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# SINGLE SNAPSHOT FREQUENCY SOURCE LOCATION WITH A HYDROPHONE ARRAY OF UNCERTAIN SHAPE: PART 1

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# Single Snapshot Target Location

## Part 1: Determination of Amplitude of Plane Wave

## by Binning by Geoffrey Sweet, 28/1/93

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Let $L$ be the length of the array, $M$ the number of hydrophones, $\lambda_0$ the wavelength of the plane wave, $a$ the amplitude of the wave, $\theta$ the angle of incidence of the wave upon the antenna, $N$ the number of bins and $P$ the power in the binning periodogram at $\lambda = \lambda_0 \sec\theta$.

### Theorem

In the limit as $M \to \infty$, and for values of $\lambda_0$ for which $L\cos\theta/\lambda_0$ is an integer $K$,

$$\frac{P}{a^2} = \frac{N^3}{2\pi^2}\sin^2(\frac{\pi}{N}) . \tag{1}$$

*E. g.*

$$\begin{aligned} \frac{P}{a^2} &= 1.0259, \quad N=3 \\ &= 1.6211, \quad N=4 \\ &= 2.1879, \quad N=5 \\ &\sim \frac{1}{2}N, \quad \textit{large } N. \end{aligned} \tag{2}$$

Thus $\frac{P}{a^2}$ is least when $N = 3$. It can be shown that this makes the estimate of $a^2$ least sensitive to noise on the signal. Hence the term 'optimal binning' when $N = 3$.

### Proof

$$p(\underline{r}, t) = a\cos(2\pi\nu_0 t - 2\pi\underline{r}.\underline{l}/\lambda_0 + \phi_0) , \tag{3}$$

where $\underline{l} = (\cos\theta, -\sin\theta)$.

Now $\underline{r} = (x, 0)$ at point $x$ on the array, therefore at time $t = t_0$

$$p(x) = a\cos(\frac{2\pi\nu}{\lambda} + \phi) = a\cos(\chi + \phi) =, \quad \lambda = \lambda_0 \sec\theta, \tag{4}$$

where

$$\begin{aligned} \chi &= \frac{2\pi x}{\lambda}, \\ \phi &= -2\pi\nu_0 t_0 - \phi_0. \end{aligned} \tag{5}$$

In binning procedure (in limit $M \to \infty$)

$$\begin{aligned} B_1 = (\overline{p})_1 = \frac{N}{2\pi}\int_0^{\frac{2\pi}{N}} a\cos(\chi+\phi)\,d\chi &= \frac{Na}{2\pi}\left[\sin(\chi+\phi)\right]_0^{\frac{2\pi}{N}} \\ &= \frac{Na}{2\pi}\left(\sin(\frac{2\pi}{N}+\phi) - \sin\phi\right). \end{aligned}$$

Therefore

$$B_1 = \frac{Na}{\pi}\sin(\frac{\pi}{N})\cos(\phi + \frac{\pi}{N}). \tag{6}$$

Similarly

$$\begin{aligned} B_2 = (\overline{p})_2 = \frac{N}{2\pi}\int_{\frac{2\pi}{N}}^{\frac{4\pi}{N}} a\cos(\chi+\phi)\,d\chi &= \frac{Na}{\pi}\sin(\frac{\pi}{N})\cos(\phi+\frac{3\pi}{N}) \\ &\vdots \\ &\vdots \\ B_n = (\overline{p})_n = \frac{N}{2\pi}\int_{\frac{2\pi(n-1)}{N}}^{\frac{2\pi n}{N}} a\cos(\chi+\phi)\,d\chi &= \frac{Na}{\pi}\sin(\frac{\pi}{N})\cos\left(\phi+\frac{(2n-1)\pi}{N}\right) \\ &\vdots \\ &\vdots \\ B_N = (\overline{p})_N = \frac{N}{2\pi}\int_{\frac{2\pi(N-1)}{N}}^{2\pi} a\cos(\chi+\phi)\,d\chi &= \frac{Na}{\pi}\sin(\frac{\pi}{N})\cos\left(\phi+\frac{(2N-1)\pi}{N}\right) \end{aligned}$$

$\Rightarrow$

$$P = B_1^2 + B_2^2 + \ldots + B_N^2 = a^2(\frac{N}{\pi})^2\sin^2(\frac{\pi}{N})\sum_{n=1}^{N}\cos^2\left(\phi+\frac{(2n-1)\pi}{N}\right) = \frac{1}{2}N \tag{7}$$

from Lemma below (all integer $N > 2$), therefore

$$\frac{P}{a^2} = \frac{N^3}{2\pi^2}\sin^2(\frac{\pi}{N}), \quad Q.\,E.\,D. \tag{8}$$

## Lemma

$$\sum_{n=1}^{N} \cos^2\left(\phi + \frac{(2n-1)\pi}{N}\right) = \frac{1}{2}N, \text{ all integer } N > 2. \tag{9}$$

## Proof

$$\begin{aligned} I_N = \sum_{n=1}^{N} \cos^2\left(\phi + \frac{(2n-1)\pi}{N}\right) &= \frac{1}{2}\sum_{n=0}^{N-1}\left(1 - \cos\left(\frac{2\phi + 2\pi(2n+1)}{N}\right)\right) \\ &= \frac{1}{2}N - \frac{1}{2}\Re\sum_{n=0}^{N-1} e^{2i\phi + \frac{2\pi i}{N}}.e^{\frac{4\pi i n}{N}} \\ &= \frac{1}{2}N - \frac{1}{2}\Re e^{2i\phi+\frac{2\pi i}{N}} \underbrace{\sum_{n=0}^{N-1} e^{\frac{4\pi i n}{N}}} \\ &\qquad\qquad = \frac{e^{\frac{4\pi i N}{N}} - 1}{e^{\frac{4\pi i}{N}} - 1} \\ &\qquad\qquad = 0, \text{ all integer } N > 2 \end{aligned}$$

$\Rightarrow$

$$I_N = \frac{1}{2}N, \text{ integer } N > 2, \text{ Q. E. D.} \tag{10}$$

*N. B.* When $L\cos\theta/\lambda_0 \neq$ integer, then $B_n \neq (\overline{p})_n$ exactly in the binning procedure; for, after taking the mean of the sums over the $K = [L/\lambda]$ wavelengths $\lambda$ (where $\lambda = \lambda_0 \sec\theta$) along the array, there is a bit of order $K$ left over at the far end.□

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