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LEARNING ABOUT POLYHEDRA THROUGH VISUAL AND TACTILE PERCEPTION AND DISCUSSION

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For José, Thiago and Filipe, without their understanding, company, and encouragement this thesis would never have been written.

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ABSTRACT

FACULTY OF EDUCATIONAL STUDIES
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This thesis is concerned with understanding in three-dimensional geometry. The study focuses on how visual perception and imagery of polyhedra are related to van Hiele levels and to Del Grande's spatial perception abilities. A written test is developed which is designed to assess van Hiele levels in three-dimensional geometry, as well as Del Grande's spatial perception abilities. Two empirical studies address how perception and imagery of polyhedra are related to geometric thought. The first of the empirical studies looks at the role of group discussion in enhancing visual perception and image formation of polyhedra. The second examines the role of purely tactile information about polyhedra and its role in forming stable models of polyhedra.

The findings of the study reveal a distinction between visual perception and image formation of polyhedra. This distinction is marked, and is not normally drawn in the mathematics education literature. Visual perception is strongly connected to van Hiele levels but image formation is not. The findings of the study also reveal a strong influence through verbal discussion on increasing student's visual perception.

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Chapter One

The study's rationale

Chapter One

The study's rationale

1.1 Introduction

This thesis addresses questions of perception and image formation in three-dimensional geometry. The aim is to understand what difficulties there might be for prospective teachers in comprehending and appropriating polyhedral models, and how discussion might enhance their geometric perception. The basic question I address is:

> *How do students own actions, the actions of others, and discussions of these actions and perceptions, assist them to gain greater insight and understanding of polyhedra?*

I was lead to this project by a previous programme in Brazil, with teachers and students on the use of concrete materials on teaching and learning geometry. This previous work was motivated by a desire to enhance the teaching and learning of geometry in north eastern Brazil. It was carried out by implementing a geometry course which emphasised transformation ideas, and utilised concrete manipulative material. 66 mathematics teachers participated from secondary state and council schools in North-eastern Brazil. The principal reasons for using concrete material extensively in this study were:

1. Geometrical constructions abound in the built environment: once it is pointed out to them, children - and teachers - can see geometric constructions everywhere, in buildings, parks, and their planned environment. Further, their everyday and formal understanding of science, engineering, and architecture utilises concrete geometric forms in many ways. The study of polyhedra is rich in mathematical content and it

provides links to several other areas such as arts (to construct, create and design shapes), chemistry (to show molecular structures), and architecture (built structures).

2. There is an interaction between students and teachers in using and building geometric material. In these interactions, and those between students, geometrical language generally improves.

3. Concrete materials assist in an understanding of geometry based on action rather than one beginning with logical deduction, as is commonly taught in Brazil. van Hiele's work shows that the latter approach is doomed to failure.

In some countries the study of 3-dimensional geometry in general, and polyhedra in particular, is educationally deprived because teachers and students are not experienced with the study of three- dimensional shapes. Brazilian schools have serious problems in the teaching and learning of mathematics in general, and geometry in particular (d'Ambrosio 1991, p. 79). As the majority of Brazilian mathematics teachers are not familiar with alternative teaching techniques, a variety of classroom procedures is rare. The conventional Brazilian mathematics lessons are mostly teacher-centred with little, sometimes no, pupil interaction. Brazilian mathematics teachers usually adopt a text book and most of them do not take into account recent changes in teaching methodology developed elsewhere. Most teachers interpret the textbook page by page to the children, with the textbooks containing both explanation and some exercises.

In the north and north-east regions of Brazil, mathematics teachers are aware of the importance of the study of geometry in Brazilian schools. However, the majority of teachers of secondary schools in these areas do not have special training in geometry, and they, in general, do not like teaching it because they do not know enough about this subject. As a result, they generally avoid teaching geometry.

In my Brazilian project, the activity of construction of geometric manipulatives was carried out by teachers constructing geometric models on the basis of given properties. Geometry was practised from the combination of properties to obtain a three dimensional shape. For example, a cube was specified as a polyhedron with 6 faces, each being a congruent square, and the internal angles all right angles. To construct the shapes by combination of properties, a mental image is useful, as is previous experience and knowledge. After having constructed the object, a description of it was required and I expected that constructing shapes would help them with a complete understanding of the shapes, by inspecting them. Each teacher used their own way to describe the three-dimensional shapes.

First, the teachers were presented with a variety of two dimensional (polydron) shapes. Teachers chose these shapes according to the specified properties of three dimensional shape to construct a new shape and to imagine the net.

Secondly, they were given nets, drawn on paper, to discover the three-dimensional shapes and then to construct them.

Thirdly, they choose the two dimensional shape to form a three dimensional shape that they wanted and then tried to describe its properties.

I also used this activity with the students and I detected more problems such as knowledge of shapes' properties and the name of the shapes. However these activities encouraged students to realise a relationship between two dimensional and three dimensional space and also to gain knowledge of abstract relationships, such as the area of the figure flattened.

The teachers are aware that geometry is important. However they were critical that only a few notions are taught in the State schools. They listed a few reasons for the lack of interest in teaching this subject:

1. Geometry is not related with every day life.

2. Lack of teaching training in mathematics, particularly geometry.

3. Lack of understanding of geometry, especially spatial relationships, transformation geometry, conservation of length, measurement, directions, and angles.

4. Lack of understanding of the connection between algebra and geometry. (Usually teachers solve problems using a formula, but do not observe connections between formulas and geometric shapes.)

The teachers expressed a concern that there is no appropriate geometric content in the school syllabus, and what there is, is not related with real life. This made it very difficult to improve the teaching of geometry in Brazilian schools. Teachers agreed that geometry is taught on a level of deduction, that students are involved in proving theorems, using rules and axioms whilst most of these students are at lower levels of thinking. One of the teachers said:

> " *Now, I observed that the use of the concrete materials such as geoboard and others in the classroom may help children through van Hiele levels. Euclidean geometry based on formal methods is not appropriate to the Brazilian mathematics curriculum.*"

The emphasis on using and thinking about concrete material in the teachers' and students' geometry course was based on these concerns.

A questionnaire was administered to the participating teachers approximately one year after the intensive course on geometry. The questionnaire indicated clearly that as polyhedra become more complex - with more faces and more intricate relationships between the faces, vertices and edges - teachers were less likely to use these models in their geometry classrooms. See Figure 1.1 below:

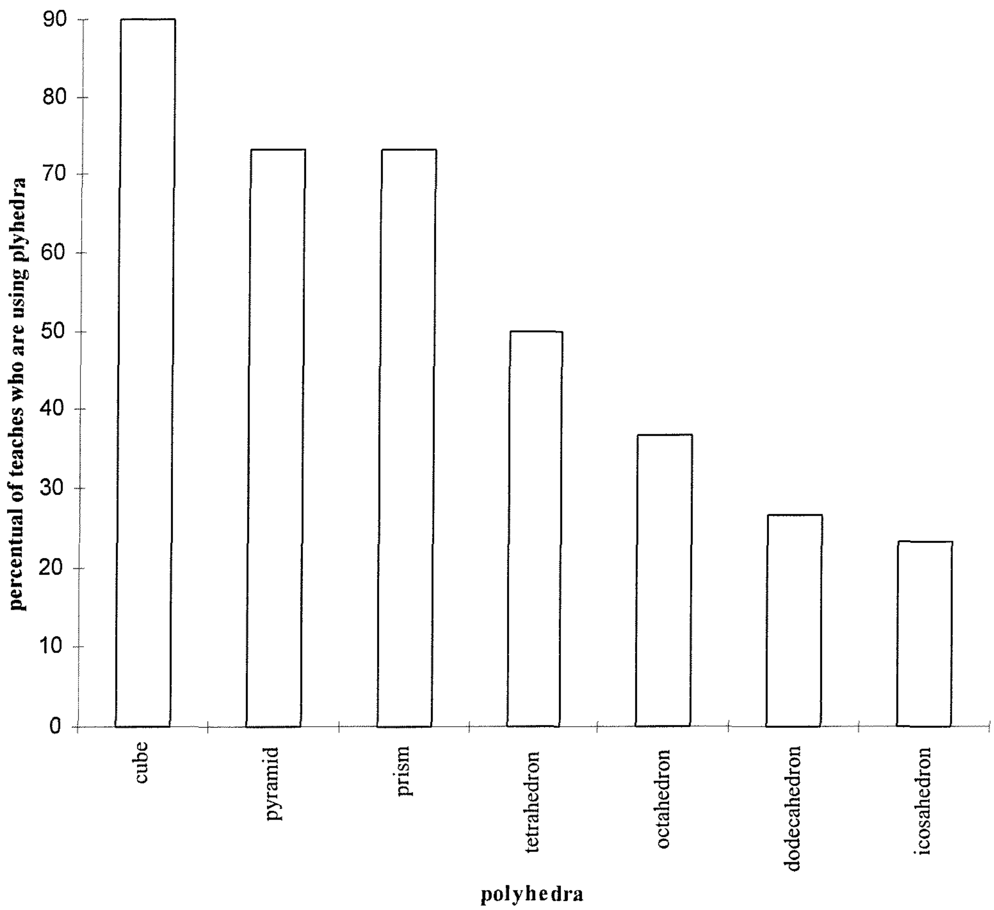


Figure 1.1 Percentage of teachers using the indicated polyhedra in their teaching (rank order)

The shape mostly used after the teaching experiment in Brazilian schools is the cube. The more sides a polyhedron has, the lower the level of use by the teachers. I hypothesise that this is due to an increasing level of complexity: the teachers find the polyhedra with more sides increasingly difficult to understand.

This created a dilemma for me. On the one hand I felt, for the reasons given above, that it was critical for the teachers to use concrete material with their students; on the other hand, the questionnaire indicated that many of the polyhedral models were difficult for the teachers to comprehend.

1.2 Research questions

The above observations were the starting point for the investigations in this thesis. I wanted to know:

* How do visual perception and imagery relate to each other?
* Can students develop greater geometric language and thought through heightened perception and explanation?
* Are some polyhedra more difficult than others to perceive - either through vision or touch, and are some polyhedra more difficult to imagine?
* Whether visual perception and imagery are related to van Hiele levels or Del Grande's categories of spatial abilities?

Some basic research questions became apparent:

* How do student teachers understand polyhedra?
* How do they form images of three-dimensional shapes?
* What information can they be expected to gain by exploring polyhedra?
* How do they develop their mathematical language through discussion in groups?
* What is visual perception in the context of polyhedra?
* What is imagery and how are visual perception and imagery related?
* How are visual perception and imagery of polyhedra related to student's level of geometric thought?
* How can discussion assist students to attain a heightened geometric knowledge of polyhedra?

Perception of polyhedra, both visual and tactile, is intimately related to a person's general level of geometric thought - as indicated, for example, by the van Hiele levels (Burger & Shaughnessy, 1986; Crowley, 1987; Fuys, Geddes & Tischler, 1988; Gutiérrez, Jaime & Fortuny, 1991; Guttiérez, 1996; Mayberry, 1983; Usiskin, 1982; Van Hiele, 1986) .

In the thesis I discuss theoretical issues of visual perception and imagery, particularly as they relate to three-dimensional geometry. Following this theoretical discussion I document and analyse two experimental studies. The first involved a number of pre-service secondary mathematics teachers in both visual perception and image formation of polyhedra. The second involved final year mathematics students' tactile perception and image formation of similar polyhedra. The reason for focusing solely on tactile perception and its relation to image formation was to separate out the role of visual perception when a student could both see and touch a polyhedron. Using a single sensory modality focuses more clearly on the nature of the information students get from that sense, and how, through discussion, they use that sensory information to build visual images.

The study of imagery and visual perception in three-dimensional geometry was carried out to discover the difficulties that students and teachers may have in understanding polyhedra. Images were used to improve students' perception and the interpretation of the images through an understanding of the three-dimensional world.

Students in both studies were given a test of geometric thinking, related to van Hiele levels, and Del Grande's categories of spatial abilities.

1.3 Outline of the thesis

An outline of the structure of the thesis follows.

Chapter One

Rationale for the thesis, including a background of previous work carried out in Brazil.

Chapter Two

This chapter contains a review of literature pertaining to three relevant areas for this thesis:

1. The use of concrete material on teaching and learning mathematics. It emphasises the fact that manipulatives by themselves do not generate mathematical experience: the information is processed in the students' mind by action and reflection. I explain what mean by concrete material and manipulatives. For a long time, manipulatives have been used to teach mathematics. I report in this chapter on the research evidence for the effectiveness, or otherwise, of manipulatives and concrete models.
2. Psychological research on visual perception and image formation. Although other authors are taken into account the theoretical perspective presented here relies heavily on the work of Kosslyn (1996). Definitions of visual perception and image formation are difficult to find in the psychological literature, largely being assumed terms, and I make an effort to explain carefully what I mean by these terms in this thesis.
3. Research on the ability to form images from verbal descriptions. This is central to the methodology of this thesis, and I present research - particularly that of Denis and Denis & Cocude -which shows that such image formation is possible.

Chapter Three

This chapter, on methodology, presents the kind of research and the area which was investigated. I gave details about the method used for data collection emphasising the use of concrete material. I present the types of concrete material used, the experimental protocols, and the nature of the student populations I give a list of question research and an overview about the results which requires experimental evidence.

Chapter Four

This chapter discusses the development of a test according to van Hiele's levels of thinking and Del Grande's spatial abilities in three-dimensional geometry. This test was designed to assess students' van Hiele levels of thought and Del Grande's spatial abilities in three-dimensional geometry. Students enrolled on this test were pre-service teachers and undergraduate of mathematics course. The reliability of this test is analysed in this chapter.

Chapter Five

This chapter presents the empirical findings from the group discussion sections. I address in depth research questions about visual perception and image formation. A small group of pre-service secondary mathematics teachers involved with the written test volunteered to participate in group discussions. I investigate how these students understand polyhedra and gain information through discussion. Analysis of data from the discussion groups shows how pre-service teachers perceive polyhedra, how they utilise imagery, and how they develop their language and ideas through discussing the shapes. A crucial distinction is made in this chapter between surface and deep visual perception.

Chapter Six

This chapter presents a method for assessing van Hiele levels from verbal data, and analyses the empirical data from the previous chapter for changes in van Hiele levels. The techniques is based on units of verbal expression which form a base for give a single unit in communication. I present dialogues which were divided up into smaller chunks. These chunks when connected were concluded by the argument of the discussion. Moreover, I understand that there are conditional relations between fragmented statements, the effect of connections that exist within chunks form cohesive units. The topics were related to a different frame such as the structure of the students' deduction which was examined and so far it was helpful to identification to assigning van Hiele levels. The analysis of part of verbal discussion shown that students will advance in van Hiele levels of thinking on basis of analyses of cohesive units. The cohesive units gave to students some insight in terms of how they can structure their questions or description.

Chapter Seven

This chapter continues the research questions about perception and image formation of polyhedron. All the polyhedra used in this experiment were the same as those in the previous, but this time were carried out with tactile experiments only. The results of the tactile perceptual tasks, the strategies used by students to describe and form images of the polyhedra are reported on this chapter.

Chapter Eight

This chapter provides discussions on the three empirical aspects of the thesis: written tests, group discussion sessions, and the tactile session.

Chapter Nine

This chapter provides conclusion for the overall study.

Chapter Two

Literature review

Chapter Two

Literature Review

2.1 Concrete materials and geometry

2.1.1 What is meant by concrete material and manipulatives in mathematics?

Both the terms “manipulative” and “concrete material”, are used in this chapter in which I discuss the importance of them in teaching and learning geometry. There is a connection between work done with the use of manipulatives, and students’ knowledge and their ability to develop strategies, alternative approaches, and reflection when they explore a mathematical task. New ideas can be stimulated by concrete material or a pictorial representation to construct meanings by reflection on a physical action.

Manipulation basically is handling a physical object to explore its characteristics. It is an action on concrete material that appears to have a purpose, as defined by the manipulator, which is often exploratory in nature. Manipulation involves an on-going process of action and reflection. Concrete materials are essentially the physical objects that students manipulate, so the terms concrete material and manipulatives are relatively interchangeable.

The usually stated purpose of using concrete material is that, for a teacher, they embody, or represent, a physical manifestation of a mathematical idea that the teacher wishes to convey. Teachers sometimes view concrete material as an external representation of an internal model they themselves have; at other times they view concrete material as the physical embodiment of a mathematical concept, task, or situation. They usually use such concrete material as cognitive tools: instruments with which a student interacts and hopefully constructs their own appropriate internal models.

Hynes (1986) defines manipulatives to be "concrete models that incorporate mathematical concepts, appeal to several senses, and can be touched and moved around by the student", and Sowell (1989), for example, views manipulatives as including both concrete and pictorial representations, and gives an operational definition of these terms:

Concrete - students work directly with materials such as beansticks, Cuisenaire rods, geoboards, paper folding, or other manipulative materials, under the supervision of a teacher.

Pictorial - students watch animated audiovisual presentation, observed demonstrations with concrete materials by teachers, or used pictures in printed materials.

2.1.2 Concrete materials and manipulatives in geometry

Concrete materials in geometry play a somewhat different role than in other parts of mathematics. For example, in dealing with the number "5" through the use of concrete objects, one can use any five objects that are regarded as instances of the same thing - 5 counters, 5 cups, 5 children, 5 chairs. However, even when children cut out triangles of different shapes from a piece of card, each concrete triangle is, for a student, a triangle, unlike 5 counters which, for abstract counters, is not "5". In geometry the concrete material is often taken to be the thing it represents. For instance, as above, a cardboard triangle is taken to be a triangle even though it is only a physical approximation to an ideal triangle. Similarly, in 3-dimensional geometry a physical model of a polyhedron is often taken to be that polyhedron rather than a physical approximation to an idealised mental model. Geometrical models are often categorical in this sense: they are physical realisations of mental objects that are determined by their defining properties.

When learners manipulate a certain object such as a cardboard net of a cube, this situation establishes a relation between the characteristics of a physical object (action) and the exploratory purpose (reflection).

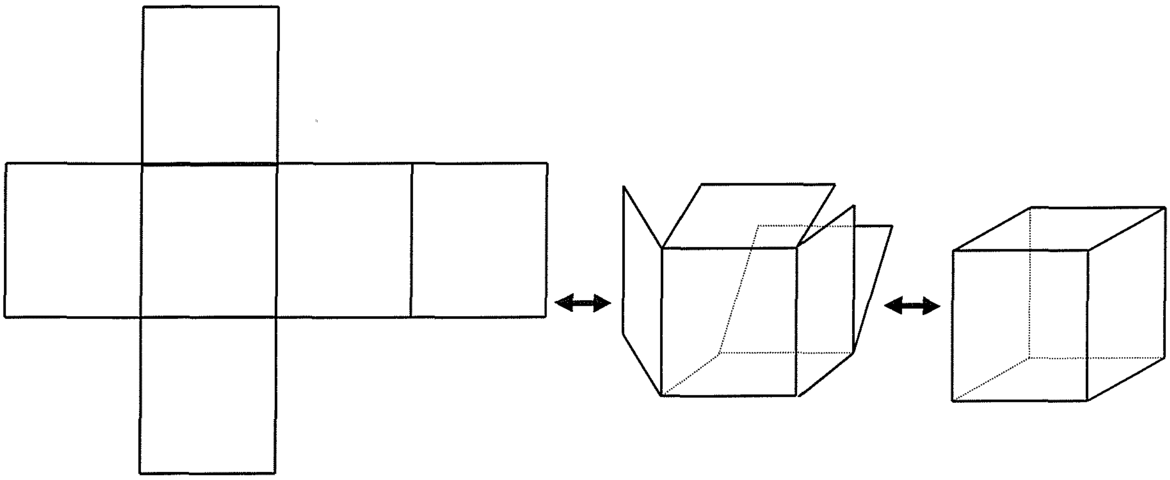


Figure 2.1 Shows the net to construct a cube

The relation between manipulatives/physical action and action/reflection are strongly connected. van Oers (1996) defines an action

"... as an attempt to change some (material or mental) object from its initial form into another form."

Krainer (1993) suggested that a high level of acting

"... refers to the initiation of active processes of concept formation which are accompanied by relevant (concept generating) and high level of reflection an important aspect of reflection refers to further questions from the learner (which in their turn could lead to a new action)."

Through this interaction between action and reflection students generate new knowledge. The effect of an action can stimulate reflection about this action.

From this point of view, when we look at the figure above, the process of action and reflection in geometry is related to shape and space: learners relate a two- dimensional shape represented by a net to a three-dimensional object (a cube). The connection between the net and a cube involves imagination. When we use imagination and image formation in this process, the action performed is the physical expression of a mental process. As an example, when learners use nets in their activities and transform a two dimensional shape into a three dimensional shape (construction of a shape - concrete action), they also may be able to imagine the results in reversible way, that is, from a cube, to imagine, the net (mental action).

However, the mere presence of concrete material is not the most important thing in a student's development in geometry. Students can develop cognitively through heightened perception - visual and tactile, but they can also develop cognitively in the absence of concrete material though talk and imagination. This is essentially a process of developing *insight*, and leads us naturally to the ideas of van Hiele who, in 1957, developed a theory providing a model of pupil's knowledge development in geometry. He was interested in finding ways in which students could develop insight in geometry.:

"In 1951 I became a teacher at a non-Montessori school, and soon I was immersed in a struggle about insight. I had understood that the learning of facts could not be the purpose of teaching mathematics, I was convinced that development of insight ought to be purpose. But the approach of the school was otherwise; it was though best that the teacher taught facts and methods; even if the pupil did not understand them." (van Hiele 1986, p. 4).

Because of this, van Hiele developed a theory of different levels of thinking in geometry. He considered that insight might be understood as the result of perception of a structure. Van Hiele (1986) developed the idea of structure from Gestalt theory, that the whole is perceived before the parts.

He observed Piaget's interviews with children who often required a knowledge of vocabulary or properties beyond the phase of learning that they were at. Consequently, he found the basis of his research in Piaget's work but went on to suggest that all students progress in their acquisition of geometrical knowledge by passing through five levels of thinking in consecutive order. He further stated that a student's level of thinking at a particular time could be identified.

The five levels are sequentially graded. Each level is characterised by a particular process of organisation of thinking matter. A student at a higher level can be expected to be competent at lower level of thinking, but not at level above. If the teaching changes and requires a level of thinking above that of the students, he/she cannot understand and is not able to carry out the necessary thought processes. The student who is only able to perform at level n cannot understand level (n+1) or higher levels.

Following van Hiele's ideas on the connections between perception, image formation, and language use in the development of geometric insight, this thesis describes the use of polyhedral manipulatives in exploring verbal and non-verbal processes for better understanding of concepts and cognitive process. Perception involves understanding of the real, visual or tactile, world. When we describe an object or scene we encode and decode visual or tactile information. Experimental approaches used in this thesis involve students in cognitive processes. Students become active learners applying prior information and experience in a new situation to acquire new knowledge. Manipulatives help learners to develop abilities to communicate using language - oral, written or drawn - and to reason more abstractly. When learners discuss, in groups, the products of their work they have opportunities to clarify their thinking, reporting their observations and expanding mathematical ideas.

Educators and psychologists had been analysing the contents and methods of teaching and improving their curriculum according to van Hiele's ideas. In order to design a new Russian curriculum van Hiele levels were used to analyse student materials for grades 1 to 8 involving children aged 7 to 15. To construct a new geometry course, Soviet research implemented the necessary modification in their methods of instruction. (Pyshkalo 1968, Wirzup, 1976). Manipulatives are essential aids in learning geometry, especially for students at lower levels in the van Hiele hierarchy (Fuys at al; 1988).

"The need now is for classroom teacher and researchers to refine the phase of learning, develop van Hiele-based materials, and implement those materials and philosophies in the classroom setting. Geometric thinking can be accessible to every one." Crowley (1987)

Research in geometry that is concentrated in the areas of visualisation, approaches to teaching, reasoning processes, and so on, are interested in the application of van Hiele theory in classrooms. Piaget's theory described mainly geometry as a science of space whereas van Hiele's theory described geometry as the science of space and a tool to demonstrate a mathematical structure.

2.1.3 Information processing

Where does the information come from for learners to acquire new knowledge in mathematics? How does this information get into the children's brain while they manipulate materials? How do manipulatives help the learner to acquire new knowledge? Do teachers think that the students get information that is in the manipulatives by looking or manipulating? as illustrated in the cartoon below:

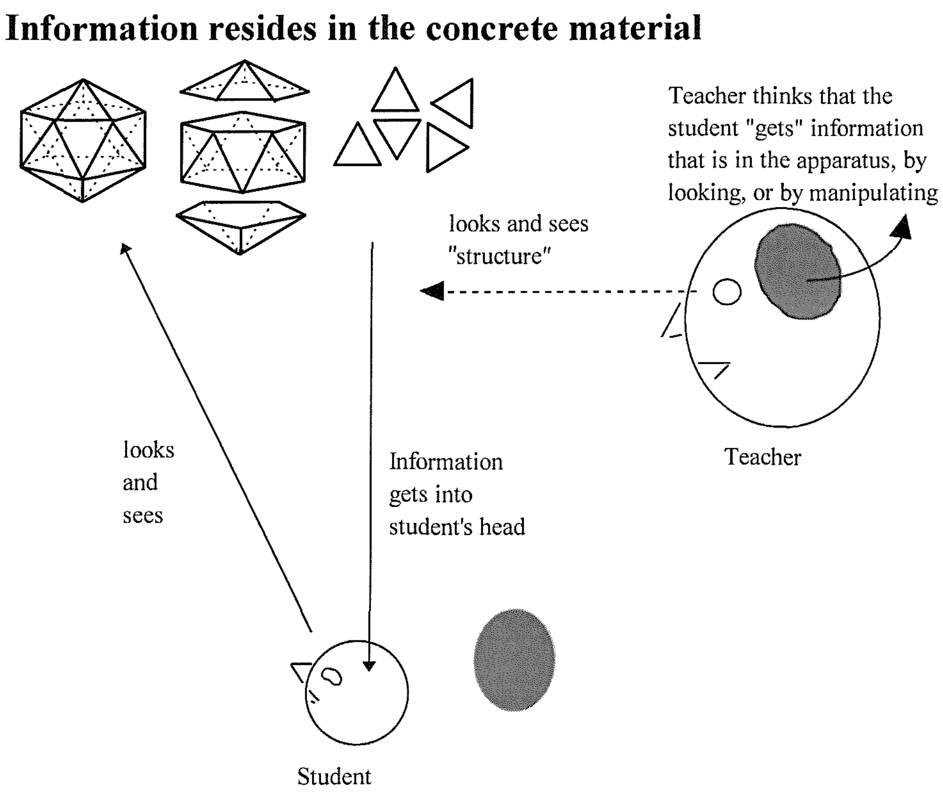


Figure 2.2 Do teachers think that the students get information that is in the manipulatives by looking?

Or do teachers think that the students get information, while seeing and manipulating materials, by acting on manipulatives and reflecting about those actions? as indicated below:

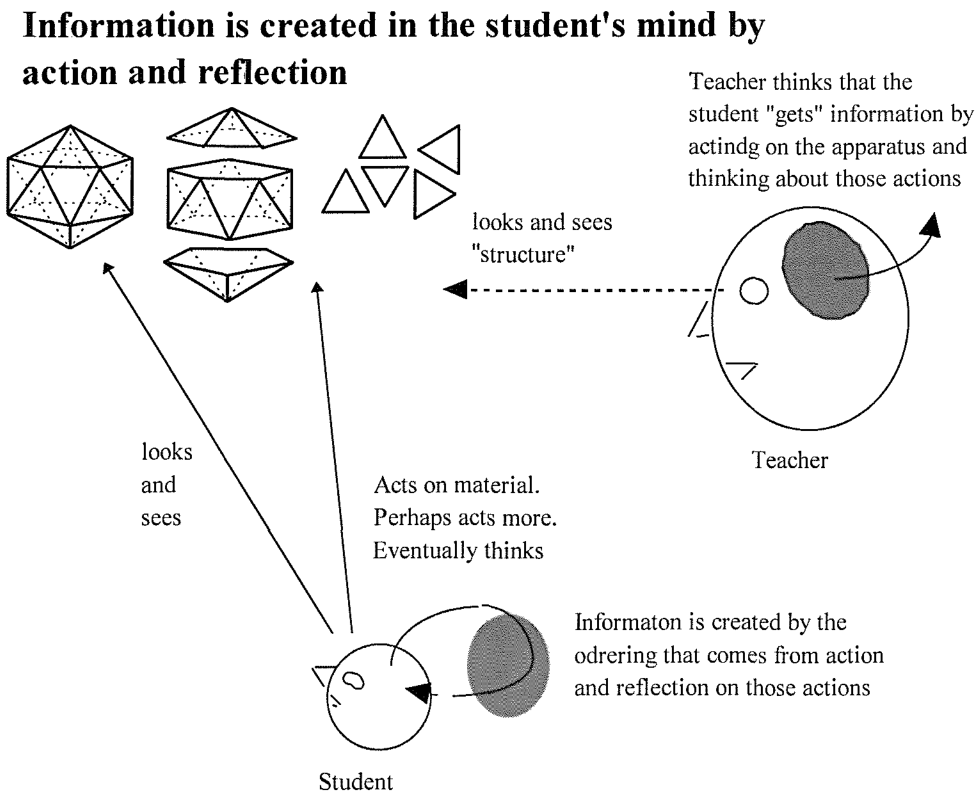


Figure 2.3 Do teachers think that the students get information that is in the manipulatives by looking, manipulating and reflecting?

The difference in view point might seem slight but is, in fact, profound. Memory is associated with thinking to access information stored in the mind to process new information. How is this new information processed? When learners use manipulatives with an exploratory purpose they are conscious of the effect of their action, but not, initially, of the process by which that effect was produced. On the other hand, they are not conscious of the organisation of cognitive structure. Crawford (1996) mentioned that

"... an action involves conscious behaviour that is stimulated by a need subordinated to a goal. An operation is an action that is transformed as a means of obtaining a result under given conditions. Operation are habits and automated procedures that are carried out without conscious intellectual effort."

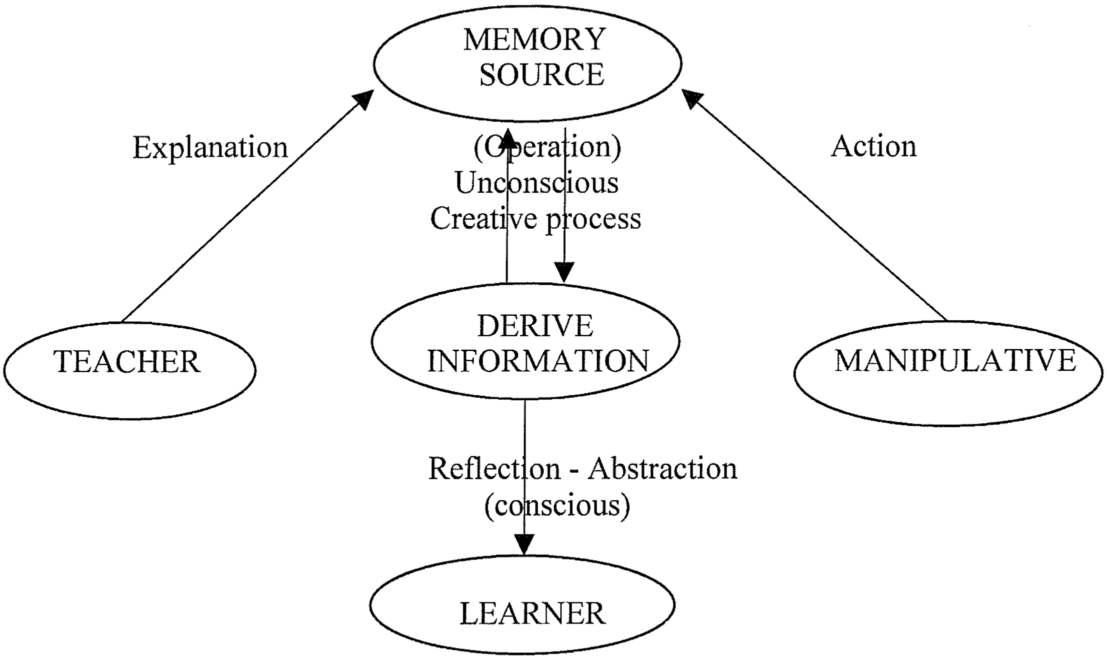


Figure 2.4 Cognitive environment

When teachers uses manipulatives in their lessons, they expect their students to understand the problem in a determined way. They expect manipulatives to stimulate a kind of perception that allows children to be conscious of their mental process. Vygotsky (1996, p. 161- 171), in his discussion about scientific concepts and implications for education and instruction, postulated that

"... these concepts are not absorbed ready-made, and instruction and learning play a leading role in their acquisition... Reflective consciousness comes to the child through the portal of concepts."

Hence, the creation of new knowledge is associated with the learners' previous experiences in which the topics explored were applicable.

Manipulatives do not present ready information by themselves. Pimm (1996 p. 13) emphasises that the use of manipulatives by itself does not contain or generate mathematical experience. People can do this with their minds and the purpose of teachers is stimulate their students' abilities to think by themselves. For students to acquire mathematical experience requires attention to their activities. Consequently, knowledge arises from the learner's reflective involvement in their activity.

Information is created in the students' mind by actions and reflections. The work with manipulatives, such as nets, where a student uses a two-dimensional shape to obtain a three-dimensional shape, implies a process of reflective thought. This involves a heuristic process of problem solving.

Piaget (1960) described geometry mainly as a science of space, and made extensive studies of children's logical thinking and of geometrical concepts with implications for the teaching of geometry. According to Piaget (1967) the manipulation of concrete objects forms the basis of people's learning about shapes. The activities that involve manipulative material are spontaneous and are essential for children to attain experience in spatial perception.

Van Hiele (1986) was convinced that geometric thinking is a relatively high level activity, and that teaching and learning conflicts arise when students have not had enough experience in thinking at lower levels. His objective was to find a method to develop insight in students' thinking, and he described a connection between thinking and learning in geometry. Van Hiele's theory uses a model that allows students to move through levels of geometric development. The significant contribution of van Hiele's studies was to engage researchers in progressive development of methodological aspects. Some researchers has been interested to develop some methodologies for geometric studies, that allow children to improve spatial thinking, visualisation and skill according to their individual capacity. Fuys et al. (1988) reported that manipulative materials are essential aids in learning geometry, specially for students at lower levels in the van Hiele hierarchy.

2.1.4 Effectiveness of concrete material and manipulatives

2.1.4.1 Negative or questioning literature

The recent literature on the effectiveness of concrete material and manipulatives in school began probably with the 1971 paper of Kieren. After reviewing a number of studies on the use of manipulatives he asked:

"For whom, for which topics, and with what materials are manipulative and play-like material valuable?" (p.232).

Kieren's paper emphasised how little was known about the effective use of manipulatives, and highlighted the need for research to answer his question, above, in light of conflicting views.

Picking up on Kieren's challenge, Friedman (1978) carried out a review of research in the years since Kieren's paper. He concluded that

"On the basis of current research, it would appear that after the first grade, where the manipulative strategy has been effective in several situations, an instructional strategy that gives pre-eminence to the use of manipulative materials is unwarranted." (p.79).

A decade later the situation was not much clearer. Baroody (1989) in an article in the Arithmetic Teacher, aimed at practising teachers, said:

"Because we are still learning about what manipulatives should be used, how to use them effectively, and when they need to be used, we must be aware of the importance of keeping an open mind about using manipulatives." (p. 5).

Baroody emphasised that manipulatives must be used appropriately for good results, and that the inappropriate use of them may make a mess of things. He was unconvinced that manipulatives necessarily stimulate reflection on the part of students. His conclusion was that the importance of manipulatives comes from the fact that physical experience is meaningful to pupils and that they are actively engaged in thinking as they use it. However, reflection on what it was the teacher intended the manipultives to engender was by no means certain.

Hiebert and Carpenter (1992) wrote about potential for instruction involving concrete material to go wrong:

... it is not simply the presence of concrete materials that provides meaning for symbols, nor is it simply the juxtaposition of materials and symbols. In order for symbols to acquire meaning, learners must connect their mental representations of written symbols with their mental representations of concrete materials. The potential for these connections to create understanding is complicated by the fact that concrete materials themselves are representations of mathematical relationships and quantities. Thus, the usefulness of concrete materials as referents for symbols depends on both their embodiments of mathematical relationships and on their connections to written symbols (p. 72).

Almost 20 years after Kieren's article, Askew and William (1995) again expressed a cautionary approach to the use of manipulatives:

"... while practical work and 'real' contexts can be useful, they need to be chosen carefully, and accompanied by careful dialogue with pupils to establish the extent of their understanding. Pupils' success on a concrete task should not be taken as an indication of understanding the abstract. Each, practical and abstract need to be explored in their own right. How links are perceived between the two needs to be the subject of considerable discussion between pupils and teachers."

Szendrei (1996), in an excellent review article aimed at researchers, repeated these concerns under the heading "How are educational material misused?" (p. 424-427). One of the important points she makes is that in the utilisation of concrete material a child may focus on features of the material that are, to the teacher, completely irrelevant. For instance, in the use of Cuisenaire rods to learn about number, a child may - and many do - focus on the colour as the relevant attribute of the material. This is similar to the problem of divergent thinking in mathematics, recently pointed out by Gray (1993), Gray & Tall (1994), and Pitta & Gray (1997). These authors have demonstrated that many children, classed as under-achieving on the basis of school tests, focus on aspects of concrete material and number that is not relevant in a mathematics lesson. For example, in talking about a "ball" these children will habitually focus on the colour of the ball, and similarly in thinking about numbers will focus on mental objects of a particular colour. In using concrete material , or images of concrete objects, these children focus on aspects of the material that is not relevant to the mathematics teachers' concerns.

2.1.4.2 Supportive literature

As the questioning literature cited above suggests, there are many reports of successful use of concrete material. At roughly the same time as Kieren's 1971 paper, Fennema (1972), citing her 1969 report (Fennema, 1969), compared instructional approaches to learn basic multiplication. She reported the effectiveness of work with strategies based on use of Cuisenaire rods. She pointed out that childrens' development of ability to use symbols effectively is associated with experience through action, and through manipulation of concrete material. Cognitive development involves use of symbols that are associated first to the use of physical actions. So, she argues, children require the use of manipulatives to make symbolic models meaningful.

Fuson and Briars (1990) used base-ten blocks to embody named value system of number words and also used digit cards to embody the positional base-ten system of numeration. They analysed children's understanding of multi-digit addition and subtraction and justifying procedures with named-value/base-ten concepts, understanding of place-value concepts, and being able to add and subtract multidigit numbers of several places, including subtraction problems with zeros in the top numbers. For them, if the practical experiment is very complex, the interpretation of the results also becomes difficult. It is suggested that we are at a level where the connection between theory and practice is necessary and it requires detailed process data on interaction between teachers and pupils. This study showed that the pupils from the first grade were able to learn using the method of base ten blocks learning, but careful evaluation is needed, so that the teaching does not go beyond their comfortable level of learning. However, it was appropriate for the pupils in the second grade as they felt no difficulty in learning the task with the support of the physical material.

Thompson (1992) used wooden blocks and also developed computerised microworlds in his research on operations with decimal numbers. Instructions were given for two groups of children to assist them in the construction of meaning for decimal numeration and construction of notational methods involving decimal numbers. They were free to develop their own schemes for solving addition and subtraction. The students who used wooden blocks understood decimal numerals and operations well, whilst the students who used microworlds were not so accurate in decimal calculation.

As we have noted in the previous section, Friedman (1978) discussed the effectiveness of manipulatives in a six year review. He concluded that manipulatives had been effective in several situations. He concluded that the use of manipulatives were favourable for young children and not necessary for older children.

Hunting & Lamon (1996) provide a positive view of the use of manipulatives, in the sense that these materials are understandable, motivational, and provide an alternative to expository teaching. Usually, students like to manipulate concrete models and they can became an activity learn manipulating physical material. These authors provide a view that the use of manipulatives require teachers' attention and responsibility in classroom instruction and also provide their students with the necessary activity to provide learning environment. So, students have an opportunity to learning constructing their reflection.

In the last decade there has been some research and commentary on the beneficial effects of the geoboard:

(1) In solving the problem of counting squares: Comela & Watson (1977), L'Heureux (1982). These papers develop techniques, geometric in nature, that involve principles of counting. Students use geoboards to count squares which are shaped with vertices at pegs. They count different n x n squares which exhibit n2 square units determined by (n +1)2 pegs.

(2) In solving the problem "How many Triangles": Moser (1985). This paper shows a problem whose solution is not directly obvious but requires the use of some mathematical ideas and skills. The problem is about a geoboard with five by five pegs and a single rubber band to form different triangles.

(3) In computing perimeters of polygons: Smith (1980). This paper explores some activities with the use of geoboards. Polygons can be constructed with perimeters of n units where n is an even number. The Pythagorean theorem figures can be applied to some. Students learn that by using the hypotenuse of a right angle triangle, they discover that the hypotenuse forms (n + r) units.

(4) In dealing with topological concepts of regions, boundary, inside and outside, combining geometry and arithmetic: Gutierrez, A. & Jaime, A. (1987). This paper emphasises the quality of a variety of activities regarding geometry in primary schools. The authors develop activities with the use of geoboards with five by five pegs. These activities induce students to carry out investigations

2.1.5 Theories of learning through concrete materials and manipulatives

2.1.5.1 Montessori

Maria Montessori (1870-1952) developed the Montessori method of teaching which has for its base the liberty of the child which comes through activity. Discipline must come through liberty and activity. Certainly, the view of discipline according to the Montessori method presents a special difference from the commonly conceived notion of discipline.

For the Montessori methodology, discipline does not occur when students are artificially quiet: the method consider this situation as unproductive.

"We claim that an individual is disciplined when he is the master of himself and when he can, as a consequence, control himself when he must follow a rule of life...it certainly embodies a lofty principle of education that is quite different from the absolute and undiscussed coercion that produces immobility." Maria Montessori pp. 86 (1967).

This method allows students to become discoverers of the world through their work in such way that they can advance in their discoveries. The essential thing is the students experience. Students can work with material using their intelligence to explore and acquire knowledge, and became independent learners through mental activity. The lessons should be given through the regime of liberty by the manifestation of the child's natural tendency in the school. The Montessori Manipulatives also were used as tools to teach arithmetic to disabled children. From this point of view, the teacher must be careful in the preparation of the environment and materials with which students will work. Therefore Montessori lessons often look like an experiment. The Montessori method uses concrete material to explore a technique of tactile learning without the materials being seen. This technique encourages children to recognise and describe objects by what they are able to feel or to distinguish as different parts of the object.

2.1.5.2 Piaget

Piaget's writings have strongly influenced our views of cognitive development, particularly in mathematics. Piagetian theory attempts to describe an individual's construction of knowledge of scientific, logical and mathematical ideas that come from organising actions through reflective abstraction. Piaget (1896-1980) studied particularly the evolution of children's thinking. He described the process of childrens' thinking and understanding by a clinical approach that involved questioning children in many problem situations. He concluded that children develop more refined thinking as they get older: they acquire more experience and knowledge, and develop more complex cognitive structures to organise that knowledge. Piaget determined that children pass through four periods/stages of cognitive development:

1. Sensori-motor period (0 to 2 years)
2. Pre-operational period (2 to 7 years)
3. Concrete operational (7 to 12)
4. Operational period 12+

Piaget and colleagues investigated children's concepts of space, and found that a child makes a progressive differentiation of various geometrical properties such as proximity, separation, and order, all of which are important in moving an object. From this work they suggested that thinking skills develop from concrete operational touch.

Piaget was principally interested in epistemological studies, and advocated that experience is the base for both learning and more abstract knowledge. Learners take the nature of things in their real world. His work describes the development of physical, logical and mathematical concepts such as number, time, space, geometry. He described geometry (Piaget, 1960) mainly as a science of space and made extensive studies of children's logical thinking and of geometric concepts with implication for teaching geometry.

Piaget believed that the process of concept emerges during specific age ranges following the stages of children development. For him, the child's cognitive development rise through stages. The phases or stages of learning, determined by Piaget, become even more complex in relation to geometry. Some examples are:

1. Development of the child's representational space was described by Piaget as a mental image of the real world . In Piaget's view, perception is the knowledge of objects resulting from direct contact with them and representation or imagination involves the evocation of objects in their absence or, it runs parallel to perception in their presence.

2. How children acquire images through the perceptual activity and relationship between the activity, perception and the ability to generate images.

3. Children communicate their understanding by using symbols such as signs, writing and drawing.

Piaget's view, that an operation is internalised action and the spatial concept results from internalised action and children's reasoning, is related to concrete experience. The characteristics of an operation are:

1. It originates from an activity.

2. It is reversible when children are able to find to logically reverse an action that negates it.

3. It is associative when children are able to think of a series of actions as a whole.

The manipulation of concrete object forms the basis of people's learning about shapes (Piaget & Inhelder 1967). The activities that involve manipulatives are spontaneous and essential for children to attain experience in spatial perception.

Piaget brought an enormous contribution to the study of the development of spatial thinking. He interpreted the development of spatial ideas by using the logical structure of mathematics itself in a sequential hypotheses: topological, projective and Euclidean.

He developed theories which are productive explaining results relating dimensions of psychological complexity.

Child (1993 p.169) mentioned some classroom applications of Piaget's works. The concrete and experimental consideration of Piaget's theory is reflected in mathematical teaching programs for primary, secondary, middle and upper schools to assist students to attain more abstract reasoning. Piaget's view of cognitive development is related to concrete experience, requiring practical experience of concrete situations. The process of visual thinking can be explored with concrete experience.

2.1.5.3 Van Hiele

The levels are called "Recognition", "Analysis". "Logical Ordering", "Deduction" and "Rigor".

Van Hiele identifies five phases of learning (a didactic approach of sequencing instruction) during which students, according to the stage they have reached within a subject, pass from one level to the next (van Hiele 1986, Crowley 1987, Burger & Culpepper). Crowley (1987) emphasises that these phases of learning are more dependent on the instruction received than on age or maturation. She stressed the method, the organisation of instruction and the material used. Hoffer (1983) reported that,

"Van Hiele specified a sequence of phase that moves from very direct instruction to the students' independence of the teachers."

These five phases of learning, through which students, progress, are:

The first phase (inquiry/information):

At this stage conversation and activities take place between teachers and students concerning objects of study. To clarify this content, observations are made and specific vocabulary is introduced. An example of this is for teachers to ask students questions such as: what is a parallelogram? what is rectangle? what is a square? do you think a rectangle could be a square? Some knowledge of the topic to be studied is given to the student by the teacher who tries ascertain how the student understand the subject and interprets the language.

The second phase (guided orientation):

The teacher gives students a sequence of activities and explores through materials, the different relations of arrangement. The teacher asks, for instance, students to construct a rhombus using a geoboard, e. g. a rhombus is constructed with equal diagonals, one that is large and another that is smaller. At this phase students are engaged in exploring objects through activities such as measuring.

The third phase (explicitation):

Students with previous experience can employ words to express their opinions about structure using accurate and technical language. The students can discuss properties of a figure, in activities (rhombus, for example), together, and with the teacher. In this stage

Hoffer (1983) related that "phase 3 has been incorrectly translated as explanation by other writers". It is essential here that students make observations explicit rather than receive lectures (explanation) from the teachers.

Fourth phase (Free orientation):

The students learn by using different ways to approach the task and gain experience in finding their own ways of solving problems. According to Hoffer (1983), and replicated by Crowley (1987), by orienting themselves in the field of investigation, many relations between the objects of the study become explicit to students.

Fifth phase (integration)

The student has considerable knowledge of the subject and can summarise and review the methods (a form of overview) without being presented anything new. They consolidate the objects and relations of their arrangements.

Van Hiele (1986) considers these stages in the study of the rhombus.

First stage: A certain figure is demonstrated, it is called a "rhombus." The pupils are shown other geometrical figures and asked if they also are rhombuses.

Second stage: The rhombus is folded on its axis of symmetry. Something is noticed about the diagonals and the angles.

Third stage: The pupils exchange their ideas about the properties of a rhombus.

Fourth stage: Some vertices and sides of a rhombus are given by position. The whole rhombus has to be constructed.

Fifth stage: The properties of a rhombus are summed up and memorised

2.1.5.4 Vygotsky

Vigotsky developed a theory that has relevance for the sociological, psychological and educational aspects of development. His theory concerning the educational field was developed further than in cognition, whilst he also gave attention to sociocultural activity. He emphasised interactions between learning and development. Vigotsky's theory stresses the emphasis on human developmental experience and social influences. He assumes that human mental activities are related as a consequence of cultural learning. (Child 1993, p. 171) in his study of Vygotsky's theory mentioned that

"The culture into which a child was born was the source of concepts to be internalised and this affected the psychological functioning of the brain."

For Vygotsky three things contribute to construction of a new development:

1. High mental functions (perception, deliberate attention, logic, memory and abstraction).
2. Cultural development .
3. Cultural development mastering one's own behavioural processes.

Vigotsky's theory involves all mental functions. To him the development of consciousness was seen as determined by the autonomous development of the single functions (Vygotsky 1996 p. 2). He devoted principal attention to the development of language in its relation to thought. Vigotsky and Piaget both paid attention to problems regarding language and thought. They developed two different points of view. Vigotsky was interested in the psychological aspect, and he investigated the origin of consciousness whereas Piaget was interested in epistemological aspects, and he investigated the origin of knowing. Both have the same view in the sense of spontaneous and non spontaneous concepts. Children have an individual construction of knowledge and meaning (by their mental effort) influenced by an instruction. Piaget focused on the growth of an individual's knowledge, whereas Vygotsky regarded a person as essentially social first, with a developing individual consciousness.

Vigotsky works embrace investigations into child's scientific concepts in comparison with spontaneous concepts. The spontaneous concept means that the concept formed by children are spontaneous and without systematic instruction. The scientific concept means that children develop concepts by their mental effort (spontaneous) and influenced by instruction (non spontaneous). He postulated that:

"... concepts cannot be assimilated by the child in a ready made form, but have to undergo a certain development. Accumulation of knowledge supports a steady growth of scientific reasoning, which in its turn favourably influences the development of spontaneous thinking." (Vygotsky 1996 p.146, 148).

For him, scientific concepts originate in classroom instruction and spontaneous concepts involve everyday life. From this it follows that teaching and learning of scientific and spontaneous concepts are central to mathematics education.

Vygotsky (1996) used manipulatives in his experimental study regarding process of concept formation in its developmental phases . The material used consist of wooden blocks varying in colour, shape, height and size. For him, practical experience is an important factor to the development of child 's thinking.

These three theories mentioned above contributed to teaching and learning mathematics in the sense of activity in school, and in task orientation, investigations, and problem solving mathematics. Also, these theories give a global contribution for curriculum development in education. Piaget stressed the development of spontaneous concepts rather than the influence of schools and maturation was a principal influence for this development. Van Hiele pointed out that insight might be understood as the result of perception of a structure, and he gave importance to the phases of learning through a didactic approach of sequencing instruction. Vygotsky emphasised instruction in schools and social interaction between children and other more qualified instructors.

2.2 Visual perception and image formation

2.2.1 Introduction

Perception involves an observation of a sequence of events in a physical world. The physical world is constituted of heterogeneous things and our perception of it is selective. We observe the environment in which we live, and the environment determines what there is to perceive. A satisfactory understanding of perception lets us present a complete description of the appearance of objects or events. This involves the ability to detect and recognise objects. Then, we are able to describe how events happens or how things appear to us. Ancient philosophers such as Aristoteles and Diogenes developed theories about imagery. Nowadays, there appears to be an increase in debates in the literature about the connections between perception and imagery (Kosslyn 1996, Paivio, 1991b, Johnson-Laird 1983, 1996).

When we refer to visual perception and mental imagery we assume that a person is able to distinguish the physical world from their internal mental world. A person perceives the physical world, and it is different from the imagined world in the person's mind. Both perception and imagery are processed in the brain however they are normally (in the absence of hallucinations) interpreted as different things by an individual: perception is interpreted as "seeing" physical things or events, whilst imagery is interpreted as an imaginative recall of perceptions, or parts of perceptions. Visual perception and mental imagery can be distinguished by the existence of objects or events that happen in the real world while imagery occurs in the space of the mind.

2.2.2 The importance of an ecological approach to visual perception

2.2.2.1 Fundamental role of units in ecological perception

Our environment provides structural units which can be captured by our perceptual system. According to Gibson (1986) units and the environment are inseparable because the units which describe the environment depend on the level of the environment described:

"The smaller units are embedded in the larger units by what I call nesting ... Units are nested within larger units... There are forms within forms both up and down scales. ... Things are components of other things." (p. 9).

"... these units tends to be repeated over the whole surface of the earth. (p. 10)."

2.2.2.2 Surface

All physical objects which are provided by our environment consist in a body located in space and they exhibit a superficial appearance. Gibson's theory of perception emphasises that surfaces are structured at various levels of metric size, and the units are nested one to another. He also stressed that in order to describe an environment we need a particular geometry. It does not necessarily mean a conventional geometry consisting of points, lines and planes, or having a co-ordinate system to locate a point or straight line in space. The space considered in ecological geometry is the space of regions that we walk such as on the street, and in the house.

Gibson (1986) considered the shape of the terrestrial environment as a layout of environment. This layout is considered permanent in some features and changing in others. Ecological geometry differs from abstract geometry and he described the difference between the terms surface and plane. Planes are colourless, perfectly transparent and not substantial, they are not textured, and they have two sides, while surfaces are never perfectly transparent, they are coloured opaque and substantial, they are textured, and have only one side. The idea of a "sheet" must not be confounded with a geometrical plane because a sheet is an object consisting of two parallel surfaces together: it may be flat or curved surface, whereas a plane is not an object - it is imaginary.

A stick must not be confounded with a geometrical line because it is an elongated object and it has a diameter. Similarly, a dihedral surface must not be confounded with a intersection of two planes. The geometry that involves points, lines and planes is an abstract geometry and the objects are distinguished by their surfaces which form an embodiment of points, lines and planes in geometry. So, a surface geometry must be distinguished from abstract geometry. A surface can be seen and an abstract geometry can only be visualised (Gibson, 1986, pp 12, 33-35).

Gibson (1974, p. 8) stressed that surfaces and edges are fundamental sensations of space, the stimuli for which need to be discovered. A continuous surface establishes as a background and the edges establish a figure contrasting the back ground. In geometry, the superficial appearance of objects means a complete boundary of a figure. A concave object, and others objects such as a pot, we consider having both an internal and external surface.

When we talk about a polyhedron, the surface means a set of faces. Two flat faces meet and form dihedral angles, and the intersection between two such faces are the edges. The edges and the dihedral angles are distinguished. The integrity of both angles and edges of a polyhedron are important in perception. Each polyhedron has its own characteristic (determined by the faces, edges and vertices, and their arrangement) which determines its shapes and differs one from another. So, the perception of a polyhedron relies on the perception of its surface. When we see a polyhedron, we can also perceive some details such as colour and texture which do not characterise the polyhedron.

2.2.3 Definition of visual perception

Despite its widespread study in psychology, the term "visual perception" is rarely defined: it is usually illustrated through examples. The Encyclopedic Dictionary of Psychology (Harré and Lamb, 1983) does, however, define visual perception as:

"the subjective experience arising from sensory stimulation of the visual system and brain."

It goes on to say:

"Perception requires not only detecting the presence or absence of features in the visual image, but also defining their relationship to one and another and assigning them to separate objects,. To achieve this end, perception cannot merely be a passive extracting process. It must also be an active integrating one, associating and organising the sensory information." (p. 655).

We also need to distinguish two further aspects of visual perception that relate to the organisation of sensory experience. First, there is what we call "surface" perception in which processing of information is carried out. This comes about as a result of stimuli that allow one to detect such things as edges of objects, corners, faces, depth and motion. Second, we also have "deep" perception which includes such things as detection of non-immediate visual properties of an object, as well as previously stored information about that object.

For example, surface visual perception is involved when a student sees that an icosahedron has faces that are triangular. Deep visual perception is involved in a student's perception that an icosahedron can be seen as a non-overlapping union of two pentagonal pyramids, and a pentagonal band of alternating triangles:

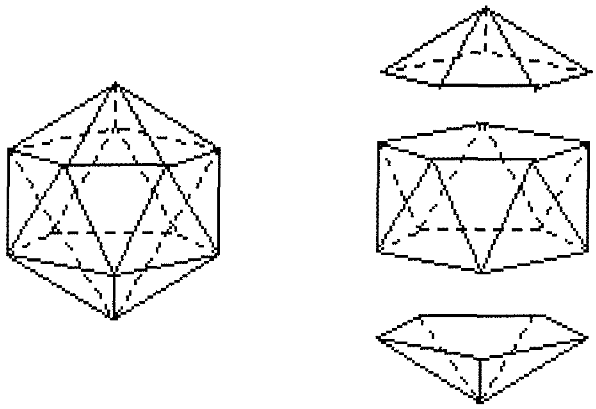


Figure 2.5. An icosahedron decomposed as a non-overlapping union of two pentagonal pyramids and a pentagonal band of triangles

Deep visual perception involves an active, conscious intelligence and is not obtained simply through staring harder. A student's eye does not simply "see" deep features of objects such as polyhedra, in the surface sense of getting them directly from visual stimuli. Seeing is an active constructive process:

"the brain does not passively record the incoming visual information. It actively seeks to interpret it, ... What you see is not what is really there; it is what your brain believes is there." (Crick, 1994, pp. 30- 31).

The term visual perception is associated to physical objects and events. People improve their sense perception as they develop and mature, thus enabling an adult to perceive objects and events better than a baby. When people start looking at new and different objects they gradually become more familiar and certain about them, so they develop a form of learning. Consequently, there may be changes in object perception resulting from experience. These changes in perception, we believe, are associated with thinking about and reflection on, perception.

We can see objects and their properties such as colour, size, texture, and we can talk about events that happen. These are results of visual perception. When we see objects we can distinguish one from another. As an example we are able to distinguish an animal sleeping from a skin coat. However mistakes in interpretation of sensory data may be made and it does not imply that people do not see an object: in this case, people may not recognise what they saw. We can see things before we can identify them. When we do not recognise what we see, we are not able to realise a correct description.

We distinguish the seeing of objects and the seeing of events by our sense of visual perception in a way that requires some knowledge of the things seen: as an example, we can see that windows are open. We can see objects closing and opening without knowing what they are, but to recognise an object or to perceive an event requires some level of identification. Cognitive psychology is interested in what people learn in their perceptual understanding, and with objects it requires us to understand that recognition and identification of objects requires some knowledge of what we see. As an example, we can recognise a three dimensional figure as a cube when, upon seeing it, the figure which we are seeing is distinguished from other three dimensional geometric figures by its properties. Firstly we receive sensory data, then utilise our perceptual sense, and then determine the meaning of what is perceived. To perceive objects and or events involves both perceptual sense and meaning. We make use of visual perception extensively in three dimensional geometry.

We recognise the incoming visual signals according to the information we already have in our brains. The visual signals are just that - simply signals, or data; they do not in themselves, constitute information. It is an active organisation by the brain, of those incoming signals, that constitutes new information. This ordering is assisted by an active, physical, ordering of visual signals.

Perception requires the active process of the observer providing information in the organisation of action. Our perception becomes precise as a result of action and movement, and our visual system plays an important role in guided action. Action guides behaviour, and action is often directed at an object and generates a movement. For example, to pick up an object and use our hands in touch exploration of this object it is necessary to be aware of its shape, size, location and orientation.

2.2.4 What is a mathematical unit?

A unit is something that is seen as a whole, to that can be repeated in action - either with actual physical material, or in the mind. A person's competence in dealing with units - arithmetic or geometric - is important in their mathematical development. Von Glasersfeld and Richards' (1983) description of a numerical unit is:

"A unit is that by virtue of which each of the things that exist is called one", (p. 1)

They stressed that the construction of number is associated to the ability to the act of abstraction from the production of unitary things. So, a child can understand numbers and the development of these processes rely the child's construction, representation and recognition of figural pattern. The understanding of organisation of units is the basis of the development in arithmetic and geometric thinking. We can use the construction of numeric ideas (regarding as units) to describe the physical construction of a certain shape such as numbers of faces, vertices, edges etc that are parts of a whole.

Wheatley and Reynolds (1996) stressed the involvement of reflective abstraction in learning about units. In their experiments using two dimensional shapes to children make tiling, they observed the children's method and thinking strategies used in this activity. So, they noted the relationship between a child's ability to construct abstract units from two dimensional shapes and also their use of ten as an abstract unit in adding and subtracting whole numbers. They conjecture that "constructing abstract units is a quite general and significant mental operation, which transcends numbers".

2.2.4.1 Difficulty with units in three dimensional geometry

In later chapters we study image formation of polyhedra. A major question in the study of image formation in 3-dimensional geometry is: What is a perceptual unit? Kosslyn (1996, pp. 114-115) details how perceptual units are processed and how they are put together to form images, but does not discuss their nature - what it is about a perceptual unit that makes it a whole.

For 3-dimensional geometric objects such as polyhedra, the possible nature of perceptual units is quite tricky. This is due, in part, to the invariant nature of perceptual units: those transformations that, applied to a perceptual unit, transform it into a perceptual unit. For instance we can scale a triangle by stretching uniformly in all directions, or we can apply a shear to a square: are these still the same perceptual units? In using pre-digested chunks of imagery, students manipulate geometric units in their heads. These units, such as triangles, squares and other polygons; pyramids; cones; cubes and cuboids, are capable of being transformed by rotation, shrinking, stretching, and so on, but retain their defining properties (vertex-edge-face relationships).

One might argue that a natural unit in the perception of a regular icosahedron is a triangular face. However triangular faces, in themselves, lack an essential feature of a mathematical unit: they are not necessarily repeatable pieces of perceptual or mental data as far as 3-dimensional object building is concerned. One needs 20 equilateral triangles to build a regular icosahedron, but recognising this only gives the triangles as numerical units: something to be counted. It does not imply that the triangles are seen as geometric units: something to be arranged in space.

The units of a three dimensional shape must have a variety of properties that arranged together form a whole shape. The variety of components characterise a certain shape. In numeration, units is a collection of individual items takes together. So there is no straightforward analogy in construction of an arithmetic and geometric units because for both there are a variety of individual properties that form an individual item, and the concept of unitary thinks that can be repeated comes from the arithmetical concepts.

2.2.5 Imagery

2.2.5.1 Definition of imagery

In this thesis it is important to state clearly what I mean by imagery. This is not an easy task, because it is not normally defined in the psychological literature, but taken for granted. In the following, I have taken the most salient points about imagery from Kosslyn (1996).

By imagery I mean, in the context of geometry, the generation of mental images, the ability to inspect these images, and to maintain and transform them in the head. This is in accord with Intons-Peterson's (1996) definition of imagery as:

"sensory-perceptual memory with spatial extent. ... we learn to label as imagery those memories that have salient sensory-perceptual and spatial features." (p.20).

For example, from a verbal description of an object such as a polyhedron with 6 equilateral congruent faces, one may form an image of a cube. With further information that the sides of faces are equal but the angles formed at adjacent edges are not right angles, one might mentally transform this image to resemble a rhomboid. This involves the processes of formation of an image, retention of that image, and inspection and transformation of the image. Note that in waking experience, images generally preserve the metric properties of space.

A perceived object is related by the senses - sight, touch, sound or taste - to a concrete object and an image is related to an abstract mental form, as for example when a verbal description activates associative memory to get enough information. This is an experience that uses imagination to form an image of an object. Concrete objects are built according to the laws of physics so that we cannot separate real objects from the physical laws to which they are subject. On the other hand, it is possible for images to be subject to alternative interpretation, as when an image of a rigid object is mentally transformed by increasing or decreasing the angles, or the size of faces, into another object. One may even imagine physically impossible objects, such as a Klein bottle, or so-called 'impossible figures'. In this sense the brain's activity is not subject - in imagination - to the physical constraints on real objects.

Both perception and image formation embody information such as:

a) Gestalt Organisation. This refers to seeing an object as a whole that means sum, combination, interaction of each part which correspond to the appearance of the shape. All of this visual information is sent to the brain which infers what, how and when these features form a determined shape. Therefore, the brain is able to identify differences between shapes

b) Association of factors that lead to a particular form such as geometric information that specifies the appearance of a particular shape. For example, metric information distinguishes shapes of the same family, such as different family hexahedra.

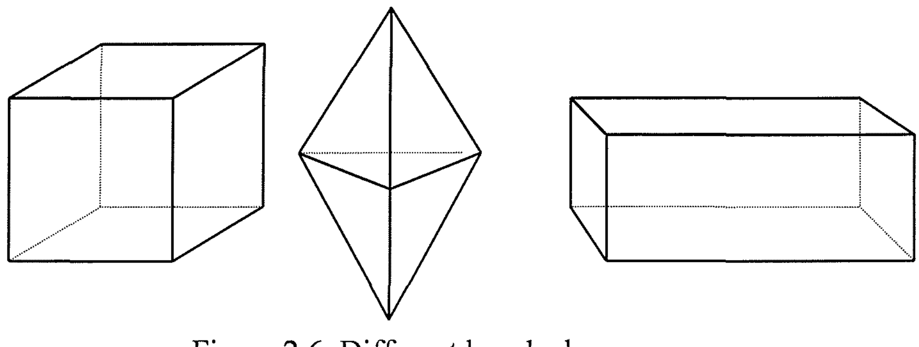


Figure 2.6 Different hexahedra.

A person can distinguish these metrically different hexahedra without conscious awareness of the way in which they were distinguished. It is the perceptual metrically different forms that impress themselves apparently directly to our vision. We say 'apparently' because as individuals are not generally aware of the micro-processes that take place in their acts of visual perception. However, there is another sense in which a person who distinguishes these hexahedra perceptually is not aware of the process, but only of the outcome: the process of distinguishing is based on visual perceptual data and not on verbal data or verbal expression. A person could say, for instance, that the second example above is 'pointer' than the cube; however we contend that this verbally expressed distinction is not part of the visual distinction that the eye-brain makes in the first instance. The hexahedra are distinguished by the eye-brain without the intervention of verbally expressed properties.

c) Organisation of units.

• There are constraints on how perceptual units can fit together to form a polyhedra: for example we can form a hexahedron from equilateral triangles or squares only or using some quadrilaterals. On the other hand we can not built a hexahedron using a hexagon.

d) Comprehension of how units may be organised:

(i) Different size

(ii) Different position.

2.2.5.2 Images as one form of mental representation

When we think about an object which is not present, we are forming and inspecting an image of this object which often contains a geometrical characteristic. Kosslyn (1996 p 3) refers to mental imagery in terms of mental representation:

" the term 'image' will refer to the internal representation that is used in information processing, not the experience itself."

Eysenck and Keane (1996) define a representation as:

"any notation or sign or set of symbols that 're-present' something to us. That is, stands for some thing in the absence of that thing." p. 204.

Things are associated to the physical world or our imagination. There are distinctions between external representations such as pictures, words, and writing and our internal mental representations of these things.

Mental representation are styles of imagination. Johnson- Laird (1996) stressed the form of representations and their process in construction. To him, the distinct forms of processing indicates the distinct format of representation. He (Johnson-Laird, 1983) emphasised three different forms of "representation": images, propositions, and mental models. These forms of representation are different but interrelated. This triple code theory stresses that propositional representations are related to expressions in a mental language structure whereas images and mental models have an analog spatial structure.

Linguistic expression builds propositional representations and then creates mental models. Propositional theory originates from language and logic and propositional representations are abstract. They are not words, they are only language-like. According to Anderson (1995)

"... a proposition is the smallest unit of knowledge that can stand a separate assertion; the smallest unit about which it makes senses to make judgement true or false."

Kosslyn defined propositional representation is a 'mental sentence' that specifies unambiguously the meaning of an assertion.

We are able to construct a mental representation of an action and an event. As an example, while we listen to a description of an object or an event or read a text we are able to construct a mental model. We rely on retaining in memory these constructed mental models and being able to and recall them. We also are often able to explain our mental models in language, when they present as images, ordering the words so listeners decode the words and reconstruct their representation of our mental models (our mental understanding).

Images are associated to both thought and sensation. The difference between reality and images is that images are abstract and present intentional properties: to construct a new image requires a mental action. It is possible for us retain images and we are able to transform them - new images result from our mental activity - and reconstruct these images. On the other hand the concrete material that we can see or touch are physical objects and we experience and distinguish them through their effects on us through perception.

2.2.5.3 Concrete models as representations of images

Kosslyn (1996) views images as representations of objects or scenes:

"The term 'image' will refer to the internal representation that is used in information processing, not the experience itself." (p. 3)

and

"According to the present theory, image generation is an extension of the kind of attentional priming discussed in chapter four. The representation of an object is accessed (by a property look-up subsystem, ... To form an image itself, the representation is primed so much that it sends its feedback to earlier areas ... . (p. 287)

This view of mental images is a consistent one in the psychological literature. We see, for example, an image of a particular face, and understand that it is an internal mental representation of a particular person's face. This is an example of how images can be mental representations of particular physical objects. Or, we may see an image of a musical event that we attended earlier. These images are mental representations of particular world objects or scenes. What, however, about images of general classes, such as 'chairs'? When we form an image of a chair, is it a general, somewhat blurred, image of a generic chair, or is it an image of a particular object - a chair we have seen before? A designer, for example, may form an image of a chair never before seen ,because they are yet to design it. This is an example of forming images from previously established perceptual units.

However, in mathematics, and particularly in 3-dimensional geometry, the situation is somewhat different. Some images are categorical: a regular icosahedron, for example, can only be imagined, correctly, in one way (taking into account differing sizes). This is different from forming an image of a dog, even a specific variety of dog such as a poodle, because there are many different dogs and poodles in the world: such an image is not categorically determined. For irregular polyhedra, the situation is more like that for forming images of dogs: for example, there is a whole class of irregular icosahedra.

So, at least for 3-dimensional geometry, we have the peculiar situation that a mental image of something as specific as a regular icosahedron is not a representation of that object: rather, concrete models of regular icosahedra are representations of the mental image. Drawings and concrete models are communicative devices, designed to represent what is seen in the mind. Hunting and Lamon (1996) made the same point in relation to concrete material:

"we shall use the term 'model' to refer to a cognitive (internal) construction, a system of quantities, relationships, operations, and representations constructed in some subjectively meaningful way, connected to the individual's existing knowledge base, and used to make sense of one's subjective world of experience. Representations (or embodiments) such as Cuisenaire rods or Dienes blocks are concrete (external) interpretations of internal models, notation systems that facilitate the communication of our models to others."

This is not to say that models are useless in assisting students to form mental images; to the contrary, they can be quite useful. However, images can be inspected for propositional information (Reisberg & Chambers, 1991): for example a person imagining a hexahedron can count the number of vertices and edges by inspecting an image.. The ambiguity, if any, is in a student's visual perception of the model.

2.2.5.4 How might images of three-dimensional geometric objects be formed?

There are two obvious ways in which people might form images of three-dimensional shapes, such as polyhedra. One is by determining vertex-edge-face relationships, and fitting together individual faces in an appropriate arrangement. The other is by forming images from larger pre-digested pieces of the image, which are then assembled.

Kosslyn indicates that images are formed from remembered units, so we expect that it is the latter mechanism by which people form images of polyhedra.

Further, limitations in working memory also made us expect that this is the way people will form polyhedral images. Kosslyn (1996, p. 291) points out that:

" .. it is better to relate parts and properties to the global image than to other parts or characteristics in the description of the shape in associative memory; if one builds detailed hierarchies of structural contingencies ... it would be rather easy to exceed working memory (i.e., limitations on how much information can be accessed at the same time) when generating an image."

In image formation, seeking information about regular sub-objects such as cubes and pyramids that might fit together in certain ways to give the entire polyhedron, is a sort of mental "chunking" procedure. It involves mentally manageable sub-units of the polyhedra which are built up, so allowing missing bits of the image to be filled in. We hypothesis that this is the way people form images of polyhedra, rather than focusing on analytic information about vertex structure. The reason is for them to reduce the cognitive load of image formation.

2.2.5.5 Why might images be useful?

One answer is that, as Kosslyn (1996) says:

"Perhaps the most basic property of visual mental imagery is that images make accessible the local geometry of objects. ... The image depicts the spatial relations among portions of an object or scene allowing one to interpret them in a novel way." (pp. 335-336)

This is one of the most useful features of images: they are manipulable in the head. Images can be mentally pulled apart and re-arranged, using parts of other images, in ways that would be difficult or impossible with concrete models. For example, a regular octahedron can be mentally transformed into an irregular octahedron, by a mental stretching of one of the constituent pyramids:

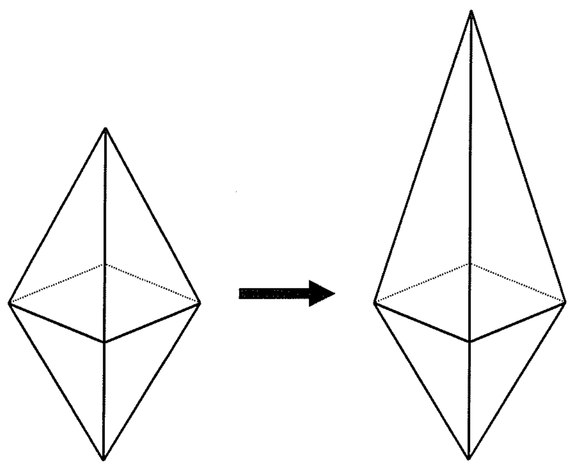


Figure. 2.7 Shows the regular and irregular octahedron.

2.3 Learning through discussion

2.3.1 Introduction

Language is a fundamental component to explore image generation. From the individual effect of linguistic description another person forms an image of an object described. To form images from description, requires that people who listen to a description from someone else first store this verbal information in their short-term memory. They then use this to retrieve knowledge stored long-term in their mind. It is this long-term knowledge that, stimulated to recall by words in short-term memory, assists in the formation of images. Kosslyn (1996 p. 336) reports that imaged objects are interpreted using the same mechanism that are used to interpret objects during perception. Denis and Cocude (1989,1996) emphasise that image generation depends on the main characteristics of the description of a scene or object. These authors designed a experiment in which people were not presented with a map at any time but a description of a map of a circular island was given. The fictitious island contained six features placed at the periphery: a harbour, a lighthouse, a creek, a hut, a beach and a cave. From the description, the location of features was based on an hour coding system. The results of these experiments indicated a correlation between response times and distance. According to Denis (1996 p. 138) the mental representation constructed from verbal descriptions contain information structured in a way similar to perceptual representation.

Kosslyn, Ball, and Reinser (1978) designed an experiment in which subjects were asked to memorise a given map of an island which contained seven objects: a hut, a tree, a rock, a well, a lake, sand and grass. There were twenty one pairs of objects positioned in different distances. The subjects learned to draw the location of each object (with considerable accuracy) on the map. Later, they were asked to close their eyes to visualise the map. When the subjects listened to the name of an object, they were required to picture the map mentally and focus on the object named. After five seconds, other objects were named. Subjects were required to scan the map and focus on these other objects mentally. They were informed to press a button when they had mentally focused on these object. The researchers reported results regarding image scanning, distance, amount of material scanned over, and the reaction time. The time taken for the subject to perform the mental operation was plotted as a function of the distance between the two objects in the first map. The amount of material scanned was kept constant and the result was that the time increased linearly according to the distance scanned.

Denis (1996) & Denis & Cocude (1989,1992) described an experiment regarding mental scanning studies which indicate the capacity of people to construct images or cognitive maps from verbal descriptions of spatial configurations. They focused in their studies on learning from a map and learning from a verbal description of a map. In the first situation people memorise a map and the location of its features. In the second, they explored the mechanisms that allow people to build up an image from communication about space. People used verbal description of scenes, and description of itineraries in unfamiliar environments.

These experiments indicate that from verbal descriptions, people can build visual images and that they scan these images in much the same way and time as they would visually scan the real objects or scenes.

2.3.2 Paivio's dual coding theory

Cognitive psychologists have been interested in studies of the processes that relate thought and words. Johnson-Laird (1983) argues that there are three kinds of representation: mental models, propositions, and images. Paivio (1986, 1991a) developed his dual code theory which proposes a basic distinction between imagery and propositional representation. The two different system for representation and processing of information are the non-verbal or imagery process which involves storage of information of non-verbal objects and events, and the verbal symbolic process which involves storage information of word meaning (linguistic information is stored in an appropriate verbal form).

There are different modalities to process both systems:

Sensorimotor System	Symbolic System	
	Verbal Linguistic world	Non verbal Non-linguistic world
Visual	Printed word	Visual object
Auditory	Spoken word	environmental sound
Hapitic	Writing	manipulate objects

Table 2.1 The modalities which involve verbal and non verbal processes.

By definition, the taste, smell and affective modalities are non verbal and there are no corresponding constructions of linguistic symbols.

The theory of the two systems, verbal and non verbal, assumes that the cognitive behaviour mediated by these two independent, and partially interconnected systems, are appropriate to encode, store and retrieve information. Information can be transferred from one system to another or either system may work in isolation. The verbal system is considered as an abstract logical mode of representation whereas the non-verbal system is concerned more with the concrete analogical mode. An object - for example a cube - has a recalled name “cube”. We recognise the shape by its recalled image, and we can link the image with the word “cube”. Paivio adopted the term “logogen” referring to a word generator (it is specific to the sense of identification of the spoken sound of the word and the visual form of the written word) and “imagen” referring to image generator. The interconnection between systems links imagens and logogens, allowing objects to be named and names to evoke images.

The dual coding theory is relevant in this work which uses both verbal and non-verbal mechanisms: both manipulatives and the their names are used as stimuli.

When people form an image of a polyhedron from a description, the image may change according to the amount of the data given. The verbal data is encoded and decoded and images are evoked from the stimuli with contributions from memory. Therefore, stimuli, memory, and imagery are connected in the generation of a visual image from a verbal description. Memory for words is quite distinct from memory for pictures of common objects and there are differences between the recall of words and of pictures. Several operations are associated to memorisation of words or visual scenes or objects. Paivio's work basically involves memory for the name of stimuli shown in a picture. In our work we are interested in more than the memory of the names of the stimuli: we are also interested in the details of three-dimensional shapes. Visual learning and learning from verbal description are fundamental in this work. They involve imagery tasks and the use of verbal codes. Verbalising visual perception is related to an individual's capacity to identify and organise sentences. In our experiment, when the polyhedron was a common, or well-known, shape its parts were described and memorised spontaneously. Students who manipulated and described the shapes encoded information using both, verbal and non verbal system: words were used to indicate things perceived. In opposition, the students who listen to descriptions, invoked memory using the verbal system alone, and for them, words are used to evoke an image.

Chapter Three

Methodological framework

Chapter Three

Methodological framework

2.1. Introduction

This thesis addresses connections between visual perception, tactile perception, imagery, language development and van Hiele levels of thought in three-dimensional geometry. The empirical research was based on a written test and on two different forms of experiments involving identification of properties of polyhedra, and the sharing of those properties with other students.

2.2 Written Test

I designed a test containing seven questions in three-dimensional geometry with each question involves various sub-items. The test initially had 7 items which were designed to generally relate to van Hiele levels of thought in geometry (van Hiele, 1986; Burger and Shaughnessy, 1986; Fuys et al, 1988; Gutiérrez et al, 1991) and to the degree of acquisition of spatial abilities (Del Grande, 1987). The ways in which each question addressed van Hiele levels and Del Grande's categories of spatial abilities is indicated in the description following that question in chapter four.

Note that I do not wish to claim the results of this test place precisely students into van Hiele levels or precise categories of spatial ability. Rather I have designed a test based around van Hiele levels and spatial abilities, the validity of which is open to interpretation.

The different questions assess a student's use of adequate language in relation to each of the van Hiele levels, their spatial abilities.

2.2.1 Del Grande's spatial perception abilities

Del Grande (1987) define spatial perception as

" the ability to recognise and discriminate stimuli in and from space and to interpret those stimuli by associating them with previous experiences."

He examined seven abilities which are relevant to cognitive development. Our test in three-dimensional geometry is related to five of these seven spatial abilities.

1. Figure-ground perception
2. Perceptual constancy
3. Position in space perception
4. Perception of spatial relationships
5. Visual discrimination

which Del Grande describes as follows:

Figure ground perception is the ability of identify a specific shape which involves intersecting lines, intersecting figures, figures completion, figure assembly. Figure assembly involves the use of two-dimensional shapes to obtain a new shape in three-dimension.

Perceptual constancy or constancy of shape and size is the ability to recognise shapes according to their appearance. The shape remains unaltered, the size of the shape does not change if the line of sight turned at different angles. However, their sensori image vary.

Position-in-space perception is the ability to determine the relationship of one object to another object and to observer. The activities regarding to this ability involve change of position, and rotation of shapes.

Perception of spatial relationship is the ability to see one or more objects in relation to oneself or in relation to each other. The activity regarding this ability involves replication of part of three-dimensional figures.

Visual discrimination is the ability to distinguish similarities and differences between objects. The activity regarding this ability involves classification of geometric shapes.

The perception of space can be explored by the interpretation of experiences of seeing, and moving. According to Del Grande (1987):

"... a person with perceptual constancy will recognise a cube seen from an oblique angle as a cube, even though the eye gets a different image when the cube is viewed from squarely in front or directly above."(p.128)

2.2.2 Background to van Hiele levels of thinking

In 1957 Pierre Marie Van Hiele and his wife Diana Van Hiele-Geldof developed a theory providing a model of pupils' knowledge development in geometry in separate doctoral dissertations at the University of Utrecht. The great interest of Van Hiele and his wife was to find ways in which their students could develop insight into geometry. Diana died shortly after her dissertation was completed; consequently Van Hiele was left to explicate and advance the theory. Van Hiele observed that in some of Piaget's interviews with children, the children often required a knowledge of vocabulary or properties beyond the phase of learning that they were at. From this, the Van Hiele's went on to suggest that all students progress in their acquisition of geometrical knowledge by passing through five levels of thinking in consecutive order. They further stated that a student's level of thinking at a particular time could be identified. The van Hiele levels are sequentially graded. Mayberry (1983) and Crowley (1987) described the lower level as being the basis of the higher level. The van Hiele levels are described below.

2.2.3 Van Hiele levels of thinking

Level 1 *(Visualisation)* Understanding basic geometric concept involves reasoning by the student at this level by means of a visual conceptualisation of ideas. Students observe objects and identify figures by comparing them with mental images but they do not identify properties of a figure. They recognise simple shapes by their appearance without distinguishing their properties. They are able to recognise similar figures with different sizes.

Level 2 *(Analysis).* A student distinguishes properties of a figure by informal analysis of the component parts. They do not grasp the relationship between different figures or between properties. The figures are recognised by their parts (faces, edges, vertices etc.). They do not grasp the relationship between different figures or between properties and they cannot explain relationships between properties.

Level 3 *(Informal deduction)* A student, through the process of definition, distinguishes the relationships between properties and figures. Example: a cube has opposite faces parallel and right angles. Students are able to classify families of solids. They can use informal argument, however, do not understand the significance of deduction as a whole. They can understand properties of concepts and form definitions, but they are not able to arrange sets of statements.

Level 4 (*Deduction*) A student understands the ordered steps involved in proving theorems, using rules and axioms, and elaborates ordered statement one from another and provides definitions. They are able to develop a proof in more than one way.

Level 5 (*Rigor*) Students is at a high level of reasoning. They can work in a variety of axiomatic systems. They carry out rigorous formal deduction. They understand geometry without a model. Geometry can be seen to be abstract.

2.2.4 Test administration

The test was administered over a two year period. Twenty five students who took part in this work were enrolled in secondary initial teacher training (PGCE). All the students had completed a first degree in mathematics All of them were volunteers from the University of Southampton, UK. 12 of the students were from the year 1995-1996 and the other 13 students were from the year 1996-1997.

2.2.5 Description of the test

The questions of the test were chosen in such a way as to indicate how the students organise their thoughts and how they use their previous knowledge and logic to build an argument. When students are required to give arguments it is necessary that they use an adequate language, according to van Hiele's theory, to explain their reasoning in three-dimensional geometry. First students must understand the question and then they must think about the solution to give an answer. According to van Hiele's theory, the development of a person (child or young adult) is dependent on language and environment.

A variety of manipulatives made from cardboard and plastic were available, and students were free to use this concrete material to assist them in answering the test questions. Some of these materials were presented in three-dimensional models and others in two-dimensional form. Each of the seven questions contained various sub-items. Pen and papers were also available to the students.

2.3 Experiment on visual perception and image formation

Van Hiele (1986) has argued that at each level of understanding in geometry, a characteristic language develops. A student learns a level of description that is appropriate to their level of understanding and geometric thought. This has two implications - first, by listening to and analysing students' language use in geometric settings we can to a large extent judge their level of geometric thought. Second, by enhancing a student's language use in geometry we automatically improve their level of geometric thought.

In three-dimensional geometry physical models play an important role in a learner's understanding of an object's properties. In examining a particular polyhedron, a learner needs to find out what specifically characterises this polyhedron: what are its properties that distinguish it from other polyhedra. This entails a process of intelligent action and reflection, through manipulating a physical model and trying to integrate sensation into a coherent perceptual mental picture.

A view of language and cognition, in the sense of how language schemes can provide a view of cognitive construction, provides us with a bridge between individual cognition and its development in social practice.

Students can relate and share their reflections during discussions, whilst we can attempt to ascertain the students' level of geometric thought from their propositional statements and questions.

2.3.1 Information gained through discussion

There are two distinct ways in which discussion can assist in the understanding of 3-dimensional geometric objects such as polyhedra. One is in the development of visual perception and the other is in image formation. If a person can see and touch a polyhedron then providing information about the properties of the polyhedron through verbal representation can assist in the development of deep visual perception. Previous knowledge, coupled with a desire to explain the polyhedron's properties, allows the possibility of a person seeing beyond superficial properties. Visual perception of 3-dimensional objects is often accompanied by tactile perception. These two forms of perception integrate to allow a student a visual-tactile perception of an object. A need to describe this perception in words to another student can crystallise perception in striking ways. The integration of surface visual perception and haptic exploration in the development of deep visual perception is not incidental. Indeed it is a dynamic *responsible* for development. As Thelen and Smith (1995) so persuasively point out:

"As a consequence of the neuroanatomy, sensory integration may be the primitive. The developmental task may not be to *construct*, but to *select* from all the multimodal associations those that represent real-life correlations of perceptions and action in the world." (p. 191)

The other way in which discussion can assist in the understanding polyhedra is through image formation dependent on verbal information. Verbal reports can be utilised by others to allow them to build mental representations of the polyhedron, usually through the formation of images. Following Johnson-Laird (1996) we view mental images as one form of mental representation, the others being propositions and mental models.

One might think, therefore, that I have confounded image formation and propositional knowledge in allowing that verbal descriptions can lead to imagery. However, there is evidence (Denis, 1996; Kosslyn, 1996; Denis and Cocude, 1989; Glushko and Cooper, 1978) that people can build images from verbal descriptions and that mental images formed as a result of these descriptions use visual mechanisms. According to Denis (1996 p. 138) mental representations constructed from verbal description contain information structured in a way similar to perceptual representation. Image formation is dependent on, but different from visual perception. Indeed, as Kosslyn (1996) points out: "... imaged objects are interpreted using the same mechanisms as perceived objects."(p. 327) and:

"Although imagery and perception rely on the same mechanisms, those mechanisms are not used identically in the two cases. In particular, images contain "previously digested" information; they are already organised into perceptual units that have been previously interpreted. In contrast, in perception one must organise the input from scratch and match it to stored representations; one does not know in advance what the object is likely to be." (p. 329)

Language is a fundamental component to explore image generation. From the individual effect of linguistic description another person forms an image of the object described. The formation of images from description requires that people who listen to the description retain verbal information and retrieve knowledge stored in the mind. Denis and Cocude (1989,1996) emphasise that image generation depends on how a scene or object characteristics is described.

2.3.2 Rationale for the task

In this study I focus on the stimulus to form images from verbal descriptions, to form deeper visual perception as a result of questioning, and the possible transition of geometric thought from one level to another as a result of verbal interactions. The task we chose was that of identification of an unseen polyhedron. One student had a polyhedron out of sight of a group of others, whose task it was to determine the nature of the polyhedron by asking questions about it. A screen separated the student with the polyhedron from the others, so that the polyhedron was hidden from their view. The student with the polyhedron could be heard by the others who could see and talk to each other.

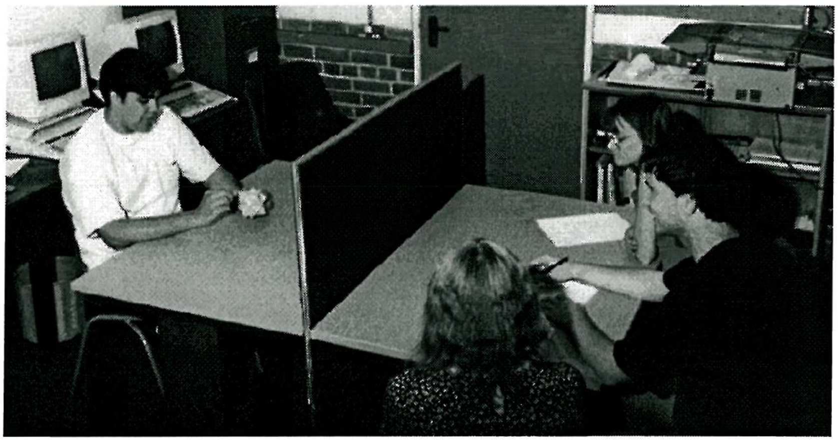


Figure 3.1. Intended arrangement of students during the task sessions

In each session we instructed the student with a polyhedron to answer questions from the other students, but not to volunteer information about the polyhedron. We also instructed the student with a polyhedron not to name it if they knew its name. The students who could not see the polyhedron were instructed to ask questions until they were satisfied they had a complete description of the hidden polyhedron. These students were permitted to talk with each other and to draw if they wished.

The student with the polyhedron will use language to describe visual perception, in answer to questions. The aim is that the other students will be stimulated by these verbal descriptions to form a mental image of the hidden polyhedron. In each session therefore we can document and analyse the visual perception of the student who has the polyhedron, the imagery of the students who cannot see it, and the verbal interactions that stimulate both.

The ability of the students to identify the hidden polyhedron to their satisfaction depends on two things. The first is their ability to form images from verbal data. The second is the ability of the person with the polyhedron to exercise appropriate visual perception and to supply other students with that data.

2.3.3 Task population

There were 7 students who participated in the task. They were volunteers from a class of 12 pre-service secondary mathematics teachers at the University of Southampton, UK, and all had completed a first degree in mathematics. Not all students could attend all sessions, so they were not arranged systematically into groups. There were 5 sessions in all, each lasting approximately an hour, and each involving the investigation of several polyhedra.

2.3.4 Manipulatives used

The polyhedra used were: cube, cuboid, rhomboid, tetrahedron, octahedron (regular and irregular), dodecahedron, icosahedron, dual cube and octahedron, dual tetrahdra, hexagonal prism, right cones with square and hexagonal bases, hexagonal bisection of a cube, and triangular prism. These polyhedra were chosen to represent a selection of easy, moderate, and difficult to recognise polyhedra. Some, such as a cube, tetrahedron, and pyramid, are commonly known, and have relatively few edges and faces, with simple arrangements of faces. At the other extreme objects such as a dual cube octahedron, dual tetrahedra, icosahedron, and a hexagonal bisection of a cube, are not commonly known and involve relatively complicated arrangements of different types of faces. The latter, in particular, were predicted to be relatively difficult to imagine.

2.4 Tactile experiment

2.4.1 Introduction

While we are generally strongly dependent on a sense of vision to detect and recognise objects, our fingers can also give us objects around us. Our hands are a sense organ that we can use to get data about shape, size, weight, temperature and so on. We also use touch with purpose to identify an objects' surface, texture, softness, hardness, smoothness and other features. Some of this sensory data, such as temperature, may not be apparent to visual perception. According to Warren and Rossano (1991) the judgement of some properties such as size and shape are generally performed more accurately with vision than with touch.

Shapes of objects are defined according to their surface arrangement and we can use our visual or tactile systems to recognise them. The task described below is a haptic exploration focusing on the ability to discriminate the location and parts that make a polyhedron. Students get tactile data and provide information about their sensations of feeling. In this tactile experiment, we provided the students with a **feely box** and some **polyhedra** which students manipulated (see the picture below). They used their hands to help form the image of the shapes by what they felt. Therefore, in this investigation, the hands' exploration was used with the purpose of acquiring information about geometric properties of the shapes. Hand movements were also used to relate to the dimensions and position of parts of the polyhedron in space. In this research we were not interested in the texture, size, hardness, or weight of the polyhedra. The motor system was used in this activity to grasp and manipulate the shape.

Figure 3.2 Students use hand movements to represent part of the polyhedron and their position in space.

The stimuli were presented in the three-dimensional form. All objects were polyhedron. When people use their fingers in a simple way to explore shapes, the skin first gets essential information of a two-dimensional form and our cognitive system provides organisation of this information to form an image of a three-dimensional shape. All the movements of the fingers are associated with the examination of shapes and knowledge.

The literature presents research on non-visual perception that emphasises perception by feeling or by enclosing an object using one's hands. Little research using this procedure seems to explore geometric properties of three-dimensional shapes. According to Gibson (1966) and Loomis & Lederman (1986) the haptic system is a perceptual system that uses both cutaneous and kinesthetic inputs which are acquired by manual exploration. Lederman & Klatzky (1987,1990), Klatzky & Lederman (1992, 1995), Klatzky *e.t al.* (1993) have done significant investigations regarding the conditions under which object dimensions are integrated during haptic exploration.

Lederman & Klatzky (1987) have established the relationship between perceptual dimensions in haptic exploration and exploratory movements. In their experiment they used a match-to-sample task, in which blindfolded subjects were presented with multdimensional objects. They were interested in information about the objects. They analysed hand movement during haptic exploration to identify the texture, hardness, temperature, weight and volume.

2.4.2 Population

Data was collected from tactile exploration among undergraduate students. Two students, in their third year of their mathematics course, are referred to in Chapter ?. These two students are called **M** and **S** and they were volunteers from the University of Southampton, UK.

We planned only one session involving investigation of polyhedra by haptic exploration. in this session the students worked together. We could identify their schemes and hand movements used by the students to get information about shapes, by having the bottom of the feely boxes open to our view, as in picture 6.1 above. The duration of the session was approximately one and a half hours. The whole session was videotaped.

2.4.3 Procedure of tactile observation

During the practical experiment each student has a feely box, and each feely box contains one cardboard shape of the same form and size: the pair of each shapes used in this experiment were congruent. All the polyhedron to be used in this experiment are the same as those to be used in the visual-tactile experiments. The shapes are placed in the feely box one each time and replaced by a different shape after students declare they are satisfied with the discussion. The shapes to be used in this experiment are a cube, cuboid, hexagonal pyramid, triangular prism, squared pyramid, tetrahedron, irregular octahedron, regular octahedron, rhomboid, dodecahedron, icosahedron, cube-octahedron, dual cube-octahedron, and dual tetrahedron.

Students are also required to represent the objects that were in their mind by drawing them (if possible). The intention in this experiment is to focus on the way students get information, form images, and the way they externalise what they feel.

In haptic perception of polyhedron, not only is the shape of the objects significant, but also their size is important to evaluate their parts. It becomes more difficult to identify a shape if one holds a very small or a very large example of it. Some difficulties in identification of the shape may occur as a consequence of constraints on our motor system. I used neither very large nor very small polyhedron. The shapes were of a size that students would likely feel comfortable whilst exploring them.

Chapter Four

Results from the written test

Chapter Four

Results from the written test

4.1 Introduction

This test described in the previous chapter was designed to assess students' geometrical and spatial abilities. The purpose of this test is provide a base-line assessment of where students are in their geometric thought when we involve them in other learning activities. We used the test results in the following chapters about visual-tactile perception and image formation experimental results and tactile-only perception and image formation.

4.2 Test questions

Question 1

a) The diagrams below represent three-dimensional figures. Can you group the diagrams that represent the same three-dimensional figure?

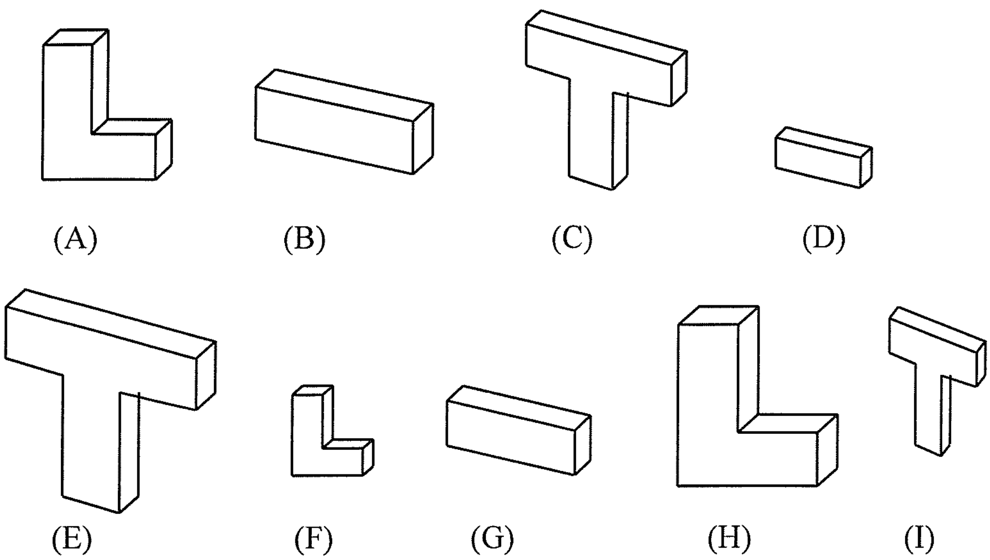


Figure 4.1 Three-dimensional shapes presented on question one.

b) What are the similarities and differences in each group?

Question 1 explores basic geometric concepts which involve visual conceptualisation of ideas. This item is associated with Del Grande's perceptual constancy of shape and size, namely that students recognise that a figure has invariant properties. The sizes are compared between figures which represent the same shape. The question requests that the same figures should be grouped by their similarity in shape irrespective of size. In this question it is not necessary to list all the properties of the shapes.

Question 2

The figures (i) and (ii) are two bi-dimensional shapes.

Is it possible to construct three-dimensional shapes using only figure (i) shown on the diagram below?

If it is possible, draw the three-dimensional figures that you think can be constructed, using only figure (i).

Name the three-dimensional shape

If it is not possible to construct three-dimensional shapes, explain why not.

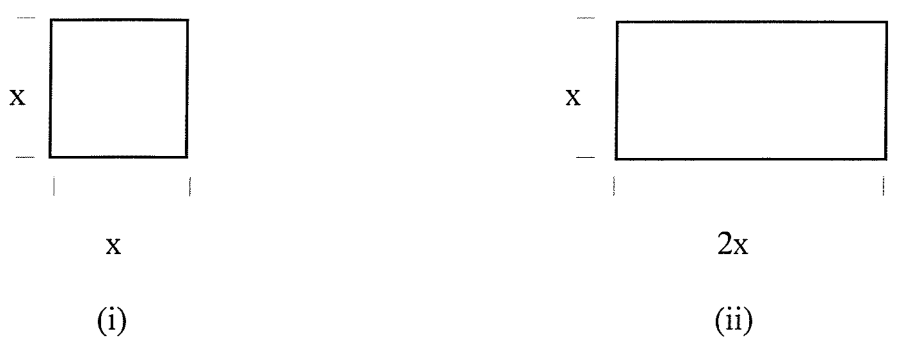


Figure 4.2 A square and a rectangle presented on question two.

Is it possible to construct three-dimensional shapes using only figure (ii) shown on the diagram above?

If it is possible, draw the three-dimensional figures that you think can be constructed, using only figure (ii).

Name the three-dimensional shapes.

If it is not possible to construct three-dimensional shapes, explain why not.

Could you list the properties of your solid(s)?

This question is related to van Hiele level 2 and Del Grande's figure ground perception. It explores a student's ability to perceive intersecting figures, figure completion, and figure assembly. On question 2, a square (figure i) and a rectangle (figure ii) (with the size of two squares together) were presented. This question explores the combination of two-dimensional shapes to produce a new shape in another dimension. Also this question works over the combinations of three-dimensional shapes to obtain another one.

To access Del Grande's figure ground perception we considered it an adequate answer if the students presented a drawing of the cube using only squares, and the drawing of the cuboid using rectangles. To get an adequate answer is not necessary to extend the plane faces.

Examples of how students might arrange two-dimensional shapes to get a three-dimensional shape are as follows:

1. With the squares with length x, one can build a cube. If one organises the shapes by extending the face plane - as two cubes together - one can build a cuboid
2. One can build a cuboid using rectangles with length 2x and width x. One can therefore construct a cube as a combination of two cuboids together.

There are many distinct ways to arrange squares or rectangles to get different cuboids and different sizes of cubes. All the new figures that are obtained from combinations of squares or rectangles have the faces in pairs lying in parallel planes. Indeed, the new solids appear by extending face planes.

Question 3

The diagrams below represent three-dimensional figures.

a) Can you group the diagrams that represent the same three-dimensional figure?.

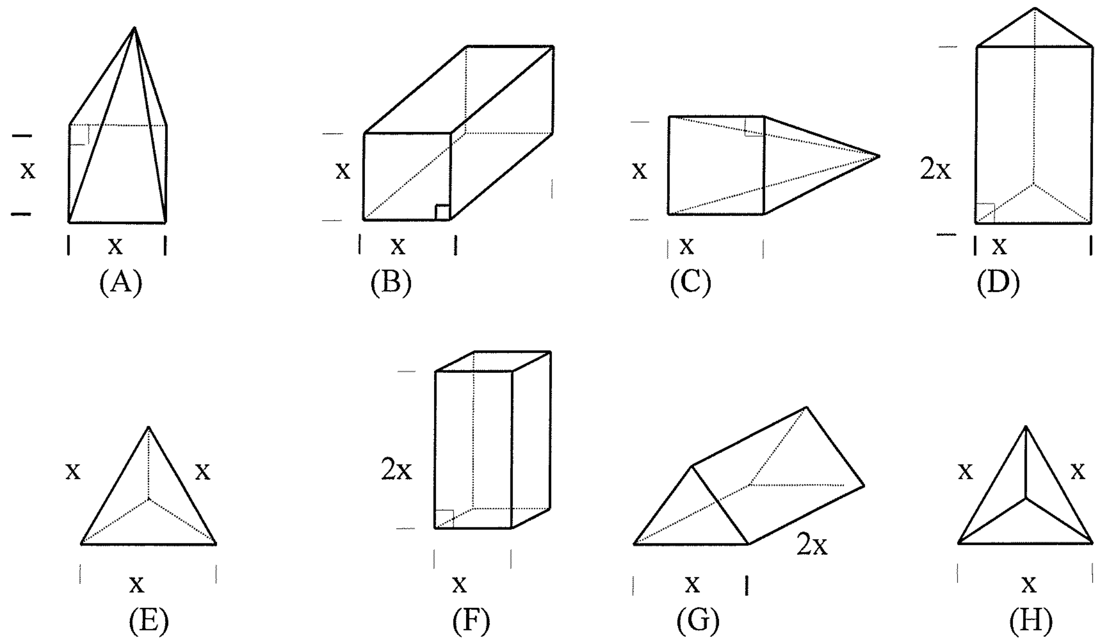


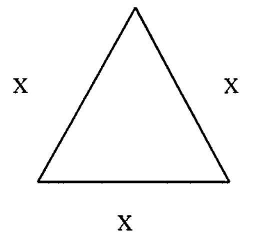
Figure 4.3 Polyhedra presented on test question three.

b) List as many properties as you can for each group of figures.

Question three requires that students group figures by their appearance. This item is linked with Del Grande position-in-space perception in which students determine the relationship of one shape to another, when they are congruent and in different positions (rotated), so in different view. Students were requested to distinguish figures of different shapes and recognise the equivalence of different views to successfully group them. The list of properties for each group are requested and these properties would be given by informal analysis of the component parts.

Question 4

The first figure shows a polygon. This polygon was used to make three solids, shown below as (A), (B), (C).



This polygon was used to make 3 solids, shown below (A), (B) and (C).

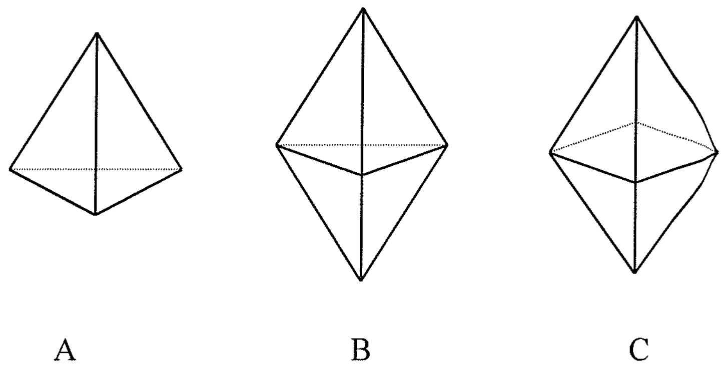


Figure 4.4. Polyhedra presented on test question four.

a) For each pair of solids list as many differences and similarities as you can.

b) A <u>regular</u> shape is one in which..

c) Which of the shapes (A), (B), (C) are regular and which are irregular? Explain your answer.

d) Draw the net of the shapes (A), (B) and (C)

In this question it is necessary to perceive and describe the figures' properties by their similarities and differences. The way to distinguish one figure from another is to recognise the set of properties in combinations. Examples are:

- all three-dimensional figures are different in the number of faces, edges and vertices, therefore, they are different shapes;
- give the numbers of faces, edges and vertices of each figure;
- these three- dimensional figures have triangular (equilateral triangle) faces;
- to each three-dimensional figure one could pay attention to the numbers of faces meet in each vertex.

However for students to attain a high van Hiele level it is necessary at least to present an argument about transformation geometry.

When we combine the shapes' properties we are describing and classifying them. Obviously, the classification of the figures in the question distinguishes the arrangement of the shapes. According to Roth and Bruce (1995)

"Concepts are mentally represented as combinations of necessary and sufficient properties which define the categories so represented." (p.24)

These authors refer to two-dimensional shapes. This theory can be applied to three-dimensional shapes. This is because when the number of faces of a figure are described- such as: the shape has six faces which means that the shape is a hexahedron-the other properties such as the faces are equilateral triangles and other necessary properties can be determined through discussion.

Question 5

The picture below shows a church.

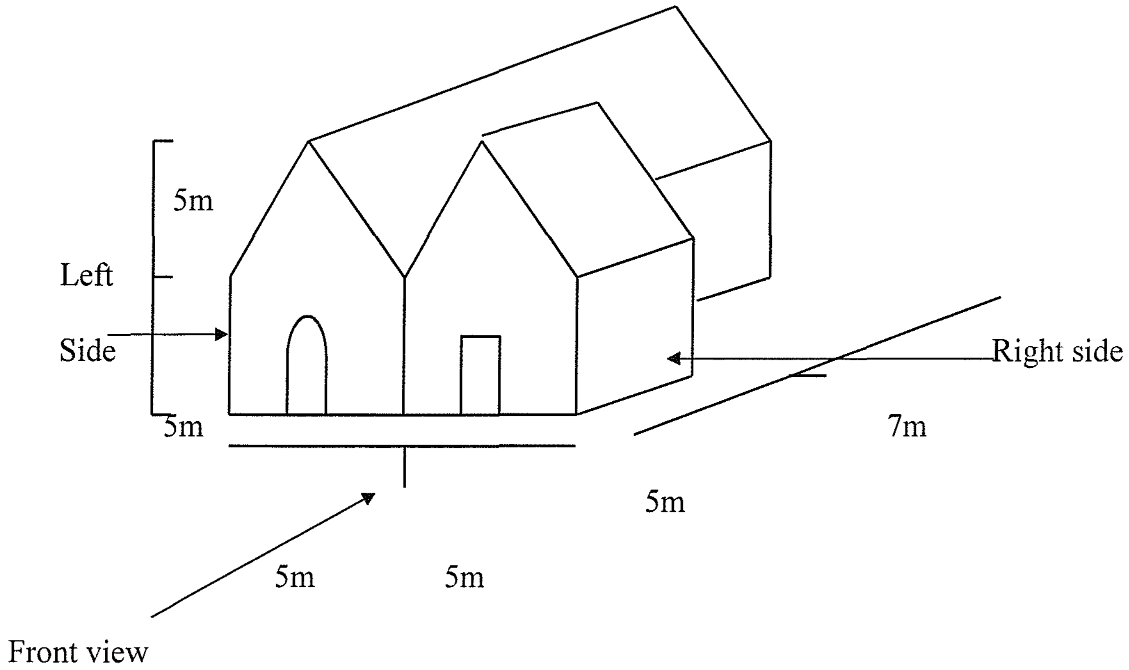


Figure 4.5 The church presented on test question five.

Could you draw the views:

a) Base.

b) Front

c) Left side

d) Right side

Question five presented a simple figure of a church. Students were requested to identify the views which are represented by a drawing of a three-dimensional picture of church. Models of the church were available if students wish to use them.

Question 6

a) Are the following conditions sufficient to uniquely determine a solid?

a. 12 edges

b. 8 vertices

c. 6 faces

b) Argument (If there is only one such solid, explain why. If there is more than one, say what *different* properties they have):

This question explores relationships between shapes by a common set of properties and is related to van Hiele level four and Del Grande's figure ground perception: the ability to identify specific figures by their components, figure completion, and figure assembly. The properties of the figures and their interrelationship provide students with opportunities to explore the family and regularity of the shapes. By an asking the students for their arguments they were stimulated to think analytically about the shapes' properties. In this way they may understand the conditions to compare shapes and to give a satisfactory definition.

Question 7.

(a) Shade in all right hand faces.

(b) Match the solids in pairs.

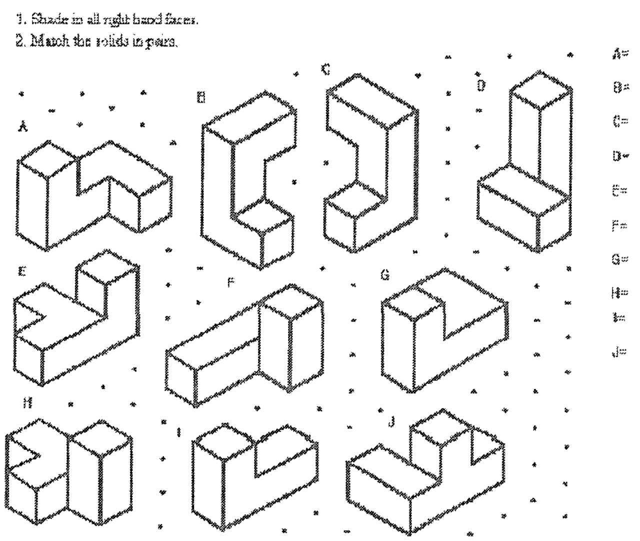


Figure 4.6 Solids presented on test question seven.

Question 7 is related to van Hiele level three and perceptual constancy which involves constancy of shape and size. It is a DIME (Giles, 1989) design which explores aspects of shapes and space in which students see the drawing on the paper which represents three-dimensional shapes. This activity is associated to the position of the solids and their vertices and it is centred on visualisation. Students need to pay attention to the faces and edges. Students need to visualise and compare the solids indicating equal pairs according to their ability to interpret congruent solids made with cubes joined together. Plastic Multilink cubes were available if students wished to use them to help identify congruent pairs. In fact, students decided do not use this material.

4.3 Relation of the questions to van Hiele levels and Del Grande's categories

The arrangement of the test questions in relation to the van Hiele levels and Del Grande categories was as follows:

		van Hiele levels of thinking			
		Level 1	Level 2	Level 3	Level 4
Del Grande Spatial Abilities	Perceptual Constancy	Question 1		Question 7	
	Figure Ground Perception		Question 2		Question 6
	Position-in-space Perception		Question 3		
	Visual Discrimination				Question 4
	Spatial Relationship			Question 5	

Table 4.1 Arrangement of written test questions according to van Hiele's theory and del Grande's perceptual abilities.

4.4 Test results

4.4.1 Del Grande's categories of spatial abilities

The ways in which students answered the questions in relation to Del Grande's categories of spatial abilities was categorised in the following scheme:

Precise: The question is answered very accurately with all relevant information included.

Adequate: The question is answered correctly, but not all relevant information is included.

Insufficient: Insufficient information, and not a full answer.

Inappropriate: Incorrect answer.

Blank: No answer.

We consider that a high qualitative understanding is required for student answers to be classified as precise or adequate.

Table 3.2 below shows the number of students' answers in each of Del Grande's spatial abilities categories.

	Del Grande Spatial Abilities Categories				
	Perceptual Constancy	Figure Ground	Position in Space	Visual Discrimination	Spatial Relationships
Precise	16	2	24	6	11
Adequate	3	17		7	9
Insufficient	3	2	1	4	2
Inappropriate	1	4		8	3
Blank	2				

Table 4.2 Numbers of students according to degree of acquisition of Del Grande's perceptual abilities.

4.4.2 Van Hiele levels

We used the following method to quantify students acquisition of a van Hiele levels of thinking. We represented to each question a section from 0 to 100% corresponding to the degree of students questions. This numerical value can distinguish the degree of attainment of each van Hiele level.

Did not attained		Attained	
Completely incorrect	Intermediate		
Blank or inappropriate	insufficient	Adequate	Precise
0%	1-59%	60-89%	90-100%

Table 3.3. The numerical value distinguishes the degree of attainment of each van Hiele level.

In this thesis I did not identify no acquisition in van Hiele level one. The results show that there were students who could attain an intermediate level. It means that the students shows insufficient answer to get acquisition of the level (the answers are partially correct). These students have proficiency at a certain level, however lack proficiency in the next level. Some students show progressive experience to attain precisely a level.

4.5 Gutierrez's profile

Gutierrez (1991) proposed the above numerical degree of acquisition of a van Hiele level and profile as following.

No acquisition	Low acquisition	intermediate acquisition	high acquisition	complete acquisition

0 15 40 60 85 100

Table 4.4 Gutierrez numerical degree of acquisition of a van Hiele levels.

Degree of acquisition				
Profile	Level 1	Level 2	Level 3	Level 4
1	Complete	Complete	Complete	≤ Low
2	Complete	Complete	≤ High	≤ Low
3	Complete	High	≤ Intermediate	≤ Low
4	Complete	≤ Intermediate	≤ Low	No acquisition
5	High or Intermediate	≤ Low	No acquisition	No acquisition
6	Low	No acquisition	No acquisition	No acquisition

This thesis shows the students according to degree of acquisition of van Hiele levels in table 4.3 in a different way than proposed for Gutierrez.

4.6 General Results according to van Hiele levels of thinking

Within the group of students tested most of them agree with the hierarchical structure of the van Hiele levels: see table 3.6 below. The profile used was similar to the profile used to assess Del Grande's spatial perception. For two of the students enrolled on this work we were not able to assign a clear van Hiele level. Gutierrez *et al* (1991) proposed a coding system to assign students to a specific degree of acquisition within each van Hiele level. We used a modification of that coding system to allocate students according their degree of acquisition.

	Not Attained		Intermediate	Attained	
	Blank	Inappropriate	Insufficient	Adequate	Precise
Level 1				5	20
Level 2			3	11	11
Level 3	2		5	10	8
Level 4		15	8		2
Level 5					

Table 4.6 Numbers of students according to degree of acquisition of van Hiele levels.

Note that to assign students to a van Hiele level is a different consideration than their acquisition of spatial abilities. For their general geometric level of thinking we consider their abilities to express accurately shapes' properties, as distinct from perceiving them.

4.7 General remarks on student answers

Question 1

A few students mentioned some properties of the shapes such as number of faces, vertices and edges. Most the students answered this question very precisely.

Question 2

Regarding question two, we will show how one of the students organised two-dimensional shapes to obtain a new figure in another dimension. The entire process was done in imagination: this student did not use the manipulatives available. They drew the shapes and named them as a cube and a diamond prism. This student visualised a cube:

a three-dimensional figure made from squares. After having drawn the cube in position (1) and rotated it to position (2) this student gave the name of shape 2, as *a diamond prism.* Note that a cube can not be drawn on flat paper using only squares. So this student drew figure (2) as a combination of polygons called rhombus or diamonds which have four sides of equal length, forming two acute and two obtuse angles. This lead to this student's conclusion that several squares can make a diamond prism.

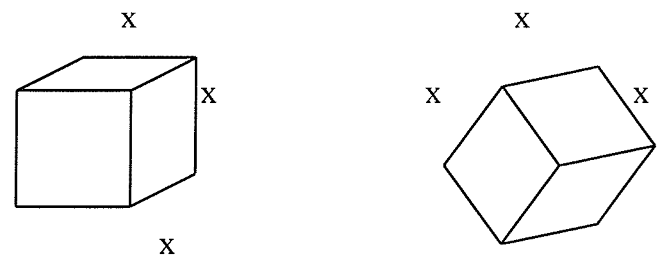


Figure 4.7. Response of one student on question two.

Question 3

To demonstrate adequate Del Grande's position-in-space perception requires the students to correctly group the figures. One of the students presented difficulty in position-in-space perception. This student did not distinguish the square pyramids and tetrahedrons. They organised them in the same group and listed the properties as "pyramid with base x."

Question 4

The students showed the similarities in the shape of the faces and differences in the number of faces. A small number of students presented differences in the numbers of vertices and in numbers of edges.

One of the students wrote:

"Solid B has twice the volume of A, solid C has four times the volume of A and C is twice the volume of B."

In reality, solid B is formed from two triangular pyramids placed together with faces in the shape of equilateral triangles. On the other hand, solid C is two square pyramids placed together with faces in the shape of equilateral triangles. So it is true that solid B has twice the volume of A, but it is not true that solid C is twice the volume of B. If we cut solid C in the vertical plane we will not have a shape formed by two pyramids with equilateral triangular faces. So this is a different shape from figure B.

On question 4b students were asked to define a regular polyhedron. Some students gave insufficient properties to define a regular polyhedron. Very few students wrote that:

All faces are regular polygons. All faces are the same and an equal number of faces meet at each vertex.

This is an effective way to describe the regular polygon since it points out these properties:

1. All faces are the same (a regular polygon).
2. Equal number of faces meet at each vertex.

The combination of these two properties is necessary to represent the concept of a regular polyhedron.

Examples of student answers:

a) "*All the 2 dimensional shapes used in the 3D shape are the same size, same lengths and same angles.*"

b) "*All faces are of the same shape and dimensions. Composed from the same regular polygon.*"

c) "*All faces are the same.*"

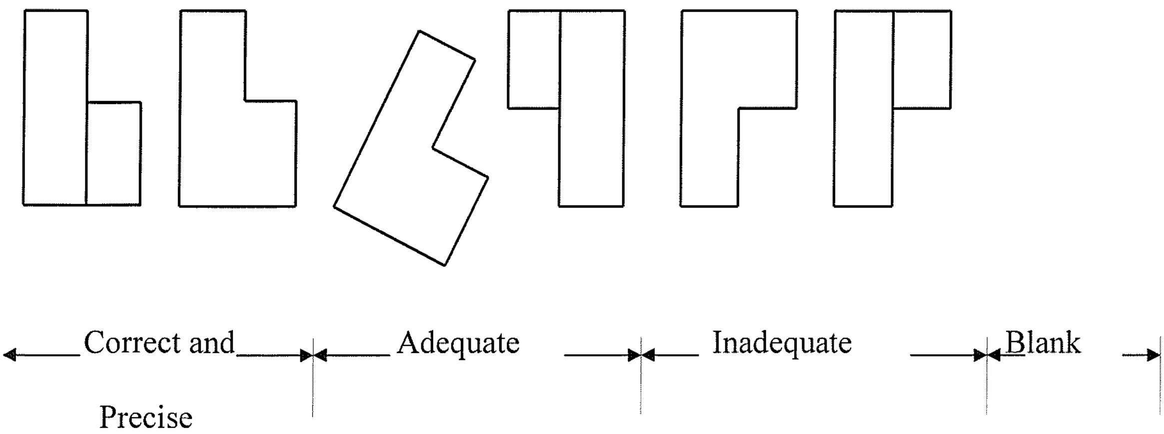
The answers a) and b) do not present a complete understanding of a regular polyhedron. Recognising parts that form a whole shape is a response indicative of van Hiele level 2. These are not sufficiently accurate statements to define a regular polyhedron. To obtain an accurate definition requires an understanding of important properties in order to give this information. So how do these students mentally represent the idea of a regular polyhedron? What strategies do they use to recognise if a shape is or is not a regular polyhedron?

Question 5

We expected better performance of the students on this question, and the answers were a surprise. Some of the students presented difficulty in this item which is connected with Del Grande's perception of spatial relationships, concerned with relating the position of objects, completing figures, and assembling parts.

All the students tried to answer this question. Different drawings of the same view were considered correct. For the different ways that students represent their answers the figures below shows the church and their views and the scheme that was used to analyse the students' correct drawings.

The picture below represents some answers for the base view of the church and how we considered the answers.



The adequate answers were less accurate than precise answers. The students did not find it easier to draw the base view as compared with drawing a front view. The facility they showed in drawing a front view correctly is due to this view being displayed on the drawing in the question. The difference between the adequate and inadequate answers are that in an adequate answer, students think of the view rotated in the same plane as the paper.

Question 6

Some students showed difficulty giving a consistent argument in relation to this question. They associated the properties presented in the question with the properties of the cube and cuboid. These answers show some ability to form mental images of two shapes. However, these students showed no evidence of understanding that the faces could be general quadrilaterals.

Question 7

In the case of a pair of congruent solids, some of the properties visible in one solid are not fully visible in the other. Some students found difficulty interpreting the diagram, perhaps because properties are hidden, or because some cubes are hidden behind others.

4.8 Reliability coefficient

In this section we calculate the Kuder-Richardson (KR 20) formula (Anastasi & Urbina, 1997, p. 97-99), inter-item consistency (briefly referred to as "reliability") of the test, in relation to its administration over two academic years to 25 students.

We start the statistics procedure by computation of the standard deviation of total scores on the set of items using the formula:

$$(SD)^2 = \frac{\sum(x - \bar{x})^2}{n-1}$$

The Kuder-Richardson formula for the interterm reliability is:

$$r_{tt} = \left(\frac{n}{n-1}\right)\frac{(SD)^2 - \sum_{1}^{n} pq}{(SD)^2}$$

where:

the total score on the test equals the sum of the scores for the separate exercises.

r_{tt} is the reliability coefficient of the hole test.

n is the number of items in the test.

SD is the Standard deviation of total scores on the test.

Σ pq is the product of the proportion of students who pass (p) and the proportion who do not pass (q) each item.

Question	1	2	3	4	5	6	7	overall
Reliability	-	0.64	0.87	0.66	<< 0.5	0.63	0.91	0.76

Table 3.7. Kuder-Richardson inter-term reliability, by question, and overall

Since all the students answered question one correctly, this question had zero variance of score and it is considered reliable. On the other hand, we removed question five from the test scores because the reliability result was considerably less than 0.5. In this question students answers fluctuated from one item to another.

Chapter Five

Results from the group discussion experiment

Chapter Five

Results from the group discussion experiment

5.1 Introduction

There were 7 students (referred to below as A, B, C, D, F, K, R) who participated in this experiment. These students were volunteers from a class of 12 pre-service secondary mathematics teachers at the University of Southampton, UK. All had completed a first degree in mathematics. Only one - student D - had any experience with polyhedra in their undergraduate mathematics courses.

All sessions were videotaped and selectively transcribed. The guiding principle in the transcription and analysis of the tapes was search for evidence for or against various issues related to visual perception and image formation, in the context of group discussion of properties of polyhedra. These issues were:

- Visual perception versus image formation
- Difficulties with visual perception
- Difficulties with image formation
- Visual perception and motor actions
- Image formation and motor actions
- Increased perception in a social setting
- Increased formation of visual images in a social setting
- Different types of visual perception and imagery
- Connections with test results

- Deepening of visual perception
- Learning through discussion

5.2 Results from the task sessions

5.2.1 Visual perception versus image formation

In the first excerpt below we illustrate a vivid example of heightened visual perception, obtained through manipulating and talking about a hexagonal bisection of a cube, during the group discussions. Note that students were shown only one half of the sliced cube.

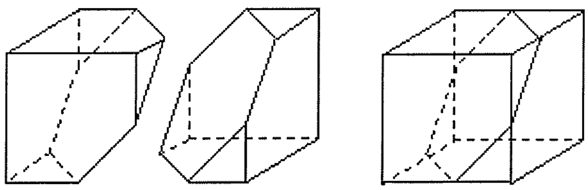


Figure 5.1 Hexagonal bisection of a cube.

Student A (test score 85%, very good spatial abilities) had been answering questions about the hexagonal bisection of a cube and trying to describe it to 3 other students. Student K had particular difficulty forming an image of the object, so A tried to describe it differently. "A" began the description and suddenly perceived the object as a slice through a cube, something that this student had not seen before. Note that in this, as in other excerpts, the student who sees and answers questions about the polyhedron is labelled "*with polyhedron*".

A (with polyhedron): Start of with a big cube K. It's the easiest way, I reckon, to draw it.

K: Big cube. Right.

A (with polyhedron): O.K. And then all you're doing is you're taking a complete slice through that cube, which will act... . Oh god! why didn't I think of saying it like that?

Student A then related the slice of the cube to a previously established view of a cube along a line through diagonally opposite vertices. This confirms that, in this student's mind, the object they saw and manipulated was indeed a slice though a cube (perpendicular to this line).

A (with polyhedron): You know how when you look … down … the apex of a cube and you've got your Y-shape, you've got your hexagon? If you slice through that Y-shape so you take out that Y ..

D: What plane of symmetry are we slicing though? Vertical or horizontal?

A (with polyhedron): Neither.

K: Right. Got my cube. Yeah.

A (with polyhedron): Right. Now you know when you've got your Y-shape.

K: Yep.

A (with polyhedron): If you … do you remember, if you slice through it ... you would see the hexagon at the back?

Two examples of image formation occurred in this group discussion when students K and D attempted to describe their understanding of the unseen hexagonal slice of a cube. In the excerpt below we see how these students used hexagons, triangles and pentagons as perceptual units, and attempted to organise those perceptual units into an image of the object they were trying to understand.

C: This isn't easy to describe.

D: I can see it in my head now, I just can't ....

K: Oh, yeah, I can imagine it going up, and then you're going like that (*waves hands*).

D: I can visualise it.

K: The triangles .. do they meet at the top?

A (with polyhedron): No.

K: Because you've got your base and the triangles going up, .. and then you've got, .. (*inaudible*) you join the two up.

A (with polyhedron): It's literally like you've taken the cube and you've taken a slice though it.

D: Can we get back to what we were talking about before, sorry. We've got our hexagon, and we've got our three triangles on alternate sides and we've folded them up a bit. Then we said from the apexes of the two triangles we've got a kind of ... forming like a midpoint between those, joined up making a pentagon on each side.

In exercising their ability to form images, students K and D utilise perceptual units that have been previously interpreted. These perceptual units are triangles, cubes and hexagons. Without such pre-digested perceptual units the students would not be able to form such complicated images at all as Kosslyn (1996) indicates.

5.2.2 Difficulties with visual perception

Student F (test score 56%, good spatial abilities, but difficulties with figure ground perception) demonstrated difficulties in visual perception of complex polyhedra. In the excerpts below, F is describing a hexagonal slice of a cube to two other students. F demonstrated some difficulties in describing this shape. F described the two-dimensional faces, but could not name an irregular pentagon as such. F did perceive that this pentagon was constructed by cutting a square, but in contrast, was not readily able to relate the three dimensional shape to a cube.

Student B had asked how many different groups of faces there were in the object, and then went on to ask of what those groups consisted. F's answer shows marked hesitation in describing the irregular pentagons formed by the slice. At this point the five-sided polygon does not seem clear to F:

B: What does each in the group have in common?

F (with polyhedron): One group is triangles, and .... uhm, the other group has one, two, three, four, five sides. (Frowns)

R: Uhm,... can you say what the groups are again please?

F (with polyhedron): Ahh, there's one group of three triangles, and there's one group of three shapes with five sides.

R: Sorry, a group of three ... ?

F (with polyhedron): Three shapes with five sides.

Notice that F does not use the term "pentagon" to describe these five-sided shapes. This is consistent with other statement's F made about irregular polygons, and indicates that F uses terms such as "pentagon" and "hexagon" to describe only regular polygons. However, with prompting from student R, F states, hesitantly, that the five-sided figures are squares with a corner cut off:

R: Uhm, ... can the five sided shapes be broken down into simple, plane simple combinations of simple shapes? For instance, like a square with a triangle stuck on top?

F (with polyhedron): Uhm, ... it's a square... with a triangle... cut off.

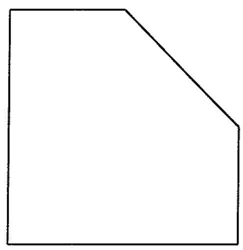


Figure 5.3. Side view of a hexagonal bisection of a cube

F shows no evidence of perception of the irregular pentagons meeting at a corner to form a cube. It is not until student R prompts F with a statement about a cube, that F begins to use the term "cube" at all. One could argue that F did see the part of a cube in the hexagonal slice, but simply did not volunteer the information without being asked. However, that is not consistent with F's volunteering information about symmetry as soon as it is perceived:

R: Uhm, so the isosceles triangle .... (long pause) ... are there, do any of these faces meet at right angles?

F (with polyhedron): Yes. The, ... the cor.., uhm, the corners of the square that haven't been cut, yeah, they all meet at ... uhm.... at right angles, all the sides of the ...uhm... adjoining faces meet at right angles on those, on those corners.

R: So you could say, you could start with a cube.

F (with polyhedron): Yep.

R: At one point.

F: Mm hmm.

...

F manipulates the polyhedron, sits it on its hexagonal face, and volunteers:

F (with polyhedron): About the corner of the cube it's very symmetrical.

R: Right, opposite that uhm, nice right angled corner of the cube, is the hexagon?

F (with polyhedron): Yep, yeah.

We infer that at this point F has perceived part of a cube in the polyhedron, and clearly perceives symmetry about a corner of this cube. This visual perception appears to be stimulated by R's question about a cube, and by F's tactile investigation, placing the hexagonal face on the table so that the corner of the cube was apparent.

Student R speculates whether the polyhedron is really part of a cube, and initially F thinks not. On reflection F perceives that building onto the hexagonal face might produce a cube:

R: If you were to take some putty or something, uhm, and fill in around where the rectangle is would it easily make into a cube

F (with polyhedron): No, I think is probably the answer to that. Uhm, ... oh maybe it does. You would have to build it onto, onto the hexagon.

What is striking about F's perception is that it seems to occur very slowly. It takes place often in response to questions from the other students, and F normally takes quite some time to respond, often turning the polyhedron over in both hands.

5.2.3 Difficulties with image formation

It is quite possible to have good visual perception but to still have relatively difficulty in forming images. The problem, it seems, is one that besets all students of mathematics, and it is the problem of establishing repeatable units: in this case, repeatable perceptual units. Student C (test score 97%, very good spatial abilities) exhibited excellent visual perception in the discussion groups. For example, in answering questions about a dual cube octahedron (see Figure 5.4) C's visual perception was of a high order.

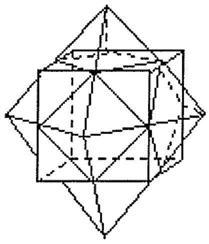


Figure 5.4. A dual cube octahedron

In particular, when answering questions about the triangular faces, and the square base pyramids embedded in the dual cube octahedron, C perceived that the isosceles right triangles formed the corners of a square:

A: ... and the isosceles form what sort of base?

C (with polyhedron): Well they don't. There's like a shape underneath ...

... ...

D: So, in essence, do we have an object with 6 square pyramids sticking out of it?

C (with polyhedron): Yeah.

... ...

C (with polyhedron): So what you need to know is what the object is underneath.

However, C showed problems in forming an image of a hexagonal section of a cube. C sat quietly throughout most of the discussion, and in response to student A, indicated a difficulty:

A: C - you're very quiet

C: Yeah ... my brain's turned to jelly.

C then asked if scissors were available to construct a model from paper:

C: I've got really bad drawing skills. I think I could *make it* if I had some scissors.

This is indicative of C's difficulty in image formation. C tried to turn the, for her, difficult task of image formation, based on verbal questioning, into a problem of visual and tactile perception by physical construction of the object.

In another episode C along with two other students, is trying to identify a rhomboid. C asks if each face is a square after A's determining that each face has four sides. We can postulate that in order to ask the question, C must have had, or been in the process of forming, an image of a square.

A: How many sides does each face have?

K (with polyhedron): Four

C: So is each face a square?

K: No

However, immediately following A's question about the angles in the faces, C asks if each face is a triangle:

A: Are all the angles of each of the faces right angles?

K (with polyhedron): No

C: Is each face a triangle?

K (with polyhedron): No

A: What, with four sides?

General laughter.

If, as Kosslyn (1996) indicates, image formation is dependent on visual perception and uses the same brain mechanisms, then it is difficult to reconcile this question about triangles with an image of a square that was constructed but a moment ago. This *might* indicate a problem with C's working memory, but even so, it still indicates that the image of the square in C's mind was far from permanent. This, of course, is likely to be a result of K's answer "No" to C's question "So is each face a square?". However we infer that C asked this question on the basis of K's statement that each face had four sides. So however we look at it, C did not have more than a transient image of a four sided figure. C may have had a *mental model* of a four sided figure (Johnson-Laird, 1996) but this mental model did not translate into an image that was maintained, inspected or transformed. Any image there may have been when C asked "So is each face a square?" appears to have vanished by the time C asks "Is each face a triangle?" (unless one wants to argue, perversely it seems to us, that a person can have an image of a square, or four-sided figure, and upon inspecting it ask "Is it a triangle?"). The point is that C's image of a square - if indeed there was one - was fragile, and vanished when K answered "No" and A asked "Are all the angles of each of the faces right angles?", bearing in mind that C had just heard K say that each side had 4 faces.

However, C was clearly capable of reasonable image formation, despite having some difficulties, as the following episodes indicate. Student A explains that if one puts a cube on the ground, such that it keeps its face on the ground in a fixed position, and in addition pushes the top, the required shape would be formed. This student did not take into account that just the lateral faces of a cube would change from a square to a rhombus, with the top and bottom remaining the same shape: a square. This was clear however to student C, who scored highly on the written test. She demonstrated an ability to understand the logic of the situation which, we infer, came from a mental image of a cube being pushed laterally:

A: Would the shape be as if you had a cube, and then you tilted it one way?

K (with polyhedron): Mmm, yeah.. if I think about it I know what you mean.

Teacher: … he said to elaborate on something like "tilt" for example, … if she wanted to.

K (with polyhedron): What do you mean by tilt? Do you mean sort of almost pushed to one side?

A: If you keep .. if you put it on the ground, if you put a sq.. a cube on the ground, and you kept the face that's on the ground absolutely in position, and then pushed from the top .... so that the whole thing sort of went out shape. Just skewed over. Does that make any sense?

C: No,... the faces at the bottom will still be a square, won't they?

Despite C's functioning at a high level on the written test (a score of 97%) - in particular, showing very good spatial abilities - overall C did not show evidence of a high level of image formation. C's visual perception was excellent, but this did not translate into a high, or even particularly good, level of image formation. This seems to indicate that image formation, whilst highly dependent on good visual perception, develops in different ways. We propose that the problem of image formation of an object such as a polyhedron is one of *integrating* local and global features. The problem of high level visual perception of a polyhedron, sufficient to describe it in words to someone else, is that of *differentiating* local parts of the object. It was in the task of integration of parts of a complex polyhedron that C seemed to have difficulties. It seems that C's analytic skills of differentiation were of a high order but the corresponding synthetic skills of integration were not so well developed.

This ability to imagine a cube being transformed, and to argue about the properties of the mentally transformed object, on the face of it indicates a high level of image formation. However what C does in this episode is to imagine a cube - a relatively simple task - and then imagine the cube transformed. This is not a task that requires integration of local and global parts.

There are two ways in which C could have mentally transformed the image of a cube. One is by generating an image of a previously seen physical cube being sheared. The other is by altering the mental image of a cube in a novel way. We hypothesise that is possible but unlikely that C had previously seen a physical cube being sheared, and that this student was, indeed, transforming the mental image of a cube, *de novo*.

5.2.4 Visual perception and motor actions

In this study we also observed numerous instances of students rolling polyhedra through their hands, rotating, and generally inspecting the polyhedra from different view points. For example student C, inspecting a dual cube octahedron, rotated it forward and backward about an axis between the hands. We infer that this was *not* used to count faces. C was asked after this rotation how many faces there were, and then carried out counting by tapping eight of the triangles and announcing "48". The rotation seemed to be a systematic way for C to visually perceive the object.

Student F rotated an icosahedron away from the body about an axis between the hands, but not so vigorously as C did for the dual cube octahedron. Then F appeared to *count* the triangular faces of the icosahedron by tapping the faces with a thumb and finger. In answer to the question: "Are they joined at all in squared based or triangular based pyramids?" F rotated the icosahedron quite vigorously about a number of different axes before answering.

The physical actions carried out by the students on polyhedra, in conjunction with low-level visual perception, are the driving force behind the relatively rapid development of high-level visual perception. Development consists of an integration of stimuli from these different modalities. This process is a selection process, not a construction process: students do not construct one particular form of high-level visual perception. Rather they form it through (probably unconscious) integration of certain selected elementary schema.

5.2.5 Image formation and motor actions

The literature on transformation of mental images makes clear that motor subsystems of the brain are involved (Kosslyn, 1996). However, as far as we can ascertain there is no substantial evidence that motor subsystems are involved in image formation. In the group discussions we observed many instances of motor activity associated with image formation. This motor activity did not seem peripheral to the formation of mental images. It appeared to be a natural result of a student trying to mentally construct an image. It was as if the hands, in particular, were reflecting an internal attempt to build the image in the mind. Some striking instances of this are presented in the table below.

The verbal transcripts and descriptions of motion come from Student D who was trying to visualise a hexagonal bisection of a cube.

Statement	Action
"So it's like a V-shape on one side"	Raises both hands to face level with hands apart, brings index fingers together along the lines of an imaginary "V" and runs the fingers back out again.
"So we're going to take the corner away?"	Shakes hands about a body width apart, then lowers hands and vibrates them in and out towards each other.
"Do these three triangles meet in an apex above the hexagon?"	Makes a tent shape with two hands, with just one finger of each hand touching.
"You've got a hexagonal base and on alternate sides you've got your isosceles triangles"	Makes a rough polygonal shape with thumbs and forefingers of both hands.
".. at the moment I'm kind of like trying to draw a net, a net, of this type of thing.. so when I fold the triangles up .."	Moves hands from approximating a polygon on the table to a folding up and down wave motion.

Table 5.1. Motor actions corresponding to verbal statements of student D

Student D was not a singular case: other students also used hand motions as they were trying to describe the formation of mental images. This motor activity is quite different to the motor activity we observed in visual perception, and seems to be a significant aid to students in their attempts to construct a coherent mental image. Its origin is unclear; however our hypothesis is that it is a significant, but not necessary, component in the integration of local and global features of an unseen 3-dimensional object. The hand movements appear to assist in bringing together the parts of an object into a global image, and seem particularly related to issues of dimensionality and position in space. For instance, D's hand movements related to making a polygonal shape, described in Table 1, appeared to indicate how D was thinking of the hexagon as a 2-dimensional object. Similarly D's making a "V" shape appears to relate to the appearance of part of a 2-dimensional face of the object. However D also indicates by the position of the hands how this face is situated in space. Whereas, D's making a tent shape with the fingers indicates that three faces meet in a certain 3-dimensional configuration. Thus, the hand movements appear to be related to dimensionality and spatial positioning, as well as to the integration of local and global features of the object.

5.2.6 Increased perception in a social setting

In the excerpts below we show how the interaction between student F and other students in the group leads F to a higher degree of perceptual awareness. Note that in the discussions reported, F has an icosahedron which the other students are trying to identify through asking questions.

F's responses lack accuracy in spatial reasoning which affect the verbal explanation. For example, F shows difficulty in understanding the arrangement of sets of perceptual units when describing the icosahedron. Before any questions are asked, F has begun to count the number of triangular faces of the icosahedron. F uses a thumb and forefinger to tap the faces whilst counting them.

K: So, what are the faces shape wise?

F (with polyhedron): Equilateral triangles.

C: How many?

F found it difficult to identify the correct number of faces, so we infer that F has difficulty in understanding the arrangement of triangles in the shape:

F (with polyhedron): Ah ... Thirty, I think.

C: It is too many .... (general laughter)

Student A asks a question about pyramids, and F manipulates the icosahedron, rotating it in various directions, apparently looking for "pyramids". F decides there aren't any:

A: Are they joined at all in squared based or triangular based pyramids?

F (with polyhedron): Uhm. Pyramids? No.

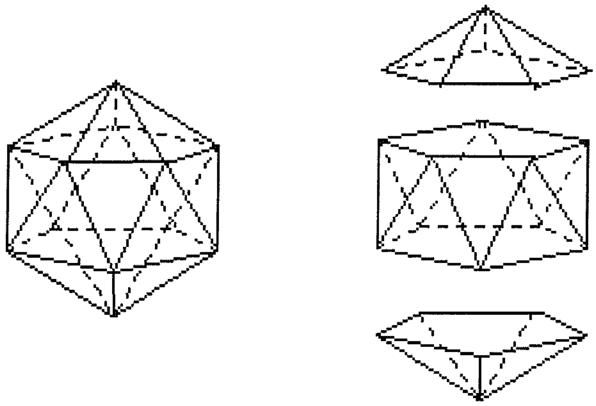
However, as result of this questioning F perceives a pentagonal base pyramid:

F (with polyhedron): Well, unless mm ... ahh ... can you have pentagon based pyramids?

A: I don't see why not.

F (with polyhedron): Yeah, well they are then.

Student D reflects on the existence of pentagonal pyramids in the icosahedron, and begins a line of questioning that allows F to see the icosahedron as a pair of pentagonal pyramids, rotated with respect to each other, and separated by a pentagonal band of triangles (see Figure 5.5).



D: Is there 6 pentagon based pyramids?

F (with polyhedron): Uhm ... (Pause of about 15 seconds).

D's question causes F some difficulty because in looking for pentagonal based pyramids, F cannot see these pyramids as disjoint sub-objects, but rather as overlapping (which indeed they are). F persists in trying to see the icosahedron as a disjoint sum of simpler parts but, because of the overlapping nature of the pentagonal pyramids, cannot:

F (with polyhedron): It's got so much symmetry, they overlap. Uhm ...

C: Oh dear! Is there any kind of basic shape, underlying it, like a cube?

F: Not that I can see.

C: Oh! Is it a totally irregular mass of triangles?

F: No its very regular, but... ahh ..so it makes it difficult to separate into... uhm ...

D asks a question about the way in which the pentagonal pyramids might sit with respect to each other. He seems to have a mental image which he "describes" using his hands - one placed at head height facing palm down, the other below it, palm up:

D: For instance, if I have like, a pentagon based pyramid. How many... is ...is... are they kind of like... say if there's one on top is there also one on the bottom?

F (with polyhedron; without hesitation): "Yes".

The fact that F answered instantly tells us that F did perceive two pentagonal pyramids situated as D describes. F could not see D's hand movements, so relied completely on D's verbal description. We infer that slightly earlier F did not perceive this arrangement of the pentagonal pyramids, because F says: "No its very regular, but... ahh ..so it makes it difficult to separate into... uhm".

Student D continues questioning, and the evidence is that F now perceives and describes the arrangement of the shape, prompted by D's question and by the information contained in that question:

D: If there is one on the left hand side is there one on the right hand side?

F (with polyhedron): There's one at the top, there is one on the bottom... There's one ...

F rotates the icosahedron in various ways apparently looking for pentagon based pyramids on the "left" and "right" hand sides. F places one finger on the "top" pyramid" and another on the "bottom" pyramid. However, F's perception is still relatively fragile as the following statement indicates:

F (with polyhedron): It's just really regular so it's difficult to split it up.

F finally gives evidence of a clear perception of the icosahedron as a disjoint sum of two pentagonal pyramids and a polygonal band of triangles (an anti-prism).

K: When you say there is one on top and one on the bottom, there's five making ... five triangles making, making the pyramid.

F (with polyhedron): Yeah.

K: ... on top, and five making the bottom, so there's four going around the outside.

F (with polyhedron): If you take the top off and the bottom off, you're not left ... with any pyramids.

D: What are you left with?

In answering D's question F places one hand over the "top" pyramid and the other hand over the "bottom" pyramid to leave the triangular band. We infer that is precisely at this point that F obtained a perception of the icosahedron as a disjoint sum of two pentagonal pyramids and a band of triangles:

F (with polyhedron): The rest of the triangles in sort of in a line joined up. So you get...

Unseen student: If you take the overall shape, will it be sort of long or round?

F (with polyhedron): Round. I'm making it harder than it is.

What is important about this episode is the gradual increase in F's visual and tactile perception of the icosahedron as a disjoint sum of simpler sub-units. This perception was prompted by the other students trying to reconcile their images of the icosahedron through questioning F. In this sense F was making sense of visual and tactile data as a result of that questioning.

5.2.7 Increased formation of visual images in a social setting

Despite student F's having difficulties on the written test with image formation, and F's almost total absence of questions during group discussions of relatively complex objects, F was able, through the questioning of other students, to produce good visual images. For example, a group of 5 students asked questions in an attempt to identify, or form images of, a hidden dual cube octahedron. In an episode lasting approximately 10 minutes, F did not ask a single question. However F produced the following drawings in the process of constructing a representation of the dual cube octahedron:

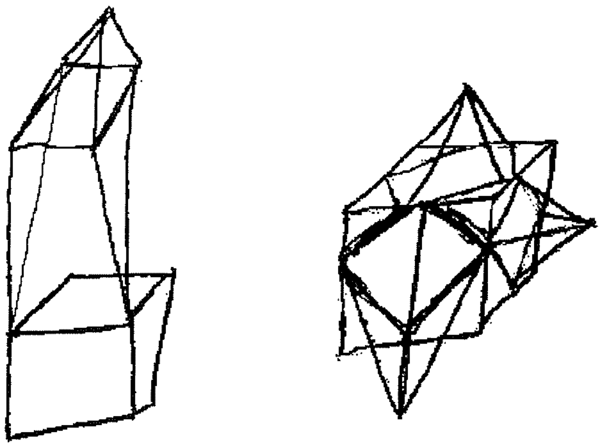


Figure 5.6. F's drawing produced in a group session to describe a dual cube octahedron

During the discussion F listened intently and nodded agreement at various points in the discussion. We are led to infer that despite F's lack of questioning, this student could organise the answers to the other students' questions, and their verbal statements of what they thought the object was, into a coherent mental image. This appears to be a clear illustration that active listening can be sufficient to form a complex mental image.

5.2.8 Different types of visual perception and imagery

In this study the students exhibited four basic types of visual perception and four types of image formation, as indicated in table 4 below. In describing a focus on spatial relations we have used the term "diectic", (Logan, 1995). This refers to a focus of visual attention on spatial relations, between objects, of top, bottom, left or right - and the term "intrinsic" - which refers to a focus of attention on relations of parts within an object.

Visual perception	Image formation
Step by step description of local parts, focusing on features of faces.	Transformation of previous images.
Focus on decomposition into 3-dimensional parts.	Focus on decomposition into 3-dimensional parts
Focus on spatial relations of faces(diectic or intrinsic visual attention - see opposite. Logan, 1995, p. 112)	Focus on decomposition into 3-dimensional parts and their spatial relations - top, bottom, right, left (diectic visual attention: Logan, 1995, p. 112)
Global-holistic focusing on 3-dimensional features, providing alternative descriptions.	Focus on metric relations such as angles and lengths, focus on combinatorial relations of edges and faces.

Table 5.2. Observed types of visual perception and image formation

The most commonly observed type of visual perception and image formation was separation of a polyhedron into different classes of faces. Another commonly occurring type of perception and image formation was a focus on decomposition of a polyhedron into 3-dimensional parts and the determination of the spatial relationships of those parts. A good example of this is recognition of an icosahedron as consisting of a pair of pentagonal pyramids, rotated with respect to each other, and connected by a pentagon of triangles. We observed only one instance in which a student tried to gain combinatorial information about the positioning of faces around the edges of a given face, and we observed only one instances of a student offering information on the number of faces meeting at a vertex. We observed no instances of a student asking for, or offering information on the number of edges meeting at a vertex.

5.2.9 Connections with test results

We expected that attainment in the written test would be connected both with visual perception and/or image formation. Whilst there does seem to be a general indicative connection between test results and visual perception, the same does not seem to be the case with regard to image formation.

The evidence suggests that students with high conceptual levels can have difficulties with image formation, whilst students with low conceptual levels can successfully form complex geometric images. For example, student C scored 97% on the written test and exhibited very good spatial abilities. This student also had good visual perception yet encountered considerable difficulties in the formation of images of complex polyhedra. Student F scored only 57% on he written test yet this student successfully formed an image of a dual cube octahedron from other students' questions. In the table below we provide some evidence that there is a general connection between test scores and visual perception.

Student	Test score (%)	Visual perception	Evidence
A	85	Very good	Hexagonal section of a cube: coming to see the corner of the cube during explanation.
B	44	Adequate	Irregular octahedron: 8 faces, not square; triangular. It's similar to two pyramids.
C	97	Excellent	Dual cube-octahedron: seeing the cube.
D	72	Good	Hexagonal prism: eight faces, subdivided down into 2 sets.
F	56	Adequate, with help	Icosahedron, hexagonal section of a cube: difficulties in seeing more than local features. Slow perception.
K	49	Adequate	Dodecahedron: describing twelve faces, all the same; regular pentagons. Agreed it was like a big ball, but described it as not round.
R	77	Good	Inverted cube (not a polyhedron): "One tangible and one intangible basic shape. ... The triangles you can touch ... the squares you can't touch ..." Hexagonal prism: 3 pairs of 4 sided faces and one pair of hexagonal faces.

Table 5.3. Relation between test scores and visual perception

5. 3 Learning through discussion

5.3.1 Introduction

Interaction between students, through asking questions and receiving answers, and in arguing about different images, is a strong stimulus to the formation of a satisfactory and stable mental image. In the group discussions there were many instances of students achieving image formation and perception that were at a higher level than indicated on written tests, and at a higher conceptual level than their written test scores indicated. We conclude that such group discussions can form a significant tool for assisting student to gain heightened geometric perception and imagery of polyhedra in a relatively short time span.

Many authors have discussed how meaning develops through interactions and interpretation (Von Glasersfeld, E. 1987; Yeackel, Cobb & Wood, 1990; Yeackel, E., Coob, P., Wheatley, G., & Merkel, G. (1990). Cobb, Yeackel, Wood, 1992; Yackel & Cobb, 1993). This work shows evidence that students can use correct understanding to access relevant information. This study also shows how group discussion can influence the knowledge between students from questions and descriptions that were understood. During discussion students exchanged information and shared knowledge. One student's description should determine what another student knew previously and what information they needed to provide.

Our perspective, therefore, is to consider the integration between questions and descriptions and how this integration can give opportunities for the students to cross the van Hiele levels of thinking. We do not intend to say that this integration can always guide students to cross the levels in a common and fast way. The study of internal action is important for this activity. Concerning geometric language, internal action occurs when students select what to say among a number of alternatives. Examples from two sessions follow:

5.3.2 Group 1: transcription of the rhomboid session

A: How many sides has the shape got?

(Relevant information directly related to the shape was required)

K: Six

D: Is that six faces?

(The question and answers previously given, were not sufficient to

determine if they were talking about a two- or three-dimensional shape).

K: Yes, six faces, sorry.

(New knowledge acquired to use the language for two-dimensional and three-dimensional shapes).

C: Are they all the same?

(With this question C in her mind tried to determine the kinds of three-dimensional shapes according to the type of faces).

Questions were significantly different and asked sequentially to distinguish shape.

A: How many sides does it face have?

(This question is related to the class of shapes)

K: four

(Students will determine in their minds to what class the bi-dimensional shape belong).

......

A: Would the shape be as if you had a cube, and then you tilted it one way?

K: What do you mean by tilt? Do you mean sort of almost pushed to one side?

A: If you keep... if you put it on the ground, if you sq... a cube on the ground, and you kept the face that's on the ground absolutely in position, and then pushed from the top... so that the whole thing sort of went out shape. Just skewed over. Does that make any sense?

This statement implies transformational equivalence. It involved an imagined transformation of part of one shape to obtain the other one. From the statement this student had the mental transformation but not in a view that involved the whole shape. In the student's mind, the transformation was not performed in a holistic manner. This is to be compared with Shepard and Cooper (1982) who, in their classic studies on mental rotation, support the notion that mental **rotation** is indeed a holistic process.

Student C, however, did not accepted A's argument.

C: No, ... the faces at the bottom will still be a square, won't they?

In these two statements, A and C report how the sequence of their images was gradually transformed. C was able to form the image of a fragmented part of the transformed shape to obtain a conclusion. This statement indicates transformational correspondence of certain parts of a shape: the process involved imagined transformation of part of one shape to obtain other one. This implies that transformational equivalence is closely connected to mental transformation.

5.3.3 Group 2: transcription of the rhomboid session

B: Is it a complex object?

(This student try to selects shapes in their mind without previous information)

F: No.

R: Is it made up of flat surfaces or curved ones?

(This student tries to select rounded shapes from flat-faced shapes)

F: Flat

B: Does it have the ...are some sides four sided?

(A descriptive question which, when presented begins a gradual increase in the complexity of the mental task).

F: Ah! They all are .

(Gradually increasing the mental task)

B: Is it a cube?

(The student with little information was not able to evaluate that the information given was not enough to predict the shape; student's were still in the stage of processing information)

F: No

B: Is it a cuboid?

(Student B tried again to predict the shape without precise details. At the time student B tried to predict the shape, they lacked relevant information and it was done in the stage of processing information)

F: No

B: How many sides does it have?

(Student B asked a question about relevant features, but not sufficient to predict the shape. This question is related to the class to which the shape belongs).

F: Six

R: What are angles between the faces?

(This student tries to keep in mind a model of a certain shape. This metric information required is a consistent pattern of his reflection, which is related to the representation of action and reasoning).

R: An approximation will do.

F Ummm....

R: 30, 45, 60, 90?

F: Uh, in one face, there are two sets of two sides which meet at 45...and uhm, the other two corners uhm...the angles are greater than 90.... and the. The sides are parallel.

(This action presented distinct characteristic because the shape' description shows different means Regarding part of the shape).

R: How many shapes are there?

(Student R tries to extend his conclusion)

F: Six

(This answer was already given previously).

B: Are all faces identical?

(Student shows that have been organising the action carried out in the activity. B had carried out a plan in questioning. This plan realises the main action.)

F: Yes.

B: Are the rectangles or cube....ahh square?

(This question shown results given in reason, he formed images of some bi-dimensional shape that he believe that can be the face of the polyhedron).

F: Neither.

B: But they are all four sided.

(conclusion was drawn from questions and descriptions. The previous knowledge, action and reason were strongly associated. It indicates that they were substantial part of the activity)

F: Yes

B: Are opposite sides,..... are they parallel?

(Certainly, B try to form the image of the shape, and the questions became gradually relevant and it indicated that his reason were influenced by hierarchical level of the action).

F: Yeah

These statements described the same shape and it is clear to notice the different structure of knowledge among the groups. The actions taken determines exactly whether the part of activity is finished and whether there is an appropriate condition to start a new part of the activity. When students completed their action they were satisfied and finished their activity. They have shown efficiency and tried to discuss in a way that avoided problems related to the understanding.

Student B starts by asking questions, but these questions were of weak relation to the related feature. Within the conversation, student B could cross the van Hiele levels asking relevant questions about the number and nature of faces, and about parallelism. Student B on the written test presented poor answers related to shapes' feature. This student also presented difficulty with the position of the shape in space perception. B could not distinguish a tetrahedron from a square based pyramid.

The effect of these experiments extended to students needing to produce an understanding and how they used the geometric language. The concrete material was useful and the effect was direct on the individual learner (the student who manipulated the shape in how this student produce construction with those that others has in their mind). Therefore, The concrete material was present on the learning environment which involved the pattern of mutual developmental understanding and mutual intellectual development, and development of social autonomy. Yackel & Cobb (1993) argue that autonomy is defined with respect to students' participation in the practices of the classroom community. Cobb 1989 & Sincler 1990 discuss that the mathematical learning is a process of activity construction. For them, the origin of meaning in students socially and culturally is situated mathematical activity.

This experience in geometric relationships as essence of external representation was significant because the students could share their interpretation. This thesis explains how students construct their geometrical knowledge interacting with each other. The new knowledge is expanded into the learning environment. The previous knowledge, reason and action guides the generation of images.

Students showed the ability to use a particular interpretation of the words they heard. Some conventional terms was used to connect the description with their imagery. Some statements were arranged individually. Their interpretative analysis was supported by their particular understanding. Cobb, Yackel & Wood (1992) said that, in social terms, the development of taken-as-shared ways of symbolising mathematical activity by the classroom community clearly facilitates mathematical communication.

The statements were offered by students in differed levels. Certainly, because one was more able than other in pointing out specific parts of the polyhedron. The student's individual purpose was share with their colleagues their understanding. They connected mental images with the use of words, and all the students had opportunities to translate what image they were forming. The interpretation is influenced by the process.

5.3.4 Significant results from group discussions

We focus on how students generate an image of a certain shape. The results show that students were capable of processing metric properties during the discussion group, and they exchanged knowledge in the construction of images of three-dimensional shapes from their verbal descriptions.

The structure of a description is associated to the cognitive capacity of the students who manipulate the shape to describe it. So, it is important to say that the structure of descriptions are not uniform. Each student adopted their own characteristics to describe the shapes. We also observed that students formulated questions which were unsuitable to predict the shapes whose description they sought. As an example we see below, in group discussion 2, the question: "is it a complex object ?" The answer for this question is not related directly to the surface features of a particular shape. It is a mere point of view which could vary from student to student.

The correct prediction of the shape described depends on the coherence of description (how people interpret their visual perception and images) and the memorisation of the description (how people integrate the given information, structured basically on language, to build a mental image). We collected evidence from the group discussion which showed that the students who listened to the description occasionally did not memorise part of the description, or forgot some information.

While the shapes were described students memorised the description. They were able to construct spatial conjectures from their previous knowledge. So, an image gradually was formed. Both, propositional and metric information may be essential in the process of image formation from verbal description. We observed in the discussion group that students generally required metric information like the number of faces, or number of sides each face contains. Propositional information such as parallelism of faces was also relevant to both group discussions.

5.3.5 Transcription and comments on a section of discussion group 1

Student K (level 2) manipulated a shape behind a screen, out of sight of students A (level 3→4), D (level 3) and C (level 4). Student A begins with a quantitative and relevant question, but uses the word "sides" rather than "faces":

A: How many sides has the shape got?

K: Six.

Student D checks that "sides" means "faces" (two-dimensional polygonal boundaries):

D: Is that six faces?

K: Yes, six faces, sorry.

C: Are they all the same?

K: Yeah.

Now student A uses "sides" to mean "one-dimensional simplicial boundary"

A: How many sides does each face have?

K: Four

A: So is each face a square?

K: No

Student A then asks for metric information:

A: Are all the angles of each of the faces right angles?

K: No

and student C, apparently not aware, or forgetting, that the faces have 4 edges, asks:

C: Is each face a triangle?

K: No

A: What, with four sides?

General laughter.

Again student A asks for pertinent information, leading her to identify the shape in her own mind:

A: Are opposite sides of the face, ahh .. parallel?

K: Yes, and they are all equal. I would say.

A: So, it's a rhombus?

K: Yes.

D: (Inaudible)

C: (To D) Each face is a rhombus.

However, further questioning by student A reveals that the identification of the shape as a "rhombus" is still somewhat confused:

A: Would the shape be as if you had a cube, and then you tilted it one way?

K: Mmm, yeah.. if I think about it I know what you mean.

Teacher: … he said to elaborate on something like "tilt" for example, … if she wanted to.

K: What do you mean by tilt? Do you mean sort of almost pushed to one side?

Student A explains that if one puts a cube on the ground, such that it keeps its face on the ground in a fixed position, and in addition pushes the top, the required shape would be formed. This student did not take into account that just the lateral faces of a cube would change from a square to a rhombus, with the top and bottom remaining the same shape: a square. This was clear to student C, who was functioning at van Hiele level 4. She demonstrated an ability to understand the logic of the situation:

A: If you keep .. if you put it on the ground, if you put a sq.. a cube on the ground, and you kept the face that's on the ground absolutely in position, and then pushed from the top .... so that the whole thing sort of went out shape. Just skewed over. Does that make any sense?

C: No,... the faces at the bottom will still be a square, won't they?

D: Kind of like you had a cube but just sheared to one side.

K: Yes. That's what I was trying to say.

Student C initiates a discussion on parallelism of opposite faces, which leads the group to agree that they have identified the object:

C: Are, ... are opposite faces parallel?

A: Does it have 3 pairs of opposite parallel faces?

A: Top and bottom, are they parallel?

K: Yes, If you were to look at it umm.. head on you could almost have a... well you have a ...... diamond shape. ....

Others: yeah, yeah.

K: That's right to say diamond shape, isn't it?

A: It's a kite, yeah? Like a kite.

K: Yes, ... 8 vertices.

A: Don't get technical on me!

10.3.2. Transcript of discussion group 2

Student F (no assigned van Hiele level: level 1→2 generally, but shows evidence of level 3 thought) manipulated a shape behind a screen, out of sight of students B (no assigned level: level 2→3, but lacks some aspects of level 2 thought) and R (level 3).

Student B begins with a question he asked several times in group discussions. This question, which is more or less subjective, does not focus on properties of the object, and is in keeping with B's lacking some aspects of level 2 thought:

B: Is it a complex object?

F: No.

Nowhere had we told the students about the flatness or otherwise of the faces of the objects, so R's next question is quite reasonable.

R: Is it made up of flat surfaces or curved ones?

F: Flat.

Student B asks a relevant question about the nature of the faces. Notice, however, that B uses "sides" in two senses. Then B asks two questions which on the basis of the known information, are merely guesses. Finally, B asks a pertinent question about the number of faces.

B: Does it have the ... are some sides 4 sided?

F: Ah They all are.

B: Is it a cube?

F: No.

B: Is it a cuboid?

F: No.

B: How many sides does it have?

F: Six.

Student R asks for metric information about angles - in keeping with the assigned van Hiele level 3:

R: What are the angles between the faces?

Teacher: (After a puzzled pause from student F) We haven't got a protractor!

R: An approximation will do.

F: Umm ...

R: 30, 45, 60, 90?

F: Uh, in one face, there are two sets of two sides which meet at 45 ... and uhm, the other two corners uhm .. the angles are greater than 90 ... and the, The sides are parallel.

Student R then asks about the number of faces despite B having already ascertained that there were 6.

R: How many faces are there?

F: Six.

Student B again asks a series of questions which indicate a functioning van Hiele level 2, in that the questions are more analytical. They also involve visual discrimination in which this student tested poorly.

B: Are all the faces identical?

F: Yes.

B: And are they rectangles or cubes... ahh, squares?

F: Neither.

B: But they are all four sided?

F: Yes.

B: Are opposite sides , (inaudible) are they parallel?

F: Yeah.

Student R makes an attempt at identification, and after being unable to draw the object, asks a question involving symmetry, which one would expect at level 4. Note that this student has good spatial abilities in all areas and very good position-in-space abilities:

R: So it's a sort of squashed cube really?

F: Yeah.

R: I can't draw it ... (inaudible)

R: If you, er, if you pick it up, and turn it around in 90 degrees and put it down again, would it, would it fit back into its previous position?

F: Yes. And there is a lot of symmetry in, in each face as well. Between, if you uhm, put a line between them ... bisecting angles

We found that the interchange of knowledge that took place during the discussions contributed to the development of the students. This led some of them to a higher van Hiele level than that observed in the written test. For example, in the written test student B presented very poor responses related to properties of shapes. This student also showed problems associated with position in space perception. On question three this student regarded tetrahedrons and square based pyramids as identical. The discussion above indicates a relatively higher achievement for this student in relation to the written test.

Comparing the above dialogues it becomes clear that group 1 used a better mathematical language than group 2. They were able to make clear when the discussion was about two-dimensional shapes (a face of the solid) or three-dimensional objects. In the beginning of the discussion, without consistent information, group 2 tried to determine the shape. Also this group used the word "side" without explaining carefully, or negotiating, what they meant.

5.3.6 Fluency of mathematical expression and thought, and its implications

It is quite possible for people to speak fluently about a scene as they describe it, either by seeing it directly or in imagination. However, we propose that such fluent verbal descriptions are related only to what we term surface features of perception or images.

Surface features are those aspects of an object or scene that require very little conscious effort to grasp. Deep features of perceptual objects, scenes, or of mental images are, to the contrary, those features that take considerable conscious effort to assimilate. For example, some surface features of an icosahedron are that it has 12 vertices and the faces are triangles. A deep feature is that an icosahedron can be seen as a non-overlapping union of two pentagonal pyramids and a pentagonal band of alternating triangles:

Apparently, the concepts of surface features and deep features are relative notions: what is a deep feature of a polyhedron, say, might be a surface feature for someone else. However, evidence presented in this thesis indicates that seeing some features of objects such as polyhedra are inherently deep features. Such deep features seem to demand practice and training, independent of the person looking at the polyhedra.

We hypothesise that when a person examines deep features of an object, a scene, or an image, and attempts to describe those deep features verbally, two things alternate: (a) speech and (b) examination of the percept or image. It is just this alternation, in our view, that produces verbal hesitancy.

Fluency of mathematical expression is not necessarily connected to fluency of thought. Thurston (1997) remarks:

"It's true that clear, articulate verbal expression can be, and often is, an important aid to understanding. However, words and articulateness can also get in the way of mathematical understanding.

"Articulateness in speech and in writing tends to be pretty variable, and not closely linked to clarity of thought. I can recall very vividly, at the age of about 5, trying to puzzle out the adult world, and arriving at the conclusion that as people age, they become much more fluent and articulate, while at the same time becoming boring, dull and fuzzy in their thinking."

In examples (1) through (4) below, pre-service secondary teachers are discussing polyhedra. Student F is holding and looking at either half a hexagonal section of a cube, or an icosahedron, in order to describe their features to other students who cannot see these polyhedra. F is asked questions by other students - R, D and C -

(1) F (with half cube): Uhm, ... it's a square ... with a triangle ... cut off.

(2) R: Uhm, so the isosceles triangle .... (long pause) ... are there? do any of these faces meet at right angles?

F (with half cube): Yes. The, ... the cor…, uhm, the corners of the square that haven't been cut, yeah, they all meet at ... uhm ... at right angles, all the sides of the ... uhm ... adjoining faces meet at right angles on those, on those corners.

(3) F (with icosahedron): Well, unless mm ... ahh ... can you have pentagon based pyramids?

(4) D: Is there 6 pentagon based pyramids?

F: Uhm ... (Pause of about 15 seconds).

F: It's got so much symmetry, they overlap. Uhm ...

C: Oh! Is it a totally irregular mass of triangles?

F: No its very regular, but... ahh ..so it makes it difficult to separate into... uhm ...

D (trying to imagine the hidden icosahedron): For instance, if I have like, a pentagon based pyramid. How many... is ...is... are they kind of like... say if there's one on top is there also one on the bottom?

Student talk in explaining and questioning is often hesitant. We hypothesise this is because students are thinking and explaining alternately. In the examples we have given, the thinking involves searching for deep features of geometric objects or images, and it seems to be just this focus of attention that inhibits speech at that moment.

Chapter Six

Estimating van Hiele levels from verbal data

Chapter Six

Estimating van Hiele levels from verbal discourse

6.1. Introduction

One of my objectives is to see if, during discussion, students use language that indicates a higher level of geometric thought - as evidenced by van Hiele levels - than they showed on the written test. This means that I have to provide a means of assigning van Hiele levels from verbal data. In this section I describe, for each van Hiele level, the type of language indicative of that level, and proto-typical examples of such language. Following that I describe how I will implement these language descriptions with transcripts of verbal data. There are two obvious ways to attempt to make an assignment of van Hiele levels from verbal data a reliable assignment. One is to use a number of experts and calculate an assignment reliability. Another is to specify, as simply as possible, criteria for assigning levels so that a relatively high level of expertise is not required. I adopt the latter approach.

6.2 The van Hiele levels and verbal discourse

Level 1 (Visualisation)

Understanding basic geometric concept involves reasoning by the student at this level by means of a visual conceptualisation of ideas. Students observe polyhedra and identify them by comparing with mental images, but they do not identify properties of a polyhedron. They recognise simple polyhedra by their appearance without distinguishing their properties. They are able to recognise similar polyhedra with different sizes. What is characteristic of this level is the ability to identify polyhedra without necessarily being able to analyse their determining components. Description of polyhedra at this level lacks information about integration of parts that form the shape.

Type of verbal discourse:

Language used at this level is very simple and describes the polyhedra as a whole. It involves an account of obvious surface features without analysis. Names are not necessarily used appropriately.

Typical examples:

"The shape has a lot of faces."

"It is round."

"Does it have a lot of points?"

Level 2 (Analysis)

A student distinguishes properties of a figure by informal analysis of the component parts. They do not grasp the relationship between different figures or between properties. The figures are recognised by their parts (faces, edges, vertices etc.). They do not grasp the relationship between different figures or between properties and they cannot explain relationships between properties.

At this level students analyse polyhedra in term of their surface feature and relationship between properties. Observation of class in what the shapes belong can happen. Each property is observed separately. Students being to understand that the several features must be integrated to form a polyhedra. Some simplistic conclusion can be observed. Students can thinking in shapes rather than as a visual gestalt. Students pointed out the angles, faces. They are able to compare the different kinds of shapes of faces that compound the polyhedra. Some interpretation of verbal description in terms of the surface feature to construct draw. Description of the class which the polyhedra belong (by its properties).

Type of verbal discourse:

Students can thinking and describe shapes using language naming the properties such as: shape of faces, vertices or corners and some others geometric properties appropriately.

Typical examples:

"It has six faces."

"Does the shape have triangular faces?"

"The shape has five vertices."

"It has twelve edges."

"Are there any square faces?"

Level 3 (Informal deduction)

A student, though the process of definition, distinguishes the relationships between properties and figures. Example: a cube has opposite faces parallel and right angles. They are able to classify families of solids. They use ordered logic to understand the properties of concepts and form definitions, but they are not able to arrange sets of statements.

The characteristic of this level is to identify and integrate a set of properties discovering new properties by deduction. Students are able to give reasons for the steps in describing how a polyhedra is arranged. They can differentiate solids by comparing their properties. They observe that rotated solids change their view point but retain their properties. They provide full definition and arguments for assumptions.

Type of verbal discourse:

Recognise that a polyhedron is regular, without necessarily giving an argument that defines regularity. Comparing properties of two polyhedra.

Typical examples:

"This polyhedron is formed from two different types of pyramids."

"It is a tetrahedron, but the previous shape was a pyramid."

"How is the pyramid arranged on the face of the cube?"

Level 4 (Deduction)

A student understands the ordered steps involved in proving theorems, using rules and axioms, and elaborates ordered statement one from another and provide definitions. They are able to develop a proof in more than one way.

This level is characterised by students formulating logically original sequences of statements that justify their conclusions. Students at this level are able to compare and contrast properties that define a certain shape. They examine and formulate definitions.

They are able to change their initial definition or argument using logic. They can create some support for their arguments Observation of a set of properties to be sure that they are enough to draw conclusions.

Type of verbal discourse:

When the description of a polyhedron is not clear for other student on the above level, rigorously and creative changes of initial argument occur relating this polyhedron as a part of other shape. This argument supports their anterior statements. Give argument when talk about shapes' transformation.

Typical examples:

Put a cube on the ground, keep the face that's on the ground fixed, and then push from the top, so that the whole thing skewed over. Will this give a rhomboid? "No... the faces at the bottom will still be a square."

"What is the plane of symmetry? Horizontal or vertical?"

"If there are 6 faces there cannot be 48 edges, because ...."

6.3 Assigning levels from transcribed data

6.3.1 Information units

When students are involved in verbal discussions to discover properties of polyhedra, they are engaged in either giving, or asking for, information. I follow Halliday (1967) and the Prague School of discourse analysis (see, Brown and Yule, 1988 ; Chapter 5) in assuming that the students verbal utterances will contain information units: smallest segments seeking or giving information about a particular aspect of a polyhedron. However, I do not follow Halliday in his analysis of information units through stresses in speech. Rather we utilise a (modified) technique of Brown and Yule (1988) who claim that information units are delineated by pauses in speech. The basis for this idea is expressed by Chafe (1979) who "regards the pause length as a function of the amount of planning which the speaker is putting into his next utterance." (Brown and Yule, 1988, p. 163). Of course, it is possible that some pauses in speech are there for the very good reason that the speaker must breathe, and this will normally result in short pauses. For the purposes of analysis I claim, following Brown and Yule (Chapter 6, 1998), that verbal utterances bounded by pauses, but internally without pauses, are *information units*. The assignment of van Hiele levels will proceed on the basis of a classification of these information units. A piece of dialogue transcribed from the experiment described in Chapter three, will be used for the purpose of describing the technique to be used for assigning van Hiele levels from transcriptions of verbal data.

6.3.2 Cohesive units

Information units are determined by pauses either side of them. Often these information units are connected to one another, in sequence, in a psycho-linguistic sense through cohesive markers of language, of which pronouns are an example (Brown and Yule, 1988). A sequence of consecutively connected information units I call a *cohesive unit.* In a general sense, all utterances in a cohesive unit are in some way connected by dealing with the same topic, as indicated by the choesive markers. Therefore, while I use information units as the basic utterances to use in the assignment of van Hiele levels, account needs to be taken of the overall sense, structure, and meaning of a cohesive unit in assigning a van Hiele level. An example of a cohesive unit is given below.

A (with polyhedron): [You know how when you look]1 .... [down]2 ... [the apex of a cube and you've got your Y-shape?]3, [you've got your hexagon?]4 [If you slice through that Y-shape so you take out that Y ...]5

D: What plane of symmetry are we slicing though? ... Vertical or horizontal?

A (with polyhedron): [Neither.]6

K: Right. Got my cube. Yeah.

A (with polyhedron): [Right.]7 [Now you know when you've got your Y-shape?]8

K: Yep.

A (with polyhedron): [If you]9 ... [do you remember?]10 [if you slice through it]11 ... [you would see the hexagon at the back?]12

A's comments 1, 2, 3, 4 and 5 individually constitute information units: they are each bounded by pauses but contain no pauses internally. Taken together, without the pauses, they constitute a consecutive set of information units that form a *cohesive unit* :

"You know how when you look down the apex of a cube and you've got your Y-shape? you've got your hexagon? If you slice through that Y-shape so you take out that Y"

The cohesiveness comes from various ties: the "down" refers back to "look" in comment 1, and forward to "the apex of a cube" in comment 3; "your hexagon" refers back to "Y-shape" in comment 3; and "that Y-shape" in comment 5, refers to the same "Y-shape" in comment 3. There are explicit formal markers of cohesiveness as well: in comment 5, for example, the first "that" refers back to the "Y-shape" in comment 3.

Other cohesive units uttered by A are:

"Neither."

"Right."

"Now you know when you've got your Y-shape?"

"If you ... do you remember? if you slice through it ... you would see the hexagon at the back?"

To these information units we can assign minimal van Hiele level: that is levels at which the student is functioning at least. They may, of course be functioning at a higher level.

6.4 Increase in van Hiele level from the written test to verbal discussion

In the analyses below I present 3 students whose van Hiele level of polyhedral thought, as indicated by their verbal utterances, was higher than that indicated by the written test.

6.4.1 Student A

Student A is visually inspecting, manipulating, and describing a hexagonal bisection of a cube.

A (with polyhedron): You know how when you look .... down ... [the apex of a cube and you've got your Y-shape?, you've got your hexagon? If you slice through that Y-shape so you take out that Y ...

I analyse the information units in this cohesive unit as follows.

1. [You know how when you look]

I observe that this student externalises their image using speech. This statement is characteristic of van Hiele level 1 because the student did not clearly indicate the general nature of what they looking at. It is, of course, possible that the student's thought process are operating at a higher van Hiele level, but this particular statement does not tell me any more.

2. [down]

This is a very general statement about direction: it does not indicate anything particular about that direction such as "look", "put", "face" etc. Of course, I know from the previous information unit, to which this one is linked, that "down" expressly refers to "look down". However, as a single, self-contained, information unit it tells us no more. As we shall see, it is only when one takes into account the cohesiveness of a sequence of information units that one can link the assigned van Hiele levels into an overall assessment. The student is seeing something and indicating the position of what they see. However, all that this information unit gives us is a sense of direction. This, by itself is characteristic of level 1. When we connect it with the first information unit however we can draw a stronger conclusion.

3. [the apex of a cube and you've got your Y-shape?]

This statement indicates a higher level of thought, van Hiele level 3 -> 4, because it is not related simply to visual perception but to an imagined object. The "Y-shape" is not a visual object, per se, rather it is a higher level construction from the visual data. This is a more complex observation than we would normally expect from someone at level 3. However it is possible that, for this student, the "Y-shape" was visually prominent, so on the basis of this information unit we should not assign more than level 3 (or possibly borderline level 4).

4. [you've got your hexagon?]

This statement expressed an aspect of a two dimensional shape without an analysis of its components. This statement is indicative of no more than a basic van Hiele level.

5. [If you slice through that Y-shape so you take out that Y]

This statement indicates a high level of thought: van Hiele level 4. The student has mentally dissected the polyhedron into other polyhedra and used those to form new mental images.

The other information units/cohesive units of student A in the reported conversation are related to van Hiele levels as follows:

* [Now you know when you've got your Y-shape?] van Hiele level 3 because the utterance deals with the arrangement of faces on a specific part of the polyhedron.

*[If you]...[do you remember?] [if you slice through it] ... [you would see the hexagon at the back?] van Hiele level 3, because there is an expression of how the parts that comprise the polyhedron are related together.

Taking all these information units which together form a single cohesive unit, we see that on the basis of this cohesive unit we assign student A to at least van Hiele 4. This student tried to explain some physical laws which are difficult to express verbally It seems that this student began from a basic level and their description became more transparent, more flexible, and more logical during conversation.

6.4.2 Student F

Student F is visually inspecting, manipulating, and describing an icosahedron:

A: Are they joined at all in squared based or triangular based pyramids?

F (with polyhedron): Uhm. Pyramids? No.

The sound "Uhm" is, superficially not a pause: that is, F indicated something by making this noise, rather than saying nothing. However, in the spirit that pauses themselves are indicative of thought (Chafe, 1979), I interpret "Uhm" as a verbal indication of a pause, carried out by F whilst she was thinking. I will use this assumption in the rest of the analysis of F's conversation.

The information unit [Pyramids? No.] is indicative only of van Hiele level 1: thinking of a shape as a whole.

I break the next utterances of student F:

F (with polyhedron): Well, unless mm ... ahh ...can you have pentagon based pyramids?

into the following information units:

[Well, unless]

[can you have pentagon based pyramids?]

The information unit [Well unless] indicates a mulling over of imagined possibilities. In this sense, I infer, it relates to imagining alternative pyramids. Note, however, that in itself, this information unit tells us nothing about student F's geometric levels of thought. It is only when it is taken in conjunction with what preceded and what follows, that we can infer it as relating to imagined alternatives to triangular and square-based pyramids.

The information unit [can you have pentagon based pyramids?] is indicative of van Hiele level 2. This is because it shows a potential understanding of a pyramid in terms of its surface features, and relationships between properties. It also indicates an understanding that several features might be integrated to form a pyramid.

The dialogue continues:

A: I don't see why not.

F (with polyhedron): Yeah, well they are then.

The information unit [Yeah, well they are then] tells us nothing, in itself, about F's level of geometric thought. However, taken together with her previous information unit [can you have pentagon based pyramids?] I infer that images of pentagonal-based pyramid were actually formed in F's mind. . Note that "they" refers to more than one pyramid, and indicates that F is beginning to see some structure in the icosahedron.

This indicates that F is thinking at the beginning of van Hiele level 3: an understanding of structure, but no evidence of informal argument.

F (with polyhedron): It's got so much symmetry, they overlap. Uhm ...

Symmetry is not a direct visual object. Therefore, in the information unit [It's got so much symmetry, they overlap] F presented logical thinking. Here we have two phrases: "It's got so much symmetry" and "they overlap". The first is indicative of level 3 because observation of symmetry, but no argument about it, is a level 3 characteristic. The second is indicative of thought at the beginning of van Hiele level 4: it indicates an understanding of structure, but does not include informal argument. The "they" in this information unit refers to the pentagonal pyramids: this is evident from the immediately preceding discussion.

F: No its very regular, but ... ahh .. so it makes it difficult to separate into... uhm ...

Here there are two information units:

[No its very regular]
[so it makes it difficult to separate into]

The information unit [No its very regular] indicates level 3 thought: it refers to the 2-dimensional faces of the icosahedron. F showed an understanding to determine the polyhedron's regularity, because F knew exactly which of the geometric properties determine regularity.

On the other hand, at this moment, F shows ingenuity in terms of forming an image of a dissected three-dimensional shape.

F (with polyhedron): There's one at the top, there is one on the bottom... There's one ...

The information unit [There's one at the top, there is one on the bottom] indicates that F was able to form the image of pentagon-based pyramids in different locations of the polyhedron in question. The following information unit [There's one] indicates she saw at least one more pentagon-based pyramid in the polyhedron. These two units are indicative of level 3 thought: they show that F saw the icosahedron as having internal structure.

On the basis of these utterances, therefore, we place student F at the beginning of van Hiele level 4.

6.4.3 Student D

Student D is listening to a description of a hexagonal bisection of a cube, and asking questions about it.

D is asking questions of student A who holds a hexagonal bisection of a cube.

D: What plane of symmetry are we slicing through? ... Vertical or horizontal?

There are two information units here:

[What plane of symmetry are we slicing through?]

[Vertical or horizontal?]

The first indicates thought at van Hiele level 4, because D presented an understanding of symmetry (non visual structure), in particular that there would be a plane of symmetry.

The second information unit also indicates functioning at van Hiele level 4. The interchange of vertical and horizontal reflects that an image of planes of symmetry can be formed.

These two temporally connected information units constitute a cohesive unit relating to a particular form of symmetry. This cohesive unit is indicative of van Hiele level 4 thought because it shows evidence that D is seeking a set of properties to be sure that they are enough to draw conclusions. Some inferences (from D's previous knowledge) like vertical or horizontal, regarding the location of the planes of symmetry, indicates that D is reasoning progressively, from a question about symmetry in general, to a specific question about the plane of symmetry.

D: Can we get back to what we were talking about before, sorry. We've got our hexagon, and we've got our three triangles on alternate sides and we've folded them up a bit. Then we said from the apexes of the two triangles we've got a kind of ... forming like a midpoint between those, joined up making a pentagon on each side.

Here there are 4 information units:

[Can we get back to what we were talking about before, sorry.]

[We've got our hexagon, and we've got our three triangles on alternate sides and we've folded them up a bit.]

[Then we said from the apexes of the two triangles we've got a kind of]

[forming like a midpoint between those, joined up making a pentagon on each side.

which we analyse as follows:

[Can we get back to what we were talking before, sorry]

This is characteristic of thinking at van Hiele level 2 because the language used indicates a requirement to analyse informally what was discussed before.

[We've got our hexagon, and we've got our three triangles on alternate sides and we've folded them up a bit].

This statement indicates thought at van Hiele level 3.

A variety of properties are described sequentially. D's statement uses relations between two-dimensional shapes (hexagon, triangles) which are part of the 3-dimensional polyhedron : the phrase "we've folded them up a bit" indicates he is thinking 3-dimensionally. The intersection of shapes in a technical language has been developed.

[Then we said from the apexes of the two triangles we've got a kind of]

This statement indicates thought at van Hiele level 2 because it lacks more observations of properties.

The statement shows only an observation of triangles and a relationship between them.

[forming like a midpoint between those, joined up making a pentagon on each side.]

This statement indicates thought at van Hiele level 3, because it indicates reasoning regarding structural relations of some of the polyhedron's properties. These properties are not complete but are logically ordered.

The cohesive unit:

"Can we get back to what we were talking about before, sorry. We've got our hexagon, and we've got our three triangles on alternate sides and we've folded them up a bit. Then we said from the apexes of the two triangles we've got a kind of ... forming like a midpoint between those, joined up making a pentagon on each side"

therefore indicates an overall functioning at van Hiele level 4.

Most of D's further utterances in this session serve only to place D at level 3. An exception is the following:

D: Kind of like you had a cube, but just sheared to one side.

This constitutes a single information unit, which indicates thought intermediate between level 3 and level 4 because there is use of transformations (level 4) but the transformation is applied inappropriately. The term "cube" evidently indicates identification of the figure. There is the possibility that D had an image of the transformed figure in mind. However, part of a relevant description for the transformation of the cube into a new polyhedron was omitted. For D to indicate thought at Hiele level 4 it would be necessary to give some discussion of transformation of most of the cube's properties.

Chapter Seven

Results from the tactile experiment

Chapter Seven

Results from the tactile experiment

7.1 Introduction

Students used the polyhedra directly to touch them and to discover their form. First, students touched the whole shape. Then, they passed their fingers very carefully for several minutes over the edges of the polyhedron with the intention of exploring the contours of the two-dimensional shapes and identifying them. It happens because these two-dimensional shapes are parts which form the polyhedron.

In this experiment, students exhibited considerable depth in tactile observations. According to Kosslyn (1996, p. 4) a given type of mental representation corresponds to a particular method used by the brain to store information. In this work we used both pattern and description to represent the same shape. Apparently, it seems to be easy to imagine and to draw a shape's determining contours, but the experiment showed evidences that it is not. There are differences between the process of recognising a given pattern and representing the image in the mind by patterns. The difficulty arose because students were required to draw the shape, and they were not given a pattern to recognise it. The images were formed in the students' minds and they tried to make a drawing corresponding to their images. The interesting point is that students understood the arrangement of certain shapes. They were able to distinguish and describe with details parts that make the shapes but they were not able to represent the image in their mind by drawing.

7.2 Achievements for haptic exploration

Most of the shapes were identified efficiently by the students through only tactile exploration.

Students were encouraged to use their interpretative skills: they used words to describe what they felt, and the data they needed to form the image of the shape. It was important for students to reach the level of performance demonstrated, not only to give the correct identification of a shape, but also to explain how they identified the shape. This experiment allowed the students to use only their hands to analyse the shape and discuss between themselves, showing their efficiency to form mental images of polyhedra. They could move their hands and the shapes during exploration, and this exploratory movement was guided and definitely related to previous knowledge about the invariant properties of shapes.

Students put their hands in the box to start their observation. They used both of their hands. The polyhedra were carefully selected and designed to be of a size that the whole shape could be felt by the hands. The palm and fingers were used to rotate the object several times. They passed their fingers carefully from one vertex to the other to feel the size of the edges. Several times they briefly closed and opened their eyes. They used their fingers to count the number of vertices to assure the kind of faces the object has, and then they counted the number of faces. They paid attention to possible parallelism of opposite faces of polyhedron. They drew the shapes that they imagined, and then touched the polyhedron again in the same way that they started the procedure of observation, apparently to assure themselves that they imagined it correctly. They verbally described their haptic perception and finally they looked at and touched the shape.

Students executed hand movements and shape movements which are associated with contacting of all the area and contours of the polyhedra. Within these movements they could feel exactly where the segments started and ended, to get an idea of the shape's surface. In this way, they could distinguish the regularity and irregularity of the shape's faces. They could detect the junction in the objects, contours, and feel the angles between faces. The level of stimulation was related to the kind of shape. As an example, a simple and common shape like a cube was discovered soon, in the first touch, and a shape like an icosahedron, required students to try very hard to identify it.

This experiment shows that the students were competent to form images and identify polyhedra. We consider the significant use of hands to investigate the shapes, the movement of their hands on stationary shapes, the style of exploration (using only one finger, more than one finger, using finger and palm), and the region covered by touch. In relation to the time taken to identify shapes, it is important to note that was a marked contrast compared to the visual field. This study presents evidence of the students' efficiency by observing motor skill activation during manipulation. Often the students used their hands to precisely determine the edges and other properties of the polyhedron. The first set of properties deliberately observed were related to the kind of faces and the number of faces. For them it represents the first step (but not sufficient information) to form the image of the whole shape.

7.3 Results of the experiment

Both students acquired a high level of thought and precision in their perceptual abilities on the written test:

Student	van Hiele levels	Spatial Abilities				
		Perceptual Constancy of Shape and size	Figure Ground Perception	Position in space perception	Visual Discrimination	Spatial Relationship
M	3 → 4	Precise	adequate	precise	adequate	precise
S	3 → 4	precise	precise	precise	adequate	precise

Table 6.1. Student results on the written test.

The graphic 6.1 below shows the time that students spend in the haptic exploration section. For each shape the time was divided into expecting shapes only, drawing + writing properties + more time expecting and time spending on discussion

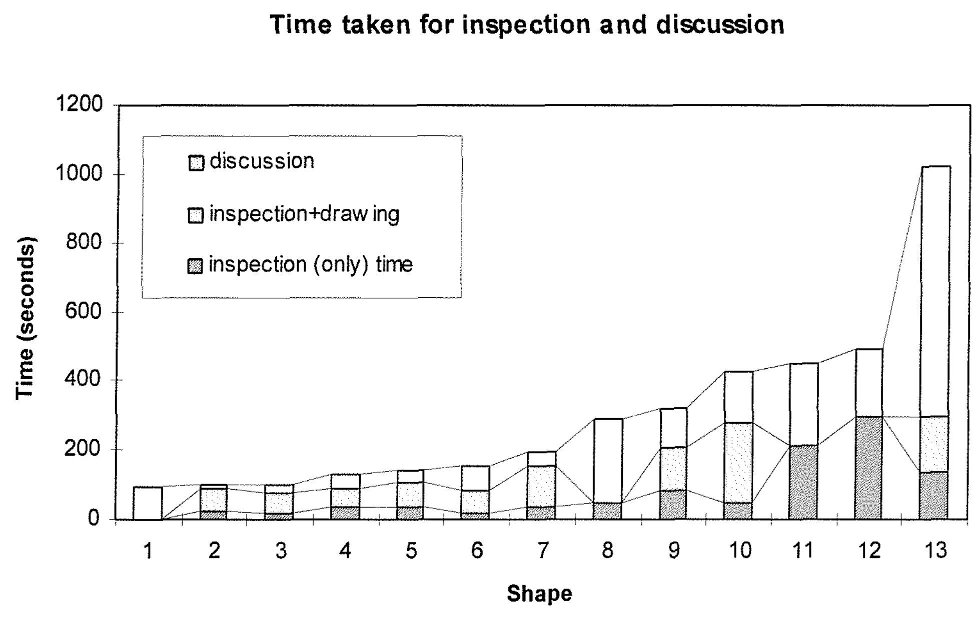


Figure 7.1 Time (second) expended in the tactile experiment. The shapes are represented on the graphic as follows: 1-Cube, 2-Squared Pyramid, 3-Triangular Prism, 4-Cuboid, 5-Regular Octahedron, 6-Irregular Octahedron, 7-Hexagonal Pyramid, 8-Dual Tetrahedron, 9-Rhomboid, 10-Dodecahedron, 11-Dual cube octahedron, 12-cubooctahedron, 13-Icosahedron.

This graph shows evidence that the cube was the easiest and the icosahedron most difficult shape of which to form an image. Two seconds after touching the cube the students reported that they knew what shape it was. On the other hand, the icosahedron was the only shape for which the students were not able to form an image; consequently they did not identify it. The easiest shape to identify was a cube and students drew it after discussion about the shape. Dual polyhedron in which shapes intersect each other, students did not try to draw.

Polyhedron	total time
cube	95
squared pyramid	101
triang. prism	103
cuboid	133
reg. octahedron	139
irreg. octahedron	153
hexag. pyramid	195
dual tetrahedron	290
rhomboid	319
dodecahedron	424
dual cube octahedron	450
truncated cube octahedron	493
icosahedron	1023

Table 7.2 Polyhedra used in the haptic experiment and total time (second)

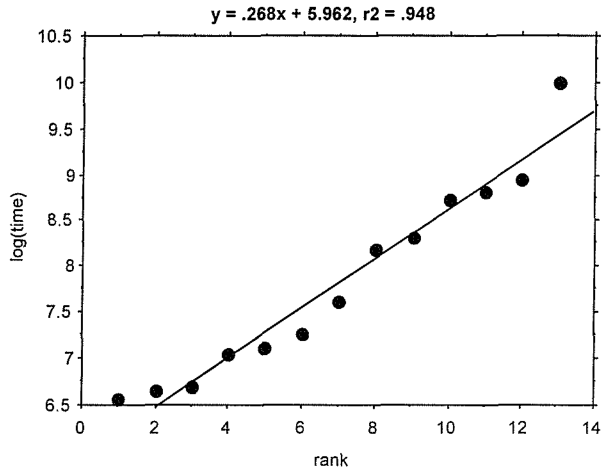


Figure 7.2 Reaction time as function of complexity of polyhedron (Haptic exploration) $r^2 = 0.943$, $p = 0.0001$

Source	Degrees of freedom:	Sum of squares:	Mean Square:	F test:
REGRESSION	1	13.114	13.114	201.209
RESIDUAL	11	.717	.065	p = .0001
TOTAL	12	13.831		

Table 7.3 Analysis of variance

The results above show that, for these students, the rank of the polyhedra, when ranked by total time taken for inspection/discussion, scales linearly with the logarithm of the total time. (Paranthetically, this suggests a way to measure the *haptic complexity* of polyhedra: Repeat this experiment with many pairs of people/students and see if the average and/or standard deviation of total time taken for inspection/discussion scales in the same way as above. If so, and it is consistent, then the rank of a polyhedron can be taken as a measure of its haptic complexity.)

7.3.1 Transcription, comments, and students' procedures during haptic exploration

We select four shapes for discussion to present. Our intention is show the scheme used by students during the discussion. The four shapes chosen were:

The **cube** because it was considered the easiest shape to recognise by only hands contact.

The **rhomboid** because students (in the two distinct experiments the visual and tactile and the only tactile) related this shape with the cube.

The **dual cube octahedron** because the polyhedron and its dual are placed together with their edges bisecting each other. This shape presents a large number of faces, and the faces present pointed arrangements which apparently are not easy task identify it by touch only. However, the students had no difficulty in identifying it.

The **icosahedron** because during tactile exploration students indicated limited understand of spatial relations. Concerning this shape, in the previous visual-tactile experiment (Chapter five) a student who manipulated it found difficulty in describing it, and this student was helped by the others who could not see the icosahedron. Working with this particular shape in both distinct experiments students showed different difficulties. Note that these difficulties are not to be merely explained in terms of the presence or absence of visual sense during experiment. The complexity of this polyhedron affected the difficulty of students' spontaneously understanding the arrangement of its surface.

For each of these four shapes, except the cube, the students in the haptic experiment felt difficulty in representing them in two-dimensional drawings.

Cube

Students spent only two seconds inspecting this shape to identify it. This episode provides evidence that generally people tend to generate images when touching an object in order to identify it.

Experimenter: Would you like to put your hands into the box?

M: We both think we know what it already is. (touching with fingertips of both hands and rotating the object)

S: We knew what it was as soon as we picked it up.

M: Yeah.

Experimenter: without naming it?

M: Without naming it? Properties? (The shapes were stationary and M's fingers passed over the edges)

Experimenter- Yeah. properties.

M: 8 vertices, 12 edges, (M reported the properties of the shape while keeping it stationary.)

S: 6 faces

M: The question is all the edges are the same length. That is what I was trying to do, feeling and then move my finger.

M rotated the cube using the hands. M used all the fingers to rotate the object, then ended the rotation. With only the fingertips of two fingers on the vertices of the cube the object was rotated again.

Experimenter: And also...what did you say about the same length?

M: The same length. Describe whether it is something or not. Which we are not allowed to say.

(continue rotating the shape touching the vertices with the fingers).

Experimenter: So what is intriguing is that you know what it is.

M: They feel as if they are of the same length. You've got to take the opposite ones.

Experimenter: So what do you think it is then?

M and S together: A cube

The students looked at the object.

Experimenter: You say you knew as soon as you picked it up?

S: Yeah. As soon as I put my hands in and picked it up I thought that was a cube.

Students reported that using only fingertips it is difficult to discover the shape: it is important to use all the hand to identify the object. They also reported that as soon as they touched the cube they felt that it had the same length on all sides. The tactile exploration also was efficient to allow students to produce an abstract description through understanding the shape's organisation.

Rhomboid

Exploring this shape the students took more time inspecting it before commenting, than they did for the cube, cuboids, prism, different kinds of pyramids, and octahedrons. The time spent inspecting the rhomboid was about one minute and twenty-five seconds before they started to draw and discuss it.

Students touched the shape using palms of both hands, and fingers. In the first step of exploration they moved their hands around the surface of the shape and used their hands to cover all the surface of the shape. Then carefully they used their fingers, in a stroking fashion, to feel the edges. Student S placed one of the flat faces of the rhomboid on the box and inspected the edges, faces and vertices. The students wrote some of the properties of the object and tried to draw it. Then they went back to feeling it to check the vertices, edges and faces.

Experimenter: This one seems to be taking you longest to write things down

M: Because it is not a common shape which is never a day to day shape until now. And also it is harder to draw in three-dimensions. If you can imagine it in your head it is still hard to get down on paper.

M moved the hand around stationary shape and then rotated it.

Experimenter: Why?

M: Pardon?

Experimenter: Why?

M: Because to draw a cube, you sort of... you know how to draw a cube in three dimensions on paper. Because you've done it so many times before. To draw a pyramid you've it done so many times before.

M retained the shape stationary in her hands without moving her hands around it for a few seconds, then started moving her hands again and finally puts the shape on the box).

From this moment both students M and S no longer used their hands to inspect the shape. We conjecture that they felt it necessary to use their hands to relate to dimensionality as well as to the position of each part of the shape. They found a way to externalise how they use imagery to explain the parts of the rhomboid. They described how they form an image of the shape.

S: (Inaudible) ... picture it in my head it... It is kind of like uhm... Because the sides feel like they are the same length. (Both students M and S used their hand to show the two-dimensional form of the rhombus)

M: Its a diamond. (M uses the hands to show the form of the rhombus)

S: It is a cube and you've done that with it... Done that and done that.

Both students, M and S used their hands to indicate the parallelism of the opposite faces of the rhomboid. They inclined their hands in a particular direction to show the transformation of the cube into the rhomboid.

M: So it will be... The face is a sort of a diamond shape now instead of rectangular.

S: So I worked out how to draw it. Because of how it felt so... How I draw a cube, it shift it sideways and tilt the side to draw in three dimensions.

Experimenter: Right.

S: It was a bit weird when I first picked it up, until I put it down so one face flat on the bottom of the box. So I could feel how... What sort of shape it was. It helped rather than turning it round in my hand.

Experimenter: So it felt the same number of faces, edges and vertices as in a cube?

M: Yeah. Six faces. Which is right, yeah.

Comparing the discussion group of the visual-tactile perception (Chapter five) and the tactile- only perception, students in the previous experiment formed the image of a rhomboid comparing its features to that of a cube. Then they transformed the cube into a rhomboid. Below we have the transcription of part of a session of the visual-tactile exploration when students compared the features of the cube to those of the rhomboid. The students A, C, D and K took part in the visual-tactile experiment.

A: Would the shape be as if you had a cube, and then you tilted it one way?

K: Mmm, yeah.. if I think about it I know what you mean.

Teacher: He said to elaborate on something like "tilt" for example, if she wanted to.

K: What do you mean by tilt? Do you mean sort of almost pushed to one side?

A: If you keep .. if you put it on the ground, if you put a sq.. a cube on the ground, and you kept the face that's on the ground absolutely in position, and then pushed from the top .... so that the whole thing sort of went out shape. Just skewed over. Does that make any sense?

C : No,... the faces at the bottom will still be a square, won't they?

D: Kind of like you had a cube but just sheared to one side.

K: Yes. That's what I was trying to say.

C : Are, ... are opposite faces parallel?

A: Does it have 3 pairs of opposite parallel faces?

A: Top and bottom, are they parallel?

Dual cube octahedron

Students spent about three minutes and thirty seconds only inspecting this shape without talking about it. Firstly they touched the shape with the palm and fingers of both hands to feel if it had a generally round form. Then they examined the faces and the form of the different pyramids that together form the shape. Both students placed the polyhedra on the bottom of the box, where they held the shape stationary, and then passed their fingers over the edges. After having rotated the shape and still keeping it on the bottom of the box, they felt the faces that form the pyramids and passed their fingers along the edges. After that, they lifted the shape and continued their examination, rotating the shape and feeling the opposite pyramids. They touched the shape at the intersection of the edges. The examination continued, with them placing the polyhedra again in the bottom of the box and lifting it several times. One of the students passed their fingers around the shape as if trying to feel a cube.

Experimenter: Is there anything you can mention about this form?

S: Well I think...

This statement underlines an important part of discussion group that is *thinking* through discussion. It seems to be supported by students' observation of the dual polyhedron, because this student seems to be genuinely reflecting.

M: It is kind of sort of big spikes and little spikes (inaudible) Four edges and three (touching and feeling the pyramids).

S: Yeah

M: It feels if it could be... It feels as if it should be two shapes which are intersecting, like the larger one is a tetrahedron and it feels that the smaller one is a cube, intersecting with something else (Rotating the shape placing their fingers on the vertices of the cube)

M showed a high level of thinking. By only touching the polyhedron this student could understand that the edges, intersecting each other, seemed to be polygons placed together, so discovering that one of the parts underlying the shape is a cube.

Experimenter: Is that what it feels like?

M: Yeah.

Experimenter: It feels that way? And you can now see that?

M: No.

S shakes the head sideways signalling "NO".

Experimenter: S can you... Can you feel a cube then?

S : Not really. I can feel that if you've got the three biggest spikes there is uh, a smaller one in between it. And if I move it round, if you've got the three big ones there, there's a smaller one in between. But... I don't... (Placing the fingers on three pyramids that are part of the octahedron and then point the smaller pyramid - part of the cube - in between them.).

M- I can definitely feel a cube. Definitely feel a cube (With eyes closed and the fingers defining the cube).

This part of the experiment showed the usefulness of classification of shapes placed together. This means relating polyhedra that have significant properties in common.

S: Yeah. Now I can (Rotating the shape until place their fingers to define the cube).

M: It feels like a big cube.

S: (inaudible) ... It felt like a cube, but its a little cube.

We can observe from the discussion between the students M and S their high level of geometric thought. The students' performance indicates to us the way that they form images of the polyhedron. What is important is that these students M and S have formed the image of parts of the polyhedron in question, namely the image of a cube. When students are able to distinguish the parts of the polyhedron they may be able to discover the whole. The dual cube octahedron proved difficult to determine by touch alone: its features were not immediately apparent to touch-only perception.

Experimenter: You felt a little cube?

S: With three vertices. So I just put my fingers on the eight ones and now it feels like a cube (With the fingers on the vertices of the cube).

M gave answers which guided S to refine her geometric performance. It seemed that this experimental study helped students to learn naturally and from each other. The information they used was stored and retrieved from memory. Interpretations of the shapes' properties provided prediction of this shape.

M: If you ignore the triang... Ignore the big ones for a minute. Then...

S: OK. I've got... I've got my fingers on the... on the vertices.

M: On the cube?

S: I'm holding the cube.

M; And ignore the big ones for a minute.

S: (inaudible) And then you've got... Its like a cube with like... Its like these pyramids stuck on each face.

Students could gradually form an image of the polyhedron in their minds. To recognise and classify a polyhedron that is composed of intersecting of polyhedra requires a development of understanding shapes. In this experiment, students acquired insight which was related to their background.

M: Yeah. That's what it feels like. Yeah.

Experimenter: Have a look.

This experiment indicates students' development in perception by only haptic exploration. The students see the shape in their minds.

M: A small cube.

Experimenter: I'm afraid first of all you said that... That... That M it felt like a cube, but you didn't see. You quite literally feel...

M: Yeah.

S: Yeah. What M said about (inaudible) my fingers (inaudible) the vertices... (inaudible)

Experimenter: Yeah she did. I heard M before.

S: (inaudible) ... the fact that I was holding the cube without the (inaudible) just going like that in each face.

Experimenter: When M was holding the cube I was about to say Here ... is. You can see the fingers sitting on the... sitting on the vertices of the cube. It was pretty clear that one.

M: Yeah. Once you ignore the... sort of the odd bits and...

It is not easy to predict this polyhedron which is a pair of intersecting dual shapes. However, one of the intersecting shapes was a cube. Students did not feel that it was easy to notice the distinction between these two intersecting shapes. It requires a high level of understanding of the complexity of the structure of the shape.

Experimenter: Yeah, you were trying to ignore the...

M: See.. I think that the only reason that I actually saw the cube was because of the last one, where it was the two intersecting tetrahedra and it suddenly made me think; Ah may be this is something that is intersecting.

Experimenter: But when you finished off you said a cube with square based pyramids stuck on the side, when you've got that description, did you have an image or not?

S and M: Yeah.

Experimenter: You did. And what about before the final verbal description or after do you think?

M: Just before.

S: Yeah. I could feel the cube sort of move my fingers on the top felt where it were. And just... I suppose you see it as you say it, if you see what I mean. You have to know what it looks like to be able to say what it is.

M: Yeah.

S: If you see what I mean. I couldn't have known that there was a square based pyramid on each face, because I didn't know that is what it was. Trying to put that into words. Sort of a split second after each other that you see it and say... you know what to say. Its not seeing it saying what do I call that?

Experimenter: Yeah.

Icosahedron

This was the longest part of the haptic exploration section. Students spent a long time and made many tentative explorations to form images. They used their hands, moving them around the icosahedron to feel that it is a convex shape. They passed their fingers from one vertex to the other over the edges. They rotated the shape several times. They placed their fingers over opposite vertices and then apparently tried to count the numbers of faces meeting in one vertex. They placed the shape flat on the bottom of the box, rotated it, and tried to draw it. Most of the time they inspected the shape carefully, and when they started to talk about it they did not show a great deal of confidence compared to their talking about the other shapes. They made considerable progress during discussion, however, they were not successful in identification of the whole the shape.

Experimenter: Can you say anything about those?

M: I can say what the faces are (Rotating the shape in the hands).

To perform a complex task, that is to identify the icosahedron by tactile exploration, involves an understanding of the shapes' arrangement that is conditional on the tactile input data. Learning arises in the experiment through the knowledge that students bring to the experimental situation, their tactile explorations, and their discussions with each other. Students geometric development requires gaining some knowledge of features which characterise the polyhedron. M is probing features of the icosahedron by relating her tactile sensations to images built from stored memories which she retrieves at the time of exploration.

Experimenter: Yes, well?...

M and S: Triangular shapes.

Experimenter: umhum.

M: From what I can see... See if I can find a top to it, but somewhere like where it sits on the ground. From each sid... edge of the top triangle, another triangle comes down from it... (inaudible)

S: How many pictures have you done?

M: And from the point... Two other points of triangles... Vertex, two other vertices come on to each point on the top (Passing the fingers from one to other three vertices of the triangle and over the edges).

In this stage of their exploration, the students tried to obtain control over the shape's features. They showed difficulties in discovering them. This experiment provides evidence that the properties of an icosahedron were not directly perceptible for them by touch data alone.

S: I'm trying to imagine the next section (with the polyhedron placed on the bottom of the box, one of the triangles was in contact with the box and S rotated it passing the fingers over the edges of the top triangle).

M: Six from the edges. And then...

S: It .. three edges and three vertices so... (rotating the polyhedra)

M: Yeah. Oh six, I was looking at the three edges.

S: Six, yeah.

In this phase of their exploration some important properties were not pointed ou by the students. They discussed some quantitative details, but lacked substantial discussion about the shapes arrangement.

Experimenter: So what sort of image do you have now?

S: Kind of like pentagons.

Students felt little difficulty in describing the images in their mind. S described a 'pentagon' which is a two-dimensional shape and which is not a face of the icosahedron. This student imagined and discussed the shape at a more abstract level: what we have termed "deep perception". S was not able to dissect the icosahedron into two pentagonal pyramids and an antiprism. However, S could "feel" the pentagon that is the base of the pentagonal pyramid around the icosahedron.

M: But triangles.

S: Triangles. I can't quite... It's like... I suppose its kind of like it was before. Where you've got... it got like the middle bit (inaudible).

M: But it seems to be more irregular.

Experimenter: Its a lot more...

M: Irregular. Its shape, than the pentagon... It feels it is just a random number of triangles stuck together to make a solid (With the shape on the hand passing palms and fingers over it).

Students presented considerable interest in sharing their experience, and the language they used was adequate to this task. However, some problems arose in the interpretation of their perception. For example: an icosahedron is a very regular shape, but the students constructed an image of an irregular shape.

M asks S: Can you find what is the top or the bottom?

S: If you put flat on the bottom of the box, yeah.

M: Then it feels two equilateral triangles on top and bottom. Isn't it?

S: Yeah.

M: Then you see what I mean when I said a triangle comes down from each edge.

S: (inaudible) ... vertices. That would make six triangles around the top and six around the bottom and two...

Experimenter: And how many is there would you say?

M: Fourteen.

S: Fourteen. I think. Not by counting them, by... If you have a triangle at the top which it has got three triangles joined on its edges. Then three at the points... That seems to make six around the top bit,

Students retrieved information that was accessible from their memory, allowing them to develop learning under the conditions of the experiment. On the basis of the statement above, students indicated an ability to express their understanding of the organisation of the triangles. This performance involves input information on memory and comparing the tactile sensory data from the concrete model to their mental representations.

M: And six around the bottom as well.

S: So it will then have six around the bottom bit. So that makes fourteen faces.

This quantitative information could be helpful when students determine the class that a particular shape belongs to. An icosahedron has a considerable number of faces, and students felt it hard to count the them. Furthermore, these students failed to understand the arrangement of the shape: around each equilateral triangle there are nine other equal triangles.

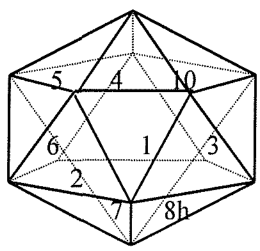


Figure 6.2 Icosahedron

Experimenter: If you put a finger on one of the vertices and you were doing something around it.

M: Yeah. That is what I was trying to work out. How many uhm. It was like. I found the top. Sort of the top triangle for example. And I found the vertices, I was trying to work out how many triangles were coming out of that. Coming into that vertex. So...

M lifted the shape, rotating it in the hands, and then placed it again on the bottom of the box, passed the fingers over the edge of the top triangle, lifted the shape again and passed the fingers over the edges of other triangles, and tried to count the number of triangles. Then, M tried to cover all the surface of the shape with both hands.

Experimenter: And what was the answer to that?

M: Three

S shakes her head sideways. The students count again.

M: There is five triangles going into each... Five triangles going into... I've got a point which has got five triangles coming into it (M pass a finger over the triangles).

S:Yeah. Five

M: (inaudible) (Trying to cover all the surface of the shape with both hands)

From their interest in knowing the sequence of the arrangement of the triangles, students came to observe the number of triangles meeting in each vertex. This was in marked contrast to students reported in Chapter five who, in trying to form images of polyhedra, never asked questions about the arrangement of faces at a vertex.

S: It sort of...

Experimenter: Have you counted the vertices? Have you counted on the other ones?

M: Uhm yeah. It is harder with this one because it is all irregular.

The students' view of regularity of the shape provides a strong constraint to the possibility of their identifying the whole shape.

S: It is like a triangle on the top, a triangle on the bottom, and then there is like a bit around the middle and they sort of slope up and there then up and there.

S left the shape on the box and used hands to explain the position of the triangles, then used the thumbs and index fingers to represent a rounded part, probably the pentagon, and then joined the middle fingers representing the top of the shape, finally placing hands parallel and join them on opposite to the representation of the top of shape.

Experimenter: Can you imagine... You said you were looking at the vertex that had five triangles coming of them. If you look straight down that vertex can you imagine what you would see?

M: Yes. It is sort of looking down at a point with a pentagon sort of around it. The... So the point will be at the centre of the pentagon. Is it a pentagon? Yeah. But then it seems to be a little point... but underneath the pentagon then it is not the same on the other side its irregular. It's underneath (Uses the fingers to represent the pentagon and then check on the shape if form the pentagon and try to cover the region of the pentagon with one hand).

Experimenter: What do you mean underneath?

M: So what you can't see. It will be hidden, like behind the pentagon.

This phase of learning shows students' increase in mastering the icosahedron, including detection of parts of the shape, but presenting difficulty in relating this part with the other parts of the icosahedron.

M: It is not sort of symmetrical. But I it could be.

M tried to feel some symmetry in the shape, but it did not seem obvious to her because she did not yet associate all the parts that form the whole shape. Consequently, there once again arose a difficulty for her to decide the regularity of the shape.

S: Does it make five edges.

M: Seems almost like there is two bits to it. There is one shape, and then half of one shape and half of another as before. I feel as if I could slice across one bit (M touch on the two pentagonal pyramids on the top and on the bottom of the icosahedron).

Experimenter: Did you say that the shape was regular or irregular? Regular or irregular?

S and M: (inaudible)

S: I feel like it should be regular. But it doesn't.

M; I don't think it is regular.

S: That makes sense. They feel like... They feel like they are equilateral triangles.

Again these students were not able to distinguish the regularity of the shape. Furthermore, they did not identify the icosahedron. On the other hand, the students' argument indicated their high level of geometric thought regarding regularity. Equilateral triangles are a very regular two-dimensional shape They recognised that the three-dimensional shape in question was formed only by equilateral triangles. They understood that it is not enough to indicate the regularity to identify the polyhedron.

Experimenter: Right.

S: So the shape should be a regular shape.

Experimenter: Where do you feel... feel the pentagon?

M: Where?

Experimenter: Where?

M: Across that ring there.

S: Around the middle. In the middle. Yeah.

M: With the point there.

Experimenter: How many pentagons did you feel?

M: How many?

Experimenter: Yes.

M: (inaudible). Its two pentagon bits with something in the middle. With a ring around the middle. And that ring is triangles. So that is going to make... five, ten...uhm.

(M tried to count opposite pentagons).

M and S count.

Experimenter: What were you saying about the ring of triangles? I think you said there were two pentagons on this. And a ring... A ring did you say?

M: Yes.

Experimenter: And then...

M: A middle section, two points with.. if you're looking from above each one, a pentagon shape, that you would see. I think towards each ... point. And then a middle section which

S says to M: Rather than having... I mean rather than have it flat at the bottom, if you turn it that way, you would see a pentagon.

M: Yeah. If you turn it with the vertex that has got five...

M counts.

S: All the vertices have (inaudible)

M: That means they all have it which ever direction you look at the...

S: If you looked at it flat...

Experimenter: Can you see this ring that...?

S: Yeah. I was looking at it so I had one of the triangles flat at the bottom of the box and trying to look at it that way,

Experimenter: Right... Yes...

S: But what M was saying, if you put one the vertices facing you then if you... say if you drew that flat you would have a pentagon, I think. That is what you saying isn't it?

M: Yeah. And for whichever vertex you take it... (Inaudible)... so whichever view you have of it, as long as you have a vertex pointing towards you, you've got a ... sort of a pentagon shape. Now how do you draw it?

Students paid attention to the vertices which indicated an important part of the shape that could help them in understand the whole. The students' minds had processed tactile sensory data which was aimed at optimising their overall performance in identifying a complex shape.

S: If it was flat... If it wasn't... If you look at it straight...

M: Straight down yeah.

S: You'd see a pentagon.

M: You'd certainly see a pentagon, but I don't know what else you would see as well as the pentagon, but you'd see a pentagon.

S: Yeah.

M: I don't know about the things that would stick up beyond it (S tried to draw the shape that she imagined)

The students see the shape.

M: Oh. It is very simple and very regular isn't it?

S: It didn't feel like that at all.

Experimenter: No? What did it feel like?

M: Very irregular.

S: Yeah. The triangles...

M: Weird. This is the ring I was talking about. And I knew there were triangles one that way, one that way, one that way... And that you had a pentagon bit there and a pentagon bit there with the ring.

7.4 Test results versus verbal discussion

In this section we compare the vocabulary used on the written test and in verbal description during the study. The students' interaction provided for an improvement in their geometrical performance. The evidence from the test results presented in table 6.1 and from students' description of polyhedra, indicates that both M and S present high geometric ability. These students are close to accessing van Hiele level four.

In the tactile experiment, as in the test responses, students paid attention to the shapes' properties, such as the kind of faces, and numbers of faces, edges and vertices. On the other hand during discussion they could express more about the shapes' properties: for example, symmetry was very little mentioned on the written test and mentioned much more during student discussion. Other examples of properties mentioned only in the discussion group were:

During discussion of a cube, M reported that all the edges are the same length.

Both M and S described the rhomboid indicating the parallelism of the opposite sides.

They described the arrangement of the faces, and how these faces intersected, for a variety of polyhedra.

In her description of the dual cube octahedron, student M described the cube and the octahedron as intersecting to form the polyhedron.

What is contrasting in the students' test responses and group discussion is the students' understanding about regularity of shapes. Neither student give a satisfactory argument to define regularity on the written test. However, M's answers were more consistent than S's answers. For example:

M answer that a regular shape is one in which:

"...symmetry exists, for example a triangular based pyramid with the same size sides as base is regular. Symmetry occurs down the middle."

M distinguished regular and irregular shapes.

S answered, a regular shape is one in which:

"... each of the edges are the same length."

On the item that required distinction between regular and irregular shapes, S gave blank answers. On the other hand, in the discussion about the icosahedron, M and S both ran their hands over the equilateral triangles and decided it was not a sufficient condition to determine the regularity of the shape.

These results indicate that in group discussion students can be involved in a strong effort to understand shapes compared to the results of written test. The evidence showed that the interaction between the students during group discussion can stimulate thinking deeply, because one student may influence the other to think with their comments or with their questions and answers.

7.5 Aspects of tactile perception and the graphic representation of the polyhedra

This section explores the graphic representation of the three-dimensional form that is the drawing of the polyhedra made by students during their haptic exploration. With this scheme we can observe the students' drawing which generally is topologically stable. By these drawings, that contain geometric information such as kind of faces and surface organisation, students are trying externalise what they have perceived. This technique in which people get information and then try to represent it graphically (when it is possible) can be a way to indicate peoples' geometric insight about the relationship between a shapes' properties obtained by haptic perception and mental image formation.

Students tried to identify three-dimensional objects and then represent them in two-dimensional patterns. This identification involves real objects and mental models to obtain a graphic representation. Familiar shapes such as a cube, cuboids, pyramids in general, and octahedrons, did not present difficulty in drawing for students. Occasionally, it is complicated to match a two-dimensional image with a three-dimensional model. Some problems arose regarding some shapes that students could identify but felt they had difficulty in drawing completely the object that was in their mind. Some of the polyhedra such as a dual cube octahedron and dual tetrahedron, which are shapes that intersect each other, students identified but did not try to draw. It occurred because the drawing approach presents projection in a perspective view that is qualitatively different from the three-dimensional model. A drawing only shows one view of the shape. For example a cube is made with all its faces square, so all its internal angles have ninety degrees. However, when we draw a cube the faces may not be represented by squares, and consequently the angles are not all ninety degrees. The projection is simple to draw in this case. In order to be specific, students presented in their drawings, a local arrangement of the parts of a polyhedron to give an oriented view. They also represented the hidden parts: the internal contours of some hidden parts that are transversal lines (representing the edges) crossing the lines that represent the parts not hidden. So, the view in perspective may produce some changes on the surface and other invariant properties of the polyhedron. This is the constraint in representing three-dimensional forms with two-dimensional forms.

Chapter Eight

Discussions of the empirical work

Chapter Eight

Discussions of the empirical work

8.1 Test results

On the written test there were several ways students indicated understanding of three-dimensional shapes. First, they imagined the two-dimensional shapes which formed parts of the three-dimensional object. Secondly, they thought about the arrangement of the two-dimensional shapes to obtain the three-dimensional object. Thirdly, they thought about the three-dimensional properties, including simple and familiar properties such as the number of vertices, and the number of edges.

In relation to those questions on the test that asked for the properties of the representations of three-dimensional shapes, some of the students gave insufficient information about the shapes' properties. This is one of the reasons for some poor results that were obtained. On the other hand, their oral description of the geometric shapes by touch and seeing were more precise than their descriptions on this test.

The results of the van Hiele test in three-dimensional geometry generally conformed to the hierarchical nature of the levels (Usiskin 1982, Mayberry 1983, Burger & Shaughnessy 1986, Gutierrez 1991) in a three-dimensional geometric setting. While it would be desirable to apply statistical procedures to test for the hierarchical nature of the levels we know of no such test. Private communication with Z. Usiskin of the University of Chicago confirmed that he too is not aware such a test. This would be a highly desirable statistical test to have, and perhaps techniques from item response modelling (*ref.* Anastasi & Urbina, 1997) could be of assistance.

However, the results presented indicate the assessment of Del Grande's spatial perception is not hierarchical. It is possible for a student to acquire the abilities pointed out by Del Grande without any regular sequence.

8.2 Group discussion sessions

8.2.1 Visual perception and imagery

The results from the group discussion sessions supported several general conclusions:

1. There is a distinction to be drawn between visual perception of polyhedra nad mental imagery of polyhedra. There was no correlation between visual perceptive ability or image formation. Visual perception was generally correlated with written test results, and thereby to van Hiele levels of geometric thought, but image formation bore no obvious relation to van Hiele levels.
2. Group discussion assisted substantially in raising visual perception and in assisting image formation. The connection between discussion and increased visual perception - so evident with student F - is particularly significant in light of the connection between visual perception and van Hiele levels.
3. Both visual perception and image formation were associated with hand gestures. The connection between hand gestures and image formation of polyhedra is significant in that, as far as we are aware, this has not been reported before in the psychological literature. This finding adds weight to Kosslyn's claim that image formation and visual perception share the same neural mechanisms. The hand gestures during image formation are probably a reflection of similar hand

gestures that would be made in handling the polyhedra during visual, and tactile, perception.

8.2.2 Perception and mental models

Strictly speaking, the visual perception of a pentagonal pyramid as a part of an icosahedron does not seem simply to be a question of perception alone, but also one of imagery. We hypothesise that a student has to form a pre-image, or mental model, of the pyramid and then see it in the physical object. It as if, upon being asked if there are pyramids in the icosahedron one forms a vague template - not quite a detailed, inspectable image - of a general pyramid, without assumptions as to the nature of its base, and then "sees" the pentagonal pyramid upon closer inspection. The pre-image is rather like what one sees when asked to imagine a bird. What sort of bird is it? It's generally no bird in particular and one cannot distinguishing fine features, because they aren't there. In the same way, in looking for a pyramid in an icosahedron we postulate that a student has a similar vague image of a paradigmatic pyramid - as a cone over a polygon - and then finds perceptual data to sharpen the image template into something with fine, detailed features.

8.2.3 "Chunking" versus edge-vertex-face description

What was striking about the student's questions was their concentration on faces of polyhedra, and the ways in which faces might fit together to form identifiable sub-objects of a polyhedra. There were almost no questions on vertices and very few on edges. There were no questions asked, or information offered, on the number of edges meeting at a vertex. This is in marked contrast to an analytic description of a polyhedron: see for example, Maeder's description of the Wythoff symbol (Maeder, 1995). One might argue that students had not been taught to look at the combinatorial structure of vertices and edges - but equally they may not have been taught to integrate local and global features. One would expect a more even distribution of approaches in that case, and it was of interest to observe the lack of questions on combinatorial structure of vertices and edges.

Both in visual perception and image formation, students seem to favour a plan of seeking information about relatively regular sub-objects such as cubes and pyramids that might fit together in certain ways to give the entire polyhedron. In image formation, this is a sort of mental "chunking" procedure in which mentally manageable sub-units of the polyhedra are built up, so allowing missing bits of the image to be filled in. We hypothesise that students do this, rather than focus on analytic information about vertex structure, so as to reduce the cognitive load of image formation. This is not surprising when we remember, as Kosslyn (1996) says, that images are formed from pre-conceived perceptual units.

8.2.4 Implications for student learning

In this study we have observed a suggestive connection between ability in 3-dimensional geometric thought, and high-level visual perception. The written test was broadly compatible with testing for van Hiele levels. The results from this test and from the polyhedra identification tasks are significant. They indicate that, in assisting students to attain a higher functioning in 3-dimensional geometry, it is important to enhance their high-level visual perception. This is an active process of "training the eye" to see deeper aspects of 3-dimensional figures. We have illustrated that one way to do this is through their providing verbal descriptions for other students. Discussion and questioning usually stimulates a student to look more deeply and to begin to visually analyse a figure. A general rise in geometric thought appears be associated with this heightened visual perception. As a result, a student is automatically propelled forwards in geometric thought, as evidenced by progress through the van Hiele hierarchy.

8.2.5 Deepening of visual perception

The students in this study were engaged in intelligent, goal-driven activity.

For example, student F wanted to be able to answer A's question about pyramids in an icosahedron It is intelligent because F had an image of a pyramid and tried to match this image with visual perception coming from further actions on the icosahedron. At first there was no match . Then something resembling a pyramid - enough like a pyramid to be iconic - appeared in perception. F then asked if one could have a pentagonal pyramid. F's intelligent activity can be described in the following diagram (reminiscent of Skemp's 1986 director schemes):

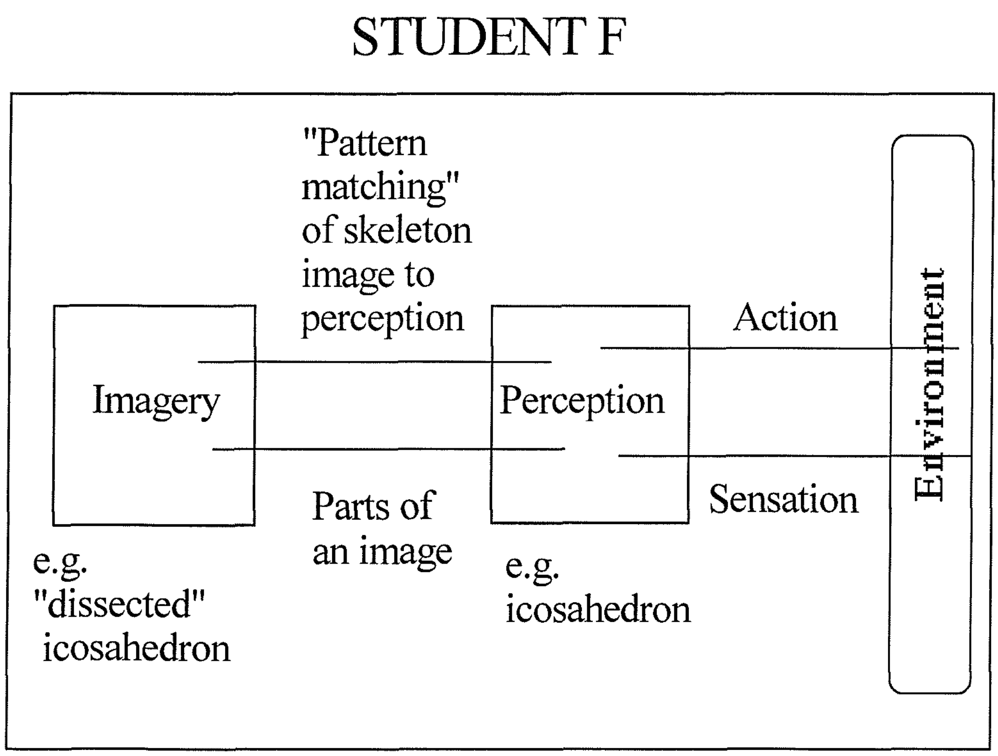


Figure 8.1 Student F's schemes

F was asked by A were there any pyramids. She said, after a pause, "No". At this point, we hypothesise F was scanning the visual perceptual form of the icosahedron for a match with an image:

It seems that F was looking, in the physical model, for something that matched one of the following patterns.

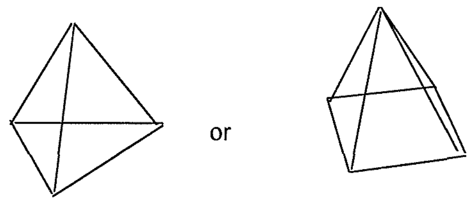


Figure 5.6 F trying to form the image of part of the icosahedron

F found no such match, and so said "No".

However, a feature of F's visual perception is iconic for a pyramid - it reminded F of one - namely, the form at a vertex:

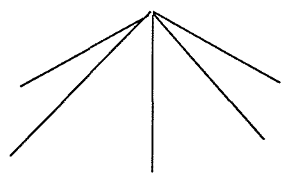


Figure 5.7. F perception of part of the icosahedron

We hypothesise this caused F to wonder if the part of the icosahedron now in focus could be called a pyramid, even though its base is a pentagon.

Once agreement was made, F was happy that there are lots of pyramids in the pentagon. In fact, F can now visually perceive pentagonal pyramids in the icosahedron easily: witness F saying there is one at the top and bottom in answer to D's question.

The perceptual procedure requires sequence and organisation of actions and reflection. Francis spontaneously realised a motor activity which contained the possibility to improve her visual perception and consequently her use of words. F procedures allowed her to form a kind of structure related to her perception of, and F's thinking about, the icosahedron form an abstract structure:

1. Attention to the correspondence between two dimensional shapes: the kind of two dimensional shape (namely triangles) and the conventional arrangement of the triangles to form the icosahedron which is a stable and conventional pattern (it is one particular icosahedron - regular - out of many possible).

2. The spatial configurations are experienced and stored in memory.

3. The spatial configuration are reorganised in F's mind. She saw the icosahedron and invoked a mental image. Mentally, she subdivided the icosahedron into two pyramids which had an antiprism in between.

Certainly, her visual-tactile experimental activity has a consequent particular sequence of mental activity: the patterns are organised and subdivided into other patterns. However, her mental activity did not seem to involve creation of units.

By this goal-directed pattern matching - scanning a visually perceived object for a match with an image - F visual perception was *deepened* . F was now seeing more than before, or possibly more than could be seen before. van Hiele would probably say she is seeing more *structure*.

With further questioning about what is between these two pentagonal pyramids, F now appears to "see" an icosahedron in dissected form. This seeing may be perceptual, but F has the possibility of bringing the dissected icosahedron back to mind as an image.

This re-collection is precisely what allows F's thinking about the icosahedron to move from perceptual to symbolic. Once a perceptual object such as an icosahedron can be brought back to memory, as an image, and operated on again in the mind, it allows a person the possibility of becoming aware of the those operations, and so acts as a symbol in Steffe's sense. In this way a sensory object in the world - a cardboard icosahedron - becomes a visual and tactile perceptual object, an image to be recalled, an image to be operated on, and then a symbolic mathematical object.

The drive - the motivating force - for students' deepening perception in this study is the goal-driven pattern matching activity of searching for matches with stored images in a perceptual object.

8.2.5 Fluency of discussion

I presented evidence that as student's are thinking about polyhedra - either inspecting them visually for features, or forming images of them - their verbal communication becomes less fluent, and more hesitant.

Is it difficult - perhaps impossible - to both talk about an object and inspect deeply an image of it, or look deeply at it for visual information, at the same time? We have found no evidence for this in the psychological literature. However, there is currently a debate about the independence of auditory and visual codes (Paivio, 1991; Brandimonte, Hitch, & Bishop, 1992; Thompson & Paivio, 1994; Partridge, 1995; Mehta, & Newcombe, 1996) that suggests the likelihood of just such a difficulty. Evidence from mathematically advanced students of varying levels of attainment shows that such difficulties occur in practice and are probably common.

Mathematics teachers are aware that their students need to learn their lessons and that it does not happen automatically. Learning is a result of understanding, and internalisation happens according to the individual's interpretation that takes place through discussion or by experiences. Teachers' performance is importance in a quality interaction between learners and their environment and teachers certainly look for an effective way to promote learning.

In teaching and learning we must consider the objective of speech. There are differences in speech concerning learning from discussion (interaction) or presenting a talk to learners. A pre-prepared lesson may present less difficulty in processing different parts of speech than an informal discussion in which the students need to interpret the questions and/or answers about an object or the scene. This is because the organisation of sentences in a pre-prepared lesson can complement the construction of new information.

The interpretation of verbalisation may happen before students understanding of a particular shape. When we are inspecting a shape and describing it we can detect more pauses or fragmented sentences or fragmented words in our speech than when we spontaneously give prepared lesson. We organise sentences in time and we tend to think while we examine a shape or event. We also think about the structure of the next sentence - thinking is a non-linguistic cognitive activity - and we use language to communicate thoughts: the results of thinking. We show evidence from our group discussions that informal descriptions are more easily affected by the organisation of the structure of a sentence than a prepared lesson in which new information is introduced constantly. Lessons based on discussion demand thinking on the part of the students.

Students have difficulty in speaking fluently as they examine deep features of objects or images. Success in verbal description usually requires the speaker time to construct their understanding What does this say about a fluent verbal description of a mathematical topic in an area that normally involves geometric images? To us, it indicates that such fluent verbal descriptions do not involve inspection or perception of deep features of images or objects. In other words, fluency of verbal description is generally incompatible with concurrent deep perception or imagery. Someone who describes such deep features fluently may have previously examined them deeply. However, they are probably not examining deep features at the same time as they talk fluently.

If we, as teachers, explain mathematics in a clear, fluent manner, the chances are we are not thinking about deep visual features of the mathematics at the same time. How then do our students learn from us to see more deeply? We suspect that in the process of organising and delivering a classroom lesson, many - perhaps most - mathematics teachers' focus of attention is on the way in which the lesson is proceeding and not on deeper features of mathematical thought. Put simply, we can prepare a lesson on polyhedra, for example, and talk fluently about the properties of polyhedra because of our preparation. What we claim, is that at that moment of fluent delivery we - the teacher - are not inspecting deep properties of the polyhedra. In other words, we are concentrating on our delivery and not on our thought processes.

Student hesitancy in explanation or questioning exists and is probably common. The reasons for this have to do with the extreme difficulty people have in talking about deeper features of objects or images at the exact same time. As a result we, as teachers, should not try to prevent students being hesitant, but rather recognise this hesitancy for what it is - prolonged deep perception or imagination. As Thurston (1997) says:

"Most students have far more mental agility than their words, or their math homework, reflects. It's important to keep a good measure of humility, and not to assume we know what's going on in their minds. Forcing them to talk in ways that are not natural for them 'can' help them think clearly, but it 'can' and often does have the effect of blocking them from thinking for themselves."

8.3 Tactile session

We have described the students hand movements during the task of discovering polyhedra. We noted that the effect of free haptic exploration of properties of common objects like a cube, cuboids in general, and squared pyramids which were identified in a considerably shorter time compared to the uncommon shapes like intersecting dual tetrahedra or dual cube octahedron.

Note that there are marked contrasts in the procedures utilised in the two kinds of free explorations: tactile-only and visual-tactile exploration. Concerning the tactile- only exploration, students explored the vertices and the edges to perceive the kind of faces of the polyhedra, whereas in the visual-tactile exploration students immediately obtained the information about the faces of the shapes and concentrated on the arrangement of the faces. Sometimes the numbers of faces was not important for the visual-tactile exploration as, for example, in the description of the icosahedron.

We have compared the hand movements from the video tapes of these two distinct experiments and we have evidences that the motor actions differ from the haptic experiment to the visual and tactile experiment. We conjecture that it happens according to the kind of information that the students get. Our visual system is more efficient than our tactile system in identifying objects. Information is processed more rapidly using the visual system than information obtained by the use of the tactile system.

We also can compare the level of verbal descriptions from both experiments and we find that students produced similar abstract descriptions. The students' perceptual levels have shown sensitivity to conform to the variety of shapes, which is dependent on a shapes' organisation. Indeed, the results of the tactile experiment showed that students could reliably identify about 93% of the shapes given to them during the session and for most of the shapes students obtained a deep features of perception. The two students could identify the same shapes and both had problem in identifying the icosahedron. With a simple identification of the polyhedron in question, it does not mean that the students attained a deep feature of perception: as an example, students identified the cube as soon as they could touch it. This was certainly identification by surface features because it required little conscious effort. However, in the student discussion both, M and S tried hard to identify the icosahedron without success. These students presented for this shape an intermediate stage between surface and deep features of perceptual objects. They could feel the pentagon around the shape. However, they could not perceive the antiprism which is the connection between the two pentagonal pyramids, one on the top and other on the bottom. Therefore, we assume that for a student to attain any perceptual level, it does not depend necessarily on complete identification of the whole shape. Excluding the cube that students identified by surface features and the icosahedron that students were not successful in identification but attained an intermediate level between surface and deep features of perception, for the other shapes, students clearly attained deep features of perception of the object.

In both tactile and visual-tactile exploration students showed proficiency in the understanding of the polyhedron presented to them. The nature of hand movements, specifically during identifying parts of a polyhedron and understanding its organisation, manipulating the shapes, in the presence or in the absence of vision, clearly was of great importance for perceptual and cognitive understanding.

Chapter Nine

Conclusions

Chapter Nine

Conclusions

9.1 General conclusions

This thesis focuses on the use of concrete manipulatives to enable students to learn by guided discovery. This approach creates an effective learning environment in which students were encouraged to use their thinking creatively. The use of manipulatives in this work motivated students to acquire new experience through sharing knowledge. The interaction between students one to another using concrete manipulatives and language as a vehicle of communication gave an important entry to students to become active learners. According to Leinkin & Zaslavsky (1997) interaction is an essential component in the process of making sense in the course of learning mathematics.

Concerning the study of perception, findings presented in this thesis verified a complexity in the perception and description of polyhedra. It is important to notice that students used a variety of styles of descriptions with different expression to describe spatial relations. They adopted an individual strategy and consequently individual difference in describing shapes that allow them to integrate information to efficiently describe polyhedra to their students. The discussion group procedure revealed to both the describer and listeners an intention to facilitate the understanding of elements that form the arrangement of described polyhedra. The structure of a polyedron's description affected the quality of image formation for the students for whom the polyhedra were hidden. So, the same shape in the same situation was described differently by the different students. We note that the strategy used in polyhedral description does not depend only the polyhedron's structure, but also depends on the students perception and their domain of previous knowledge. According to Denis (1996)

" ...apt describers are not only characterised by adequate linguistic aptitudes but also by their capacity of constructing a coherent visuospatial representation of the object described." (p. 189)

Some findings regarding comparison of results from different perceptual experiences, considering different conditions of stimulation, were reported in previous chapters. As an example, the human visual system is able to produce a visual image of a certain shape without intentional reasoning about the parts of the shape whereas the use of a "tactile- only system" the students showed interest in the parts that constitute the whole shapes. This tactile procedure is based on generating a numbers of images such as the two-dimensional forms that are parts of the polyhedron, the numbers and kind of two-dimensional shapes meet in each vertex before forming the image of the three-dimensional object.

We have shown in this thesis that understanding geometric shapes are all complex processes. The use of manipulatives can have a considerable effect on teaching and learners when used appropriately, taking into account both teachers and students perceptual abilities. Some polyhedra are definitely more complex than others, and obtaining deep perception of these complex polyhedra is not an easy task. Group discussion that involves students in visual and/or tactile exploration can contribute substantially to an increase in students' geometric knowledge and appropriate

This thesis has pointed out the importance of visual learning, tactile learning and learning through discussion that involves visual perception and image formation. Evidence shows that verbalised visual perception or verbalised tactile perception are associated with students' individual capacity to mentally organise the features of polyhedra and organise coherent sentences.

This work shows evidence that students are usually hesitant while questioning, answering, and arguing about polyhedra. It happens because they were thinking and explaining alternatively. Evidence presented in this thesis indicates that while a student inspected a certain polyhedron and attained deep features of perception they made an effort to verbalise coherently those deep features. Therefore they inspected their perceptual sense, and generated images to produce coherent. As a result, fluency in verbal descriptions are related to the perceptual sense and to images. When students saw or felt some features of a polyhedra, it often indicated a deep feature of perception.

Students often presented difficulty in speech fluency as they examined the deep feature of a polyhedron or image. Interpretation to construct sentences may happen before the students' understanding. Consequently students need time to have success in verbal description.

This work presents evidence that during group discussion students reveal a sufficient understanding of the spatial model presented to them and during conversation, some students became competent to give a more elaborate description of their observations of a particular polhedron, using adequate geometric vocabulary. A deeper analysis of polyhedra promoted students to use different and detailed arguments. They were able to use different procedures and to stimulate the group to attain a high level of thought.

The interpretation of the discussion group indicated that some consistent questions and explanations contained inferences from students' previous knowledge. Students' request to predict a certain shape, showed their development of geometric knowledge and consequently the development of geometric language, because the students' argumentation involved action, movement and progress between conjectures.

The reflexive relationship among students and between students and subject was both the purpose of analyses based on observation of students behaviour and cognition. The important cognitive aspects are: How students perceive and generate images of arrangement of the parts of polyhedra, how they integrate local and global features., what process the students used to generate images, and how they develop linguistic description. The cognitive development arose both in individuals and in the group as a whole.

All the students who took part on this study had opportunites to develop their polyhedra understanding and consequently their linguistic description and language structure. The situation created by them while they described shapes and interaction between them prompted to them to converge on a polyhedron's interpretation The analysis of group discussion indicates that often students shared general geometric meaning. In addition the kind of students interaction in which they were engaged they had opportunities to be influenced by the others. As a result they were engaged in and they could produce mutual learning opportunites.

9.2 Proposals for future research

Previous chapters in this thesis provide results of investigations concerning students' cognitive development. This study examines students' abilities to represent their images through linguistic information and by drawing. Special sections show integration of students' linguistic development with students' improvement of perceptual abilities. The linguistic analysis is related to spatial orientation and propositional information, meaning that students described verbally what they perceived.

In this section we provide a proposal for future work which is related to the research in this thesis. Some tasks will be outlined focusing on three-dimensional geometry. Tasks will be associated to **construction of tactile perceptual models and visual-tactile perceptual models**. The purpose of this study is to consider the nature of students' explanation of their visual and tactile perception. The language used during students' discussion groups will be the source for interpretation of the perceptual models. Continuing the work with three-dimensional geometry focusing on tactile perception, visual perception, image formation, imagery, and language development, the interest for the future investigation will correlate these studies to investigate a hierarchy of perception and how this hierarchy is associated to the van Hiele levels. All the tasks intended to be used involve manipulation of concrete models and verbal information. The experiments reported in this thesis analysed the motor action during verbal description, and indicated how this is related to geometric thinking. For the future experiments it is also expected that students will express their geometric ideas using hand movements.

When students describe their perception they are explaining their understanding, which not happen randomly: understanding depends on previous knowledge. The basic idea of the future work is to obtain evidence of different models which underlie perception in geometry.

In the following section we describe the method which we intend to use to observe the characteristics of students' answers.

9.2.1 Structure of the Perceptual Models

We intend to construct a scheme of observation of perceptual data. This scheme of observation will provide a delineation of "phases". This will be perceptual phases, similar to the van Hiele phases, as described in Chapter One. We will compare the relation between students thinking and their propositional statements. Instruction and experience will aid the students in observing properties of polyhedra. We will start from simple identification and recognition of shapes, moving onto the analysis and description of them.

The hypothesised phases are:

PHASE 1- Students differentiate polyhedra without being able to describe them in any geometric way. They identify shapes as a whole by their appearance. In this phase, students can differentiate one shape from another without providing any reasoning for their distinction. For example, they would be able to distinguish a triangular based pyramid from a square based pyramid.

PHASE 2 -Students are able to see and recognise the faces of a polyhedron and are competent to describe them. This phase requires adequate description of different faces from one to another. For example square pyramids have triangular and square faces, and a description of each different face is required.

PHASE 3- Students begin to describe arrangements of faces and/or edges. This phase requires extending explanations to the complex organisation of the parts that constitute the shape, and comparing properties of different three-dimensional shapes. Correct and logical justification of relations between different polyhedron of the same class are also characteristic of this phase.

PHASE 4- Students can see and describe symmetries. This phase requires particular attention to proportionality and regularities presented in certain shapes. Therefore students must be able to compare symmetrical properties of two different polyhedron. An example is comparing the Platonic solids with other semi-regular solids or irregular solids and describing their perceptions.

PHASE 5- Seeing and describing integrated parts of a polyhedron. For example, perceiving that an icosahedron can be seen as a non-overlapping union of two pentagonal pyramids, and an anti-prism, or perceiving a dual cube-octahedron as an intersecting cube and octahedron.

PHASE 6- Full description of the polyhedron analytically. Theoretical interpretation formulating a correct argument with precise details through the use more formal language explaining the geometric aspect of the shapes. For example, interpreting transformations, using argument to explain relationships between shapes. Relationship between shapes differing widely in their properties: students must explain their conclusions.

To be placed in a particular phase students must provide evidence of their understanding using structured language. The validity of the perceptual models will be analysed according to students answers during interview. For each of the models, the vocabulary that students use will be analysed, to ascertain if this vocabulary is suitable to the phase. To analyse the students' answers and to assist their perceptual progress, each task will be correlated to the different models and the interviewer will have a list of questions to assess the students vocabulary and the use of language. We might measure the experiment validity for different schools to detect if groups are heterogeneous or not, so that we will have validity from a large population.

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