UNIVERSITY OF SOUTHAMPTON
FACULTY OF SOCIAL SCIENCES

YOUNG CHILDREN'S IDEAS ABOUT NUMBER WORDS AND SCRIPTS AND THE CONNECTION WITH THEIR PROGRESS IN ARITHMETIC

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ABSTRACT

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This thesis is concerned with the learning of the conventional system of decimal numeration by children in the earliest years of school. Conforming to a constructivist view of learning it has the premise that young children elaborate early ideas on how conventional number words and numerals are organised. A basic tenet of this thesis is that external systems of spoken and written numeration have to be acted upon to be known by a child.

More than one decade ago Sinclair and Sinclair (1984) set forth the question about the possible links between the knowledge that pre-school children develop about written numerals in their environment and the development of number concepts. The findings and conclusions of this thesis advance some answers to the above question. Following a longitudinal investigation with two lines of study this research set out to explore children's theories on written and spoken numeration and how these theories might affect their progress in arithmetical knowledge. The first line of study interpreted children's progress in arithmetic according to an established constructivist model of stages of conceptualisation. The second line of study explored children's ideas upon number words and numerals and their possible transformation during the investigation. Nine case studies were selected from a reception class of a school in the South of England - 4 to 5 year-old children. The fieldwork was conducted in two stages during a one-year-long investigation. Findings revealed that children's conjectures on written and spoken numeration are established and reorganised towards increasing compatibility with conventional numeration. Second, findings of this study suggest that children's reasoning in arithmetic becomes increasingly sophisticated due to two systems of constructions that they start establishing from an early age. One system concerns their reflections on their actions of counting and the other concerns their abstractions of a system of spoken and written signs in social use. Although these two systems of constructions are initially separate, they integrate to empower children's growing sophistication in mathematics.

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List of Abbreviations and Other References

- PS: Perceptual stage according to the model of learning stages
- FS: Figurative stage according to the model of learning stages
- INS: Initial number sequence stage
- TNS: Tacitly nested sequence stage
- ENS: Explicitly nested sequence stage
- When a numeral is in between inverted commas, e.g. "234" it refers to the conventional reading aloud of the numeral. Hence, "234" should be read "two hundred and thirty four".
- When a numeral is in between single inverted commas, e.g. '234' it refers to the number script or numeral.
- In the transcripts or episodes presented in the thesis, "I" is the interviewer, that is, the researcher. All children's names are referred with their initial or the initial of their surname if necessary. For example, Eleanor D is "D" and Eleanor W is "W".

Child	Initial used in the transcripts
Tom	T
Johnny	J or Jy (when the interview is with Joe=J)
Jack	J
Eamon	E
Stephanie	S
Alice	A
Eloise	E
Eleanor W	W
Eleanor D	D

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Summary of Findings and Conclusions

Some of the major findings and conclusions of this thesis follow:

1. Written and spoken numeration are taken as objects of reflection by young children in their attempt to organise their experience. Children's constructions on the system of number words and number scripts are not merely idiosyncratic: they concern the use of these systems of signs in their socio-cultural settings.

2. External systems of number representation have to be acted upon to be known by the child: children are active learners of the systems of number words and numerals used in their environment. Conventional signs are not introduced by school, they introduce themselves in so far as children try to make sense of them and use them in their every day experience.

3. External signs of number representation become to symbolise composite arithmetic units through a complex system of constructions that could not be explained solely by children's abstractions of the activity of counting. Hence, understanding of the system of decimal numeration does not come through reorganisation of counting activity only.

4. The principle of place value that governs conventional written number notation is appropriated in gradual approximations which concern both the development of part-to-whole reasoning and a system of constructions on written and spoken signs of number representation.

5. Children's reasoning in arithmetic becomes inextricably linked to their theories on how written and spoken numeration are organised. Conventional numeration systems mediate cognition through children's theories upon the external signs.

6. Children's mathematical ideas are not solely based on their abstractions on actions of counting but are inextricably related to abstractions using the signs of their culture.

7. Progressive sophistication in children's arithmetic knowledge is dependent both on the child's intellectual activity and on the societal presentations encountered in their experience.

8. Children come to conceptualise arithmetic units and conventional numeration from two initially independent lines of constructive activity. When these two lines become integrated the child attains higher sophistication in arithmetic.

1. Introduction

1.1 The Research Problem

> "When you've got your way of maths, and you've done the big numbers...you have to do sums completely different...So you have to, so it's like you have to change your brain actually" (Jack, 6 years)

The learning of numerical concepts is inextricably related to the learning of a pre-existingt cultural system of representation: the system of decimal numeration. Number words and numerals are signs used in the young child's environment. A vast body of research in early number learning has focused on the learning of mathematical concepts embedded in the system of numeration and how children become capable of certain competencies when dealing with the system (e.g. Steffe, L. 1992[b]; Jones et al., 1994; Gray, 1991; Fuson et al. 1997[b]). Nevertheless, extensive research in elementary arithmetic seems to put aside the observation that both written and spoken numeration are complex systems of signs available to young children's inspection and use before they enter school. Research has acknowledged that there are intuitive or informal ideas that children acquire before school instruction (e.g. Sinclair and Sinclair, 1984; Hughes, 1986; Carraher and Schielmann, 1990). However, there is a paucity of studies on how children initially come to establish an organisation of the external sign systems of number representation. Particularly in the case of written numeration, research studies have barely focused on how young children apprehend meanings of numerals. As is known, learning mathematics in primary school is concerned with the learning of a conventional system of signs: the decimal system of numeration. This is a lengthy process and this thesis is concerned only with the initial steps. As children grow older, they generally employ more sophisticated methods to solve arithmetic tasks they could not solve before. These methods involve the use of a system of numeration of their culture which they start learning from an early age. Pedagogical problems concerning the teaching of this cultural artefact have been the rationale of a vast body of research related to how schools can facilitate its learning. The aforementioned pedagogical problems are still the object of further research and subject to a variety of recommendations stemming from them. However, if one grants that children enter school with their own set of insights we must agree that we know very little in the domain of children's initial thoughts on written and spoken number signs -i.e. systems of words and numerals. Nevertheless, anyone who shares time with young children knows about their fascination with "numbers". In particular, large numbers can be the object of their comments and talk. For example, one has wondered how their ideas might be established and develop when a 5-year-old child says he knows that infinity is "an eight turned sideways" or that "if it has more zeros it is a big number".

From a widely accepted point of view, knowledge grows from already existing knowledge. Hence, what one can regard as mathematical knowledge must be connected with the child's coherent, inferential thinking that takes place, often implicitly, in everyday life. There is much that children know as a result of participating in a social setting. The child enters the classroom with their own set of insights, experiences, and relationships. It was the premise of this study that reflection on external systems of spoken and written numeration was a source of meaning for young children scarcely investigated. Whereas the first aim was to map out young children's ideas on written and spoken numeration, the second question was whether these ideas were related to children's mathematical knowledge. Hence, in relation to the informal knowledge children acquire in their social environments, the question of the study was twofold:

- First, what do young children already know about these systems of external signs, number words and written numbers, as they enter school and before they are told about the mathematical structure of conventional numeration?

- Second, the study sought to reveal whether these presumed ideas pre-school children may acquire, from their use of number words and scripts in their social environments, have any role in their subsequent mathematical progress.

The present thesis is the result of a constructivist study which followed the learning history of a selected group of pre-school children during a longitudinal investigation.

1. 2 An Overview of the Chapters

This thesis includes 8 Chapters. Following the present Introduction, Chapter 2 presents an overview of the previous research related to the research problem and sets the grounds for the formulation of the two-line investigation. Chapter 3 develops a two-part theoretical framework. Chapter 4 develops methodological issues of the study and presents the design of a one-year-long investigation conducted in two stages of fieldwork with nine case studies. Chapters 5 and 6 present the findings of the investigation and set the ground for the general discussion of Chapter 7. Chapter 7 undertakes a general discussion in the light of the findings of the two lines of investigation. Finally, Chapter 8 presents the concluding remarks of the thesis.

*

2. An Overview of the Previous Research

2.1 Introduction

This chapter presents a summary of what is known in relation to the research problem. The aim is to examine the problematic learning of numeration while giving an account of the vast body of research in the field through the lenses of the author. A refinement of the research questions concludes this account and sets the grounds for the theoretical framework of the study which is developed in Chapter 3. What follows is a discussion on what researchers in the field of psychology and mathematics education have focused on in their studies of children's learning of numeration. Four sections organise the presentation: 1. Numeration as representational systems; 2. Numeration as a socio-cultural system of signs; 3. Numeration and arithmetic; and 4. A focus on written numeration. The chapter concludes with a discussion on the review of the literature in the light of a constructivist perspective and sets forth the research questions of the study.

2.2 Numeration as Representational Systems

As Kaput (1987[a], p. 20) put it, the actual school work in primary mathematics is not about numbers, but about "a particular representation system for numbers", the base-ten placeholder system or the decimal system of numeration. According to this idea of representational system, one can isolate a represented world and a representing world which are related by the representational system[1]. In mathematics, the represented world is a mathematical structure (Kaput, 1987[b]). Hence, one can propose in the case of decimal numeration, that the represented world comprises a system of natural numbers and arithmetic operations on them, while the representing world can be the written decimal system or the system of number-names in a given language.

Skemp (1986, p. 147) has argued that numeration "means the naming of numbers" and that there are systems of numeration better than others. He did not focus on the

[1] Kaput distinguished three more elements: the aspects of the represented world that are being represented, the aspects of the representing world that are doing the representing, and the correspondence dynamic between the two worlds.

distinction between the system of number-names and the system of numerals but this is a distinction that researchers have ubiquitously brought to the fore to address the learning of numeration (e.g. Fuson, 1990[a, b]; Fuson et al., 1997[a,b]; Carraher, 1985; Bell, 1990; Lerner and Sadovsky, 1995; Sinclair et al., 1992; Seron et al., 1992; Wright, 1998; Giroux and Lemoyne, 1998; Pimm, 1995). As a consequence of different treatments in the literature, the system of number-names and the system of written numbers have been termed respectively, e.g., "spoken number words and written number marks" (Fuson, 1990[b]), "oral and written numeration" (Sinclair et al., 1992), "alphabetic and Arabic" (Seron et al., 1992), or "spoken and written numeration" (Lerner and Sadovsky, 1995). In all these treatments, the underlying assumption is that these are "external" representational systems.

2. 2. 1 Internal and external representations

The view that distinguishes between internal and external representations has been set forth by Lesh et al. (1987) and Hiebert and Wearne (1992). The latter argue that

> "building connections between external representations supports more coherent and useful internal representations (Hiebert and Wearne, 1992, p. 99)

Janvier (1987) agrees with the same view when he argues that external representations are called to represent certain mathematical realities. In distinguishing external from internal representations he proposes that the former concern external symbolic organisations and the latter concern internal formulations one constructs of reality. In this thesis, the term numeration alludes to two forms of external representation of numbers, the systems of number words in a given language on the one hand, and the system of written Hindu-Arabic numerals on the other. The terms "words" and "scripts" or "names" and "marks" will be used to refer to particular elements of the system. For example, '567' is referred to as a number script and "twenty five" is referred to as a number name in English. Generally, when implying the organisation or the system of number names and number scripts, the terms written and spoken numeration will be used as a shortcut for the system of number words and system of number scripts.

2. 2. 2 Written and spoken numeration

A salient difference between spoken and written numeration is the positional characteristic of the latter. Irrespective of the language of the spoken system, written numbers are organised by the principle of place value whereby the value represented by a given digit is dependent on the relative place or position in the number script. For example, in the script '457', '4' represents four hundred. A second characteristic of the system is the recursive regrouping in base ten which yields units of different order which are powers of the base (e.g. 10, 100, 1000 or 10^1, 10^2, 10^3). Hence, in '457', '4' represents 40 groups of ten groups of ten or 4 hundreds[2]. Finally, a third feature of written numeration is the use of a finite set of marks (0 to 9) to represent any number. In contrast, spoken numeration has different names for units of different powers of the base.

The difference between spoken and written numeration has been discussed in the literature (e.g. Fuson, 1990[b], Lerner and Sadovsky, 1995). Fuson refers to "named-value words and unnamed position-value written marks". In the named-value system of number words, the value of the units of different order are generally explicitly stated[3] whereas in the unnamed position-value system of written marks they are implicit in the position of the marks. In our example of 457 the unit "hundred" is explicitly said but in writing is implicit in the third place from right to left. Lerner and Sadovsky[4] argue that the positional feature of written numeration renders it more hermetic than spoken numeration, something that presents a challenge for children's interpretation. They argue that non positional systems like additive systems can be seen as transparent because one can add the marks to obtain the value of the represented number. For example, in the Egyptian system, which was also a decimal system, '|' represents one, '∩' represents ten irrespective of their position in the

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[2] Another characteristic of the system concerns the use of '0' as the sign that "holds the place" in a given script when there are not units of a certain order of the base.

[3] This in turn depends on the language of the spoken system. For example, in the most explicit case of Asian languages, '234' is said "2 hundreds 3 tens 4". Spoken numeration in English contains irregularities in the values of lower order units (ones and decades). For example, whereas in Korean '17' is said "one ten seven", in English it is said "seventeen" with no explicit naming of decades and ones.

[4] Lerner and Sadovsky (1995) conducted their research project with Spanish speaking children.

script, and there were new symbols for hundred and thousand[5]. Hence, the script '∩∩∩|||||||' represents thirty seven irrespective of the relative position of the marks. In their study of children's acquisition of written numeration these researchers highlight that children's production of number scripts can be seen as having an additive feature. They argue that children seem to base their productions of written numbers on their correspondence with spoken numbers. For example, young children write '200030' for "two thousand and thirty" because they know the conventional scripts for "two thousand" and for "thirty", and they concatenate the scripts in correspondence to the number words. In a sense, one can see the Hindu-Arabic system as less transparent or hermetic because the value of a mark is not absolute but relative to the position it has in the script. Nevertheless, the positional system gains in economy because it requires only ten marks to record any number (i.e. 0 to 9). It also gains in efficiency of computation because the Hindu-Arabic system

> "makes use of the fact that if we add a set of 2 and a set of 3, we get a set of 5, whether the elements of these sets are singles objects or themselves sets of objects or even sets of sets" (Skemp, 1986, p. 148)

To follow Skemp's statement one can consider his example and compare the addition of two numbers written in the Roman and in the Hindu-Arabic system[6].

XXIV	24
+ XXXIX	+ 39

When calculating the total of the above two numbers one may experience a "natural" translation of the Roman numerals into the Hindu-Arabic system to compute the result. As Skemp has put it, indication of the number of sets of different orders of the base are depicted in the Hindu-Arabic system and this has resulted in a more efficient system for calculation. Efficiency and transparency are not independent features of a system of numeration. The more efficient a system is the less transparent it becomes. One can see the Egyptian number scripts as a direct translation of the actions of counting and regrouping and the Hindu-Arabic numerals as hiding these actions behind the characteristics of the system. The consequence of this is the indisputable

[5] Casell (1960) and Lerner and Sadovsky (1995)

[6] In the Roman system I=1, X=10, and V=5 but if 'I' precedes 'X' or 'V' it is subtracted from them. So IX=9, and IV=4

economy and efficiency of the latter. This idea of trade-off between efficiency and comprehension has been discussed by Hughes when he argues that

> "There is little doubt that the system we currently use is an extremely efficient and powerful tool, the result of a long period of development. However, this increase in power and efficiency has been achieved by incorporating features -such as fully ciphered numerals, place value and operator signs- which do not feature naturally in the representation of young children." (Hughes, 1986, p. 94)

A final observation concerning the difference between written and spoken numeration is made by Steffe (1994) drawing upon Menninger's research. Menninger (1969) argues that the rules that govern numerals (ordering and regrouping) do not correspond to the rule of the verbal sequence which is a stepwise gradation. Further, it is remarked that Roman and Indo-European number word sequences differ in minor details but their systems of (written) numeration differ radically. Therefore, Menninger concludes, the writing of numerals is not a mere representation of the number word sequence. This, Steffe argues, demonstrates the independence of verbal and written systems and gives a perspective on why there can be an unresolved tension between children's verbal number sequences and written numerals.

2. 2. 3 The use of manipulatives in teaching place value numeration

A rich understanding of place value ranks as one of the most relevant goals in the teaching of primary mathematics. However, there is a great deal of controversy about what this understanding is, how it is accomplished and what teaching strategies are suitable to apply (e.g. Perry et al., 1994; Thompson P, 1992; Hiebert and Wearne, 1992; Fuson and Briars, 1990; Bednarz and Janvier, 1988, 1982; Jones et al., 1993, 1994, 1996; Kamii, 1985, 1986, 1989; Thomas et al., 1994, 1996). Considerable research and pedagogical effort has been made in the last thirty years to address this problem. Physical embodiments have been designed to promote the learning of numeration in primary mathematics usually in the hope that numerical relations of the system become "concretised", or "readily apprehensible". Examples of these concrete materials, often called "manipulatives", are Cuisenaire Rods, Dienes blocks or Multilink cubes[7]. Typically, these materials consist in a set of different physical

[7] More recently computerised microworlds with structured blocks (e.g. Thompson P, 1992).

objects that represent units of different orders of the base[8]. Other pedagogical efforts are less systematic or commercialised and consist in employing bundles of different sizes, making the grouping actions explicit or using different pictures or different colours to represent the different grouping sets (e.g. red for ones, blue for tens, green for hundreds). Manipulatives have been used to "bridge" a "concrete stage", manipulating the blocks, with a "symbolic stage", using the Hindu-Arabic system (Bohan and Shawaker, 1994; Hiebert, 1984; Bednarz and Janvier, 1988). A considerable body of literature advocates their use in primary mathematics (e.g. Fuson et al., 1997[a, b]; Bohan and Shawaker, 1994; Fuson and Briars, 1990; Sowell, 1989; Fennema, 1972). However, research literature has put the rationale for the use of these concrete materials under criticism (e.g. Thompson I, 1998; Perry et al., 1994; Baroody, 1989, 1990; Wigley, 1997; Maclellan, 1997[a]; Deloache et al., 1998; Lerner and Sadovsky, 1995; Gravemeijer, 1997; Hunting and Lamon 1993; Friedman, 1978). This ambiguity seems to result from addressing different issues or from holding different standpoints. On the one hand, whereas in some reports the focus is on the learning of mathematical concepts and operations, in the other accounts researchers focus on the learning of the very system of written representation. Wigley finds no sense in the view of making a mathematical idea concrete because there is no obvious connection between what is done with manipulatives and how we write numbers.

> "Ten-rods clearly look different from unit cubes, whereas 'place value' is a reference to the fact that the same mark (digit) is read differently according to its position in a number" (Wigley, 1994, p. 5)

He argues that place value is not a property of any physical material and that the notion of place value "has to make sense in itself, not through some external reference". Pedagogical efforts to teach place value through the use of concrete materials have resulted in a paradox:

> "in order to make the positionality of the system comprehensible for children, the very property of positionality is taken away" (Lerner and Sadovsky, p. 138).

Precisely, when 345 is represented by 3 flats, 4 longs and 5 small cubes or any other similar representation, the property of place value is absent because irrespective of the position of the items, 345 is signified by the total of the values of the items. In a

[8] For example, a small wooden cube (1 cm x 1 cm x 1 cm) represents a one, a "long" represents a ten (10 cm x 1 cm x 1 cm), a "flat" represents a hundred (10 cm x 10 cm x 1 cm), and a big cube represents

sense, a representation of this sort brings to mind the Egyptian system, a decimal but not positional system. Some literature advocating the use of concrete materials holds a "representational view of the mind" according to which

> "students gradually construct mental representations that accurately mirror the mathematical features of external representations." (Cobb et al., 1992, p. 3)[9]

Against this view Gravemeijer (1997) has pointed out that

> "the mathematical concepts embodied in the didactical representations are only there for the experts who already have those concepts available to see. For the students there is nothing more to see than the concrete materials." (p. 318)

Arguing from a constructivist standpoint Hunting and Lamon (1993) have suggested that concrete materials are only one possible source of reasoning that might lead to the construction of mathematical knowledge. Cobb et al. (ibid) propose the term "pedagogical symbol systems" to emphasise their "symbolising role in individual and collective mathematical activity". In accordance with this view Meira (1998) has suggested that physical embodiments should be conceived as "conversational pieces" or the motive for engagement in conversation and argumentation. In brief, the use of manipulatives for the teaching of numeration can be justified from different perspectives. However, instead of working with artificial pedagogical symbol systems or conversational pieces, children can use, frequent, talk, and reflect upon the written and spoken numeration of their social worlds. One can propose that the conversational pieces can be the very number words and number scripts that children encounter in their everyday experience.

2. 3 Numeration as Socio-Cultural Systems of Signs

One of the most cited contributions of Vygotsky to the study of children's learning is the concept of *mediation* (Wertsch, 1985, 1998) and the attendant idea that sign systems play a central role in the development of cognitive thought. According to this view children appropriate or internalise pre-existent cultural systems of signs by using them and by participating in their social environments. Vygotsky's view of learning focuses on social influences in the developing intellect of a child through two major notions: internalisation and zone of proximal development. Internalisation must not

a thousand (10 cm x 10 cm x 10 cm).

[9] This is not Cobb and co-researchers' view.

be understood as a copy of the external or the social but a process of transformation whereby the individual actively masters or appropriates external sign systems (Wertsch, 1985, p. 66-67). The zone of proximal development is defined as the

> "distance between a child's actual developmental level as determined by independent problem solving and the higher level of potential development as determined through problem solving under the guidance or in collaboration with more capable peers" (Wertsch, 1985, p. 67-68).

Vygotsky's learner is a "social agent" (Bickhard, 1995) but nevertheless is an active organiser of their experience (Becker and Varelas, 1995; Cobb, 1996; Steffe, 1998; Jones et al., 1994). Systems of signs used in their culture become psychological tools or tools of thinking for the child and "pre-existent knowledge of the culture is made part of the developing intellect of the individual" (Becker and Varelas, 1993). When referring to systems of signs Vygotsky focused primarily on language but he mentioned other systems of signs such as number systems (Confrey, 1995[a], Wertsch, 1998). Vygotsky argued that "[s]igns and words serve children first and foremost as a means of social contact with other people" (Vygotsky, 1978, p. 28). Hence, "[a] sign is always originally a means of influencing others, and only later becomes a means of influencing oneself" (Vygotsky, 1981; cited in Wertsch, 1985, p. 92). According to this view children do not need to conceptualise the underlying structure that the sign system is called to represent, in order to start using it.

> "children do not have to know the full cultural analysis of a tool to begin using it" [...] The child's appropriation of culturally devised tools comes about through involvement in culturally organised activities in which the tool plays a role" (Newman et al., 1993, p. 63)

2. 3. 1 Materiality of mediational means

One of the characteristic of mediational means is its materiality (Wertsch, 1998). External systems of signs like spoken and written numeration are "physical objects" or "artefacts" that continue to exist even when not incorporated in the flow of action.

> "external, material properties of cultural tools have important implications for understanding how internal processes come into existence and operate. Such internal processes can be thought of as skills in using particular mediational means. The development of such skills requires acting with, and reacting to, the material properties of cultural tools." (Wertsch, ibid, p. 31).

This idea is akin to the notion of external representational system outlined above and the attendant notion that "physical properties of symbols can be resources for reasoning" (Greeno, 1991, p. 196). A particular type of external representation can be an object of active interpretation which can present strengths and weaknesses or constraints and support (Kaput, 1991). The idea that cultural tools mediate cognitive development seems to be held by culture-based investigations in mathematics education. One type of study focuses on how the use of different languages can affect cognition and the second type focuses on how understanding can be accounted for in culturally organised activities outside school.

2. 3. 2 Different-language-naming systems of numeration

The first line of study argues that differences in the number naming system of different languages promote differences in conceptualisation of numeration (e.g. Miura et al., 1988; Fuson and Kwon, 1992[a, b]; Bell, 1990; Fuson, 1990[a, b]). These studies argue that European children are at a disadvantage with Asian children because Asian languages have more regular naming systems that makes apparent the units of ten and one in a given number[10]. For example, in Korean 19 is said "ten nine" and 27 is said "two ten seven" whereas English and Spanish children need to acquire irregular names for segments of the number sequence[11]. These studies assess children's solution procedures in a set of tasks which are taken to indicate a certain level of conceptual understanding. Conclusions suggest that Asian children achieve higher levels of conceptualisation than their English peers. In view of this, for example Bell proposes that

> "two-digit numbers can be satisfactorily modelled with 10 blocks, and reinforced with simultaneous use of decimalised number names such as 'two tens and three' and the corresponding numeric symbols. Such representations may be helpful in overcoming the linguistic disadvantages for children enculturated to a non-transparent standard number word sequence." (Bell, 1990, p. 12)

Hewitt and Brown (1998) reported on a study in which the English naming system of numbers was "regularised"[12]. On the other hand, Fuson (1990[a]) has warned against delaying English speaking children's use of numbers in the hundreds and thousands because it is the segment where the number names are regularised and more clearly represent the units embedded in the system[13]. The underlying "transparency" view

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[10] e.g. Korean, Vietnamese, Mandarin, as Asian languages, and English, French, Spanish, as European.
[11] Irregularities differ with the language. For example, in English there are new names for 11, 12, and the rest of the teens do not express the name in terms of ten and units. In Spanish, there are new names for 11, 12, 13 14, 15 but from 16 to 19, the names can be translated as "10 and 6", "10 and 7", and so on.
[12] The regularisation of English number names of the first decade was as follows 10=one-ty; 11=one-ty one; 12= one-ty two; 13=one-ty three; and so on.
[13] In Fuson's studies the use of base-ten blocks is advocated to rectify the irregularities of the spoken system (e.g. Fuson[a,b]; Fuson and Kwon, 1992[a])

can be put under scrutiny in the light of considerations made in the previous section. The most regular spoken system differs from written numeration in the same way as concrete materials do. A number name such as "two tens and three" is not written as '2103'. Once more the question is whether pedagogical endeavours address the construction of numerical concepts such as "unit of ten" or whether it addresses the learning of written numeration.

2. 3. 3 Culture and cognitive development

The second line of study is exemplified by paradigmatic studies that report children's mathematical understanding in culturally organised practices outside school (e.g. Carraher, 1985, 1988; Nunes and Bryant, 1996; Carraher et al., 1987; Carraher and Schliemann, 1990; Saxe, 1988, 1995). The salient finding of these investigations is that young children who did not perform well in school tasks using conventional written numeration and standard algorithms demonstrated understanding of numerical concepts and operations in culturally organised practices such as selling candies in the street using their culture currency notes and coins. For example, Saxe (1988) argues that 10 and 12 year old children in Brazil operated with large numbers although they were not able to represent these numbers with Hindu-Arabic numerals. Nunes and Bryant (ibid) argue that the ability to write numbers conventionally is not a pre-requisite for understanding addition of units of different values. They acknowledge that learning the Hindu-Arabic system can open up new possibilities for children but claim that the understanding of units and addition is primary in the understanding of written numeration. These insightful investigations have shown that children's mathematical conceptualisations occur in the course of or as a result of cultural activities in which spoken and written numeration are used and play a role. However, the learning of written and spoken numeration as systems of representation remains, in a sense, unexplored because the focus is on the mathematical concepts and operations that the system is called to represent. That is why Nunes and Bryant (1996) and Saxe (1988) remark that is not necessary for children to know written numeration to account for certain mathematical knowledge. Nonetheless, if one conceives written numeration[14] as an object of knowledge which is primarily an object of social use, and subsequently an object of school instruction, one is left with the question of how children organise their experience related to the use of number scripts and number words. These systems of signs are carriers of mathematical meaning - among other meanings. Nevertheless, as systems of conventional signs, written and spoken numeration can be taken as the very object of knowledge that children explore and intellectually act upon changing their viewpoint as their ideas of the object are organised and reorganised in the course of using the signs. This perspective will not particularly focus on mathematical meaning but whatever meaning can be inferred to be the viewpoint of the child.

2. 4 Numeration and Arithmetic

Numbers and arithmetic operations on numbers can be seen as the mathematical knowledge embedded in the system of numeration. The literature acknowledges arithmetic knowledge as the heart of elementary school mathematics (e.g. Wood et al., 1995). Research in the last twenty years has traced the origins of arithmetic knowledge -including the concept of number and basic arithmetic operations, in the activity of counting (e.g. Steffe et al, 1983, Steffe and von Glasersfeld, 1983; Steffe and Cobb, 1988; Steffe, 1988[b], 1992[a,b], 1994; Maclellan, 1993, 1995, 1997[b]; Foster, 1994; Thompson I, 1995, 1997[b, c]; Secada et al., 1983; Carpenter and Moser, 1984; Anghileri, 1997; Gray, 1991, 1993, 1994, 1995, 1997; Fuson and Fuson, 1992; Fuson, 1982, 1988; Fuson et al., 1982; Cobb and Wheatley, 1988; Gray and Tall, 1994; Wright, 1991[a, b]; Baroody, 1987; Pepper and Hunting, 1998). Despite differences of perspective, all accounts of children's counting include a sense of progress in the strategies they use to solve tasks related to cardinality, addition and subtraction. This notion of progress is often expressed in the literature through terms such as sophistication, fluency, flexibility, efficiency or competency. Analyses of children's counting solutions to arithmetic tasks have demonstrated that as they grow older, they abandon or transform certain counting routines and employ more sophisticated methods to solve the same tasks or to solve tasks they could not solve before.

[14] Written and spoken numeration

2. 4. 1 Interpreting progress in arithmetic knowledge through counting

Researchers have distinguished two qualitatively different counting strategies. Essentially, when given a collection of "n" items and another collection of "m" items and asked to figure out the total of items, the young child can take one of the following strategies: 1. Count all the items using the standard number word sequence, starting from one; or 2. Take "n" or "m" as a base for counting and count on, starting with the next number word until all the items are counted. These two types of strategies are respectively called "counting all" and "counting on". The latter is thought to indicate a conceptual advance in the child's knowledge (e.g. Fuson, 1982, 1988; Secada et al., 1983; Steffe et al., 1983; Wright, 1991[a]; Gray, 1993, 1995; Maclellan, 1995; Carpenter and Moser, 1984). Fuson (1988) argues that the fundamental insight consists in the child noticing that the counting word "n" said for the last item in the first collection is the same number word "n" that tells the cardinality of that addend. Likewise, Steffe et al. (1983) distinguish a stage in children's counting where a number word signifies the act of counting from one up to and including that word. In general terms, this conceptual achievement is indicated by the child no longer counting from one but proceeding to count on from a given number word[15]. Gray (1995) claims that the concept of number is formed from the compression of counting actions and that

> "Children's growing sophistication in handling the addition of numbers can be seen as steady compression from counting processes to handling number concepts." (p. 37)

Gray (1993) argues that there is a conceptual shift in the compression of counting-all procedures into counting-on strategies. The notion of compression or encapsulation of actions into objects of thought has been recently discussed in mathematics education and the compression of counting is presented as an example of this general idea (e.g. Gray and Tall, 1994). Arithmetic knowledge grows from children's shifting focus on objects, actions on objects and results of their actions. The proceptual interpretation of mathematical symbols set forth by Gray and Tall (1994) alludes to the flexible meaning of the symbol as a process or an object to be further operated upon. The

[15] Steffe et al distinguish counting on solutions as the result of the elaboration of the number word sequence (the ability to utter consecutive names in the sequence) from counting on as a result of the operation of integration (the reflective ability to take as one what can be also considered as separate unitary items). This point will be further developed in Chapter 3

notion of compression of counting into arithmetic procepts is compatible with a constructivist view according to which symbols "point at" previous abstractions:

> "The answer lies in the fact that we can 'package' generative procedures and treat them as unitary conceptual entities." (von Glasersfeld, 1995^{b}, p. 173)

In agreement with the notion of progressive sophistication, other types of strategies are included in models of conceptual hierarchy and increasing efficiency in simple arithmetic. For example, Carpenter and Moser (1984) set forth an outline of children's types of addition strategies for verbal arithmetic problems[16] as follows: 1. Strategies based on direct modelling with fingers or physical objects; 2. Strategies based on the use of counting sequences including count on from first addend and count on from larger addend; and 3. Strategies based on recalled number facts including known facts and derived facts from known ones. In the first level, the child represents two sets of physical objects or fingers and counts the whole collection of items from one. In the second level, the child counts on from the first number word given or counts on from the larger number of the two addends. For example, if the situation is interpreted as '4+8' the child could count on from 4 or decide to count on 4 from 8. Finally, in the third level the child drops counting routines and either recalls the number fact or derives the solution to a given problem from a known sum. In a similar manner, Gray (1991) proposes a hierarchy of strategies as follows: 1. Count-all; 2. Count-on; 3. Derived fact; 4. Known fact. In both Gray's and Carpenter and Moser's hierarchies, "known fact" strategies allude to solutions to addition problems with no apparent counting. As has been mentioned "derived fact" strategies allude to children's responses that base the unknown answer from a known sum. For example, if a child interprets the problem as '4+9', they reason that the answer is one less than 14, knowing that '4+10=14'. Likewise, Fuson (1982) reported that the children of their studies used four different addition solution procedures: counting all, counting on, recall of facts, and solutions derived from known facts. She argues that

> "The use of counting on replaces the use of counting all, and as the number of recalled facts grows, counting on may move from a principal means of sum finding to a 'backup' means used when a certain addition fact cannot be recalled." (p. 79)

[16] Carpenter and Moser (1984) present a hierarchy for subtraction strategies: 1. Modelling strategies including separating from, adding on and matching; and 2. Counting strategies including counting down from and counting up from a given number.

In these researchers' accounts there is an implied conceptual progress indicated by the gradual detachment from actions on perceptual objects and actual counting routines. Nevertheless, there does not seem to be an account for the mechanism that explains that progress. Despite Gray's reference to "compression" of counting procedures, a better explanation of how children drop initial counting strategies and resort to more sophisticated solutions remains unaccounted for. This is of paramount concern for research because there are some children, often characterised as "low achievers" (e.g. Gray, 1991; Pitta and Gray, 1997) who seem to rely on counting procedures for too long, often in detriment to more efficient strategies. It seems that abandoning initial counting routines is a necessary step to achieve increasing sophistication:

> "having begun arithmetic by learning to count, it then becomes necessary to discourage counting! [...] Problems with larger numbers, where trial and improvement methods are needed, can be used to promote intelligent exploration and use of the number system." (Wigley, 1994, p. 146)

Once children abandon counting-all solutions, counting-on procedures are usually used for a prolonged period of elementary school (Fuson, 1982). The literature has documented a wide range of children's strategies to solve addition and subtraction problems in elementary arithmetic (e.g. Thompson I, 1995; 1998; Carpenter et al., 1996; Cobb and Wheatley, 1988). Children's strategies usually combine counting on with other types of solutions including reasoning strategies (e.g. using derived facts) or partitioning strategies (e.g. 15+4=19 because 5: 6, 7, 8, 9) or by incorporating counting on by tens and ones.

A model of construction of arithmetic units and operations

The aforementioned progression in children's counting procedures and gradual detachment from counting actual items have been the object of study by Steffe et al. (1983) and Steffe and Cobb (1988). They have set forth a model of counting types and arithmetic units and operations which has been discussed and applied in further studies that have validated the model (e.g. Steffe, 1983, 1988[b], 1992[a, b], 1994; Wright, 1988, 1990, 1991[a, b]; Pepper and Hunting, 1998; Wiegel, 1998; Cobb and Wheatley, 1988; Kornilaki, 1994, cited in Nunes and Bryant[17], 1996; Nunes, 1997; Steffe and

[17] Nunes and Bryant, 1996, pp. 57-59.

von Glasersfeld, 1983; Pearn and Merrifield, 1998; Biddlecomb, 1994; Cobb, 1987, 1995). Central to the model of counting types (Steffe et al., 1983) is the idea of unit.

> "[T]he development of the concept of unit, that forms the basis of all numerical operations and arithmetic skills, is a reliably regular process that goes through a sequence of stages and can be mapped in detail." (Steffe et al., 1983, p. 114).

According to this view, a number, as an arithmetical object, is a unit which is itself composed of units. A child's concept of number is a product of their intellect and involves a double act of abstraction.

> "There is a first act of abstraction that produces units from sensory-motor material; i.e., unitary 'things', [...]. There is also a second act of abstraction that takes these units as the material for the construction of a unit that comprises them" (Steffe et al., ibid, pp. 115-116).

The conceptual creation of number as a composite unity is traced in the activity of counting which is conceived as a complex activity comprising: 1. The ability to produce a sequence of conventional number words, 2. The ability to produce perceptually or otherwise entities that can be counted, and 3. The ability to co-ordinate the previous two activities in such a way that a number word corresponds to a countable entity. Each instance of counting is called "a counting act". Counting is seen as the "instrumental mechanism" whereby the child constructs more complex mathematical concepts and the operations of addition and subtraction (Steffe et al., 1983, p. 114). Children's reflection on their counting activity transforms the object of their awareness and sophisticates their counting methods as it sophisticates their knowledge of counting. The counting types model distinguishes five increasingly sophisticated types of units that children create when they count: perceptual, figural, motor, verbal and abstract. Children can be classified as counters of a particular type on the basis of the most sophisticated type of unit they can count. The model interprets the process by which children become able to operate with abstract units. According to Steffe et al. (ibid) this happens through the developing of counting and through the shifting awareness of the units that are being counted. The child, through reflective abstraction, gradually becomes aware of different aspects of the counting procedure and some of these aspects become superfluous. First, the child can only count objects that are present in their immediate experiential field. Later, when objects are not immediately available, the child can re-present[18] or re-create the non perceptually available items. In this case the child can count visualised items (e.g. dots, bricks), motor items (e.g. finger movements), or spoken number names. Counters of perceptual, figural, motor or verbal units need to carry out sensory-motor activity or create figural representations to give meaning to the number words uttered. These four types of counting are thought to be figurative because they depend on some form of sensory-motor experience (actual or re-presented). There is a paramount change when the child reflects on their actions to construct the fifth unit type, an abstract unit. Counters of abstract units can reflect on the re-presented unitary items and unite them in thought thereby constructing a unit, integrating in thought what can be also considered as separate entities. This intellectual operation of reflective abstraction is termed "integration" or "unitising" and can be inferred from the child's solution to the following problem:

> "There are seven marbles in this cup. (rattling marbles in the cup). Here are four more marbles (placing four marbles on the table). How many marbles are there in all?" (Steffe et al., 1983, p. 42)

If the child counts from one the seven marbles of the cup - by visualising them or by replacing them with fingers - seven does not exist in their experience until the child counts from one to seven and it ceases to exist once the counting activity has been completed. In this sense

> "numbers are transitory entities that have to be made and remade by actually counting and do not exist independent from the activity of counting." (Cobb and Wheatley, 1988, p. 3)

On the other hand, if the child says there are seven in the first cup and subsequently counts the additional marbles "eight, nine, ten, eleven", that is an indication of abstraction and independence of sensory-motor experience. "Seven" carries the force of counting from one to seven, but the child no longer needs to carry out the count. In this counting-on solution, one can infer the child's operation of integration as a result of which "seven" stands for the activity of carrying out the count from one to seven, including seven. This action has become superfluous for the child and this explains his counting-on strategy. When children can construct abstract units, the child's counting is said to be operative rather than figurative because they can reflect on their

[18] From this perspective, re-presenting alludes to making an experience present, hence, to re-create the experience (von Glasersfeld, 1991[a], 1995[b])

counting actions. According to a subsequent extension of this model (Steffe and Cobb, 1988) the first four counting types (i.e. perceptual, figural, motor, and verbal) constitute a stage in children's counting which pertains to the domain of non numerical counting (i.e. figurative counting). Generally, children's actions and reflections that depend directly on sensory-motor experience are not considered as belonging to the realm of number. Nevertheless, sensory-motor experience is indispensable for the construction of number as an arithmetical unit thereby rendering sensory-motor action as a source of arithmetic knowledge. Steffe (1983) draws a fundamental distinction between figurative and operative counting. There is a qualitative shift when the child becomes a counter of abstract units. Abstract single units can become themselves objects of reflection and can be unitised through new integrations[19]. Children's sophisticated reasoning strategies can be explained by their further operations of integration.

In summary, the model of counting types and arithmetic strategies not only distinguishes levels of sophistication in children's solutions but gives a viable account of how this sophistication comes about. One can say it views the construction of the decimal numeration system as the process of construction of arithmetic units and operations on them. Kamii (1986) has postulated that to understand mathematical knowledge embedded in the system of decimal numeration the child has to construct part-whole operations. The model of Steffe and Cobb (1988) gives an account on how part-whole operations become part of the child's knowledge from their genesis in the actions of counting perceptual items. For this reason the model can be used as a framework to interpret progress in children's arithmetic.

2. 4. 2 The number word sequence

The conception of counting as a complex activity is important because the literature distinguishes two types of counting. The first type concerns the mere recitation of a culture specific sequence of number names in a standard order and the second type concerns the accompanying correspondence with a collection of objects. Pimm (1995) has termed these two types intransitive and transitive counting respectively.

[19] There are five learning stages of counting which will be discussed in Chapter 2.

The distinction is important because counting as mere recitation is seen to concern the acquisition of a culture specific sequence of number names as a distinct process, isolated from the process of mathematical conceptualisations that come about with the use of the sequence. This is clear in Fuson et al.'s (1982) distinction between the acquisition and elaboration of the number word sequence. These are seen as two distinct although overlapping phases. For them, the initial acquisition concerns the learning of the conventional sequence of number words and the elaboration phase concerns a process of differentiation of the words in the sequence and the establishment of relations among these words. Five stages in the elaboration of the sequence are distinguished: string, unbreakable list, breakable chain, numerable chain, and bi-directional chain. The levels are related to different sequence production skills (e.g. forward and backward sequences), to the ability to comprehend and produce relations among the words in the sequence (e.g. words before, after, or between) , and to the ability to use the words in counting solutions (e.g. counting-on, counting-up or down to, counting-from). Development is reported in relation to different kinds of abilities that children can demonstrate at different ages but Fuson et al. (1982) acknowledge that they did not focus on the process by which a child moves from one level to another.

The counting types model of Steffe et al. (1983) accounts for the transformation of counting units until they become arithmetic objects or abstract units for the child. They state that the segment of the sequence they dealt with when constructing the model of counting types was from one to twenty. They refer to the necessary proficiency of the initial segments of the sequence in order to study children's counting. Theoretically, children's standard verbal sequence is a number sequence when it refers to abstract units and it contains records of counting. This means that for figurative counters, their counting sequence is not a numerical sequence. Most important is the alluded independence of the acquisition of the standard verbal sequence seen as "linear arrangements of conventional number words in their standard order" (Steffe et al., 1983, p. 25) and the remark that children may construct relations among elements of the sequence independently from the activity of counting[20]. Steffe et al. (1983) argue that the progress in learning the sequence is logically independent from the progress in the construction of different types of units. On this point, these researchers refer to Fuson et al.'s (1982) phase of elaboration of the sequence. The relations that children establish within the sequence, it is argued, are independent from counting because they are relations within the sequence, qua sequence. However, what seems yet not addressed are children's elaboration of relations among number words qua *system* rather than qua *sequence*. In this sense, the study of the acquisition of number words as a system of signs in use compels us to study children's ideas on number words and their elaborations upon a system of numeration in use. Further, it urges us to contemplate children's use and reflection on segments of the sequence not limited to twenty but including a version of the sequence as extended as it is in their cultural environments. This approach conceives the acquisition of spoken numeration as a conceptual task for the child who is seen to change their viewpoint of the object of reflection in interaction with the object. Children's abstractions and organisation of their experience have not yet been studied from this perspective. In the model of Steffe and Cobb (1988) numbers as units of units are not yet part of the experience of a figurative counter. For abstract counters, number words - or written numbers - are thought to symbolise children's operations in the activity of counting and therefore they can be said to refer to numbers. However, the model does not contemplate children's reflections on conventional number words and scripts outside the activity of counting and the possible contribution of these in their arithmetic progress. The child's viewpoint of number words as constituting a system where some words seem to have particular interest or are objects of particular use has not been focused on. As constituting a pre-existing cultural system of representation, number words can be taken by the child as an object of reflection not solely in the context of counting. Based on the premise that children elaborate stable ideas on spoken numeration, the question is twofold. First, can these ideas be isolated from children's behaviour, and second, can they influence or affect children's progress in arithmetic knowledge?

[20] A number sequence is a sequence of abstract unit items but is not necessarily a counting sequence. A number sequence can be thought of as a sequence of abstract units that contains records of counting (Steffe, 1992[a])

Steffe and co-researchers' writings do not attend to the system of numeration per se but to children's actions and operations using their counting sequence that can be taken to betoken their arithmetic knowledge. It can be argued that for Steffe, children use a standard number word sequence, not a system of number words. Numeration notions are subsidiary to the abstract idea of unit constructed by children in the activity of counting and little attention is paid to children's reflection on conventional number sign systems.

2. 5 A Focus on Written Numeration

Place value and the rule of recursive grouping in base ten constitute the hallmarks of written numeration. The controversy surrounding the teaching and learning of place value has been discussed above. From an expert point of view Fuson (1990[b]) identifies the features of the system of written and spoken numbers and argues that whole numbers are expressed by these systems which use multiunits to build up large numbers. Hence, children have to learn the multiunits used by these systems through understanding the characteristics of the system of spoken number words and the system of written number marks. Fuson contends that

> "understanding of these systems requires understanding these features, each feature can be considered a conceptual structure that any given individual may or may not have." (Fuson, 1990[b], p. 344)

In the above sense, if children have not acquired these conceptual structures, their performance in addition and subtraction of large numbers is inadequate. Hence, children need to acquire multiunit conceptual structures for the spoken number words and written number marks. As has already been mentioned, in certain language-spoken systems the multiunits are regularly indicated by the number names. For example, the Korean number name "two *hundred* five *ten* six"[21] indicates how many multiunits of "ten" and "hundred" form the number. Conversely, English spoken numeration is irregular and multiunits are not clearly indicated until the hundreds. The idea is that English spoken numeration presents an obstacle for the construction of adequate conceptual structures and therefore, the acquisition of multiunit concepts can be supported by physical collectible multiunits like base-ten blocks (e.g. Fuson,

[21] This is the literal translation for '256' in Korean.

1990[a]; Fuson et al., 1997[a]). Fuson contends that school instruction does not ensure that these conceptual structures are connected to the written marks because it focuses on procedural rules that dictate what to do with the written marks. This produces a "concatenated single-digit conceptual structure" whereby the marks '4' and '3' in '341' are not seen as standing for multiunits of hundred and ten. For Fuson et al. (ibid) written and spoken numbers are "mathematical referrers" of quantities and operations, which are the "mathematical referents". Children initially have a "unitary conceptual structure" in which the referents of number words and number marks are single objects or collections; subsequently, more abstract, abbreviated and integrated conceptual structures develop and number words and marks refer to increasingly larger multiunits related to ten. In Fuson's studies, an expert analysis of the presumed mathematical referents is set forth and researchers set out to investigate how adequate children's conceptual structures are in relation to this a priori analysis. One can take a different perspective and investigate with a fresh approach children's viewpoints on number scripts. As with spoken numbers, one can see the acquisition of written numeration as a conceptual task and attempt to study what regularities and views of the system of written numbers children can establish when using the system. In this sense, two remarks made by Fuson (1990[a]) deserve further consideration. These are, first, the idea that irregularities of English spoken numeration constitute an obstacle for children to comprehend that the system has multiunits of different kinds and second, that the right-to-left increasing order is an arbitrary feature of written numbers and that it "must be shown to children; they cannot deduce it" (Fuson, ibid, p. 386). These two contentions must be put under scrutiny when one conceives the acquisition of written and spoken numeration as conceptual tasks which start well before children enter school. In the sense of Hughes (1986), one must clear the mind of old assumptions and look afresh at the question of how children learn these systems of conventional signs.

2. 5. 1 Number scripts from the viewpoint of the child

Because young children might not interpret printed numerals as standing for units of units, in the sense of Steffe, one can refer to them as "number scripts". For example, number scripts like '3', '1000', and '45' are present in children's everyday experience and they may constitute the object of their reflection. Older children and adults use these scripts in presence of younger learners before they enter school and outside school settings when they are in the first or second year of primary education. When number scripts play a role in these situations young children hypothesise about the meanings of these signs. In this sense, conventional number signs are not introduced by school - rather they introduced themselves insofar as children try to make sense of them. Wright has recently pointed out that despite significant research activity in the last twenty years[22],

> "research in early arithmetic does not focus on how children acquire knowledge of numerals [...], research endeavours in the area of written arithmetic have focused on learning and using the symbols for operations in conjunction with numerals rather than what might be viewed as an earlier topic, i.e. learning and using numerals per se". (Wright, 1998, p. 201)

Nevertheless, some research literature has documented young children's production of written numerals albeit this not being the focus of investigation. Some studies have focused on children's ideas of written numbers (e.g. Nunes and Bryant, 1996; Lerner and Sadovsky, 1995; Seron et al., 1992; Wright, 1998; Hughes, 1986; Sinclair and Sinclair, 1984, 1986; Nunes, 1997; Bergeron and Herscovics, 1990; Sinclair et al., 1983; Sinclair et al., 1992; Sinclair and Scheuer, 1993; Comiti and Bessot, 1987). For example, Hughes (ibid) has documented young children's[23] representation of quantity and has set forth four categories of representations: idiosyncratic, pictorial, iconic, and symbolic. Only the latter refers to children's spontaneous use of conventional number scripts. Hughes found that symbolic responses were the most common among older children but younger children also used this form of representation. Similarly, Sinclair et al. (ibid) identified six notation types to represent quantity among 4 to 6 year olds: global representation of quantity, representation of the object-kind, one-to-one correspondence with non-numerals, one-to-one correspondence with numerals, cardinal value alone, and cardinal value and object-kind. These studies presented collections of up to 6 objects, and findings suggest that very young children do not spontaneously use conventional number scripts to indicate quantity but that they eventually use these signs to convey this particular type of numerical information. In this sense, Bergeron and Herscovics

[22] This refers to research in arithmetic in the early years (from birth up to 7 or 8 years of age).
[23] English children from 3 to 7 years of age

(ibid)[24] argue that kindergarteners living in urban centres can demonstrate quite extensive numerical knowledge. In their study, the majority of children used numerals to convey information of quantity. One point that must be noted is that the aforementioned studies involved different-language speaking children using the conventional Hindu-Arabic system of written representation[25]. Sinclair and Sinclair (1984) conducted a study to investigate how young children interpret written numbers of their environment[26]. In particular, they were interested in the "function of meaning" children attribute to numerals. They presented 4 categories of responses: description of numeral, global function, specific function, and tag. These findings suggest that young children recognise that numerals are used to convey information. This is not surprising since number scripts are used in cultural practices for a wide range of purposes such as to indicate measurements, labels, ordinalities, and cardinalities (Fuson, 1988; Sinclair and Sinclair, 1984, 1986). On the other hand, other studies have shown that young children do not comprehend that parts of the notation of a number accounts for parts of the quantity represented (e.g. Kamii M, 1982 cited in Kamii C, 1985; Kamii C, 1985; Sinclair et al., 1992). Results suggest that even older children do not demonstrate this comprehension after 3 or 4 years of schooling and teaching of place value. Kamii has argued that

> "Place value is taught in first grade, but first graders cannot understand it and, therefore, do not learn it" (Kamii, 1985, p. 111)

Young children have not yet developed the necessary logico-mathematical reasoning that is required to comprehend place value and as a consequence Kamii has recommended that the teaching of place value should be delayed until children have worked with the series and comprehended part-whole relations. However, she argues that reading and writing numerals are activities that children are keen on and should be encouraged because

> "children do not have trouble acquiring this social, conventional knowledge in first grade" (Kamii C, ibid, p. 53).

Kamii's contention is that numerals and words constitute social knowledge and that children have no trouble *acquiring* it (Kamii, 1986). On the other hand, she argues

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[24] French and English speaking children of 5-6 years of age

[25] These languages include amongst others, English, Spanish, and French.

[26] French speaking children aged 4 to 6.

that place value involves logico-mathematical knowledge which children need to *construct*. One must notice her distinctive use of the terms "acquire" and "construct" for two different types of knowledge, social and logico-mathematical, respectively. Notwithstanding, it has already been argued that the acquisition of the system of numerals can be conceived as a conceptual task, even from an early age, and in this sense, the view of "construction" needs to be reinstated.

2. 5. 2 A step-by-step gradation at school

There is a prevalent pedagogical view according to which school work with numbers has to be segmented in steps from smaller to larger numbers. This view includes the use of a set of "pre-number" activities for pre-school children that comprise classifying, sorting, and matching objects according to different criteria[27]. Subsequently, school programs include working with numbers up to 10, successively up to 100 and then extending the work beyond 1000 in the succesive years of school. Generally, when work with written numbers is extended beyond 10, instructional devices or manipulatives are employed in order to support the learning of the idea of ten as a counting set and the same approach is followed to present a new order of the base in subsequent years of school (Sinclair et al., 1992; Wigley, 1994; Thompson I, 1997[a]; Aubrey, 1997; Lerner and Sadovsky, 1995; Leino, 1990). The alluded segmentation of the series of numbers might restrict children from using number words and scripts in culture-like situations (e.g. use of currency coins or notes) or simply as conversational pieces in no particular context. However, children's participation in everyday practices is permeated with experiences with number signs.

2. 5. 3 Children's ideas about number scripts

Research has shown that children elaborate conjectures about number scripts well before they can be said to comprehend place value or the underlying mathematical structure that written numeration represents. This claim is based on a pervasive findings in young children's productions. A number of studies have documented that young children often write large numbers they have not been taught in a non

[27] The rationale for this pedagogical approach rests on historical developments in mathematics associated with set theory (Thompson, 1997[a])

conventional manner (e.g. Nunes, 1997; Nunes and Bryant, 1996; Lerner and Sadovsky, 1995; Seron et al., 1992). Typically, these productions have been termed "lexicalisation" (Seron et al., 1992); "literal productions" (Nunes and Bryant, 1996); "concatenations" (Lerner and Sadovsky, 1995); or "fully extended representations" (Nunes, 1997). These concatenated forms consist in children's productions of non conventional number scripts by juxtaposing known scripts according to the number name of the dictated number. For example, "two thousand three hundred" is written as '2000300' because children assume that numbers are written by concatenating the scripts of the number words forming two thousand three hundred. This type of "error" is systematic among young children of different European languages[28]. Nunes and Bryant (1996) argue that these productions demonstrate that children are active learners of written numeration and that they use their "own system" to produce written numbers in a relatively consistent manner. Nevertheless, these researchers have not explored how these initial productions change towards conventional forms. In other words, they see children's incorrect written numbers as driven by an underlying rule but they do not study how this rule transforms towards the conventional writing of numbers. In a sense, children's own theories have not been studied by these researchers in order to document how they must be transformed when they are no longer viable. Findings of different studies have shown that children's rule of concatenation is not a one-or-nothing rule and that it is subject to variations. Lerner and Sadovsky (ibid) argue that children construct other hypotheses apart from the rule of concatenation. For example, the idea that the more figures a number has, the larger the number being annotated[29]. This hypothesis is elaborated in interaction with written numeration and more or less independently of the development of the spoken series because children use it in situations when they do not know the conventional name for the number. For example, when they compare '15674' and '47' they sustain that the former is the larger because it has "more numbers" although they cannot read '15674'. Research also suggests that children become somehow aware of the importance of the relative position of the digits as an indicator of the value being represented.

[28] These include English, French, Spanish, Italian and Portuguese (Nunes and Bryant, 1996)
[29] This is also indicated by other studies (e.g. Sinclair and Scheuer, 1992) and confirmed in an exploratory study undertaken with 5 to 6 year old children from a primary school of the South of England in April 1998 (Silveira, 1998).

In summary, although they have been scarcely studied, young children's ideas on written numbers are fairly consistent rather than merely idiosyncratic. The hypotheses of concatenation and the conjecture related to the amount of figures are examples of these ideas and have been documented in the literature. However, a study of how these ideas transform and how they might influence children's mathematical knowledge has not been carried out.

2. 6 A Discussion from a Constructivist Stance

The initial learning of the system of numeration in its written and spoken forms of conventional representation is the focus of this thesis. The aim of the study is twofold. First, the aim is to explore and chart young children's ideas upon written and spoken numbers seen as systems of conventional signs used in their everyday experiences and second, the study aims at connecting children's ideas on the signs to their progress in mathematical knowledge. The study holds a constructivist view of learning. The following subsections detail some relevant aspects of a constructivist stance while discussing a number of issues addressed by the research literature.

2. 6. 1 A constructivist standpoint

The major tenet of constructivism[30] is that knowledge is not passively received but build up by the cognizing subject. Simple as it may appear, this view represented a challenge to behaviourist educational research at the time of its emergence and it still represents a view subject to scrutiny (Ernest, 1994). Since then, a myriad of research studies have been published holding constructivism as their theoretical framework. However, other papers have been published which show a great deal of controversy about its epistemological and philosophical foundations and its suitability for framing research in mathematics education (e.g. Steffe and Gale, 1995; Lerman, 1996; Steffe, 1998; Kilpatrick, 1987). As a consequence of these theoretical debates the literature has distinguished research perspectives which are seen to hold constructivist tenets but incorporate aspects of other research traditions in psychology or socio-cultural

[30] Constructivism in mathematics education emerged formally in 1987 in the Eleventh International Conference of the Psychology of Mathematics Education in Montreal (Ernest, 1995, p. xi)

studies. These perspectives have been termed, among others, "trivial constructivism", "social constructivism", and "radical constructivism". Generally, all current accounts of constructivist research programmes draw upon the studies of Jean Piaget and Lev Vygotsky although some theoretical papers have warned against the problematic combination of both research programmes (e.g. Lerman, 1996; Confrey, 1994, 1995[a, b, c]; Castorina, 1993; Bickhard, 1995). It is prudent in the present discussion to avoid the intricacies of this on-going debate. Instead, the discussion will principally draw upon the constructivist views of knowledge and learning set forth by von Glasersfeld and Sinclair and will adopt the view whereby

> "We are left to justify the choice of a particular perspective because it allows us to solve certain problems - to 'see' certain things while being blinded to others." (Konold, 1995, p. 180)

Knowledge and learning from a constructivist viewpoint

Constructivism has been defined by von Glasersfeld (1995[b]) as a way of thinking about knowledge and the act of knowing that can be presented in two basic principles:

1. Knowledge is not passively received but built up by the cognizing subject;
2. The function of cognition is adaptive and serves the organisation of the experiential world, not the discovery of ontological reality.

These basic assumptions account for learning as the transformation of the viable theories that young children construct in their attempt to organise their experiential world. Sinclair has posited that

> "at all levels the subject constructs 'theories' (in action or thought) to make sense of his experience; as long as these theories work the subject will abide by them." (Sinclair, 1990, p. 20)

In this sense, as children have new experiences they will partly conserve and partly transform their ideas when they are no longer adequate. The view of adequate knowledge corresponds to von Glasersfeld's idea of viable knowledge and it is expressed in Sinclair's claim that "as long as these theories work the subject will abide by them". The notion of viable knowledge replaces the idea of absolute knowledge (von Glasersfeld, 1995[b]) and it defines knowledge as partly stable, partly transitory. von Glasersfeld (ibid) argues that the view of viable knowledge is twofold. First, it concerns the ideas and theories that work, in the sense stated by Sinclair. Second, it concerns the internal coherence of one's own set of ideas.

> "limited systems of reasoning lead to different possible answers for one and the same problem and the subject becomes aware of this discrepancy." (Sinclair, 1990, p. 21)

von Glasersfeld suggests that

> "one should think of knowledge as a kind of compendium of concepts and actions that one has found to be successful, given the purposes one had in mind." (von Glaserfeld, 1995[a], p. 7)

In another sense, the notion of viability of knowledge is intricately related to its "correspondence to the knowledge other people have constructed" (Sinclair, 1987, p. 29). This is particularly true for the learning of conventional systems of signs like spoken and written numeration because for knowledge of these systems to be viable, it needs to be discussed and checked by others. Therefore the construction of knowledge is dependent both on the child's intellectual activity and on the societal presentations children encounter in their experience (Sinclair, 1990).

A constructivist view of learning holds a fundamental view of continuity of knowledge whereby new knowledge grows from already existing knowledge (e.g. Ackermann, 1998; Sinclair, 1990; Leino, 1990; Duit, 1995). The view that the learner already knows sets out the rationale for investigating the standpoint of children's learning of a particular object at different points in time. In doing this, constructivist research programmes have set forth models of children's learning and knowledge which interpret their progress over a certain period of time. This is the case of Steffe et al.'s (1983) model of counting types which accounts for transformation in children's counting methods until they can be said to have constructed the idea of abstract unit or "number". In this sense, children's counting strategies are studied to infer transformation in children's nascent and changing arithmetic knowledge. The model is a way of interpreting such transformation which accounts for increasingly sophisticated organisation of children's mathematical experience.

Some theoretical definitions

As the basis of the constructivist perspective of learning, Piaget's theory of knowledge constitutes a system of theoretical constructs comprising action schema, assimilation, perturbation, equilibration, accommodation, and abstraction. These have been discussed extensively in the literature and particularly by von Glaserfeld who has remarked that Piaget's "notion of scheme is not a simple affair" (von Glasersfeld, 1995[b], p. 66). It follows that Piagetian theory is complex and consequently, it is no wonder that particular constructs of the theory have been redefined to address particular problems in educational research. von Glasersfeld (1995[c]) has argued that Piaget's epistemology is an attempt to account for how individuals come to organise their experience - or how they come to know - by establishing regularities in their experiential world. There is the fundamental assumption that the cognizing subject has the ability and the tendency to establish recurrences in their flow of experience. The contention of Piagetian epistemology is that children's conceptions, ideas or schemes are not purely idiosyncratic but that there may be commonly occurring features in children's notions that can be isolated and mapped out. This mapping out yields a model of an "epistemic subject". Piaget stated that knowing an object does not mean copying it but acting on it. Action has to be understood as any kind of behaviour that brings about a change in the world or in the cognizing subject view of the world (Sinclair, 1987). The notion of schema which can be defined as "whatever is repeatable and generalisable in an action" (Piaget, 1970, cited in Confrey, 1994) was set forth to account for the child's own organisation of their experiential world - that is, their knowledge. One primary feature of knowledge is its structure. Based on the aforementioned notion of action, it follows that a schema is a theoretical construct to account for the presumed cognitive structures through which the subject comes to know. These cognitive structures - action schemes, concepts, rules, or theories - can be inferred when children talk, act, solve a problem or justify their solution in a relatively stable manner. It follows that constructivists employ the notion of schema in a variety of ways (Konold and Johnson, 1991) and therefore it can be interpreted to be a child's particular solution method or a child's more or less stable idea or system of ideas that can be inferred at one particular point in time[31]. What is relevant is that the theory states that while these ideas or methods are viable, the subject assimilates their experience, but when they are no longer viable, the subject can experience tension or perturbation which may give rise to transformation of the current ideas or

[31] An analysis of the notion of schema is set forth by von Glasersfeld (1995[b]) who distinguished three elements in a schema: 1. The recognition of a particular situation; 2. An activity associated with the situation; 3. An expectation of certain result associated with the activity.

methods. The aforementioned tension can be interpreted as disequilibration and the transformation of ideas can be called accommodation which results in restoring the equilibrium of the subject's ideas. The establishing of actions and ideas are a consequence of the subject's tendency to abstract regularities from their experience and co-ordinate elements of their experience. Thus, a primary mechanism of learning is abstraction. Abstraction as an assumed capability is connected with another assumption of Piagetian epistemology which is the faculty of re-presenting to oneself a prior piece of experience and the ability of make judgements of similarity and difference (von Glasersfeld, 1995[b]). This notion of re-presentation is the hallmark of von Glasersfeld's account of constructivism because it serves the explanation of how a child comes to think or reflect upon their prior experience. Based on the aforementioned view that children's theories transform over time, there can be different kinds of abstraction depending on the actions or ideas on which they are performed. Although the distinction is problematic, the literature has distinguished two main kinds of abstraction - empirical abstraction from reflective abstraction[32]. The former concerns sensory-motor experience and the latter concerns co-ordinations of sensory-motor or other kind of experience. More generally, Piaget (in von Glasersfeld, 1995[b]) stated that the former concerns observables and the latter concerns co-ordinations. Kamii (1985) argues that simple or empirical abstraction concerns the child's focus on particular properties of objects (e.g. the colour red from a collection of different items) and reflective or constructive abstraction concerns relations that the child establishes among the objects (e.g. the relation of similarity among a set of objects). She argues that for the young child, one type of abstraction cannot take place independently of the other but as the child grows older, reflective abstractions are possible with independence from empirical abstractions. von Glasersfeld relates these two levels of abstraction to two domains of action or knowledge. These are the domains of the "figurative" and the "operative". Generally speaking, the former concerns "physical actions" and "sensory motor experience" and the latter concerns "mental operations" or co-ordinations of actions (von Glasersfeld, 1991[a], 1995[b,c]). As an explanatory construct, reflective abstraction has appeal for the constructivist

[32] Piaget subdivided reflective abstraction in three types: reflecting, reflected and pseudo-empirical abstractions (von Glasersfeld, 1995[b] pp. 103-105)

researcher in mathematics education because it can be used to account for the process by which actions are reified or objectified[33] and become mental mathematical objects that can in turn be acted upon (Cobb, 1996). The mechanism or process of reflecting is essential to the construction of mathematical ideas.

Studying the viewpoint of the child

It follows from the constructivist argument that for research to know children's learning in any area, there is a need to infer their viewpoint about the object of their knowledge. If they establish theories of the object in their attempt to organise their experience, these ideas can be mapped out and their transformations can be studied. It has been argued that transformations in children's ideas occur when they experience a contradiction. But contradictions may occur only after children have established some theories or set of ideas that are viable. Most relevant is that the response of a child to a contradiction can be of different forms. First, a child might ignore the contradiction and keep their initial idea of scheme; second, they might deal with the contradiction by holding both contradictory elements separate; and third, they might construct a new notion that resolves the contradiction. It is the latter case that serves best the construction of knew ideas or schemes.

Another example is expressed in Steffe's (1983) argument that children's counting methods can be conceived as schemes. In studying children's counting methods and how they transform Steffe et al. (1983) and Steffe and Cobb (1988) have set forth a model of learning which accounts for the construction of increasingly sophisticated arithmetic knowledge. Arithmetic knowledge is traced in the child's activity of counting and, following a Piagetian framework, sensory-motor action is seen as the source of arithmetic knowledge.

Some remarks on the learner

It has been argued that the account of learning that derives from the Piagetian theory

[33] Objectivation of processes or actions in the construction of mathematical knowledge has been a recent theme of discussion in mathematics education (e.g. Sfard, 1991; Dubinsky, 1991; Tall et al., 1999; Gray and Tall, 1994)

of knowledge fails to account for learning as it may occur in social practices. The individual young learner is seen as interacting with an object of knowledge and constructing their own view of the object.

> "With every step forward in knowledge that brings the subject nearer to his object, the latter retreats so that the successive models elaborated by the subject are no more than approximations that despite improvements can never reach the object itself, which continues to possess unknown properties" (Piaget, 1980, cited in Sinclair, 1987, p. 29).

The learner is conceived as an epistemic subject whose knowledge transforms from a less to a more sophisticated state. It has been argued that this posture cannot account for the relevant role of the interactions with others in social settings and the role of systems of signs in the learner's constructions. In this latter sense the learner is a social agent[34] and research that incorporates socio-cultural perspectives has brought further light into the study of learning in educational settings. As a consequence of the aforementioned critique, constructivist accounts have incorporated Vygotskian notions, such as the zone of proximal development, to account for the crucial role of others in the learner's intellectual progress. Other significant notions are the previously discussed role of mediation of systems of conventional signs[35] and the appropriation of these systems which express pre-existent cultural knowledge. As has already been argued, children can be seen as "appropriating" their cultural inheritance by using and mastering the mediational means of their culture. It must be noticed that from a socio-cultural perspective the very use of mediational means transforms the subject's actions. Appropriation can be seen as the process of making one's own the pre-existent knowledge of the culture in interaction with more competent peers (Werstch, 1998). These insights acknowledged, one can readopt a constructivist position and conceive the appropriation of written and spoken numeration as the transformations of the child's point of view on those systems, seen as their object of knowledge or the known. In this sense, as has already been argued, the assumption is that children construct theories in their interaction with the object that can be isolated and mapped out. Thus,

[34] Bickhard, 1995

[35] These are referred to as mediational means and they refer to language but include other system of signs such as systems of numeration

> "Though Piaget was not interested in the child's mastery of [conventional sign systems][36], we believe it can also be studied in the frame of his constructivist theory and that such studies contribute to a deeper understanding of the development of mathematical competence." (Sinclair and Sinclair, 1986, p. 67)

Hence, this study adopts a constructivist view to investigate both children's ideas about the system of conventional number scripts and words and children's increasingly sophisticated mathematical schemes. The latter develop from the activity of counting and the former can be interpreted as a more or less coherent set of conjectures that children establish to organise their experience with these external signs. Therefore, the constructivist stance is a viable framework for the purpose of this study[37]. Notwithstanding, as has been mentioned above, one must realise that the choice of perspective allows us to see certain things, while being blinded to others.

2. 6. 2 Some remarks on the literature review

The review of the literature presented in this chapter has revealed that the learning of the conventional system of numeration can be explained by numerous aspects and factors. These include individual constructive aspects and socio-cultural circumstances. All these factors contribute to account for emergent conceptualisations and competence in a child's mathematical career. It goes without saying that the learning of numeration has to be seen as a prolonged process and a complex problem to study since as an object of learning it concerns the construction of increasingly sophisticated mathematical ideas. The present investigation is solely addressing the initial steps. In particular, the study is concerned with the learning of pre-schoolers and children in the first year of school. Extensive research in elementary arithmetic does not seem to focus on the observation that both written and spoken numeration are complex systems of signs available to young children's inspection and use before the previously alluded step-by-step presentation of the series in schools. Written and spoken numeration are viewed as systems of socio-cultural knowledge (e.g. Carraher, 1985), as objects of school instruction (e.g. Fuson et al., 1997[a,b]) or as an object of mathematical learning (Kamii, 1985). However, there seems to be a paucity of

[36] Sinclair uses "conventional symbols" instead but the proposed change is thought to maintain the meaning intended in the paper.

[37] In this sense, constructivism is subject to its own view of viable knowledge.

investigations on how children establish and transform initial ideas upon written and spoken numeration. This paucity of information on children's initial ideas about conventional numerical signs had been addressed by Sinclair's[38] studies which have shown that young children elaborate ideas on the communicational function of numerical print. Lerner and Sadovsky (1995) have undertaken a pedagogical research project which has suggested that children elaborate conjectures about the system of written numeration which may transform as they interact with the system in situations where written numeration plays a role. Although undertaking a study of children's ideas on these signs may contribute to an elucidation of the reasons for their difficulties in mathematics, none of these studies seem to focus on both the ideas of the system of signs and their mathematical ideas. As will be argued below, Sinclair suggests that the construction of the system of numerical notation and the construction of numerical concepts are separate in their genesis but both system of constructions are later integrated in the child's mathematical experience.

One can agree with the socio-cultural view according to which written and spoken numeration constitute material mediational means in the pre-existent culture. According to this view sign systems are appropriated by children while using the system in cultural activities and in interaction with adults and more competent peers. Although the child can be seen as a social agent, it is possible to hold a constructivist interpretation of the process of appropriation whereby the child is seen as elaborating their own ideas on these pre-existent systems of signs of their culture. These original ideas may or may not be viable and may transform in the course of their constructive appropriation. Therefore, this study adopts the view that written and spoken numeration can be taken as objects of reflection by young children in their attempt to organise their experience. Consequently it sets out to study the more or less stable ideas children elaborate upon these signs. In this sense, rather than focusing in an a priori analysis of the system as representations of mathematical structures or concepts or as an irregular system of words, this study aims at inferring children's establishment of regularities in their view of the written and spoken system of

[38] Sinclair and Sinclair, 1984; Sinclair and Scheuer, 1993; Sinclair et al, 1992

numeration. However, there is a second aim of the study which relates to children's mathematical knowledge.

2. 6. 3 Two lines of investigation

The former considerations are particularly relevant because as systems of conventional signs, written and spoken numeration are, at the same time, objects of social use, objects of the child's reflection, and means of expressing mathematical ideas. From a constructivist stance, studying how children appropriate these systems of conventional signs involves studying children's points of view on the systems of written and spoken numbers, the regularities they may establish in the use of the system, the rules they may construct and the viability of those initial theories. In this sense, Sinclair and Sinclair (1986) have argued that young children work on two complementary aspects of these environmental signs: their form and their meaning. Findings set forth by Lerner and Sadovsky (1995) support the contention that

> "Conceptual work on the form of written numerals is thus from a very early age orientated towards finding a *system*" (Sinclair and Sinclair, 1986, p. 69; italics in the original) .

Hence, one line of the present study aims to investigate children's ideas on "their" system of written and spoken numeration. Sinclair's remark on the conceptual nature of the understanding of written numeration finds expression in research evidence suggesting that children construct original ideas on both written and spoken numeration. She argues that the work they carry out is conceptual in nature but draws heavily on interactions with their environment and knowledge brought to being by other children and adults. Sinclair goes further and contends that children's conceptual work on the system of written and spoken numeration is the result of constructing environmental information into more or less coherent systems, which in order to be viable, need to fit their nascent concepts of number and arithmetic operations. Thereby, the present study involves a second line of investigation which aims to interpret the progress in children's mathematical ideas. The study of mathematical ideas - such as number and arithmetic operations - can be undertaken by studying children's progress in their counting knowledge as seen by Steffe et al. (1983) and Steffe and Cobb (1988). Their model can be used to interpret children's knowledge of arithmetic through the use of increasingly sophisticated counting strategies.

The design of two lines of investigation is based on Sinclair's argument expressed in the following paragraph:

> "To our mind, trying to understand the written system activates conceptual schemes similar to those at work in the construction of the number concepts themselves. Although initially these schemes are separate, gradually they lead to their reciprocal assimilation." (Sinclair and Sinclair, 1986, p. 72)

Following Sinclair's proposal, this study holds the view that children's counting methods can be interpreted to be the child's approximation to number concepts (i.e. notions of arithmetic unit). Children's arithmetic knowledge is conceived as initially separate from the set of ideas that they establish to make sense of written and spoken numbers in their environment. The claim that the latter can influence or interact with the former is implied in Sinclair's analysis but has not been elaborated or articulated with research studies. Further, these two types of constructions, those which are taken to account for children's arithmetic knowledge and those which are taken to account for children's theories upon written and spoken numeration have not been studied in any longitudinal investigation of young children. A study of this kind can shed light into the question of how these two types of constructions may interact or be integrated in the child's organisation of their mathematical experience. Such a study can shed light into the comprehension of children's difficulties in constructing empowering mathematical ideas.

2. 6. 4 The research questions

As has already been mentioned this study has two lines of investigation. The first line will follow children's advances in their mathematical knowledge. To interpret these advances in children's competence, the study will use the constructivist model of five learning stages set forth by Steffe and collaborators. The second line of the study will set out to explore and map out children's ideas on written and spoken numbers. These explorations will be based in the assumptions and research findings of Lerner and Sadovsky and Sinclair's studies. It follows that the theoretical framework for the study comprises two parts with a common constructivist base. The constructivist theoretical framework to undertake the aforementioned two lines of investigation is developed in the next chapter. The longitudinal study will attempt to address the following research questions:

- What are the ideas, theories, conjectures, young children establish upon written and spoken numbers?
- How do these ideas transform, change, over a period of time?
- Can this set of ideas be said to be orientated to finding a system?
- How do children's ideas on written and spoken numeration, interact and influence, their progress in arithmetic knowledge?
- How is this initial knowledge of written and spoken numeration linked to later conceptualisation in mathematics?

2. 7 Summary

This chapter has presented a scenario of previous research in relation to the learning of numeration. The review of the literature has revealed that the learning of the conventional system of numeration is multifaceted and that the least studied of these aspects has been the original ideas children construct upon written and spoken conventional systems of signs. The literature has shown that the learning of numeration can be interpreted from different perspectives from which the constructivist stance is only one of a number of viable alternatives. Children organise their experiential world by establishing action schemes, ideas, concepts, and methods that can be studied and followed during a period of time. There can be two types of constructions isolated from children's learning. On the one hand, children's ideas on number scripts and number words can be mapped out and followed during a period of time. On the other hand, children's counting methods can be taken to indicate their increasingly sophisticated arithmetic concepts. It is the aim of this study to follow these two lines of investigation from a constructivist perspective, to show whether there are any relevant connections for the elucidation of difficulties in elementary mathematics.

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3. A Constructivist Theoretical Framework

3. 1 Introduction

Based on the preceding chapter, Chapter 3 develops a two-part construcitivist framework for the analysis of children's ideas. The first part addresses the theory of counting types and learning stages to analyse children's progress in arithmetic. A brief discussion concludes the formulation of this part. The second part addresses the study of children's ideas on spoken and written numeration. The chapter concludes with a discussion on how this two-part framework will address the research questions presented in the preceding chapter and sets the grounds for the methodological aspects of the study which are unfolded in Chapter 4.

3. 2 Part One: A Framework for Interpreting Children's Progress in Arithmetic[1]

Based on longitudinal studies of first and second graders[2] in the United States, Steffe and Cobb (1988) have set forth a model of five learning stages in the construction of the number sequence. This model accounts for children's progress in their arithmetic knowledge which is traced in their activity of counting and explains the construction of addition and subtraction of arithmetic units. Children's thinking is seen as developing from figurative to operative characteristics through the process of re-presentation and reflective abstraction (von Glasersfeld, 1991[a], 1995[b]). The model of learning stages reformulates a previous model of counting types (Steffe et al., 1983). The following section addresses fundamental aspects of this basic model.

3. 2. 1 Children's counting types

In the model of counting types, which was briefly introduced in Chapter 2, counting comprises three elements: 1. The counter's ability to produce the standard number word sequence; 2. The counter's ability to produce countable entities; and 3. The counter's ability to co-ordinate the previous two activities. Children are seen to

[1] This section develops a framework for the analysis and classification of children's progress in arithmetic. If the reader is familiar with the model of learning stages (Steffe and Cobb, 1988) they can proceed reading Part two of this chapter.
[2] Children aged 5 to 7 years old.

perfect the standard sequence from 1 to 20 and the quality of the items that they conceive while counting undergoes a developmental change. The counting types distinguished by the model are: perceptual, figural, motor, verbal, and abstract. They are based on the nature of the units the child is seen to re-present[3] while counting. Hence, a child may be called a perceptual counter if they are thought to count perceptual units. Because the quality of the items children create in the activity of counting is always a matter of inference, the following outline of the five types of units includes observable indicators which may allow those inferences.

1. *Perceptual units.* The child distinguishes actual discrete objects to count (e.g. bricks). Evidence for classifying a child as a "perceptual unit counter" is the indispensability of actual items to count.
2. *Figural units.* Subsequently, this dependency decreases and the child can count imagined or visualised items. Indication for this is that the child can count hidden objects. At this moment the child can be classified as a "figural unit counter". The child re-presents the perceptual units that constituted previous countable items.
3. *Motor units.* When the child becomes aware of the motor acts (e.g. pointing, tapping) that are initially involved in counting, they, themselves, are the countable items or the "tokens of counting". This brings more control over the counting activity, because counting no longer depends on perceptual or figural patterns. The child counts movements they create while counting. Indication of this is, for example, when the child counts the pointing acts as actual items that "could" be pointed at.
4. *Verbal units.* Subsequently, the child's awareness shifts to include the vocal production of the number words. The utterances of number words are taken as countable items.
5. *Abstract units.* Abstract units are items stripped from sensory-motor qualities. When the child becomes aware that each number word implies the potential counting activity from one and including that word, the child is thought to count abstract units because the abstraction required for this type of counting strips countable items from its sensory-motor qualities. The child no longer needs to

[3] Re-present is used in the sense of von Glasersfeld (1991[a], 1995[b]) and has been explained in the previous chapter.

carry out the count from one to a given number word because they can take that number word as implying that count.

A child can be classified as a counter of a particular type according to the most sophisticated unit type they can count. That is, theoretically, a perceptual counter can only count actual items but a motoral counter can count motor items, figural items or perceptual items[4]. The only way of inferring abstract counting is through the validated presumption of children's reflective thinking. The validation of any inference needs to be done on the basis of further interaction with the child. It follows that sophistication in counting is a matter of increasing detachment from the child's immediate experience and the shifting awareness of different aspects of counting and the items that are being counted. Progress depends on reflective thought whereby some of these aspects become superfluous. There is a major re-organisation of counting experience that comes about when the child enters the domain of abstract counting. In the four preceding counting types - i.e., perceptual, figural, motor, and verbal - the child attribution of meaning to the counting words requires sensory-motor experience or re-presented activity. This is the reason why they always count-all from one rather than coutn-on from a given number when solving the unknown total of two collections. In his discussion of the model of counting types, Cobb argues that

> "Only at the fifth level, abstract counting, can number words or numerals signify conceptual entities that appear to exist independently of the child's actual or re-presented sensory-motor activity." (Cobb, 1987, p. 168)

Put it in different words, Steffe et al. (1983) contend that when children are counters of abstract unit items, they are capable of creating numbers. This fundamental re-organisation concerns the construction of arithmetical units in the child's experience and it opens the pathway to novel constructions through reflecting on their initial idea of number. This, in time, relies on the reflective thinking of young children who are capable of the uniting operation of "integration". Integration is "the act of uniting what one may also consider distinct unitary items" (Steffe et al. Ibid). An act of integration occurs when the child's use of the word "4" implies the potential act of counting from 1 to 4. In brief, children commence counting only those objects which are in their immediate experience. Second, through re-presentation and reflection on

[4] Typical responses which lead to classification of counting types can be found in Appendix 1.

past counting experience they elaborate more sophisticated ideas of countable items, until they can operate with abstract countable items or abstract units. Wright (1988) has proposed that the five counting types can be organised in three stages as follows: the perceptual stage, where children can count perceptual unit items only; the figurative stage, where children can count figural, motor, or verbal unit items, and finally, the abstract stage, where children can count abstract unit items. What is relevant is that a child's conceptual knowledge which is interpreted to be in the perceptual or figurative stage is considered non-numerical whereas if it is in the abstract stage is considered numerical. Further investigations by Steffe and collaborators distinguished two more shifts in children's re-organisation of the counting sequence once they have become abstract counters. The following subsection presents the reformulated model of five learning stages which is based on the basic model of counting types.

3. 2. 2 The learning stages

The conceptualisation of the number sequence is seen as the foundation material for all arithmetic. When children become aware, through reflecting on their counting activity, that a given number word implies the potential activity of carrying the count from one and including that word, the spectrum of possible abstractions is widened. A child can now take that segment as material for further operating and results of this operating can itself become abstracted through novel acts of integration. The model of learning stages includes the perceptual and figurative stages proposed by Wright (ibid) and is completed with three more stages that account for transformations in children's number sequence once they have become abstract counters. The three stages of abstract counting are: Initial number sequence, Tacitly nested number sequence, and Explicitly nested number sequence. Thus, the five stages include the perceptual and figurative stages mentioned above, and the three stages of abstract counting. These account for children's re-organisation of their counting scheme from the perceptual stage to the elaboration of the Explicitly nested number sequence. Each stage represents an achievement and the beginning of new constructive activity. As with the characterisation of counting types, the outline of the learning stages includes behavioural indicators that may allow inferences regarding children's knowledge of counting and the number sequence. Each stage concerns qualitatively different ways of conceptual operating. Although each stage distinguishes conceptual re-organisations these are indicated by a variety of observable counting strategies. The following account of the model of learning stages[5] has been adapted from Steffe and Cobb (1988) and draws upon consecutive studies and essays about the original presentation (e.g. Steffe, 1988[a], 1992[a, b], 1994; Wright, 1988, 1990, 1991[a, b]; Wiegel, 1998; Cobb and Wheatley, 1988; Pearn and Merrifield, 1998; Biddlecomb, 1994). In the present formulation, transitional phases are included based on the presentation of Wiegel[6] (1998). The rationale to include transitional phases responds to the same criteria as why the formulation includes behavioural indicators for each particular stage. Since this is a model to interpret progress in children's knowledge, and as such, is a matter of inferential decisions, one must rely on observable indicators to make those decisions. Therefore, transitional phases contribute to the clarification of what re-organisations define a stage and what observable indicators contribute to the interpretation of children's organisation of knowledge. The five learning stages are: 1. Perceptual counting, 2. Figurative counting, 3. Initial number sequence, 4. Tacitly nested number sequence, and 5. Explicitly nested number sequence[7].

1. *Perceptual counting (PC)*

Children can count experiential objects in the immediate here-and-now: they must see or feel what they are to count. Thus, they cannot find the total of a partially screened collection and they count from one to find the total of a segmented visible collection.

Transition from perceptual to figurative counting stage:

Children may be able to count objects that are not in their immediate range of action or perception. They create perceptual items to count that serve as substitutes for the figurative unit items that are only imagined. The figurative unit items (the imagined objects) are countable but the child still needs to create experiential unit items to count. Children typically use fixed finger patterns. An indicator of this transition is a slight perturbation when not being able to keep track of the counting acts. Thus, a child is able to count partially screened collections in some contexts.

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[5] A stage in the construction of the number sequence responds to four criteria (Steffe and Cobb, 1988, pp. 7-8).

[6] Wiegel presented transitional phases from perceptual to figurative stages and from figurative to the INS stages. Here, two more transitional phases are incorporated.

[7] Appendix 1 presents theoretical examples and an extended description of the learning stages.

2. *Figurative counting (FC)*

The child can create and count the items of a figurative collection. This stage includes cases in which children can create and count motor acts. The meaning of a number word is the experience of counting visualised, motor or verbal unit items. Children at this stage no longer depend on direct sensory experience but typically re-create a sensory experience when counting. The counter appears to visualise the items that cannot be seen and this need for visualisation is a relatively severe restriction on the facility with which counting can take place. Theoretically, the child can find the total of partially screened collections in all contexts but always counts-all the items presented from one.

Transition from figurative to initial-number-sequence counting:

When a child is able to monitor their continuation of counting to find the number of screened objects, one can presume that the child's reflective thinking affords the construction of numerical ideas. We can attribute to the child the execution of the operations that are necessary to make numbers: the operations of re-presentation and integration. This happens when a child is able to re-present the results of counting and when the child can run through the items of the figurative pattern holding them at a distance. Before that, a number word does not necessarily stand for the segment from one up to and including that number word. That is why before this re-organisation occurs, the child counts all items from one. During the transition from figurative to initial number sequence, the child can count on in some contexts, but they do not count on from a given number word in an anticipatory manner: the child may monitor the continuation of counting acts during the activity itself.

3. *Initial number sequence (INS)*

Children are able to count-on to find the total of partially screened collections in all contexts and may be able to find missing addends in partially screened collections. They can count-off from a number word, to find differences. For children at this stage, the meaning of number words has undergone a significant change. Presented with the task of finding the sum of two subcollections of screened objects, children will start by stating the sum of the first subcollection and then count on. For example, the child would commence by saying "6" and then, say "7, 8, 9". Stating the initial number implies the activity of having counted the first six items (figural, motor, or verbal), but re-counting when presented with an additional subcollection to add on is now unnecessary. Children at this level are operational rather than figurative because they no longer depend on the links to re-presented experiences as in stage 2. A particular number word stands for the possible counting activity and refers to the individual number words of the segment from 1 and up to that particular number word. It can symbolise the individual number words in sequence, but it is yet to symbolise a unit containing that sequence. Children have not yet constructed a "one more" relation (order with inclusion). A number word of the INS symbolises its initial segment, it can be used in such a way that it symbolises counted items (but not counted and countable).

Transition from the INS to the TNS:

An indication that a child, still in the Initial number sequence stage, is on the transition to the TNS is the ability to reflect on their counting acts. This happens when the child counts their counting acts in an anticipatory manner to find out the missing addend of a partially screened collection. Being able to find the missing addend by counting on is a characteristic of the INS but when the child intentionally counts the counting acts of the continuation of counting with the intention of uniting them in thought, there is indication of a further re-organisation.

4. *Tacitly nested number sequence (TNS)*

Children are able to focus on the collection of unit items as one thing as well as on the individual abstract unit items. They are able to form "composite units". A number word refers to the verbal number sequence from 1 up to and including that word as the constituents of a composite unit. This is a "nested" sequence or an inclusive sequence because children are aware that a subsequent number word means one more unit and a composite unit from one to that given word. Children can count-on and count-down, and are able to select the more appropriate strategy depending on the problem to be solved. A number word of the TNS symbolises a unit containing its initial segment. It can be used in such a way that it symbolises both countable and counted items. It is recursive in that the results of counting can be reconstituted as a situation of counting (it is not a one way scheme like the INS). The child's concept of a number word contains records necessary to make an abstract composite unit.

Transition from TNS to the ENS:

A transitional phase between the TNS and the ENS can be characterised by the child's emerging awareness of the relation between addition and subtractions operations in constituting composite units. Children may be able to use thinking strategies to short-cut a counting routine. In the TNS children can co-ordinate two different number sequences (that is why they can double count) whereas in the ENS they can have the two number sequences and understand that one can be included in the other.

5. *Explicitly nested number sequence*

Numerical part-to-whole reasoning is the identifying characteristic of the ENS. Children are simultaneously aware of two number sequences and can disembed the smaller composite unit from the containing composite unit and compare them. For example, the child at this stage is simultaneously aware of the two distinct number sequences that are only implied at stage 4. Therefore they can disembed the smaller composite unit from the containing composite unit and compare them. They can conceptualise the whole, a part of the whole and the remainder. The child is aware of the inclusion relation and can intentionally disembed a segment of the explicitly nested number sequence from its inclusion in the containing sequence and treat is as a unit in its own right. There is understanding of the relationship between addition and subtraction Children at this level are able to import knowledge into a problem in order to simplify or shortcut a counting routine. These solutions are based on thinking strategies such as compensation (one more, one less).

A relation between the types of counting set forth by the foundation model and the learning stages presented above is depicted in the following table.

Learning Stages and Counting Types

	LEARNING STAGES	COUNTING TYPE
NON NUMERICAL COUNTING	1. Perceptual counting	Perceptual
	2. Figurative counting	Figural, motor, verbal
NUMERICAL COUNTING	3. Initial number sequence	Abstract
	4. Tacitly nested number sequence	Abstract
	5. Explicitly nested number sequence	Abstract

Some theoretical remarks on the formulation of stages

The account of transformation of children's counting outlined above rests on the constructivist construct of reflective abstraction. As has been argued, one can distinguish different types or levels of abstractions, for example, depending on the nature of the material on which the abstraction operates. In the initial stages of counting, children can elaborate and re-present figurative patterns (arrays of dots, finger patterns). These re-presented patterns can be the material for the child to operate on through reflection. Particularly, the operation of integration or unitising is the theoretical mechanism of construction of numerical patterns from figurative patterns.

> "A numerical finger pattern is created by applying the unitising operation to a figurative finger pattern. Numerical finger patterns have a property of mobility and are not limited to specific finger patterns." (Steffe, 1994, p. 140)

As has been previously mentioned, integration is the reflective abstraction whereby the child unites in thought what may be also considered distinct unitary items. In the INS stage, the integration operation is applied to sensory-motor material. For example, when children count-on to find the total of a partially screened collection, they apply the integration operation to the sub-collection of visible items. This integration can be applied sequentially and because of this it is termed sequential integration operation. In the TNS, the child can apply the integration operation to material that is the result of a previous integration. For example, children's alternative to count down to solve certain subtraction problems (e.g. 22-17) indicates that the composite unit of the subtrahend is itself seen as a composite unit tacitly embedded in the minuend. This is a progressive integration because the subtrahend - the result of an integration operation - is taken as material for further operating. Finally, in the ENS, part-whole operations are the results of further abstractions termed disembedding operations. Disembedding is the application of the integration operation to a part of a composite unit or number word sequence while remaining aware of the two created composite units that constitute the initial whole. This disembedding operation implies awareness of the reversibility of counting and the relation between addition and subtraction. In the above situation of a subtraction problem (i.e. 22-17), this awareness endows the child with the alternative to count-up-to.

In brief, reflective abstractions that explain the transformation of children's number sequences concern the operation of integration. First, this operation can be applied sequentially to figurative material. Second, it can be applied progressively to the results of a previous integration. Finally, it can be applied to part of a composite unit while remaining aware of the parts composing the initial whole. It follows that children make progress in the spectrum of problems they are able to solve and the range of methods they can use to solve the problems. The following table presents the relation between the stages and the kind of operations theoretically available to the counter.

Learning Stages and Operations

	STAGES*	OPERATIONS AVAILABLE
Non-numerical counting	PS	------
	FS	Figurative re-presentations
INTEGRATION OPERATION →		
Numerical counting	INS	Sequential integrations
	TNS	Progressive integrations
	ENS	Disembedding

*PS= Perceptual counting; FS= Figurative counting; INS= Initial number sequence; TNS= Tacitly nested number sequence; ENS= Explicitly nested number sequence.

The above table highlights the threshold marked by the operation of integration. This is a major re-organisation of children's counting because it gives rise to the construction of initial numerical ideas. The reflective thinking that gives rise to the INS is marked by the anticipatory manner in which a child counts-on to solve a counting problem of the type described further above. By way of contrast, the absence of this type of reflection is indicated by a child's insistence on counting-all the items from one. In Steffe's words,

> "A prime indication of lack of numerical structure is when a child always starts counting at 'one' even when the first few of a collection of countable items are hidden." (Steffe, 1983, p. 119)

The transformation of the counting scheme from the figurative stage to the elaboration of the initial number sequence is related to the two domains of knowledge. These are figurative and operative knowledge. As has been argued, figurative knowledge relies on sensory-motor experience and physical actions whereas operative knowledge concerns "mental operations" (Steffe, 1983). Mental operations are actions that can be carried out in thought. It follows that perceptual and figurative counting belong to the domain of the former, and the elaboration of the number sequence - i.e. INS, TNS, and ENS - with the construction of abstract units and composite units, belong to the domain of the latter. Two other major considerations complete the framework for an analysis of children's progress in arithmetic. First, there is the theoretical analysis of the construction of the idea of ten. Second, there is the emergence of thinking strategies that children are seen to use to solve arithmetic problems. The following subsections address these two issues in turn and their relationship with the learning stages.

A theoretical analysis of the idea of ten.

Although the learning of the numeration system is not the focus of the model of learning stages, Steffe's study of children's counting scheme addresses the construction of the idea of ten for counters at different stages[8] (Steffe and Cobb, 1988). Further investigations of the same age children[9] (Cobb and Wheatley, 1988) are the basis for the present theoretical formulation of children's transforming notion of ten. In these investigations, supportive material was designed in the problems presented to children. For example, strips of 10 squares and single squares were used to formulate a variety of problems. Once the child has counted the squares in a string, several situations can be posed. One alternative can be to present strips one by one and ask the child to work out how many squares there are in total whenever an additional string is presented. Another alternative is to present 3 visible strips and 2 hidden strips and 5 hidden single squares. The possible procedures are:

1. To ask the child how many squares there are if there are 2 strips and 5 squares under the cover.
2. To ask the child to find out how many squares there are hidden if there are 55 squares in total.

It follows from the previous considerations about the construction of abstract units and composite units that, the idea of ten as a unit can only be constructed once the child has at least become a counter of abstract units: that is, once the child has at least

[8] Ten is seen as a composite unit with special role in the numeration system.
[9] Children of this study were second graders in the United States (age 7-8).

elaborated the most advanced type of individual units. This means that before the INS stage, a child's idea of ten belongs to the domain of figurative knowledge. "Ten" may mean a finger pattern or the actual counting of ten objects from 1 to 10. When a child has constructed the INS "ten" may mean the potential count from 1 up to and including 10 but the child cannot take this segment as one thing for further operating. The child can only focus on the single abstract items constituting ten but not on that structure as a unit. This idea of ten is termed "numerical composite". The simultaneous focus on "ten" as one unit and as a composite unit emerges during the TNS when progressive integrations are possible and it gives way to a notion of ten as an "abstract composite unit". An analysis of children's counting by tens and ones has yielded a difference between image-dependent and image-independent concepts of ten[10]. Abstract composite units of ten typically rely on supportive material (e.g. strips and squares) for their creation. These units of ten need to be perceptually present or imagined to be counted. Independence from supportive material suggests a more sophisticated idea of ten, termed "iterable unit". To distinguish it from an abstract composite unit, one can say that iterable units of ten and one are created in anticipation while counting by tens and ones to solve a particular problem. This can be done with the flexibility of reversible counting by tens and ones and it is possible when the child is a counter of the ENS.

In summary, one can distinguish four increasingly sophisticated ideas of ten. The following outline includes observable indicators to infer qualitatively different ideas of ten.

1. *Ten as a perceptual collection or figurative pattern*

The child needs to count from one a finger pattern of ten or a collection of ten actual objects to give meaning to "ten". The child may co-ordinate a verbal sequence by tens with figurative patterns of ten.

2. *Ten as a numerical composite*

Children focus on the constituent elements of the idea of ten, that is, the individual 10 items rather than on the ten as one single entity. Children cannot take ten as a single

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[10] Cobb, 1995

discrete thing itself composed of10 items. If a child counts suitable material by tens, e.g. strips of 10 squares, the child is actually counting the strips by one, using a verbal sequence in tens.

3. *Ten as an abstract composite unit*

Children's idea of ten is a single entity composed of 10 items but this idea has still important limitations. Each act of counting by tens means a unit composed of ten ones but not necessarily increasing ten ones. "39" can be regarded as "3 units of 10 items and 9 items" but not as "39" items simultaneously . To count by tens and ones in order to solve a problem a child needs material of some kind to be present or imagined (i.e. image-dependent manner), for example, strings of 10 squares and single squares. This image-dependent solution indicates this type of idea of ten. The units of ten and one, as it were, need to be re-presented or imagined to be counted.

4. *Ten as an iterable unit*

Children can anticipate that they could solve a task by counting by tens and ones in the absence of actual material (i.e. image-independent manner). Ten is a single entity composed of 10 items and it can be used in either way as required. Counting by ten means increasing ten items or a unit of 10 items. Children create this unit of ten as they count (i.e. the unit of ten does not need to be perceptually available or re-presented as with the abstract composite unit of ten). For example, if the child counts 51-61-71-81, they can count simultaneously 10-20-30, or 1 ten, 2 tens, 3 tens. The idea of ten is image-independent and affords reversing the count by tens into a count by ones if the problem requires it. For example, '34-...=16', can be solved by subtracting two tens (e.g. counting backwards 24, 14) and adding two ones (e.g. by counting 15, 16). It is relevant to include another type of image-independent idea of ten called "abstract collectible unit". According to Cobb and Wheatley (1988), the difference with an iterable ten resides in ten as an iterable unit being constructed by reorganising the activity of counting by ones whereas an abstract collectible unit of ten is constructed on the basis of abstractions of suitable collections. An example of a solution to the problem '37+24' using ten as an abstract collectible unit, can be the following: 3 tens and 2 tens is 5 tens and 7 ones and 4 ones is 1 ten and 1 one, so the answer is 61. A solution to the same problem using ten as an iterable unit can be as follows: 37, 47, 57-58, 59, 60, 61. The table below organises the above considerations and relates the learning stages to the ideas of ten at different stages.

Learning Stages and the Concept of Ten

	STAGE*	Idea of Ten	General Indicator
Non- numerical counting	PS	Enactive idea	Count from 1 to 10 a collection of 10 actual objects.
	FG	Enactive or figurative idea.	Count perceptual or figurative pattern of 10 items.
Numerical counting	INS	Numerical composite	Count individual items of suitable material**
	TNS	Abstract composite unit	Co-ordinate count by ten and by one starting in the middle of a decade with suitable material***
	ENS	Iterable unit	Co-ordinate count by ten and by one without suitable material.

*PS= Perceptual stage, FS= Figurative stage; INS= Initial number sequence; TNS= Tacitly nested number sequence; ENS= Explicitly nested number sequence.
**Rows of 10 squares and single squares
***Present or imagined

Ten as an iterable unit is a more sophisticated idea than ten as a numerical composite or ten as an abstract composite unit because it betokens further reorganisation of the activity of counting by ones. An account of sophistication of children's solution methods to arithmetic problems cannot be completed without an account of the wide range of thinking strategies children use to solve addition and subtraction situations.

Children's thinking strategies

Research studies have documented how children use original methods to solve simple arithmetic problems (e.g. Steffe and Cobb, 1988; Foster, 1994; Thompson I, 1995, 1997[b]; Carpenter and Moser, 1984; Gray, 1991, 1997; Carpenter et al., 1996). As has been argued, there is an implication of increasing sophistication in children's methods which initially rely on actual counting procedures and subsequently depend on strategic reasoning. Gray (1991) argues that derived fact methods are used by children who use a known sum to solve an unknown sum. For example, to solve the problem '6+5' a child may count-on from 6 while keeping track of 5 counting acts. However, a more efficient method can be constructed by using the known sum 5+5 and reasoning that the unknown sum is 1 more. According to the model of learning stages children can only reason strategically when number words symbolise prior experience and the result of prior operations of integration[11]. It follows, that children can start using

[11] According to Steffe and Cobb (1988), thinking strategies are the result of the co-ordination of symbols (i.e. number words that stand for prior constructions).

thinking strategies at the stages of the TNS and the ENS. The kinds of thinking strategies distinguished by Steffe and Cobb (1988) are outlined below.

1. *Addend increasing-decreasing*

A known sum is altered to solve an unknown one. It is altered by decreasing or increasing it and then changing the given sum by altering one addend in the suitable way. For example, 10+6=16 (known), hence 9+6=15 (derived), is an addend decreasing strategy.

2. *Compensation*

The two addends of a known sum are altered to solve the unknown sum increasing and correspondingly decreasing the given addends. For example, 10+6=16 (known), hence 9+7=16 (derived).

3. *Inverse*

In an unknown difference, the subtrahend is used as one of the addends whose sum is the minuend. For example, 12-7=5 (derived) because 7+5=12 (known).

4. *Minuend or subtrahend variation*

A known difference is altered to solve an unknown one by altering the minuend or the subtrahend and the known difference in a suitable way. For example, 12-7=5 (derived) because 12-6=6 (known)

As has been mentioned, Steffe and Cobb endow the use of thinking strategies to abstract counters who have elaborated the TNS and the ENS. They distinguish between local and spontaneous thinking strategies. The former refers to strategies involving sums just worked out and the latter refers to strategies involving sums known without solving. In this sense, spontaneous thinking strategies are similar to the derived facts strategies set forth by Carpenter and Moser (1984) and Gray (1991). Derived or deduced facts are the more advanced strategies in an implied conceptual hierarchy which is presented below and is adapted from the work of Gray.

Conceptual Hierarchy of Counting-Gray 1991

	STRATEGY	The child may
	DERIVED FACT	use any other known number facts
	KNOWN FACT	know directly
PROCEDURAL	COUNT-ON	conceptualise the value of at least one set and use the appropriate counting procedure
	COUNT-ALL	know how to count

In Gray's analysis, derived fact solutions are higher order strategies because they indicate growth of proceptual thinking[12]. In Steffe and Cobb's analysis, the child can reason strategically because number words symbolise or stand for prior constructions that can be further operated upon. In the present analysis, what counts as sophistication is the availability of alternative strategies which relies on children's knowledge. For example, when a child is asked to solve the problem '11+5', a child limited to counting-all procedures is the least advanced in a hypothetical progression. A child that uses counting-on strategies may or may not have available alternative solutions. A short-cut of a counting-on solution to solve the given sum can be a compensation strategy derived from '10+6'. Therefore, sophistication concerns the alternative, more efficient strategies available to the child in an anticipatory manner.

3. 2. 3 Discussion on the framework to analyse children's arithmetic

This theoretical framework serves as a viable explanation of children's increasingly sophisticated solutions to simple arithmetic problems. Children's solution methods transform as has been documented by the literature. A constructivist stance takes the view that it is the underlying reorganisation of knowledge that supports the use of more efficient strategies to solve simple arithmetic problems. Therefore, the presented framework affords the interpretation of children's progress in their arithmetic knowledge and this progress is accounted for by the inferred reorganisation of knowledge attained by reflective abstractions. The framework comprises the five learning stages together with the considerations concerning the elaboration of the idea of ten and the use of thinking strategies that have shown to develop in the later stages.

The preceding considerations can be summarised in the following table.

[12] Proceptual thinking concerns the compression of counting procedures into procepts. A procept concerns the flexible ability to see a mathematical symbol as a concept and as a process (Gray, 1991).

Model of Learning Stages

	STAGE	OPERATIONS AVAILABLE	IDEA OF TEN	THINKING STRATEGIES
NON-NUMERICAL COUNTING	PS	Unitising[13]	Enactive	No
	FS	Re-presenting	Figurative	No
NUMERICAL COUNTING	INS	Sequential integrations	Numerical composite	No
	TNS	Progressive integrations	Abstract composite unit	Yes
	ENS	Disembedding	Iterable unit/Collectible unit	Yes

Although children's arithmetic knowledge is seen as emerging from their counting activity, the alluded progress is betokened by the abandoning of actual counting procedures. As has been argued, thinking strategies or derived facts gradually dispense with actual-counting solutions. It goes without saying that children still count to solve arithmetic problems but their solutions involve reorganisation of counting by ones. The major of these reorganisations is when the child can count by tens as a shortcut for counting by ones. Nevertheless, it can be argued that the efficiency attendant on thinking strategies depends on abandoning counting methods. As a matter of fact, it is the availability of alternative strategies, as has been previously suggested, that indicates sophistication. As research has documented, children some times continue using less efficient methods to solve simple arithmetic problems despite having demonstrated their capability to use more efficient solutions. In this respect, being efficient prior to action suggests higher sophistication than being efficient in action. This has been suggested with the term "anticipatory manner" in the previous section. The anticipatory nature of children's solutions is an indication of the sophistication that comes about with progress and it can only be inferred from children's flexibility in solving problems. The possibility of anticipation is viewed as the result of past reflective abstractions which are taken as the generators of the child's knowledge. Knowledge, in this sense, is conceived as

> "the schemes of action and the schemes of operating that are functioning reliably and effectively" (von Glasersfeld, 1980, in Steffe, 1983, p. 118).

[13] Unitising in the sense of creating discrete countable entities.

he framework of learning stages with the analysis of the concept of ten and thinking strategies takes the view that the child's progress in arithmetic cannot be explained without the construct of reflective abstraction. Reflection is the paramount explanatory construct to account for increasing sophistication in a child's particular solution. The event of an act of reflection is always a matter of interpretation. Steffe and von Glasersfeld (1988) point out that reflective abstraction requires that the subject has something to reflect upon. In this sense, the reflecting subject "holds an object at a distance" (Steffe, 1994). This object of reflection is taken as a given because is the result of a prior construction. What must be noted is that the child's ability to reflect is an assumption of the theoretical framework. However, the very instance of a particular event of reflective abstraction is a matter of an observer's interpretation. Therefore, one can already foresee methodological implications for the above mentioned statement. Suffice to say at this point that for those inferences to be made, there is the crucial necessity of intensive and long term interaction with children.

Having decided one's position on what constitutes progress in arithmetic knowledge and having given an account to interpret that progress, it is of paramount importance to discuss the assumptions and limitations of this interpretative framework.

Some considerations on the genesis and transformation of mathematical meaning.

The major contention of Steffe's model of children's mathematics is that

> "mathematical concepts will be created by means of abstraction that reflects upon actions rather upon sensory impressions (Steffe and von Glasersfeld, 1988, p. 16).

According to this view the genesis of mathematical meaning or the psychological existence of arithmetical objects has to be found in prior reflections on the activity of counting. As it has been discussed, counting activity does primarily require sensory or re-presented experience but subsequent constructions are taken as givens and the material for further reflecting. This recursive nature of reflecting is what constitutes increasingly sophisticated mathematical ideas from less sophisticated ideas[14]. In this context, the child's meaning for a number word can be mathematical or non-mathematical; or else, as has been argued, numerical or non-numerical. Theoretically, abstractions that depend only on sensory experience are not considered as belonging to the realm of number. However, they become part of it through the process of reflective abstraction. Thus, figurative counters cannot "mean" a numerical structure when they use number words in counting. This is indicated by the indispensability of enacting the count "to give meaning to the number word". Meaning - from a constructivist stance - is tied to what the subject can re-present. It follows that to "mean" mathematically, a child needs to be aware of a numerical structure. As has been discussed, this only starts to occur when the child unites in thought what may be considered as individual unitary items. In the context of the learning stages, meaning is related to "where the child is at" or "what the child is aware of" (Steffe and Cobb, 1988). Therefore, sources of meaning for a number word can be found in previous constructions, schemes, actions or operations that are functioning reliably. For a figurative counter the sources of meaning can be found in finger patterns or an enacted or re-presented count whereas for an abstract counter the sources of meaning rest on what they can do upon those re-presentations.

The aforementioned view of meaning is compatible with a more comprehensive constructivist account of meaning according to which

> "what a word means is always something an individual has abstracted from his or her own experience" (von Glasersfeld, 1995[b]).

One can take the view that those abstractions from experience mention by von Glasersfeld do not solely rest on the activity of counting. Put in other words, the sources of meaning for number words can be found in activities other than counting. In particular, since number words and number scripts are signs used in the child's social environment, there is reason to believe - from a constructivist stance - that the child actively tries to make sense of these signs. It can be granted, as the preceding

[14] In this sense, Steffe's theoretical framework is compatible with Piaget's genetic epistemology because it answers the question of how one can go from a state of less sufficient knowledge to a state of higher knowledge (von Glaserfeld, 1995[b], Castorina, 1993).

theoretical framework has done, that the possible mathematical meaning of number words needs to be traced in the child's activity of counting. This is a viable explanation as previous research has shown. However, it is a different thing to argue that the *only* source of meaning for number words is to be found in the child's activity of counting. In fact, as has been anticipated in the previous chapter, research has shown that children elaborate abstractions from their experience when interacting with numerical print. By the same token, it has been documented that children reflect on the number words used in their social environments. The following part of the chapter develops a framework to investigate the nature of these constructions and the question of their possible integration with constructions deriving from counting activity. Before going into part two of this Chapter, there still remain two points of discussion which concern the assumptions of the framework of learning stages.

The framework of learning stages is taken as a viable model to organise the observation of children's counting methods and the interpretation of their progress in arithmetic knowledge. Nevertheless, there is a particular assumption on which the framework is based that needs to be further considered. This is the assumption whereby

> "Eventually the children's number sequences were not restricted to the number words preceding 'twenty'. The children became adept with the number words in the decades as symbols for number sequences, and quickly developed number sequences up to 'one hundred'". (Steffe and Cobb, 1988, p. 223)

It must be noted that the original model of counting types, on which the framework of learning stages is based, states that children need to perfect the number word sequence from 1 to 20 in order to study children's counting methods. The "perfection" of the sequence is a given as is the "eventual proficiency" with number words up to 100. The point that needs to be made is that children might have already made conjectures about number words and scripts in the hundreds or thousands as well as in the first two decades of the sequence. These constructions concern their ideas of the system of number words and number scripts and have remained barely studied. The value of mapping out these ideas rests on their possible connection with children's arithmetic progress. It is the premise of this study that these ideas can be mapped out and it is one of the questions of this study whether there is an integration of these constructions with the system of constructions deriving from counting.

Finally, there is the nature of the problems presented to develop the model of counting types and learning stages. These are counting tasks presented to children who are seen interpreting a particular problem, according to their prior constructions. Novel abstractions are inferred and isolated to account for further reorganisation of the child's mathematical - or figurative - experience. In undertaking this approach, Steffe and collaborators have set forth a powerful theoretical tool to interpret children's arithmetic. However, as Konold (1995) pointed out, once one takes a given perspective, one may see certain things while remaining blinded to others. In this sense, the counting tasks are used as exploratory tools to study children's arithmetic knowledge but there are other constructions that children elaborate in situations other than counting that might be crucial to "see" in order to understand their mathematical progress. The following section presents the framework to analyse children's abstractions from their experiential world that concern their ideas on conventional systems of numerical signs.

3. 3 Part Two: A Framework for Studying Children's Theories on Spoken and Written Numeration

Vygotsky's contention that pre-existent cultural systems of signs do not just facilitate individual cognition but fundamentally transform it[15], has raised controversial themes of discussion in constructivism. A pertinent example of this pre-existent knowledge is expressed by the system of written and spoken numeration. For the constructivist theorist, the mere conception of pre-existent knowledge exposes a problematic view because, in principle, knowledge cannot pre-exist for the cognizing subject. The expression "cultural knowledge" can be controversial because as individual construction, knowledge is conceived as one's own view of the world[16]. As has been succinctly discussed in the previous chapter, Wertsch (1998) has argued that individuals encounter a set of external systems of signs that they are to master or

[15] This contention is related to the notion of mediation discussed in the previous chapter (e.g. Werstch, 1985, 1998).
[16] According to a constructivist conception, knowledge is the individual organisation of experience. It can be contrasted with others' knowledge but never shown to be the same (von Glasersfeld, 1995[b]).

appropriate[17] in their participation with more competent peers of their socio-cultural settings. The construct "taken-as-shared" has been used by constructivist researchers in mathematics education to refer to those meanings that a social group may consider as culturally accepted at a given point in time (e.g. Yackel and Cobb, 1996; Cobb et al., 1992)[18]. While maintaining the view of individual organisation of experience, the "taken-as shared" expression accounts for knowledge that one, as an observer, may conceive as pre-existent to the young learner. However, the child's knowledge can only be said to be more or less compatible with the pre-existent knowledge of their culture. Most relevant is the active role of the individual in constructing this pre-existing knowledge of their culture. As Becker and Varelas have argued,

> "the Vygotskian approach is similar to the Piagetian one in contending that the individual constructs knowledge and this cannot be done for the individual." (Becker and Varelas, 1995, p. 443).

Hence, the constructivist argument, can be expressed as follows: the young child constructs the pre-existent knowledge of their culture by modifying their own knowledge. From a constructivist perspective one can concede that children encounter and are to master an existing set of cultural tools. Moreover, it seems indisputable that children are to appropriate conventional systems of signs. It cannot be a surprise that constructivist researchers have incorporated the terms "mastery" and "appropriation" to account for the learning of cultural system of signs (e.g. Vergnaud, 1987; Sinclair and Sinclair, 1986; Lerner and Sadovsky, 1995). In particular, Lerner and Sadovsky (1995) have undertaken a didactical research project that addresses the appropriation of written numeration. On the other hand, Sinclair and Sinclair (1986) have documented children's ideas on numerical print in their studies of children's mastery of written of written numerals. However, these studies maintain the view of individual organisation of experience related to these external systems of signs that express the pre-existent knowledge of the culture. These researchers bring to the fore the crucial importance of social interaction and the use of the system of signs for the child's ideas on the systems to emerge. However, the constructivist view of a knower-

[17] Werstch draws a theoretical distinction between appropriation and mastery whereby one can master a set of cultural tools without appropriating it (Werstch, 1998).

[18] The social group can be the classroom mathematical community, the culturally accepted knowledge is said to be "negotiated" by the participants. (Yackel and Cobb, 1996).

known dialectic relation is the theoretical standpoint to account for children's ideas on the conventional signs.

It follows from a constructivist perspective that external systems of spoken and written numeration have to be acted upon to be known by a child. Irrespective of any kind of instruction, children elaborate ideas on these systems in so far as they interact with them and they encounter situations in which to establish regularities, construct viable ideas and modify their initial ideas when contradictions are experienced. In this sense, this constructivist framework pursues the question of what are the vicissitudes in the child's points of view of the systems of external representation of number words and number scripts. These systems of conventional signs are focused on rather than as mediational means, as the piece of taken-as-shared knowledge children are to construct in their experiential worlds.

3. 3. 1 The exploratory nature of the second line of investigation

Before discussing relevant aspects of this part of the framework, it must be noticed that the study of young children's ideas on written and spoken numeration is of an exploratory nature. Unlike the first line of investigation which begins with a viable model to interpret children's progress in arithmetic, there is no such framework available to interpret children's progress in their knowledge of the system of number scripts and number words. Therefore, the study of young children's ideas on number words and number scripts is of an exploratory nature and involves theory construction. This point will be further developed in the consecutive chapters. Suffice to say at this moment, that the present framework is formulated on the basis of previous investigations and other researchers who did not focus on the same problem but nevertheless have documented findings relevant to the present investigation.

3. 3. 1 Children's theories

As Sinclair (1987, 1990) has put it, children construct viable theories to make sense of their experience. These theories are regularities that a cognizing subject tends to establish in their experience (von Glasersfeld, 1995[b]). Theoretically, learning occurs when current viable constructions are modified by means of an experienced perturbation or contradiction inherent in the present system of ideas which may trigger a viable modification. Sinclair clarifies the notion of perturbation in the following terms,

> "limited systems of reasoning lead to different possible answers for one and the same problem and the subject becomes aware of this discrepancy." (Sinclair, 1990, p. 20)

Now, before contradictions can arise, theories or ideas that the learner constructs have to be established. This is explained by Fosnot when she argues that,

> "contradictions are constructed only secondarily, after learners first search for similarities between experiences (called affirmations) and attempt to organise each experience with their present schemes." (Fosnot, 1996, p. 16).

As has been argued, the way of accounting for new ideas or modification of previous ideas of the learner is by reflective abstraction. The "affirmations" referred to by Fosnot in the above citation are the viable theories conceived by Sinclair. It follows from the preceding considerations that one must strive to isolate children's current ideas in order to be able to infer any modification provoked by a perturbation. Theoretically, there are three possible types of compensation arising from a perturbation which have been briefly exposed in the previous chapter. These three types of responses to a contradiction have been set forth by Inhelder et al. (1974) and discussed by Fosnot (1996) and Lerner and Sadovsky (1995). What follows is a recapitulation of the three types of responses accompanied by a formulation of examples[19] for each particular case. What must be held in mind is that children establish ideas about written numeration that come to contradict other established theories and as a consequence of this experienced tension they may transform their initial schemes. Different responses depend on how children manage the aforementioned tension. For example, it has been stated that children believe that the number of digits of a script is related to the quantity that is being annotated. Thus, the more digits the scripts has, the larger the number. Children also believe that number scripts are written by juxtaposing other scripts already known. For example, "250" is written as '20050'. On the other hand, they may know from their previous experience with money that, for example, "three hundred" is more than "two hundred and fifty". From the point of view of an observer, the child arrives at a contradiction when they maintain that '20050' is higher than '300' according to the criterion of digits after

[19] These examples have been adapted from Lerner and Sadovsky (1995).

admitting that "three hundred" is higher than "two hundred and fifty" in the context of money. The three types of possible answers follow.

1. Children may ignore the contradiction and persevere with the initial scheme or idea. These means that the child may consider one idea or circumstance at a time, thereby avoiding the contradiction. By dissociating the problem they can deal with two answers that would be contradictory if they could face them simultaneously. For example, in the situation set forth above, the child admits that '20050' is higher than '300' but in the situation with money, three hundred is more than a two hundred and fifty.

2. Children seem to admit a difficulty, they might hesitate, holding both theories simultaneously and dealing with the contradiction by making local compensations. This response is not an anticipatory solution and serves only the particular situation in which the contradiction has arisen. The first sign that children are becoming aware of the conflict is apparent perplexity or dissatisfaction. In the example given, the child feels uneasy with their own production and this leads them to make corrections in order to reduce the number of digits in their written form - i.e. '20050' - for example by deleting some of the noughts that form the script. Or else, they may reinterpret their script as "two million and fifty" rather than "two hundred and fifty". This type of adjustment represents a compromise solution and although children may find it more or less satisfactory, it does not serve as an anticipatory solution. The child can only make adjustments after writing the number and after facing the contradiction. Therefore, they need to tackle the problem in each new situation when the conflict arises.

3. Children may construct a new, more encompassing, notion that explains and resolves the prior contradiction. The child can see the contradiction and feels uncomfortable about it. Therefore they are prepared to find a way to solve the problem. The solution, in turn, will serve as a new anticipatory form of knowledge applicable to other situations and not only to the particular one where the contradiction has occurred. This is the type of compensation that yields a new piece of knowledge as a result of solving a problematic situation[20]. In the given example, a

[20] It has been argued that only with this type of answer can one talk about cognitive conflict.

child may want to control the "length" of particular number scripts, *before* writing it. Some children seem to realise that the number scripts end up being too long when they are guided by the concatenation rule. When they try to control their number scripts beforehand, one may infer that they are aware of the contradiction produced by their current ideas. In the given example, the child can use the known script '200' as a guide to control the number of digits in "230". The child now is using the known script of "two hundred" in a different form. Rather than to juxtapose it to the script of "50" it is used to control the number of digits in "230".

In brief, children start constructing theories that, although previously viable, may turn out to be contradictory. To become aware of the discrepancy or inconsistency of their own set of ideas seems to be the necessary path to construct newly viable systems of ideas. It must be noticed that, in the case of numeration, more stable and viable ideas are those compatible with conventional or pre-existent knowledge. Those constructions which are inconsistent with conventional knowledge are likely to encounter societal presentations that prove them incoherent or unpracticable. Hence, children's constructions on the system of number words and number scripts are not necessarily idiosyncratic. They concern the uses of these systems of signs in the socio-cultural settings of the child.

Children's ideas on number words and scripts

As Sinclair's studies have shown, children focus on number scripts as they isolate them from the mass of graphic material that surrounds them (e.g. Sinclair and Sinclair, 1984; Sinclair et al., 1983; Sinclair et al., 1992; Sinclair and Scheuer, 1993). They come to know that written numeration is an ideographic system and not an alphabetic one like written language[21]. Among other researchers, Sinclair and Sinclair (1984) have argued that written numbers are used in the child's environment holding cardinal or ordinal properties, referring to measurements or functioning as labels for different forms of classification. Older children and adults comment on this numerical print and the young child participates in situations where these signs are

[21] This means that a number mark rather than standing for a particular sound (like a letter of the alphabet) stands for an idea of quantity (Sinclair and Sinclair, 1984).

used. Children elaborate hypotheses about the properties of the notational system based on their use and interaction with it. Since number scripts and words are signs used in the child's social environment, it follows that the child actively tries to make sense of these signs. Learning numeration comprises reflective work on the form of external signs, the rules for their combination and use, the link between verbal and written counterparts and the integration and co-ordination of the information provided by adults on the particular function of numerals and number names in their environment. The conventional system of written numeration is an object of knowledge for young children and it has been argued that research has scarcely focused on its construction by the young child (e.g. Wright, 1998, Sinclair and Sinclair, 1984). The same can be argued about number words. Although children's knowledge of number words has been the focus of a wide range of studies, it seems that a focus on children's ideas on the system of number words that encompass their initial use and conjectures about "a system" has not been included in these studies. Young children gradually come to conceptualise written and spoken numeration as systems of representation which they start learning well before school instruction. The conceptualisation of these systems is a prolonged process which includes the construction of mathematical objects and operations. However, the initial ideas on the system of words and scripts have been uncharted. Investigations of young children's learning rarely focus on number words and scripts higher than ten, twenty, or a hundred. Findings documented by Lerner and Sadovky (1995) suggest that children elaborate conjectures on the system of written numbers. These researchers have conducted a didactical research project which does not ignore the fact that written numeration is not only a cultural product but also an object of every day use. As a consequence of this, their study assumes that children have opportunities to elaborate knowledge upon these systems well before they enter school.

Lerner and Sadovsky[22] adopt a constructivist approach and set out to investigate what are the aspects that children consider relevant in their reflection on number scripts used in their environment. Put in other words, these researchers set out to explore

[22] This didactical research project was undertaken in three schools of an urban area in Argentina (Lerner and Sadovsky, 1995).

what things they learn about written numeration. They designed situations to allow opportunities for children to put into play their conceptions about written numbers. The pedagogical approach was developed to favour the confrontation of current ideas, the detection of inconsistencies, and finally lead children to reformulate their previous thoughts so to be gradually compatible with conventional notation. These researchers argue that the production and interpretation of written numbers are activities inherent to the work with a system of representation like written numeration. They were driven by the goal of creating situations that allow children to unveil the organisation of the system as well as the way in which the system expresses the numerical structure that it is supposed to represent (Lerner and Sadovky, 1995). Their premise is that written numeration is a carrier of mathematical meaning - numbers, order relation, arithmetic operations - and therefore ordering and comparing as well as operating are essential aspects of the use of number scripts. This is in accordance with Sinclair's (1986) claim that children work on two complementary aspects of these environmental signs: their form and their meaning. Findings of the aforementioned research that constitute the basis for the present study are formulated and discussed in the following three points.

1. Children construct specific criteria to compare number scripts. These criteria are constructed relatively independent from their knowledge of number names and is seen as emerging from children's interaction with written numbers. This presumption is viable since children can use these criteria to compare number scripts despite not being able to read the scripts in a conventional way. First, children believe that the number of digits is related to the quantity being represented. This criterion can be expressed, in children's words, argue "the more numbers the bigger the number" [23]. Second, children believe that the position of the digits has a relevant role to play in writing numbers. This criterion can be expressed, also in children's words, "the first one is the boss".

[23] Citations of children's justifications are translated from Spanish to best maintain their original meaning.

Two considerations are relevant before turning to the next point. First, one must contemplate the possibility that when children argue "the more numbers the bigger the number" they are in fact referring to more marks or squiggles and not to any aspect of a system of representation per se. However, Lerner and Sadovsky maintain that the criterion related to the quantity of digits concerns a property of the conventional system of written numeration. Should the children of their study be living in Roman times, they argue, this criterion could not be established. Since Roman notation is not strictly positional nor is it ruled by a recursive regrouping with a base, the criterion "the more digits the bigger the number" is not viable. In the Roman system, 'MC' represents a larger quantity than 'LXXXIII' although it has fewer figures than the latter[24]. Unfortunately, one cannot interview Roman children to find out whether they would construct other criteria but not the conjecture inferred for the children of Lerner and Sadovsky's study.

The second consideration to be made is that children's conjectures have shown to be more or less consistent. In particular, children's criteria can be used to compare scripts in particular intervals of the series but not others, or can be overridden by other ideas. For example, a child who has previously used the criterion "the first one is the boss" may subsequently maintain that '199' is higher than '200' "because 9 is higher than zero" and the former script has more nines. This idea can be termed "face value" and it has been shown to emerge in children's justifications.

2. Children initially believe that numbers[25] are written by concatenating the form of number scripts already known. This finding, as it has been previously argued, is documented by other researchers addressing disparate research questions (e.g. Nunes, 1997; Nunes and Bryant, 1996; Lerner and Sadovsky, 1995; Seron et al., 1992). Concatenated forms such as '2000300' for "2300" haven shown to be systematically produced by young children and have been termed "fully extended representations" as opposed to "compact representations" exemplified by conventional forms (Nunes,

[24] In the Roman System, MC = 1100, and LXXXIII = 83.
[25] Steffe, I suggest, would argue that for figurative counters, "numbers" are not part of the child's knowledge, and hence, one could not talk about writing numbers, but writing number names by producing the corresponding number scripts.

1997). What is relevant to point out is that Lerner and Sadovsky have shown how these ideas may transform towards being increasingly compatible with conventional notation. Without this aim in mind, Seron et al. (1992) have documented how, as children grow older, their productions are partially concatenated as opposed to fully concatenated. For example, instead of producing '20010' for "210" older children write '2010', possibly based on their knowledge of conventional forms of 201, 202, 203, and so on. Lerner and Sadovsky have found that children may produce fully or partially concatenated forms and may use their concatenation rule in some intervals of the series but not others. For example, they may write '100020' for "1020" but they write '100023' for "1023" instead of '1000203'. By the same token, children may concatenate numbers in the thousands but write conventional forms for number scripts in the hundreds.

3. Children's appropriation of the conventional writing of number scripts does not follow the order of the numerical series. This means that children may be able to write number scripts for, "200", for example, in a conventional form before they are able to write the script for "43". Lerner and Sadovsky argue that the conventional form for round numbers or knots (e.g. 100, 200, 300, 1000, 2000, 3000) are appropriated before the forms for the numbers within the intervals of the knots. As has been argued in point 2, children produced concatenated forms for numbers such as "2300" based on their knowledge of the conventional form for the knots (i.e. 2000 and 300). The point that needs to made is that this knowledge seems to depend on children's social settings[26]. One can presume that if children do not encounter situations where these numerals are used, there is little chance that they may know these forms.

A relevant consideration concerning the study of Lerner and Sadovsky relates to a methodological aspect which will be further addressed in the consecutive chapter. This point concerns the manner in which children's ideas are inferred. In particular, the reference is to children's criteria that serves the comparison on number scripts whose conventional reading the child may not know. In an exploratory phase, these

[26] In the case of the children of Lerner and Sadovsky's study, children were familiar with the currency notes of the time in Argentina which involved numerals in the hundreds and thousands.

researchers conducted clinical interviews with 6 year olds based on designed tasks which aimed at putting children's conjectures into play, as stated above. Typically, a child was asked to solve a task involving the production, or interpretation, of number scripts and was asked to justify their solution or production. It was this verbal justification, together with the actual solution or production which counted as the source for inferring children's current ideas on number scripts.

Children's ideas related to the number of digits or the importance of the position of digits in a script are shown to be viable criteria to put numbers in order or to compare numbers. However, it goes without saying that children do not initially know that "the first one is the boss" because it represents units of ten - in a two digit script -, or that "the more numbers the bigger the number" because the system requires more figures to represent increasingly higher powers of the base. Nevertheless, children's criteria are viable interpretations of a particular system of representation: the Hindu-Arabic system. In this sense, children construct regularities of the system before they construct the reasons which have generated them. Lerner and Sadovsky's findings have shown how children may start learning these systems of external signs. The present study is not didactical research that advocates a particular form of teaching written or spoken numeration. A step behind such a proposal, this study charts children's ideas on these systems in order to establish their relevance to children's arithmetic progress.

One final point that needs to be remarked upon is that the regularities children may abstract on the system of number scripts depend on their interaction with an extended interval of the series. Although the above considerations were made principally in relation to written numeration, they apply to children's conjectures upon spoken numeration. This means that, contrary to typical school work in the beginnings of primary education, young children are supposed to interact with number scripts and number words in the hundreds and thousands. Since written - and spoken - numeration are seen as objects of social use, the series in children's environment are not segmented for the sake of instruction. The complexity of social use brings to the fore the possibility of children's hypotheses to interpret numerals and number words.

3. 3. 3 Knowledge as gradual approximations

The aforementioned segmentation of school work upon the series of numbers has been said to hold the view of knowledge as a ready-made body that needs to be conveyed in small and easy steps in order to smooth the learning process (e.g. Leino, 1990). Although a constructivist conception of knowledge is contrary to this view, a rationale for this step-by-step gradation can be found in the view that children construct numbers from small to larger ones (Kamii, 1985)[27]. This graduation seems indisputable and it is the practice of most curriculum programmes in several countries[28]. Hence, when children arrive at school, the exploration of written and spoken numeration systems is somehow interrupted because school work imposes work with "numbers" up to 10, subsequently up to 20, and so on. Contrary to this view, this study will incorporate situations in which number names and scripts in the hundreds and thousands are in play. The rationale for this decision is twofold. First, there is the assumption that children construct conjectures on the system of number words and number scripts stemming from their interaction with these systems in their complex social environments. Second, there is the hypothesis that children will establish regularities in these systems when they interact with extended intervals of the series. By segmenting the series children are left with little chance of abstracting regularities which can be the material for further conceptualisation. The idea is that the urge to simplify or to withhold the complexity of the social use of written and spoken numbers may undermine children's possibilities to construct the systems of written and spoken numeration. The incorporation of the extended intervals of the series implies the conception of knowledge as provisional in nature. It goes without saying that the use of high number words and scripts by young children does not allow per se inferences of their mathematical meaning for the child. Children's use of number words and number scripts may be not conventional and, from an observer point of view, may seem erroneous, or contradictory. However, it can allow inferences on their conceptions of a system of number words and scripts and eventually, their ideas on how these systems express their mathematical ideas. Thus,

[27] In Steffe's terms, children need to construct an abstract single unit in order to be able to construct an abstract composite unit.

[28] Refer to the subsection "Step-by-step gradation" of Chapter 2

it is not a prerequisite to understand the base ten structure of the system to start using and thinking about written numeration. Mathematical constructions may arise as solutions to problems with written and spoken numeration and number scripts can be the originators of conceptualisation of the base ten structure. In this sense, Lerner and Sadovsky propose to use written numbers as the very didactical material instead of designing artificial systems of representation. To use number scripts and number words means to produce and interpret number scripts, to establish comparisons among them and rely on them to solve arithmetic problems.

Informal knowledge of pre-school children

One can argue that what children learn from their use of written and spoken numbers in their environment constitutes their informal knowledge of the system as opposed to what they may be explicitly taught in school. Pre-school children elaborate notions whose underlying logic needs to be explored, and related to their constructions of arithmetic knowledge. What needs to be considered is that these ideas may remain implicit in the sense that they are neither considered nor made explicit when the child is taught about numeration at school. For example, at some point, children are told, irrespective of what their informal knowledge might be, that in '14', '1' means one group, bundle, or tower of ten things, and '4' means four of these things. Some children may be aware of the importance of the position of digits and might have abstracted that '1' is in front of all the teen scripts. However, for some children this might be an utter novelty, with no connection with their previous ideas. Moreover, since they have not abstracted that '1' is in front of all the teen scripts, what the teacher tells them may make no sense to them at all. Initial knowledge of number words and scripts, albeit not necessarily mathematical and stemming from different system of ideas, may be eventually integrated with children's construction of numerical notions.

What children learn from their interaction with number words and scripts in their social environments seems to depend on the nature and quality of this interaction. Related to this, is the use of number signs in the child's environment. As an example, one can mention the different forms of reading aloud in English, scripts such as '1900'. This numeral can be read as "nineteen hundred" or as "one thousand nine hundred". Reflections on these uses may lead to children's elaborations on ideas how number words and number scripts are organised. It has been argued that what children know about number words of the sequence is highly dependent on the opportunities they have to practice the sequence (e.g. Fuson, 1988). Diversity in what children know must be foreseen. However, regularities in what they consider relevant aspects of these signs may also be found.

3. 4 General Discussion

It has been suggested that young children think about how hundreds, thousands and millions are written and organised. As has been argued, children's knowledge about the organisation of number names and number scripts has been scarcely investigated. The premise of this study is that young children make conjectures about number words and scripts in the hundreds or thousands as well as in the first two decades of the sequence. These constructions concern their ideas of the system of number words and number scripts. Limited as this set of ideas may be, they are the first abstractions on which further abstractions can be made. By "set of ideas" one is not referring to the mathematical structure that an expert user can see expressed by the conventional representation, but to the constructions that children are thought to make in their approximation to their object of knowledge. In children's worlds, written and spoken numeration ideas consist of successive approximations which pass through definitions and redefinitions. This study focuses on the early steps of these approximations.

Socially constructed systems of signs such as written and spoken numeration, are part of the environment young children interact with and it is through this interaction that children can start conceptualising them. They do not need to comprehend them thoroughly to start using them. The comprehension of the system develops as they use and reflect upon it in their participation in culturally organised situations. Kamii (1985) has claimed that reading and writing numbers are activities in which young children are at ease. Her claim is that young children find no trouble in acquiring this social, conventional knowledge and this framework agrees with this position. One has witnessed young children's fascination with high number words as well as their inquisitive talk about them. However, what Kamii seems not to focus on, is the view that children approach the learning of written and spoken numeration as a conceptual task. Children's ideas on number words and scripts are not a simple matter of social transmission or imitation of adults. From this standpoint, the present investigation sets out to study what is it that children appropriate from the societal presentations they interact with. Written and spoken numeration are presented to children as an object of thought which they need to appropriate, by which I mean construct through reflective abstraction. Hence, this study holds the view that this appropriation or construction of knowledge is a result of general processes of abstraction on the one hand, and on the other hand, of the way objects are presented to the young child with their societal meanings (Sinclair, 1990). As has already been argued, children elaborate ideas on these societal presentations of external systems of signs. These constructions constitute a form of organised knowledge about the external systems of signs. In the case of written numeration, it has been shown that the young child, who has not to invent a system but to understand the system of their culture, abstracts regularities of the system prior to the conceptualisation of the reasons that explain those regularities. Children's comparative criteria are precisely the construction of those regularities. For example, they may know that numbers written with more digits are larger than those written with fewer digits. However, they do not know why this is the case. The reasons that explain why children's criteria "work" are to be found in constructions of a different nature, namely, those that lead to mathematical ideas. Precisely, to know that '345' is higher than '34' is to know that '3' in '345' represents 3 units of a hundred and '3' in '34' represents 3 units of ten.

The construction of the notion of unit of units has been discussed in the first part of this framework and one can now argue that this type of constructions are of a different kind compared to those related to the elaborations on how number words and number scripts are organised. However, mathematical progress in one's culture is incontestably dependent on the construction of systems of external representation. Therefore, as has been anticipated, this study aims at investigating what is the relation, between these two kinds of constructions: on the one hand, those stemming from the organisation of the system of number words and number scripts, and on the other hand those stemming from children's activity of counting. The position that sets forth constructions of different nature, has been discussed in the previous chapter. It emerges from a theoretical consideration postulated by Sinclair and Sinclair (1986).

They have argued that the construction of written and spoken numeration triggers conceptual schemes similar to those at play in the construction of numerical concepts. One must recall the conclusion of their claim,

> Although initially these schemes are separate, gradually they lead to their reciprocal assimilation." (Sinclair and Sinclair, 1986, p. 72)

They do not elaborate on this position nor has there been research evidence to support it. In another report, the position is stated in the following manner,

> "Understanding of written numerical notations is a construction process that is *necessary* to the understanding of our numeration system, and it participates in and directly influences mathematical cognition. The grasp of numerical notation is thus deserving study in its own right, and is not to be approached exclusively as means of representing knowledge acquired in other domains (cognition, counting, computation)." (Sinclair and Scheuer, 1993, p. 203, emphasis in the original).

The present study is based on Sinclair's postulate and aims to elaborate further the claim that understanding written (and spoken) numeration "directly influences mathematical cognition". Therefore the present framework comprises two lines of investigation. The first will study children's progress in arithmetic, by using the model of learning stages to interpret increasingly sophisticated arithmetic knowledge. The second sets out to investigate children's organisation of number words and number scripts through the study of their ideas and conjectures of the systems of written and spoken numeration. The second line does not come with a model to organise observations. In this sense, the enterprise involves the activity of theory construction. Nevertheless, part two of the previous discussion has set out a preliminary framework to undertake this line of investigation.

The rationale for the second line of investigation is twofold. First, if knowledge grows from already existing knowledge, the question is what young children already know about these systems of external signs as they enter school and before they are told about the organisation of the system of written and spoken numeration. Second, it is the question of this study whether the system of ideas children elaborate on written and spoken numeration has any role in their mathematical progress. Therefore this study pursues the question of how children's constructions of external numeration systems relates to their increasing competence in arithmetic. It will elaborate on the connections of findings stemming from separate lines of the study.

Some considerations about the connections of the two lines of the study

Both parts of the theoretical framework are based on a constructivist standpoint. Precisely, the common element is the view that what counts is the interpretation of the child. Reflection is a common explanatory construct to account for the transformation of children's ideas. As Steffe has put it in his analysis of children's mathematics, reflection is understood as requiring the reflecting subject to hold an object at a distance (Steffe, 1994), or as the result of distinguishing what thinks from what is being thought. (von Glasersfeld, 1995[b]). Reflection will be incorporated in the analysis and discussion as a common explanatory construct. The "object" of reflection may differ in the two lines of study. Precisely, the object of reflection for the child is a matter of inference for an observer. Nevertheless, one can anticipate that in the first line of the study, reflecting will be on re-presented collections of countable units or actions on these collections whereas in the second line of investigation, reflecting will be on physical aspects of the external signs, or on co-ordinations of previously established ideas.

Another common aspect of the proposed two-part investigation involves the implication of a longitudinal study to answer the research questions. Long-term investigations allow phases of exploration, establishment of conjectures and design of situations to validate those hypotheses. Related to the view that what counts is the view of the child, is another methodological aspect which concerns the intensive interaction with children in order to be able to formulate viable inferences on their thinking. These are methodological aspects which will be further developed in Chapter 4. Before turning to methodological considerations, there is one final point of discussion.

Steffe's studies of children's counting have the assumption that they cannot understand the numeration system until they construct part-whole operations. But written and spoken numeration can be objects of reflection, upon which children elaborate conjectures from an early age. According to a constructivist view, children hold theories about their objects of knowledge and in the case of written and spoken numeration these can be crucial to understand their progress in arithmetic. The point that needs to be made is that reference to a conventional organisation of numbers expressed in naming systems (Skemp, 1986) is hardly focused on in Steffe's studies. Nevertheless, Steffe has stated that a major reorganisation of counting is when the child can count by tens as a shortcut of counting by ones. By the same token, in the study of children's meaning for number words, the meaning of "ten" is given closer inspection because of "its importance in the decimal numeration system" (Steffe, 1988[b], p. 95). Hence, although there is indirect reference to numeration, as organised by the notion of ten, *Steffe's studies focus on children's abstractions taking the use of a pre-existent cultural naming spoken system as a given.* As for the written system, Steffe's theoretical enterprise to chart the mathematics of children emerged from the concern on the incorporation of written numeration in the teaching of arithmetic. His argument is that the "place value approach" or the "cardinal approach" used by school mathematics programmes puts great demand on children's uniting operation (Steffe, 1994). According to his studies of children's number sequences, to see 11 as a unit composed of one ten and one more is the result of a progressive integration operation that might not be available to children when they are taught about tens and ones. This place value approach is supported by the use of written notation. Steffe's position seems to warn against it because young children's schemes cannot provide the knowledge required to comprehend written numeration. This position is clarified if one thinks that what children do when they write number scripts cannot be seen as annotating "numbers", in the sense of composite units. Of course one knows that children write and speak about numbers and that number scripts and number words can be reflected upon to organise cultural representation systems. Now, the initial ideas that children elaborate upon these systems of signs may not be considered "mathematical knowledge". It is the purpose of this study, to address how the abstractions on the external system of signs may be integrated with children's mathematical knowledge. The focus on children's construction of written and spoken numeration aims to be related to their arithmetic knowledge. The suspicion is that these two kinds of constructions, although initially separate, eventually integrate to empower children's mathematics.

3. 5 Summary

This chapter has unfolded a constructivist theoretical framework. With the question of how children's initial theories of number words and scripts impinge upon their arithmetic progress, this study will undertake two lines of investigation. The first line will interpret children's arithmetic progress during a one-year long period and the second line will explore children's knowledge of written and spoken numeration during this same period. Whereas the first line of investigation in based on the model of learning stages to organise observations, the second line is of an exploratory nature and rests on previously documented findings. Based on these findings, the aim is to isolate children's ideas on number words and scripts and ascertain how their system of ideas may influence their mathematical sophistication. Chapter 4 addresses methodological considerations relevant to these two lines of investigation.

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4. Methodological Considerations

4.1 Introduction

Having outlined the purpose of the study and its theoretical foundations, the present chapter unfolds methodological aspects that justify the design of the study. It is organised in three sections as follows. The first section addresses general considerations of a constructivist methodology; the second section lays out the design of the study as a longitudinal teaching experiment with two stages of fieldwork; and finally the third section discusses particularities of the tasks developed to undertake the two lines of investigation. The chapter concludes with an outline of the subsequent Chapters 5 and 6 which present the findings of the present two-line investigation.

4.2 Preliminary Considerations

There are some general considerations that are relevant to the formulation of the present methodological framework. These concern the generation and nature of data in constructivist studies of young children. The sources of data in constructivist studies are principally the recordings of task-based interviews with children. The nature of these interviews derives from the Piagetian clinical method[1] but has been transformed by constructivist researchers in mathematics education to empower the role of the investigator using the Vygotskyan notion of zone of proximal development. This methodological framework has been termed "constructivist teaching experiment" (Steffe, 1991, 1992[a], 1995; Steffe and Cobb, 1983; Hunting, 1983, 1996; Davis and Hunting, 1991; Wright, 1988; von Glasersfeld, 1991[b], 1995[b]). It must be remarked, that although the researcher has a set of conjectures in mind about the particular piece of knowledge to be investigated, there is no purpose of "teaching" in the traditional sense. Rather, the research enterprise depends to a great extent on the ability to de-centre from one's own point of view and to try to see the perspective of the child. In the interview sessions or experiential encounters[2] there is typically a discussion centred upon a task or problem presented to the child. The tasks have been designed to

[1] Subsequently known as "critical exploration" (Inhelder et al., 1974)

[2] Steffe, 1988[a], also referred to as experimental encounters.

give the child every opportunity to display some form of behaviour from which one can infer their ideas, schemes or manner of thinking. Generally, there is a request for the child to explain their response or solution. Some interventions aim at bringing about reorganisation of children's thinking. In this sense, the reformulation of the methodology conceives a researcher who is a crucial participant in interactive communication with the child. The constructivist teaching experiment is not only an exploratory technique but also a means of influencing children's current conceptualisations. It follows that there is a constant dependency between the inferences made by the researcher to the responses of the child. This method of interviewing can be summarised as actively probing children as they reason about an act or an aspect of their environment. Children's solutions come to confirm or disprove one's assumptions or conjectures as well as constituting the basis on which further conjectures are formulated. The product of the researcher's inferences becomes a viable version or model of children's own ideas or thinking. The components of this model are precisely the regularities the researcher abstracts from the variability of children's responses. These abstractions require longitudinal observations to ascertain the validity of the inferences made. Precisely, to test the viability of one's conjectures about children's thinking, there is the need to create situations in which their hypothetical ways of thinking will produce certain observable behaviour (von Glasersfeld, 1991[b]).

This research methodology has been termed second-order model building to highlight the inferential nature of the theoretical accounts of children's knowledge[3]. An example of a second-order model is the model of learning stages[4] and the first line of this study will analyse children's solutions to counting and arithmetic tasks using this established model. In this sense, the model of learning stages will function as an organiser of data.

The second line of the study has no systematic model to organise the observations but relies on preliminary conjectures and assumptions. These assumptions are:

[3] The child's knowledge is considered a first order model and the researcher's model of the child's knowledge is a second order model (Steffe et al., 1983, Steffe, 1995, 1998). Refer to Glossary.
[4] Steffe and Cobb, 1988.

1. Children elaborate criteria to compare number scripts and to organise number words that they encounter in their every day environment.
2. If the series of number words and number scripts is used without curricular restrictions, children can display behaviour from which their current ideas about the organisation of number words and scripts can be inferred.

In the second line of investigation, which is framed by the above assumptions, there is a purpose of theory construction. The methodology of a teaching experiment is adapted from Steffe's (1991) and Steffe and Cobb (1983) but it does not - necessarily - pursue an account of children's mathematical knowledge. It does pursue an account of children's knowledge of their system of number words and scripts as they enter formal education. Two crucial points need to be made. First, the teaching experiment aims at giving an account of children's knowledge of the external system of number words and scripts. Second, it aims at articulating this account with findings stemming from the first line of investigation framed by the model of learning stages.

According to Steffe (1995), there are three aspects of a second-order model:

1. The model must exist in a living, experiential form.
2. There must be an abstracted form of the model that explains certain crucial aspects of the living experience (e.g. the child's knowledge and the child's modification of that knowledge)
3. The model is a model of the epistemic child, not of particular individual children[5]

A longitudinal constructivist investigation

To recapitulate, the basic features of a constructivist teaching experiment are twofold: 1. It involves a longitudinal investigation, and 2. It requires intensive interaction with children (e.g. Sinclair, 1990; Steffe, 1991, 1994). Both elements respond to the need to formulate inferences about children's thinking and to validate these inferences in interaction with children. This follows from the view that children's ideas are not

[5] In this thesis, a child is a social subject who has an epistemic relation to the system of signs of their culture -the system of spoken and written numeration.

static and to study their relative stability and transformation one needs to make educated guesses that are to be confirmed in intensive and extensive interaction with children. Although these inference always remain as conjectures, after sufficient time has been spent with the child and after a sufficient variety of situations have been explored, it is possible to determine whether a particular advance has taken place (von Glasersfeld, 1995[c]). Therefore, longitudinal teaching experiments implicate the close study of a small number of cases during a long-term study[6].

For researchers who work in a constructivist framework there is the expectation of the occurrence of certain regularities in the child's responses or solutions. These recurring patterns are the material for building second-order models. However, the existence of the hypothetical states, ideas, or schemes of children should be seen as subject to the observer's frame of reference as well as the frame of reference of the observed (Steffe, 1995). The researcher in a teaching experiment has the major interest of hypothesising about children's current elaborations on a certain piece of knowledge - i.e. the system of number words and scripts - and of fostering further conceptualisation of these initial elaborations. For the study of the modification of current hypothetical ideas or schemes, the methodology aims at provoking hypothetical instances of tension, discomfort, or perturbation. Precisely, this methodology is directed towards understanding the progress children make over an extended period of time. The interventions may take the form of critical questions or suggestions in crucial moments of the session. However, for any possibility for facilitation of conflict, there has to be hypothetical stable schemes or ideas of children which are contradictory from the researcher's point of view. Hence, first there is the need to establish the nature and stability of children's ideas, before subsequently being able to ponder scenarios in which children might consider them inconsistent.

The need to adopt the child viewpoint can place a great demand on the researcher's role and hence, previous planning and cautious scripting of the interview session is considered a fundamental feature of the methodology (Davis and Hunting, 1991).

[6] A long-term study an be from 6 weeks to 2 years (Steffe and Cobb, 1983)

This point is to be discussed in the third section of this chapter which exposes essential characteristics of the tasks designed to undertake the two lines of study.

The methodology of the teaching experiment was used during the study to accomplish the following purposes:

1. Development of tasks.
2. Identification and classification of behavioural strategies exhibited by children which allow the inference of the child's stable theories.
3. Selection of children whose hypothetical ideas are of a different type, who have or have not constructed certain theories and who are representative of a wider group of children.

The following section addresses more closely the design of the study and the particularities of the fieldwork.

4. 3 The Study

The design of this study can be characterised as a one-year-long teaching experiment with two stages of fieldwork separated by a period of preliminary analysis. Nine children were selected from a Reception class of a primary school in the South of England to develop the case studies involved. The School was located in an urban area, in the vicinity of a university main campus. It had one class for each year of primary education including a reception class and it was ranked at a high position according to the government's inspection of schools[7]. Children were of different cultural origins, considering the religion and language of their parents. Children's ages ranged from 4 years-8 months to 5 years-7 months at the start of the study.

The first stage of fieldwork took place in the Summer term of 1998 (April-July 1998), that is, the last term of children's Reception Year. The second and final stage of fieldwork took place after the school's Summer holiday, when the children were in the second term of Year 1 (January-March 1999). The period of five months that separated the two stages of fieldwork comprised a phase of preliminary analysis of the

[7] OFSTED inspections.

data gathered in the first stage and the preparation and planning of the interview sessions of the second stage of fieldwork. This period lasted during both children's Summer holidays and their first term in Year 1.

A series of task-based interviews with children were video taped and a total of approximately 25 tapes and their subsequent transcription constitute the raw data of the study. It must be noted that although the thesis involves nine case studies, the development of these cases followed a two-phase selection. Before the final selection of 9 cases, interviews were held with 35, and subsequently 14, children. This process of selection will be discussed in more detail below. What is important to remark now is that the video recordings were initially of 35 children. Subsequently, in the last part of the first stage of fieldwork, only 14 children were targeted. Finally, during the second stage of fieldwork, only the 9 case studies were targeted.

According to the nature of the study, each stage of fieldwork comprised two lines of investigation. An overview of the study can be organised in three phases depicted in the following table:

Design of Study

PHASE	FIRST LINE OF INVESTIGATION: Children's arithmetic	SECOND LINE OF INVESTIGATION: Children's knowledge of external signs
First stage of fieldwork (April-July 1998) Last term of Reception Year.	• Initial assessment of children's counting. • Tasks designed to produce a first classification. • First selection of 14 children	• Exploratory phase • Tasks designed to establish patterns of behaviour that could allow the inference of children's ideas on number words and scripts.
Preliminary Analysis (August-December 1998) Summer holidays and First term of Year 1.	• Refinement of the classification of children's learning stages based on the initial assessment of children's counting. • Final selection of 9 case studies to follow in the second stage of fieldwork.	• Organisation and analysis of data, elaboration of hypotheses of children's criteria and ideas. • Preparation of tasks to validate this conjectures. • Preliminary conjectures to connect the two lines of investigation.
Second stage of fieldwork (January-March 1999) Second term of Year 2	• Final assessment of children's learning stage with special focus on their thinking strategies.	• Validation phase with focus on the viability of conjectures and possibility of observing modifications.

4. 3. 1 Sources of data

There are two sources of data for the present study. First, and foremost, are the video recordings of task-based interview sessions which were held outside the children's classroom in a separate room of the School. Tasks or scenarios were planned and scripted, and interview sessions based on them were video taped for subsequent viewing and transcription. Segments of these transcriptions were selected to illustrate claims of observable patterns that suggested children's hypothetical thinking. The protocols or transcripts presented in the analysis of findings are generally termed "episodes" or "research episodes". It must be noticed that whereas interview sessions or research sessions are the actual events in time and space, episodes are fragments of transcriptions selected for illustration of the analysis.

According to the nature of the two lines of investigation, task-based interviews were planned according to their objectives. For example, in order to assess children's counting schemes, the interviews involved a series of situations adapted from previous studies of children's counting guided by the model of learning stages[8]. On the other hand, to explore children's ideas on number words and scripts, a scenario where these signs might play a role had to be developed and a set of possible questions had to be carefully planned. The particularities of the interviews planned for each line of investigation are addressed in the following section. What is relevant to remark at this point is that the interviews were planned either for an individual child, as one-to-one interaction with the researcher, or as interviews with pairs of children to promote conversation between the interviewees guided by prompts of the researcher. Although some interviews of the first stage of fieldwork were undertaken with pairs of children, the case studies were of particular children. Another point to be remarked is that, although a given interview session was planned for a particular line of investigation, this did not rule out considering data generated by this interview in the other line of investigation. An example that is pertinent to consider is the Dice-game interview which was planned for the second line of investigation. Two children had to annotate the scorings of a dice game which consisted in throwing the dice in turns and totalling

[8] e.g. Steffe and Cobb, 1988[b], 1992[a,b], 1994; Wright, 1988, 1990, 1991[b]; Wiegel, 1998.

the points obtained until certain given score was reached - e.g. 20 -. The aim was to focus on children's writing of numerals as a purposeful activity to record quantities and to find themes of discussion about the number scripts written or yet to be written. However, children's ways of interpreting and playing the game - i.e. their methods for totalling the points obtained - could also be taken as indication of their counting schemes - a focus of the first line of investigation - although this interview was not designed for that specific purpose.

The second source of data was a diary of fieldwork. A diary of fieldwork was kept from April 1998 to April 1999 to annotate classroom activities, informal interactions[9] with the children of the study and written notes related to the research sessions. Each annotation was dated and contextualised. The diary of fieldwork had a twofold purpose. First, to keep a record of specific comments, notes, or observations of the researcher, during or after an interview with a particular child took place. These annotations could relate to a change in children's disposition, a change of space where the session took place, or any significant feature that could aid the interpretation of data generated in a particular session. The second purpose was to document what activities children were doing in the classroom with their teacher or teachers. By the same token, the dynamics of the mathematics classroom was documented. This involved documenting the manner in which children worked - e.g. as a whole discussion led by the teacher or in groups. Participant observations[10] of mathematics classes were undertaken by the researcher so there was always a record of the activities that children were doing at school. In particular, during the first term of Year 1, the researcher systematically participated in mathematics classrooms until the second stage of fieldwork started in the second term. This avoided the problem of losing familiarisation with the children of the study as well as providing a source of information for the activities children were doing in school. This latter function of the diary of fieldwork was sometimes crucial to the interpretation of data generated in the interview sessions. For example, the diary of fieldwork documented that until the end of Reception class children worked with numbers up to 10, although for some

[9] i.e. Outside the interview sessions.

[10] In the sense that the researcher participated in parts of the activities, specifically when children were working in groups.

advanced children special workbooks were prepared with sums up to 20. By the same token, it was documented that in December 1998, at the end of the first term of Year 1, children were told about "tens and units" and the first activities related to tens and units were incorporated in the classroom work. The diary of fieldwork was a valuable source of information to analyse the data generated in the interview sessions because it documented informal interactions with children of the study and examples of activities they solved in class. The diary of fieldwork offered a space for documentation of relevant aspects of the classroom work both during Reception Year and Year 1. For instance, children had two teachers in Reception class whereas in Year 1 they had only one. A relevant aspect of school work that changed in Year 1 was the placement of children in level-groups according to their level of performance in a particular subject - e.g. reading, writing, mathematics. However, it was documented that the aim of the teacher was to propose all and every activity to the whole class. The researcher participated in regular classroom activities, particularly in periods where data was not being collected. Often, the researcher had opportunity to interact with a particular child who was one of the case studies. These interactions and relevant comments documented in the diary confirmed assumptions and conjectures stemming from the analysis of the interview sessions. An example that is pertinent to consider was an observation of one of the activities proposed by the teacher to one of the figurative counters of the study at the time of the observation. The child was asked to use coins of 1p, 2p, and 5p, to "make 10p". On no occasion did the child count on from the first number to complete the sum to 10. Her solution was at each new combination, putting up from 1 as many fingers indicated by the coins selected. Hence, she chose 2p and put 2 fingers up, then 2p and put up 2 more fingers, and so on. This observation in class with this particular child, confirmed the interpretation of data generated in the interview sessions. Informal conversations with the teacher were the source of information on what activities had been planned for a particular week. These were also documented in the diary of fieldwork with the purpose that has been previously discussed.

Some relevant considerations

The above mentioned three phases of the study - i.e. first stage of fieldwork, preliminary analysis, and second stage of fieldwork - suggest a clear distinction of empirical and theoretical work (Steffe and Cobb, 1983). However, it must be noticed that on-going analysis framed by current conjectures was part of each stage of data collection. This was particularly the case in the preparation of subsequent tasks during the year long study. The task-based interviews were not planned a priori but during the investigation. In particular, during the exploratory phase of the second line of investigation, tasks were developed according to data generated by previous interviews as well an on-going analysis of these data. In the second stage of fieldwork, the planning ahead of the tasks was a possibility and it resulted from the phase of preliminary that separated the two stages of data collection. The simultaneity of empirical and theoretical work is somehow lost in the linear presentation of the thesis, however, it is pertinent to make the point. As a consequence of this on-going planning of tasks based on previously collected data, there were some weeks of suspension of data collection within the first stage of fieldwork. In particular, after the initial interviews of the 35 children of the class, there was a period of suspension of data collection. This was preceded and followed by weeks of intensive work of data collection - i.e. everyday interview sessions. After the final selection of case studies was completed, in the second stage of fieldwork, work with the nine children was intensive and data was generated for only the nine case studies. This means that the average frequency with which the researcher worked with the nine children of the study varied from the first to the second stage of the fieldwork.

Another point worth discussing in more detail is the rationale for undertaking the investigation with children from Reception class. As has been anticipated in the previous chapters, the aim of this study is to chart the knowledge that young children appropriate from their everyday experiences and this involves informal -pre-school- ideas. Appropriation of knowledge is here understood as constructive organisation of one's own experience, but an experience of a social object in use: the numeration system in its written and spoken forms. Children had to be young enough to investigate their informal knowledge[11] but old enough to have participated in situations where external representation of numeration played a role and where adults or older children commented on these external signs. Children of Reception class seemed to be the appropriate subjects for the investigation with the provision that the study commenced in the last term of Reception class and was planned to follow the case studies into their first year of primary school. As for the assessment of their progress in arithmetic knowledge, the model of learning stages has been used in previous research for longitudinal analysis of pre-school children mathematical development (Wright, 1990, 1991[b]). Therefore, the two lines of investigation were undertaken with a group of children of a Reception class who were followed into Year 1 of primary education.

A final remark concerns the new hypothetical constructions that were presumed to be observed in the second stage of fieldwork - when children were in Year 1. These were a product of children's interactions and reflections on their in and out of school experiences with written and spoken numeration. This was also the case for their reorganisation of their arithmetic knowledge and it included children's experiences during the research sessions. However, in relation to the second line of investigation, school experiences were not likely to involve situations with number scripts and words higher than 20 albeit for some children of the study (the most advanced ones) classroom work could involve numbers up to 100. Therefore, it can always be sustained that children's hypothetical elaborations were a product of their interaction with a system of external conventional signs used in their out-of-school everyday physical and social environments. Certainly, this was a reasonable supposition for the first stage of fieldwork - when children were in Reception class - since they worked only with number marks and words up to 10.

4. 3. 2 Selection of the case studies

The nine case studies were selected in two phases. In the first stage of fieldwork (April-July 1998), an initial counting assessment and general exploration of children's ideas on number words and scripts were undertaken with the whole class of 35

[11] Formal instruction involving "place value activities" commences in Year 1 with two digit numbers.

children. These initial interview sessions allowed the selection of 14 children from the wider group with whom further and extended work took place during the first stage of fieldwork. The criteria for the first selection targeted children who:

1. Were keen to work with the researcher.
2. Were reasonably keen to work with "numbers" they did not work with at school.
3. Were from a spectrum of the classification according to the model of learning stages.

The first two points were basic requirements of the methodology. First, it must be noted, all children were generally motivated to work with the researcher. This was shown by their alertness and excitement in the moments when the researcher was to choose who was going to work with her. Nevertheless, some children were not selected because they appeared timid during the interview session in the moments when verbal justification was required[12]. Second and in connection with the previous point, some children were keen to solve the counting problems but when asked about large number words and scripts they expressed that they did not know those numbers because they only knew up to 10 in school. Children who did not turn timid at this point were preferred. Finally, there was an initial assessment of counting which allowed a classification of children in three groups according to the first three learning stages, including two transitional cases. Six children were perceptual counters, with two in the transition to a figurative scheme; five children were figurative counters; and finally, three children were abstract counters. The following table distinguishes the fourteen cases after the first selection of children.

Selection of 14 children

PERCEPTUAL	FIGURATIVE	ABSTRACT
Eloise (5-6) [13]	Alice (5-0)	Joe (5-4)
Eleanor W (4-8)	Jack (5-6)	Johnny (5-7)
Eleanor D (4-11)	Eamon (5-6)	Tom (5-7)
Duncan (4-11)	Sam (5-2)	
Becky (5-1) →	Hettie (5-6)	
Stephanie (4-11) →		

→ = Transition P→F

[12] This criterion did not rule out cases which were considered valuable according to more powerful criteria. For example, Stephanie, a rather shy girl, was nevertheless chosen because of her place in the classification according to the two lines of investigation.

[13] Age is given in years and months.

The subsequent and final selection of the nine case studies took place after the first stage of fieldwork was completed and during the period of preliminary analysis that separated the first and the second stages of fieldwork (August-December 1998). In addition to the general criteria used in the first selection, the criteria that justify the final selection were tightly related to the two lines of investigation. From the group of fourteen children, nine were chosen according to two basic points:

1. Children were from a spectrum of classification in relation to having or not established certain ideas in the domain of number words or number scripts.
2. Children were from a spectrum of classification in relation to the model of learning stages.

For the final selection of case studies substantial information had been gathered in relation to the second line of investigation. Criteria to maintain or drop cases were tightly connected with the findings and analysis of the data concerning children's ideas about number words and scripts. This will be the subject of the subsequent chapters. Although the final selection was primarily based on findings related to the second line of investigation, the refined classification of the case studies at the end of the Summer term of 1998 according to the model of learning stages was the other essential component of the selection. In brief, nine children were chosen to develop the case studies in the second stage of the fieldwork. According to the classification of children of the initial counting assessment, the following table presents the case studies that were followed during the one-year-long study from April 1998 to March 1999.

Selection of 9 case studies

PERCEPTUAL	FIGURATIVE	ABSTRACT
Eloise (5-6) [14]	Alice (5-0)	Johnny (5-7)
Eleanor W (4-8)	Jack (5-6)→	Tom (5-7)
Eleanor D (4-11)	Eamon (5-6) →	
Stephanie (4-11) →		

→ = Transition P→F or F→ INS

[14] Age is given in years and months.

4. 4 The Task-based Interview Sessions

The task-based interviews developed during the study corresponded to the objectives of the two lines of investigation. On the one hand, the first line of investigation aimed at interpreting children's knowledge of arithmetic and their progress during the year long study. On the other hand, the second line of the investigation aimed at mapping out children's stable ideas on number scripts and words. Hence, there were specific tasks for each line of investigation. Nevertheless, as it has been already pointed out, data generated for a particular line of investigation could be interpreted in the light of the other line of investigation. Before going into the particularities of the different types of tasks, it is worth addressing some general considerations.

The video recordings of the interview sessions functioned as a witness of the interaction with children and allowed reconstruction of the sessions over the length of the study. All interview sessions were planned to last for approximately 25 minutes but the duration of the sessions depended on the child's engagement with the task and their general will to stay and work with the researcher. Thus, the actual time spent with a child or a pair of children in each session varied from 25 minutes to 50 minutes. Some interviews comprised two sessions, particularly when the situation involved completing a writing task and discussion that emerged during the session delayed the completion of the task (e.g. Board of 100 seats). Splitting the session in two parts was sometimes planned because the research work had to adjust to the School timetable.

The other general issue of the interview sessions was the problem of power imbalance[15] in the researcher-child interaction. Certain norms of classroom work assumed by children had to be detected and renegotiated with them. The child could transfer their idea of the nature of teacher-pupil relationship to the researcher-child interaction. This was particularly relevant if one considers that children may look for cues, and seek to find hints in the adult's behaviour to show them that they are right or wrong as they are solving a given situation. Constant checking of the level of engagement and understanding of the child was ensured and the view of the

[15] Hunting, 1996.

communication with the child was that of a negotiation of meaning during the sessions. Particularly relevant was to note that - specifically during Year 1- the dynamic of classroom work did not form the children into constant discussion, explaining one's point of view and listening to the other's explanation. This partly discouraged the design of subsequent interview sessions with pairs of children, although the decision to undertake one-to-one sessions in the second stage of fieldwork was based on the close study of the cases selected. This particular point will be discussed further below in the subsection that addresses the tasks developed for the second line of investigation.

Related to the question of power imbalance was the need to develop techniques to ascertain how fragile or robust children's hypothetical ideas were. This was done by constant questioning or probing in the same situation or in a variety of contexts. The technique of requestioning the child or using counter suggestions even when the child was right, was adjusted in interview session of both lines of investigation. However, it was particularly relevant in the case of the second line of investigation where the aim was to chart children's original ideas.

In the presentation of the findings - which are the subject of the two subsequent chapters - a brief description of a given task always precedes the presentation of an episode which is the focus of analysis. The full scripts for all the interview sessions undertaken during the study are presented in the Appendix. The following subsections address the particularities of the tasks developed for each line of investigation.

4. 4. 1 Tasks developed for the first line of investigation

The theory of learning stages formulated in the previous chapter was the framework that guided the planning, conducting, and analysis of the interview sessions of the first line of investigation. Children were presented with a particular task, but what counted for analysis was the interpretation of the child. What was focused on was not the range of the child's success per se but the form in which the child interpreted and attempted to solve the tasks presented. Although difficult to transcribe, facial expressions, a pause, or particular movements of a child were of paramount importance to the researcher's interpretation. Thus, for the counting tasks in the first line of investigation, body movements and gestures during the actual solution, were particularly crucial to the observation apart from the previous or posterior comments or justification that the child offered.

Steffe and co-researchers[16] have repeatedly indicated that a classification of a child's solution has to respond to stable patterns of behaviour observed in a variety of tasks. This is particularly true when the enterprise is to build a model. However, the view held in the first line of investigation was that of the use of an established model of children's arithmetic. In this sense, the model of learning stages was used as an organiser of data and a cautiously designed range of tasks was used to produce the classification. This view is supported by findings of previous research which used similar tasks for classification[17]. However, no task by itself could serve as a rigid criterion for a particular classification.

Therefore, for the first line of investigation, the design of tasks were adaptations from those used in previous investigations[18] and the analysis relied on the comparison of original and subsequent formulations of the model of learning stages and constant revisiting of documented classifications. Particular importance was given to behavioural indicators and their interpretation in previous research using the model, as well as revisiting the video recordings in the light of new insights or ambiguities. This is partly the reason why transitional phases were included in the formulation of the model of learning stages presented in the previous chapter. As was previously mentioned, behavioural indicators were essential to handle the difficulties of the process of classification. It must be noticed that, initially, the formulation of the model focused on the perceptual, figurative and abstract stage - i.e. without the distinction of number sequences[19]. This was pertinent, given the age of the children at the start of the investigation. The expectation was that the majority of the children

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[16] Steffe et al. (1983)
[17] e.g. Wright, 1988, 1990, 1991[a, b]; Wiegel, 1998; Cobb and Wheatley, 1988; Pearn and Merrifield, 1998; Biddlecomb, 1994; Pepper and Hunting, 1998.
[18] This include original investigations (Steffe et al, 1983; Steffe and Cobb, 1988) and subsequent investigations (see previous note)
[19] The number sequence stages are: INS, TNS, and ENS.

were going to be at the perceptual or figurative stage. However, as the investigation refined the analysis, there was the need not only to incorporate the subsequent stages but also to delineate transitional phases between stages to distinguish subtle differences in the children's organisation of counting and initial knowledge of arithmetic.

Features of children's language use were also contemplated. For example, explanations that the children gave were carefully distinguished in relation to the statements used. The expression "I counted on" was compared to "I added". If patterns of specific word used were found at different moments, these were detected and attempts to explain them were explored.

The tasks which focused on children's progress in arithmetic emerging from their counting activity were presented at three moments of the longitudinal study. A series of counting tasks and simple arithmetic situations was presented in each of the following sessions:

1. Initial counting assessment (At the start of the study)
2. End of Summer term assessment (At the end of the first stage of fieldwork)
3. Final counting assessment, including the Rows and squares interview (In the second stage of fieldwork)

In the three series of tasks, a set of situations was presented to the child ranging from counting problems to simple arithmetic sums presented in written form. All interview sessions planned for the first line of investigation were carried out on a one-to-one basis with the child. Finally, it must be remarked that data gathered in interview sessions of the second line of investigation were also taken into consideration for the interpretation of children's progress in their arithmetic knowledge. This is a relevant point because written sums that incorporated number words and scripts in the range of hundreds or thousands provided another insight into children's arithmetic knowledge. Previous studies framed by the model of learning stages document tasks and solutions involving two digit numbers only.

4. 4. 2 Tasks developed for the second line of investigation

Unlike the first line of investigation, which was based on a viable model to interpret children's progress in arithmetic, there was no framework available to interpret children's progress in their knowledge of the system of number scripts and number words. The elaboration of an account of how this appropriation of external signs occurred - for the children of the study - was one of the major aims of the thesis. It follows that tasks of the second line of investigation had to be developed as exploratory means to establish the grounds for further study. It must be pointed out, once again, that the two basic assumptions that framed the design of the initial interviews were:

1. Children build criteria to interpret and use number scripts and words they encounter in their every day environment despite not grasping the mathematical organisation of the number system.
2. Children can display behaviour from which their current ideas about the organisation of number words and scripts can be inferred, particularly if the series of number words and number scripts is put into play in all possible extension.

With these two basic assumptions, scenarios were developed as an excuse to use number scripts and number words in specific contexts. To use number scripts and number words means to produce and interpret these signs, to establish comparisons among them and eventually to rely on them to solve arithmetic problems. Scenarios designed to frame a given interview session modelled a pretended situation in which writing numbers had a purpose of enumerating items, recording quantities, or annotating sums. The aim was to promote discussion about the number scripts and words produced. It must be noticed that often children engaged in conversation about number scripts and words without the need of a context or scenario. Therefore interview sessions were planned which involved writing number scripts by dictation, discussing these scripts, putting number scripts in order and other similar tasks. An example of these tasks was a simple dictation of numbers and a subsequent discussion about the scripts produced. By way of contrast, an example of a context or scenario was a dice game where children had to annotate their scorings. Writing numbers had a purpose in the activity - i.e. recording of points - but what was aimed at was themes of discussion about the number scripts produced. In brief, the aim was to put children's conjectures into play. In the case of number scripts, the child was asked to solve a task involving the production, or interpretation of number scripts and was asked to justify their solution or production. In the case of number words, children were asked to compare given numbers, put them in order, and explain the reason for their response. Their verbal justification, together with the actual solution or production, counted as the source for inferring children's current ideas on number words and scripts. As has been discussed in the previous chapter, the study had the premise that written numeration is a carrier of mathematical meaning - e.g. numbers, order relation, arithmetic operations - and therefore ordering and comparing and, inevitably, operating were unavoidable aspects of the use of number scripts. Hence, written productions, interpretations and children's solution to tasks, as well as prior comments and posterior justification were the key to the analysis. For this line of investigation it was also imperative to revisit the video recordings and notes from the diary of fieldwork. In addition, revisiting children's written productions in the light of new questions was also crucial for the analysis. The scenarios presented during the study, covered a wide range of situations in which number words and scripts were used by children. Children's responses were expected to be task-sensitive but the aim was to look for patterns of behaviour which could be considered stable across situations. This was particularly true for the second line of investigation.

In accordance with previous research[20], tasks were designed for which reading aloud conventionally was not a necessary pre-requisite since the objective was to bring to light the strategies and hypotheses which children construct in their early attempts to understand number scripts. To ask young children to write or read number scripts other than the first ten of the number series meant to conceive children's unorthodox productions as a window to their ways of thinking about these external signs. Errors from the point of view of an observer needed to be interpreted according to the viewpoint of the child, particularly when errors seemed to be systematic.

[20] Sinclair and Scheuer, 1993; Lerner and Sadovsky, 1995.

It has been said that written and spoken numeration are seen as conventional external systems of signs which are appropriated in interaction with others in purposeful culturally organised practices. Less has been said about the argument according to which ideas that children hold about numeration may be implicit[21]. According to this argument, children's conjectures about these external signs must be brought into play if one's aim is to study them and map them out. As this point is considered at the moment of task planning, one must ponder another documented finding related to pre-school children's aptitudes or abilities as they enter formal education. Research has suggested that young children's abilities - e.g. counting, reciting the number sequence - depend on the extent to which they have come across opportunities to participate in certain activities[22]. Therefore, one must expect diversity when studying what young pre-school children know about number words and scripts. The important point, however, is to establish regularities in this diversity. As has been previously discussed, the aim to chart children's ideas of these systems and how these ideas transform over a prolonged period of time, implies the fundamental importance of theory construction.

The problem that the researcher may face when children look for cues in the adult's behaviour has already been discussed. The interview sessions were held with the aim of constantly probing the child's responses in order to ascertain the extent to which children were engaged in a particular situation. Similarly, probing was ensured in order to ascertain the nature of the situation or problem they seemed engaged with. One form of probing was to provide counter suggestions after a child had answered and justified a certain task irrespective of whether the answer was right or wrong from an observer's point of view. Another technique consisted in asking the child to pretend they had to explain what they knew, or how they solved a situation, to another child[23]. This technique had a twofold purpose. First, it was a window to the child's

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[21] In the sense of Lerner and Savosky (1995): knowledge that is put into play in the form of viable theories; or in the sense of Ferreiro (1994), knowledge that a children bring with them to formal instruction.
[22] For example, Fuson (1988) suggests that the acquisition of the number sequence in young children depends on the opportunities that children have to practice the sequence at home.
[23] Teach back technique discussed by Ackermann (1995), Kamii (1985), and Hughes (1986)depends on the opportunities that children have to practice the sequence at home.

current manner of thinking and second, it provided opportunities to the child to reflect on their own thinking. In the former sense, Hughes (1986) has argued that,

> "children seem to be more expressive and articulate in a role whereby they imagine that they are helping someone smaller and less knowledgeable than themselves." (Hughes, 1986, p. 104)

In the latter sense, Kamii (1985) has suggested that,

> "children sometimes correct themselves when they try to explain their reasoning to someone else. In the context of trying to make sense to another person, children often make relationships at a higher level than before." (Kamii, 1985, p. 110).

Both senses are contained in von Glasersfeld's statement:

> "To verbalise what one is doing ensures that one is examining it. And it is precisely during such examination of mental operating that insufficiencies, contradictions, or irrelevancies are likely to be spotted." (von Glasersfeld, 1991[b], p. xviii)

Whereas Kamii and Hughes refer their arguments to the study of young children, von Glasersfeld is concerned with a general epistemological position[24]. The important point to make in this context is that the present study pondered the turning point in children's development of logic approximately at age 7 (Kamii, 1985) that yields a fundamental change in children's reasoning. On the other hand, children as young as 4 find that their judgements can be contradicted by others (Inhelder et al., 1974). This is a relevant point to consider, in relation to children's ability to consider two points of view or ideas and put them into a relationship. Further, this consideration is tightly related to the possibility of classroom norms which involve the activities of discussion and exchange of viewpoints as a manner of working. Explaining one's point of view and listening to the other's, seemed not be a norm in the children's regular classroom. In this study, however, this manner of working was negotiated during the interview sessions that were conducted with pairs of children. When the interview sessions were on a one-to-one basis with the child, discussion was generated by the researcher only and it depended on the responses of the child. In these sessions the problem of power imbalance was of particular concern because children were aware that the researcher "knew" how to read and write numbers. Hence, answering questions such as "is this how you write a hundred?" was a decision to be made according to the particular problem in hand. Generally, when this kind of situation emerged, children were encouraged to find out for themselves in a social carrier of numerical print[25].

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[24] i.e. Constructivism.
[25] e.g. Rulers, books, calendars.

This "devolution" technique had a constructivist rationale. As previous literature has suggested, intellectual autonomy ranks as one of the major goals of development (e.g. Kamii, 1985). In turn, this technique was a means to explore the strategies that were available to children to "find" a number script in a given social carrier. A final probing technique was the questioning routine developed to explore whether a child had been told a particular idea or whether the idea was of their own. After a statement of justification, the child was simply asked whether they had found out "that" for themselves or whether someone had told them. This technique included questions like: "Did you figure it out or did somebody tell you that?" or "Who told you that?". In occasions the child would respond in a straight forward manner (e.g. "My mummy told me") or would look puzzled but subsequently respond that they had figured "that" out on their own. For example, this technique was extremely valuable for systematically appearing statements of the form: "ten times ten is a hundred" in the second stage of fieldwork.

Children's theories are the affirmations or regularities children establish to organise their experience. If these theories enter in contradiction children may modify their current, no longer viable, theories. However, as was previously discussed, for perturbations to occur, theories or ideas need to be established. Therefore, the fundamental objective of the second line of investigation was to map out these theories. One could not anticipate whether during the length of the study situations of tension or conflict would arise. Hence, to design situations in which established theories could enter in contradiction was a subsidiary aim of the study. Moreover, according to the framework, modifications in children's ideas by reflective abstraction and states of perturbation cannot always be "observed". However, there are observations from which one can infer those modifications have occurred although one may not be able to establish when they actually happened.

The interview sessions for the second line of investigation can be organised in three moments as follow:

1. Sessions undertaken before the first selection of children in the first part of the first stage of fieldwork.

2. Sessions undertaken after the first selection of 14 children in the last part of the first stage of fieldwork.
3. Sessions undertaken in the second stage of fieldwork.

The interviews of the second line of investigation that took place between April and July 1998 - i.e. first stage of fieldwork - were predominantly planned and conducted with pairs of children. Children were matched according to their same type of counting according to a preliminary analysis of the initial assessment[26]. Pairs were not always maintained in subsequent sessions due to unplanned school-absences. The rationale for undertaking interviews in pairs has already been discussed in relation to the aim of discussion and consideration of conflicting points of view. In the first stage of fieldwork, this seemed the appropriate form of interview to undertake the exploration of children's ideas. In the second stage of fieldwork, all interview sessions for both lines of investigation were conducted individually. As has already been mentioned, the decision was principally related to the development of the nine case studies. The child's interlocutor was now the researcher and the emergence of discussion was covered by her.

Interviews planned for the second stage of fieldwork had the major goal of validating conjectures stemming from the preliminary analysis of data. Scenarios presented to children could be the context for different task-based interviews. For example, a scenario of a cinema auditorium that was designed for the first stage of fieldwork was used in the second stage of fieldwork for a new interview session. Relations that children established between the two interviews, their comments or justifications were the base for further insights into children's ideas.

Before presenting the overview of the fieldwork including all interview sessions conducted during the study, it is worth remarking that specific interview sessions were commonly sources of data for both lines of investigation. For example, the interview "Laptop sums" was both presenting arithmetic problems as well as involving the use of external signs. Children's solutions and explanations in relation to how they solved

[26] This was undertaken in the Initial assessment interview session, which was conducted with individual children.

a particular task, focused not only on their methods - from which one could infer their learning stage - but also on their theories upon number scripts and words and the form of organisation they may have established for those systems of representation. The following table presents an overview of all interview sessions conducted during the one year long study.

Overview of interview sessions of the study

FIRST STAGE OF FIELDWORK (March-July 1998)			
Reception Class: 35 children			
• Diary of fieldwork (First stage): March 1998-July 1998			
Interview session	**Line of investigation**	**Type of interaction**	**Date**
• Initial counting assessment	First line	Individual	April-May 1998
• Numbers at home and outside school	Second line	In pairs	May 1998
• Cinema scenario 1	Second line	In pairs	June 1998
• Dictation of number scripts	Second line	In pairs	June 1998
FIRST SELECTION OF 14 CHILDREN			
• Dice game	Second line	In pairs	July 1998
• Board of 100 seats	Second line	In pairs	July 1998
• End of term counting assessment	First line	Individual	July 1998
• Number scripts	Second line	Individual	July 1998
SECOND STAGE OF FIELDWORK (Jan-Mar 1999)			
FINAL SELECTION OF THE 9 CASE STUDIES			
• Diary of fieldwork (Second stage): September 1998-March 1999			
• Number scripts	Second line	Individual	January 1999
• Cinema scenario 2	Second line	Individual	February 1999
• Cinema tickets	Second line	Individual	February 1999
• Final counting assessment	First line	Individual	February 1999
• Paper-clip boxes	Second line	Individual	February 1999
• Regularities in number words and scripts	Second line	Individual	March 1999
• Laptop sums	Second line/First line	Individual	March 1999
• Number in between	Second line	Individual	March 1999
• Rows and squares	First line	Individual	March 1999

As one can see in the table above, the Laptop sums interview was undertaken in March 1999, the last month of the study. This interview is placed in the second line of investigation but it was actually a source of data for both lines of investigation. Indeed, as the findings and discussion will elaborate in the following chapters, it was the aim of connecting both lines of investigation which guided the analysis of data generated during the last part of the study. The two lines of investigation yielded data which was addressed as analytically separate. However, in the second stage of the fieldwork, the aim was to find points of connection whereby findings related to children's progress in arithmetic could be integrated with findings stemming from the investigation of their knowledge of the external systems of representation. These will be discussed in the final part of the thesis.

In the presentation of the findings of the two lines of study, the analysis includes pertinent descriptions of the tasks when a given episode is used to illustrate the analysis. The full scripts for each interview session are presented in Appendix 2. However, as has already been discussed, the data include answers to questions which were not scripted a priori, but which emerged in the interviewing process.

4. 4. 3 Remarks on the analysis of data

The aim of the analysis was to find behavioural patterns that could be the base for inferring children's ideas on number scripts and number words. Conjectures are made upon these inferences and subsequent behaviour of children is thought to confirm or invalidate these inferences. As has been mentioned, this was mostly the case for the second line of investigation which had the aim of giving an account on how young children appropriate written and spoken systems of numeration. For the first line of the study, the aim was to interpret their arithmetic knowledge using an established model. The analysis of data was therefore of a different nature for each line of investigation. Whereas the analysis of children's counting can be characterised as an assessment for classification, the analysis of children's ideas on number scripts and number words can be characterised as an exploration with empirical and theoretical phases. In the latter case, the analysis relied on the viewing of the recordings, the scanning of transcriptions and constant search for stable patterns of behaviour across the situations presented. As has been previously discussed, there was a period of preliminary analysis of data collected in the first stage of fieldwork. This preliminary analysis was revised and reorganised for the writing of this thesis. The second and final analysis took place at the end of the fieldwork and during the writing process of the present thesis.

The findings

The following two chapters present the findings of the two lines of investigation. This form of presentation may not portray accurately the dynamic of the analysis which yielded the final version of the findings presented in this thesis. However, it is presumed, this is compensated for by the clarity of presentation of the arguments put forth. Hence, Chapter 5 presents the findings of the first stage of fieldwork according to the two lines of investigation. Chapter 6 presents the findings of the second stage of fieldwork and sets the grounds for the general discussion which is the subject of the final two chapters of the thesis.

4. 5 Summary

This chapter has discussed a general methodological framework for the present two-line longitudinal investigation. It has presented the design of two lines of study. Whereas the first line relies on an established model of children's arithmetic to interpret their progress during the year long study, the second line has the aim of studying children's ideas on the conventional systems of written and spoken numeration. In this sense, the second line of investigation is of an exploratory nature and its purpose is to produce and account of children's knowledge of these conventional systems as they grow more competent in arithmetic. The following two chapters present the findings of the two lines of study and set the grounds for a general discussion.

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5. Findings of the First Stage of the Fieldwork

5. 1 Introduction

This chapter presents the findings of the first stage of the fieldwork according to the two lines of investigation. It is organised in two parts followed by a general discussion. Part one presents the findings of the first line of investigation. The case studies were classified according to their learning stage at the commencement of the study and their modification during the first stage of the fieldwork. Part two presents the findings of the second line of investigation. The case studies were classified in three groups according to their ideas in the domain of number words and number scripts. The chapter concludes with a discussion on the findings of the first stage of fieldwork and the perspectives for undertaking the second and final stage of fieldwork. This discussion draws upon the classification of case studies both according to children's arithmetic and to their ideas on number scripts and words.

5. 2 Part One: An Analysis of Children's Counting.

5. 2. 1 Initial assessment of the case studies

The initial classification of the case studies is presented in the following table. Two children were classified as abstract counters. The other seven children were either figurative of perceptual counters.

Initial Counting Assessment-April 1998

Child	Type of unit	Learning stage
Johnny	abstract	INS
Tom	abstract	INS
Jack	figurative (verbal)	FS→INS
Eamon	figurative (verbal)	FS→INS
Alice	figurative (verbal)	FS
Stephanie	→figurative	→FS
Eloise	perceptual	PS
Eleanor D	perceptual	PS
Elearnor W	perceptual	PS

Three children were placed in transitional phases as can be observed in the table. For this reason, the following sections address the initial classification of the case studies in four groups: Johnny and Tom as the more advanced counters; Jack and Eamon as the counters on the transition to the INS; Alice and Stephanie as the figurative counters; and Eloise, Eleanor W and Eleanor D as the perceptual counters.

The more advanced counters: Johnny and Tom

Tom and Johnny had at least constructed the INS when the study commenced. They both seemed to count abstract units while counting. Although they were seen to create motor units when solving the tasks (e.g. movement of fingers), they used this material as a base for further operation. They used the operation of integration, an operation only possible at the stage of the INS. The transcripts below present their solution to a counting task. Tom and Johnny were presented with 6 bricks. After they counted the bricks, the interviewer placed 3 more bricks on the table and covered them with a box. The problem was to work out how many bricks were on the table in total.

T: (stares at the 6 visible cubes in silence for two seconds) "9!"
I: "How did you know so quickly?"
T: "'cause I counted 3 of them and I counted 6: 7, 8, 9"

J: (after two seconds staring ahead in concentration) "9"
I: "Good! How did you work that out?"
J: "In my head"
I: "How did you do it in your head?"
J: "Uhm....I counted"
I: "How did you count?"
J: "I counted 7, 8, 9"

Both children used a counting-on strategy to solve the task with apparent ease. This behaviour indicated that each number word (e.g. "6") signified the segment from 1 to that number word and there was no need for them to re count all the bricks from 1. Confirmation for this inference was found during other solutions to the counting tasks, for example, when the children had to find out how many from 10 bricks had been hidden. The following transcript presents Johnny's solution to this task.

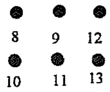
J: (counts the 6 visible bricks and proceeds to count on subvocally synchronous with putting 4 fingers up, then he grabs the 4 fingers together with his other hand) "4!"
I: "How did you do that?"
J: "I counted in my head, in my head"
I: "Can you count out loud for me?"
J: "7, 8, 9, 10"

Johnny seemed to anticipate that he had to keep track of his counting acts in order to find the answer to the problem. He created motor units (i.e. putting up fingers) but he subsequently run through these to unite them in thought, he looked at his four fingers (result of his activity) and used the integration operation. Tom's solution is below.

T: (promptly) "4"
I: "How do you know?"
T: (upset) "Let me think again...(he looks down to the table for a second), 2!" [I uncovers the 4 cubes] (disgruntled) "I *said* that the first time!"

The interviewer decided to present Tom with a similar problem: 5 bricks were hidden from a total of 12. Tom counted 13 bricks due to an error of co-ordination.

T: (after counting the 7 visible cubes) "8, 9, 10, 11, 12, 13 (while pointing at imaginary places on the table)...So 1, 2, 3, 4, 5, 6, 7 (pointing at the visible cubes)...8 is 1, 12 is 2....3!"



Tom's pointing acts on the table

Although Tom's procedure was inefficient (e.g. miscounting the total of bricks, uttering 12 after 8), his solution betokened his flexibility to solve the task through double counting. However, for both Johnny and Tom it was not evident that they could co-ordinate two number sequences, a characteristic of the TNS. For this reason, they were classified as counters with an INS scheme.

On the transition to the construction of abstract units: Jack and Eamon

Jack and Eamon were initially classified as counters on the transition to the INS stage. For example, in the following episode Eamon had to find the total of 6 visible bricks and 3 hidden bricks.

E: "Er...[looks up in concentration] I don't know how many these are" [points at the 6 visible bricks that he has just counted]
I: "Can you count them?"
E: [counts the 6 visible bricks in silence, then stares ahead in concentration for 2 seconds, with no visible movements] "9"
I: "Well done! How did you work that out?"
E: "You see, you can use fingers and your head....you know you have got a brain, you see?"

It was not evident that Eamon counted-on to find the total of 9 because he needed to count the bricks again to solve the task. When the interviewer asked Eamon if he had seen "pictures" or fingers in his brain, it did not seem that this was the way he had "used his brain". A more reasonable inference was that he inwardly uttered three more number words (i.e. 7, 8, 9), in which case he would be thought of creating verbal unit items, a characteristic of a counter in the verbal period. Jack's solution to the same problem follows:

J: [looks up and down in concentration for 3 seconds, then smiles] "9"
I: "How did you know? How did you work it out?"
J: "I just started from 6 and went up 3"

That Jack spontaneously "started from 6" indicated that he used a counting-on strategy. However, it must be noted that a "verbal curtailment" of the number word sequence was an alternative explanation to the result of the operation of integration. Jack did not say "I had 6 in my mind" but "I started from 6", the last number word he had uttered immediately before the question was posed. He seemed to be inwardly uttering "7, 8, 9". For this reason, Jack was also classified as a counter in the verbal period, on the transition to the INS stage. The classification of Jack and Eamon was confirmed when they had to find how many (namely 4) from 10 bricks were hidden. Both looked perplexed at the problem. Jack said *"I tried to think but I didn't even see... it was quite hard"*. The fact that neither of the two children attempted to count-on to solve this problems, produced a different classification from that of Johnny and Tom.

The counters of figurative units: Alice and Stephanie

Alice was classified as a counter of verbal units. She was not placed with Eamon and Jack, because she clearly counted from 1 to solve partially screened collections. For example, after Alice counted 6 visible bricks, 3 more were covered under the box.

I: "How many are there all together, if I put these with these?
A: (utters while staring at the box) "1, 2, 3, (pause, then points at the 6 bricks 1 by 1), 4, 5, 6, 7, 8, 9"

No movements were apparent when Alice uttered "1, 2, 3". This indicated that the number words were countable items for her, but she needed to start from 1.

Stephanie's solution to the same problem is presented below. First, she "guessed" that there were 10 bricks all together. The interviewer asked Stephanie if she could work it out and prompted her again by asking how many she could see and how many were hidden. After she responded, "6" and "3", respectively, the following episode occurred.

I: "So how many would there be altogether?"
S: "9"
I: "How do you know, 9? Why did you say 9? You said 10 before"
S: (silence)

Stephanie's seemed unaware of how she solved the problem. However, she responded at the insistence of the interviewer. The 3 bricks were still hidden under the box.

I: "Can you tell me how you worked it out? How did you do it? How did you know there are 9?"
S: "'cause they are 3 more from 6"
I: "Right, and did you know that quickly or did you count?"
S: "I know quickly"

At no time did Stephanie give any indication that she was counting. However, her hands were under the table when she answered "9". She could have imagined spatial or finger patterns of 6, 3 and 9. That is why she said "3 more from 6". However, these were not numerical patterns, rather they seemed fixed perceptual items. For this reason, Stephanie was classified as a perceptual counter on the transition to figurative counting.

The counters of perceptual items: Eloise, Eleanor D, and Eleanor W.

Eloise, Eleanor D, and Eleanor W[1] could not solve any of the partially screened collection problems. They were classified as being at the perceptual stage. Solutions of the three children to the same counting task are presented below. In each episode the child had counted the 6 visible bricks and agreed that 3 were hidden under the box.

I: "If I put these with those, how many would there be all together?"
E: "Uhm......(touches the 6 visible bricks). Uhm...er...I don't know"

I: "So if we put these with these, all together, how many would there be all together?
W: (looks at the box without moving her fingers for a few seconds) "10"

I: "If I put these with these, how many would there be all together?
D: (stares ahead for a few seconds in silence) Uhm.....(subvocally counting the visible bricks) 1, 2, 3, 4, 5, 6,....(3 second silence), 7"

In the three cases, the child had no strategy to count the hidden bricks as part of the whole collection. Eleanor D seemed to count the visible bricks and the box as another perceptual item.

5. 2. 2 Discussion on the initial assessment

At the time of the Initial assessment, all 34 children of the Reception class were interviewed and the classification of children was based mainly on their solutions to three counting tasks. It was a tentative classification, subsequently refined with an extended set of tasks and further analysis which took place three months later. Some of the children later selected as case studies did not always display counting routines to solve the tasks. This posed problems for the classification which had not been foreseen. Children's counting solutions to problems in the context of other interview sessions[2] assisted the analysis when the classification was doubtful. The purpose of the initial conceptual assessment was to produce a classification of children's counting in terms

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[1] In the transcripts "W" is Eleanor W, "D" is Eleanor D, and "E" is Eloise

[2] For example during the interview session: Dice game, of the second line of investigation.

of what types of units they seemed to count and what learning stage they were at. The tasks of the initial assessment were hoped to function as "crucial experiments" in order to place a child as a counter of perceptual, figurative, or abstract units. In other words, in order to distinguish major differences in their awareness of counting. Repeated viewing of the video recordings and transcripts and review of the theoretical model were necessary to ascertain from behavioural indicators what learning stage the children were in, particularly between the figurative stage and the stage of the INS. Reporting transitional periods became necessary because for two cases (i.e. Jack and Eamon) no conclusive classification in terms of stages seemed appropriate with the initial data. Precisely, this is the nature of a stage classification: some children were bound to be in transition. A wider range of tasks seemed to be necessary to refine the classification of the case studies. The end of term assessment was an extended session with a variety of counting tasks which aimed to undertake this refinement.

5. 2. 3 The end of term assessment

The tasks designed for the second assessment were extensions of those presented in the first assessment. New tasks were incorporated in the hope of discerning conceptual differences between children's solutions and to clarify doubtful classifications of the initial assessment. The analysis took advantage of the child's solutions to other problems presented during the study to confirm or disprove current inferences. Before presenting the analysis of the case studies, an outline of the classification made in the end of term assessment is shown in the table below.

End of Term Counting Assessment- July 1998

Child	Type of unit	Conceptual stage
Johnny	abstract	INS→TNS
Tom	abstract	INS→TNS
Jack	abstract	INS
Eamon	abstract	INS
Alice	abstract	INS
Stephanie	figurative	FS
Eleanor W	figurative	FS
Eleanor D	perceptual	PS
Eloise	perceptual	PS

The following presentation groups the case studies as follow: Tom and Johnny as the more advanced counters, Jack, Eamon and Alice as the counters of the INS; Stephanie and Eleanor W as the figurative counters, and Eleanor D and Eloise as the perceptual counters.

The more advanced counters: Tom and Johnny

Three months after the commencement of the study Tom and Johnny displayed solutions which were deemed to be thinking strategies characteristic of the TNS stage. In this stage, not only is a child able to perform counting-on solutions but they are capable of taking a number word as standing for the acts of counting from one up to that number word and to take that segment as one entity. Likewise, they can co-ordinate two different number sequences, i.e. they can double count to monitor their continuation of counting to solve a missing addend problem. The following episodes illustrate that both Tom and Johnny were on the transition to the TNS or have constructed the TNS. Tom developed a curious method to monitor the continuation of counting in missing addend problems. He moved his teeth or his head in anticipation to count his counting acts. For example, the following episode presents his solution to the written sum "8+....=13".

T: [stares ahead in concentration for a few seconds, moving his jaw from side to side] "5!"
I: "How did you do that?"
T: "'cause you go (moves his jaw from side to side) 1, 2,...(starts again) 8: 9, 10, 11, 12, 13, 4...Ough!. (starts again) 8: 9, 10, 11, 12, 13, and if it ends up with this one (points to the right hand side of his mouth) that means it's an odd number and then I'm counting in my head (touches his head), so it's 5!"

Tom anticipated the production of movements in order to count them. Hence, it carried the force of double counting. Further, he talked about the number "5" which he created by applying the operation of integration to the five motor items. Similarly, he solved "13 + ... = 36" with the force of double counting. Tom showed that he was not restricted to counting-off-from, a solution typical of a counter of the INS, when solving subtraction problems. For example, he solved the sum "22-17" as follows.

T: (stares ahead in silence, moving his jaw 5 times from side to side) "5"
I: "How did you work it out?"
T: (promptly) "Because it's 5 *to* 22 so if you take 17 away it will be 5"

Not only was Tom not troubled by the subtraction 22-17, but he adopted a more efficient strategy to solve the problem. This was immediately after he solved "11 - 4" by counting off 4 from 11. He did not seem to count down to but viewed the problem as "17 + ... = 22" and counted on 5 from 17. This strategic reasoning (an inverse strategy for subtraction) was shown in other tasks. Tom saw the composite unit 17 “nested” in 22, a characteristic of the TNS. Further, his solution indicated some form of awareness of the relation between addition and subtraction. Tom’s use of thinking strategies is also illustrated by the following episode. After he counted 4 and 5 bricks in separate lots, the bricks were covered under two boxes and Tom was asked to work out how many bricks there were all together.

T: (promptly) “9”
I: “How did you know that so quickly?”
T: (promptly) “5 and 5 is 10 and 4 and 5, that means it’s 9”

This use of an addend decreasing strategy is representative of Tom’s solutions to tasks that he had formerly solved by counting.

In the case of Johnny, his solutions to missing addend problems indicated that he was a counter at least on the transition to the TNS. This classification was based on his spontaneous use of compensation thinking strategies or use of double counting when solving missing addend problems. For example, when 4 bricks were hidden from a total of 10, he figured out the hidden subcollection with a compensation strategy.

J: (looks down counting the 6 visible bricks subvocally, then stares ahead for 2 seconds) “4”
I: “How did you do that?”
J: (points to the visible bricks) “I just counted these: 1, 2, 3, 4, 5, 6, and there must be 4 under there”
I: “Why there must be 4 under there?”
J: “Because if there is 5 there, then there must be 5 there, because 5 and 5 is 10”
I: “Oh, but there’s 6 there....”
J: “Yeah, there’s one more here, so there must be one less there.”

Johnny used other compensation and addend decreasing strategies to solve tasks he had formerly solved by counting routines. Like Tom, he solved missing addend problems by double counting in anticipation. For example, he solved “8 + ... = 13” and “23+...= 36” with no hesitation, putting fingers up in anticipation to specify the second addend.

Counters of the INS: Jack, Eamon, and Alice

A crucial behavioural indicator of the INS stage is that a child no longer needs to count the first subcollection of a partially screened collection to solve the total. This is because the number word for the first subcollection can stand for the sequence from one to that number word and no longer needs to be re enacted. In the case of Jack, the following episode illustrate his type of answer when solving the total of 7 visible bricks visible and 4 hidden under a box.

J: (with his hand on his mouth, he stares ahead for 3 seconds) "11"
I: " Well done (lifts the box) How did you work it out?"
J: "I remember how many there were...and then I pretended that there was 4 pieces in my head....I was crossing them...ticking them off and that's how I got 11"

Jack's use of the operation of integration for the first 7 bricks is betokened by his comment "I remember how many there were". On the other hand, he seemed to need to visualise the "four pieces" to be counted. Similarly, Jack imagined "squares" that he had to "tick off" when solving "11 - 4 =". However, Jack never double counted to find missing addend, an indication that he had constructed the INS only. In the following transcript Jack had to solve the sum "8 + ... = 13".

J: (lies his head and arms on the table, but apparently in concentration) "4?"
I: "How did you work it out?"
J: (sits up on his chair) "I counted from 8"

Although he started counting from 8, he could not "double count" to count his counting acts after 8. The interviewer suggested to him to do it again, and Jack came up with the same answer: "4".

Like Jack, Eamon counted on to solve all problems with partially screened collections. He did not use, in anticipation, any strategy of double counting to find a missing addend task. He could operate with abstract units, but he was classified as a counter of the INS stage. Despite the aforementioned remarks, when Eamon lacked a counting strategy with which to solve a problem he seemed to resort to a thinking strategy typical of the TNS stage. For example, when he had to solve how many bricks (namely 4) from a total of 10 were hidden under the box, Eamon's reasoning seemed to carry the force of a compensation thinking strategy.

E: (pensive, with the 6 bricks in his hands) "So I think is 5, 6....Before you were hiding some of these how many were there?
I: "10"
E: (reasoning) "So there's 5 here and 5 here..."
I: "But those in your hand are not 5"
E: (pensive, without looking at the visible bricks) "So there's 4 there and 6 there"

Eamon's explanation seemed to carry the force of Tom's and Johnny's reasoning "one less from 5 must require one more from 5 in the second addend to make 10". Although Eamon's verbalisation was not the same as Johnny's, the inference was not unreasonable because this was not an isolated strategy. In another problem, he had to find how many bricks from a line of 9 bricks were hidden. Eamon moved his finger along the 6 visible bricks, presumably counting them in silence.

E: (smiles) "3"
I: "How did you know?"
E: "When I said 4 and 5, it would be 4 and 3 to make a less number you see..."

Eamon seemed to derive 6+3 from the known 4+5. This is a reasonable inference because just before the above episode, he solved that 4 and 5 bricks hidden under two boxes were "9" all together. Moreover, the inference that he did not double count is based on his response to the interviewer's subsequent prompt.

I: "Did you count 1, 2, 3, 4, 5, 6, and then say 7, 8, 9?"
E: (seems to nod reluctantly) "I've counted these (pointing to the 6 visible cubes)...Well, I didn't count these ones" (tapping the box)...

Eamon solved all problems involving partially screened collection using his INS scheme. For example, when he had to solve the total of 10 and 13 bricks covered by two boxes, he counted on from 13 and arrived at "23". In explanation he said *"I had 13 in my head and I added 10 with them"*. However he was in trouble with missing addend problems because he was apparently unable to double count. As was shown above, in some occasions he resorted to a thinking strategy to "avoid" counting.

In the case of Alice, it must be recalled that she was initially classified as a counter of the figurative stage. Three months later, her counting scheme had transformed because she no longer counted from 1 when solving problems with partially screened collections. In all occasions she counted-on in anticipation and was now classified as a counter of the INS stage. For example, when Alice solved the total of 7 visible bricks and 4 hidden ones, she stared ahead in concentration before responding "11" and explained:

A: "I just said it in my mind"
I: "Yeah, but how did it go? How was it? Did you start it at 7?"
A: "I started with 7 and then I added some 4 and then I realised it was 8, 9, 10, 11, four" (she puts 4 fingers one by one and then shows 4 all together).

Alice used numerical finger patters of 5 and 10 to solve missing addend problems. For example, she solved the horizontal sum: "8 + ... = 13" by putting fingers up and showing her hand open. This, however, was not a double counting strategy in anticipation. Similarly, she solved "23 + ...= 36":

A: (counts staring at her fingers on the table for a few seconds, then she waves 5 fingers, another 5 fingers and then 3 fingers with the other hand) "5, 5, 3....10: 11, 12, 13, 13"

Alice's reflection upon her finger patterns is evident in the above passage. She united them in thought, using the operation of integration. What it is not evident is her awareness of two number sequences, that is, a sequence from 1 to 13 and a sequence from 11 to 23. That she could not double count nor had she established the TNS was evident when she attempted to solve "22 - 17" by counting off 17 from 22.

A: "I think I can count backwards" (counts backwards subvocally while putting fingers up, when she runs out she starts putting the same finger up again, after a few seconds she looks a bit puzzled) "I didn't know whether I counted to 17...about 17, I counted from 22...."

Like Jack and Eamon, Alice was classified as a counter of abstract units who had constructed the INS.

The figurative counters: Stephanie and Eleanor W

Only two children who were initially classified as perceptual counters had now constructed a figurative scheme: Eleanor W and Stephanie. Both children could count the total of a partially screened collection but they always counted from 1. Stephanie had been initially classified in transition to a figurative counting scheme and her classification remained ambiguous in the second assessment as will be seen below. Stephanie rarely counted out loud when prompted to explain whether she "guessed" or whether she "counted" to solve a given task. "Guessing" or "estimating" was her typical spontaneous response. She was classified as a counter in transition to a figurative stage on the basis of two crucial episodes. That Stephanie could create figurative units and count them was suggested by her solution to "11-4=".

S: (looks ahead for a few seconds with her hands on the table, her fingers folded in both hands) "15".
I: Did you do 11 add 4?"
S shakes her head (meaning "no") shyly

Stephanie seemed to interpret the situation as 11 + 4, and seemed to "feel" and count all her fingers up to 11 and 4 more. Her lack of awareness of what she had actually done was apparent. When she was asked again whether she solved 11 add 4 or 11 take away 4, Stephanie responded "11 add 4" with an expression of realisation on her face. It was apparent that Stephanie counted all 11 and 4 fingers and could not solve "11-4" by counting backwards. In another episode, she was presented with lots of 4 and 5 bricks. After she counted the bricks of two separate collections, the two lots were covered and she was asked to work out how many bricks there were all together. After she seemed to "guess", "8", the interviewer suggested to her to work it out.

I: "Can you work it out with fingers?"
S seems reluctant but she nods
I: "Go on then"

S: (puts 1 finger up, then puts the 5 fingers up on the table and 4 fingers up on the other hand, she then stares at the open hand and then at the 4 other fingers for a few seconds) "9"

Stephanie seemed to use finger pattern for "5" and "4" to solve the problem. These finger patterns were re-presentations of the hidden bricks. Although she solved the problem with the intervention of the interviewer, she seemed to substitute the bricks for her finger patterns. Whether the fingers were figural representatives or the actual perceptual items that Stephanie counted remained ambiguous. Nevertheless, Stephanie solved the problem with the hidden subcollections. It was inferred that her original answer "8" was the result of some reflection (because she looked up in concentration for a few seconds). That is, it was inferred that Stephanie visualised the hidden bricks and counted them without knowing when to stop. Hence, Stephanie was classified as a counter in transition to a figurative counting scheme.

In the case of Eleanor W, it was apparent that she had modified her perceptual scheme. For example, she solved the total of 7 visible bricks and 4 hidden bricks.

W: Uhm, (points to the 7 bricks counting them subvocally and then stops and raises her eyebrows) "9?"
I: "Is that a guess?"
W: (shakes her head slightly) "No" (counts subvocally the visible cubes from 1 and then very carefully she puts 4 fingers 1 by 1, counting subvocally 8, 9, 10, 11, then she raises her eyebrows) "11?"

Eleanor arrived at the answer "11" in apparent concentration. When the 4 bricks were uncovered she actually counted the 11 bricks and seemed extremely pleased at her achievement. Similarly, she solved the total of two hidden subcollections of 4 and 5 bricks as the following transcript documents:

E: (stares ahead in concentration for a few seconds, then puts 5 fingers up 1 by 1 while uttering 1, 2, 3, 4, 5, then pauses and re starts at one, in the same manner, again stopping at 5) "How many was in there? 4 (answers to herself and puts 3 and 1 more finger up on the other hand in silence), 9?"

Like Stephanie, Eleanor W did not solve any of the missing addend problems and always counted from one to count partially screened collections. She was classified as a counter with a figurative scheme.

The perceptual counters: Eloise and Eleanor D

Eloise and Eleanor D had been classified as counters of perceptual units in the initial assessment and were confirmed to be at this same stage three months later. A child in the perceptual stage can only count items that are in their immediate experiential field. Once part of a given collection was screened, the child seemed unable to re-present the screened items nor the activity of counting them. In the case of Eloise, the following episode presents her typical solution to a problem with hidden items. She was asked to find the total of 7 visible bricks and 4 hidden ones.

E: (puts 7 fingers up and then touches the 7 visible cubes with her hands, she then puts 4 fingers up, then she seems to start again and puts 7 fingers up and counts them subvocally as she touches them 1 by 1 with her nose, then she seems to count 4 of the displayed cubes) "11"
C: "How did you work it out?"
E: "I counted... Even though I didn't have enough fingers I used some of the bricks here"
C: "Right, very good...and how did you start?"
E: "Well, I started with 5 here and 2 here (putting 7 fingers up), then I got 2 and 2 here, and then I put these back up again (showing her 7 fingers) and then I counted.

Eloise counted the subcollection of 7 and the subcollection of 4 and then counted the whole collection from 1. This was apparent in her observable behaviour and subsequent explanation. Fingers or cubes had to be there to be counted. This inference was confirmed in another episode, where Eloise had to find the total of two hidden subcollections of 4 and 5 bricks after she had just counted the actual bricks.

E: (smiles after the bricks are screened) "I don't know"
I: "Don't know? Can you do something to work it out? You know, there's how many there?"
E: "5"
I: "And how many here?"
E: "4" (puts 4 and 5 fingers up on each hand while staring at her fingers, counts subvocally from 1) "9"

In the above task, Eloise created fixed finger patterns of 4 and 5 to count. It is presumed that these were not yet figurative re-presentation of items to count but the primary material for counting. This type of response is still of the domain of perceptual counting.

Like Eloise, Eleanor D counted only items that were in her immediate experiential field. When part of a collection was screened, she looked perplexed and resorted to "guessing". For example, after Eleanor had counted 7 bricks, she was presented with 4 bricks which were hidden under a box.

I: "If I put those with these, all together, how many would there be?"
D: (touches the visible bricks and looks ahead for a few seconds) "8?"
I: "How did you do that?"
D: "I guessed"

That Eleanor D could not re-present hidden items nor the activity of counting them was confirmed by her responses to other tasks. For example, after she had counted 4 and 5 bricks and the 2 lots were subsequently hidden, she had to find the total.

D: (2 seconds) "12?"
I: "Is that a guess again?"
D: "Yeah"

I: "Can you do something to work it out?"
D: (visibly uncomfortable) "No"

Both Eleanor D and Eloise were still at the perceptual stage at the end of the first stage of fieldwork.

5. 3 Discussion on the Findings of the First Line of Investigation

At the end of the first stage of the fieldwork, the analysis concluded with a major classification of the case studies. There were two groups of children: counters with a numerical scheme and counters with a non numerical scheme. The former could count abstract units and the latter could only count perceptual or figurative units. Tom, Johnny, Alice, Jack, and Eamon were counters of abstract units and were seen using the operation of integration. Tom and Johnny were classified as counters in transition to the TNS because they could count their counting acts to solve missing addend problems. They were seen as capable of progressive integration operations. Eamon, Jack and Alice were classified as counters of the INS because they always counted on to solve partially screen collections and were seen as using sequential integration operations. By the end of the first stage of fieldwork, these children could act on re-presented figurative material or material that was a product of a previous integration. Their counting scheme was numerical. Conversely, for the second group of children their counting was non numerical. Stephanie, Eleanor W, Eloise, and Eleanor D were counters of perceptual or figurative units. Conceptually, they were less aware of their activity of counting, hence, their available schemes were less sophisticated when solving the problems presented. They depended on perceptual or figurative material to solve the counting problems or they were limited to re-presentation of this material and the activity of counting. Their focus was still on the figurative material, not on the activity of counting nor on the counting acts. The classification of Stephanie was not conclusive: she did not solve crucial tasks involving counting a partially screen collection, and hence, she was placed in transition to a figurative stage, her counting remaining as a perceptual scheme. Eleanor W was thought of having constructed a figurative scheme at the end of the first stage of fieldwork and, in her case, the analysis was conclusive. She could count partially screened collections but was not yet seen to use the operation of integration and, hence, she always started counting from one. Finally, Eleanor D and Eloise were classified as counters of the perceptual stage because they could only count sensory available items. For the latter four children their counting sequence was a non numerical scheme. The common feature of their behaviour was a lack of reflection upon the actual or re-presented counting activity.

Using the model of counting types and conceptual stages was propitious to interpret major differences in children's conceptualisation of counting and the genesis of their arithmetic knowledge. The construction of a numerical scheme seemed to be a milestone in children's history of construction of arithmetic knowledge because it empowered children's methods in the context of counting tasks. The first group of children gave evidence of reflective activity of a kind which was absent for the second group of counters until the end of the first stage of the study. An overview of the classification of the case studies shows that there were important changes in the three months of the first stage of fieldwork as the following graph summarises.

Children's Conceptualisation of the Counting Sequence: (April-July 1998)

	STAGES	INITIAL ASSESSMENT	SUMMER ASSESSMENT (JULY 1998)
NUMERICAL SCHEMES INTEGRATION OPERATION	**INS→TNS**		**TOM JOHNNY**
	INS	**JOHNNY TOM**	**ALICE JACK EAMON**
NON NUMERICAL SCHEMED FIGURATIVE AND PERCEPTUAL	**FS→INS**	**JACK EAMON**	
	PS→FS	**ALICE STEPHANIE**	**ELEANORW STEPHANIE**
	PS	**ELEANORW ELEANORD ELOISE**	**ELOISE ELEANORD**

One can observe 3 cases moving up the boundary of non numerical counting: Eamon, Jack, and Alice were classified as abstract counters three months after their initial assessment. The case of Alice was of particular importance because Eamon and Jack had been previously classified in transition to abstract counting. Alice's solutions, on the other hand, had rapidly undergone a significant change. Johnny and Tom were confirmed as the more advanced counters moving up to the TNS, while Stephanie, Eloise, Eleanor W, and Eleanor D remained in the bottom sector of the graph. One can conceive 3 groups of counters: the more advanced counters with Tom and Johnny; the intermediate counters with Eamon, Jack and Alice; and finally the non-numerical counters with Eleanor W, Stephanie, Eloise, and Eleanor D.

The aim of the counting assessment was to interpret children's conceptual progress in arithmetic. It was evident at end of the first stage of fieldwork that the crucial difference between our 9 counters was the extent to which they did not use actual counting routines. It seemed apparent that some children were intentionally avoiding or by passing counting through the use of strategic reasoning. This was the case for the advanced counters, Tom and Johnny, and seemed to include Eamon, a counter of the intermediate group. An alternative to actual counting routines involved thinking strategies which were not observed with children classified as figurative or perceptual counters.

This preliminary analysis provided a classification of the case studies according to their learning stage, to contrast with the findings of the second line of investigation. The findings for the second line of investigation are presented below, in part two of the present chapter. This part presents the exploration of the children's ideas on number scripts and number words, and looks at their reflective activity in the context of producing and interpreting spoken and written conventional signs.

5. 4 Part Two: An Exploration of Children's Ideas on Number Scripts and Number Words

In the second line of study, the working assumption was that young children reflect on the myriad of number words and scripts that they encounter in every day situations and the aim was to explore what kind of reflections they make. The focus was on what aspects they attended to when they talked about, used, and produced number words and scripts in conversation in a variety of situations. It was soon apparent that children had elaborated conjectures about number words and scripts which were not taught at school.

5. 4. 1 The notion of rules or regularities

How did children seem to appropriate number scripts and number words? Children of the study seemed to elaborate more or less stable and explicit conjectures on the systems of spoken and written numerical signs. These rules often had the form of criteria to solve a task. These were more or less conventional and relatively explicit - i.e. prone to be verbalised: the more conventional, the more stable a rule seemed to be. Before going into the analysis of the case studies, it is pertinent to present an outline of children's main rules[3]. The table below introduces the rules more relevant to the analysis.

Children's Rules in Number Words and Number Scripts

Rule	In number	Characterisation
of the digits	scripts	the more digits the bigger the number (conventional)
of the zeros	scripts	zeros- noughts or "ohs", make a big number (partially conventional)
of the front digit	scripts	the front digit has a leading role in the meaning of the number (conventional but relative to other rules)
of the number sequence	scripts/words	children read the scripts and refer to the number sequence to decide which of the scripts is higher
of hundred and thousand scripts	scripts	hundreds are written with 3 digits, thousands are written with 4 digits (conventional)
of the two zeros, or three zeros	scripts	hundreds are written with 2 zeros and thousands are written with 3 zeros (partially conventional)
of the place	scripts	the 3rd place from right to left tells the hundreds and the 4th place tells you the thousands (conventional)
of importance of position of digits	2-digit scripts/words	children know that a change in the position of the figures forming a script, transforms the name of the number (conventional)
of the decades	2-digit scripts	the decade knots are written with a digit from 1 to 9 and a zero
of the recurrence of 1 to 9	2-digit scripts	children know that number words up to 99 are written with a recursive cycle from 1 to 9 in the right hand place
of the sequence in tens	words	children produce a sequence in tens with the awareness of omitting number words, beyond 100.
of the counting hundreds	words	children produce a sequence in hundreds (i.e. "1 hundred, 2 hundred- also in thousands and millions)
of numbers of the same kind	words	children establish numbers of the same kind (e.g. hundreds, thousands) and add or subtract them together like ones.
of concatenation	words/scripts	numbers are written as they are spoken: 230 is written '20030' (unconventional)

From an observer's point of view, the closer to conventional rules children's criteria seemed to be, the fewer the chances of contradiction. When the rules were partially conventional or unconventional, they were subject to contradictions with other established rules or with the use of signs in children's environment. On some

[3] A more comprehensive description of all the rules isolated from children's behaviour is presented in the Appendix.

occasions, children created local rules or justifications, specific to a particular situation. Stable conjectures were termed regularities or rules and children were seen to establish regularities in the conventional systems of spoken and written numbers. Hence, the term "rule" is not here used in a sense of "imparted rule" or "taught algorithm". It refers to the inference of children's original and relatively stable elaborations or ideas on number words and scripts. The term regularities is often used to avoid the former sense of the word.

It was found that children of the study could be placed in three groups, according to whether they had established relatively stable rules upon number words and number scripts. In the following sections there is a presentation of the case studies and the findings of the first stage of the fieldwork in relation to the children's appropriation of number words and scripts. Depending on whether children were seen elaborating conventional rules in number scripts or number words; in both of these or in none of these, the case studies were placed in three groups. The table below summarises this classification.

Establishing Regularities in Number Words and Scripts-First Stage of Fieldwork

Group 1: Establishing regularities or rules in number words and scripts	Group 2: Establishing regularities or rules either in number words or in number scripts		Group 3: No apparent establishment of regularities or rules
	In number words	In number scripts	
TOM	EAMON	JACK	ELOISE
JOHNNY		STEPHANIE	ELEANOR W
ALICE			ELEANOR D

5. 3. 2 Group 1: Establishing rules in number words and scripts

The case of Tom

From a very early stage during the fieldwork, Tom seemed eager to show the interviewer what he knew about number words and scripts. He had established the rules of the two zeros and the rule of the digits. For example, when he put successfully in a line the scripts: 3-56-100-2500-7800-89567-100000, he justified why '100000' was higher than '89567'.

T: "Because it's got more"

I: "More what?"
T: "Lett…numbers!"

Tom used this rule systematically despite misleading questioning by the interviewer. He also established partially conventional rules, such as the rule of the two zeros whereby '100', '2500', and '7800' were all read as "hundreds". This suggested that for Tom, there different "kinds" of numbers which were written in different forms: "the hundreds" were written with two zeros whereas the 2-digit numbers were of a different kind. In the same task, Tom showed that he had constructed the rule of the front digit.

I: (pointing at 2500 and 7800) "These two have the same amount of numbers: 1, 2, 3, 4, 1, 2, 3, 4, how do you know which one is bigger?"
T: (pointing at 7800 and then at 2500 and then back to 7800) "Because the seven's in the front, because that means that's *two* hundred, and that means that's *seven* hundred".
I: "So the front number is quite important…."
T: "Yeah, 'cause that's a hundred, that means, it says one hundred, and that's *two* hundred" (pointing at 2 in 2500)

Tom showed every indication that he had appropriated the scripts for hundred and thousand knots[4] and interpreted scripts with three zeros as "thousands". For example, he described the writing of "one hundred" as *"one, oh, oh"* and "five hundred as *"five, oh, oh"*, and he wrote all hundred and thousand knots conventionally, including a spontaneous production of "10 thousand".

T: "I can write ten thousand" (he writes '10000') […]
I: "Is that ten thousand?"
T: "Yeah, 'cause look: ten, and three more"

However, Tom produced non conventional regularities for "102", "103", and "104". Having abstracted a pattern in the number names, Tom produced '121', '131', and '141'. Tom's reflection on how numbers were written was apparent during the sessions. For example, in the following episode, Tom and Jack had to write "110".

T: " 'cause…one, one, zero" (he writes '110')
J: "One, zero, one (he writes '101') […]
T: "No! This is how you do it 'cause one hundred and eleven is…" (he writes '111')

Yet, Tom wrote '110' when he was dictated "10 hundred" in the context of writing all hundred knots. When "10 hundred" was dictated Jack wrote '1000' following the written pattern of all the hundred knot scripts (e.g. '300', '900'). Tom visibly hesitated and finally wrote '110'.

[4] Knots are round numbers: e.g. 100, 200, 1000, 2000.

T: "Ten hundred, look" (points at his '110')
I: "But you told me that that's one hundred and ten"
T: "One hundred and ten...(he looks puzzled)...He's got it wrong as well" (pointing at J's '1000')
I: "I think that's ten hundred...or is that a thousand?"
T: "A thousand"

For Tom '1000' was the script for "a thousand", that is why he did not write '1000' for "ten hundred" and wrote '110' instead. However, when he was reminded that he had said '110' was "110", he reluctantly wrote '1000' for "10 hundred". Similarly, Tom established regularities in 2-digit number scripts and on copious occasions he readily expressed his ideas about the number scripts produced or yet to be produced. He had constructed the rule of the importance of the position of digits, the rule of the decades and of the recurrence of 1 to 9. For example, in the following episode, Tom was playing a dice game with Alice. She had to write the score of 3 after her previous score of 3. Alice wrote the two scripts '3' and '3' next to each other. The interviewer asked whether the children could read that script '33'.

T: "Thirty three" [...] " ...'cause thirty three is three...for thirty....and then is another three (gesturing in the air as he was writing it)...because twenty three, thirty three...."

After writing, '33', the children were asked which of the two threes told you "thirty". Tom volunteered the following reflection:

T: "That one (he points to the ten digit '3' in his '33'). Because it's in front of all of these...and then it's a three like thirty one, thirty two, thirty three" (pointing at the scripts 31, 32, 33)

These type of reflections were widespread in the sessions with Tom. On various occasions he expressed that one has to write the '2' in front after 19, or the '1' in front when writing numbers up to 19. For example, in the following episode, Tom corrected his partner by saying that "15" was written as '15' not as '51'.

I: "How do you know that one is fifteen and that one is fifty one?"
T: "It's because one's before the five...'cause...ten, it has a one in front of the zero, so it keeps on going until [nineteen]"

Similarly, in the context of writing numbers up to 100[5], he explained to his partner *"twenty has a two at the beginning...thirty has a three at the beginning... "*.

Like with number scripts, Tom showed evidence that he was reflecting on the system of number words. He seemed to have constructed "kinds" of numbers. Hundreds, thousands, or millions were different "kinds" of numbers and they seemed to be dealt

[5] Board of 100 seats session

with as countable entities. A rule to count hundreds was also apparent from Tom's behaviour. For example, in the context of a cinema auditorium[6], Tom and Jack were told that a hundred people could sit in each of the sections of the cinema. Tom counted in hundreds, from "1 hundred" to "11 hundred" while pointing at parts of the cinema auditorium. The interviewer intervened to limit the countable parts to 9:

T: "Right (pointing to the 9 parts), one, two, three, four five, six, seven, eight, nine"
I: "So, how many people?"
J: "Ten"
T: "Nine hundred"
J: "Ten, ten hundred"
I: "Ten hundred, do you think?"
T: (disgruntled) I'm going to [count them] one, two, three,...(pointing at the 3 lateral parts), four, five, six,...(pointing at the 3 middle parts),...seven, eight, nine" (pointing to the other 3 lateral parts
I: "How many?"
T: *"Nine"*
I: "How many people?...Nine? Nine people?"
T: "Yeah. [annoyed] No! Nine hundred!"

Tom clearly seemed to have different "meanings" for *9* people and *9 hundred* people. In the last transcript, he counted in ones the 9 parts of the cinema but he kept the meaning of "9 hundred people". *"Hundred"* seemed to have the meaning of potentially counting from one to one hundred. The conjecture that Tom saw "hundreds" as countable entities of a kind, was confirmed in subsequent episodes. For example, in the following transcript, the children wrote '100' on each of 9 cards to "tell the owner of the cinema" how many people fitted in each part. In the following episode, the interviewer had taken away all but one of the cards.

I: "How many people are there now in the cinema?"
J: "One hundred!"
T: [with J] "One hundred"
I: "A hundred, what happens if ten people come back?"
J: "Ten hundred"
I: [overlaps with J] "Ten more people. [After J] No, only ten more people come. How many people?"
T: "One hundred and ten!"

Unlike Jack, Tom did not seem to follow a mere linguistic pattern. For Tom "ten more" was not "ten hundred". The consistency of Tom's answers confirmed the conjecture that for him, *hundreds* were countable groups. The inference was that Tom had appropriated that in our conventional system of numbers, numbers of a kind (same order units) can be added or counted. Similarly, Tom used this rule to converse about thousands and millions.

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[6] Cinema scenario 1 session

Another regularity that Tom established during the first stage of the fieldwork was his sequence in tens. In the initial assessment Tom counted "in tens" when asked by the interviewer and stopped spontaneously at 100. He uttered the decade words with apparent ease visibly moving his head at each counting act. During the first stage of the fieldwork, Tom was prompted on various occasions to continue counting after "a hundred". Tom continued after 100 as "101, 102, 103" but stopped spontaneously, expressing "that's not right". This meant that he was actually reflecting on his sequence in tens as a different sequence from his sequence in ones. Finally, on another occasion, he was prompted to continue counting in tens after 100.

T: "200, 300, 400, 500..."
I: [interrupting] "Is that counting in tens?"
T: [pensive] "No. A hundred and ten, a hundred and twenty, 130, 140, 150, 160, 170, 180, 190, [pause], 200, 210, 220, 230, 240, 250, 260, 270, 280, 290, 300"

It was conjectured that Tom had elaborated a different meaning for his sequence in tens. It was not like his sequence in ones. By the same token, Tom's sequence in tens was different from counting "in hundreds". Tom was aware of different "rules" to count. He could count in ones, he could count in tens, and he could count in hundreds.

The case of Johnny

Like Tom, Johnny seemed to use specific criteria to solve the tasks involving number words and scripts. In the domain of number scripts, it was soon apparent that Johnny had appropriated the conventional form for 2-, 3-, and 4- digit numbers. He had constructed the rule of the digits, and the rule of the front digit. For example, when he put in a line the scripts 3, 56, 100, 2500, 7800, 89567, and 100000, Johnny completed the task conventionally and systematically and justified his line as the following transcript documents[7].

I: "That's brilliant! But how did you know how to do this?"
J: "Because 3 has only got one number in it, and that's got 2 [digits] (56), and that's got 3 (100), that's got 4 (2500), but it starts with 2, a lower number than 7 [in 7800]. And that's 4 numbers and that's got 5 numbers [89567] and that's got (pauses while he re-counts subvocally) 6 numbers [100000]".
I: "Right. Is it the more numbers the bigger the number?"
J: "Yeah"

[7] Johnny systematically chose the lowest of all to put in his line. Moreover, he overtly counted the digits in 7800, 89567 and 100000.

As well as the rule of the digits, Johnny articulated his rule of the front digit when the interviewer asked him how he had worked out which one from '2500' and '7800' was the highest. Johnny did this with no hesitation as is evident in the following transcript.

J: "Because that's got 2 at the start (2500) and that's got 7 at the start (7800) and 2 is a lower number than 7"
I: "Right, but why do you look at the *start* number, at the number at the front?"
J: "Because that's....er....er ...thousands it is"

From the episode above, it was reasonable to presume that Johnny had appropriated the rules of the place and of the hundred and thousand scripts. This presumption was corroborated in an immediately subsequent episode.

J: "that's a thousand (pointing at 2500), and there's three more numbers, two thousand and three more numbers, and there's three numbers in a hundred and more to a thousand and that is a thousand, five hundred"
I: "Ah...How do you read that number (2500)?"
J: "Two thousand five hundred"

Johnny was able to "read" the number scripts '2500' and '7800' using his criteria on number scripts. His rule of the thousand scripts was yet to be revisited for scripts of more than 4 digits. When Johnny was asked whether '89567' was "a thousand number" he showed hesitation and could not decide whether this was "acceptable". His elaboration of the rule of the place was indicated by the order of 342, 350, 351, 360, 367, and 380 in a line and the subsequent conventional reading of these scripts. Johnny explained to the interviewer how he knew all those numbers in the following episode.

J: "'cause they've all got 3 at the front"
I: "And how do you know that they are not three thousand?"
J: "Because a thousand's got more than, more than three numbers in ..."

Most remarkable in Johnny's case was the complete absence of concatenated forms for 3 and 4 digit numbers during the first stage of fieldwork. For example, Johnny wrote conventionally, in dictation, the following scripts: '110', '120', '138', '1001', despite his companion having written concatenated forms: e.g. '10010' for 110 and '10020' for 120. Johnny did not engage in discussion about his companion's different written forms. Despite the aforementioned confidence in writing 3 and 4 digit number scripts, Johnny visibly hesitated during a dictation of "hundreds". After he and Joe had written all hundred knots, including "900", "10 hundred" was dictated.

Jy starts writing '1', pauses, looks at J's sheet and completes '1000'. J writes '1000'
I: "That's very good. Do you know that number has a different name?"
Jy: [instantly] "A thousand"

Johnny manifested uneasiness before writing "ten hundred" because he anticipated that the script '1000' was "a thousand". This was why he paused before completing the production of '1000'. Johnny seemed to accept that the script '1000' had "a different name". However, he knew the "proper" name for '1000'. The tension between "10 hundred" and "a thousand" was going to be raised again during the study.

Rules that Johnny had constructed put him in other situations of tension. For example, in the context of a dice game where each dot was worth "one hundred", Johnny had accumulated "12 hundred" points. When he had to write this number, he was in disagreement with his partner Joe.

J: [interrupting] "One, two, oh, oh"
I: "All right, go on then Jonathan"
Jy: "No...I think it is one, two, oh"
J: [joins Jy]....two, oh, oh, one, two, oh, oh"
Jy: [pensive] "That's ..er...twelve thousand"
J: [shakes his head] "No"
I: "Well, how do you think it is Jonathan?"
Jy writes '120'

The reason why Johnny maintained '120' was 12 *hundred* and '1200' was 12 *thousand* seemed to be his rule of hundred and thousand scripts. According to this rule, 4 digits made a "thousand number": "12 hundred" was, for Johnny, a *hundred* number, which should be written with 3 digits, potentially contradicting the conventional writing of "120". Johnny's reluctance to write "12 hundred" with 4 digits was similar to his uneasiness to write "10 hundred" with 4 digits. Only when the interviewer granted that '1000' had "another name" did Johnny wrote '1000' for "10 hundred". The rule of the hundred and thousand scripts was so powerful that they lead Johnny to contradict conventional forms he seemed to have appropriated. This was why he stubbornly maintained that '1200' was "twelve thousand" and not "twelve hundred" like his companion was stating.

Johnny had constructed stable ideas in the domain of number words, including the rules of the counting hundreds, of the numbers of a kind, and of the sequence in tens.

For example, during the dice game with dice that scored in hundreds[8]. Johnny solved at ease all the scoring of hundreds until he had to solve "600+600". Co-ordination of numbers of different kinds presented a challenge like the following episode illustrates.

Jy: "Two thousand…it's two thousand"
I: "Is that two thousand?"
Jy: "Yeah…ten hundred is a thousand"
I: "Yeah?"
Jy: [pensive] "So two more equals two…"
I: "But two more *hundreds* is it…?"
Jy: [hesitant] "Oh!"

Johnny de-composed the "12 hundred" into 10 and 2 hundred because he clearly stated that 10 hundred was "a thousand" and that there were "2 more". Despite the apparent realisation at the end of the above transcript, Joe's intervention misled Johnny in his attempt to solve the problem. In fact, further tension arose when Johnny attempted to write his scoring of "12 hundred" as it has been documented further above.

Johnny showed every indication that he had established a sequence in tens with a different meaning from his sequence in ones. This was suggested by his monitoring of the continuation of his sequence in tens after 100. For example, after he had counted successfully up to 200, the interviewer prompted him to continue. Johnny uttered "201, 202" but immediately re-started with "210". His monitoring of the sequence in tens after 200 suggested that counting in tens was an alternative to counting in ones. For example, this was suggested in one episode when he and Joe had completed a board with 100 seats for a cinema auditorium[9].

I: "Ok….if you are in fifty and you go to sixty, how many more do you move?"
Jy: "Ten!"
I: "How did you work it out?"
Jy: "Because ten to twenty is ten…to sixty…so fifty to sixty is ten"

Johnny was using his sequence in tens with the meaning of "ten seats apart". In a subsequent episode, this was corroborated by Johnny's response.

I: "Right…..what about if you go from fifty to seventy?" [J starts counting the seats]
Jy: "Er…[one second silence, looks ahead in concentration] Twenty!"

Johnny seemed delighted that he "knew" without counting - unlike his companion

[8] Dice game session with the blue dice: each dot was worth one hundred
[9] Board of 100 seats interview

Joe. He was actually short-cutting the count by ones, using his sequence in tens.

The case of Alice

Like Tom and Johnny, Alice seemed to use specific criteria to put in order number scripts whose names she did not know. For example, when she was presented with the numerals 3, 56, 100, 2500, 7800, 89567, and 100000, she put them in a line in a successful and enthusiastic manner. Alice seemed in deep concentration when she solved this type of task but when the interviewer prompted her to explain, she said "*I guessed*". She subsequently expressed that 100000 was the highest because it had "the most numbers". The conjecture was that she had constructed the rule of the digits. Similarly, she explained how she knew which one of 2500 and 7800 was higher.

A: (giggles and looks at the numbers for a couple of seconds) "I just work it out!, (pointing) seven, eight and five...two, eight....five....
I: "Why do you look at those numbers first?"
A: "Because they are the first two"

Alice was aware that the first rule (of the digits) was not sufficient to explain her line. A second criterion, the rule of the front digit or the front digits, is apparent in her explanation which she subsequently used to put 320, 350, 360, 367, and 380 in a line. Alice systematically placed the next higher number on the line. In the following transcript, the interviewer asked her whether she could read those numbers.

A stares at '320' for a few seconds in apparent concentration.
A: "Three hundred...(pauses and looks at the I)...and twenty" [more comfortably and fluently], 350, 360, 367, 380"

The fact that Alice took some time to "read" the number scripts indicated that she had not read them previous to putting them in order and attended to the fact that all scripts had '3' at the front. Alice appropriated the scripts for the hundred and thousand knots during the first stage of the fieldwork as well as she constructed the rule of the two and three zeros. For example, after they had written all hundreds knots including "900", "10 hundred" was dictated.

L[10] starts writing '10'
A: "No! That's a thousand!
I: "How do you know?" [A giggles but does not respond] "All right, write a thousand then"
A: "I can write a thousand"

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[10] In the transcript, L = Isobel and I = Interviewer

Contrary to Isobel, Alice was visibly troubled by the writing of "ten hundred". She appeared pensive and did not write until a new prompt from the interviewer who asked to write "10 hundred". Isobel wrote '1010' and Alice reluctantly wrote '1000':

A: (pointing at L's '1010') "But that's ten and ten"
I: "That's 10 and 10. Well, may be, may be Alice is right and *that* is ten hundred ('1000') because it looks like eight hundred, nine hundred and ten hundred...But Alice said: but that's a thousand!"
A: "Yeah"
I: "Can we read a thousand....can we read that as a thousand?"
L: "No..."
A: (pensive with the pen in her mouth) "I don't know"

Like Johnny and Tom, Alice seemed to accept, provisionally at least, that "ten hundred" was written as "a thousand". Although Alice appropriated the script for the thousand knots by the end of the first stage of fieldwork, this was not an immediate finding. Initially, Alice produced some unconventional forms (e.g. 31000 for "3000") which she soon dropped in interaction with other children. However, for the numbers within the hundred knots Alice produced concatenated forms. For example, she wrote "5010" as '500010' and "5005" as '50005'. In the domain of 2-digit scripts, Alice had constructed important regularities like the rule of the importance of the position of digits and of the decades. In the episode below Alice and Hettie were writing numbers on the board of 100 seats, "32" was going to be written as '23'

I: "Does it matter if we change the order of the numbers?"
H: "No" (she writes '32')
A: (staring ahead, pensive) "It *does* matter...because it's the wrong number"

Although Alice never articulated her ideas on number scripts as explicitly as Tom, she gave every indication of her reflections on how numerals were written. For example, in the above episode, she seemed well aware of an essential aspect of written numeration, that is, the order of digits have a leading role in the "meaning" of the number. Alice pointed out that *"it does matter"* which order you write the digits because you could write *"the wrong number"*. As for the decade scripts, Alice wrote and read all decade scripts conventionally up to 100 in a variety of situations. For example, when she was presented with a printed scale in tens on a school ruler.
In a similar way to Tom, Alice seemed to have established certain rules to deal with number words. She had established the rule of the counting hundreds. Whether it was in the context of dice game or in the cinema scenario, *hundreds* seemed to refer to

groups of "one hundred". In the following episode of the cinema auditorium[11], Alice was eager to show that she could count "hundreds".

I: "In this part a hundred people can fit (pointing to the 3 middle sections consecutively), and a hundred people more here, and a hundred people more here...how...?
A: (interrupting) "That's three hundred!"

Alice's enthusiasm to show that she *knew* about "hundreds" was apparent in several situations. She responded swiftly that there had to be *"six hundred"* (people) in the 6 lateral parts. In the following episode the interviewer asked how many people could fit in the cinema altogether: 9 x 100. Isobel ventured *"a hundred and nineteen"* and Alice was visibly perplexed at Isobel's response. Prompted to explain, she expressed:

A: "Easy" [...] (pointing at the 3rd side section) "Three hundred, (pointing at the other 3 side sections) four hundred, five hundred, six hundred, seven...seven hundred (pointing to the first middle section), eight hundred (pointing to the 8th section), [pause]
L: "Nine hundred!" (looking at A)
A: [pointing to the 9th part] "Nine hundred"

In various occasions after the first presentation, Alice consistently explained to others how she worked out that there were "900" in total in the cinema if there were "100" in each part. In addition she knew that the 10th hundred was "one thousand" when she produced a sequence in hundreds. Further, Alice seemed to have established the rule of numbers of a kind, whereby she counted thousands or millions together but not hundreds with thousands.

Alice volunteered to count in tens beyond '100' at a very early stage during the study. Counting in tens was something that she felt at ease with. Although she did not know the scripts for the decades within the hundreds, her rule of the sequence in tens aided the interpretation of the scripts of a ten-scale-ruler from 10 to 300. At 110, she hesitated but continued reading from '120' in a conventional manner, pausing again at '200' but completing the reading successfully until '300'. Similarly, after completing 40 of the 100 seats of cinema auditorium[12], Alice pointed at 10, 20, 30, and 40, showing her companion that this was like "counting in tens". Although the other decades had yet not been written, Alice pointed at imaginary separate points to show the position of the seats number 50, 60, 70, 80, 90, and 100. This suggested that for her, counting in tens had a meaning of "jumping 10 seats". In this sense, Alice

[11] Cinema scenario 1 session
[12] Board of 100 seats session

seemed to be aware of two different manners of counting up to 100: by ones and by tens.

5. 4. 3 Discussion on the case studies of Group 1

Tom, Johnny and Alice had constructed original theories to deal with external numerical signs. They were grouped in a different category from Jack, Eamon, and Stephanie, because, it was in both the domain of number scripts and number words that their criteria were isolated. Despite the differences of these three case studies, their established regularities seemed to conflict in one particular case: the name for the script '1000'. On the one hand, according to the rule of the two zeros, "10 hundred" was written as '1000'. On the other hand, according to their appropriation of thousand knots, '1000' was the script for "1 thousand". The co-ordination of these two names (10 hundred and 1 thousand) was going to play a significant play in these children's history of construction of the conventional system of signs. For all these three cases, stable rules such as the rule of the digits, the rule of the front digit, and the rule of the numbers of a kind had to be co-ordinated in a coherent manner.

5. 4. 4 Group 2: Establishing regularities either in number words or scripts

The case of Eamon

Eamon's behaviour across all situations presented indicated that he had constructed conjectures about how number words are organised. In contrast, he had not seemed to establish stable ideas about number scripts. In the domain of spoken numeration, Eamon had constructed the rule for the counting hundreds, the rule of numbers of a kind, and the rule of the sequence in tens. He used the rule of the counting hundreds in a consistent manner despite counter-suggestions and contrary opinions of his partner. For example, in the context of a cinema scenario[13], Sam had expressed that 100+100+100 was "130".

I: "A hundred people can sit in all over this section and a hundred people *more* here, and a hundred people *more* here (pointing at the 3 middle sections). So how many people in total?"
S: (excited) "A hundred and thirty"
E: (assertively) Three hundred
I: Oh! Three hundred...or a hundred and thirty? You have to decide now"
S: "It's a hundred and thirty"
I: "What do you think Eamon?"
E: "Three hundred"

[13] Cinema scenario 1 session

Hundreds seemed to refer to groups of people and they seemed to be numbers of the same kind. This was indicated by Eamon's solution to problems posed in the cinema scenario with hundreds of people leaving and coming back to the cinema. In the following episode, there was only "100" people left in the cinema.

I: "What if only fifty people come back? Only fifty people... Not a hundred, fifty people come back?"
S: "It would be fifty hundred"
I: "Fifty hundred, do you think?"
S: "Yeah!"
E: [pensive] "Fifty hundred? [...] That would be more far than about thousands!"

The interviewer prompted Eamon to express his answer to the problem and he responded that "100" and "50" would be "150". Eamon's reflective attitude during this episode was apparent. The interviewer probed the independence and consistency of Eamon's answer by sowing seeds of doubt.

I: "But Samuel said...what did you say, fifty hundred?" [S nods]
E: "He said fifty hundred and then he said sixty hundred, that's what *he* said"
I: "Right Eamon, what do *you* think? A hundred plus fifty [is]?"
E: [assertive] "A hundred and fifty"

The consistency of Eamon's answers indicated that he had established the rule of the numbers of a kind. After responding that "100" and "20" was "120" and other problems of similar type[14], Eamon paused and distanced himself from the verbal pattern "a hundred and" to respond that "1 hundred" and "1 hundred" was "2 hundred". Similarly, he used million and thousand words according to the rule of numbers of a kind. For example, in the following episode, Eamon had recently expressed that 1 million and 2 million was 3 million people.

I: "And what if a thousand more come?"
E: [pensive] "One thousand more....er..." (shakes the pen in his hand)
S: [perplexed, turns to E] "One..."
E: "Well...It's three million one thousand"
S: [repeating] "Three million and one thousand" [...]
I: "And what if you have three million and ten more come, ten more people?"
E: "Three million and ten"

Eamon did not count *hundreds* together with *thousands* because they were not numbers of the same kind. On the other hand, that Eamon's use of hundreds involved a sense of groups of 100 people was confirmed by an unexpected episode. In the context of making the board of 100 seats[15] Eamon asked how one said "a hundred years". The interviewer, who did not comprehend Eamon's question, responded "a

[14] 100+80, 100+88
[15] Board of 100 seats session

century". A few minutes later during the session, Eamon expressed "I think I'm going to take 4 centuries to do this". Eamon's spontaneous contribution was in the context of completing 100 seats and numbering them. Eamon seemed to use “century” with the meaning of a group of “a hundred years” but also as a “one” because he referred to “4 centuries”, that is, four groups of 100 years. Further, Eamon had appropriated a number word sequence in hundreds with two novel elements. First, the 10^{th} name of the sequence was “1 thousand” and not “10 hundred”. Second, his sequence “in hundreds” seemed to be an alternative to his sequence in ones. He seemed to be aware of two forms to “navigate” the sequence up to “1 thousand”: in ones and in hundreds. This was a reasonable conjecture because he knew that the number just before “1000” was "999" and not "900". Further, he responded with apparent ease that before 999 was 998 and 997. Hence, a sequence in ones and a sequence in hundreds seemed to be two ways of reaching 1 thousand.

Eamon's establishment of his sequence in tens was apparent from a very early stage during the study. In an unexpected manner, he counted in tens from 420 to show that he could count up to “more than two thousand” in tens. On one occasion, Eamon counted up to 1200 in tens starting from 800. He also volunteered his *meaning* for counting in tens when he expressed "I know I can make it bigger with tens". This suggested that uttering the decade words was not a mere recitation: Eamon was “making” bigger “numbers” and could do this faster when counting in tens. Producing the sequence in tens had a purpose, and it was different than producing the sequence in ones. It was apparent that Eamon was “looking down” to the sequence of decade words, taking it as an object of reflection. He was organising them in “cycles” of “hundreds”. In the following episode, Eamon was helping Jack to count in tens.

E: “and then you continue all the same as how we start to a hundred you see?…Again and again and then one thousand and then it’s up to hundreds and *then* you count to thousands, you see?

The fact that he could start from “800” and he could explain to his partner how he counted in tens confirmed the conjecture that his sequence in tens was his object of reflection.

As previously stated, Eamon's knowledge of conventional number scripts was in contrast to his exploration of number words. Although in some occasions, Eamon's solutions suggested that he was elaborating the rule of the digits, his justifications were easily destabilised. His elaborations upon number scripts were local, and often partially conventional. For example, he had established the criteria according to which the more zeros a script had, the bigger the number. Moreover, these regularities in number scripts were typically aided by his knowledge of spoken numbers. For example, when he read with Alice the ten-scale from 10 to 300 on a school ruler, he seemed driven by his sequence in tens which he seemed to co-ordinate with the scripts on the ruler[16]. Nevertheless, Eamon indicated at different moments that he was actively exploring the way particular "numbers" were written. For example, he volunteered that *"a thousand's got three zeros"* and a million was *"a one a six zeros"*. Eamon was not keen on writing numbers and usually was stumped when he had to annotate numbers he seemed at ease with when he solving problems with hundreds and thousands. At the start of the study, he had not established the rule of the importance of the position of digits because he produced "10" as '01' and "12" as '21'. Eamon's behaviour in writing tasks suggested that he had practised and reflected less on number scripts than he had done in the domain of number words. However, discussion with his companions were sources of learning for Eamon. By the end of the first stage of fieldwork, Eamon seemed to have appropriated the scripts for hundred and thousand knots. He was eager to tell the interviewer that he had found out how to write "1000" because he had a puzzle with 1000 pieces. This occurred in a dictation of hundreds, after writing all hundred knots, including "900". Eamon was reluctant to write "10 hundred" because, like the children of Group 1, he seemed to anticipate that '1000', the script for 10 hundred following the pattern of the hundred knots, was in fact the script for "1 thousand". Moreover, Eamon knew that the 10^{th} hundred was "1 thousand" when he counted in hundreds.

The case of Jack

Unlike Eamon, there was clear indication that Jack had established criteria to deal with number scripts. However, he had not seemed to establish rules in number words. In particular, Jack had elaborated rules in the domain of 2-digit number scripts. His

[16] Also his interpretation of 2500 and 7800 as the same numbers because they had two zeros (i.e. rule of numbers of a kind), and his reference to the number word sequence when putting scripts in order.

criteria to deal with 3 and 4 digit number scripts were also stable, despite some inconsistencies during the first stage of fieldwork. There was early indication of Jack's elaboration of the rule of the digits when he explained that his companion's "number" on the calculator was bigger than his because it had "more numbers". This presumption was confirmed when Jack put in order the scripts: 3, 56, 100, 2500, 7800, 89567, and 100000. Despite being initially puzzled at the question of how he knew, he justified his line using the rule of the digits.

J: "That's 8 and that's a 1 and that's got more numbers that that....[points at 8 in 89567 and 1 in 100000 and then at 100000 and 89567]
I: "Right, so the more numbers the bigger the number?"
J nods

Jack seemed unaware of the redundancy of using the rule of the front digit with 89567 and 100000. Nevertheless, he expressed that 100000 had "more numbers". With 2500 and 7800 he seemed to use a version of the rule of the front digit, because he referred to both 2 and 5 in 2500 and 7 and 8 in 7800. That Jack was using a criterion to put the scripts in order was apparent when he successfully put in order the scripts 320, 342, 350, 351, 360, 367, and 380. He said he worked it out "by the middle one". Jack put the scripts in a line without reading them aloud. When asked whether he could read those numbers, he was stumped. However, a prompt by the interviewer suggested that Jack was appropriating other regularities of number scripts.

I: "Can you read those numbers?"
J: [perplexed] "I can't"
I: "What number do you think that is? [320] Do you think is around a hundred?"
J: [pensive for a few seconds] "Three hundred and twenty".

Jack subsequently read all the remaining scripts in a conventional manner and seemed pleased at his achievement. In addition he explained how he had worked it out.

I: "How did you work it out Jack?"
J: [points at 3 in 380] "Because that's ....that is in the number of the hundreds place"
I: "How did you know that?"
J: [points at a place after 0 in 380] "Because if there was another number after that 'oh', it would mean it was in the thousands place"

His justification suggested that he was establishing the rules for the hundred and thousand scripts and the rule of the place. He was able to read the 3-digit scripts because he had appropriated that 3-digit scripts were "hundreds" and that the third place from the right in any script was the place of the hundreds. However, the stability of Jack's rule of hundred and thousand scripts was to be gained during the first stage of fieldwork. Although Jack seemed acquainted with 2-digit scripts, the writing of hundred and thousand knots seemed a novelty for him at the commencement of the study. This was shown in his inconsistency when writing and interpreting hundred knots. For example, he could produced a written pattern in correspondence to a verbal pattern (e.g. '200', '300', '400' for "200", "300", and "400") but this written pattern could be local and unstable (e.g. '121', '131', instead of '200', '300'). Jack was producing a pattern and not reflecting on it. This was apparent when he wrote "10 hundred" following the pattern of hundred knots. Unlike Tom, Jack did not seem aware that '1000' was the script for "another number". On the other hand, Jack had established the rule of the importance of the position of digits in 2-digit scripts, he had appropriated the regularity of the decade scripts. Not only could he write and interpret the decades but also he could talk about them, and characterise them before putting them in writing. For example, in the context of writing the numbers for the cinema seats, Eamon was having trouble with writing "40".

J: "He's writing all the round numbers, he could look up there [points at 20], that would remind him"
E: [a bit puzzled] "Well, I'm not looking at....I think I know....[pensive]
I: "Why could he look up there to remind him [points at 20]?"
J: "Because that [20]....because it's always going to be the same but that's going to be...that's simply a four [points at 2 in 20] instead of a two"

The decade scripts seemed to be a marker for Jack. In the above excerpt, he expressed how '20' and '40' are "the same". Similarly, Jack suggested that Eamon should look at his previous "13" in order to write "14".

In the domain of number words Jack's responses differed from those of Eamon and the children of Group 1. He did not use hundreds as if they were countable groups. When asked how many people could fit in all the cinema if there was 100 in each part, Jack counted "one hundred, a hundred and one, a hundred and two". Jack's justifications were easily destabilised which suggested that his verbal patterns were local and not an indication of a stable rule. On the other hand, "counting in tens" had no apparent meaning for Jack even though he had appropriated the form of decade scripts as previously stated. During the initial assessment interview, Jack could not count in tens, despite the interviewer's prompts. Subsequently, on other occasions he counted in a "tee fashion". For example, in the following episode Jack was invited to show how he could count in tens.

J: "I *can* count in tens"
I: "Yeah...Can you show us? Just a little bit"
J: "[swinging his head] Ten, twentee, thirtee, fortee, fiftee, sixtee, seventee, eightee, ninetee, twenty" [stops, perplexed]

To his own surprise, Jack had "come back" to twenty. This perplexity suggested that he experienced some perturbation about the outcome of his "tee sequence in tens". At the end of the first stage of fieldwork, and when he and Eamon had completed a board with 100 seats[17], Jack followed the count in tens as a different count from ones because he pointed at the decade scripts until 100 (with no intervention of the interviewer). In this episode Jack seemed to focus simultaneously on the decade scripts, the decade words, and the number line, and this might have lead him to a new organisation of his decade words. By the end of the first stage of fieldwork, Jack gave indication that he could engage with reflections of children of Group 1.

The case of Stephanie

Like Jack, Stephanie seemed to have established regularities in number scripts but not in number words. Unlike Jack's case, Stephanie's rules seemed restricted to 2-digit number scripts. Writing numbers that were not taught at school, within the segment of 1-100, seemed not a novelty for Stephanie. For example, in an early session she read all decade scripts up to 100 from the ten scale of a ruler. She uttered "101" for '110' but stopped spontaneously at '120'. When writing a board of 100 seats[18] Stephanie's comments betokened the regularities that she had established for written numbers. For example, when the two girls were writing the thirties, the interviewer asked why the '3' was always at the front.

S: "Because it's the same as this one [points to the end of the previous line: 17, at the teen numbers)....because this thirty changes to three to go thirty one, thirty two...because it's...because you...the thirty sounds like it's the three sounds like it's 'thir'...so the one...the number that's after...it's second one"
I: "Oh, how is this similar to this....you said this is similar to this" [points to 17 again]
S: "Uhm...because...because it's like you say...the three is before just the same as the one...[smiles with satisfaction]

Stephanie referred to the '1' in front of all the teen scripts, a remark that had come up during a previous episode. She smiled complacently because she had appropriated the rule of the importance of the position of digits and the rule of the recurrence of 1 to 9.

[17] Board of 100 seats session
[18] Board of 100 seats session

With 3 or 4-digit scripts, Stephanie's rules were partially conventional and unstable. For example, she expressed that '100000' was higher than '100' because it had "more noughts". However, her solution indicated that she had not established the rule of the digits nor the rule of the front digit.

Like Jack, in the domain of number words, Stephanie seemed to have no meaning for "counting in tens". When asked to count in tens, Stephanie refused and expressed that they were only "allowed to count in 2's". One episode lead to Stephanie's realisation that her knowledge of the decade scripts could help her to count in tens. In the following transcript, she had refused to count in tens.

I: "Can you read these numbers?" [showing S the 10-scale ruler]
S: "We've only count...we've only allowed to count in threes, twos, and ones" [I points at 10 in the ten scale of the ruler and moves her finger along the scale] Ten, twenty, thirty, forty, fifty, sixty, seventy, eighty, ninety, twen...a hundred"

Stephanie could read conventionally all the decade scripts, including 100. The interviewer prompted Stephanie to do it again but this time without looking.

S: [stares ahead in concentration] "Ten, 20, 30, 40, 50, 60, 70, 80, 90, a hundred" [proud]
I: "Very good, that's counting in tens, isn't it?"
S looks puzzled and nods

Stephanie seemed to visualise the printed scale and uttered all the decade words. For the first time she seemed to establish a meaning for counting in tens. However, her apparent perplexity at questions with hundreds and thousands in the context of the cinema auditorium or the dice game indicated that Stephanie had not established rules such as that of numbers of the same kind.

5. 4. 5 Discussion on the case studies of Group 2:

Unlike the children of Group 1, Eamon, Jack and Stephanie had established stable rules in number words or in number scripts but not in both domains. In general, whereas Eamon seemed more comfortable with number words, Jack and Stephanie seemed to prefer writing numbers and talking about these scripts. Some criteria, like the rule of the zeros were partially conventional and hence, subject to contradictions. For example, Eamon admitted that '100000' was higher than '89567' because it had more digits but then admitted that '89567' was higher because '8' was higher than '1'. The co-ordination of the rules of the digits and of the front digit was not resolved since their justifications and responses were easily destabilised. Nevertheless, it was conjectured that these children had appropriated some rules in the course of the first stage of the fieldwork. For example, Jack and Stephanie seemed to have established a regularity in the decade scripts that they could associate with counting in tens. Writing and reading number scripts seemed to be a pivotal activity for reflection. Eamon, on the other hand, attended to regularities of number words. He had appropriated that numbers of the same kind could be added together or, in his own words, that "big numbers are the same as little numbers".

5. 4. 6 Group 3: Children who had not established rules

The case of Eloise

Eloise never refused to read or write "big numbers". In the initial assessment, she read '13' as "33", '31' and "63", and '51' as "65". However, she did not seem to adopt a reflective attitude during the sessions and each situation seemed a new and isolated instance. It was apparent that writing 2-digit number scripts was a novel task for Eloise. She often "copied" the scripts from another child or waited for them to be "dictated" to her by her more knowledgeable companion. Various situations indicated that Eloise attempted to write what she was "hearing". For example, she would write "1 hundred" like '100', "a hundred and 2" like '002', "eight-teen" like '81'. In the following episode she explained how she figured out how to write "44".

I: "How do you know that forty four is four four?"
E: "Forty four...because I heard the sounds and it was four, four..."

Her focus on the "sounds", yielded concatenated forms such as '0099' for "199". But this concatenations were unstable, for example, she annotated alternatively '122' and '002' for "102". She seemed unaware of the recurrence of 1 to 9 in the decades or of any criteria related to script's digits. In the context of writing numbers up to 100[19], and aided by her companion, Eloise wrote number scripts she had not written before. After writing all the precedent decades, Eloise wrote '91' for "91" but then wrote '29' for "92". This suggested that she had not abstracted the constant position of '9'. Eloise produced unconventional names like "thirty-ten" after "39" and wrote it according to her criteria of concatenating the sounds: '310'. When 3- and 4-digit scripts were involved, Eloise expressed that, if a script was "longer", it had to go "the

[19] Board of 100 seats session

last" in her line but subsequently stated that if it had more "ohs", i.e. zeros, it was a higher number. When pointed out that '2500' had more zeros than '89567' she placed '2500' after '89567' in her line of scripts. In the domain of number words, when Eloise was asked to count in tens, she counted backwards in ones starting from 10. Although she could produce a verbal pattern such as "1 hundred, 2 hundred, 3 hundred", her answers indicated that she had not elaborated the rule of the numbers of the same kind. For example, in the context of a dice game, where each dot was worth 1 hundred, she promptly responded that 2 dots was "2 hundred". However, when she was asked how she knew, she explained: *"a one is one hundred so all the other numbers are hundreds too"*. When she had to work out the scoring of "4 dots", Eloise responded untroubled: *"just four dots is four"*.

The case of Eleanor W

Eleanor W seemed to annotate number scripts as isolated instances and abstractions were rarely made. Number scripts seemed to be marks on the paper hardly related to one another or to the number words involved. In a similar manner to Eloise, Eleanor W always had a go at interpreting number scripts she did not know. For example, during the initial assessment she read '13' as "23", '14' as "24" and '12' as "22". In subsequent sessions it was apparent that writing and reading 2 digit numbers were a novelty for Eleanor W. It was also apparent from her behaviour that she had not constructed stable ideas about number scripts. Eleanor W seemed to "draw" or "copy" each number script, usually prompted by the interviewer. Despite interventions by the interviewer, she did not seem to abstract regularities from a series of scripts. For example, in the context of writing the numbers for the 100 cinema seats, Eleanor W and her companion were stumped after they finished the seat number 10. Hence, it was suggested that she use a calendar of June (from 1 to 30) and a book to work out which number came next. Eleanor copied each number script as a new mark, apparently unaware of the patterns of the scripts: for example, she wrote '91' for "19" and after writing '30' and '31' she proposes "2-1" for "32". To the interviewer's surprise, neither children abstracted that '3' was in front of all the thirties: when '37' was not printed on the page, the children were at a loss. Eleanor W proposed, untroubled, '77' and they continued with the next number scripts. Verbal patterns like "ty-one to ty-nine" were produced but the interviewer had to intervene on several occasions to provide the new decade word. Like Eloise, Eleanor W proposed "twenty-ten" for the number after "29" but subsequently accepted the conventional form. Further, verbal patterns seemed to play an important part in Eleanor W's answers: for example, after writing '99' and pronouncing "99", she proposed "20" as the next number. Similarly, she proposed "nineteen" after writing '80' -i.e. "eightee[n]". She did not know the conventional form for "100" and she produced non conventional regularities for the hundred knots. She wrote, untroubled, '22', '33', '44', for "200", "300" and "400". With respect to 3- and 4-digit scripts, Eleanor W attended to the length of the script. For example, when she had to explain why her number script '51431' was bigger than her companion's, she expressed, " *'cause it's longer than the other one*". On other occasions, she attended to the zeros of the scripts. In one episode, she explained that *"if [a number script] has lots of zeros is making a higher number"*. For example, in the following transcript she explained her line '3000- 4000-1000'

I: "Which one is the lowest of all?"
W: "Easy...That one! (3000)"
I: "And then?"
W: "That, and then that one" (4000 and 1000)
I: "How did you work it out?"
W: "'cause that's a three, a four and a one. Because I know that's the lowest number, and then it gets higher but then I know that one's a hundred and....I know that's the highest number I can ever get"

In the domain of number words, Eleanor W never seemed uncomfortable and always responded with excitement. However, her responses did not indicate any stable conjectures upon the number words. For example, in the context of the cinema scenario[20] and after discussing that there were a hundred people in each of the 9 parts of the cinema, she breathed in, and proposed that *"two thousand and three billion"* could fit in. Unreflective "guesses" were the typical response of Elearnor W. Likewise, when she was asked whether she could count in tens, Eleanor produced a "tee sequence" after the usual prompt: "10, 20". For example, in the context of reading the 10-scale of a ruler, she uttered: "ten, twenty, thirtee, fourtee, fiftee, sixtee, seventee, eightee, ninetee, twenty" and continued: "21, 22".

[20] Cinema scenario 1 session.

Eleanor D

Eleanor D's responses were similar to Eleanor W's and Eloise's. In the domain of number scripts, Eleanor D did not know 2-digit number scripts and in the writing tasks she tended to copy from her companion's scripts or to copy from a carrier of printed numerals (e.g. calendar). The absence of abstractions upon aspects of the scripts was as striking as in the case of Eleanor W. Like Eleanor W, Eleanor D used the printed numerals on the book pages to copy a squiggle anew in each particular instance. She read '41' as "24" after having recited "39, 40, 41" with the intervention of the interviewer. When Eleanor D was interviewed alone, her answers suggested local criteria to deal with number scripts. For example, she explained that '56' was higher than '3' *"because it hasn't got one number!"*. However, she subsequently placed '2500' after '89567'. Like Eleanor W, Eleanor D's responded in an idiosyncratic manner to the recognition questions of the initial assessment. For example, she read '13' as "33", '15' as "45" and '12' as "32". In various occasions, Eleanor D's comments suggested that her reflections on number signs were related to events in her family life. For example, she said she knew how to write "12" because her sister was 12. However, she did not build up on this to write "13" or any of the other teen scripts. In the realm of number words, Eleanor D produced a "tee sequence" in the same fashion as Eleanor W and Eloise. With respect to hundred words, Eleanor D always responded in a "this-is-my-best-guess" manner, like it is apparent in the following episode with Hettie in the context of the cinema scenario[21].

I: "[I]f we have one hundred, 100, and 100 (pointing each time), how many people altogether?"
H: "Uhm....Two hundred?"
I: "Two hundred do you think? How many do you think Eleanor?"
E: "Uhm....Twelve hundred?"

Eleanor D had counted the 9 parts of the cinema, and in one occasion stated that there were 3 people in the 3 middle parts of the cinema thereby counting the parts of the cinema rather than the hundreds of people.

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[21] Cinema scenario 1 session

5. 4. 7 Discussion on the case studies of Group 3

Although the cases of Group 3 differ in their particular features, they all fell in the category "no apparent construction of stable rules". When there was a criterion that seemed to underlie the child's explanation or solution to a task, this was local, or particular to the situation and therefore prone to be replaced in a new situation. In addition to their inconsistency, these criteria were unconventional or partially conventional and subject to destabilisation by the interviewer's counter-suggestions. For all three children, writing numbers up to 100 was evidently a tedious task and writing these marks hardly seemed to engage them in discussion about how "numbers" were written. Number words and number scripts did not seem to be organised according to stable rules. As Eloise said on one occasion, counting up to 100 *"would probably take one whole day and one whole night"*. The three children made similar remarks involving family events when writing and reading number scripts. For example, Eleanor W expressed that her brother was *"nearly going to be ten"* when she was doing the board of 100 seats[22], or when she played with the dice she commented: *"you get 2 like baby 2"*; or Eleanor D commented *"I'm 5"* after getting a score of 5 in the die game.

5. 5 Discussion on the Findings of the First Stage of Fieldwork

Number work at school during the first stage of fieldwork - i.e. last term of reception year - can be characterised as adding and taking away sums up to 10, typically accompanied by pictures of objects. With the exception of some children (e.g. Tom, Johnny and other children of the class) all these activities drew upon numerals and words only up to 10. For those children who the teacher considered "more advanced" their worksheets incorporated numbers up to 20. Therefore, children's regularities or rules in number words and scripts had not been imparted to them at school. That "one thousand" is written as '1000' is a piece of "social knowledge"[23]. However, that "all thousands are written with 3 zeros" is an original elaboration by young children who interact with numerals and the way they are used in their social settings. Therefore, children's rules were seen as original elaborations or constructions which they had

[22] Board of 100 seats session

[23] Kamii, 1986.

established or appropriated from their reflections upon these external signs and their use in their everyday environment. The question was whether by exploring children's stable theories about external signs one could unveil relevant implications on children's progress in arithmetic. But before drawing possible connections between the findings of the two lines of investigation, it was necessary to examine the categorisation of the cases in the three groups presented above.

Findings of the first stage of the fieldwork indicated that all children could produce regularities or justifications for their use of these external signs. However, it was only some of the children who had constructed stable theories about number words and scripts. For others, their rules were idiosyncratic and unconventional. The former were termed "rule makers" and depending on the domain in which they showed to construct their rules they were termed "rule makers in number words" or "rule makers in number scripts". This suggested that there were different systems of number words and scripts available to children . Children's rules indicated that they were establishing a frame of organisation where a particular number word or script could be referred to. A given number word could be related to others and by the same token a particular script could be referred to other scripts via the use of a rule or a regularity. The set of rules that children elaborated constituted the child's system of number words or scripts. Children of Group 1 were rule makers in number words and scripts: it seemed that rules in number scripts were increasingly integrated to rules in number words. For example, the rule of numbers of a kind was coherent with the rule of hundred and thousand scripts: children believed that numbers of a kind were similarly written -e.g. with the same quantity of zeros, and different kinds of numbers were written differently.

However, as was previously discussed, situations of tension emerged when current ideas seemed to produce incompatibilities. Children of Group 1 had appropriated that there were numbers of the same kind - e.g. hundreds or thousands. They believed that these were written in a similar manner - e.g. hundreds were written with 2 zeros and thousands with 3 zeros. Hence, when "10 hundred" was dictated, a perturbation was experienced. This situation had not been anticipated. It seemed to arise from the particular way of reading 4-digit numbers in English. For example, '1900' can be read as "1 thousand 9 hundred" or as "19 hundred". The "equivalence" of reading

'1000' as "10 hundred" or "1 thousand" was yet to be explored in the second stage of fieldwork. A prospective solution of this tension could offer a new plane of reflection on the organisation of "numbers". Moreover, this tension was particularly relevant in the subsequent stage of the study, due to its connection with children's arithmetic. In contrast to Group 1, children of Group 3, seemed to have no stable frame of organisation of the external numerical signs. Because their criteria seemed unstable, it seemed on occasions that every occurrence of a number word or script was an isolated squiggle or word referring to a particular event - e.g. somebody's age, a quantity of bricks. Children of Group 2 were also rule makers but the history of construction of their system of conjectures indicated that their reflections were initially and predominantly in either number words or number scripts. The subsequent stage of the study needed to establish whether rules from the other domain would be elaborated. It must be remarked that the perturbation caused by the two readings of the script '1000' did not arise for the children of Group 2 nor for the children of Group 3. This confirmed the assumption that for perturbations to occur, a set of viable theories need to be established. In this case, children's rules in both number scripts and number words seemed to be necessary for this perturbation to arise. In reality, as was documented above, Eamon seemed uncomfortable when having to write "10 hundred". He was not a rule maker of Group 1, but nevertheless, this indicates that he was establishing conjectures in both number words and number scripts by the end of the Summer term. This had to be confirmed in the subsequent stage of fieldwork.

Despite the stable rules found in Groups 1 and 2, it must be remarked that (with the exception of Johnny) there was one pervasive finding: children also produced concatenated forms. It was conjectured that for children of Group 1, the rule of numbers of a kind and the rule of hundred and thousand scripts were not compatible with these concatenations. Precisely, if a 3-digit script was a hundred number, and a 4-digit script was a thousand number, children could not accept that "2300" is written as '2000300'. This meant that confirmation for the establishment of the aforementioned rules and the consequent letting go to concatenated forms had to be explored in the final stage of the study.

The rule of the numbers of a kind deserves further consideration. Children who seemed to have constructed this rule had appropriated that, in our naming system of

numbers, units of the same order can be operated on as single units or ones.
As was previously discussed, Skemp pointed out that the Hindu-Arabic system makes use of the fact that when one adds 2 sets with 3 sets, the result is 5 sets irrespective of whether the sets are single objects, sets of objects, or sets of sets[24]. Remarkably, children of the study had apparently appropriated this essential characteristic of numeration from their reflections on spoken numbers. Precisely, they seemed to have constructed this criterion using number names where numbers of different kinds - i.e. units of different orders of the base - are explicit: for these children, 2 hundred plus 3 hundred was 5 hundred because 2 plus 3 was 5 and because hundreds were numbers of the same kind. Similarly, children's sequence in tens was also a relevant construction: the idea of ten seemed intuitively important as they spoke and wrote the decade signs. For children who had established the rule of the sequence in tens, counting in tens had a different meaning from counting in ones because they paused, puzzled, at 100 and reflected about the continuation of the count. One of the questions for the second part of the study was the implication of this presumed rule in children's arithmetic knowledge.

Although children were organising their number word system as well as making sense of written numbers independently of the interventions and guidance during interview sessions, it was apparent that the research sessions constituted situations of learning for the children of the study. As has been discussed, Eamon and Jack seemed to engage with their more knowledgeable companions. The second stage of the study needed to establish what rules seemed to be independently used and could be confirmed as stable for a particular child. This meant that in the final stage of the study the interview sessions had to be carried out individually.

Drawing connections between the two lines of the study

Children's progress in arithmetic is indicated by their ability to drop actual counting routines with physical objects. They must construct thinking strategies through the use of external conventional signs that come to stand for systems of integrations or composite units. The model of learning stages circumscribes the explanation of

[24] Skemp, 1986, refer to Chapter 1.

children's arithmetic knowledge to the schemes of action and operation that are functioning reliably and effectively in the activity of counting. It was the question of the present study whether knowledge of systems of number words and scripts has any part in children's acquisition of arithmetic sophistication. The model of learning stages does not seem to account for the significant constructions that children abstract when using number words and scripts. In terms of rules upon number words and scripts that children construct, does this knowledge integrate with children's arithmetic knowledge? Based on the findings of the two lines of the investigation, it was apparent that for the nine case studies, those children who had a more advanced organisation of their counting activity, were in Groups 1 and 2. Children who had at least constructed the INS, that is, children who had a numerical scheme, were rule makers in number words or in number scripts. Stephanie was the only rule maker, in number scripts, who was a counter with a non numerical scheme. However, it must be remarked, Stephanie's rules were restricted to 2-digit scripts. The following table presents the results of the first stage of fieldwork regarding children's organisation of counting and their ideas on number words and scripts.

Children's Counting and their Rules on Number Words and Scripts- July 1998

		Group 1: Rules in number word and scripts	**Group 2: Rules in number words or scripts:**		**Group 3: No stable rules**
			words	**scripts**	
NUMERICAL COUNTING	TNS→ENS	TOM JOHNNY			
	INS	ALICE	EAMON	JACK	
NON NUMERICAL COUNTING	PS→FS			STEPHANIE	ELEANOR W ELOISE
	PS				ELEANOR D

One can coarsely conclude that rule makers were the more advanced counters, in particular, counters with a numerical scheme. However, it must be noted that Alice, Jack and Eamon were figurative counters at the start of the study whose counting scheme transformed by the end of term assessment. In particular, Alice, who was initially a counter of the figurative stage and a rule maker in both spoken and written numeration was - three months later - a counter of the INS. It was reasonable to conjecture that children's constructions in the activity of counting and their rules in number words and scripts were initially relatively independent systems but as they integrated, they empowered children's progress in arithmetic. If this was the case, the following stage of fieldwork should confirm that rule makers in both number scripts and number words would transform their counting methods into the use of more sophisticated strategic reasoning. Second, those children who had not established rules would tend to continue in the stage of non numerical counting.

That children can notice patterns in written numbers and verbal patterns in number names has been said to be a product of *empirical* abstraction as opposed to *reflective* abstraction (this latter leading to mathematical knowledge). Children can identify, for example, that 1 is in front of all the teen scripts or learn how to write all 2-digit numbers by repeating a cyclical order. However, upon these abstractions children can conjecture about the reasons for these patterns. If these abstractions have not been made, it seems unreasonable to "teach" them that 1 in the teens means 10. Without claiming that the identified rules constituted mathematical knowledge, the present thesis sets forward the conjecture that children's rules were original constructions upon the conventional signs. A child could be shown that '100' is read "100" and '1000' is read "1000" but children of the study were not told that the more digits a script had the higher the number, nor were they told that if a "number" had 3 digits it was a "hundred number". Findings of the first stage of fieldwork suggested that the abstraction of regularities of number words and scripts was inextricably linked to the meaning a child conferred to a particular rule. These two aspects could not be easily separated. For example, for Eamon, counting in tens meant making higher numbers faster. Moreover, Eleanor W, Eleanor D, and Eloise did not seem able to abstract regularities of 2-digit numbers despite prompts by the interviewer. This suggested that abstracting regularities was not a mere matter of empirical abstraction, which any 5 year old is capable of. Rather, children tended to interpret the products of their empirical abstractions and constructed theories with which they interpreted the external signs. Children's rules constituted children's knowledge of an hypothesised system of number signs. They appropriated conventionalities through the elaboration of viable criteria to organise external numerical signs.

5.6 Summary

This chapter has presented the findings of the first stage of the fieldwork. According to the first line of study, the nine case studies were classified in two major groups: numerical and non numerical counters. According to the second line of the study, the case studies were classified in three groups depending on their establishment of stable rules upon number words and scripts. It was conjectured that if children's ideas upon conventional signs integrated with their system of mathematical constructions, children would achieve further sophistication. Two main aspects had to be focused on in the final stage of the fieldwork: first, there was the need to confirm the establishment of the inferred rules in Groups 1 and 2 and to explore the construction of stable criteria for children in Group 3; and second, there was the need to establish levels of sophistication in children's arithmetic knowledge focusing on their use of strategic reasoning. Finally, there was the goal to ascertain the extent to which children's rules integrated with their arithmetic knowledge.

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6. Findings of the Second Stage of the Fieldwork

6. 1 Introduction

This chapter presents the findings of the final stage of fieldwork. Like the previous chapter, it is organised in two parts according to the two lines of investigation. In part one, the analysis presents the transformation of the counting schemes of the nine children. This analysis focuses on their concept of ten, their use of thinking strategies and their learning stage. In part two, the analysis focuses on the confirmation of children's rules inferred in the first stage of fieldwork. Second, it examines the transformation of these rules through the establishment of new criteria and rules. Finally, it examines the possible development of rules for children in Group 3. The chapter concludes by setting the grounds for a general discussion of the findings.

6. 2 Part One: Children's Progress in Arithmetic

Findings from the first stage of the study showed that for Tom and Johnny, and subsequently, for Jack, Eamon, and Alice, a wider range of alternative methods seemed available to them through their reflective activity. These children were able to count abstract units and in some occasions they resorted to thinking strategies to solve the counting tasks. They continued employing counting routines to solve the counting tasks, but they demonstrated alternative solutions: for example, using an addend decreasing strategy to solve 5+6[1]. On the other extreme, Stephanie, Eloise, Eleanor W, and Eleanor D seemed unable to count abstract units. They seemed limited to their immediate experience and needed actual or visualised objects to interpret and solve the given problems. One goal of the second part of the fieldwork was to look closer at the former group of case studies and their increase in sophistication in their arithmetic knowledge. Sophistication was indicated by their ability to reason with thinking strategies. Concomitant with this was the children's ability to reason with abstract composite units, in particular, units of ten.

[1] 5+6 is 11 because 6 + 6 is 12

Although part one of this chapter does not deal with the analysis of children's rules, these may be referred to in the following sections, including new rules that were identified in the second stage of the study[2]. In the following sections, there is a final analysis of children's counting activity according to the model, and particularly, an examination of children's growth in sophistication in solving arithmetic tasks. The sections address the case studies in their original groups: the more advanced counters, the intermediate counters, and the figurative or perceptual counters. Any transformations are discussed at the end of each section.

6. 2. 1 The more advanced counters

The case of Tom

The more striking characteristic of Tom's responses during the final counting assessment was the absence of counting actual items by ones when solving the tasks presented. He had previously - and occasionally - used addend decreasing strategies in the first stage of fieldwork but now, thinking strategies were his predominant solutions. For example, Tom used a compensation strategy to find out the total of 9 visible bricks and 4 hidden ones. He swiftly responded "13" and explained: *" 'cause I just did it with 3 and 10, 'cause I just needed to do one less"*. Likewise, Tom used a recursive compensation strategy when writing sums totalling a given number. For example, for 20: '19+1', '18+2', '17+3'. In explanation, he expressed *"you have to add one more on to there and it has to be one lower there"*. Addend decreasing or addend increasing strategies also prevailed in Tom's solutions. For example, he solved '15+....=21' based on "15+5=20" and '5+...=22' based on '15+5=20'. Tom's explanation for the latter solution was *"I knew 15 plus 5 equals 20 so I just needed to add two"*. The latter strategy suggested part-to-whole reasoning. Tom disembedded 2 from the answer "22", and 20 and 2 were recombined as 5, 15, and 2 and subsequently as 5 and 17.

Tom's use of ten in all situations during the second stage of fieldwork, indicated that he had constructed ten as an iterative unit, and also as an abstract collective unit. For example, when he solved 57+....=89, he explained *"I went 67, 77, 87, and then added*

[2] Part two of this chapter will specifically address children's rules in number words and scripts.

two more units to that make thirty two". Tom used a flexible sequence in tens[3] in anticipation and in the absence of any actual material. He kept track of 3 units of ten and 2 single units, which he transformed into 32 single units. Another example of Tom's use of an iterable ten was documented when he had to solve the sum "24 + 15". He said swiftly that it was "39" and explained, *"'cause I just did 29, adding on 5 and I add a 10, that's 39"*. It must be observed that Tom often expressed that he had "added" instead of "counted". Evidence that Tom's sequence in tens and flexible sequence in tens implied increment or decrement of 10 ones was found in his solutions and explanations. For example, he responded untroubled that the tickets '72' and '82' were "10" seats apart[4] and explained: *"I know because it's ten more"*. In a similar manner, he solved that '82' and '52' were 30 apart and declared that he had *"done it with 5 and 8"*. Focusing on the digits of the scripts was a recurrent feature of Tom's responses. In the context of working with cinema tickets[5] he expressed that '70' and '100' were "30 seats apart" and claimed: *" I did it with 3 and 10"*. He seemed to focus on the decade digits suggesting an idea of ten as a collective unit. For example, he responded that the answer to "12+...=23"[6] was "11" because *"there had to be another ten"* and another *"one"*. It was apparent that he did not count by tens from 12 but he seemed to focus on the units of tens and single units. A reversible co-ordination of units of ten and single units was also another characteristic of Tom's solutions. He solved at ease sums like '45-....=36', or '62+....=71' by adding or subtracting a unit of ten and then compensating by subtracting or adding a single unit. For example, for the former sum, he stated that "9" was the answer and explained: *"5 and 5 would be 10 so 6 here, it means that it has to be 1 less there to make that number"*. After using a unit of ten ones he acted on it to subtract single one: 45 - 10 = 35, hence, 45 - 9 = 36. Further, this type of solution involved strategic reasoning, i.e. a subtrahend variation strategy: 45-9=36 based on 45-10=35.

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[3] Refer to part two of the present chapter: a flexible sequence in tens can start at any number word.
[4] Cinema tickets session
[5] Cinema tickets session
[6] Rows and squares session: 1 row and 2 squares visible and 23 squares all together.

The case of Johnny

Johnny used thinking strategies as opposed to counting routines but he often used counting-on, and double-counting solutions until the situation was too burdensome. The first problem that seemed to trouble him was the written sum '22 - 17'. He seemed to realise that counting off 17 was a very tedious option. Johnny "counted back the 10" but without performing the count. Then he expressed he had to count back 7. When asked why he had counted back the ten, he explained: *"Because I know 20 take away 10 is 10 [...] and add the two on the ten"*. This suggests that he disembedded the 2 from 22 and added it to 10. Further, Johnny indicated that he was aware of another alternative solution. The interviewer asked him whether he could count on to solve the problem and he engaged with this suggestion by counting on 5 from 17 and asserting: *"counting on is easier"*. This episode suggested that Johnny was capable of part-to-whole reasoning and that he was aware of the relation between addition and subtraction. On the other hand, Johnny often used thinking strategies to solve arithmetic tasks. For example, he responded untroubled that the total of 6 and 7 hidden bricks was "13". In explanation he expressed, *"'cause I know what 6 and 6 is and I added on the one"*. Similarly, he reasoned that the answer to '15+....=21' was "6" because *"15 add 5 is 20 but I needed 21 so I added a one to it so it would be 6"*. Further, Johnny reasoned strategically with units of ten. For example, when he solved "70+60", he explained *"I know 60 and 60 is a hundred and twenty, a hundred and thirty is, I add another 10 is a hundred and thirty"*. It was also conjectured that he was using his rule of numbers of the same kind because of his knowledge of the double of 6 documented further above (if 6 and 6 is 12, then 60 and 60 is 120). Like Tom, Johnny used compensation strategies, for example, when he solved that the answer to '49+...=100' was "51" because *"49 is one less than 50 so 50 would be one more"*. Similarly, he compensated 5 less with 5 more to solve "45+....=100'.

Johnny used ten as a unit itself composed of 10 items. Ten could be a unit but the meaning could shift to the composition of ten items according to the situation. He could count by tens and ones in anticipation to solve a range of non contextualised problems. This meant that he had constructed ten as an iterable unit. The following episode presents Johnny's explanation of his answer to the sum '37+52'.

J: "I started at 37 and then I added another 10, that's 47, and then I added another 10, that's 57, and another 10, that's 67, and another 10, that's 77, and another 10 is 87, and add the 2 is 89"

In a similar fashion, to solve the problem "48+.....=72" Johnny reasoned *"48 there, 58, 68, er, **twenty four**"*. He put 2 fingers up 1 by 1 to keep track of the counting acts by ten, hence, 2 counting acts by ten were twenty and then he added 4 to 72. His use of a flexible sequence in tens in a forward or backward fashion was a typical strategy. Moreover, Johnny's use of a bi-directional flexible sequence in tens in the domain of hundreds and thousands. For example, in the context of writing a cinema ticket which were 10 seats apart from '580', he wrote untroubled '570'. When suggested that another friend wanted a seat 10 seats apart from '580', he wrote '590'. Similarly, he solved at ease 713+30, 834+10, and 615+20[7]. Johnny's use of his sequence in tens or flexible sequence in tens indicated that he increased or decreased by 1 ten or 10 ones. His use of decades as if they were ones is illustrated by his solution to "90 + 50".

J: [stares ahead in concentration for a few seconds] "A hundred and forty"
I: "How did you figure it out?"
J: "'cause I pretended that was 5 and that was 9 and then I counted in ones"
I: "Why did you pretend that?"
J: "So it's easier"

A reversible use of units of tens and ones was also an advanced strategy. For example, when he had to solve the sum '57+...=83', he used a flexible sequence in tens "67, 77, 87" and monitored the subtraction of 4 single units.

6. 2. 2 Summary of the cases of the more advanced counters

Tom's and Johnny's use of thinking strategies and disembedding operation indicated that they were both operating at the level of the ENS or were on the transition to establishing the ENS. This new reorganisation of their number sequence implied the commencement of part-to-whole reasoning and awareness of the relation between addition and subtraction. Their use of a unit of ten in an image-independent manner indicated that they had constructed ten as an iterable unit, the most advanced type of composite unit of ten. Both children used a flexible sequence in tens in forward or

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[7] Cinema tickets session.

backward direction which could start at any point of the series. Children's flexible sequence in tens together with their solutions to arithmetic tasks involving hundreds and thousands will be further discussed in part two of the present chapter.

6. 2. 3 The intermediate group

The case of Jack

Jack showed a notable transformation of his former counting routines. He could now solve missing addend problems through the use of strategic reasoning. For example, he used a compensation strategy to solve how many from 12 bricks had been hidden when 7 were displayed.

J: [counts the 7 visible cubes] "Five"
I: "How did you work it out?"
J: "I counted these and there was 7 and I knew 6 and 6 is 12 so if you took away 1 from 6, so it would be 7 plus 5"

Similarly, he used an addend decreasing strategy to solve that 6 bricks were hidden from a total of 11 when 5 were displayed. After a few seconds of concentration he responded "6" and explained: *"[t]here's 5 here, and 6 and 6, and take a 1 and you don't add it on to the other number and you get 5 plus 6 and I've got 5 and you've got 6 and it's 11 all together, there's 11, you've got the 6".* Jack derived 5+6 from 6+6. Thinking strategies were also used with numbers of different kinds like Jack explained when he wrote sums to 100 and 1000. For example, Jack wrote '50+50' and '60+40' and explained he had taken *"one away from the fives"*. That "one" was unit of 10, because Jack distinguished situations that required single units: for example, to solve '49+....=100' Jack wrote '51' and justified *"you have to add one more to fifty"*. New sophistication in Jack's solutions were evidenced by his reversible use of tens and ones which indicated his ability to perform progressive integrations. For example, when he had to solve the problem '58+....=91', his reasoning indicated that he had added 4 tens and subtracted 7 ones to obtain "33". Similarly, he solved the problem '36+....=45' by adding 1 ten and subtracting 1 single unit; or '52-...=37' by subtracting 20 and then adding 5. Moreover, by the end of the study Jack showed indication of part-to-whole reasoning as it was apparent from his solutions to a range of arithmetic tasks. For example, in one occasion he solved the problem '20-...=12' and he reasoned aloud:

J: "Well, if you have 20 take away 10 it would be 10, I add another two then it would be twelve, I take two from ten so it would be 8.

Jack viewed 20 as two parts of 10 and then disembedded 2 from one 10 and added it on to the other 10. Hence, he recombined 20 as 12 and 8. This sophistication was concomitant with his advanced unit of ten. Although this was not an immediate finding of the final stage of the fieldwork, Jack's solutions to arithmetic tasks evidenced a rapid progress into dealing with ten as an iterable unit. It must be noted that he started the second part of the fieldwork using a novel sequence in tens[8]. By the end of the study, he seemed to have established a flexible sequence in tens which he could use from any point of the series in anticipation and in an image-independent manner. This meant that his most advanced unit of ten was iterable or an abstract collective unit but, it must be remarked, this was not an immediate achievement. In February 1999 Jack gave indication of being able to co-ordinate a count by tens and ones only in an image-dependent manner. For example, in the context of the cinema tickets, he seemed to visualised rows of 10 seats to be able to tell how many seats apart there were between two given tickets[9]. By the end of the study, his use of a flexible sequence in tens meaning an increment of decrement of 10 was apparent. For example, he solved at ease the sum '72+...=96' by *"adding the tens first"* and then *"adding the 4"*. Like Tom, Jack dealt with decades as if they were ones. For example, when he had to solve how many clips there were in two boxes with '30' and '70'clips, he responded *"one hundred"*.

I: "How did you do that? I didn't see you counting"
J: "I know, I didn't count"
I: "How did you do it?"
J: "Well, I take away the nought (covering the zeros in 30 and 70), and add 3 more from 7, and what do you get? 10! Add the nought and I've got the number a hundred"

Jack often referred to the digits of the scripts to support his explanations. One could presume that he was taking the decades as if they were ones. The above episode suggests that Jack could re-establish the meaning of composed ones by "adding the nought" back. Jack seemed to use ten as a unit itself composed by ten items and with the meaning of incrementing or decrementing by ten. For example, when he solved

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[8] Jack was in transition to establish a sequence in tens by the end of the first stage of fieldwork; refer to part two of this chapter.
[9] Cinema tickets session

'48+...=72'[10], he subtracted 4 tens from 72 and subsequently subtracted the 8 because he explained: *"I took away eight from thirty two"*.

The case of Eamon

Eamon could now explicitly double count to solve missing addend problems. For example, the following episode documents how he solved how many bricks were hidden from a total of 11 bricks when 5 were displayed.

E: [holds his head and looks ahead in concentration] "Six?"
I: "How did you know?"
E: "I found here there were 5 and then there were....and then I counted in my head 6, one, 7, two, 8, three and then 8, three, what was it....9, four, and 5 and 11, six!"

Eamon solved other problems by double counting by ones but he also resorted to more efficient strategies. For example, when he was presented with the sum "23+...=36" he attempted to count by ones and kept track of his counting acts. When the interviewer asked him whether he could count in tens, he commented: *"23 add 10 was 33"* and he needed *"3 more"*. This suggested that he was aware of a flexible sequence in tens because he responded swiftly that 23 add 10 was 33. Likewise, in a subsequent episode he responded that if he started with 22 and he finished with 12, he would have taken away 10. Apart from double counting strategies, Eamon often used addend decreasing or addend increasing strategies. For example, he solved "6+7=13" based on 7+7=14, "9+5=14" based on 10+5=15, or '65+...=71' based on 65+5=70. Similarly, he reasoned *"50+60 is a hundred and 10 because 50 and 50 is a hundred and 50 and 60 is one more 10"*. However, a major difference with the more advanced counters was that Eamon seemed incapable of reversible co-ordination of units of tens and ones. Until the end of the study he seemed at a loss when the strategy of adding tens and subtracting ones was proposed. For example when he had to solve the problem '37+....=52' he seemed troubled and did not engage with subtracting 5 ones after adding 2 tens. He seemed unable to subtract units of ten and single ones in anticipation to solve, for example, '61-....=49'. Eamon's most advanced co-ordination of counting in tens and ones was in the presence of suitable material. By the end of the study he successfully solved problems with rows of 10 squares and

[10] Rows and squares session: 4 rows and 8 squares visible and 72 squares all together.

single squares[11] which indicated his idea of ten as abstract composite unit. Eamon could use a flexible sequence in tens to count tens and ones in the presence of suitable material. This was not an immediate achievement, since in February 1998 he could only use his old sequence in tens. For example, in the context of 100 seats he solved that '70' and '100' were "30 seats apart" and explained: *"I counted on from 80 to a hundred and I saw how many were in my head, it was 3"*. Eamon used his sequence in tens but could not use a flexible sequence in tens to tell how many seats there were between '11' and '21'. Nevertheless, his solution to '52' and '82' indicated his flexibility because he counted the rows of seats by tens, using a sequence in tens and responded "30 seats apart"; subsequently he expressed: " *"I added **three** more tens"*. However, his use of ten was not iterative because when he was asked how many seats there were between '65' and '75' he proceeded to count the seats on the board. By the end of the study, Eamon seemed able to use a flexible sequence in tens by starting at any point of the series to count in tens and ones in presence of suitable material. Nevertheless, in the absence of suitable material, he was troubled: for example, when he had to solve sums like '37+....=52. That he had constructed ten as an abstract composite unit was indicated by his co-ordination of counting by tens and ones in problems with rows and squares. For example, in the following episode, 5 rows and 7 squares were displayed and he worked out how many were hidden from a total of 91.

E: [pointing at the visible rows and squares] "81, 71, 61, 51, 41, 40, 39, 38, 37, 36, 35, thirty four!"
I: "Shall we see?" [uncovers the 3 rows and 4 squares]
E: [delighted] "Thirty four, yeah"

When suitable material was absent Eamon often solved the problem aided by the interviewer. For example, with the written sum '37+...=52', he added 10 and then 5 under the guidance of suitable prompting.

The case of Alice

Alice relied on counting routines by ones unless this became too tedious. In these cases, she derived sums from other known sums, using thinking strategies. Further, she showed indication of being somehow aware of the relation between addition and subtraction. For example, she solved '22-17' by counting on 5 from 17. In explanation she said *"I did 17 plus 5 equals 22 and I took 17 away"*. Alice generally counted on

[11] Rows and squares session

by ones to solve partially screened collections and double counted when solving missing addend problems or similar problems such as written sums (e.g. '8 +...=13' or '23+...=36'). She also reasoned strategically to find the answer to a missing addend problem. For example, when she figured out how many bricks from a total of 12 bricks were hidden if 7 were displayed. Alice seemed to derive the sum 7+5 from her known 8+4. Likewise, she used addend decreasing strategies or compensation strategies. For example, in the context of writing sums to 20, Alice derived 4+16 from a precedent 6+14 because she explained: *"I took away 2 from, er, 6 and then I added 2 from 14"*. This was a spontaneous compensation strategy. Similarly, Alice used compensation strategies with sums up to 100 by *"going up and down in tens"*. For example, she wrote '50+50', '60+40' and '70+30'. However, until the end of the study Alice was unable to co-ordinate counting by tens and ones in anticipation. Her most advanced concept was ten as an abstract composite unit. This meant that Alice co-ordinated counting by tens and ones to solve situations when suitable material was available. In episodes towards the end of the study, she seemed able to count by tens and ones using a flexible sequence in tens but this was in an image-dependent manner. For example, in the context of a rows and squares problem she solved the total of 22 squares hidden and 1 row and 6 visible squares. She reasoned *"32- 33, 34, 35, 36, 37, 38"*. Alice counted by tens from 22 to 32 because there was 1 row of 10 and then she proceeded to count on the 8 single squares by ones. However, when she was presented with a missing addend problem she was visibly uncomfortable. On one occasion 4 rows and 7 squares were displayed and Alice had to find out the number of hidden squares from a total of 79 (47+...=79): she counted by tens and ones the 47 squares and then stopped in apparent puzzlement. In a similar episode she attempted to solve '57+....=91"[12].

A: [counts in tens and ones the visible rows and squares] "Fifty seven"
I: "Uh-hu. And how many here?" [points at the blue cover]
A: [clapping hands] "58, 59, 60...58, 59...60, 70, 80, 90, 91....so that is...how many tens did I count?"

Alice counted by ones and tens, using her sequence in tens, using her acts of clapping but could not subsequently keep track of how many tens and how many ones she had counted. Prompted by the interviewer she repeated her solution, this time keeping track of her counting acts by ones and by tens using her fingers. She uttered *"58, 59,*

[12] Rows and squares session

...60, 70, 80, 90...-91" and next she declared *"That's thirty three"*. Despite omitting 1 single unit, she solved the problem co-ordinating counting by tens and ones with the support of suitable material. A noticeable feature was that she counted in ones first and then proceeded to use her old sequence in tens adapting it to the problem but she did not use a flexible sequence in tens (e.g. "67, 77, 87"). However, occasionally Alice used a flexible sequence in tens to solve other situations. For example, she solved '37+....=47' and declared: *"I know 10 plus 10 plus 10 and I know how to count from 7 in tens, 7-17-27-37-47-57-67"*. Nevertheless, this was not an immediate achievement, since in the context of the cinema tickets[13], she counted the seats by ones from 11 to 21 to respond that there were 10 seats apart.

6. 2. 4 Summary of the cases of the intermediate group

Jack seemed to reorganise his number sequence dramatically during the second stage of fieldwork. At the end of the study he could iterate tens and reversibly co-ordinate units of tens and single units. He seemed capable of part-to-whole reasoning and that placed him in the ENS with Tom and Johnny or at least in transition to the ENS. On the other hand, Alice and Eamon could only operate with abstract composite units of ten. This meant that they were capable of co-ordinating their sequence in tens and ones with the support of suitable material. Their thinking strategies indicated that they were capable of progressive integrations and were now in the TNS. With respect of the second line of the study, it will be discussed how Jack had established new rules in number words. When solving arithmetic problems he used a novel sequence in tens and a flexible sequence in tens. This meant that he no longer was a case of Group 2 as defined in the first stage of analysis because he had established rules in both number words and number scripts. On the other hand, Eamon and Alice were using the previously established rules to solve arithmetic problems, in particular, their sequence in tens.

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[13] Cinema tickets session

6. 2. 5 The group of counters of the figurative or perceptual stage

The case of Stephanie

Stephanie showed clear indication that she no longer needed to count from 1 when solving partially screened collections. She was now a numerical counter and her reorganisation of counting was evidenced by her typical counting-on solutions. For example, she solved how many clips were in total in two boxes with '5' and '9' clips.

S: [counts on 5 subvocally, putting fingers up 1 by 1) "14"
I: "How did you do that?"
S: "I just did the same [as I did] with the other questions. I said I had 5 and I just added 9 on, I just carried on from 5: 6, 7, 8, 9, 10, 11, 12, 13, 14" [putting fingers up again]

Stephanie also displayed other strategies that bypassed counting routines. These were thinking strategies similar to those displayed by Eamon in the first stage of the fieldwork. For example when she was told that there were 11 bricks all together and only 5 were visible, she reasoned: *"I counted these [5] and I knew 5 and 5 is 10 so I just added another one on!"*. Stephanie's use of addend decreasing strategies was found in other contexts. For example, when she worked out that 9 visible bricks together with 4 hidden ones made a total of "13". In explanation she expressed *"I know 10 and 4 is 14 and then I just took away 1 of them and it was 13"*. Although Stephanie solved missing addend problems by counting on her fingers, it was presumed that this was not done in anticipation with the force of double counting. For example, she solved "8+....=13" and stared at her 5 fingers to answer "5" but had trouble to solve "23+...=36". After a few guesses and prompted by the interviewer, she counted on her fingers and carefully stared at two patterns of 5 and 3 fingers, responding "13". Stephanie united her finger patterns of 10 and 3 performing the operation of integration. However, these were sequential integrations and this meant that she had established the INS only. This was confirmed by her trouble in solving "22-17" by counting off 17 on her fingers. Although Stephanie wrote sums up to 100 using apparent compensations (e.g. 98+2, 99+1), she never referred to adding and subtracting by ones or tens like the children of the TNS. Moreover, she was unable to solve '49+...=100' based on 50+50.

Stephanie showed her appropriation of a new sequence in tens but there were some limitations to its use. It was evident that she talked about tens as if they were ones and that she maintained the meaning of "differentiated ones". However she seemed troubled when she had to co-ordinate the two meanings. For example, when

Stephanie had to solve how many clips were in two boxes with '50' and '60' clips, she responded hesitantly, after a few seconds of deep concentration:

S: [uncomfortable] "A hundred and one?"
I: "A hundred and one?"
S: "I knew 50 plus 50 is a hundred and then I added another 10 and I got a hundred and one"

This type of episode suggested that co-ordination of tens as units and composites was problematic. It must be noted that Stephanie generally solved these problems using her sequence in tens and keeping track of the counting acts to obtain the answer. For example, she counted *"90, 100, 110, 120"* to respond that '80' and '40' was "120". It was when she had to co-ordinate the two meanings of ten as units and as composites that Stephanie encountered limitation. Stephanie's solutions indicated that she was using ten as a numerical composite. For example, when Stephanie was presented with 2 rows and 5 squares and she was told that there were 45 squares all together. After she established 25, she put up her 10 fingers 1 by 1 twice, as she was counting on subvocally to 45. She responded there were 20 squares under the cover. In a similar manner, when she was presented with the problem '34+....=54', Stephanie counted in ones putting her 10 fingers up twice and wrote '20' for the second addend. However, some solutions indicated her flexibility in using her sequence in tens and ones when suitable material was available. For example, when 10 rows of 10 single squares were displayed and 1 square was taken away, Stephanie responded that there were "99" squares left; and when 1 row was taken away, she responded that there were "89" squares left.

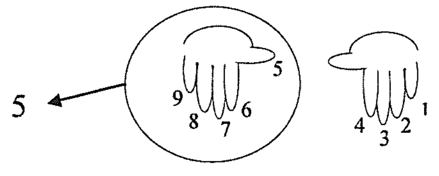
The case of Eloise

Eloise's counting remained non numerical because until the end of the study she typically counted from 1. Eloise often used a fixed re-presentation of objects (e.g. bricks, fingers) when collections were screened from her view. For example, in the following episode, she had to solve how many bricks were hidden all together under two covers with 6 and 7 bricks respectively. Eloise put 6 fingers up and then put 7 fingers up, and she repeated these actions, always staring at her fingers.

E: "First I had 8, no, 6, and then I had....[repeats the finger procedure with 6 and 7 fingers]. Sixteen"
I: "How did you figure that one?"
E: "I went 1, 2, 3, 4, ...5, 6, [putting up and staring at her 6 fingers and starting again] and then I counted it again and I came up with 16"

Eloise did not visualise the hidden objects, rather she seemed to re-instate anew the perceptual objects to be counted because she put 6 fingers up and counted them from 1 and then put 7 finger up and counted them from 1. Then she seemed to count the whole collection from 1 but this was unsuccessful because she could not keep track of how many fingers she had to count. In another occasion, Eloise seemed to visualise the hidden items. For example, when she had 9 visible bricks and 4 were hidden, she re-counted from 1 the 9 bricks and continued the count until 12, and then 13. In explanation she said she had "drawn" the bricks with her fingers "on the table". That Eloise needed to have the actual objects in place was apparent when she was asked to solve the written sum "5+6". She counted out 6 bricks and placed her hand wide open next to the bricks. Then she counted the whole collection from 1 by pointing at the bricks and her 5 fingers. By the end of the study, and in the context of writing sums[14], Eloise always counted her fingers in order to write sums to 10. Each sum seemed an isolated situation which she solved by putting fingers up and counting them to state the first addend and subsequently by counting from 1 the remaining folded fingers. For example, to write '2+8=10', she put 2 fingers up and counted from 1 the remaining 8 folded fingers. She repeated this procedure anew to write '3+7=10'. Eloise was troubled when she was asked to write sums to 20: after a few failed attempts to count her fingers, she gave up. She could solve situations like '3+....=10' by counting her fingers, but when the total surpassed 10, she was stumped. Her finger patterns were fixed figurative patterns. Her arduous solution to'4+....=9' indicates this:

E: "1, 2, 3, 4, [points at the I's fingers], plus [points to the 5th finger and carries on] 5, 6, 7, 8, [pause] nine" [...] "Plus....nine...equals...no [starts counting I's fingers again] 1, 2, 3, 4, [jumps to the 6th finger, so counting all 5 fingers of the other hand] 5, 6, 7, 8, 9....Four plus five equals nine"



Eloise's use of fingers when solving '4+...=9'

Eloise could only solve the problem when she counted a fixed pattern of "5" on the interviewer's hands. Her figurative counting scheme anticipated that her concept of ten had to refer to figurative patterns. For example, to solve the problem of '10' and

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[14] Laptop sums session

'10' paper clips in 2 boxes, she counted her 10 fingers from 1 to 10, and subsequently waved her two hands open again and counted all 10 fingers again up to 20. However, when she had to solve the total of '20' and '10' paper clips she was stumped. She solved the problem by counting out 10 bricks, then another 20 bricks and then counting the whole collection from 1 to 30. Eloise was at the level of figurative counting until the end of the study. In coherence with this, her idea of ten was related to a collection of ten actual objects, or a figurative ten. When she had to write her ticket so she could sit 10 seats apart from '60' she counted the seats, aided by the interviewer, arriving at seat '70'. However, when asked what ticket she had to write, she responded "ten". This episode occurred after it was apparent that Eloise had interpreted the task. In the context of the rows and squares situation, Eloise had counted 10 squares in 1 row and the next problem presented 2 rows. Eloise proceeded to count by one from 1 all the squares to respond "20". In a similar manner she counted by ones from 1 the squares of 3 rows, and 4 rows. When she was presented with the situation with 10 rows, she seemed overwhelmed and expressed: *"I'd never be able to count them!"*.

The case of Eleanor W

Eleanor W's organisation of counting had not undergone a significant change. She was able to count partially screened collections but her counting remained figurative because she typically counted all items from 1. For example, when solving how many bricks were in total with two hidden subcollections of 6 and 7 bricks, Eleanor W counted from 1 to 6 while putting fingers up 1 by 1 and continued counting subvocally and putting fingers up until she eventually stopped and responded "16". In another episode, she solved the total of 9 visible bricks and 4 hidden ones by counting from 1 to 9 and continuing to 13. It was presumed that she replaced the 4 hidden bricks with her fingers thereby indicating her figurative counting scheme. When she had to solve the total of '5' and '9' clips inside two boxes, she put up 9 fingers and carried on counting fingers until she paused, hesitant, and responded "15". On this occasion, it was presumed, she could not use a flexible pattern of five, to carry on the count. That is, when she carried on counting after the ninth finger, she counted an extra finger of her fixed finger pattern of 5. Likewise, her responses in the context of writing sums to 5 and 10 indicated that each sum was an independent event. She used her finger pattern of 5 to write '3+2', '2+3', and '4+1', each one solved anew. For example, for the first sum she put up 3 fingers and stared at the remaining 2 before writing the sum. In the same manner, she wrote '4+6', '6+4', '10+0', and '7+3'. When Eleanor W was asked to write sums to 20 she wrote '20+0' and then said that there were *"no more"*. The interviewer prompted her by writing the first addend - '19' - and she completed the sum by adding '1'. In explanation she said: "after ninetee[n] it goes twenty". This was an important episode to examine because Eleanor wrote '18+2', '17+3', and '16+4'. She explained that she was counting *"back from 20, 19, 18 and you keep on going"*. However, she did not refer to any compensation until she wrote '4+11' after she had written '5+10' for a sum up to "15". In this occasion she explained "after four it goes five so if it's just one less than 10, it must be a higher". Eleanor's insight was thought to be local to the context of the laptop task which gave her immediate feedback. Nevertheless, Eleanor was not seen to perform numerical integrations. She relied extensively on finger patterns which she used to solve problems like '20+...=30'. After a few failed attempts while she was waving her hands, she used her finger patterns of 5 twice in apparent concentration, and wrote '10' as the answer. In explanation she said *"I just counted to twenty, and then counted on, and then as I was counting with my fingers I added on 5 and then another 5 makes 30"*. Eleanor W's solutions to some of the rows and squares problems were apparent counting-on solutions. For example, when she was told that 14 squares were hidden, and 1 row and 6 squares were visible, she counted-on by ones the visible 16 squares from "14". However, this was a verbal curtailment because Eleanor W continued using her counting-all routines in other contexts. Her "counting-on" solution did not indicate a reorganisation of counting nor did it betoken an integration operation. As a counter of the figurative stage, Eleanor W's idea of ten was a figurative ten, supported by her finger patterns, or rows of 10 squares. In the situation with the cinema tickets Eleanor W counted the seats by ones to solve all the situations presented. For example, she counted the seats between 10 and 20 after placing the tickets on the board. Immediately after this, she was presented with the tickets '20' and '30' which she placed on the board. Then, she proceeded to count the seats from 20 to 30 and responded, with puzzlement: *"[t]en again"*. In the context of the rows and squares situation, Eleanor W used her "tee-sequence in tens" and a moment of tension arose from its use. First, she responded that 2 rows on the screen were 20 squares because

"10 plus 10 is 20" and seemed delighted at the fact that she had not counted. When she had to find the total of squares of 3 rows she whispered *"10, 20"* and then counted on in ones up to 30 while pointing at the single squares of the third row. In the same manner, when 4 rows were presented, she counted by ones from 21 to 40 while pointing at the individual squares of the 3rd and 4th rows. After this, she interpreted that there were 10 rows, and produced a tee-sequence to respond, hesitantly, that there were *" [t]wenty"* squares in total. Perplexed by the contradiction that there were 20 squares in 2 rows and 20 squares in 10 rows, Eleanor W proceeded to count by ones from 1 to 10 the squares of the 1st row and then continued with the 2nd row by ones. At this point, she paused, and continued reciting her tee-sequence, thereby arriving at "twenty" after counting 10 rows. She solved her conflict locally by stating that there were "ninet[ee] squares in 10 rows because 20 was not possible. Eleanor W realised that she could use this sequence when counting the squares of the rows, since it would produce the "same number" as if she counted in ones. Therefore, "thir[ee]" was a name in a verbal sequence. This meant that she was in fact counting the rows in ones, using a different recitation of words, she was not counting in a different way, like Tom or Eamon.

The case of Eleanor D

Like Eleanor W and Eloise, Eleanor D solved the counting tasks depending on actual objects or figurative items. On occasions, she provided justifications that indicated counting-on procedures. Nevertheless, it was apparent from Eleanor D's solutions across all situations that she could not perform numerical integrations. For example, when she had to solve the total of 9 visible bricks and 4 hidden ones, she guessed "12". Prompted to work it out, she took some seconds in concentration to respond "13". In explanation she said *" I counted in my head, from 9"*. Her counting-on seemed to refer to a verbal shortcut and not to the result of an integration. This presumption was confirmed when she solved a subsequent problem by a counting-all routine. To solve the total of 6 and 7 bricks screened from her view under two covers, she put 7 fingers up and put 3 more in silence, she paused, touched 3 fingers again with her face and then declared *"After 12, I can't remember the number"*. Eleanor counted all 6 and 7 fingers from 1 as replacements of the hidden bricks. This was a counting routine typical of the figurative stage. Similarly, to solve "11-4" Eleanor put up her 10 fingers and then put 4 fingers down 1 by 1. She finally responded "6" after folding down 4 of her fingers. This indicated her use of fixed finger patterns, like Eleanor W and Eloise. Her limitation to count actual objects explains her trouble to solve the problem of '9' and '5' clips hidden inside two boxes. Eleanor D used extended figurative finger patterns to solve some problems. For example, when she was asked to solve '8+....=13' she put up 8 fingers and staring at her hands open responded hesitantly *"5?"* and explained *"I know my fingers"*. When the interviewer asked her whether she could count on to solve the problem she responded *"[n]o"*. Likewise, Eleanor D's answers to the laptop sums were identical to those of Eloise. She needed the fixed patterns of fingers to solve a given situations. For the sums up to 10, she wrote '5+5, '4+6', '2+8', '3+7' and '9+1', in that order. She used her finger pattern of 10 which she partitioned conveniently for each sum. Like Eleanor W, Eleanor D looked ahead in concentration when she was asked to write sums to 20 and wrote '19+1=20' and '10+10=20'. When she was prompted to do more sums, she wrote '18+2=20' and '17+3=20'. Each sum was written after Eleanor D took some seconds to count to 20 from the first number of the sum. However, this was presumed to be a verbal curtailment for the same reason discussed above. When Eleanor was presented with the problems with tens, she resorted to "guessing", in an uninvolved manner. For example, when she was asked how many clips if she had '10' and '10' in two boxes, she said "20" and explained: *"my mummy told me"*. Eleanor D's figurative ten was illustrated by her solutions to the tickets situation. She counted by ones the seats to find out how many seats between '10' and '20' and '20' and '30' and she counted the seats between 40 and 50 to write the ticket 10 seats apart from '40'. Eleanor D arrived at the same conflicting situation as Eleanor W when using her tee-sequence in tens. In the context of a blind board of 10 by 10 seats[15] she counted from 1 to point at seat number 10. Subsequently she pointed at the 20th place to point at the seat number 20 and then she recited her tee-sequence in tens while pointing at the "decade seats".

E: "'cause I knew that, you count from thir[ee] all the way to fourt[ee] and then from fourt[ee] to fift[ee] [points at the rows] [...] Then you count on and you go to sixt[ee] then you count on and you get sevent[ee], then you count on and you get eight[ee], and then you count on and you get ninet[ee], and then you count on and you get to twenty...er, uh! [perplexed] I said, that one's.... [re-counts subvocally] Yeah, twenty"

[15] Cinema tickets session

Eleanor counted again using her tee-sequence in tens to make sure the last number was "twenty". On the other hand, she "knew" that there were "100" seats on the board. Although she admitted the last seat was 100, her perturbation remained unresolved because she used her tee-sequence in tens in subsequent episodes.

6. 2. 6 Summary of the case studies of the group of figurative counters

The most revealing finding was that from the four children, only Stephanie had progressed in her organisation of counting and was now at the INS. Eloise, Eleanor W and Eleanor D were still counters with a non numerical counting scheme. Stephanie showed flexibility to bypass counting by the use of strategic reasoning, a typical feature of the TNS. However this could only imply a transitional phase because she was unable to operate with abstract units of ten or perform progressive integrations. Eloise, Eleanor W, and Eleanor D had not reorganised their counting activity and they all used a figurative idea of ten. Most relevant was the use of a tee-sequence in tens which seemed to lead the two latter children to an uncomfortable situation when solving counting tasks.

6. 3 Discussion on the Case Studies in Relation to their Learning Stage

Three cases had significantly reorganised their counting activity in the course of one year. First, Alice, a counter formerly in the INS, was now capable of progressive integrations and was at the stage of the TNS. It must be noted that at the start of the study Alice was classified as a figurative counter together with Eloise. One year later, her solutions had undergone a significant change whereas Eloise remained in the stage of non numerical counting. Second, Stephanie, formerly on the transition to figurative counting, was now a counter of abstract units and had at least constructed the INS. Her solutions indicated independence from counting actual objects and she was also seen using thinking strategies. Third, Jack, a counter formerly in the INS was now reasoning strategically, reversibly co-ordinating units of ten and one, and started thinking with part-to-whole operations. It must be remarked, Jack was classified in transition to the INS at the start of the study which suggests that he was the case who more drastically transformed his counting during the one-year investigation. On the other hand, Tom and Johnny continued being the group of more advanced counters, now operating at the level of the ENS. At the other extreme, Eloise, Elearnor D and

Eleanor W remained as non numerical counters for the duration of the one-year investigation. These remarks in relation to the transformation of the case studies are particularly relevant in connection with the children's constructions upon number words and scripts. It was apparent that the broadening of the conceptual gap in arithmetic knowledge between two similar cases at the start of the investigation was concomitant with their different frames of organisation for number words and number scripts. Precisely, whereas Eloise was a case of Group 3, Alice was a case of Group 1. This meant that Alice, who had elaborated rules in number words and number scripts, had progressed in her arithmetic knowledge. Conversely, Eloise, who had not established stable conventional rules, remained in the stage of non numerical counting. As has been anticipated, part two of the chapter will deal with children's rules in the final stage of the study. The previous sections have suggested that children had constructed new rules and that they were using these rules to solve arithmetic tasks. In particular, there was a whole range of tasks involving hundreds and thousands whose solutions indicated children's growth in arithmetic sophistication and their use of rules in number words and number scripts.

The three moments of children's assessment during the year long study are presented in the graph below.

Development of the Case Studies according to their Learning Stage

	STAGES	INITIAL ASSESSMENT (APRIL 1998)	SUMMER ASSESSMENT (JULY 1998)	FINAL ASSESSMENT (FEB-MARCH 1999)
	→ENS			TOM JOHNNY JACK
NUMERICAL SCHEMES INTEGRATION OPERATION	**TNS**		TOM-G1 JOHNNY-G1	ALICE EAMON
	INS→	JOHNNY TOM	ALICE -G1 JACK-G2 EAMON-G2	STEPHANIE
	FS→INS	JACK EAMON		
NON NUMERICAL SCHEME FIGURATIVE AND PERCEPTUAL	**→FS**	ALICE STEPHANIE	ELEANORW-G3 STEPHANIE-G2 ELOISE-G3	ELEANORW ELOISE ELEANORD
	PS	ELOISE ELEANORW ELEANORD	ELEANORD-G3	

First stage of fieldwork (Initial and Summer Assessment) — Second stage of fieldwork (Final Assessment)

The graph depicts the observed tendency of the rule makers - i.e. Groups 1 and 2 - to move towards higher sophistication in arithmetic. In contrast, children who were classified in Group 3 remained in the stage of non numerical counting. Nevertheless, the first classification of the case studies in Groups 1, 2, and 3 had to be revised according to the findings of the second stage of fieldwork. In the following sections, there is an examination of the stability and transformation of children's rules in the final stage of fieldwork. Second, there is a discussion of the use of their rules in solving arithmetic problems.

6. 3 Part Two: Stability and Transformation of Children's Rules

Findings of the first stage of fieldwork produced a classification of the case studies in 3 groups. Group 1 were the children who were seen establishing rules in number words and number scripts with no restriction of the system of numeration. That is, they had criteria to make sense of hundred and thousand words and scripts. Group 2 were children who seemed to focus their reflection either in the domain of number words or scripts. Eamon, for example, had stable ideas about hundred and thousand word scripts but he had not yet established regularities in the domain of number scripts. On the other hand, Jack and Stephanie seemed to have elaborated stable rules for number scripts but had not established rules in the domain of number words. Jack's rules encompassed rules with no restriction of the sequence but Stephanie's regularities seemed limited to 2-digit scripts. Children of Group 3 seemed to have no stable ideas on number words and scripts. When they manifested criteria or produced regularities these were unconventional and local to the particular context. The final stage of fieldwork had two main goals in relation to the second line of investigation. First, the validation of the inferences of the first stage of fieldwork. Tasks were designed to confirm the use and explicitness of children's rules, and their possible transformations. If children had elaborated certain rules, they had to solve certain tasks at ease, providing relatively explicit explanations. Further, if children had elaborated certain rules upon the number scripts or words they could violate these rules if prompted to do so. Likewise, if children had established regularities, they could reflect on their possible meaning. For example, if a child had abstracted that '2' was in front of all the twenties, they could reflect on "why" this was the case. Finally, new rules could be constructed on the base of former elaborations. The second goal was to examine the use of these inferred rules in solving basic arithmetic problems. Questions to be answered were:

1. Did children of Groups 1 and 2 use the rules inferred in the first stage of fieldwork?
2. Had children of Group 1 and 2 elaborated new rules in number words or scripts?
3. Had children of Group 3 elaborated any stable rules?
4. Did children use their rules when solving arithmetic problems?

In the following sections there is an analysis of children's solutions to the tasks designed to probe their construction of rules. The case studies are presented in three sections which respect the classification of the original Groups 1, 2, and 3. Any transformations are discussed at the end of each section.

6. 4. 1 The cases of Group 1: Tom, Johnny and Alice

The case of Tom

Tom no longer hesitated to interpret hundreds or thousand because he had constructed the rule of the hundred and thousand scripts. He no longer produced concatenated forms because he knew that hundreds were written with 3 digits and thousands with 4 digits. Likewise, he no longer interpreted '2500' and '100' as hundreds, because his new rule reorganised his former rule of the two zeros. He had established the rule of the place, whereby he could work out which of two scripts with the same amount of digits represented a higher number. For example, he expressed that he had looked at the front digit in '2500' and '7800' because they *"tell you if there's more thousands"*. His rule of the hundred scripts and rule of the place, was evident when he stated that "2 hundreds" were the numerals with *"three digits and [that] had a two at the beginning"*. This meant that '2009' was no longer "209" because it had 4-digits and hence, it was a "thousand number". Tom now reasoned about his former abstraction *"the 2 goes in front of all the twenties like the 3 goes in front of all the thirties"*. For example, he responded that '2' was written 10 times in front of all the twenties and justified: *"'cause there are ten twenties"*. Further, he expressed that the "2" in front of all the twenties meant *"two tens"*. Similarly, he worked out that the line '17-37-57-77" could be explained as *"every two tens it goes to a 7"* or that the line was going in "twenties". Tom interpreted number scripts in the hundreds and thousands with his previous and novel rules. For example, he worked out that in the line '2010-2120-2230-2340-2450-2560-2670-2780-2890' the *"numbers"* were *"adding another*

***hundred** and ten"*[16]. After carefully reading the scripts he justified: *"they were going up in tens, and they were going up in hundreds"* while pointing respectively at the decade and hundred digits of the scripts. His rule of the place was not a mere matter of empirical abstraction because Tom held in thought "ten" and "hundred" to form *"a hundred and ten"*. Further, he worked out the next number of his line.

T: "Two thousand ten hundred"
I: "That's a very strange number....two thousand ten hundred"
T: "Three thousand!"
I: "How did you work out it was three thousand? That was a guess!" [...]
T: "No, it wasn't a guess"
I: "Explain it to me then"
T: "Because I know that another ten on, ten hundred, would be another thousand"

Tom co-ordinated decades, hundreds, and thousands to answer "three thousand". He added decades together and obtained 10 decades which he exchanged for 1 more hundred, and, finally, he added hundreds together to form 10 hundreds. At this point, his number was "2 thousand 10 hundred". Confronted with the interviewer's comment, Tom exchanged 10 hundreds into 1 thousand and added the thousands together to obtain "3 thousand". The rule underlying these co-ordinations was the new rule of exchange whereby 10 numbers of the same kind were exchanged for a number of a different kind. Similarly, he worked out that 12 centuries were *"1 thousand 2 hundred years"* and that 20 centuries made *"two millenniums"*. In relation to his rule of the sequence in tens, Tom had now constructed a flexible sequence in tens whereby he could count in tens from any given number word and in any backward or forward fashion. His new sequence was not restricted to 2-digit numbers and could be used to solve, for example how many seats apart were tickets '347' and '357'[17]. Tom could reason about his former sequence in tens because he could use it at a distance and as an object of reflection. For example, Tom responded at ease that he would count 10 times to 100 and 20 times to 200 if he counted in tens. More impressively, he worked out that it would be "25" five times to 250 and explained: *'cause it's just two to twenty and then you just do 5 tens of 50, so that means you add five more"*. Tom's reversible use of tens as units and as composites was documented in several episodes of part one of this chapter. Similarly, in the context of making towers of 10 bricks, he

[16] Regularities session
[17] Cinema tickets session

responded untroubled that 11 towers would be "110" bricks. Subsequently, he worked out how many towers 120 bricks would make: he responded swiftly "12" and explained: *"110 were 11 towers so one more tower would be 12"*.

The case of Johnny

Like Tom, Johnny had constructed new rules in addition to his former rule of the digits and rule of the front number. He elaborated the rule of the hundred and thousand scripts whereby he interpreted 3- and 4-digit scripts. It must be noted, Johnny was the only case who never produced concatenated forms in the first stage of fieldwork. However, there was no clear indication that he had constructed explicit criteria to interpret hundreds and thousands. In the final stage of the fieldwork his rules were explicitly formulated and he could read or produce 3- and 4-digit numbers conventionally. Johnny could also reason upon abstractions on number scripts. For example, he responded untroubled that there were "10" twos in front of all the tiles with twenties. In justifying his answer, Johnny alluded to the digits from 1 to 9 plus the nought in 20. This meant that Johnny anticipated that there were 10 "numbers" in the twenties. It was apparent that Johnny could reason about abstracted aspects of 2-digit scripts with a new flexible sequence in tens. For example, he explained that '8-18-28' were "10 apart" and that '17-37-57-77-97' were "20 apart". His explanation follows:

J: [looks ahead in concentration] "If that was 7 and that was thirty [seven] [points at 17 and 37], then they'd be thirty....they'd be thirty apart, but that's 17 it's got...uhm, a ten before so it's twenty apart"

Johnny seemed to view 17 nested in 37 and disembed it to explain they were 20 apart. Similarly, he explained that 77 and 97 were also "20 apart" because *"70 and 90 are 20 apart"* and if you *"add the 7 on they are still twenty apart"*. Johnny could now reason about his former sequence in tens. He responded without performing the count that he would count 10 times to 100, 20 times to 200 and 100 to 1000. For only explanation he expressed *"ten tens in a hundred"*. More strikingly, he worked out that it would be 25 times to 250 and 250 to 2500. Prompted to explain further, Johnny appeared troubled, like his companion Tom. When dealing with hundred and thousand scripts he also gave evidence of reasoning upon the regularities abstracted. He constructed a new rule of exchange and found no trouble to view '1000' as "10 hundred" or "1 thousand". For example, when he put in a line the scripts 6110-6210-

6310-6410-6510-6610-6710-6810-6910[18], he declared: *"it changes every hundred"* and expressed that they were *"a hundred"* apart. Johnny read the scripts in apparent concentration, presumably monitoring his reading with the rule of the thousand scripts. Then he worked out which number would be next in his line.

J: "Uhm....seven thousand...no, yeah, seven thousand and ten"
I: "Brilliant Johnny! How did you figure that one out?"
J: [promptly] "Because 9 [points at 9 in 6910], it would be 10, and that would be 7 [thousand] and then you put the 10 on.

Johnny exchanged 10 hundreds into 1 thousand. According to his rule of numbers of a kind, 6 thousand and 1 more thousand were 7 thousand. Finally, he dissembedded 10 from 6910 and recomposed it with his last integration of "7 thousand *and ten*".

The case of Alice

Alice used the rules presumably established in the first part of the fieldwork: the rule of the digits and the rule of the front digit. She was now able to explicitly co-ordinate her rules to justify her solution to a given task. For example, when she had to justify why she had put '2500' before '7800' in her line, she expressed *"'cause I looked at the whole number and they were both four [digits], and then you have to look at this one"*. Alice logically co-ordinated her criteria to justify her answer, something that seemed implicit in the first stage of the fieldwork. Prompted to explain why she had looked at the front digit, she explained the front digit told you *"how many thousands"*. This assertion suggested that she had constructed the rule of the hundred and thousand scripts and the rule of the place. When Alice was presented with the tiles with twenties, she counted all the tiles from 21 to 29 to respond that the '2' was 9 times in front of the twenties. When she was pointed out that she had omitted the 2 in 20, she agreed that there were 10 twos in front of the twenties. However, she responded swiftly that there were 10 fours in front of the forties. In contrast to Tom, Alice responded that the '2' of the twenties meant *"twenty"* and the '4' of the forties meant *"forty something"*. When Alice was presented with the scripts of the decades, she put them in a line and prompted to explain what "10, 20, 30" meant, she expressed that it was "*counting in tens"*. In the same manner, Alice explained that to have '8-18-28-38-48-58-68-78-78-88-98' one had to *"start at 8, and count in tens"*. She

[18] Regularities session

expressed this, after spontaneously pointing out that the next number was "108". This latter response suggested a new flexible sequence in tens, however, it was apparent from another episode that Alice could not easily reason about this flexible sequence in tens because she was unable to explain the line '17-37-57-77-97'[19]. Another difference with Tom and Johnny was that Alice could not engage with tasks involving thousand or hundred scripts nor did she seem able to co-ordinate decades, hundreds, and thousands. For example, when Alice was presented with the scripts from '6110' to '6910' in hundreds, she put the scripts in a line and read them conventionally, presumably using her rule of the thousand scripts. However, she seemed unable to figure out the reason these scripts were put together[20]. Other solutions indicated her elaboration of the new rule. When she put the scripts '8000-8010-8020-8030-8040-8050-8060-8070-8080-8090' in a line, she read the scripts conventionally and declared that they were *"counting in tens"*. Then she responded in apparent concentration that the next number was *"five thousand"*. Alice seemed aware that 1 more of the same kind (tens) would produce a number of a different kind. However, she did not co-ordinate the exchange 10 decades-1 hundred but 10 decades-1 thousand. Alice's use of her sequence in tens, indicated that she was able to reason about it. She responded swiftly that it was 10 times to 100 when you counted in tens and 20 times to 200. However, she seemed troubled when she had to solve how many times it would be to 1000 and proceeded to count in tens in deep concentration, unable to keep track of how many times she had counted. However, she worked out that it was "25" times to 250 and explained: *"I counted 10, 20, then I added on the 5 and I added on the 2 hundred"*. Alice's explanation indicated that co-ordinations of numbers of different kind were problematic for her.

6. 4. 2 Summary of cases of Group 1

The three cases of Group 1 were confirmed as rule makers in the domain of number words and scripts. Children solved similar and new tasks presented using previous rules and indicated their awareness of features of written and spoken numeration. Not only were they aware of regularities in number words and scripts but they could start

[19] Like 3
[20] Like 3

reasoning about them. Novel rules, deriving from previous constructions seemed to underlie children's solutions in the final stage of fieldwork. First, there was the new rule of exchange. They had appropriated that "10 hundred" was "one thousand". This rule seemed to be constructed as a solution to the tension between the script of "10 hundred" and "1 thousand" documented in the first stage of the fieldwork, and it was also used to co-ordinate decades and hundreds. The second novel rule was termed the rule of the flexible sequence in tens. Children could produce a sequence in tens starting from any given number word. This rule was a modification of the rule of the sequence in tens established in the first stage of the fieldwork. Third, the rule of the hundred and thousand scripts was inferred in situations of interpretation and production of 3- and 4-digit scripts. According to this rule children no longer produced concatenated forms and could interpret 3 and 4 digit scripts.

6. 4. 3 The cases of Group 2: Jack, Eamon and Stephanie

The case of Jack

In addition to his rules in number scripts, Jack had now constructed rules in number words. The clear articulation of his rules and reasoning were remarkable features of his case. He used the rule of the digits, the rule of the front digit, and the novel rule of the hundred and thousand scripts. For example, in the following episode he explained how he had worked out the order of: '4999', '2500' and '7800' in his line.

J: "Well, first I look at if there are 4 numbers in that one and 4 numbers in that one" [2500 and 4999]
I: "Right..."
J: "So it's the same kind...it's one of the thousands"
I: "Right....right, that's good"
J: "So er...if it's one of the thousands: so I've got to count the first number...[pensive]
I: "Why?"
J: [pensive] "Because...er...This is a bit hard to explain"

Jack was verbalising the rule of the digits and the rule of the front digit and he considered them in logical order to justify his answer. Subsequently he explained why he attended to the front digit: *"that number tells you if it's in the hundred or the thousand"*. This was his articulation of the rule of the place. The rule of numbers of the same kind transpired in his reference to *"thousands"* as a "*kind of number*". Similarly, Jack had constructed the rule of the sequence in tens and his responses suggested that he could reason upon his novel sequence. Further, by the end of the study he had established a flexible sequence in tens. By the same token, he had established the rule of the counting hundreds. Jack's responses suggested that he had also established the novel rule of exchange whereby 10 numbers of a kind formed a new number of a different kind. In particular, "10 hundred" was "1 thousand". It was presumed that his elaborations on the organisation of number words originated from his rules in number scripts. His criteria to make sense of number scripts were primary in the history of construction, as has been documented in the first stage of the fieldwork. In the following excerpt, there is indication that he had not only established a sequence in tens but could establish a flexible sequence in tens.

I: "Can you count in tens, starting from... seven?"
J: "Oh....[smiles, alert] That's a bit tricky, isn't it?" [...] "Well, seventeen....twenty seven, 37, 47, 57, 67, 77, 87, 97, one hundred and seven"
I: "Oh!. And after one hundred and seven?"
J: [smiles, confident] "One hundred and seventeen, 127, 137, 147, 157, 167, 177, ..."

Jack was interrupted at "177" and was prompted to explain how he figured it out so fast, after he thought it was a "tricky" question. Jack expressed pensive:

J: "Well, you don't really bother about counting. Ok. you bother about counting the seven...you keep on counting in tens, but the last number it's always seven".

His explanation indicated that his flexible sequence in tens was based on his novel sequence in tens. Jack was able to reason upon his newly established regularities in number words. For example, he responded swiftly that he had to count "ten" times to a hundred and, pensive, explained that there were *"ten tens in a one hundred"*. Subsequently, he found no trouble to work out that it would be 20 times to 200 and 100 times to 1000. His explanation came unexpectedly:

J: "When you've got your way of maths, and you've done the big numbers...you have to do sums completely different [laughs] so you have to, so it's like you have to change your brain actually".

Jack's reflection indicated that he had not counted but made use of the new abstractions he had constructed upon number words and scripts. Like Tom, Jack responded that there were "10" twos in front of the twenties, and ten fours in front of the forties[21]. Remarkably, he also responded prompted that there would be *"one hundred*" twos in front of all two hundred numbers. In relation to Jack's novel rule

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of exchange, he responded with apparent ease that "12 centuries" was *"1 thousand 2 hundred years"*. This rule was apparently based on the tension whereby the scripts for "10 hundred" and "1 thousand" were same. It must be noted that Jack had now established new rules in number words. Nevertheless, Jack's explanations referred to his original rules in number scripts: *"10 hundred, when you write it, is a thousand"*.

The case of Eamon

As Jack had constructed rules in number words, Eamon had established rules in number scripts, including the rule of the digits, the rule of the front digit, and the logical co-ordination of the two to justify the order of number scripts in a line. Likewise, he used his former rules in number words with the addition of *"thousands"* and *"ty-thousands"* and *"hundred thousands"*. These rules were put into play when he read conventionally and in deep concentration the scripts '89567', '100000' and '10200' in separate occasions. Also, Eamon had constructed the rule of the hundred and thousand scripts which he articulated to justify his solutions to tasks. For example, he stated that in the line '6110-6210-6310-6410-6510-6610-6710-6810-6910' they were all hundreds and worked out in concentration that for *"these numbers you have, at least, to count in hundreds"* and subsequently worked out that the next number in the line was *"six thousand one thousand and ten"*. He then commented: *"there's no such thing as six thousand one thousand"* and reasoned that the next number was *"one thousand from six thousand"*. Despite further prompts, Eamon did not reinstate the "10" in "7010", like Johnny did. Nevertheless, his overt reasoning during the episode indicated that he had used a novel rule of exchange whereby 10 hundreds were exchanged for 1 thousand. The rule of numbers of the same kind, whereby hundreds could be added together, also underpinned his answer. Similarly, in a line of number scripts "going in 5 hundreds" he worked out that the next number after 9500 was "10 thousand" by reasoning that *"5 hundred and 5 hundred is one thousand"* and another thousand from 9 thousand *"must make ten thousand"*. In the case of Eamon, the rule of exchange stemmed from his rule of the counting hundreds and his awareness that the 10th hundred was 1 thousand. However, it must be noted that Eamon had now established rules in number scripts as well as in number words. His rule of the sequence in tens was used to solve a variety of tasks, however, it was unclear that he had established a flexible sequence in tens. On one occasion Eamon was asked to count in tens starting at 7. He counted from 7 to 17, using his fingers, then raised his head in apparent realisation, and continued fluently: *"27, 37, 47, 57, 67, 77, 87, 97"*. In another episode[22], he solved the "secret reason" in the line 8-18-28-38-48-58-68-78-98: he attended to the decade digits and responded that it was *"to count in tens"*. Similarly, with '17-37-57-77- 97', he attended to the decade digits and reasoned that these were *"counting in twenties"*. His justification follows:

E: "Well I figured out that....you know, you see fifty....and sixty, seventy [holds 57], and sixty, seventy, you need both tens from there, and there is one in between, that's, if there's one in between, ten and ten is twenty, that's how I know!"

Eamon's explanation referred explicitly to his former sequence in tens rather than a flexible sequence in tens. He could reflect on this former rule because he responded without performing the count that he would count 10 times by tens to 100 and 20 times to 200. He seemed troubled when he had to work out how many times to 1000 but he stated that he had to know *"ten times ten"* and realised this was *"100"*. In the same manner, he was troubled to work out how many times it would be to 250.

The case of Stephanie

Stephanie had established novel rules in the domain of number words as well as the regularities she had established in number scripts, but it was evident that these were restricted to 2-digit number signs. She had established a novel sequence in tens but had not established novel stable ideas in the domain of number scripts. Her system of rules in number words and scripts seemed restricted to 100. For example, in one episode she explained that she had placed '3788' in between '2500' and '7800' because it had *"the same amount of numbers"* and because *"25 is lower than 37 and 78 is higher than 37"*. Although the rule of the digits seemed to be her criterion it was apparent that Stephanie also resorted to her original rule of the noughts whereby "the more noughts, the higher the number". This was apparent, when she placed '300' at the end of her line '320-342-350-351-360-367-380'. She justified her solution by saying *"Because it's got two zeros and none of the others have got two zeros so I think it must be higher [than] the rest"*. It must be noted that Stephanie did not read these scripts before or after completing the task. This indicated that she had not constructed the rule of the hundred and thousand scripts. This was confirmed when

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she was unable to tell which numbers were two hundred numbers from a set of scripts given - 200-202-205-2000-2005-2009-2100 - nor could she sort number scripts of 3, 4-, and 5-digit numbers in piles of hundreds and thousands. It was apparent that she was establishing regularities upon the scripts because she selected '200-205-2100' as two hundreds because they had *"two at the front"*. However, since she had not established the rule of hundred and thousand scripts Stephanie had no criteria to monitor the writing nor the reading of scripts. As a consequence of this, Stephanie was still producing concatenated forms, for example, she wrote '20010' for "2 hundred and 10". This meant that there were potential conflicts. On one occasion she had agreed that "300" was higher than "210" and she was stumped when she was confronted with the fact that '20010' had more zeros than '300' - the scripts she had written for "210" and "300" respectively. Although Stephanie finally wrote the conventional form for "210" this was a local solution and it was apparent from subsequent episodes that she was unaware of the rule of hundred scripts. An important finding was that Stephanie had elaborated a novel rule in number words: her new sequence in tens. Despite reading and writing decades at ease, Stephanie had previously no meaning for "counting in tens". Now, she was able to produce the decade words beyond a hundred. Although she spontaneously stopped at 190, she seemed aware that her sequence was different from her sequence in ones:

S: "Well, if you count in ones....you just get all the numbers, all the numbers that there are, [very fast] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.....if you count in tens 10, 20, 30, 40, 50, 60, 70, 80, you don't get the numbers in the middle of the tens..."

Her response suggested that she could reason upon her new sequence in tens. Although she found trouble to explain how she knew that she would count 10 times by tens up to 100, she counted to show that it was true. Subsequently, she worked out that it was 20 times to 200, and explained: *"I just added another ten on"*. Further, when she worked out that the line with the cards '10-20-30-40-50-60-70-80-90-100' was "counting in tens", she responded that there were *"ten cards"* in total. With the line '8-18-28-38-48-58-68-78-88-98', she hesitated but ventured that it was *"counting in tens"* and responded that the next numbers would be "108", "118", and "128". This indicated that she might have started to establish a flexible sequence in tens. Despite her novel sequence in tens, Stephanie had not appropriated important regularities like the rule of numbers of a kind or of the counting hundreds. Although she counted in hundreds to establish the total of clips in 4 boxes of 100 clips, it was apparent that her rule had not the force of the rule of numbers of the same kind. For example, when 1 lollipop was extracted from a jar of 200, she responded that there were "one hundred" left in the jar.

6. 4. 4 Summary of the cases of Group 2

Jack and Eamon were now cases of Group 1 because they showed indication of their elaboration of rules in both number words and scripts. Jack had constructed criteria for number words as well as number scripts. Eamon had now constructed stable ideas on number scripts as well as in number words. Eamon and Jack had also established, as the children of Group 1, the new rule of exchange. Jack's case showed striking similarity to those of Group 1 because he also was seen to use a flexible sequence in tens. Although Eamon did not use a flexible sequence in tens at ease from the start of the second stage of fieldwork, it was presumed that he was establishing this new regularity based on his solutions to tasks at the end of the study. On the other hand, Stephanie, was seen to establish regularities in number words and scripts but these were restricted to number words and scripts up to a hundred. Stable rules did not seem to encompass hundreds or thousand words nor 3- and 4-digit scripts. Stephanie had not established the rule of exchange nor had she appropriated rules in number words and scripts involving hundreds and thousands.

6. 4. 5 The cases of Group 3: Eloise, Eleanor W, and Eleanor D

Although these three cases differed in particular features, their similarities were predominant. The following presentation justifies that the children had not established rules in the domain of number scripts nor in number words. The first subsection addresses the three cases in relation to their responses involving number scripts; and the second subsection addresses the three cases in relation to their responses involving number words.

No stable rules in number scripts

Children's responses when putting scripts in a line from the lowest to the highest indicated that they had local and unstable criteria to interpret the scripts. For example, one of Eloise's line was '3-56-2500-100-7800-89567-100000'. She expressed that

'100000' was the highest because *"it's got more hundreds"*. When '100000' was taken away from her line, she expressed that '89567' was the highest because *"all of those numbers are high, quite high"*. When she was given the script '3999' to put in her line, she placed it in between '89567' and '100000' and explained *"'cause I know ninety nine, a hundred, so, nineteen ninety nine, then one hundred and a hundred"*. In the same task, Eleanor W's solution was '3-56-100-100000-2500-7800-89567'. She placed '3999' in between '2500' and '7800' and explained: *'cause it has 3 in it*". Similarly, she placed '1999' just before '2500' and expressed *"it goes 1, 2, 3"*. However, her criteria was destabilised when pointed that '1999' had nines which were high numbers. This seemed to intrigue her and she admitted that '1999' was higher than '2500'. Likewise, Eleanor D, who initially put the scripts in conventional order, justified that '100000' was the highest of all *"because it's got loads of zeros"* . She subsequently placed '3999' in between '89567' and '100000'. In another task, Eloise's line was '351-342-367-320-350-380-360" and expressed: *" I guessed the zeros make the numbers bigger"*. Eleanor D's solution was strikingly similar: '342-367-351-320-350-360-380' and said that the "numbers with the noughts were high". However, the criteria whereby zeros made the numbers bigger was unstable because she opted to change '320' at the beginning of her line. Eleanor W's line was '351-320-342-350-360-367-380' (it was presumed that she interpreted '351' as "315"). Precisely, when she was asked whether she could read those numbers, she stared in apparent concentration and read the scripts conventionally with the exception of '351'. Subsequently, she was given '300' to put in her line.

W: "This is a higher number, so it must... [looks at the end of her line but then stares at her line], It must go first, 'cause that's the lowest number in all these" [places 300 at the beginning]
I: "How do you know?"
W: "Because it's three hundred and I know there's not any little numbers in three hundreds"

Eleanor W's deliberation indicated that she was driven by two contradicting hypotheses. On the one hand, according to her idea that zeros make a high number, '300' was a "higher number" and it had to go at the end of her line. On the other hand, she seemed to attend to her recent conventional reading of the scripts and decided that "300" was the lowest of all. Although this was an insightful solution it was soon apparent that it was local to the situation and that Eleanor W had not established other rules crucial to the organisation of number scripts. All three children were unable to choose the "2 hundreds" from a bunch of number scripts[23]. Eleanor W asked: *"Is it the ones which got 2 at the front?"* and subsequently declared *"Only two hundreds if has zeros in them"*. After she had completed the solution, her pile of "two hundreds" was '200-2100-238-206-202-250-222-102-210-1002-002-2000-2005' and the "not two hundreds" were '20050-10200-2009-205- 245-1200-214-265-276-296'.

I: "How did you work it out?"
E: "'cause these ones [not 2 h's] have zeros, but these [2 h's] are the shortest numbers and these have twos in"
I: "These are the shortest numbers?"
E: "And have twos in, and I know those have twos in [not 2 h's], but I just knew that they go there"

In a similar manner, Eloise and Eleanor D found the task puzzling because they had elaborated no rules to deal with this problem. By the same token, neither of the children could successfully sort the hundred and thousand scripts. They appeared hesitant first, and then perplexed at the task. This was not surprising since they had not appropriated the hundred and thousand knots nor had they constructed the rule of hundred and thousand scripts. They could recognise some scripts (e.g. 101) but they produced unstable concatenations. For example, Eleanor W wrote '3010' for "310", Eleanor D wrote '20010' for "210", and Eloise wrote '20030' for "230". The absence of conventional stable rules was also true for 2-digit scripts. When the numbered tiles with twenties was presented, none of the children recognised the tiles as the "twenties" but as "twos" as opposed to the other bunch of "fours" - the forties. Moreover, Eloise and Eleanor W put '20' at the end of their line as follows: '21-22-23-24-25-26-27-28-29-20'. Eloise read '21' as "12" and expressed there were *"numbers missing"*. Other prompts yielded surprising responses.

I: "What comes before when you count, 20 or 29? What comes first in the number line?"
E stays in silence but places 20 after 28 [28-20-29]

Eloise proposed that "thirty" came after "twenty" but prompted to count, she counted from 1 and guided by the interviewer she reorganised her line from 20 to 29. This was a local arrangement, because when she put the forties in a line, she placed '40' at the *end* of her line. Eleanor W's response was identical suggesting that they had not established conventional regularities of number scripts. In a similar manner, the three children made the following type of line with decade scripts: '30-40-50-60-70-80-90-20' and uttered their tee-sequence in tens: "fortee-fiftee-fiftee-sixtee-seventee-eihtee-

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ninetee-twenty". When asked what came after that, Eleanor expressed *"we don't have any twenty ones"* while holding up the card with '100'.

No rules in number words

As it had been documented in part one of this chapter, the three children used a fixed recitation from twenty to twenty termed "tee-sequence in tens". It was typically triggered by the prompt "Can you count in tens, like: 10, 20" or by the children's spontaneous reading of '10' and '20' (like in one episode with the ten-scale on the ruler). This recitation seemed to be a verbal pattern acquired by counting the teens up to 20. A remarkable feature was that all three children produced this sequence and seemed to experience no lack of comfort when arriving "back to 20". Neither of Eleanor D or Eloise counted in 2's or 5's despite prompts given by the interviewer. Eleanor W paused and in apparent concentration she pronounced steadily the sequence in 2's up to "28" and then she continued with "29". In the same manner, she proceeded to recite the sequence in 5' until "90" after which she pronounced "a hundred". Prompted to continue, she was stumped but ventured "a hundred and eighty". The three children were still disconcerted when dealing with hundreds. For example, in the cinema scenario, when they were asked how many people there were in total if there was 100 in each of the 9 parts, they breathed in deeply and responded with a "big number": e.g. *"twelve hundred and sixty eight"* or *"ten thousand"*. This indicated that they had not appropriated that they count hundreds as ones of a different kind. Likewise, in a task[24] with a jar of "200" lollipops, one lollipop was taken away and their responses were strikingly similar. The following excerpt, documents Eloise's response.

I: "How many do you have to take away to have one hundred lollies?"
E: [puzzled] "Er...I don't know..."
I: "What about if I take one away, how many are there inside?" [takes one lollipop away]
E: [touches the jar, pensive] "One hundred in there"

6. 4. 6 Summary of the case studies of Group 3

All three children of Group 3 had not established novel stable conventional ideas on number words or scripts. However, this did not mean that they did not express their ideas on the numerical signs. Precisely, for example, Eloise and Eleanor W were

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remarkably keen to provide justifications for their solutions to tasks. However, these ideas were local or specific to the context. In addition, these ideas seemed partially conventional, for example, the rule whereby the more zeros a script had, the higher the "number" was. This meant that children produced concatenated forms and were establishing partially conventional criteria. However, they had not established the rule of the digits, the rule of the front digit, or the rule of the recurrence of 1 to 9 in 2-digit scripts. In the domain of number words, they had not established a sequence in tens nor the crucial rule of numbers of a kind.

6. 5 Discussion on the Findings of the Two Lines of Investigation

As has been anticipated in the above sections, there was a transformation of the rules children had constructed in the final stage of the fieldwork. This produced a redefinition of Group 2 which was now hosting the only case of Stephanie who had constructed rules in number words and number scripts restricted to 100. Group 1 and Group 3 remained with the same definition, but Group 1 had now the cases of Jack and Eamon as well as Johnny, Tom, and Alice. Finally, Eloise, Eleanor D and Eleanor W were still classified as the cases who had not constructed stable rules. At the end of the study, the table presenting the classification of the case studies in relation to their arithmetic knowledge and their elaboration of rules had changed as follows:

Classification of the Case Studies at the End of the Study

		Group 1	Group 2	Group 3
	→ENS	TOM JOHNNY JACK		
NUMERICAL COUNTING	TNS	ALICE EAMON		
	INS→		STEPHANIE	
FIGURATIVE COUNTING	→FS			ELEANOR W ELEANOR D ELOISE

Group 1: Rules in number words and scripts; Group 2: Rules in number words and scripts restricted to 100; Group 3: No stable rules.

The graph below shows the development of the case studies during the one-year study according to the findings of both lines of investigation.

Development of the Case Studies According to both Lines of Investigation

	STAGES	INITIAL ASSESSMENT (APRIL 1998)	SUMMER ASSESSMENT (JULY 1998)	FINAL ASSESSMENT (FEB-MARCH 1999)
NUMERICAL SCHEMES INTEGRATION OPERATION	→ENS			TOM-G1 JOHNNY-G1 JACK-G1
	→TNS		TOM-G1 JOHNNY-G1	ALICE-G1 EAMON-G1
	INS→	JOHNNY TOM	ALICE -G1 JACK-G2 EAMON-G2	STEPHANIE-G2
NON NUMERICAL SCHEME FIGURATIVE AND PERCEPTUAL	FS→INS	JACK EAMON		
	→FS	ALICE STEPHANIE	ELEANORW-G3 STEPHANIE-G2 ELOISE-G3	ELEANORW-G3 ELOISE-G3 ELEANORD-G3
	PS	ELOISE ELEANORW ELEANORD	ELEANORD-G3	

First stage of fieldwork (Initial and Summer Assessments) — Second stage of fieldwork (Final Assessment)

In the graph, the development of the nine cases in relation to their arithmetic progress can be appreciated. The findings according to the second line of investigation are summarised by their classification in Groups 1, 2, 3 - G1, G2, G3. First, Tom and Johnny were the more advanced counters from the commencement of the study. They started as counters of the INS and were in transition to the ENS or had established the ENS at the end of the study. Jack started as a case in transition to the INS and was by the end of the study, a counter in transition to the ENS together with Johnny and Tom. Likewise, Eamon had progressed significantly since the initial classification as a counter in transition to the INS: one year later he had constructed the TNS. Alice, made rapid progress in one year because she was at the figurative stage at the start of the study and was at the stage of the TNS when the fieldwork was completed. Stephanie was also a non-numerical counter at the commencement of the investigation but her counting scheme had undergone significant progress: she had at least constructed the INS by the end of the study. Finally, and at the other extreme, during the one year long study, Eleanor W, Eleanor D, and Eloise had not become abstract counters. When the fieldwork was completed, they were still at the figurative stage.

What were these children's ideas on number words and scripts? Had they elaborated stable hypotheses and rules upon number words and scripts during the one year long investigation? As has been documented, Tom, Johnny and Alice had a history of construction of rules upon words and scripts which were increasingly organised. Particular rules were co-ordinated with others to construct novel criteria which were seen in action when children solved the situations presented to unveil their theories. In the case of Jack and Eamon, their stable conjectures seemed to originate in a particular domain. Jack's rules were predominantly in number scripts and this delayed important rules in number words, like his sequence in tens, and the rule of numbers of the same kind. In the case of Eamon, his appropriation of rules in number words were primary and his reflections on number scripts came with posteriority. Nevertheless, both children were seen elaborating rules in both number words and scripts in the final stage of fieldwork and hence, they were now cases in Group 1 together with Tom, Johnny, and Alice. Stephanie was a rule maker in number scripts but the analysis found that her stable rules were restricted to 2-digit numbers. In the final stage of the fieldwork she had constructed novel regularities in number words: the rule of the sequence in tens, but her restriction to 100 was apparent. This justified having a second category, Group 2, now redefined as the case who had elaborated stable regularities in number scripts and number words restricted to one hundred. Finally, Eleanor W, Eleanor D, and Eloise were classified as not rule makers in the first stage of the fieldwork. Their unconventional ideas demonstrated they had not constructed stable theories compatible with the conventionalities of spoken and written numeration. By the end of the study, there were no significant transformations in the three cases. Eleanor W and Eleanor D were still using a tee-sequence in tens.

These findings seemed to confirm that children's theories upon written and spoken numeration, and their constructions in the activity of counting are initially independent, but growth in arithmetic sophistication comes with the integration of these two systems of constructions. Precisely, for those cases who were rule makers in number words and number scripts, their learning stages were the most advanced ones: TNS and ENS. Stephanie, whose rules were seen restricted to one hundred, was a counter of the INS. At the other extreme, Eleanor W, Eleanor D and Eloise, who had not constructed stable theories were still in the stage of non numerical counting.

But how did children's rules empower their arithmetic knowledge? Which rules were used mathematically in solving the tasks presented? This question will be addressed in the general discussion of the two lines of investigation. The preceding findings have anticipated how children used the rules of numbers of the same kind, the rule of the hundred and thousand scripts, and the novel rule of exchange to solve simple arithmetic problems. However, children's solutions to a variety of tasks involving hundreds and thousands were not readily analysed with the model of learning stages as it has been formulated in this thesis. It was necessary to incorporate the rules children had constructed in order to explain their behaviour.

6. 6 Summary

Findings of the two lines on investigation of the final stage of fieldwork confirmed that rule makers had achieved higher sophistication in their arithmetic knowledge. At the other extreme, the case studies classified in Group 3 were still counters of the figurative stage. Chapter 7 presents a general discussion concerning findings of the two lines of investigation. Part of this discussion will address how children's theories were put into play when solving arithmetic problems.

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7. Discussion of the Two Lines of Investigation

7. 1 Introduction

The preceding two chapters have discussed the findings of the two lines of investigation. On the one hand, the nine case studies were placed in a spectrum of levels of conceptualisation according to their sophistication in arithmetic. On the other hand, the findings of the second line of the study have suggested that children's rules in number words and number scripts became integrated with their arithmetic knowledge. The present chapter discusses how this integration was seen to occur through examining children's use of their rules in solving a variety of arithmetic problems. It is organised in two main sections. The first, addresses the cases studies in relation to their use of rules when solving arithmetic situations involving hundreds and thousands. The second, develops a general discussion of the case studies.

7. 2 The Use of Rules when Solving Arithmetic Problems

Children of Group 1 and the newly defined Group 2 with the case of Stephanie had elaborated stable rules compatible with the conventionalities of spoken and written numeration. These rules had been seen to develop from the early start of the study and into the novel rules of exchange and of hundred and thousand scripts found in the second stage of the study. Children who were seen constructing regularities in the first stage of fieldwork were on the transition to the INS or had constructed the INS. On the other hand, children of Group 3, who had not constructed stable theories compatible with the conventionalities of written or spoken numeration, depended on actual objects to solve basic arithmetic tasks. For these cases, there seemed to be no "theories" that they could use in anticipation. In the second stage of the fieldwork it was found that the cases of Group 3 had not transformed their figurative counting scheme. They continued counting-all from 1 to solve partially screened collections and although they could curtail the verbal recitation of the sequence they could not perform numerical integrations. With respect to their elaborations of spoken and written numeration it was found that their criteria to interpret number scripts was unstable and contradictory. For example, they believed that the more zeros a script had the bigger the number. They had not established the rule of the digits nor the rule of the front digit. In contrast, for the cases of Group 1 and 2, their rules seemed to underlie their solutions to a variety of arithmetic tasks, in particular, those involving hundreds and thousands. The model of learning stages could not frame the analysis of children's solutions when solving this type of arithmetic problems because it could not account for abstractions children made when dealing with external signs. These abstractions, in turn, seemed to underlie their operations of integration when solving arithmetic problems. Two systems of constructions - one stemming from their reflections upon written and spoken numeration - and the other stemming from their activity of counting, were seen in action when solving arithmetic problems dealing with numbers, in particular, when children solved situations involving decades, hundreds, and thousands. Hence, the first part of the discussion of the case studies in the light of the two lines of investigation, addresses children's solutions to arithmetic tasks using the inferred rules in number words and number scripts. But before starting with the case studies in turn, it is pertinent to recall the main rules that children had established. First, there was the rule of numbers of a kind, which was a rule more general than the counting hundreds. This rule betokened the appropriation of an essential characteristic of the decimal system of numeration, namely the fact that 'n' sets plus 'm' sets of the same class produce 'n+m' sets of the same class irrespective whether the sets are single units, or units of a certain order of the base -e.g. decades, hundreds, or thousands. Hence, 2 hundreds plus 3 hundreds make 5 hundred and 6 thousands plus 1 thousand make 7 thousands. The second rule was the rule of exchange whereby 10 numbers of a kind produced a number of a different kind. This rule seemed to stem partly from children's reflections on the scripts for hundreds, on the one hand - e.g. they are written with two zeros like '100' - and on the scripts for thousands - e.g. they are written with three zeros like '1000'. Hence, the tension that children experienced when they had to write "10 hundred" which could be the result of adding hundreds through their rule of numbers of a kind. Third, there was the rule of hundred and thousand scripts which included all number scripts of the kind of hundreds - i.e. all 3 digit numbers - and all number scripts of the kind of thousands - i.e. all 4 digit numbers -. According to this rule, annotating "two thousand three hundred and fifty" could be done in a conventional manner because children knew that although parts of the name were '2000', '300' and '50', the number script was necessarily written with 4 digits -i.e. '2350'. In a similar manner, they could interpret

3 and 4 digit scripts and they could interpret that parts of the script stood for "hundreds" or "thousands". It must be remarked that this rule was intrinsically connected to the rule termed rule of the place, according to which children knew which digit was "telling you the hundreds or thousands". Finally, there was the rule of the sequence in tens and a new elaboration of the rule whereby children could produce a sequence in tens starting at any point of the series. Children had appropriated that their sequence in ones could be organised or said in a different way. The following sections address the rule makers' use of their theories in solving arithmetic problems involving decades, hundreds, and thousands.

The case of Tom

Sums like '2000+400', '600 + 400' and '2000 - 400' were solved by Tom with apparent ease. His solution to the latter sum is documented in the following episode.

T: [pensive] "It's one thousand and.....one thousand and six...
I: And six?
T: [disgruntled] No! One thousand six hundred!

Tom's solutions indicated that his rule of exchange was underlying his reasoning. The co-ordination of 10 units with a unit of a different kind was also an elaboration of his rule of numbers of a kind. This meant that 10 hundreds could be exchanged for 1 thousand or 10 tens could be converted into 1 hundred. This was clear when Tom took *"one away from a hundred"* to solve '170 - 80'. His reasoning indicated that he had subtracted one unit of ten from 1 hundred after subtracting 70 from 170. That Tom was dealing with "ones" that betokened a given "kind" of number, was apparent in another episode when he solved "90 + 50" by responding *"[a] hundred and forty 'cause I just do 9 and 5"*. In the same manner, for '80+60' he reasoned *"you take away 2 to make a hundred, twenty, so, you have 4 left"* to make "140". His flexible use of "two" as "twenty" is apparent in Tom's explanation, and the use of "8 and 2" to make "10 decades" or "a hundred" is also implied. Tom's rule of the numbers of the same kind was apparent when he explained how he solved there were "400" clips in 4 boxes with 100 clips each.

I: "How do you know? You didn't count 1 by 1"
T: [smiles] "No! That would take ages, I just did it in hundreds: one hundred, two hundred, three hundred, four hundred".

Tom was clearly aware that one *could* count the clips "1 by 1" but he used his rule of the counting hundreds to solve the total of clips. His justification indicated that each *hundred* implied a potential count by ones but he was taking the hundreds as units. When he was told that 6 boxes were hidden, he concluded that there would be a thousand clips in total and explained: *"'cause I just know 10 hundred's a thousand and 6 and 4 makes 10"*. Tom's rule of exchange was used to solve sums like '4560+500'. First, he read the sum in deep concentration and wrote '5060' as the answer. Prompted to explain he said *"500 and 500 makes another thousand that means I've got to change that to one, and that's sixty so I put sixty there"*. Tom integrated numbers of the same kind and used the rule of exchange to transform 10 units of a kind into a unit of another kind. For example, to solve '4690+110' he explained: *"I know ten and ninety make another hundred and so I added er, eight hundred"*. His co-ordination of decades, hundreds and thousands using the rule of exchange was evidenced in his solution to '1000-...=340'. After a few moments of concentration Tom wrote '660' and explained with satisfaction that he had done *"10 take away 6 equals 4"* and that he then *"used the hundreds"*. His reasoning indicated that he simultaneously co-ordinated 10 decades with 1 hundred and 10 hundreds with 1 thousand. Tom's rules of hundred and thousand scripts underlined the interpretation of scripts before solving sums of the type "2010+...=2030" and '2638-....=2038'. For the latter, he reasoned:

T: [reads] " Two thousand six hundred and thirty eight....two thousand and thirty eight" [pensive, stares ahead for 2 seconds then writes 600 and clicks to see 2038]
I: "Brilliant! How did you figure that one so quickly?"
T: "I just need to take that 6 away [in 2638] to make it a zero but that means I have to take six hundred away".

It must be remarked that Tom had not been taught conventional algorithms and his solutions relied on his own reflections on the numerals. Another novel rule was a re-elaboration of his former sequence in tens. Tom could count in tens and ones in anticipation from any point of the series using a flexible sequence in tens. He could co-ordinate his different sequences - e.g. in tens or in hundreds - to work out the difference between, say, '301' and '601', or to solve situations like 905+50 or 905-50.

The case of Johnny

Johnny used his novel rule of exchange and former rule of numbers of the same kind when solving problems involving hundreds and thousands. For example, when he was presented with 6 visible boxes of 100 clips and 4 hidden ones, he responded that there were *"a thousand"* clips in total after overtly counting *"in hundreds"* from 600 and clarifying that *"ten hundred"* was the same as *"a thousand"*. His reasoning implying hundreds as units was indicated by his solution to a missing addend problem[1] where he had 3 visible boxes and had to find out how many boxes were hidden if there were 700 clips in total. He responded after a few seconds of concentration and explained:

J: "I know 3 and 4 is 7 and I've got 3 here, then, and there's 7 all together so I know there's 4 in there"
I: "I said 700 all together, how do you know there are 7 boxes?"
J: "'cause a hundred in each one"

Johnny's use of hundreds as "ones" suggests that he was viewing hundreds as numbers of the same kind. Like Tom, he co-ordinated this rule with his rule of exchange to solve that there were 10 small 100-clip boxes covered when he had 2000 clips all together and a big box of 1000 was displayed. In justification he expressed: *"10 hundred is a thousand and there's a hundred in each box and so there must be ten"*. The use of the rule of exchange was manifest when no actual material was available. For example, when Johnny solved the written sums '1000 - 500'.

J: "That's nine....Five hundred!"
I: "How did you figure it out so quickly?"
J: "'cause I know 5 plus 5 is 10, but you have to take away 5 from 10, but it's a thousand take away 5 hundred, it's five hundred"

Similarly, he solved '2000 - 400' "in ones" but next he shifted the meaning to "hundreds and thousands".

J: "I know 6 add 4 is 10, but 16 add 4 is 20, so you take away....the four....and then it's 16 but you have to add....thousands, the hundreds so it must be one thousand six hundred"

This was possible because Johnny reasoned with numbers of the same kind. Johnny solved problems like '920+...=1000' and always justified his answer. In this case, he expressed: *"I know 20 add 80 is a hundred"*. When he wrote '300' as the ticket 70 seats apart from '230'[2], he explained: *"I know seven and three makes ten, so I add a hundred on"*. Failed attempts made transparent Johnny's focus on digits and his rules. For example, he wrote '430' as the answer for '670+....=1000'[3] but after having immediate feedback from the computer -i.e. '670+430=1100', he modified his answer to '330'. Further, he solved '560+...=1000' successfully in anticipation.

J: [stares at the screen in concentration for 2 seconds] "Uhm..." [writes 440 and clicks to see 1000]

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[1] Paper-clip boxes session
[2] Cinema tickets session
[3] Laptop sums session

I: "Oh, brilliant. You did it in the first go. How did you figure it out this time?"
J: "Because, uhm...., because I knew that was a sixty [560], and I know, so that's nearly a hundred, sixty is nearly a hundred, so I put the four hundred and then fif...and then if that was a seventy then I add three but that is a sixty so I add four so that makes a thousand"

Like Tom, Johnny solved sums like or '2478-...=2408' after carefully reading the number scripts involved. In this occasion he explained:

J: "You just take away the seven"
I: "Right. But you didn't take away seven, you took away seventy"
J: "And the seven stands for seventy"
I: "How do you know that?"
J: "Because you can't fit an 'oh' after every single seventy, or eighty, or ninety, a hundred, and so on"

Johnny's reason "you can't fit an oh" is an expression of his knowledge that hundreds are written with 3 digits and thousands with 4 - i.e. his rule of the hundred and thousand scripts -. Johnny knew that you didn't write '70' to mean "seventy" but if '7' was in that particular script in that particular position, it meant "70". By the same token, Johnny solved '2478-....=2078' or situations like 301+300.

The case of Jack

By the end of the study Jack had made significant advances in relation to his rules. He constructed rules in number words which were not established in the first stage of the fieldwork. He had appropriated the crucial rule of numbers of a kind which he had not constructed before. In addition, he had constructed the novel rule of exchange, a flexible sequence in tens and his rule of the hundred and thousand scripts based on his former rules in number scripts. Jack often referred to features of the involved number scripts to justify his answers. For example, he explained how he knew that 1000 and 1000 was 2000: *"you take away the nought and do1 plus1, then you put the noughts back and that's 2 thousand"*. Similarly, he explained how he knew that there were 400 clips in 4 boxes of 100[4].

I: "But you didn't count 400 one by one, how do you know there are 400?"
J: "You don't count 1, 2, 3, 4, 5, ...[the count dies down] you count in...one hundreds so one hundred, [putting boxes 1 by1], two hundred, three hundred, four hundred"

In a consecutive episode, Jack found out how many boxes were hidden when 2 boxes of 100 were visible and there were 900 clips in total. First he made sure he had to figure out how many *"paper-clip boxes"* and then responded "7". He justified this by saying, *"I started with 900 and I took away 200"*. This indicated that he distinguished

[4] Paper-clip boxes session

two senses of hundreds: as units and as composites. His new rule of exchange was betokened by his solution to the total of clips when 6 boxes of 100 were visible and 4 were hidden under a cover.

J: [leans back in concentration for a few seconds] "A thousand paper-clips"
I: "That's brilliant, how did you figure it out?"
J: "Well, there's 400 [points to the cover], 500, 600, 700, 800, 900, 10 hundred" [points to the boxes]
I: "Yes, how did you work out that was a thousand?"
J: "Well, ten hundred, when you write it, is a thousand"

Jack was prompted to decide whether "10 hundred" was "the same" as "a thousand".

J: "I mean it's not the same when you say it but when you write it, it is [...] Look! You write it like this [writes '1000'] you write 10 hundred like that"
I: "But do you think that when you have ten hundred, you have a thousand?"
J: "Well, er...yeah"
I: [...] "How many hundreds do you need to have a thousand?"
J: "How many hundreds?" [I nods] "Ten"
I: "And if you had 2 thousand how many hundreds would you need?"
J: "Twenty"

The genesis of Jack's rule of exchange was in his rules for number scripts. However it was only when he had established the rule of numbers of a kind and of the counting hundreds that he constructed the novel rule of exchange. Although he had indicated an awareness of the relation between 10 hundred and 1 thousand, Jack did not use his rule of exchange at ease and in anticipation in problems like '560+...=1000'. He added 420 to obtain 980 and subsequently added 20 more to obtain the total wanted. Prompted to "do it in one go", Jack solved '670+....=1000' manifesting his awareness of a rule of exchange related to 10. He wrote '430', that is, 4 as the complement for 6 and 3 as the complement for 7. On the other hand, Jack solved problems like '2638-...=2038' using his rule of hundred and thousand scripts and his rule of the place. In justification he said: *"I just take away the six, and they are in hundreds"*. Prompted to explain why 6 meant 6 hundred, he said *"because there's two numbers after"*.

The case of Eamon

Eamon used his rules in number words in solving problems with hundreds. For example, he responded that there were 400 clips in 4 boxes of 100. In justification he said: *"I added hundreds 1 by 1"*. Next, the interviewer took 1 clip from one box and asked him how many clips he had. Eamon responded pensively: *"Three hundred...and ninety nine"*. His response indicated that counting in hundreds was not a mere recitation but that he was counting numbers of the same kind. When 6 boxes were hidden he solved that the total was *"ten hundred or a thousand"* and expressed he had worked it out with *"4 and 6"*. These 4 and 6 were units of 100 different from single ones because when the interviewer took 1 clip away he responded there were 999 clips.

His rule of numbers of the same kind and the novel rule of exchange were put into play when he solved how many small boxes of 100 were hidden if there were 2000 clips in total and a big box with 1000 was displayed.

E: "Well, I knew 10 hundred was a thousand, and then I added another 10 boxes, another 10 hundred [touching the big box], and that's 2 thousand, [...] and then I had to take away 1 thousand and it was 1 thousand" [touches the cover]

However, he found trouble to solve how many clips were hidden if there were 2000 clips in total and 4 small 100 boxes were displayed, i.e. 400+...=2000. He counted in hundreds from 400 and paused hesitantly. Eamon expressed: *"I'm a bit confused, because there's going to be 10 in my mind"* apparently unable to transform those 10 hundreds into 1 thousand and complete the solution. Nevertheless, he solved other missing addend problems in the context of clip-boxes, e.g., 500+...=1500. His new rule of hundred and thousand scripts monitored his response to the following situation. Eamon manifested that his dad had told him a *"special way of doing maths"* which was the column algorithm to calculate sums with 2-digit numbers. This method seemed like his rule of numbers of the same kind whereby he added 9 and 5 to solve 50+90. Eamon was invited to solve 82+98 using his method with paper and pencil.

E: "This is fun....9 and 8 was 17 [writes 17], 8 and 2 is 10 but....Where shall I put it? Otherwise it will make a 4 digit number! [writes 1, so he's got 171] My dad says that that part is a bit tricky, I do not know that part"

Eamon's "monitoring" of the script for the total of the sum is remarkable: he did not write "1710" because that would be *"a 4 digit number"*, i.e., a thousand number. The taught *"special way of doing maths"* was being monitored by his own knowledge of number scripts. Based on this knowledge he solved '4010+...=4210'.

The case of Alice

Like Eamon, Alice appeared troubled with missing addend problems involving hundreds and thousands. She seemed unable to engage with problems of the type '2000-400'. However, she was seen to use her rules of the counting hundreds and numbers of the same kind. For example, she counted on in hundreds to solve that there were one thousand clips if 6 boxes of 100 were visible and 4 boxes were hidden.

I: "How did you figure that out?"
A: [giggles] 5 hundred [piling up the boxes], 6 hundred, plus [lifts the cover] 7 hundred, 8 hundred, 9 hundred, and I know the next hundred is a thousand! You've got a thousand paper-clips to use!

Her justification indicated her rule of the counting hundreds but it was not apparent whether for Alice "10 hundreds" were "a thousand". When she had 5 small boxes of 100 and a big box of 1000, she was asked how many small boxes were hidden if there were 2000 clips in total, she worked out that there were 5 boxes of 100 covered.

A: [counts subvocally the small boxes] "1, 2, 3, 4, 5 [touches the cover]. Five"
I: "Why?"
A: "Because that makes 10....because if there's 5 here and that's a thousand [touches the big box] so this must be 5 to make another...."

Alice seemed to use a rule of exchange whereby she needed 10 boxes "to make another" number. However, the use of her rule was limited because she could not solve "1200 +...= 1600" in the context of the clip-boxes. Her solution to '700+400' suggested that Alice found the co-ordination of hundreds and thousands problematic.

A: looks ahead in concentration for 2 seconds] "A thousand!"
I: "Why?"
A: "'cause I know 7 plus 4 is 10, I mean 7 plus 4 is 11 so it must be a thousand....[raises her eyebrows pensive] I mean a thousand and....ten"

The case of Stephanie

As has been discussed in the previous chapter, Stephanie's rules seemed restricted to number words and number scripts up to 100. However, she seemed to be aware of some features of number words that she had not established in the first part of the fieldwork. For example, when she was asked how many clips there were in 4 boxes of 100 clips, Stephanie responded promptly "400".

I: "Very good. Did you count 1 by 1 quickly?"
S: "No...."
I: "How did you do it?"
S: "[...] I did it in hundreds....I counted in hundreds"

When the interviewer prompted her to count "in hundreds" Stephanie recited the sequence in hundreds, including 10-11- and 12-hundred. The idea that the 10^{th}

hundred was "a thousand" was absent as were crucial rules like the rule of hundred and thousand scripts. Moreover, her sequence in hundreds was not differentiated from the sequence in ones because when she was asked how many lollipops there were left if the interviewer took 1 out of a jar with 200, Stephanie responded "1 hundred". This indicated that her idea of hundreds was not yet an idea of groups and ones simultaneously. Although she seemed to use the rule of exchange implicitly in the context of tens and hundreds, this was not the case in the context of hundreds and thousands. She could solve problems using her novel sequence in tens but occasionally bypassed counting by tens and reasoned strategically. For example, she solved '240+60' carefully counting by tens and pausing after 290 to respond "300". But when she had '40+60' Stephanie replied immediately that was "a hundred" because *"6 plus 4 is 10"*.

Nevertheless, the co-ordination of decades as units and hundreds was problematic. For example, she had trouble to solve 40+....=120.

S: "40 and 60 is 100 so if I added 2 more tens on, it would be two hundred...a hundred and two..."

As has been documented Stephanie had not appropriated important rules in number scripts. Although she solved '100+100' she did not know how to write "200". In the same manner, she could not write "10 hundred" as the answer to '600+400' or '900+100'. However, when '500+500' was presented, she responded "a thousand" and wrote '1000'. In a subsequent episode, she solved '980+20' as "10 hundred" and proceeded to write '1000'. The interviewer prompted her to see the problem.

I: "Is it a thousand or ten hundred?"
S: [puzzled, silence]
I: Which one is higher: a thousand or ten hundred?
S: [hesitant] "A thousand..."
I: "How do you write a thousand?"
S writes '1000'
I: "They are written in the same way, how can they be one higher than the other, if they are written in the same way? What do you reckon? Which one is higher?"
S: [hesitant] "They are both the same...."

It was apparent that Stephanie's answer did not came as a solution to the same tension that Tom or Johnny had experienced in the first part of the fieldwork. She did not seem to experience a genuine conflict because she had not established important rules such as the hundreds are written with 3 digits and the thousands with 4 digits nor the rule of the numbers of a kind. Not surprisingly, she was stumped by the sum '203+...=263'.

The cases of Group 3: Eleanor D, Eleanor W, and Eloise

Children were equally stumped by the problems with hundreds, even when there was suitable material available to them. The episode below illustrates Eloise's response to the problem with 4 small boxes of 100 clips.

I: "These are all the same. Can you tell me how many I've got now, all together?"
E: [looks down at the 4 boxes] "Four....boxes"
I: "Yeah, and how many paper-clips? Remember that there's one hundred in each one"
E: [2 second silence, uncomfortable] "Er....I don't know that sum yet"
I: "But it's a lot of paper-clips"
E: [nodding] "And I've only got 10 fingers"

Likewise, Eleanor W was perplexed at the same task. After agreeing that there were "100" clips in one box, she breathed in and ventured that there were "11 hundred and sixteen" clips in all 4 boxes. In a similar fashion, Eleanor D responded *"a thousand"*. The three cases had strikingly similar responses in the task with a jar of 200 lollipops. For example, the following episode documents Eleanor D's response.

I: "How many do you think we would have to take out of the jar to have one hundred inside?"
D: [raises her eyebrows] "One"
C: [...] "If I take one lolly away [takes 1 lollipop out], how many would we have left inside?"
D: "Er.....one hundred"

In a different context, when Eloise volunteered to write a sum that equalled "a hundred" she proposed *"2 hundred take away 1 hundred is 1 hundred"* and wrote '200-1'. Her astonishment when she saw '199' on the laptop screen was apparent.

I: "Oh! What's that number?"
E: [stunned] "Two [puts 2 fingers up] take away one [puts 1 finger down] ...two hundred take away....[puzzled] Say you've got two hundred on here [puts 2 finger up] and...these two hundred, and take away....[one] hundred [puts one finger down] ***is*** one hundred"

Evidently, for Eloise, like for Eleanor D and Eleanor W, "one hundred" was nothing but one perceptual unit.

7. 3 General Discussion of the Case Studies

The second stage of analysis aimed to ascertain the extent to which progress in arithmetic knowledge was attendant on children's elaboration of regularities in written and spoken numeration. If this were the case, findings should confirm that rule makers in both number scripts and number words would transform their counting methods into the use of strategic reasoning, progressive integrations and part-to-whole reasoning. Moreover, rule makers upon written and spoken numeration should use their rules in solving arithmetic problems. Second, those children who had not established rules in the first part of the fieldwork would tend to continue in the stages of non numerical counting, unless they had started to construct stable theories upon written and spoken numeration.

Findings of the second stage of fieldwork corroborated the inference of children's theories through their solutions to carefully designed tasks and through the unexpected construction of new rules that overcame former conflictive ideas. It was found that Eamon and Jack, who were seen as rule makers in a particular domain were subsequently rule makers in both written and spoken numeration. They were cases of Group 1, with Alice, Tom, and Johnny who also had established novel rules upon their former rules in number words and scripts. It was found that all the cases of Group 1 were in transition to the TNS or in transition to the ENS. This meant that they reasoned strategically and they could use a composite unit of ten, that is, they had re-organised their system of ones and constructed a system of tens. Conceptual differences between Johnny, Tom, and Jack, on the one hand, and Alice and Eamon, on the other hand, were found in the reversible co-ordination of tens and ones of the former, and the inability to do this of the latter. This was explained by the concept of ten they could use in anticipation. For Alice and Eamon, it was an abstract composite unit of ten, which depended on visualised material to co-ordinate units of tens and ones. For Tom, Johnny and Jack, it was an iterable unit of ten which allowed them to use their flexible sequence in tens to reversibly co-ordinate units of ten and ones in anticipation. Although Eamon and Alice were seen in the process of establishing a flexible sequence in tens, they tended to use their former sequence in tens involving the decade knots alone - e.g. 10, 20, 30. Similarly, Johnny and Tom's use of the rule of exchange was used to solve sums with hundreds and thousands and demonstrated their part-to-whole reasoning. On the other hand, Eamon and Alice's use of the sequence yielded striking responses like Eamon's solution to '6910+100'. Despite the establishment of different levels of arithmetic knowledge - i.e. some children were at the TNS and some in transition to the ENS - all children of Group 1 were appropriating an essential characteristic of the conventional system of number words: the base ten organisation. Decades, hundreds and thousands had become intuitively important. Their rules were not imparted to them but they were original constructions of the societal presentation of the conventional systems of written and spoken numeration. These constructions started to emerge well before Johnny and Tom developed part-to-whole reasoning and as a relatively independent system of constructions. However, the inferred rules could account for their solutions to arithmetic problems involving decades, hundreds, and thousands in the second part of the fieldwork. Their part-to-whole and strategic reasoning became interwoven with their constructions upon written and spoken numeration. Similarly, in the case of Eamon and Alice, their progressive integrations and creation of composite units became interwoven with the system of constructions in number words and number scripts. For Jack, his new rules in number scripts and number words indicated that he was in transition to the ENS.

These two system of constructions can be seen as relatively independent because reflections upon written and spoken signs were not seen to correspond to any particular learning stage. Nevertheless, although children of the study might have constructed rules when they were at the figurative stage, these children had soon constructed the INS. This is illustrated by the case of Alice who was a figurative counter in April 1998. According to the second line of investigation, she was classified as a rule maker of Group 1 and it was found that she had soon transformed her figurative counting stage to the INS - July 1998. Stephanie, on the other hand, was the only figurative counter in July 1998 who was considered a rule maker in number scripts. Her counting scheme had undergone an important transformation in January 1999 because she was now at the stage of the INS. Also, she had now constructed the rule of the sequence in tens. Although she had not constructed important rules, it was apparent that she had started to encounter contradictions when dealing with number scripts. She was seen using her novel sequence in tens to solve arithmetic problems but although she surprised the interviewer with her strategic answers, she was still at the INS. The case of Jack suggested that his reflections upon written numeration might have had a pivotal role in his progress in arithmetic. Although he was a counter in transition to the INS in April 1998 and a rule maker in number scripts, three months later he had constructed the INS and at the end of the study he had constructed relevant rules in written and spoken numeration which he used to demonstrate methods typical of the ENS. Jack used aspects of number scripts - e.g. number of digits, position of digits - to support this reasoning and justifications, and this was a characteristic that prevailed in the second stage of fieldwork. It must be noted that he had not been taught any paper-and-pencil algorithm nor had children been taught terms like tens and units when he was first seen reflecting on number scripts. In the case of Tom and Johnny, they were the most advanced counters from the start to the end of the study and they were always seen as rule makers of Group 1.

Eloise, Eleanor D, and Eleanor W were the three cases who seemed unable to perform numerical integrations even at the end of the study. They had constructed a figurative counting scheme and although they occasionally curtailed the verbal sequence to count on partially screened collections they showed counting-all solutions in subsequent contexts. It was also apparent that although they could establish criteria to deal with number scripts, their justifications were unstable and local to specific contexts. They had not appropriated conventional rules like those which Tom and Johnny had been able to use twelve months earlier. For these three cases the classification Group 3-Non numerical counting scheme remained stable during the one-year-long study. Detachment from counting routines involving actual objects seemed not to be possible for Eloise, Eleanor D, and Eleanor W. For example, for Eloise, "doing it in her head" had the meaning of silently counting items from the distance "without pointing", and counting to a hundred would "take a whole day and a whole night". What was actually being done "in their heads" by Jack, or Tom, was not available to any of the above three children nor were the rules that these children elaborated upon the system of number words and scripts. Several situations during the final stage of fieldwork demonstrated that children's abstractions on external signs were not just a mere matter of empirical abstraction of which any 5 year old is capable. It was documented how Eleanor W and Eleanor D could not write e.g. '37' after having written all the row of the thirties. On the other hand, it was documented how Tom had abstracted that there was a '2' at the front of all the twenties in the first stage of the study. He had established that this was also the case for '1' in front of all the teens and '4 in front of all the forties. In the second stage of fieldwork he demonstrated reasoning upon this abstraction by explaining that 2 in the twenties meant "two tens" or by spontaneously justifying his answer by saying that there were "ten numbers" in the twenties. At the other extreme of progression in arithmetic, Eloise - or any of the other two cases of Group 3 - expressed that the tiles with the

20's had been put together "because they had all the twos" rather than "the twenties". Further, these children proceeded to count the 2-scripts when they had to find out how many '2's' were in front of the tiles. When they were asked how many times was the '4' in front, they proceeded to count anew. This indicates that abstractions children make upon conventional number scripts are in fact driven by their previous theories upon how "numbers" are organised. These theories concern both empirical abstractions upon the external signs, and meanings children construct on those regularities. These meanings, in time, come to be numerical integrations. For example, Tom knew that tens were numbers of a kind, that they could be added together as units and he also knew that ten was a composite of ten single ones. In contrast, children of Group 3 had not yet established the rule of the importance of the position of digits in 2-digit scripts. On the other hand, when dealing with number words, the three children seemed to use a verbal pattern which was termed tee-sequence in tens. It has been documented how this fixed verbal recitation stimulated situations of conflict in the context of counting single squares and rows of squares. A tension caused by the use of her tee-sequence was documented for the case of Eleanor W in particular. One could conjecture that it is in the context of counting objects and using her verbal recitation that she could start reflecting upon her actions when counting single objects.

7. 3. 1 Contribution of the second line of investigation

Under the assumption that children establish regularities when interacting with the mass of external numerical signs of their environments, the second line of investigation found that children establish theories upon how number scripts come to convey meaning. Given the categories produced for the findings of the second line of investigation it was apparent that rules upon written and spoken numeration that children constructed were not immediately stable and conventional. Three main phases can be discerned for the appropriation of number scripts and number words:

1. First, children's ideas can be partially conventional or unconventional and they can be used in particular contexts. They seem to work in dealing particularly with number scripts in specific situations. An example is the idea according to which the more zeros a script had, the bigger the number. These ideas cannot be used

simultaneously with other ideas. They are used locally, in a particular context but they could be contradicted by the use of another local criteria in a different context.

2. In a second phase, partially conventional ideas are changed so as to be viable in all situations: for example, the idea that the more zeros a script has, the higher the number is replaced by the rule of the digits. Further, rules can be co-ordinated. For example, the rule of the digits is co-ordinated with the rule of the front digit and the child can explain that to order the number scripts one must attend to the quantity of figures. If it is the same quantity, one must attend to the front digit to distinguish which number script is higher.
3. In a third phase, ideas that seemed to be formerly contradictory are reorganised to define a new rule. For example, the rule of exchange solves the tension of having a script -1000- with two different number names according to former criteria. These novel rules seem to be solutions to genuine conflicts experienced by children.

It follows from the categories established in the second line of investigation that phase 3 involves the construction of stable rules compatible with the conventionalities of spoken and written numeration of our culture. On the other hand, in phases 1 and 2 stable theories can predominate in one particular domain, for example, only in number words or in number scripts. Previous research has documented a particular and systematic type of justification that children give when comparing number scripts. In this thesis, this kind of behavioural pattern has been termed rule of the digits. In contrast to previous research where the rule of the digits is seen as "an isolated principle or rule" (Sinclair and Scheuer, 1993; Giroux and Lemoyne, 1998), this study has documented how children co-ordinate the rule of the digits with the rule of the front digits and how they construct novel rules based on their former criteria.

Tom, Johnny, and Alice were in phase 2 and 3 during the year-long investigation. Jack and Eamon were in phases 1 and 2 at the start of the study but their reflections were of phase 3 at the end of the study. Stephanie, despite her new rule of the sequence in tens, had not established the rule of numbers of the same kind, nor had she appropriated the conventional forms for hundreds and thousands. Hence, the conflict between "10 hundred" and "1 thousand" was not possible, her rules were in phase 1 and 2. Finally, Eloise, Eleanor D and Eleanor W were in phase 1 during the whole investigation. The three phases presented above do not imply a hierarchy of development nor do they pretend any strict correspondence with the model of learning stages. Children started constructing ideas upon number words and number scripts before they had constructed numerical counting schemes - i.e. the INS. However, as has been discussed, there was an apparent correspondence of progress in arithmetic and constructions of new rules for the children who had attained the TNS or were in transition to it or the ENS. For these children, the rules they had established became themes of reflections. One can conclude that rules were firstly viable theories because they worked without knowing why. Subsequently they became co-ordinated with other rules and novel rules were constructed based on these co-ordinations. New rules betokened children's reflections on the reasons why previous rules worked. Children were sure that the more digits a script had, the higher the number being annotated. But only when they established numbers of a kind, and they appropriated that the 3rd digit stood for groups of hundreds and the 4th digit stood for thousands, could they start to explain why 4-digit numbers were higher than 3-digit numbers. Precisely, the novel rule of the place was an elaboration of the rule of digits and the rule of front digit. That is, it came to explain why if a number had more digits it was higher than another one. Children's rule of the hundred and thousand scripts was another novel construction that permitted them to write conventional forms for number scripts formerly concatenated. Precisely, in the first stage of the fieldwork, children wrote concatenated forms, e.g. '20050' for "250", but in the second stage of fieldwork, they had established that hundreds were written with 3 digits. Moreover, they knew that hundreds were "numbers of a kind", different from the kind of thousands. Children's system of rules were in constant re-organisation because they became themes of reflection. Number words and number scripts were used to signify addition and subtraction operations and counting routines that seemed no longer necessary. The table below presents a possible correspondence of the three phases outlined above and the learning stages of the model.

Phases of Construction of Rules and Learning Stages

Learning Stage	Phase
NON NUMERICAL COUNTING	*One:* Unconventional local criteria
TRANSITION TO INS	*Two:* Conventional rules, co-ordination of rules
ABSTRACT COUNTING: INS-TNS-ENS	*Three:* Novel rules based on former contradictions

The aforementioned considerations allow the claim that the sophistication that came about with strategic reasoning in arithmetic did not solely depend on children's abstractions on their actions of counting but also on their abstractions on the system of number words and number scripts. There was no child who could operate at the level of the INS, TNS, or ENS who had not constructed important conventionalities of the systems of spoken and written numeration. This should not be surprising because the sophistication that comes about with dropping routines of counting by ones depends on knowing a system of numeration which shapes the ways our culture has come to operate on numbers.

7. 3. 2 Mediation of written and spoken numeration

The problem of mediation of systems of cultural signs has been addressed at the start of this thesis. It can now be discussed in relation with the connection of the two lines of investigation. The second line of investigation aimed at exploring how children construct regularities in the systems of spoken and written numeration of their culture. In this sense, written numeration and spoken numeration were viewed as the object of knowledge children come to organise. With an epistemology of individual constructivist activity, the question of how written and spoken numeration mediate their mathematical cognition is the question of how children's rules on number word and number scripts may have a role in their progress in arithmetic knowledge. In other words, the conventional systems of spoken and written numeration may be seen mediating children's construction of a system of numbers, not of a system of numeration that is the system of numeration of their culture. The second line of investigation has documented how children started appropriating a paramount characteristic of the system of written notation of numbers: the principle of place value. First, they established that the more digits a script had the higher the quantity being annotated. Second, they established that there were numbers of different kinds and that in a given script, digits stood for "hundreds" or "thousands". Of course, for these children, numerical integrations were possible and their construction of a system of composite units became integrated with their elaborations on how these conventional signs were organised. It follows that external signs came to mean composite units through a complex system of constructions that could not be explained solely by children's abstractions in the activity of counting. The principle of place value that governs written notation is appropriated in gradual approximations which concern both the development of part-to-whole reasoning and a system of rules on number scripts and number words. By the same token the genesis of the concept of ten as a composite unit is intrinsically interwoven with children's theories about number words and number scripts. It seems that the idea of ten as a unit is a product of a process of the appropriation of number words which starts well before an iterative ten can be said to be constructed. Eamon understood at the commencement of the study that one could make bigger numbers faster when counting in tens. In this sense, he regarded counting by tens without the use of fingers as a better strategy than counting by ones. This meant that he had intuitively appropriated that there were different ways of counting and some were better ways than others. Eamon's awareness of "better ways" was also suggested by his will to use the standard algorithm to solve a sum with 2-digit numbers. However, his system of rules monitored the use of a method he had not yet appropriated. Conventionalities of spoken and written numeration were appropriated in different phases and these intertwined with children's system of numerical integrations. The more sophisticated approximations to the use of written and spoken numeration were those of Tom or Johnny. However, as one knows, their approximations were not completed or final: they constituted the system of constructions that this study could identify at the end of the present investigation.

The model of counting stages - and the conceptual analysis that it allows - confines the explanation of children's mathematics to the "schemes of action and operation that are functioning reliably and effectively" in the activity of counting. This thesis has shown that children's reasoning in arithmetic becomes inextricably linked to their theories on how written and spoken numeration are organised. In this sense, numeration mediated cognition through children's theories upon the external signs. The model of learning stages can explain how children progress from a figurative counting scheme to a numerical counting scheme. However, children's more sophisticated methods and insights become highly related to constructions which are relatively independent from their counting activity. Children's mathematical ideas cannot be based solely on their abstractions on actions but are inextricably related to abstractions using the system of signs of their culture. When children abstract that 10 hundreds equals 1 thousand and they use this in solving problems like '6910+100', their complex reasoning becomes apparent.

This thesis acknowledges the constructivist view that children must construct a system of tens on a system of ones[5] but it claims that children's mathematical constructions are inescapably related to their use of conventional signs. Becker and Varelas[6] have proposed two components of knowledge construction: "top-down" and "bottom-up". Strictly speaking they are both bottom-up because they both concern the constructive abstractions of the learner. However, the former highlights the role of the pre-existent knowledge of the culture in shaping or modifying the learner's abstractions, and the latter focuses on the original acts of reflecting of the learner or the so called "endogenous processes"[7]. In this sense, Sinclair has also claimed that learning in any domain depends both on endogenous process of abstraction and societal presentations that children encounter in their environment. Precisely, the notion that 10 hundreds make 1 thousand originated from two conventional readings of 4 digit numbers. Hence understanding of the system of decimal numeration does not come through reorganisation of counting activity only. Although part-to-whole reasoning and an iterable unit of ten allows a viewpoint of the system which is highly sophisticated, there are previous phases of approximation to the system of written numeration. In this sense, the view that "meaning should be developed before conventional symbols

[5] Steffe et al 1988; Kamii, 1986.
[6] Becker and Varelas, 1995.
[7] Sinclair, 1987.

are introduced" cannot stand without controversy[8]. Conventional signs are not introduced, they introduce themselves in so far as children try to make sense of them and use them in their social environments. As Kaput (1991) has put it, conventional symbols limit and constrain the construction of meaning. Furthermore, meanings are not a once-and-for-all phenomenon but are constant systems in transformation. Hence, developing the concepts first, and then introduce conventional signs seems an awkward pedagogical principle. Of course, this does not mean to recommend the teaching of standard algorithms at an early age. This is perhaps what has been criticised by the principle which advocates for the prior "development of the concept" before introducing its external representation. However, this thesis has shown how reflections on external systems of signs can drive children's thinking from an early age. By the same token the view that children first develop spoken numeration and subsequently written numeration cannot be accepted without challenge. Steffe has set forth an ordinal approach to number following children's counting methods as opposed to a cardinal approach (or place value approach) that he denounces to be the current approach held by school text books and instruction. He argues that

> "children's number sequences in a given culture cannot be taken as being imbued with all of their cultural meaning, if so for no other reason than the operations that they symbolise can be quite different among children to say nothing of the differences between children and adults. It is quite significant that the verbal number sequence was built up before a mature system of written numerals became established." (Steffe, 1994, p. 165).

He quotes Menninger (1969) to point out that the rules that govern early numerals - ordering and regrouping - do not correspond to the rule of the sequence which is a stepwise gradation. Menninger remarks that systems of writing numbers are not a mere representation of the number word sequence, something that has been addressed in this study in the treatment of two different systems of representation. Steffe draws upon Menninger's remark concerning the independence of spoken and written systems and uses this to advocate his proposed ordinal approach to number.

> "Whatever those numerical operations are that are available to children, they are initially symbolised by their verbal number sequences." (p. 166)

However, Steffe seems to leave unstudied those reflections and abstractions stemming from children's use of written numeration not imposed by school instruction. These include children's reflections on spoken numeration and refer to their use of number

[8] Cobb and Wheatly, 1988

words and scripts in contexts other than counting. A "cardinal approach" is taken spontaneously by children when they abstract that 10 hundreds equal 1 thousand or that the 3rd place in a 3 digit number stands for "hundreds". Hence, the approach is simultaneously cardinal and ordinal when children abstract regularities on written and spoken numeration. Literature has documented that the notion initially grasped by young children's interpretation of 2-digit numerals is one of partition and not one of regrouping (e.g. Sinclair and Scheuer, 1993; Sinclair et al., 1992). For example, '2' in '23' means "20". This was found clearly, at an early stage of the fieldwork, in the case of Alice or Tom. Subsequently, children come to grasp an important aspect of our written system of numeration: that parts of cardinalities are noted in parts of the notation, and that digits of a number script conserve their primary meaning. For example, in '23', '2' stands for *two* units of ten. The findings of this thesis agree with previous research and contribute to our understanding of how the aforementioned comprehension comes about. Lerner and Sadovsky (1995) have also claimed that the notion of regrouping is not the way children first approach written numeration. This thesis concords with their findings and with these researchers claim that establishing regularities in written notation has a twofold importance. First, it allows the posing of problems directed to explicitate the organisation of the system. But to formulate questions about the reasons which explain the regularities - i.e. the regrouping in base 10 - has no sense if these regularities have not yet been established. Hence, as Lerner and Sadovsky have pointed out, for those who have to construct a system that is already made, the regularities are abstracted before the reasons which have generated them. The second point of importance is that the establishment of rules - e.g. the rule of hundred and thousand scripts - allows progress in the use of written numbers. Precisely, children who have elaborated viable rules on the system of words and scripts can control their writing and interpretation of these signs.

For Steffe, written numeration is not seen as the material or the context within which children construct their primary ideas about numbers. Numbers are conceptual entities that come into being in the activity of counting[9]. This is demonstrated when one views young children repeatedly counting-all from one to solve the total of partially

[9] Cobb and Wheatley, 1988

screened collections. But when children become abstract counters there is a wide range of constructions that seem to intertwine with their reflections upon their actions of counting. Numbers can no longer be seen existing solely in the context of counting activities because number words and number scripts come to stand for counting actions and they are in turn subject to reflections within a system of signs in social use. In Steffe's model of learning stages, numeration is a subsidiary object of knowledge: the focus is on children's actions and operations to construct systems of composite units and operations. Abstractions on the naming systems - written and spoken - play no part in children's construction of mathematical knowledge. It can be argued that the model concentrates on the bottom-up component conceived by Becker and Varelas and neglects the top-down component. In constructing a system of numerical integrations, children use a system of names of their culture. The top-down component of knowledge, that is, children's rules upon cultural systems of representation, intervenes despite not being analysed by Steffe[10]. This study has shown that, even with an epistemology of individual construction, the top-down component is inherent in children's system of abstractions upon counting actions.

Bickhard (1995, p. 262) has discussed the problem involved with the theory of mediation. He argues that "mediation is not enough" because constructive mechanisms are not explained. Although, as suggested at the beginning of this thesis[11], constructivism has not focused on the epistemic relation of the subject with the system of signs of their culture, the present investigation has centred on children's learning of external representational systems of number. In its second line of investigation, it has endorsed the view that

> " Culture works as a teaching machine, and we learn from interacting with both the world and others. The artefacts that others have constructed literally become our worlds." (Ackermann, 1995, p. 353)

This study offers one way to reinterpret findings of cross-cultural investigations that document different methods of solution for Asian-language children compared to their

[10] In Mind as Action, Werstch (p. 109, 1998) distinguishes two senses of social. One sense of social has to do with whether one or more individuals participate in a given action in the immediate context. This is the social-interactional sense. The second sense of social refers to the fact that sociocultural mediational means -systems of signs- are part of any sociocultural setting. This is the sociocultural sense that can be seen as the one which concerns the top-down component of knowledge.
[11] Sinclair (1986, p. 67), on Piaget's absent focus on systems of signs.

English speaking peers when solving arithmetic tasks. As has been discussed, according to these studies, different naming systems of their language mediate children's cognition and put Asian students at an advantage compared to their English peers. According to these studies this is because Asian spoken numeration is more "transparent" than the English system. In relation to this point, Fuson has stated that

> "Delaying (U.S) children's opportunities for interacting with numbers in the hundreds and thousands may only delay opportunities to construct multiunit conceptual structures for multidigit numbers" (Fuson, 1990[a] p. 277)

According to this thesis, it is through children's inspection, use, and construction of rules on the system of number words that their systems of numerical constructions become integrated with their rules in spoken numeration. Systems of number words are not per se the mediators but the system of rules upon them may integrate with their numerical integrations.

7. 4 On the Way to Conclusions

In the history of construction of abstract units there is a different constructive activity relatively independent from the activity of counting. This reflective activity might originate prior to the INS stage. However, it is at this stage that it seems to empower children's increasing sophistication in arithmetic. Children's awareness and reflection upon the systems of spoken and written numeration of their culture integrates with their reflections on counting activity.

In discussing the status of writing symbols, von Glasersfeld has argued that,

> "from the perspective of mathematical understanding, any manipulation of symbols (external signs) is an empty activity unless the user of the symbols has already access to the mental operations they indicate" (von Glasersfeld, 1995[c], p. 382)

In this discussion von Glasersfeld seems to endow an absolute status to the mental operations "indicated" by external signs (symbols). He seems to argue that unless the user of the signs has those mental operations available, the writing or manipulation of external signs does not constitute mathematical activity. However, from his own constructivist stand, any "manipulation" of external signs can be seen as suggesting some form of construction for the "user" because users tend to establish regularities in their experience concerning these external signs. One can interpret "empty activity" in the above citation as "non mathematical" activity. But one can argue that there are other types of abstractions that relate to the use of conventional system of signs which, albeit not necessarily mathematical, can affect concomitant or posterior mathematical constructions. The point that needs to made is that, as a constructivist standpoint allows one to see, mathematical operations, presumably indicated by the signs, are defined and redefined in the subject's experience. During this time, the child uses external signs and establishes other regularities that come to be subsequently integrated with their mathematical abstractions. These regularities are the product of active reflection on the systems of number signs - spoken or written. Children are eager to know about these signs and what they mean, because they are "the numbers" for them. Elaborating regularities or rules within these systems of signs is part of this reflective activity. These elaborations may well be seen partly as the product of empirical abstractions but it is ultimately reflective abstractions, i.e. co-ordinations on previous rules, and counting activity, which gives numerical meaning to the signs.

The appropriation of a system of rules is not seen as a "copy" of what is already there, i.e. the conventional system of numeration in its spoken or written form, because it is a process of original construction. As has been discussed, the appropriation of these systems of signs is not a once-and-for-all process. It is different for any particular child's history of construction: conjectures on either spoken or written signs might prevail at the genesis of this process of appropriation. Nevertheless, children's theories are not idiosyncratic or unique because they are shaped by their very objects of knowledge: the pre-existent systems of naming numbers. The rules that children elaborate and use successfully are consensual with the rules of conventional numeration. Hence, one can claim that they are "appropriated". However, children's rules are the product of their intellectual action on the external signs. When Jack solved the sum '2638-…=2038' he knew that 6 meant 600 in '2638'. This was not the product of an idiosyncratic rule but a conventional one, that is why it was appropriated by him. Nevertheless, the claim is that Jack had discovered it through the co-ordination of former rules and that it had not been "transmitted" to him. In this sense, one must attend to children's claims, for example, Johnny's assertion upon his knowledge of numerals: "I learnt it for myself".

Mathematical use of external systems of signs involves the construction of abstract units and composite units, but one cannot remain blind to the fact that children's reflective activity does not solely occur in the context of counting. There is a system of abstractions constructed relatively independent from the activity of counting. Young children's reflections on the actions of counting collections of objects is of paramount importance, but increasing awareness of a system of signs in cultural use seems to have a part in children's progress in arithmetic.

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8. Conclusions

8. 1 Two Systems of Constructions

It was the premise of the study and a working hypothesis that reflection on the material system of spoken and written numbers was a source of meaning scarcely investigated. Was children's knowledge of number words and scripts as indicated by their rules connected to their progress in arithmetic knowledge? This thesis has found that children's rules of the their systems of number words and scripts were attendant on their increasing sophistication in arithmetic indicated by their detachment from actual counting routines. Conjectures of the preliminary analysis after completion of the first stage of the fieldwork were confirmed by a final analysis of children's solution to a variety of problems presented in the second stage of fieldwork. Findings of this thesis concord with previously documented evidence that children conjecture how numerals come to communicate meaning in their environments. However, it has shown that their conjectures conform to a system of rules which becomes integrated with their reflective abstractions stemming from their activity of counting and whereby the model of learning stages identifies increasingly sophisticated concepts of abstract units and composite units. The second line of investigation followed the transformation of the children's abandoning their idea of concatenation of number scripts through the establishment of the rule of hundred and thousand scripts. Children who appropriated the conventional scripts for numbers which were not taught at school, did so through the elaboration of conventional rules. School instruction does not seem to operate on children's conjectures on the system of written and spoken numeration used in their cultural environments. With an epistemology of individual construction, this study has shown that children establish stable ideas on how written and spoken numeration work and come to communicate meaning. These ideas are not necessarily mathematical ideas but they are notions on how these system of external signs are organised and they are subsequently integrated with the product of reflective abstractions stemming from children's counting experience. Although there can be two systems of constructions which are relatively independent, children's arithmetic knowledge in stages of higher sophistication becomes interwoven with their rules in number scripts and number words which, in turn, integrate mathematical meaning.

8. 2 On the Way to Didactical Recommendations

The view according to which knowledge is actively constructed by an individual allows one to see certain things, while being blinded to others. In this sense, the study on how children come to know a cultural system of signs which embed mathematical meaning was not conducted in the context of a classroom setting nor in a specific cultural setting. By following nine case studies during a one year investigation, this thesis has set forth an account on how children start learning the cultural systems of written and spoken numeration before formal school instruction. Didactical recommendations have been beyond the focus of this research. Notwithstanding, research stemming from this type of studies can inform research which addresses problems involving learning in schools. In this sense, one concords with Inhelder when she argues that

> "Although learning studies certainly do not close the gap between cognitive psychology and classroom practice, they constitute a link in the chain that may eventually unite the two." (Inhelder et al., 1974, p. 30).

This thesis has addressed a research interest widely discussed by constructivist researchers who found the study of the learning of conventional systems of signs a problematic topic. In one discussion about their research on the function of meaning of numerals in children's environment, Sinclair and Sinclair set forth the following question,

> "does the knowledge pre-school children develop about written numerals in their environment interact with the development of number concepts? It would be surprising if this were not the case. [...] For the moment, we can do no more than speculate about the possible links." (Sinclair and Sinclair, 1984, p. 183).

This thesis has come to give some answers to premises and speculations of former research and, in turn, new questions can be posed. Findings documented in this research can be used to design didactical research projects which contemplate children's previous constructions on conventional systems of signs as well as reflections on their actions of counting objects. Current curriculum programmes do not view the use of written numeration as material for children's reflection. Initial knowledge about written and spoken numeration constructed by young children can be taken into account to develop further knowledge of the system including the mathematical notions expressed by the system. Therefore, the recommendation is not to return to the teaching of standard written algorithms which use conventionalities of the system but to use the rules children establish in order to design conditions for the possibility of learning more sophisticated notions involved in the use of the system of written and spoken numeration.

Steffe, amongst other constructivist researchers in mathematics education, has stated that reflection is the mechanism whereby physical actions become mental operations. This seems to be a viable and powerful explanation of the genesis of mathematical knowledge. However, reflection, as a constructive mechanism, can also be defined as the process through which simple ideas become co-ordinated with other ideas or theories. Hence, reflective abstraction accounts for a complex system of constructions that children come to demonstrate through their increasingly sophisticated methods in solving elementary arithmetic problems. In analysing the findings of both lines of the investigation, it was apparent that children do elaborate sophisticated theories on the system of signs as a relatively independent constructive activity. These reflections on the system of signs, its organisation and its use, transform children's solution strategies to early arithmetic problems and become intrinsically linked to children's concepts of composite units and reversible operations. The elaboration and reflection on the system of words and numerals contributes to the construction of abstract units and it is not a subsidiary process but is intrinsically related to that construction. Children come to conceptualise units and a conventional numeration system from two independent lines of constructive activity. When these two lines become integrated the child attains higher sophistication in arithmetic. It is children's reflection on the system of external signs of their culture which seems to empower the construction of systems of increasingly sophisticated arithmetic units.

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*

Glossary

First-order model: The hypothetical models the observed subject constructs to order, comprehend, and control their own experience -i.e. the subject's knowledge (Steffe et al, 1983; Steffe, 1995)

Flexible sequence in tens: A flexible recitation of the decade words which can start at any point of the number sequence. For example: "17, 27, 37, 47".

Integration: A reflective abstraction whereby one unites in thought what one may also consider single unitary items. Sequential integrations concern figurative material but progressive integrations concern the result of a previous integration.

Knots: Round numbers, e.g. hundred knots are 100, 200, 300.

Local rule or regularity: It stems from its use in a particular context or situation but does not seem to be a generalised elaboration that the child applies in all presented situations. A local rule is fragile, and even if conventional can be easily destabilised after its use with a second prompt by the interviewer.

Numerical finger patterns: re-presented finger patterns that have been stripped from their sensory qualities. They are not fixed like figural or perceptual finger patterns.

Partially screened collection: A situation in a counting task where one part of a collection of objects is hidden or screened, the other part being visible.

Rule of exchange: One of children's novel rule in the second stage of fieldwork whereby ten numbers of a kind formed a number of a different kind. For example, 10 hundred was 1 thousand.

Rule: (in number scripts or in number words) An original construction which is a relatively stable theory on how external signs are organised. It is typically compatible with conventional features of spoken numeration

Second-order model: The hypothetical models observers may construct of the subject's knowledge in order to explain their observations of the subject's states and activities (Steffe et al, 1983; Steffe, 1995).

Tee-sequence in tens: A fixed recitation that starts at ten and finishes at twenty as follows: ten-twentee-thirtee-fourtee-fiftee-sixtee-seventee-eightee-ninetee-twenty. It seemed to originate from the recitation of the teens.

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1. Appendix 1: The Model of Learning Stages

Typical responses for the classification of counting types

If 4 bricks are visible and 3 bricks are covered, a child's solution to finding the total may indicate what type of unit items they can create while counting. Typical responses for classification follow:

1. *Perceptual counter:* The child counts the visible bricks from one and is puzzled when the cloth is reached. They may point at the cloth and utter "5", that is, counting the cloth as a fifth perceptual item.
2. *Figural counter:* The child counts the visible bricks from one and stares intently ahead carrying on up to "7". The inference is that the child imagined the three hidden bricks and counted them.
3. *Motor counter:* The child counts the visible bricks from one and then points to imaginary places in the air, three times, stopping at "7".
4. *Verbal counter:* The child utters with no apparent movements "1, 2, 3" and then carries on counting the visible items up to "7".
5. *Abstract counter:* The child counts on from 4, keeping track of three counting acts, for example, by putting up fingers 1 by 1, until "7" is reached.

These responses are simply illustrations of theoretical considerations but interpreting a child's solution to claim any progress in their knowledge of counting is always a matter of inference which needs to be confronted with new observations. To exemplify how controversial a classification can be, one can suppose, in the above problem, that a child proceeds to count the visible bricks while uttering "4, 5, 6, 7". One can infer the child counted on because they thought that "3" implies the counting act from "1 to 3" and therefore the child is an abstract counter. However, several alternatives are possible, all of which do not afford a classification as an abstract counter. For example, a child may have verbally curtailed uttering the words "1, 2, 3" without carrying the force of the reflective abstraction required for abstract counting. The only way of inferring abstract counting is through the validated presumption of children's reflective thinking. The validation of any inference needs to be done on the basis of further interaction with the child.

Examples of children's responses for the classification of their learning stage

Theoretical examples of children's solutions given below are based on the kind of counting problems presented by Steffe and Cobb (1988) to 5 to 7 year-old children. These consist of a variety of problems based on counting partially or totally screened partitioned collections of items. For example, if 6 bricks are visible and 5 are screened, the child is presented with the following tasks:

1. Count the visible bricks being told that 5 bricks are covered. Find out how many bricks there are all together.
2. There are some bricks hidden and there are 11 bricks all together with the visible bricks. Find the number of hidden bricks.

These tasks are referred as finding the total of a partially screened collection and finding the missing addend of a partially screened collection, respectively.

As a child re-organises their counting scheme, different problems can be presented and the present formulation of the learning stages includes theoretical examples of these.

1. *Perceptual counting*

Children can count experiential objects in the immediate here-and-now: they must see or feel what they are to count. Thus, they cannot find the total of a partially screened collections and they count from one to find the total of a segmented visible collection. Typically, they count from 1 each subcollection and subsequently start counting from 1 the whole collection. Actual objects (e.g. fingers, bricks) need to be perceptually present to be counted. Number words seem to refer to the transitory experience of counting collections of perceptual items.

Transition from perceptual to figurative counting stage:

Children may be able to count objects that are not in their immediate range of action or perception. They create perceptual items to count that serve as substitutes for the figurative unit items that are only imagined. The figurative unit items (the imagined objects) are countable but the child still needs to create experiential unit items to count. During this transition, the child may put up 4 and 3 fingers as *replacement* for the hidden items and then count them as a collection of perceptual items. These finger patterns are still in the domain of perceptual counting because they are taken as collections of perceptual unit items for counting. Each pattern is counted from one and then the whole collection is recounted from one. There is a subtle change when the child sequentially puts up fingers to complete the finger patterns synchronous with uttering the number words, that is the child might utter 1, 2, 3, 4, and after pausing, 5, 6, 7, synchronous with putting up fingers. This is an indicator of figurative counting because the fingers stand for the screened objects. Creating finger patterns seems to be a path towards the figurative counting stage but this is not an immediate achievement: it only occurs when both the activity of counting and the items to be counted can be re-presented. Some children with perceptual counting can produce figurative collections, but their counting scheme remains perceptual because they cannot independently create countable items in the activity of counting that are substitutes for the figurative representatives they intend to count. That is why perceptual counters in the transition to figurative counting cannot yet keep track of counting. An indicator of this transition is a slight perturbation when not being able to keep track of the counting acts. Thus, child is able to count partially screened collections in some contexts.

2. *Figurative counting*

The child can create and count the items of a figurative collection. This stage includes cases in which children can create and count motor acts. The meaning of a number word is the experience of counting visualised, motor or verbal unit items. Children at this stage no longer depend on direct sensory experience but typically re-create a sensory experience when counting. The counter appears to visualise the items that cannot be seen and this need for visualisation is a relatively severe restriction on the facility with which counting can take place. All movements are important (e.g. hand waving over hidden objects). Children at this level count from one when solving addition problems with screened collections and will often point at imaginary objects to aid counting. When required to count two collections of 6 visible items and 3 screened items, the child typically counts the 6 items first, from 1 to 6 (to give meaning to "six") and then, counts the 3 other items by waving 3 times over the screen that covers the 3 items. Theoretically, the child can find the total of partially screened collections in all contexts but always counts-all the items presented from one. When counting the screened part, the child often keeps track of the counting acts with finger patterns and can focus on the screened part to figure out the number of screened objects.

Transition from figurative to initial-number-sequence counting:

When a child is able to monitor their continuation of counting to find the number of screened objects, one can presume that the child's reflective thinking affords the construction of numerical ideas. We can attribute to the child the execution of the operations that are necessary to make numbers: the operations of re-presentation and integration. This happens when a child is able to re-present the results of counting and when the child can run through the items of the figurative pattern holding them at a distance. Before that, a number word does not necessarily stand for the segment from one up to and including that number word. That is why before this re-organisation occurs, the child counts all items from one. During the transition from figurative to initial number sequence, the child can count on in some contexts. However, they do not count on from a given number word in an anticipatory manner in all situations. The child may monitor the continuation of counting acts related to a partially screened collection during the activity itself.

3. *Initial number sequence (INS)*

Children are able to count-on to find the total of partially screened collections in all contexts and may be able to find missing addends in partially screened collections. They can count-off from a number word, to find differences. For children at this stage, the meaning of number words has undergone a significant change. Presented with the task of finding the sum of two subcollections of screened objects, children will start by stating the sum of the first subcollection and then count on. In the example of stage 2 above, the child would commence by saying "6" and then, say "7, 8, 9". Stating the initial number implies the activity of having counted the first six items (figural, motor, or verbal), but re-counting when presented with an additional subcollection to add on is now unnecessary. Children at this level are operational rather than figurative because they no longer depend on the links to re-presented experiences as in stage 2. A particular number word stands for the possible counting activity and refers to the individual number words of the segment from 1 and up to that particular number word. It does not, however, refer to a unit containing these number words. A number word has yet to symbolise the number sequence up to and including that word as one thing. It can symbolise the individual number words in sequence, but it is yet to symbolise a unit containing that sequence. Children have not yet constructed a "one more" relation (order with inclusion). A number word of the

INS symbolises its initial segment, it can be used in such a way that it symbolises counted items (but not counted and countable). It is a one way scheme.

Transition from the INS to the TNS:

An indication that a child, still in the Initial number sequence stage, is on the transition to the Tacitly nested number sequence is the ability to reflect on their counting acts. This happens when the child counts their counting acts in an anticipatory manner to find out the missing addend of a partially screened collection. Being able to find the missing addend by counting on is a characteristic of the INS but when the child intentionally counts the counting acts of the continuation of counting with the intention of uniting them in thought, there is indication of a further re-organisation.

4. *Tacitly nested number sequence (TNS)*

Children are able to focus on the collection of unit items as one thing as well as on the individual abstract unit items. They are able to form "composite units". A number word refers to the verbal number sequence from 1 up to and including that word as the constituents of a composite unit. This is a "nested" sequence or an inclusive sequence because children are aware that a subsequent number word means one more unit and a composite unit from one to that given word. Children can count-on and count-down, and are able to select the more appropriate strategy depending on the problem to be solved. For example, they generally count-down instead of counting-off to solve subtraction problems like '22-17'. In this case, the child would typically count down to 17 rather attempting to count off 17 from 22 (a strategy typically used in stage 3, because the counter at this stage just can not regard 17 as a composite unit to count down to). The number sequence from 1 to 17 is nested in the larger sequence from 1 to 22 but there is no explicit awareness of the two distinct number sequences. A further indicator of this stage can be that the child is able to establish the counting acts as countable items and thus, the child is able to double count to keep track of the counting acts. A number word of the TNS symbolises a unit containing its initial segment. It can be used in such a way that it symbolises both countable and counted items. It is recursive in that the results of counting can be reconstituted as a situation of counting (it is not a one way scheme like the INS). The child's concept of a number word contains records necessary to make an abstract composite unit.

Transition from TNS to the ENS:

A transitional phase between the TNS and the ENS can be characterised by the child's emerging awareness of the relation between addition and subtractions operations in constituting composite units. Children may be able to use thinking strategies to short-cut a counting routine. In the TNS children can co-ordinate two different number sequences (that is why they can double count) whereas in the ENS they can have the two number sequences and understand that one can be included in the other.

5. *Explicitly nested number sequence*

Numerical part-to-whole reasoning is the identifying characteristic of the ENS. Children are simultaneously aware of two number sequences and can disembed the smaller composite unit from the containing composite unit and compare them. For example, the child at this stage is simultaneously aware of the two distinct number sequences that are only implied at stage 4. Therefore they can disembed the smaller composite unit from the containing composite unit and compare them. They can conceptualise the whole, a part of the whole and the remainder. The child is aware of the inclusion relation and can intentionally disembed a segment of the explicitly nested number sequence from its inclusion in the containing sequence and treat is as a unit in its own right. There is understanding of the relationship between addition and subtraction Children at this level are able to import knowledge into a problem in order to simplify or shortcut a counting routine. These solutions are based on thinking strategies such as compensation strategies. For example, a child may work out that '4+6' is 11 based on their knowledge of '5+5' by subtracting 1 from 5 and adding 1 to 5. In this stage, any number word symbolises a corresponding initial segment in 2 ways: as a composite and as a unit. A child can simultaneously view the composite unit as part of the sequence as well as a number in its own right (the number sequence is graduated). The inclusion relation is now made explicit. A number word now can symbolise a unit that contains a TNS. An example of a solution for '12+.....=19' at this stage can be the following: "10 plus 9 is 19, I take away the 2, I mean 10 plus 2 is 12 and 9 take away 2 is 7". The child decomposed 19 into 10 and 9 and composed them in 12 and 7. The units 9 and 10 are still part of the whole unit 19, and they are material for further numerical operating.

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2. Appendix 2: Interview Sessions Scripts

Interview session: Initial counting assessment-April 1998. (Individual)

ASSESSMENT RECORD[1]

Name: Age: Date:

1. Counting forwards.

Ask: Can you count starting at 1? If the child does not start, prompt: "Like 1, 2, 3..."
Ask: Can you count starting at 8: If the child does not start, prompt: "Like 8, 9,..."

	Prompt		Up to	Omitted
By ones from 1	yes	no		
By ones from 8	yes	no		

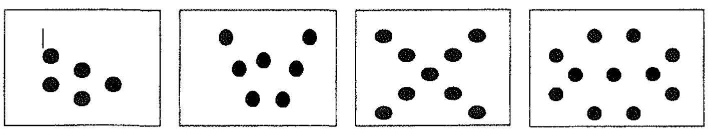
2. Counting backwards by ones.
Ask: Can you count backwards starting at 20? If the child responds successfully, omit next question.
Ask: Can you count backwards starting at 10?

	Prompt		Up to	Omitted
From 20	yes	no		
From 10	yes	no		

3. Counting forwards.
Ask: Can you count by twos/in twos? If the child does not respond, prompt: "Like 2, 4,..."
Ask: Can you count by tens/in tens? If the child does not respond, prompt: "Like 10, 20, ..."
Ask: Can you count by fives/in fives? If the child does not respond, prompt: "Like 5, 10, ..."

	Prompt		Success		Up to	Omitted
By twos from 2	yes	no	yes	no		
By fives from 5	yes	no	yes	no		
By tens from 10	yes	no	yes	no		

4. Patterns.
Present the child with the following flash-cards (15 cm x 10 cum) and ask: Can you tell me how many dots there are on this card?



[1] Adapted from: Initial Clinical Assessment Procedure Mathematics, Pearn and Merrifield, 1995. The tables were used to organise the initial results. Full transcription of the answers to the counting tasks was undertaken.

5	yes	no	Instant answer	Physical count
7	yes	no	Instant answer	Physical count
9	yes	no	Instant answer	Physical count
11	yes	no	Instant answer	Physical count

5. Number sequence.

Ask: Can you tell me the numbers between 6 and 12? If the child does not respond, prompt: You know, when you count: 1, 2, 3, what are the numbers between 6 and 12?
Ask: What is the number before 15?
Ask: What is the number after 11? If the child has failed in the previous two questions ask:
What is the number after 4?
What is the number before 9?

Numbers	successful		Answer	
between 6 & 12	yes	no	Instant	Physical count
Before 15	yes	no	Instant	Physical count
After 11	yes	no	Instant	Physical count
After 4	yes	no	Instant	Physical count
Before 9	yes	no	Instant	Physical count

6. Physical counting.

Present the child with a bunch of counters and ask: Can you count out 14 counters?
If the child is unsuccessful (e.g. does not stop at 14), ask:
Can you count out 8 counters?

	Successful	
14 beads	yes	no
8 beads	yes	no

7. Numeral recognition.
Present the child with flash-cards with the following numerals on and ask:
Can you tell me what this number is?

Numeral	Successful		Reading as
13	yes	no	
31	yes	no	
14	yes	no	
41	yes	no	
15	yes	no	
51	yes	no	
7	yes	no	
8	yes	no	
12	yes	no	

Counting types.

Present the child with a bunch of 6 counters (wooden bricks of 1cm x 1cm x 1 cm) and ask them to count them. Put 3 more on the table and say: I've got 3 more here, can you see? (put them under a cover) But I'm going to hide them under this cover. Can you tell me how many counters there are all together? If the child seems to be puzzled, prompt: If I put these, together with these, how many would there be all together?

	Successful		Answer		
6 + 3= 9	yes	no	Physical count	Count on	Instant answer

Present the child with a bunch of 10 counters and ask them to count them. Place 4 under a cover and say: I'm going to put some under this cover. Can you tell me how many I put under the cover?

	Successful		Answer		
10= 6 + *4*	yes	no	Physical count	Count on	Instant answer

9. Sharing

Present the child with 3 dolls and 12 counters and say: Let's pretend that these dolls have worked hard in the garden and they deserve to have some biscuits. Let's pretend that these (the counters) are the biscuits. Can you give away the biscuits so they all have a fair share? Remember that they don't want to fight over who's got more biscuits, they all have to have the same amount.
When the child has finished, ask: Have they all got a fair share?
After the child has responded, cover all but 1 of the lots of counters.
Pointing to the visible lot, ask: How many has this doll got?
Pointing to one of the hidden lots, ask: How many has this doll got?

	Successful		Answer	
12/3 = 4	yes	no	Physical count	Instant answer

Interview session: Numbers at home and outside school- May 1998. (In pairs)

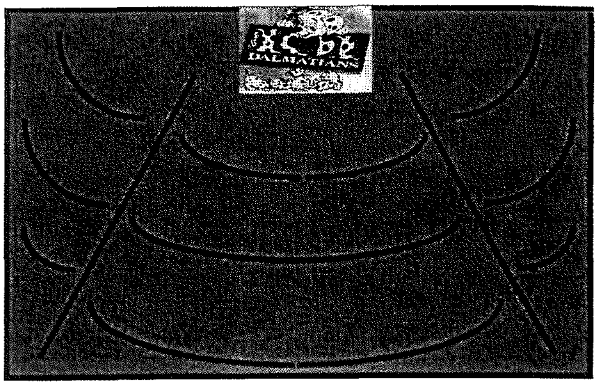
Guiding questions and tasks:

- The other day we were doing some counting. Today I'd like to talk about numbers with you. Do you talk about numbers at home? What sort of numbers do you talk about? Who do you talk about numbers with? When do you talk about numbers? What do you say about numbers? Do you do anything with numbers?
- Do you count at home with your mum or dad? Do you count with anybody at home? What do you count for?
- What numbers do you know? Where do you see numbers written? Anywhere else? Can you find numbers written on the road? Can you find numbers in a house? Where else can you find numbers? Can you see numbers on the television? What kind of numbers can you see there? What are these numbers for?
- The other day you went to visit Safeway supermarket: did you find numbers there? What about when you had to queue up to buy ham and cheese? Where did you find those numbers? Where they high numbers? Can you write any of those numbers that you saw?
- Have you ever seen the film 101 Dalmatians? (If yes) Had you ever heard that number before? Can you think of a bigger number than that?
- Can you write 101? (Ask the children to write the number they had proposed as bigger than 101). Can you write the highest number you can think of?
- Where else do you think you can find big numbers?
- What other big numbers do you know? How do you know these numbers? Where did you learn them? Did you learn them at home?
- Can you find numbers on a calendar? What sort of numbers can you find on a calendar? What do you think they tell us? Do you talk about this numbers at home?
- If someone gave you a lot of money, what would be the thing you would like to buy? How much would that cost? (When both children have responded). What do you think is more expensive? Can you write those numbers? (provide paper and pencil or a calculator).
- Can you find numbers in a book? What sort of numbers can you find in a book? What do you think they tell you?
- Suppose somebody from another planet in the universe came here and asked you about (big) numbers: how would you explain what a (big) number is? Suppose he asked you: where should I go to see written numbers? What would you tell him?
- Present the children with a 30 cm ruler with a scale in tens: Can you read these numbers?
- Give the children calculators and ask: Can you write the highest number you can think of? (After both children have responded) Do you know that number? Which one is the highest? How can you tell?

Interview session: Cinema scenario 1-June 1998. (In pairs)

(In connection with Interview session: Dictation of number scripts-June 1998)[1].

Present the children with the picture of a cinema auditorium (64 cm x 45 cm cardboard sheet):



Tell the children that this is a big (huge) cinema and that they are showing a new film of 101 Dalmatians. Explain that the sections are parts of the cinema where people seat, and that the picture shows the cinema screen; explain that they cannot see the seats but they have to imagine that they are there. Tell them that there are 3 sections (point at them) on one side, and 3 sections on the other side, and 3 sections in the middle.

1. Ask them if they had seen 101 Dalmatians before. Ask how many puppies there were in the film. Talk about the number 101. For example: Do you know that number: 101? What comes before 101? And before that? (continue); What comes after 101? Which one is higher: a hundred or a hundred and one? How can you tell? Do you know a higher number than 101?

2. Ask them how many people they think can fit in the cinema. For example: How many people do you think can fit in this part? And in this part? If they say 10, or 20 people, say: Higher than that, this is a very big cinema and lots of people can come and see the new film. Negotiate how many people can seat in each part of the cinema. If possible lead the children to agree with a hundred people in each and all sections of the cinema. If this is not possible, tell them that a hundred people can seat in each part of the cinema. For example: A hundred people come and seat in this part of the cinema (point at 1 section), a hundred people more come and seat in this part of the cinema, and another hundred here (always point at the section).

3. Tell the children (see above) that a hundred people come and seat in each part of the cinema. Tell the children that the manager wants to know how many people come and see the new film of 101 Dalmatians so they have to figure out how many people there are all together in all the parts of the cinema. If children count the 9 sections of

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[1] This was in fact a two-session interview that included the cinema scenario and dictation of number scripts.

the cinema, and respond "9", ask: Nine people?. If children respond "9 hundred", ask: How did you figure it out?

4. (This question was typically asked after the children had written '100' on each of the cinema sections, see Interview session: Dictation of number scripts). Tell the children that film is very long so there is a break in the middle. Say that all the people sitting in the middle parts go out for a drink (point to the 3 middle sections). Ask: How many people stayed in their seats? How many people went for a drink? Always ask: How did you figure it out?

5. Tell the children that one day the cinema was not completely full and all the middle sections were empty, with no people at all. Ask how many people there were in the cinema. Ask how they figured it out.

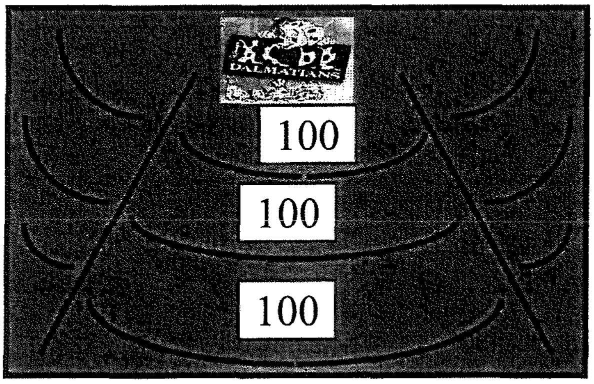
6. Tell the children that one day a thousand people came to see the film and the next day another thousand people came. Ask: How many people came to see the film all together? (After they respond) And if another thousand come and see the film? Always ask how they figured it out.

7. Similar question to 6 but now with a million.

Interview session: Dictation of number scripts-June 1998. (In pairs)

(In connection with Interview session: Cinema scenario 1-June 1998.

- Present the children with the cinema scenario (see Interview session: Cinema scenario 1). After questions 1. to 3. are asked, say: The manager of the cinema wants to know how many people come and see the film so you have to tell him. Perhaps if you write it down for him, he can know how many people fit in each section. How many people did we say fit in each part? (After the children responded a hundred, give cards and pencils). Can you write a hundred on each of these cards so the manager of the cinema knows that there are a hundred people in each part?
- Invite the children to place the cards on top of each section:



- When all 9 cards with '100' on them are placed on the board, ask: How many people were there in the cinema all together?
- (see point 4 of Interview session Cinema scenario). Take all but the 3 central cards off the board and say: All these people go during the brake to have a drink. How many people stay in the cinema? If the children say "3", ask: 3 people?; if the children say "3 hundred", ask: How did you figure it out?
- (Always putting cards back on the board or taking them away at the same time) Ask: What happens if these people come back? What happens if all these people go?
- Give both children a blank piece of paper and pencil, and say: Now you have to write all these numbers down for the owner of the cinema? So: How many people were there in each part of the cinema? How many people were there in total? How many people stayed when all these went for a break? And when these people came back? (Always wait for the children to write, if the scripts are different, discuss which one they think is how you write the number).
- Dictation: (after all the above, or earlier, if children are engaged), so how do you write 800? I bet you can write 10 hundred (wait for written productions and discussion.
- The numbers that are dictated are 100, 200, 300, 400, 500, 600, 700, 800, 900 and "ten hundred". If children write '1000' (their form for 10 hundred) is "a thousand", they are asked to write 2000, 3000, and 5000.
- Other numbers that are dictated are: 101, 102, 103, 104, 105, 106, 107, 108, 109, and 110. Also: 150, 120, 250, 2300. Always compare scripts produced and discuss if they are different for the "same number".

Interview session: Dice game-July 1998. (In pairs)

There are three dice for this session. The first dice (used in all interviews) is a standard dice of dimensions 2.5 cm x 2.5 cm x 2.5 cm (with this dice, each dot is worth 1 point). The other two dice are used in some interviews, depending on the engagement of children. The second dice (the blue dice, made of cardboard) is of dimensions 5 cm x 5 cm x 5 cm (with this dice, each dot is work a hundred). The third dice (black, with white dots) is of dimensions 4 cm x 4 cm x 4 cm (with this dice each dot is worth ten)

First dice:

- Tell children that they are going to play a dice game. Ask them whether they had played before. If they respond no, play with the two children, for two rounds to show the goal of the game: to accumulate points. If they say that they had played before, explained the rules: tell the children that this game is about getting points, as many as the dice tells you and the first one who reaches 10 is the winner (then 20).
- Ask them what they can use to remember the scorings. There is a box full of bricks on the table. Suggest pencil and paper to annotate the scorings.
- Ask children to play in turns and tell them that the first one who reaches 10 is the winner.
- Ask children to play up to 20, or up to the number they suggest. If children annotate the number of each round, suggest that they can stop the game to find out the total.
- Discuss when a child cannot annotate a script, the other child disagrees with the writing. Always ask how they know how to write these numbers.

Second dice:

- Tell the children that with the blue dice, each dot is worth a hundred. Ask the children how many points they would get with 2, 3, 4, dots. Ask how they figured it out. Ask the children if they want to play with this dice. Remind them that they have to annotate the scorings. If they play, always stop to discuss the number scripts produced.

Third dice:

- Similar to previous point.

Interview session: Board of 100 seats-July 1998. (In pairs-two sessions)

- Present the children with the cinema scenario (see Interview session: Cinema scenario 1) and talk about how many people there were in each part. Ask the children how many people there were in total and ask how they figured it out. If children do not remember, tell them that there were a hundred people in each part.

- Present the children with a blank piece of cardboard (64 cm x 45 cm-same green colour as the original, with the screen picture at the top) and tell them that they had to put all hundred seats for the people to sit down in the part just in front of the screen. Ask the children if they can draw cinema seats and write their numbers from 1 to 100. Talk about why they need to write the numbers (so people know where to sit down in the cinema when looking at their tickets).

- Negotiate the turns of writing and the manner they are going to draw the seats (top to bottom, left to right). Suggest children to do one seat each at a time, but this can be negotiated between the children according to their pace of writing and if they know how to write the number scripts. Always stop when children disagree, discuss and probe for answers on how they know how to write these numbers.

- At some point (40) ask whether they have completed half of the seats. Ask how they could tell.

- When they are at 90 or 95, ask how many seats there are left to do. Ask how could they tell.

- When they have completed the 100 seats, ask to find seat number: 10, 20, 30, and so on. Ask about these numbers, ask if they know them (e.g. when they count in tens, e.g. they all have a nought). Ask to find seat number: 38, 76, etc. Ask how they knew how to find them.

Interview session: Assessment of Counting Units- End of Summer Term- July 1998. (Individual)

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1. Show 7 counters on the table and ask: "How many counters are there on the table?" When the child has answered show 4 more counters and a box lid and say: "I've got 4 counters here and I'm going to put them under this box: How many counters are there altogether?" Always after the child has responded say: "How did you do that?" or "How did you work it out?"

2. Display a lot of cubes on the table (20) and ask: "Can you count out 10 counters?" When the child has done this, ask: "How many counters have you got?" When the child has answered, cover all the counters and display only 6, covering the rest (4) under the box and saying: "I'm going to cover some of your counters. How many did I cover?".

3. Display a lot of 5 counters and a lot of 4 counters on the table and ask: "How many counters are there here (5)?" and "How many counters are there here (4)?" When the child has responded cover the two lots of counters with two box lids and say: "I'm going to cover these counters and these ones with these boxes". Then put the box lids together and ask: "How many are there all together?"

4. Display 6 counters in a line and cover 3 under a box (following the line) and say: "There are 9 counters altogether on the table: how many are there under the box?".

5. Show the child the two paper-clip boxes (with the numerals 5 and 9 on them) and say: "There are some paper clips left in each box: it says on the box how many there are left in each one. Can you work out how many paper clips there are altogether?"

6. Display the card "8 + ... = 13". Invite the child to read the sum. If the child does not read the sum say: "This sum says that we've got 8 add another number and that that equals or makes 13. Can you work out what number goes there? (or "Can your work out the answer?"). After the child has responded say: "How did you do that?"

7. Display the card "23 + ... = 36". Invite the child to read the sum. If the child does not read the sum say: "This sum says that we've got 23 add another number and that that equals 36. Can you work out what number goes there?"

8. Display the card "11 - 4 =......". Invite the child to read the sum. If the child does not read the sum say: "This sum says 11 take away 4 equals.....Can you work out the answer?"

9. Display the card "22 - 17 =......". Invite the child to read the sum. If the child does not read the sum say: "This sum says 22 take away 17 equals.....Can you work out the answer?" If the child responds, suggest to find out in another way (e.g. if the child counted on from 17, suggest if they can count down)

Interview session: Number scripts-July 1998 and January 1999. (Individual)

- Give the child 6 cards with the following numbers written on them 3, 56, 100, 2500, 77800, 89567, and 100000 and say: "These are some numbers that I wrote on these cards: are they all the same? Do you know some of them?" Wait and see if the child attempts to read the scripts, all of them or some of them. When the child has said that they are not the same, ask which one is the highest (or biggest), and which one is the lowest (or smallest).

- Ask: "Can you put them in order? Can you put them in a line from the lowest (or smallest) to the highest (or the biggest)?" When the child has done this, ask: "How did you do that?" Ask: (with 2500 and 7800) How did you know here which one was the highest? Can you read these numbers?

- If it can be established from the previous conversation that "the more numbers the bigger the number", ask: "What happens if they have the same amount of numbers?" Depending on the response, display the following cards: 200, 300, and 800 and ask: "Can you put them in order? After they respond say: "How did you know?". Ask: "Can you read these numbers?".

- Display the following cards: 380, 350, and 390 and ask: "Can you put these in a line from the lowest to highest?" After the child has completed the task, ask: "Can you read these numbers?", "How did you know which one was the highest?"

Interview session: Cinema scenario 2-February 1999. (Individual)

- Present the child with the cinema scenario (see Interview session: cinema scenario 1). Talk about the situation to see what the child remembers. For example, ask: Do you remember the cinema with these sections for the seats? How many people were there in each of those parts? If the child does not remember, tell the child there were 100 people in each part.

- And how many people were there in the middle part: a hundred plus another hundred plus another hundred? (point at the sections when asking)

- And how many people were there altogether in the cinema? Always ask how they figured it out, probe for stability of the answer by asking various questions of the type: What happened when the people here (in the middle parts) left? How many people were left in the cinema?

- So how many people are there all together? (When the child has responded 900). Suppose the owner of the cinema wants to make it even bigger and puts a hundred more seats. How many people can come and see the film? Always ask: How did you figure it out?

- The owner of the cinema has a problem and he can only get 50 seats to put in the cinema. How many people can come and see the film?; How did you figure it out?

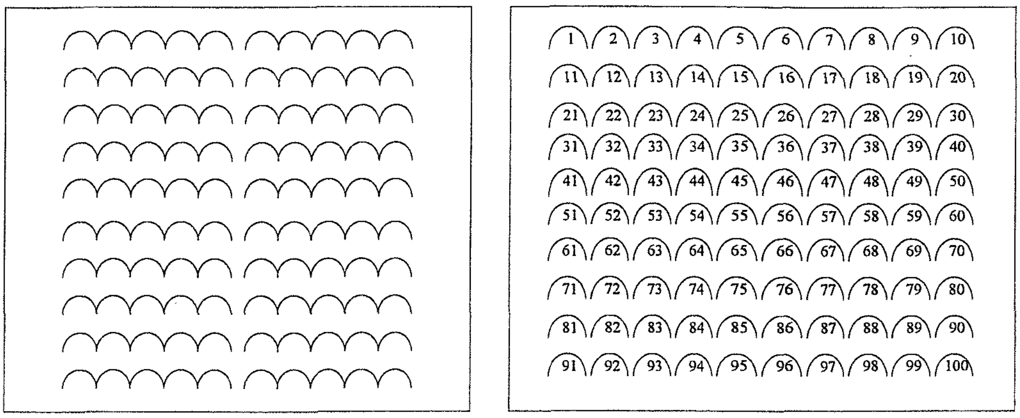
- After all, he can put 50 more seats, how many people can come and see the film?

- Do you know how to write all these numbers? (e.g. 900, 950, 1000), can you tell me how you write these numbers?

Interview session: Cinema tickets-February 1999. (Individual-two parts)

Part one:

- Draw a 10 by 10 arrangements of cinema seats in a sheet of cardboard (64 cm x 45 cm) without numbering them (this is the blind board). Draw the same arrangement on the other side of the sheet of cardboard, now numbering the seats from 1 to 100 starting from the top and from left to right (this is the numbered board):



- Present the child -briefly, with the cinema scenario (see Interview session: Cinema scenario 1). Then ask how many seats there were in each part. If the child does not remember, suggest 100. Remind the child about the session about the board of 100 seats. For example, ask: Do you remember when you made the hundred seats? You wrote all the numbers so people could know where to sit down when they bought their tickets.

- Present the blind board. Say: Here are all the hundred seats but with no numbers written on them. The owner of the cinema put the seats in rows. How many seats did he put in each row? After the child has counted the seats, ask: Where would you write 10?

- Then ask: Where would you write 20, 30, 40, 50, 90, 100? How did you know?

- Turn the sheet of cardboard over (present the numbered board) and ask: Were you right? Is 10-20-30, where you said? How did you know?

- Let's pretend that you and a friend go to the cinema and you buy the tickets. If you have a ticket with "6" and your friend has a ticket with "7", are you going to be sitting far away from or close to your friend? (give the tickets with numerals 6 and 7 written on them). Suggest the child to put them on the seats and try to agree that there are one seat away. Then ask: How many seats away from each other would you be sitting if you have "6" and your friend has "10"? (give the tickets to the child) If the child does not respond, suggest to count the seats from 6 to 10. Agree that they are 4 seats away or apart.

- Tell the child that they are going to have two tickets each time, one for their friend and one for them. Tell the child to find out how many seats apart they will be each time. If the child puts the tickets on the seats ask whether they can do say how many seats apart "as fast as possible". If the child does not visibly count, ask the child, how they figured it out. Present the child with the following pairs of tickets: 10-20; 20-30; 40-60; 60-70; 20-50; 60-90; 50-100; 70-100.

- Present the child with the following pairs of tickets: 5-10; 25-35; 15-20; 65-70; 85-90; 95-100; 35-40. Always ask how they figured out how many seats apart the tickets are.

- Present the child with the following pairs of tickets: 11-21; 31-41; 61-71; 81-91.

- Present the child with the following pairs of tickets: 22-32; 72-82; 52-62.

- Present the child with the following pairs of tickets: 24-34; 52-82; 65-75.

- Place a pile of blank tickets on the table and say: Let's pretend that we go to the cinema together) This is my ticket (write '10' on it). Can you write the number for your ticket so you are 2 seats away from me? (After the child writes their ticket, ask how they figured it out)

- Ask the child to write their ticket 10 seats apart from the researcher; show the following tickets: (guiding example): 30-65-90-74-83. (Always ask how they figured it out).

- After the child has responded one of the previous questions, say: Another friend comes and also wants to seat 10 seats away from me. Can you write his ticket? If the child gives the friend the same ticket as their own, suggest: but they can't sit with you. Can you give him another ticket so he's also 10 seats apart from me? Always ask how they figured it out.

- Ask the child to write their ticket so they are 5 seats apart from the researcher: (guiding examples): 30-65-90-74. Also ask the question about the friend.

- Ask the child to write their ticket 30 seats apart-50 seats apart: (examples): 20-30-50

Interview session: Cinema tickets-February 1999. (Individual-two parts)

Part two: (this session was not presented to all children, this depended on their engagement with Part one; some children were presented only with this part)

- Present the child -briefly, with the cinema scenario (see Interview session: Cinema scenario 1). Then ask how many seats there were in each part. If the child does not remember, suggest 100. Remind the child about the session about the board of 100 seats. For example, ask: Do you remember when you made the hundred seats? You wrote all the numbers so people could know where to sit down when they bought their tickets. Then ask: But how many people were there all together in the cinema?

- Present the child with the numbered board (i.e. the cinema scenario with 9 stickers, each one on each part of the cinema and each one numbered from 1 to 100, 101 to 200, and so on, up to 801 to 900). Say: Suppose that we are in the big cinema with all 900 seats in it. So we can have any ticket up to 900.

- Let's pretend that you and a friend go to the cinema and you buy the tickets. If you have a ticket with "6" and your friend has a ticket with "7", are you going to be sitting far or close to your friend? (give the tickets with numerals 6 and 7 written on them). Agree that the tickets are one seat away. Then ask: How many seats away from each other would you be sitting if you have "6" and your friend has "10"? (give the tickets to the child). Agree that they are 4 seats away or apart.

- Tell the child that they are going to have two tickets each time, one for their friend and one for them. Tell the child to find out how many seats apart they will be each time. Ask the child to tell "as fast as possible". Present the child with the following pairs of tickets: 100-200; 500-510; 200-500; 600-900; 100-150; 230-240; 430-450; 625-685; 258-268; 730-750; 340-400; 244-254; 432-542; 382-392; 301-601; 320-347; 100-890; 1-999; 240-60; 380-120. (Also: 10-20; 20-30; 40-60; 60-70; 20-50; 60-90; 50-100; 70-100). Always ask how they figured it out. If the child is puzzled, suggest to use the seats on the board

- Place a pile of blank tickets on the table and say: Let's pretend that we go to the cinema together. I'm going to write my ticket and you are going to write yours. For example: I'm going to be seat number 200 and you are going to be sitting 10 seats away from me: Can you write your tickets so you're 10 seats away from me?

- After the child has responded the previous questions, say: Another friend comes and also wants to seat 10 seats away from me. Can you write his ticket? If the child gives the friend the same ticket as their own, suggest: but they can't sit with you. Can you give him another ticket so he's also 10 seats apart from me? Always ask how they figured it out.

- Similar questions with: (guiding examples): (300)-[90]; (550)-[50]; (400)-[180]; (205)-[25]; (467)-[10]; (501)-[75]; (620)-[280] Always ask how they figured it out. Include 5, 10 and multiples of these in the questions of how many seats apart.

Interview session: Final counting assessment-February 1999. (Individual)

1. Show 9 counters on the table and ask: "How many counters are there on the table?" When the child has answered show 4 more counters and a box lid and say: "I've got 4 counters here and I'm going to put them under this box: How many counters are there altogether?" Always after the child has responded ask: "How did you do that?" or "How did you work it out?"

2. Display a group of 12 cubes on the table and ask the child to count them. Hide 5 cubes and ask: "I put some cubes under the cover, how many did I hide?".

3. Display a group of 6 counters and a group of 7 counters on the table and ask: "How many counters are there here (6)?" and "How many counters are there here (7)?" When the child has responded cover the two lots of counters with two box lids and say: "I'm going to cover these counters and these ones with these boxes". Then put the box lids together and ask: "How many are there all together?"

4. Display 5 counters in a line and cover 6 under a box (following the line) and say: "There are 11 counters altogether on the table: how many are there under the box?".

5. Show the child the two paper-clip boxes (with the numerals 5 and 9 on them) and say: "There are some paper clips left in each box: it says on the box how many there are left in each one. Can you work out how many paper clips there are altogether?"

6. Display the card "8 + ... = 13". Invite the child to read the sum. If the child does not read the sum say: "This sum says that we've got 8 add another number and that that equals or makes 13. Can you work out what number goes there? (or "Can your work out the answer?"). After the child has responded say: "How did you do that?"

7. Display the card "23 + ... = 36". Invite the child to read the sum. If the child does not read the sum say: "This sum says that we've got 23 add another number and that that equals 36. Can you work out what number goes there?"

8. Display the card "11 - 4 =......". Invite the child to read the sum. If the child does not read the sum say: "This sum says 11 take away 4 equals.....Can you work out the answer?"

9. Display the card "22 - 17 =......". Invite the child to read the sum. If the child does not read the sum say: "This sum says 22 take away 17 equals.....Can you work out the answer?"

10. After all the above are presented, tell the child that these are some sums they had to solve as fast as they can. Present the child with an A4 cardboard sheet with the following sums written:

- 11 + … = 16
- 15 + … = 21
- 24 + 15 = …
- 55 + 15 =
- 75 + 5
- 37 + … = 52
- 24 - 15 =
- 76 - 48 =
- 35 - 49 =
- 600 + 400
- 1000 - 500
- 2000 - 500
- 700 + 400
- 170 - 80

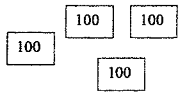
11. Present the child with the 2 paper-clip boxes and change the label each time. Say: Lets' pretend there are these many clips left in this box, and these many clips left in this box: How many are there all together in both boxes? Always ask how they figured it out.

- 70 + 40
- 10 + 90
- 80 + 50
- 20 + 70
- 90 + 50
- 50 + 60
- 80 + 60
- 70 + 60
- 70 + 80
- 80 + 90

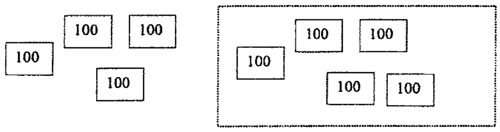
Interview session: Paper-clip boxes- February 1999. (Individual)

- Present the child with one paper-clip box, and say: I've got this paper clip box here, how many clips has it got? It says here on the label. (after the child has read 100). Open the box and take 1 clip away, then ask: If I take one away, how many clips are there left?

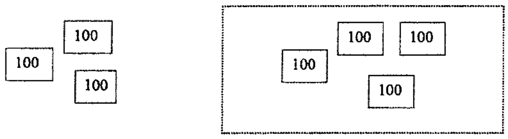
- Display 4 boxes and ask: If you have all these boxes, how many paper clips would you have all together? Ask how they figured it out. After they answer, take 1 clip away from one of the boxes and ask: How many clips are there now?



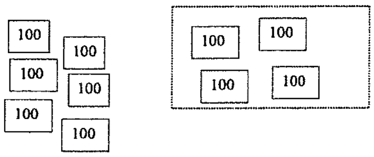
- Hide 5 boxes under a cover and say: I've put 5 more of these boxes under this cover, how many clips are there all together? Ask how they figured it out.



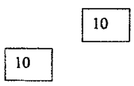
- Ask the child to closer their eyes and put 4 boxes under a cover, leaving 3 boxes visible, then ask: There are 700 paper-clips all together now. How many boxes did I hide under the cover? Always ask how they figured it out.



- Ask the child to close their eyes and put 4 boxes under the cover, leaving 6 boxes visible. Then ask: How many paper clips are there now? (After the child responds 600), say: I put 4 boxes under the cover. How many paper-clips are there all together? Ask the child how they figured it out.



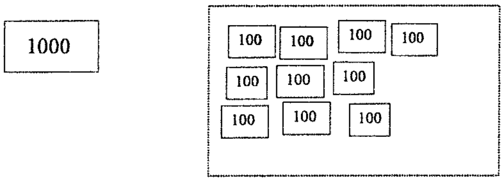
- Present the child with 2 paper clip boxes and say: Let's pretend that we've used some of the clips and we have only a few left in each of these boxes. To remind ourselves how many clips there are left in each box, we stick a label, see? Can you tell me how many clips all together if we have 10 clips and another 10 clips left?



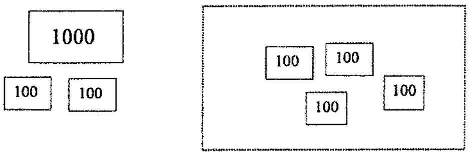
- Ask a similar question with the following labels: 10-20; 40-60; 90-10; 30-20; 70-30; 50-50; 60-40; 90-50, 60-50; 80-40. Always ask how they figured out the answer.

- Present the child with the big box of 1000 clips (this box is the container of 10 small boxes of 100 paper-clips), and say: I've got this bigger box which has this many clips in (show '1000' on the box and wait for the child to read the script: if the child reads a thousand or ten hundred, use this in the consecutive talk). Then ask: Can you tell me how many of the 100 clip boxes we need to fill in this box? How do you know? Did you count?

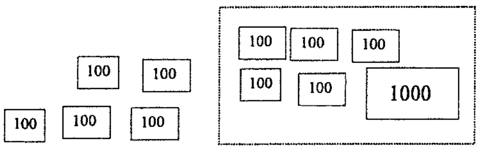
- Display the 1000-box and cover the ten 100-boxes. Ask: There are 2000 paper-clips altogether, how many small boxes did I hide? Always ask how they figured it out.



- Display the 1000-box and two 100-boxes and cover four 100-boxes. Ask: There are one thousand six hundred paper-clips all together. How many small boxes did I hide?



- Display five 100-boxes and cover the 1000-box and five 100-boxes. There are two thousand paper-clips all together. How many paper-clips are there under the cover?



- Let's suppose that we have one two thousand paper-clips here. What if I use 120 paper-clips, how many would there be left?

- If we had a box of 500 clips, how many boxes could we put in this big box?

Interview session: Regularities in number words and number scripts-March 1999. (Individual-two part session)

Part one: ordering and sorting number scripts.

Ordering numbers:

- Present the number scripts: 3-56-100-2500-7800-89567-100000 and say: Here I've got some numbers written. Can you put them in a line from the lowest to the highest? After the child has finished, ask: How did you work it out? How do you know this one is higher? To prompt: Is it because...? To probe answer: Depending on the response, give another/s scripts, e.g. 3999, 400; and ask to place it in their line. Ask how they knew where to put it. Ask the child if they could read those numbers.

- If criterion of the digits has been established, ask to write 230 (expecting a concatenated form: '20030') and 300. Probe to raise a situation of contradiction.

- Ask to solve similar task as the first one with scripts: 342-250-351-360-367-380. Ask for reasons, and then give 300 to place in the line to probe children's answers.

- Present the child with hundred knots and thousand knots-scripts. Ask to read these numbers, ask how they know which numbers they are. Ask them to put them in a line (the hundreds first, the thousands then; then both together).

Sorting numbers:

- Present the child with a mixed lot of hundred and thousand knots and ask to put the hundreds in a pile and the thousands in another pile. Ask how they work out how to do this. Probe for stability of answers.

- Same as before but present with the mixed lot of hundreds and <u>thousands</u>: 133-245-250-338-345-950-938-<u>1999-2460-3100-4678-5555-5620-8540</u>. Ask the child to justify their answer.

- Present the child with a lot of number scripts and ask to make a pile of "less than thousands", another pile of "<u>thousands</u>" (explain that it can be any number in the thousands), and a pile of "more than thousands": 45-56-88-265-604-750-<u>1998-3600-3819-7000</u>-20050-30050-50006-70050-90001-90050-90123-234569-1000000. Ask to justify their answer.

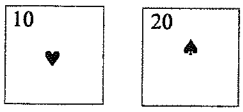
- Present the child with the cards: 200-202-205-206-210-214-222-238-245-250-276-296-284-002-102-1002-1200-2000-2005-2009-2100-10200-20050. Ask: Can you put in a pile all the numbers that are two hundreds and in another pile the ones that are not? Ask to justify their answer.

- Give the child a blank card and ask: Can you write a number that is not a thousand? How do you know that is not a thousand? Similar question for a number that is not a hundred.

Working out rules:

- Tiles: Present the child with pile of numbered tiles (2.5 cm x 2.5 cm) with twenties and a pile of tiles with forties (all decades). Tell that a child of year 2 has sorted these tiles and that they have to figure out what he was thinking, what was in his mind when he put them in different piles. When the child has responded with a reason why the 20's and 40's are in different piles, ask how they figured it out. Then, ask the child to put the twenties in a line from the lowest to the highest. When the child has completed the line, ask how many times the 2 is in front (or at the beginning of) of all the twenties. Ask the same for the forties before putting them in a line. Ask, what does the 2 mean? How do you know?

- Cards 1: Present the child with game cards (15 cm x 10 cm) with all the decades up to 100. Tell the child that a child from year 2 has sorted the cards together and ask whether they can find out what this child was thinking when he put all these cards together. If there is no answer, suggest to put the numbers in a line first. After the child finds a reason, ask how they figured it out. Ask which number would come next in the line. Ask how they figured it out.



- Cards 2: Present the child with game cards in tens: 8-18-28- and so on up to 98 and ask the same questions as in the previous task. Also ask what comes after 98 after the child has given a reason. Ask how they have worked it out.

- Cards 3: Present the child with game cards: 17-37-57-77-97 and ask the same questions as in 12. Also ask what comes next and probe how they have figured the answer out.

- Numbers 1: Present the child with cards with the following number scripts: 510 to 590 in tens. Ask the same question as in the previous task. Always suggest to put the numbers in a line first if the child looks puzzled. Also ask what comes next after the child has figured out and answer.

- Numbers 2: Present the number scripts: 5200-5210 in tens up to 5290. Same as previous task.

- Numbers 3: Present the number scripts: 4810 up to 4890 in tens. Same as previous task.

- Numbers 4, 5, and 6: Same as previous task with 6110 to 6910 in hundreds; with 2010 up to 2890 in 110's; and with 2500 up to 9500 in 500's.

Interview session: Regularities in number words and number scripts-March 1999. (Individual-two part session)

Part two: counting (guiding questions)

- Can you count in tens? Can you count in tens starting at 80? Starting at 7?
- How many times do you count, when you count in tens up to 100? 200? 250? 1000? 2500? How did you work it out? How do you know?
- Why do you think people count in tens? What does it mean to count in tens? What happens when you count in tens?
- Can you count in 2's? 5's? 50's? 20's?
- How many times do you count in 2's when you count up to 100?
- Do you know how we say a hundred years? (If the child does not know, tell them). How many years are there in 12 C?, 20 C? How do you know? How did you figure it out?

Lollipops:

- Present the child with a big jar with 200 lollipops (it reads in a label). Ask: How many lollies do you think are there in the jar? After they estimate the number, say: It says here, (show label: '200'). (After the child has read 200 conventionally, ask to read the price: 5p each (also on the label)
- Take out 1 lollipop and ask: If I take one away, how many are there now left?
- Ask the child to take away more lollies so there are 190 left. Ask how they worked it out. If the child counts backwards until 190, ask: How many lollipops are there on the table?
- Ask the child how many would we have to take away so there are 50 left inside the jar. Ask how they worked it out.

Interview session: Laptop sums-March 1999. (Individual)

On excel spreadsheets, prepare tasks as follows:

Writing sums up to:

- Present the child with the spreadsheet and say: "Have you ever worked with a computer? This is a laptop computer and this is a worksheet. We are going do sums on this worksheet. These rectangles are called cells. I'll show you" (moving the mouse and showing the child). "If I want to write a sum that equals 5 I can put in 2 and 3 in these cells and when I click on this cell (the third cell), the computer tells you the answer" (the spreadsheet is prepared to give the total of the previous two cells; as in a calculator, immediate feedback is given by the computer). "Can you write another sum that equals 5?" Wait for the child to get used to using the mouse and the keyboard. Ask again to write another sum to 5.

- When the child seems familiarised with the computer and the mouse use, ask: "Can you write all the sums that you can think of up to 10? You are meant to do this as fast as you can." Ask reasons for solutions if possible.

- Propose the same task with 20; 100; 1000.

Completing the sum: addition

- Say: "Now I'm going to put in a number here in the first cell (put in 4) and below here (another cell on the spreadsheet, separate from the line of the sum), I'm going to write which number I want to be the answer of your sum" (write 10). "Which number do you have to write here in the second cell so the answer is 10?" [The problem is: 4 + ...= → 10; where 4 is in the 1st cell, '....' is the 2nd cell, and '→' shows the answer to be put in the 3rd cell; but '10' is not written in the third cell, because it is given by the child when they complete the second cell (the 3rd cell is the total of the 1st and 2nd cell)

- Ask the same question with: (examples): 3→10; 6→10; 2→10; 7→10; 8→10; 8→12; 7→15; 10→21; 9→17; 15→22; 20→64; 22→54; 36→45; 37→52; 27→37; 48→98; 30→42; 65→71; 82→57

- Ask the same questions with (examples): 100→120; 238→248; 297→307; 325→330; 845→854; 721→741; 632→641; 256→368

- Ask the same questions with (examples): 1000→2000; 1010→1020; 2130→2530; 3408→3428; 2615→2815; 1000→2500; 3000→5000; 4010→4210; 2610→2615; 2590→2610; 3299→3300

Completing the sum: subtraction

- Explain the child that now they are going to do take away sums. As with previous tasks, present: 10→2; 10→7; 10→10→5; 20→12; 20→17; 20→6; 20→15; 98→16; 45→36; 54→22; 22→15; 23→5; 76→48; 65→40

- Same as before with (examples): 170→90; 120→30; 1000→500; 2000→400; 3000→2690; 2638→2038; 3515→3505

Write the answer: (This part was not done with all children)

- A new sheet has a written sum using 3 cells. A 4th cell reads "yes" or "no" according to whether the 3rd cell is the right answer to the sum. Write the sum '3+4=' and ask the child to write the answer. E.g.: "That sums says 3 plus 4 equals, can you write the answer here?" When the answer (7) is written, say: "Well done,the computer says 'yes', so you are right. Now I'm going to put other sums and you've got to write the answer in the 3rd cell and click to see if you're right.

- Examples of sums: 100+100, 120+80, 240+60, 1000+500, 1000+1000, 1600+400, 500+500; 980+20.

Interview session: Number in between-March 1999. (Individual)

- On a spread sheet design three big cells with labels below that read from left to right: "lower", "in between", "higher". The middle bottom cell reads "in between if the number written is in between the left-hand and the right-hand numbers.

100	200	300
lower	in between	higher

3	20	7
lower	no	higher

- Present the child with the screen "3-20-7" and say: In this cell I put 3 and in this other cell I wrote 7. It says 3, lower and 7, higher. Why do you think it says "no" for 20? If the child does not find the wanted answer, say: Because 20 is not in between 3 and 7, when you count. Can you write a number there that is in between 3 and 7? (show the child that they have to click below to check if the number is in between)

- Can you put another number that is in between 3 and 7? (remind the child to click on the cell below)

- Ask: Can you put another number that is not in between? (remind the child to click on the cell below)

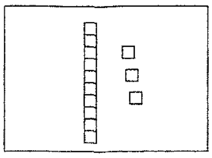
- Say: Now I'm going to change the lower number and the higher number and you have to write a number in between. (Always ask for reason if possible after solution). Present the child with: 15-20; 10-20; 12-20; 10-18; 20-30; 20-40; 50-60; 60-80; 10-90; 90-100; 98-100; 80-82; 100-300; 100-200; 90-120; 200-400; 340-400; 540-545; 598-600; 251-260; 918-980

- Similar task with: 2520-2600; 2699-3720; 9998-10000; 9000-20000; 4875-5000; 6297-6499 (If the child write the consecutive number, re pose the question for the same pair, for example: can you write a number that is in between and is 10 more, 100 more, etc.?)

Interview: Rows and squares-March 1999. (Individual)

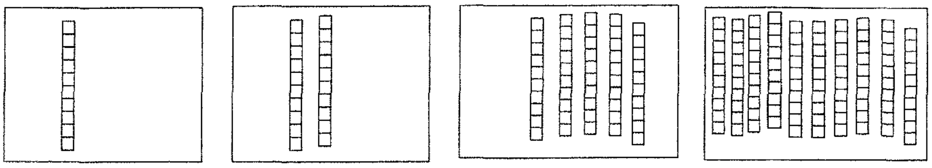
Design all the screens in word as follow.

- Present the child with this screen and ask: How many squares are there here? How many squares in a row? (Try to establish the number of squares in a row)

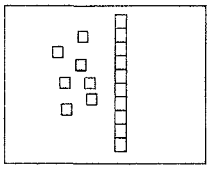


13 squares

- Ask a similar question with the following screens. (Always ask how they have figured it out if possible after solution). Present 1 to 10 rows depending on the child an the interview: For example, present 10 - 20 - 50 - 100.



Ask similar questions with the following screens. (Ask how they have figured it out when possible)



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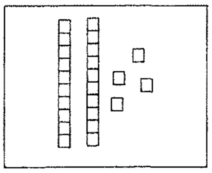

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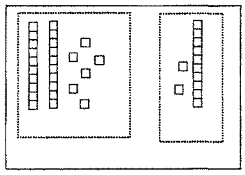
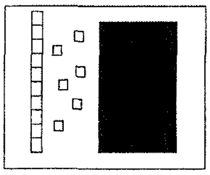


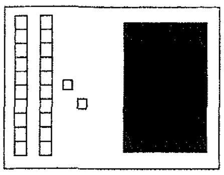
26 + 12

- Present the child with this screen and say: There are 4 rows and 2 little squares underneath this blue cover, how many squares are there all together? (there is a blue box on the screen that says "4 rows and 2 squares"). Always ask how they worked it out.



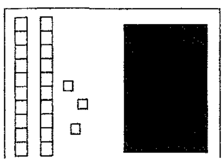
16 + 42

- Present the child with this screen and say: There is 1 row and 5 squares under the blue cover, how many squares are there all together? (there is a blue box on the sheet that reads "1 row and 5 squares")



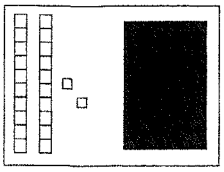
22 + 15

- Present the child with this screen and say: There are 3 rows and 8 squares under the blue cover, how many squares are there all together? (there is a blue box on the sheet that reads "3 rows and 8 squares")



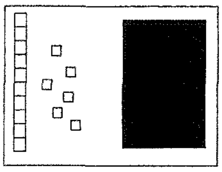
23 + 38

- Present the child with this screen and say: There is 1 row and 9 squares under the blue cover, how many squares are there all together? (There is a blue box on the sheet that reads "1 row and 9 squares")



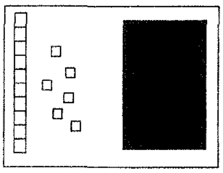
22 + 19

- Say: There are 22 little squares under the blue cover, how many squares are there all together? (the blue box reads "22")



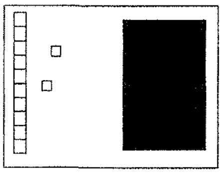
16 + 22

- Say: There are 14 little squares under the blue cover, how many squares all together? (the blue box reads "14")



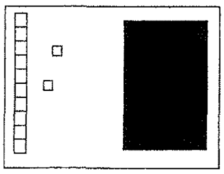
16 + 14

- Say: There are 24 little squares under the blue cover, how many squares are there all together? (the blue box reads "24")



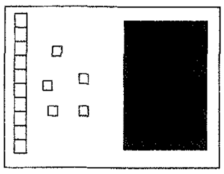
12 + 24

- Say: This one is different: there are 23 squares all together, how many did I put under the blue cover? (There is a white box that reads "23")



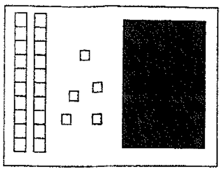
12 + … = 23

- Say: There are 25 squares all together, how many did I put under the blue cover? (the white box reads "25")



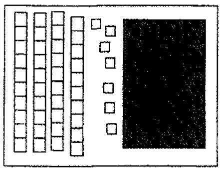
15 + … = 25

- Say: There are 45 squares all together, how many did I put under the blue cover? (the white box reads "45")



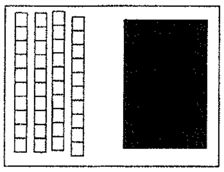
25 + … = 45

- Say: There are 79 squares all together, how many did I put under the blue cover? (the white box reads "79")



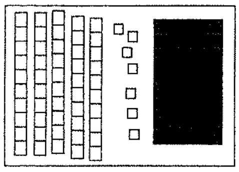
47 + … = 79

- Say: There are 57 squares all together, how many did I put under the blue cover? (the white box reads "57")



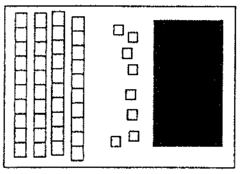
40 + … = 57

- Say: There are 91 squares all together, how many did I put under the blue cover? (the white box reads "91")



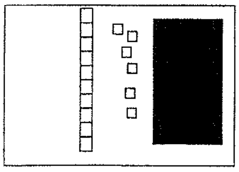
57 + … = 91

- Say: There are 72 squares all together, how many did I put under the blue cover? (the white box reads "72")



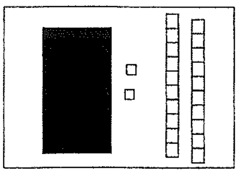
48 + … = 72

- Say: There are 4 rows and 2 little squares under the blue cover. How many little squares are there all together?



4 rows and 2 squares + 16 = …

- Say: There is 1 row and 5 little squares under the blue cover. How many little squares are there all together?



1 rows and 5 squares + 22 = …

3. Appendix 3: Rules According to the Preliminary Analysis

Rules inferred in situations of production and interpretation of n-digit scripts

- **Rule 1** (conventional and stable) or **rule of the digits**: the more digits the bigger the number. This rule is stable across scripts of 2, 3, or more digit number scripts. For example, '89567' is higher than '2500' because it has more numbers.
- **Rule 1A** or **rule of the zeros** (partially conventional, subject to contradictions): zeros or noughts or "ohs", make a big number. For example, '300' is higher than '342' because it has zeros, and zeros make a big number.
- **Rule 1B** or **rule of the nines** (conventional): nines in all places of the script make the highest number possible for that script.
- **Rule 2** or **rule of the front digit** (conventional and stable if used with Rule 1, subject to contradiction if used independently from Rule 1): the front digit has a leading role in discerning the meaning (value) of the number. For example, '7800' is higher '2500' because "7" is higher than "2". When the front digit of two scripts is the same, the value of the number script is judged by the second front digit, or by the remaining part of the script. For example, '367' is higher than '342' because "67" is higher than "42". This rule indicates children's appropriation of an essential characteristic of the written system of numbers, that is, the importance of the place of the figures in a number script to determine the value of a number. The importance being from left to right.
- **Rule 2A** or **rule of the front number** (like Rule 2) : the "number" at the front is used to discern the value of the number script, for example '7800' is higher than '2500' because "78" is higher than "25".
- **Rule 3** or **rule of the number sequence** (conventional and stable, often limited to the known segment of the number sequence): children interpret the number scripts by naming the script and refer to the number sequence to decide which of the scripts is higher. For example, "56" is higher than 3 because when you count is comes after.
- **Rule 4** or **rule of the hundred and thousand scripts** (conventional and stable): hundreds are written with 3 digits, thousands are written with 4 digits. Scripts are interpreted as different "kind" of numbers, that is, hundred or thousand numbers. Children that use this rule in conjunction with Rule 2 can read and write 3 and 4 digit scripts conventionally. For example, '2500' is "two thousand".
- **Rule 4A** or **rule of hundred and thousand knots**[1] (conventional): hundred knots are written with 2 zeros and thousand knots are written with 3 zeros.
- **Rule 4B** or **rule of the two zeros** (partially conventional, subject to contradiction): if a script has two zeros, the script is interpreted as "hundred". For example, '2500' is "25 hundred". This rule is subject to contradiction if the child has also established Rule 4. For example, '7800' is interpreted as "78 hundred" (a kind of conventional reading) but it is also a "thousand number" because it has 4 digits.
- **Rule 5** or **rule of the place** (conventional and stable, often used with Rule 4): the third place from right to left is the place of the hundreds or tells you the hundreds and the fourth place tells you the thousands. This rule has found to be established previous to "place value instruction" at school (i.e. instruction about "tens" and "units").

Rules inferred in situations of production and interpretation of 2 digit scripts

- **Rule 6** or **rule of importance of position of digits** (conventional): children realise that a change in the position of the figures forming a script, transforms the name of the number. Children discern two different readings of a script according to the position of the digits. For example, '12' is a different number from '21'. The different reading may or many not be manifestly associated to the meaning of the different scripts (e.g. a different value of the number)
- **Rule 7** or **rule of the decade scripts** (conventional when used together with Rule 7): children have appropriated the scripts for the decade knots as formed by a digit from 1 to 9 and a zero (i.e. 10, 20, 30, and so on, up to 90). The rule of the decade scripts can or cannot be accompanied by the

[1] Knots are round figures, for example, 100, 200, 300; 1000, 2000, and 3000.

appropriation of any of any spoken "sequence in tens". In effect, some children of the study were seen to have appropriated the decade scripts (i.e. could read and write decade scripts conventionally) well before they established any meaning for "counting in tens".

- **Rule 8** or **rule of the recurrence of 1 to 9 in the decades and 1 to 9 within each decade** (conventional and not often verbalised): This rule requires the previous construction or establishment of Rules 6 and 7. Children who had established this rule could produce and interpret conventionally 2-digit scripts. Explicitation of the recursive appearance of the figures is not pervasive in the transcripts, but characterised some of the cases of appropriation. For example, a child that uses Rule 8 to produce number scripts for 2 digit numbers has established that 4 is in front of all the forties, and the forties are written as 4-zero, 4-one, 4-two, and so on, up to 4-nine.
- **Rule 9** or **rule of ten in a row**: children who have established this rule realise that the scripts of the twenties form a row of ten as well all the scripts of the other decades.
- **Rule 10** or **rule of the next script**: children who have established this rule, realise that they can discern the name of a given script by interpreting the script which is next to it or by interpreting a script so many scripts away from the one given. This rule implies a one-to-one correspondence between the sequence of scripts and the standard spoken number sequence as well as the awareness of an abstract occurrence of "a number" per each number script or each number word. For example, if a given script is not "readable", a child may read the script just before that one or the script just after that one in order to interpret the given script.

Rules or regularities in number words

- **Rule 11** or **rule of the sequence in tens** (conventional and stable when the sequence is uttered up to 100): children that had established this rule were able to produce the sequence in ones with the recursive utterance of the "ty-one to nine" pattern with apparent ease. Counting in tens, meant, it was inferred , omitting the number words within the decade words. To probe this inference, the child was asked to continue after a hundred. If the child could conventionally count in tens beyond a hundred, or if they experienced conscious trouble at the moment of answering, the child is thought of having elaborated this regularity of the system of words. Depending on the explanation and extension of the sequence, the child was thought to be implicitly or explicitly aware of the recursive occurrence of the decade words, the hundred words (and the thousand words) and a simultaneous co-ordination of these "cycles".
- **Rule 12** or **rule of the counting hundreds** (partially conventional and stable): children produce a spoken sequence "in hundreds" by producing the sequence in ones and using the word "hundred" as if it were a countable entity. For example, children counted "one hundred, two hundred, three hundred, and so on, up to ten hundred". Eleven hundred and twelve hundred were often part of the sequence. Likewise, children could count thousands, or billions. Children that had established this rule were said to count "in hundreds", as a "different" counting sequence, that is, not associated with the same "meaning" of the sequence in ones.
- **Rule 13** or **rule of (addition and subtraction of) numbers of the same "kind"** (conventional but subject to contradiction when the sum is 10 or more): children add "numbers" of the same kind (e.g. hundreds, thousands, millions). For example, all hundred numbers were added as if they were ones. Two hundred and three hundred is five hundred because two and three is five. This rule was stable when the kind of numbers varied. For example, when thousands or millions were part of the situation. On the one hand, this rule suggests an implicit awareness of the "multiplicative syntax" of the number word system (3000 is 3 x 1000, i.e. three thousand). English language speakers (and Spanish) produce the name "three thousand". Children seemed to elaborate the rule "n" of a kind and "m" of the same kind produces "n + m" of the same kind, irrespective of the "kind of number" in play (e.g. hundreds or thousands). The stability of this rule was probed further in the analysis by taking a close look at children's solutions to problems such as, how many is two hundred and fifty more, or two million and five more. Children who were seen to have established **rule 13** shifted the use of the "multiplicative syntax" to an "additive syntax" to solve the new problem. The former syntax implied the "addition" of numbers of the same kind whilst the latter implied the "addition" of numbers of a different kind. Hence, two million and five more, was not "seven million", but "two million and five". On the other hand, the inferred elaboration of this rule suggests that children had "appropriated" an essential feature of the decimal conventional system, that is, higher numbers (i.e. "big numbers" or units of higher order) behave like lower numbers (i.e. "little numbers" or units of lower order). Hence, children "notice" that if one and one is two, then one hundred and one hundred

is two hundred, or if five and five is ten, then fifty and fifty is a hundred. However, it was presumed that "two hundred" had a different meaning from "two" for the child[2].

- **Rule 14** or **rule of the organising hundreds** (conventional): children seemed to have appropriated a sequence in hundreds, that is, one hundred, two hundred, and so on, up to one thousand. The tenth hundred was "a thousand".
- **Rule 15** or **rule of making units** (conventional): children who used this rule were aware of the capability of "renaming" a number of things. In doing this, they were thought to be uniting intellectually a number of entities which they could also have considered separately. For example, a century meant "a hundred years", or 100 minutes was a whole hour.
- **Rule 16** or **rule of the successor structure** (conventional): children who were thought of having established this rule used and interpreted the next number word of the spoken sequence as meaning "one more". For example, a hundred and one was one more than a hundred, because first is a hundred, and then a hundred and one.
- **Rule 17** or **rule of the sequence in fives** (conventional): some children were able to produce the spoken sequence when asked to "count in fives". This sequence is thought to be a different sequence from the sequence in ones when it was seen to be used with the inferred meaning of "five more" or "five less".
- **Rule 18** or **rule of the halves** (conventional): children appropriated regularities of the system of spoken numbers based on their reflection on regularities of number scripts. This rule refers to children's explanation such as the following fifty is half of a hundred like five is half of ten.

Non conventional regularities with number scripts and words

- **Rule 19** or **rule of concatenation** (subject to contradiction with all conventional rules): number scripts are written in a "concatenated" form. Children conjectured that number words are written as they are spoken. Children appropriate the number scripts for certain number words. For example, they may know that "a hundred" is written as '100' and they know that "twenty" is written as '20'. For the child, parts of the spoken number corresponded to parts of the number script so for example, they wrote "a hundred and twenty" as '10020'.
- **Rules 20** or **rule of regularities in scripts**: when a verbal pattern was noticed by the child in number words, the child spelled out this pattern in writing. For example, "two hundred", "three hundred" and "four hundred" were written as '102', '103', and '104'. The child reproduced a pattern in writing but this was not a conventional regularity.

[2] See Skemp's explanation of this feature of the numeration system in Skemp (1986, p. 148)