space representation (2.24)-(2.25). The starting point of the following analysis is to assume that the process is stable along the pass. Now assume that the process is free of control inputs and rewrite the dynamics of the process in the form of (5.3). Now since, by assumption, the process is stable along the pass, theorem 4.7 holds and we have

$$\rho(z_1, z) = \begin{vmatrix} I_n - z_1 A & -z_1 B_0 \\ -zC & I_m - zD_1 \end{vmatrix} \neq 0, \quad \text{for } (z_1, z) \in \overline{U}^2. \tag{5.92}$$

The definition of a stability margin for 2D discrete systems described by the Roesser state-space model was first introduced in (Agathoklis et al., 1982) as a criterion for characterising the spatial domain performance of a stable 2D system. Here (see also (Walach and Zeheb, 1982; Swamy et al., 1981)) a stability margin is defined as the shortest distance between the singularities of the system and the boundary of the stability region, which is the boundary of the unit bidisc. This is the largest bidisc where $\rho(z_1, z)$ has no roots, i.e.

$$\begin{aligned} &\rho(z_1, z) \neq 0 \text{ in } U^2_{\sigma_1} = \{(z_1, z) : |z_1| < 1 + \sigma_1, |z| < 1\} \\ &\rho(z_1, z) \neq 0 \text{ in } U^2_{\sigma_2} = \{(z_1, z) : |z_1| < 1, |z| < 1 + \sigma_2\} \\ &\rho(z_1, z) \neq 0 \text{ in } U^2_{\sigma} = \{(z_1, z) : |z_1| < 1 + \sigma, |z| < 1 + \sigma\}. \end{aligned} \tag{5.93}$$

Then the conditions $\sigma_1 > 0$, $\sigma_2 > 0$, $\sigma > 0$, are necessary and sufficient conditions for stability along the pass of discrete processes described by (2.24) and (2.25).

A review of the literature indicates that considerable effort has been directed towards the development of algorithms for computing σ_1, σ_2 and σ. As a result of this, numerous algorithms are available based on different approaches/starting points. For example, (Walach and Zeheb, 1982; Agathoklis et al., 1982; Hertz and Zeheb, 1987) are based on minimizing the distance between the roots of $\rho(z_1, z)$ and the boundary of the unit bidisc $T^2 = \{(z_1, z) : |z_1| = 1, |z| = 1\}$ and (Roytman et al., 1987) introduces algorithms based on the so-called resultant matrix.

5.7 A Lyapunov Approach to Stability Margin Analysis

A different approach is based on the premise that it is not always necessary to know the exact value of the stability margin. Instead it suffices to know that they are greater than certain lower limits. This concept is illustrated for the equivalent 1D case in figure 5.1. Examples of such limits can be obtained as functions of the positive definite solution to the 2D Lyapunov equation which has been extensively discussed in chapter 4. Here we present one such limit as a function of the solution of (4.93) (see also (Agathoklis, 1985; Agathoklis, 1988) for the case of 2D systems with dynamics described by the Roesser state-space model).

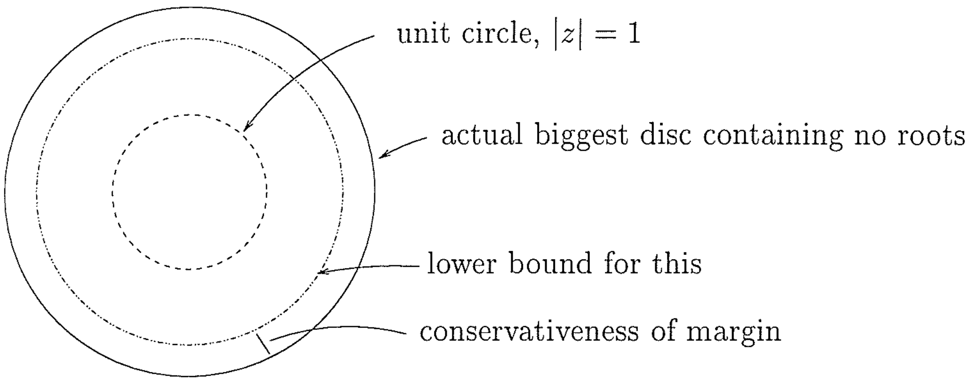

Figure 5.1: Lower Bound for 1D Stability Margin

Consider again the unforced stable along the pass subclass of discrete linear repetitive processes of (5.3). In addition, assume that there exists a positive definite solution pair $\{W, Q\}$ to the 2D Lyapunov equation (4.93), which can be rewritten in the form

$$W - \Phi^T W \Phi = Q \tag{5.94}$$

with

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}. \tag{5.95}$$

Note once again that the satisfaction of the 2D Lyapunov equation is a stronger requirement to that of stability along the pass alone.

Now pre and post multiply equation (5.94) by $(\beta_1 I_n \oplus \beta_2 I_m)$ where β_1 and β_2 are real positive scalars to yield

$$\begin{bmatrix} \beta_1^2 W_1 & 0 \\ 0 & \beta_2^2 W_2 \end{bmatrix} - \hat{\Phi}^T \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} \hat{\Phi} = \begin{bmatrix} \beta_1^2 Q_1 & \beta_2\beta_1 Q_2 \\ \beta_1\beta_2 Q_2^T & \beta_2^2 Q_3 \end{bmatrix} \tag{5.96}$$

where

$$\hat{\Phi} = \begin{bmatrix} \beta_1 A & \beta_2 B_0 \\ \beta_1 C & \beta_2 D_1 \end{bmatrix}. \tag{5.97}$$

Then adding W to both sides of (5.96) and rearranging gives

$$W - \hat{\Phi}^T W \hat{\Phi} = \overline{Q} \tag{5.98}$$

where

$$\overline{Q} = \begin{bmatrix} \beta_1^2 Q_1 + (1-\beta_1^2) W_1 & \beta_2\beta_1 Q_2 \\ \beta_1\beta_2 Q_2^T & \beta_2^2 Q_3 + (1-\beta_2^2) W_2 \end{bmatrix}. \tag{5.99}$$

Since W is positive definite, a sufficient condition for the stability along the pass of discrete processes with augmented plant matrix $\hat{\Phi}$ is that $\overline{Q}$ is positive definite, which is in turn a sufficient condition for

$$\hat{\rho}(z_1, z) = \det(\mathrm{diag}\{I_n, I_m\} - \mathrm{diag}\{z_1 I_n, z I_m\}\hat{\Phi}) \neq 0 \quad \forall\ (z_1, z) \in \overline{U}^2. \tag{5.100}$$

The relationship between the zeros of $\rho(z_1, z)$ and $\hat{\rho}(z_1, z)$, denoted (z_1^s, z^s) and $(\hat{z}_1^s, \hat{z}^s)$ respectively, is established with the following lemma,

Lemma 5.5 *(Agathoklis, 1988) If (z_1^s, z^s) and $(\hat{z}_1^s, \hat{z}^s)$ denote the roots of $\rho(z_1, z)$ and $\hat{\rho}(z_1, z)$ respectively, then the following relationship holds*

$$(\hat{z}_1^s, \hat{z}^s) = (\beta_1 z_1^s, \beta_2 z^s). \tag{5.101}$$

Consider now the roots of the characteristic equation $\hat{\rho}(z_1, z)$ as a function of β_1 and β_2 and refer to the process with augmented plant matrix $\hat{\Phi}$ as the 'new process'. Clearly if $\beta_1 = \beta_2 = 1$, $\hat{\rho}(z_1, z) = \rho(z_1, z)$ and the new process is stable along the pass. If $\beta_i > 1$, $i = 1, 2$, the roots of $\hat{\rho}(z_1, z)$ move away from (z_1^s, z^s) towards infinity and the new process remains stable along the pass, while for $\beta_i < 1$, $i = 1, 2$, the roots move towards the boundary of the unit bidisc and, eventually, some move to within the unit bidisc. Hence a sufficient condition for the roots of $\hat{\rho}(z_1, z)$ to be outside the unit bidisc for a certain value of (β_1, β_2) is that $\overline{Q}$ is positive definite for that value. Consequently the range of (β_1, β_2) for which $\overline{Q}$ remains positive definite is clearly related to the distance between the roots of $\rho(z_1, z)$ and the boundary of the unit bidisc. Note that, since the satisfaction of the 2D Lyapunov equation is a sufficient but not necessary condition for stability along the pass, $\overline{Q}$ not being positive definite does not imply that $\hat{\rho}(z_1, z)$ has a root within the unit bidisc and hence is unstable along the pass. Hence this approach can give lower bounds only for the stability margins, and, in general, not the exact values.

A lower bound for σ_1

From the definition of the stability margin bidisc $U_{\sigma_1}^2$ in (5.93) it follows that a lower bound for σ_1 can be obtained from the range of β_1 for which $\overline{Q}$ is positive definite with $\beta_2 = 1$. Setting $\beta_2 = 1$ in (5.99) yields

$$\overline{Q} = \begin{bmatrix} \beta_1^2 Q_1 + (1 - \beta_1^2) W_1 & \beta_1 Q_2 \\ \beta_1 Q_2^T & Q_3 \end{bmatrix}. \tag{5.102}$$

Since Q_3 is positive definite, we require

$$\beta_1^2 Q_1 + (1 - \beta_1^2) W_1 - \beta_1^2 Q_2 Q_3^{-1} Q_2^T > 0. \tag{5.103}$$

Then a lower bound for σ_1 can be obtained as

$$\sigma_1 \geq \beta_1^{\text{limit}} - 1 \tag{5.104}$$

where

$$\beta_1^{\text{limit}} = \sqrt{\frac{\min \lambda(W_1)}{\max \lambda(W_1 - Q_1 + Q_2^T Q_3^{-1} Q_2)}}. \tag{5.105}$$

Similar bounds for σ_2 and σ can be obtained as follows

$$\sigma_2 \geq \beta_2^{\text{limit}} - 1 \tag{5.106}$$

where

$$\beta_2^{\text{limit}} = \sqrt{\frac{\min \lambda(W_2)}{\max \lambda(W_2 - Q_3 + Q_2^T Q_1^{-1} Q_2)}} \tag{5.107}$$

and

$$\beta = \beta_1 = \beta_2, \quad \sigma \geq \beta^{\text{limit}} - 1 \tag{5.108}$$

where

$$\beta^{\text{limit}} = \sqrt{\frac{\min \lambda(W)}{\max \lambda(W - Q)}}. \tag{5.109}$$

An alternative lower bound for σ_1 can be obtained using the state vector of (2.52), i.e.

$$X_k(p) = [x_k(p)^T, \ z_k(p)^T]^T \tag{5.110}$$

and introducing

$$A_1 = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & B_0 \\ 0 & D_1 \end{bmatrix}. \tag{5.111}$$

Then a bound for σ_1 can be derived from the range of β_1 for which $\overline{Q}$ is positive definite with $\beta_2 = 1$ as follows. The aim is to find a lower bound for β_1 such that $\hat{\Phi}$ with $\beta_2 = 1$ satisfies the 2D Lyapunov equation and hence is stable along the pass. Now, since both Φ and $\hat{\Phi}$ admit positive definite solutions to the 2D Lyapunov equation (5.94), we can write

$$X_k^T(p)\left\{(W - \Phi^T W \Phi) - (W - \hat{\Phi}^T W \hat{\Phi})\right\} X_k(p) \leq X_k^T(p)\left\{W - \Phi^T W \Phi\right\} X_k(p). \tag{5.112}$$

After extensive but routine manipulations we obtain,

$$\begin{aligned} X_k^T(p)\{(\beta_1 - 1)^2 A_1^T W A_1 \ + \ (\beta_1 - 1) A_1^T W \Phi + (\beta_1 - 1)\Phi W A_1^T\} X_k(p) \\ \leq X_k^T(p)\left\{W - \Phi^T W \Phi\right\} X_k(p). \end{aligned} \tag{5.113}$$

Then (see (Tzafestas et al., 1992) for the case of 2D discrete linear systems described by the Roesser model) a sufficient condition for $\overline{Q}$ to be positive definite is

$$(\beta_1 - 1)^2 \|A_1\|^2 \|W\| + 2(\beta_1 - 1)\|A_1\| \ \|W\| \ \|\Phi\| \leq \lambda_{\min}(W - \Phi^T W \Phi) = \lambda_{\min}(Q). \tag{5.114}$$

Hence the stability margin is given by

$$\sigma_1 = \beta_1 - 1 \geq \hat{\beta}_1 \tag{5.115}$$

where

$$\hat{\beta}_1 = \frac{-||\Phi||\,||W|| + \sqrt{||W||^2||\Phi||^2 + ||W||\lambda_{\min}(Q)}}{||A_1||\,||W||}. \tag{5.116}$$

The lower bound for σ_2 is derived by routine changes to the analysis just completed for σ_1. Hence only the final result is given here, i.e.

$$\sigma_2 = \beta_2 - 1 \geq \hat{\beta}_2 \tag{5.117}$$

where

$$\hat{\beta}_2 = \frac{-||\Phi||\,||W|| + \sqrt{||W||^2||\Phi||^2 + ||W||\lambda_{\min}(Q)}}{||A_2||\,||W||}. \tag{5.118}$$

The lower bound for σ follows from considering the case where $\beta_1 = \beta_2 = \beta$. The final result is

$$\sigma = \beta - 1 \geq \hat{\beta} \tag{5.119}$$

where

$$\hat{\beta} = -1 + \sqrt{1 + \frac{\lambda_{\min}(Q)}{||\Phi||^2||W||}}. \tag{5.120}$$

5.8 Minimum Spectral Norms

Clearly the lower bounds for the stability margins presented in the previous section depend upon the choice of matrices W and Q, i.e. different bounds are obtained for different pairs of matrices. In (Lu et al., 1986) it is shown that the least conservative bounds, i.e. those closest to the actual value of the stability margin, can be obtained with a pair $\{W, Q\}$ corresponding to the minimum norm of the augmented system matrix Φ.

Suppose that the example under consideration is stable along the pass and that it admits a positive definite solution pair $\{W, Q\}$ to the 2D Lyapunov equation (5.94). Then the minimum spectral norm μ for the process is defined as

$$\mu = \min_T ||T\Phi T^{-1}|| \tag{5.121}$$

where $T = T_1 \oplus T_2$, and T_1 and T_2 are $n \times n$ and $m \times m$ matrices respectively.

The pair $\{W, Q\}$ obtained sets $W = \hat{T}^{-1}\hat{T}$ where $\hat{T}$ is the matrix which minimises (5.121) and corresponds to the minimum norm μ of Φ. This choice of W gives the best lower bounds for the stability margins. Algorithms for obtaining a regular $\hat{T}$ are discussed in (Lodge and Fahmy, 1981) and properties of μ are discussed in (Lu et al., 1986).

5.9 The Poles of a Repetitive Process

As yet in this thesis no discussion has been given on the subject of a repetitive process version of the well known 1D linear systems theory concept of a pole (or zero). The stability dependence on two complex variables precludes the numerical definition of a pole and hence the singularities are no longer isolated points (as in 1D linear systems theory) but multidimensional manifolds.

Within this section, the poles of a discrete unit memory linear repetitive process are studied using the behavioural approach (Wood et al., 2000). Behavioural theory uses a high level of abstract algebra, and consideration of the subject of poles (and, more recently zeros, - see (Zaris et al., 2000) for the details) of multidimensional (and hence 2D) systems is a recent advance in this area. Hence only a brief outline of the ideas as applicable to linear repetitive process theory is given - for further details see (Rogers et al., 2000b) - and the area remains open for future research.

Consider the discrete linear repetitive process with state-space representation (2.24)-(2.25). Since the state vector on pass 0, i.e. $x_0(p)$, $0 \le p \le \alpha$, plays no role in the process dynamic evolution, it is convenient to relabel the state trajectories $x_{k+1}(p) \mapsto x_k(p)$ (keeping of course the same interpretation). (Note that this is equivalent to introducing a change of variables, eg. $z_k(p) := x_{k+1}(p)$, and proceeding from there.)

The repetitive process dynamics are now described by the following state-space representation over $0 \le p \le \alpha$, $k \ge 0$,

$$\begin{aligned} x_k(p+1) &= A\, x_k(p) + B\, u_{k+1}(p) + B_0\, y_k(p) \\ y_{k+1}(p) &= C\, x_k(p) + D_1\, y_k(p). \end{aligned} \tag{5.122}$$

Using the terminology of (Wood et al., 2000), the behaviour $\mathcal{B}_{x,u,y}$ of this system can now be given by the kernel representation,

$$\begin{pmatrix} z_1 I_n - A & -zB & -B_0 \\ -C & 0 & zI_m - D_1 \end{pmatrix} \begin{pmatrix} x \\ u \\ y \end{pmatrix} = 0, \tag{5.123}$$

where z_1 denotes the along the pass shift operator, eg. z_1 applied to $x_k(p)$ gives

$$(z_1 x_k)(p) := x_k(p+1), \tag{5.124}$$

and z denotes the pass to pass shift operator, as follows

$$(zy_k)(p) := y_{k+1}(p). \tag{5.125}$$

Note that the components of the solutions of the system can be considered as functions from $\mathbb{Z}_+^2$ to $\mathbb{R}$, though for purposes of interpretation they are cut off in one dimension at the pass length α (a key difference from the standard 2D linear systems case).

The poles of this system are defined as the characteristic points of the zero-input behaviour $\mathcal{B}_{x,0,y}$ (i.e. the unforced, or nominal, discrete process obtained by setting $u_k(p) \equiv 0 \forall k \geq 0,\ 0 \leq p \leq \alpha$), that is the set of all trajectories which can arise when the input vanishes. The zero-input behaviour is given to within trivial isomorphism by

$$\begin{pmatrix} z_1 I_n - A & -B_0 \\ -C & zI_m - D_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0. \tag{5.126}$$

Following (Wood et al., 2000), the poles of the process can be defined as

Definition 5.1 (Poles of a Discrete Linear Repetitive Process) *The poles of the discrete linear repetitive process (2.24)-(2.25) are the points in 2D complex space where the matrix on the left hand side of (5.126) fails to have full rank. That is, they are given by the set*

$$\mathcal{V}(\mathcal{B}_{x,0,y}) = \{(a_1, a) \in \mathbb{C}^2 \mid \rho(a_1, a) = 0\}, \tag{5.127}$$

where

$$\rho(z_1, z) = \det \begin{pmatrix} z_1 I_n - A & -B_0 \\ -C & zI_m - D_1 \end{pmatrix}. \tag{5.128}$$

The set $\mathcal{V}$ is called the pole variety of the system.

Since, in this case, the pole variety is given by the vanishing of a single 2D non-unit polynomial, it is guaranteed to be a one-dimensional geometric set in 2D complex space, i.e. a curve. In particular, the pole variety cannot be zero-dimensional (i.e. finite). This corresponds to the fact that proper principal ideals in the ring $\mathbb{C}[z_1, z]$ have codimension 1. Note also that the pole variety is a complex variety, even though the entries of the matrices A, B_0, C and D_1 are generally assumed to be real. This is essential in order to capture the full exponential-type dynamics of the system.

Poles can be interpreted in terms of exponential trajectories (Wood et al., 2000), which in the case of repetitive processes have a clear physical interpretation. Assume therefore that $(a_1, a) \in \mathbb{C}^2$ is a zero of $\rho(z_1, z)$, and write it in the form $a_1 = r_1 e^{i\theta_1}, a = re^{i\theta}$ (with $\theta_1 = 0$ for $a_1 = 0$ and $\theta = 0$ for $a = 0$). The existence of such a zero guarantees (see (Wood et al., 2000) for the details) the existence of a so-called exponential trajectory in the system having the form

$$\begin{aligned} x'_k(p) &= x_{00}^1 r_1^p r^k \cos(\theta_1 p + \theta k) + x_{00}^2 r_1^p r^k \sin(\theta_1 p + \theta k), \\ y'_k(p) &= y_{00}^1 r_1^p r^k \cos(\theta_1 p + \theta k) + y_{00}^2 r_1^p r^k \sin(\theta_1 p + \theta k), \\ u'_k(p) &= 0, \end{aligned} \tag{5.129}$$

where $x_{00}^1, x_{00}^2 \in \mathbb{R}^n$, $y_{00}^1, y_{00}^2 \in \mathbb{R}^m$, and at least one of these four is non-zero. This form of exponential trajectory has been characterised algebraically by Oberst (Oberst, 1990). Conversely, the existence of such a trajectory implies that $\rho(r_1 e^{i\theta_1}, re^{i\theta}) = 0$, i.e. the so-called frequency $(r_1 e^{i\theta_1}, re^{i\theta})$ is a pole of the repetitive process.

In the case where $(a_1, a) \in \mathbb{R}^2$ it is straightforward to construct such trajectories from the zeros of the characteristic polynomial $\rho(z_1, z)$. Take a_1 and a to be real numbers satisfying $\rho(a_1, a) = 0$. There must then exist a non-zero vector $(x_{00}, y_{00}) \in \mathbb{R}^{n+m}$ satisfying

$$\begin{pmatrix} a_1 I_n - A & -B_0 \\ -C & aI_m - D_1 \end{pmatrix} \begin{pmatrix} x_{00} \\ y_{00} \end{pmatrix} = 0 \tag{5.130}$$

System trajectories can now be obtained by extending (x_{00}, y_{00}) to give

$$\begin{aligned} x'_k(p) &= x_{00} a_1^p a^k, \\ y'_k(p) &= y_{00} a_1^p a^k, \\ u'_k(p) &= 0 \end{aligned} \tag{5.131}$$

Then it can easily be shown that

$$\begin{aligned} x'_k(p+1) &= a_1 x_{00} a_1^p a^k \\ &= A x_{00} a_1^p a^k + B_0 y_{00} a_1^p a^k \\ &= A x'_k(p) + B u'_{k+1}(p) + B_0 y'_k(p) \end{aligned} \tag{5.132}$$

$$\begin{aligned} y'_{k+1}(p) &= a y_{00} a_1^p a^k \\ &= D_1 y_{00} a_1^p a^k + C x_{00} a_1^p a^k \\ &= D_1 y'_k(p) + C x'_k(p), \end{aligned} \tag{5.133}$$

proving that (5.131) indeed describes a solution of the system.

Returning to the general case (5.129), we see that if $\bmod a = r > 1$ then we have a non-zero exponential (or sinusoidal) state-output trajectory in the system, which tends towards infinity as the pass number increases (but may remain stable along any given pass). Conversely, if $\bmod a = r \leq 1$ for all poles (a_1, a), then no trajectory tends to infinity for a given value of p as the pass number increases, but there may be trajectories tending to infinity along the pass. Thus we again run up against the distinction between asymptotic stability and stability along the pass. In order to avoid having trajectories of the form (5.129) which are unstable either along the pass or in the k-direction we also need to avoid poles (a_1, a) with $\bmod a_1 > 1$. In other words, we need that the characteristic variety (5.127) of the zero-input behavior lies in the closed unit polydisc

$$\overline{\mathcal{P}}_1 = \{(a_1, a) \in \mathbb{C}^2 \mid \bmod a_1 \leq 1, \quad \bmod a \leq 1\} \tag{5.134}$$

It can be shown that the characteristic polynomial characterisation of stability along the pass of theorem 4.7 is equivalent to the condition that no poles of the system lie outside $\overline{\mathcal{P}}_1$. Equivalently, with zero input there should be no exponential/ sinusoidal state-output trajectories which tend to infinity either in the pass to pass direction or along the pass.

Valcher has obtained similar results for the more general setting of stability of 2D behaviours over the lattice $\mathbb{Z}^2$ (Valcher, 2000).

Note finally that poles can be decomposed into controllable and uncontrollable, observable and unobservable poles, as described in (Wood et al., 2000). The only one of these sets which can be easily described for repetitive processes is the set of unobservable poles, which give the (2D) frequencies which can occur in the state when both input and output vanish. These are given by the rank-loss points of the matrix

$$\begin{pmatrix} z_1 I_n - A \\ -C \end{pmatrix} \tag{5.135}$$

and so indeed describe the defect of observability.

5.10 Summary

Within this chapter an initial investigation into the area of stability robustness of discrete linear repetitive processes has been undertaken.

Firstly the subject of bounds on the size of parameter variations which are allowable to avoid instability of a stable along the pass process are considered. Two different types of perturbations have been considered:

(i) structured, where the perturbation model structure and bounds on the individual elements of the perturbation matrix are known; and

(ii) unstructured, where at most a spectral norm bound on the perturbation is known.

It has been shown in chapter 4 that stability along the pass can be characterised in terms of a 2D Lyapunov equation, but that the resulting condition is sufficient but not necessary (except in certain special cases - see chapter 4 for the details). Here it has been shown how this potential conservativeness is offset by the availability of robustness measures using the 2D Lyapunov equation as a starting point which are not available from other characterisations of stability along the pass.

Using this approach, lower bounds on the unstructured type of perturbations of case (ii) above have been presented in section 5.4. The two bounds derived, v_b^1 and v_b^2, are lower bounds for v, the exact bound for stable along the pass perturbations. Clearly in respect of a given example, evaluation of v is the preferred option. This however can be computationally intensive, and so it may be acceptable to look towards a suitable alternative. The Lyapunov bounds give sufficient conditions on the minimum norm requirement of the perturbation matrix $\Delta\Phi$ but require a solution to the 2D Lyapunov equation (4.93). Clearly $v_b^1 \leq v$ and $v_b^2 \leq v$, but also a comparison of the equivalent bounds for 2D discrete systems described by the Roesser state-space model in (Lu, 1994b) reveals that

$$v_b^2 \leq v_b^1. \tag{5.136}$$

Other bounds are also possible. Clearly further development is needed here, in particular on the development of alternative approaches and on comparing these bounds in terms of conservativeness and related factors. In particular, it is anticipated that individual bounds on each element of the augmented plant perturbation matrix (5.6) are possible. To distinguish this work from its standard 1D linear systems counterpart emphasis should be placed on the two repetitive interaction terms ΔB_0 and ΔD_1. In addition, the approach has looked at the unstructured class of perturbations only. Class (i) type perturbations provide additional information on the structure of the perturbations, hence it is expected that the resulting bounds will be tighter.

Section 5.5 presents stability robustness analysis using a Fornasini-Marchesini model as a starting point for two classes of structured and unstructured perturbations. Note that to fully exploit these results, the least conservative set for a particular example should obviously be used.

The second type of robustness analysis considered within this thesis is stability margins. Here it is shown how the definitions of stability margins for discrete linear repetitive processes are the natural generalisation of the corresponding terms from 1D linear systems theory. Note that, however, since a fundamental difference between systems in one dimension and those in n, $n \geq 2$, is that the singularities are no longer isolated poles, but are multidimensional manifolds, the stability margin for a repetitive process is given in terms of analytic regions in the (z_1, z) plane.

As a result of this, numerous algorithms are available for evaluating the stability margins based on different starting points (see section 5.6.1 for the details). An alternative approach is based on the premise that it is not always necessary to know the exact value of the stability margin. Instead it suffices to know that it is greater than certain lower limits. Examples of such limits have been presented in section 5.7, which uses the existence of a positive definite solution pair $\{W, Q\}$ to the 2D Lyapunov equation of chapter 4 as a starting point. The analysis here is presented in (Benton et al., 1999). The margins presented depend explicitly on the matrices W and Q of the solution to the 2D Lyapunov equation. Clearly different W and Q give different lower bounds for the margins. It is shown in section 5.8 that the least conservative lower bound corresponds to the minimum norm of the augmented plant matrix Φ.

Finally, section 5.9 has provided an initial discussion on the extension to linear repetitive processes of some very recent results on the definition of the concept of a pole of a multidimensional system using the behavioural approach. A pole has been defined (Wood et al., 2000) as an element of $\mathbb{C}^2$ space which is a zero of the characteristic polynomial $\rho(z_1, z)$. The potential strength of this approach is that the poles can be interpreted in terms of so-called exponential trajectories which, in the case of discrete linear repetitive processes, have a real physical meaning. Clearly this fact has major implications regarding the development of robustness measures for these processes - particularly for stability margins - since a major criticism of approaches used to date has been the lack of any strong 'physical meaning'. For these reasons, this highly promising area is one in which immediate research effort should be directed.

Chapter 6

Controller Structures

6.1 Introduction

The unique control problem associated with linear repetitive processes is the possible presence of oscillations in the output sequence of pass profiles which increase in amplitude from pass to pass (i.e. in the k direction). This behaviour is apparent in the long-wall coal cutting example via the presence of severe undulations in the newly cut floor profile caused by the machines weight as it comes to rest and has been illustrated in figure 2.3. The minimum aim, therefore, of any control scheme for these processes is stabilisation.

As indicated, repetitive processes clearly introduce control problems which are outside the scope of existing 1D linear systems theory, and hence the question as to when and under what conditions does a basic physically realisable stabilising controller exist is complicated by the fact that the process dynamics depend explicitly on two complex variables. Research into controller design for linear repetitive processes is still in its infancy, but the 'obvious' staring point is to look at available structures from conventional 1D linear systems theory and see to what degree they may be applied here.

A general control problem can be formulated with the following aims:

(i) to set (or define) objectives;

(ii) specify control structures (such as feedback control schemes); and

(iii) the development of design algorithms (ideally within a computer aided control system design environment).

Some effort has been directed towards the development of suitable control objectives for differential and discrete linear repetitive processes (Smyth, 1992), where, clearly, an obviously necessary feature of any practically feasible control scheme is stability along the pass.

Additional design considerations can be based on performance specifications regarding, for example, the limit profile for the process. So-called limit profile based strategies have the following type of considerations as elements of the control objective:

(i) specifications for the dynamics of y_∞, i.e. in addition to stability, the limit profile dynamics should satisfy such additional 1D linear systems performance criteria as deemed appropriate (standard linear control measures can be applied);

(ii) requirements on the rate of approach of the output sequence of pass profiles $\{y_k\}_{k\geq 1}$ to the limit profile y_∞, i.e. the output sequence must be within a specified 'band' of the limit profile after a specified number of passes, say k^*, have elapsed and remain within it $\forall\, k > k^*$;

(iii) bounds on the error $y_k - y_\infty$, $k \geq 0$, on a given pass, i.e. the error should be 'acceptable'.

These points have been addressed in (Smyth et al., 1994) using detailed simulation studies where the following general purpose specification for the form of the limit profile has been formulated (in addition to the obvious requirement of stability along the pass):

Drive the output sequence of pass profiles $\{y_k\}_{k\geq 1}$ to a limit profile y_∞ with 'acceptable' along the pass dynamics. 'Practical' convergence should occur in a 'reasonable' number of passes and simultaneously 'tolerable' errors on any pass k should be guaranteed.

The interpretation of the terms in quotation marks clearly should be refined into design criteria appropriate to the particular application under consideration.

Control structures for linear repetitive processes can be classified, in general terms, under the following two headings,

(i) those which explicitly use information from the current pass only, termed memoryless controllers,

(ii) those which explicitly use information from the current pass and/or previous pass profiles, state vectors and input vectors - so-called controllers with memory.

Memoryless schemes clearly have the simpler structure in terms of implementation and of data which must be logged/stored. Hence the potential of such schemes should be fully evaluated prior to the consideration of those with a potentially more complex structure, such as those in class (ii) above or alternatives. Consideration of the specification of control structures for differential and discrete linear repetitive processes is the subject of this chapter.

(Smyth, 1992) makes initial progress in the development of design algorithms for implementation within a computer aided control system design environment. This area is beyond the scope of this thesis, but is the subject of an on-going research program into the development and design of MATLAB toolboxes by Gramacki et al., see for example (Gramacki et al., 1999).

This chapter first introduces so-called memoryless feedback control schemes, which use information from the current pass only. The application of purely 1D control structures fails in general (apart from a few restrictive special cases - see below) since linear repetitive processes introduce control problems which are outside the scope of existing theory. Recent studies have indicated that a state/output feedback (Rogers and Owens, 1993; Smyth, 1992) or feedback and feedforward (Amann et al., 1996) approach may make some progress towards the controller problem, but it will not succeed in all cases. A return-difference theory is then developed which acts as a natural counterpart to the corresponding 1D theory. Using the standard linear systems case as motivation, it is to be expected that much valuable insight into the general area of controller design can be gained by considering subclasses with certain special structural properties. In section 6.4 and onwards, the feedback structures introduced in section 6.2 are applied to certain classes of these so-called benchmark problems. Here it is shown that, for one subclass of practical interest, a 1D control action is all that is required for acceptable systems performance. This theory is novel and provides the basis for the paper (Benton et al., 1998a). A discussion of the effective use of controllers with memory is included in section 6.6, which is then illustrated by looking at a subclass of second order differential processes. In section 6.8 discrete processes are considered. Here it is shown how a discrete linear repetitive process can be regarded as being derived from a differential process under fast sampling conditions, as shown in (Benton et al., 2000b). A control scheme is then designed for a benchmark class of discrete processes - so-called multivariable discrete first order lags. In section 6.9 the 2D Lyapunov equation of chapter 4 is used in the design of a current pass state feedback control law which has been augmented by 'feedforward' previous pass action. This is a type of controller with memory, and hence is an example of the schemes introduced in section 6.6. Finally, the chapter concludes by noting some areas for future work.

6.2 Memoryless Feedback Control Schemes for Linear Repetitive Processes

The starting point in the development of memoryless control schemes for differential linear repetitive processes is to consider feedback type control structures for which a full 1D control theory is readily available and see to what extent they may be applicable here. These schemes are the natural generalisation of a corresponding scheme for the derived conventional linear system $L_D(A, B, C)$ of the process (2.11)-(2.12). In particular, they reduce to this scheme under application of the three actions below:

(i) any previous pass terms are deleted;

(ii) the pass subscript $k+1$ is dropped; and

(iii) the concept of a pass length is ignored.

Consider the subclass of differential linear repetitive processes with state-space model (2.11)-(2.12) (for the discrete case see, for example, (Rogers and Owens, 1992b)). To introduce the first of these schemes, first note that the standard state feedback law for the derived conventional linear system $L_D(A, B, C)$ of the process (i.e. the system obtained from (2.11)-(2.12) by applying points (i) to (iii) above) has the following form over $t \geq 0$,

$$u(t) = F\,x(t) + G\,r(t) \tag{6.1}$$

where F and G are constant $l \times n$ and $l \times m$ matrices respectively and $r(t)$ is the new $m \times 1$ external reference input.

The natural generalisation of this state feedback law to the full differential process (2.11)-(2.12) is then as follows (Rogers and Owens, 1992b; Smyth, 1992) for $k \geq 0$, $0 \leq t \leq \alpha$,

$$u_{k+1}(t) = F\,x_{k+1}(t) + G\,r_{k+1}(t) \tag{6.2}$$

with F and G defined as above, and $r_{k+1}(t)$ denoting the $m \times 1$ external reference vector on pass $k+1$. This scheme is termed 'current point' since it uses information from the current time instant on the current pass only, and is an example of a so-called memoryless controller.

Applying the feedback controller (6.2) to the differential process (2.11) yields the following closed loop system over $0 \leq t \leq \alpha$, $k \geq 0$,

$$\begin{aligned}\dot{x}_{k+1}(t) &= (A + BF)\,x_{k+1}(t) + BG\,r_{k+1}(t) + \sum_{j=1}^{M} B_{j-1}\,y_{k+1-j}(t)\\ y_{k+1}(t) &= C\,x_{k+1}(t) + \sum_{j=1}^{M} D_j\,y_{k+1-j}(t).\end{aligned} \tag{6.3}$$

Clearly this closed loop system has an identical structure to the differential process (2.11) (with the new external reference vector $r_{k+1}(t)$ replacing the control input vector $u_{k+1}(t)$) and hence the process (2.11) is said to be closed under the control action (6.2). Therefore all stability conditions which have been derived for the differential process (and which have been presented in chapter 3) may be applied to the closed loop system (6.3). In particular, the asymptotic stability and stability along the pass results of theorems 3.4 and 3.6 respectively may be used to assess the stability of the closed loop system.

Note that the state feedback law (6.2) does not affect the D_j, $1 \leq j \leq M$, matrices in the closed loop system (6.3) (which is also true for the case of processes with the dynamic boundary conditions of (2.18)), and hence the asymptotic stability of the process is unaffected by the controller. This is a direct result of the fact that the output pass profile $y_{k+1}(t)$, $k \geq 0$, does not explicitly depend on the input vector $u_{k+1}(t)$, $k \geq 0$, on a given pass, i.e. there is no 'direct feedthrough' between input and output. Therefore we have the situation that the property of asymptotic stability is invariant under memoryless state feedback, and hence an asymptotically unstable system cannot be stabilised by a multipass causal feedback control scheme (for a further discussion of this point, see the conclusions section of this chapter). How to overcome this problem remains an open area. For now, we use the argument that, in practical terms, asymptotic stability is always present due to the stabilising influence of resetting the initial conditions at the beginning of each pass. In addition, it should also be noted that the observation of industrially oriented cases leads to the conclusion that the de-stabilising influences (in these cases) arise due to the along the pass dynamics only - see (Smyth, 1992) for the details of this point. For these reasons, for the remainder of this chapter we will assume asymptotic stability holds.

The state feedback control law (6.2) requires the availability of all elements of the state vector $x_{k+1}(t)$, which may not always be possible, due to, for example, physical/financial constraints. In such cases, by analogy with the standard 1D approach, state estimators/observers may be employed. With this in mind, an alternative to the state-activated feedback control scheme presented above is to consider classes of output feedback control schemes (see, for example, (Rogers and Owens, 1993) and (Rogers and Owens, 1995a) for the discrete/differential cases respectively).

Consider the output sequence of pass profiles $\{y_k(t)\}_{k\geq 1}$ from the differential non-unit memory linear repetitive process with state-space form (2.11)-(2.12). Then at time t on pass k the information in the following set is causal (as already illustrated in figure 2.5) and can be used for output feedback control,

$$\begin{aligned} Y &= Y_1 \cup Y_2 \\ Y_1 &= \{y_k(\tau) : \ 0 \leq \tau \leq t\} \\ Y_2 &= \{y_r(\tau) : \ 0 \leq \tau \leq \alpha, \ 0 \leq r \leq k-1\}. \end{aligned} \tag{6.4}$$

From an implementation viewpoint, control schemes which use information from the current time instant t on pass k clearly have the simplest structure in terms of information which must be stored/logged, and hence it is logical to see what progress can be achieved using these so-called current point controllers prior to the consideration of those with a potentially more complex structure.

Suppose, therefore, that $r_{k+1}(t)$ is an external $m \times 1$ vector representing the desired behaviour of the process on pass $k+1$, and define the so-called current pass error vector over $0 \le t \le \alpha$ as

$$e_{k+1}(t) = r_{k+1}(t) - y_{k+1}(t), \quad k \ge 0. \tag{6.5}$$

Then a memoryless dynamic unity negative feedback controller for (2.11)-(2.12) constructs the input $u_{k+1}(t),\ \ k \ge 0$, as the output from the state-space system

$$\begin{aligned} \dot{x}^C_{k+1}(t) &= A^C\, x^C_{k+1}(t) + B^C\, e_{k+1}(t) \\ u_{k+1}(t) &= C^C\, x^C_{k+1}(t) + D^C\, e_{k+1}(t), \quad 0 \le t \le \alpha, \ k \ge 0, \end{aligned} \tag{6.6}$$

where $x^C_{k+1}(t)$ is the $n_1 \times 1$ internal state of the controller on pass $k+1$. The resulting control scheme describes a memoryless dynamic unity negative error actuated output feedback control scheme for (2.11)-(2.12) and, in effect, (6.6) describes a standard 1D forward path controller applied on pass $k+1$. Specific choices of the matrices in (6.6) can now be made to yield a number of special cases of control laws which are the natural generalisation of their extensively used conventional linear systems counterpart.

At this stage, introduce the so-called augmented state vector

$$X^A_{k+1}(t) = [x_{k+1}(t)^T,\ x^C_{k+1}(t)^T]^T \in \mathbb{R}^N, \quad N = n + n_1. \tag{6.7}$$

It then follows that the state-space models describing the forward path and closed loop systems of (2.11) under (6.5)-(6.6) are given over $0 \le t \le \alpha,\ k \ge 0$, by

$$\begin{aligned} \dot{X}^A_{k+1}(t) &= \hat{A}\, X^A_{k+1}(t) + \hat{B}\, e_{k+1}(t) + \sum_{j=1}^{M} \hat{B}_{j-1}\, y_{k+1-j}(t) \\ y_{k+1}(t) &= \hat{C}\, X^A_{k+1}(t) + \sum_{j=1}^{M} \hat{D}_j\, y_{k+1-j}(t) \end{aligned} \tag{6.8}$$

and

$$\begin{aligned} \dot{X}^A_{k+1}(t) &= (\hat{A} - \hat{B}\hat{C})\, X^A_{k+1}(t) + \hat{B}\, r_{k+1}(t) + \sum_{j=1}^{M} (\hat{B}_{j-1} - \hat{B}\hat{D}_j)\, y_{k+1-j}(t) \\ y_{k+1}(t) &= \hat{C}\, X^A_{k+1}(t) + \sum_{j=1}^{M} \hat{D}_j\, y_{k+1-j}(t) \end{aligned} \tag{6.9}$$

respectively, where

$$\hat{A} = \begin{bmatrix} A & BC^C \\ 0 & A^C \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} BD^C \\ B^C \end{bmatrix}, \quad \hat{B}_{j-1} = \begin{bmatrix} B_{j-1} \\ 0 \end{bmatrix}, \quad 1 \le j \le M,$$
$$\hat{C} = \begin{bmatrix} C & 0 \end{bmatrix}, \quad \hat{D}_j = D_j, \ 1 \le j \le M. \tag{6.10}$$

Both (6.8) and (6.9) are closed in that they have an identical structure to the open loop model (2.11). Hence known stability theory again may be applied. In particular, note that the matrices D_j, $1 \le j \le M$, in (6.9) are once again invariant under this scheme, and hence the conclusions drawn earlier on asymptotic stability also apply here for this class of output feedback control structures. Note that, here, the process is closed under a (memoryless) cascade connection. It can also be seen that closure also holds under a parallel (feedforward) connection and with non-unity negative feedback loops with memoryless controllers in the feedback paths.

In order to proceed with a return-difference type analysis, it is first necessary to describe memoryless dynamic unity negative feedback control in 2D transfer-function terms. First recall from section 2.7 that the 2D transfer-function matrix description of (2.11) is

$$Y(s,z) = G(s,z)\,U(s,z) \tag{6.11}$$

where the $m \times l$ 2D transfer-function matrix $G(s,z)$ is given by

$$G(s,z) = \left(I_m - \sum_{j=1}^{M} G_j(s) z^{-j} \right)^{-1} G_0(s) \tag{6.12}$$

with

$$G_0(s) = C(sI_n - A)^{-1}B \tag{6.13}$$

and

$$G_j(s) = C(sI_n - A)^{-1}B_{j-1} + D_j, \quad 1 \le j \le M. \tag{6.14}$$

Then note that the 2D transform versions of (6.5) and (6.6) are

$$e(s,z) = R(s,z) - Y(s,z) \tag{6.15}$$

and

$$U(s,z) = K(s,z)\,e(s,z) \tag{6.16}$$

respectively, where

$$K(s,z) \equiv K(s) = C^C(sI_{n_1} - A^C)^{-1}B^C + D^C. \tag{6.17}$$

Further, applying these transforms to (6.8), after some manipulation yields

$$\begin{aligned} Y(s,z) &:= Q(s,z)\, e(s,z) \\ &= G(s,z)K(s,z)\, e(s,z) \end{aligned} \tag{6.18}$$

i.e. the 2D transfer-function matrix $Q(s,z)$ of the forward path system is just the product of that for the plant and the forward path controller.

Substituting in for $e(s,z)$ gives

$$Y(s,z) = H(s,z)\, R(s,z) \tag{6.19}$$

where the $m \times m$ 2D closed loop transfer-function matrix $H(s,z)$ is given by

$$H(s,z) = (I_m + Q(s,z))^{-1}Q(s,z). \tag{6.20}$$

The block diagram interpretation of (6.19) is given in figure 6.1, where it can be seen that this scheme is clearly the natural generalisation of its conventional linear systems counterpart.

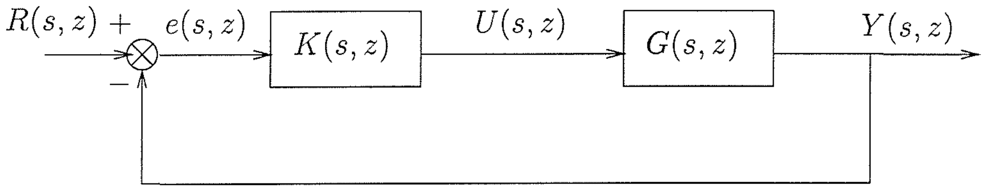

Figure 6.1: Forward Path Memoryless Controller

Now that the memoryless dynamic unity negative feedback control scheme has been expressed in 2D transfer-function matrix terms, we can proceed with a return-difference type analysis.

6.3 Return-Difference Theory

In 1D linear systems theory, the return-difference operator generates the difference between the injected and returned signals. Within this section, the concept of a return-difference theory for linear repetitive processes is considered.

Consider the 1D derived conventional linear system $L_D(A, B, C)$ of the differential linear repetitive process (2.11)-(2.12) with transfer-function matrix $G_0(s)$ and subject to dynamic unity negative feedback control with forward path controller transfer-function matrix $K(s)$. Further, let $\rho_o(s)$ and $\rho_c(s)$ denote the characteristic polynomials of the open loop forward path system and the closed loop system respectively, and denote the return-difference matrix by $T(s)$. Then we have

$$T(s) = I_m + G_0(s)K(s). \tag{6.21}$$

The standard 1D linear systems result, as shown in any relevant text, is

$$|T(s)| = \frac{\rho_c(s)}{\rho_o(s)}. \tag{6.22}$$

This relationship acts as a basis for a large number of design techniques currently available in conventional linear systems theory.

For the case of the differential linear repetitive process with state-space model (2.11)-(2.12) under memoryless dynamic unity negative feedback control, the natural definition of a return-difference matrix is

$$T(s, z) = I_m + G(s, z)K(s, z). \tag{6.23}$$

To link this matrix to closed loop stability along the pass, it is necessary to first introduce the concept of a characteristic polynomial for the process. As in the case of its 1D linear systems counterpart, this should contain all of the information necessary to determine the stability nature of the process. Consequently, an obvious candidate for this open loop is

$$\rho_o(s, z) = P_a(z)A_P(s, z) \tag{6.24}$$

where, from definitions 3.5 and 3.6, $P_a(z)$ and $A_P(s, z)$ are the asymptotic stability and stability along the pass polynomials respectively.

Further, by Schurs formula,

$$\rho_o(s, z) = \begin{vmatrix} sI_n - A & -B(z) \\ -C & Q(z) \end{vmatrix} \tag{6.25}$$

with $Q(z)$ as in (3.34) and

$$B(z) = \sum_{j=1}^{M} B_{j-1}\, z^{-j}. \tag{6.26}$$

Then the following result characterises stability along the pass in terms of its characteristic polynomial,

Theorem 6.1 (Characteristic Polynomial - Stability along the Pass) *(Rogers and Owens, 1992b) Suppose that the assumptions of theorem 3.5 hold. Then the extended linear repetitive process $S(E_\alpha, W_\alpha, L_\alpha)_{\alpha \geq \alpha_0}$ generated by (2.11)-(2.12) with $\alpha \geq \alpha_0$ is stable along the pass if, and only if,*

$$\rho_o(s,z) \neq 0 \text{ in the region } \{s : Re\{s\} \geq 0\} \cup \{z : |z| \geq 1\} \tag{6.27}$$

where $\rho_o(s,z)$ denotes the characteristic polynomial of the process, defined by (6.25).

Under the assumption that (2.11) (and hence (6.9)) is asymptotically stable, the following result expresses stability along the pass under memoryless dynamic feedback control in terms of the matrices $T(s)$ and $T(s,z)$, where $T(s)$ and $T(s,z)$ are the return-difference matrices of the derived conventional linear system $L_D(A, B, C)$ and the full process respectively,

Theorem 6.2 (Return-Difference Matrix - Stability along the Pass) *(Rogers and Owens, 1995a) Suppose that the differential linear repetitive process (2.11)-(2.12) is asymptotically stable and subject to the memoryless dynamic unity negative feedback control scheme described by (6.5)-(6.6). Then the extended linear repetitive process $S(E_\alpha, W_\alpha, L_\alpha)_{\alpha \geq \alpha_0}$ generated by the closed loop state-space model (6.9) with $\alpha \geq \alpha_0$ is stable along the pass if, and only if,*

(a) $|T(s)| \neq 0, \quad Re\{s\} \geq 0,$ *and*

(b) $|T(s,z)| \neq 0, \quad Re\{s\} \geq 0, \quad |z| \geq 1,$

where the return-difference matrices $T(s)$ and $T(s,z)$ are defined by (6.21) and (6.23) respectively.

Note that the version of this result for the discrete subclass of processes can be found in (Rogers and Owens, 1993).

Return now to $T(s,z)$ of (6.23) and let $\rho_c(s,z)$ denote the closed loop polynomial, i.e.

$$\rho_c(s,z) = \begin{vmatrix} sI_n - A + BC & -(B(z) - BD(z)) \\ -C & Q(z) \end{vmatrix}, \tag{6.28}$$

with $B(z)$ and $Q(z)$ as for (6.25) and

$$D(z) = \sum_{j=1}^{M} D_j \, z^{-j}. \tag{6.29}$$

Then it can easily be shown that

$$\frac{\rho_c(s,z)}{\rho_o(s,z)} = |T(s,z)|. \tag{6.30}$$

Given that $T(s,z)$ is the natural generalisation of the return-difference matrix $T(s)$ for conventional linear systems, it can be conjectured that $T(s,z)$ should play a similar role in the design of control schemes for, say, closed loop stability along the pass. This subject remains an open area for future research.

6.4 Application to Benchmark Problems I. Multivariable First Order Lags

Within section 6.2 candidate memoryless feedback controller schemes for differential linear repetitive processes have been introduced. To illustrate the potential of this general approach, these structures are applied here to subclasses of processes possessing certain special properties - so-called benchmark problems - which provides a starting point for the analysis of more complex cases. The work in this and the subsequent section forms the basis for the paper (Benton et al., 1998a) and is novel.

Consider the subclass of differential unit memory processes where the state-space triple (A, B, C) in (2.13) takes the structure of a multivariable first order lag (Owens, 1975), i.e. $m = l = n$ and the first Markov parameter is nonsingular. First order lags are the multivariable equivalent of the scalar first order lag $\frac{k_0}{Ts+1}$ and a full control theory exists for them. Relevant results are presented within the appendix section A.6.

Hence, given (2.13)-(2.14) with $m = l = n$, $|CB| \neq 0$ and $D_1 \equiv 0$, a simple (current pass) state transformation yields the equivalent description over $k \geq 0$, $0 \leq t \leq \alpha$,

$$\dot{y}_{k+1}(t) = -A_0^{-1} A_1 \, y_{k+1}(t) + A_0^{-1} \, u_{k+1}(t) + B_0 \, y_k(t) \tag{6.31}$$

where A_0, A_1 and B_0 are real constant $m \times m$ matrices with $|A_0| \neq 0$.

As a first attempt at controller design, consider the proportional forward path controller

$$u_{k+1}(t) = K\, e_{k+1}(t) = (\rho A_0 - A_1)\, e_{k+1}(t) \tag{6.32}$$

where $\rho > 0$ is a real scalar gain and the $m \times 1$ current pass error vector $e_{k+1}(t)$ on pass $k+1$ is as defined in (6.5). This control scheme is an example of the error actuated feedback laws introduced in section 6.2, and can be obtained from (6.6) by setting $A^C \equiv B^C \equiv C^C \equiv 0,\ D^C \equiv K = (\rho A_0 - A_1)$.

Application of this control action to (6.31) yields the following closed loop system,

$$\dot{y}_{k+1}(t) = -\rho I_m\, y_{k+1}(t) + (\rho I_m - A_0^{-1} A_1)\, r_{k+1}(t) + B_0\, y_k(t). \tag{6.33}$$

Since both (6.31) and (6.33) are subclasses of differential linear repetitive processes, i.e. (6.31) is closed under (6.32), the results of theorems 3.4 and 3.6 may be applied to assess stability. Now, as the 'D_1' matrix in (6.33) is identically zero, the closed loop process is automatically asymptotically stable. In terms of theorem 3.6 for stability along the pass, condition (b) clearly holds $\forall \rho > 0$. Now consider the closed loop interpass transfer-function matrix of (6.33) which can easily be seen to have the form

$$G_1(s) = \frac{1}{s+\rho} B_0. \tag{6.34}$$

Condition (c) of theorem 3.6 then translates to the requirement that all eigenvalues of $G_1(s)$ have modulus strictly less than unity $\forall s = i\omega,\ \omega \geq 0$. Suppose now that the eigenvalues of $G_1(s)$ are denoted by $\lambda_j(s),\ 1 \leq j \leq m$. Then it follows that

$$\lim_{\rho \to +\infty} \sup_{\omega \geq 0} |\lambda_j(i\omega)| = 0, \tag{6.35}$$

and hence the following result is obtained,

Theorem 6.3 *(Benton et al., 1998a) Suppose that the differential linear repetitive process (6.31) is subject to memoryless proportional unity negative feedback control defined by (6.5) and (6.32). Then the resulting closed loop system (6.33) is stable along the pass for all*

$$\rho > r(B_0). \tag{6.36}$$

This result shows that a differential linear repetitive process (2.13)-(2.14) with state-space triple (A, B, C) having the structure of a multivariable first order lag can be stabilised by a 1D control action provided a high enough gain is applied.

Under the control action of theorem 6.3, the closed loop limit profile of the process is described by the 1D state-space model

$$\dot{y}_\infty(t) = (-\rho I_m + B_0)\, y_\infty(t) + (\rho I_m - A_0^{-1}A_1)\, r_\infty(t), \quad 0 \le t \le \alpha. \tag{6.37}$$

Since $\rho > r(B_0)$, this closed loop limit profile is stable in the standard 1D sense.

As $\rho \longrightarrow +\infty$, the limit profile dynamics approach those of the system

$$\dot{y}_\infty(t) = -\rho\, y_\infty(t) + \rho\, r_\infty(t), \tag{6.38}$$

which is a stable, totally non-interacting 1D linear system with zero steady state error in response to a unit step applied at $t = 0$ any channel. In particular, 'high gain' produces closed loop stability and low static and dynamic interaction between loops.

The question remaining is how to find an admissible finite gain ρ, and compute information on the rate of approach of the output sequence of pass profiles $\{y_k\}_{k\ge1}$ to the limit profile y_∞ in terms of bounds on the error $e_k = y_k - y_\infty$.

In what follows it is shown that these questions can be answered by replacing the necessary and sufficient condition on gain for stability along the pass of theorem 6.3 by a sufficient but not necessary alternative. This analysis uses some basic results from the theory of nonnegative matrices which are summarised in the appendix section A.1.

Suppose now that the eigenvalues of the 'A' matrix have strictly negative real parts (a necessary condition for stability along the pass) - in other words that the derived conventional linear system is stable in the standard 1D sense. Suppose also that the simulation-based route of section 3.5 is adopted and that $W(t)$ denotes the step response matrix of $G_1(s)$ and note that, given (6.34), each element in this matrix is monotonic and sign-definite. Therefore, the maximal value of $W(t)$ occurs at $t = +\infty$, and hence by the final value theorem we can write,

$$\|W\|_p := \|W(+\infty)\|_p = \|G_1(0)\|_p = \frac{1}{\rho}\|B_0\|_p. \tag{6.39}$$

It follows then that

$$\sup_{\omega\ge0} r(G_1(i\omega)) \le \sup_{\omega\ge0} r(\|G_1(i\omega)\|_p) \le r(\|W\|_p), \tag{6.40}$$

and hence we obtain the following sufficient condition for closed loop stability along the pass,

Theorem 6.4 *(Benton et al., 1998a) Suppose that the differential linear repetitive process (6.31) is subject to memoryless proportional unity negative feedback control defined by (6.5) and (6.32). Then the resulting closed loop system is stable along the pass if*

$$\rho \geq r(||B_0||_p). \tag{6.41}$$

Turning now to the estimate of convergence rates, the following result is proved in (Smyth, 1992) where, without loss of any generality, the initial pass profile has been set equal to zero,

Theorem 6.5 (Performance Bounds) *(Smyth, 1992) Suppose that the underlying function spaces are L_∞ spaces. Suppose also that the condition of theorem 6.4 holds for a given value of ρ and that the reference signal is pass independent, i.e. $r_k \equiv r_\infty$ for all values of k. Then*

$$||y_k - y_\infty||_p \leq m_k := ||M_k||_p ||r_\infty||_p \tag{6.42}$$

where

$$||M_k||_p = (I_m - ||W||_p)^{-1} ||W||_p^k \tag{6.43}$$

and $||r_\infty||_p$ has i^{th} element $\sup_{t \geq 0} |r_\infty^i(t)|, \ 1 \leq i \leq m.$

The performance information made available from this result indicates that the output sequence of pass profiles approaches the limit profile at a geometric rate determined by a computable scalar $\gamma \in (r(||W||_p), 1)$. Also the i^{th} element of the output vector on a given pass k, denoted y_k^i, lies (point-wise) in the range defined by

$$y_\infty^i(t) - m_k^i \leq y_k^i \leq y_\infty^i(t) + m_k^i \quad \forall \ t \geq 0 \tag{6.44}$$

where m_k^i can easily be calculated from theorem 6.5.

This bound for y_k^i has a simple graphical interpretation, as shown in figure 6.2, where it can be seen that y_k^i lies in a band of width $2m_k^i$ which approaches zero geometrically as $\rho \to +\infty$.

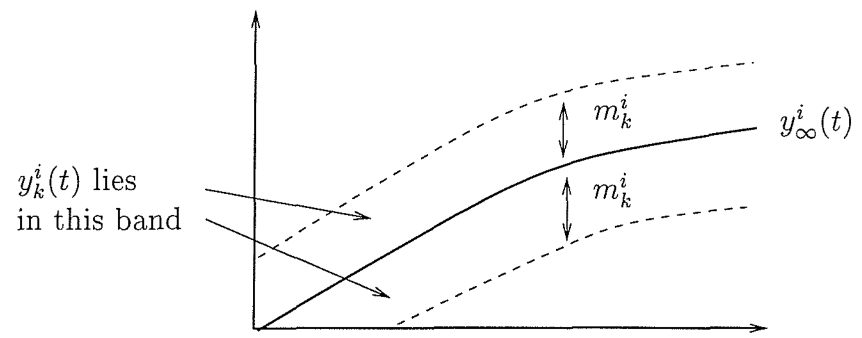

Figure 6.2: Bounds on rates of approach of the i^{th} element of the output sequence of pass profiles to the limit profile

The overall conclusion of this analysis is that under 'high gain' (i.e. $\rho \to +\infty$) the limit profile of the closed loop system can be reached to within arbitrary accuracy on the first pass. In applications terms, this level of performance will not be achievable except when the value of the scalar gain ρ actually employed is physically implementable. This situation where the required gain ρ is outside the available range can arise in several ways, for example the ideal choice of parameter may be physically unavailable or unrealistic due to eg. an inaccurate plant model, financial restrictions or structural/data uncertainties. Two alternative approaches to this analysis are introduced in the following section.

6.5 Extensions

The paper (Benton et al., 1998a) presents two refinements to the analysis presented here. If the value of the scalar gain ρ that can actually be implemented is outside the range required to give the desired level of performance then, by analogy with standard 1D linear systems theory, an alternative is to include dynamics within the forward path controller. This section begins by generalising the results of the previous section in this respect. Also the possibility of further generalisation to the case where the state-space triple (A, B, C) only approximates the structure of a multivariable first order lag in a well defined sense is considered. In this situation, it is shown how a reduced order model of the dynamics may be used in the design of a controller for the process in many cases of practical interest.

6.5.1 A More General Parametric Controller

In practice, the ideal choice of scalar gain ρ may be physically unavailable or unrealistic, in which case a more general parametric form of the controller (6.32) is to replace the real constant matrix K by

$$K(s) = A_0 \operatorname{diag}\{\rho_j(s)\}_{1\leq j\leq m} - A_1 \tag{6.45}$$

where the $\rho_j(s)$, $1 \leq j \leq m$, are proper minimum phase transfer-functions. This forward path controller can be realised by the state-space model of (6.6) and it can easily be seen that the application of this controller yields the closed loop system

$$\begin{aligned} Y(s,z) &= \operatorname{diag}\left\{\frac{1}{s+\rho_j(s)}\right\}_{1\leq j\leq m} (\operatorname{diag}\{\rho_j(s)\}_{1\leq j\leq m} - A_0^{-1}A_1)\, R(s,z) \\ &\quad + z^{-1} \operatorname{diag}\left\{\frac{1}{s+\rho_j(s)}\right\}_{1\leq j\leq m} B_0\, Y(s,z). \end{aligned} \tag{6.46}$$

It can then be easily seen that condition (b) of the stability along the pass theorem 3.6 is governed by the zeros of the so-called scalar return differences

$$r_j(s) = 1 + s^{-1}\rho_j(s), \quad 1 \leq j \leq m. \tag{6.47}$$

Also, the closed loop interpass transfer-function matrix takes the form

$$G_1(s) = \operatorname{diag}\{(s+\rho_j(s))^{-1}\}_{1\leq j\leq m}\, B_0. \tag{6.48}$$

Hence, if the $\rho_j(s)$, $1 \leq j \leq m$, have been chosen so as condition (b) of theorem 3.6 holds, then the closed loop system is stable along the pass if, and only if, all eigenvalues of $G_1(s)$ have modulus strictly less than unity $\forall s = i\omega$, $\omega \geq 0$. This condition can be tested via standard 1D linear systems techniques.

To consider the use of theorems 6.4 and 6.5 in this case, first note that the entries in the step response matrix $W(t)$ are no longer guaranteed to be monotonic and sign definite. Hence the closed form expression for $||W||_p$ is no longer available. Instead the elements of this matrix must be computed numerically using the techniques and software detailed in (Smyth, 1992). Here it suffices to note that the analysis of the previous section generalises in a natural manner and that the associated computations are numerically reliable and efficient.

6.5.2 Approximation Method

The second refinement to the work presented here considers the case where the state-space triple (A, B, C) does not exactly fit the multivariable first order lag model (6.31). In conventional linear systems controller design applications, low order models play an important role due to the presence of approximate pole-zero cancellation in the system transfer-function matrix. In such situations, controller design can be based on a simplified model of the complex plant dynamics, and this reduced order model can provide insight into the system structure. Bearing this in mind, although at a first glance the structure of the multivariable first order systems introduced in the previous section appears restrictive, it is natural to consider the use of reduced order models for which a known analytic design method exists.

Within this section a subclass of differential processes is introduced where the state-space triple (A, B, C) does not exactly fit the multivariable first order lag model (A.57). It is shown how, in such situations, controller design may be based on a simplified model of the plant dynamics (such as, in this case, a multivariable first order model) to achieve acceptable systems performance, provided a contraction mapping condition is satisfied. Such an approach exploits the simple structure of the low order model controllers (with known analytic design techniques) whilst being applicable to systems of a more complex nature. The approximate model can be of arbitrary dynamic complexity - the first order model presented here is the simplest possible (and, as shown in (Edwards and Owens, 1977), is adequate for controller design in many cases of practical interest), but more complex models can be obtained from identification experiments. The first order model can be estimated from the plant model or from experimental transient tests. (Note, however, that the model contains no information on the plant zero structure.)

Following the analysis in (Owens, 1978) for multivariable systems, consider the case where the state-space triple (A, B, C) in (2.13)-(2.14) has the $m \times m$ minimum phase transfer-function matrix $G(s)$ of the form

$$G^{-1}(s) = sA_0 + A_1 + H_a(s), \quad |A_0| \neq 0, \tag{6.49}$$

where $H_a(s)$ is stable and strictly proper. It then follows that $H_a(0)$ is finite, and replacing A_1 by $A_1 + H_a(0)$, $H_a(s)$ by $H_a(s) - H_a(0)$ and defining $A_0 H(s) = H_a(s) - H_a(0)$ yields

$$G^{-1}(s) = sA_0 + A_1 + A_0 H(s), \quad |A_0| \neq 0, \quad H(0) = 0. \tag{6.50}$$

Using theorem A.4 and the identity $G^{-1}(s)G(s) = I_m$, it is easily verified that $A_0CB = I_m$ and hence that $|CB| \neq 0$, $A_0 = (CB)^{-1}$ and clearly $A_1 = G^{-1}(s)|_{s=0}$.

It is intuitively reasonable that if $H(s)$ is 'small' in some well defined sense then $G(s)$ can be approximated by the first order model obtained by neglecting $H(s)$ in equation (6.50), i.e.

$$G_A^{-1}(s) = sA_0 + A_1, \tag{6.51}$$

where the results of section 6.4 can then be used to construct a control scheme $K(s)$ for the system. The precise mathematical justification of these ideas uses the theory of functional analysis in the form of Banach spaces of analytic functions and the contraction mapping theorem (see (Freeman, 1973) for the details).

Now (as in (Owens, 1978)) define the norm, $||H||$, of $H(s)$

$$||H|| := \max_{s\in D} \max_{1\leq i\leq m} \sum_{j=1}^{m} |H_{ij}(s)| \tag{6.52}$$

where D is the contour in the complex plane consisting of the imaginary axis $s = i\omega$, $|\omega| \leq R$, and the large semicircle $|s| = R$ in the right half complex plane.

Application of the procedure outlined in section 6.4 to design a memoryless unity negative controller for the process then proceeds as follows. Select the forward path controller as

$$K(s) = \rho A_0 - A_1 \tag{6.53}$$

where ρ is a positive real scalar. Then by applying this control action to the approximate multivariable first order lag process

$$G^{-1}(s,z) = A_0 s + A_1 + A_0 H(s) - z^{-1}A_0 B_0, \quad H(0) = 0, \tag{6.54}$$

after some manipulation, gives a closed loop system of the form

$$Y(s,z) = G_0(s)\, R(s,z) + z^{-1}G_1(s)\, Y(s,z) \tag{6.55}$$

with the transfer-function matrices $G_0(s)$ and $G_1(s)$ given by

$$\begin{aligned} G_0(s) &= (I_m + \frac{1}{s+\rho}H(s))^{-1}\frac{\rho}{s+\rho}(I_m - \frac{A_0^{-1}A_1}{\rho}), \quad \text{and} \\ G_1(s) &= \frac{1}{s+\rho}(I_m + \frac{1}{s+\rho}H(s))^{-1}B_0. \end{aligned} \tag{6.56}$$

Here $G_0(s)$ represents the derived conventional linear system of the approximate multivariable first order lag repetitive process under the forward path proportional controller of (6.5), (6.32) and (6.53) and $G_1(s)$ is the closed loop interpass transfer-function matrix.

Considering each of the conditions of theorem 3.6 in turn, the stability along the pass of the closed loop system can now be assessed. Firstly, the closed loop system is asymptotically stable since the 'D_1' matrix of condition (a) is identically zero.

For condition (b), we require the derived conventional linear system to be stable in the standard 1D sense. In order to assess this, consider the 1D linear system with open loop transfer-function matrix

$$Y(s) = G(s)\,U(s) \tag{6.57}$$

and closed loop transfer-function matrix

$$Y(s) = G_0(s)\,R(s) \tag{6.58}$$

with $G_0(s)$ as in (6.56).

The stability of this 1D linear system can be assessed using the approach given in Edwards and Owens (Edwards and Owens, 1977). The method utilises the techniques of Freeman (Freeman, 1973) in the form used by Owens (Owens, 1974).

Then, by defining

$$\begin{aligned} Q(s) &:= G(s)K(s), \quad \text{and} \\ Q_A(s) &:= G_A(s)K(s) \end{aligned} \tag{6.59}$$

with $G(s)$ and $G_A(s)$ as in (6.50) and (6.51), after some manipulation we can write

$$\begin{aligned} Y(s) &= G_0(s)\,R(s) \\ &= -Q^{-1}(s)\,Y(s) + R(s) \\ &= (I_m + Q_A^{-1}(s))^{-1}[\{Q_A^{-1}(s) - Q^{-1}(s)\}Y(s) + R(s)]. \end{aligned} \tag{6.60}$$

Let D be the usual Nyquist contour as defined earlier in this section, and consider $R \longrightarrow +\infty$. Assume that $Q^{-1}(s)$ and $Q_A^{-1}(s)$ are bounded on D and analytic in it's interior and that $(I_m + Q_A^{-1}(s))^{-1} = (I_m + Q_A(s))^{-1}Q_A(s)$ is stable. Then a sufficient condition for closed loop stability can be obtained by application of the contraction mapping theorem (Owens, 1974). That is to say, we require

$$\max_{1\leq i\leq m} \sup_{s\in D} \sum_{j=1}^{m} |(I_m + Q_A^{-1}(s))^{-1}(Q_A^{-1}(s) - Q^{-1}(s))_{ij}| < 1. \tag{6.61}$$

Noting that

$$\begin{aligned}(I_m + Q_A^{-1}(s))^{-1}(Q_A^{-1}(s) - Q^{-1}(s)) \\ &= \{K(s) + G_A^{-1}(s)\}^{-1}K(s)K^{-1}(s)\{G_A^{-1}(s) - G^{-1}(s)\} \\ &= \frac{(-1)}{s+\rho}H(s),\end{aligned} \tag{6.62}$$

expression (6.61) becomes

$$\max_{1\leq i\leq m} \sup_{s\in D} \sum_{j=1}^{m} \left|\frac{H_{ij}(s)}{s+\rho}\right| < 1. \tag{6.63}$$

This result states that if the minimum phase multivariable system $G(s)$ of (6.50) is approximated by the minimum phase reduced model $G_A(s)$ of (6.51) and a forward path controller $K(s)$ is designed to ensure that the reduced order closed loop system $\{I_m + Q_A(s)\}^{-1}\{Q_A^{-1}(s)\} = \{I_m + Q_A^{-1}(s)\}^{-1}$ is stable in the standard 1D sense, then application of $K(s)$ to $G(s)$ yields a 1D stable closed loop system provided that expression (6.63) is satisfied.

It is easily verified that it is always possible to choose $\rho > 0$ to satisfy this, and hence guarantee the stability of the derived conventional linear system, by choosing

$$\rho > \max_{1\leq i\leq m} \sup_{s\in D} \sum_{j=1}^{m} |H_{ij}(s)| \tag{6.64}$$

i.e.

$$\rho > \|H\|. \tag{6.65}$$

Prior to a discussion of this result, a few points should be made regarding the evaluation of $\|H\|$. To calculate $\|H\|$ we need to search the right half s-plane to see where it attains it's maximum. Since $H(s)$ is stable and strictly proper, its derivative, $H'(s)$, exists and is strictly proper. Therefore $H(s)$ is analytic and non-constant and hence the maximum modulus theorem may be applied, which states that $|H(s)|$ achieves its maximum on the boundary of the contour D. Any point on the semicircular section of this contour may be written in the polar form $s = Re^{i\theta}$, with $-\pi/2 < \theta \le \pi/2$. So, since $H(s)$ is strictly proper, $|H(Re^{i\theta})| \longrightarrow 0$ as $R \longrightarrow +\infty$. Hence the maximum value of $|H(s)|$ is achieved on the imaginary axis. So, $\|H(s)\| = \max_{1\le i\le m} \sup_{s=i\omega} \sum_{j=1}^{m} |H_{ij}(s)|$, which can be evaluated by searching the imaginary axis.

Now consider again the high gain condition (6.65). In particular, note that if $H(s)$ is 'small' in the sense that the right hand side of (6.65) is small then $G_A(s)$, the transfer-function matrix of the derived conventional linear system obtained by deleting the term $H(s)$ in (6.50), will be a good approximation to $G(s)$ in both the closed and open loop system. In more general situations, however, $H(s)$ may be significant and higher gains are required to ensure stability. Note that this technique will not cope with any general system, since the transfer-function matrix $G^{-1}(s)$ of the derived conventional linear system must be of the form (6.50), and it may be that the control gains required to satisfy (6.65) are too high for practical application.

Now return to condition (c) of the stability along the pass theorem 3.6. If the scalar gain ρ has been chosen so that condition (b) of this result holds, i.e. if $\rho > \|H\|$, then the closed loop system is stable along the pass if, and only if, the eigenvalues of $G_1(s)$ of (6.56) have modulus strictly less than unity $\forall\, s = i\omega$, $\omega \ge 0$. This condition can be tested via standard linear system techniques.

The closed loop limit profile of the process can be represented by

$$(sI_m + \rho(I_m - \frac{1}{\rho}(B_0 - H(s)))Y_\infty(s) = \rho(I_m - \frac{A_0^{-1}A_1}{\rho})R_\infty(s). \tag{6.66}$$

Notice for high gain (i.e. as $\rho \longrightarrow +\infty$) this is equivalent to

$$Y_\infty(s) \simeq \frac{\rho}{s+\rho} R_\infty(s) \tag{6.67}$$

which is the same limit profile for the exact multivariable first order lag repetitive process case (as outlined in (Rogers and Owens, 1992b) and as can be seen from taking Laplace transforms of (6.38)). Hence, in the limit, the output of the system under the controller based on the reduced order model approaches that of the exact multivariable first order lag process.

6.6 Effective use of Memory Terms

The control structures of the previous sections all possess the so-called memoryless property in that information is used from the current pass only. These memoryless schemes have a simple structure and relatively low demands on information which must be logged/stored, and hence the majority of research to date has been focussed on this class of control scheme. Problems arise, however, when one or more of the control objectives cannot be met by a current pass controller. Then one way forward is to introduce controllers with memory, i.e. those which explicitly use information from the current pass and/or previous pass profiles, state vectors and input vectors. Such controllers utilise data from the Y_2 set in the definition of causal information of (6.4).

(Rogers and Owens, 1992b) analyses so-called proportional repetitive minor loop compensation schemes which constitute a subclass of all possible control schemes with memory. Here, with respect to the differential process (2.11)-(2.12), a memoryless linear state feedback law with proportional repetitive minor loop compensation has the structure

$$u_{k+1}(t) = F\,x_{k+1}(t) + G\,r_{k+1}(t) - \sum_{j=1}^{M} K_j\,y_{k+1-j}(t), \;\; 0 \leq t \leq \alpha, \;\; k \geq 0, \quad (6.68)$$

where F, G and K_j, $1 \leq j \leq M$, are $l \times n$, $l \times m$ and $l \times m$ matrices respectively, and $r_{k+1}(t)$ is the new $m \times 1$ external reference vector on pass $k+1$. Figure 6.3 shows a schematic diagram of this control action. Note that this reduces to the memoryless forward path controller (6.2) if the previous pass contribution terms are deleted.

Applying the control scheme (6.68) to the differential process (2.11) yields the closed loop state-space model

$$\dot{x}_{k+1}(t) = (A + BF)\,x_{k+1}(t) + BG\,r_{k+1}(t) + \sum_{j=1}^{M}(B_{j-1} - BK_j)\,y_{k+1-j}(t)$$

$$y_{k+1}(t) = C\,x_{k+1}(t) + \sum_{j=1}^{M} D_j\,y_{k+1-j}(t), \quad 0 \leq t \leq \alpha, \;\; k \geq 0, \quad (6.69)$$

which is closed in the sense that it has an identical structure to (2.11). Hence known stability theory applies. Note once again that the D_j, $1 \leq j \leq M$, matrices are invariant under this control action, and hence it is necessary to assume open

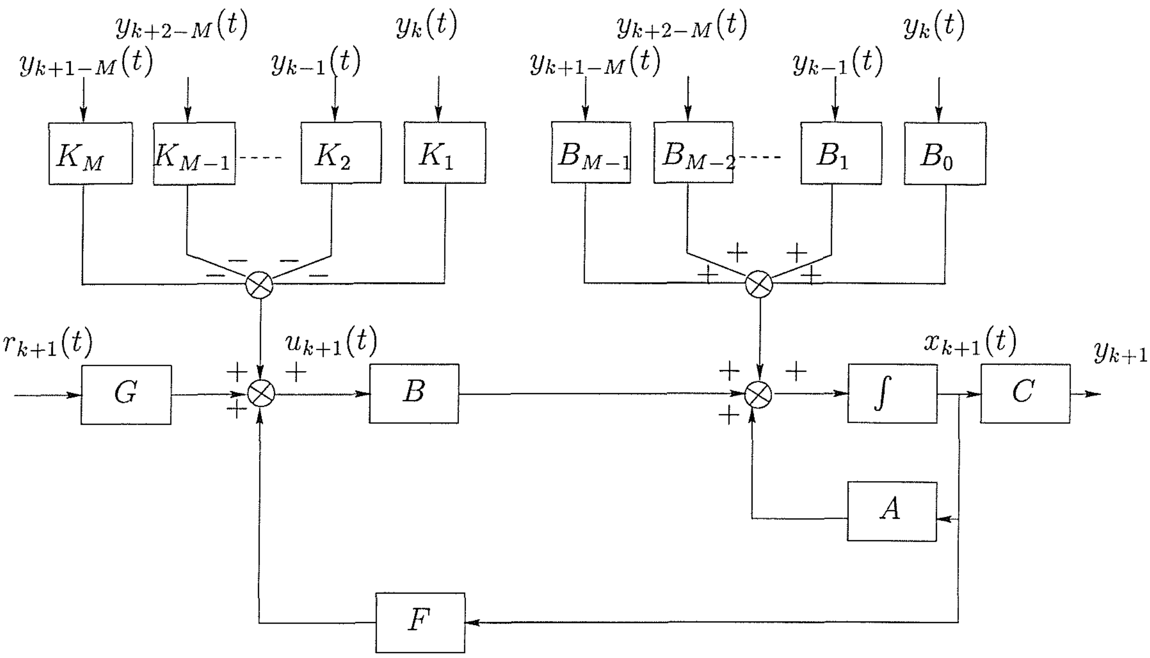

Figure 6.3: Structure of a memoryless linear state feedback controller with proportional repetitive minor loop compensation

loop asymptotic stability. The extra design freedom achieved by implementing a controller with memory is clearly the choice of the matrices K_j, $1 \leq j \leq M$.

Returning now to output feedback control schemes, a memoryless dynamic unity negative feedback controller with proportional repetitive minor loop compensation for (2.11) constructs the input $u_{k+1}(t)$, $k \geq 0$, as

$$u_{k+1}(t) = y^C_{k+1}(t) - \sum_{j=1}^{M} K_j\, y_{k+1-j}(t), \quad 0 \leq t \leq \alpha, \ k \geq 0, \tag{6.70}$$

where K_j, $1 \leq j \leq M$, is a $l \times m$ matrix and $y^C_{k+1}(t)$ is the output from

$$\begin{aligned} \dot{x}^C_{k+1}(t) &= A^C\, x^C_{k+1}(t) + B^C\, e_{k+1}(t) \\ y^C_{k+1}(t) &= C^C\, x^C_{k+1}(t) + D^C\, e_{k+1}(t), \quad 0 \leq t \leq \alpha, \ k \geq 0, \end{aligned} \tag{6.71}$$

where $x^C_{k+1}(t)$ is the $n_1 \times 1$ internal state of (6.71) and the current pass error vector $e_{k+1}(t)$ is again defined by (6.5). Note that this scheme reduces to (6.5)-(6.6) if K_j, $1 \leq j \leq M$, are chosen to be identically zero.

Then, using the augmented state vector definition of (6.7), application of the controller (6.70)-(6.71) to the differential process (2.11) yields the following composite state-space model describing the forward path system,

$$\dot{X}^A_{k+1}(t) = \hat{A}\, X^A_{k+1}(t) + \hat{B}\, e_{k+1}(t) + \sum_{j=1}^{M} \hat{B}_{j-1}\, y_{k+1-j}(t)$$

$$y_{k+1}(t) = \hat{C}\, X^A_{k+1}(t) + \sum_{j=1}^{M} \hat{D}_j\, y_{k+1-j}(t), \quad 0 \le t \le \alpha,\ k \ge 0, \qquad (6.72)$$

where $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}_j$, $1 \le j \le M$, are as in (6.10), but here

$$\hat{B}_{j-1} = \begin{bmatrix} B_{j-1} - BK_j \\ 0 \end{bmatrix}, \quad 1 \le j \le M. \qquad (6.73)$$

Further, combining (6.5) and (6.72) yields the closed loop state-space model

$$\dot{X}^A_{k+1}(t) = (\hat{A} - \hat{B}\hat{C})\, X^A_{k+1}(t) + \hat{B}\, r_{k+1}(t) + \sum_{j=1}^{M} (\hat{B}_{j-1} - \hat{B}\hat{D}_j)\, y_{k+1-j}(t)$$

$$y_{k+1}(t) = \hat{C}\, X^A_{k+1}(t) + \sum_{j=1}^{M} \hat{D}_j\, y_{k+1-j}(t), \quad 0 \le t \le \alpha,\ k \ge 0. \qquad (6.74)$$

Both (6.72) and (6.74) are closed in the sense that they have an identical structure to (2.11). Hence known stability can once again be applied, but note that the D_j, $1 \le j \le M$, matrices are again invariant. Hence it is necessary to once again assume open loop asymptotic stability. As with the state feedback scheme introduced earlier in this section, the extra design freedom associated with this scheme is the choice of design parameters K_j, $1 \le j \le M$. These terms only influence the interpretation of condition (c) of theorem 3.6 in that they effect the previous pass driving terms in the state equation only and hence are referred to as having the so-called separation property. As a result of this, such controllers should be of particular use in terms of the so-called repetitive systems disturbance decoupling problem (see later). Within the next section, this scheme is applied to a class of benchmark problems - so-called multivariable second order lags.

6.7 Application to Benchmark Problems II. Multivariable Second Order Lags

Despite the introduction of an approximation term into the model of the state-space triple (A, B, C), multivariable first order lags do not describe all the dynamics effects observed in differential linear systems. In particular, the multivariable first order lag model does not admit the modelling of oscillations in the closed loop system. (Owens, 1975) introduces the concept of a multivariable second order type system as a useful vehicle for illustrating oscillation in multivariable feedback systems. Note that, in general, a higher order model will produce less conservative results at the potential expense of working with a more complex model. Within this section, previously discussed control structures are applied to this second class of benchmark problems.

Consider the subclass of differential linear repetitive processes where the state-space triple (A, B, C) in (2.13) takes the structure of a multivariable second order lag (Owens, 1975). A second order structure (termed 'restrictive' in the literature) is be defined by analogy with the second order inverse transfer-function $g^{-1}(s) = s(sa_0 + a_1)$, $a_0 \neq 0$, and has the derived conventional $m \times m$ invertible linear system $L_D(A, B, C)$ with inverse transfer-function matrix,

$$\begin{aligned} G^{-1}(s) &= s\{sA_0 + A_1\}, \quad |A_0| \neq 0 \\ \text{or} \quad G(s) &= \frac{1}{s}\{sA_0 + A_1\}^{-1} \end{aligned} \tag{6.75}$$

i.e. the outputs are simply integrated outputs from the first order lag $\{sA_0 + A_1\}^{-1}$.

It can easily be verified that the process whose derived conventional linear system has this structure can be described by the following second order differential equation with matrix coefficients,

$$\begin{aligned} & A_0 \frac{d^2 y_{k+1}}{dt^2}(t) + A_1 \frac{dy_{k+1}}{dt}(t) = u_{k+1}(t) + A_0 B_0\, y_k(t), \;\; 0 \leq t \leq \alpha, \;\; k \geq 0, \\ \Longrightarrow \quad & \frac{d^2 y_{k+1}}{dt^2}(t) + A_0^{-1} A_1 \frac{dy_{k+1}}{dt}(t) = A_0^{-1}\, u_{k+1}(t) + B_0\, y_k(t). \end{aligned} \tag{6.76}$$

This process can be written in the form of a state-vector model by introducing the variables

$$\begin{aligned} x_{k+1}^{(1)}(t) &= y_{k+1}(t) \\ x_{k+1}^{(2)}(t) &= \dot{y}_{k+1}(t) \end{aligned} \tag{6.77}$$

where $x_{k+1}^{(1)}(t)$ and $x_{k+1}^{(2)}(t)$ are $m \times 1$ vectors, and writing

$$\begin{aligned}
\begin{pmatrix} \dot{x}_{k+1}^{(1)}(t) \\ \dot{x}_{k+1}^{(2)}(t) \end{pmatrix} &= \begin{pmatrix} 0 & I_m \\ 0 & -A_0^{-1}A_1 \end{pmatrix} \begin{pmatrix} x_{k+1}^{(1)}(t) \\ x_{k+1}^{(2)}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ A_0^{-1} \end{pmatrix} u_{k+1}(t) \\
&\quad + \begin{pmatrix} 0 & 0 \\ B_0 & 0 \end{pmatrix} \begin{pmatrix} x_k^{(1)}(t) \\ x_k^{(2)}(t) \end{pmatrix} \\
y_{k+1}(t) &= \begin{pmatrix} I_m & 0 \end{pmatrix} \begin{pmatrix} x_{k+1}^{(1)}(t) \\ x_{k+1}^{(2)}(t) \end{pmatrix}, \quad 0 \le t \le \alpha, \ k \ge 0, \qquad (6.78)
\end{aligned}$$

or, introducing the augmented state vector $X_{k+1}^A(t) = (x_{k+1}^{(1)}(t)^T,\, x_{k+1}^{(2)}(t)^T)^T$ gives,

$$\begin{aligned}
\dot{X}_{k+1}^A(t) &= \hat{A}\, X_{k+1}^A(t) + \hat{B}\, u_{k+1}(t) + \hat{B}_0\, X_k^A(t) \\
y_{k+1}(t) &= \hat{C}\, X_{k+1}^A(t), \quad 0 \le t \le \alpha, \ k \ge 0, \qquad (6.79)
\end{aligned}$$

where

$$\begin{aligned}
&\hat{A} = \begin{pmatrix} 0 & I_m \\ 0 & -A_0^{-1}A_1 \end{pmatrix}, \ \hat{B} = \begin{pmatrix} 0 \\ A_0^{-1} \end{pmatrix}, \ \hat{B}_0 = \begin{pmatrix} 0 & 0 \\ B_0 & 0 \end{pmatrix}, \\
&\text{and} \quad \hat{C} = \begin{pmatrix} 0 & I_m \end{pmatrix}. \qquad (6.80)
\end{aligned}$$

At this point a brief introduction to the well known disturbance decoupling with stability problem for 1D linear systems is given. Consider, then, the system

$$\begin{aligned}
\dot{x}(t) &= A\,x(t) + B\,u(t) + D\,q(t) \\
y(t) &= C\,x(t), \quad t \ge 0, \qquad (6.81)
\end{aligned}$$

where $x(t)$ is the $n \times 1$ state vector, $y(t)$ is the $m \times 1$ output vector, $u(t)$ is the $l \times 1$ vector of control inputs and $q(t)$ is a $v \times 1$ vector representing a disturbance which is assumed not to be directly measurable by the controller. Further, suppose that the linear state feedback law $u(t) = F\,x(t)$ is applied. Then the disturbance decoupling problem is to find a suitable F such that the disturbance $q(t)$ has no influence on the controlled output $y(t)$. Equivalently, the closed loop system is said to be disturbance decoupled relative to the pair $\{y(t), q(t)\}$ if, for each $n \times 1$ initial condition $x(0)$, the output $y(t)$, $t \ge 0$, is identical $\forall\, q(t) \in \mathbb{R}^v$.

In the field of 1D systems theory, much research effort has been invested in this control problem. The fact that the resulting conditions do not ensure closed loop stability (in that the eigenvalues of $(A+BF)$ have strictly negative real parts) has led to the introduction of the so-called disturbance decoupling with stability problem.

Returning now to repetitive processes, it is clear that a similar problem can be formulated. In particular, consider the differential unit memory case (all results generalise in a natural manner to the non-unit memory case) and interpret the previous pass term $y_k(t)$, $0 \le t \le \alpha$, $k \ge 0$, as a disturbance which is not directly measurable by the controller on pass $k+1$. Suppose also that the current pass controller

$$u_{k+1}(t) = F\, x_{k+1}(t), \quad 0 \le t \le \alpha, \;\; k \ge 0, \tag{6.82}$$

is applied. Then the first requirement of the so-called repetitive systems disturbance decoupling with stability problem (RSDDSP) is that, given any initial condition $x_k(0) = d_k \in \mathbb{R}^n$, the closed loop output $y_k(t)$ is the same for all $y_{k-1}(t) \in \mathbb{R}^m$. Hence repetitive systems disturbance decoupling simply means that the contribution of the previous pass profile to the current one is zero, $0 \le t \le \alpha$, $k \ge k^* \ge 1$, i.e. the previous pass profile is regarded as a disturbance to be rejected. Clearly the optimal choice of k^* here is $k^* = 1$. The second requirement of the RSDDSP is that (as a basic minimum) all eigenvalues of $A + BF$ have strictly negative real parts. It therefore appears that there exists some strong structural similarities between the RSDDSP and its conventional linear systems counterpart. As a result of this link, it appears that an extension of 1D approaches such as using geometric concepts such as (A, B)-invariant subspaces may make progress in tackling the problem. This general area remains a subject for future work.

For now, we use the memoryless dynamic unity negative feedback controller with proportional repetitive minor loop compensation introduced in section 6.6 to solve the RSDDSP in the special case of the subclass of differential processes whose derived conventional linear system has the structure of a multivariable second order lag.

Return, therefore, to the differential process (6.79) and consider the application of the following forward path controller over $0 \le t \le \alpha$, $k \ge 0$,

$$u_{k+1}(t) = K(r_{k+1}(t) - y_{k+1}(t)) - A_0 B_0\, y_k(t) \tag{6.83}$$

with $K > 0$. (6.83) is an example of the memoryless dynamic unity negative feedback control action with proportional repetitive minor loop compensation of (6.5) and (6.70)-(6.71) introduced in section 6.6.

Then, application of (6.83) yields the following closed loop system

$$\begin{aligned} \dot{X}^A_{k+1}(t) &= \hat{A}_C\, X^A_{k+1}(t) + \hat{B}_C\, r_{k+1}(t) \\ y_{k+1}(t) &= \hat{C}\, X^A_{k+1}(t), \quad 0 \le t \le \alpha, \;\; k \ge 0, \end{aligned} \tag{6.84}$$

with $X_{k+1}^A(t)$ and $\hat{C}$ defined as in (6.79),

$$\hat{A}_C = \begin{pmatrix} 0 & I_m \\ -A_0^{-1}K & -A_0^{-1}A_1 \end{pmatrix} \quad \text{and} \quad \hat{B}_C = \begin{pmatrix} 0 \\ -A_0^{-1}K \end{pmatrix}. \tag{6.85}$$

Clearly, the repetitive interaction terms have disappeared and hence this is just a standard (second-order) 1D system. Hence in (6.74) the pass profile $y_k(t)$, $0 \leq t \leq \alpha$, $k \geq 0$, is independent of the pass profiles $y_{k-j}(t)$, $1 \leq j \leq M$, for all passes $k \geq 1$. Equivalently, the repetitive systems disturbance decoupling problem is achieved in this case with an optimum choice of $k^* = 1$.

The closed loop limit profile in this case is described in transfer-function matrix terms by

$$Y_\infty(s) = \hat{C}(sI_m - \hat{A}_C)^{-1}\hat{B}_C\, R_\infty(s) \tag{6.86}$$

which is just the transfer-function matrix of the derived conventional linear system under the memoryless control scheme obtained by deleting the repetitive interaction terms in (6.83). Hence the design exercise can be completed by using appropriate 1D techniques to choose the K in (6.83) to meet the required specifications.

6.8 Discrete First Order Models for Linear Repetitive Processes

Within this section, discrete linear repetitive processes are considered. Initially it is shown how a unit memory differential linear repetitive process can be successfully sampled to obtain a linear time-invariant discrete repetitive process, provided that the sampling rate is high enough. The multivariable first order lag model introduced in section 6.4 for differential processes is extended to define an equivalent formulation for discrete sampled data processes. Finally, it is shown how a multivariable discrete first order lag, in many cases of practical interest, is a quite adequate approximation for the purpose of controller design provided that the plant is minimum phase and satisfies a contraction mapping condition. The analysis introduction in this section is novel and can be found in (Benton et al., 2000b).

6.8.1 Fast Sampling of Linear Repetitive Processes

Due to advances in computer technology, systems today tend to be viewed as sampled data systems. Within this section it is shown how the discrete unit memory linear repetitive process, denoted S_d, described by the following state-space model over $0 \leq p \leq \alpha$, $k \geq 0$,

$$\begin{aligned} x_{k+1}(p+1) &= \hat{A}\, x_{k+1}(p) + \hat{B}\, u_{k+1}(p) + \hat{B}_0\, y_k(p) \\ y_{k+1}(p) &= \hat{C}\, x_{k+1}(p) \end{aligned} \tag{6.87}$$

can be regarded as being derived from a differential unit memory process, denoted S, of the following form over $0 \leq t \leq \alpha$, $k \geq 0$,

$$\begin{aligned} \dot{x}_{k+1}(t) &= A\, x_{k+1}(t) + B\, u_{k+1}(t) + B_0\, y_k(t) \\ y_{k+1}(t) &= C\, x_{k+1}(t) \end{aligned} \tag{6.88}$$

with initial conditions $x_{k+1}(0) = d_{k+1}$, $k \geq 0$, and $y_0(t) = \hat{y}(t)$, $0 \leq t \leq \alpha$.

Now subject (6.88) to synchronous digital control with sampling period h, where

$$x_{k+1}^q := x_{k+1}(qh) \tag{6.89}$$

and where, for integer q, $0 \leq q \leq \frac{\alpha}{h}$, and piecewise continuous input

$$\begin{aligned} u_{k+1}^q &:= u_{k+1}(qh) \\ &= u_{k+1}(t) \quad \text{on } qh \leq t < (q+1)h. \end{aligned} \tag{6.90}$$

In addition, note that, under fast sampling conditions (i.e. $h \longrightarrow 0$), $y_k(t)$ on the interval $[qh, (q+1)h)$ can be approximated by $y_k(qh)$, $0 \leq q \leq \frac{\alpha}{h}$, $k \geq 0$. This approximation improves as $h \longrightarrow 0$, and we have

$$\lim_{h \to 0^+} y_k(\tau) = y_k(qh), \quad \text{on } [qh, (q+1)h). \tag{6.91}$$

Note that this is equivalent to the assumption that the previous pass profile is piecewise continuous.

Then, if the differential linear repetitive process (6.88) is subject to the sampling scheme described by (6.89) and (6.90), a discrete linear repetitive process of the form (6.87) is obtained with

$$\hat{A} = e^{Ah}, \ \hat{B} = \hat{A} \int_0^h e^{-A\tau} B\, d\tau \text{ and } \hat{B}_0 = \hat{A} \int_0^h e^{-A\tau} B_0\, d\tau. \tag{6.92}$$

The derivations of the result given here is lengthy and hence has been included as an appendix (section B). It should be noted, however, that the approximation in the final term of (6.92) improves as $h \longrightarrow 0$, i.e. under fast sampling conditions, and that the differential model is recovered in the limit.

6.8.2 Application to Benchmark Problems III. Multivariable Discrete First Order Lags

Consider the subclass of discrete unit memory processes where the state-space triple (A, B, C) in (2.24)-(2.25) takes the structure of a multivariable discrete first order lag (Owens, 1979). In the cited reference, an $m \times m$ discrete first order lag is defined to be a controllable and observable m-input m-output discrete time system with inverse transfer-function matrix

$$G_A^{-1}(z_1) = (z_1 - 1)\Delta_0 + \Delta_1 \tag{6.93}$$

where Δ_0 and Δ_1 are real $m \times m$ matrices and $|\Delta_0| \neq 0$.

Given (2.24)-(2.25) with $m = l = n$ and $|CB| \neq 0$, a simple (current pass) state transformation yields the equivalent description over $0 \leq p \leq \alpha$, $k \geq 0$,

$$y_{k+1}(p+1) \quad = \quad (I_m - \Delta_0^{-1}\Delta_1)\, y_{k+1}(p) + \Delta_0^{-1}\, u_{k+1}(p) + B_0\, y_k(p), \tag{6.94}$$

where Δ_0, Δ_1 and B_0 are real constant $m \times m$ matrices with $|\Delta_0| \neq 0$. This state-space representation has 2D transfer-function matrix $G_A(z_1, z)$, where

$$G_A^{-1}(z_1, z) = (z_1 - 1)\Delta_0 + \Delta_1 - z^{-1}\Delta_0 B_0. \tag{6.95}$$

As a first attempt at controller design, consider the memoryless proportional forward path controller of the general parametric form

$$U(z_1, z) = K(z_1)\, E(z_1, z) = (\text{diag}\{1 - \rho_j(z_1)\}_{1 \leq j \leq m}\Delta_0 - \Delta_1)\, E(z_1, z), \tag{6.96}$$

where the $\rho_j(z)$, $1 \leq j \leq m$, are proper minimum phase transfer-function matrices, the $m \times 1$ current pass error vector $e_{k+1}(p)$ on pass $k+1$ has 2D transform

$$E(z_1, z) = R(z_1, z) - Y(z_1, z) \tag{6.97}$$

and where $R(z_1, z)$ is the 2D transform of the new external reference input.

This control scheme is a discrete example of the differential error actuated feedback systems introduced in section 6.2. Application of this control action to (6.94) yields the closed loop system,

$$Y(z_1,z) = G_0(z_1)\,R(z_1,z) + z^{-1}G_1(z_1)\,Y(z_1,z) \tag{6.98}$$

where

$$\begin{aligned} G_0(z_1) &= \operatorname{diag}\left\{\frac{1}{z_1-\rho_j(z_1)}\right\}_{1\le j\le m}\left(\operatorname{diag}\{1-\rho_j(z_1)\}_{1\le j\le m} - \Delta_0^{-1}\Delta_1\right) \\ G_1(z_1) &= \operatorname{diag}\left\{\frac{1}{z_1-\rho_j(z_1)}\right\}_{1\le j\le m} B_0. \end{aligned} \tag{6.99}$$

Application of theorem 3.6 for stability along the pass then proceeds as follows. Clearly, both the open and closed loop systems are asymptotically stable. Now for condition (b) of theorem 3.6 to hold, we require that the derived conventional linear system $L_D(A,B,C)$ is stable in the standard 1D sense.

The closed loop transfer-function matrix of $L_D(A,B,C)$ is given by $G_0(z_1)$ in (6.99) above, which is clearly stable if, and only if, $|\rho_j(z_1)| < 1,\ 1 \le j \le m\ \forall\, |z_1| = 1$.

In particular, the closed loop derived conventional linear system possesses small steady state errors and small interaction effects in response to unit step demands only if the elements of the matrix $\Delta_0^{-1}\Delta_1$ are 'small enough'. It can be seen that this is not a severe restriction on the practical application of the results by regarding the discrete process (6.94) as being derived from a differential process of the form (6.87) under the sampling scheme described by (6.89) and (6.90). Clearly the discrete process S_d is a discrete model of the differential process S which is parameterised by the sampling interval h. Hence $G_A(z_1)$ of (6.93) is also parameterised by h in the sense that the choice of Δ_0 and Δ_1 will depend explicitly on this sampling interval. Then, on comparing the matrices of (6.94) with (6.92), it follows that

$$\lim_{h\to 0^+} \Delta_0^{-1}\Delta_1 = \lim_{h\to 0^+}\{I_m - e^{Ah}\} = 0 \tag{6.100}$$

and hence the closed loop derived conventional linear system will possess small interaction effects and steady state errors in response to unit step demands if the sampling rate is fast enough.

Assuming, for simplicity, that A has a diagonal canonical form with eigenvalues denoted by $\lambda_j,\ 1 \le j \le m$, and eigenvector matrix E. Then

$$\hat{A} = E\,\operatorname{diag}\{e^{\lambda_j h}\}_{1\le j\le m}\,E^{-1} \tag{6.101}$$

suggesting that a necessary condition for $\Delta_0^{-1}\Delta_1$ to be small is that

$$|\lambda_j h| \ll 1, \quad 1 \leq j \leq m. \tag{6.102}$$

Equivalently, the sampling rate must be fast in comparison to the poles of the underlying continuous open loop plant.

Now returning to the stability along the pass theorem 3.6, the closed loop interpass transfer-function of (6.94) has the form

$$G_1(z_1) = \text{diag}\left\{\frac{1}{z_1 - \rho_j(z_1)}\right\}_{1\leq j\leq m} B_0. \tag{6.103}$$

Therefore, if the $\rho_j(z_1)$, $1 \leq j \leq m$, have been chosen so as condition (b) of theorem 3.6 holds, then the closed loop system is stable along the pass if, and only if, all eigenvalues of $G_1(z_1)$ have modulus strictly less than unity for all real frequencies z_1 satisfying $|z_1| = 1$. This condition can be tested via standard 1D linear systems techniques.

The design method can be extended to the case where the state-space triple (A, B, C) in (2.24)-(2.25) only approximates the structure of a discrete multivariable first order lag.

Consider the subclass of discrete linear repetitive processes whose derived conventional linear system $L_D(A, B, C)$ has the approximate structure of a multivariable discrete multivariable first order lag. In this case $L_D(A, B, C)$ has an open loop $m \times m$ invertible, minimum phase transfer-function matrix $G(z_1)$ of the form

$$G^{-1}(z_1) = (z_1 - 1)\Delta_0 + \Delta_1 + \Delta_0 H(z_1) \tag{6.104}$$

where $H(z_1)$ is strictly proper, $H(1) = 0$ and $|\Delta_0| \neq 0$.

Then, in a method analogous to the one presented in section 6.5.2 for differential processes, the discrete first order lag model $G_A^{-1}(z_1)$ of (6.93) can be used as a reduced order model for the purpose of controller design provided that $H(z_1)$ satisfies a contraction mapping condition.

The relevant matrix is

$$\begin{aligned} L(z_1) &= [K(z_1) + G_A^{-1}(z_1)]^{-1}[G_A^{-1}(z_1) - G^{-1}(z_1)] \\ &= (-1)\,\text{diag}\left\{\frac{1}{z_1 - \rho_j(z_1)}\right\}_{1\leq j\leq m} H(z_1) \end{aligned} \tag{6.105}$$

and a sufficient condition for closed loop stability is that $||L(z_1)|| < 1$.

Since the analysis here is just the discrete counterpart to that presented in section 6.5.2, the details are omitted - see for example (Owens, 1979) for the case of 1D multivariable discrete systems.

6.9 Controller Design using a 2D Lyapunov Equation Approach

Within this section the 2D Lyapunov equation of chapter 4 is used in the design of a so-called current pass state feedback control law augmented by 'feedforward' previous pass output action, which is an example of a controller with memory discussed earlier in this chapter.

Consider a discrete linear repetitive process with state-space representation (2.24) and (2.25). Then for this process, such a control law has the form

$$u_{k+1}(p) = -F\, x_{k+1}(p) + S\, y_k(p), \quad 0 \leq p \leq \alpha, \;\; k \geq 0. \tag{6.106}$$

Application of this control action to the process yields the following closed loop system over $0 \leq p \leq \alpha$, $k \geq 0$,

$$\begin{aligned} x_{k+1}(p+1) &= (A - BF)\, x_{k+1}(p) + (B_0 + BS)\, y_k(p) \\ y_{k+1}(p) &= C\, x_{k+1}(p) + D_1\, y_k(p). \end{aligned} \tag{6.107}$$

The closed loop augmented plant matrix for (6.107) is defined by

$$\Phi_c := \begin{bmatrix} A - BF & B_0 + BS \\ C & D_1 \end{bmatrix} = \Phi - \hat{B}\hat{S} \tag{6.108}$$

where Φ is the augmented plant matrix of the uncontrolled system and

$$\hat{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \hat{S} = \begin{bmatrix} F & -S \end{bmatrix}. \tag{6.109}$$

Following the analysis in (Lu and Lee, 1985) for the case of discrete linear systems described by the Roesser 2D state-space model, we replace Φ in the 2D Lyapunov equation (4.93) by Φ_c to give

$$\begin{aligned} & (\Phi - \hat{B}\hat{S})^T W(\Phi - \hat{B}\hat{S}) - W = -Q \\ \Longrightarrow \quad & (\hat{B}^T W\Phi)^T \hat{S} + \hat{S}^T(\hat{B}^T W\Phi) - \hat{S}^T \hat{B}^T W\hat{B}\hat{S} + (W - \Phi^T W\Phi - Q) = 0 \end{aligned} \tag{6.110}$$

or

$$\hat{\Phi}^T \hat{S} + \hat{S}^T \hat{\Phi} - \hat{S}^T \hat{D}\hat{S} + \hat{Q} = 0 \tag{6.111}$$

where

$$\begin{aligned} \hat{\Phi} &= \hat{B}^T W\Phi \\ \hat{D} &= \hat{B}^T W\hat{B} \\ \hat{Q} &= W - \Phi^T W\Phi - Q. \end{aligned} \tag{6.112}$$

Now consider (temporarily) the case when $l = N := n+m$. Then in this case, (6.111) is a matrix Riccati equation with $\hat{D} \geq 0$ and $\hat{Q}^T = \hat{Q}$. This leads immediately to the following results on invoking the 2D Lyapunov equation as a sufficient condition for closed loop stability along the pass.

Theorem 6.6 *Consider $S(E_\alpha, W_\alpha, L_\alpha)$ generated by (2.24)-(2.25) subject to the control law (6.106) in the case when $l = N$. Then the resulting closed loop system is stable along the pass if $\exists$ two $N \times N$ matrices $W = W_1 \oplus W_2 > 0$ and $Q > 0$ such that the Riccati equation (6.111) has a real solution $\hat{K}$.*

To solve this Riccati equation, first construct the $2N \times 2N$ matrix

$$\hat{M} = \begin{bmatrix} \hat{\Phi} & -\hat{D} \\ -\hat{Q} & \hat{\Phi}^T \end{bmatrix} = \begin{bmatrix} \hat{B}^T W\Phi & -\hat{B}^T W\hat{B} \\ -(W - \Phi^T W\Phi - Q) & -\Phi^T W\hat{B} \end{bmatrix}. \tag{6.113}$$

Also let a_i be the $2N \times 1$ eigenvector of $\hat{M}$ corresponding to the eigenvalue λ_i, $1 \leq i \leq 2N$, and partition it as

$$a_i = \begin{bmatrix} b_i \\ c_i \end{bmatrix}, \quad 1 \leq i \leq 2N, \tag{6.114}$$

where b_i and c_i are $N \times 1$ vectors. Then we have the following result from (Rogers et al., 2000a),

Theorem 6.7 *Suppose that $a_1, \cdots, a_N$ are eigenvectors of the matrix $\hat{M}$ of (6.113) corresponding to eigenvalues $\lambda_1, \cdots, \lambda_N$. Suppose also that $\begin{bmatrix} b_1 & \cdots & b_N \end{bmatrix}^{-1}$ exists. Then, if $\overline{\lambda}_i \neq -\lambda_j$, $1 \leq i, j \leq N$,*

$$\hat{K} = \begin{bmatrix} c_1 & \cdots & c_N \end{bmatrix} \begin{bmatrix} b_1 & \cdots & b_N \end{bmatrix}^{-1} \tag{6.115}$$

is a solution of the Riccati equation (6.111). Also if the eigenvectors a_i, $1 \leq i \leq N$, are real then the matrix $\hat{K}$ here is a real solution of this Riccati equation.

Hence we have the following theorem,

Theorem 6.8 *Consider $S(E_\alpha, W_\alpha, L_\alpha)$ generated by (2.24)-(2.25) with $l = N$ and subject to the control law (6.106). Then the resulting closed loop system is stable along the pass if $Q > 0$ and $W = W_1 \oplus W_2 > 0$ can be chosen such that $\hat{M}$ of (6.113) has N real eigenvectors $a_1, \cdots, a_N$ corresponding to the eigenvalues $\lambda_1, \cdots, \lambda_N$, with $\overline{\lambda}_i \neq -\lambda_j$, $1 \leq i, j \leq N$, and $\begin{bmatrix} b_1 & \cdots & b_N \end{bmatrix}^{-1}$ exists.*

Example 6.1 *As an example, consider the case when*

$$\Phi = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{6.116}$$

Hence we can set $W = \beta_1 I_2$ and $Q = \beta_2 I_2$ with $\beta_i > 0$, $i = 1, 2$.

Then, in this case,

$$\hat{M} = -\beta_1 \begin{bmatrix} -1 & -2 & 2 & -2 \\ 1 & 2 & -2 & 2 \\ q-1 & -1 & 1 & -1 \\ -1 & q-2 & 2 & -2 \end{bmatrix} := -\beta_1 \tilde{M} \tag{6.117}$$

where $q = 1 - \frac{\beta_2}{\beta_1}$.

Now note that $\det(\lambda I - \tilde{M}) = \lambda^2(\lambda^2 - (4q-1))$, and hence choosing $\beta_1 = 1$, $\beta_2 = 1/2$ gives the eigenvalues of $\hat{M}$ as $\lambda_{1,3} = 0$, $\lambda_{2,4} = \pm 1$. Also the eigenvectors for $\lambda_1 = 0$ and $\lambda_2 = 1$ are

$$a_1 = \begin{bmatrix} 1 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 1 \\ -1 \\ 1/2 \\ 1/2 \end{bmatrix}. \tag{6.118}$$

Hence

$$\hat{K} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \tag{6.119}$$

which is a real symmetric solution of (6.111).

Also note that

$$\Phi_c = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \tag{6.120}$$

which corresponds to a stable along the pass process.

Now consider the (more realistic) case of when $l < n+m$ and let the matrix $\hat{K}$ be of the form $\hat{K} = P_1 P_2$, where P_2 has dimensions $N \times N$. Then (6.111) takes the form

$$\tilde{\Phi}^T P_2 + P_2^T \tilde{\Phi} - P_2^T E P_2 + \hat{Q} = 0 \tag{6.121}$$

where

$$\tilde{\Phi} = P_1^T \hat{B}^T W \Phi, \quad E = P_1^T \hat{B}^T W \hat{B} P_1 \tag{6.122}$$

and hence the above stabilisation can also be applied in the general case.

Example 6.2 *As an example, consider the case when*

$$\Phi = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{6.123}$$

Here set $P_1 = \begin{bmatrix} 1 & -1 \end{bmatrix}$ *and then*

$$\hat{B} P_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{6.124}$$

Also, from the previous example, (6.121) has a real solution

$$P_2 = \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix} \tag{6.125}$$

and hence in this case

$$\hat{K} = P_1 P_2 = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}. \tag{6.126}$$

6.10 Summary and Conclusions

This chapter has presented some of the available control structures for linear repetitive processes which have been considered to date. Control schemes for differential and discrete processes inherently fall into two different categories,

(i) those which explicitly use information from the current pass only, termed memoryless controllers, and

(ii) those which explicitly use information from the current pass and/or previous pass profiles, state vectors and input vectors - so-called controllers with memory.

Differential and discrete linear repetitive processes clearly have strong structural links with standard (1D) differential and discrete linear systems respectively. In fact, it has been shown in chapter 3 that the stability theory for these two subclasses of processes can be tested by direct application of well known 1D linear systems tests. This raises the natural question of what exactly can be achieved by standard (1D) feedback control schemes in this context, e.g. is it possible to use standard unity negative feedback control policies to stabilise these processes?

As a starting point in answering this question, section 6.2 has presented classes of state feedback (see, for example, (Smyth, 1992; Rogers and Owens, 1992b)) and output feedback (see, for example, (Rogers and Owens, 1993) and (Rogers and Owens, 1995a) for the discrete/differential cases respectively) control laws, which are examples of so-called current point schemes. The application of both types of structure results in a closed system, and hence known stability theory can be applied. It is found, however, that the property of asymptotic stability remains invariant under these control policies, and, in fact, under all multipass causal feedback control schemes (Rogers and Owens, 1992b) for the following reasons:

(i) for the simple boundary conditions case, asymptotic stability depends only on the matrices $D_j, 1 \leq j \leq M$. For the case of dynamic boundary conditions, the result just noted is a necessary condition for asymptotic stability of processes with these boundary conditions, and hence the same conclusion can be drawn;

(ii) for eg. differential processes, the output $y_{k+1}(t)$ does not explicitly depend on the input $u_{k+1}(t)$, $0 \leq t \leq \alpha$, $k \geq 0$, (i.e. there is no 'direct feedthrough' between input and output on any pass).

How to overcome this systems theoretic problem is not clear, and remains a topic for further research. For now, we use the argument that, in practical cases, asymptotic stability is always present due to the stabilising influence of resetting the initial conditions on each pass, and hence for this chapter asymptotic stability has been assumed to hold.

An additional point should be made about the state feedback schemes already mentioned. Such structures require the availability of all elements of the state vector, and hence current pass state feedback laws can, in general, only be implemented with an observer structure. Observer theory for differential and discrete linear repetitive processes is not covered here and remains an open area for future research.

To illustrate the potential of this general approach, section 6.4 sees the application of memoryless feedback control schemes to processes possessing a certain special structure - so-called benchmark problems. Here it is shown that the question of as to what exactly can be achieved using a standard (1D) memoryless feedback control scheme has a solution in one case of practical interest with the added benefit of 'high' performance in a meaningful sense. The analysis presented here is novel and provides the basis for the paper (Benton et al., 1998a). The general philosophy adopted in this work is in the spirit of (Sebek and Kraus, 1995) for other classes of 2D linear systems, i.e. the use of 'simple' structure controllers. In contrast to this previous work which can only consider stability, here the design of the controller for stability and performance can be achieved in one step. The analysis replaces the necessary and sufficient condition on gain for stability along the pass of theorem 6.3 by a sufficient but not necessary alternative. This potential conservative is offset by the availability of strong information on performance along a given pass from this result at no extra computational cost, which is not available from Nyquist-like alternatives.

Section 6.5 has introduced some refinements to the analysis of the previous section. If the required value of the scalar gain ρ is outside the available range, by analogy with standard 1D linear systems theory, an approach is to include dynamics within the forward path controller. The application of a general parametric form of a proportional controller has been given in section 6.5.1. For the inclusion of an integral control element, see for example (Owens, 1978). Section 6.5.2 has considered the use of reduced order models in the design of controllers for linear repetitive processes. Here it has been shown that a first order model may be used to achieve acceptable systems performance for processes possessing a certain structure, provided a contraction mapping condition is satisfied. This ensures that if the approximation term $H(s)$ is 'small' in some well defined sense, then the reduced order model will be a good approximation for both the closed and open loop system dynamics. In more general situations, however, $H(s)$ may be significant and higher gains are required to ensure stability.

When one or more of the control objectives cannot be met by a current pass controller, one way forward is to introduce controllers with memory, i.e. those which explicitly use information from the current pass and/or previous pass profiles, state vectors and input vectors. Within section 6.6, as an example of a controller with memory, a so-called memoryless linear state feedback law with proportional repetitive minor loop compensation has been presented. This type of control structure has been applied to a benchmark class of processes whose derived conventional linear system has the structure of a multivariable second order lag, where it has been shown to successfully give a solution to the so-called repetitive systems disturbance decoupling with stability problem.

In section 6.8 discrete processes have been considered. It has been shown how a discrete linear repetitive process can be regarded as being derived from a differential process under fast sampling conditions. The analysis presented here is novel and can be found in (Benton et al., 2000b). The structures of the previous sections have then been applied to a benchmark class of discrete processes - so-called discrete multivariable first order lags.

Finally, in section 6.9 the 2D Lyapunov equation of chapter 4 has been used in the design of a current pass state feedback law with 'feedforward' of the previous pass output action, which is an example of a control action with memory. This leads to the 2D Lyapunov equation being used as a sufficient condition for closed loop stability along the pass.

The controller structures presented within this chapter are by no means exhaustive. Research into available control schemes for linear repetitive processes remains in its early stages, and only certain aspects of the general problem area have been addressed. Iterative learning control remains an application where the most progress has been made today in terms of the development of control schemes for differential and discrete processes - see, for example, (Amann et al., 1996) for feedback and feedforward actions or (Amann et al., 1998) for 2D predictive control. In terms of repetitive processes, this chapter and other work (eg. (Rogers and Owens, 1993; Rogers and Owens, 1995a)) has demonstrated the potential strength of feedback control structures. In addition, the relative simplicity of the schemes implies that their potential should be fully investigated prior to the consideration of those with a more complex structure. An open area where future research effort should be directed is the development of meaningful optimal control policies for linear repetitive processes. This is discussed further in the final chapter of this thesis.

Chapter 7

Conclusions and Further Work

The aim of this thesis has been to extend the existing systems theory for linear repetitive processes with a constant pass length α. Within this chapter each of the major areas covered are discussed including a summary of what has been presented, novel contributions made and ideas on how the work performed may be extended. Finally some directions for short to medium term future research are discussed.

7.1 Stability

Within chapter 3 the rigorous stability theory for linear repetitive processes with a constant pass length α developed by Rogers and Owens has been presented. The theory is based on an abstract model in a Banach space setting and includes all such processes as special cases, and hence provides a powerful general base for the control related study of these processes. In particular, asymptotic stability and stability along the pass results for the subclasses of differential and discrete processes have been given, which are the subject of this thesis.

Despite its apparent weakness, there are cases where asymptotic stability is all that is needed for acceptable systems performance, see for example (Amann et al., 1996; Owens and Rogers, 2000), or indeed all that is achievable, see for example (Roberts, 1996; Roberts, 2000), but in the majority of cases it is the stronger condition of stability along the pass which is required.

The problem of testing a differential/discrete process for stability along the pass reduces to three conditions which can all be tested via well known 1D linear systems stability techniques (and hence can be implemented into a computer aided analysis environment). These tests however provide no really 'useful' information concerning expected systems performance and, in particular, about the behaviour of the output sequence of pass profiles as the process evolves from pass to pass. In addition, the third condition for both subclasses involves the computation of the eigenvalues of a potentially high dimensioned transfer-function matrix for all points on the unit circle in the complex plane for the discrete case and the imaginary axis of the complex plane for the differential case, which may result in a very high computational load.

With this motivation, simple structure stability tests have been presented in section 3.6 which, for the discrete subclass of processes, replace the computational complexity of this final stability along the pass condition with the one-off computation of the eigenvalues of a matrix with constant entries. This work is novel and forms the basis for the paper (Benton et al., 1998b). The analysis exploits the basic properties of nonnegative matrices and provides alternative sufficient conditions for stability along the pass. Although the sufficient nature of the tests means that the results will not produce a conclusive result for all examples, they act as a simple low-computational load 'acceptance' criterion in some cases. To offset this conservativeness, the tests provide, at no extra computational cost, strong information on performance along a given pass, which is not available from the Nyquist-like characterisations of stability along the pass. Similar results to the above have been presented for systems described by the Roesser / Fornasini-Marchesini 2D state-space models, but there are no Roesser / Fornasini-Marchesini alternatives possible for these performance measures.

In chapter 2 it is highlighted how the boundary conditions $x_{k+1}(0) = d_{k+1}$, $k \geq 0$, are sometimes not strong enough to adequately model the process dynamics. With this motivation, so-called dynamic boundary conditions have been proposed in (Owens and Rogers, 1999) for the differential case (and (Rogers et al., 1998) for the discrete case). In the same paper it has been shown how the dynamic boundary condition term effects the bounded linear operator which governs the process dynamics, and hence the stability of the process is affected. In fact, the incorrect modelling of boundary conditions could lead to an asymptotically unstable process being misinterpreted as asymptotically stable. For the differential subclass of processes with dynamic boundary conditions the asymptotic stability result can no longer be tested by using well known 1D systems theory techniques. The problem of developing computationally efficient stability tests here is still an open problem. For a certain subclass of discrete processes with dynamic boundary conditions, however, the resulting conditions can be tested for using 1D techniques. The route is via a 1D equivalent linear systems state-space model of the process dynamics and further details of this and the issues arising here due to the inclusion of dynamic boundary conditions can be found in, for example, (Galkowski et al., 2000).

Discrete linear repetitive processes have strong structural links with 2D discrete linear systems described by the Roesser and Fornasini-Marchesini state-space models. Several key differences exist however. Repetitive processes are uniquely characterised by a finite pass length - this is the key distinction between these processes and the classes of continuous-discrete and discrete-discrete systems reported in the literature. Another point to note is that not all linear repetitive processes have an equivalent Roesser / Fornasini-Marchesini state-space model interpretation (such as processes with interpass smoothing) - hence linear repetitive processes are not, in general, a subclass of 2D systems having a Roesser / Fornasini-Marchesini type dynamic representation. For these reasons, the well developed 2D linear systems theory cannot be directly applied here, such as what is meant (if anything) by controllability for these processes. However, it is still feasible to exploit such theory for examples for which a Roesser / Fornasini-Marchesini interpretation of the process dynamics exists. For example, chapter 2 includes Roesser / Fornasini-Marchesini interpretations of the dynamics of a subclass of discrete processes. The Fornasini-Marchesini model presented is singular, however, but it is concluded in (Galkowski et al., 1999b) that the singularity is not an intrinsic feature of the process.

Chapter 3 includes stability results obtained from well known 2D theory using these representations of the process dynamics as starting points. The 2D systems interpretations have led to the following advances in terms of systems theory for discrete linear repetitive processes:

(i) for the standard (nonsingular) model, a formal equivalence has been shown to exist between stability along the pass and the BIBO stability of the Roesser (and therefore Fornasini-Marchesini) interpretations; and

(ii) the singular model has led to the development of a transition matrix (or fundamental matrix sequence) and hence a general response formula which leads to a characterisation of certain reachability/controllability properties.

These conclusions are drawn for the simplest boundary conditions case, but a generalisation to the case of dynamic boundary conditions should be possible (Owens and Rogers, 2000), and this stands as a short term future research area.

Finally, in this chapter, a Volterra approach to the stability analysis of discrete linear repetitive processes has been introduced. The powerful theory of the Volterra operator has only recently started to be applied to the area of linear repetitive processes and so, as yet, no conclusive appraisal of the approach can be made. It is anticipated, however, that this route will have a major role to play in the onward development of a mature systems theory for linear (and nonlinear) repetitive processes, and hence is an area for short to medium term research. In particular, the approach has already been used (Dymkov et al., 2000) to produce significant new results on controllability for these processes.

7.2 Lyapunov Equations

As a result of the equivalence between standard 2D stability concepts and the stability along the pass of certain subclasses of discrete linear repetitive processes, it is natural to consider the application of well known 2D techniques. Within chapter 4, the question of to what extent a Lyapunov equation based approach to the stability analysis of linear repetitive processes is applicable has been addressed. The aim here is to give a suitable extension to existing 1D theory and provide an alternative route to obtaining performance prediction information than the time-domain (also termed simulation-based) tests of chapter 3. A review of the literature reveals that, for 2D linear systems described by the Roesser / Fornasini-Marchesini state-space models, essentially two different types of equation have been considered:

(i) the 1D Lyapunov equation, so-called because the equation has an identical structure to that for discrete linear time-invariant systems, but with matrices which are functions of a complex variable; and

(ii) the so-called 2D Lyapunov equation, defined in terms of matrices with constant entries.

Initially the 1D equation approach has been considered. The necessary and sufficient stability along the pass conditions presented are implemented via computations on matrices with constant entries and provide an alternative to the Nyquist-like stability along the pass tests of chapter 3. In addition, the test produces at no extra computational cost performance measures in the form of computable information concerning the convergence of the output sequence of pass profiles under stability along the pass to the resulting limit profile (for which, for the discrete subclass, there are no Roesser / Fornasini-Marchesini alternatives). The 1D Lyapunov equation approach, however, has not been useful in providing measures of robustness such as stability margins, which have been discussed in chapter 5.

A discussion on methods of solving the 1D Lyapunov equation has been provided in section 4.3. Here it has been shown how the solution of the 1D Lyapunov equation (and hence stability tests only involving computations on matrices with constant entries) in the general case requires the use of the Kronecker product for matrices. The solution involves the requirement that a Hermitian matrix $P(s)$ evaluated on the imaginary axis, $s = i\omega$, satisfies the so-called axis positivity property of Šiljak (Šiljak, 1971).

Within section 4.4, a 1D Lyapunov equation has been developed for a subclass of differential linear repetitive processes possessing a special structure of dynamic boundary conditions, which is of particular interest in terms of links with delay-differential systems and also repetitive control schemes. The analysis presented here provides the basis for the papers (Benton et al., 2000c) and (Benton et al., 2000d). In chapter 3 it was highlighted how the first condition of the test for stability along the pass (i.e. the asymptotic stability condition) for processes with dynamic boundary conditions cannot be tested using well known 1D linear systems techniques. Thus the aim of the analysis here has been to develop a 1D Lyapunov equation interpretation of this condition for differential processes with this special class of dynamic boundary conditions, and hence supplying a computationally viable testing method. Strict positive realness based tests have been given for the new stability conditions which indicate how the condition is equivalent to testing for positive realness of a certain 1D rational transfer-function matrix. Hence a 1D characterisation of stability along the pass has been obtained for this subclass of differential processes.

The 2D Lyapunov equation differs from the 1D case in that it provides sufficient but not necessary conditions for stability along the pass (except in a number of special cases - see the text for the details). The analysis given on this approach is presented in (Benton et al., 1999) and subsequently extended in (Benton et al., 2000a). The analysis uses the theory and results of strictly bounded real matrices, and in particular uses the bounded real lemma. A counter-example is given which shows that there exists stable along the pass processes for which no solution pair $\{W, Q\}$ to the 2D Lyapunov equation exists. Despite this conservativeness, the 2D Lyapunov equation has a (potentially) major role to play in the analysis of discrete linear repetitive processes in terms of the provision of strong computable performance information (see section 4.9) for a given pass, and in providing a starting point in the evaluation of stability margins and robust stability theory, which have been discussed in chapter 5.

In section 4.7, a 2D Lyapunov equation has been derived for a class of discrete processes using a Fornasini-Marchesini representation of the process dynamics as a starting point. The resulting new sufficient stability along the pass conditions involve the computation of generalised eigenvalues. Two new algorithms for giving a positive definite solution pair to the 2D Lyapunov equation have been introduced. In both, the equation has been reduced to solving simultaneously three expressions. The first algorithm is based on the use of spectral factorisation and utilises well known 1D methods at each step. The second algorithm replaces the use of spectral factorisation with the need to solve a Riccati-type equation to determine W_1.

Finally, it should be noted that the analysis presented using the 2D Lyapunov equation approach has only been for the discrete subclass of processes. The development of a 2D Lyapunov equation for the differential subclass of processes remains an open area.

7.3 Robustness

Within chapter 5 an initial investigation into the area of robust stability theory for linear repetitive processes has been made. When analysing a process it is important to not only determine stability, but also obtain some indication of how robust the process is to perturbations in the system. In particular, within this thesis, the subject areas of allowable parameter variation bounds and stability margins have been investigated.

Given a stable along the pass discrete linear repetitive process, the first of these areas considers how the process stability is affected by perturbations within the process system matrices, which may arise due to model inaccuracy or measurement noise for example. Two different types of perturbation in the matrices which define the state-space model have been looked at:

(i) structured, where the perturbation model structure and bounds on the individual elements of the perturbation matrix are known; and

(ii) unstructured, where at most a spectral norm bound on the perturbation is known.

The aim of the analysis here then has been to find methods of determining the minimum norm of the matrix $\Delta\Phi$ such that the perturbed process remains stable along the pass. A discussion of some of the methods available for determining this exact bound have been given - in many cases, however, a good lower bound often suffices. In section 5.4, a Lyapunov approach to finding lower bounds for this exact minimum norm bound has been presented. The analysis uses the existence of a positive definite solution pair to the 2D Lyapunov equation as a starting point, and hence is an application area of part of the analysis of chapter 4. In addition, the availability of these robust stability measures using the 2D Lyapunov equation offsets some of the inherent conservativeness due to the equations sufficient but not necessary nature.

It should be noted that the bounds obtained

(i) are not available for all stable along the pass discrete processes (since there exist processes which are stable along the pass and for which there exists no solution to the 2D Lyapunov equation), and

(ii) depend explicitly on the matrices W and Q which provide the solution to the 2D Lyapunov equation.

It has been shown in section 5.8 that the least conservative lower bound corresponds to the minimum norm of the augmented plant matrix Φ.

Finally in this section, robustness analysis has been presented using a Fornasini-Marchesini representation of the process dynamics as a starting point. Different bounds are obtained, and clearly to fully exploit these results, the least conservative set for a particular example should be used.

Clearly further development is needed here, in particularly in terms of the development of alternative approaches and on comparing these bounds in terms of conservativeness and related factors.

Stability margins provide an indication as to what extent the singularities of a system may be 'moved' before the process becomes unstable. Given a stable along the pass process, the stability margin has been defined as the shortest distance between the singularities of the system and the boundary of the stability region - for discrete linear repetitive processes, this is the boundary of the unit bidisc. Then a necessary and sufficient condition for stability along the pass of these processes is that this measure, σ say, is greater than zero.

Different methods for evaluating these stability margins have been discussed, and once again it has been shown how a 2D Lyapunov equation approach can be used to obtain good lower bounds for the margins.

A valid criticism on the work to date on stability margins has been the lack of a 'transparent' link to resulting systems performance. With this motivation, in section 5.9 some very recent results on the definition of a pole of a multidimensional system using the behavioural approach have been interpreted for a subclass of discrete linear repetitive processes. Here a pole has been defined as an element of $\mathbb{C}^2$ space which is a zero of the characteristic polynomial $\rho(z_1, z)$ of the process. The potential strength of this approach is that the poles can be interpreted in terms of so-called exponential trajectories of the process which, in the case of discrete linear repetitive processes, have a well defined physical meaning. In effect, these exponential trajectories form the 'building blocks' of the process dynamics, and hence this has major implications regarding the analysis of these processes. In particular, it is anticipated that the application of this approach to stability margin analysis will result in a 'transparent' link to expected systems performance. Hence this highly promising area is one in which immediate future research effort should be directed.

7.4 Controller Structures

The unique control problem associated with linear repetitive processes is that the output sequence of pass profiles can contain oscillations which increase in amplitude from pass to pass. This behaviour can be seen in the long-wall coal cutting example via the presence of severe undulations in the newly cut coal floor wall which have to be removed manually, and hence this is a key reason behind the 'stop/start' typical cutting pattern in a working coal mine.

In (Smyth, 1992; Smyth et al., 1994) objectives for the control of linear repetitive processes have been formulated, together with the development of design algorithms. Here we have concentrated on the specification of controller structures for these processes, which can be classified under the two general headings:

(i) memoryless controllers, which explicitly use information from the current pass only; and

(ii) so-called controllers with memory which explicitly use information from the current pass and/or previous pass profiles, state vectors and input vectors.

Memoryless schemes clearly have the simpler structure in terms of implementation and of data which must be logged/stored and hence the initial work in this area has concentrated on such schemes. Differential and discrete linear repetitive processes have strong structural links with 1D differential and discrete linear systems. This raises the natural question of what can be achieved using standard 1D feedback control schemes. Such schemes use data from the current time instant on the current pass only and as such are termed current point controllers.

Section 6.2 has introduced current pass state feedback and output feedback control laws. It has been shown that linear repetitive processes are closed under such control actions, and hence known stability theory may be applied. It is shown here that the property of asymptotic stability is invariant under memoryless state and output feedback, i.e. an asymptotically unstable system cannot be stabilised by a memoryless multipass causal feedback control scheme. This is due to the facts that, under all multipass causal feedback control schemes,

(i) asymptotic stability only depends on D_1, and

(ii) the output $y_{k+1}(t)$ does not explicitly depend on the input $u_{k+1}(t)$ on a given pass - i.e. there is no 'direct feedthrough' of the input to the output.

How to overcome this problem remains an open area for future research. For now the argument is used that asymptotic stability is practically inherent. Note that the state feedback control schemes can in general only be implemented with an observer structure. Observer theory for linear repetitive processes remains an open research area.

Valuable insight into the general area of controller design can be gained by studying subclasses of processes with certain special structural properties - so-called benchmark problems. The application of purely 1D control actions tend to fail, except in a few certain special cases, since the process dynamics depend explicitly on two independent variables. Here, however, it is shown how for one subclass of practical interest (so called multivariable first order lags) a 1D control action is all that is required for acceptable systems performance under certain requirements on gain. The work presented here is novel and can be found in (Benton et al., 1998a). The analysis replaces the necessary and sufficient stability along the pass condition on gain with a sufficient only alternative. To offset this potential conservativeness, strong information on performance along a given pass is available from the tests at no extra computational cost, which is not available from the Nyquist-like characterisations of stability along the pass. Two refinements to this analysis have also been presented which extend the scope of application of the theory.

When one or more of the control objectives cannot be met by a current pass controller, a way forward is to look at controllers with memory. Within section 6.6, a so-called memoryless linear state feedback law with proportional repetitive minor loop compensation has been presented and applied to a class of benchmark problem where it has been shown, in this case, to give a solution to the so-called repetitive systems disturbance decoupling with stability problem.

In section 6.8 discrete processes have been considered. Here it has been shown how a discrete process can be regarded as being derived from a differential process under fast sampling conditions. The analysis presented here can be found in (Benton et al., 2000b).

Finally within this chapter, the 2D Lyapunov equation of chapter 4 has been used in the design of a current pass state feedback law with 'feedforward' of the previous pass output action, which is an example of a control action with memory. This provides an additional application area of the 2D Lyapunov equation analysis of chapter 4, thus offsetting part of its overall conservative nature.

The controller structures presented in this chapter do not provide an exhaustive list.

Research into available control schemes for these processes remains in its early stages and only certain aspects of the general problem have been addressed. Clearly much future work must be performed before a realistic assessment of available techniques can be made. Iterative learning control remains an application area where the most progress has been made to date in terms of the development of control schemes for differential and discrete processes, and is one area where current research effort is being focussed.

As a final point, the development of optimal control schemes for linear repetitive processes remains open. In (Jones and Owens, 1981) an initial attempt at the numerical optimisation of multipass processes was given, but little progress has been made since, leaving this subject open for future research.

7.5 Final Remarks

Before concluding this thesis, a few final remarks should be made. Firstly note that alternative approaches to the analysis of repetitive processes are also possible. For example, in (Johnson et al., 1996), analysis generalising the Rosenbrock systems matrix theory (Rosenbrock, 1970) for these processes has been performed. Similarly, there is much scope for the use of the behavioural approach here, as noted in the robustness section above.

Finally, the subject of the implementation of the stability tests of chapter 3 and controller design algorithms of chapter 6 into a computer aided design environment is beyond the scope of this thesis. This is the subject of an ongoing research programme into the development and design of MATLAB toolboxes by Gramacki et al., see for example (Gramacki et al., 1999).

Appendix A

Background Results and Theory

A.1 Some Results from Functional Analysis and the Theory of Matrices

The analysis within this thesis uses results from the theory of matrices and functional analysis which are summarised below. The proofs of the results can be found in any relevant text and so are omitted.

Definition A.1 (Spectral Value, Spectrum, Spectral Radius) *A complex number λ is said not to be a spectral value of L_α if, and only if, the bounded linear operator $\lambda I - L_\alpha$, where I is the identity operator in E_α, has range dense in E_α and a bounded inverse $(\lambda I - L_\alpha)^{-1}$. Then the set $\sigma(L_\alpha)$ of all spectral values of L_α is called the spectrum of L_α and its spectral radius is defined to be the finite positive number*

$$r(L_\alpha) := \sup_{\lambda \in \sigma(L_\alpha)} |\lambda| \tag{A.1}$$

or equivalently

$$r(L_\alpha) = \lim_{k \to +\infty} ||L_\alpha^k||^{1/k}. \tag{A.2}$$

If E_α is finite dimensional, then L_α can be represented by a complex square matrix. Then $r(L_\alpha)$ is the maximum of the moduli of the eigenvalues.

Definition A.2 (Partial Ordering, Nonnegative Matrix) *The partial ordering $\leq$ on $n_1 \times n_2$ matrices is defined by the relation*

$$A \leq B, \quad \text{if, and only if,} \quad A_{ij} \leq B_{ij} \;\; \forall\; i, j. \tag{A.3}$$

Further, the absolute value of an $n_1 \times n_2$ matrix A is defined to be the $n_1 \times n_2$ real, or so-called nonnegative, matrix

$$||A||_p = \begin{bmatrix} |A_{11}| & |A_{1n_2}| \\ & \\ |A_{n_1 1}| & |A_{n_1 n_2}| \end{bmatrix}. \tag{A.4}$$

Lemma A.1 (Properties of Nonnegative Matrices) *The absolute value, $||A||_p$, of an $n_1 \times n_2$ matrix A has the following 'norm-like' properties,*

(a) $||A||_p \geq 0$,

(b) $||\gamma A||_p = |\gamma|\; ||A||_p$, *for all complex scalars* γ,

(c) If B is another $n_1 \times n_2$ matrix then $||A + B||_p \leq ||A||_p + ||B||_p$,

(d) If B is another matrix compatible for pre-multiplication by A then $||AB||_p \leq ||A||_p ||B||_p$,

(e) If A and B are square matrices then $0 \leq ||A||_p \leq B \Rightarrow r(A) \leq r(||A||_p) \leq r(B)$.

Lemma A.2 *If A is an $n_1 \times n_1$ matrix then $(I_{n_1} - ||A||_p)^{-1}$ exists and is nonnegative if, and only if,*

$$r(||A||_p) < 1. \tag{A.5}$$

Definition A.3 (Absolute Value of a Vector) *Let X be a Banach space and X^d its d^{th} Cartesian product regarded as the linear vector space of columns $X = (x_1,\, x_2,\, \cdots,\, x_d)^T$ of elements of X. Then the absolute value of $x \in X^d$ is defined as*

$$||x||_p = (||x_1||,\, ||x_2||,\, \cdots,\, ||x_d||)^T \in \mathbb{R}^d \tag{A.6}$$

where $||\cdot||$ *denotes the norm in* X. *Further the norm in* $\mathbb{R}^q$ *is defined as*

$$||x||_q = \max_{1\leq i\leq q} |x_i| \tag{A.7}$$

where $x \in \mathbb{R}^q$ *is regarded as the column* $x = (x_1, x_2, \cdots, x_q)^T$, *and the norm in* X^d *is defined as*

$$||x|| = \max_{1\leq i\leq d} ||x_i||. \tag{A.8}$$

Definition A.4 (Absolute Value of an Operator) *Let* $B(X^{d_2}, X^{d_1})$ *denote the space of bounded linear operators mapping* X^{d_2} *into* X^{d_1}. *Further, represent* $L \in B(X^{d_2}, X^{d_1})$ *as*

$$Y = Lx \tag{A.9}$$

or

$$y_i = \sum_j L_{ij} x_j \tag{A.10}$$

where the L_{ij} *are bounded linear operators in* X. *Then the absolute value of* L *is defined to be*

$$||L||_p = \begin{bmatrix} ||L_{11}|| & ||L_{1d_2}|| \\ & \\ ||L_{d_1 1}|| & ||L_{d_1 d_2}|| \end{bmatrix} \tag{A.11}$$

where $||\cdot||$ *is also used to denote the operator norm induced by the vector norm in* X.

Theorem A.1 (Banach-Steinhaus (Uniform Boundedness) Theorem) *(Kreyszig, 1978) Let* $\{T_n\}$ *be a sequence of bounded linear operators* $T_n : X \longrightarrow Y$ *from a Banach space* X *into a normed space* Y *such that* $\{||T_n x||\}$ *is bounded for every* $x \in X$, *say,*

$$||T_n x|| \leq c_x, \qquad n = 1, 2, \cdots \tag{A.12}$$

where c_x *is a real number. Then the sequence of the norms* $\{||T_n||\}$ *is bounded, that is, there is a* c *such that*

$$||T_n|| \leq c, \qquad n = 1, 2, \cdots . \tag{A.13}$$

Definition A.5 (Contraction) *Let $X = (X, d)$ be a metric space. A mapping $T : X \longrightarrow X$ is called a contraction on X if there is a positive real number $\beta < 1$ such that $\forall\, x, y \in X$,*

$$d(Tx, Ty) \leq \beta\, d(x, y). \tag{A.14}$$

Geometrically this means that any points x and y have images which are closer together than the points x and y.

Definition A.6 (Kronecker Product) *The Kronecker product of two matrices of appropriate dimensions takes the form*

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & \ddots & & \\ a_{m1}B & & \cdots & a_{mn}B \end{bmatrix}. \tag{A.15}$$

Definition A.7 (Positive Definiteness) *The matrix A is positive semidefinite, denoted $A \geq 0$, if the quadratic form $x^T Ax \geq 0\ \forall x$. If equality holds only when $x \equiv 0$, we say that A is positive definite, denoted $A > 0$. Note that because $2x^T Ax = x^T(A+A^T)x + x^T(A-A^T)x = x^T(A+A^T)x$, we usually assume that A is symmetric.*

Lemma A.3 (Properties of Positive Definite Matrices) *The following properties of positive definite matrices hold:*

(a) A symmetric matrix A is positive definite (semidefinite) if, and only if, all it's eigenvalues are positive (nonnegative);

(b) A is positive semidefinite if, and only if, it can be written in the factored form $A = TT^T$ for some matrix T, known as a square root of A;

(c) If A is positive definite, then all its principal submatrices are positive definite. In particular, all the diagonal entries are positive;

(d) If A is positive definite then the factorization $A = LDM^T$ exists and $D = diag(d_1, \cdots, d_n)$ has positive diagonal entries; and

(e) If $A \in \mathbb{R}^{n\times n}$ is positive definite, and $X \in \mathbb{R}^{n\times k}$ has rank k, then $B = X^T AX \in \mathbb{R}^{k\times k}$ is also positive definite.

Lemma A.4 (Cholesky Factorization) *If $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite, then there exists a unique lower triangular $G \in \mathbb{R}^{n\times n}$ with positive diagonal entries such that $A = GG^T$.*

Definition A.8 (Normal Matrices) *If for matrix $A \in \mathbb{R}^{n\times n}$*

$$A^T A = AA^T, \tag{A.16}$$

then A is said to be normal.

Lemma A.5 *$A \in \mathbb{C}^{n\times n}$ is normal if, and only if, there exists a unitary $Q \in \mathbb{C}^{n\times n}$ such that*

$$Q^* AQ = diag\{\lambda_1, \cdots, \lambda_n\} \tag{A.17}$$

where $\lambda_i,\ 1 \leq i \leq n$, are the eigenvalues of A.

A.2 A Formal Derivation of the 2D Transfer-Function Representation

In order to introduce a transfer-function matrix description for linear repetitive processes described by, say, the differential non-unit memory subclass of processes with the state-space representation (2.11)-(2.12), some formal definitions are first required. These definitions can be regarded as the natural generalisation of the associated 1D concepts from the well known differential/discrete linear systems theory and here just the main results are stated without proof. For a complete discussion (plus related proofs etc.) see, for example, (Rogers and Owens, 1992b; Rogers and Owens, 1989a).

Definition A.9 (z-Transform) *The 'z-transforms' of the sequences $u_{k+1}(t), x_{k+1}(t)$ and $y_{k+1}(t),\ 0 \leq t \leq \alpha,\ k \geq 0$, are defined by*

$$\begin{aligned} U(t,z) &= u_1(t) + z^{-1}u_2(t) + z^{-2}u_3(t)\cdots \\ X(t,z) &= x_1(t) + z^{-1}x_2(t) + z^{-2}x_3(t)\cdots \qquad \textit{and} \\ Y(t,z) &= y_1(t) + z^{-1}y_2(t) + z^{-2}y_3(t)\cdots \end{aligned} \tag{A.18}$$

respectively.

Results on the convergence and existence properties of the equations (A.18) are contained in the following result,

Lemma A.6 *Suppose that the terms in (A.18) are bounded in the sense that there exists real numbers* $M_i > 0,\ \lambda_i > 0,\ 1 \le i \le 3$, *such that*

$$\begin{aligned} \|u_k(\cdot)\| &\le M_1\lambda_1^{k-1},\ k \ge 1 \\ \|x_k(\cdot)\| &\le M_2\lambda_2^{k-1},\ k \ge 1 \quad \text{and} \\ \|y_k(\cdot)\| &\le M_3\lambda_3^{k-1},\ k \ge 1 \end{aligned} \tag{A.19}$$

where $\|\cdot\|$ *is chosen as any suitable norm in* E_α. *Then (A.18) converge absolutely in the regions* $|z| > \lambda_1,\ |z| > \lambda_2$ *and* $|z| > \lambda_3$ *respectively.*

Define $\frac{\partial}{\partial t}X(t,z)$ as

$$\frac{\partial}{\partial t}X(t,z) := \frac{\partial}{\partial t}x_1(t) + z^{-1}\frac{\partial}{\partial t}x_2(t) + z^{-2}\frac{\partial}{\partial t}x_3(t) + \cdots \tag{A.20}$$

and consider, without loss of generality, the special case of zero initial pass profiles and state initial conditions on each pass, i.e.

$$\begin{aligned} &y_{1-j}(t) = 0, \quad 0 \le t \le \alpha, \quad 1 \le j \le M, \\ &d_{k+1} = 0, \quad k \ge 0. \end{aligned} \tag{A.21}$$

Hence $X(0,z) = 0$, and the 'z-transform' of (2.11)-(2.12) in this case is easily shown to be

$$\begin{aligned} \frac{\partial}{\partial t}X(t,z) &= (A + B(z)(I_m - D(z))^{-1}C)\,X(t,z) + B\,U(t,z) \\ Y(t,z) &= (I_m - D(z))^{-1}C\,X(t,z) \end{aligned} \tag{A.22}$$

where

$$B(z) = \sum_{j=1}^{M} B_{j-1}z^{-j}, \quad D(z) = \sum_{j=1}^{M} D_j z^{-j} \tag{A.23}$$

and the term $(I_m - D(z))$ is always invertible since

$$\lim_{|z|\to+\infty}(I_m - D(z)) = I_m. \tag{A.24}$$

One method for solving for $X(t,z)$ (and hence for $Y(t,z)$) in (A.22)-(A.23) is to use a Laplace transform approach. The potential problem here is that the variables

$u_j(t)$, $x_j(t)$ and $y_j(t)$, $j \geq 1$, of the series $U(t,z)$, $X(t,z)$ and $Y(t,z)$ respectively are only defined on the finite interval $[0,\alpha]$. The use of the Laplace transform however, requires the variables to be defined over $[0,+\infty)$. This problem can be overcome by noting that, due to multipass causality, the result will be unaffected if the Laplace transform is applied to arbitrary extensions of the variables from $[0,\alpha]$ to $[0,+\infty)$ (provided, of course, that these extensions satisfy the necessary existence conditions).

Then, assuming that the variables $u_j(t)$, $x_j(t)$ and $y_j(t)$, $j \geq 1$, have been suitably extended from $[0,\alpha]$ to $[0,+\infty)$, the Laplace transforms can be defined as follows,

Definition A.10 (s-Transform) *The 's-transforms' of the series $U(t,z)$, $X(t,z)$ and $Y(t,z)$, $0 \leq t \leq \alpha$, $k \geq 0$, are defined by*

$$\begin{aligned} U(s,z) &= \mathcal{L}U(t,z) = \mathcal{L}u_1(t) + z^{-1}\mathcal{L}u_2(t) + z^{-2}\mathcal{L}u_3(t)\cdots \\ X(s,z) &= \mathcal{L}X(t,z) = \mathcal{L}x_1(t) + z^{-1}\mathcal{L}x_2(t) + z^{-2}\mathcal{L}x_3(t)\cdots \quad \textit{and} \\ Y(s,z) &= \mathcal{L}Y(t,z) = \mathcal{L}y_1(t) + z^{-1}\mathcal{L}y_2(t) + z^{-2}\mathcal{L}y_3(t)\cdots \end{aligned} \tag{A.25}$$

respectively, where $\mathcal{L}$ denotes the Laplace transform with respect to the along the pass variable t.

Results on the convergence and existence properties of the equations (A.25) are contained in the following lemma,

Lemma A.7 *Suppose that there exists real numbers $M_i > 0$, $\beta_i > 0$, $\lambda_i > 0$, $1 \leq i \leq 3$, such that*

$$\begin{aligned} \|u_j(t)\| &\leq M_1\, e^{\beta_1 t}\lambda_1^{j-1}, \ k \geq 1 \\ \|x_j(t)\| &\leq M_2\, e^{\beta_2 t}\lambda_2^{j-1}, \ k \geq 1 \quad \textit{and} \\ \|y_j(t)\| &\leq M_3\, e^{\beta_3 t}\lambda_3^{j-1}, \ k \geq 1 \end{aligned} \tag{A.26}$$

respectively, $j \geq 1$, $\forall\, t \geq 0$, where $\|\cdot\|$ denotes any suitable vector norm. Then the series of (A.25) converge absolutely in the regions $\{|z| > \lambda_1,\ Re\{s\} > \beta_1\}$, $\{|z| > \lambda_2,\ Re\{s\} > \beta_2\}$ and $\{|z| > \lambda_3,\ Re\{s\} > \beta_3\}$ respectively.

The results and definitions presented here are for the differential subclass of processes. Equivalent results for discrete processes are presented in (Rogers and Owens, 1992b) and, since the results generalise in a natural manner, the details are omitted here.

A.3 Mathematical Background for Simulation-Based Stability Tests

This section introduces the background results necessary for the simulation-based stability tests and subsequent analysis of section 3.5. Further details of these results can be found in, for example, (Owens and Chotai, 1983) and (Rogers and Owens, 1990b) and the relevant references therein. The section begins with the following result, known as the total variation lemma,

Lemma A.8 (Total Variation Lemma) *Suppose that $g \in L_1(0,T)$, d is a real scalar and*

$$f(t) := d + \int_0^t g(\tau)d\tau \tag{A.27}$$

is bounded and continuous on the infinite open interval $0 < t < +\infty$ with local maxima and minima at times $t_1 < t_2 < \cdots$ satisfying $\sup t_j = +\infty$ in the extended half-line $t > 0$. Then with $t_0 = 0$,

$$N_T(f) = |d| + \int_0^T |g(t)|dt \tag{A.28}$$

where

$$N_T(f) := |f(0+)| + \sum_{k=1}^{k^*} |f(t_k) - f(t_{k-1})| + |f(T) - f(t_{k^*})|, \tag{A.29}$$

k^* is the largest integer k such that $t_k < T$, and

$$N_\infty(f) := \sup_{T \geq 0} N_T(f). \tag{A.30}$$

The quantity $N_T(f)$ is the norm of f regarded as a function of the bounded variations on the half-open interval $0 < t \leq T$ (for each function f). Hence it is termed the total variation of f. $N_T(f)$ is a continuous function of T and is monotonically increasing. Hence $N_\infty(f)$ can be obtained as

$$N_\infty(f) = \lim_{T \to +\infty} N_T(f). \tag{A.31}$$

Further, $N_T(f)$ can easily be computed from simple graphical operations on $f(t)$ as can been seen via figure A.1.

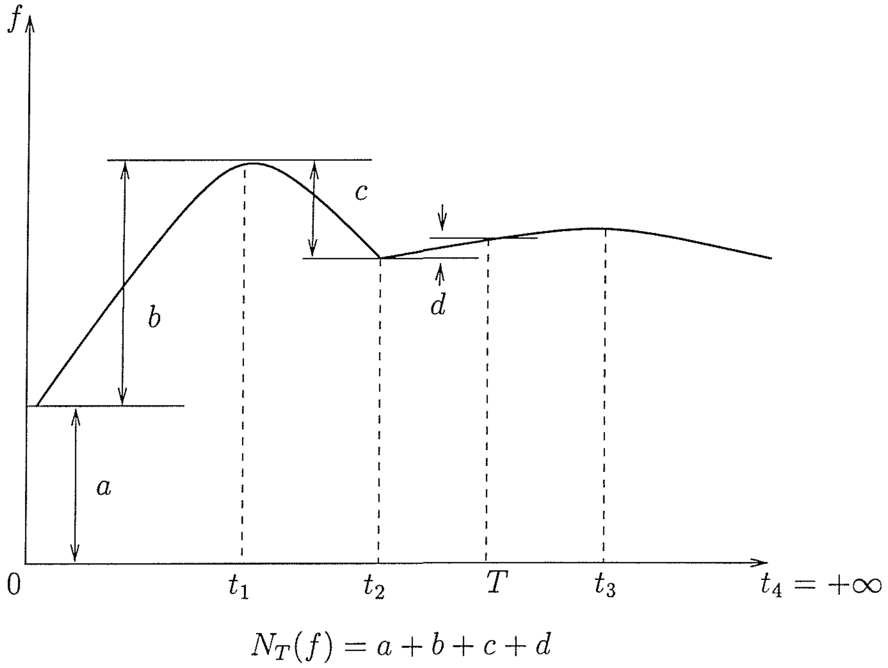

Figure A.1: $N_T(f)$ - The Total Variation of f

These operations can be implemented into a CAD environment (for further details see (Smyth, 1992)). Note also that

$$\lim_{T \to +\infty} |N_\infty(f) - N_T(f)| = 0, \tag{A.32}$$

and consequently $N_\infty(f)$ can be accurately estimated using data on a 'long enough' time interval $0 < t < T$.

Note that an equivalent discrete result to lemma A.8 can be found in (Rogers and Owens, 1992b).

The following analysis requires some basic results from the theory of nonnegative matrices which are reviewed in the appendix section A.1. In particular, use will be made of the special case $X = L_\infty(0, +\infty)$ in definitions A.3 and A.4 as follows,

Definition A.11 (Extended Space, Natural Projection) *The extended space of $X^d = L_\infty^d(0, +\infty)$ is denoted by X_e^d. Further, the natural projection of $L \in X_e^d$ into $X_{(0,T)}^d = L_\infty^d(0, T)$ regarded as a subspace of X^d is denoted by $P_T L$.*

Lemma A.9 *Consider $L \in B(X^{d_2}, X^{d_1})$ and suppose that its elements L_{ij} have the convolution form*

$$(L_{ij}x_j)(t) = d_{ij}x_j(t) + \int_0^t H_{ij}(t-\tau)x_j(\tau)d\tau. \tag{A.33}$$

Then $P_T L_{ij}$ has induced norm

$$||P_T L_{ij}|| = |d_{ij}| + \int_0^T |H_{ij}(\tau)|d\tau \tag{A.34}$$

in $L_\infty(0,T)$.

Further, use will be made of the following results,

Lemma A.10 *Suppose that $L \in B(X^{d_2}, X^{d_1})$ has elements of the form (A.33) and denote the step response matrix of L by $Q(t)$ with elements $Q_{ij}(t)$. Then*

$$||P_T L_{ij}|| = N_T(Q_{ij}), \quad 1 \le i \le d_1, \ 1 \le j \le d_2, \ \forall \ T > 0, \tag{A.35}$$

and hence

$$||P_T L||_p = \begin{bmatrix} N_T(Q_{11}) & \cdots & N_T(Q_{1d_2}) \\ \vdots & \ddots & \\ N_T(Q_{d_1 1}) & & N_T(Q_{d_1 d_2}) \end{bmatrix} \quad \forall \ T > 0. \tag{A.36}$$

Theorem A.2 *Suppose that the elements of $L \in B(X^{d_2}, X^{d_1})$ have the structure of (A.33). Then $\forall \ T > 0$,*

$$\begin{aligned} ||P_T L|| &= ||(||P_T L||_p)|| = \max_{1 \le i \le d_1} \sum_{j=1}^{d_2} N_T(Q_{ij}) \\ &\le ||L|| = ||(||P_\infty L||_p)|| = \max_{1 \le i \le d_1} \sum_{j=1}^{d_2} N_\infty(Q_{ij}). \end{aligned} \tag{A.37}$$

A.4 Two-Dimensional Systems : A Review of Basic Concepts

This section introduces some of the well established theory for 2D linear systems. For a comprehensive treatment see, for example, (Dudgeon and Mersereau, 1984).

Initially we introduce the following partial ordering scheme for ordered pairs of integers (i, j), with $i \geq 0,\ j \geq 0$,

$$\begin{aligned}(h,k) &\leq (i,j) \text{ if, and only if, } h \leq i \text{ and } k \leq j\\ (h,k) &= (i,j) \text{ if, and only if, } h = i \text{ and } k = j\\ (h,k) &< (i,j) \text{ if, and only if, } (h,k) \leq (i,j) \text{ and } (h,k) \neq (i,j).\end{aligned} \tag{A.38}$$

Then a two-dimensional linear shift-invariant system, in general, can be described by a convolution of the input $u(m, n)$ and the impulse response function $h(m, n)$. Here, however, it is only necessary to consider initially the special case of scalar systems whose input/output map is described by the recursive structure

$$y(m,n) = \sum_{k=0}^{K}\sum_{l=0}^{L} a(k,l)\, u(m-k, n-l) - \sum_{i=0}^{I}\sum_{j=0}^{J} b(i,j)\, y(m-i, n-j), \tag{A.39}$$

for $(i, j) \neq 0$. This difference equation describes, in effect, a quarter plane 2D digital filter, which is said to be spatially causal over the quadrant $(i, j) > 0$ since $y(m, n)$ depends only on input and output variables at points $(i, j) \leq (m, n)$.

Applying the 2D z-transform to (A.39) (where, using 2D systems convention, z_1 and z are regarded as 'backwards' shift operators) yields the 2D transfer-function matrix description

$$G(z_1, z) = \frac{A(z_1, z)}{B(z_1, z)}, \tag{A.40}$$

where

$$A(z_1,z) = \sum_{k=0}^{K}\sum_{l=0}^{L} a(k,l) z_1^k z^l, \quad B(z_1,z) = \sum_{i=0}^{I}\sum_{j=0}^{J} b(k,l) z_1^i z^j, \tag{A.41}$$

and, for notational simplicity, we take $b(0, 0) = 1$.

Now, since $b(0, 0) = 1$, $B(z_1, z) \neq 0$ in some neighbourhood U_ϵ^2 of $(0, 0)$, where

$$U_\epsilon^2 := \{(z_1, z) : |z_1| < \epsilon,\ |z| < \epsilon\}. \tag{A.42}$$

Hence, in U_ϵ^2, the function $G(z_1, z)$ is analytic and has the power series expansion

$$G(z_1, z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h(m,n) z_1^m z^n. \tag{A.43}$$

As in the 1D case, the only truly useful systems are those which are stable. We say that a system is BIBO (bounded input / bounded output) stable if its output sequence remains bounded whenever its input sequence is bounded. Then the linear shift-invariant 2D system (A.39) is said to be BIBO stable if, and only if,

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|h(m,n)| < +\infty, \tag{A.44}$$

i.e. if the impulse response is absolutely summable.

At this stage, a fundamental difference between 1D and 2D systems theory should be noted (which is one of the major reasons why, in the analysis of 2D systems, a simple extension of the 1D results is often incorrect). Given two functions $P(p)$ and $Q(p)$, consider $\frac{P(p)}{Q(p)}$. Then when the dimension of $p > 1$, even if $P(p)$ and $Q(p)$ are relatively prime, their zero sets may intersect resulting in a 'bad' type of singularity called a nonessential singularity of the second kind. This type of singularity is only encountered in systems of dimension ≥ 2 and has no one-dimensional counterpart. Note that a zero of $Q(p)$ which is not simultaneously a zero of $P(p)$ is called a nonessential singularity of the first kind (which is analogous to a pole of a 1D system). In (Goodman, 1977) it has been shown (via clever counter-examples) that the existence of nonessential singularities of the second kind on the boundary of the unit polydisc in the z-plane can cause problems. This has the unexpected result in that the stability problem is influenced not only by the denominator polynomial but also by the numerator polynomial. For the remainder of this section, however, this problem is avoided by assuming that $A(z_1, z)$ and $B(z_1, z)$ are mutually coprime and have no nonessential singularities of the second kind.

The following then is the basic result for BIBO stability of systems given by (A.39) due to Shanks,

Lemma A.11 (Shanks BIBO Stability Test) *(Shanks et al., 1972) The 2D system (A.39) with 2D transfer-function matrix $G(z_1, z)$ of (A.40) is BIBO stable if, and only if,*

$$B(z_1, z) \neq 0, \quad |z_1| \leq 1, \;\; |z| \leq 1. \tag{A.45}$$

Since this result is computationally intensive to check, lemma A.11 cannot be tested in all but a very few simple cases. This problem can be overcome, however, by using the following equivalent standard result due to Huang,

Lemma A.12 (Huang BIBO Stability Test) *(Huang, 1972) The 2D system (A.39) with 2D transfer-function matrix (A.40) is BIBO stable if, and only if,*

(a)

$$B(z_1, 0) \neq 0 \quad \forall \ |z_1| \leq 1, \tag{A.46}$$

and

(b)

$$B(z_1, z) \neq 0 \quad \forall \ |z_1| = 1, \ \ |z| \leq 1. \tag{A.47}$$

Note that the conditions of lemma A.12 are interchangeable in terms of z_1 and z.

A.5 Some Properties of the Volterra Operator

Within this section, some properties of the Volterra operator are established which are required for the analysis presented in section 3.9. For proofs of the results see, for example, (Dymkov et al., 1999).

Let E be a finite dimensional normed linear space over the complex field $\mathbb{C}$ with norm $||\cdot||_E$ and let $\mathbb{Z}_+$ be the set of nonnegative integers. Also let $S(\mathbb{Z}_+, E)$ be the linear space of all sequences on E, i.e. the functions $f : \mathbb{Z}_+ \longrightarrow E$. Then $S(\mathbb{Z}_+, E)$ is a locally convex Hausdorff topological space when equipped with the topology of uniform convergence on finite sets, i.e. the family of neighbourhoods is defined as

$$U_{M,\epsilon} = \{f : f \in S(\mathbb{Z}_+, E), \ ||f(k)||_E < \epsilon, \ k \in N\} \tag{A.48}$$

where N is the set of all finite subsets from $\mathbb{Z}_+$, and ϵ ranges over the set $\mathbb{R}_+$ of all positive real numbers.

Suppose now that $B(\mathbb{Z}_+, E)$ denotes the subspace of $S(\mathbb{Z}_+, E)$ of all bounded functions, i.e. $f : \mathbb{Z}_+ \longrightarrow E$ such that $\sup_{k\in\mathbb{Z}_+} ||f(k)||_E < +\infty$. Then it is a standard fact that $B(\mathbb{Z}_+, E)$ is dense in $S(\mathbb{Z}_+, E)$ with respect to the topology of uniform convergence over finite sets. Also $B(\mathbb{Z}_+, E)$ is a Banach space under a suitable norm definition, eg. $||f|| = \sup_{k\in\mathbb{Z}_+} ||f(k)||_E$.

The Volterra operator used within this thesis, $V_0 : B(\mathbb{Z}_+, E) \longrightarrow B(\mathbb{Z}_+, E)$, is defined by

$$(V_0 f)(s) := \sum_{i=0}^{s} A_i f(s-i), \quad s \in \mathbb{Z}_+ \tag{A.49}$$

where $A_i : E \longrightarrow V$, $i \in \mathbb{Z}_+$, are given linear operators. Operators of this form are known as discrete Volterra operators or shift operators of the second type.

Suppose now that there exists some fixed bases in E and V. Then the linear operators A_i, $i \in \mathbb{Z}_+$, can be interpreted as matrices on the complex field $\mathbb{C}$. Also associate with each function $x \in B(\mathbb{Z}_+, E)$ the analytic function $x(z)$ defined by the power series

$$x(z) = \sum_{i=0}^{\infty} x_i z^i \tag{A.50}$$

which converges in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Then it can easily be shown that the mapping $x \longrightarrow x(z)$ is bijective.

Now associate with each Volterra operator V_0 its representation $V_0(z)$ in the ring of power series defined by

$$V_0(z) = \sum_{i=0}^{\infty} A_i z^i, \quad z \in \mathbb{C}. \tag{A.51}$$

Then the mapping $V_0 \longrightarrow V_0(z)$ is injective between operators on the form (A.49) and the set of formal matrix series whose members are of the form (A.51), and it can easily be seen that the matrix function $V_0(z) : E \longrightarrow V$ is a linear map for each $z \in U$.

Suppose now that the matrices A_i, $i \in \mathbb{Z}_+$, are such that the power series (A.51) converges in some domain which contains the unit disc U. Then it follows immediately that

$$\sum_{i=0}^{\infty} ||A_i|| < +\infty, \tag{A.52}$$

and hence, for each function $f \in B(\mathbb{Z}_+, E)$,

$$\left\| \sum_{i=0}^{s} A_i f(s-i) \right\|_V \leq \left(\sum_{i=0}^{\infty} ||A_i|| \right) ||f||. \tag{A.53}$$

Hence, under these assumptions, V_0 is a bounded linear operator.

Let V_1, $V_2 : B(\mathbb{Z}_+, E) \longrightarrow B(\mathbb{Z}_+, E)$ be Volterra operators. Then the composition $V_1V_2 : B(\mathbb{Z}_+, E) \longrightarrow B(\mathbb{Z}_+, E)$ is also a Volterra operator and its representation in the ring of power series is given by $V_1V_2 \longrightarrow V_1(z)V_2(z)$. Also if $\beta \in B(\mathbb{Z}_+, E)$ then the image $V_0\beta \in B(\mathbb{Z}_+, E)$ corresponds to the analytic function $V_0(z)\beta(z)$.

The following result now characterises the inverse operator of $V_0(z)$,

Lemma A.13 (Inverse Volterra Operator) *If $E = V$ and $\det V_0(z) \neq 0$, $|z| \leq 1$, $z \in \mathbb{C}$, then the Volterra operator V_0 is invertible.*

It can be shown that the matrix $V_0(z)$ can be transformed (or factored) by applying appropriate elementary operations to obtain the following

$$V_0(z) = \sigma_1(z)p(z)\sigma_2(z), \tag{A.54}$$

where $\sigma_1(z)$ and $\sigma_2(z)$ are square matrices of appropriate dimension which are analytic in the unit disc U and have nonzero determinants at all points of the closed unit disc $\overline{U}$, and the matrix $p(z)$, which has the same dimensions as $V_0(z)$, and has elements which are all are zero except, possibly, for those on the leading diagonal which are monic polynomials with roots in the closed unit disc $\overline{U}$.

Without loss of generality, it is assumed that the nonzero diagonal elements $p_1(z), \cdots, p_l(z)$ of the matrix $p(z)$ are in the first l rows with the property that each nonzero polynomial $p_j(z)$ divides $p_{j+1}(z)$, $1 \leq j \leq l-1$. Then, the matrix $p(z)$ can be written in the form

$$p(z) = \begin{pmatrix} p_1(z) & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & p_2(z) & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & p_l(z) & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}. \tag{A.55}$$

The following result establishes that the Volterra operators Q_1 and Q_2 generated by the matrices $\sigma_1(z)$ and $\sigma_2(z)$ respectively are invertible,

Lemma A.14 (Bijective Volterra Operators) *The Volterra operator $V_0 : B(\mathbb{Z}_+, E) \longrightarrow B(\mathbb{Z}_+, E)$ is bijective if, and only if, rank $V_0(z) = n \, \forall z \in \mathbb{C}$, $|z| \leq 1$, where $n = \dim E$.*

The spectrum of the Volterra operator can be characterised by the following result,

Theorem A.3 (Spectrum of the Volterra Operator) *The spectrum $\Sigma(V_0)$ of the Volterra operator V_0 is given by*

$$\Sigma(V_0) = \bigcup_{|z|\leq 1} \sigma(V_0(z)) \tag{A.56}$$

where $\sigma(V_0(z))$ denotes the eigenvalues of the matrix $V_0(z)$.

A.6 Theory of the Multivariable First Order Lag

The theory of standard differential linear systems with the structure of a multivariable first order lag can be found in (Owens, 1978) and the references therein. There follows a brief summary of the main facts.

Definition A.12 (Multivariable First Order Lag) *(Owens, 1978) An m-input /m-output strictly proper system described by the $m \times m$ transfer-function matrix $G_A(s)$ is said to be a multivariable first order lag system if, and only if, $|G_A(s)| \not\equiv 0$ and*

$$G_A^{-1}(s) = A_0 s + A_1 \tag{A.57}$$

where A_0 and A_1 are real constant matrices with $|A_0| \neq 0$.

The term first order lag is motivated by the analogy with the classical first order lag defined by the transfer-function

$$g^{-1}(s) = a_0 s + a_1, \quad a_0 \neq 0. \tag{A.58}$$

Writing

$$G_A(s) = \{A_0 s + A_1\}^{-1} = \{sI_m + A_0^{-1}A_1\}^{-1}A_0^{-1}, \tag{A.59}$$

it can be seen that $G_A(s)$ has a state-space realisation specified by $A = -A_0^{-1}A_1$, $B = A_0^{-1}$, $C = I_m$ and $n = m$. This is formalised in the following theorem,

Theorem A.4 *(Owens, 1978) An m-input/ m-output strictly proper, controllable and observable system specified by the state-space triple (A, B, C) is a multivariable first order lag if, and only if, $n = m$ and $|CB| \neq 0$.*

Using the series expansion of $G_A(s)$ for large values of $|s|$ we have

$$G_A(s) = C(sI_m - A)^{-1}B = \frac{1}{s}CB + \frac{1}{s^2}CAB + \frac{1}{s^3}CA^2B + \cdots \tag{A.60}$$

and hence

$$I_m = G_A^{-1}(s)G_A(s) = \{A_0 + \frac{1}{s}A_1\}\{CB + \frac{1}{s}CAB + \cdots\} \tag{A.61}$$

and, by equating powers of s^{-1}, it follows that $A_0CB = I_m$, i.e. $A_0 = (CB)^{-1}$. Also it can be seen that $A_1 = \lim_{s \to 0} G_A^{-1}(s)$.

Extending the analogy, a differential unit memory linear repetitive process whose derived conventional linear system $L_D(A, B, C)$ takes the form of a multivariable first order lag has the state-space model

$$\begin{aligned} \dot{x}_{k+1}(t) &= -A_0^{-1}A_1\, x_{k+1}(t) + A_0^{-1}\, u_{k+1}(t) + B_0\, y_k(t) \\ y_{k+1}(t) &= x_{k+1}(t). \end{aligned} \tag{A.62}$$

Appendix B

Sampling Result Derivation

Within this section, the fast sampling of linear repetitive processes result of section 6.8.1 is derived.

Consider the discrete unit memory linear repetitive process described by the state-space model

$$\begin{aligned} x_{k+1}(p+1) &= \hat{A}\, x_{k+1}(p) + \hat{B}\, u_{k+1}(p) + \hat{B}_0\, y_k(p), \\ y_{k+1}(p) &= \hat{C}\, x_{k+1}(p), \quad 0 \le p \le \alpha, \;\; k \ge 0, \end{aligned} \tag{B.1}$$

and regard this process as being derived from a differential unit memory process of the form

$$\begin{aligned} \dot{x}_{k+1}(t) &= A\, x_{k+1}(t) + B\, u_{k+1}(t) + B_0\, y_k(t), \\ y_{k+1}(t) &= C\, x_{k+1}(t), \quad 0 \le t \le \alpha, \;\; k \ge 0, \end{aligned} \tag{B.2}$$

with initial conditions $x_{k+1}(0) = d_{k+1}$, $k \ge 0$, and $y_0(t) = \hat{y}(t)$, $0 \le t \le \alpha$.

Now subject (B.2) to synchronous digital control with sampling period h, where

$$x^q_{k+1} := x_{k+1}(qh), \tag{B.3}$$

and where, for integer q, $0 \le q \le \frac{\alpha}{h}$, and piecewise continuous input

$$\begin{aligned} u^q_{k+1} &:= u_{k+1}(qh) \\ &= u_{k+1}(t) \quad \text{on } [qh, (q+1)h). \end{aligned} \tag{B.4}$$

As a starting point to the following analysis, first note that (B.2) has the following solution for $k \geq 0,\ 0 \leq t \leq \alpha$,

$$y_{k+1}(t) = C \int_0^t e^{A(t-\tau)} \{B_0\, y_k(\tau) + B\, u_{k+1}(\tau)\}\, d\tau + Ce^{At}\, d_{k+1}. \tag{B.5}$$

At the time instant $t = qh$, the solution (B.5), $k \geq 0$, of (B.2) is

$$\begin{aligned} y_{k+1}(qh) &= C \int_0^{qh} e^{A(qh-\tau)} \{B_0\, y_k(\tau) + B\, u_{k+1}(\tau)\}\, d\tau + Ce^{Aqh}\, d_{k+1} \\ \Longrightarrow\ x_{k+1}(qh) &= e^{Aqh} \left\{ \int_0^{qh} e^{-A\tau} \{B_0\, y_k(\tau) + B\, u_{k+1}(\tau)\}\, d\tau + d_{k+1} \right\}. \end{aligned} \tag{B.6}$$

Similarly, at the time instant $t = (q+1)h$ we have, for $k \geq 0$,

$$\begin{aligned} x_{k+1}((q+1)h) &= e^{A(q+1)h} \left\{ \int_0^{(q+1)h} e^{-A\tau} \{B_0\, y_k(\tau) + B\, u_{k+1}(\tau)\}\, d\tau + d_{k+1} \right\} \\ &= e^{Ah} e^{Aqh} \left\{ \int_0^{qh} e^{-A\tau} \{B_0\, y_k(\tau) + B\, u_{k+1}(\tau)\}\, d\tau \right. \\ &\quad \left. + \int_{qh}^{(q+1)h} e^{-A\tau} \{B_0\, y_k(\tau) + B\, u_{k+1}(\tau)\}\, d\tau + d_{k+1} \right\} \\ &= e^{Ah} \left\{ x_{k+1}(qh) + e^{Aqh} \int_{qh}^{(q+1)h} e^{-A\tau} \{B_0\, y_k(\tau) + B\, u_{k+1}(\tau)\}\, d\tau \right\}. \end{aligned} \tag{B.7}$$

Now consider each term in this expression in turn.

Firstly look at the term involving $u_{k+1}(t)$ in (B.7). Due to the fact that the input is piecewise continuous, i.e. $u_{k+1}(t) \equiv u_{k+1}(qh)$ on $[qh, (q+1)h)$, we can write for $k \geq 0,\ 0 \leq q \leq \frac{\alpha}{h}$,

$$\begin{aligned} e^{Ah} e^{Aqh} \int_{qh}^{(q+1)h} e^{-A\tau} B\, u_{k+1}(\tau)\, d\tau &= e^{Ah} e^{Aqh} e^{-Aqh} \int_0^h e^{-A\tau} B\, d\tau\, u_{k+1}(qh) \\ &= e^{Ah} \int_0^h e^{-A\tau} B\, d\tau\, u_{k+1}^q. \end{aligned} \tag{B.8}$$

Now consider the $y_k(t)$ term in (B.7). Initially note that, under fast sampling conditions (i.e. under $h \longrightarrow 0$), $y_k(t)$ on the interval $[qh, (q+1)h)$ can be approximated by $y_k(qh)$, $0 \leq q \leq \frac{\alpha}{h}$, $k \geq 0$. This approximation improves as $h \longrightarrow 0$, and we have

$$\lim_{h \to 0^+} y_k(\tau) = y_k(qh), \quad \text{on } [qh, (q+1)h). \tag{B.9}$$

This is equivalent to the assumption that the previous pass profile is piecewise continuous.

Hence, under this assumption, the $y_k(t)$ term in (B.7) can be written

$$e^{Ah}e^{Aqh}\int_{qh}^{(q+1)h} e^{-A\tau}B_0\, y_k(\tau)\, d\tau = e^{Ah}\int_0^h e^{-A\tau}B_0\, d\tau\, y_k^q. \tag{B.10}$$

Then combining (B.8) and (B.10) and introducing the notation of (B.3) enables (B.7) to be written

$$x_{k+1}^{q+1} = e^{Ah}x_{k+1}^q + e^{Ah}\int_0^h e^{-A\tau}\left(B\, d\tau\, u_{k+1}^q + B_0\, d\tau\, y_k^q\right). \tag{B.11}$$

Comparing this result with (B.2) gives

$$\hat{A} = e^{Ah},\ \ \hat{B} = \hat{A}\int_0^h e^{-A\tau}B\, d\tau \ \text{ and } \ \hat{B}_0 = \hat{A}\int_0^h e^{-A\tau}B_0\, d\tau \tag{B.12}$$

as required.

In the following analysis, it is shown that the discrete linear repetitive process (B.1) obtained via the synchronous sampling scheme defined by (B.3) and (B.4) becomes a differential linear repetitive process of the form (B.2) in the limit $h \longrightarrow 0^+$.

Following the approach in (Ackermann, 1985), from (B.11) we can write,

$$\begin{aligned}\lim_{h\to 0^+}\frac{1}{h}\left(x_{k+1}^{q+1} - x_{k+1}^q\right) &= \lim_{h\to 0^+}\frac{1}{h}\left\{\left(e^{Ah} - I_n\right)x_{k+1}^q + e^{Ah}\int_0^h e^{-A\tau}B\, d\tau\, u_{k+1}^q\right. \\ &\quad \left. + e^{Ah}\int_0^h e^{-A\tau}B_0\, d\tau\, y_k^q\right\}. \end{aligned} \tag{B.13}$$

Now consider each term in this expression in turn.

Firstly look at the term involving x_{k+1}^q. Using the power series expansion for e^{Ah}, we can write

$$\begin{aligned}\lim_{h\to 0^+}\frac{1}{h}(e^{Ah} - I_n)\, x_{k+1}^q &= \lim_{h\to 0^+}\frac{1}{h}\left\{I_n + Ah + O(h^2) - I_n\right\}\, x_{k+1}^q \\ &= A\, x_{k+1}^q \end{aligned} \tag{B.14}$$

where $O(h^2)$ represents terms involving h^2 or higher powers of h.

Similarly, for the term involving u_{k+1}^q we can write,

$$\begin{aligned}
&\lim_{h\to 0^+} \frac{1}{h}\left\{e^{Ah}\int_0^h e^{-A\tau}B\,d\tau\,u_{k+1}^q\right\}\\
&= \lim_{h\to 0^+}\frac{1}{h}\left\{\left(I_n + Ah + O(h^2)\right)\int_0^h \left(I_n + A\tau + O(\tau^2)\right)B\,d\tau\,u_{k+1}^q\right\}\\
&= \lim_{h\to 0^+}\frac{1}{h}\left\{\left(I_n + Ah + O(h^2)\right)\left[h + O(h^2)\right]B\,u_{k+1}^q\right\}\\
&= B\,u_{k+1}^q. \qquad\qquad (B.15)
\end{aligned}$$

Finally for y_k^q we have

$$\begin{aligned}
&\lim_{h\to 0^+} \frac{1}{h}\left\{e^{Ah}\int_0^h e^{-A\tau}B_0\,d\tau\,y_k^q\right\}\\
&= \lim_{h\to 0^+}\frac{1}{h}\left\{\left(I_n + Ah + O(h^2)\right)\int_0^h \left(I_n + A\tau + O(\tau^2)\right)B_0\,d\tau\,y_k^q\right\}\\
&= \lim_{h\to 0^+}\frac{1}{h}\left\{\left(I + Ah + O(h^2)\right)\left[h + O(h^2)\right]B_0\,y_k^q\right\}\\
&= B_0\,y_k^q. \qquad\qquad (B.16)
\end{aligned}$$

Also note that, in the limit $h \longrightarrow 0^+$, u_{k+1}^q and y_k^q become continuous variables. Hence, combining these three results, and noting that, in the limit,

$$\lim_{h\longrightarrow 0^+}\frac{1}{h}\left(x_{k+1}^{q+1} - x_{k+1}^q\right) = \dot{x}_{k+1}^q, \qquad (B.17)$$

we can write

$$\dot{x}_{k+1}^q = A\,x_{k+1}^q + B\,u_{k+1}^q + B_0\,y_k^q, \qquad (B.18)$$

which, as $h \longrightarrow 0^+$, approaches

$$\dot{x}_{k+1}(t) = A\,x_{k+1}(t) + B\,u_{k+1}(t) + B_0\,y_k(t), \qquad (B.19)$$

which is just the differential linear repetitive process (B.2) as required.

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