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Quantum Aspects of Target Space Duality

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ABSTRACT

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Quantum Aspects of Target Space Duality

by Peter John Hodges

This thesis explores the phenomenon of target space duality at the quantum level. Aspects of Abelian and non-Abelian variants are considered in the context of Batalin–Vilkovisky quantisation in order to define path integrals. Obstacles to the construction of non-Abelian duals are overcome and the quantum nature of Abelian models is considered; neither investigation relying upon any supersymmetry.

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Chapter 1

Quantization of Field Theories

1.1 Introduction

Field theory is the mainstay of modern particle physics. Lagrangian and Hamiltonian formulations of interacting fields have provided the most promising framework to date describing the observed fundamental forces. Whilst a Lagrangian density is a familiar object to all individuals working with field theories, it does not follow that it is always a conceptually simple matter to discuss such objects in a Quantum theoretical framework. One should not forget that the Lagrangians from which we draw the majority of our inspiration are Classical objects; to be regarded as limits of the Quantum theory that we aspire to describe.

The framework in which the most progress has been made in describing field theories was introduced by Feynman and Kacs. This elegant description unifies one's approach to statistical mechanics and field theory. Physical observables pertaining to a set of generic fields ϕ, are derived from what is referred to as a generating functional, or path integral, $\mathcal{Z}$

$$\mathcal{Z} = \int [\mathcal{D}\phi] \; e^{-S[\phi]}$$

This is only a formal statement of how one calculates the generating functional; it is tacitly assumed that one can synthesize a suitable quantum action $S[\phi]$ by some undiscussed means. This is not necessarily a simple task. A pertinent question to ask is: what can possibly go wrong?

1.2 Symmetries of the Classical Action

Since all of our understanding is derived from the existence of a Classical limit, one's starting point is the Classical action. Is it sufficient to insert such an object into the path integral? How might a Classical action disrupt the calculation of the path integral? One such failure arises if the action contains a redundancy in terms of the number of fields describing the physical system, i.e. gauge symmetries make the transition from Classical to Quantum theoretical descriptions problematic. Why so? Imagine a local transformation of the fields labelled by a set of objects $\{\Lambda\}$ which form a group under the action of a composition law

$$\mathcal{T}^{\Lambda} : \phi(x) \longrightarrow \phi^{\Lambda}(x)$$

Such that

$$\mathcal{T}^{\Lambda} : S[\phi] \longrightarrow S\left[\phi^{\Lambda}\right] = S[\phi]$$

Under such transformations the action is unchanged, which is what we mean when we refer to a symmetry of the action. Such a degeneracy of the action may well interfere in the computation of the path integral, since it leads to an overcounting of physical states.

Under this set of circumstances a means of control is required to define the required sum over histories. Roughly speaking, one ought to define a new measure of integration that 'divides out' the redundant integrations; mathematicians refer to this as determination of the Haar measure.

Having had this possible inconvienience drawn to one's attention it is perhaps obvious how one controls such overcounting; one should foliate the summation over the phase space defined by the generic fields ϕ in such a way that, in any one such sheet, there is contained only one configuration corresponding to any particular physical state of the system so described. Having stated the means by which one intends to attack the problem, there remains the matter of devising a suitable method to implement this approach.

The prefered method in this publication, Batalin–Vilkovisky quantization, is possibly not familiar to many readers. One proposes to look at a familiar tool in order to gain some insight into the reasons why one requires more refined methods.

1.3 The De-Witt–Fadeev–Popov Method

Most introductory level quantum field theory textbooks make reference to the De-Witt–Fadeev–Popov Method [14, 22, 29, 3, 37], and so readers will be reassured when one considers functionals of the form

$$\mathcal{Z} = \int [\mathcal{D}\phi]\, e^{-\mathcal{S}[\phi]} B\,[f]\,\det \mathcal{F} \tag{1.1}$$

The objects within this functional have certain properties which one shall describe.

Under the action of a gauge transformation

$$\mathcal{T}^{\Lambda} : [\mathcal{D}\phi]\, e^{-\mathcal{S}[\phi]} \quad \longrightarrow \quad \left[\mathcal{D}\phi^{\Lambda}\right] e^{-\mathcal{S}\left[\phi^{\Lambda}\right]}$$

$$= \quad [\mathcal{D}\phi]\, e^{-\mathcal{S}[\phi]}$$

These objects; the components of the measure of integration and the exponentiated action, are invariant under the action of the gauge group. Often it is true that these objects are separately invariant.

The functions $\{f^{\alpha}[\phi; x]\}$ are a set of gauge variant functions of the fields; there are as many f as there are parameters to describe the gauge group, it is in this sense that one might hope to factor out the redundancy inherent in a gauge field theory. These f are employed in 'gauge fixing' the action when inserted into the functional $B[f]$. In addition, the f give rise to a Faddeev–Popov determinant in the path integral

$$\det \mathcal{F} = \left\| \mathcal{F}^{\alpha;y}_{\beta;x} \Big|_{\Lambda=\mathcal{I}} \right\| = \left\| \frac{\delta f^{\alpha}\left[\phi^{\Lambda}; x\right]}{\delta \Lambda^{\beta}(y)} \bigg|_{\Lambda=\mathcal{I}} \right\| \tag{1.2}$$

Where ϕ^{Λ} continues to represent the gauge transformed fields, Λ_{β} represent the parameters that allow one to describe elements of the gauge group. The Faddeev–Popov determinant is evaluated at the identity element of the gauge group, $\mathcal{I}$. From this point one might ordinarily introduce a pair of Faddeev–Popov ghost fields in order to promote the matrix $\mathcal{F}^{\beta}_{\alpha}(x, y)$ into the action.

In many cases it is true to say that the path integral, so assembled, is independant of the gauge fixing functionals f_{α} and depends only on the choice of functional $B[f]$ through an irrelevant, constant normalizing factor.

Let us examine this construction in greater detail and expose those problems which may arise. We begin with a trivial change of variables in (1.1) by an arbitrary gauge transformation

$$\mathcal{Z} = \int [\mathcal{D}\phi^{\Lambda}] \, e^{-\mathcal{S}[\phi^{\Lambda}]} B\left[f\left[\phi^{\Lambda}\right]\right] \det \mathcal{F}\left[\phi^{\Lambda}\right] \tag{1.3}$$

Such a changing of variables naturally has no effect on the integration. Now we may make use of the invariance of the action and $[\mathcal{D}\phi]$ under local gauge transformations in order to write the path integral partly in terms of the original variables.

$$\mathcal{Z} = \int [\mathcal{D}\phi] \, e^{-\mathcal{S}[\phi]} B\left[f\left[\phi^{\Lambda}\right]\right] \det \mathcal{F}\left[\phi^{\Lambda}\right] \tag{1.4}$$

The choice of gauge transformation was arbitrary, so it follows that the left hand side of (1.3) does not depend upon it. We are therefore free to integrate over all possible gauge transformations Λ with some appropriately chosen weighting functional $\rho[\Lambda]$

$$\mathcal{Z} \int [\mathcal{D}\Lambda] \, \rho[\Lambda] = \int [\mathcal{D}\phi] \, e^{-\mathcal{S}[\phi]} C[\phi] \tag{1.5}$$

Where $C[\phi]$ is given by the expression

$$C[\phi] = \int [\mathcal{D}\Lambda]\, \rho[\Lambda]\, B\left[f\left[\phi^{\Lambda}\right]\right] \det F\left[\phi^{\Lambda}\right] \tag{1.6}$$

Equation (1.2) states that

$$\mathcal{F}^{\alpha;y}_{\beta;x}\left[\phi^{\Lambda}\right] = \left.\frac{\delta f^{\alpha}\left[\left(\phi^{\Lambda}\right)^{\Lambda'};x\right]}{\delta \Lambda'^{\beta}(y)}\right|_{\Lambda'=\mathcal{I}} \tag{1.7}$$

Now, Λ and Λ' are elements of a group, it follows that they can be composed to form another element of the same group

$$\Lambda \cdot \Lambda' = \Lambda''\left(\Lambda, \Lambda'\right) \tag{1.8}$$

From which it follows that one may write the twice transformed field ϕ in (1.7) as

$$\left(\phi^{\Lambda}\right)^{\Lambda'} = \phi^{\Lambda''\left(\Lambda,\Lambda'\right)} \tag{1.9}$$

Having made this observation, one may re-write the matrix $\mathcal{F}$

$$\mathcal{F}^{\alpha;y}_{\beta;x} = \mathcal{G}^{\alpha;z}_{\gamma;x}\mathcal{H}^{\gamma;y}_{\beta;z}$$

Where we employ the (DeWitt) convention of summing over all repeated indices including the spacetime variety. In order to construct this product one uses the chain rule and defines

$$\mathcal{G}^{\alpha;z}_{\gamma;x}[\phi,\Lambda] \equiv \left.\frac{\delta f^{\alpha}\left[\phi^{\Lambda''};x\right]}{\delta \Lambda''^{\gamma}(z)}\right|_{\Lambda''=\Lambda}$$

$$= \frac{\delta f^{\alpha}\left[\phi^{\Lambda}; x\right]}{\delta \Lambda^{\gamma}} \tag{1.10}$$

and

$$\mathcal{H}^{\gamma y}_{z\beta} = \left.\frac{\delta \Lambda''^{\gamma}(z; \Lambda, \Lambda')}{\delta \Lambda'^{\beta}(y)}\right|_{\Lambda'=\mathcal{I}} \tag{1.11}$$

It then follows that

$$\det \mathcal{F} = \det \mathcal{G} \cdot \det \mathcal{H}$$

One observes that $\det \mathcal{G}$ appears explicitly in (1.10) as the necessary Jacobian for the transition from integration over the parameters describing the group to integration over the gauge fixing functionals $[\mathcal{D}f]$.

If it were then possible for one to make the choice for the weighting functional $\rho\,[\Lambda]$ in the measure such that

$$\rho\,[\Lambda] = \frac{1}{\det \mathcal{H}}$$

It would then be true that the expression (1.6) would be an irrelevant constant determined by ones choice of the functional $B\,[f]$. This much is standard lore in quantum field theory. It is important that we remember how we arrived at this point, however. Recall (1.8); we have made use of one of the properties of group elements under composistion, in addition we have assumed that this mapping is unique. That is to say that for every Λ' acting upon an arbitrary group element Λ one will arrive at a distinct element of the same group Λ''. This is not the only possible complication, we also assume that the generators of the group are linearly independent. If this fails to be the case, then the determinant of $\mathcal{H}$ will be singular and hence the proposed choice of weighting functional will be inadmissable.

The appearance of additional redundancy in one's description of a given symmetry lead to complications in quantization of a field theory. This is far from a hopeless state of affairs to arrive at. What one ought to do, is 'gauge fix the gauge fixing'. It follows that one's experience with the DeWitt–Faddeev–Popov method is far from wasted, all that one needs to do is extend (1.1) in order to foliate the path integral correctly. This is where the Batalin–Vilkovisky formalism comes to the fore.

1.4 Batalin–Vilkovisky Formalism

The Anti-field or Batalin–Vilkovisky formalism is a well established means of describing quantum field theories that contain arbitrary degrees of reducible symmetry. There are certainly many papers that utilise and describe the implementation of this method [4, 5, 6]. One intends to discuss this method in some detail as it will be of considerable significance elsewhere in this thesis.

Let us begin by describing a theory which at the classical level contains a set of fields labelled by $\{\phi^i\}$; these fields will, in general, be a mixture of Grassmann odd and even objects. These fields will be permitted to undergo a set of local, field dependent transformations, where an arbitrary infinitesimal change can be written

$$\delta\phi^i = R^i_{\alpha_0}(\phi)\omega^{\alpha_0} \quad i = 1, \ldots, n(= n_+ + n_-) \tag{1.12}$$

Where one denotes a set of gauge parameters by ω^{α_0} and one continues to employ a De-Witt summation convention; the index α_0 representing a set of internal and space-time indices, which are summed over on repetition. If the Classical action, $\mathcal{S}_c(\phi)$ is left invariant under (1.12) then this is, of course, a symmetry of the theory. If the set of generators $R^i_{\alpha_0}$ are linearly independent in the vicinity of stationary points of the theory, then the theory is said to possess an irreducible symmetry. One may feel free to use the Fadeev-Popov procedure in order to quantize such a theory. In Batalin–Vilkovisky terminology such a class of theory would be known as a 'zero stage' theory, at this point one may well understand the purpose of the numerical subscript attached to the gauge parameter. A first stage theory is such that $R^i_{\alpha_0}(\phi)$ possesses non-trivial zero eigenvectors $Z^{\alpha_0}_{\alpha_1}(\phi)$ with respect to the index α_0. One can extend this notion to a theory of second stage reducibility, should $Z^{\alpha_0}_{\alpha_1}(\phi)$ possess non-trivial zero eigenvectors with respect to the index α_1; furthermore should subsequent eigenvectors also possess non-trivial zero eigenvectors then one can continue this notion indefinitely to an n^{th} stage of reducibility. For the purposes of this thesis it will be sufficient to restrict the discussion to first stage theories only.

1.4.1 Notation and Conventions

Before one elaborates on this notion it would be wise to introduce some conventions that will be employed in discussing this formalism. In order to discuss fields of Grassmann odd or even character one introduces the operator

$$\begin{aligned} \epsilon_i &= \epsilon(\beta^i) \\ &= \begin{cases} 0 & \text{If } \beta^i \text{ is Grassmann even} \\ 1 & \text{If } \beta^i \text{ is Grassmann odd} \end{cases} \end{aligned} \tag{1.13}$$

In this instance β^i is intended to represent a field, a generator or any combination of these objects; it is often convienient to refer to the Grassmann character of an object via reference to the index attached to it. Similarly one shall employ an operator $gh(\beta^i)$ which reveals the ghost number of β^i.

The formalism demands that one calculates functional derivatives of the fields. When the Grassmann character of an object is not specified then it is useful to employ two distinct derivatives

$$\frac{\delta_r}{\delta\beta^i} \quad \text{Which acts from the right hand side.} \tag{1.14}$$

$$\frac{\delta_l}{\delta\beta^i} \quad \text{Which acts from the left hand side.} \tag{1.15}$$

The notation employed readily allows one to discuss various operations in terms of actions carried out upon matrices, as is common practise in quantum field theory. For example, if one wishes to determine whether a propagator is invertible, one must consider the rank of such an operator. The rank of a matrix is, as always, the maximal size of its invertible square minor. For an even parity matrix, **X**, the rank of such a matrix may be decomposed into the ranks of its Bose-Bose and Fermi-Fermi blocks.

$$Rank\ \mathbf{X} = Rank_+\ \mathbf{X} + Rank_-\ \mathbf{X}$$

$Rank_\pm$ refers to the Bose-Bose and Fermi-Fermi blocks respectively.

In general one's sets will contain objects of Grassmann odd and Grassmann even character

$$\{\beta^i\} \quad \text{i=1,2,...,n}$$

However, when one wishes to distinguish between these objects then we will make further use of $\pm$ to separate Grassmann even and Grassmann odd quantities respectively, always remembering that

$$\{\beta^i\} \quad \text{i=1,2,}\ldots,n_+ + n_-$$

It is now appropriate to begin describing the framework in which one shall operate.

1.4.2 First Stage Reducible Theories

It is assumed the the classical action $\mathcal{S}_c(\phi)$, up to gauge transformations, possesses at least one stationary point ϕ_0

$$\left.\frac{\delta \mathcal{S}_c}{\delta \phi^i}\right|_{\phi_0} = 0 \qquad (1.16)$$

In other words one requires the action to be capable of describing phenomena that can be understood at the classical level. This is essential to our interpretation irrespective of the formulation employed to place the theory on the quantum level. It is necessary to be able to define clear in and out states which one can observe at the macroscopic level, as this is the scale at which any experimentalist would exist and, as such, forces (1.16) upon one. In addition we expect the action to be infinitely smooth (differentiable) in the vicinity of ϕ_0.

It has already been expressed (1.12) that one begins with the expectation that the theory will be invariant under the action of a local gauge transformation; therefore we also assume the existence of $m_0 (= m_{0+} + m_{0-})$ Noether identities in a differentiable neighbourhood of the classical solution ϕ_0

$$\frac{\delta_r \mathcal{S}_c}{\delta \phi^i} R^i_{\alpha_0}(\phi) = 0\,, \qquad \alpha_0 = 1, \ldots, m_0 \tag{1.17}$$

In addition, the generators of the transformations $R^i_{\alpha_0}$ are taken to be regular and differentiable. The Grassman parity of the generator is easily deduced when one considers (1.12)

$$\epsilon\left(R^i_{\alpha_0}(\phi)\right) = \epsilon_i + \epsilon_{\alpha_0} \pmod 2 \tag{1.18}$$

This guarantees that gauge transformed fields have the correct Grassman character, relative to the initial set of fields.

One shall be discussing theories classified by a first stage reducible symmetry; which, as previously stated, implies the existence of a set of non-trivial zero eigenvectors for the generators of the gauge transformation, $R^i_{\alpha_0}(\phi)$

$$R^i_{\alpha_0} Z^{\alpha_0}_{\alpha_1}\Big|_{\phi_0} = 0 \tag{1.19}$$

The index α_1 labels the set of non-trivial zero eigenvectors and is such that $\alpha_1 = 1, \ldots, m_1 (= m_{1+} + m_{1-})$. One sees in general that one can write the parameters for a local gauge transformation

$$\omega^{\alpha_0} = \hat{\omega}^{\alpha_0} + \upsilon^{\alpha_1} Z^{\alpha_0}_{\alpha_1}(\phi) \tag{1.20}$$

All gauge transformations connected by the new gauge parameter υ^{ω_1} are seen to be equivalent, with respect to transformations acting upon the fields ϕ, upon examination of (1.12) and (1.19). The Grassmann character of $Z^{\alpha_0}_{\alpha_1}$ is seen, after examination of (1.20) to be

$$\epsilon\left(Z^{\alpha_0}_{\alpha_1}\right) = \epsilon_{\alpha_0} + \epsilon_{\alpha_1} \pmod 2 \tag{1.21}$$

One has stated in advance that the symmetry is first stage reducible. This implies that there cannot be found a set of non-trivial zero eigenvectors for $Z^{\alpha_0}_{\alpha_1}$

$$\nexists \ \left\{X^{\alpha_1}_{\alpha_2}(\phi)\right\} \quad \Big| \quad Z^{\alpha_0}_{\alpha_1}(\phi)X^{\alpha_1}_{\alpha_2}(\phi)\Big|_{\phi_0} = 0 \tag{1.22}$$

If the statement (1.22) is not satisfied, then the theory will have a symmetry that is at least second stage reducible.

Having established what one means by first stage reducibility, it is simple to re-state this information in the language of matrices

$$Rank_{\pm}\ R^i_{\alpha_0}\Big|_{\phi_0} = (m_0 - m_1)_{\pm} \qquad m_{0\pm} > m_{1\pm} \tag{1.23}$$

Equation (1.23) makes transparent the fact that gauge transformations are unique only after factoring out the action of $Z^{\alpha_0}_{\alpha_1}$ upon the parameters of the transformation (1.20). The inequality in (1.23) must also be inserted, were it not to hold the gauge group would have a trivial action; it being possible to identify all transformations with the identity element by (1.20).

Having insisted that (1.22) is true, then by construction

$$Rank_{\pm}\ Z^{\alpha_0}_{\alpha_1}\Big|_{\phi_0} = m_{1\pm} \tag{1.24}$$

Having gathered this information, one is in a posistion to make a statement about an operator of particular interest in field theory

$$Rank_{\pm}\ \frac{\delta_l \delta_r \mathcal{S}_c}{\delta\phi^i \delta\phi^j}\bigg|_{\phi_0} = n_{\pm} - (m_0 - m_1)_{\pm} \qquad n_{\pm} > (m_{0\pm} - m_{1\pm}) \tag{1.25}$$

The inequality in (1.25) is necessary in order for the action to describe a physically interesting system. Were the inequality not satisfied then all field configurations could be identified with a constant set of fields via the transformation (1.12).

It is already known that the field configurations are physically equivalent up to gauge transformations (1.12). In addition gauge transformations are only unique up to the action of $Z^{\alpha_0}_{\alpha_1}$ upon the gauge parameters (1.20). Having factored out the action of $Z^{\alpha_0}_{\alpha_1}$, then it is seen that there are only $(m_0 - m_1)$ true gauge parameters. Factoring out true gauge transformations from the field configurations allows one to construct physical objects such as propagators. A properly gauge fixed action will have m_0 gauge fixing conditions and m_1 conditions to fix the gauge fixing; this will lead to (1.25) possessing maximal rank in the gauge fixed action, and thus invertible propagators for the physical degrees of freedom. These two sets of necessary gauge fixing conditions will lead to two families of ghost fields being introduced into the action; ghosts and 'ghosts-for-ghosts', so named for the reasons outlined above.

One can go on to describe fields with higher degrees of reducibility. One might guess that further families of ghosts would appear, associated with additional redundancy in the description of the gauge symmetry; and one would be correct in this assumption.

1.5 Gauge Fixing by the Anti-Field Method

Central to the formalism of Batalin–Vilkovisky in the gauge fixing of a Classical action is the notion of the anti-field. These objects will be introduced into the action in order to insert gauge fixing conditions in a logical fashion. In order to gauge fix an action it is necessary to enlarge the space of fields employed to accomodate the ghosts. The Classical fields ϕ^i will be members of a larger set Φ^A, the content of which is to be determined i. e.

$$\phi^i \subset \Phi^A$$

It is to the larger set $\{\Phi^A\}$ that one associates the set of anti-fields $\{\Phi^*_A\}$. The anti-fields possess opposite Grassmann character to the corresponding field

$$\begin{aligned} \epsilon(\Phi^A) &= \quad \epsilon_A \\ \epsilon(\Phi^*_A) &= \epsilon_A + 1 \pmod 2 \end{aligned} \tag{1.26}$$

The anti-fields will implement gauge fixing, this is achieved by selecting surface on the phase space of Φ, Φ^* described by

$$\Sigma : \Phi^* = \frac{\delta \Psi(\Phi)}{\delta \Phi^A} \tag{1.27}$$

The functional $\Psi(\Phi)$ is of Grassmann odd character, and is known as the gauge fixing fermion.

Having introduced two abstract sets of fields and anti-fields, one defines the action of the anti-bracket which may act upon two arbitrary functionals X,Y

$$(X\ ,\ Y) \equiv \frac{\delta_r X}{\delta \Phi^A} \cdot \frac{\delta_l Y}{\delta \Phi^*_A} - \frac{\delta_r X}{\delta \Phi^*_A} \cdot \frac{\delta_l Y}{\delta \Phi^A} \tag{1.28}$$

With these tools in hand, one can proceed to develop the means to determine how one extends the Classical action $\mathcal{S}_c(\phi)$ to incorporate the ghosts and anti-fields. Consider a bosonic functional $\mathcal{W}(\Phi, \Phi^*)$ that satisfies the constraint

$$\Delta\ \exp\left(-\frac{\mathcal{W}}{\hbar}\right) = 0, \quad \text{where } \Delta \equiv \frac{\delta_r}{\delta\Phi^A}\cdot\frac{\delta_l}{\delta\Phi^*_A} \tag{1.29}$$

Observe that one has not adopted a system of natural units in which $\hbar = 1$, as the Planck constant will be employed later as a counting parameter. The compact equation (1.29) can be re-expressed

$$\Delta\ \exp\left(-\frac{\mathcal{W}}{\hbar}\right) = \sum_{n=0}^{n=\infty}\frac{1}{n!}\left(\frac{\delta_r}{\delta\Phi^A}\left[-\frac{n}{\hbar}\left(\frac{-\mathcal{W}}{\hbar}\right)^{n-1}\frac{\delta_l\mathcal{W}}{\delta\Phi^*_A}\right]\right)$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\begin{pmatrix}\frac{n(n-1)}{\hbar^2}\left(\frac{-\mathcal{W}}{\hbar}\right)^{n-2}\frac{\delta_r\mathcal{W}}{\delta\Phi^A}\frac{\delta_l\mathcal{W}}{\delta\Phi^*_A}\\ -\frac{n}{\hbar}\left(\frac{-\mathcal{W}}{\hbar}\right)^{n-1}\frac{\delta_r}{\delta\Phi^A}\frac{\delta_l\mathcal{W}}{\delta\Phi^*_A}\end{pmatrix}$$

Ignoring zero valued terms,

$$= \sum_{n=0}^{n=\infty}\frac{1}{n!}\left(\frac{-\mathcal{W}}{\hbar}\right)^{n}\begin{pmatrix}\frac{1}{\hbar^2}\frac{\delta_r\mathcal{W}}{\delta\Phi^A}\frac{\delta_l\mathcal{W}}{\delta\Phi^*_A}\\ -\frac{1}{\hbar}\Delta\,\mathcal{W}\end{pmatrix} \tag{1.30}$$

That is,

$$\Delta\ \exp\left(\frac{-\mathcal{W}}{\hbar}\right) = \exp\left(\frac{-\mathcal{W}}{\hbar}\right)\left(\frac{1}{2\hbar^2}(\mathcal{W}\,,\,\mathcal{W}) - \frac{1}{\hbar}\Delta\,\mathcal{W}\right) \tag{1.31}$$

This manipulation (1.31) allows one to re-express the condition on $\mathcal{W}$ (1.29) in the following form

$$\frac{1}{2}(\mathcal{W}\,,\,\mathcal{W}) = \hbar\Delta\,\mathcal{W} \tag{1.32}$$

Equation (1.32) shall be referred to as the *quantum master equation.*

It would not be unnatural to ask why one might be particularly interested in functionals satisfying (1.32). In order to answer this question one should consider a path integral of the functional $\mathcal{W}$ constrained as in (1.27)

$$\mathcal{W}_\Sigma(\Phi) \equiv \mathcal{W}\left(\Phi, \Phi^*_A = \frac{\delta\Psi}{\delta\Phi}\right) \tag{1.33}$$

So that one may write the path integral

$$\mathcal{Z}_\Psi = \int [\mathcal{D}\Phi] \exp\left(\frac{-\mathcal{W}_\Sigma(\Phi)}{\hbar}\right) \tag{1.34}$$

Having so expressed the path integral, let us consider the consequences of changing the gauge fixing fermion by an infinitesimal amount

$$\Psi\,(\Phi) \longrightarrow \Psi\,(\Phi) + \psi\,(\Phi)$$

It is sufficient to restrict attention to such infinitesimal deformations of the functional Ψ, since finite deformations may be obtained by integration of such infinitesimal transformations. One simultaneously makes a transformation of the fields, Φ which for infinitesimal transformations may be written as

$$\begin{aligned} \Phi &\longrightarrow \tilde{\Phi} = \Phi + \delta\Phi \\ \delta\Phi^A &= -\frac{1}{\hbar}(\,\Phi^A\,,\,\mathcal{W}\,)\psi\,(\Phi)\Big|_\Sigma \\ &= -\frac{1}{\hbar}\frac{\delta_l\mathcal{W}}{\delta\Phi^*_A}\psi\Big|_\Sigma \end{aligned} \tag{1.35}$$

This transformation is the generalization of the Becchi-Rouet-Stora-Tyutin (**BRST**) transformation.

Performing a change of variables in a path integral requires one to calculate a Jacobian. The associated matrix is obtained by differentiating (1.35)

$$\frac{\delta_l \tilde{\Phi}^A}{\delta \Phi^B} = \delta^A_B - \frac{1}{\hbar}\frac{\delta_l}{\delta \Phi^B}\left(\frac{\delta_l \mathcal{W}}{\delta \Phi^*_A}\cdot \psi\right)\Bigg|_\Sigma \tag{1.36}$$

From which the Jacobian is a simple matter to obtain for infinitesimal ψ

$$\left\|\frac{\delta_l \tilde{\Phi}}{\delta \Phi}\right\| = 1 - \frac{1}{\hbar}\left(\frac{\delta_l}{\delta \Phi^A}\cdot\frac{\delta_l \mathcal{W}}{\delta \Phi^*_A}\right)\psi - \frac{1}{\hbar}\frac{\delta_l \mathcal{W}}{\delta \Phi^*_A}\cdot\frac{\delta \psi}{\delta \Phi^A}\Bigg|_\Sigma + \mathcal{O}\left(\psi^2\right) \tag{1.37}$$

One can make use of the Grassmann character of the gauge fixing fermion and the functionally differentiated action to write

$$\begin{aligned} \frac{\delta_l}{\delta \Phi^A}\cdot\frac{\delta_l \mathcal{W}}{\delta \Phi^*_A}\cdot\psi &= \psi\cdot\frac{\delta_l}{\delta \Phi^A}\cdot\frac{\delta_r \mathcal{W}}{\delta \Phi^*_A} \\ &= -\Delta\mathcal{W}\cdot\psi \end{aligned} \tag{1.38}$$

The final observation may be made after examining the definition of Δ, (1.29), and making use of the fact that both ψ and $\Delta\mathcal{W}$ are of Grassmann odd character.

It is also useful to make the observation that

$$\begin{aligned} \frac{\delta_l \mathcal{W}}{\delta \Phi^*_A}\cdot\frac{\delta \psi}{\delta \Phi^A}\Bigg|_\Sigma &= \frac{\delta_l \mathcal{W}}{\delta \Phi^*_A}\cdot\delta\Phi^*_A\Bigg|_\Sigma \\ &= 0 \end{aligned} \tag{1.39}$$

Which follows from the definition of the surface, Σ (1.27). The two observations (1.38), (1.39) can be used to simplify the expression for the Jacobian when substituted into (1.37); explicitly one can write that for infinitesimal transformations

$$\left\| \frac{\delta_l \tilde{\Phi}}{\delta \Phi} \right\| = 1 + \frac{1}{\hbar} \Delta \mathcal{W} \cdot \psi \tag{1.40}$$

One should also relate the transformed action to the original functional

$$\delta \mathcal{W} = \frac{\delta_r \mathcal{W}}{\delta \Phi^A} \cdot \delta \Phi^A + \frac{\delta_r \mathcal{W}}{\delta \Phi^*_A} \cdot \delta \Phi^*_A \Big|_{\Sigma} \tag{1.41}$$

One may substitute the explicit form for the transformation of the fields Φ using (1.35). Also, making use of the definition of the surface Σ, (1.27), one arrives at the expression

$$\begin{aligned} \delta \mathcal{W} &= -\frac{1}{\hbar} \frac{\delta_r \mathcal{W}}{\delta \Phi^A} \cdot \frac{\delta_l \mathcal{W}}{\delta \Phi^*_A} \cdot \psi \Big|_{\Sigma} \\ &= -\frac{1}{2\hbar} \left(\frac{\delta_r \mathcal{W}}{\delta \Phi^A} \cdot \frac{\delta_l \mathcal{W}}{\delta \Phi^*_A} + \frac{\delta_l \mathcal{W}}{\delta \Phi^*_A} \cdot \frac{\delta_r \mathcal{W}}{\delta \Phi^A} \right) \cdot \psi \\ &= -\frac{1}{2\hbar} \left(\frac{\delta_r \mathcal{W}}{\delta \Phi^A} \cdot \frac{\delta_l \mathcal{W}}{\delta \Phi^*_A} - \frac{\delta_r \mathcal{W}}{\delta \Phi^*_A} \cdot \frac{\delta_l \mathcal{W}}{\delta \Phi^A} \right) \cdot \psi \\ &= -\frac{1}{2\hbar} (\mathcal{W}\,,\ \mathcal{W}) \cdot \psi \end{aligned} \tag{1.42}$$

The final form follows after the change in sign that results from changing a right to a left fermionic derivative when acting upon a bosonic functional. In addition the definition (1.28) was also employed to arrive at (1.42).

One may write the path integral in the transformed fields in the form

$$\mathcal{Z}_{\Psi+\delta\psi} = \int [\mathcal{D}\Phi] \frac{1}{\left\| \frac{\delta_l \tilde{\Phi}}{\delta \Phi} \right\|} \exp\left(-\frac{\mathcal{W}_\Psi}{\hbar}\right) \cdot \exp\left(-\frac{\delta \mathcal{W}}{\hbar}\right) \tag{1.43}$$

It is straightforward to substitute the expressions (1.40) and (1.42) into the path integral, which will allow one to examine the effect of selecting a different gauge fixing fermion.

$$\begin{aligned} \mathcal{Z}_{\Psi+\delta\psi} &= \int [\mathcal{D}\Phi]\, e^{\left(-\frac{\mathcal{W}_\Psi}{\hbar}\right)} \cdot \left(1 - \frac{1}{\hbar}\Delta\mathcal{W}\cdot\psi\right) \cdot \left(1 + \frac{1}{2\hbar^2}(\mathcal{W}\,,\,\mathcal{W})\cdot\psi\right) \\ &= \int [\mathcal{D}\Phi]\, e^{\left(-\frac{W_\Psi}{\hbar}\right)} \left(1 + \frac{1}{\hbar^2}\left(\frac{1}{2}(\mathcal{W}\,,\,\mathcal{W}) - \hbar\Delta\mathcal{W}\right)\cdot\psi\right) \end{aligned} \tag{1.44}$$

Since one confines the discussions to functionals satisfying the quantum master equation (1.32) it is true that

$$\begin{aligned} \mathcal{Z}_{\Psi+\delta\psi} &= \int [\mathcal{D}\Phi] \exp\left(-\frac{W_\Psi}{\hbar}\right) \\ &= \mathcal{Z}_\Psi \end{aligned} \tag{1.45}$$

One sees that the path integral, $\mathcal{Z}$, is independent of the choice of gauge fixing fermion Ψ, provided that it satisfies the quantum master equation (1.32). It follows that the choice of gauge fermion is completely arbitrary, provided that the path integral is non-degenerate following the selection of this gauge fixing device.

Chapter 2

Dualities in Quantum Field Theories

2.1 Introduction

Duality is a word seen frequently in the literature in the fields of elementary particle physics [19, 35, 18, 36] and statistical mechanics [24]. Duality, as may perhaps surprise some readers, was recognized as a powerful concept some time ago in both field theory [13, 28] and statistical mechanics.

In some instances it is possible to discover two formulations that describe a physical system. If it is possible to find two such complementary perspectives then the system is said to exhibit *duality*.

There are numerous examples of dual formulations. The most well known example of a duality relation is the identification between the two dimensional Sine-Gordon and massive Thirring models [11, 26]. The duality transformation relates the free bosons of the Sine-Gordon model with the free fermions of the Thirring model. Two features of this example stand out as characteristic of duality relations. Strong coupling in one model is exchanged for weak coupling in the other. Fundamental objects in the weakly coupled theory are identified with solitonic excitations of the dual construction.

Amongst the many categories that exist labelling dualities, one shall consider target space, or T- , duality in particular [23]. In particular, it is one's aim to consider duality within the context of quantum, rather than Classical, field theory. This is a major departure from those discussions that are already located within the literature.

Whilst Lagrangian densities are familiar objects to all individuals working with field theories, it does not follow that it is always a conceptually simple matter to discuss such objects in a Quantum theoretical framework. One should not forget that the Lagrangians from which one draws the majority of our inspiration are Classical objects; to be regarded as limits of the Quantum theory that one aspires to describe.

The framework in which the most progress has been made in describing field theories was introduced by Dirac [28], Feynman [15, 16] and Kacs. This elegant description unifies one's approach to statistical mechanics and field theory. Physical observables pertaining to a set of generic fields ϕ, are derived from what is referred to as a generating functional, or path integral, $\mathcal{Z}$

$$\mathcal{Z} = \int [\mathcal{D}\phi] \; e^{-S[\phi]}$$

This is only a formal statement of how one calculates the generating functional; it is tacitly assumed that one can synthesize a suitable quantum action $S[\phi]$ by some undiscussed means. This is not necessarily a simple task.

2.2 Description of T-Duality

T-Duality can be understood in the context of the two dimensional sigma model construction[8, 10, 7, 30]. Sigma models map one from a base to a target space, with coordinates in the target space identified with scalar excitations from the perspective of the base. The action of the sigma model contains quantities that one identifies as possessing geometrical significance in the target space. In particular, the target space is endowed with a metric and torsion.

T-duality connects target spaces with apparently different characteristics. T-duality is a symmetry which relates physical measurements made within a large space-time radius to those made within space-times of small radii.

2.2.1 Abelian Discussion

There are a number of ways of demonstrating T-duality. Buscher [8, 10, 7] describes a manifold containing a metric, torsion and dilaton field One requires that the manifold contains at least one Abelian symmetry, in this instance. The action of the symmetry is such that the fields are left invariant, up to an exterior derivative in the case of the torsion. One may choose a system of coordinates adapted to the abelian symmetry group, with each Abelian symmetry acting purely upon one coordinate. In order to familarise oneself with the practice, consider the most general dualizable bosonic non-linear sigma-model, defined on a manifold **M** of dimension $(n+1)$ and containing an Abelian isometry:

$$\mathcal{S} = \frac{1}{4\pi\alpha} \int d^2\sigma \left(\sqrt{\gamma}\gamma^{\mu\nu} g_{ab} \partial_\mu x^a \partial_\nu x^b + \epsilon^{\mu\nu} h_{ab} \partial_\mu x^a \partial_\nu x^b + \alpha R^{(2)} \phi(x) \right) \quad (2.1)$$

Where Greek indices are understood to reside on the world-sheet and

Roman characters belong to **M**. The fields present are the dilaton ϕ coupled to the two dimensional Ricci scalar $R^{(2)}$ derived from metric $\gamma_{\mu\nu}$.

The metric g and torsion potential h on **M** are understood to be constructed within a coordinate system adapted to the isometry and will be independant of one of the coordinates, x^0 say. We begin by 'gauging' the isometry, replacing world-sheet partial derivatives acting on x^0 with a covariant operator $\mathcal{D}$, i.e. :

$$\partial_\mu x^0 \longrightarrow \mathcal{D}_\mu x^0$$

$$\mathcal{D}_\mu x^0 = \partial_\mu x^0 + A_\mu \tag{2.2}$$

In addition, we restrict the new field via a Lagrange multiplier term, $\mathcal{S}_L$:

$$\mathcal{S}_L = \frac{1}{8\pi\alpha}\int \epsilon^{\mu\nu}\hat{x}^0 F_{\mu\nu} \tag{2.3}$$

$F_{\mu\nu}$ is the familiar field strength tensor;

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.4}$$

The Lagrange multiplier is $\hat{x}^0$. Constraining the field A by Lagrange multipliers in this fashion allows these fields to return the derivatives of the original model. Integrating over the multipliers returns one to the original state of affairs, integrating over the vector fields constructs the dual model. Explicitly the modified action is written;

$$\mathcal{S} = \frac{1}{4\pi\alpha}\int d^2\sigma \begin{pmatrix} \sqrt{\gamma}\gamma^{\mu\nu}\left(g_{00}A_\mu A_\nu + 2g_{0i}A_\mu\partial_\nu x^i + g_{ij}\partial_\mu x^i\partial_\nu x^j\right) \\ +\epsilon^{\mu\nu}\left(h_{0i}A_\mu\partial_\nu x^i + h_{ij}\partial_\mu x^i\partial_\nu x^j + \hat{x}^0\partial_\mu A_\nu\right) + \alpha\sqrt{\gamma}R^{(2)}\phi \end{pmatrix} \tag{2.5}$$

This with the understanding that summation over i, j does not include x^0. Furthermore, one has used the gauge freedom available to reduce expressions involving covariant derivatives to eliminate explicit reference to $\partial_\mu x^0$. If one does integrate over the vector field, then the following result is arrived at;

$$0 = \sqrt{\gamma}\gamma^{\mu\nu}\left(2g_{00}A_\nu + 2g_{0i}\partial_\nu x^i\right) + \epsilon^{\mu\nu}\left(h_{0i}\partial_\nu x^i - \partial_\nu\hat{x}^0\right) \tag{2.6}$$

Substituting this result back into (2.5) leads to the following to the identification of a dual sigma model;

$$\hat{\mathcal{S}} = \frac{1}{4\pi\alpha}\int d^2\sigma\sqrt{\gamma}\gamma^{\mu\nu}\hat{g}_{ab}\partial_\mu\hat{x}^a\partial_\nu\hat{x}^b + \epsilon^{\mu\nu}\hat{h}_{ab}\partial_\mu\hat{x}^a\partial_\nu\hat{x}^b + \alpha\sqrt{\gamma}R^{(2)}\phi \tag{2.7}$$

The new field set are to be related to the previous model like so;

$$\hat{x}^a = \left\{\hat{x}^0, x^i\right\} \tag{2.8}$$

The dual metric is given by;

$$\hat{g}_{00} = \tfrac{1}{g_{00}} \quad \hat{g}_{0i} = \tfrac{h_{0i}}{g_{00}} \quad \hat{g}_{ij} = g_{ij} - \tfrac{(g_{0i}g_{0j} - h_{0i}h_{0j})}{g_{00}} \tag{2.9}$$

With the torsion potential described by;

$$\hat{h}_{0i} = -\hat{h}_{i0} = \frac{g_{0i}}{g_{00}} \quad \hat{h}_{ij} = -\hat{h}_{ji} = h_{ij} + \frac{(g_{0i}h_{0j} - h_{0i}g_{0j})}{g_{00}} \tag{2.10}$$

The duality transformation is then seen, by (2.9), to act in the fashion discussed earlier, inverting radii.

Having absorbed this observation made at the Classical level, it is intended that one's thesis will expand upon this notion and discuss T- Duality at the Quantum level, using a path integral Batalin–Vilkovisky formulation. In addition, the T - Duality symmetry group will be further expanded beyond the Abelian, and further perturbative Quantum field theoretical results presented. Significantly the field theory that will be considered will contain Bosonic and Fermionic sectors but will not be augmented by any amount of Supersymmetry; which further sets this work apart from that of others.

2.3 Non-Abelian Discussion

The act of implementing non-Abelian duality in two dimensional sigma models results unavoidably in an additional reducible symmetry. The Batalin–Vilkovisky formalism is employed to handle this new symmetry. Valuable lessons are learnt here with respect to non–Abelian duality. We emphasise, in particular, the effects of the ghost sector corresponding to this symmetry on non–Abelian duality.

Duality transformations have understandably brought about a surge of new interests in string theory. The importance of these transformations lies in their ability to connect seemingly different string backgrounds. This might shed some light on one of the longstanding problems in superstring theory, namely the non–uniqueness of the low energy physics expected from this theory. As it is well–known, the phenomenology predicted by superstring theory depends upon the way the extra six dimensions are compactified. Hence, if the space on which one carries out the compactification are related to each other by duality transformations, then their corresponding low energy physics should also be related. This is also the idea behind mirror symmetry [33] which might well be another manifestation of duality transformations [2, 20].

The duality transformation that concerns us here is the so–called T–duality [9]. These can be understood as canonical transformations on the phase space of a sigma model [25]. There is, however, a well defined procedure at the level of the Lagrangian which allows the construction of dual theories [31]. It consists in gauging an isometry group of a non–linear sigma model and at the same time restricting, by means of a Lagrange multiplier, the gauge field to be pure gauge. The integration over the gauge fields (without a kinetic term) leads to the dual theory.

The duality transformation is termed Abelian or non–Abelian depending on whether the isometry group is Abelian or not. Abelian duality has proved to be of crucial importance in string [1] and membrane [34, 17] theories. On the other hand, its non–Abelian counterpart has not yet been fully exploited [17]. This is because non–Abelian duality is hampered by conceptual problems. In particular, performing the transformation twice does not return the original model [21, 32], and as such the term duality can only be understood to refer to the Lagrange term that connects the original and the dual model. One of the issues in non–Abelian duality is the appearance, as explained below, of a new local symmetry in the action [27].

As stated, it is our aim to deal with the quantisation of such theories and hence it will be necessary to deal with this new symmetry. The understanding of this symmetry is crucial to any possible exploitation (and probably to the understanding of the other issues) of non–Abelian duality. We outline below the manifestation of this symmetry. As this symmetry is reducible we will employ the previously discussed Batalin–Vilkovisky method in order to arrive at a sensibly defined path integral [5]. Recalling previous discussion it should then be obvious that the new symmetry will have an impact upon the ghost sector of a properly quantized action.

Suppose that one has a two-dimensional theory described by an action $S(\varphi)$ which is invariant under some global symmetry for the generic fields φ. Let us also assume that the generators of this symmetry form a closed Lie algebra $\mathcal{G}$. Furthermore it is also assumed that one can gauge these symmetry in an anomaly-free way. It is then straightforward to find the dual of this theory at the classical level. This is found by considering the gauge invariant action [31]

$$\begin{aligned} I(\varphi, A, \Lambda) &= S(\varphi, A) + \int \mathrm{d}^2 x \mathrm{tr}(\Lambda F) \\ F &\equiv \epsilon^{\mu\nu} F_{\mu\nu} \ . \end{aligned} \tag{2.11}$$

Here $S(\varphi, A)$ is the gauged version of $S(\varphi)$. The gauge field A_μ takes value in the Lie algebra $\mathcal{G}$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu\ ,\ A_\nu]$ is the corresponding field strength. The trace tr is the invariant bi-linear form of the Lie algebra $\mathcal{G}$ such that $\mathrm{tr}(XY) = \eta_{ab} X^a Y^b$.

The new field Λ is a Lagrange multiplier which, at the classical level, imposes the constraints $F_{\mu\nu} = 0$. This is then solved by $A_\mu = g^{-1}\partial_\mu g$, where g is an element in the Lie group corresponding to $\mathcal{G}$. Recall now that A_μ and

Λ transform as

$$\begin{aligned} A_\mu &\longrightarrow hA_\mu h^{-1} - \partial_\mu h h^{-1} \\ \Lambda &\longrightarrow h\Lambda h^{-1} \end{aligned} \tag{2.12}$$

where h is the Lie algebra valued gauge function. Of course, the transformation of the generic field φ is also governed by this same function. Using this gauge freedom, we can choose a gauge such that $g = 1$. Hence, in this gauge, the gauge field vanishes and the action $I(\varphi, A, \Lambda)$ is classically equivalent to the original action $S(\varphi)$.

At the classical level, the dual theory is obtained by keeping the Lagrange multiplier and eliminating instead the gauge fields by their equations of motion. We are supposing that the gauge fields appear quadratically at most and without derivatives in the gauged action $S(\varphi, A)$. To get the right degrees of freedom in the dual theory a gauge fixing condition must be chosen.

The issues that concerns us in this paper are those necessary to implement the duality transformation at the quantum level. This is a well-known procedure if the Lie algebra $\mathcal{G}$ is Abelian. However, if $\mathcal{G}$ is non-Abelian then the matter must be considered carefully. This is mainly because the action (2.11) now has another local symmetry which must be taken into account in the path integral. Due to the properties of the trace, the gauge invariant action I is also invariant under

$$\begin{aligned} \Lambda &\longrightarrow \Lambda + [\xi\ ,\ F] \\ A_\mu &\longrightarrow A_\mu\ ,\quad \varphi \longrightarrow \varphi\ , \end{aligned} \tag{2.13}$$

where ξ is the new local gauge function corresponding to this extra symmetry. It should be noted that if the gauge function ξ takes value in the centre (or maximal ideal) of the Lie algebra $\mathcal{G}$, then the transformation of Λ vanishes; thus the new symmetry is reducible (i.e., not all the components of Λ enter the transformation). This fact will have consequences, as we will see, on the Faddeev-Popov ghosts required to gauge fix this new symmetry. In the rest of the paper and for simplicity, we will consider only the case when $\mathcal{G}$ is semi-simple (that is, no maximal ideals are present in $\mathcal{G}$); hence the new transformation is reducible only when ξ is proportional to F. In this case in the formalism of Batalin-Vilkovosky, which suitably deals with reducible symmetries, our symmetry is first-stage reducible. We will apply this formalism to quantise the extra symmetry.

To obtain the dual theory, we have to perform the path integral over the ϕ, A_μ and Λ in the action (2.11). There are, therefore, two symmetries that one needs to gauge fix. The first one is the usual local gauge transformation in (2.12) and the second is the extra symmetry in (2.13). Since the two symmetries are completely independent and different in nature, it is therefore essential to keep one symmetry intact if the other is being fixed.

We choose first to fix the extra symmetry in (2.13) keeping the gauge symmetry in (2.12) intact. This is easily achieved if we choose a gauge fixing condition for the symmetry (2.13) which transforms covariantly with respect to the local gauge transformation (2.12).

2.4 Construction of Batalin–Vilkovisky Actions

The Batalin-Vilkovisky formalism manages theories with reducible symmetries. The Faddeev-Popov procedure is, in general, not sufficient for such theories. A simplistic use of the Becchi-Rouet-Stora-Tyutin (BRST) quantisation is also inappropriate in this case. We will expand the formal discussion of the earlier chapter to provide some useful results for later use.

Let $\mathcal{S}$ be a classical action for some generic fields ϕ^i, $i = 1, \ldots, n$ (fermionic or bosonic in nature). The equations of motion of this gauge action are assumed to possess at least one solution ϕ_0. Let m_0 be the number of gauge parameters (fermionic and bosonic) of this gauge invariant action; hence m_0

Noether identities hold

$$\frac{\partial_r \mathcal{S}}{\partial \phi^i} R^i_{\alpha_0} = 0 \ , \ \ \alpha_0 = 1, \ldots, m_0 \ . \tag{2.14}$$

$R^i_{\alpha_0}(\phi)$ are the generators of the gauge transformations and are supposed to be regular functionals of the fields ϕ^j. These transformations are written as $\delta\phi^i = R^i_{\alpha_0}\delta\theta^{\alpha_0}$, where θ^{α_0} are the gauge parameters. We will denote by ∂_r and ∂_l the right and left functional derivatives, respectively. We also use the de Witt convention that summation over repeated indices includes an integration over spacetime.

The gauge symmetry is then reducible if there exists (at least on-shell) a set of m_1 zero-eigenvalue eigenvectors $Z^{\alpha_0}_{(1)\alpha_1}$ such that

$$R^i_{\alpha_0} Z^{\alpha_0}_{(1)\alpha_1}|_{\phi_0} = 0 \ , \ \ \alpha_1 = 1, \ldots, m_1 \ . \tag{2.15}$$

The symmetry is said to be first-stage reducible if the null vectors $Z^{\alpha_0}_{(1)\alpha_1}$ are independent. We will consider here only symmetries such as these.

The fields ϕ^i are part of a larger set of fields Φ^A, $A = 1, \ldots, N$ (the rest of the fields being the different ghosts and some Lagrange multipliers necessary for gauge fixing). The Batalin-Vilkovisky formalism associates with each field Φ^A an anti-field Φ^*_A possessing opposite statistics. These anti-fields are just tools for constructing a BRST invariant action. If we denote by $\epsilon\left(\Phi^A\right) \equiv \epsilon_A$

the statistics of the field Φ^A, then the fermion number of the anti-field is $\epsilon\left(\Phi_A^*\right) = \epsilon_A + 1\,(\text{mod}2)$.

It is then guaranteed that there exists a BRST invariant quantum action $\mathbf{S}\left(\Phi, \Phi^*\right)$ which satisfies the two requirements [5]

$$\mathbf{S}\left(\Phi, \Phi^*\right)|_{\Phi^*=0} = \mathcal{S}\left(\phi\right)$$
$$\left(\mathbf{S}, \mathbf{S}\right) \equiv \frac{\partial_r \mathbf{S}}{\partial \Phi^A}\frac{\partial_l \mathbf{S}}{\partial \Phi_A^*} - \frac{\partial_r \mathbf{S}}{\partial \Phi_A^*}\frac{\partial_r \mathbf{S}}{\partial \Phi^A} = 0 \ , \tag{2.16}$$

The first expression demands that one can retrieve thet correct classical field theory. The second equation is what is known as the master equation and its solution will be our main concern.

The minimum number of fields contained within a first-stage reducible theory is the number of fields in $\Phi_{\text{min}}^A = \left\{\phi^i, C_{(0)}^{\alpha_0}, C_{(1)}^{\alpha_1}\right\}$ plus Φ_{min}^*. The fields $C_{(0)}^{\alpha_0}$ are assigned a ghost number equal to 1 and are the usual Faddeev-Popov ghosts, whilst $C_{(1)}^{\alpha_1}$ are the ghosts-for-ghosts fields and have ghost number equal to 2. Of course, the field ϕ^i has zero ghost number. The statistics of a field, or anti-field, is the sum of the statistics of its index and the absolute value of its ghost number. The first stage in constructing a BRST invariant theory is to associate an action $\mathbf{S}\left(\Phi_{\text{min}}, \Phi_{\text{min}}^*\right)$ with this minimum set of fields. This action can be expanded in powers of the anti-fields, where each term in the expansion has zero ghost number. The leading terms in this expansion are of the form [5]

$$\begin{aligned}
\mathbf{S}\left(\Phi_{\min}, \Phi^*_{\min}\right) &= \mathcal{S} + \phi^*_i R^i_{\alpha_0} C^{\alpha_0}_{(0)} + C^*_{(0)\alpha_0}\left[Z^{\alpha_0}_{(1)\alpha_1} C^{\alpha_1}_{(1)} + T^{\alpha_0}_{\beta_0\gamma_0} C^{\gamma_0}_{(0)} C^{\beta_0}_{(0)}\right] \\
&+ C^*_{(1)\alpha_1}\left[A^{\alpha_1}_{\beta_1\alpha_0} C^{\alpha_0}_{(0)} C^{\beta_1}_{(1)} + F^{\alpha_1}_{\alpha_0\beta_0\gamma_0} C^{\gamma_0}_{(0)} C^{\beta_0}_{(0)} C^{\alpha_0}_{(0)}\right] \\
&+ \phi^*_i \phi^*_j \left[B^{ji}_{\alpha_1} C^{\alpha_1}_{(1)} + E^{ji}_{\alpha_0\beta_0} C^{\beta_0}_{(0)} C^{\alpha_0}_{(0)}\right] \\
&+ 2C^*_{(0)\alpha_0}\phi^*_i \left[G^{i\alpha_0}_{\alpha_1\beta_0} C^{\beta_0}_{(0)} C^{\alpha_1}_{(1)} + D^{i\alpha_0}_{\beta_0\gamma_0\delta_0} C^{\delta_0}_{(0)} C^{\gamma_0}_{(0)} C^{\beta_0}_{(0)}\right] \\
&+ \ldots \qquad (2.17)
\end{aligned}$$

There are no more terms in this expansion for the usual first-stage reducible theories.

The master equation then imposes the following conditions on the different coefficients in the above expansion

$$\frac{\partial_r \mathcal{S}}{\partial \phi^i} R^i_{\alpha_0} C^{\alpha_0}_{(0)} = 0 \ , \qquad (2.18)$$

$$R^i_{\alpha_0} Z^{\alpha_0}_{(1)\beta_1} C^{\beta_1}_{(1)} - 2\frac{\partial_r \mathcal{S}}{\partial \phi^j} B^{ji}_{\beta_1} C^{\beta_1}_{(1)} (-1)^{\epsilon_i} = 0 \ , \qquad (2.19)$$

$$\frac{\partial_r R^i_{\alpha_0} C^{\alpha_0}_{(0)}}{\partial \phi^j} R^j_{\beta_0} C^{\beta_0}_{(0)} + R^i_{\alpha_0} T^{\alpha_0}_{\beta_0\gamma_0} C^{\gamma_0}_{(0)} C^{\beta_0}_{(0)} - 2\frac{\partial_r \mathcal{S}}{\partial \phi^j} E^{ji}_{\beta_0\gamma_0} C^{\gamma_0}_{(0)} C^{\beta_0}_{(0)} (-1)^{\epsilon_i} = 0 \qquad (2,20)$$

$$\begin{aligned}
&\frac{\partial_r T^{\alpha_0}_{\beta_0\gamma_0} C^{\gamma_0}_{(0)} C^{\beta_0}_{(0)}}{\partial \phi^j} R^j_{\delta_0} C^{\delta_0}_{(0)} + 2T^{\alpha_0}_{\beta_0\gamma_0} C^{\gamma_0}_{(0)} T^{\beta_0}_{\delta_0\mu_0} C^{\mu_0}_{(1)} C^{\delta_0}_{(0)} + Z^{\alpha_0}_{(1)\beta_1} F^{\beta_1}_{\beta_0\gamma_0\delta_0} C^{\delta_0}_{(0)} C^{\gamma_0}_{(0)} C^{\beta_0}_{(0)} \\
+ \ & 2\frac{\partial_r \mathcal{S}}{\partial \phi^j} D^{j\alpha_0}_{\beta_0\gamma_0\delta_0} C^{\delta_0}_{(0)} C^{\gamma_0}_{(0)} C^{\beta_0}_{(0)} (-1)^{\epsilon_{\alpha_0}} = 0 \ , \qquad (2.21)
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial_r Z^{\alpha_0}_{(1)\beta_1} C^{\beta_1}_{(1)}}{\partial \phi^j} R^j_{\delta_0} C^{\delta_0}_{(0)} + 2T^{\alpha_0}_{\beta_0\gamma_0} C^{\gamma_0}_{(0)} Z^{\beta_0}_{(1)\delta_1} C^{\delta_1}_{(1)} + Z^{\alpha_0}_{(1)\beta_1} A^{\beta_1}_{\gamma_1\beta_0} C^{\beta_0}_{(0)} C^{\gamma_1}_{(1)} \\
+ \ & 2\frac{\partial_r \mathcal{S}}{\partial \phi^j} G^{j\alpha_0}_{\gamma_1\beta_0} C^{\beta_0}_{(0)} C^{\gamma_1}_{(1)} (-1)^{\epsilon_{\alpha_0}} = 0 \ . \qquad (2.22)
\end{aligned}$$

Here $\epsilon_i = \epsilon\left(\phi^i\right)$, whilst ϵ_{α_0} is the Grassmann parity of the gauge parameter.

The minimum sets of fields Φ_{min} and of anti-fields Φ^*_{min} can be enlarged to include more fields and their corresponding anti-fields. The master equation implies that, if $\mathbf{S}\left(\Phi_{\text{min}}, \Phi^*_{\text{min}}\right)$ is a solution, then

$$\mathbf{S}\left(\Phi, \Phi^*\right) = \mathbf{S}\left(\Phi_{\text{min}}, \Phi^*_{\text{min}}\right) + \bar{C}^{*\alpha_0}_{(0)}\Pi_{(0)\alpha_0} + \bar{C}^{*\beta_1}_{(1)}\Pi_{(1)\beta_1} + C'^{*}_{(1)\beta_1}\Pi'^{\beta_1}_{(1)} \tag{2.23}$$

is also a solution. The new fields may be employed in gauge fixing as we will see shortly, and are assigned the ghost numbers

$$\begin{aligned} &\text{gh}\left(\Pi_{(0)\alpha_0}\right) = \text{gh}\left(C'^{\alpha_1}_{(1)}\right) = 0 \\ &\text{gh}\left(C^{\alpha_0}_{(0)}\right) = -\text{gh}\left(\bar{C}_{(0)\alpha_0}\right) = -\text{gh}\left(\Pi_{(1)\alpha_1}\right) = \text{gh}\left(\Pi'^{\alpha_1}_{(1)}\right) = 1 \\ &\text{gh}\left(C^{\alpha_1}_{(1)}\right) = -\text{gh}\left(\bar{C}_{(1)\alpha_1}\right) = 2 \ . \end{aligned} \tag{2.24}$$

The fields with a star denote their corresponding anti-fields.

The anti-fields are not physical fields and should be eliminated from the theory. This is achieved through the introduction of what is known as the gauge-fixing fermion $\Psi\left(\Phi\right)$. This is a functional of odd statistics and having a ghost number equal to -1. The anti-fields in the full action (2.23) are then replaced by

$$\Phi^*_A = \frac{\partial\Psi}{\partial\Phi^A} \ . \tag{2.25}$$

The functional Ψ has to satisfy certain conditions in order to make all the ghost propagators invertible. The simplest choice of functional Ψ for first-stage reducible theories takes the form

$$\Psi(\Phi) = \bar{C}_{(0)\alpha_0}\chi^{\alpha_0} + \bar{C}_{(1)\beta_1}\Omega^{\beta_1}_{\alpha_0}C^{\alpha_0}_{(0)} + \bar{C}_{(0)\alpha_0}\Sigma^{\alpha_0}_{\beta_1}C'^{\beta_1}_{(1)} \ , \tag{2.26}$$

where $\chi^{\alpha_0}(\phi^i)$ is an admissible gauge condition for the classical fields ϕ^i. The matrices $\Omega^{\beta_1}_{\alpha_0}$ and $\Sigma^{\alpha_0}_{\beta_1}$ are some suitable maximal rank matrices which remove the degeneracy of the kinetic term of the ghosts $C^{\alpha_0}_{(0)}$ and $\bar{C}_{(0)\alpha_0}$.

Note that the integration in the path integral over the Π's of (2.23) leads to three sets of gauge conditions. These conditions are in the form of δ-functions. To obtain the usual quadratic gauge-fixing Lagrangian (the 't Hooft method), a linear term in the Π's is added to Ψ. In the simplest cases the following gauge fermion leads to to a quadratic gauge-fixing Lagrangian

$$\widetilde{\Psi} = \Psi + \frac{1}{2}\left[\bar{C}_{(0)\alpha_0}\Gamma^{\alpha_0\beta_0}\Pi_{(0)\beta_0} + \bar{C}_{(1)\alpha_1}\Theta^{\alpha_1}_{\beta_1}\Pi'^{\beta_1}_{(1)} - (-1)^{\epsilon_{\alpha_1}}\Pi_{(1)\alpha_1}\Theta^{\alpha_1}_{\beta_1}C'^{\beta_1}_{(1)}\right] \ , \tag{2.27}$$

where Ψ is given in (2.26) and $\Gamma^{\alpha_0\beta_0}$ and $\Theta^{\alpha_1}_{\beta_1}$ are some invertible matrices assumed to contain no derivatives. The integration over the Π's will give Gaussian averages of gauge conditions instead of δ-functions. This issue will be of considerable relevance when we consider non-Abelian duality in sigma models.

To end this brief review of the Batalin-Vilkovisky formalism, we provide a means to determine the BRST transformations of the different fields. A generic quantity $P(\Phi, \Phi^*)$ having statistics ϵ_P, has a BRST transformation given by

$$\delta P = (-1)^{\epsilon_P} (P, \mathbf{S}) \quad . \tag{2.28}$$

This transformation is nilpotent ($\delta^2 P = 0$) by virtue of the master equation satisfied by $\mathbf{S}$. This definition of the BRST transformation guarantees that $\mathbf{S}$ is, by construction, BRST invariant. The factor $(-1)^{\epsilon_P}$ has been chosen to enforce graded Leibniz rules for δ.

Upon elimination of the anti-fields through (2.25), the action $\mathbf{S}\left(\Phi, \Phi^* = \frac{\partial \Psi}{\partial \Phi}\right)$ is still BRST invariant. In general, however, the nilpotency of the BRST transformation holds only when the equations of motion of the quantum action $\mathbf{S}\left(\Phi, \Phi^* = \frac{\partial \Psi}{\partial \Phi}\right)$ are used.

We are now at a stage where we can apply the Batalin-Vilkovisky formalism to theories of the form given in (2.11).

2.5 Application of the Batalin-Vilkovisky Formalism

In order to become familiar with the general ideas of the anti-field formalism, let us start by quantising the action (2.11). We will deal with the symmetry (2.13) leaving the usual gauge symmetry (2.12) untouched throughout the procedure. This may be regarded as a preliminary exercise before one tackles more complicated cases.

The variation of this action with respect to Λ leads to the equation of motion

$$F^a \equiv \epsilon^{\mu\nu} F^a_{\mu\nu} = 0 \ , \tag{2.29}$$

where we have written $A_\mu = A^a_\mu T_a$, $F_{\mu\nu} = F^a_{\mu\nu} T_a$ and $\Lambda = \Lambda^a T_a$. The T_a are the generators of the Lie algebra $\mathcal{G}$ such that $[T_a \ , \ T_b] = f^c_{ab} T_c$.

The set of classical fields is $\phi^i = \left\{\varphi, A^a_\mu, \Lambda^a\right\}$. The transformation we are dealing with is Abelian and closes off-shell; hence the structure constants $T^{\alpha_0}_{\beta_0\gamma_0}$ vanish. Let us now investigate which of the coefficients of the expansion (2.17) survive in this case.

The tranformation (2.13) leads to $R^i_{\alpha_0}$ which are nonzero only when the index i refers to the field Λ^a

$$R^{a(x)}_{b(y)} = f^a_{bc} F^c(x)\, \delta(x-y) \quad , \tag{2.30}$$

where the index $i = \{a, x\}$ and $\alpha_0 = \{b, y\}$. Due to the anti-symmetry of the structure constants f^a_{bc}, the null vectors of $R^i_{\alpha_0}$ are given by

$$Z^{b(y)}_{(1)(z)} = F^b(y)\, \delta(y-z) \quad , \tag{2.31}$$

where the index $\beta_1 = \{z\}$. It is clear that these null vectors are linearly independent off-shell.; hence this theory is said to be first–stage reducible. Since $T^{\alpha_0}_{\beta_0\gamma_0}$, $R^i_{\alpha_0}$ and $Z^{\alpha_0}_{(1)\beta_1}$ do not depend on the field Λ^a, a solution to the master equation is obtained by setting all the other coefficients in (2.17) to zero.

Hence, keeping the Batalin-Vilkovisky notation, we are left with

$$\mathbf{S}\left(\Phi_{\text{min}}, \Phi^*_{\text{min}}\right) = \mathcal{S}(\Phi) + \phi^*_i R^i_{\alpha_0} C^{\alpha_0}_{(0)} + C^*_{(0)\alpha_0} Z^{\alpha_0}_{(1)\alpha_1} C^{\alpha_1}_{(1)} \quad . \tag{2.32}$$

The full quantum action is then written in the suggestive form

$$\begin{aligned} \mathbf{S}\left(\Phi, \Phi^*\right) &= \mathcal{S}(\phi) + \mathcal{S}_{\text{ghost}} + \mathcal{S}_{\text{gauge}} \\ \mathcal{S}_{\text{ghost}} &\equiv \frac{\partial \Psi}{\partial \Lambda^i} R^i_{\alpha_0} C^{\alpha_0}_{(0)} + \frac{\partial \Psi}{\partial C^{\alpha_0}_{(0)}} Z^{\alpha_0}_{(1)\alpha_1} C^{\alpha_1}_{(1)} \\ \mathcal{S}_{\text{gauge}} &\equiv \frac{\partial \Psi}{\partial \bar{C}_{(0)\alpha_0}} \Pi_{(0)\alpha_0} + \frac{\partial \Psi'}{\partial \bar{C}_{(1)\beta_1}} \Pi_{(1)\beta_1} + \frac{\partial \Psi}{\partial C'^{\beta_1}_{(1)}} \Pi'^{\beta_1}_{(1)} \quad . \end{aligned} \tag{2.33}$$

The anti-fields have been eliminated using the gauge-fixing fermion Ψ.

The next step in determining the full quantum action is to construct the gauge-fixing fermion Ψ. As mentioned earlier, we would like to gauge fix the transformation (2.13) without breaking the usual gauge symmetry in (2.12). This can be achieved by choosing a gauge fixing condition which transforms covariantly under (2.12). A gauge fixing condition which has this property is given by

$$\chi^a = f^a_{bc}\Lambda^b F^c \quad . \tag{2.34}$$

This is a set of [dim$\mathcal{G}$ − rank$\mathcal{G}$] equations which are compatible with the transformation (2.13). The gauge fermion then takes the form

$$\Psi = \int \mathrm{d}^2x \left[\bar{C}_{(0)a} f^a_{bc}\Lambda^b F^c + \bar{C}_{(1)}\Omega_a C^a_{(0)} + \bar{C}_{(0)a}\Sigma^a C_{(1)}\right] \quad . \tag{2.35}$$

Under the gauge transformations (2.12), the ghost fields are obviously required to transform in the adjoint representation of $\mathcal{G}$. The matrices $\Omega^{\beta_1}_{\alpha_0}$ and $\Sigma^{\alpha_0}_{\beta_1}$ are chosen such that the gauge covariance (2.12) is maintained. These matrices are also assumed to be independent of the field Λ.

The ghost action is therefore given by

$$\mathcal{S}_{\text{ghost}} = \int \mathrm{d}^2x \left[\bar{C}_{(0)a} f^a_{bc} f^b_{de} F^c F^e C^d_{(0)} + \bar{C}_{(1)}\Omega_a F^a C_{(1)} + \bar{C}_{(0)\,c} f^c_{ad} F^d \bar{C}_{(0)\,e} f^e_{bg} F^g \eta^{bh} f^a_{hi} F^i\right] \tag{2.36}$$

It is clear that $\mathcal{S}_{\text{ghost}}$ is invariant under

$$\begin{aligned} C^a_{(0)} &\longrightarrow C^a_{(0)} + \alpha F^a \\ \bar{C}_{(0)_a} &\longrightarrow C_{(0)_a} + \bar{\alpha}\eta_{ab}F^b \end{aligned} \qquad (2.37)$$

where α and $\bar{\alpha}$ are two local Grassmanian parameters. In this sense the ghost action is degenerate (that is, the gauge fixing did not remove all the symmetries of our theory). It is the role of the gauge fixing Lagrangian to remove all the degeneracies.

The integration over the II's in $\mathcal{S}_{\text{gauge}}$ leads to three conditions

$$f^a_{bc}\Lambda^b F^c + \Sigma^a C'_{(1)} = 0 \ \ , \ \ \Omega_a C^a_{(0)} = 0 \quad \bar{C}_a \Sigma = 0 \ . \qquad (2.38)$$

The first condition fixes the gauge transformation in (2.13) and eliminates $C'_{(1)}$. Multiplication by $\eta_{ad}F^d$ of the first equation yields $\eta_{ad}F^d\Sigma^a C'_{(1)} = 0$. This is sufficient to eliminate $C'_{(1)}$ provided that $\eta_{ad}F^d\Sigma^a$ does not vanish identically. The remaining two conditions fix the ghost transformation mentioned in (2.37). We found that the two matrices

$$\Omega_a = \eta_{ab}F^b \ \ , \ \ \Sigma^a = F^a \qquad (2.39)$$

satisfy all the above mentioned requirements.

In this way we have constructed a BRST invariant quantum theory. If one wishes to eliminate the anti-fields using the gauge fermion Ψ then the

BRST transformations are given by

$$\delta_\Psi \Phi^A = (-1)^{\epsilon_A} \frac{\partial_l \mathcal{S}}{\partial \Psi^*_A}\bigg|_{\Phi^* = \frac{\partial \Psi}{\partial \Phi}} \tag{2.40}$$

It is then a simple matter to write down the BRST transformations for the fields

$$\begin{aligned}
\delta_\Psi \Lambda^a &= f^a_{bc} F^c C^b_{(0)} \\
\delta_\Psi C^a_{(0)} &= -F^a C_{(1)} \\
\delta_\Psi \bar{C}_{(0)\,a} &= -\Pi_{(0)\,a} \\
\delta_\Psi \bar{C}_{(1)} &= \Pi_{(1)} \\
\delta_\Psi C^{'}_{(1)} &= \delta_\Psi \Pi_{(0)\,a} = \delta_\Psi \Pi_{(1)} = \delta_\Psi \Pi^{'}_{(1)} = 0\ .
\end{aligned} \tag{2.41}$$

It then follows that the BRST transformations are nilpotent.

Finally, we would like to investigate a point which is relevant to non-Abelian duality. This concerns the addition of linear terms in the Π's to the gauge fermion Ψ. In this case the new gauge fermion takes the form

$$\widetilde{\Psi} = \Psi + \frac{1}{2} \int \mathrm{d}^2 x \left[\bar{C}_{(0)a} \Gamma^{ab} \Pi_{(0)b} + \bar{C}^{'}_{(1)} \Theta \Pi^{'}_{(1)} - \Pi_{(1)} \Theta C_{(1)} \right] \ , \tag{2.42}$$

where Ψ is the gauge fermion given in (2.35). In order to maintain covariance under (2.12), a simple choice for the two matrices $\Gamma^{\alpha_0 \beta_0}$ and $\Theta^{\alpha_1}_{\beta_1}$ is

$$\Gamma^{ab} = n \eta^{ab} \quad , \quad \Theta = m \ , \tag{2.43}$$

where η^{ab} is the inverse of η_{ab} and n and m are two constant parameters.

The integration over the Π's results in the quadratic gauge-breaking Lagrangian

$$\begin{aligned} \mathcal{S}_{\text{gauge}} &= \int \mathrm{d}^2x \left[-\frac{1}{2n} \left(f^a_{bc} \Lambda^b F^c \right) \eta_{ad} \left(f^d_{rs} \Lambda^r F^s \right) - \frac{1}{m} \bar{C}_{(0)a} F^a \eta_{bc} F^c C^b_{(0)} \right. \\ &- \left. \frac{1}{2n} C'_{(1)} F^a \eta_{ab} F^b C'_{(1)} \right] \quad . \end{aligned} \tag{2.44}$$

This is the usual Gaussian gauge fixing Lagrangian. The first term removes the gauge freedom of the original action while the second term removes the degeneracy of the ghost Lagrangian (2.36). The last term is required for BRST invariance and is a characteristic of the anti–field formalism.

This completes the quantisation of the new symmetry (2.13). Let us now list the consequences of our work on non–Abelian duality.

2.6 Conclusions

We have shown in here that the procedure by which non–Abelian duality is implemented in sigma models naturally leads to the prescence of a reducible symmetry. We have dealt with this symmetry using the Batalin–Vilkovisky formalism. This unavoidably introduces new fields into the theory. Some of these fields are bosonic in nature $\left(C_{(1)},\ \bar{C}_{(1)} \text{ and } C'_{(1)} \right)$ and could play a rôle similar to that of the Lagrange multiplier Λ. Recall also that as far as the usual gauge transformations (2.12) is concerned, these new fields transform in the adjoint representation of the gauge group $\mathcal{G}$. This fact strengthens the above statement about these fields.

In order to proceed further in the determination of the dual theory one must carry out an integration over the gauge fields in the full action (2.33). However, this is no more straightforward as this action includes terms quadratic in the field strength of the gauge fields. This fact is worsened if we consider the gauge fermion $\tilde{\Psi}$ instead of Ψ. The integration over the gauge fields would lead to a dual theory containing non–local terms. The latter can no longer be interpreted as a sigma model corresponding to a string background. This issue, in fact, is particularly specific to our choice of gauge fixing condition which contains the field strength. It is possible to find a gauge breaking term which does not contain any gauge fields. These types of gauge are discussed later and involve only the sigma model fields φ and the Lagrange multiplier Λ.

In this paper we have started by quantising the symmetry (2.13) keeping manifest the usual gauge symmetry (2.12). It is then natural to address the following question: could we have started the other way around? That is, to quantise first the symmetry in (2.12). This is an important issue. Let us simply mention that there are two ways in which to gauge fix the symmetry (2.12). The first is, for instance, to choose a standard gauge of the Landau type $\partial^\mu A^a_\mu = 0$. This could be solved by setting $A^a_\mu = \partial_\mu \lambda^a$ and leads to a non–vanishing field strength. Therefore, this type of gauge fixing does not break the new symmetry in (2.13). The second type of gauge fixing is a non–standard one and involves setting some fields (φ and Λ) to zero. In general, however, this gauge automatically breaks the new symmetry in (2.13). This is the type of gauge fixing which has been considered in the literature on non–Abelian duality.

Chapter 3

Perturbative Investigation

In this chapter we will investigate the duality procedure in the context of perturbative quantum field theory. For reasons of clarity and time we will concern ourselves with two simple models which will contain both a bosonic and a fermionic sector. Both theories will of course be constructed to be covariant on the two dimensional base space so as to maintain the geometrical interpretation of the duality process which was previously discussed. Having chosen our candidate theories we will then construct dual versions of these theories. Using the Batalin–Vilkovisky technique we shall isolate interesting quantum features arising from the duality transformation.

3.1 Free Field Theory

The simplest theory containing bosons and fermions that one can imagine is a free theory, with no interactions between bosons or fermions and indeed excluding any iteraction within either sector. For later convenience, we will compose the bosonic fields of the free theory into a single complex valued scalar,

$$\mathcal{S}_{\text{free}} = \int d^2x \frac{1}{2} |\partial\phi|^2 + \bar{\psi} i \not{\partial} \psi \tag{3.1}$$

Note that with this casting (3.1) the bosonic coordinate is dimensionless, whilst the fermionic field has the dimension of $[\text{length}]^{-1/2}$ in the base space. Note also that whilst we will only consider two scalar fields this action should always be regarded as a sub-sector of a larger theory.

Free theories are entirely soluable; construction of the integration measure is trivial in the absence of local gauge symmetries and integration of the generating function is to act upon a set of Gaussian fields. Given this one can write a generating function in terms of minimally coupled source terms and conventional field propagators and compute any process without difficulty. Furthermore the lack of interaction means that the theory has no interesting behaviour with changing energy scales and does not require renormalisation; without loop diagrams there is no opportunity to introduce infinities by naieve action of functional derivatives. Given the apparently trivial form of the candidate field theory one might question why it would be selected for investigation. The answer is entriely because of its simple structure, in constructing the dual to such a theory one might hope to isloate features which are in some sense 'pure' and so not inhibit any discussion of this duality procedure in the quantum context. Having gained these first principles one could then be more ambititious and introduce coupling between and within bosonic and fermionic sectors.

The choice of coordinates implies a change in the manifestation of the symmetry which we act upon to construct the dual representation. Since we have effectively chosen a set of Cartesian, rather than polar, coordinates (ϕ) in the target space rather than polar, the action of rotation now manifests itself in the following fashion.

$$\phi \longrightarrow e^{i\alpha}\phi \tag{3.2}$$

Having observed this distinction, we can now implement Abelian duality by adding a gauge like field, A_μ, and a Lagrange multiplier, λ, to ensure equivalence of the new and old (3.1) action

$$\tilde{S}_{\text{free}} = \int d^2x \, |(\partial - iA)\phi|^2 + \bar{\psi}(i \not{\partial} + \not{A})\psi + \lambda F \tag{3.3}$$

The gauge like field, as will be familiar from the study of electromagnetism responds to the U(1) transformation (3.2) by acting as a connection to the partial derivative, minimally coupling the matter fields to the gauge group.

$$A_\mu \longrightarrow A_\mu - i\partial_\mu \alpha \tag{3.4}$$

Note that from the outset, we have chosen for the fermionic sector to transform under the action of this symmetry acting with the prejudice that the fermionic coordinates are partners in this duality. Whilst this is hardly the weakest assumption one could make in constructing a dual theory, it is perhaps obvious that there would be little point in considering such fields if we were not to adopt this position. In the abscence of any interaction these fermions would enjoy the same role on either side of the duality procedure and would add no further complications to the discussion and as such one could well have ignored such fields and selected a purely bosonic theory as a starting position. The fermions are chosen to transform in the same representation of the rotation group as the scalar fields and have a transformation equivalent to (3.2)

$$\psi \longrightarrow e^{i\alpha}\psi \tag{3.5}$$

Having so constructed the action (3.3) with the intention of integrating over the connecting field A in the path integral it would now be appropriate to consider gauge fixing issues in order for the operation to be well defined. Since we have constrained ourselves to consideration of Abelian T–duality this theory is irreducible in the Batalin–Vilkovisky sense and as such will have a simpler ghost sector than non–Abelian theories (2.32). The enlarged quantum action, prior to elimination of anti-fields takes on the simpler form

$$\mathcal{S}(\Phi, \Phi^*) = \tilde{\mathcal{S}}(\phi, \psi)_{\text{free}} + \phi^* \eta^\dagger + \phi^{\dagger *}(-1)\eta + \psi^* \bar{\alpha} + \bar{\psi}^*(-1)\alpha \tag{3.6}$$

The simple form of the quantum action (3.6) requires only two ghost fields, η nd α with the Grassmann characters

$$\begin{aligned} \epsilon(\eta) &= 1 \\ \epsilon(\alpha) &= 0 \end{aligned} \tag{3.7}$$

In summary, the simplicity of the minimal field action is entirely due to the generators of the (U(1)) symmetry possessing no null directions and as such the path integral can be foliated into 'slices' of a particular gauge by simply adding one ghost degree of freedom for each field participating in the transformation.

The gauge fixing fermion Ψ can be selected in a fashion that does not depend upon the gauge like field and has a simple geometric interpretation. Bearing in mind that we maintain a concept of interchanging radii as a fundamental of T–duality, we can select a gauge which focuses upon rotations such a concept remaining coherent on either of our target spaces. Specifically we can choose a gauge fermion of a form such as

$$\Psi = \psi(\phi - \phi^{\dagger}) \tag{3.8}$$

This simple form of the gauge fixing fermion will align the scalar excitation along the real direction, naturally this choice of direction is entirely arbitray and other directions could be selected equally well. Having chosen our gauge fixing scheme, we may proceed to eliminate the anti-fields from (3.6) using (3.8) and the usual definition of the physical surface of field/anti-field phase space (1.27).

$$\mathcal{S}(\Phi) = \tilde{\mathcal{S}}(\phi, \psi)_{\text{free}} + \bar{\alpha}(\phi - \phi^{\dagger}) \tag{3.9}$$

Notice that this choice of gauge retains only the minimal number of ghost fields for the symmetry, the other factors integrating out to irrelevant volume factors. Furthermore, this choice of gauge will lead to no additional complications in integrating out the gauge like field since it introduces no further operators carrying A_μ dependence, gathering like terms we can write the action in the following fashion

$$\begin{aligned} \mathcal{S}(\Phi) &= \frac{1}{2}(\partial\phi)^2 + \frac{1}{8\phi^2}(\partial\lambda)^2 + \bar{\psi} i \not{\partial}\psi + \bar{\alpha}(\phi - \phi^{\dagger}) \\ &+ \frac{|\phi|^2}{2}\left(A_\mu + \left[j_\mu - \frac{\epsilon_{\mu\nu}}{2|\phi|^2}\partial^\nu\lambda\right]\right)^2 \\ &- \frac{|\phi|^2}{2}\left[j_\mu - \frac{1}{2|\phi|^2}\epsilon_{\mu\nu}\partial^\nu\lambda\right]^2 \end{aligned} \tag{3.10}$$

Where the current appearing in (3.10), j_μ, is defined by

$$j_\mu = \bar{\psi}\gamma_\mu\psi + \frac{i}{2}\left(\phi^{\dagger}\partial_\mu\phi - \phi\partial_\mu\phi^{\dagger}\right) \tag{3.11}$$

The appearance of j_μ is natural since this would be the sum of the bosonic and fermionic currents were we to be considering a dynamical A, i.e. electromagnetism.

The choice of gauge fixing (3.8) leads to a delta functional appearing in the path integral, additionally we see that we may make a linear and irrelevant change of coordinates in the gauge like field, A_μ, resulting in a simple Gaussian contribution of this field which generates an ignorable normalization. Since the introduction of the former integrand guarantees the definition of the latter we may proceed and write down the dual action.

$$\begin{aligned} \tilde{S} &= \frac{1}{2}(\partial\phi)^2 + \frac{1}{8\phi^2}(\partial\lambda)^2 + \bar{\psi} i \not\partial \psi \\ &+ \frac{1}{2\phi^2}(\bar{\psi}\gamma_\mu\psi)^2 + \frac{1}{\phi^2}\epsilon_{\mu\nu}\bar{\psi}\gamma^\mu\psi\partial^\nu\lambda \end{aligned} \tag{3.12}$$

Where the scalar field ϕ is now reduced to real values by virtue of the gauge fixing and the Lagrange multiplier has been endowed with dynamic qualities as might have been anticipated. In addition we see that this dual model has been endowed with a pair of interactions. A Thirring style fermionic self-coupling and an exchange between fermionic current and the momentum of the new dynamic field. We could envisage rotations in the scalar direction giving rise to a compensating flow of fermionic charge. Since interactions have appeared where previously there were none, one might become concerned about the renormalized behaviour of the pair of theories. Free field theory has no interesting dynamic behaviour with variations in energy scale, this is most apparently not the case in the interacting model.

From the priviledged position which we occupy having arrived at (3.12) it is known that this is simply a free field theory cast in a perverse system of coordinates. This can be revealed by either repeating the previous steps, since the duality transformation is reversible or by identifying the line element on the plane cast in polar coordinates. By applying such techniques one can reduce the theory to the free case. It is not always a simple matter to reduce a collection of fields to the simplest possible description of their degrees of freedom, in particular when less trivial interactions are present within the action. We will proceed with the dual model action and calculate results based upon the fields as presently described to produce results consistent with the underlying free field theory. Without the benefit of the mechanism of duality the results that we shall arrive at would appear surprising.

3.2 One Loop Behaviour of Dual Model

The form of the dual action (3.12) contains terms which appear to make a perturbative analysis difficult. However, if we are willing to permit the original scalar excitation to gain a vacuum expectation value then it will be possible to analyse the model in perturbation theory. We could conceive of adding a symmetry preserving potential to the action which would generate a large expectation value for ϕ^2, in principle one large enough to eliminate quantum fluctuations.

$$\phi = R + \alpha \tag{3.13}$$

Equation (3.13) is a background field expansion of ϕ based upon the above modification of the dual theory. This expansion is in terms of the classical expectation value for the coordinate radius, R, and the quantum fluctuation, α. We assume that the background radius R is large enough to permit a power series expansion of ϕ in (3.12), so to lowest order the action becomes.

$$\begin{aligned} \mathcal{S} &= \frac{1}{2}(\partial\alpha)^2 + \frac{1}{8R^2}(\partial\lambda)^2 + \bar{\psi} i \not{\partial}\psi \\ &- \frac{1}{2R^2}\left((\epsilon^{\mu\rho}\partial_\rho\lambda + \bar{\psi}\gamma^\mu\psi)\bar{\psi}\gamma_\mu\psi\right) \end{aligned} \tag{3.14}$$

To lowest order in this expansion the variation in the original bosonic field is decoupled from the dual coordinate and the fermionic sector, restoring to an extent the original form of the theory. The interactions in this expansion are then associated with rotations within the dual target space. The classical radius of the original bosonic field remains associated to the new interaction and can be used in a perturbative evaluation of the theory in $1/R$. This perturbative expansion then is best understood for large radius spaces, i.e. flatter target manifolds.

In order to perform calculations within this theory, one connects the dynamic field to a set of external currents, upon which one may act with functional derivatives in order to construct amplitudes for interesting processes.

In perturbation theory, one takes further advantage of the external sources and uses functional calculus in order to build the ineracting theory from free field theory. Generically, for a theory containing a set of field Φ with a classical action that may be composed in the following fashion

$$\mathcal{S}_{\text{interacting}}(\Phi) = \mathcal{S}_{\text{free}}(\Phi) + \mathcal{S}_{\text{interacting}}(\Phi) \tag{3.15}$$

If we minimally couple these fields to a set of currents $\mathbf{J}$ one might write the path integral in the following fashion

$$Z[\mathbf{J}] = \int [\mathcal{D}\Phi] \exp\left(i\mathcal{S}_{\text{interacting}}\left(\frac{\delta}{\delta J}\right)\right) \cdot \exp(i\left(\mathcal{S}_{\text{free}}(\Phi) + \mathbf{J}\cdot\Phi\right)) \tag{3.16}$$

If one then can write the propagator for the free theory as $\mathbf{P}$, by which we may mean either the bosonic or fermionic object depending upon which element of Φ we are considering, then it would appear that we may integrate (3.16) and obtain the perturbatively useful expression

$$Z[\mathbf{J}] = \exp\left(i\mathcal{S}_{\text{interacting}}\left(\frac{\delta}{\delta\mathbf{J}}\right)\right)\cdot\exp\left(\mathbf{J}\cdot i\mathbf{P}\cdot\mathbf{J}\right) \tag{3.17}$$

The expression (3.17) is of a form amenable to a perturbative analysis, if there is a parameter contained within the interacting component of the action 'small enough' to justify the expansion of the exponentiated operations as a series which may be truncated after a finite number of steps. Naturally this ignores any transgressions which one may have committed in compounding functional derivatives which is why one has to renormalise such calculations. The preceding discussion should suggest that we are proposing that the classical radius, or rather its reciprocal, is the desired parameter about which to perform a perturbative investigation. We will ignore the radial field α from henceforth as it does not contribute renormalising terms. If the fields are renormalised with factors Z_ψ and Z_α

$$\begin{aligned} \psi &\longrightarrow Z_\psi^{1/2}\psi \\ \lambda &\longrightarrow Z_\lambda^{1/2}\alpha \end{aligned} \tag{3.18}$$

Similarly, the renormalisation of the three point and four point interaction will be denoted Z_3 and Z_4 respectively. The renormalised action is then expressed as

$$\begin{aligned} \mathcal{S} &= \frac{1}{2} Z_\lambda (\partial\lambda)^2 + Z_\psi \bar{\psi} i \not{\partial} \psi \\ &- \frac{1}{R} Z_3 \epsilon^{\mu\nu} \bar{\psi}\gamma_\mu \psi . \partial_\nu \lambda - \frac{1}{2} Z_4 \left(\frac{1}{R}\right)^2 (\bar{\psi}\gamma_\mu\psi)^2 \end{aligned} \tag{3.19}$$

Examining the expression (3.19) the behaviour of the coupling is of interest under renormalisation. We see that the renormalised couplings flow in the following fashion

$$\begin{aligned} \left(\frac{1}{R_0}\right)_3 &= \frac{1}{R} \frac{Z_3}{Z_\psi Z_\lambda^{1/2}} \\ \left(\frac{1}{R_0^2}\right)_4 &= \frac{1}{R^2} \frac{Z_4}{Z_\psi^2} \end{aligned} \tag{3.20}$$

These two distinct behaviours are a significant problem. In order to preserve the geometric interpretation of the dual theory we require that these two flows be identical, i.e.

$$Z_3^2 = Z_\lambda Z_4 \tag{3.21}$$

If (3.21) can be verified then the relation between coupling and geometric radius of the target space is preserved after quantum corrections. It is interesting to note that the role of the fermion is implicit. What follows is a one loop test of this requirement.

3.2.1 Evaluating Graphs

If the dynamic fields are coupled to external fields $\{\alpha, \bar{\alpha}, J\}$ which have the appropriate scalar or fermionic characteristics then expectation values of any given field configuration can be calculated by

$$\begin{aligned} \langle 0 \mid \Phi(x_0) \ldots \Phi(x_n) \mid 0 \rangle &= \frac{\delta}{\delta \mathbf{J}(x_0)} \cdots \frac{\delta}{\delta \mathbf{J}(x_n)} \\ &\cdot \exp\left(g_1 \frac{\delta_r}{\delta \alpha} \gamma_\mu \frac{\delta_l}{\delta \bar{\alpha}} \epsilon^{\mu\nu} \partial_\nu \left(\frac{\delta}{\delta J} \right) - \frac{1}{2} g_2^2 \left(\frac{\delta_l}{\delta \alpha} \gamma_\mu \frac{\delta_r}{\delta \bar{\alpha}} \right)^2 \right) \\ &\cdot \exp\left(\bar{\alpha} i \not{S}_f \alpha + J i \Delta_f J \right)\big|_{(\bar{\alpha}, \alpha, J)=0} \end{aligned} \tag{3.22}$$

Where $\mathbf{\Phi}$ refers to any dynamic field and $\mathbf{J}$ to the appropriate external current. Remembering to act on the left when differentiating with respect to $\bar{\alpha}$ and on the right when differentiating with respect to α. Right and left action is not distinguished for the Grassman even current, J.

The operators $\not{S}_f$ and Δ_f are the usual Feynman free field fermion and scalar propagators respectively, in my cavalier notation these are written as

$$
\begin{aligned}
\not{S}(k)_f &= \frac{1}{\not{k} + i\epsilon} \\
\Delta(k)_f &= \frac{1}{k^2 + i\epsilon}
\end{aligned}
\tag{3.23}
$$

Note that for the purposes of this investigation two abstract couplings g_1 and g_2 have been introduced, the hope being that they may be set equal at one-loop level as we have previously had them in the classical theory.

The contributions to any one expectation value can be composed as a sum of Feynman diagrams, such diagrams and accompanying rules provide a useful mnemonic to generated amplitudes and minimise the number of direct functional derivatives that one need perform upon the generating functional. It is my intention throughout these perturbative calculations to be exceedingly lax in the use of momentum conserving delta functions and simply to apply these rules, which will make notation a little more compact, any related integration over momentum must be considered to have been implictly performed. The rules for evaluating the Feynman graphs for evaluating amplitudes of (3.22) are the following

1. Draw each distinct diagram for the corellator at the desired perturba-

tive order.

2. To each three point vertex associate a factor

$$g_1\gamma^\mu\epsilon^{\mu\nu}k_\nu$$

3. To each four point vertex associate a factor

$$2(-ig_2^2)\gamma^\mu\otimes\gamma^\mu$$

4. To each internal line associate the correct propagator.

5. Integrate over internal momenta $d^Dk/(2\pi)^D = ḓ^Dk$

6. For each internal fermionic loop attach a factor of -1 and trace over spinor indices.

7. Contract external spinor indices with an outgoing spinor carrying the correct momentum.

8. Divide each diagram by the size of the permutation group of vertex-fixed lines.

Anticipating the need to renormalise this model a method for regularising it must be selected. Dimensional regularisation will be the preferred method here. In order to regularise we introduce an arbitrary mass scale, μ, and a controlling parameter, ϵ, that defines how far an expression is being evaluated away from the two dimensional space, $D = 2 - \epsilon$. The counterterms that will be introduced will be minimal, that is only the pole terms will be eliminated by the selected renormalising factors. Results that are useful in evaluating amplitudes can be found in the appendix.

3.2.2 Scalar Renormalisation

At one loop there is only one diagram, generated by two three-point interactions that contributes to renormalisation of the scalar propagator. In order to calculate the renormalisation of the scalar propagator, Z_λ, one must evaluate the following expectation value

$$\begin{aligned} \delta\Delta_f(k) &= -g_1^2 \eta_{\mu\nu} \epsilon^{\mu\rho} \epsilon^{\alpha\beta} \eta_{\alpha\lambda} 2k_\rho k_\beta . \mathrm{Tr} \int d^D p \gamma^\nu i\, S_f(k+p) \gamma^\lambda i S_f(-p) \\ &= -g_1^2 \cdot \frac{-i}{4\pi} k^2 \cdot \frac{2}{\epsilon} \mathrm{Tr}(1) \\ &= g_1^2 \frac{i}{\pi\epsilon} k^2 \end{aligned} \tag{3.24}$$

Therefore the minimal renormalsation of the scalar field is given by

$$Z_\lambda = 1 - g_1^2 \frac{1}{\pi\epsilon} \tag{3.25}$$

3.2.3 Fermion Renormalisation

At one loop there are two diagrams which potentially could lead to renormalisation of the fermion field. The first is due to contraction of a pair of legs on a four point interaction.

$$\delta S_f^4 = \bar{u}(p)\gamma_\mu u(p)(-ig_2^2)\mathrm{Tr}\int d^D k\gamma_\mu \frac{i}{\not k} \tag{3.26}$$

However this contribution vanishes and the only contribution to fermion renormalisation arises from a contraction of two three-point interactions, with an inner bosonic loop.

$$\begin{aligned}\delta S_f &= g_1^2 \int d^D p\bar{u}(k)\gamma^\mu i S_f(p+k)\gamma^\nu u(k) \cdot \frac{i}{p^2+i\epsilon}\eta_{\mu\rho}\epsilon^{\rho\alpha}p_\alpha \eta_{\nu\lambda}\epsilon^{\lambda\beta}p_\beta \\ &= g_1^2 \cdot \frac{i\not k}{4\pi} \cdot \frac{2}{\epsilon}\end{aligned} \tag{3.27}$$

The desired one loop renormalisation of the fermion is then

$$Z_\psi = 1 - g_1^2 \frac{1}{2\pi\epsilon} \tag{3.28}$$

3.2.4 Three Point Renormalisation

There are two diagrams that could modify the three point interaction. Firstly a contraction of two fermionic legs of a three-point interaction with two legs of the four point interaction.

$$\delta\mathcal{I}_3^{3,4} = 2ig_2^2 g_1 \epsilon^{\mu\alpha} p_\alpha \mathrm{Tr}\left[\gamma^\mu \int d\!\!/^D k i S_f(q+k)\gamma^\nu i S_f(k-r)\right] \bar{u}(q)\gamma_\nu u(r) \tag{3.29}$$

This diagram is finite and can be ignored. The remaining contribution is a QED-like vertex correction composed of three three-point vertices.

$$\begin{aligned} \delta\mathcal{I}_3^{3\times 3} &= -g_1^3 \int d\!\!/^D q \bar{u}(p)\epsilon_{\mu\nu} q^\mu \gamma^\nu i S_f(p+q)\epsilon_{\rho\lambda} k^\rho \gamma^\lambda i S_f(r-q) \\ &\cdot \quad \epsilon_{\alpha\beta} q^\alpha \gamma^\beta i\Delta_f(q) u(r) \\ &= -\frac{g_1^3}{\pi\epsilon}\bar{u}(p)\epsilon_{\lambda\rho}\gamma^\lambda k^\rho u(r) \end{aligned} \tag{3.30}$$

From which the one-loop three-point renormalisation is seen to be

$$Z_3 = 1 + \frac{g_1^2}{\pi\epsilon} \tag{3.31}$$

3.2.5 Four Point Renormalisation

There is only one diagram that can renormalise the four-point interaction, it is formed by contracting to four-point interactions along two legs.

$$\begin{aligned}\delta\mathcal{I}_4 &= \frac{(2g_2^2)^2}{2^2}\bar{u}(r)\gamma^\mu u(s)\bar{u}(p)\gamma^\nu u(q)\mathrm{Tr}\left[\gamma^\nu\int d\!\!\!/^D k iS_f(p+k)\gamma^\mu iS_f(q-k)\right] \\ &= g_2^4\frac{i}{\pi\epsilon}\bar{u}(r)\gamma_\mu u(s)\bar{u}(p)\gamma_\mu u(q) \qquad (3.32)\end{aligned}$$

To one-loop we renormalise the four-point interaction like so

$$Z_4 = 1 - \frac{g_2^2}{\pi\epsilon} \qquad (3.33)$$

3.2.6 Geometric Interpretation After One-Loop

The preceding calculations have yielded enough information for the relation (3.21). Using (3.31),(3.25) and (3.28) we can calculate

$$\begin{aligned}Z_3^2 &= \left(1-\frac{g_1^2}{\pi\epsilon}\right)^2 \\ &= 1-\frac{2g_1^2}{\pi\epsilon}+\mathcal{O}(g_1^4) \qquad (3.34)\end{aligned}$$

Similarly the right hand side of (3.21) can be evaluated.

$$\begin{aligned} Z_\lambda Z_4 &= \left(1-\frac{g_1^2}{\pi\epsilon}\right)\left(1-\frac{g_2^2}{\pi\epsilon}\right) \\ &= 1-\frac{g_1^2+g_2^2}{\pi\epsilon}+\mathcal{O}(g^4) \end{aligned} \tag{3.35}$$

We then see that (3.34) and (3.35) are equal if and only if $g_1 = g_2$. That is the geometric interpretation is preserved to one loop and we may safely replace g_i with $1/R$. Having established that the notion of target space radius is preserved after these quantum corrections, we may wish to see how the radius varies with changing energy scale. We may select either of the two expressions in (3.20) in order to see how the radius scales

$$\begin{aligned} \frac{Z_4}{Z_\psi^2} &= \frac{1-\frac{1}{\pi R^2\epsilon}}{\left(1-\frac{1}{2\pi R^2\epsilon}\right)^2} \\ &= \left(1-\frac{1}{\pi R^2\epsilon}\right)\left(1+\frac{1}{\pi R^2\epsilon}\right)+\mathcal{O}(R^{-4}) \\ &= 1+\mathcal{O}(R^{-4}) \end{aligned} \tag{3.36}$$

To one-loop order, not only does the theory preserve the interpretation of the target space radius, but this radius remain invariant. Quantum corrections appear then to renormalise the dual scalar field and the fermions that it interacts with, however the cumulative effect is to preserve the target space.

3.3 Interacting Model

Having observed some success working with the target space dual to a free field theory, it is tempting to attempt to introduce some interactions into the original model. Starting from a clean slate, one has considerable freedom of choice as to which interactions to introduce. To take a concrete example, we will insert additional coupling between the original scalar excitations with a pair of fermionic fields, $\psi = \{L, R\}$.

$$\begin{aligned} \mathcal{S} &= \frac{1}{2}\left|\partial\phi\right|^2 + \bar{\psi}i\ \partial\psi \\ &+ \phi\bar{\psi}M_-\psi + \phi^\dagger\bar{\psi}M_+\psi \end{aligned} \tag{3.37}$$

Writing the fermions as column vectors, the operators $M_\pm$ which acts to select members of ψ, i.e.

$$\begin{aligned} M_+ &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ M_- &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{3.38}$$

So that $P = M_- + M_+$ is the permutation matrix. The purpose behind the exercise is to construct another action which will remain invariant under rotations within the target space. If we permit the fermions to carry a charge under the action of rotation, such that

$$\begin{aligned} \phi &\longrightarrow e^{i\theta}\phi \\ L &\longrightarrow e^{i\theta/2}L \\ R &\longrightarrow e^{i\theta/2}R \end{aligned} \qquad (3.39)$$

Then (3.37) is invariant under rotations in the target space. Furthermore, these simple three point interactions will require no modification under the duality transformation and will be passed directly into the dual theory. Once more we introduce the gauge-like field A and dual scalar λ, if in addition we also introduce a U(1) charge operator Q then the original action may be written as

$$\begin{aligned} \mathcal{S} &= \frac{1}{2}(\partial_\mu + iA_\mu)\phi \cdot (\partial_\mu - iA_\mu)\phi + \bar{\psi}i(\not{\partial} + i \not{A}Q)\psi \\ &+ \lambda F\phi + \bar{\psi}M_-\psi + \phi^\dagger\bar{\psi}M_+\psi \end{aligned} \qquad (3.40)$$

Furthermore we may choose employ the same gauge fixing scheme as in the previous model $\phi - \phi^\dagger = 0$. So that the dual model, in the background field expansion, reads

$$\begin{aligned} \mathcal{S}_D &= \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(\partial\lambda)^2 + \bar{\psi}i\not\partial\psi + \phi\bar{\psi}P\psi \\ &+ \frac{1}{2R^2}(\bar{\psi}\gamma^\mu Q\psi)^2 + \frac{1}{R}\epsilon_{\mu\nu}\bar{\psi}\gamma_\mu Q\psi\partial_\nu\lambda \end{aligned} \tag{3.41}$$

Examining (3.41) we see that the action decomposes into two distinct interacting sectors. That which is common between the original and dual models and a sector of new interactions between the dual scalar and the fermionic sector. For the purposes of this investigation the interaction between the original scalar and the flavour changing fermion term are irrelevant as it adds no new graphs in perturbation theory on the dual side of the formulation.

That leaves those terms with which the dual scalar interacts with the pair of fermions. The form of the interactions with the dual scalar is similar to that of the free field dual, including a Thirring style interaction and a transverse coupling to the scalar. The new interactions contain only the charge operator and as such the fermionic fields act as two independent sectors where graphs generated from these interactions are concerned. Since there is no coupling between these sectors and since we would simply be drawing graphs that are topologically identical to those considered in the free field case, we can evaluate the graphs simply by allowing for the modified charge of the fermions. It follows that we arrive back at the condition (3.35) and then since we only require the running charges to be equal when squared all of the previous observations apply. At the one loop level, the fields and interactions will receive renormalisations such that the radius of the target space is left invariant.

3.4 Conclusions

In this chapter we have considered quantum aspects of Abelian target space duality. Beginning with a free field theory of scalars and fermions we were led to a dual model containing interactions between a dual scalar and fermions. Expanding the theory at large radius in the original model it was seen that the geometric interpretation of the transformation was unspoiled at the one loop level (3.21),(3.34),(3.35). Most significantly it was seen that whilst fields and new interactions received perturbative renormalisations their combined effect was to leave target space radii invariant (3.36).

Suitably encouraged by the free field results, an interacting model was constructed bearing interactions between a pair of fermions and a pair of scalars. A demand made of the interacting model was that it remain invariant under rotations within the target space and this in term implied that the fermions each gained a novel charge $\pm 1/2$ under the action of target space rotation. In effect, the fermions were wrapped twice around the scalar axial direction. Having arrived at a suitable model calculation of the dual theory was performed in the familiar fashion and examined. The free field primer with which we began the chapter was found to be extremely useful as a labour saving device. The dual interacting model could be decomposed into an invariant and a dual sector; the invariant sector requiring no further attention in this context. The dual sector was deeply similar to that of the dual free model and a one loop evaluation of the theory could be performed with no real effort, it was apparent that (3.21) also held for the interacting model. Therefore, target space radii are also invariant for this class of interacting model at the one loop level.

Chapter 4

Conclusions

This material has made extensive use of the Batalin–Vilkovisky technique in order to develop well defined path integrals from classically inspired theories. We have demonstrated in the preliminary material that these techniques are well developed for attacking theories which contain symmetries which are reducible. The formalism provides a simple and convenient method for progressing from classical actions and preferred gauge fixing conditions to full quantum actions complete with any necessary ghost degrees of freedom.

Having developed the means to examine theories with reducible symmetries, we turned our attention to a non-Abelian generalisation of target space duality. Having been inspired by the classical method [10, 12], we identify the conceptual difficulty of implementing this technique at the quantum level. Inherent in the process of calculating the non-Abelian dual of an initial model is the prescence of reducible symmetry, which we attempted to resolve by the previously developed techniques. Whilst it is possible to arrive at a well defined quantum action the choice of gauge fixing can obscure the underlying theory.

Returning to the Abelian case we considered applying a novel gauge fixing condition to the action. The choice of gauge fixing whilst avoiding the ambiguities of the previous work broke covariance within the original target space, focusing upon a preferred direction in that space. Proceeding at a low order and to one-loop in the dual model we discovered that quantum corrections in the dual model do not interfere with the radius of the target space. It was then found that for a class of interacting model that the same assertion could be made. As interesting as these results are, it should always be bourne in mind that they were only computed to a single loop in perturbation theory and that this action in turn was based upon the simplest possible background field approximation. It remains to be seen whether more general results can be arrived at. Furthermore, assuming that a more in-depth investigation reveals this to be a fruitful area of study there is still the matter of non-Abelian duality to be considered in this context. One might go on to look at similar classes of gauge fixing on target spaces with more complex symmetries and look for more complex conservation of geometry but this is significantly beyond the scope of this present work.

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Appendix A

Momentum Space Integrals

In a purely Euclidean D - space the following integral can be obtained [29]

$$\int d^D k \frac{1}{(k^2 + 2k \cdot p + M^2)^A} = \frac{\Gamma(A - D/2)}{(4\pi)^{D/2}\Gamma(A)} \frac{1}{(M^2 - p^2)^{A-D/2}} \tag{A.1}$$

The (A.1) expression can be used as a generating function for a series of useful results, by differentiating with respect to p_μ. Furthermore the results can be analytically continued back to Minkowski space by Wick rotation $idk^0_{\mathrm{Euc}} = dk^0_{\mathrm{Mink}}$.

1. No momentum in numerator

$$\int d^D k \frac{1}{(k^2 - 2k\cdot p - M^2)^A} = \frac{i(-1)^A}{(4\pi)^D/2} \frac{\Gamma(A-D/2)}{\Gamma(A)} \frac{1}{(p^2+b^2)^{A-D/2}} \tag{A.2}$$

2. One power of momentum in numerator

$$\int d^D k \frac{k_\mu}{(k^2 - 2k\cdot p - b^2)^A} = \frac{i(-1)^A}{(4\pi)^{D/2}} \frac{\Gamma(A-D/2)}{\Gamma(A)} \frac{p_\mu}{(p^2+b^2)^{A-D/2}} \tag{A.3}$$

3. Two powers of momentum in numerator

$$\begin{aligned} \int d^D k \frac{k_\mu k_\nu}{(k^2 - 2k\cdot p - b^2)^A} &= \frac{i(-1)^A}{(4\pi)^{D/2}} \frac{1}{\Gamma(A)} \\ &\times \left[\begin{array}{c} \frac{\Gamma(A-D/2) p_\mu p_\nu}{(p^2+b^2)^{A-D/2}} \\ -\frac{\Gamma(A-1-D/2)\eta_{\mu\nu}}{2(p^2+b^2)^{A-1-D/2}} \end{array} \right] \end{aligned} \tag{A.4}$$

4. Three powers of momentum in numerator

$$\begin{aligned} \int d^D k \frac{k_\mu k_\nu k_\rho}{(k^2 - 2k\cdot p - b^2)^A} = \frac{-i(-1)^A}{2(4\pi)^{D/2}} \frac{1}{\Gamma(A)} \times \\ \left[\begin{array}{c} \Gamma(A-1-D/2) \frac{\eta_{\mu\rho} p_\nu + \eta_{\nu\rho} p_\mu + \eta_{\mu\nu} p_\rho}{(p^2+b^2)^{A-1-D/2}} \\ -2\Gamma(A-D/2) \frac{p_\mu p_\nu p_\rho}{(p^2+b^2)^{A-D/2}} \end{array} \right] \end{aligned} \tag{A.5}$$

Appendix B

Gamma Matrix Results

The Dirac gamma matrices are such that

$$\{\,\gamma^\mu\,,\,\gamma^\nu\,\} = 2\eta^{\mu\nu} \tag{B.1}$$

The defining relationship (B.1) allow one to simplify products of gamma matrices.

$$\gamma^\mu\gamma_\mu = D \tag{B.2}$$

$$\gamma^\lambda\gamma^\mu\gamma_\lambda = (2-D)\gamma^\mu \tag{B.3}$$

$$\gamma^\lambda\gamma^\mu\gamma^\nu\gamma_\lambda = 2\gamma^\nu\gamma^\mu - (2-D)\gamma^\mu\gamma^\nu \tag{B.4}$$

$$\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\lambda = -4\gamma^\rho\gamma^\nu\gamma^\mu \tag{B.5}$$

$$+ 4(\eta^{\rho\nu}\gamma^\mu - \eta^{\mu\rho}\gamma^\nu + \eta^{\mu\nu}\gamma^\rho) \tag{B.6}$$

$$+ (2-D)\gamma^\mu\gamma^\nu\gamma^\rho \tag{B.7}$$

$$\mathrm{Tr}(1) = D \tag{B.8}$$

$$\mathrm{Tr}\left(\prod_i^{i=2n+1}\gamma^{\mu_i}\right) = 0 \tag{B.9}$$