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THE EFFECTS OF BORROWER AND LENDER REPUTATION IN CREDIT MARKETS

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ABSTRACT

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Mainstream neoclassical economics predicts that financial markets will operate in a frictionless manner, with transactions occurring directly between savers and borrowers. However, in reality there exists a variety of informational imperfections and incentive problems which disrupt financial markets. In this thesis, I explore how borrowers' and lenders' concerns about their reputations can mitigate these incentive problems.

In chapters one and two of the thesis, I formulate a new theory of financial intermediation and explain the general structure of credit markets. Borrowers without established credit histories have incentives to repudiate their debt obligations, and are therefore unable to issue debt directly. Banks exist in order to provide finance for this class of borrowers. Banks can curtail borrowers' incentives to default on debt by building a reputation for liquidating defaulters. However, over time, borrowers' concerns about reputation improve their incentives, such that they are able to issue debt directly.

In chapter three of the thesis, I analyze the dynamics of firms' credit ratings. Borrowers with short credit histories face the poorest incentives, and (depending on initial conditions) for these borrowers debt repayment can only be enforced by the threat of liquidation. However, over time if borrowers repay debt on all dates, they will establish a good credit history. This may improve their incentives, such that they will repay debt because they are concerned about their reputations for being a good credit risk, even if they face no threat of liquidation if they do default.

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Contents

0.1 Introduction

Mainstream neoclassical economics predicts that financial markets will operate in a frictionless manner, with transactions occurring directly between savers and borrowers, and with the Walrasian auctioneer clearing markets instantaneously. However, the empirical evidence contradicting this view of the world is overwhelmingly. There exists a variety of informational imperfections which plagues financial markets, causing serious impediments to the flow of funds from savers to borrowers. In particular, these informational imperfections give rise to several moral hazard problems, in which borrowers have incentives to undertake certain actions which increase their own pay-offs, but only at the expense of their creditors' pay-offs. For example, borrowers may have incentives to choose excessively risky investment projects. Or they may have incentives to default on their debt repayments. However, markets have developed a plethora of methods for overcoming these incentive problems. The existing literature on informational imperfections in financial markets examines how incentive problems can affect the optimal contractual form. It also analyzes how incentive problems can be mitigated when borrowers and lenders transact on a repeated number of occasions in the market. Furthermore, there is a substantial literature which explains the existence of financial intermediaries as an endogenous mechanism to mitigate the information and incentive problems which disrupt financial markets.

In this thesis, I explore how borrowers' and lenders' concerns about their reputations can mitigate these incentive problems. Chapters one and two answer the following questions:

1) Why do banks exist?

2) Why do new borrowers (i.e. firms which have only recently begun trading) tend to borrow using bank finance, whilst established borrowers tend to issue debt directly?

These questions have been addressed by several papers in the literature. However, my research improves upon the existing literature, in that it proposes a new theory of financial intermediation, which explains certain features about bank and borrower behaviour in the credit market which the existing literature is unable to account for. In particular, it explains why banks have a reputation for aggressive liquidation, and why small businesses borrow almost exclusively from banks and pay an interest rate premium on bank debt. I model a dual incentive problem with two-sided uncertainty in which neither borrowers nor lenders can commit to first-best actions. I show how concerns about reputation are effective in alleviating the incentive problems, and the implications of these reputation effects on the nature of financial intermediation and the structure of the credit market. I explore the evolution of borrower and lender reputation effects over time, and show that the pattern of evolution is such that borrower and lender reputation effects are mutually reinforcing.

Chapter three answers the following questions:

1) Why do firms' credit ratings improve over time?

2) Why do aggregate shocks have less impact on the risk of default of well-established firms, compared to recent start-up firms?

In this chapter, I analyze the dynamics of firms' credit ratings, in the context of a multi-period moral hazard problem, in which borrowers have incentives to default on their debt obligations. Borrowers with short credit histories face the poorest incentives, and (depending on initial conditions) for these borrowers debt repayment can only be enforced by the threat of liquidation. However, over time if borrowers repay debt on all dates, they will establish a good credit history. This may improve their incentives, such that they will repay debt because they are concerned about their reputations for being a good credit risk, even if they face no threat of liquidation if they do default.

Much of the literature on informational imperfections in financial markets is based on Akerlof's (1970) 'Market for Lemons' paper, in which sellers have private information about a product's quality, which is unobservable to buyers. This may drive down the market price, which reflects the average quality of the pool of products on the market, such that high quality products are driven out of the market, a phenomenon called adverse selection. If the degree of adverse selection is severe, the market may actually break down altogether.

A key paper based on the 'Market for Lemons' model is Stiglitz and Weiss (1981). In their model, asymmetric information arises because the risk that a borrower's project will fail is unobservable to lenders. Lenders issue funds on the basis of a standard debt contract, which specifies a fixed interest rate if the project succeeds, and a low (possibly zero) pay-off if the project fails. Risky projects are undesirable to lenders, due to the high probability of project failure. Although risky projects yield higher returns if the project succeeds, lenders do not benefit from this upside gain, due to the fixed interest rate specified by the debt contract. Under certain conditions, credit will be rationed in equilibrium, i.e. some borrowers will receive loans, whilst other observationally equivalent borrowers will not. The reason for this is as follows. If lenders charge a high interest rate, then it will not be worthwhile for borrowers with safe projects to invest, hence they are driven out of the market. Thus the average riskiness of projects remaining in the market rises. If the lender compensates for this by increasing the interest rate even further, then this drives out more safe projects, such that lenders' expected returns may actually fall. Hence, lenders may actually be better off by charging a lower interest rate, even though this means demand for funds exceeds supply, i.e. there exists a credit rationing equilibrium.

Another important strand of the literature examines how incentive problems affect contractual form. Townsend (1979) analyzes the case in which borrowers have incentives to misreport project outcome, in order to avoid repaying creditors. Monitoring the project outcome is costly to lenders. In this case, the optimal contract is a debt contract, which specifies a fixed interest repayment. If the borrower defaults on the repayment, the lender will monitor the project outcome. The debt contract is optimal because it means that the lender need not monitor in all states of nature (which a variable 'equity' contract would require), but only in the 'default' state in which the project fails.

The role of debt as an instrument of control has received much attention in the recent literature, e.g. Hart and Moore (1994) and (1996), Aghion and Bolton (1992). In

Hart and Moore (1996), borrowers have incentives to repudiate their debt obligations. Lenders are unable to enforce repayment by writing a comprehensive contingent contract, i.e. contracts are incomplete. Hence, lenders use debt contracts to enforce repayment. The contract specifies that if borrowers default on debt, then ownership and control of project assets will be transferred from the borrower to the lender. Lenders can thus threaten to liquidate project assets if borrowers repudiate. If there is still a long time before the project is completed, i.e. the project has a high continuation value, then the threat of liquidation will be effective in deterring borrowers from repudiation. However, if the project is almost completed and therefore has a low continuation value, borrowers will not be deterred from repudiation by the threat of liquidation. Although debt is useful in enforcing repayment by borrowers, it can also lead to inefficient liquidation. Borrowers may experience temporary cashflow problems, such that they are forced to default on debt. Even if the project has a high continuation value, but has a low liquidation value, lenders may prefer to liquidate, because they know that if they allow the project to continue, then its liquidation value will fall even further, and hence they will be unable to deter borrowers from repudiating when the project has been completed.

Another strand of the literature examines how incentive problems can be mitigated when borrowers and lenders transact on a repeated number of occasions in the market. Stiglitz and Weiss (1983) show how borrowers' incentives can be improved if lenders threaten to terminate their credit. In their model, borrowers have incentives to choose risky projects. In the one shot problem, if lenders compensate for the higher risk of default by increasing the interest rate, this exacerbates borrowers' incentives to choose risky projects. However, in the two period problem, lenders have a tool to improve borrowers' incentives. If borrowers default on date 1, lenders can punish them on date 2, by charging a higher interest rate, or under certain conditions, by terminating credit. This mitigates borrowers' incentives to choose risky projects on date 1.

Townsend (1982) shows how borrowers' incentives can be improved by writing a multi-period contract with lenders. In his model, borrowers are risk-averse and face a stochastic endowment stream. Lenders are risk-neutral, hence gains from trade exist if lenders agree to insure away the risk faced by borrowers. In the full information case, the optimal contract is a one period contract, in which borrowers are fully insured. However, if there exists asymmetric information, i.e. only borrowers observe the state of nature, then borrowers have an incentive to misreport the outcome, i.e. to claim always that the 'default' state has occurred, in order to receive the insurance payout. Consequently, lenders will refuse to write a one period contract. However, incentives can be improved by writing a two period contract, and making future exchanges contingent on present claims. Hence, if borrowers claim that the 'default' state has occurred on date 1, then the lender will pay out on date1, but will not pay out if the borrowers claims again on date 2. Hence, the borrower has an incentive to report the state of nature truthfully on date 1, so that he can benefit from insurance on date 2.

Gertler (1992) also shows how borrowers' incentives can be improved by writing a multi-period contract with lenders. In his model, borrowers have incentives to misreport project output, which is unobservable to lenders, in order to reduce the payment to the lender. However, capacity utilization is observable, and incentive compatibility can be achieved, but only by the lender requiring that capacity utilization must be very low, if the 'bad' productivity state of nature arises, in order to eliminate borrowers' incentives to misreport the state. Hence, there exists an agency cost of borrowing, which arises due to the underemployment of capital in the 'bad' productivity state. But this agency cost can be reduced if the borrower and lender write a multi-period contract. If the borrower claims the bad state arises and defaults on date 1, the lender can simply reschedule the repayment to date 2. Hence, the borrower's incentive to misreport the state is mitigated, and thus the underemployment of capital to achieve incentive compatibility on date 1 need not be as severe. Gertler's model also shows how the existence of multi-period contracts can propagate macroeconomic fluctuations. The amount of debt which can be rescheduled to date 2 depends on the borrower's 'financial capacity', i.e. his ability to repay the debt on date 2 , which depends on his expected earnings. If a negative productivity shock occurs on date 2, this reduces the borrower's financial capacity, which therefore increases the agency cost of borrowing on date 1. Through this mechanism, Gertler shows that 'small but persistent shifts in macroeconomic fundamentals may induce large fluctuations in financial constraints, which in turn are transmitted into potentially large fluctuations in output'.

There are also several papers on dynamic models of incentive problems which show how concerns about reputation can be effective in improving agents' incentives, if they face a long horizon. In the sovereign debt literature, e.g. Grossman and van Huyck (1988), Cole and Kehoe (1996), governments have the opportunity to repudiate repayment of foreign debt, given that the power of sovereignty means that the creditors are unable to enforce the debt contract. However, governments may have an incentive to repay debt in order to build a reputation as an honest borrower and thereby avoid exclusion from future borrowing. Similarly, in Holmstrom (1982), a manager whose ability is uncertain has an incentive to increase his effort in order to convince the market that he is of high ability, and thus increase future wages. These models have in common the implication that reputation effects are strongest at the start of the agent's life, when the amount of information an action can reveal about his type is highest, and the horizon is longest.

Diamond (1989) formulates a model of debt markets in which borrowers' incentives improve over time. His model specifies distinct borrower types who are observationally equivalent, and analyzes the ability of reputation to eliminate borrowers' incentives to choose risky projects. If, initially, there is a severe problem of adverse selection (a large proportion of borrowers with undesirable characteristics), reputation will be too weak to improve the incentives of borrowers with short track records. Over time, as adverse selection diminishes, reputation effects strengthen such that borrowers with a good credit rating, i.e. a record of non-default, will no longer have an incentive to select risky projects. Alternatively, if adverse selection is initially quite mild, reputation can begin to work immediately. The model predicts that firm credit ratings will improve over time, since the risk of liquidity default falls as the degree of adverse selection falls and incentives to choose risky projects diminish.

There is also a substantial literature which explains the existence of financial intermediaries as an endogenous mechanism to mitigate the information and incentive problems which disrupt financial markets. Fama (1980) shows that in an environment of frictionless competitive financial markets, banks and other financial institutions are simply veils over real economic behaviour, and have no intrinsic role to perform.

Fama (1985) notes that borrowers who obtain bank loans tend to pay an interest rate premium compared to the market interest rate on directly-placed debt. He infers from this that bank debt is 'special' is some sense, for certain classes of borrowers. The role of the bank is motivated in that the bank develops a comparative advantage in gathering information about borrowers, which helps to mitigate informational imperfections.

In Boyd and Prescott (1986), the role of the bank is to evaluate the quality of borrowers' projects, and hence mitigate the problem of adverse selection. In their model, all agents initially have an endowment of wealth and a project, but in equilibrium, only those with high quality projects become borrowers, whilst those with low quality projects become savers.

In Moore (1987), banks lend to borrowers repeatedly and write multi-period contracts, which helps to minimize informational distortions, and thus generate efficiency gains. The special role of banks is that by taking deposits from a large number of savers, and lending to a large number of borrowers, banks diversify away the liquidity risk faced by individual borrowers and savers. If transactions were to occur directly between borrowers and savers, then the risk that multi-period arrangements would be disrupted are much higher, and thus efficiency gains from long term contracts would not be realized.

In Diamond (1984), the intermediary takes on the role of delegated monitor. Monitoring is required since borrowers have an incentive to misreport project outcome, and monitoring by the intermediary is the most efficient solution because the alternatives are either duplication of effort if each lender monitors independently, or a free-rider problem in which case no lender monitors. The problem of monitoring the monitor does not arise, since by holding a heavily diversified portfolio the intermediary can make non-contingent payments to creditors.

Diamond (1991) formulates a theory of the dynamics of demand for bank lending. It provides an explanation for why borrowers with poor credit ratings tend to borrow from banks whilst those with established track records borrow directly by issuing publicly-traded debt. In his model, banks provide a special service by monitoring borrower actions, which alleviates moral hazard over project choice and therefore enables them to charge a lower interest rate than non-bank lenders. Borrowers with short track records are charged high interest rates if they borrow directly, hence reputation effects are too weak to improve their incentives, which means they are better off borrowing from the bank. However, borrowers with established track records are charged low interest rates, hence reputation is effective in improving their incentives, hence they will prefer to borrow directly.

Chemmanur and Fulghieri (1994) also develops a theory of bank lending by attributing a monitoring role to banks. Specifically, banks undertake costly evaluation of financially distressed firms, in order to determine whether to liquidate or renegotiate and thus minimize inefficient liquidation. Unlike Diamond (1991), where the specialness of banks arises from the assumption that banks have access to a monitoring technology, whilst bondholders do not, in their model, all lenders have the ability to monitor. However, due to their multi-period nature, banks have a greater incentive to evaluate distressed firms, in order to build a reputation for eschewing inefficient liquidation. Firms with a greater probability of facing financial distress therefore prefer bank loans over directly-placed debt.

Petersen and Rajan (1995) looks at long term relationships between firms and banks, and analyze whether a competitive credit market is inimical to the value of such relationships. In their model, they consider a problem of adverse selection for new firms, as in Diamond (1989). The problem with a competitive credit market is that, due to the adverse selection problem, banks must charge young firms a high interest rate, which can severely distort the firm's incentives, and may result in the firm not receiving credit at all. But the situation changes if we assume that banks have local monopoly power. Banks can charge young firms a lower interest rate than the competitive rate, and hence improve their incentives. Given their local monopoly power, banks are able to charge a mark up on the competitive interest rate to the firm in the second period. Although their is no longer a problem of adverse selection, the firm cannot seek a lower interest rate elsewhere, and hence is tied to the bank for the second period. Thus the bank is able to subsidize the first period loan, because it knows it will be able to recoup its costs in the second period. The paper examines data on small businesses and finds it supports the hypothesis that credit constraints are less binding if the degree of credit market competition is lower.

Franks and Nyborg (1996) look at how the efficiency of financial restructuring is affected by the distribution of control rights under the UK insolvency code. They introduce the possibility that creditors may actually have private benefits in allowing the firm to continue trading rather than liquidating, e.g. a trade creditor gains rents from doing business with the firm which disappear if the firm is liquidated. The paper questions the widespread belief that the UK insolvency code causes inefficient liquidation. It finds that inefficiency actually depends on the debt structure and whether the controlling creditor has private benefits.

Rajan (1992) focuses similarly on banks' ability to make flexible financial decisions over liquidation versus renegotiation. He argues that banks have a natural advantage over bondholders in this role due to the inside information they gather about the firm during the lending process, which is unavailable to arms-length bondholders. However, bank lending also bears a cost, in that banks have bargaining power over the firm's profits, once projects have begun.

James (1987) examines the empirical evidence on whether banks provide some special service to borrowers not available through direct finance. He finds the evidence is supportive in two respects. First, bank borrowers bear the cost of reserve requirements on

CDs, rather than the CD holders themselves, suggesting that these borrowers are willing to pay a premium for the service provided by the bank. Second, he finds that the stock price effect of the announcement of new bank credit agreements is larger than the stock price effect of announcements of public debt offerings.

Boot and Thakor (1993) address the following two questions. First, why would a firm raising external capital wish to issue multiple types of claims against its cashflows? Second, why do firms pool individual assets into a portfolio and then partition the portfolio cashflows? They show that, in an asymmetric information environment, the issuer's expected revenue is enhanced by such cashflow partitioning because it makes informed trade more profitable.

0.2 Introduction to Chapters One and Two

Chapters one and two formulate a new theory of financial intermediation and explain the general structure of credit markets. In these chapters, I explain why new borrowers (i.e. firms which have only recently begun trading) tend to borrow using bank finance, whilst established borrowers tend to issue debt directly. I analyze in a model with two-sided uncertainty a dual incentive problem in which neither borrowers nor lenders can commit to first-best actions. I show how concerns about reputation are effective in alleviating the incentive problems, and the implications of these reputation effects on the nature of financial intermediation and the structure of the credit market. I explore the evolution of borrower and lender reputation effects over time, and show that the pattern of evolution is such that borrower and lender reputation effects are mutually reinforcing.

These chapters provide answers to the following questions:

1) Why do banks exist?

2) Why do new borrowers (i.e. firms which have only recently begun trading) tend to borrow using bank finance, whilst established borrowers tend to issue debt directly?

Mainstream neoclassical models of financial markets predict that transactions will occur directly between savers and borrowers, there being absolutely no need for a financial intermediary, or middleman, of any kind. But we know that banks do exist, and have existed for 500-600 years, ever since the inception of capitalism itself. So the very existence of banks poses an intriguing puzzle, and an interesting topic for research.

I model the problem as follows. Borrowers have an investment project, for which they seek finance. They have two potential sources of finance: a bank, or direct lenders (i.e. issuing debt directly). Suppose borrowers choose to borrow from direct lenders. We assume that contracts are incomplete. This creates the following moral hazard problem for borrowers. Because contracts are incomplete, they have the opportunity to repudiate their debt obligations, in order to renegotiate with the lender and thus decrease the level of debt repayment. Lenders are powerless to enforce the contract. Even though they know that the borrower's project has succeeded, and he has the funds to repay the debt, the project outcome is unverifiable by a third party, hence the lender cannot ask the courts to enforce repayment.

But direct lenders do have the ability to liquidate, if borrowers repudiate the debt. However, liquidation is very costly, as it totally destroys project output. Thus, direct lenders would actually be better off agreeing to renegotiate, since this means they get a partial repayment, rather than nothing at all.

Although borrowers have the opportunity to repudiate, this doesn't necessarily mean they have the incentive to do so. However, new borrowers with short credit records do have strong incentives to repudiate, for the following reason. I assume two different borrower types with different investment projects. 'Safe' borrowers' projects have zero probability of failure, but 'risky' borrowers' projects have a high probability of failure, such that lenders will prefer not to lend to them. Borrower type is unobservable to lenders. Since new borrowers lack an established credit history, lenders are unable to distinguish whether they are safe or risky. To compensate for the risk of default, lenders pool all new borrowers together and charge them high interest rates. This gives new borrowers an incentive to repudiate, in order to renegotiate down the level of debt repayment. But direct lenders know that new borrowers have an incentive to repudiate. Given that they are unable to enforce the debt contract, and given that liquidation is too costly, it will be optimal for them to refuse to lend to any new borrowers.

So new borrowers are unable to issue debt directly. But what about bank finance? The bank is willing to lend to new borrowers. But the bank faces the same problems as the direct lenders, in that contracts are incomplete, and liquidation is very costly. But the bank is different to direct lenders in the following way. Direct lenders lend for a single period only, then they die off. The bank, however, is long-lived, and is a multi-period player in the credit market. This gives the bank the ability to build a reputation for being tough on borrowers who repudiate. Hence, if borrowers repudiate today, the bank will liquidate. This convinces borrowers that the bank is tough, and thus if they repudiate again tomorrow, they believe that the bank will probably liquidate again, in which case they would receive a project return of zero. Hence, borrowers will prefer to repay the debt.

But it is only worthwhile for the bank to build a reputation if the cost of building the reputation (i.e. the cost of liquidation) can be offset by a stream of rents generated by the reputation. But where do these rents come from? We assume that the bank is a monopolist[1]. Given this, and given that direct lenders will refuse to lend to any new borrowers, the bank has market power and is therefore able to charge a premium on interest rates. Hence, it will be optimal for the bank to liquidate if borrowers repudiate, in order to be able to charge interest rate premiums on future dates. Thus, the threat of liquidation is credible, and is an effective deterrent to repudiation.

Hence, the answer to the question, why do banks exist, can be summarized as follows. Banks exist in order to lend to new borrowers with short credit histories. These borrowers are unable to issue debt directly, because direct lenders expect them to repudiate their debt obligations. The bank is able to lend, because it is a multi-period player in the credit market, which allows it to build a reputation for being tough and thus deter borrowers from repudiation.

But what role do direct lenders play in the credit market? Over time, if borrowers continue to repay debt on all dates, they establish a good credit history. This convinces direct lenders that the borrower is probably a safe borrower, and thus they will charge lower interest rates, to reflect the lower risk of default. But now that borrowers are charged low interest rates, their incentives change. If they repudiate, lenders will think they are risky borrowers, and will refuse to lend to them anymore. Hence, these borrowers risk losing access to cheap credit, which they have earned by building up a spotless credit history. Hence, they will prefer to repay the debt, because they are concerned about maintaining their reputation as a good credit risk. Direct lenders know that established

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[1]This assumption is not crucial to the model. We explain later how the market structure can be derived from first principles. There will still be a concentrated market structure in equilibrium, although not necessarily a monopoly. But the important point is that banks will be able to charge interest rate premiums, because direct lenders will refuse to lend, and hence these premiums will not be competed away.

borrowers will repay for this reason, hence they are willing to lend to them. This explains why established borrowers are able to issue debt directly, whereas new borrowers are not.

The focus of the recent literature has been to explain why certain classes of borrowers choose bank finance rather than issuing debt directly. Most papers have approached this issue by claiming that certain borrowers prefer bank finance because banks provide some special service which direct lenders are either unable or unwilling to offer. For example, in Diamond (1991), banks monitor borrowers' project choices, which reduces the degree of adverse selection and moral hazard, and hence allows banks to charge a lower interest rate than direct lenders, to borrowers without established credit histories. In Chemmanur and Fulghieri (1994), banks are willing to evaluate the prospects of firms in financial distress, and are thus less likely to liquidate inefficiently than direct lenders. Hence borrowers who are more prone to financial distress are actually willing to pay an interest rate premium on bank finance, compared to the rate on directly-placed debt, in order to benefit from the evaluation service that the bank provides.

My model, however, provides an entirely different explanation for the existence of banks. Direct lenders refuse to lend to borrowers with short credit histories, because these borrowers have incentives to repudiate their debt obligations. Direct lenders cannot compensate for the risk of repudiation by charging even higher interest rates, as borrowers will renegotiate downward the debt repayment in any case. Therein lies the function of the banking sector: banks lend to new borrowers who are unable to obtain finance elsewhere. There are two key differences between the bank and direct lenders. First, the bank is a multi-period player in the credit market, and therefore has the ability to build a reputation for toughness. Second, there exists a concentrated market structure in the banking sector, whereas the market for directly-placed debt is competitive. This is crucial to the model, since it allows the bank to charge interest rate premiums. Note that in my model, there is nothing intrinsically different about bank debt and directly-placed debt. Although borrowers with short credit histories would prefer to borrow from direct lenders, they are unable to, and are forced instead to borrow from the bank at a premium rate.

Hence, these chapters provide a different explanation for the existence of banks. But it is also an improvement on the existing theories for the following reasons. It explains the following two important stylized facts of the credit market, which the existing literature has been unable to account for:

1) Banks have a reputation for aggressive liquidation.

2) Small businesses borrow almost exclusively from banks and pay an interest rate premium on bank debt[2].

The first stylized fact is most relevant to the US and UK banking systems, in which banks have a reputation for being tough on firms in financial distress, tending to liquidate immediately rather than arranging rescue packages and allowing time for financial restructuring, which is more typical of the Continental and Japanese banking systems (see Frankel and Montgomery (1991)). Note that Chemmanur and Fulghieri (1994) predicts the opposite result. In their model, banks are willing to be flexible if borrowers default, rather than liquidating without hesitation.

The second fact concerns small businesses' dependence on bank finance (the main UK clearing banks provide around 90 per cent of all small firm lending (Batchelor 1989)), and the existence of interest rate premiums on bank debt. Note that Diamond (1991) actually predicts the opposite, that banks charge a lower interest rate to new borrowers than direct lenders. Although Chemmanur and Fulghieri (1994) does predict that banks charge interest rate premiums, in their model borrowers are actually willing to pay an interest rate premium on bank finance, in order to benefit from the evaluation service

[2]The evidence on how high bank interest rates are is mixed. However, the headline interest rates charged by banks is not so important. What matters is the overall cost of bank debt. There is substantial evidence whcih suggest this is high for small businesses. For example, the Cruickshank Report concluded that banks in the UK use their considerable market power to overcharge small businesses on interest rates, and to impose high non-interest charges (e.g. banks insist that small business customers also maintain their current accounts (which levy high transactions costs) with the bank if they wish to obtain finance). The model I use generalizes easily to consider these 'hidden charges', and the results are unaffected. The important point is the bank has market power, and is therefore able to impose high non-interest charges.

that the bank provides. However, there is a wealth of survey evidence (e.g. Binks et al. (1992), Storey et al. (1988), Cowling et al. (1991), Bank of England (1996)), which suggests that this does not explain what is actually going on. Small businesses in the UK do not give the impression that banks perform a special service for which they are willing to pay a premium. On the contrary, they complain about overcharging on interest rates and the lack of availability of alternative sources of finance. But this is exactly what my model predicts. New borrowers are unable to borrow directly, and must therefore rely on bank finance, and are forced to pay whatever the bank charges.

My model also generates two important implications for financial policy. Firstly, the small business lobby in the UK has long argued that banks sometimes act precipitately by appointing receivers to a struggling business without giving it the opportunity to explore alternatives such as restructuring. These concerns have been voiced loudly, especially during the 1990-92 recession, in which period relations between small firms and banks reached an all time low. Given the widely accepted view that the small business sector is an important engine of growth, innovation and employment creation, the government has taken these problems seriously and has responded by undertaking reforms to corporate insolvency procedures which curb creditors' powers to liquidate small businesses (Bank of England 1996). However, although such reforms may reduce the incidence of inefficient liquidation, my model suggests that they may be counter-productive. If banks' powers to liquidate are curtailed, then borrowers face less of a deterrent to repudiation, which may affect banks' willingness to lend to small businesses. Hence, it is important that reforms to financial policy recognize the trade-off between deterring debt repudiation and minimizing inefficient liquidation.

Secondly, suppose the government imposed an interest rate ceiling to prevent banks from charging high interest rates. In my model, this would mean that if borrowers repudiated, the bank would have no incentive to liquidate and thus build a reputation for toughness, as there would be no future rents to offset the cost of liquidation. Hence liquidation would no longer be a credible threat, and borrowers would not be deterred from repudiation. Thus, the bank would be better off not lending to new borrowers. Hence, measures to cut interest rates charged by banks could be counter-productive.

Chapter 1

Borrower Reputation Effects and the Market for Directly-Placed Debt

1.1 Introduction

In this chapter, we solve for equilibrium when borrowers borrow directly on the open market. We also analyze the dynamics of borrower reputation effects. We begin the model with a distribution of observationally equivalent, infinitely-lived borrowers with different project types and with no credit history, and a large number of direct lenders and one bank lender with no previous history of lending. The proportion of risky borrowers in the market is initially quite high, such that market interest rates are too high for borrower reputation to be effective in deterring repudiation. Direct lenders will therefore refuse to lend, given that they face debt repudiation if they do lend. However, the bank is willing to lend, because it is able to deter repudiation by building a reputation for toughness. But the bank faces a finite horizon for the following reason. Every period, there exists a positive probability that risky borrowers will be forced to default due to project failure, thus revealing their type and resulting in credit termination. This means for any borrower who repays the debt, the probability that it is a safe borrower rises, and thus it is charged a lower interest rate on future dates. This implies that the value of repaying debt and thus retaining access to credit rises, i.e. the net value of borrower reputation increases over time. On some date, the net gain from retaining access to credit exceeds the pay-off from repudiation, even if the borrower faces zero probability of liquidation. Hence on this date, direct lenders can lend, given that borrowers' concerns about maintaining their reputation will police their incentives, and thus the competitive market for directly placed debt becomes open. Borrowers with a good credit history will prefer to borrow directly, given that the bank charges a premium on interest rates, and thus on this date they will switch from bank finance to direct finance. Hence, the bank faces a finite horizon.[1]

The structure of the chapter is as follows. In section 1.2 we set out the model. In section 1.3, we solve for borrowers' optimal repayment strategies, and show that if interest rates are too high, borrowers will prefer to repudiate debt. We then show that as the market interest rate falls over time, the net value of borrowers' reputations rises and thus incentives to repay debt improve.

We show that safe borrowers value future borrowing more highly than risky borrowers. This implies that it could be possible that on some date, safe borrowers prefer to repay debt, whilst risky borrowers prefer to repudiate, resulting in the separation of borrowers. However, we prove in section 1.4 that there exists no such separating equilibrium.

We then show in section 1.5 that the threshold date on which borrowers switch from bank finance to direct finance does actually exist, and hence borrowers are not tied to the bank for the rest of their lives. We also prove that once borrowers switch to direct finance, they never switch back again to bank finance on a later date. Finally, we derive the threshold date. We show that it is the first date on which the pay-off from repaying debt and retaining access to future credit exceeds the pay-off from repudiation, for both borrower types.

[1]If we introduce entry of new cohorts of borrowers each period, the bank will no longer face a finite horizon and will continue to lend in the model's steady state. If we assume that the new cohorts face the same initial degree of adverse selection as the original cohort of borrowers, and similarly that the new borrowers have no prior credit history, then they will also be unable to borrow directly and must rely on bank finance.

1.2 The Model

Borrowers

Borrowers are risk neutral and infinitely-lived. They receive no endowment but have access to an investment project each period. There are two types:

- type A borrowers have one riskless project each period. They can invest I units and receive $Y(I)I$ units with probability 1, where $Y(I) = a - bI$, with $a > 0$, $b > 0$ hence the investment technology exhibits diminishing returns,

- type B borrowers have one risky project each period, which returns $Y(I)I$ with probability π and zero with probability $1 - \pi$, where $\pi Y(I) < 1$, hence type B projects yield negative net present value to lenders.[2]

Borrowers must invest funds in their projects, they cannot invest instead in the riskless asset or consume funds directly.

The initial population of borrowers contains a publicly observable fraction f_A of type A's and $f_B(= 1 - f_A)$ of type B's. Borrower type is private information and all borrowers are initially observationally equivalent. Project returns are independently distributed. I assume that although the project outcome is observed by both borrower and lender, it is not observable by any third party, including other lenders. Hence lenders are unable to enforce repayment by writing a comprehensive contract, i.e. contracts are incomplete, and the only enforceable contract is a debt contract which specifies a fixed payment of r per unit loan, where r is the gross interest rate, and entitles the creditor to liquidate the project in the event of default. However, I assume that lenders are unable to commit ex ante to liquidation. This gives rise to the following incentive problem. Borrowers can repudiate their debt contracts in order to negotiate downward the level of debt repayment.

[2]We assume that the type B project return in the successful state is the same as the type A project return. This is purely for analytical convenience. The nature of the results are unaffected if we assume type B has a different project output function $Z(I) \neq Y(I)$, providing that $\pi Z(I) < 1$ holds.

Although I do not explicitly model the bargaining process, I assume that after debt repudiation, the lender still requires the borrower's cooperation to deliver project returns, hence both the borrower and lender have some bargaining power over the project output. We set the renegotiated payment to lenders at $(1-\mu)\,Y\,(I)\,I$, where μ parameterizes the borrower's bargaining power, and $(1-\mu)\,Y\,(I) < 1$, hence the following condition holds

$$(1-\mu)\,Y\,(I)\,I < rI \qquad \forall r \geq 1 \tag{1.1}$$

i.e. lenders are worse off after contract renegotiation even when the interest rate on debt is at its lowest level, equal to the riskless rate.

Borrowers maximize discounted expected consumption, given by $\sum_{t=1}^{\infty} \beta^{t-1} E(c_t)$, where c_t is period t consumption, and β is the discount factor with $0 < \beta < 1$. Consumption of all agents must be non-negative each period, hence borrowers have limited liability. This means that type B borrowers can earn a positive expected return on their projects, given that $\pi(Y\,(I) - r)I + (1-\pi)0 \geq 0$ where $r \leq Y\,(I)$, even though type B projects yield negative net present value to lenders. I also assume that borrowers cannot save. This allows us to focus on the importance of reputation effects in improving borrowers incentives, and abstracts from issues of optimal capital structure (i.e. what mix of debt and internal equity should borrowers use). Although there exists a countable infinity of both borrowers and direct lenders, we assume borrowers' projects are in relatively short supply, and hence the riskless asset is in use in any equilibrium. Hence, in the competitive credit market, direct lenders will lend to any borrower who offers a debt contract with an expected return no less than the riskless rate of return.

Lenders

Lenders are risk neutral and receive an endowment of consumption goods each period. They have a choice over how to invest this:

1) they can invest in a riskless asset, which has a gross rate of return of R, or

2) they can lend to borrowers who have access to investment projects.

For simplicity, we set the riskless rate of interest to zero, i.e. $R = 1$, and assume that lenders do not discount the future. This does not in any way affect the results, but is used for analytical convenience.

There are two distinct categories of lenders:

1) The bank. I assume that the bank is a monopolist, and is infinitely-lived.

2) Direct lenders. These live for a single period only, hence borrowers face a new generation of direct lenders each period. In order to model the competitive structure of the market for directly-placed debt, I assume there are a large number (a countable infinity) of small direct lenders, each of whom has no individual influence over the market interest rate.

I assume that the bank is observationally distinct from direct lenders, hence direct lenders cannot masquerade as bank lenders, and vice-versa.

Both the bank and direct lenders have a liquidation technology which gives them the ability to liquidate borrowers' projects if they default on debt repayment. Both the bank and direct lenders receive a pay-off of $(1 - \mu)Y(I_t)I_t$ if they renegotiate. Direct lenders receive a pay-off of zero from liquidation.

I assume the following information structure. At each point in time, lenders can observe borrowers' entire credit histories. Specifically, it tells them whether a borrower has ever defaulted on a loan, but does not reveal whether default was strategic (i.e. repudiation) or forced by project failure. Also, at each point in time borrowers can observe the bank's history of actions, namely on which dates it has chosen to lend in the credit market, and whether it liquidated or renegotiated when borrowers have defaulted.

We model the problem as the repeated play of an extensive form stage game with incomplete information, using the concept of sequential equilibrium. We describe the one period stage game for any given date t' below.

We describe first the stage game in which borrowers borrow from the bank. All bank pay-offs are given net of the return available from the riskless asset. The stage game consists of the following sequence of steps:

Step (1): The bank decides whether to lend to borrowers in the credit market on date t', or to invest in the riskless asset. If it does not lend, then the date t' stage game ends, the bank receives a net pay-off of zero, and all borrowers receive a pay-off of zero. If it lends, the game proceeds to step (2).

Step (2): The bank sets the interest rate, and also decides whether to ration the amount of credit it will lend to each borrower.

Step (3): Nature determines the outcome of a randomizing device, which is observable to all borrowers but not to the bank.

Step (4): Borrowers offer debt contracts to the bank, specifying the interest rate and the amount they wish to borrow.

Step (5): The bank chooses which debt contracts to accept. If the bank refuses to lend to any given borrower, then the date t' stage game ends for that borrower and it and the bank receive a pay-off of zero.

Step (6): Nature determines the outcome of type B projects. If a borrower's project fails, then its date t' stage game ends, it receives a pay-off of zero and the bank receives a net pay-off of $-I_t$. If its project succeeds, then it proceeds to step (7).

Step (7): Borrowers choose whether to repay or repudiate. If a borrower repays, then its date t' stage game ends, it receives a pay-off of $(Y(I_{t'}) - r_{t'})I_{t'}$, and the bank receives a net pay-off of $(r_{t'} - 1)I_{t'}$. If it repudiates, then the game proceeds to step (8).

Step (8): The bank decides whether to liquidate or renegotiate with each borrower who repudiated. If it liquidates, it receives a net pay-off of $q(w) - I_t$ (if it is class W) or $(r_{t'} - 1)I_{t'}$ (if it is class S), and the borrower receives a pay-off of zero. If it renegotiates, it receives a net pay-off of $-(1 - (1 - \mu)Y(I_{t'}))I_{t'}$, and the borrower receives a pay-off of $\mu Y(I_{t'})I_{t'}$.

At steps (4) and (7), the borrower is at an information set regarding the bank's class and type. At steps (1), (2), (5), and (8), the bank is at an information set regarding the borrower's type, and at steps (5) and (8) it is at an information set regarding the outcome of the randomizing device.

The bank plays this stage game against every borrower. The game is structured such that the bank plays step (5) sequentially against all borrowers before it moves on to step (8), which it also plays sequentially against each borrower.

The stage game in which direct lenders lend is identical to the above, except that step (2) is omitted, given that direct lenders have no control over the interest rate and are unable to ration credit.

It is useful to set out the definitions of the strategies of players, how beliefs are formed, and the equilibrium concept used.

A strategy for the borrower (type A or B) is the following:

a) a choice of debt contract, specifying the interest rate and the amount of funds they wish to borrow, and the length of the contract

b) a decision on whether to repay or repudiate

A strategy for the bank is the following:

a) a decision on whether to lend to borrowers in the credit market on date t', or to invest in the riskless asset

b) a choice of the interest rate r_t, and a decision on whether to ration the amount of credit it will lend to each borrower

c) a decision on whether to accept any given borrower's debt contract

d) a decision on whether to liquidate or renegotiate

A strategy for the direct lender is the following:

a) a decision on whether to lend to borrowers in the credit market on date t', or to invest in the riskless asset

b) a decision on whether to accept any given borrower's debt contract

c) a decision on whether to liquidate or renegotiate

The concept of sequential equilibrium must satisfy the following conditions:

1) At each information set, the player with the move must form a belief about which node in the information set he is at, by assigning a probability to the event that he is at any given node.

2) Players must be sequentially rational, i.e. players' strategies at any given information set must be optimal, given their beliefs and given their subsequent strategies and the other players' subsequent strategies.

3) Beliefs must be consistent with equilibrium strategies, i.e. beliefs must be updated using Bayes' Rule.

4) Beliefs for information sets off the equilibrium path must be consistent with some small perturbation of the equilibrium strategy profile, such that these informations sets are actually reached during play.

1.3 The Dynamics of Borrower Reputation Effects

Figure 1 illustrates the borrower's moral hazard problem. New borrowers without an established credit history face a problem of adverse selection, i.e. there is a high proportion of type B borrowers in the market, but all borrowers are observationally equivalent. Hence, all new borrowers are pooled together and charged high interest rates, to reflect the high risk of default. This gives these new borrowers the incentive to repudiate, in order to renegotiate downward the level of debt repayment. If borrowers decide to repudiate, it is optimal for the direct lender to renegotiate, because its pay-off from renegotiation $(1-\mu)Y(I_t)$ exceeds the pay-off from liquidation (zero). Suppose the lender threatens to liquidate, in order to try to deter the borrower from repudiation. But borrowers know that direct lenders will prefer ex post to renegotiate, hence the threat is not credible and hence they will not be deterred from repudiation. Direct lenders know that it is optimal for borrowers to repudiate, hence they will refuse to lend, and thus borrowers are forced to borrow from the bank.

On every date, lenders form an assessment of each borrower's type, on the basis of its track record for debt repayment. On the basis of this assessment, lenders decide whether to re-lend to a borrower, or to terminate credit. If the track record implies that the

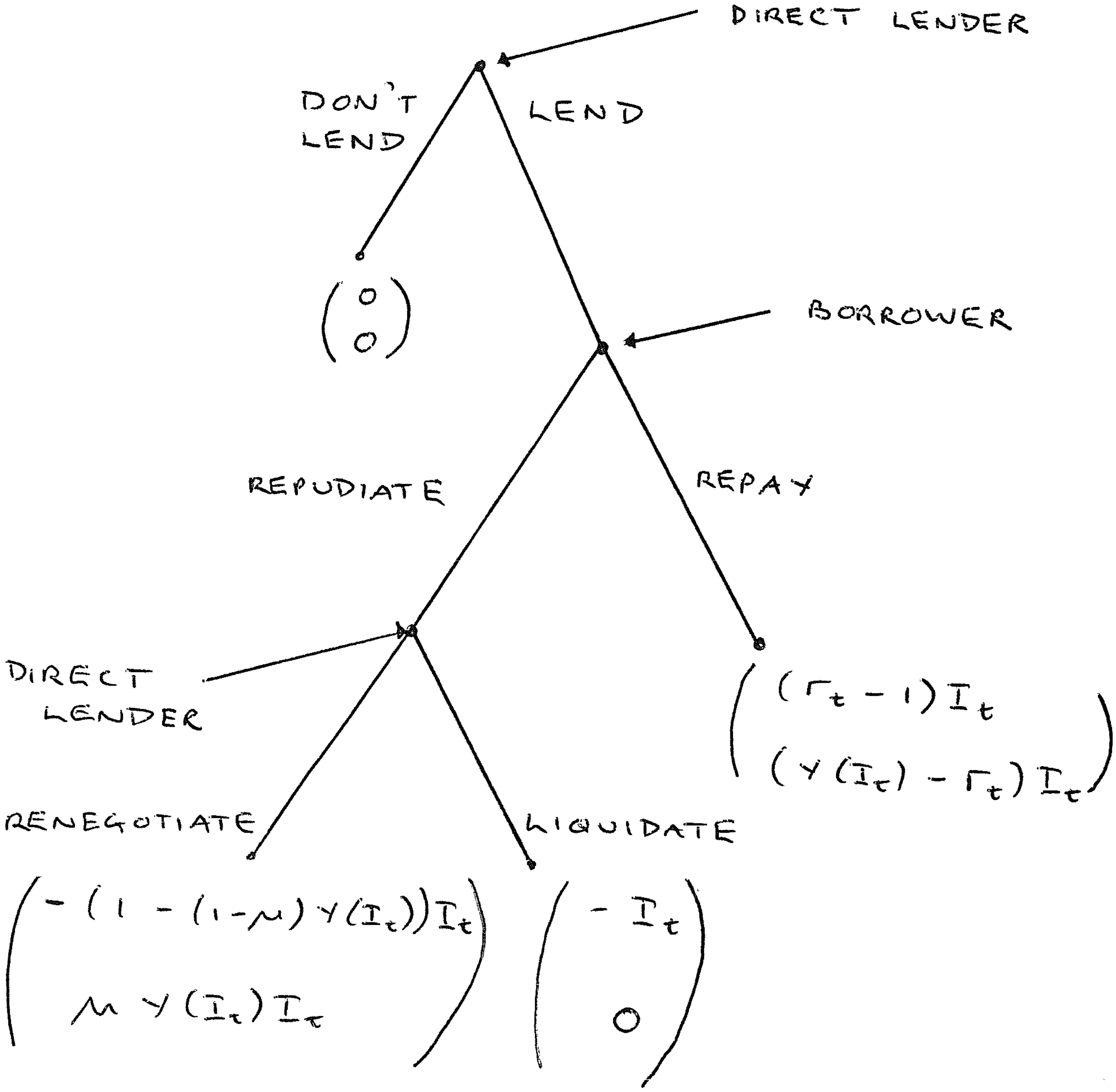

FIGURE 1 : THE BORROWER'S MORAL HAZARD PROBLEM

probability that the borrower is type B is high, then direct lenders will terminate credit. If borrowers continue to repay debt on all dates, they establish a good credit history, and thus the lender's assessment of the probability that the borrower is type A rises (since only type B projects fail). Hence, direct lenders will re-lend to these borrowers and will charge them lower interest rates, to reflect the lower risk of default. But now that borrowers are charged low interest rates, their incentives change. We illustrate this in Figure 2. Suppose a borrower whose project has succeeded decides to repudiate. How will lenders revise their belief about the borrower's type? Note that lenders cannot distinguish between a borrower who defaults due to project failure, and a borrower who defaults strategically (hence we have the information set), because project outcome is not publicly observable. Hence, if lenders form the belief that if the borrower defaults, then it must be a type B borrower whose project failed (i.e. they are at node n_2 in the information set), their optimal response will be to terminate this borrower's credit. Hence, borrowers with good credit histories will prefer not to repudiate, because they don't want to jeopardize their reputations for being a good credit risk, and thus have their credit terminated, given that they are now able to borrow cheaply from direct lenders. This is why direct lenders are willing to lend to borrowers with established credit histories, because these borrowers are concerned about their reputations, and thus do not have incentives to repudiate debt.

We begin this section by deriving type A and B borrowers' optimal repayment strategies if they borrow directly, as a function of current and future market interest rates. First, we must analyze what are direct lenders' equilibrium re-lending strategies. We prove below that in equilibrium, direct lenders' beliefs about borrower type, if a borrower defaults, are that he is type B, and hence it is optimal to terminate his credit. Hence, in equilibrium, if a borrower defaults, the present discounted value of his future pay-offs is zero.

It is an optimal strategy for type A borrowers to repay debt if and only if the following holds:

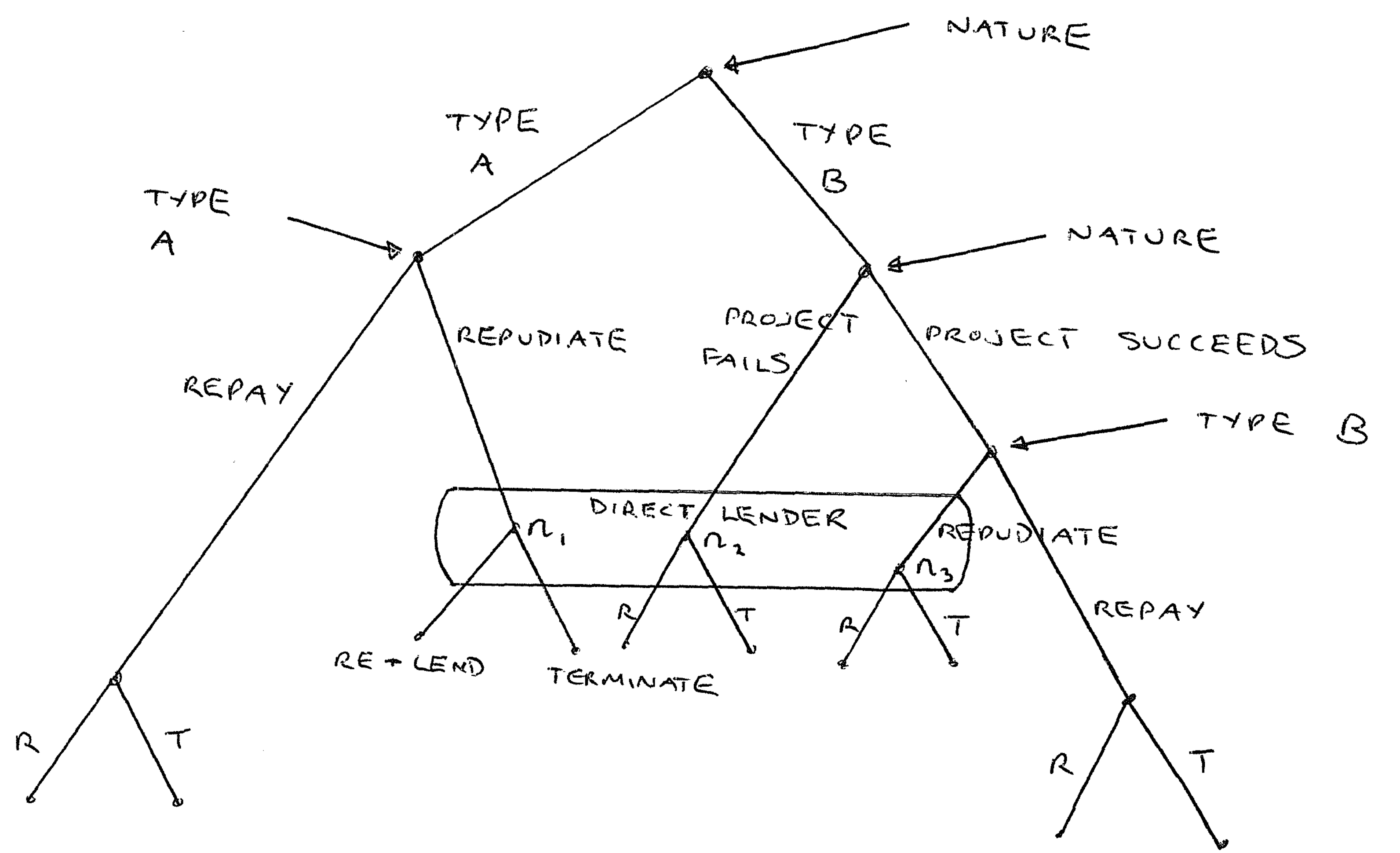

FIGURE 2: INCENTIVES OF BORROWERS WITH ESTABLISHED CREDIT HISTORIES

$$V_{At}(rep) \geq V_{At}(def)$$

$$\Leftrightarrow (Y(I_t) - r_t)I_t + \beta V_{At+1} \geq \mu Y(I_t)I_t$$

$$\Leftrightarrow (r_t - (1-\mu)Y(I_t))\, I_t \leq \beta V_{At+1} \tag{1.2}$$

where $V_{At}(rep)$ and $V_{At}(def)$ are the expected present discounted value of repaying and defaulting on debt respectively on date t, and V_{At+1} is the expected future value to type A borrowers of retaining access to credit on future dates, given future interest rates. Hence condition (1.2) states that if interest rates are low enough, and thus the value of retaining access to credit on future dates is high enough, the return to repaying debt exceeds the pay-off from repudiation, and hence reputation is effective for type A borrowers.

Similarly for type B borrowers, it is an optimal strategy to repay debt if and only if the following holds:

$$V_{Bt}(rep) \geq V_{Bt}(def)$$

$$\Leftrightarrow (Y(I_t) - r_t)I_t + \beta\pi V_{Bt+1} \geq \mu Y(I_t)I_t$$

$$\Leftrightarrow (r_t - (1-\mu)Y(I_t))\, I_t \leq \beta\pi V_{Bt+1} \tag{1.3}$$

We now derive direct lenders' optimal lending strategies. We assume that the market for directly-placed debt is competitive, hence no individual direct lender has any influence over the market interest rate. Direct lenders will therefore accept any debt contract which offers an expected return no less than the riskless interest rate. Suppose both type

A's and type B's (if their projects succeed) repay debt on date t. Then

$$Eu_{Dt} = F_{At}r_t + F_{Bt}\pi r_t \geq 1$$

$$\Leftrightarrow r_t \geq \frac{1}{F_{At} + F_{Bt}\pi} = r_t^D \tag{1.4}$$

where F_{At} and F_{Bt} are the proportion of type A and B borrowers respectively in the market at the start of date t, and Eu_{Dt} is the lender's expected return on date t.

Hence lenders will accept any debt contract which offers an interest rate $r_t \geq r_t^D$. Given that borrowers are relatively scarce, they will offer the lowest rate necessary to ensure that the contract is accepted. Hence in equilibrium, $r_t = r_t^D$. Given the market interest rate r_t^D, borrowers will borrow amount I_t^D given by $I(rep, r_t^D)$, which is the borrower's demand for funds function (which is solved for in section 2.4 below).

But suppose both borrower types repudiate debt. Then

$$Eu_{Dt} = (F_{At} + F_{Bt}\pi)(1 - \mu)Y(I_t)$$

Hence $Eu_{Dt} < 1$, given that $(1 - \mu)Y(I_t) < 1$ by assumption, and hence it is optimal for lenders to refuse to accept debt contracts. Thus, on dates when borrower reputation is ineffective, direct lenders will refuse to lend.

Both the bank and direct lenders will refuse to lend to any borrower who is revealed to be type B. Type B's expected project return is $\pi Y(I) < 1$. Hence, even if lenders charge the maximum possible interest rate $r = Y(I)$, they receive an expected return lower than the return on the riskless asset. Hence, it is optimal to terminate credit to any borrower who is revealed to be type B.

We assume that f_B, the initial proportion of type B borrowers in the credit market is sufficiently high as to drive up interest rates such that on date $t = 1$, conditions (1.2) and (1.3) fail to hold. However, over time, the proportion of type B borrowers in the market declines, as they default and thus reveal their type, hence resulting in their exclusion from future borrowing. Hence, F_{Bt} falls over time, and from (1.4), the interest rate charged by direct lenders falls over time. As interest rates fall, the value of repaying debt and thus retaining access to future borrowing opportunities increases, i.e. the net value of borrower reputation increases.

On dates when interest rates are low enough such that conditions (1.2) and (1.3) hold, borrowers will prefer to borrow directly at the market interest rate given by (1.4), rather than from the bank, given that the bank charges an interest rate premium (as we prove in section 2.5 below). Given that (1.2) and (1.3) hold, borrowers will prefer to repay debt, and hence, direct lenders will be willing to lend. This means that in equilibrium, the only borrowers who default are type B borrowers who are forced to default due to project failure. Hence, lenders' beliefs that default implies type B are consistent with borrowers' equilibrium strategies.

We now derive the dynamics of the competitive market interest rate series. On any date t, every type B borrower faces a probability of project failure of $1-\pi$. If its project does fail, the borrower is revealed to be type B, and it is denied future credit. Hence the evolution of F_{At} and F_{Bt},the proportions of type A and B borrowers remaining in the market at the beginning of date t, is given as follows:

$$F_{At} = \frac{F_{At-1}}{F_{At-1} + \pi F_{Bt-1}}, \qquad F_{Bt} = \frac{\pi F_{Bt-1}}{F_{At-1} + \pi F_{Bt-1}} \tag{1.5a}$$

Hence, $\theta_t = F_{At} + \pi F_{Bt}$ is monotonically increasing over time, given that $F_{At+1} > F_{At}$ and $F_{Bt+1} < F_{Bt}$ for all t. This implies that, from (1.4), $r^D_{t+1} < r^D_t$ for all t. Hence, interest rates are monotonically decreasing over time.

Let us define date $T+1$ as the earliest date on which reputation is effective for type B borrowers, i.e. condition (1.3) holds. But if reputation is effective for type B's, then it must also be effective for type A's. This is because for all t, $V_{At+1} > V_{Bt+1}$, given that type A's project has a higher expected return then type B's, hence if (1.3) holds then (1.2) must also hold. Hence direct lenders will lend on date $T+1$, because both borrower types will prefer to repay the debt. But the fact that $V_{At+1} > V_{Bt+1}$ for all t implies that there exists some date $t_A < T+1$ on which (1.2) holds, but (1.3) fails to hold. Hence on this date, if direct lenders were to lend, type A's would repay the debt but type B's would prefer to repudiate. Direct lenders would have to charge a higher interest rate than (1.4), given by r'_t, to compensate for the lower (renegotiated) repayments from type B borrowers, so that they receive an expected return no lower than the return on the riskless asset. Hence lenders will accept debt contracts from borrowers

$$\Leftrightarrow F_{At}r'_t + F_{Bt}\pi(1-\mu)Y(I_t) \geq 1$$

Now that suppose with interest rate r'_t, (1.2) still holds. Then the equilibrium outcome would be for direct lenders to lend, type A's to repay the debt, and type B's to repudiate. Thus type A and B borrowers would separate. However, we now show that there is no such separating equilibrium.

1.4 Analysis of the Existence of a Separating Equilibrium

This section proves that there exists no separating equilibrium, in which on some date $\tilde{t}$ type A borrowers repay debt and type B's repudiate. We solve for the equilibrium on dates $t > \tilde{t}$ after borrowers have separated. Borrowers who repudiate are revealed to be type B, and are therefore excluded from future borrowing. Borrowers who repay are revealed to be type A. We then show that direct lenders will also refuse to lend to borrowers who are known to be type A. We do this by first solving for equilibrium assuming that only direct lenders lend. In this case, there exists two equilibria, one in which borrowers' strategy is to repay debt on all dates and hence it is optimal for direct lenders to lend on all dates (the repayment equilibrium), and the other in which borrowers' strategy is to repudiate on all dates, and hence it is optimal for direct lenders to refuse to lend on all dates (the no lending equilibrium). We then analyze what happens when we allow the bank to lend. In the no lending equilibrium, providing its reputation is intact, the bank can deter repudiation by credibly threatening to liquidate defaulters. Hence the bank will lend, and will charge a monopoly premium on interest rates given that no other lender will lend. In the repayment equilibrium, we show that the effect of allowing the bank to lend is that direct lenders will refuse to lend, hence the bank will again be the sole lender and will charge a monopolist premium on interest rates. This occurs because direct lenders can only deter repudiation by threatening to terminate credit if borrowers repudiate, given that they cannot commit to liquidation. However, when we allow the bank to lend, the threat of credit termination by direct lenders becomes ineffective, since the bank will re-lend to borrowers who repudiate. We show that it may be optimal for borrowers to default on directly-placed debt, even if they are forced to pay the bank's monopoly premium on interest rates forever after. We then show that, if this fails to hold, borrowers can offer a long term debt contract to the bank, which specifies future interest rates for a fixed length of time. Providing the contracted interest rates are low enough, borrowers will again have an incentive to repudiate on directly-placed debt, and then borrow from the bank on future dates. Hence direct lenders will refuse to lend.

Thus the unique equilibrium following separation is one in which direct lenders do not lend and hence type A borrowers are forced to borrow from the bank. This reduces significantly type A's pay-off from separation, such that, providing a certain condition holds, it will not be optimal on any date for type A borrowers to separate from type B borrowers.

Suppose type A's do separate from type B's on some date $\tilde{t} < T + 1$. We now solve for the post-separation equilibrium on dates $t > \tilde{t}$. We begin by solving for equilibrium assuming that only direct lenders lend.

Post-separation equilibrium with no bank lending

Given that there is no longer incomplete information about borrower type, we seek a subgame perfect equilibrium. Lemma 1 proves that there are in fact two subgame perfect equilibria.

Lemma 1 *Assuming the bank does not lend, in the post-separation game there exists the following two equilibria:*

1) The repayment equilibrium. Direct lenders' strategies are to lend on date $\tilde{t}+1$, and to re-lend on all subsequent dates to any type A borrower who has not defaulted since date $\tilde{t}+1$, on any debt contract which offers an interest rate $r_t \geq 1$. If a borrower has defaulted at least once, direct lenders terminate credit on all subsequent dates. Type A borrowers' strategies are to repay every period, but after the first default, to default on all subsequent dates. In the repayment equilibrium, direct lenders will lend on all dates and borrowers will repay on all dates.

2) The no lending equilibrium. Direct lenders' strategies are to refuse to lend on all dates. Borrowers' strategies are to repudiate on all dates.

Proof. We begin with the repayment equilibrium. We first prove that the equilibrium strategies form a Nash equilibrium in the game as a whole. Given that borrowers repay on all dates, it is optimal for direct lenders to lend on all dates to any borrower who has not defaulted since date $\tilde{t}+1$. As noted above, direct lenders will accept any debt contract which offers an expected return no less than the riskless interest rate. Given that type A projects have zero probability of failure, direct lenders will accept debt contracts from type A's which offer the riskless interest rate. Given that default provokes credit termination, it is optimal for borrowers to repay on all dates

$$\Leftrightarrow \quad V_{At}(rep) > V_{At}(def)$$
$$\Leftrightarrow \quad (1-(1-\mu)Y(I^*))\,I^* < \lim_{T\to\infty}[\sum_{i=1}^{T-t}\beta^i(Y(I^*)-1)I^*] \tag{1.6}$$

where $I^* = \frac{a-1}{2b}$ (from (2.9) below). Assuming that (1.12) holds, from (1.11) and given that $\lim_{T\to\infty}[\sum_{i=1}^{T-t}\beta^i(Y(I^*)-1)I^*] > \lim_{T\to\infty}[\sum_{i=1}^{T-t}(\beta\pi)^i(Y(I^*)-1)I^*]$, (1.6) must also hold.

To prove that this is a subgame perfect equilibrium, we must prove that the equilibrium strategies form a Nash equilibrium in every subgame. Given that on any date, the set of possible subgames is equivalent to the set of possible histories, we can classify subgames into two types, those in which borrowers have repaid debt on every date $t \geq \tilde{t}+1$, and those in which borrowers have repudiated at least once. In the first type of subgame, given that the continuation game is identical to the game as a whole, and that the equilibrium strategy profile specifies strategies which are a Nash equilibrium in the game as a whole, they must also constitute an equilibrium in this subgame. In the second type of subgame, the equilibrium strategy profile specifies that direct lenders terminate credit on all future dates, given that borrowers have defaulted at least once. This is an optimal strategy, given that borrowers' strategies are to repudiate on all future dates. Borrowers' strategies are also optimal, given that direct lenders terminate credit anyway, hence borrowers' repayment strategies have no effect on their equilibrium pay-offs after the first default. Hence these strategies constitute a Nash equilibrium in the second type of subgame also. Hence the repayment equilibrium is subgame perfect.

We now turn to the no lending equilibrium. The strategy profile in the no lending equilibrium is equivalent to the strategies specified in the repayment equilibrium for the continuation game after borrowers repudiate. We proved above that these strategies constitute a Nash equilibrium. In any subgame, given that the continuation game is identical to the game as a whole, and that the strategy profile specifies strategies which are a Nash equilibrium in the game as a whole, they must also constitute an equilibrium in the subgame. Hence, the no lending equilibrium is also subgame perfect.

Q.E.D.

Post-separation equilibrium with bank lending

We now analyze what happens when we allow the bank to lend. We assume that the bank has retained its reputation up to the separation date (we explain in section

2.2 below how the bank loses its reputation), and thus continues to lend, charging the monopolist interest rate. Lemma 2 shows that when we introduce bank lending, the repayment equilibrium as described above cannot exist. Hence the unique equilibrium is one in which only the bank lends.

Lemma 2 *In the post-separation game, there is a unique equilibrium in which the bank lends on all dates, charges a monopoly premium on interest rates, and liquidates all defaulters, direct lenders do not lend on any date, and borrowers issue bank debt and repay on all dates.*

Proof. First, we claim that direct lenders will not lend in equilibrium, and prove that the bank's optimal strategy is to lend on all dates, charge a monopoly premium on interest rates, and liquidate all defaulters, and borrowers' optimal strategies are to issue bank debt and repay on all dates. We give here a sketch of this part of the proof (see the appendix for the full proof). Suppose borrowers' strategies are to repudiate on all dates. We show that this cannot be an equilibrium strategy. Consider some arbitrary date $\tilde{t} < T' < \infty$ after borrowers have separated. Suppose for the moment that borrowers do repudiate on all dates $t > T'$. This would mean that no class W bank would have an incentive to lend on dates $t > T'$, and thus date T' is effectively the class W bank's horizon. The equilibrium is therefore similar to the finite horizon bank lending equilibrium described in chapter 2. In this equilibrium, class W banks have an incentive to liquidate borrowers in order to build a reputation. However, reputation effects weaken as the horizon approaches, i.e. the bank will prefer to renegotiate with defaulters unless its liquidation costs are low, since their are fewer lending opportunities to defend. But if T' is sufficiently large, there exists some date $\hat{t} < T'$ on which the horizon is long enough such that if borrowers default, all bank types will prefer to liquidate, no matter how high their liquidation costs. Given that borrowers face liquidation with probability 1 if they default, they will prefer to repay debt. Now suppose $T' \to \infty$. Given that the horizon is now infinite, all bank types will prefer to liquidate on all dates.

We now prove that direct lenders will prefer not to lend in equilibrium. Consider the repayment equilibrium as described in lemma 1, in which it is optimal for direct lenders to lend and for borrowers repay on all dates. If borrowers repudiate on any date, direct lenders' equilibrium strategies are to terminate credit, and borrowers' equilibrium strategies are to repudiate on all future dates. But given these strategies, we know from the proof above that it is optimal for the bank to lend. Hence if borrowers repudiate, although they will not be able to borrow directly, they have the option of borrowing from the bank on future dates. We now prove that it is optimal for borrowers to repudiate on directly-placed debt, and then borrow from the bank on future dates.

It is optimal for borrowers to repudiate on directly-placed debt if and only if the following holds:

$$V_{At}(rep) < V_{At}(def)$$

$$\Leftrightarrow \left(Y(I^D) - 1\right) I^D + \beta V_{At+1}(r^D) < \mu Y(I^D) I^D + \beta V_{At+1}(r^M)$$

$$\Leftrightarrow \frac{1}{1-\beta}\left(Y(I^D) - 1\right) I^D < \mu Y(I^D) I^D + \frac{\beta}{1-\beta}\left(Y(I^M) - r^M\right) I^M \qquad (1.7)$$

where $V_{At+1}(r^D)$ is the present discounted value from borrowing directly at the riskless interest rate, $V_{At+1}(r^M)$ is the present discounted value from borrowing from the bank at the monopolist interest rate and $r^M = \frac{a+1}{2}$ (the interest rate set by the bank, derived in section 2.5), $I^D = \frac{a-1}{2b}$, $I^M = \frac{a-1}{4b}$ (from the borrower's demand for funds function). Hence if condition (1.7) holds, i.e. if the pay-off from borrowing from the bank on all future dates is not too much lower than the pay-off from retaining access to direct credit, then it is optimal for borrowers to repudiate on directly-placed debt and thus direct lenders will not lend.

Suppose condition (1.7) fails to hold. We now show that it is still optimal for borrowers to repudiate on directly-placed debt.

Suppose borrowers borrow directly on date $\hat{t}$, but before making the repayment decision, they offer a long term debt contract to the bank, which specifies the interest rate for the following n periods. Then borrowers will default on date $\hat{t}$ and borrow from the bank on future dates if and only if the following condition holds:

$$\left(Y(I^D)-1\right)I^D+\beta V_{A\hat{t}+1}(r^D)<\mu Y(I^D)I^D+\sum_{i=1}^{n}\beta^i\left(Y(I^M_{\hat{t}+i})-r^M_{\hat{t}+i}\right)I^M_{\hat{t}+i}+\beta^{n+1}V_{A\hat{t}+n+1}(r^M)$$

$$\Leftrightarrow\left(1-(1-\mu)Y(I^D)\right)I^D>\beta\left(V_{A\hat{t}+1}(r^D)-V_{A\hat{t}+1}(r^M_{\hat{t}+1}\dots r^M_{\hat{t}+n},r^M)\right)\qquad(1.8)$$

where $V_{A\hat{t}+1}(r^D)$ is the present discounted value from borrowing directly at the riskless interest rate, $V_{A\hat{t}+n+1}(r^M)$ is the present discounted value from borrowing from the bank at the monopolist interest rate, and $r^M_{\hat{t}+1}\dots r^M_{\hat{t}+n}$ is the interest rate series specified in the contract. Condition (1.8) states that borrowers will repudiate if and only if the contracted interest rates are low enough to compensate for the reduction in future pay-offs (compared to the pay-offs available if borrowers were to repay debt and thus retain access to direct credit) after the contract expires, when borrowers will be forced to pay the monopolist interest rates. It is optimal for the bank to accept the contract if and only if $\hat{V}_{M\hat{t}+1}>0$, where $\hat{V}_{M\hat{t}+1}$ is the bank's present value of future pay-offs (net of the return on the riskless asset) during and after the contracted time period on date $\hat{t}+1$. Suppose the contract specifies interest rates lower than the riskless rate, such that it offers negative net present value to the bank. Providing that the bank is able to recoup the cost of the subsidized contracted interest rates by charging the monopolist interest rate after the contract expires, it will still be optimal to accept the contract. But what happens if the bank accepts the contract and then borrowers repay debt? Borrowers will then be able to enjoy the subsidized contracted interest rates without having to pay the monopolist interest rates after the contract expires, given that by not repudiating they retain access to direct credit. Hence the bank will not accept the contract unless it knows that borrowers will default on date $\hat{t}$, so that they are locked out of the direct credit market after the contract has expired. It will be optimal for borrowers to default on date $\hat{t}$ if and only if the following condition holds:

$$
\begin{aligned}
& \left(Y(I^D) - 1\right) I^D + \sum_{i=1}^{n} \beta^i \left(Y(I^M_{\hat{t}+i}) - r^M_{\hat{t}+i}\right) I^M_{\hat{t}+i} + \beta^{n+1} V_{A\hat{t}+n+1}(r^D) \\
< \; & \mu Y(I^D) I^D + \sum_{i=1}^{n} \beta^i \left(Y(I^M_{\hat{t}+i}) - r^M_{\hat{t}+i}\right) I^M_{\hat{t}+i} + \beta^{n+1} V_{A\hat{t}+n+1}(r^M) \\
\Leftrightarrow \; & \beta^{n+1} \left(V_{A\hat{t}+n+1}(r^D) - V_{A\hat{t}+n+1}(r^M)\right) < \left(1 - (1-\mu) Y(I^D)\right) I^D \qquad (1.9)
\end{aligned}
$$

Condition (1.9) states that, provided that n is large enough, i.e. the contract is long enough, it will be optimal for borrowers to default on date $\hat{t}$. The reason for this is that the longer the contract, the further away the date on which borrowers must start paying the monopolist interest rate, hence the lower is the discounted value of repaying debt and thus retaining access to direct credit. Given that (1.9) holds, if the contract specifies interest rates $r_t = 1$ for $\hat{t} + 1 \leq t \leq \hat{t} + n$ then (1.8) will also hold and thus it will be optimal for borrowers to offer the contract. $\hat{V}_{M\hat{t}+1} > 0$ also holds in this case, given that for the duration of the contract the bank gets a net per period pay-off of zero, but is able to charge the monopolist interest rate when the contract expires, hence it is optimal for the bank to accept the contract. Thus borrowers will repudiate on directly-placed debt, which means that direct lenders will refuse to lend in equilibrium. The unique equilibrium following separation is therefore one in which the bank lends on all dates, charges a monopoly premium on interest rates, and liquidates all defaulters, direct lenders do not lend on any date, and borrowers issue bank debt and repay on all dates.

Q.E.D.

Hence lemma 2 proves that following separation, type A borrowers are forced to borrow from the bank at the monopolist interest rate because they cannot commit to repaying directly-placed debt, which reduces their future pay-offs significantly. Proposition 1 proves that, if a certain condition holds, type A borrowers will prefer not to separate from type B's in equilibrium.

Proposition 1 *There exists no separating equilibrium in which type A borrowers repay* *and type B borrowers repudiate directly-placed debt, if the following condition holds:*

$$\left(1-(1-\mu)Y(I^D)\right)I^D > \frac{\beta}{1-\beta}\left(Y(I^M)-r^M\right)I^M \tag{1.10}$$

Proof. It is optimal for type A borrowers to repay debt on date $\tilde{t}$

$$\Leftrightarrow V_{A\tilde{t}}(rep) > V_{A\tilde{t}}(def)$$

This holds only if it holds for the lowest possible interest rate $r_{\tilde{t}} = 1$, i.e. only if

$$\left(Y(I^D)-1\right)I^D + \beta V_{At+1} > \mu Y(I^D)I^D$$

From lemma 2, if type A borrowers separate, then they must borrow from the bank at the monopolist interest rate on all future dates, hence $V_{At+1} = \frac{1}{1-\beta}\left(Y(I^M)-r^M\right)I^M$. Hence $V_{A\tilde{t}}(rep) > V_{A\tilde{t}}(def)$ only if

$$\left(1-(1-\mu)Y(I^D)\right)I^D < \frac{\beta}{1-\beta}\left(Y(I^M)-r^M\right)I^M$$

Hence if condition (1.10) holds, this implies $V_{At}(def) > V_{At}(rep)$, hence type A borrowers will repudiate on date $\tilde{t}$. Hence there exists no separating equilibrium, no matter what interest rate type A borrowers offer on date $\tilde{t}$.

Q.E.D.

Proposition 1 has proved that, providing condition (1.10) holds, there exists no separating equilibrium, in which on some date, type A borrowers have incentives to repay debt, whilst type B borrowers will prefer to repudiate. This is because the incentive to repay debt is that borrowers retain access to future borrowing. However, this incentive is removed if type A's separate from type B's, because type A's will be forced to borrow from the bank after separation, which reduces significantly the value of future borrowing opportunities. We have thus shown that the threshold date on which direct lenders enter the market and borrowers switch from bank debt to directly-placed debt is date $T + 1$, which is the earliest date that (1.3) holds and hence reputation is effective for both borrower types, providing that such a date exists. We must now prove that such a date does indeed exist, i.e. that borrower reputation will eventually be effective, and that direct lenders will enter the market at some stage, and hence borrowers will not be forced to borrow from the bank forever.

1.5 The Existence of the Market for Directly-Placed Debt

Consider some date $t \geq T + 1$. In order for reputation to be effective at some stage, and thus for direct lenders to lend, we must prove that it is optimal for the bank not to re-lend to any borrower who repudiates. We assume that the bank's beliefs about any borrower who defaults are that default implies that the borrower is type B. Hence the bank's optimal strategy is to refuse to re-lend to any borrower who defaults. This means that if borrowers borrow directly and repudiate, they will not be able to borrow from the bank afterwards, hence repudiation will result in credit termination. Thus given that for dates $t \geq T + 1$ conditions (1.2) and (1.3) hold, both borrower types will have an incentive to repay debt, i.e. reputation is effective, and therefore direct lenders will lend. The only borrowers who will default are those type B's constrained to default due to project failure. Hence the bank's beliefs that default implies that the borrower is type B are consistent.

Now suppose that borrowers can offer the bank multi-period debt contracts as in the post-separation equilibrium described above. Does reputation become ineffective as previously? Suppose the bank's beliefs are that 'offer of contract implies borrower is type B'. Then it is optimal for the bank not to accept the contract. Hence repudiation will provoke credit termination, and thus reputation is effective. Given that in equilibrium,

neither borrower type will find it profitable to deviate and offer the bank a multi-period debt contract, the bank's beliefs about borrower type if a borrower offer such a contract are consistent. This specification of beliefs also satisfies the Cho-Kreps intuitive criterion, given that type A borrowers can offer no contract whose pay-off exceeds the equilibrium pay-off from repaying debt, such that the contract's pay-off is also equilibrium dominated by type B borrowers' pay-offs from repaying debt. This is because, for any given contract which satisfies (1.8),
$(1-(1-\mu)Y(I^D))\,I^D > \beta\pi\left(V_{B\bar{t}+1}(r^D) - V_{B\bar{t}+1}(r^M_{\bar{t}+1}\dots r^M_{\bar{t}+n}, r^M)\right)$ must also hold, given that
$\beta\left(V_{A\bar{t}+1}(r^D) - V_{A\bar{t}+1}(r^M_{\bar{t}+1}\dots r^M_{\bar{t}+n}, r^M)\right) > \beta\pi\left(V_{B\bar{t}+1}(r^D) - V_{B\bar{t}+1}(r^M_{\bar{t}+1}\dots r^M_{\bar{t}+n}, r^M)\right)$,
hence it will also be optimal for type B borrowers to offer the contract. If type A's offer a contract such that (1.8) fails to hold, then the Cho-Kreps criterion disallows the inference that only type A's could have offered such a contract, given that it is equilibrium dominated by the pay-off from repaying debt.

This result illustrates the importance of borrower reputation effects in ensuring the existence of the competitive market for directly-placed debt. As long as there is a small degree of uncertainty over borrower type, borrowers' concerns about maintaining their reputation will ensure that they repay debt, as all lenders believe that only type B borrowers default, and will therefore refuse to re-lend to defaulters. If type A borrowers separate from type B's, then this source of uncertainty is eliminated, and credit termination is no longer a credible threat because lenders know that any borrower who defaults cannot be type B. Hence the bank will re-lend to defaulters, and thus borrowers will be unable to commit to repaying directly-placed debt, i.e. borrowers' reputation has no value and thus direct lenders will exit the market.[3]

We now need to prove that date $T+1$ actually exists, i.e. that when borrowers have

[3]There does exist another equilibrium. If the bank believes that the offer of a deviant contract implies nothing new about borrower type, i.e. implies the pool of current borrowers in the market, then it will accept the contract. Hence, this equilibrium would be the same as the post-separation equilibrium in which direct lenders refuse to lend.

established a long enough track record for repaying debt, interest rates will be low enough for reputation to be effective, i.e. for (1.3) to hold. We do this by deriving a necessary condition for reputation to be effective for type B's. We then prove that this condition is also a sufficient condition for reputation to be effective for both type A and B borrowers.

Lemma 3 derives a necessary condition for reputation to be effective for type B borrowers.

Lemma 3 *A necessary condition for reputation to be effective for type B borrowers on some date* $t < \infty$ *is* $\beta > \frac{1-(1-\mu)Y(I^*)}{\pi\mu Y(I^*)}$.

Proof. Reputation is effective, hence type B's will repay debt

$$\Leftrightarrow (r_t - (1-\mu)Y(I_t))\, I_t < \beta\pi V_{Bt+1} \text{ which holds only if}$$

$$(1-(1-\mu)Y(I^*))\, I^* < \lim_{T\to\infty}[\sum_{i=1}^{T-t}(\beta\pi)^i(Y(I^*)-1)I^*] \tag{1.11}$$

$$\Leftrightarrow \beta > \frac{1-(1-\mu)Y(I^*)}{\pi\mu Y(I^*)} \tag{1.12}$$

where $I^* = \frac{a-1}{2b}$, from the borrower's demand for funds function (which is solved for in section 2.4 below).

Q.E.D.

Line 1 of the proof is simply condition (1.3), the condition for reputation to be effective for type B borrowers. In order to derive a necessary condition for reputation to be effective, line 2 derives the condition for (1.3) to hold when conditions for borrowers to repay debt are the best they could possibly be, i.e. when interest rates are at their lowest possible level on every date, which is equal to the riskless rate (since as explained above, lenders will never lend at interest rates lower than this), hence $r_t = 1 \quad \forall t$.

Lemma 3 shows that if the discount factor is too low and thus condition (1.12) fails to hold, reputation will never be effective in enforcing debt repayment for type B borrowers, no matter how low interest rates fall over time. Intuitively, if borrowers do not value future pay-offs highly, then the short-term pay-off from repudiation will exceed the long-term stream of pay-offs available if borrowers repay debt and thus retain access to future credit.

Lemma 4 below proves that condition (1.12) is also a sufficient condition for reputation to be effective for type B borrowers on some date $t < \infty$.

Lemma 4 *There exists some date $t < \infty$ on which reputation is effective for type B borrowers, if and only if condition (1.12) holds.*

Proof. See appendix.

Lemma 4 shows that over time, interest rates will fall sufficiently low such that reputation will be effective for type B borrowers remaining in the market. As explained above, if reputation is effective for type A's, then it must also be effective for type B's.

We have now proved that date $T+1$, the threshold date on which direct lenders enter the market and thus borrowers switch from bank debt to directly-placed debt, does exist. Before we can actually derive this date, we must prove that it is unique, i.e. borrowers do not switch back and forth between bank and directly-placed debt over time. We do this by proving that the dynamics of borrower reputation effects are monotonic, i.e. the net value of borrower reputation (the present discounted value of project returns net of debt repayments) is strictly increasing over time, hence incentives to repay debt improve over time. Hence, if on any given date t' borrower reputation is ineffective and thus direct lenders will not lend, then given that incentives deteriorate going backwards in time, reputation must also be ineffective on all dates $t < t'$ and thus direct lenders will also refuse to lend on all dates $t < t'$. Hence it must be true that on all dates $t \leq T$, borrowers use bank finance, and for all dates $t \geq T+1$, they use direct finance.

Proposition 2 proves that the dynamics of type B borrower reputation effects are monotonic.

Proposition 2 *If reputation is ineffective for type B borrowers on date $\hat{t}$, then reputation is ineffective on all dates $t < \hat{t}$, if the following condition holds*

$$(F_A + F_B\pi) > \frac{1+\mu}{a} \tag{1.13}$$

Proof. Suppose that reputation is effective on date $\hat{t}+1$. Hence, from condition (1.3) we have $(r_{\hat{t}} - (1-\mu)Y(I_{\hat{t}}))\, I_{\hat{t}} - \beta\pi V_{B\hat{t}+1} \leq 0$. We now prove that $V_{B\hat{t}} < V_{B\hat{t}+1}$. Given that reputation is ineffective on date $\hat{t}$, we have $V_{B\hat{t}} \leq \mu Y(I_{\hat{t}})I_{\hat{t}}$. As shown in section 2.8 below, $I_t < I_{t'}$ for all $t < t'$. This implies that $\mu Y(I_{\hat{t}})I_{\hat{t}} < \mu Y(I_{\hat{t}+1})I_{\hat{t}+1}$. Given this, and that $V_{B\hat{t}+1} \geq \mu Y(I_{\hat{t}+1})I_{\hat{t}+1}$ holds (since reputation is effective on date $\hat{t}+1$), then $V_{B\hat{t}} \leq \mu Y(I_{\hat{t}})I_{\hat{t}}$ implies $V_{B\hat{t}} < V_{B\hat{t}+1}$. We now examine how $(r_t - (1-\mu)Y(I_t))\, I_t$ evolves over time. From section 2.8 below, we know that $r_{\hat{t}-1} > r_{\hat{t}}$ and $I_{\hat{t}-1} < I_{\hat{t}}$. We now show that, providing (1.13) holds, $\frac{\partial(r-(1-\mu)Y(I))I}{\partial r} > 0$ holds, hence $\left(r_{\hat{t}-1} - (1-\mu)Y(I_{\hat{t}-1})\right) I_{\hat{t}-1} > (r_{\hat{t}} - (1-\mu)Y(I_{\hat{t}}))\, I_{\hat{t}}$ must hold. Given that $I(r) = \frac{a-r}{2b}$, we have

$$(r - (1-\mu)Y(I))\, I = \left(\frac{1}{2}(1+\mu)r - \frac{1}{2}(1-\mu)a\right)\frac{a-r}{2b}$$

$$\frac{\partial\,(r-(1-\mu)Y(I))\, I}{\partial r} = \left(\frac{1}{2}(1+\mu)r - \frac{1}{2}(1-\mu)a\right)\left(-\frac{1}{2b}\right) + \frac{1}{2}\frac{a-r}{2b}(1+\mu)$$

and

$$\frac{\partial^2\,(r-(1-\mu)Y(I))\, I}{\partial r^2} = -\frac{1}{2b}(1+\mu)$$

Hence

$$\frac{\partial\,(r-(1-\mu)Y(I))\, I}{\partial r} = 0 \Leftrightarrow r = \frac{a}{1+\mu}$$

Given that $\frac{\partial^2(r-(1-\mu)Y(I))I}{\partial r^2} < 0$, if $r < \frac{a}{1+\mu}$ then $\frac{\partial(r-(1-\mu)Y(I))I}{\partial r} > 0$. From (1.4),

$$r_t < \frac{a}{1+\mu} \Leftrightarrow (F_{At} + F_{Bt}\pi) > \frac{1+\mu}{a}$$

But if (1.13) holds, then given that $(F_{At} + F_{Bt}\pi) \geq (F_A + F_B\pi)$ for all t, $r_t < \frac{a}{1+\mu}$ for all t, hence given that $r_{t-1} > r_t$ for all t, this implies

$$\left(r_{t-1} - (1-\mu)Y(I_{t-1})\right) I_{t-1} > \left(r_t - (1-\mu)Y(I_t)\right) I_t \text{ for all } t \tag{1.14}$$

Hence $V_{B\hat{t}} < V_{B\hat{t}+1}$ and $\left(r_{\hat{t}-1} - (1-\mu)Y(I_{\hat{t}-1})\right) I_{\hat{t}-1} > \left(r_{\hat{t}} - (1-\mu)Y(I_{\hat{t}})\right) I_{\hat{t}}$ imply

$$\left(r_{\hat{t}-1} - (1-\mu)Y(I_{\hat{t}-1})\right) I_{\hat{t}-1} - \beta\pi V_{B\hat{t}} > \left(r_{\hat{t}} - (1-\mu)Y(I_{\hat{t}})\right) I_{\hat{t}} - \beta\pi V_{B\hat{t}+1} > 0$$

Hence from condition (1.3), reputation is ineffective on date $\hat{t}-1$. Similarly, $V_{B\hat{t}-1} \leq \mu Y(I_{\hat{t}-1})I_{\hat{t}-1}$ and $\mu Y(I_{\hat{t}-1})I_{\hat{t}-1} < \mu Y(I_{\hat{t}+1})I_{\hat{t}+1}$ imply that $V_{B\hat{t}-1} < V_{B\hat{t}+1}$. From (1.14), $\left(r_{\hat{t}-2} - (1-\mu)Y(I_{\hat{t}-2})\right) I_{\hat{t}-2} > \left(r_{\hat{t}} - (1-\mu)Y(I_{\hat{t}})\right) I_{\hat{t}}$, hence

$$\left(r_{\hat{t}-2} - (1-\mu)Y(I_{\hat{t}-2})\right) I_{\hat{t}-2} - \beta\pi V_{B\hat{t}-1} > \left(r_{\hat{t}} - (1-\mu)Y(I_{\hat{t}})\right) I_{\hat{t}} - \beta\pi V_{B\hat{t}+1} > 0$$

hence reputation is also ineffective on date $\hat{t}-2$. By recursion on this process, it must be true that reputation is ineffective on any date $t < \hat{t}$.

Q.E.D.

The intuition behind this proof is as follows. There are two factors which determine whether or not reputation is effective: the present discounted value of future borrowing (given by $\beta\pi V_{Bt+1}$), and the excess cost of repaying debt in full today, over and above the renegotiated debt repayment (given by $\left(r_t - (1-\mu)Y(I_t)\right) I_t$). On date $\hat{t}$, reputation is ineffective. The only ways that reputation can become effective on some date $t < \hat{t}$ is if $\beta\pi V_{Bt+1}$ increases or $\left(r_t - (1-\mu)Y(I_t)\right) I_t$ falls as we go backwards in time. However, $\beta\pi V_{Bt+1}$ is strictly decreasing as we go backwards in time, because borrowers face higher interest rates on all future dates. Also, the excess cost of repaying debt is strictly increasing as we go backwards in time, providing that condition (1.13) holds. Hence when borrower incentives deteriorate sufficiently such that reputation is no longer effective, there is no way that reputation can become effective again on some earlier date.

We can now derive date $T+1$. We start with date $t=1$ and check whether $\left(r_1^D - (1-\mu)Y(I_1)\right) I_1 \leq \beta\pi V_{B2}$ holds, where V_{B2} is computed under the claim that borrowers borrow directly at the market interest rate and repay debt on all dates, hence $V_{B2} = \sum_{t=2}^{\infty} \left(\beta^{t-2}(Y(I_t) - r_t^D)I_t\right)$. If this fails to hold we iterate on this process until we find date $T+1$ such that $\left(r_t^D - (1-\mu)Y(I_t)\right) I_t \leq \beta\pi V_{Bt+1}$ holds. From proposition 2, we know that this must then hold for all $t \geq T+1$, and thus the claim that borrowers borrow directly at the market interest rate and repay debt on all dates is fulfilled.

Chapter 2

The Bank Lending Equilibrium

2.1 Introduction

In this chapter, we solve for the equilibrium on dates $t \leq T$, on which borrowers are unable to borrow directly and are thus forced to borrow from the bank. On each of these dates, the bank plays the following stage game against each borrower, consisting of the following sequence of steps (bank pay-offs are given net of the return available from the riskless asset):

Step (1): The bank decides whether to lend to borrowers in the credit market on date t', or to invest in the riskless asset. If it does not lend, then the date t' stage game ends, the bank receives a net pay-off of zero, and all borrowers receive a pay-off of zero. If it lends, the game proceeds to step (2).

Step (2): The bank sets the interest rate, and also decides whether to ration the amount of credit it will lend to each borrower.

Step (3): Nature determines the outcome of a randomizing device, which is observable to all borrowers but not to the bank.

Step (4): Borrowers offer debt contracts to the bank, specifying the interest rate and the amount they wish to borrow.

Step (5): The bank chooses which debt contracts to accept. If the bank refuses to lend to any given borrower, then the date t' stage game ends for that borrower and it and the bank receive a pay-off of zero.

Step (6): Nature determines the outcome of type B projects. If a borrower's project fails, then its date t' stage game ends, it receives a pay-off of zero and the bank receives a net pay-off of $-I_t$. If its project succeeds, then it proceeds to step (7).

Step (7): Borrowers choose whether to repay or default. If a borrower repays, then its date t' stage game ends, it receives a pay-off of $(Y(I_{t'}) - r_{t'})I_{t'}$, and the bank receives a net pay-off of $(r_{t'} - 1)I_{t'}$. If it defaults, then the game proceeds to step (8).

Step (8): The bank decides whether to liquidate or renegotiate with each borrower who defaulted. If it liquidates, it receives a net pay-off of $q(w) - I_t$ (if it is class W) or $(r_{t'} - 1)I_{t'}$ (if it is class S), and the borrower receives a pay-off of zero. If it renegotiates, it receives a net pay-off of $-(1 - (1 - \mu)Y(I_{t'}))I_{t'}$, and the borrower receives a pay-off of $\mu Y(I_{t'})I_{t'}$.

This chapter solves for the equilibrium of this stage game on any given date $t \leq T$, and solves by backward induction starting with step 8 through to step 1. This involves solving sequentially the following set of problems:

1) The bank's liquidation versus renegotiation problem, given that the borrower has repudiated.

2) The borrower's repayment versus repudiation problem.

3) The borrower's demand for funds.

4) The bank's decision on what interest rate to charge.

5) The bank's decision on whether or not to lend.

Before we embark on the formal solution of the stage game equilibrium, it is instructive to give an intuitive description of the equilibrium.

I model uncertainty over the bank's pay-offs from liquidation as follows. I assume that there exists two classes of banks:

a) Class S banks. This class has pay-off $q(s)$ if it liquidates when borrowers repudiate, where $q(s) > (1 - \mu)Y(I_t)I_t \ \forall t$, hence class S banks prefer liquidation to renegotiation.

We make the stronger assumption that $q(s) = r_t I_t \ \forall t$, hence class S banks are indifferent between borrowers repaying and repudiating debt. This is purely for analytical convenience, and does not affect the main results in the paper, which will obtain providing $q(s) > (1-\mu)Y(I_t)I_t \ \forall t$ holds.

b) Class W banks. We assume a continuum of types of class W banks, where each type has pay-off $q(w)$ from liquidation, where w is the realization of random variable ω which is uniformly distributed on $[0,1]$, and distributed independently across all banks. The pay-off function is given by $q : [0,1] \rightarrow (-\infty, 0)$ where $q(w)$ is a strictly increasing and continuous function of w.

The bank's class and type are both private information. We assume that borrowers' assessment of the prior probability that a bank is class S is given by $\frac{\delta}{1+\delta}$, where δ is small.

Given a one period horizon, if borrowers repudiate, a tough bank will prefer to liquidate, but a weak bank will prefer to renegotiate, since for weak banks liquidation is relatively costly. Hence if borrowers know that the bank is tough, they will prefer to repay the debt, since if they repudiate the bank is sure to liquidate and thus borrowers get a zero pay-off. However, if borrowers know that the bank is weak, they will repudiate, since they know that the bank will agree to renegotiate anyway. But what happens with a multi-period horizon? If borrowers repudiate, weak banks might have an incentive to liquidate, even though liquidation is costly, in order to persuade borrowers that it is tough. The bank's class is private information, hence borrowers do not know whether they face a tough or weak bank. But if the bank liquidates today, borrowers might think that they are actually facing a tough bank (i.e. they are at node n_1 in the information set in Figure 3). Hence, it is optimal for them to repay the debt tomorrow, since the bank is likely to liquidate again if they repudiate tomorrow. And thus, a weak bank has an incentive to liquidate today, in order to build a reputation for being tough, and thus deter borrowers from repudiating on future dates.

But we assume that the prior probability that the bank is tough is actually very small, and this is common knowledge. Hence, at the beginning of the game when the

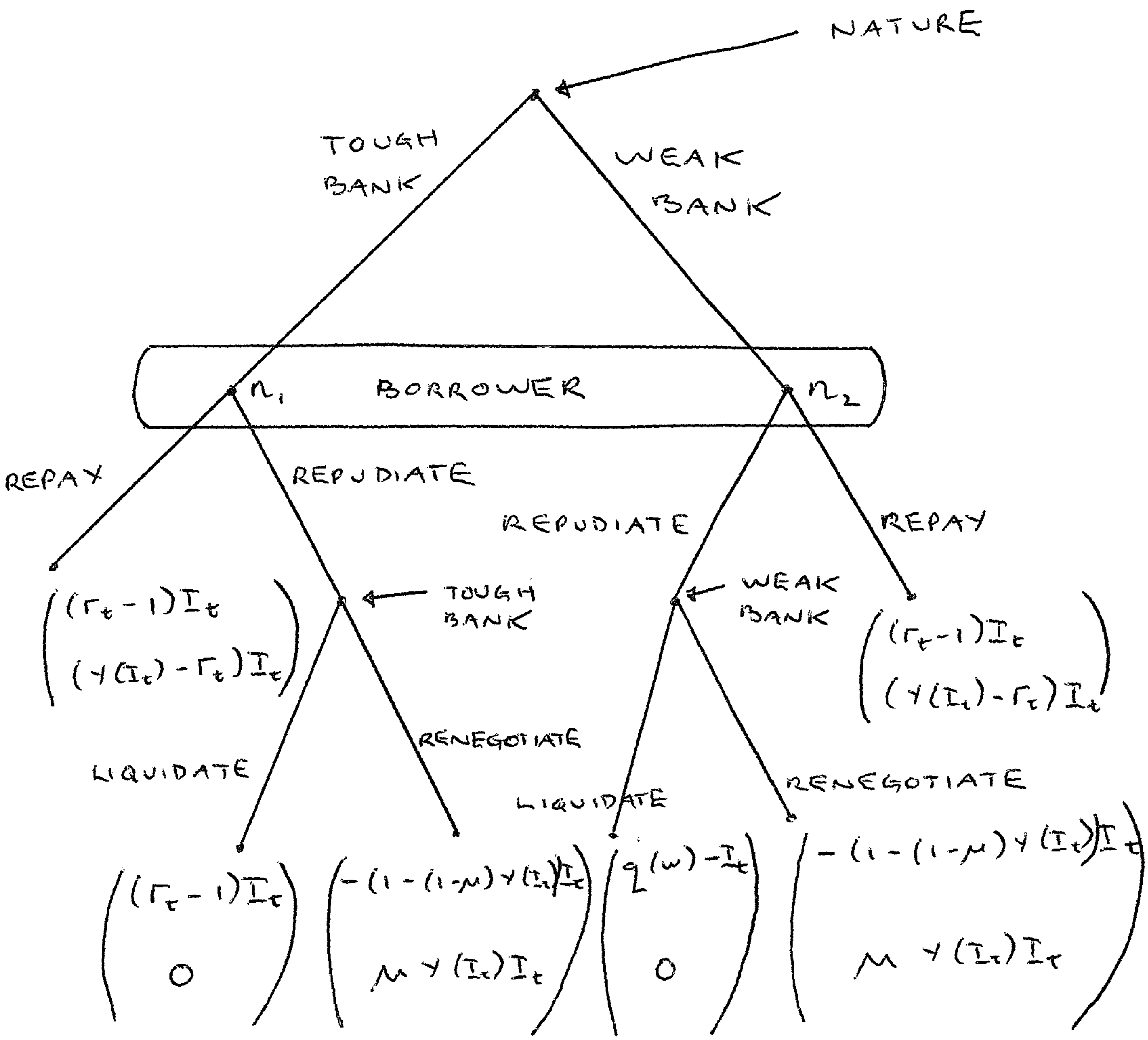

FIGURE 3 : THE BANK LENDING STAGE GAME

bank has not yet built its reputation, borrowers believe that they are most probably at node n_2 rather than n_1 in the information set. However, borrowers will still be deterred from defaulting even though they believe the bank is probably weak for the following reason. A weak bank will have an incentive to liquidate defaulters, because if it fails to do so, it reveals its type to borrowers and thus borrowers will always repudiate if the bank ever lends again. And thus the bank would lose the profitable opportunities to lend to these new borrowers at a premium interest rate on future dates. Hence even though the prior probability that the bank is tough is small, the threat of liquidation is credible because borrowers know that the bank has a long horizon and thus has many future lending opportunities to defend.

As we show in the formal solution of the stage game equilibrium, on the equilibrium path, there are two possibilities on any given date:

1) all borrowers (whose projects succeeded) repay debt, or

2) all borrowers (whose projects succeeded) randomize between repayment and repudiation (the mixed strategy equilibrium).

What this means is, we must solve each step of the stage game for both these cases.

In this chapter, we also explore the evolution of borrower and bank reputation effects over time, and show that the pattern of evolution is such that borrower and bank reputation effects are mutually reinforcing. At the start of its life, the bank has no previous history of lending, and hence no track record for liquidating borrowers who repudiate. Over time, borrowers might repudiate on a series of dates, and the bank will decide whether to liquidate or renegotiate. If borrowers repudiate during the earlier stages of the bank's life, even if liquidation is very costly, the bank will prefer to liquidate in order to build a reputation as a tough bank, and thereby deter default on later dates. However, if repudiation occurs closer to the bank's horizon, the bank will probably prefer to renegotiate with defaulters (unless its liquidation costs are very low), since there are fewer lending opportunities to defend. In other words, bank reputation effects weaken over time. If on any date before the horizon the bank renegotiates with defaulters, it loses its reputation permanently, and is consequently unable to lend on future dates, i.e. the bank fails. Given that borrower reputation effects will not yet be strong enough to discourage repudiation, direct lenders will not be able to lend, and thus the entire credit market breaks down. However, the mutual reinforcement of borrower and bank reputation effects acts to reduce the probability that the credit market breaks down in this way. During the early stages of their lives when track records for credit repayment are relatively short, borrowers' incentives to repudiate are strong. But it is exactly at this point in the bank's life that incentives to liquidate are strongest. The high probability that borrowers who repudiate will be liquidated may be sufficient to deter repudiation. But over time, as the horizon gets closer, the bank's incentives to liquidate weaken. However, as the horizon approaches, the net value of borrower reputation rises, given that the lemons premium on the interest rate falls, and given that borrowers will soon be able to borrow directly instead of being forced to borrow from the bank at a premium interest rate, since they have established a good credit history. Hence, at the stage when the bank's incentives to liquidate are weakest, it is fortunately the case that the bank's reputation might not be tested anyway, because borrowers' incentives to repudiate are much weaker. It is this 'contraflow' in the system of reputation dynamics which tends to reduce the probability that the bank fails.

The structure of the chapter is as follows. Sections 2.2-2.6 solve the stage game equilibrium for the case in which borrowers repay debt. Section 2.7 solves the stage game in the mixed strategy equilibrium. Section 2.8 solves the multi-period equilibrium, and section 2.9 derives the properties of the multi-period equilibrium. Section 2.10 analyzes the dynamic interaction of bank and borrower reputation effects.

2.2 The Bank's Liquidation versus Renegotiation Problem

Consider the event that all borrowers, both type A's and type B's whose projects succeeded, repudiate on some date $t' \leq T$. What is the bank's optimal strategy? To determine whether it is optimal for the bank to liquidate or renegotiate, we need to consider two factors:

1) the bank's stage game pay-off on date t' if it liquidates or renegotiates

2) the effects of the bank's action on date t' on its future pay-offs.

The second factor is important for the following reason. The bank's future pay-offs are a function of borrower repayment decisions on future dates. Borrowers' decisions on whether to repay or repudiate depend on their assessment of the probability that the bank will liquidate if they repudiate. More specifically, on each date borrowers form an assessment of the probability that the bank is class S, conditional upon the history of the bank's actions. We define this assessment to be the bank's reputation x, meaning that if the bank has reputation x, borrowers' beliefs are that the bank is class S, or class W with $w \geq x$ (where w denotes the class W bank's type, which specifies its pay-off from liquidation). Borrowers map all observable information on the bank's actions into the reputation variable x. For example, if the bank liquidates on date t', borrowers might believe that this indicates that the bank has a lower cost of liquidation than they previously thought (i.e. its type w is higher than previously assessed). Hence the bank's reputation is revised upwards. This means that the probability that the bank will liquidate on future dates is assessed to be higher, and thus borrowers will be less likely to repudiate, which means that the bank's future pay-offs will be higher.

First, we solve for a class S bank's optimal strategy. We start with date T, the bank's terminal date. Since there are no future pay-offs to consider, we need only consider the bank's stage game pay-off on date T. For a class S bank, the pay-off from liquidation $q(S) = r(I_T)I_T$ exceeds the pay-off from renegotiation $(1 - \mu)Y(I_T)I_T$ by assumption, hence it is optimal to liquidate borrowers who default. What is the bank's optimal strategy on date $T-1$? Suppose the bank's action on date $T-1$ affects borrowers' beliefs such that borrowers will prefer to repay on date T. How does this affect the bank's pay-off on date T? If borrowers repay on date T, the bank gets a net pay-off of $(r(I_T)-1)I_T$. But if borrowers repudiate, the bank will liquidate and hence it still gets a pay-off of $(r(I_T)-1)I_T$. Hence the bank's action on date $T-1$ has no effect on its future pay-offs. Thus to solve for the bank's optimal strategy on date $T-1$, we only need to consider the bank's stage game pay-off, and as for date T, it is optimal to liquidate if borrowers repudiate. By recursion, this argument applies for all dates $t < T$. Hence it is an optimal strategy for a class S bank to liquidate on all dates, independent of its reputation x.

We now solve for a class W bank's optimal strategy. On date T, the bank's stage game pay-off from renegotiation $(1-\mu)Y(I_T)I_T$ exceeds its pay-off from liquidation $q(w)$ for all types w by assumption, hence it is optimal to renegotiate if borrowers default on date T. What is the bank's optimal strategy on dates $t' < T$? We need to determine the effects of the bank's date t' action on its future pay-offs, by solving for borrowers' equilibrium beliefs about the bank's type, given its decision to liquidate or renegotiate on date t'. What are borrowers' beliefs if the bank renegotiates on date t'? As explained above it is never optimal for a class S bank to renegotiate, hence renegotiation must imply that the bank is class W. How does this affect borrowers' future actions? Borrowers know that a class W bank will renegotiate on date T, hence they will repudiate on date T, given that $V_T(rep) < V_T(def)$ if the probability that borrowers face liquidation is zero. Hence the bank will refuse to lend on date T. However, through the Chain Store Paradox, borrowers will repudiate on all future dates and thus the bank will prefer not to lend on all future dates, if it is revealed to be class W. This is because, if the bank refuses to lend on date T, then date T-1 becomes its terminal date, and thus it will be optimal to renegotiate on date T-1. Hence borrowers will prefer to repudiate on date T-1, which implies that the bank will not lend on date T-1. By recursion, this argument holds for all dates after t'. What this means is that, if the bank ever renegotiates, it loses its reputation forever after. Hence its optimal strategy will be to renegotiate on all future dates, and thus it will be unable to lend to borrowers with short credit histories ever again and must instead invest in the riskless asset, i.e. the bank fails. Hence if the bank renegotiates on date t', it receives a stage game pay-off of $(1-\mu)Y(I_{t'})I_{t'}$, and a present value of future pay-offs (net of the return on the riskless asset) of zero. Suppose the bank has never renegotiated in the past, and has therefore retained its reputation. If it liquidates on date t', it retains its reputation and is therefore able to lend on future dates. The expected present value of the bank's future pay-offs (net of the return on the riskless asset) is given by $V_{Mt'+1}(w, x_{t'+1})$, bank's value function on date $t'+1$. We explain how this is derived in section 2.8 below. If the gain from liquidation (net of the cost of liquidation $q(w)$) on date t' in terms of being able to lend on future dates exceeds the pay-off from repudiation, then it will be optimal to liquidate, i.e. liquidation is an optimal strategy if and only if condition (2.1) below holds:

$$q(w) + V_{Mt'+1}(w, x_{t'+1}) > (1-\mu)Y(I_{t'})I_{t'} + 0 \tag{2.1}$$

where $x_{t'+1}$ is the bank's ex post reputation after liquidating on date t'. Since $q(w)$ is increasing in w by assumption and $V_{Mt+1}(w, x_{t+1})$ is increasing in w and x_{t+1}(we prove this in section 2.8), we can define a critical value of reputation $\hat{x}_t$ which satisfies the following condition:

$$\hat{x}_t \equiv \inf\{x \in [0,1] \mid w > x \Rightarrow q(w) + V_{Mt+1}(w, w) > (1-\mu)Y(I_t)I_t\} \tag{2.2}$$

The interpretation of condition (2.2) is as follows. If the bank's type w exceeds $\hat{x}_t$, then its optimal strategy will be to liquidate on date t. However, if the bank's type w is less than $\hat{x}_t$, then its optimal strategy will be to renegotiate on date t.

Revision of reputation

We now show how the bank's reputation is updated on date $t+1$, given its actions during date t. We assume that the bank begins on date $t = 1$ with reputation $x = 0$, and if it ever renegotiates and thus reveals its class to be W, its reputation falls to $x = -1$. If the bank ever reveals its class to be S, it is accorded reputation $x = 1$. Suppose the bank enters date t with reputation $x_t = -1$ or $x_t = 1$. Then its reputation on date $t+1$ is also $x_{t+1} = -1$ or $x_{t+1} = 1$ respectively, independent of its actions on date t. Suppose the bank enters date t with reputation $-1 < x_t < 1$. Then its reputation on entering date $t+1$ is given in Table 1 below.

If $-1 < x_t < 1$:

Outcome on date t	Reputation entering date $t+1$
borrowers repay	$x_{t+1} = x_t$
bank liquidates	$x_{t+1} = \max(x_{t+1}, \hat{x}_t)$
bank renegotiates	$x_{t+1} = -1$

Table 1: Revision of reputation

We must now prove that reputation is updated in accordance with Bayes Rule, i.e. equilibrium beliefs are consistent with the bank's equilibrium strategies as set out above. As explained above, given that a class S bank will never renegotiate, the belief that renegotiation implies that the bank is class W is consistent with equilibrium strategies. Also, if the bank performs an action which reveals its class (as S or W), then given that the uncertainty over the bank's class is eliminated, its reputation will stay at -1 or 1 on all future dates, regardless of its future actions. Hence the belief that $x_t = -1(1) \Rightarrow x_{t+1} = -1(1)$ (independent of the bank's actions on date t) is consistent.

If borrowers repay on date t, then there is no new information about the bank, hence $x_{t+1} = x_t$ is consistent.

If borrowers repudiate and the bank liquidates, how should reputation be updated given the bank's equilibrium strategies? If the bank entered date t with reputation x_t, then borrowers believe that its type is $w \geq x_t$. There are two possible cases.

First, if $x_t > \hat{x}_t$, then

$$q(w) + V_{Mt+1}(w, x_t) \geq q(w) + V_{Mt+1}(x_t, x_t) > (1-\mu)Y(I_t)I_t \tag{2.3}$$

for all $w \geq x_t$. The first inequality holds given that $w \geq x_t$, and that $q(w)$ and $V_{Mt+1}(w, x_t)$ are increasing in w. The second inequality follows from the definition of $\hat{x}_t$ and given that $x_t > \hat{x}_t$ and that $V_{Mt+1}(w, x_t)$ is increasing in x. Hence condition (2.3) states that it is optimal for all bank types $w \geq x_t$ to liquidate on date t. Thus borrowers' equilibrium beliefs must be that liquidation on date t implies that the bank is type $w \geq x_t$, i.e. reputation stays at $x_{t+1} = x_t$.

Second, if $x \leq \hat{x}_t < 1$, then the fact that $q(w)$ and $V_{Mt+1}(w, x)$ are increasing in w implies that

$$q(\hat{x}_t) + V_{Mt+1}(\hat{x}_t, \hat{x}_t) \gtreqqless q(w) + V_{Mt+1}(w, \hat{x}_t) \tag{2.4}$$

as $\hat{x}_t \gtreqqless w$. If $0 < \hat{x}_t < 1$, then from (2.2), the left-hand side of the inequality is equal to $(1-\mu)Y(I_t)I_t$. Hence condition (2.4) states that it is optimal for the bank to liquidate if and only if it has type $w \geq \hat{x}_t$. Thus to be consistent with equilibrium strategies, borrowers' beliefs must be that liquidation on date t implies that the bank is type $w \geq \hat{x}_t$, i.e. reputation is updated to $x_{t+1} = \hat{x}_t$. If $\hat{x}_t = 0$, then the left-hand side is greater than $(1-\mu)Y(I_t)I_t$, hence all bank types prefer to liquidate, and if $\hat{x}_t = 1$ it will only be optimal for a class S bank to liquidate, and thus borrowers' beliefs are consistent.

We have thus proved that reputation is updated in accordance with Bayes' Rule.

To simplify the analysis, we assume the existence of a public randomizing device, such that in equilibrium, borrowers will repudiate en masse[1]. At the beginning of every period, this device transmits a signal which says 'repay' or 'repudiate'. The signal is immediately observable to all borrowers, but not observable to the bank or any other lender until the end of the period, after borrowers have made their repayment decisions.

[1]If we assume independent randomization instead, the nature of the results in this paper are unchanged.

Hence, in the mixed strategy equilibrium we describe in section 2.7 below, borrowers will coordinate their actions according to the signal outcome.

One might think that $\hat{x}_t$ should be a function of borrowers' equilibrium strategies on date t. For example, if the pure strategy equilibrium holds in which all borrowers repay debt, if an individual borrower deviates and repudiates, then is it optimal for the bank to liquidate regardless of its type (in which case we would set $\hat{x}_t = 0$), given that the cost of liquidating a single borrower is small compared to the loss of reputation if the bank renegotiates? Hence, there would be potential multiplicity of equilibria, in that if a given borrower believes that no other borrowers intend to repudiate, it might prefer not to repudiate on its own, if it believes that the bank will be sure to liquidate. However, if it believes that all other borrowers also intend to repudiate, it will be less deterred from repudiation, given that it will be much more costly for the bank to liquidate in this case. However, we rule out this multiplicity as follows.

We assume that each borrower cannot observe the repayment decision of other borrowers before making its own decision, it can only observe the public signal. Also, the bank must make its liquidation vs. renegotiation decision sequentially, it cannot wait until all borrowers have made their repayment decisions, and then decide whether or not to liquidate based on the number of borrowers who repudiate. We specify the bank's beliefs as follows: if any one borrower repudiates then this implies that the signal outcome is 'repudiate', hence the bank's belief is that all borrowers will repudiate. Given these beliefs, the bank's equilibrium liquidation strategy will not be contingent upon the number of borrowers who repudiate. In the mixed strategy equilibrium described below, if the signal outcome is 'repudiate', then all borrowers will repudiate. Hence, it will be optimal for the bank to liquidate if and only if its type is $w > \hat{x}_t$, as described above. In the repayment equilibrium, if an individual borrower deviates and repudiates, then the bank's belief is that the signal outcome is 'repudiate', in which case it believes that all borrowers will repudiate. Hence, it will be optimal for the bank to liquidate if and only if its type is $w > \hat{x}_t$, as in the previous case. Hence, $\hat{x}_t$ is not a function of borrower equilibrium strategies. The bank's beliefs are consistent with borrower equilibrium strategies, given that in equilibrium, borrowers will repudiate if and only if the signal outcome is 'repudiate'.

We now turn to the borrower's repayment versus repudiation problem.

2.3 The Borrower's Repayment versus Repudiation Problem

We now solve for the borrower's optimal strategy on whether to repay or repudiate. The borrower's optimal strategy is a function of the bank's reputation. The greater is the bank's reputation, the higher is the probability that the bank will liquidate if borrowers repudiate, hence the stronger are borrowers' incentives to repay the debt. We solve for the optimal repayment strategy on date t by solving for the critical level of reputation (called x_t^*), such that type B borrowers are indifferent between repaying and repudiating. We then prove that if reputation is greater than x_t^*, then both type A and B borrowers will prefer to repay debt, and if reputation is less than x_t^*, then both type A and B borrowers will prefer to repudiate.

We solve for the borrower's optimal repayment strategy, given his choice of debt contract (i.e. given the interest rate r_t and the amount he has borrowed I_t). But first we need to derive the bank's equilibrium beliefs about borrower type, if a borrower defaults. The bank knows whether default is strategic or forced by project failure, because it can observe project output. Hence, if a borrower defaults due to project failure, the bank's belief must be that the borrower is type B, given that type A projects have zero probability of failure by assumption. Hence the bank's optimal strategy is to terminate credit. But what if default is strategic? Let us first consider the pure strategy equilibrium in which all borrowers repay debt. Suppose an individual borrower deviates and repudiates. Suppose that the bank's belief in this case is that the borrower is type B, hence it is optimal to terminate credit. Hence, if a borrower repudiates, its pay-off is given by

$$V_t(def) = P(x_t).0 + (1 - P(x_t))\mu Y(I_t)I_t$$

where $P(x_t)$ is the probability that the bank will liquidate on date t given reputation x_t, and $V_{t+1}(def) = 0$ given that repudiation results in credit termination. If type A and B (assuming their projects succeeded) borrowers repay debt, their respective pay-offs are given by

$$\begin{aligned} V_{At}(rep) &= (Y(I_t) - r_t)I_t + \beta V_{At+1} \\ V_{Bt}(rep) &= (Y(I_t) - r_t)I_t + \beta\pi V_{Bt+1} \end{aligned}$$

Hence it is optimal for type A borrowers to repay, if and only if

$$(r_t - (1 - (1 - P_t(x))\mu)Y(I_t))\, I_t < \beta V_{At+1}$$

and it is optimal for type B's to repay if and only if

$$(r_t - (1 - (1 - P_t(x))\mu)Y(I_t))\, I_t < \beta\pi V_{Bt+1} \tag{2.5}$$

$P(x_t)$ is derived from the bank's equilibrium liquidation strategy as set out in section 2.2 above. Hence,

$$\begin{aligned} P(x_t) &= 1 \text{ if } x_t > \hat{x}_t \\ &= \frac{\delta + (1 - \hat{x}_t)}{\delta + (1 - x_t)} \text{ if } 0 \leq x_t \leq \hat{x}_t \\ &= 0 \text{ if } x_t = -1 \end{aligned} \tag{2.6}$$

We need to derive the critical value of $P(x_t)$, denoted P_t^*, such that type B borrowers are indifferent between repaying and repudiating. Hence P_t^* is determined such that

$$(Y(I_t) - r_t)I_t + \beta\pi V_{Bt+1} = (1 - P_t^*)\mu Y(I_t)I_t \tag{2.7}$$

We know that $0 < P_t^* < 1$ for all $t \leq T$, given that $V_{Bt}(def)$ is a decreasing function of $P(x_t)$ and that if $P_t^* = 1$, $V_{Bt}(def) = 0$ hence $V_{Bt}(rep) > V_{Bt}(def)$, and if $P_t^* = 0$ then $V_{Bt}(rep) < V_{Bt}(def)$ (we know this holds for all $t \leq T$ since in section 1.5 we derived date $T + 1$ as the first date on which $V_{Bt}(rep) > V_{Bt}(def)$ when $P(x_t) = 0$).

We now solve for x_t^*, the corresponding critical value of x_t such that $P(x_t) = P_t^*$, from (2.6), which gives

$$x_t^* = \frac{\hat{x}_t - (1 - P_t^*)(1 + \delta)}{P_t^*} \tag{2.8}$$

Hence $x_t > x_t^* \Rightarrow P(x_t) > P_t^* \Rightarrow V_{Bt}(rep) > V_{Bt}(def)$. Given that $V_{At}(rep) > V_{Bt}(rep)$ for all t, and $V_{At}(def) = V_{Bt}(def)$, $V_{Bt}(rep) > V_{Bt}(def)$ implies $V_{At}(rep) > V_{At}(def)$. Hence if $x_t > x_t^*$, it is optimal for both borrower types to repay. Therefore the bank's belief that repudiation implies the borrower is type B is consistent, since in equilibrium, neither borrower type will repudiate[2]. Note that $x_t^* < \hat{x}_t$, since $0 < P_t^* < 1$. Also, note that for a given P_t^*, if $\hat{x}_t$ is sufficiently low then x_t^* is negative and thus borrowers will prefer to repay debt even if the bank has reputation $x_t = 0$.

Hence on all dates on which $x_t > x_t^*$ holds, in equilibrium all borrowers (whose projects succeeded) will repay debt. But what happens on dates on which $x_t < x_t^*$? $x_t < x_t^* \Rightarrow P(x_t) < P_t^* \Rightarrow V_{Bt}(rep) < V_{Bt}(def)$. As we prove in section 3.3 below, $x_t < x_t^* \Rightarrow V_{At}(rep) < V_{At}(def)$ also, hence it is optimal for both borrower types to repudiate. We show in section 2.7 below that in this case, the unique equilibrium is a mixed strategy equilibrium in which borrowers randomize between repayment and repudiation.

We now analyze the borrower's demand for funds.

[2]There exists another equilibrium in which the bank's belief is that repudiation implies the pool of current borrowers in the market. In this case, it would be optimal to re-lend to borrowers who repudiate. Hence, borrowers' pay-off from repudiation would be higher, and thus x_t^* would need to be higher.

2.4 The Borrower's Demand for Funds

In this section, we determine how much borrowers will borrow, given the interest rate which is set by the bank, i.e. we solve for the borrower's demand for funds function.

Suppose that borrowers face no restrictions on their demand for funds. Then borrowers will prefer to borrow different amounts depending on whether they intend to repay or repudiate. Let $I^*(def)$ and $I^*(rep)$ be the borrowers' demand functions when they choose to repudiate and repay respectively. Then it is optimal to borrow $I^*(def)$ and repudiate if

$$V_t(def, I^*(def), x_t) > V_t(rep, I^*(rep), r_t)$$

and it is optimal to borrow $I^*(rep)$ and repay debt if $V_t(rep, I^*(rep), r_t) > V_t(def, I^*(def), x_t)$. If type A borrowers choose to repay, they solve for $I^*_{At}(rep)$ as follows:

$$I^*_{At}(rep) = \arg\max\left((Y(I_t) - r_t)I_t + \beta V_{At+1}\right)$$

Given that the choice of I_t has no intertemporal effects, i.e. period t project returns are independent of past levels of investment, this becomes

$$\begin{aligned} I^*_{At}(rep) &= \arg\max\left((Y(I_t) - r_t)I_t\right) \\ &= \frac{a - r_t}{2b} \end{aligned} \qquad (2.9)$$

This is type A's demand for funds function, given type A's decision to repay debt. If type A's decide to repudiate, they solve for $I^*_{At}(def)$ as follows:

$$\begin{aligned} I^*_{At}(def) &= \arg\max\left((1 - P_t(x))\mu Y(I_t)I_t\right) \\ &= \frac{a}{2b} \end{aligned}$$

Type B's demand functions are identical to type A's, given that

$$\begin{aligned} I_{Bt}^*(rep) &= \arg\max\left(\pi(Y(I_t) - r_t)I_t + (1-\pi)0\right) \\ &= \arg\max\left((Y(I_t) - r_t)I_t\right) \\ &= \frac{a - r_t}{2b} \end{aligned}$$

hence $I_{At}^*(rep) = I_{Bt}^*(rep) = I^*(rep, r_t)$ and similarly $I_{At}^*(def) = I_{Bt}^*(def) = I^*(def)$.

But we are faced with the following problem given these strategies. If borrowers choose amount $I^*(def)$, then the bank knows that they intend to repudiate. Hence, the bank might have an incentive to ration credit to the amount $I^*(rep, r_t)$, in order to ensure that borrowers will prefer to repay debt. We now explain this in further detail.

We define x_t^1 as the critical x_t such that

$$V_{Bt}(def, I^*(def), x_t^1) = V_{Bt}(rep, I^*(rep), r_t) \tag{2.10}$$

and x_t^2 as the critical x_t such that

$$V_{Bt}(def, I^*(rep, r_t), x_t^2) = V_{Bt}(rep, I^*(rep, r_t), r_t) \tag{2.11}$$

where $x_t^1 > x_t^2$ given that the choice of $I^*(def)$ optimizes $V_{Bt}(def, x)$, hence it must be true that $V_{Bt}(def, I^*(def), x) > V_{Bt}(def, I^*(rep, r_t), x)$.

If $x_t > x_t^1$, then from (2.10), $V_{Bt}(def, I^*(def), x) < V_{Bt}(rep, I^*(rep), r_t)$ hence type B borrowers' optimal strategy will be to borrow $I^*(rep, r_t)$ and repay debt. Given that $V_{At}(rep, r_t) > V_{Bt}(rep, r_t)$ for all r_t, type A borrowers' optimal strategy will also be to borrow $I^*(rep, r_t)$ and repay debt. Hence, if $x_t > x_t^1$, the bank has no incentive to ration credit, since borrowers will prefer to borrow $I^*(rep, r_t)$ and repay debt anyway.

If $x < x_t^2$, then from (2.11)
$V_{Bt}(def, I^*(def), x_t) > V_{Bt}(def, I^*(rep, r_t), x_t) > V_{Bt}(rep, I^*(rep, r_t), r_t)$ hence it is optimal for type B borrowers to borrow $I^*(def)$ and repudiate. Type A borrowers' pay-

off from repudiation is given by $V_{At}(def, x_t) = (1 - P(x_t))\mu Y(I_t)I_t + P(x_t)\beta V_{At+1}$, given that if all borrowers repudiate, the bank's equilibrium beliefs are that repudiation implies the pool of current borrowers, hence it is optimal to re-lend to borrowers who repudiate. However, the bank can only re-lend if it liquidates, since renegotiation results in loss of reputation, hence type A borrowers' future value is given by $P_t(x)\beta V_{At+1}$. Type A borrowers' pay-off from repaying debt is given by $V_{At}(rep, r_t) = (Y(I_t) - r_t)I_t + P_t(x)\beta V_{At+1}$. If the bank fails to liquidate, then the only remaining source of borrowing is the open market. Given that type B's optimal strategy is to repudiate, by repaying debt, the borrower reveals itself to be type A. However, for simplicity, we assume that the 'no lending' equilibrium, as defined in section 1.4 above, holds, hence direct lenders will be unwilling to lend, given their beliefs that borrowers' strategy is to repudiate on all dates. Hence type A borrowers' future value from repaying debt is again given by $P_t(x)\beta V_{At+1}$. It is optimal for type A borrowers to borrow $I^*(def)$ and repudiate if and only if $V_{At}(def, I^*(def), x) > V_{At}(rep, I^*(rep, r_t), r_t) \Leftrightarrow (1 - P_t(x))\mu Y(I^*(def))I^*(def) > (Y(I^*(rep, r_t)) - r_t)I^*(rep, r_t)$. From (2.11), we know that $(1 - P_t(x))\mu Y(I^*(rep, r_t))I^*(rep, r_t) > (Y(I^*(rep, r_t)) - r_t)I^*(rep, r_t)$. Hence given that $Y(I^*(def))I^*(def) > Y(I^*(rep, r_t))I^*(rep, r_t)$, we have
$V_{At}(def, I^*(def), x) > V_{At}(rep, I^*(rep, r_t), r_t)$. Hence if $x < x_t^2$, both borrower types will borrow $I^*(def)$ and repudiate.

If $x_t < x_t^1$, from (2.10), $V_{Bt}(def, I^*(def), x) > V_{Bt}(rep, I^*(rep), r_t)$, hence type B borrowers will prefer to borrow $I^*(def)$ and default. But suppose $x_t^2 < x < x_t^1$ and suppose the bank rations credit to $I_t \leq I^*(rep, r_t)$. Given this restriction, type B's optimal strategy will now be to borrow $I^*(rep, r_t)$ and repay debt, given that from (2.11), $V_{Bt}(rep, I^*(rep, r_t), r_t) > V_{Bt}(def, I^*(rep, r_t), x_t)$ for $x_t > x_t^2$. With similar reasoning to above, type A's optimal strategy will also be to borrow $I^*(rep, r_t)$ and repay debt. We now show that it is an optimal strategy for both class S and W banks to ration credit. Class S is indifferent about whether borrowers repay or default, since its pay-offs are the same in any case, hence to ration credit is an optimal strategy. Class W will strictly prefer to ration credit, given that the stage game pay-off if borrowers repay exceeds the pay-off if they repudiate, whether the bank liquidates or renegotiates. Borrowers' beliefs are unaffected if the bank fails to ration credit, hence the bank's reputation does not change, and thus there are no gains in future pay-offs. Hence, it is optimal for a class W bank to ration credit to $I_t \leq I^*(rep, r_t)$ if $x_t^2 < x < x_t^1$, in order to eliminate borrowers' incentives to repudiate.

Hence in this section, we have shown that if $x_t > x_t^2$, borrowers' demand for funds function is given by $I^*(rep, r_t)$ and the equilibrium outcome is for all borrowers to repay debt. Hence, the critical value of reputation x_t^* defined in section 2.3 above is actually given by x_t^2, i.e. x_t^* is the level of reputation such that type B borrowers are indifferent between repaying and repudiating, given that they have borrowed amount $I^*(rep, r_t)$. Since $I^*(rep, r_t)$ is a function of the interest rate r_t, then so is x_t^*, hence we define the critical value of reputation on date t as $x^*(r_t)$.

If $x_t < x_t^2$, rationing credit to $I_t \leq I^*(rep, r_t)$ will not stop borrowers from defaulting, since from (2.11), $V_{Bt}(rep, I^*(rep, r_t), r_t) < V_{Bt}(def, I^*(rep, r_t), x_t)$, hence even if borrowers are restricted to borrowing amount $I_t \leq I^*(rep, r_t^*)$, they will still prefer to default. In this case, the mixed strategy equilibrium holds. We solve for borrowers' demand for funds in this case in section 2.7 below.

2.5 The Bank's Determination of the Interest Rate

We now turn to the bank's determination of r_t^*, the equilibrium interest rate. We actually solve instead for I_t^*, the amount of funds the bank supplies to each borrower in equilibrium, but this is equivalent to solving for r_t^*, given that the bank has monopoly control over setting the interest rate. Thus the bank solves for I_t^*, given the borrower's demand for funds function. First, we consider the equilibrium in which $x_t > x^*(r_t)$ and hence borrowers will repay debt. In section 2.7 below, we solve for the equilibrium interest rate in the mixed strategy equilibrium when $x_t < x^*(r_t)$.

Consider the class S bank's problem. The class S bank chooses I_t^* to maximize its stage game pay-off only, because its choice of I_t^* has no effect on future pay-offs. This is because the only intertemporal variable is its reputation x_t, and the class S bank's pay-offs are independent of its reputation, as explained in section 2.2 above. The class S bank's optimization problem is therefore

$$\max_{I_t} \left(\theta_t r(I_t) - 1\right) I_t$$

$$\text{subject to } r(I_t) = a - 2bI_t$$

where $r(I_t)$ is the borrower's demand for funds function given that borrowers will repay debt, and $\theta_t = F_{At} + F_{Bt}\pi$. The bank optimizes over $\theta_t r(I_t)$, the expected debt repayment, given that type B borrowers' projects will fail with probability π. The opportunity cost of lending is given by the return on the riskless asset, which is equal to one.

F.O.C.

$$\theta_t r(I_t) - 1 + \theta_t r'(I_t) I_t = 0$$

$$\Leftrightarrow I_t^* = \frac{a - 1/\theta_t}{4b}, \text{ and } r_t^* = \frac{1}{2}(a + 1/\theta_t)$$

To rule out corner solutions in which $r_t^* > Y(I_t^*)$, in which case the bank would set $r_t = Y(I_t)$, we assume that $r_t^* < Y(I_t^*)$ for all t, which holds if and only if

$$\begin{aligned} \frac{1}{2}(a + 1/\theta_t) &< a - bI_t^* \Leftrightarrow \\ a &> \frac{1}{\theta_t} \end{aligned} \tag{2.12}$$

Recall that the interest rate charged by direct lenders on any date t is given by $r_t^D = \frac{1}{\theta_t}$.

Hence

$$\begin{aligned}
r_t^* &> r_t^D \Leftrightarrow \\
\frac{1}{2}(a + 1/\theta_t) &> \frac{1}{\theta_t} \Leftrightarrow \\
a &> \frac{1}{\theta_t}
\end{aligned}$$

Hence, given that (2.12) holds, we have proved that the bank charges an interest premium on all dates, compared to the competitive market interest rate.

Hence we have solved for the class S bank's optimal choice of the interest rate. But what is the optimal choice of interest rate for a class W bank? Given that all borrowers will repay debt in the equilibrium under consideration, the class W bank's optimization problem is identical to the class S problem. The stage game pay-off function is identical for both classes. Also, the class W bank has no incentive to choose a different interest rate, in order to affect its reputation and thus influence future pay-offs. In fact, the unique specification of borrowers' beliefs about such a deviation are that it reveals the bank is class W, hence reputation would fall to -1. (This is explained in more detail in section 2.7 below). Hence a class W bank will set the same interest rate r_t^* as a class S bank.

2.6 The Bank's Lending Decision

We now turn to the bank's decision on whether or not to lend on any given date. Under what circumstances would the bank prefer not to lend? Suppose the bank's reputation is low, such that borrowers will not repay debt with certainty, but will repudiate with positive probability. If liquidation is very costly to the bank, then it will prefer to renegotiate if borrowers repudiate, in which event it will receive a net pay-off less than the return on the riskless asset. Hence, if the probability of repudiation is quite high, then it will be optimal for the bank to refuse to lend to borrowers, and invest instead in the riskless asset.

A class S bank will prefer to lend on all dates, irrespective of borrowers' repayment strategies, given that its pay-offs are independent of borrower actions, as explained above. Hence, if the bank fails to lend on any date, the unique equilibrium beliefs are that the bank is class W. Hence, its reputation falls to $x = -1$, and stays at $x = -1$, irrespective of its future actions.

What is the optimal lending strategy for a class W bank? It is optimal for a class W bank to refuse to lend on date t if $V_{Mt}(w, x_t) < 0$, i.e. if the pay-off from lending in the credit market is lower than the pay-off from investing in the riskless asset. We must consider two cases.

First, suppose that the bank's reputation at the end of date $t-1$ (before it lends on date t) is $x' > x(r_t^*)$. Then the following equilibrium holds. The bank will lend regardless of its type w. Borrowers will repay debt with certainty. If the bank lends, its reputation stays at x'. It is an optimal strategy for all bank types to lend, given that borrowers will repay debt, hence $V_{Mt}(w, x') \geq 0$ holds for all w (this is because the bank's stage game net pay-off is positive, and the present value of its future pay-offs can be no less than zero given that the bank behaves optimally on all future dates, since if future pay-offs were negative, the bank's optimal strategy would be to exit the credit market and invest in the riskless asset instead). It is an optimal strategy for all borrowers to repay debt given that $x' > x(r_t^*)$ (see section 2.3 above). The revision of reputation if the bank lends satisfies Bayes' Rule. In equilibrium, all bank types w will lend. Hence the action of lending on date t reveals no new information about the bank's type, hence reputation does not change.

Second, suppose that the bank's reputation before it lends on date t is $x' < x^*(r_t^*)$. Then the following equilibrium holds. If the bank has type $w \geq x^*(r_t^*)$, it will lend. If it has type $w < x^*(r_t^*)$, it will not lend. Borrowers will randomize, repudiating with probability α_t, which is determined such that the following holds

$$(1-\alpha_t)\left[(\theta_t r_t^* - 1)\, I_t^* + V_{Mt+1}(x^*(r_t^*), x^*(r_t^*))\right] - \alpha_t\left(1 - (1-\mu)Y(I_t)\right) I_t = 0 \qquad (2.13)$$

If the bank lends, its reputation increases to $x^*(r_t^*)$. If it does not lend, its reputation falls to -1. Note that α_t is determined such that type $w = x^*(r_t^*)$ is indifferent between lending and not lending. We can see this as follows. If the outcome of randomization is that borrowers repay debt, then bank type $w = x^*(r_t^*)$ receives a net pay-off of $(\theta_t r_t^* - 1)\, I_t^* + V_{Mt+1}(x^*(r_t^*), x^*(r_t^*))$. If, however, borrowers repudiate, then the type $x^*(r_t^*)$ bank will prefer to renegotiate (given that from (2.8), $x^*(r_t^*) < \hat{x}_t$), and hence receives a net pay-off of $(1-(1-\mu)Y(I_t))\, I_t$. Hence the type $x^*(r_t^*)$ bank's expected pay-off if it lends is equal to zero, which is its net pay-off if it doesn't lend. Note also that $V_{Mt+1}(w, x^*(r_t^*)) = V_{Mt+1}(x^*(r_t^*), x^*(r_t^*))$ for all $w < \hat{x}_t$ (this is proved later in section 2.8), hence all types $w < \hat{x}_t$ are indifferent between lending and not lending. It is therefore an optimal strategy for types $x^*(r_t^*) \leq w \leq \hat{x}_t$ to lend and for types $w < x^*(r_t^*)$ not to lend. Types $w > \hat{x}_t$ will strictly prefer to lend, given that for $w > \hat{x}_t$, $V_{Mt}(w, x^*(r_t^*)) =$
$(1-\alpha_t)\left[(\theta_t r_t^* - 1)\, I_t^* + V_{Mt+1}(w, x^*(r_t^*))\right] + \alpha_t\left(q(w) - I_t + V_{Mt+1}(w, \hat{x}_t)\right) > 0$ from (2.13), and given that $V_{Mt+1}(w, x^*(r_t^*)) > V_{Mt+1}(x^*(r_t^*), x^*(r_t^*))$, and that $q(w) + V_{Mt+1}(w, \hat{x}_t) > (1-\mu)Y(I_t)I_t$ from the definition of $\hat{x}_t$. Hence we have proved that the bank's equilibrium strategy is optimal. We prove in section 2.7 below that it is an optimal strategy for borrowers to randomize, repudiating with probability α_t. Given that the bank will lend if and only if it has type $w \geq x^*(r_t^*)$, the revision of reputation as stated above is consistent with the bank's equilibrium strategy.

We must also prove that this is the unique equilibrium when $x' < x^*(r_t^*)$. Does there exist an equilibrium in which for some y where $x' \leq y < x^*(r_t^*)$, types w where $y < w < x^*(r_t^*)$ also lend? If the bank lends, its reputation rises to y. However, given that $y < x^*(r_t^*)$, borrowers' optimal strategy will be to repudiate with probability 1. Types $w < \hat{x}_t$ will prefer to renegotiate, and thus if they lend they receive a net pay-off of $-(1-(1-\mu)Y(I_t))\, I_t$. Hence these types will prefer not to lend, and hence this cannot be an equilibrium.

Does there exist an equilibrium in which for some y' where $y' > x^*(r_t^*)$, only types $w \geq y'$ lend? In this case, if the bank lends, its reputation will jump to y'. However, given that $y' > x^*(r_t^*)$, borrowers' optimal strategy will be to repay with probability 1. But this means that if types $w < y'$ were to lend, they would receive a pay-off greater than the return on the riskless asset, hence these types would prefer to lend also. Hence, this is not an equilibrium, and thus the equilibrium set out above is unique.

Now, suppose $x' < x^*(r_t^*)$ holds. Given that $V_{Bt}(rep, I^*(rep), r_t)$ is decreasing in r_t and $V_{Bt}(def, I^*(rep, r_t), x_t)$, is decreasing in x_t, from (2.11), $x(r_t^*)$ is decreasing in r_t. Hence, the bank might have an incentive to reduce the interest rate, such that borrowers prefer to repay debt, i.e. such that $x' > x^*(r_t^*)$ holds. However, we show in section 2.7 below that this cannot be an equilibrium. Hence, if $x' < x^*(r_t^*)$, the unique equilibrium is the mixed strategy equilibrium.

2.7 Mixed Strategy Equilibrium

In this section, we solve for the stage game equilibrium for the case $x' < x^*(r_t^*)$, in which we have a mixed strategy equilibrium. This involves solving by backward induction the same set of problems as for the pure strategy equilibrium above.

The solution to the bank's liquidation vs. renegotiation problem is exactly the same as in the case when borrowers repay debt, as explained in section 2.2 above. We now solve for borrowers' demand for funds function, and decisions on whether to repay or repudiate, in the mixed strategy equilibrium.

Borrowers' demand for funds and optimal repayment strategy

Before we can derive borrowers' pay-off functions if they repay or repudiate, we must determine the bank's optimal strategy on whether to re-lend or terminate credit if a borrower repudiates. Hence, we must derive the bank's equilibrium beliefs about borrower type, if a borrower repudiates.

The bank observes the public randomization signal at the end of the period, and its beliefs about borrower type are a function of the signal outcome and the borrower's action, as set out in Table 2 below.

	Signal outcome: repay	**Signal outcome: repudiate**
Borrower repays	F_{Bt}	F_{Bt}
Borrower repudiates	1	F_{Bt}

Table 2: Bank's beliefs

Table 2 shows the bank's assessment of the probability that the borrower is type B, given its action and the signal outcome on date t. Hence, if the signal indicates 'repudiate' and the borrower repudiates, the bank's assessment is that the probability that the borrower is type B is unchanged, given that it believes that all borrowers will repudiate. If the signal indicates 'repay' and the borrower repudiates, it believes that the borrower is type B. We prove below that in the mixed strategy equilibrium, it is optimal for all borrowers to repay if the signal indicates 'repay' and to repudiate if the signal indicates 'repudiate'. Hence these beliefs are consistent with borrowers' equilibrium strategies.

The implication of these beliefs are that, if the signal indicates 'repudiate' and any given borrower repudiates, it will be optimal for the bank to re-lend to him, given that repudiation implies the pool of current borrowers. However, if the signal indicates 'repay' and any given borrower repudiates, it will be optimal for the bank to terminate his credit, given that in this case repudiation implies that the borrower is type B.

We now prove that it is an optimal strategy for all borrowers to repay if the signal indicates 'repay', and to repudiate if the signal indicates 'repudiate'. We also solve for borrowers' demand for funds function, given this repayment strategy.

Borrowers observe the signal outcome before they choose how much to borrow, hence it is feasible for them to condition their demand for funds upon the signal outcome. However, lemma 5 below proves that in equilibrium, the demand for funds function is independent of the signal outcome. Lemma 6 then proves that the demand for funds function in the mixed strategy equilibrium must satisfy $I(r_t) \leq I^*(rep, r_t)$ for all r_t. Subject to these constraints, proposition 3 then proves that $I_t = I^*(rep, r_t)$ is the unique equilibrium demand for funds function. Hence the demand for funds function in the mixed strategy equilibrium is identical to the equilibrium demand function in the pure strategy equilibrium.

Lemma 5 below proves that in the mixed strategy equilibrium, the demand for funds function is independent of the signal outcome.

Lemma 5 *In the mixed strategy equilibrium, the demand for funds function is independent of the signal outcome, hence $I^*(rep, r_t, \alpha_t) = I^*(def, r_t, \alpha_t) = I^*(r_t, \alpha_t)$ where α_t is the probability that the signal outcome on date t is 'repudiate'.*

Proof. Consider an equilibrium in which borrowers borrow $I^*(rep, r_t)$ if the signal says 'repay' and $I^*(def, r_t)$ if the signal says 'repudiate'. But if borrowers demand $I^*(def, r_t)$, this reveals to the bank that the signal outcome is 'repudiate'. Hence the bank believes that borrowers intend to repudiate, even though it doesn't observe the signal outcome. This means that bank types $w < \hat{x}_t$ will refuse to lend if borrowers demand $I^*(def, r_t)$, given that they will receive a pay-off lower than the riskless interest rate if they do lend. Hence these bank types will lend if and only if borrowers demand $I^*(rep, r_t)$, from which they infer that borrowers will repay debt. But this means that $V_{Mt}(x(r_t^*), x(r_t^*)) = (1 - \alpha_t)\left[(\theta_t r_t^* - 1) I_t^* + V_{Mt+1}(x^*(r_t^*), x^*(r_t^*))\right] + \alpha_t 0 > 0$ (from (2.13)). However, for equilibrium to hold, type $w = x(r_t^*)$ must be indifferent between lending and not lending, which is not the case here. Hence equilibrium fails to hold. Thus, for equilibrium to hold, $I^*(rep, r_t, \alpha_t) = I^*(def, r_t, \alpha_t) = I^*(r_t, \alpha_t)$ must be true.

Q.E.D.

Hence, in order not to reveal the signal outcome, borrowers' demand for funds must be independent of their repayment intentions. Given that in the mixed strategy equilibrium, the signal indicates 'repay' with probability $(1-\alpha_t)$ and 'repudiate' with probability α_t, the first best choice of demand function would be $I^*(r_t, \alpha_t)$, where $I^*(r_t, \alpha_t)$ solves the following problem:

$$\max_{I_t}(1-\alpha_t)V_t(rep, r_t) + \alpha_t V_t(def, x(r_t)) \tag{2.14}$$

However, lemma 6 proves that $I^*(r_t, \alpha_t)$ cannot be an equilibrium. This is because $I^*(r_t, \alpha_t) > I^*(rep, r_t)$, and in fact, lemma 6 proves that any demand function $I(r_t) > I^*(rep, r_t)$ cannot be an equilibrium.

Lemma 6 *Borrowers' demand for funds function in the mixed strategy equilibrium must satisfy* $I(r_t) \leq I^*(rep, r_t)$ *for all* r_t.

Proof. See appendix.

The intuition behind the proof is as follows. Although ex ante borrowers' first best demand function is given by $I^*(r_t, \alpha_t)$, they are unable to commit to this choice after observing the signal outcome. If the signal outcome is 'repay', borrowers will actually prefer to borrow amount $I^*(rep, r_t)$ rather than $I^*(r_t, \alpha_t)$, given that $I^*(rep, r_t)$ is the optimal demand function if borrowers intend to repay.

We have thus far shown that the demand for funds function must be independent of the signal outcome, and must satisfy $I(r_t) \leq I^*(rep, r_t)$ for all r_t. Subject to these constraints, proposition 3 below proves that $I_t = I^*(rep, r_t)$ is the unique equilibrium demand function. This proposition also proves that it is an optimal strategy for all borrowers to repay if the signal indicates 'repay', and to repudiate if the signal indicates 'repudiate'.

Proposition 3 *Borrowers' unique equilibrium strategies in the mixed strategy equilibrium are given as follows. Borrowers' demand for funds function is given by* $I_t =$

$I^*(rep, r_t)$. *If the signal indicates 'repay', borrowers will repay, and if the signal indicates 'repudiate', borrowers will repudiate.*

Proof. See appendix.

We have thus solved for the borrowers' demand for funds function and proved that it is an optimal strategy for all borrowers to repay if the signal indicates 'repay', and to repudiate if the signal indicates 'repudiate', in the mixed strategy equilibrium.

The bank's determination of the interest rate

We now solve for the bank's determination of the interest rate r_t^*, in the mixed strategy equilibrium. As in the case when borrowers repay debt, the bank chooses I_t^* optimally, subject to the borrowers' demand for funds function. First, we solve the class S bank's optimization problem:

$$\begin{aligned} & \max_{I_t} \left(\theta_t(1-\alpha_t)r(I_t) - 1\right) I_t + \theta_t \alpha_t q(s) \\ \Leftrightarrow \quad & \max_{I_t} \left(\theta_t(1-\alpha_t)r(I_t) - 1\right) I_t + \theta_t \alpha_t r(I_t) I_t \\ \Leftrightarrow \quad & \max_{I_t} \left(\theta_t r(I_t) - 1\right) I_t \end{aligned}$$

$$\text{subject to } r(I_t) = a - 2bI_t$$

Hence given that class S's pay-off is independent of borrowers' repayment strategy, and given that the borrowers' demand function in the mixed strategy equilibrium is identical to the demand function in the equilibrium in which borrowers repay, class S's optimal choice of I_t^* is the same as in the equilibrium in which borrowers repay.

We now consider a class W bank's optimization problem. If the bank has type $w \geq \hat{x}_t$, it will liquidate if the outcome of randomization is that borrowers repudiate, hence receiving a stage game pay-off of $q(w) - I_t$. Hence, the bank would choose I_t to solve the following problem:

$$\max_{I_t} \left(\theta_t(1-\alpha_t)r(I_t) - 1\right) I_t + \theta_t\alpha_t q(w)$$

$$\text{subject to } r(I_t) = a - 2bI_t$$

If the bank has type $w < \hat{x}_t$, it will renegotiate if the outcome of randomization is that borrowers repudiate, hence receiving a stage game pay-off of $-(1-(1-\mu)Y(I_t))I_t$. Hence the bank would choose I_t to solve the following problem:

$$\max_{I_t} \left(\theta_t(1-\alpha_t)r(I_t) + \theta_t\alpha_t(1-\mu)Y(I_t)I_t - 1\right) I_t$$

$$\text{subject to } r(I_t) = a - 2bI_t$$

Hence in both cases, the class W bank's optimal choice of I_t differs from class S's optimal choice of I_t^*. However, if the bank chooses $I_t \neq I_t^*$, it reveals that it is a class W bank and thus loses its reputation. In fact, the unique specification of equilibrium beliefs which satisfies the Cho-Kreps intuitive criterion is the following: if the bank chooses $I_t = I_t^*$, its reputation stays at x_t (its reputation after having chosen to lend on date t). But if the bank chooses $I_t \neq I_t^*$, its reputation falls to -1. Given these beliefs, it is optimal for a class W bank to choose $I_t = I_t^*$, and receive an expected pay-off $V_{Mt}(w, x_t) \geq 0$, rather than to choose $I_t \neq I_t^*$, which would yield a pay-off of $-(1-(1-\mu)Y(I_t))I_t$. In equilibrium, no bank type will choose $I_t \neq I_t^*$, hence these beliefs are consistent with equilibrium strategies. But we must also show that beliefs off the equilibrium path satisfy Cho-Kreps. The Cho-Kreps intuitive criterion disallows inferences from actions off the equilibrium path which imply that some type undertook an action which is dominated by the proposed equilibrium pay-off. A class S bank would never choose $I_t \neq I_t^*$, since it gets a strictly lower stage game pay-off from setting $I_t \neq I_t^*$, and whatever the implied effects on its reputation, its future pay-offs are unaffected. Hence any $I_t \neq I_t^*$ is strictly dominated by $I_t = I_t^*$ for the class S bank, and thus Cho-Kreps requires that the unique 'off the equilibrium path' beliefs are that choosing $I_t \neq I_t^*$ implies the bank is class W.

Suppose $x' < x^*(r_t^*)$ holds. Given that $V_{Bt}(rep, I^*(rep), r_t)$ is decreasing in r_t and $V_{Bt}(def, I^*(rep, r_t), x_t)$, is decreasing in x_t, from (2.11), $x(r_t^*)$ is decreasing in r_t. Hence, as mentioned in section 2.6 above, a class W bank might have an incentive to reduce the interest rate, such that borrowers prefer to repay debt, i.e. such that $x' > x^*(r_t^*)$ holds. But this requires that the bank sets $I_t \neq I_t^*$, which reveals it to be class W. Hence, there exists no equilibrium in which the bank sets a lower interest rate in order to improve borrowers' incentives to repay debt.

2.8 The Multi-Period Equilibrium

In sections 2.2-2.7 above, we have solved for the stage game equilibrium on dates $t \leq T$, on which borrowers borrow from the bank. This section solves for the T-period equilibrium. Essentially, this involves deriving the series $[\hat{x}_t]_{t=1}^T$ and $[x_t^*]_{t=1}^T$.

It is useful to give a brief illustration of what happens along the equilibrium path in the multi-period equilibrium. Consider some date $T - a$. Suppose that on entering date $T - a$, the bank has reputation $x < x_{T-a}^*$. This means that in equilibrium, bank types $w < x_{T-a}^*$ will cease to lend, and thus their reputation falls to $x = -1$ (as shown in Figure 4). Bank types $w \geq x_{T-a}^*$ will continue to lend, and their reputation rises to x_{T-a}^*. Borrowers' equilibrium strategy is to randomize. If the outcome of randomization is that borrowers repay, then the bank is not tested, and thus its reputation stays at x_{T-a}^*. If borrowers repudiate, then bank types $x_{T-a}^* \leq w < \hat{x}_{T-a}$ will renegotiate, and thus their reputation falls to -1. Bank types $w \geq \hat{x}_{T-a}$ will liquidate, and thus their reputation rises to $\hat{x}_{T-a}$. With its reputation boosted, the bank is able to deter further repudiation by borrowers on a succession of dates. However, the bank will eventually face mixed strategy repudiation again, on some date $T - b$, which is the first date after $T - a$, on which $\hat{x}_{T-a} \leq x_{T-b}^*$. Hence, on this date the process of borrower randomization, and liquidation versus renegotiation recurs. The reason why the bank faces repudiation

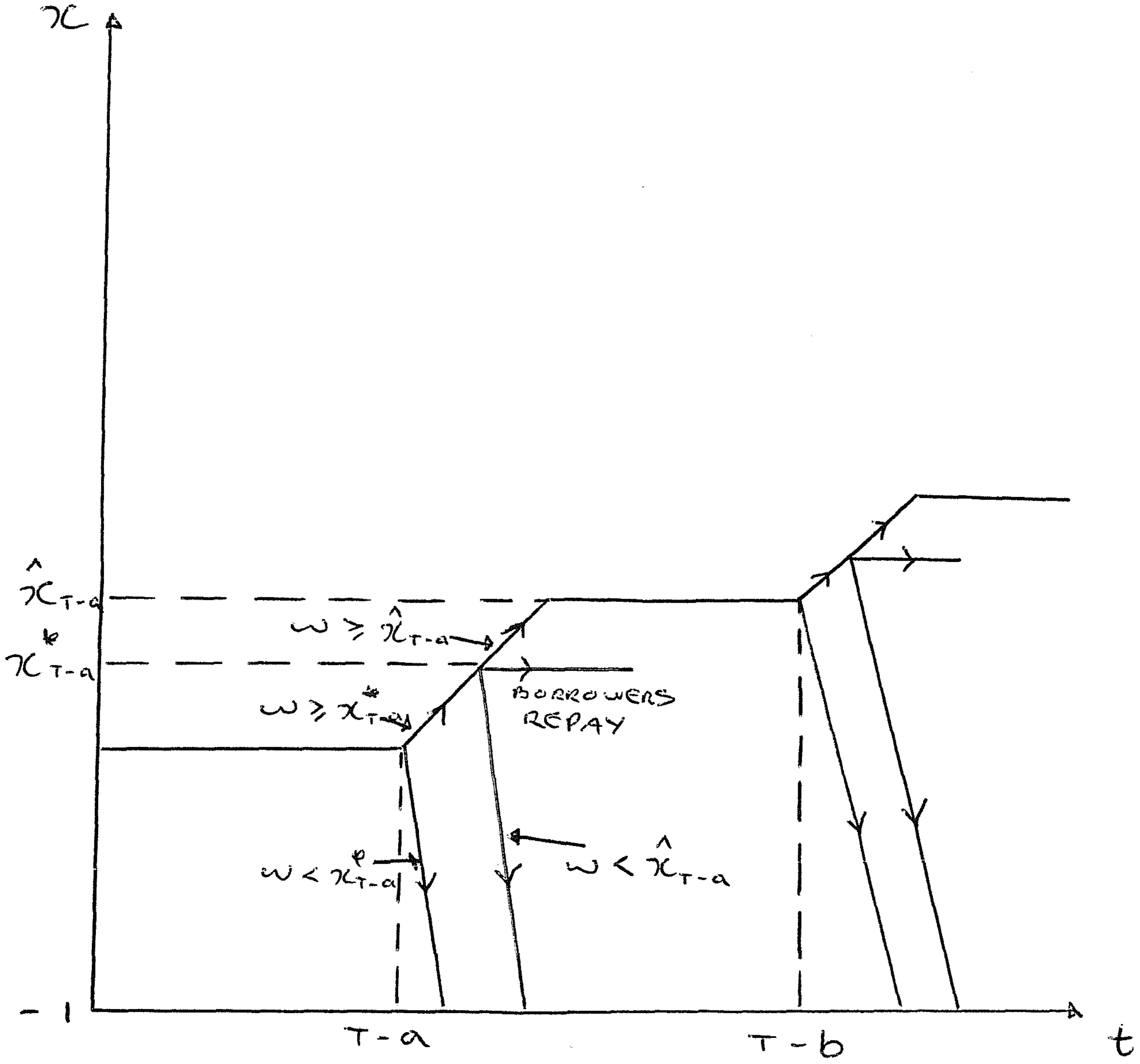

FIGURE 4 : THE MULTI-PERIOD EQUILIBRIUM

again, even after it has liquidated on some previous date is as follows. As the horizon gets closer, the bank's incentives to liquidate weaken, given that there are fewer future pay-offs at stake. As the bank's incentives weaken, the probability that borrowers face liquidation if they repudiate falls. Hence, liquidation becomes less of a deterrent, and although the bank's reputation was sufficient to deter repudiation on previous dates, this is no longer the case.

In the solution described in this section, we show that these episodes of mixed strategy repudiation, followed by liquidation/renegotiation, occur at regular intervals. In fact, we characterize the equilibrium by splitting the T-period time span into several zones. Each zone is defined such that, if repudiation occurs on some date, and the bank liquidates, then the bank will face no further repudiation within the same zone.

The evolution of the bank's stage game pay-off over time

Before we can derive the series $[\hat{x}_t]_{t=1}^T$ and $[x_t^*]_{t=1}^T$, we need to analyze how $(\theta_t r_t^* - 1) I_t^*$, the bank's stage game pay-off on date t if borrowers repay, changes over time. On any date t every type B borrower faces a probability of project failure of $1 - \pi$. If its project does fail, the borrower is revealed to be type B, and it is denied future credit. Hence the evolution of F_{At} and F_{Bt},the proportions of type A and B borrowers remaining in the market on any given date t, is given as follows:

$$F_{At+1} = \frac{F_{At}}{F_{At} + \pi F_{Bt}}, \qquad F_{Bt+1} = \frac{\pi F_{Bt}}{F_{At} + \pi F_{Bt}} \tag{2.15}$$

Hence $\theta_t = F_{At} + \pi F_{Bt}$ is monotonically increasing over time, given that $F_{At+1} > F_{At}$ and $F_{Bt+1} < F_{Bt}$ for all t. (2.15) holds on all dates, whether in equilibrium borrowers repay, or randomize between repayment and repudiation. Suppose that the outcome of randomization is that borrower repudiate. Given that the bank can distinguish between repudiation, and default forced by project failure, it will only re-lend to borrowers whose projects succeeded. Hence those type B borrowers whose projects failed are unable to 'pool' with borrowers who repudiated, and will therefore still have their credit terminated.

Thus the proportion of type B's in the market still falls, as given by (2.15). Hence given that

$$I_t^* = \frac{a - 1/\theta_t}{4b}, \text{ and } \theta_t r_t^* = \frac{1}{2}(\theta_t a + 1)$$

for all $t \leq T$ (from section 2.5), it follows that for any $t'' > t'$, $I_{t''}^* > I_{t'}^*$, and $\theta_{t''} r_{t''}^* > \theta_{t'} r_{t'}^*$, hence $(\theta_{t''} r_{t''}^* - 1) I_{t''}^* > (\theta_{t'} r_{t'}^* - 1) I_{t'}^*$. Hence the bank's stage game pay-off (if borrowers repay) is monotonically increasing over time.

Solving for zone 1

We derive the series $[\hat{x}_t]_{t=1}^T$ and $[x_t^*]_{t=1}^T$ by backward induction. Hence, we must start by solving for zone 1, which begins on date T (we number the zones in reverse order).

We start by solving for $\hat{x}_T$, the ex post reputation accorded to the bank if it liquidates on date T. However, as explained in section 2.2 above, given that there are no future periods on which the bank can lend, class W will renegotiate on date T, but class S will liquidate. Hence $\hat{x}_T = 1$. We can now solve for x_T^*. From (2.8), we have

$$x_T^* = \frac{1 - (1 - P_T(x^*))(1 + \delta)}{P_T(x^*)}$$

We now solve for $\hat{x}_{T-1}$. As explained in section 2.2, $\hat{x}_{T-1}$ is determined such that the following holds

$$q(\hat{x}_{T-1}) + V_{MT}(\hat{x}_{T-1}, \hat{x}_{T-1}) = (1 - \mu)Y(I_{T-1})I_{T-1}$$

i.e. such that type $w = \hat{x}_{T-1}$ is indifferent between liquidation and renegotiation on date $T - 1$. Suppose that if the bank liquidates on date $T - 1$, its reputation increases sufficiently such that borrowers are deterred from repudiating on date T, i.e. $\hat{x}_{T-1} > x_T^*$ holds. If this holds, then $V_{MT}(\hat{x}_{T-1}, \hat{x}_{T-1}) = (\theta_T r_T^* - 1) I_T^*$. Hence, there must exist $x_T^* < z < 1$, such that

$$q(z) + (\theta_T r_T^* - 1) I_T^* = (1 - \mu)Y(I_{T-1}^*)I_{T-1}^* \tag{2.16}$$

holds. If this holds, then the unique equilibrium on date $T-1$ is $\hat{x}_{T-1} = z$, where $\hat{x}_{T-1} > x_T^*$. Borrowers will repay on date T, hence $V_{MT}(\hat{x}_{T-1}, \hat{x}_{T-1}) = (\theta_T r_T^* - 1)\, I_T^*$.

But what if $z < x_T^*$ holds instead? In this case, $\hat{x}_{T-1} = z$ cannot be an equilibrium. If the bank liquidates on date $T-1$, then it enters date T with reputation $\hat{x}_{T-1} < x_T^*$. As explained in section 3.5 above, in equilibrium borrowers will randomize between repayment and repudiation, and bank type $w = \hat{x}_{T-1}$ will not lend on date T, and hence $V_{MT}(\hat{x}_{T-1}, \hat{x}_{T-1}) = 0$. But this means that $q(\hat{x}_{T-1}) + 0 < (1-\mu)Y(I_{T-1})I_{T-1}$ (given that $q(w) < 0$ for $w < 1$), hence type $w = \hat{x}_{T-1}$ is not indifferent between liquidation and renegotiation on date T-1. Thus $\hat{x}_{T-1} = z$ cannot be an equilibrium. In fact, following the same reasoning, any $\hat{x}_{T-1} < x_T^*$ cannot be an equilibrium.

$\hat{x}_{T-1} > x_T^*$ also cannot be an equilibrium for the following reason. If the bank liquidates on date T-1 and enters date T with reputation $\hat{x}_{T-1} > x_T^*$, borrowers optimal strategy will be to repay, which means $V_{MT}(\hat{x}_{T-1}, \hat{x}_{T-1}) = (\theta_T r_T^* - 1)\, I_T^*$. But given that (2.16) holds for $z < x_T^*$, and given that $\hat{x}_{T-1} > z$ and that $q(w)$ is strictly increasing in w, this implies that

$$q(\hat{x}_{T-1}) + V_{MT}(\hat{x}_{T-1}, \hat{x}_{T-1}) > (1-\mu)Y(I_{T-1})I_{T-1}$$

Hence type $w = \hat{x}_{T-1}$ is not indifferent between liquidation and renegotiation on date T-1. Thus $\hat{x}_{T-1} > x_T^*$ cannot be an equilibrium.

If $z < x_T^*$, the unique equilibrium is in fact $\hat{x}_{T-1} = x_T^*$, and borrowers' equilibrium strategy on date T is the following: if the bank liquidates on date T-1 and enters date T with reputation x_T^*, borrowers will randomize, defaulting with probability α_T, where α_T is determined such that the following holds

$$q(x_T^*) + (1-\alpha_T)\,(\theta_T r_T^* - 1)\, I_T^* - \alpha_T\,(1 - (1-\mu)Y(I_T))\, I_T = (1-\mu)Y(I_{T-1})I_{T-1} \quad (2.17)$$

Hence α_T is determined such that type $w = x_T^*$ is indifferent between liquidation and renegotiation on date T-1. We know that there exists $0 < \alpha_T < 1$ such that (2.17) holds, given that $(1-\alpha_T)(\theta_T r_T^* - 1) I_T^* - \alpha_T (1-(1-\mu)Y(I_T)) I_T$ is monotonically decreasing in α_T, that $q(x_T^*) + (1-\alpha_T)(\theta_T r_T^* - 1) I_T^* - \alpha_T (1-(1-\mu)Y(I_T)) I_T < 0$ if $\alpha_T = 1$ and that

$q(x_T^*) + (1-\alpha_T)(\theta_T r_T^* - 1) I_T^* - \alpha_T (1-(1-\mu)Y(I_T)) I_T =$

$q(x_T^*) + (\theta_T r_T^* - 1) I_T^* > (1-\mu)Y(I_{T-1})I_{T-1}$ if $\alpha_T = 0$. The updating of reputation in this equilibrium is consistent with the bank's optimal strategy, given that

$$q(\hat{x}_{T-1}) + V_{MT}(\hat{x}_{T-1}, \hat{x}_{T-1}) \gtreqless q(w) + V_{MT}(w, \hat{x}_{T-1})$$

as $\hat{x}_t \gtreqless w$. Randomization is an optimal strategy for borrowers, given that the bank enters date T with reputation x_T^*, hence

$V_{Bt}(rep, I^*(rep, r_t), r_t) = V_{Bt}(def, I^*(rep, r_t), x(r_t))$ by definition.

In the above derivation of $\hat{x}_{T-1}$, we have illustrated two different possibilities for the equilibrium path. In the first case, liquidation on date $T-1$ boosts the bank's reputation sufficiently such that it will face no further repudiation from borrowers. In the second case, however, liquidation on date $T-1$ does not increase reputation sufficiently to deter positive probability of repudiation on date T. Recall the definition of the zones. We define each zone, such that in equilibrium, borrowers will repudiate on no more than one date within each zone. This means that if borrowers repudiate on any date in zone 1, and the bank liquidates, then its reputation will be sufficiently increased such that it will not encounter repudiation on any future date during zone 1, nor for the rest of its lifetime (given that zone 1 is the ultimate zone). Hence, similar to the first case described above for $\hat{x}_{T-1}$, for all dates $T-a \in Zone$ 1, $\hat{x}_{T-a}$ is determined such that the following condition holds:

$$q(\hat{x}_{T-a}) + \sum_{i=0}^{a-1} \left(\theta_{T-i} r_{T-i}^* - 1\right) I_{T-i}^* = (1-\mu)Y(I_{T-a})I_{T-a}$$

where $\hat{x}_{T-a} > \max \left[x_{T-i}^*\right]_{i=0}^{a-1}$, and hence if the bank liquidates on date $T-a$, then it will face no further repudiation during zone 1. Given that borrowers will repay on all future dates, the bank's future pay-offs if it liquidates on date $T-a$ are given by $V_{MT-a+1}(\hat{x}_{T-a},\hat{x}_{T-a}) = \sum_{i=0}^{a-1}\left(\theta_{T-i}r^*_{T-i}-1\right)I^*_{T-i}$.

Given $\hat{x}_{T-a}$, we derive x^*_{T-a} using (2.8). This describes the equilibrium for all dates in zone 1. We now need to solve for when zone 1 ends and zone 2 begins. (We define the end of a zone to be the date furthest away from date T, and the beginning of a zone to be the date closest to date T). We do this by using the following algorithm:

1) Set $a=1$.

2) Derive x^*_{T-a+1} (from (2.8)).

3) Derive z_{T-a} such that the following holds

$$q(z_{T-a}) + \sum_{i=0}^{a-1}\left(\theta_{T-i}r^*_{T-i}-1\right)I^*_{T-i} = (1-\mu)Y(I_{T-a})I_{T-a}$$

If $z_{T-a} \leq \max\left[x^*_{T-i}\right]_{i=0}^{a-1}$, then $x^{*\max}_{Z1} = \max\left[x^*_{T-i}\right]_{i=0}^{a-1}$ and zone 2 begins on date $T-a$. If $z_{T-a} > \max\left[x^*_{T-i}\right]_{i=0}^{a-1}$, then set $\hat{x}_{T-a} = z_{T-a}$ and return to step (2) and repeat for $a=a+1$.

We iterate on this process until we arrive at date $T-j_1$, the start date of zone 2, where j_1 is the smallest a such that the following condition holds:

$z_{T-j_1} < \max\left[x^*_{T-i}\right]_{i=0}^{j_1-1}$ where z_{T-j_1} solves the following

$$q(z_{T-j_1}) + \sum_{i=0}^{j_1-1}\left(\theta_{T-i}r^*_{T-i}-1\right)I^*_{T-i} = (1-\mu)Y(I^*_{T-j_1})I^*_{T-j_1} \tag{2.18}$$

We define the highest critical level of reputation in zone 1 as $\max\left[x^*_{T-i}\right]_{i=0}^{j_1-1} = x^*_{T-k_1} = x^{*\max}_{Z1}$. The explanation for why date $T-j_1$ is in zone 2 and not zone 1 is as follows. If the bank liquidates on date $T-j_1$, its reputation rises to $\hat{x}_{T-j_1}$. If $T-j_1$ is in zone 1, then it must be true that reputation rises sufficiently to deter any future repudiation, i.e. $\hat{x}_{T-j_1} > \max\left[x^*_{T-i}\right]_{i=0}^{a-1}$ must hold. But this cannot hold, because if it did hold, then the value of the bank's future pay-offs would be $V_{MT-j_1+1}(\hat{x}_{T-j_1},\hat{x}_{T-j_1}) = \sum_{i=0}^{j_1-1}\left(\theta_{T-i}r^*_{T-i}-1\right)I^*_{T-i}$, hence $\hat{x}_{T-j_1}$ would solve the following

$$q(\hat{x}_{T-j_1}) + \sum_{i=0}^{j_1-1} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i} = (1-\mu) Y(I^*_{T-j_1}) I^*_{T-j_1} \tag{2.19}$$

But we know that $z_{T-j_1} < \max\left[x^*_{T-i}\right]_{i=0}^{j_1-1}$ where z_{T-j_1} solves (2.18), which is identical to (2.19), hence it cannot be true that $\hat{x}_{T-j_1} > \max\left[x^*_{T-i}\right]_{i=0}^{a-1}$, and thus $T - j_1$ cannot be in zone 1.

We continue by solving for equilibrium in zone 2.

Solving for zone 2

Zone 2 is defined such that if the bank liquidates on any date during zone 2, it will face no further repudiation before it enters zone 1. We now solve for equilibrium on dates in zone 2. First, we need to prove the following lemma.

Lemma 7 *For any $\hat{x}_{T-g-h}$ which solves*

$$q(\hat{x}_{T-g-h}) + \sum_{i=g}^{g+h-1} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i} = (1-\mu) Y(I^*_{T-g-h}) I^*_{T-g-h} \tag{2.20}$$

and for any $z_{T-g-h-m}$ which solves

$$q(z_{T-g-h-m}) + \sum_{i=g+m}^{g+h+m-1} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i} = (1-\mu) Y(I^*_{T-g-h-m}) I^*_{T-g-h-m} \tag{2.21}$$

where $m > 0$, it must be true that $z_{T-g-h-m} > \hat{x}_{T-g-h}$, if the following set of conditions holds

$$\left(\theta_{t+1} r^*_{t+1} - 1\right) \theta_t^2 \left(\theta_{t+1} - \theta_t\right) > (1-\mu) Y(I^*_t) \theta_{t+1}^2 \left(\theta_t - \theta_{t-1}\right) \qquad \textit{for all } t \leq T \tag{2.22}$$

Proof. See appendix.

We now solve for equilibrium on all dates in zone 2. First, we solve for equilibrium on dates $T-(j_1+k_1) \leq t \leq T-j_1$. Proposition 4 proves that for all these dates, $\hat{x}_t$ is identical and is given by $[\hat{x}_{T-i}]_{i=j_1}^{j_1+k_1} = x^*_{T-k_1}$. We then derive an algorithm which solves for the remainder of zone 2, and also derives the dates on which zone 2 ends and zone 3 begins.

Proposition 4 *The unique equilibrium on dates* $T-(j_1+k_1) \leq t \leq T-j_1$ *is* $[\hat{x}_{T-i}]_{i=j_1}^{j_1+k_1} = x^*_{T-k_1}$.

Proof. We give here a sketch of the proof (see the appendix for the full proof). We know that $[\hat{x}_{T-i}]_{i=j_1}^{j_1+k_1} > x^*_{T-k_1}$ cannot be an equilibrium for the following reason. Consider $\hat{x}_{T-j_1}$. Recall $\max\left[x^*_{T-i}\right]_{i=0}^{j_1-1} = x^*_{T-k_1} = x^{*\max}_{Z1}$, i.e. $x^*_{T-k_1}$ is the highest critical value of reputation in zone 1. Hence, if $\hat{x}_{T-j_1} > x^*_{T-k_1}$, then if the bank liquidates on date $T-j_1$, it will face no further repudiation before the horizon, given that . But this would mean that date $T-j_1$ is in zone 1 (from the definition of zone 1), hence this cannot be an equilibrium. This argument generalizes for all dates $T-(j_1+k_1) \leq t \leq T-j_1$, as shown in the full proof.

We know that $[\hat{x}_{T-i}]_{i=j_1}^{j_1+k_1} < x^*_{T-k_1}$ cannot be an equilibrium for the following reason. Consider $\hat{x}_{T-j_1}$ and consider the bank type $w = \hat{x}_{T-j_1}$. If $\hat{x}_{T-j_1} < x^*_{T-k_1}$, then if type $w = \hat{x}_{T-j_1}$ liquidates on date $T-j_1$, the furthest it can go and keep on lending is date $T-k_1-1$. On date $T-k_1$, given that the bank has reputation $\hat{x}_{T-j_1} < x^*_{T-k_1}$, borrowers' equilibrium strategy will be to randomize, and in equilibrium, type $w = \hat{x}_{T-j_1}$ will not lend from date $T-k_1$ onwards (given that it receives a pay-off of zero if it continues to lend, and thus it is indifferent between lending and not lending). Now, suppose that for dates $T-j_1+1 \leq t \leq T-k_1-1$, $x^*_t < \hat{x}_{T-j_1}$. Hence, with reputation $\hat{x}_{T-j_1}$, borrowers will repay with probability 1 on all dates $T-j_1+1 \leq t \leq T-k_1-1$. Hence, given this, and given that type $w = \hat{x}_{T-j_1}$ will not lend after date $T-k_1-1$, its future pay-offs are given by $V_{MT-j_1+1}(\hat{x}_{T-j_1}, \hat{x}_{T-j_1}) = \sum\limits_{i=k_1+1}^{j_1-1} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i}$ (the argument we construct here is generalized in the full proof to consider all possible equilibria, e.g. if in equilibrium the bank were only to lend up to date $T-k_1-2$). Hence, for equilibrium to hold, $\hat{x}_{T-j_1}$

must solve the following:

$$q(\hat{x}_{T-j_1}) + \sum_{i=k_1+1}^{j_1-1} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i}. = (1-\mu) Y(I^*_{T-j_1}) I^*_{T-j_1} \tag{2.23}$$

Note that $\hat{x}_{T-j_1}$ is a decreasing function of the sum of future pay-offs.
If $\sum_{i=k_1+1}^{j_1-1} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i}$ increases, then $q(\hat{x}_{T-j_1})$ must fall for (2.23) to hold, i.e. $\hat{x}_{T-j_1}$ must fall since q is increasing in $\hat{x}_{T-j_1}$. Note also that, following liquidation on date $T - j_1$, the number of dates for which type $w = \hat{x}_{T-j_1}$ will continue to lend, without facing repudiation on any date, is $j_1 - k_1 - 1$. But there also exists a date in zone 1, $T - (j_1 - k_1 - 1)$, such that if the bank liquidates on date $T - (j_1 - k_1 - 1)$, the number of dates for which it will continue to lend, without facing repudiation on any date, is $j_1 - k_1 - 1$. In this case, $\hat{x}_{T-(j_1-k_1-1)}$ is determined such that the following holds:

$$q(\hat{x}_{T-(j_1-k_1-1)}) + \sum_{i=0}^{j_1-k_1-2} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i}. = (1-\mu) Y(I^*_{T-(j_1-k_1-1)}) I^*_{T-(j_1-k_1-1)} \tag{2.24}$$

Hence, given that the sum of future pay-offs is similar on date $T - (j_1 - k_1 - 1)$ to date $T - j_1$, $\hat{x}_{T-j_1}$ which solves (2.23) will be close to $\hat{x}_{T-(j_1-k_1-1)}$ which solves (2.24). However, from above, for any $t'' > t'$, $I^*_{t''} > I^*_{t'}$, and $\theta_{t''} r^*_{t''} > \theta_{t'} r^*_{t'}$, hence $\left(\theta_{t''} r^*_{t''} - 1\right) I^*_{t''} > \left(\theta_{t'} r^*_{t'} - 1\right) I^*_{t'}$. Hence, $\sum_{i=0}^{j_1-k_1-2} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i} > \sum_{i=k_1+1}^{j_1-1} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i}$ must hold. So for (2.24) to hold, (ignoring any difference in the RHS) $\hat{x}_{T-j_1}$ would need to be higher than $\hat{x}_{T-(j_1-k_1-1)}$. Lemma (7) actually states that the difference between the future pay-offs in (2.23) and (2.24) dominates any difference in the RHS. Hence, it must be the case that $\hat{x}_{T-j_1} > \hat{x}_{T-(j_1-k_1-1)}$. But since date $T - (j_1 - k_1 - 1)$ is in zone 1, we know that $\hat{x}_{T-(j_1-k_1-1)} > x^*_{T-k_1}$ holds. Hence, $\hat{x}_{T-j_1} > \hat{x}_{T-(j_1-k_1-1)}$ implies that $\hat{x}_{T-j_1} > x^*_{T-k_1}$. But this contradicts our original claim that $\hat{x}_{T-j_1} < x^*_{T-k_1}$, hence $\hat{x}_{T-j_1} < x^*_{T-k_1}$ cannot be an equilibrium. This argument generalizes for all dates $T - (j_1 + k_1) \leq t \leq T - j_1$, as shown in the full proof.

Since $\hat{x}_{T-j_1} > x^*_{T-k_1}$ cannot be an equilibrium, and $\hat{x}_{T-j_1} < x^*_{T-k_1}$ cannot be an equilibrium, the unique equilibrium is $\hat{x}_{T-j_1} = x^*_{T-k_1}$. Given that max $\left[x^*_{T-i}\right]_{i=0}^{j_1-1} = x^*_{T-k_1}$, we know that $\hat{x}_{T-j_1} > x^*_t$ for $T - j_1 + 1 \leq t \leq T - k_1 - 1$, hence borrowers will repay on all these dates. For equilibrium to hold, borrowers must randomize on date $T - k_1$, such that type $w = \hat{x}_{T-j_1} = x^*_{T-k_1}$ is indifferent between liquidation and renegotiation on date $T - j_1$. If the outcome of randomization is that borrowers repudiate, type $w = \hat{x}_{T-j_1}$'s optimal strategy will be to renegotiate, given that $\hat{x}_{T-j_1} = x^*_{T-k_1} < \hat{x}_{T-k_1}$. Hence, in equilibrium borrowers will repudiate with probability α_{T-k_1}, where α_{T-k_1} solves the following:

$$q(\hat{x}_{T-j_1}) + \sum_{i=k_1+1}^{j_1-1} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i} + V_{MT-k_1}(\hat{x}_{T-j_1}, \hat{x}_{T-j_1}) =$$

$$(1-\mu) Y(I^*_{T-j_1}) I^*_{T-j_1}$$

where

$V_{MT-k_1}(\hat{x}_{T-j_1}, \hat{x}_{T-j_1}) = (1 - \alpha_{T-k_1}) \left[\sum_{i=0}^{k_1} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i}\right]$
$-\alpha_{T-k_1} \left[1 - (1-\mu) Y(I^*_{T-k_1}) I^*_{T-k_1}\right]$. Also, if the bank liquidates on date $T - j_1$, its reputation on entering date $T - k_1$ will be $\hat{x}_{T-j_1} = x^*_{T-k_1}$, hence randomization will be an optimal strategy for borrowers. This argument generalizes for all dates $T - (j_1 + k_1) \leq t \leq T - j_1$, as shown in the full proof.

Q.E.D.

We now need to solve for equilibrium in the remainder of zone 2, and also derive the dates on which zone 2 ends and zone 3 begins. The following algorithm does this.

1) Derive $\left[x^*_{T-i}\right]_{i=j_1}^{j_1+k_1-1}$

2) Set $a = 1$, $b = 0$.

3) Derive $x^*_{T-(j_1+k_1+a+b-1)}$.

4) Derive $z_{T-(j_1+k_1+a+b)}$ such that the following holds

$$q(z_{T-(j_1+k_1+a+b)})+\sum_{i=k_1+b+1}^{j_1+k_1+a+b-1}\left(\theta_{T-i}r^*_{T-i}-1\right)I^*_{T-i}=(1-\mu)Y(I_{T-(j_1+k_1+a+b)})I_{T-(j_1+k_1+a+b)} \tag{2.25}$$

If $z_{T-(j_1+k_1+a+b)} \leq \max\left[x^*_{T-i}\right]_{i=k_1+b+1}^{j_1+k_1+a+b-1}$, then go to step (5). If $z_{T-(j_1+k_1+a+b)} > \max\left[x^*_{T-i}\right]_{i=k_1+b+1}^{j_1+k_1+a+b-1}$, then the equilibrium for date $T-(j_1+k_1+a+b)$ is $\hat{x}_{T-(j_1+k_1+a+b)} = z_{T-(j_1+k_1+a+b)}$. Return to step (3) and repeat for $a = a+1$, $b = b$.

5) If $\max\left[x^*_{T-i}\right]_{i=k_1+b+1}^{j_1+k_1+a+b-1} \in$ *Zone* 2, then $x^{*\max}_{Z2} = \max\left[x^*_{T-i}\right]_{i=k_1+b+1}^{j_1+k_1+a+b-1}$ and Zone 3 begins on date $T-(j_1+k_1+a+b)$. If $\max\left[x^*_{T-i}\right]_{i=k_1+b+1}^{j_1+k_1+a+b-1} \in$ *Zone* 1, then let $\max\left[x^*_{T-i}\right]_{i=k_1+b+1}^{j_1+k_1+a+b-1} \equiv x^*_{T-k_1-b-c}$ where $1 \leq c \leq j_1-k_1-b-1$. Then the equilibrium for dates $T-(j_1+k_1+a+b+c-1) \leq t \leq T-(j_1+k_1+a+b)$ is $[\hat{x}_{T-i}]_{i=j_1+k_1+a+b}^{j_1+k_1+a+b+c-1} = x^*_{T-k_1-b-c}$. Return to step (3) and repeat for $a = a$ and $b = b+c$.

We iterate on this process until we arrive at date $T-(k_1+j_1+j_2+n_1)$, the start date of Zone 3, where j_2 is the smallest a and n_1 is the smallest b such that the following two conditions hold:

1) $z_{T-(j_1+k_1+j_2+n_1)} \leq \max\left[x^*_{T-i}\right]_{i=k_1+n_1+1}^{j_1+k_1+j_2+n_1-1}$ where $z_{T-(j_1+k_1+j_2+n_1)}$ solves the following

$$\begin{aligned} & q(z_{T-(j_1+k_1+j_2+n_1)})+\sum_{i=k_1+n_1+1}^{j_1+k_1+j_2+n_1-1}\left(\theta_{T-i}r^*_{T-i}-1\right)I^*_{T-i} \\ = \ & (1-\mu)Y(I_{T-(j_1+k_1+j_2+n_1)})I_{T-(j_1+k_1+j_2+n_1)} \end{aligned}$$

and

2) $\max\left[x^*_{T-i}\right]_{i=k_1+n_1+1}^{j_1+k_1+j_2+n_1-1} \in$ *Zone* 2 where $j_2 \geq 1$ and $0 \leq n_1 \leq j_1-k_1-1$

In step (4) we can prove that $\hat{x}_{T-(j_1+k_1+a+b)} = z_{T-(j_1+k_1+a+b)}$ is an equilibrium, given that (2.25) holds, and $V_{MT-(j_1+k_1+a+b)+1}(\hat{x}_{T-(j_1+k_1+a+b)}, \hat{x}_{T-(j_1+k_1+a+b)}) = \sum_{i=k_1+b+1}^{j_1+k_1+a+b-1}\left(\theta_{T-i}r^*_{T-i}-1\right)I^*_{T-i}$, since $z_{T-(j_1+k_1+a+b)} > \max\left[x^*_{T-i}\right]_{i=k_1+b+1}^{j_1+k_1+a+b-1}$ implies that with reputation $z_{T-(j_1+k_1+a+b)}$, the bank will face no further repudiation before date $T-(k_1+b+1)$. Also, we know that $z_{T-(j_1+k_1+a+b)} < x^*_{T-k_1-b}$, given that from step (5) there exists $z_{T-(j_1+k_1+a'+b)} < x^*_{T-k_1-b}$ which solves

$$q(z_{T-(j_1+k_1+a'+b)}) + \sum_{i=k_1+b+1}^{j_1+k_1+a'+b-1} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i} = (1-\mu) Y(I_{T-(j_1+k_1+a'+b)}) I_{T-(j_1+k_1+a'+b)}$$

where $a' < a$, hence $\sum_{i=k_1+b+1}^{j_1+k_1+a+b-1} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i} > \sum_{i=k_1+b+1}^{j_1+k_1+a'+b-1} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i}$ and $(1-\mu)Y(I_{T-(j_1+k_1+a+b)})I_{T-(j_1+k_1+a+b)} < (1-\mu)Y(I_{T-(j_1+k_1+a'+b)})I_{T-(j_1+k_1+a'+b)}$ which implies that $z_{T-(j_1+k_1+a+b)} < z_{T-(j_1+k_1+a'+b)} < x^*_{T-k_1-b}$, hence in equilibrium type $z_{T-(j_1+k_1+a+b)}$ will cease to lend on date $T-k_1-b$.

In step (5), the proof that $[\hat{x}_{T-i}]_{i=j_1+k_1+a+b}^{j_1+k_1+a+b+c-1} = x^*_{T-k_1-b-c}$ is the same as the proof of proposition 4.

The above solution of equilibrium in zone 2 also proves that $V_{Mt+1}(w,x)$ is weakly increasing in both w and x. The higher is the bank's type w, the lower is its cost of liquidation, and hence the bank can survive for a longer period of time with its reputation intact, before its incentives to liquidate deteriorate sufficiently such that it is no longer optimal to liquidate, and hence the bank ceases to lend. Hence, higher is the bank's type w, the greater are its future pay-offs. The higher is the bank's reputation x, the longer is the period of time it can lend without facing repudiation, and hence the greater are its future pay-offs.

Solving for zone τ

We now give a general solution of the multi-period equilibrium by solving for the generic zone τ. We solve recursively for equilibrium in zone τ, given the equilibrium for zone $\tau-1$, and derive the dates on which zone τ ends and zone $\tau+1$ begins.

Proposition 5 *Suppose zone τ begins on date $T-(k_{\tau-2}+\sum_{y=1}^{\tau-1} j_y + n_{\tau-2})$. Let*

$\max\left[x_t^* \mid x_t^* \in Zone\ \tau - 1\right] \equiv x_{T-k_{\tau-1}}^*$. Then equilibrium for dates $T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1) \leq t \leq T-(k_{\tau-2}+\sum_{y=1}^{\tau-1} j_y+n_{\tau-2})$ is given by $[\hat{x}_{T-i}]_{i=k_{\tau-2}+\sum_{y=1}^{\tau-1} j_y+n_{\tau-2}}^{k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1} = x_{T-k_{\tau-1}}^*$. The following algorithm then solves for equilibrium on the remaining dates in zone τ.

1) Derive $\left[x_{T-i}^*\right]_{i=k_{\tau-2}+\sum_{y=1}^{\tau-1} j_y+n_{\tau-2}}^{k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-2}$

2) Set $a=1$, $b=0$.

3) Derive $x^*_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-2+a+b)}$.

4) Derive $z_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b)}$ such that the following holds $q(z_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b)})+$

$$\sum_{i=k_{\tau-1}+b+1}^{k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-2+a+b} \left(\theta_{T-i} r_{T-i}^* - 1\right) I_{T-i}^* = (1-\mu) Y(I_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b)}) I_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b)}$$

If $z_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b)} \leq \max\left[x_{T-i}^*\right]_{i=k_{\tau-1}+b+1}^{k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-2+a+b}$, then go to step (5).

If $z_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b)} > \max\left[x_{T-i}^*\right]_{i=k_{\tau-1}+b+1}^{k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-2+a+b}$, then the equilibrium for date $T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b)$ is $\hat{x}_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b)} = z_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b)}$. Return to step (3) and repeat for $a=a+1$, $b=b$.

5) If $\max\left[x_{T-i}^*\right]_{i=k_{\tau-1}+b+1}^{k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-2+a+b} \in Zone\ \tau$, then $x_{Z\tau}^{*\max} = \max\left[x_{T-i}^*\right]_{i=k_{\tau-1}+b+1}^{k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-2+a+b}$ and Zone $\tau+1$ begins on date $T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b)$. If $\max\left[x_{T-i}^*\right]_{i=k_{\tau-1}+b+1}^{k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-2+a+b} \in Zone\ \tau-1$, then let $\max\left[x_{T-i}^*\right]_{i=k_{\tau-1}+b+1}^{k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-2+a+b} \equiv x_{T-k_{\tau-1}-b-c}^*$. Then the equilibrium for dates $T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-2+a+b+c) \leq t \leq T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b)$ is $[\hat{x}_{T-i}]_{i=k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b}^{k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-2+a+b+c} = x_{T-k_{\tau-1}-b-c}^*$. Return to step (3) and repeat for $a=a$ and $b=b+c$.

Proof. The proof that $[\hat{x}_{T-i}]_{i=k_{\tau-2}+\sum_{y=1}^{\tau-1} j_y+n_{\tau-2}}^{k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1} = x^*_{T-k_{\tau-1}}$ is the same as the proof of proposition 4. The proofs that $\hat{x}_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b)} = z_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b)}$ in step (4) and $[\hat{x}_{T-i}]_{i=k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b}^{k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-2+a+b+c} = x^*_{T-k_{\tau-1}-b-c}$ in step (5) are the same as the proofs of step (4) and (5) in the algorithm above which solves for zone 2.

Q.E.D.

The algorithm derives the dates on which zone τ ends and zone $\tau+1$ begins as follows. Zone τ ends on date $T-(k_{\tau-1}+\sum_{y=1}^{\tau} j_y + n_{\tau-1}-1)$ and zone $\tau+1$ begins on date $T-(k_{\tau-1}+\sum_{y=1}^{\tau} j_y + n_{\tau-1})$, where $j\tau$ and $n_{\tau-1}$ are the smallest integers such that the following two conditions hold:

1) $z_{T-(k_{\tau-1}+\sum_{y=1}^{\tau} j_y+n_{\tau-1})} \leq \max\left[x^*_{T-i}\right]_{i=k_{\tau-1}+n_{\tau-1}-1}^{k_{\tau-1}+\sum_{y=1}^{\tau} j_y+n_{\tau-1}-1}$ where $z_{T-(k_{\tau-1}+\sum_{y=1}^{\tau} j_y+n_{\tau-1})}$ solves the following

$$q(z_{T-(k_{\tau-1}+\sum_{y=1}^{\tau} j_y+n_{\tau-1})}) + \sum_{i=k_{\tau-1}+n_{\tau-1}+1}^{k_{\tau-1}+\sum_{y=1}^{\tau} j_y+n_{\tau-1}-1} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i} =$$

$$(1-\mu)Y(I_{T-(k_{\tau-1}+\sum_{y=1}^{\tau} j_y+n_{\tau-1})}) I_{T-(k_{\tau-1}+\sum_{y=1}^{\tau} j_y+n_{\tau-1})}$$

and

2) $\max\left[x^*_{T-i}\right]_{i=k_{\tau-1}+n_{\tau-1}-1}^{k_{\tau-1}+\sum_{y=1}^{\tau} j_y+n_{\tau-1}-1} \in Zone\ \tau$ where $j_\tau \geq 1$ and $0 \leq n_{\tau-1} \leq k_{\tau-2} - k_{\tau-1} + \sum_{y=1}^{\tau-1} j_y + n_{\tau-2}$.

We continue to solve for the equilibrium in each zone using this algorithm until either we reach date $t=1$, or we reach some date $T-t'$ such that lemma 8 holds.

Lemma 8 *If there exists j' large enough such that*

$$q(z_{T-t'}) + \sum_{i=t'-j'}^{t'-1} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i} > (1-\mu)Y(I_{T-t'}) I_{T-t'}$$

where $z_{T-t'} = 0$, $\left[x^*_{T-i}\right]^{t'-1}_{i=t'-j'} \leq 0$, *and* $x^*_{T-(t'-j')+1} > 0$ *then* $[\hat{x}_i]^{T-t'}_{i=1} = 0$.

Proof. Given that $q(w) + V_{MT-t'+1}(w, 0) > (1-\mu)Y(I_{T-t'})I_{T-t'}$ for $w \geq 0$, $\hat{x}_{T-t'} = 0$ is consistent with the bank's equilibrium strategy. $\hat{x}_{T-t'} = 0$ implies that $x^*_{T-t'} \leq 0$, hence it must be true that

$$q(0) + \sum_{i=t'-j'}^{t'} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i} > (1-\mu)Y(I_{T-t'-1})I_{T-t'-1}$$

given that $\sum_{i=t'-j'}^{t'} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i} > \sum_{i=t'-j'}^{t'-1} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i}$ and $(1-\mu)Y(I_{T-t'-1})I_{T-t'-1} < (1-\mu)Y(I_{T-t'})I_{T-t'}$, hence $\hat{x}_{T-t'-1} = 0$. By recursion on this process, we get $[\hat{x}_i]^{T-t'}_{i=1} = 0$.

Q.E.D.

Lemma 8 shows that, as the bank's horizon gets longer, there comes a point when the sum of future pay-offs available to the bank if it retains its reputation grows so large that all bank types $w \geq 0$ will prefer to liquidate, if borrowers repudiate, no matter how high are their costs of liquidation. Hence, $q(\hat{x}_t) + V_{Mt+1}(\hat{x}_t, \hat{x}_t) > (1-\mu)Y(I_t)I_t$ holds even if $\hat{x}_t = 0$. If this is the case, in equilibrium we set $\hat{x}_t = 0$.

2.9 Properties of the Multi-Period Equilibrium

In chapter one, we explained the process of how start-up firms without a credit history gradually evolve over time into well-established 'blue chip' firms with good credit ratings, who are able to borrow directly on the open market. Essentially, we are modelling the transition of an emerging market economy into a fully-fledged market economy over time. For the initial conditions, we have a cohort of small firms who have never invested in the past, and a newly set up bank which has no previous experience of lending. In the steady state, we have a fully-fledged market economy, with a bank which has an established track record for lending in the credit market, and a cohort of well-established 'blue chip' firms with good credit ratings. Hence, it is not the case that the bank develops first, and then lends to these small firms, who can then grow over time. Instead, the evolution of the bank and the evolution of the small firms is coincident. In section 2.2 above, we explained that if the bank ever lost its reputation, it would never be able to lend again, i.e. it would fail. But given this coincident evolution, if the bank ever fails, then this puts a brake on the development of the small firms. These firms rely on bank finance, since they are unable to borrow directly. But if the bank fails, then they are unable to continue to borrow and invest, and hence will never be able to develop to a point at which they are able to borrow directly. Hence, it is important to analyze whether the bank will survive up to the point at which firms have developed sufficiently such that they no longer need the bank, because they are able to borrow directly. We do this by deriving the following two properties of the multi-period equilibrium:

1) The upper and lower bounds on the bank's ex ante probability of failure, i.e. the probability on date $t = 0$ when the bank has no track record (and before nature has determined its class and type) that it will lose its reputation on some date before the horizon.

2) The maximum number of dates on which the bank might face repudiation during its lifetime.

The first property gives us an idea of the probability that the bank will survive and continue to lend right up to date T. It would be much more complicated to derive the exact probability of failure (instead of the upper and lower bounds), as this would be a function of the actual equilibrium outcomes, i.e. the outcomes of borrower randomizations (for each possible permutation for the sequence of outcomes, there would be a different corresponding probability). Also, the derivation of the upper and lower bounds is a more interesting result, because it allows us to illustrate clearly how the dynamics of the bank's and borrowers' reputation effects are mutually re-inforcing, which tends to lower the probability that the bank fails, as we explain in section 2.10 below.

The second property is measured by the total number of zones, given that in equilibrium, borrowers will never repudiate more than once within any given zone. It is interesting because it gives a measure of the costs of reputation building for the bank. The greater is the total number of zones, the higher is the maximum number of repudiations that the bank might encounter during its lifetime, hence the greater are its potential costs of liquidation. Note that the total number of zones only gives an upper bound on the number of repudiations, given that for each zone there exists a positive probability that no repudiation occurs at all, and in the first zone (closest to date $t = 1$), there could actually be zero probability of repudiation. These statements are proved in proposition 6 below.

First, we prove that the maximum number of repudiations is given by the total number of zones. Proposition 6 below proves this. But before we can prove proposition 6, we must prove lemma 9 below. This proves that the series $[\hat{x}_t]_{t=1}^{T}$ is (weakly) increasing over time.

Lemma 9 *If $T - t' \in$ Zone τ and $T - t'' \in$ Zone $\tau - 1$, then $\hat{x}_{T-t'} < \hat{x}_{T-t''}$.*

Proof. See appendix.

We now prove proposition 6.

Proposition 6 *Suppose there are ζ zones in total. Then for all zones $\tau < \zeta$, the occurrence of more than one repudiation within each zone is a zero probability event, but the event of exactly one repudiation occurs with positive probability. In zone ζ, the occurrence of any repudiation at all is a zero probability event if and only if $x_{z\zeta}^{*\max} \leq 0$, in which case the maximum number of repudiations is $\zeta - 1$. If $x_{z\zeta}^{*\max} > 0$, the maximum number of repudiations is ζ.*

Proof. We give here a sketch of the proof (see the appendix for the full proof). First, we prove that for all zones $\tau < \zeta$, the event of exactly one repudiation occurs with positive probability, i.e. although the bank might not encounter repudiation at all in any given zone, it is not the case that the bank will escape repudiation with certainty. From the construction of each zone, we know that in equilibrium, the bank cannot enter zone τ with a reputation $x > x_{z\tau}^{*\max} = x_{T-k_\tau}^{*}$. Hence, if the bank has not yet encountered repudiation and liquidated before date $T - k_\tau$, its reputation will still be $x \leq x_{T-k_\tau}^{*}$ on date $T - k_\tau$. Hence, in equilibrium borrowers will randomize on date $T - k_\tau$, and thus repudiation will occur with positive probability.

We now prove that the occurrence of more than one repudiation within each zone is a zero probability event. From the construction of each zone, we know that for all dates in zone τ, $\hat{x}_t > x_{z\tau}^{*\max}$. If $\hat{x}_{t'} < x_{z\tau}^{*\max}$, then from the algorithm given in proposition 5, date t' cannot belong to zone τ. Hence, after the first time that borrowers repudiate in zone τ, if the bank liquidates, its reputation rises sufficiently such that it will face no further repudiations. Given that $\hat{x}_t > x_{z\tau}^{*\max} = x_{T-k_\tau}^{*}$, borrowers will repay on date $T - k_\tau$, and given that $x_t^* \leq x_{z\tau}^{*\max}$ for all other dates in zone τ, borrowers will repay on all other dates.

We now prove that in zone ζ, the occurrence of any repudiation at all is a zero probability event if and only if $x_{z\zeta}^{*\max} \leq 0$, in which case the maximum number of repudiations is $\zeta - 1$, and that if $x_{z\zeta}^{*\max} > 0$, then the maximum number of repudiations is ζ. If $x_{z\zeta}^{*\max} \leq 0$, then this means that in zone ζ, the bank's horizon is very long, hence the value of its future pay-offs (if it retains its reputation) is very high. Hence, if borrowers repudiate, most bank types will prefer to liquidate, even if liquidation is very costly. In fact, the probability that borrowers face liquidation, even if the bank has not yet built a reputation (i.e. $x = 0$), is very high, such that borrowers will prefer to repay. Hence, although the bank begins its life with $x = 0$, it will face no repudiation during the first zone, zone ζ. Hence, in this case, given that there are ζ zones in total, the maximum number of repudiations is $\zeta - 1$. However, if $x_{z\zeta}^{*\max} > 0$, then the bank will encounter repudiation with positive probability during zone ζ, hence the maximum number of repudiations is ζ.

Q.E.D.

We can now give a general characterization of equilibrium. The bank begins on date $t = 0$ with reputation $x = 0$. There are two possibilities:

1) $x_{z\zeta}^{*\max} > 0$, or

2) $x_{z\zeta}^{*\max} = 0$.

If $x_{z\zeta}^{*\max} > 0$ then on the first date t' in zone ζ that $x_{t'}^* > 0$, if the bank's type is $w < x_{t'}^*$ then it will cease to lend. If its type is $w \geq x_{t'}^*$ it will lend and its reputation is revised upwards to $x = x_{t'}^*$, and borrowers will randomize. If the outcome of randomization is that borrowers repay debt, then the bank is not tested and its reputation stays at $x = x_{t'}^*$. The bank then faces potential repudiation again on the first date t'' that $x_{t'}^* < x_{t''}^*$. If, however $x_{t'}^* = x_{z\zeta}^{*\max}$ then the bank will face no further defaults during zone ζ.

If the outcome of randomization is that borrowers repudiate, then the bank will liquidate if and only if its type is $w \geq \hat{x}_t$, and thus its reputation is revised upwards to $x = \hat{x}_t$. It will then face no further repudiation during zone ζ. If the bank's type is $w < \hat{x}_t$ it will renegotiate and thus lose its reputation, and hence it will not lend on any subsequent date.

This process then recurs during each zone.

If $x_{z\zeta}^{*\max} = 0$, then there is no repudiation during zone ζ. For zones $\tau < \zeta$, the equilibrium path is as described above for the case $x_{z\zeta}^{*\max} > 0$.

We now derive the upper and lower bounds on the bank's ex ante probability of failure, in proposition 7 below.

Proposition 7 *The upper and lower bounds on the ex ante probability that the bank will fail before date T are given by $\frac{\hat{x}_{T-k_1}}{1+\delta}$ and $\frac{x^*_{T-k_1}}{1+\delta}$ respectively.*

Proof. We give here a sketch of the proof (see the appendix for the full proof). To derive the lower bound on the probability that the bank fails, we need to solve for the critical bank type w', such that whatever the sequence of actual outcomes along the equilibrium path, we know that types $w \leq w'$ will always lose their reputation before date T, but there exists some outcomes in which types $w > w'$ will retain their reputation up to date T. Recall that w is the realization of random variable ω which is uniformly distributed on $[0,1]$. Hence, if we take as given that the bank is class W, then w' is the lower bound on the probability that the bank fails. But ex ante there is also a probability that the bank is class S, given by $\frac{\delta}{1+\delta}$. Hence, the lower bound on the ex ante probability that the bank fails is given by

$$\frac{\delta}{1+\delta}(0) + \frac{1}{1+\delta}(w') = \frac{1}{1+\delta}w'$$

From the proof of proposition 6, we know that within any given zone τ, there exists $x^*_{T-k_\tau} \equiv x^{*\max}_{z\tau}$, such that if the bank evades repudiation before date $T-k_\tau$, it will face repudiation with positive probability on date $T-k_\tau$. In equilibrium, types $w < x^*_{T-k_\tau}$ will cease to lend on this date, and will thus lose their reputation. What about types $w \geq x^*_{T-k_\tau}$? These types will lend on date $T-k_\tau$, and will evade repudiation if the outcome of borrower randomization is that borrowers repay. So, is $\frac{1}{1+\delta}x^*_{T-k_\tau}$ the lower bound on the probability of failure? Reputation jumps to $x^*_{T-k_\tau}$, but this is not high enough to prevent further repudiation. There will be a date t' in zone $\tau-1$, for which $x^*_{T-k_\tau} < x^*_{t'}$, and hence on this date types $w < x^*_{t'}$ will cease to lend. Hence $\frac{1}{1+\delta}x^*_{T-k_\tau}$ is not the lower bound on the probability of failure. So we must solve for the key date, after which the bank will face no further repudiations. This must be the date in the T-period timespan on which x^*_t is highest. From lemma 9, we know that this is date $T-k_1$ in zone 1, with $x^*_{T-k_\tau} = x^{*\max}_{z1}$. If the bank evades repudiation in zone 1 before date $T-k_1$, it will face repudiation with positive probability on date $T-k_1$. In equilibrium, types $w < x^*_{T-k_1}$ will cease to lend on this date, but types $w \geq x^*_{T-k_1}$ will lend. If the outcome of borrower randomization is that borrowers repay, then reputation will jump to $x^*_{T-k_1}$, and given that $x^*_{T-k_1} \geq x^*_t$ for all t, the bank will face no further repudiations. Hence, providing that borrowers repay on date $T-k_1$, then if the bank has type $w \geq x^*_{T-k_1}$, it will not fail before date T. Thus the lower bound on the probability of failure is $\frac{1}{1+\delta}x^*_{T-k_1}$.

We now prove that the upper bound on the probability that the bank fails is $\frac{1}{1+\delta}\hat{x}_{T-k_1}$.

We need to solve for the critical bank type w'', such that whatever the sequence of actual outcomes along the equilibrium path, we know that types $w > w''$ will always retain their reputation up to date T, but there exists some outcomes in which types $w \leq w''$ will lose their reputation before date T. Hence, $\frac{1}{1+\delta}w''$ is the upper bound on the probability that the bank fails. To solve for w'', we need to find the date on which the bank's incentives to liquidate are weakest, i.e. the date on which along the equilibrium path, $\hat{x}_t$ is highest. From lemma 9, we know that $\hat{x}_t$ is weakly increasing over time, hence over the entire T-period time span, the maximum $\hat{x}_t$ is $\hat{x}_T$. However, we must only include dates on which there exists some sequence of actual outcomes, such that borrowers might repudiate on these dates, and hence the bank's incentives to liquidate are relevant. But as we just explained above, if the bank evades repudiation during zone 1 before date $T - k_1$, then borrowers will randomize on date $T - k_1$. If the outcome is that borrowers repudiate, then in equilibrium only types $w \geq \hat{x}_{T-k_1}$ will liquidate, their reputation rises to $\hat{x}_{T-k_1}$ and they will face no further repudiation. If the outcome is that borrowers repay, then reputation stays at $x^*_{T-k_1}$, and the bank will face no further repudiation. Hence, there exists no sequence of outcomes in which borrowers will repudiate and the bank's incentives to liquidate are tested beyond date $T - k_1$. Hence, it is irrelevant that $\hat{x}_T > \hat{x}_{T-k_1}$, and thus $\frac{1}{1+\delta}\hat{x}_{T-k_1}$ is the upper bound on the probability of bank failure.

Q.E.D.

2.10 The Dynamic Interaction of Bank and Borrower Reputation Effects

In this section, we illustrate how the dynamics of bank and borrower reputation effects are mutually re-inforcing, which helps to reduce the probability that the bank fails. The reason for this is as follows. During the early stages of their lives when track records for credit repayment are relatively short, borrowers are charged high interest rates, and hence their incentives to repudiate are strong, i.e. borrower reputation effects are weak. But it is exactly at this point in the bank's life that incentives to liquidate are strongest, because the bank has a long horizon, and thus it pays to liquidate and build a reputation, in order to secure future lending opportunities. Hence, if borrowers repudiate, the probability that they will be liquidated is very high, which tends to deter them from repudiation. However, over time, as the horizon gets closer, the bank's incentives to liquidate weaken. But by this time, the value of borrower reputation has increased significantly, for the following two reasons. Firstly, the proportion of type B borrowers in the market has fallen, and hence the lemons premium on the interest rate has fallen. Secondly, borrowers will soon be able to borrow directly instead of being forced to borrow from the bank at a premium interest rate, since they have established a good credit history. Hence, borrowers will have a strong incentive to repay debt, in order to retain access to future credit. Hence, at the stage when most bank types would prefer to renegotiate, it is fortunately the case that the bank's reputation will not be tested anyway, because borrowers will prefer not to repudiate. This tends to reduce the probability that the bank loses its reputation. We can see this illustrated more clearly in Figure 5. Notice that the series $[\hat{x}_t]_{t=1}^T$ is weakly increasing over time, reaching 1 on date T. Hence, it becomes increasingly more difficult for the bank to liquidate as the horizon gets closer. But notice also that P_t^*, the critical probability of liquidation which makes type B borrowers indifferent between repayment and repudiation, is monotonically decreasing over time. This reflects the fact that the value of borrower reputation increases over time. It also affects the series $[x_t^*]_{t=1}^T$ in the following way. x_t^* tends to increase as $\hat{x}_t$ increases over time. But the rising value of borrower reputation tends to offset any increase in x_t^*. This may have the effect that x_t^* does not continue to rise right up to date T, but peaks at $x_{t_{\max}}^*$ on some date $t_{\max} < T$. What this means is that, if the bank attains reputation $x \geq x_{t_{\max}}^*$ before or on date $t_{\max}$, then it will never again face repudiation after this date, since reputation will exceed x_t^* on all dates after $t_{\max}$, which as explained in section 2.3, means that borrowers will prefer to repay debt. Thus the bank's reputation will not be tested after date $t_{\max}$, when $\hat{x}_t$ is

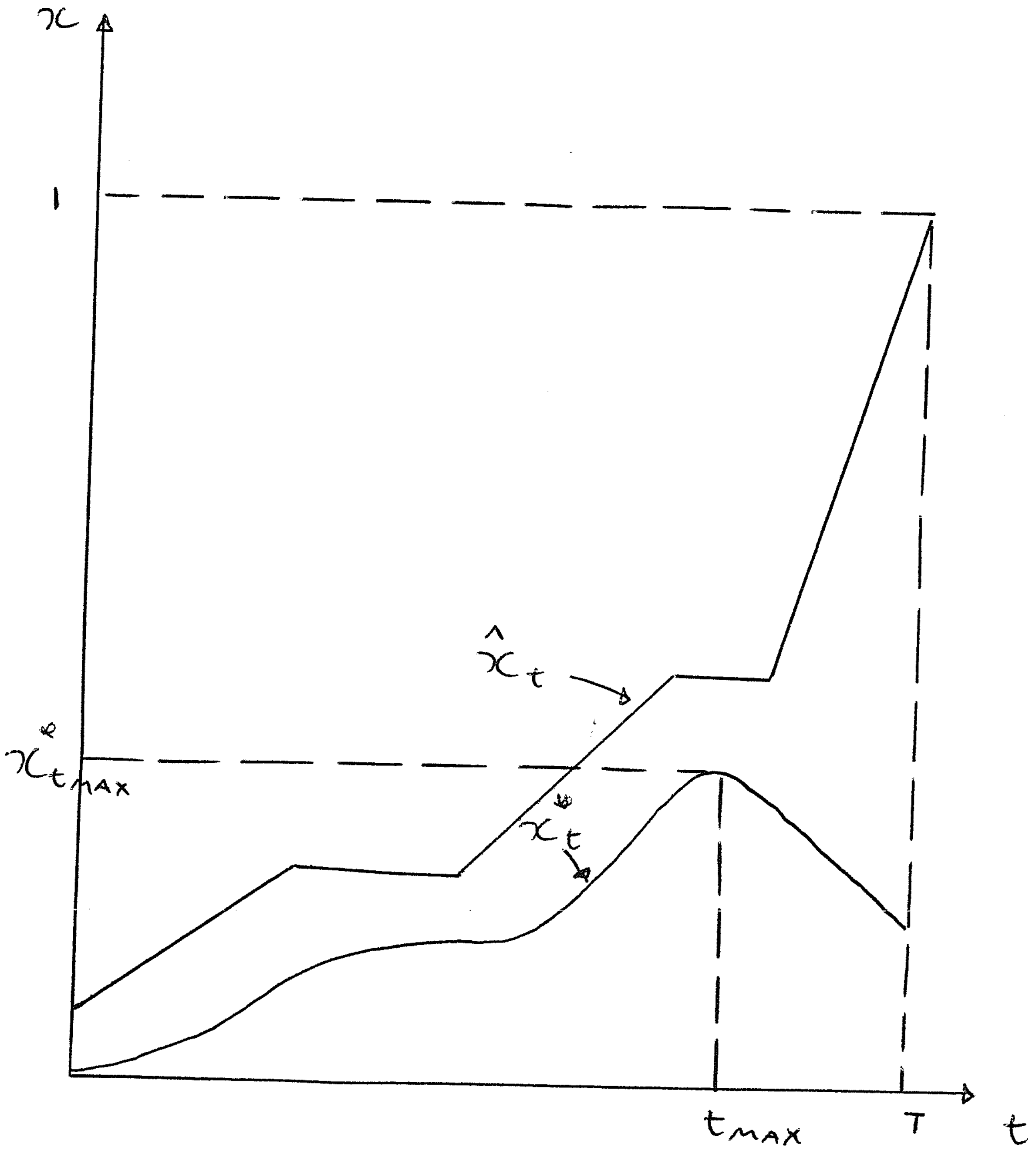

FIGURE 5 : THE DYNAMIC INTERACTION OF BANK AND BORROWER REPUTATION EFFECTS

very high, and thus when most bank types will prefer not to liquidate.

First, we must prove that the series P_t^* (the critical probability of liquidation such that type B borrowers are indifferent between repayment and repudiation) is monotonically decreasing over time. Proposition 8 below proves this.

Proposition 8 *$P_{t-1}^* > P_t^*$ for all $t \leq T$ if the following holds*

$$(F_A + F_B\pi) > \frac{1+\mu}{(1-\mu)a} \tag{2.26}$$

Proof. From (2.7), P_t^* is determined such that the following holds

$$(r_t^* - (1-\mu(1-P_t^*))Y(I_t^*))\, I_t^* = \beta\pi V_{Bt+1}$$

From $I(r_t)$, the borrower demand for funds function, we know that $(Y(I(r_t)) - r_t)I(r_t) < (Y(I(r_{t'})) - r_{t'})I(r_{t'})$ if $r_t > r_{t'}$. Given that $V_{Bt+1} = \sum_{i=0}^{\infty}(\beta\pi)^i(Y(I_{t+1+i}) - r_{t+1+i})I_{t+1+i}$ and $V_{Bt} = \sum_{i=0}^{\infty}(\beta\pi)^i(Y(I_{t+i}) - r_{t+i})I_{t+i}$,
$V_{Bt+1} - V_{Bt} = \sum_{i=0}^{\infty}(\beta\pi)^i\left((Y(I_{t+1+i}) - r_{t+1+i})I_{t+1+i} - (Y(I_{t+i})) - r_{t+i})I_{t+i},\right)$. Given that $(Y(I_{t+1+i}) - r_{t+1+i})I_{t+1+i} - (Y(I_{t+i})) - r_{t+i})I_{t+i} > 0$ for all $i > 0$ (since $r_{t+1+i} < r_{t+i}$), we have $V_{Bt+1} - V_{Bt} > 0$. We will now prove that $(r_{t-1} - (1-\mu)Y(I_{t-1}))\, I_{t-1} > (r_t - (1-\mu)Y(I_t))\, I_t$ if (2.26) holds. On dates $t \leq T$, the interest rate is set by the bank, given by $r_t^* = \frac{1}{2}(a + 1/\theta_t)$ from above. Hence $\theta_{t-1} < \theta_t \Rightarrow r_{t-1}^* > r_t^*$. From proposition 2, if $r < \frac{a}{1+\mu}$ then $\frac{\partial(r-(1-\mu)Y(I))I}{\partial r} > 0$. Hence $\frac{\partial(r-(1-\mu)Y(I))I}{\partial r} > 0$ if $\frac{1}{2}(a+1/\theta) < \frac{a}{1+\mu} \Leftrightarrow \theta > \frac{(1+\mu)}{(1-\mu)a}$. Hence $(r_{t-1} - (1-\mu)Y(I_{t-1}))\, I_{t-1} > (r_t - (1-\mu)Y(I_t))\, I_t$ if (2.26) holds.

P_{t-1}^* is determined such that the following holds

$$\left(r_{t-1}^* - (1-\mu(1-P_{t-1}^*))Y(I_{t-1}^*)\right) I_{t-1}^* = \beta\pi V_{Bt} \tag{2.27}$$

From above, $V_{Bt} < V_{Bt+1}$ and $(r_{t-1} - (1-\mu)Y(I_{t-1}))\, I_{t-1} > (r_t - (1-\mu)Y(I_t))\, I_t$. Hence $\left(r_{t-1}^* - (1-\mu(1-P_t^*))Y(I_{t-1}^*)\right) I_{t-1}^* > \beta\pi V_{Bt}$. Given that $\frac{\partial(r-(1-\mu(1-P^*))Y(I))I}{\partial P^*} = -\mu Y(I)I < 0$, and $Y(I_{t-1}^*)I_{t-1}^* < Y(I_t^*)I_t^*$, it must be true that $P_{t-1}^* > P_t^*$ for (2.27) to hold.

Q.E.D.

The intuition of the proof is as follows. As the proportion of type B borrowers in the market falls over time, the interest rate charged by the bank is monotonically decreasing over time. This means that the future value to borrowers of retaining access to credit on future dates increases over time, and thus borrowers' incentives to repay debt strengthen over time, i.e. borrower reputation effects strengthen over time. Thus the critical probability of liquidation required to make type B borrowers indifferent between repayment and repudiation is monotonically decreasing over time.

The result shown in figure 5 depends on the fact that although $\hat{x}_t$ is monotonically increasing over time, x_t^* may decrease over time if the effects of stronger borrower reputation exceeds the effects of weaker bank reputation. This is proved formally in proposition 9 below.

Proposition 9 *For any $t' < t''$, $\hat{x}_{t'} \leq \hat{x}_{t''}$ must hold. There exists date $t_{\max} \leq T$ such that $x_{t'}^* < x_{t\max}^*$ for all $t > t_{\max}$.*

Proof. First, we prove that for any $t' < t''$, $\hat{x}_{t'} \leq \hat{x}_{t''}$ must hold. From lemma 9 above, we know that if $T - t' \in$ *Zone* τ and $T - t'' \in$ *Zone* $\tau - 1$, then $\hat{x}_{T-t'} < \hat{x}_{T-t''}$. We must now prove that $\hat{x}_{t'} \leq \hat{x}_{t''}$ holds for any $t' < t''$ within a given zone. ■Take any date $T-(k_{\tau-1}+\sum\limits_{y=1}^{\tau-1} j_y-1+a+b)$ in zone τ. From the solution algorithm derived in proposition 5, we know that on this date either step (4) or step (5) of the algorithm fails to hold. Suppose step (4) fails to hold. Then as explained in proposition 5, we set $\hat{x}_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b)} = z_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b)}$. What is the equilibrium on date $T-(k_{\tau-1}+\sum\limits_{y=1}^{\tau-1} j_y-1+a+b-1)$? This is given by $\hat{x}_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b-1)} = z_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b-1)}$ where $z_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b)}$ solves the following: $q(z_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b-1)}) + \sum\limits_{i=k_{\tau-1}+b+1}^{k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-2+a+b} \left(\theta_{T-i} r_{T-i}^* - 1\right) I_{T-i}^* =$

$(1-\mu)Y(I_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b-1)})I_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b-1)}$. Hence from lemma 7, it must be true that $\hat{x}_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b)} < \hat{x}_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b-1)}$.

Suppose step (5) fails to hold. Then as explained in proposition 5, we set $\hat{x}_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b)}$ $x^*_{T-k_{\tau-1}-b-c}$. What is the equilibrium on date $T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y - 1 + a + b - 1)$? This is given by $\hat{x}_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b-1)} = x^*_{T-k_{\tau-1}-b-c}$, hence $\hat{x}_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b)} = \hat{x}_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1+a+b-1)}$. Hence by recursion, for any $t' < t''$ within any given zone , $\hat{x}_{t'} \leq \hat{x}_{t''}$ must hold.

We now prove that there exists date $t_{\max} \leq T$ such that $x^*_{t'} < x^*_{t\max}$ for all $t > t_{\max}$. From the solution algorithm for zone 1 given above, we know that for any date in zone 1, $\hat{x}_t > x^*_{t\max}$ must hold. Hence, if there exists $x^*_{t'} > x^*_{t\max}$ for some date $t' > t_{\max}$, then there exists some date t'' in zone 1 for which $\hat{x}_{t''} < \max\left[x^*_{T-i}\right]_{i=0}^{j_1-1}$, in which case date t'' cannot be in zone 1, which contradicts the original assertion.

Q.E.D.

We can now illustrate that the dynamics of bank and borrower reputation effects are mutually reinforcing, with the effect that the probability of bank failure is reduced. Consider any date $T-a \in \mathit{Zone}\ \tau$ and any date $T-b \in \mathit{Zone}\ \tau-1$. Given that $T-a < T-b$, from proposition 8 we have $P^*_{T-a} > P^*_{T-b}$. But suppose $P^*_{T-a} = P^*_{T-b}$. From lemma 9, we know that $\hat{x}_{T-a} < \hat{x}_{T-b}$, and from (2.8), $\hat{x}_{T-a} < \hat{x}_{T-b} \Rightarrow x^*_{T-a} < x^*_{T-b}$. Hence as we go backwards in time, although borrower reputation effects weaken, bank incentives to liquidate grow stronger (hence $\hat{x}_{T-a} < \hat{x}_{T-b}$), and thus for a given reputation, the probability that the bank will liquidate if borrowers repudiate increases. This tends to offset the deterioration in borrower incentives (due to the higher interest rates they are charged), making it less likely that borrowers will repudiate for any given bank reputation, i.e. it tends to reduce x^*_t. As mentioned in the proof of proposition 7, the maximum level of x^*_t defines the lower bound on the probability of bank failure. Hence, going backwards in time, bank reputation effects grow stronger, which (partially) offsets any increase in

x_t^* due to the fact that borrower reputation effects grow weaker, which therefore tends to reduce the lower bound on the probability of bank failure.

Consider any two dates t and $t+c$, where $c > 0$. Suppose $\hat{x}_t = \hat{x}_{t+c}$. Then from (2.8), $P_{t+c}^* < P_t^* \Rightarrow x_{t+c}^* < x_t^*$. Hence, although over time $\hat{x}_t$ increases, and thus bank reputation effects become less effective in mitigating borrower incentives to repudiate, this is offset to an extent by the rising value of borrower reputation, which tends to offset any increase in x_t^* over time, which therefore tends to reduce the lower bound on the probability of bank failure. Furthermore, for any date $t > T - k_1$, the increase in the value of borrower reputation compared with date $T - k_1$ is sufficiently large as to offset completely the deterioration in the bank's reputation effects, such that the lower bound on the probability of bank failure increases no further. This explains why x_t^* peaks on date $T - k_1$, instead of continuing to increase up to date T. As explained in the proof of proposition 7, this means that the bank will face no further repudiation beyond this date. Hence, even though $\hat{x}_t$ continues to rise, eventually reaching 1 on date T, the upper bound on the probability of bank failure does not rise above $\frac{1}{1+\delta}\hat{x}_{T-k_1}$. The intuition behind this result goes as follows. Towards the horizon, the bank's incentives to liquidate are weakest, because there are not many future pay-offs to be gained if the bank retains its reputation and continues to lend. Hence, if borrowers repudiate at this stage, it is very unlikely that the bank will liquidate. However, it is exactly at this stage that borrowers' concerns about their reputation are strong, such that they will prefer not to repudiate. Thus, the dynamics of borrower reputation effects are such that the bank's reputation is not tested at the time when it is most fragile (i.e. when most bank types will prefer to renegotiate), and hence the upper bound on the probability of failure increases no further.

2.11 Conclusion to Chapters One and Two

In these chapters, I have formulated a new theory of financial intermediation and have explained the general structure of credit markets.

Essentially, these chapters have answered the question, why do banks exist? Banks exist in order to lend to new borrowers with short credit histories. These borrowers are unable to issue debt directly, because direct lenders regard them as too risky, and expect them to repudiate their debt obligations. The bank is able to lend, because it is a multi-period player in the credit market, which allows it to build a reputation for being tough and thus deter borrowers from repudiation.

These chapters have explained several important empirical features of the credit market. They have explained why new borrowers (i.e. firms which have only recently begun trading) tend to borrow using bank finance, whilst established borrowers tend to issue debt directly. Established borrowers are charged lower interest rates. Consequently, they are concerned about maintaining their reputation for being a good credit risk, and will thus refrain from repudiation, in order to prevent direct lenders from terminating their credit. Hence, direct lenders are willing to lend to them.

These chapters have also explained why banks in the US and UK have a reputation for being tough on firms in financial distress, tending to liquidate immediately rather than arranging rescue packages and allowing time for financial restructuring. In my model, banks liquidate borrowers who default in order to build a reputation for being tough and thus deter other borrowers from repudiation on future dates.

These chapters have also explained why small businesses borrow almost exclusively from banks and pay interest rate premiums on bank debt, even though survey evidence suggests that banks do not perform a special service for which they are willing to pay a premium. In my model, new borrowers (e.g. small businesses) are unable to issue debt directly, and must therefore rely on bank finance, and are forced to pay whatever the bank charges.

My model also generates two important implications for financial policy. Firstly, it suggests that measures to curb banks' powers to liquidate may be counter-productive. Although this may reduce the incidence of inefficient liquidation, the cost is that liquidation would be less of a deterrent to borrower repudiation.

Secondly, my model predicts that measures to cut interest rates charged by banks could be counter-productive. If borrowers repudiated, the bank would have no incentive to liquidate and thus build a reputation for toughness, as there would be no future rents to offset the cost of liquidation. Hence liquidation would no longer be a credible threat, and thus borrowers would not be deterred from repudiation. Thus, the bank would be better off not lending to new borrowers at all.

There are two main directions for future research. Firstly, it would be useful to introduce entry of new cohorts of borrowers. In the current model, on the threshold date when borrowers have established a long credit history and thus reputation becomes effective in policing their incentives, they switch from bank finance to direct finance. Hence, after this date the bank ceases to lend. But if we introduce entry of new cohorts of borrowers each period, the bank will continue to lend in the model's steady state. If we assume that the new cohorts face the same initial degree of adverse selection as the original cohort of borrowers, and similarly that the new borrowers have no prior credit history, then they will also be unable to borrow directly and must rely on bank finance.

Secondly, it would be interesting to derive the market structure of the banking sector from first principles, rather than assuming a monopoly bank as I have done here. In this extension to the model, I assume that all lenders are initially identical, i.e. there is no distinction between bank lenders and direct lenders. However, in equilibrium, only a small number of lenders become banks, whilst the remainder become direct lenders. The reason for this is as follows. Banks Cournot compete by choosing the amount of funds they wish to supply in the credit market, and the market determines the interest rate. Hence the greater the number of banks which enter the credit market, the lower the interest rate premiums they are able to charge. During early periods, there will be some mixed strategy defaults, and banks will liquidate in order to build a reputation for toughness, and thus deter borrowers from future default, and also in order to deter further entry of banks into the credit market, and thus prevent their interest rate premiums from being competed away. Entry will occur up to the point that further entry would sufficiently reduce interest rates, such that the rents generated by the reputation no longer offset the cost of entry (which is endogenously determined as the cost of liquidating defaulters), i.e. liquidation costs serve as a barrier to entry.

This extension makes two important contributions to the literature. Firstly, it explains the existence of a highly concentrated market structure in the banking sector, and the evidence that banks wield a considerable degree of market power in the US and UK economies, something which no other paper in the literature has done. In the UK, the main clearing banks provide around 90 per cent of all small firm lending (Batchelor 1989). In the US, although market concentration is low for the economy as a whole, banks have a considerable degree of local market power, especially in banking services which require a local presence, such as lending to small businesses (see Frankel and Montgomery (1991)).

Secondly, it makes an important theoretical contribution to the literature on industrial organization. The standard approach in modelling reputation games in this literature is to model an incumbent long-lived monopolist which faces repeated entry by a sequence of other (long-lived) monopolists (e.g. Kreps-Wilson (1982a), Fudenberg-Kreps (1987)). However, in my approach, I make no initial assumptions about the market structure, it is derived from first principles.

Chapter 3

The Dynamics of Firms' Credit Ratings

3.1 Introduction

This chapter analyzes the dynamics of firms' credit ratings, in the context of a multi-period moral hazard problem, in which borrowers have incentives to repudiate their debt obligations. Borrowers with short credit histories face the poorest incentives, and (depending on initial conditions) for these borrowers debt repayment can only be enforced by the threat of liquidation. However, over time if borrowers repay debt on all dates, they will establish a good credit history. This may improve their incentives, such that they will repay debt because they are concerned about their reputations for being a good credit risk, even if they face no threat of liquidation if they do default.

The chapter provides answers to the following two questions:

1) Why do firms' credit ratings improve over time?

2) Why do aggregate shocks have less impact on the risk of default of well-established firms, compared to recent start-up firms?

A firm's credit rating is a measure of its overall default risk, which is composed of both liquidity default (in which firm is unable to meet its debt obligations due to cash flow constraints), and strategic default (in which the firm is able to repay debt, but chooses to divert the cash flows instead). This chapter contributes to the literature by analyzing jointly the risks of liquidity and strategic default and their implications for the dynamics of credit ratings. It also generates predictions which are consistent with the following two stylized observations:

1) firm credit ratings improve over time

2) aggregate shocks have less impact on the risk of default of well-established firms, compared to recent start-up firms.

The first observation reflects the empirical evidence that firms which have only recently started trading have lower credit ratings than firms which have been trading for a long time. As for the second observation, in the event of a negative aggregate shock, evidence suggests that an established firm with a good track record is less likely to have its credit rating downgraded than a recent start-up, without an established credit record. This is particularly true of the 1990-92 UK recession, characterized by a credit crunch which, it is suggested, impacted most severely on recent start-ups and small businesses (e.g. see Keasey and Watson (1994)).

Standard models of reputation, as in the sovereign debt literature and Holmstrom (1982), actually predict that agents' incentives deteriorate over time, hence they do not explain either of these observations. Diamond (1989) does explain the first observation, but not the second.

The following papers in the empirical literature on firm credit ratings are of interest. Altman and Kao (1992) examines the data on the dynamics of corporate bond ratings. Using a sample of over 7000 bonds issued in the 1970-88 period they obtain the following results. A-rated bonds appear to be more stable than AAA-rated bonds. BB-rated bonds are the least stable. Furthermore, bonds initially rated A and above have a greater tendency to be downgraded than to be upgraded. Among the investment grades, only bonds initially rated BBB tend to be upgraded more than they are downgraded. As for non-investment grade 'junk' bonds, there does not appear to be a tendency toward either upgrades or downgrades in the sample period.

Hand, Holthausen and Leftwich (1992) examines the effects of bond rating agency announcements on bond and stock prices. They consider two types of announcements: warnings of possible rating downgrades via additions to Standard and Poor's Credit Watch List, and actual rating changes announced by Moody's and Standard and Poor's. Bond price effects are identified for both types of announcements. The stock price effects of agency announcements are also examined and contrasted with the bond price effects.

Hite and Warga (1997) also analyze the effects of bond rating agency announcements on bond prices. A unique feature of their study is the use of a data set containing recent firm-specific information that also provides a long event window. Downgraded firms reveal a significant announcement effect in both the announcement month and preannouncement period. Upgrade effects are much weaker in magnitude and significance.

Gale and Hellwig (1985) show in a model of borrowing and lending with asymmetric information that the optimal incentive-compatible contract is the standard debt contract. They obtain the following four results. First, they show that the optimal contract takes the form of a standard debt contract with bankruptcy. This requires that the firm makes a fixed debt repayment when it remains solvent, and that the firm is declared bankrupt if it does not make the repayment in full, in which case the creditor is allowed to recoup as much of the debt as possible from the firm's liquidated assets. Second, they derive the conditions under which credit rationing occurs. They show that the equilibrium investment level never exceeds and typically falls short of the first best level. Third, they compare optimal contracts with the result of interest-rate-taking behaviour, and find that this tends to reduce the optimal loan size and interest rate. Fourth, they show that there exists a non-monotonic relationship between liquidity and investment.

The following theoretical literature on reputation games is also of interest. Milgrom and Roberts (1982) formulate a reputation game to explain the apparently irrational practice of predatory pricing. They show that although predation is costly in the short run, it is optimal for a long-lived monopolist if the short run cost of predation is outweighed by the long run benefits in terms of reducing the probability that entry occurs on future dates. The model resolves the Chain Store Paradox by introducing uncertainty over the incumbent's pay-offs from predation. Specifically, there exists two types of incumbents, weak and tough. The tough type can only prey. The weak type can either prey or accommodate. But providing the horizon is long enough, the weak type has an incentive to prey, on order to build a reputation for being a tough type, and thus deter future entry.

Fudenberg and Kreps (1987) extends the work on reputation games by considering a game in which the incumbent competes with several entrants simultaneously, rather than sequentially as in Milgrom and Roberts (1982), and Kreps and Wilson (1982). Their model yields the following two results. First, there exists a trade-off between the short run cost of maintaining a reputation, and the longer run benefits that accrue from it, as in the model with sequential entry. However, the results differ from the sequential entry model for the following reason. As the number of entrants increases, the reputation of the incumbent may no longer dominate, if the short run costs rise more quickly than the long run benefits. Second, they compare the equilibrium under information linkage (in which case entrants observe the past play of the incumbent against all other entrants) with the equilibrium under information isolation (in which case entrants observe only the past play of the incumbent with themselves). They find that even if the incumbent's reputation dominates in the information linkage case, she may actually prefer the information isolation case. The reason is that, in the information linkage case, although the short run costs of defending a reputation are high, the long run losses from losing the reputation are even higher.

Fudenberg and Levine (1989), (1992) generalize the previous literature on reputation effects in a game in which a single long run player faces an infinite sequence of short run players. They suppose that short-run players assign positive prior probability to the long run player's being one of several different commitment types, each of which plays a fixed stage game strategy each period. Each commitment type corresponds to a particular reputation that the long run player might build. Instead of solving explicitly for the equilibrium strategies, they derive the upper and lower bounds on the long run player's pay-offs that hold in any Nash equilibrium of the game. The Stackelberg strategy is defined as the strategy of the long run player which yields his most preferred pure strategy profile of the stage game, given that the short run player chooses an optimal response to his strategy. The paper finds that as the discount factor goes to 1, the upper bound on the long run player's Nash equilibrium pay-off converges to the Stackelberg pay-off.

Celentani, Fudenberg, Levine and Pesendorfer analyze a reputation game between a patient player 1 and a nonmyopic but less patient opponent, player 2. Player 2 is uncertain of player 1's type, in particular player 1 may be a commitment type which plays a fixed stage game strategy every period. They show that the game with a non-myopic player differs from the game with a myopic player for the following reason. Given that the non-myopic player cares about future pay-offs, the Stackelberg pay-off is no longer necessarily the highest attainable pay-off. For instance, a higher pay-off could be gained by using a punishment strategy such as 'tit for tat'. They find that player 1's equilibrium pay-off is bounded below by what he could get through commitment in the repeated game.

This chapter also has interesting implications for the recent literature on financial structure in an incomplete contracting framework, and can be broadly interpreted as a repeated version of the one shot situation analyzed in Hart-Moore (1997), in which inefficient liquidation arises in equilibrium as an inevitable consequence of borrowers' incentives to repudiate debt repayments. Although I consider a different set up, my analysis shows that given a sufficiently long horizon, borrowers' concerns about their reputations improve their incentives to repay debt, such that inefficient liquidation is mitigated.

I model the problem as follows. Borrowers have an investment project, for which they seek finance. I assume that contracts are incomplete. This gives borrowers the opportunity to default strategically, i.e. to default on debt repayment even if their projects are successful, and they are able to repay the debt. But lenders do have the ability to liquidate, if borrowers default on debt. However, lenders are unable to commit to liquidating borrowers if it is not optimal ex post to do so. If the borrower's project succeeded, then it is optimal to liquidate. But if the project failed, then the lender would prefer not to liquidate, because liquidation is costly and there is no project output for it to be worth liquidating. The problem is that project outcome is unobservable to lenders, so a borrower can default strategically by claiming that his project failed, in order to deter the lender from liquidating.

Although borrowers have the opportunity to default strategically, this doesn't necessarily mean they have the incentive to do so. However, new borrowers with short credit track records may have strong incentives to default strategically, for the following reason. I assume two different borrower types with different investment projects. 'Safe' borrowers' projects have zero probability of failure, but 'risky' borrowers' projects have a high probability of failure, such that lenders will prefer not to lend to them. Borrower type is unobservable to lenders. Since new borrowers lack an established credit history, lenders are unable to distinguish whether they are safe or risky. To compensate for the risk of default, lenders pool all new borrowers together and, if there is a high proportion of risky borrowers in the market, charge them high interest rates. This gives new borrowers an incentive to default strategically.

Over time, if borrowers continue to repay debt on all dates, they establish a good credit history. This convinces lenders that the borrower is probably a safe borrower, and thus they will charge lower interest rates, to reflect the lower risk of default. If the borrower is the safe type, now that it is charged low interest rates, its incentives change. If it defaults strategically, lenders will think it is a risky borrower, and will refuse to lend to it anymore. Hence, these borrowers risk losing access to cheap credit, which they have earned by building up a spotless credit history. Hence, they will prefer to repay the debt, because they are concerned about maintaining their reputation as a good credit risk.

A given borrower's risk of default decreases over time (and hence its credit rating improves over time) for two reasons. Firstly, every period, there exists a positive probability that risky borrowers will be forced to default due to project failure, thus revealing their type and resulting in credit termination. Hence, the proportion of risky borrowers in the market falls over time, and thus the risk of default due to project failure for borrowers remaining in the market falls over time. Secondly, safe borrowers incentives to repay debt improve over time, as they become concerned about reputation, and hence the risk of strategic default falls over time as well.

We explain why aggregate shocks have less impact on firms' risk of default, as they develop a good credit history, as follows. If borrowers default during early periods when they have yet to establish a track record for repaying debt, lenders will be reluctant to liquidate, because there is a high proportion of risky borrowers remaining in the market. Hence, if a borrower defaults, it is likely to be a risky borrower whose project has failed, rather than a safe borrower who has defaulted strategically. Hence, the lender's expected pay-off from liquidation will be low. Since borrowers infer this, they have a greater incentive to default strategically, because given that project outcome is unobservable, they can 'hide' their type by pretending to be a risky borrower whose project has failed, in which case lenders will be reluctant to liquidate. However, over time, the proportion of risky borrowers remaining in the market falls. Suppose that a negative aggregate shock occurs, which worsens borrowers' incentives to repay debt. If a borrower has established a good credit history, then if he defaults, lenders will believe that he is defaulting strategically, rather than a risky borrower whose project has failed, since it is unlikely that a risky borrower could have developed such a good credit history. This means that the lender will have a greater incentive to liquidate, which deters the borrower from defaulting strategically. Effectively, this means that borrowers with short credit histories are more affected by a negative shock than borrowers with established credit histories.

The structure of the chapter is as follows. Section 3.2 sets out the model. Section

3.3 solves the one period problem. Sections 3.4 and 3.5 solve for borrowers' optimal repayment strategies and lenders' optimal liquidation and lending strategies in the multi-period equilibrium. Section 2.7 solves for equilibrium interest rates. Section 3.7 proves the existence of reputation equilibrium. Section 3.8 analyzes the dynamics of reputation along the equilibrium path, and provides a characterization of the general model. Section 3.9 gives a summary of comparative statics, and analyzes the effects of a temporary interest rate shock.

3.2 The Model

Lenders are risk-neutral and receive an endowment of consumption goods each period. They have a choice over how to invest this:

1) they can invest in a riskless asset, which has a gross rate of return of R, or

2) they can lend to borrowers who have access to investment projects.

We assume that lenders exist for one period only, hence borrowers face a new generation of lenders each period. This allows us to focus on reputation effects as the only intertemporal enforcement mechanism. The effects of reputation building by lenders in enforcing debt repayment are analyzed in chapters 1 and 2.

Borrowers are also risk-neutral, and live for T periods. They receive no endowment but have access to an indivisible investment project each period. There are two types:

- type G borrowers have one high expected return, safe project each period. They can invest one unit and receive G with certainty.

- type B borrowers have one low expected return, risky project each period, which returns B with probability p_B (where $B > G$ and $p_B B < R$) and zero with probability $1 - p_B$.

Hence, only type G projects yield positive net present value and are attractive to lenders.

Borrowers must invest funds in their projects. They cannot invest instead in the riskless asset.

The initial population of borrowers contains a publicly known fraction f_G of type G's and $f_B(= 1 - f_G)$ of type B's. A borrower's type is private information and all borrowers are initially observationally equivalent. Project returns are independently distributed and are also private information. The optimal contract is therefore a debt contract, which specifies a fixed payment of r_t per unit loan.

Borrowers maximize discounted expected consumption, given by $\sum_{t=1}^{T} \beta^{t-1} E(c_t)$, where c_t is period t consumption, β is the discount factor with $0 < \beta < 1$, and T is finite but large. Consumption of all agents must be non-negative each period, hence borrowers have limited liability. We assume borrowers' projects are in relatively short supply, and hence the riskless asset is held in equilibrium. Hence, lenders will lend to any borrower who offers a debt contract with an expected return no lower than the return on the riskless asset.

Borrowers operate a two-stage project, stage one being the production of a capital good, and stage two being the transformation of the capital good into a consumption good. By project failure, we mean that stage one is unsuccessful. However, whether or not stage one has succeeded is observable only to the borrower, not to the lender. This gives rise to the following incentive problem: borrowers can claim that a successful project has failed, and thus divert the entire project returns rather than repaying the debt.

Suppose the borrower defaults on debt repayment, claiming that stage one of the project has failed. If the lender believes this is not true, why not simply wait for the borrower to complete stage two, at which point the project output becomes observable? The problem with this is that although at this stage the project output is observable to the lender, it is not verifiable by a third party, i.e. contracts are incomplete. Lenders are unable to enforce repayment by taking the borrower to court. Hence the only enforceable contract is a debt contract which transfers ownership and control of project assets from the borrower to the creditor in the event of default.

There are two main formal bankruptcy procedures that creditors can follow in the US and UK: Chapter 7 (liquidation in the UK) and Chapter 11 (administration in the UK). In Chapter 7, the firm's assets are sold off (piecemeal) in a cash auction. In Chapter 11, however, creditors attempt to restructure the firm, in order to allow it to continue to trade. During the process, all creditors' claims are frozen, and an administrator is appointed to run the company.

In our model, we consider bankruptcy in the form of Chapter 11. After stage one of the project, the only project asset is the capital good, which is project specific, and hence its opportunity cost is zero. Thus a cash auction as Chapter 7 entails would be inefficient[1]. Hence, the lender can only extract value from the project by allowing it to continue. However, the lender's key problem is who to appoint to continue the project: should he allow the entrepreneur to remain in place (we denote this as project continuation), or replace him with an outside manager (we denote this as project liquidation)?

The first-best solution would be to allow the entrepreneur to continue the project, whether stage one has succeeded or failed. If stage one has succeeded, then the lender will be able to continue the project (i.e. complete stage two) without the entrepreneur, by appointing an outside manager. But to do this he must incur a fixed cost C, which represents an efficiency cost, given that the outside manager knows less about running the project than the entrepreneur. Hence the final project return is $G - C$ or $B - C$, depending on the project type. However, if stage one of the project has actually failed, then only the entrepreneur, who has specific human capital for producing the capital good, can rescue it. This yields a final return of $(1 - \gamma)B$ where $0 < \gamma < 1$ (recall that only type B projects can fail). If the lender tries to complete the project himself, the

[1]Another disadvantage of using Chapter 7 in this environment is that the entrepreneur, with private information about the project's true value, would be the sole bidder in a cash auction, thus readmitting the incentive problem. Even if credit-constrained, or forbidden from participating in the auction, he could arrange finance from a third party who could bid for him.

gross project return is zero, but the net return (given the fixed cost C) is $-C$.

The borrower's and lender's pay-offs are summarized in Table 3 below. We give the pay-offs for a type B project, the pay-offs for a type G project are the same as for a successful type B project, except that B is replaced with G.

	project liquidation	project continuation
project failure	$-C, 0$	$0, (1-\gamma)B$
project success	$B-C, 0$	$0, B$

Table 3: Lender and borrower pay-offs (lender pay-offs given first)

Although the first-best outcome is for the lender to allow the borrower to remain in place and complete the project, this gives the borrower the opportunity to divert the entire project returns, thus yielding the lender a pay-off of zero. If the project outcome were observable, the lender's optimal strategy would be to liquidate (take over) the project if it had succeeded, and thus prevent the borrower from diverting the project returns. If the project had failed, then it would not be optimal to liquidate, given that liquidation is costly, but there would be no project output for it to be worthwhile to liquidate. But the lender's problem arises from the fact that the project outcome is unobservable. This creates the following moral hazard problem for the borrower: he can claim that the project has failed and thus default strategically. The lender will be reluctant to liquidate, given that if the borrower's claim is true, then liquidating would yield a lower pay-off than not liquidating, given the fixed costs of liquidation.

We make the following parameter restrictions: $p_B B > C$ and $R + C < G$. We also set $\gamma = 1$, hence the type B borrower receives a pay-off of zero if his project fails, whether the lender liquidates, or allows the borrower to continue. This has no effects on the results, and is assumed purely for analytical convenience.

We assume the following information structure. At each point in time, lenders can observe borrowers' entire credit histories. Specifically, it tells them whether a borrower has ever defaulted on a loan. It does not, however, tell them whether default was strategic or forced by project failure, i.e. it does not reveal the project outcome. This information is known to the borrower, and only to the lender if he liquidates.

We model the problem as the repeated play of an extensive form stage game with incomplete information, using the concept of sequential equilibrium. The stage game on every date t' consists of the following sequence of steps:

1) Borrowers offer debt contracts to lenders, specifying the interest rate at which they are willing to borrow.

2) Lenders decide which contracts to accept. If a lender refuses to lend to any given borrower, then the date t' stage game ends for that borrower and it receives a pay-off of zero.

3) Nature determines the outcome of type B projects, which is observed by the borrower only. If the project succeeds, the borrower proceeds to step (4). If it fails, the borrower proceeds to step (5).

4) Borrowers choose whether to repay or default. If a borrower repays, then its date t' stage game ends, it receives a pay-off of $G - r_{t'}$ (if type G) or $B - r_{t'}$ (if type B) and the lender receives a pay-off of $r_{t'}$. If a borrower defaults, then it proceeds to step (5).

5) Lenders decide whether to liquidate (i.e. take over the project), or allow the borrower to continue. Pay-offs are as given in Table 3.

At step (5), the lender is at an information set regarding both the borrower's type, and the project outcome.

It is useful to set out the definitions of the strategies of players, how beliefs are formed, and the equilibrium concept used.

A strategy for a borrower (type G or B) consists of the following:

a) a choice of debt contract, specifying the interest rate.

b) a choice of how much debt to repay (between 0 and G (B for type B borrowers))

A strategy for a lender consists of the following:

a) a decision on whether or not to accept any borrower's offer of a debt contract

b) a decision on whether to liquidate (i.e. take over the project), or allow the borrower to continue

The concept of sequential equilibrium must satisfy the following conditions:

1) At each information set, the player with the move must form a belief about which node in the information set he is at, by assigning a probability to the event that he is at any given node.

2) Players must be sequentially rational, i.e. players' strategies at any given information set must be optimal, given their beliefs and given their subsequent strategies and the other players' subsequent strategies.

3) Beliefs must be consistent with equilibrium strategies, i.e. beliefs must be updated using Bayes' Rule.

4) Beliefs for information sets off the equilibrium path must be consistent with some small perturbation of the equilibrium strategy profile, such that these informations sets are actually reached during play.

We begin by solving the one period problem.

3.3 The One Period Problem

In the one period problem, borrowers have no incentives to repay debt in order to maintain their reputation, because there are no future borrowing opportunities which they are concerned about losing if lenders terminate credit. Hence, the threat of liquidation is the only possible enforcement mechanism. The one period problem consists of the extensive form stage game outlined above, which we solve by backward induction, beginning with the last step, the lender's decision on whether or not to liquidate, if a borrower defaults.

The lender faces the following problem: if a borrower defaults, given that borrower type and project outcome are unobservable, he does not know whether it is a type G or B borrower who are defaulting strategically, or a type B borrower whose project failed, i.e. the lender is at information set I (see Figure 6). Hence, the lender must form an assessment of which node in this information set he is at. If he believes he is at node X_1 or X_3, i.e. default is strategic, then it is optimal to liquidate, since liquidation yields a pay-off of $G-C$ or $B-C$, whereas not liquidating yields a pay-off of zero. However, if he believes he is at node X_2, i.e. default was forced by project failure, then it is optimal not to liquidate, since liquidation yields a pay-off of $-C$, whereas not liquidating yields a pay-off of zero. We now show that there can be no pure strategy equilibrium in the one period game, in which lenders liquidate or don't liquidate, with probability one, and borrower repay or default with probability one. First, suppose that in equilibrium, both borrower types default strategically, and the lender's strategy is to liquidate with probability one. But this means that borrowers' pay-off from default is zero, whereas their pay-off from repaying debt is $G - r_{t'}$ (if type G) or $B - r_{t'}$ (if type B), which is greater than zero. Hence, default is not an optimal strategy and thus this is not an equilibrium..

We now consider the equilibrium in which both borrower types default strategically, and the lender's strategy is not to liquidate. We now show that, providing condition (3.1) below holds, then it is optimal for the lender to liquidate, hence this is not an equilibrium.

$$f_G > \frac{C - p_B B}{G - p_B B} \tag{3.1}$$

We need to determine what are the lender's beliefs about borrower type and project outcome, if the borrower defaults, i.e. what are the conditional probabilities that the lender is at nodes x_1, x_2, or x_3, in information set I. In any equilibrium, this information set is on the equilibrium path, since there is a non-zero probability that any given borrower is a type B borrower who will default due to project failure. Hence, as required by sequential equilibrium, lenders' beliefs are determined by Bayes' Rule and are consistent with borrowers' equilibrium strategies. Consider an equilibrium in which both borrower types default strategically. The lender's beliefs, derived using Bayes Rule, are therefore:

$\mu(x_1) = f_G$

$\mu(x_2) = (1 - f_G)(1 - p_B)$

$\mu(x_3) = (1 - f_G)p_B$ where $\mu(x_1)$ is the conditional probability that the lender is at node x_1, given that he is at information set I etc.

Hence, the lender's expected pay-off if it liquidates is given by:

$Eu^L(liq) = \mu(x_1)(G - C) + \mu(x_3)(B - C) - \mu(x_2)C$

and its pay-off if it if it doesn't liquidate is given by:

$Eu^L(no\ liq) = 0$

Hence, it is optimal not to liquidate if and only if the following holds:

$$\begin{aligned} Eu^L(liq) &< Eu^L(no\ liq) \Leftrightarrow \\ \mu(x_1)(G - C) + \mu(x_3)(B - C) - \mu(x_2))C &< 0 \Leftrightarrow \\ f_G &< \frac{C - p_B B}{G - p_B B} \end{aligned}$$

Hence, providing that condition (3.1) holds, it is optimal for lenders to liquidate, and thus the pure strategy equilibrium described above does not hold .

We now consider the pure strategy equilibrium in which both borrower types repay, and the lender liquidates, if a borrower defaults. Given that borrower equilibrium strategies are to repay, if their project succeeds, the lender's belief if a borrower defaults must be that default was forced by project failure. Hence, liquidation yields a pay-off of $-C$, whereas not liquidating yields a pay-off of zero. Thus, lenders will prefer not to liquidate, and hence this is not an equilibrium.

We now consider the pure strategy equilibrium in which both borrower types repay, and the lender does not liquidate, if a borrower defaults. But given that they do not face liquidation, it will be optimal for borrowers to default, hence this is not an equilibrium.

Hence, there exists no equilibrium in pure strategies. The intuition for this is as follows. Suppose lenders liquidate with probability one if borrowers default. But then borrowers will never want to default strategically if they face certain liquidation, and will only default if forced to by project failure. However, this means that lenders will be unable to commit to liquidation, given the belief that default must imply project failure, and hence it is not worth liquidating. Thus, for equilibrium to hold, borrowers must default strategically with positive probability, so that liquidation will be ex post an optimal response for lenders. Lenders, in turn, must not liquidate with probability one, so that borrowers will not be deterred from defaulting strategically. Hence, we seek a mixed strategy equilibrium, in which borrowers randomize between repayment and default, and lenders randomize their liquidation response.

We now show that the unique sequential equilibrium in mixed strategies in the one period game is as follows. Type G borrowers randomize, defaulting strategically with probability π^G. Type B borrowers repay if their projects succeed. Lenders randomize, liquidating with probability α.

In equilibrium, the probability that the lender liquidates is determined such that type G borrowers are indifferent between repaying and defaulting. Hence, α is determined such that

$Eu^G(repay) = Eu^G(default)$

$\Leftrightarrow G - r = \alpha.0 + (1-\alpha)G$

$\Leftrightarrow \alpha = \frac{r}{G}$

where $Eu^G(repay)$ and $Eu^G(default)$ are the type G borrower's expected pay-offs from repaying and defaulting respectively.

In equilibrium, the probability that type G defaults strategically is determined such that lenders are indifferent between liquidating and not liquidating. Hence π_G is determined such that

$Eu^L(liq) = Eu^L(no\ liq)$

$\Leftrightarrow \mu(x_1)(G - C) - (1 - \mu(x_1))C = 0$

$\Leftrightarrow \mu(x_1) = \frac{C}{G}$

From $\mu(x_1) = \frac{\pi^G f_G}{\pi^G f_G + (1-p_B)(1-f_G)}$, given that lenders' beliefs are consistent with equilibrium strategies, we get $\pi_G = \frac{C(1-p_B)(1-f_G)}{(G-C)f_G}$.

To complete the proof that this is an equilibrium, we must also show that type B's decision to repay, if their projects succeed, is optimal, i.e. we must show

$$\begin{aligned} Eu^B(repay) &> Eu^B(default) \Leftrightarrow \\ B - r &> \alpha.0 + (1-\alpha)B \Leftrightarrow \\ \alpha &> \frac{r}{B} \end{aligned} \tag{3.2}$$

which holds, given that $B > G$.

We now prove that this equilibrium is unique. The only possible alternative equilibria are the following:

1) Type B's randomize, and type G's repay with probability one.

2) Type B's randomize, and type G's default with probability one.

Consider the first equilibrium. For equilibrium to hold, the probability that lenders liquidate must be such that type B's are indifferent between repaying and defaulting. From (3.2), this means that $\alpha = \frac{r}{B}$ must hold. But this implies that $Eu^G(default) = \alpha.0 + (1-\alpha)G = \frac{(B-r)G}{B}$. But it is optimal for type G's to repay if and only if

$$\begin{aligned} Eu^G(repay) &> Eu^G(default) \Leftrightarrow \\ G - r &> \frac{(B-r)G}{B} \Leftrightarrow \\ G &> B \end{aligned}$$

which is not true by assumption. Hence, it is optimal for type G's to default and hence this equilibrium fails to hold.

We now prove that the second equilibrium cannot hold, providing that condition (3.3) below holds:

$$f_G > \frac{(1-p_B)C}{(G-p_BC)} \tag{3.3}$$

Type B's must default with probability π_B, such that lenders are indifferent between liquidating and not liquidating, i.e. such that the following holds

$$\begin{aligned} Eu^L(liq) &= Eu^L(no\ liq) \Leftrightarrow \\ \mu(x_1)(G-C)+\mu(x_3)(B-C)-\mu(x_2))C &= 0 \Leftrightarrow \\ f_G(G-C)+(1-f_G)p_B\pi_B(B-C)-(1-f_G)(1-p_B)C &= 0 \end{aligned}$$

But note that even when $\pi_B = 0$, and thus $Eu^L(liq)$ is minimized, we have $Eu^L(liq) = f_G(G-C)-(1-f_G)(1-p_B)C$. Hence,

$$\begin{aligned} Eu^L(liq) &> Eu^L(no\ liq) \Leftrightarrow \\ f_G &> \frac{(1-p_B)C}{(G-p_BC)} \end{aligned}$$

Thus, given that condition (3.3) holds, lenders strictly prefer to liquidate, and thus equilibrium fails to hold.

We now consider step 4, the borrower's repayment decision, in more detail. Let $r \leq G-C$. Considering the type G borrower, given that he has chosen to repay, it is optimal to repay the full amount r, i.e. $Eu^G(r) > Eu^G(r-\varepsilon)$ and $Eu^G(r) > Eu^G(r+\varepsilon)$ where $\varepsilon > 0$. If the borrower defaults partially by repaying $r-\varepsilon$, this reveals that the project was successful and it is therefore optimal for the lender to liquidate with certainty, yielding a pay-off of zero to the borrower. Repaying more than r is never optimal, given that it reduces the current period pay-off, and has no signalling value for future pay-offs, given the one period horizon. However, suppose $r > G-C$. The borrower has no incentive to repay more than $G-C$, since this is the maximum pay-off that the lender can expect from liquidation. This result also holds for type B borrowers. Although liquidation would yield a higher pay-off of $B-C$, type B's have no incentive to reveal their type by repaying more than $G-C$.

We now turn to step 2, the lending decision. Lenders will accept any debt contract which offers an expected return no less than the riskless interest rate R. Hence, lenders will accept a debt contract which offer interest rate r if and only if

$Eu^L \geq R$

$\Leftrightarrow f_G(1-\pi_G)r + (1-f_G)p_B r \geq R$

$\Leftrightarrow r \geq R/[f_G + \frac{(1-f_G)}{G-C}(p_B G - C)] = r^*$

Hence, lenders will accept any debt contract which offers an interest rate $r \geq r^*$.

We now turn to step 1, the borrower's choice of debt contract. Borrowers will offer the lowest interest rate necessary to ensure that the contract is accepted. Given that borrowers are relatively scarce, the equilibrium interest rate is driven down to $r = r^*$.

Note that if $r^* > G - C$, i.e. the lowest feasible interest rate exceeds the maximum that borrowers are willing to repay, the credit market is closed, since lenders can earn more through investing in the riskless asset. This gives us the following open market condition:

$$f_G > \frac{R + C - p_B G}{G(1-p_B)} = P^{crit} \tag{3.4}$$

3.4 The Multi Period Problem

We must now solve for the multi-period equilibrium. We solve the T-period problem as the repeated play of the stage game outlined above. In this section, we solve each step of the extensive form stage game.

Borrowers' Repayment Strategies and Lenders' Liquidation Strategies

This section solves for step (4), borrowers' optimal repayment strategies, and step (5), lenders' optimal liquidation strategies.

In the one period problem considered above, the only discipline device which deters borrowers from defaulting strategically is the threat of liquidation. In the multi-period problem, however, borrowers may prefer to repay debt for the following reason. Borrowers who have established a track record for repaying debt are charged lower interest rates. Hence, these borrowers will prefer not to default, because they don't want to ruin their reputations for repaying debt and thus risk having their credit terminated, given that they are now able to borrow cheaply. Hence, borrower reputation effects enforce debt repayment, rather than the threat of liquidation. If this is the case, we say that reputation equilibrium holds.

We must begin by analyzing whether reputation equilibrium ever exists for either borrower type. We consider type B borrowers first. Proposition 10 below proves that reputation equilibrium never exists for type B borrowers.

Proposition 10 *Reputation effects never exist for type B borrowers.*

Proof. Let V_{Bt} be the type B borrower's value function on date t. Then reputation equilibrium requires $V_{Bt}(h_t^r) \geq V_{Bt}(h_t^d, \alpha_t = 0)$ where $V_{Bt}(h_t^r) = B - r_t + \beta p_B V_{Bt+1}$ and $V_{Bt}(h_t^d, \alpha_t = 0) = B$. Hence reputation effects exist for type B's $\Leftrightarrow r_t \leq \beta p_B V_{Bt+1}$.

A necessary condition for reputation effects is that $V_{Bt}(h_t^r) \geq V_{Bt}(h_t^d, \alpha_t = 0)$ holds for $r_t = R \quad \forall t$ and $T \to \infty$, i.e. when the expected value of remaining in the credit market is maximized. Under these conditions, we require

$$R \leq \lim_{T\to\infty} \left[\sum_{i=1}^{T-t} (\beta p_B)^i (B - R) \right]$$

$$\Leftrightarrow \lim_{T\to\infty} \left[\sum_{i=0}^{T-t-1} (\beta p_B)^i (R - \beta p_B B) + (\beta p_B)^{T-t} R \right] \leq 0$$

However, this cannot hold, given that $p_B B < R$ by assumption.

Q.E.D.

Hence even when the value of future borrowing is maximized, i.e. borrowers face an infinite horizon and interest rates are at their lowest possible level, equal to the riskless interest rate, type B's will still prefer to default strategically. The reason for this is as follows. It is not worth repaying debt in order to avoid credit termination tomorrow, because the value of future borrowing is very low to type B borrowers, given that type B projects yield a low expected return. Repayment by type B borrowers can therefore only be enforced by the threat of liquidation.

We now analyze whether reputation effects ever exist for type G borrowers. If reputation is ever to have value (i.e. be effective in deterring default), it must have value when $r_t = R \quad \forall t$ and $T \to \infty$. Lemma 10 derives a necessary condition for reputation to be effective for type G's.

Lemma 10 *A necessary condition for reputation effects to exist for type G borrowers on some date is* $\beta \geq \frac{R}{G}$.

Proof. Type G reputation equilibrium exists

$$\Leftrightarrow V_{Gt}(h_t^r) \geq V_{Gt}(h_t^d, \alpha_t = 0)$$

$$\Leftrightarrow r_t \leq \beta V_{Gt+1} \text{ which holds only if}$$

$$R \leq \lim_{T\to\infty} [\sum_{i=1}^{T-t} \beta^i (G-R)]$$

$$\Leftrightarrow R \leq \frac{\beta(G-R)}{1-\beta}$$

$$\Leftrightarrow \beta \geq \frac{R}{G} \tag{3.5}$$

Q.E.D.

Hence, if the discount factor is too low, reputation equilibrium will never exist. Intuitively, if borrowers do not value the future highly, exclusion from future borrowing is not an effective discipline in deterring strategic default.

Lemma 11 provides a sufficient condition for the existence of reputation equilibrium for type G borrowers on some date t, providing that interest rates on all future dates satisfy a given upper bound.

Lemma 11 *If, for all* $t \in [\hat{t}, T]$, $r_t < \beta G$, *then there exists* $T < \infty$ *such that reputation has value for type G borrowers on date* $\hat{t}$. *This bound on future interest rates specifies feasible rates if and only if condition (3.5) holds.*

Proof. Reputation has value on date t $\Leftrightarrow V_{Gt}(h_t^\tau) > G$. Given that it is always feasible for type G's to repay debt each period, and that $Eu_{Gt} = G - r_t$, i.e. type G's expected pay-off each period is independent of its repayment decision in equilibrium (given that if ex ante type G's find it optimal to default, in equilibrium lenders will set α_t such that the expected pay-off from default equals the pay-off from repaying debt), and given that we know for $t \geq \hat{t}$ that $r_t < \beta G$, we can state

$$V_{G\hat{t}}(h_t^\tau) = \sum_{t=\hat{t}}^{T} (G - r_t)\beta^{t-\hat{t}} > \sum_{t=\hat{t}}^{T} (G - \beta G)\beta^{t-\hat{t}}$$

Taking the limit of the final expression as $T \to \infty$:

$$\lim_{T\to\infty} \left[\sum_{t=\hat{t}}^{T} G(1-\beta)\beta^{t-\hat{t}} \right] = G$$

Hence $\lim_{T\to\infty} V_{G\hat{t}}(h_{\hat{t}}^\tau) > G$. Given that we can find $T' < \infty$ such that

$$\lim_{T\to\infty} V_{G\hat{t}}(h_{\hat{t}}^\tau) - \lim_{T\to T'} V_{G\hat{t}}(h_{\hat{t}}^\tau) < \varepsilon$$

for all $\varepsilon > 0$, reputation has value at date $\hat{t}$ for $T < \infty$. To prove that $r_t < \beta G$ specifies feasible rates if and only if condition (3.5) holds,

$$\beta G > R \Leftrightarrow \beta > \frac{R}{G}$$

which is condition (3.5).

Q.E.D.

Lemma 11 shows that providing that the horizon is long and that interest rates are low enough, reputation will have value for type G borrowers.

We now consider how borrowers' optimal repayment strategies vary with interest rates, and what are the lenders' corresponding optimal liquidation responses.

First, we must analyze what are lenders' equilibrium re-lending strategies. We prove in section 3.5 below that in any equilibrium, lenders' beliefs about borrower type, if a borrower defaults, are that there is a sufficiently high probability that the borrower is type B, such that it is optimal to terminate his credit. Hence, in any equilibrium, if a borrower defaults, the present discounted value of his future pay-offs is zero.

We now show that there exists no equilibrium in which both borrower types default strategically, irrespective of interest rates. The reason is the same as in the one period game. If both borrower types default, then providing condition (3.1) holds, it is optimal for the lender to liquidate with probability one. Hence, borrowers would prefer to repay, and hence this cannot be an equilibrium.

We now show that if $r_t \leq \beta V_{Gt+1}$ holds, then there exists the following unique equilibrium. Type G borrowers will repay debt. Type B borrowers will randomize between repayment and default, defaulting strategically with probability π^*_{Bt}, where $0 < \pi^*_{Bt} < 1$. If a borrower defaults, lenders will randomize their liquidation response, liquidating with probability α^*_{Bt}, where $0 < \alpha^*_{Bt} < 1$.

First, we must specify lenders' beliefs about borrower type, if a borrower defaults. In this equilibrium, type G's always repay debt. Hence, lenders' beliefs must be that default implies that the borrower is type B, and hence it is optimal to terminate credit to any borrower who defaults. Suppose lenders do not liquidate if borrowers default. Then type G's pay-offs from repaying and defaulting respectively are:

$V_{Gt}(h_t^r) = G - r_t + \beta V_{Gt+1}$

$V_{Gt}(h_t^d) = G + 0.$

Hence, it is optimal for type G's to repay debt if and only if

$$\begin{aligned} V_{Gt}(h_t^r) &\geq V_{Gt}(h_t^d) \Leftrightarrow \\ r_t &\leq \beta V_{Gt+1} \end{aligned}$$

Hence, if $r_t \leq \beta V_{Gt+1}$ holds, then type G's will prefer to repay debt even if there is no threat of liquidation. If this is the case, we say that reputation equilibrium holds. The intuition is as follows. When interest rates are very low, then the net value of future borrowing exceeds the one off payment from default, and thus type G borrowers would prefer to repay, in order to maintain their reputations and thus retain access to future credit.

Is it also optimal for type B's to repay? Proposition 10 proved that no matter how low are interest rates, type B's will prefer to default, if there is no threat of liquidation. Hence, there exists no equilibrium in which both borrower types repay with probability one. But neither does there exist an equilibrium in which type B's default with probability one. If type B's strategy is to default strategically with probability one, then lenders' specification of beliefs in information set I is given by $\mu_t(\frac{X_1}{h_t^d}) = 0$, $\mu_t(\frac{X_2}{h_t^d}) = (1 - p_B)$, $\mu_t(\frac{X_3}{h_t^d}) = p_B$. Hence, it is optimal for lenders to liquidate

$$\begin{aligned} &\Leftrightarrow \; Eu_t^L(liq) > Eu_t^L(no\ liq) \\ &\Leftrightarrow \; p_B(B - C) - (1 - p_B)C > 0 \\ &\Leftrightarrow \; p_B B > C \end{aligned} \tag{3.6}$$

But we know that (3.6) holds by assumption, hence lenders will prefer to liquidate. But this means it is not optimal for type B's to default, hence this is not an equilibrium. Hence, there exists no pure strategy equilibrium in which type B's either repay or default with probability one. But there does exist the following mixed strategy equilibrium.

For mixed strategy equilibrium to hold, lenders must liquidate with probability α_{Bt}^*, such that type B borrowers are indifferent between repayment and strategic default, i.e. such that the following holds:

$$V_{Bt}(h_t^r) = V_{Bt}(h_t^d)$$

$$\Leftrightarrow B - r_t + \beta p_B V_{Bt+1} = (1-\alpha_t)B + \alpha_t 0$$

$$\Leftrightarrow \alpha_t = (r_t - \beta p_B V_{Bt+1})/B = \alpha_{Bt}^* \tag{3.7}$$

where $0 < \alpha_{Bt}^* < 1$. Type B's must default strategically with probability π_{Bt}^*, which makes lenders indifferent between liquidation and no liquidation, i.e. such that the following holds:

$$Eu_t^L(liq) = Eu_t^L(no\ liq)$$

$$\Leftrightarrow \mu_t(\frac{X_3}{h_t^d})(B-C) - (1-\mu_t(\frac{X_3}{h_t^d}))C = 0$$

$$\Leftrightarrow \mu_t(\frac{X_3}{h_t^d}) = \frac{C}{B} \tag{3.8}$$

Lenders' equilibrium beliefs are given by

$$\mu_t(\frac{X_3}{h_t^d}) = \frac{\pi_{Bt}p_{Bt}}{\pi_{Bt}p_{Bt} + (1-p_{Bt})} \qquad \mu_t(\frac{X_2}{h_t^d}) = \frac{1-p_{Bt}}{\pi_{Bt}p_{Bt} + (1-p_{Bt})} \tag{3.9}$$

Hence (3.8) and (3.9) imply

$$\pi_{Bt}^* = \frac{(1-p_B)}{p_B}\frac{C}{(B-C)} \tag{3.10a}$$

where $0 < \pi_{Bt}^* < 1$.

We now solve for equilibrium when $r_t > \beta V_{Gt+1}$. In this case, type G's will no longer prefer to repay debt if lenders do not liquidate, hence reputation equilibrium no longer holds. Lemmas **??** and 13 below solve for equilibrium when $r_t > \beta V_{Gt+1}$.

Lemma 12 *If $\beta V_{Gt+1} < r_t \leq \frac{\beta}{B-G}(BV_{Gt+1} - GV_{Bt+1})$, the unique equilibrium is as follows. Type G's will repay debt, type B's will randomize, defaulting strategically with probability π^*_{Bt}, and lenders will randomize, liquidating with probability α^*_{Bt}.*

Proof. Suppose lenders' strategy is not to liquidate if borrowers default. Given that $r_t > \beta V_{Gt+1}$, reputation is not effective for either borrower type, hence both types will default strategically. Hence, the lenders' strategy is not optimal and hence this is not an equilibrium. In equilibrium, lenders must randomize their liquidation response, such that one of the borrower types is indifferent between repayment and default. For equilibrium to hold, lenders must liquidate with probability α^*_{Bt}, given by (3.7), such that type B's will be indifferent, and type B's must default strategically with probability π^*_{Bt}, given by (3.10a), such that lenders are indifferent. Type G's will find it optimal to repay if and only if

$$\begin{aligned}
V_{Gt}(h^r_t) &\geq V_{Gt}(h^d_t) \Leftrightarrow \\
G - r_t + \beta V_{Gt+1} &\geq (1 - \alpha^*_{Bt})G \Leftrightarrow \\
r_t &\leq \frac{\beta}{B-G}(BV_{Gt+1} - GV_{Bt+1})
\end{aligned}$$

Hence, if $\beta V_{Gt+1} < r_t \leq \frac{\beta}{B-G}(BV_{Gt+1} - GV_{Bt+1})$ holds, although reputation equilibrium does not hold, type G borrowers will repay in equilibrium because the probability of liquidation is sufficiently high to deter default.

We now show that the alternative strategy profile, in which type G's randomize, cannot be an equilibrium. If type G's randomize, it must be true that

$$V_{Gt}(h^r_t) = V_{Gt}(h^d_t)$$

$$\Leftrightarrow G - r_t + \beta V_{Gt+1} = (1-\alpha_t)G$$

$$\Leftrightarrow \alpha_t = (r_t - \beta V_{Gt+1})/G = \alpha^*_{Gt} \tag{3.11}$$

But it will be optimal for type B's to repay

$$\begin{aligned} &\Leftrightarrow \quad V_{Bt}(h^r_t) \geq V_{Bt}(h^d_t) \\ &\Leftrightarrow \quad B - r_t + \beta p_B V_{Bt+1} \geq (1-\alpha^*_{Gt})B \\ &\Leftrightarrow \quad r_t \geq \frac{\beta}{B-G}(BV_{Gt+1} - GV_{Bt+1}) \end{aligned}$$

which does not hold. Hence, type B's will default with probability 1 in this equilibrium. Type G's must randomize, defaulting with probability π^*_{Gt},such that lenders are indifferent. But as shown above, given that (3.6) holds, lenders will strictly prefer to liquidate even if $\pi^*_{Gt} = 0$. Hence, this cannot be an equilibrium.

Q.E.D.

Lemma 13 *If $r_t \geq \frac{\beta}{B-G}(BV_{Gt+1} - GV_{Bt+1})$ holds, in equilibrium type B's will repay, type G's will randomize, defaulting strategically with probability π^*_{Gt}, and lenders will randomize, liquidating with probability α^*_{Gt}*

Proof. If lenders liquidate with probability α^*_{Gt} (given by (3.11)), it is optimal for type B's to repay if and only if the following holds:

$$r_t > \frac{\beta}{B-G}(BV_{Gt+1} - GV_{Bt+1})$$

(see proof of lemma **??** above). Type G's will default with probability π^*_{Gt}, determined such that lenders are indifferent. The specification of lender beliefs in information set I, given borrowers' equilibrium strategies is as follows: $\mu_t(\frac{X_1}{h^d_t}) = \frac{\pi^*_{Gt}F_{Gt}}{\pi^*_{Gt}F_{Gt}+(1-p_B)(1-F_{Gt})}$,

$\mu_t(\frac{X_2}{h_t^d}) = \frac{(1-F_{Gt})(1-P_B)}{\pi^*_{Gt}F_{Gt}+(1-p_B)(1-F_{Gt})}$, $\mu_t(\frac{X_3}{h_t^d}) = 0$. Lenders are indifferent if and only if

$$\begin{aligned} Eu_t^L(liq) &= Eu_t^L(no\ liq) \Leftrightarrow \\ \mu(x_1)(G-C) - \mu(x_2))C &= 0 \Leftrightarrow \\ \pi^*_{Gt} &= \frac{C(1-p_B)(1-F_{Gt})}{(G-C)F_{Gt}} \end{aligned}$$

We now show that the alternative strategy profile, in which type B's randomize, cannot be an equilibrium. For type B's to randomize, lenders must liquidate with probability α^*_{Bt}, given by (3.7), such that type B's are indifferent between repayment and default. Type G's will prefer to repay if and only if

$$V_{Gt}(h_t^r) \geq V_{Gt}(h_t^d)$$

$$\Leftrightarrow G - r_t + \beta V_{Gt+1} > (1-\alpha^*_{Bt})G$$

$$\Leftrightarrow r_t \leq \frac{\beta}{B-G}(BV_{Gt+1} - GV_{Bt+1})$$

Hence, type G's will prefer to default. However, from the analysis of the one period game, providing that condition (3.3) holds, if type G's default with probability one, then lenders will strictly prefer to liquidate, hence this cannot be an equilibrium.

3.5 The Lending Decision

This section solves for step (2) of the stage game, the lender's decision on whether to re-lend or terminate credit to a borrower, given his credit history.

Lenders will never lend to any borrower who is known to be type B, because the expected return on a type B project is less than the riskless interest rate. Every period, lenders observe each borrower's entire credit history, which tells them in which periods they defaulted on debt. They map this information into an assessment that the given borrower is of type G. Only if this implies a sufficiently high probability that the borrower is type G will the lender agree to lend to him this period.

Let $H_t = [h_1, h_2, \ldots h_{t-1}]$ be a borrower's credit history at the beginning of date t, where

$h_i = h^r$ if he repays in period i, and

$h_i = h^d$ if he defaults in period i.

Then $P_t(H_t)$, the lender's date t assessment that the borrower is type G given his credit history, is a function $P : \theta \longrightarrow [0, 1]$, where θ is the set of all possible histories.

Section 3.4 showed that if $r_t \leq \beta V_{Gt+1}$ holds, and if lenders beliefs are that default implies that the borrower is type B, then reputation equilibrium holds, and type G borrowers will repay debt because they fear the threat of credit termination. But suppose we specify lenders' beliefs differently. For example, suppose lenders believe that even if a borrower defaults, the probability that he is type G is still sufficiently high such that it is optimal to re-lend to him. Given these beliefs, type G borrowers no longer have an incentive to repay debt, because lenders will still re-lend to them if they default. Hence, reputation equilibrium no longer exists. However, proposition 11 below proves that in equilibrium, the unique specification of lenders' beliefs about a borrower who defaults is that the probability that he is type B is sufficiently high such that it is optimal to terminate his credit.

Proposition 11 *If a borrower defaults on date t', the unique equilibrium strategy is for all lenders to terminate credit for all $t > t'$.*

Proof. There is an extremely large number of different arbitrary specifications for lenders' beliefs about borrower type, given that a borrower has defaulted, for the T-period equilibrium path as a whole. For example, lenders might believe on all dates that default implies a high probability that the borrower is type G, hence it is optimal to re-lend if borrowers default on any date. Or they might believe that on some dates, default implies a high probability that the borrower is type G, hence it is optimal to re-lend, but on other dates, default implies that the borrower is type B, hence it is optimal to terminate credit. However, we now prove that the unique specification of beliefs is that the probability that he is type B is sufficiently high such that it is optimal to terminate his credit, on all dates.

It is optimal to lend on date $t+1$ to a borrower who defaults on date t if and only if

$$P_{t+1}(h_t^d) > P_{t+1}^{crit} \tag{3.12}$$

P_{t+1}^{crit} is the lowest probability that a borrower is type G for which the market is open. If $P_{t+1}(h_t^d) < P_{t+1}^{crit}$, then lenders earn an expected return less than the riskless interest rate, even if they charge the maximum feasible interest rate $r_{t+1} = G - C$. Hence it is optimal not to lend.

Suppose lenders' strategy is to re-lend to all borrowers who default on date t. Then, reputation effects will be irrelevant, and thus the multi-period game collapses to the one period game. In section 3.3, we saw that it is optimal for type G's to default with probability π_{Gt}^* and for type B's to repay. The probability that a borrower who defaults is type G is given by the lender's assessment that he is at node X_1 in information set I (see Figure 6), which is given by $P_{t+1}(h_t^d) = \mu(x_1) = \frac{C}{G}$.

We must now derive P_{t+1}^{crit}. This depends on borrowers' strategies on date $t+1$, and hence lenders' beliefs on date $t+1$ about borrowers who default. If lenders' beliefs are the same as for date $t+1$, they will re-lend on date $t+2$ also if borrowers default on date $t+1$. Hence, the equilibrium will be the same as for date t, where type G's default with probability π_{Gt}^* and type B's repay. Hence $P_{t+1}^{crit} = \frac{R+C-p_BG}{G(1-p_B)}$ (from 3.4 above). But $\frac{C}{G} < \frac{R+C-p_BG}{G(1-p_B)}$, hence (3.12) fails to hold. This means it is optimal for lenders not to re-lend if borrowers default on date t.

But suppose lenders' beliefs on date $t+1$ are that default implies the borrower is type B, and hence it is optimal to terminate credit if borrowers default. In this case, then as we saw in section 3.4, if $r_t > \frac{\beta}{B-G}(BV_{Gt+1} - GV_{Bt+1})$ holds, then the equilibrium is the same as before, hence (3.12) fails to hold. However, if $r_t \leq \frac{\beta}{B-G}(BV_{Gt+1} - GV_{Bt+1})$ holds, then it is optimal for type G's to repay and type B's to default strategically with probability π^*_{Bt+1}on date $t+1$. Hence

$$Eu^L_{t+1} = F_{Gt+1}r_{t+1} + F_{Bt+1}p_B(1 - \pi^*_{Bt+1})r_{t+1}$$

which implies

$$r^*_{t+1} = R\left[F_{Gt+1} + F_{Bt+1}\left(\frac{p_B B - C}{B - C}\right)\right]^{-1}$$

A necessary, but not sufficient, condition for the market to be open on date $t+1$ is $r^*_{t+1} < G$, or equivalently,

$$F_{Gt+1} > \frac{R(B-C) - (p_B B - C)G}{GB(1-p_B)} = \hat{P}^{crit}_{t+1}$$

The following is therefore a necessary condition for it to be optimal for lenders to lend to borrowers who defaulted on date t

$$\frac{C}{G} > \frac{R(B-C) - (p_B B - C)G}{GB(1-p_B)}$$

$$\Leftrightarrow C > \frac{RB - p_B BG}{B(1-p_B) + R - G} \tag{3.13}$$

Given that $C < p_B B$, (3.13) holds only if

$$p_B B > \frac{RB - p_B BG}{B(1-p_B) + R - G}$$

$$\Leftrightarrow p_B B > R$$

which cannot be true, by assumption. Hence, (3.12) fails to hold in this case also, and thus it is optimal for lenders not to re-lend if borrowers default on date t.

Q.E.D.

Proposition 11 proves that even though default does not necessarily imply that the borrower is type B, it will always result in credit termination. On some dates when reputation is ineffective, type G's will default with positive probability π^*_{Gt}. However, given the threat of liquidation, this probability is sufficiently low such that the pool of borrowers who default contains a high proportion of type B borrowers who were constrained to default due to project failure. Given that lenders cannot distinguish between type G's who defaulted strategically, and type B's whose projects failed, it is optimal for lenders not to re-lend to any of these borrowers.

Proposition 11 also implies that the date t lending decision depends only on the date $t-1$ outcome. If the borrower repaid on date $t-1$, this implies that he must have repaid on all previous dates. Hence

$$P_t(H_t) \equiv P_t(h_{t-1})$$

3.6 Solving for Equilibrium Interest Rates

This section solves for step (1) of the stage game, borrowers' offers of debt contracts to lenders, specifying the interest rate at which they are willing to borrow. As in the one period game, borrowers will offer the lowest possible interest rate r_t^* each period such that their debt contracts are accepted, i.e. such that lenders earn an expected return equal to the riskless interest rate.

Furthermore, in equilibrium all borrowers pay the same interest rate. This simply reflects the fact that credit is terminated to any borrower who defaults, and therefore all borrowers who remain in the market have identical credit histories (i.e. a track record for repaying debt on all dates). Hence only the publicly observable fraction of type G's remaining in the market is relevant in determining the equilibrium interest rate time path. The date t equilibrium interest rate is derived such that lenders' expected return equals the riskless interest rate, i.e. such that the following holds

$$Eu_t^L = R$$

$$\Leftrightarrow F_{Gt}(1-\pi_{Gt}^*)r_t + (1-F_{Gt})p_B(1-\pi_{Bt}^*)r_t = R$$

$$\Leftrightarrow r_t = R\left[F_{Gt}(1-\pi_{Gt}^*) + (1-F_{Gt})p_B(1-\pi_{Bt}^*)\right]^{-1} = r_t^* \tag{3.14}$$

where F_{Gt} is the proportion of type G's in the market who have not defaulted by the start of date t. Note from (3.14) that interest rates are a function of borrower repayment decisions, i.e. interest rates are increasing in π_{Gt}^* and π_{Bt}^*. Hence, we need to examine the dynamics of borrower repayment strategies before we can explicitly derive an equilibrium time path for interest rates. However, we do know that $F_{Bt} < F_{Bt-1}$ holds for all t. This is true regardless of borrowers repayment strategies along the equilibrium path. For example, suppose type G's default strategically with probability π_{Gt}^* in equilibrium, and type B's repay. From proposition 11, we know that $1-p_B > \pi_{Gt}^*$ for all t. If this did not hold, then the pool of borrowers who defaulted on date t would contain a higher proportion of type G's than the pool of borrowers in the market at the start of date t. This would mean that it would be optimal for lenders to re-lend to borrowers who defaulted, but we know from proposition 11 that this cannot be true. Hence, $1-p_B > \pi_{Gt}^*$ must hold and hence it must be true that the fraction of type B's in the credit market falls over time, regardless of borrowers' equilibrium strategies.

3.7 Proving the Existence of Reputation Equilibrium

Lemma 11 provides a sufficient condition for reputation equilibrium to exist. In this section, we prove that providing that borrowers' horizon is long enough, then this sufficient condition will hold, and thus reputation equilibrium will exist on some dates.

As shown in the analysis of the one period game, on date T (the terminal date) reputation effects do not exist for type G's. However, we now show that, as we go backwards in time, there exists some date $\hat{t} < T$ on which type G borrowers' horizon is long enough, such that reputation equilibrium exists.

We define the endgame as the time period until date T that begins when reputation loses value for type G's (i.e. reputation equilibrium fails to hold). Hence the endgame begins on date τ, where τ is the smallest t for which $r_t > \beta V_{Gt+1}$ that occurs after some date $\hat{t}$ where $r_t \leq \beta V_{Gt+1}$. The endgame is bounded if there exists $K < \infty$ such that $T - \tau < K$ as $T \to \infty$. Hence, proving that there exists some date $\hat{t} < T$ on which reputation equilibrium exists is equivalent to proving the existence of a bounded endgame.

Proposition 12 *If $r_1 \leq G - C$, then the endgame is of bounded length as $T \to \infty$ if and only if 3.5 holds.*

Proof. We first show that $r_t \to R$ as $t \to \infty$. Suppose type G's default strategically along the equilibrium path. The time path of interest rates is given by

$$r_t = R\left[F_{Gt}(1 - \pi^*_{Gt}) + F_{Bt}p_B\right]^{-1} \tag{3.15}$$

$$\text{where } \pi^*_{Gt} = \frac{C(1 - p_B)F_{Bt}}{(G - C)F_{Gt}}$$

$$\text{and } F_{Gt} = \frac{(1 - \pi^*_{Gt-1})F_{Gt-1}}{(1 - \pi^*_{Gt-1})F_{Gt-1} + p_B F_{Bt-1}} \qquad F_{Bt} = \frac{p_B F_{Bt-1}}{(1 - \pi^*_{Gt-1})F_{Gt-1} + p_B F_{Bt-1}}$$

We know from the argument given above that $F_{Bt} \to 0$ as $t \to \infty$. This implies that $\pi^*_{Gt} \to 0$ and hence $r_t \to R$. This holds irrespective of borrowers' equilibrium repayment strategies, i.e. whether reputation has value or not.

If $T \to \infty$, there will be an arbitrarily long horizon in which, for all $\varepsilon > 0$, $r_t < R + \varepsilon$ for $\hat{t} < t < T$, for some date $\hat{t}$. The profile of future interest rates from date $\hat{t}$ onwards therefore satisfies the condition set out in lemma 11 for the existence of a bounded endgame as $T \to \infty$.

Q.E.D.

We have therefore shown that 3.5, the necessary condition for reputation to have value is also a sufficient condition if $T \to \infty$.

3.8 The Dynamics of Reputation along the Equilibrium Path

We know that reputation equilibrium will exist on some date $\hat{t} < T$, where T is large, providing that (3.5) holds. We now analyze the dynamics of borrower reputation effects. If reputation has value on date $\hat{t}$, does this imply that reputation has value on all dates $t \geq \hat{t}$ before the end game? Or does reputation repeatedly lose and regain value over time? In fact, the dynamics of reputation effects are as follows. During an initial period, if adverse selection is high, then reputation effects will not exist. But over time, reputation will increase in value, such that at some point, reputation equilibrium will exist. Reputation equilibrium will then hold up to the endgame, after which it will never exist. The intuition for these dynamics is as follows. Given the condition for reputation equilibrium to hold, $r_t \leq \beta V_{Gt+1}$, the dynamics of reputation depend purely on the time path of $r_t - \beta V_{Gt+1}$. We prove below that interest rates are strictly decreasing over time. Hence, this tends to increase the value of reputation over time. However, over time the horizon shortens, which tends to reduce the value of reputation. If reputation is ineffective during initial periods, then the reason is high interest rates, rather than a short horizon. Over time, interest rates fall, but the horizon shortens also. However, given that the terminal date is very far off, the effect of a shorter horizon is discounted significantly, and thus the effects of lower interest rates dominates. Hence, reputation becomes effective after time. However, as the horizon gets very close, the effects of a shorter horizon dominates the effects of lower interest rates, and thus reputation loses value. This result is proved formally in proposition 13 below.

Proposition 13 *If reputation has value on date t^+, but does not have value on some date $t' < t^+$, then reputation does not have value for all $t < t'$.*

Proof. Let date $\hat{t}$ be the largest t that is less than t^+ on which reputation has no value, which implies $V_{\hat{t}} < G$ and $r_{\hat{t}} - \beta V_{\hat{t}+1} > 0$. But reputation has value at $\hat{t}+1$, which implies $V_{\hat{t}+1} \geq G$. Hence $V_{\hat{t}} < V_{\hat{t}+1}$. Given that $F_{B\hat{t}-1} > F_{B\hat{t}}$, we know $r_{\hat{t}-1} > r_{\hat{t}}$. This implies

$$r_{\hat{t}-1} - \beta V_{\hat{t}} > r_{\hat{t}} - \beta V_{\hat{t}+1} > 0$$

Hence reputation has no value at date $\hat{t} - 1$, and because $r_t > r_{\hat{t}}$ and $V_t < V_{\hat{t}+1}$ for all $t < \hat{t}$, recursion implies that reputation has no value for all $t < \hat{t}$.
Q.E.D.

Suppose $r_1 \leq \beta V_2$. Then providing (3.5) holds, reputation works immediately, and has value on all dates before the endgame. However, suppose $r_1 > \beta V_2$. Reputation does not work initially, but as interest rates fall over time, reputation gains value at some later date, and from proposition 13, reputation equilibrium holds until the endgame.

We now consider the dynamics of borrowers' repayment decisions on dates when reputation equilibrium does not exist. As shown in section 3.4 above, in equilibrium, one of the borrower types will find it strictly optimal to repay, whilst the other will randomize between repayment and default. If type G or type B repays, we call this a type G or type B repayment equilibrium respectively. Proposition 14 summarizes the dynamics of repayment equilibrium.

Proposition 14 *If type G repayment equilibrium exists on date t^+, but type B repayment equilibrium exists on some date $t' < t^+$, then type B repayment equilibrium exists on all dates $t < t'$.*

Proof. Let $\hat{t} \geq t'$ be the largest t that is less than t^+ on which type B's repay. Hence on date $\hat{t}$,

$$\alpha_{G\hat{t}} \geq \alpha_{B\hat{t}} \tag{3.16}$$

However, on date $\hat{t}+1$, type G's repay and type B's randomize, hence

$$\alpha_{G\hat{t}+1} < \alpha_{B\hat{t}+1} \tag{3.17}$$

(3.16) and (3.17) imply

$$\alpha_{B\hat{t}} - \alpha_{B\hat{t}+1} < \alpha_{G\hat{t}} - \alpha_{G\hat{t}+1} \tag{3.18}$$

From lemma 13, type B repayment equilibrium exists

$$\Leftrightarrow r_t - \frac{\beta}{B-G}(BV_{Gt+1} - GV_{Bt+1}) > 0$$

$$BV_{G\hat{t}} - GV_{B\hat{t}} < BV_{G\hat{t}+1} - GV_{B\hat{t}+1}$$

$$\Leftrightarrow G(V_{B\hat{t}+1} - V_{B\hat{t}}) < B(V_{G\hat{t}+1} - V_{G\hat{t}})$$

$$\Leftrightarrow G\left[(1-\alpha_{B\hat{t}+1})B - (1-\alpha_{B\hat{t}})B\right] < B\left[(1-\alpha_{G\hat{t}+1})G - (1-\alpha_{G\hat{t}})G\right]$$

$$\Leftrightarrow BG(\alpha_{B\hat{t}} - \alpha_{B\hat{t}+1}) < BG(\alpha_{G\hat{t}} - \alpha_{G\hat{t}+1})$$

which holds given 3.18.

The following argument proves that $r_{\hat{t}-1} > r_{\hat{t}}$. Suppose that borrower repayment strategies are unchanged from date t to t-1. Given that we know $F_{Bt-1} > F_{Bt}$, it follows that $r_{\hat{t}-1} > r_{\hat{t}}$. Now suppose instead that type G's decide to repay on date t-1. However, the fall in interest rates resulting from this change in repayment behaviour must be large enough to make it optimal for type G's to repay. But if this were true, then type G's would also find it optimal to repay in period t (given that all borrowers will offer the lowest feasible interest rate), which is not true. Hence $r_{\hat{t}-1} > r_{\hat{t}}$ holds.

Hence given that $BV_{G\hat{t}} - GV_{B\hat{t}} < BV_{G\hat{t}+1} - GV_{B\hat{t}+1}$, and that $r_{\hat{t}-1} > r_{\hat{t}}$, it must be true that

$$r_{\hat{t}-1} - \frac{\beta}{B-G}(BV_{G\hat{t}} - GV_{B\hat{t}}) > r_{\hat{t}} - \frac{\beta}{B-G}(BV_{G\hat{t}+1} - GV_{B\hat{t}+1}) > 0$$

hence type B's will repay on date $\hat{t}-1$. Furthermore, given that $r_{\hat{t}-j} > r_{\hat{t}}$, and that $BV_{G\hat{t}-j} - GV_{B\hat{t}-j} < BV_{G\hat{t}+1} - GV_{B\hat{t}+1}$ for all $j > 0$, recursion implies that type B's repay on all dates $t < \hat{t}$.

Q.E.D.

Proposition 14 shows that if interest rates are high enough for borrowers with short track records, incentives are actually worse for type G borrowers than for type B's. This is because, although type G's always value future borrowing more highly than type B's, if interest rates are very high, then future borrowing is severely discounted, and current pay-offs matter much more. Given that type B's have a higher pay-off (if their projects succeed) than type G's, they fear the threat of liquidation more than type G's, and thus have a greater incentive to repay debt. However, as interest rates fall over time, incentives to repay grow differentially stronger for type G's, given that they value the future more highly than type B's, and the credit market evolves from type B to type G repayment equilibrium. As interest rates fall further, repayment equilibrium transforms into reputation equilibrium. However, let $\tilde{t}$ be the largest t such that $r_{\tilde{t}} - \frac{\beta}{B-G}(BV_{G\tilde{t}+1} - GV_{B\tilde{t}+1}) > 0$ holds. If for $\tilde{t}+1$, $r_{\tilde{t}+1} \leq \beta V_{G\tilde{t}+2}$ then type B repayment equilibrium evolves directly into reputation equilibrium, without the intermittent stage of type G repayment equilibrium.

We can now characterize equilibrium for different initial populations of borrowers in terms of exogenous parameters. To do this, we need to solve for the equilibrium time path of interest rates $[r_t^*]_{t=1}^T$. As shown above, interest rates are a function of borrowers' equilibrium repayment strategies, which are in turn a function of interest rates. However, we know that on any date, borrowers' equilibrium repayment strategies take one of two forms, either type G's repay and type B's randomize, or vice versa. Solving for $[r_t^*]_{t=1}^T$ is therefore equivalent to solving for date $\tilde{t}$, the smallest t for which type G's repay in equilibrium. We solve for date $\tilde{t}$ by using the following algorithm.

We begin with the conjecture that type G's repay on all dates $t = 1 \ldots \tau$. We then derive the equilibrium interest rate time series based on this conjecture using $r_t^* = R\left[F_{Gt} + F_{Bt}P_B(1-\pi_{Bt}^*)\right]^{-1}$, and call this $[r_t(1)]_{t=1}^T$. We then use this series to derive $\hat{V}_{G2}$ and $\hat{V}_{B2}$. If $r_1(1) \leq \frac{\beta}{B-G}(B\hat{V}_{G2} - G\hat{V}_{B2})$, then from propositions 13 and 14, $r_t(1) \leq \frac{\beta}{B-G}(B\hat{V}_{Gt+1} - G\hat{V}_{Bt+1})$ for all $t < \tau$, which means the initial conjecture that type G's repay on all dates before the endgame is true, hence $[r_t^*(1)]_{t=1}^T$ is the equilibrium interest rate series. However, if $r_1(1) > \frac{\beta}{B-G}(B\hat{V}_{G2} - G\hat{V}_{B2})$, then type G's will randomize on date $t = 1$, which contradicts our initial conjecture. We then derive a revised interest rate series $[r_t(2)]_{t=1}^T$, assuming that type G's randomize on date $t = 1$, but repay on all dates $t = 2 \ldots \tau$. If $r_2(2) \leq \frac{\beta}{B-G}(B\hat{V}_{G3} - G\hat{V}_{B3})$, then $[r_t(2)]_{t=1}^T$ is the equilibrium interest rate series. If not, we iterate on this process until we find $\left[r_t(\tilde{t})\right]_{t=1}^T$such that $r_{\tilde{t}}(\tilde{t}) \leq \frac{\beta}{B-G}(B\hat{V}_{G\tilde{t}+1} - G\hat{V}_{B\tilde{t}+1})$. Providing that (3.5) holds, we know from proposition 12 that there exists $\tilde{t} < \tau$ for $T < \infty$.

The equilibrium interest rate time series is given by

$$\begin{aligned}
r_t^*(\tilde{t}) &= R\left[F_{Gt}(1-\pi_{Gt}^*) + F_{Bt}p_B\right]^{-1} \text{ for } t < \tilde{t} && (3.19)\\
\text{where } \pi_{Gt}^* &= \frac{C(1-p_B)F_{Bt}}{(G-C)F_{Gt}} \text{ and} && (3.20)\\
F_{Gt} &= \frac{(1-\pi_{Gt-1}^*)F_{Gt-1}}{(1-\pi_{Gt-1}^*)F_{Gt-1} + p_B F_{Bt-1}}, \quad F_{Bt} = \frac{p_B F_{Bt-1}}{(1-\pi_{Gt-1}^*)F_{Gt-1} + p_B F_{Bt-1}} \\
\text{and } r_t^*(\tilde{t}) &= R\left[F_{Gt} + F_{Bt}p_B(1-\pi_B^*)\right]^{-1} \text{ for } \tilde{t} \leq t < \tau && (3.21)\\
\text{where } \pi_B^* &= \frac{(1-p_B)}{p_B}\frac{C}{(B-C)} \text{ and} && (3.22)\\
F_{Gt} &= \frac{F_{Gt-1}}{F_{Gt-1} + p_B(1-\pi_B^*)F_{Bt-1}}, \quad F_{Bt} = \frac{p_B(1-\pi_B^*)F_{Bt-1}}{F_{Gt-1} + p_B(1-\pi_B^*)F_{Bt-1}}
\end{aligned}$$

Now that we have solved for borrowers' equilibrium repayment strategies over time, we can solve for the dynamics of interest rates on the equilibrium path. We do this in proposition 15 below. Proposition 15 proves that, away from the endgame, interest rates are monotonically decreasing over time, and that the interest rate converges to the riskless rate as t gets very large.

Proposition 15 *For any date t, where $t-i < t < \tau$, $r_t < r_{t-i}$ must be true. Furthermore, $r_t \rightarrow R$ as $t \rightarrow \infty$.*

Proof. We begin by proving the first part of the proposition. From (3.19) and (3.21), we know that interest rates are a function of F_{Bt}, the fraction of type B borrowers remaining in the market, and π^*_{Gt} and π^*_{Bt}, the probabilities that type G and B borrowers default strategically. As explained in section 3.6 above, we know that $F_{Bt} < F_{Bt-i}$ holds for all $t > t-i$, hence from (3.19) and (3.21) and holding π^*_{Gt} and π^*_{Bt} constant, we know that $r_t < r_{t-i}$ must be true.

For dates $t < \tilde{t}$ there exists type B repayment equilibrium, hence the interest rate is given by (3.19). From (3.20), and given that $F_{Bt} < F_{Bt-i}$ holds for all $t > t-i$, we know that $\pi^*_{Gt} < \pi^*_{Gt-i}$ holds for all $t > t-i$, hence $r_t < r_{t-i}$ must hold for all $t > t-i$. For dates $t \geq \tilde{t}$ there exists type G repayment equilibrium or reputation equilibrium, hence the interest rate is given by (3.21). From (3.22), we know that $\pi^*_{Bt} = \pi^*_{Bt-i}$ holds for all $t > t-i$, hence $r_t < r_{t-i}$ must hold for all $t > t+i$.

We now prove that $r_t \rightarrow R$ as $t \rightarrow \infty$. We know from the argument in section 3.6 above that $F_{Bt} \rightarrow 0$ as $t \rightarrow \infty$. From (3.20), this implies that $r_t \rightarrow R$ as $t \rightarrow \infty$.

Q.E.D. ■

Type G's value function on date t is given by

$$V_{Gt}(\tilde{t}) = \sum_{j=0}^{\tau-t-1} \left[\beta^j (G - r_{t+j}(\tilde{t}))\right] + \beta^{\tau-t} V_{G\tau}(\tilde{t})$$

As $T \rightarrow \infty$, the existence of a bounded endgame implies $\tau \rightarrow \infty$, hence the final term approaches zero, and we can therefore approximate $V_{Gt}(\tilde{t})$ by

$$\bar{V}_{Gt}(\tilde{t}) = \sum_{j=0}^{\infty} \beta^j (G - r_{t+j}(\tilde{t}))$$

Similarly for type B's

$$\bar{V}_{Bt}(\tilde{t}) = \sum_{j=0}^{\infty} \beta^j p_B^j (B - r_{t+j}(\tilde{t}))$$

Proposition 16 summarizes the characterization of equilibrium for different initial conditions.

Proposition 16 *For any fixed $\tau < \infty$, there exists $T < \infty$ such that, for all $t < \tau$, the following conditions hold:*

1. If $r_1(1) \le \beta V_{G2}(1)$ and $r_1(1) \le G - C$ (i.e. f_B is sufficiently low and (3.5) holds), then reputation equilibrium holds immediately and for all $t < \tau$, and equilibrium interest rates are given by $r_t^*(1)$.

2. If $\beta V_{G2}(1) < r_1(1) \le \frac{\beta}{B-G}(BV_{G2}(1) - GV_{B2}(1))$, and $r_1(1) \le G - C$, the initial equilibrium is a type G repayment equilibrium, which evolves into reputation equilibrium over time. Equilibrium interest rates are once again given by $r_t^*(1)$.

3. If $r_1(1) > \frac{\beta}{B-G}(BV_{G2}(1) - GV_{B2}(1))$, and $r_1(1) \le G - C$, the initial equilibrium is a type B repayment equilibrium. This evolves into type G repayment equilibrium over time, which in turn becomes reputation equilibrium.

3.9 Summary of Comparative Statics

An increase in f_G implies lower interest rates at all dates, hence $r_1(1) - \beta V_{G2}(1)$ falls. For a large enough increase in f_G, initial equilibrium switches from repayment to reputation equilibrium.

A decrease in G has no effect on interest rates if $\tilde{t} = 1$, but $V_{G2}(1)$ falls, hence reputation effects are weakened. If $\tilde{t} > 1$, given that π^*_{Gt} is a decreasing function of G, a fall in G implies higher interest rates on all dates $t < \tilde{t}$ for which type G's randomize, and higher rates on all other dates given that F_{Bt} is larger than otherwise, because a higher proportion of type G's default in early periods. A further effect is that the increase in $B - G$ increases type B's return from repaying debt, relative to type G's, which may effect a switch from type G to type B repayment equilibrium.

A decrease in B increases π^*_{Bt}, implying higher interest rates on all dates for which type B's randomize. However, given that F_{Bt} falls at a faster rate, the net effect on interest rates is ambiguous.

An increase in C increases π^*_{Bt}, which implies that F_{Bt} falls at a faster rate. Hence the net effect on $V_{G2}(1)$ is ambiguous, depending on the size of the discount rate. A decrease in p_B is also ambiguous, for the same reason.

A change in β has several effects. A decrease in β implies $V_{G2}(1)$ falls, hence reputation effects are weakened. Furthermore, if β is sufficiently low, (3.5) no longer holds, hence reputation equilibrium never exists. The size of β also affects the initial equilibrium configuration. If β is sufficiently high such that (3.5) holds comfortably, reputation equilibrium will hold immediately, no matter how high the initial interest rate, providing of course the market is open. However, if (3.5) only just holds, initial equilibrium will be characterized by type G or type B repayment equilibrium for high interest rates, with type G's acquiring a reputation over time.

A permanent increase in R, the riskless interest rate, has a similar effect to a fall in f_G, with interest rates rising on all dates.

To model the effects of a temporary aggregate shock, we analyze the effects of a temporary increase in R to R' on some date t_2 which lasts for n periods. Suppose that the increase in R is sufficient to shift equilibrium on date t_2 from reputation equilibrium, to type B repayment equilibrium, in which type Gs will default strategically with probability $\pi^*_{Gt_2}$. Suppose also that there exists date $t_1 < t_2$ on which type B repayment equilibrium holds and the following also holds

$$r_{t_1} - \beta V_{Gt_1+1}(R) < r_{t_2} - \beta V_{Gt_2+1}(R') \tag{3.23}$$

where $V_{Gt_1+1}(R)$ is the value function given the original riskless interest rate, and $V_{Gt_2+1}(R')$ is the value function given the temporary shock which increases the riskless interest rate to R' on dates t_2 to t_2+n. From $\pi^*_{Gt} = \frac{C(1-P_B)F_{Bt}}{(G-C)F_{Gt}}$, and given that $F_{Bt_1} > F_{Bt_2}$, it must be true that $\pi^*_{Gt_2} < \pi^*_{Gt_1}$. Hence, even though from (3.23) incentives to repay debt are worse for type G borrowers on date t_2 than on date t_1, the equilibrium probability that type G's will default on date t_2 is actually lower than on date t_1. The reason for this is as follows. If type G borrowers default during early periods when they have yet to establish a track record for repaying debt, lenders will be reluctant to liquidate, because there is a high proportion of type B borrowers remaining in the market. Hence, if a borrower defaults, it is likely to be a type B borrower whose project has failed, rather than a type G who has defaulted strategically. Hence, the lender's expected pay-off from liquidation will be low. Since type G borrowers know this, they have a greater incentive to default, because given that project outcome is unobservable, they can 'hide' their type by pretending to be a type B borrower whose project has failed, in which case lenders will be reluctant to liquidate. However, over time, the proportion of type B's remaining in the market falls. Hence, if a borrower defaults, it is less likely to be a type B borrower whose project has failed. This means that the lender will have a greater incentive to liquidate, and thus in equilibrium, the probability that type Gs default falls, even if incentives to repay debt are weaker than on previous dates. Effectively, this means that borrowers with short credit histories are more affected by a negative interest rate shock than borrowers with established credit histories. Hence, this result provides an explanation for the stylized observation noted above, that aggregate shocks have less impact on firms' risk of default, as firms develop a good credit history.

3.10 Conclusion

In chapters one and two of this thesis, I have formulated a new theory of financial intermediation and explained the general structure of credit markets.

Essentially, I have answered the questions, why do banks exist, and why do new borrowers (i.e. firms which have only recently begun trading) tend to borrow using bank finance, whilst established borrowers tend to issue debt directly? Banks exist in order to lend to new borrowers with short credit histories. These borrowers are unable to issue debt directly, because direct lenders expect them to repudiate their debt obligations. The bank is able to lend, because it is a multi-period player in the credit market, which allows it to build a reputation for being tough and thus deter borrowers from repudiation.

Established borrowers are charged lower interest rates. Consequently, they are concerned about maintaining their reputation for being a good credit risk, and will thus refrain from repudiation, in order to prevent direct lenders from terminating their credit. Hence, direct lenders are willing to lend to them.

In chapter three of this thesis, I have answered the question, why do firms' credit ratings improve over time? There are two reasons. Firstly, if firms establish a track record for repaying debt over time, this convinces lenders that they are probably a good credit risk (a safe borrower). Every period, there exists a positive probability that risky borrowers will be forced to default due to project failure, thus revealing their type and resulting in credit termination. Hence, the proportion of risky borrowers in the market falls over time, and thus the risk of default due to project failure for borrowers remaining in the market falls over time. Secondly, safe borrowers' incentives to repay debt improve over time, as they become concerned about their reputation, and hence the risk of strategic default falls over time as well.

I have also answered the question, why do aggregate shocks have less impact on the risk of default of well-established firms, compared to recent start-up firms? If borrowers default during the early stages of their lives when they have yet to establish a track record for repaying debt, lenders will be reluctant to liquidate, because there is a high proportion of risky borrowers remaining in the market. Hence, if a borrower defaults, it is likely to be a risky borrower whose project has failed, rather than a safe borrower who has defaulted strategically. Hence, the lender's expected pay-off from liquidation will be low. Since borrowers know this, they have a greater incentive to default strategically, because given that project outcome is unobservable, they can 'hide' their type by pretending to be a risky borrower whose project has failed, in which case lenders will be reluctant to liquidate. However, over time, the proportion of risky borrowers remaining in the market falls. Suppose that a negative aggregate shock occurs, which worsens borrowers' incentives to repay debt. If a borrower has established a good credit history, then if he defaults, lenders will believe that he is defaulting strategically, rather than a risky borrower whose project has failed, since it is unlikely that a risky borrower could have developed such a good credit history. This means that the lender will have a greater incentive to liquidate, which deters the borrower from defaulting strategically, even though his incentives have worsened. Effectively, this means that borrowers with short credit histories are more affected by a negative shock than borrowers with established credit histories.

3.11 Appendix

Proof of lemma 2

We claim that direct lenders will not lend in equilibrium, and prove that the bank's optimal strategy is to lend on all dates, charge a monopoly premium on interest rates, and liquidate all defaulters, and borrowers' optimal strategies are to issue bank debt and repay on all dates.

Consider some date $T' < \infty$. Suppose borrowers' strategy is to default on all dates $t > T'$. What is the bank's optimal strategy on date T' ? Effectively, date T' is the bank's horizon, since borrowers will default on all future dates, hence type W banks will not liquidate defaulters on date T'. We therefore solve for equilibrium on dates $\tilde{t} < t \leq T'$ using the same procedure as given in chapter 2, in the case where there is no separation. Consider what happens as $T' \rightarrow \infty$. Given that $V_{At}(rep)$ and $V_{At}(def)$ are constant for $t > \tilde{t}$, $P(x_t^*)$ is also constant, where $P(x_t^*)$ is the critical probability that the bank liquidates defaulters, such that $V_{At}(rep) = V_{At}(def)$. As T' grows larger, $V_{Mt+1}(\hat{x}_t, \hat{x}_t)$ grows larger, and from the equilibrium condition

$$q(\hat{x}_t) + V_{Mt+1}(\hat{x}_t, \hat{x}_t) = (1-\mu)Y(I_t)I_t$$

$\hat{x}_t \rightarrow 0$, hence $x_t^* \rightarrow 0$ also. Call t' the largest t such that $x_t^* = 0$. Given that $T' - t'$ is bounded, $t' \rightarrow \infty$ as $T' \rightarrow \infty$. This means that for any given reputation $x \geq 0$ on date $\tilde{t}$, the bank's strategy is to liquidate defaulters with probability 1 on all future dates, and borrowers' optimal strategy is therefore to repay on all future dates.

We proved that this was an equilibrium by first claiming that borrowers default on all dates $t > T'$. In order to prove that this is the unique equilibrium, we must prove that the equilibrium is unchanged if we assume some other sequence of future actions by borrowers. Suppose we assume some arbitrary sequence of actions other than default on all dates, e.g. borrowers repaying on some dates and defaulting on others. Given that this sequence of actions would yield a higher pay-off to the bank than the original sequence in which borrowers always default, and given that we proved that it is optimal for all bank types to liquidate under the original sequence, then it must also be optimal to liquidate under the new sequence. Hence the equilibrium is unchanged, and is thus unique.

Proof of lemma 4

First, we prove that if interest rates are low enough for a sufficiently long period of time, then reputation will be effective for type B's, if and only if condition (1.12) holds. If, for all $t \in [\hat{t}, t']$, $r_t < \beta\pi Y(I^*) + (1-\beta\pi)(1-\mu)Y(I^*)$, then there exists $t' < \infty$ such that reputation is effective for type B borrowers on date $\hat{t}$. This bound on future interest rates specifies feasible rates if and only if condition (1.12) holds.

Reputation is effective on date t $\Leftrightarrow$ $V_{Bt}(rep) \geq \mu Y(I_t)I_t$. Given that type B's will repay debt on all dates, we have

$$\begin{aligned} V_{B\hat{t}}(rep) &= \sum_{t=\hat{t}}^{t'} (Y(I_t) - r_t)I_t(\beta\pi)^{t-\hat{t}} + (\beta\pi)^{t'+1-\hat{t}} V_{Bt'+1}(rep) > \\ & \sum_{t=\hat{t}}^{t'} (Y(I^*) - (\beta\pi Y(I^*) + (1-\beta\pi)(1-\mu)Y(I^*)))I^*(\beta\pi)^{t-\hat{t}} \end{aligned}$$

Taking the limit of the final expression as $t' \rightarrow \infty$:

$$\lim_{t' \rightarrow \infty} \left[\sum_{t=\hat{t}}^{t'} \mu Y(I^*)I^*(1-\beta\pi)(\beta\pi)^{t-\hat{t}} \right] = \mu Y(I^*)I^*$$

Hence $\lim\limits_{t' \rightarrow \infty} V_{B\hat{t}}(rep) > \mu Y(I^*)I^*$. Given that we can find $t'' < \infty$ such that

$$\lim_{t' \rightarrow \infty} V_{B\hat{t}}(rep) - \lim_{t' \rightarrow t''} V_{B\hat{t}}(rep) < \varepsilon$$

for all $\varepsilon > 0$, reputation is effective on date $\hat{t}$ for $t' < \infty$. Note that $r_t < \beta\pi Y(I^*) + (1-$

$\beta\pi)(1-\mu)Y(I^*)$ specifies feasible rates if and only if

$$\begin{aligned}\beta\pi Y(I^*)+(1-\beta\pi)(1-\mu)Y(I^*) &> 1 \Leftrightarrow \\ \beta &> \frac{1-(1-\mu)Y(I^*)}{\pi\mu Y(I^*)}\end{aligned}$$

which is condition (1.12).

We now prove that this bound on future interest rates will be satisfied in equilibrium, if and only if condition (1.12) holds, and hence there does exist some date $t<\infty$ on which reputation is effective for type B borrowers. From (1.5a), we see that $F_{Bt}\to 0$ as $t\to\infty$. This implies that $r_t\to 1$ as $t\to\infty$. If $t'\to\infty$, there will be an arbitrarily long horizon in which, for all $\varepsilon>0$, $r_t<1+\varepsilon$ for $\hat{t}<t<t'$, for some date $\hat{t}$. The profile of future interest rates from date $\hat{t}$ onwards therefore satisfies the condition set out above for reputation to be effective for type B's on some date $t<\infty$.

Q.E.D.

Proof of lemma 6

Taking the first order condition of (2.14), we have

$$(1-\alpha_t)\frac{\partial V_t(rep,r_t)}{\partial I}+\alpha_t\frac{\partial V_t(def,x(r_t))}{\partial I}=0$$

Borrowers' static pay-off functions for date t are given by

$$\begin{aligned}u_t(rep,r_t) &= (Y(I_t)-r_t)I_t \\ u_t(def,x(r_t)) &= (1-P_t(x(r_t)))\mu Y(I_t)I_t\end{aligned}$$

Hence

$$\frac{\partial V_t(rep,r_t)}{\partial I} \geq 0 \text{ and } \frac{\partial V_t(def,x(r_t))}{\partial I}>0 \quad \text{for } I_t\leq I^*(rep,r_t) \tag{3.24}$$

$$\frac{\partial V_t(rep,r_t)}{\partial I} < 0 \text{ and } \frac{\partial V_t(def,x(r_t))}{\partial I}\leq 0 \quad \text{for } I_t\geq I^*(def) \tag{3.25}$$

Therefore, from (3.24) and (3.25), it must be true that
$I^*(rep, r_t) < I^*(r_t, \alpha_t) < I^*(def, x(r_t))$ for all r_t. We now prove that any demand function $I(r_t)$ where $I(r_t) > I^*(rep, r_t)$ for some r_t cannot be an equilibrium.

Consider an equilibrium with the demand function $I'(r_t)$ where $I'(\hat{r}_t) > I^*(rep, \hat{r}_t)$ for some arbitrary value $\hat{r}_t$. In equilibrium, $x(\hat{r}_t)$ is determined such that

$$V_{Bt}(rep, I'(\hat{r}_t), \hat{r}_t) = V_{Bt}(def, I'(\hat{r}_t), x(\hat{r}_t))$$

and for equilibrium to hold, it must be optimal for borrowers to borrow $I'(\hat{r}_t)$ and repay if the signal outcome is 'repay', and repudiate if the signal outcome is 'repudiate'. The bank will refuse to lend if and only if the borrower offers a contract which implies it is optimal for the borrower to default with certainty, i.e. it is optimal for the borrower to default even if the signal outcome is 'repay', i.e. $\Leftrightarrow V_t(rep, I(r_t), r_t) < V_t(def, I(r_t), x(r_t))$ where $V_t(def, I(r_t), x(r_t)) = (1 - P_t(x(r_t)))\mu Y(I_t)I_t$ given that the bank's belief when the signal outcome is 'repay' is that default implies type B, hence credit will be terminated if a borrower defaults. But there exists a profitable deviation from the borrowers' strategy, which satisfies the bank's constraints on lending. Borrowers will prefer to borrow $I^*(rep, r_t)$ rather than $I'(r_t)$ if the signal outcome is 'repay', given that $I^*(rep, r_t) = \arg\max V_t(rep, r_t) \Rightarrow V_t(rep, I^*(rep, r_t), r_t) > V_t(rep, I'(r_t), r_t)$ and the bank will be willing to lend given that $V_t(rep, I^*(rep, r_t), r_t) > V_t(rep, I'(r_t), r_t) \Rightarrow V_t(rep, I^*(rep, r_t), r_t) > V_t(def, I'(r_t), x(r_t))$, since in equilibrium $V_{Bt}(rep, I'(r_t), r_t) = V_{Bt}(def, I'(r_t), x(r_t))$, and $V_t(def, I'(r_t), x(r_t)) > V_t(def, I^*(rep, r_t), x(r_t))$ must be true given that $\frac{\partial V_t(def, x(r_t))}{\partial I} > 0$ for $I^*(rep, r_t) \leq I_t < I^*(def, x(r_t))$, hence
$V_t(rep, I^*(rep, r_t), r_t) > V_t(def, I^*(rep, r_t), x(r_t))$ must hold.

Q.E.D.

Proof of proposition 3

Lemma 5 rules out any demand function in which the level of borrowing is a function of the signal outcome. Lemma 6 further rules out any demand function for which $I(r_t) >$

$I^*(rep, r_t)$ for some r_t. Hence only demand functions which satisfy $I(r_t) \leq I^*(rep, r_t)$ $\forall$ r_t are feasible in equilibrium, and borrowers' optimization problem becomes

$$\max_{I_t}(1-\alpha_t)V_t(rep, r_t) + \alpha_t V_t(def, x(r_t))$$

$$\text{subject to } I_t \leq I^*(rep, r_t)$$

However, given that for any demand function $I'(r_t)$ where $I'(\hat{r}_t) < I^*(rep, \hat{r}_t)$ for some $\hat{r}_t$, $I^*(rep, r_t) = \arg\max V_t(rep, r_t) \Rightarrow V_t(rep, I^*(rep, r_t), r_t) > V_t(rep, I'(r_t), r_t)$, and $V_t(def, I^*(rep, r_t), x(r_t)) > V_t(def, I'(r_t), x(r_t))$ given that $\frac{\partial V_t(def, x(r_t))}{\partial I} > 0$ for $I_t < I^*(def, x(r_t))$, therefore the following must be true
$(1-\alpha_t)V_t(rep, I^*(rep, r_t), r_t) + \alpha_t V_t(def, I^*(rep, r_t), x(r_t)) >$
$(1-\alpha_t)V_t(rep, I'(r_t), r_t) + \alpha_t V_t(def, I'(r_t), x(r_t))$. Hence borrowers will strictly prefer $I^*(rep, r_t)$ to any $I_t < I^*(rep, r_t)$. Hence if $I^*(rep, r_t)$ is an equilibrium demand function, it is the unique equilibrium. We now prove that it is an equilibrium for borrowers to borrow $I^*(rep, r_t)$ and repay if the signal outcome is 'repay', and borrow $I^*(rep, r_t)$ and repudiate if the signal outcome is 'repudiate'. In equilibrium, $x(r_t)$ is determined such that

$$V_{Bt}(rep, I^*(rep, r_t), r_t) = V_{Bt}(def, I^*(rep, r_t), x(r_t)) \tag{3.26}$$

where $V_{Bt}(def, I^*(rep, r_t), x(r_t)) = (1 - P_t(x(r_t)))\mu Y(I^*(rep, r_t))I^*(rep, r_t)$. If the signal outcome is 'repudiate', there are no profitable deviations through borrowing $I'(r_t) \neq I^*(rep, r_t)$ and defaulting. If $I'(r_t) < I^*(rep, r_t)$, $V_t(def, I'(r_t), x(r_t)) < V_t(def, I^*(rep, r_t), x(r_t))$ given that $\frac{\partial V_t(def, x(r_t))}{\partial I} > 0$ for $I_t < I^*(def, x(r_t))$. If $I'(r_t) > I^*(rep, r_t)$, $V_t(def, I'(r_t), x(r_t)) > V_t(def, I^*(rep, r_t), x(r_t))$ must be true given that $\frac{\partial V_t(def, x(r_t))}{\partial I} > 0$ for $I_t < I^*(def, x(r_t))$, and $V_t(rep, I^*(rep, r_t), r_t) > V_t(rep, I'(r_t), r_t) \Rightarrow V_t(def, I'(r_t), x(r_t)) > V_t(rep, I^*(rep, r_t), r_t)$, since (3.26) holds in equilibrium, hence $V_t(def, I'(r_t), x(r_t)) > V_t(rep, I'(r_t), r_t)$ must be true, which means that the bank will refuse to lend. There are also no profitable deviations through borrowing any $I'(r_t)$ and repaying. Given that the signal outcome is 'repudiate', the bank's beliefs are that default implies the pool of current borrowers.

Therefore

$V_{Bt}(def, x(r_t)) = P_t(x(r_t))\beta\pi V_{Bt+1} + (1 - P_t(x(r_t)))\mu Y(I_t))I_t$

$V_{Bt}(rep, r_t) = (Y(I_t) - r_t)I_t + P_t(x(r_t))\beta\pi V_{Bt+1}$

$V_{At}(def, x(r_t)) = P_t(x(r_t))\beta V_{At+1} + (1 - P_t(x(r_t)))\mu Y(I_t))I_t$

$V_{At}(rep, r_t) = (Y(I_t) - r_t)I_t + P_t(x(r_t))\beta V_{At+1}$, given that if the bank liquidates, it will re-lend to defaulters, and if the bank renegotiates, it will lose its reputation, hence the credit market closes and there are no future borrowing opportunities. Then from (3.26), and given that the signal outcome is 'repudiate', $V_t(def, I^*(rep, r_t), x(r_t)) > V_t(rep, I^*(rep, r_t), r_t) > V_t(rep, I'(r_t), r_t)$, hence there are no gains if borrowers repay when the signal outcome is 'repudiate'.

If the signal outcome is 'repay', borrowers have no incentive to borrow $I'(r_t) \leq I^*(rep, r_t)$ and default, given that $x(r_t)$ is determined such that (3.26) holds, and $\frac{\partial V_t(def, x(r_t))}{\partial I} > 0$ for $I_t < I^*(def, x(r_t))$, hence

$V_t(rep, I^*(rep, r_t), r_t) \geq V_t(def, I'(r_t), x(r_t))$. Borrowers cannot deviate by borrowing $I'(r_t) > I^*(rep, r_t)$, since as shown above, $V_t(def, I'(r_t), x(r_t)) > V_t(rep, I'(r_t), r_t)$, hence the bank will not lend.

Q.E.D.

Proof of lemma 7

We prove by induction, proving first for the case $m = 1$. We begin by proving that

$$\left(\theta_{T-(g+h-1)} r^*_{T-(g+h-1)} - 1\right) I^*_{T-(g+h-1)} - \left(\theta_{T-(g+h)} r^*_{T-(g+h)} - 1\right) I^*_{T-(g+h)} >$$

$$(1 - \mu)\left(Y(I^*_{T-(g+h)}) I^*_{T-(g+h)} - Y(I^*_{T-(g+h+1)}) I^*_{T-(g+h+1)}\right) \tag{3.27}$$

We need to derive the change in

$$\left(\theta_{t+1} r^*_{t+1} - 1\right) I^*_{t+1} - (1 - \mu) Y(I^*_t) I^*_t$$

for a given change in θ_t and θ_{t+1}, which is given by

$$\left(\left(\theta_{t+1}r^*_{t+1}-1\right)\frac{\partial I^*_{t+1}}{\partial\theta_{t+1}}+I^*_{t+1}\frac{\partial(\theta^*_{t+1}r^*_{t+1})}{\partial\theta_{t+1}}\right)(\theta_{t+1}-\theta_t)-$$
$$(1-\mu)\left(Y(I^*_t)\frac{\partial I^*_t}{\partial\theta_t}+I^*_t\frac{\partial Y(I^*_t)}{\partial\theta_t}\right)(\theta_t-\theta_{t-1})$$

Given that $\frac{\partial I^*}{\partial\theta}=\frac{1}{4b}\left(\frac{1}{\theta^2}\right)$, $\frac{\partial(\theta^*_{t+1}r^*_{t+1})}{\partial\theta_{t+1}}>0$, $\frac{\partial Y(I^*_t)}{\partial\theta_t}<0$, then

$$\left(\theta_{t+1}r^*_{t+1}-1\right)\theta_t^2\left(\theta_{t+1}-\theta_t\right)>(1-\mu)Y(I^*_t)\theta^2_{t+1}\left(\theta_t-\theta_{t-1}\right)\Rightarrow$$

$$\left(\left(\theta_{t+1}r^*_{t+1}-1\right)\frac{\partial I^*_{t+1}}{\partial\theta_{t+1}}+I^*_{t+1}\frac{\partial(\theta^*_{t+1}r^*_{t+1})}{\partial\theta_{t+1}}\right)(\theta_{t+1}-\theta_t)-$$
$$(1-\mu)\left(Y(I^*_t)\frac{\partial I^*_t}{\partial\theta_t}+I^*_t\frac{\partial Y(I^*_t)}{\partial\theta_t}\right)(\theta_t-\theta_{t-1})>0$$

Hence if (2.22) holds then (3.27) holds to a first order approximation. When $m=1$, (2.20) – (2.21) gives $q(\hat{x}_{T-g-h})-q(z_{T-g-h-1})+\left(\theta_{T-(g+h-1)}r^*_{T-(g+h-1)}-1\right)I^*_{T-(g+h-1)}$ $-\left(\theta_{T-(g+h)}r^*_{T-(g+h)}-1\right)I^*_{T-(g+h)}-(1-\mu)(Y(I^*_{T-(g+h)})I^*_{T-(g+h)}-$ $Y(I^*_{T-(g+h+1)})I^*_{T-(g+h+1)})+\sum_{i=g}^{g+h-2}\left(\theta_{T-i}r^*_{T-i}-1\right)I^*_{T-i}-\sum_{i=g+1}^{g+h-1}\left(\theta_{T-i}r^*_{T-i}-1\right)I^*_{T-i}=0$. From (3.27), and given that for any $t''>t'$, $\left(\theta_{t''}r^*_{t''}-1\right)I^*_{t''}>\left(\theta_{t'}r^*_{t'}-1\right)I^*_{t'}$, it must be true that $q(\hat{x}_{T-g-h})<q(z_{T-g-h-1})\Leftrightarrow z_{T-g-h-1}>\hat{x}_{T-g-h}$.

We now prove for the case $m=m'+1$, given that $z_{T-g-h-m'}>\hat{x}_{T-g-h}$ holds. $z_{T-g-h-m'}$ and $z_{T-g-h-(m'+1)}$ are determined such that the following hold

$$q(z_{T-g-h-m'})+\sum_{i=g+m'}^{g+h+m'-1}\left(\theta_{T-i}r^*_{T-i}-1\right)I^*_{T-i}=(1-\mu)Y(I^*_{T-g-h-m'})I^*_{T-g-h-m'}\quad(3.28)$$

$$q(z_{T-g-h-(m'+1)})+\sum_{i=g+m'+1}^{g+h+m'}\left(\theta_{T-i}r^*_{T-i}-1\right)I^*_{T-i}=(1-\mu)Y(I^*_{T-g-h-(m'+1)})I^*_{T-g-h-(m'+1)}\quad(3.29)$$

Given that

$\left(\theta_{T-(g+h+m'-1)} r^*_{T-(g+h+m'-1)} - 1\right) I^*_{T-(g+h+m'-1)} -$
$\left(\theta_{T-(g+h+m')} r^*_{T-(g+h+m')} - 1\right) I^*_{T-(g+h+m')} >$
$(1-\mu)\left(Y(I^*_{T-(g+h+m')}) I^*_{T-(g+h+m')} - Y(I^*_{T-(g+h+m'+1)}) I^*_{T-(g+h+m'+1)}\right)$ and that for any $t'' > t'$, $(\theta_{t''} r^*_{t''} - 1) I^*_{t''} > (\theta_{t'} r^*_{t'} - 1) I^*_{t'}$, (3.28) – (3.29) implies that $q(z_{T-g-h-(m'+1)}) < q(z_{T-g-h-m'}) \Leftrightarrow z_{T-g-h-(m'+1)} > z_{T-g-h-m'} > \hat{x}_{T-g-h-m'}$.

Q.E.D.

Proof of proposition 4

We begin by proving that $[\hat{x}_{T-i}]_{i=j_1}^{j_1+k_1} > x^*_{T-k_1}$ cannot be an equilibrium. We know that $\hat{x}_{T-j_1} > x^*_{T-k_1}$ cannot be an equilibrium, given that with reputation $\hat{x}_{T-j_1}$ the bank proceeds to date T without facing any further defaults, hence $V_{MT-j_1+1}(\hat{x}_{T-j_1}, \hat{x}_{T-j_1}) = \sum_{i=0}^{j_1-1} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i}$. But from the definition of $x^{*\max}_{Z1}$, $z_{T-j_1} < x^{*\max}_{Z1}$ where z_{T-j_1} solves the following

$$q(z_{T-j_1}) + \sum_{i=0}^{j_1-1} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i} = (1-\mu) Y(I^*_{T-j_1}) I^*_{T-j_1}$$

Hence $\hat{x}_{T-j_1} > x^{*\max}_{Z1}$ implies

$$q(\hat{x}_{T-j_1}) + V_{MT-j_1+1}(\hat{x}_{T-j_1}, \hat{x}_{T-j_1}) > (1-\mu) Y(I^*_{T-j_1}) I^*_{T-j_1}$$

thus $\hat{x}_{T-j_1} > x^{*\max}_{Z1}$ cannot be an equilibrium. We now prove that $\hat{x}_{T-j_1-n} > x^{*\max}_{Z1}$ cannot be an equilibrium, for $n = 1, 2 \ldots k_1$. We have $\max\left[x^*_{T-i}\right]_{i=0}^{j_1-1+n} = x^{*\max}_{Z1}$ given that in equilibrium $\left[\hat{x}_{T-(j_1-1+i)}\right]_{i=1}^{n} = x^{*\max}_{Z1}$, hence $\left[x^*_{T-(j_1-1+i)}\right]_{i=1}^{n} < x^{*\max}_{Z1}$. With reputation $\hat{x}_{T-j_1-n} > x^{*\max}_{Z1}$ the bank proceeds to date T without facing any further repudiation, hence $V_{MT-j_1-n+1}(\hat{x}_{T-j_1-n}, \hat{x}_{T-j_1-n}) = \sum_{i=0}^{j_1+n-1} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i}$. Given the definition of z_{T-j_1}, and given that $(1-\mu) Y(I^*_{T-j_1-n}) I^*_{T-j_1-n} < (1-\mu) Y(I^*_{T-j_1}) I^*_{T-j_1}$, $\hat{x}_{T-j_1-n} > x^{*\max}_{Z1}$ implies

$$q(\hat{x}_{T-j_1-n}) + V_{MT-j_1-n+1}(\hat{x}_{T-j_1-n}, \hat{x}_{T-j_1-n}) > (1-\mu) Y(I^*_{T-j_1-n}) I^*_{T-j_1-n}$$

thus $\hat{x}_{T-j_1-n} > x^{*\max}_{Z1}$ cannot be an equilibrium for $n = 1, 2...k_1$.

We now prove that $[\hat{x}_{T-i}]_{i=j_1}^{j_1+k_1} < x^*_{T-k_1}$ cannot be an equilibrium. Suppose $j_1 = k_1+1$. Then if $\hat{x}_{T-j_1} < x^*_{T-k_1}$, it would be optimal for type $w = \hat{x}_{T-j_1}$ to drop out on date $T-k_1$, hence $V_{MT-j_1+1}(\hat{x}_{T-j_1}, \hat{x}_{T-j_1}) = 0$. Given that

$$q(\hat{x}_{T-j_1}) + 0 < (1-\mu)Y(I^*_{T-j_1})I^*_{T-j_1}$$

$\hat{x}_{T-j_1} < x^*_{T-k_1}$ cannot be an equilibrium. Now suppose $j_1 > k_1 + 1$ and $[\hat{x}_{T-i}]_{i=j_1}^{j_1+k_1} < x^*_{T-k_1}$. If bank type $\hat{x}_{T-j_1}$ liquidates on date $T - j_1$, it could continue to lend without encountering further default for no more than $j_1 - k_1 - 1$ periods, after which on date $T - k_1$, it will drop out given that $\hat{x}_{T-j_1} < x^*_{T-k_1}$, hence $V_{MT-j_1+1}(\hat{x}_{T-j_1}, \hat{x}_{T-j_1}) \leq \sum\limits_{i=k_1+1}^{j_1-1} \left(\theta_{T-i}r^*_{T-i} - 1\right) I^*_{T-i}$, where $\sum\limits_{i=k_1+1}^{j_1-1} \left(\theta_{T-i}r^*_{T-i} - 1\right) I^*_{T-i}$ is the future pay-off available to the bank with type $\hat{x}_{T-j_1}$ and reputation $\hat{x}_{T-j_1}$ if borrowers repay with probability 1 on dates $T - j_1 + 1 \leq t \leq T - k_1 - 1$. Similarly for $n = 1, 2, \ldots k_1$, after liquidation on date $T - (j_1 + n)$, bank type $\hat{x}_{T-(j_1+n)}$ could continue for no more than $j_1 - k_1 - 1 + n$ periods before dropping out on date $T - k_1$, hence $V_{MT-(j_1+n)+1}(\hat{x}_{T-(j_1+n)}, \hat{x}_{T-(j_1+n)}) \leq \sum\limits_{i=k_1+1}^{j_1-1+n} \left(\theta_{T-i}r^*_{T-i} - 1\right) I^*_{T-i}$. Now, suppose for some $p > j_1$ there exists $z_{T-p} < x^*_{T-k_1}$ such that the following holds

$$q(z_{T-p}) + \sum_{i=k_1+1}^{p-1} \left(\theta_{T-i}r^*_{T-i} - 1\right) I^*_{T-i} = (1-\mu)Y(I^*_{T-p})I^*_{T-p}$$

We now prove that it cannot be true that $p \leq j_1 + k_1$. Suppose $p \leq j_1 + k_1$. Then z_{T-p} solves

$$q(z_{T-p}) + V_{MT-p+1}(z_{T-p}, z_{T-p}) = (1-\mu)Y(I^*_{T-p})I^*_{T-p}$$

where $V_{MT-p+1}(z_{T-p}, z_{T-p}) \leq \sum\limits_{i=k_1+1}^{p-1} \left(\theta_{T-i}r^*_{T-i} - 1\right) I^*_{T-i}$, as explained above. From the definition of $x^{*\max}_{Z1}$, we know that there exists $\hat{x}_{T-n} > x^{*\max}_{Z1}$ for $n = 1, 2...j_1 - 1$ which satisfy

$$q(\hat{x}_{T-n}) + V_{MT-n+1}(\hat{x}_{T-n}, \hat{x}_{T-n}) = (1-\mu)Y(I^*_{T-n})I^*_{T-n}$$

where $V_{MT-n+1}(\hat{x}_{T-n}, \hat{x}_{T-n}) = \sum_{i=0}^{n-1} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i}$. Hence for $p = j_1+1, j_1+2...j_1+k_1$ there exists a corresponding $n = p - j_1$ such that

$$V_{MT-p+1}(z_{T-p}, z_{T-p}) \leq \sum_{i=k_1+1}^{p-1} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i} < V_{MT-n+1}(\hat{x}_{T-n}, \hat{x}_{T-n})$$

Hence from lemma 7, $z_{T-p} > \hat{x}_{T-n} > x^{*\max}_{Z1}$, which contradicts the original assertion that $z_{T-p} < x^{*\max}_{Z1}$. Therefore $[\hat{x}_{T-i}]_{i=j_1}^{j_1+k_1} < x^{*\max}_{Z1}$ cannot be an equilibrium.

We now prove that $[\hat{x}_{T-i}]_{i=j_1}^{j_1+k_1} = x^{*\max}_{Z1}$ is an equilibrium. For equilibrium to hold for $n = 0, 1, 2, \ldots k_1$, borrowers must randomize on date $T - k_1$ such that type $w = x^{*\max}_{Z1}$ is indifferent between liquidation and renegotiation if borrowers default on date $T - j_1 - n$. Hence borrowers must default on date $T - k_1$ with probability $\alpha^{T-j_1-n}_{T-k_1}$ which is determined such that the following holds

$$q(\hat{x}_{T-j_1-n}) + \sum_{i=k_1+1}^{j_1-1+n} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i} + V_{MT-k_1}(\hat{x}_{T-j_1-n}, \hat{x}_{T-j_1-n}) =$$

$$(1-\mu)Y(I^*_{T-j_1-n})I^*_{T-j_1-n} \tag{3.30}$$

where
$V_{MT-k_1}(\hat{x}_{T-j_1-n}, \hat{x}_{T-j_1-n}) = (1 - \alpha^{T-j_1-n}_{T-k_1}) \left[\sum_{i=0}^{k_1} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i}\right]$
$-\alpha^{T-j_1-n}_{T-k_1} \left[1 - (1-\mu)Y(I^*_{T-k_1})I^*_{T-k_1}\right]$ and $V_{MT-k_1}(\hat{x}_{T-j_1-n}, \hat{x}_{T-j_1-n}) > 0$. We know that there must exist $0 < \alpha^{T-j_1-n}_{T-k_1} < 1$, given that $\alpha^{T-j_1-n}_{T-k_1} = 0$ implies that

$$q(\hat{x}_{T-j_1-n}) + V_{MT-j_1-n+1}(\hat{x}_{T-j_1-n}, \hat{x}_{T-j_1-n}) > (1-\mu)Y(I^*_{T-j_1-n})I^*_{T-j_1-n}$$

given that from the definition of Zone 1, $z_{T-j_1} < x^{*\max}_{Z1}$ where z_{T-j_1} solves the following

$$q(z_{T-j_1}) + \sum_{i=0}^{j_1-1} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i} = (1-\mu)Y(I^*_{T-j_1})I^*_{T-j_1}$$

and $\alpha_{T-k_1}^{T-j_1-n} = 1$ implies that

$$q(\hat{x}_{T-j_1-n}) + V_{MT-j_1-n+1}(\hat{x}_{T-j_1-n}, \hat{x}_{T-j_1-n}) < (1-\mu)Y(I^*_{T-j_1-n})I^*_{T-j_1-n}$$

given that from the proof above that $[\hat{x}_{T-i}]_{i=j_1}^{j_1+k_1} < x_{Z1}^{*\max}$ cannot be an equilibrium, there exists $z_{T-j_1-n} > x_{Z1}^{*\max}$ which solves

$$q(z_{T-j_1-n}) + \sum_{i=k_1+1}^{j_1-1+n} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i} = (1-\mu)Y(I^*_{T-j_1-n})I^*_{T-j_1-n}$$

hence

$$q(x_{Z1}^{*\max}) + \sum_{i=k_1+1}^{j_1-1+n} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i} < (1-\mu)Y(I^*_{T-j_1-n})I^*_{T-j_1-n} \tag{3.31}$$

Since $V_{MT-k_1}(\hat{x}_{T-j_1-n}, \hat{x}_{T-j_1-n})$ is continuous and decreasing in α_{T-k_1}, there exists $0 < \alpha_{T-k_1}^{T-j_1-n} < 1$ such that (3.30) holds, and from (3.31) we know that $V_{MT-k_1}(\hat{x}_{T-j_1-n}, \hat{x}_{T-j_1-n}) > 0$ must hold.

The revision of reputation if the bank liquidates on date $T - j_1 - n$ is consistent with the bank's equilibrium strategies, given that

$$q(w) + V_{MT-j_1-n+1}(w, \hat{x}_{T-j_1-n}) \gtreqless (1-\mu)Y(I^*_{T-j_1-n})I^*_{T-j_1-n}$$

as $w \gtreqless \hat{x}_{T-j_1-n}$. Hence borrowers' belief that liquidation on date $T - j_1 - n$ implies that bank type must be $w \geq \hat{x}_{T-j_1-n}$ is consistent with the bank's equilibrium strategy, since only bank types $w \geq \hat{x}_{T-j_1-n}$ find it optimal to liquidate. Finally, borrowers' belief that if the bank lends on date $T - k_1$ then this implies reputation is unchanged at $\hat{x}_{T-j_1-n}$ is consistent with the bank's equilibrium strategy, since it is optimal for all bank types $w \geq \hat{x}_{T-j_1-n}$ to lend on date $T - k_1$ given that $V_{MT-k_1}(\hat{x}_{T-j_1-n}, \hat{x}_{T-j_1-n}) > 0$ as shown above. Hence $x_{T-k_1} = x^*_{T-k_1}$ which means randomization is an optimal strategy for borrowers. Note that borrowers' equilibrium strategy on date $T - k_1$ is a function of both

x_{T-k_1} and the liquidation outcome on dates $T-(j_1+k_1) \leq t \leq T-j_1$. If the bank's post-entry reputation is $x^*_{T-k_1}$ and if borrowers default on date $T-j_1-n$ and the bank liquidates, borrowers' strategy is to default on date $T-k_1$ with probability $\alpha^{T-j_1-n}_{T-k_1}$. If the bank's reputation after entering date $T-k_1$ is $x^*_{T-k_1}$ and there is no default on dates $T-(j_1+k_1) \leq t \leq T-j_1$, borrowers' strategy is to default on date $T-k_1$ with probability $\alpha^*_{T-k_1}$, as defined above.

Q.E.D.

Proof of lemma 9

We begin by proving that $\hat{x}_{T-t''} > x^*_{T-k_{t-1}}$ where $x^*_{T-k_{\tau-1}} \equiv x^{*\max}_{z\tau-1} = \max\left[x^*_{\tilde{t}} \mid \tilde{t} \in Zone\ \tau-1\right]$. For $T-t'' \in Zone$ 1, we know that $\hat{x}_{T-t''} > x^{*\max}_{z1}$ from the definition of $x^{*\max}_{z1}$. For zones $\tau-1>1$, first we consider $t''>k_{\tau-1}$. Suppose $\hat{x}_{T-t''} \leq x^*_{T-k_{\tau-1}}$. For $\hat{x}_{T-t''}$ to be an equilibrium, it must be true that for some $T-t''' \in Zone\ \tau-2$, there exists $z_{T-t''}$ which solves the following

$$q(z_{T-t''}) + \sum_{i=t'''+1}^{t''-1} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i} = (1-\mu) Y(I_{T-t''}) I_{T-t''}$$

where $z_{T-t''} < x^*_{T-k_{\tau-1}}$ and $x^*_{T-k_{\tau-1}} = \max\left[x^*_{T-i}\right]^{t''-1}_{i=t'''+1}$. But from proposition 5, $z_{T-t''} < x^*_{T-k_{\tau-1}}$ implies that $T-t'' \in Zone\ \tau$. Hence it must be true that $\hat{x}_{T-t''} > x^*_{T-k_{\tau-1}}$. Now we consider $t'' \leq k_{\tau-1}$. We know from (2.8) that $\hat{x}_{T-k_{\tau-1}} > x^*_{T-k_{\tau-1}}$. From proposition 5, $T-k_{\tau-1} \in Zone\ \tau-1$ implies that either of the following must hold:

1) $\hat{x}_{T-k_{\tau-1}} = x^*_{T-k_{\tau-2}-b}$ where $b \geq 0$ and $T-k_{\tau-2}-b \in Zone\ \tau-2$, or

2) $\hat{x}_{T-k_{\tau-1}} = z_{T-k_{\tau-1}}$ where $z_{T-k_{\tau-1}}$ solves the following

$$q(z_{T-k_{\tau-1}}) + \sum_{i=k_{\tau-1}+b+1}^{k_{\tau-1}-1} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i} = (1-\mu) Y(I_{T-k_{\tau-1}}) I_{T-k_{\tau-1}} \tag{3.32}$$

where $b \geq 0$ and $T-k_{\tau-2}-b \in Zone\ \tau-2$.

If (1) is true, then the following must hold

$$q(x^*_{T-k_{\tau-2}-b}) + V_{MT-k_{\tau-1}+1}(x^*_{T-k_{\tau-2}-b}, x^*_{T-k_{\tau-2}-b}) = (1-\mu)Y(I_{T-k_{\tau-1}})I_{T-k_{\tau-1}} \quad (3.33)$$

where $V_{MT-k_{\tau-1}+1}(x^*_{T-k_{\tau-2}-b}, x^*_{T-k_{\tau-2}-b}) = \sum_{i=k_{\tau-2}+b+1}^{k_{\tau-1}-1} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i} +$
$(1-\alpha^{T-k_{\tau-1}}_{T-k_{\tau-2}-b})(V_{MT-k_{\tau-2}-b}(x^*_{T-k_{\tau-2}-b}, x^*_{T-k_{\tau-2}-b}))$
$-\alpha^{T-k_{\tau-1}}_{T-k_{\tau-2}-b}\left[1-(1-\mu)Y(I^*_{T-k_{\tau-2}-b})I^*_{T-k_{\tau-2}-b}\right]$ and $V_{MT-k_{\tau-2}-b}(x^*_{T-k_{\tau-2}-b}, x^*_{T-k_{\tau-2}-b}) >$
0. Now consider $\hat{x}_{T-k_{\tau-1}+1}$. Suppose $\hat{x}_{T-k_{\tau-1}+1} < x^*_{T-k_{\tau-2}-b}$. Then
$V_{MT-k_{\tau-1}+2}(\hat{x}_{T-k_{\tau-1}+1}, \hat{x}_{T-k_{\tau-1}+1}) = \sum_{i=k_{\tau-2}+b+1}^{k_{\tau-1}-2} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i}$ since in equilibrium $\hat{x}_{T-k_{\tau-1}+1} < x^*_{T-k_{\tau-2}-b}$ implies that type $w = \hat{x}_{T-k_{\tau-1}+1}$ must drop out on date $T-k_{\tau-2}-b$. But given (3.33) and given that $Y(I_{T-k_{\tau-1}})I_{T-k_{\tau-1}} < Y(I_{T-k_{\tau-1}+1})I_{T-k_{\tau-1}+1}$, this implies that

$$q(\hat{x}_{T-k_{\tau-1}+1}) + V_{MT-k_{\tau-1}+2}(\hat{x}_{T-k_{\tau-1}+1}, \hat{x}_{T-k_{\tau-1}+1}) < (1-\mu)Y(I_{T-k_{\tau-1}+1})I_{T-k_{\tau-1}+1}$$

hence $\hat{x}_{T-k_{\tau-1}+1} < x^*_{T-k_{\tau-2}-b}$ cannot be an equilibrium, which means that $\hat{x}_{T-k_{\tau-1}+1} \geq \hat{x}_{T-k_{\tau-1}} > x^*_{T-k_{t-1}}$

If (2) holds, then $\hat{x}_{T-k_{\tau-1}} < x^*_{T-k_{\tau-2}-b}$, hence from the solution algorithm $\hat{x}_{T-k_{\tau-1}+1} \leq x^*_{T-k_{\tau-2}-b}$. If $\hat{x}_{T-k_{\tau-1}+1} = x^*_{T-k_{\tau-2}-b}$, then $\hat{x}_{T-k_{\tau-1}+1} > \hat{x}_{T-k_{\tau-1}} > x^*_{T-k_{t-1}}$. If $\hat{x}_{T-k_{\tau-1}+1} < x^*_{T-k_{\tau-2}-b}$, then it must be true that the following holds

$$q(\hat{x}_{T-k_{\tau-1}+1}) + \sum_{i=k_{\tau-1}+b+1}^{k_{\tau-1}-2} \left(\theta_{T-i} r^*_{T-i} - 1\right) I^*_{T-i} = (1-\mu)Y(I_{T-k_{\tau-1}+1})I_{T-k_{\tau-1}+1}$$

Hence from (3.32) and given that $Y(I_{T-k_{\tau-1}+1})I_{T-k_{\tau-1}+1} > Y(I_{T-k_{\tau-1}})I_{T-k_{\tau-1}}$, it must be true that $\hat{x}_{T-k_{\tau-1}+1} > \hat{x}_{T-k_{\tau-1}} > x^*_{T-k_{t-1}}$. We have thus proved that $\hat{x}_{T-k_{\tau-1}+1} > x^*_{T-k_{t-1}}$. By recursion on the process above, the proof follows that $\hat{x}_{T-k_{\tau-1}+i} > x^*_{T-k_{t-1}}$ for all $T-k_{\tau-1}+i \in Zone\ \tau-1$.

We must now prove that $\hat{x}_{T-t'} \leq x^*_{T-k_{\tau-1}}$. We have $[\hat{x}_{T-i}]_{i=k_{\tau-2}+\sum_{y=1}^{\tau-1} j_y+n_{\tau-2}}^{k_{\tau-1}+\sum_{y=1}^{\tau-1} j_y-1} = x^*_{T-k_{\tau-1}}$

(the proof is the same as the proof of proposition 4). For $[\hat{x}_{T-i}]_{i=k_{\tau-1}+\sum_{y=1}^{\tau-1}j_y}^{k_{\tau-1}+\sum_{y=1}^{\tau}j_y+n_{\tau-1}-1}$, from the algorithm used to solve for Zone 2, we know that either the condition in step (4) or step (5) fails to hold. If step (4) fails to hold, then we set $\hat{x}_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1}j_y+a+b)} = z_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1}j_y+a+b)}$ where $z_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1}j_y+a+b)}$ solves

$$q(z_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1}j_y+a+b)}) + \sum_{i=k_{\tau-1}+b+1}^{k_{\tau-1}+\sum_{y=1}^{\tau-1}j_y+a+b-1} \left(\theta_{T-i}r_{T-i}^{*} - 1\right) I_{T-i}^{*} =$$
$$(1-\mu)Y(I_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1}j_y+a+b)})I_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1}j_y+a+b)}$$

The proof that $z_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1}j_y+a+b)} < x_{T-(k_{\tau-1}+b)}^{*}$ is the same as the proof above that $z_{T-(j_1+k_1+a+b)} < x_{T-k_1-b}^{*}$. Given that $x_{T-(k_{\tau-1}+b)}^{*} \leq x_{T-k_{t-1}}^{*}$ for $b \geq 0$, since $x_{T-k_{t-1}}^{*} = \max\left[x_{T-i}^{*} \mid x_{T-i}^{*} \in Zone\ \tau-1\right]$, we have $\hat{x}_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1}j_y+a+b)} < x_{T-k_{t-1}}^{*}$. If step (5) fails to hold, we set $\hat{x}_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1}j_y+a+b)} = \max\left[x_{T-i}^{*}\right]_{i=k_{\tau-1}+b-1}^{k_{\tau-1}+\sum_{y=1}^{\tau-1}j_y+a+b-1}$, where $\max\left[x_{T-i}^{*}\right]_{i=k_{\tau-1}+b-1}^{k_{\tau-1}+\sum_{y=1}^{\tau-1}j_y+a+b-1} < x_{z\tau-1}^{*\max}$ by definition, hence $\hat{x}_{T-(k_{\tau-1}+\sum_{y=1}^{\tau-1}j_y+a+b)} < x_{T-k_{t-1}}^{*}$.

Q.E.D.

Proof of proposition 6

First we prove that for all zones $\tau < \zeta$, the occurrence of more than one repudiation within each zone is a zero probability event. Suppose the first repudiation in zone τ occurs on date t'. If the bank liquidates, its ex post reputation must be greater than or equal to $\hat{x}_{t'}$. Given that from lemma 9 $\hat{x}_{t'} > x_{z\tau}^{*\max}$, borrowers will not default on any further date in zone τ. Next we prove that the event of exactly one repudiation occurs with positive probability. First we show that if the bank enters Zone τ with reputation $x \leq x_{z\tau}^{*\max}$, then it will face repudiation with positive probability during Zone τ. If the bank has not yet experienced repudiation during zone τ, then on date $T-k_\tau$, if $x < x_{z\tau}^{*\max}$ types $w < x_{z\tau}^{*\max}$ will drop out, and thus if the bank lends its reputation will rise to $x_{z\tau}^{*\max}$, and as shown above, if $x = x_{z\tau}^{*\max}$, borrowers' equilibrium strategy is randomize, hence repudiation will occur with positive probability. We now prove by induction that the bank will enter each Zone τ with reputation $x \leq x_{z\tau}^{*\max}$, and hence will face repudiation with positive probability. First we claim that the bank will enter Zone $\tau+1$ with reputation $x \leq x_{z\tau+1}^{*\max}$. If repudiation occurs on some date t'' during Zone $\tau+1$ and the bank liquidates, then its reputation rises to $\hat{x}_{t''}$. From above, there is zero probability of further repudiation, hence the bank leaves Zone $\tau+1$ and enters Zone τ with reputation $\hat{x}_{t''}$. If there is no repudiation during Zone $\tau+1$, then the bank leaves Zone $\tau+1$ and enters Zone τ with reputation $x^{*\max}_{z\tau+1}$, as explained above. Given that from lemma 9 $x^{*\max}_{z\tau+1} < \hat{x}_{t''} \leq x^{*\max}_{z\tau}$, it must be true that the bank enters Zone τ with reputation $x \leq x^{*\max}_{z\tau}$. If $x^{*\max}_{z\zeta} > 0$, then given that the bank begins the game with reputation $x = 0$, it is true for Zone ζ that the bank enters with reputation $x \leq x^{*\max}_{z\zeta}$, and thus it is true by induction that the bank will enter every Zone τ with reputation $x \leq x^{*\max}_{z\tau}$. If, however, $x^{*\max}_{z\zeta} \leq 0$ then there is zero probability of repudiation in Zone ζ, hence the bank will enter Zone $\zeta - 1$ with reputation $x = 0$, thus once more it is true by induction that the bank will enter every Zone τ with reputation $x \leq x^{*\max}_{z\tau}$.

We now prove that in zone ζ, the occurrence of any repudiation at all is a zero probability event if and only if $x^{*\max}_{z\zeta} \leq 0$. If $x^{*\max}_{z\zeta} \leq 0$, then borrowers will prefer not to default during zone ζ even though the bank's reputation is unchanged at $x = 0$. If $x^{*\max}_{z\zeta} > 0$, then if the bank's reputation is unchanged at $x = 0$ on date $T - k_\zeta$, borrowers will randomize.

We conclude by proving that zone ζ is the only zone for which the occurrence of repudiation can be a zero probability event. Suppose there exists some zone $\tau < \zeta$ for which $x^{*\max}_{z\tau} \leq 0$, hence the occurrence of repudiation is a zero probability event. From proposition 5, zone $\tau+1$ begins on date $T-(k_{\tau-1}+\sum\limits_{y=1}^{\tau} j_y+n_{\tau-1})$ where $j\tau$ and $n_{\tau-1}$ are the smallest integers such that conditions (1) and (2) given in proposition 5 hold. However, conditions (1) and (2) require that $z_{T-(k_{\tau-1}+\sum_{y=1}^{\tau} j_y+n_{\tau-1})} < x^{*\max}_{z\tau}$, but $x^{*\max}_{z\tau} \leq 0$, hence this is impossible. Hence it must be true that $T-(k_{\tau-1}+\sum\limits_{y=1}^{\tau} j_y+n_{\tau-1}) \in \mathit{Zone}\ \tau$, and either:

1) $x^*_{T-(k_{\tau-1}+\sum\limits_{y=1}^{\tau} j_y+n_{\tau-1})} > 0$, in which case the occurrence of repudiation is not a zero probability event in *Zone* τ, or

2) $x^*_{T-(k_{\tau-1}+\sum\limits_{y=1}^{\tau} j_y+n_{\tau-1})} \leq 0$. We then iterate on the above process until either (1) holds or we reach $t = 1$, in which case $\mathit{Zone}\ \tau = \mathit{Zone}\ \zeta$.

Q.E.D.

Proof of proposition 7

To prove that $\frac{1}{1+\delta}x^*_{T-k_1}$ is the lower bound on the ex ante probability of bank failure, we must show that in any equilibrium, types $w < x^*_{T-k_1}$ will always lose their reputation before date T. From lemma 9 we know that $x^*_{T-k_1} \equiv x^{*\max}_{z1} > x^{*\max}_{z\tau}$ for all zones $\tau > 1$. Suppose the bank enters date $T - k_1$ with reputation $x \leq x^*_{T-k_1}$. In equilibrium, types $w < x^*_{T-k_1}$ will drop out, types $w \geq x^*_{T-k_1}$ will enter and their reputation rises to $x^*_{T-k_1}$, and borrowers will randomize. Hence the ex ante probability of bank failure can be no lower than $\frac{1}{1+\delta}x^*_{T-k_1}$. To prove that $\frac{1}{1+\delta}\hat{x}_{T-k_1}$ is the upper bound on the ex ante probability of bank failure, we must show that in any equilibrium, types $w \geq \hat{x}_{T-k_1}$ will always retain their reputation up to date T. From lemma **9** we know that $\hat{x}_{T-t'} < \hat{x}_{T-t''}$ where $T - t'' \in \mathit{Zone}\ 1$ and $T - t' \in \mathit{Zone}\ \tau$ for all $\tau > 1$. From the algorithm above used to solve for $x^{*\max}_{z1}$, we know that for $T - t'' \in \mathit{Zone}\ 1$, $\hat{x}_{T-t''} = z_{t''}$ where $z_{t''}$ solves the following

$$q(z_{T-t''}) + \sum_{i=0}^{t''-1} \left(\theta_{T-i}r^*_{T-i} - 1\right) I^*_{T-i} = (1-\mu)Y(I_{T-t''})I_{T-t''}$$

Hence for $T-t'' < T-k_1$, given that $\sum_{i=0}^{t''-1} \left(\theta_{T-i}r^*_{T-i} - 1\right) I^*_{T-i} > \sum_{i=0}^{k_1-1} \left(\theta_{T-i}r^*_{T-i} - 1\right) I^*_{T-i}$ and $Y(I_{T-t''})I_{T-t''} < Y(I_{T-k_1})I_{T-k_1}$, it must be true that $\hat{x}_{T-t''} < \hat{x}_{T-k_1}$. Similarly, for $T - t'' > T - k_1$ it must be true that $\hat{x}_{T-t''} > \hat{x}_{T-k_1}$. If borrowers were to default on some date $T - t'' > T - k_1$, then types $\hat{x}_{T-k_1} \leq w < \hat{x}_{T-t''}$ would fail to liquidate and would thus lose their reputation. However, this event never occurs. From proposition 6, we know that the bank enters Zone 1 with reputation $x \leq x^*_{T-k_1}$. This means that either borrowers repudiate on some date $T - t'' \leq T - k_1$ or they do not repudiate at all during zone 1. From proposition 6, if borrowers default on some date $T - t'' \leq T - k_1$ and the bank liquidates, it will face no further defaults, hence given that $\hat{x}_{T-t''} \leq \hat{x}_{T-k_1}$ for $T - t'' \leq T - k_1$, types $w \geq \hat{x}_{T-k_1}$ will retain their reputation up to date T. If borrowers do not default by date $T - k_1$, they will not default after date $T - k_1$, given that if the bank enters on date $T - k_1$ its reputation is revised to $x = x^*_{T-k_1}$ and that $x^*_{T-t''} < x^*_{T-k_1}$ for $T - t'' > T - k_1$. Hence, it is irrelevant that $\hat{x}_{T-t''} > \hat{x}_{T-k_1}$ for $T - t'' > T - k_1$ and thus $\frac{1}{1+\delta}\hat{x}_{T-k_1}$ is the upper bound on the probability of bank failure.

Q.E.D.

3.12 Bibliography

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