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TRAPPED INTERNAL WAVES PRODUCED BY A SUBMERGED SLENDER BODY MOVING IN A STRATIFIED FLUID

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Nomenclature

F_n	Froude number $= \frac{U}{\sqrt{gL}}$
g	Gravity acceleration $= (0, 0, -g)$
h_i	Vertical separation of the origin and the rigid lid
h_j	Vertical separation of the rigid lid and the j^{th} interface
$H(x)$	Heaviside unit step function
i	Layer number in which source point is located, $i = 1$ is the uppermost layer
j	Layer number in which field point is located
$\boldsymbol{k}$	Horizontal wavenumber vector $= (k_1, k_2, 0)$
k	Horizontal wavenumber scalar $= \sqrt{k_1^2 + k_2^2}$
k_1	wavenumber in the x direction
k_2	wavenumber in the y direction
L	Characteristic length, for example length of body
$\mathcal{L}$	Differential operator
N	Brunt-Väisälä frequency $= \sqrt{-\frac{g}{\rho(z)}\frac{\partial\rho}{\partial z}}$
N_i	Brunt-Väisälä frequency of i^{th} layer
$\tilde{N}$	Non-dimensional Brunt-Väisälä frequency $= \frac{NL}{U}$
$o(x)$	Landau order symbol
$O(x)$	Landau order symbol
$Oxyz$	Moving reference coordinate system located at the singularity or the body's centroid
p	Pressure distribution
$\boldsymbol{r}$	$= (r_1, r_2, r_3)$
r_1	$= x - \xi$
r_2	$= y - \eta$
r_3	$= z - \zeta$
t_1	Thickness of upper layer
t_2	Thickness of middle layer
$\boldsymbol{U}$	Translational velocity of body $= -\{U, V, W\}$
$\boldsymbol{u}$	Parametric disturbance velocity vector $= (u, v, w)$
$\boldsymbol{V}$	Disturbance velocity vector
$\boldsymbol{x}$	Position vector of the field point (x, y, z)
α_n	n_{th} root of $D(k_1, k_2) = 0$

β	$= -\frac{\tilde{N}(z \mp \zeta)}{\sqrt{r_2^2 + (z \mp \zeta)^2}} \quad (z < \zeta < 0)$
γ	$= \sqrt{1 - \frac{\tilde{N}^2}{k_1(k_1 - i\epsilon)}}$
γ_i	$= \sqrt{1 - \frac{\tilde{N}_i^2}{k_1(k_1 - i\epsilon)}}$
$\delta(x)$	Dirac delta function
δ_{ij}	Kronecker delta function
ϵ	A small parameter introduced as an artificial damping mechanism, a radiation condition is obeyed when $\epsilon \to 0$
κ_2	Imaginary part of the complex wavenumber $K_2 = k_2 + i\kappa_2$
μ	Kinematic viscosity coefficient
$\boldsymbol{\xi}$	Position vector of source point (ξ, η, ζ)
ρ	Density stratification of fluid medium
ϕ	Function describing the fluid disturbance created by the source singularity
Φ	Function describing the fluid disturbance created by the prolate spheroid
ψ	Function describing the fluid disturbance created by the source singularity
Ψ	Function describing the fluid disturbance created by the prolate spheroid
$\frac{D}{Dt}$	Total derivative $= \frac{\partial}{\partial t} + \boldsymbol{V} \cdot \boldsymbol{\nabla}$
$\boldsymbol{\nabla}$	$= (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$
$\boldsymbol{\nabla}_h$	$= (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0)$

Abstract

Analytical solutions are obtained for the disturbance generated by a singularity moving horizontally in a layer of a three layer fluid, each layer possessing a constant Brunt-Väisälä frequency. A radiation condition is enforced using an artificial damping mechanism. The singularity solution is developed into a continuous source/sink line distribution which is used to model a prolate spheroid. The disturbance velocity field generated by the body displays the characteristics associated with the propagation of trapped internal waves. The disturbance calculated on the fluid's surface is compared with those obtained using a constant density three layer fluid model and a constant Brunt-Väisälä frequency model. The patterns produced by the current model described herein display significant departures from previous patterns.

1 Introduction

The vertical density profile present in most of the worlds oceans provides a mechanism through which internal waves may propagate. These waves have a very low frequency and do not decay rapidly like surface waves. These properties may permit the detection of a deeply submerged body moving in the density profile. Due to the presence of the density profile the fluid has rotational characteristics and therefore may no longer be described using a single potential flow function. It is necessary to model (simultaneously) the rotational characteristics of the fluid using an appropriate density profile. Mathematical models exist for several different density approximations. The simplest modelled density profiles are those created using constant density fluid layers, see Price and Westlake(1993). Although the rotational property of the fluid is absent the layered models provide an approximation to the disturbance created by the body via the interactive processes occurring between the body, the free surface and the interface(s). The rotational aspect of the fluid can be included if the fluid posesses a constant Brunt-Väisälä frequency, see Price and Westlake(1994a, 1994b). However, although a true internal wave is now being generated the wave is not *trapped* in the fluid and can propagate freely behind the body. This report details the theory used to model true trapped body generated internal waves using a three layer fluid. Each possesses a constant Brunt-Väisälä frequency. The rotational aspect of the fluid is retained though it remains incompressible. The continuous density profile may be represented fairly accurately by the three layer constant Brunt-Väisälä frequency model. It is shown that an internal wave may be trapped in the middle layer, that is, the middle layer acts as a waveguide.

2 Formulation of the fundamental solution

2.1 The equations of motion

It is assumed that the fluid structure interaction experienced by a rigid, arbitrary shaped body moving in a prescribed density stratified fluid can be described with reference to a body fixed coordinate system. For generality this moves with a translational velocity $-\boldsymbol{U}(t) = -\{U(t), V(t), W(t)\}$ and the equations of motion describing the velocity of the fluid disturbance $\boldsymbol{V}(t)$ in a stratified fluid with viscosity μ and density ρ are of the form (see Batchelor(1967), page 147)

Equation of momentum

$$\rho\frac{D\boldsymbol{V}}{Dt} + \epsilon\boldsymbol{V} = \rho\left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}\right] + \epsilon\boldsymbol{V} = -\boldsymbol{\nabla}p + \rho\boldsymbol{g} + \boldsymbol{\nabla}^2(\mu\boldsymbol{V})$$

$$-\boldsymbol{V}\boldsymbol{\nabla}^2\mu - (\boldsymbol{\nabla}\times\boldsymbol{V})\times\boldsymbol{\nabla}\mu + \frac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{V}) + \frac{2}{3}(\boldsymbol{\nabla}\cdot\boldsymbol{V})\boldsymbol{\nabla}\mu + \rho\dot{\boldsymbol{U}} \tag{1}$$

Here p denotes the pressure in the fluid, $\boldsymbol{g} = (0, 0, -g)$ and the variable μ denotes the kinematic viscosity coefficient. ϵ is an artificial damping coefficient which is used to enforce the radiation condition (e.g. equivalent to Rayleigh damping). An overdot denotes an acceleration and $\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$.

Equation of Conservation of Mass or Continuity

$$\frac{D\rho}{Dt} + \rho\boldsymbol{\nabla}\cdot\boldsymbol{V} = \frac{\partial\rho}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\rho + \rho\boldsymbol{\nabla}\cdot\boldsymbol{V} = 0 \tag{2}$$

It is assumed that the fluid is incompressible and no heat transfer occurs, that is

$$\boldsymbol{\nabla}\cdot\boldsymbol{V} = 0 \tag{3}$$

The substitution of equation 3 into equation 1 gives

$$\rho\frac{D\boldsymbol{V}}{Dt} + \epsilon\boldsymbol{V} = -\boldsymbol{\nabla}p + \rho\boldsymbol{g} + \boldsymbol{\nabla}^2(\mu\boldsymbol{V}) - \boldsymbol{V}\boldsymbol{\nabla}^2\mu - (\boldsymbol{\nabla}\times\boldsymbol{V})\times\boldsymbol{\nabla}\mu + \rho\dot{\boldsymbol{U}} \tag{4}$$

and its substitution into equation 2 gives

$$\frac{\partial\rho}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\rho = 0 \tag{5}$$

Let us assume that the variables describing the fluid structure interaction can be expressed in the form

$$\begin{array}{rcccc}
p(x,y,z,t) & = & p_0(z,t) & + & p_1(x,y,z,t) \\
\rho(x,y,z,t) & = & \rho_0(z,t) & + & \rho_1(x,y,z,t) \\
\mu(x,y,z,t) & = & \mu_0(z,t) & + & \mu_1(x,y,z,t) \\
\boldsymbol{V}(x,y,z,t) & = & \boldsymbol{U}(t) & + & \boldsymbol{u}(x,y,z,t)
\end{array}$$

where p_1, ρ_1 and μ_1 and $|\boldsymbol{u}|$ are all small quantities compared to p_0, ρ_0 and μ_0 and $|\boldsymbol{U}|$ respectively. Under these assumptions the equation of momentum, equation 4, describing the parametric disturbances becomes

$$(\rho_0+\rho_1)\frac{D(\boldsymbol{U}+\boldsymbol{u})}{Dt} + \epsilon(\boldsymbol{U}+\boldsymbol{u}) = -\boldsymbol{\nabla}(p_0+p_1) + (\rho_0+\rho_1)\boldsymbol{g} + (\rho_0+\rho_1)\dot{\boldsymbol{U}}$$

$$+\boldsymbol{\nabla}^2\left[(\mu_0+\mu_1)(\boldsymbol{U}+\boldsymbol{u})\right] - (\boldsymbol{U}+\boldsymbol{u})\boldsymbol{\nabla}^2(\mu_0+\mu_1) - \left[\boldsymbol{\nabla}\times(\boldsymbol{U}+\boldsymbol{u})\right]\times\boldsymbol{\nabla}(\mu_0+\mu_1)$$

from which the first order terms produce

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} + \epsilon \boldsymbol{u} = \rho_0 \left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{U}\cdot\boldsymbol{\nabla})\boldsymbol{u}\right] + \epsilon \boldsymbol{u} = -\boldsymbol{\nabla} p_1 + \rho_1 \boldsymbol{g} + \boldsymbol{\nabla}^2(\mu_0 \boldsymbol{u}) - \boldsymbol{u}\boldsymbol{\nabla}^2 \mu_0 - (\boldsymbol{\nabla}\times\boldsymbol{u})\times\boldsymbol{\nabla}\mu_0 \quad (6)$$

It is interesting to note that to the chosen order of approximation this equation is not dependent on the parametric viscosity variation μ_1. Similarly equation 5 becomes

$$\frac{\partial \rho_1}{\partial t} + (\boldsymbol{U}\cdot\boldsymbol{\nabla})\rho_1 + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\rho_0 = 0 \quad (7)$$

and equation 3 becomes

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \quad (8)$$

Furthermore, for a body travelling horizontally with a steady translational velocity $\boldsymbol{U}(t) = \boldsymbol{U} = (U, 0, 0)$ and assuming $\rho_0(z,t) = \rho_0(z)$, equation 6 in component form becomes

$$\rho_0\left(\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x}\right) + \epsilon u = -\frac{\partial p_1}{\partial x} + \mu_0 \boldsymbol{\nabla}^2 u + \frac{\partial \mu_0}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \quad (9)$$

$$\rho_0\left(\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x}\right) + \epsilon v = -\frac{\partial p_1}{\partial y} + \mu_0 \boldsymbol{\nabla}^2 v + \frac{\partial \mu_0}{\partial z}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \quad (10)$$

$$\rho_0\left(\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x}\right) + \epsilon w = -\frac{\partial p_1}{\partial z} + \mu_0 \boldsymbol{\nabla}^2 w - \rho_1 g + 2\frac{\partial \mu_0}{\partial z}\frac{\partial w}{\partial z} \quad (11)$$

Equation 7 becomes

$$\frac{\partial \rho_1}{\partial t} + U\frac{\partial \rho_1}{\partial x} + w\frac{\partial \rho_0}{\partial z} = 0, \quad (12)$$

and equation 8,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (13)$$

The combination $\frac{\partial}{\partial y}$ [9] - $\frac{\partial}{\partial x}$ [10] gives

$$\left[\rho_0 \frac{D}{Dt} + \epsilon - \mu_0 \boldsymbol{\nabla}^2 - \frac{\partial \mu_0}{\partial z}\frac{\partial}{\partial z}\right]\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = 0 \quad (14)$$

that is, there exists a function $\phi(x,y,z)$ such that

$$u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y} \quad (15)$$

is a solution of equation 14 and from equation 13 we obtain the result

$$\boldsymbol{\nabla}_h^2 \phi = -\frac{\partial w}{\partial z} \quad (16)$$

where $\boldsymbol{\nabla}_h = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0\right)$.

The inclusion of density stratification into the mathematical model destroys the concept of irrotational fluid motion and this influence is considered to be of far greater importance than viscous effects in this linearised theory. For this reason, we shall examine a simplified mathematical model adopting linear equations in the absence of viscosity, $\mu_0 = 0$, but with density stratification $\rho_0(z)$ and hence rotational fluid motion. In this case, the equations describing the fluid disturbance can be expressed as

$$\left.\begin{array}{ll} [\,9\,] \text{ and } [\,10\,] & \rho_0 \frac{D\phi}{Dt} + \epsilon\phi + p_1 = 0 \\ [\,11\,] & \rho_0 \frac{Dw}{Dt} + \epsilon w + \frac{\partial p_1}{\partial z} + g\rho_1 = 0 \\ [\,13\,] & \frac{D\rho_1}{Dt} + w\frac{\partial \rho_1}{\partial z} = 0 \\ [\,16\,] & \nabla_h^2\phi + \frac{\partial w}{\partial z} = 0 \end{array}\right\} \quad (17)$$

Equation 17 can be written in non-dimensional form using the non-dimensionalising variables L, U, ρ_0 and g. That is

$$\begin{array}{ccc} \phi = UL\phi' & p_1 = \rho_0 U^2 p_1' & w = Uw' \\ (x,y,z) = L(x',y',z') & t = \frac{L}{U}t' & \frac{D}{Dt} = \frac{U}{L}\frac{D}{Dt'} \\ \nabla_h^2 = \left(\frac{1}{L}\right)^2 \nabla_h'^2 & \rho_1 = \rho_0\rho_1' & N^2(z) = \frac{g}{L}N'^2(z) \end{array}$$

and equation 17 becomes

$$\frac{D\phi'}{Dt'} + \epsilon\phi' + p_1' = 0$$

$$\frac{Dw'}{Dt'} + \epsilon w' + \frac{\partial p_1'}{\partial z'} - N'^2(z)p_1' + \frac{\rho_1'}{F_n^2} = 0$$

$$\frac{D\rho_1'}{Dt'} - w'N'^2(z) = 0$$

$$\nabla_h'^2\phi + \frac{\partial w'}{\partial z'} = 0$$

By eliminating p_1' and ρ_1' and dropping the superscripts in the subsequent analysis but retaining the understanding that the equations and variables are expressed in non-dimensional form, the following coupled equations are derived

$$\left.\begin{array}{c} \nabla_h^2\phi + \frac{\partial w}{\partial z} = 0 \\ \frac{D}{Dt}\left(\frac{D}{Dt} + \epsilon\right)\left(w - \frac{\partial\phi}{\partial z} + N^2(z)\phi\right) + \left(\frac{N(z)}{F_n}\right)^2 w = 0 \end{array}\right\} \quad (18)$$

2.2 Low speed approximation

If we let N_m^2 to denote the maximum value of the non dimensional parameter $N^2(z)$ and restrict the analysis to the range $N_m^2 < 1$, $F_n < 1$ such that $\frac{N_m^2}{F_n^2} \sim O(1)$ then a low speed approximation can be introduced into the analysis by assuming that $\frac{N_m^2}{F_n^2} \gg N_m^2$. That is, the variables w and ϕ can be written as

$$w = w_0 + N_m^2 w_1 + (N_m^2)^2 w_2 + \ldots$$

$$\phi = \phi_0 + N_m^2\phi_1 + (N_m^2)^2\phi_2 + \ldots$$

Substituting w and ϕ into equation 18 a zero order theory is described by the equations

$$\nabla_h^2\phi_0 + \frac{\partial w_0}{\partial z} = 0$$

$$\frac{D}{Dt}\left(\frac{D}{Dt} + \epsilon\right)\left(w_0 - \frac{\partial\phi_0}{\partial z}\right) + \tilde{N}^2(z)w_0 = 0$$

For the steady state case we now have $\frac{D}{Dt} = \frac{\partial}{\partial x}$, we find that the zeroth order approximation becomes

$$\left.\begin{array}{c} \nabla_h^2\phi_0 + \frac{\partial w_0}{\partial z} = 0 \\ \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + \epsilon\right)\left(w_0 - \frac{\partial\Phi_0}{\partial z}\right) + \tilde{N}^2(z)w_0 = 0 \end{array}\right\} \quad (19)$$

where $\tilde{N}(z) = \frac{N(z)}{Fn}$.

These equations together with the relationship between ϕ and the velocities, equation 15, form the starting point for any investigation into fluid with stratified characteristics. We may set the Brunt-Väisälä frequency to a constant value and seek analytic solutions for these equations, ie $\tilde{N}(z) = \tilde{N}$.

2.3 Unbounded forms of the functions ϕ and ψ

The introduction of a variable ψ, related to the vertical velocity component w in the form

$$w = \frac{\partial \psi}{\partial z}$$

modifies equations 19

$$\nabla_h^2 \phi + \frac{\partial^2 \psi}{\partial z^2} = 0 \tag{20}$$

$$\frac{\partial}{\partial x}(\frac{\partial}{\partial x} + \epsilon)\left(\frac{\partial \psi}{\partial z} - \frac{\partial \phi}{\partial z}\right) + \tilde{N}^2 \frac{\partial \psi}{\partial z} = 0. \tag{21}$$

The introduction of a body force takes the form of a dirac delta function, this is introduced on the right hand side of equation 20. This equation is derived from the incompressibility condition, i.e. fluid is neither created or destroyed. The delta function singularity introduces fluid into the domain and therefore equation 20 must be modified. That is,

$$\nabla_h^2 \phi + \frac{\partial^2 \psi}{\partial z^2} = \delta(\boldsymbol{r}) \tag{22}$$

$$\frac{\partial}{\partial x}(\frac{\partial}{\partial x} + \epsilon)\left(\frac{\partial \psi}{\partial z} - \frac{\partial \phi}{\partial z}\right) + \tilde{N}^2 \frac{\partial \psi}{\partial z} = 0. \tag{23}$$

Integrating equation 23 with respect to z we obtain

$$\frac{\partial}{\partial x}(\frac{\partial}{\partial x} + \epsilon)(\psi - \phi) + \tilde{N}^2 \psi = 0 \tag{24}$$

and the adjoint of this equation is given by

$$\frac{\partial}{\partial x}(\frac{\partial}{\partial x} - \epsilon)(\psi - \phi) + \tilde{N}^2 \psi = 0 \tag{25}$$

Equations 22 and 25 form the basis equations describing the fluid disturbance caused by the moving slender body.

The velocity components u, v and w are directly related to the functions ϕ and ψ. The derivation of these functions will then allow the velocity components to be obtained.

Eliminating ϕ using $\frac{\partial}{\partial x}(\frac{\partial}{\partial x} - \epsilon)$ [22] and ∇_h^2 [25] gives a partial differential equation in ψ only,

$$\frac{\partial}{\partial x}(\frac{\partial}{\partial x} - \epsilon)\nabla^2 \psi + \tilde{N}^2 \nabla_h^2 \psi = \frac{\partial}{\partial x}(\frac{\partial}{\partial x} - \epsilon)\delta(\boldsymbol{r}) \tag{26}$$

and eliminating ψ using $\frac{\partial^2}{\partial z^2}$ [25] gives

$$\frac{\partial}{\partial x}(\frac{\partial}{\partial x} - \epsilon)\nabla^2 \phi + \tilde{N}^2 \nabla_h^2 \phi = \frac{\partial}{\partial x}(\frac{\partial}{\partial x} - \epsilon)\delta(\boldsymbol{r}) + \tilde{N}^2 \delta(\boldsymbol{r}) \tag{27}$$

Equations 26 and 27 may be written in the matrix form

$$\mathcal{L}\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}(\frac{\partial}{\partial x} - \epsilon) + \tilde{N}^2 \\ \frac{\partial}{\partial x}(\frac{\partial}{\partial x} - \epsilon) \end{pmatrix} \delta(\boldsymbol{r}) \tag{28}$$

where the differential operator $\mathcal{L}$ is given by

$$\mathcal{L}(\,) = \left\{\frac{\partial}{\partial x}(\frac{\partial}{\partial x} - \epsilon)\boldsymbol{\nabla}^2 + \tilde{N}^2\boldsymbol{\nabla}_h^2\right\}(\,)$$

By the application of Fourier transforms we can rewrite equation 28 into the form

$$\mathcal{L}\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 1 - \frac{\tilde{N}^2}{k_1[k_1 - i\epsilon]} \\ 1 \end{pmatrix}\delta(z-\zeta)\frac{e^{i\boldsymbol{r}\cdot\boldsymbol{k}}}{2\pi} \tag{29}$$

where $\mathcal{L}$ is now given by

$$\mathcal{L}(\,) = \left\{\frac{\partial^2}{\partial z^2} + \left[\frac{\tilde{N}^2}{k_1[k_1 - i\epsilon]} - 1\right]k^2\right\}(\,),$$

$\boldsymbol{k} = (k_1, k_2, 0)$ and $k = \sqrt{k_1^2 + k_2^2}$.

Let us now examine the equation

$$\left[\frac{\partial^2}{\partial z^2} - (k\gamma)^2\right] G(k_1, k_2, z; \boldsymbol{\xi}) = \delta(z-\zeta) \tag{30}$$

where $\gamma = \sqrt{1 - \frac{\tilde{N}^2}{k_1[k_1 - i\epsilon]}}$.

This equation has the following unbounded solution

$$\begin{array}{ll} \frac{e^{(z-\zeta)k\gamma} - e^{-(z-\zeta)k\gamma}}{2k\gamma} + A(k_1, k_2)e^{zk\gamma} + B(k_1, k_2)e^{-zk\gamma} & 0 > z > \zeta \\ A(k_1, k_2)e^{zk\gamma} + B(k_1, k_2)e^{-zk\gamma} & z < \zeta < 0 \end{array} \tag{31}$$

where $A(k_1, k_2)$ and $B(k_1, k_2)$ are obtained through the application of boundary conditions.

2.4 The coordinate system

Figure 1 illustrates the location and orientation of the orthogonal, right handed, coordinate axis system chosen to simplify the mathematics of the three layer system. For example, for a body in the upper layer the rigid lid is the plane $z = 0$, the x-axis lies along the direction of motion of the body and the positive y-axis is chosen to satisfy the orthogonality condition. The origin of the coordinate system lies vertically above the body's centroid. For a body in one of the lower layers the undisturbed interface above the body becomes the plane $z = 0$.

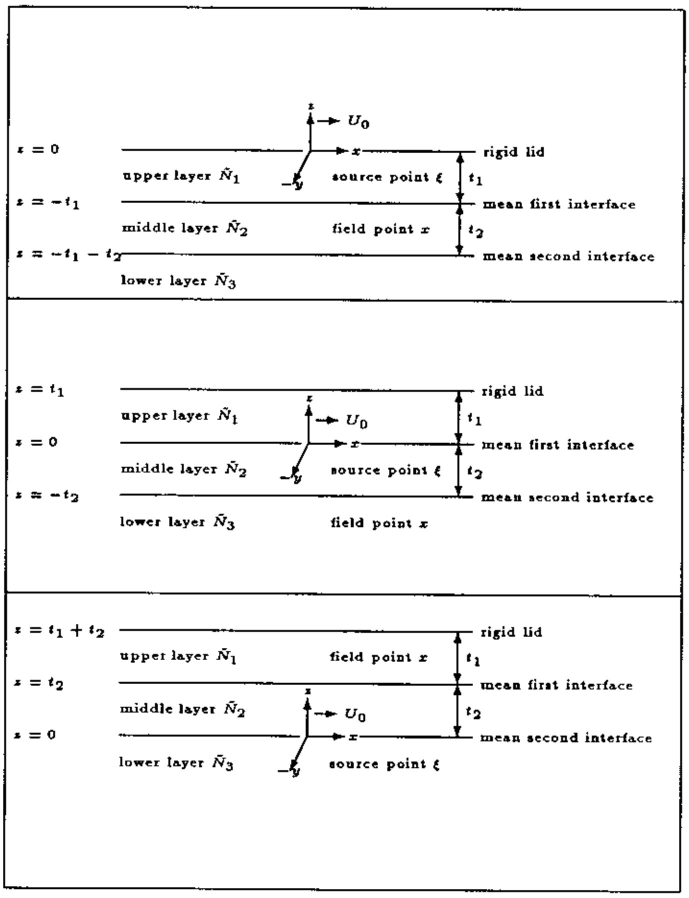

Figure 1: The coordinate system for the three layer fluid model

2.5 Application of the unbounded solution to the three layer model

The fluid disturbance in the j^{th} layer created by the steady motion of the impulse moving in the i^{th} layer is given by the two functions ϕ_{ij} and ψ_{ij}. The uppermost layer is layer one. Extending the unbounded solution, equation 31, for the layered model and applying inverse Fourier transforms the function ψ_{ij} has the form

$$\psi_{ij}(\boldsymbol{x},\boldsymbol{\xi}) = \frac{\delta_{ij}H(z-\zeta)}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{(z-\zeta)k\gamma_j} - e^{-(z-\zeta)k\gamma_j}}{2k\gamma_j}e^{-i(r_1k_1+r_2k_2)}dk_1dk_2$$

$$+\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[A_{ij}(k_1,k_2)e^{zk\gamma_j} + B_{ij}(k_1,k_2)e^{-zk\gamma_j}\right]e^{-i(r_1k_1+r_2k_2)}dk_1dk_2 \quad (32)$$

where $\gamma_j = \sqrt{1 - \frac{\tilde{N}_j^2}{k_1(k_1 - i\epsilon)}}$, δ_{ij} is the Kronecker delta function and $H(x)$ is the Heaviside unit step function.

Using equation 29 ϕ_{ij} is given by

$$\phi_{ij}(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{\delta_{ij} H(z-\zeta)}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \gamma_j^2 \frac{e^{(z-\zeta)k\gamma_j} - e^{-(z-\zeta)k\gamma_j}}{2k\gamma_j} e^{-i(r_1 k_1 + r_2 k_2)} dk_1 dk_2$$

$$+ \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \gamma_j^2 \left[A_{ij}(k_1, k_2) e^{zk\gamma_j} + B_{ij}(k_1, k_2) e^{-zk\gamma_j} \right] e^{-i(r_1 k_1 + r_2 k_2)} dk_1 dk_2 \qquad (33)$$

2.6 Boundary Conditions

The linearised boundary conditions imposed to describe the fluid disturbance in the j^{th} layer created by the steady motion of the impulse moving in the i^{th} layer of a three layer fluid system are

1. The rigid lid condition

$$\frac{\partial \psi_{i1}}{\partial z} = 0 \quad \text{on } z = h_i \quad \text{for } i = 1, 2, 3 \qquad (34)$$

2. The interface conditions

$$\frac{\partial \psi_{i(j-1)}}{\partial z} = \frac{\partial \psi_{ij}}{\partial z} \quad \text{on } z = h_i - h_j \quad \text{for } i = 1, 2, 3 \quad \text{and } j = 2, 3 \qquad (35)$$

$$\frac{\partial^2 \phi_{i(j-1)}}{\partial x^2} = \frac{\partial^2 \phi_{ij}}{\partial x^2} \quad \text{on } z = h_i - h_j \quad \text{for } i = 1, 2, 3 \quad \text{and } j = 2, 3 \qquad (36)$$

3. The bottom condition

$$\nabla \phi_{i3} \rightarrow 0 \quad \text{as } z \rightarrow -\infty \quad \text{for } i = 1, 2, 3$$

$$\nabla \psi_{i3} \rightarrow 0 \quad \text{as } z \rightarrow -\infty \quad \text{for } i = 1, 2, 3 \qquad (37)$$

4. The radiation condition

$$\phi_{ij} = \begin{cases} O(\frac{1}{|\boldsymbol{x}|}) & \text{for } x > 0 \\ o(1) & \text{for } x < 0 \end{cases} \quad \text{as } |\boldsymbol{x}| \rightarrow \infty \quad \text{for } i = 1, 2, 3 \quad \text{and } j = 1, 2, 3$$

$$\psi_{ij} = \begin{cases} O(\frac{1}{|\boldsymbol{x}|}) & \text{for } x > 0 \\ o(1) & \text{for } x < 0 \end{cases} \quad \text{as } |\boldsymbol{x}| \rightarrow \infty \quad \text{for } i = 1, 2, 3 \quad \text{and } j = 1, 2, 3 \qquad (38)$$

where $O(x)$ and $o(x)$ denote the Landau order symbols as defined by Erdelyi(1956) and $\delta(x)$ denotes the Direc delta function. h_i is the vertical separation of the origin and rigid lid and h_j is the vertical separation of the rigid lid and j^{th} interface.

Equation 34 confines the fluid's upper surface to be plane, ie no waves. Equation 35 ensures the vertical velocity is continuous across the interfaces. Equation 36 describes a combination of the kinematic and dynamic conditions applied on the interface. The kinematic condition ensures that the fluid particles cannot pass through the disturbed interface, the dynamic condition ensures the pressure in the fluid across the disturbed interfaces is continuous. The application of this boundary condition allows the presence of wave systems on the interfaces. Equation 37 imposes the condition that the disturbance decays with increasing depth. Equation 38 requires that the waves only exist behind the body and no waves propagate upstream.

The method of solution involves the substitution of the unbounded forms of the functions ψ_{ij} and ϕ_{ij}, equations 32 and 33 into the boundary conditions 34 - 36, thus forming a system of equations from which the unknown coefficients $A_{ij}(k_1, k_2)$ and $B_{ij}(k_1, k_2)$ can be determined. Immediately we can set $B_{i3}(k_1, k_2) = 0$ to comply with the condition stated in boundary condition 37. The radiation condition is fulfilled when $\epsilon \rightarrow 0$ later in the analysis.

2.6 Derivation of the functions ϕ_{ij} and ψ_{ij}

2.7 Derivation for the impulse in the upper layer

The substitution of equations 32 and 33 into the boundary conditions 34-36 when $i = 1$ produces the system of equations

$$\begin{bmatrix} k\gamma_1 & -k\gamma_1 & 0 & 0 & 0 \\ \gamma_1 e^{-t_1 k\gamma_1} & -\gamma_1 e^{t_1 k\gamma_1} & -\gamma_2 e^{-t_1 k\gamma_2} & \gamma_2 e^{t_1 k\gamma_2} & 0 \\ \gamma_1^2 e^{-t_1 k\gamma_1} & \gamma_1^2 e^{t_1 k\gamma_1} & -\gamma_2^2 e^{-t_1 k\gamma_2} & -\gamma_2^2 e^{t_1 k\gamma_2} & 0 \\ 0 & 0 & \gamma_2 e^{-(t_1+t_2)k\gamma_2} & -\gamma_2 e^{(t_1+t_2)k\gamma_2} & -\gamma_3 e^{-(t_1+t_2)k\gamma_3} \\ 0 & 0 & \gamma_2^2 e^{-(t_1+t_2)k\gamma_2} & \gamma_2^2 e^{(t_1+t_2)k\gamma_2} & -\gamma_3^2 e^{-(t_1+t_2)k\gamma_3} \end{bmatrix} \begin{Bmatrix} A_{11} \\ B_{11} \\ A_{12} \\ B_{12} \\ A_{13} \end{Bmatrix} = \begin{Bmatrix} -\cosh \zeta k\gamma_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Solving for $A_{1j}(k_1, k_2)$ and $B_{1j}(k_1, k_2)$ then substituting into equation 32, we obtain the results:

$$\psi_{11}(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{H(z-\zeta)}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{(z-\zeta)k\gamma_1} - e^{-(z-\zeta)k\gamma_1}}{2k\gamma_1} e^{-i(r_1 k_1 + r_2 k_2)} dk_1 dk_2$$

$$-\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{\gamma_1 (\gamma_3 \sinh t_2 k\gamma_2 + \gamma_2 \cosh t_2 k\gamma_2) \sinh(z + t_1)k\gamma_1 + \gamma_2 (\gamma_2 \sinh t_2 k\gamma_2 + \gamma_3 \cosh t_2 k\gamma_2) \cosh(z + t_1)k\gamma_1\} \frac{\cosh \zeta k\gamma_1}{k\gamma_1 D(k_1, k_2)} e^{-i(r_1 k_1 + r_2 k_2)} dk_1 dk_2 \qquad 0 \geq z \geq -t_1$$

$$\psi_{12}(\boldsymbol{x}, \boldsymbol{\xi}) = -\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{\gamma_2 \sinh(z + t_1 + t_2)k\gamma_2 + \gamma_3 \cosh(z + t_1 + t_2)k\gamma_2\} \frac{\gamma_1 \cosh \zeta k\gamma_1}{k\gamma_2 D(k_1, k_2)} e^{-i(r_1 k_1 + r_2 k_2)} dk_1 dk_2 \qquad -t_1 \geq z \geq -(t_1 + t_2)$$

$$\psi_{13}(\boldsymbol{x}, \boldsymbol{\xi}) = -\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\gamma_1 \gamma_2 e^{zk\gamma_3} \cosh \zeta k\gamma_1}{k\gamma_3 D(k_1, k_2)} e^{-i(r_1 k_1 + r_2 k_2)} dk_1 dk_2 \qquad -(t_1 + t_2) \geq z > -\infty$$

where

$$D(k_1, k_2) = \gamma_2 \sinh t_1 k\gamma_1 [\gamma_2 \sinh t_2 k\gamma_2 + \gamma_3 \cosh t_2 k\gamma_2] + \gamma_1 \cosh t_1 k\gamma_1 [\gamma_3 \sinh t_2 k\gamma_2 + \gamma_2 \cosh t_2 k\gamma_2]$$

2.8 Derivation for the impulse in the middle layer

The substitution of equations 32 and 33 into the boundary conditions 34-36 when $i = 2$ produces the system of equations

$$\begin{bmatrix} k\gamma_1(e^{-2t_1k\gamma_1}-1) & -k\gamma_2 & k\gamma_2 & 0 \\ k\gamma_1^2(e^{-2t_1k\gamma_1}+1) & -k\gamma_2^2 & -k\gamma_2^2 & 0 \\ 0 & \gamma_2 e^{-t_2k\gamma_2} & -\gamma_2 e^{t_2k\gamma_2} & -\gamma_3 e^{-t_2k\gamma_3} \\ 0 & \gamma_2^2 e^{-t_2k\gamma_2} & \gamma_2^2 e^{t_2k\gamma_2} & -\gamma_3^2 e^{-t_2k\gamma_3} \end{bmatrix} \begin{Bmatrix} B_{21} \\ A_{22} \\ B_{22} \\ A_{23} \end{Bmatrix} = \begin{Bmatrix} \cosh\zeta k\gamma_2 \\ -\gamma_2\sinh\zeta k\gamma_2 \\ 0 \\ 0 \\ 0 \end{Bmatrix}.$$

Solving for $A_{2j}(k_1, k_2)$ and $B_{2j}(k_1, k_2)$ then substituting into equation 32, we obtain the results:

$$\psi_{21}(\boldsymbol{x}, \boldsymbol{\xi}) = -\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\gamma_2\sinh(\zeta+t_2)k\gamma_2 + \gamma_3\cosh(\zeta+t_2)k\gamma_2\}\,\frac{\gamma_2\cosh(z-t_1)k\gamma_1}{k\gamma_1 D(k_1,k_2)}e^{-i(r_1k_1+r_2k_2)}dk_1dk_2 \qquad t_1 \geq z \geq 0$$

$$\psi_{22}(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{H(z-\zeta)}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{(z-\zeta)k\gamma_2} - e^{-(z-\zeta)k\gamma_2}}{2k\gamma_2}e^{-i(r_1k_1+r_2k_2)}dk_1dk_2$$

$$-\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\gamma_2\sinh(z+t_2)k\gamma_2 + \gamma_3\cosh(z+t_2)k\gamma_2]\,[\gamma_1\cosh t_1k\gamma_1\cosh\zeta k\gamma_2 - \gamma_2\sinh t_1k\gamma_2\sinh\zeta k\gamma_2]\,\frac{1}{k\gamma_2 D(k_1,k_2)}e^{-i(r_1k_1+r_2k_2)}dk_1dk_2 \qquad 0 \geq z \geq -t_2$$

$$\psi_{23}(\boldsymbol{x}, \boldsymbol{\xi}) = -\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\gamma_1\cosh t_1k\gamma_1\cosh\zeta k\gamma_2 - \gamma_2\sinh t_1k\gamma_1\sinh\zeta k\gamma_2]\,\frac{\gamma_2 e^{(z+t_2)k\gamma_3}}{k\gamma_3 D(k_1,k_2)}e^{-i(r_1k_1+r_2k_2)}dk_1dk_2 \qquad -t_2 \geq z > -\infty$$

where

$$D(k_1,k_2) = \gamma_2\sinh t_1k\gamma_1\,[\gamma_2\sinh t_2k\gamma_2 + \gamma_3\cosh t_2k\gamma_2] + \gamma_1\cosh t_1k\gamma_1\,[\gamma_3\sinh t_2k\gamma_2 + \gamma_2\cosh t_2k\gamma_2]$$

2.9 Derivation for the impulse in the lower layer

The substitution of equations 32 and 33 into the boundary conditions 34-36 when $i = 3$ with the body in the lower layer produces the system of equations

$$\begin{bmatrix} \gamma_1 e^{-t_2 k\gamma_1}(e^{-2t_1 k\gamma_1} - 1) & -\gamma_2 e^{t_2 k\gamma_2} & \gamma_2 e^{-t_2 k\gamma_2} & 0 \\ \gamma_1^2 e^{-t_2 k\gamma_1}(e^{-2t_1 k\gamma_1} + 1) & -\gamma_2^2 e^{t_2 k\gamma_2} & -\gamma_2^2 e^{t_2 k\gamma_2} & 0 \\ 0 & k\gamma_2 & -k\gamma_2 & -k\gamma_3 \\ 0 & k\gamma_2^2 & k\gamma_2^2 & -k\gamma_3^2 \end{bmatrix} \begin{Bmatrix} B_{31} \\ A_{32} \\ B_{32} \\ A_{33} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \cosh\zeta k\gamma_3 \\ -\gamma_3 \sinh\zeta k\gamma_3 \end{Bmatrix}$$

Solving for $A_{i3}(k_1, k_2)$ and $B_{i3}(k_1, k_2)$ then substituting into equation 32, we obtain the results :

$$\psi_{31}(\boldsymbol{x}, \boldsymbol{\xi}) = -\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\gamma_2\gamma_3 e^{\zeta k\gamma_3} \cosh(z - t_1 - t_2)k\gamma_1}{k\gamma_1 D(k_1, k_2)} e^{-i(r_1 k_1 + r_2 k_2)} dk_1 dk_2 \qquad t_1 + t_2 \geq z \geq t_1$$

$$\psi_{32}(\boldsymbol{x}, \boldsymbol{\xi}) = -\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{\gamma_1 \cosh t_1 k\gamma_1 \cosh(z - t_2)k\gamma_2 - \gamma_2 \sinh t_1 k\gamma_1 \sinh(z - t_2)k\gamma_2\} \frac{\gamma_3 e^{\zeta k\gamma_3}}{2k\gamma_2 D(k_1, k_2)} e^{-i(r_1 k_1 + r_2 k_2)} dk_1 dk_2 \qquad t_1 \geq z \geq 0$$

$$\psi_{33}(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{H(z - \zeta)}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{(z-\zeta)k\gamma_3} - e^{-(z-\zeta)k\gamma_3}}{2k\gamma_3} e^{-i(r_1 k_1 + r_2 k_2)} dk_1 dk_2$$

$$-\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{\gamma_2 [\gamma_2 \sinh t_1 k\gamma_1 \sinh t_2 k\gamma_2 + \gamma_1 \cosh t_1 k\gamma_1 \cosh t_2 k\gamma_2] \cosh\zeta k\gamma_3$$

$$-\gamma_3 [\gamma_2 \sinh t_1 k\gamma_1 \cosh t_2 k\gamma_2 + \gamma_1 \cosh t_1 k\gamma_1 \sinh t_2 k\gamma_2] \sinh\zeta k\gamma_3\} \frac{e^{z k\gamma_3}}{k\gamma_3 D(k_1, k_2)} e^{-i(r_1 k_1 + r_2 k_2)} dk_1 dk_2 \qquad 0 \geq z > -\infty$$

where

$$D(k_1, k_2) = \gamma_2 \sinh t_1 k\gamma_1 [\gamma_2 \sinh t_2 k\gamma_2 + \gamma_3 \cosh t_2 k\gamma_2] + \gamma_1 \cosh t_1 k\gamma_1 [\gamma_3 \sinh t_2 k\gamma_2 + \gamma_2 \cosh t_2 k\gamma_2]$$

2.10 Generalisation of Functions

The functions can be rearranged into the general form

$$\psi_{ij}(\boldsymbol{x},\boldsymbol{\xi}) = -\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{F_{ij}(k_1,k_2)}{k\gamma_j^2 D(k_1,k_2)}e^{-i(r_1k_1+r_2k_2)}dk_1dk_2$$

$$\phi_{ij}(\boldsymbol{x},\boldsymbol{\xi}) = -\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{F_{ij}(k_1,k_2)}{kD(k_1,k_2)}e^{-i(r_1k_1+r_2k_2)}dk_1dk_2$$

where the functions $D(k_1,k_2)$ and $F_{ij}(k_1,k_2)$ are detailed below. When $i=j$ the functions are shown for the $z<\zeta$ case, for $z>\zeta$ these functions can be obtained through exchanging z and ζ.

$$D(k_1,k_2) = \frac{\sinh t_1k\gamma_1}{\gamma_1}\left[\gamma_2\sinh t_2k\gamma_2+\gamma_3\cosh t_2k\gamma_2\right]+\cosh t_1k\gamma_1\left[\gamma_3\frac{\sinh t_2k\gamma_2}{\gamma_2}+\cosh t_2k\gamma_2\right]$$

$$F_{11}(k_1,k_2) = \cosh\zeta k\gamma_1\left\{\gamma_1\sinh(z+t_1)k\gamma_1\left[\gamma_3\frac{\sinh t_2k\gamma_2}{\gamma_2}+\cosh t_2k\gamma_2\right]+\cosh(z+t_1)k\gamma_1\left[\gamma_2\sinh t_2k\gamma_2+\gamma_3\cosh t_2k\gamma_2\right]\right\}$$

$$F_{12}(k_1,k_2) = \cosh\zeta k\gamma_1\left[\gamma_2\sinh(z+t_1+t_2)k\gamma_2+\gamma_3\cosh(z+t_1+t_2)k\gamma_2\right]$$

$$F_{13}(k_1,k_2) = \gamma_3 e^{zk\gamma_3}\cosh\zeta k\gamma_1$$

$$F_{21}(k_1,k_2) = \cosh(z-t_1)k\gamma_1\left[\gamma_2\sinh(\zeta+t_2)k\gamma_2+\gamma_3\cosh(\zeta+t_2)k\gamma_2\right]$$

$$F_{22}(k_1,k_2) = \left[\gamma_2\sinh(z+t_2)k\gamma_2+\gamma_3\cosh(z+t_2)k\gamma_2\right]\left[\cosh t_1k\gamma_1\cosh\zeta k\gamma_2-\gamma_2\frac{\sinh t_1k\gamma_1}{\gamma_1}\sinh\zeta k\gamma_2\right]$$

$$F_{23}(k_1,k_2) = \gamma_3 e^{(z+t_2)k\gamma_3}\left[\cosh t_1k\gamma_1\cosh\zeta k\gamma_2-\gamma_2\frac{\sinh t_1k\gamma_1}{\gamma_1}\sinh\zeta k\gamma_2\right]$$

$$F_{31}(k_1,k_2) = \gamma_3 e^{\zeta k\gamma_3} \cosh(z-t_1-t_2)k\gamma_1$$

$$F_{32}(k_1,k_2) = \gamma_3 e^{\zeta k\gamma_3} \left[\cosh t_1 k\gamma_1 \cosh(z-t_2)k\gamma_2 - \gamma_2 \frac{\sinh t_1 k\gamma_1}{\gamma_1} \sinh(z-t_2)k\gamma_2\right]$$

$$F_{33}(k_1,k_2) = \gamma_3 e^{zk\gamma_3} \left\{\cosh \zeta k\gamma_3 \left[\gamma_2 \frac{\sinh t_1 k\gamma_1}{\gamma_1} \sinh t_2 k\gamma_2 + \cosh t_1 k\gamma_1 \cosh t_2 k\gamma_2\right] - \gamma_3 \sinh \zeta k\gamma_3 \left[\frac{\sinh t_1 k\gamma_1}{\gamma_1} \cosh t_2 k\gamma_2 + \cosh t_1 k\gamma_1 \frac{\sinh t_2 k\gamma_2}{\gamma_2}\right]\right\}$$

When $\tilde{N}_1 = \tilde{N}_2 = \tilde{N}_3 = \tilde{N}$ this formulation reduces to the constant N model, see Price and Westlake(1994b), ie

$$\psi(\boldsymbol{x},\boldsymbol{\xi}) = -\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{(\zeta-z)k\gamma} + e^{(\zeta+z)k\gamma}}{2k\gamma} e^{-i(r_1k_1+r_2k_2)} dk_1 dk_2 \qquad 0 > z > \zeta$$

$$\psi(\boldsymbol{x},\boldsymbol{\xi}) = -\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{(z-\zeta)k\gamma} + e^{(z+\zeta)k\gamma}}{2k\gamma} e^{-i(r_1k_1+r_2k_2)} dk_1 dk_2 \qquad z < \zeta < 0$$

3 The integration process

The double integral required to transform the solutions obtained in the wavenumber plane back to a spacial plane are

$$\psi_{ij}(\boldsymbol{x},\boldsymbol{\xi}) = -\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{F_{ij}(k_1,k_2)}{k\gamma_j^2 D(k_1,k_2)}e^{-i(r_1k_1+r_2k_2)}dk_1dk_2$$

$$\phi_{ij}(\boldsymbol{x},\boldsymbol{\xi}) = -\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{F_{ij}(k_1,k_2)}{kD(k_1,k_2)}e^{-i(r_1k_1+r_2k_2)}dk_1dk_2$$

The presence of $D(k_1,k_2)$ on the denominator introduces poles into the integrand which precludes direct integration if a complete solution is sought.

3.1 Poles in the integrand

To complete the inverse Fourier transforms it is now necessary to investigate the location and nature of the zeros of $D(k_1,k_2)$ before ϵ is set to zero. Direct substitution of $\epsilon = 0$ at this stage results in a symmetric solution being obtained which does not satisfy the radiation condition. Some manipulation of the integrand is required before the effect of ϵ is realised. It is found that the zeros of $D(k_1,k_2)$ are located at all values of k_2 and in the k_1 intervals between the lowest ($\tilde{N}_{min}$) and the highest ($\tilde{N}_{max}$) Brunt-Väisälä frequencies chosen for the three layers. If the wavenumber k_2 is taken to be complex this behaviour appears graphically as

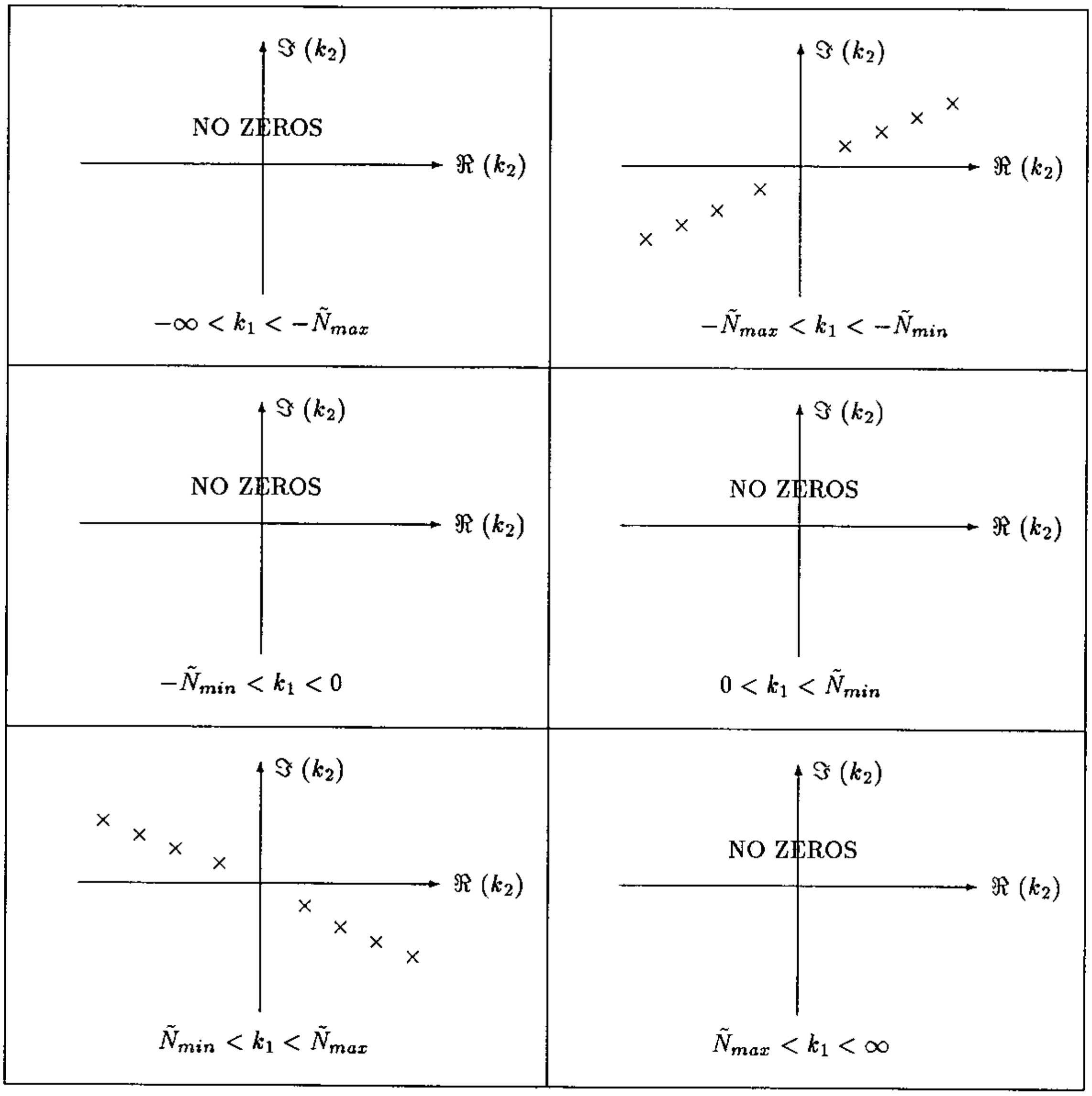

Figure 2: The zeros of $D(k_1, k_2)$

Note the antisymmetric behaviour displayed in the zeros locations for both wavenumber axes. As $\epsilon \rightarrow 0$ the zeros return to the $\Re(k_2)$ axis, this is similar to the behaviour of the zeros detailed by Lighthill(1979), page 365. As the integrand has poles contour integration is required in the intervals $-\tilde{N}_{max} < k_1 < -\tilde{N}_{min}$ and $\tilde{N}_{min} < k_1 < \tilde{N}_{max}$, $-\infty < k_2 < \infty$. The use of contour integration in the remaining intervals is not necessary, the integration may be carried out directly. However it is found that the integrand and hence the integration process is simplified if contour integration is applied here also.

3.2 The contour integral

A conventional large radius contour circumventing any poles present using normal techniques may not be used in this case as the integrand possesses a complex square root. The presence of this square root and its inherent discontinuous nature necessitates additional paths to be added to the contour. The complex square root $\sqrt{z} = \sqrt{a + ib}$ has the cartesian result

$$\sqrt{a+ib} = \left[\frac{\sqrt{a^2+b^2}+a}{2}\right] + i\ \mathrm{sign}(b)\left[\frac{\sqrt{a^2+b^2}-a}{2}\right] \tag{39}$$

as the square root of a positive real number is defined to be positive. It is the sign function which causes the discontinuity, if $a < 0$ and $b \to 0_+$ the result is $i\sqrt{a}$, if $a < 0$ and $b \to 0_-$ the result is $-i\sqrt{a}$. Thus the rule describing the line along which the discontinuity exists is

$$\Re(z) < 0$$
$$\Im(z) = 0$$

A simple example of this technique is detailed in appendix A.

3.3 Validation against the constant N model

This technique now established can be extended to a more complex problem, the constant N model. It is necessary to demonstrate that the result obtained using a standard line integral can also be obtained in the constant N model using contour integration. In Price and Westlake(1994b) the function ψ was obtained through the use of line integration giving a Bessel type solution, then epsilon was set to zero (page 6). Now we attempt the contour integration before ϵ is set to zero., then as $\epsilon \to 0$ the results should concur. In the constant N model it was not necessary to use contour integration as no poles existed in the integrand, however for the layered constant N model poles exist and contour integration is therefore necessary to obtain a complete solution. The contours used in both cases are identical therefore validation using the constant N model is desirable. For the constant N model ψ is

$$-(2\pi)^2\psi(\boldsymbol{x}\ ,\boldsymbol{\xi}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{((\zeta-z)k\gamma} + e^{((\zeta+z)k\gamma}}{2k\gamma} e^{-i(r_1k_1+r_2k_2)}dk_1dk_2 \qquad 0 > z > \zeta$$

$$-(2\pi)^2\psi(\boldsymbol{x}\ ,\boldsymbol{\xi}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{(z-\zeta)k\gamma} + e^{(z+\zeta)k\gamma}}{2k\gamma} e^{-i(r_1k_1+r_2k_2)}dk_1dk_2 \qquad z < \zeta < 0$$

The term $k\gamma$ must be considered to be a single square root as the integral may not be convergent if k and γ are considered as separate roots when one of the wavenumbers becomes complex. This is an implicit assumption in the fundamental solution.

$$k\gamma = \left[(k_1^2+k_2^2)\left(1 - \frac{\tilde{N}^2}{k_1(k_1-i\epsilon)}\right)\right]^{\frac{1}{2}}$$

Either k_1 or k_2 must be transformed into a complex variable, examination of the root indicates that the transformation of k_2 will provide a simpler expression. Let $K_2 = k_2 + i\kappa_2$ substituting into $k\gamma$ we have

$$k\gamma = \frac{1}{\sqrt{k_1^2+\epsilon^2}}\left\{(k_1^2+k_2^2-\kappa_2^2)(k_1^2+\epsilon^2-\tilde{N}^2) + \frac{2k_2\kappa_2\epsilon\tilde{N}^2}{k_1} + i\left[2k_2\kappa_2(k_1^2+\epsilon^2-\tilde{N}^2) - \frac{\tilde{N}^2\epsilon}{k_1}(k_1^2+k_2^2-\kappa_2^2)\right]\right\}^{\frac{1}{2}}$$

Applying the techniques previously established the discontinuity exists on the line $\Im(k\gamma) = 0$, ie

$$\kappa_2 = \frac{-2k_2(k_1^2+\epsilon^2-\tilde{N}^2) \pm \sqrt{4k_2^2(k_1^2+\epsilon^2-\tilde{N}^2)^2 + \frac{4\epsilon^2\tilde{N}^4}{k_1^2}(k_1^2+k_2^2)}}{\frac{2\tilde{N}^2\epsilon}{k_1}}$$

When $k_2 = 0$ $\kappa_2 = \pm k_1$ and when $\epsilon \to 0$ $\kappa_2 = 0$ or ∞. The interval on this line on which $\Re(k\gamma) < 0$ requires numerical examination.

Figure 3 shows graphically the discontinuities indicated by the thick lines.

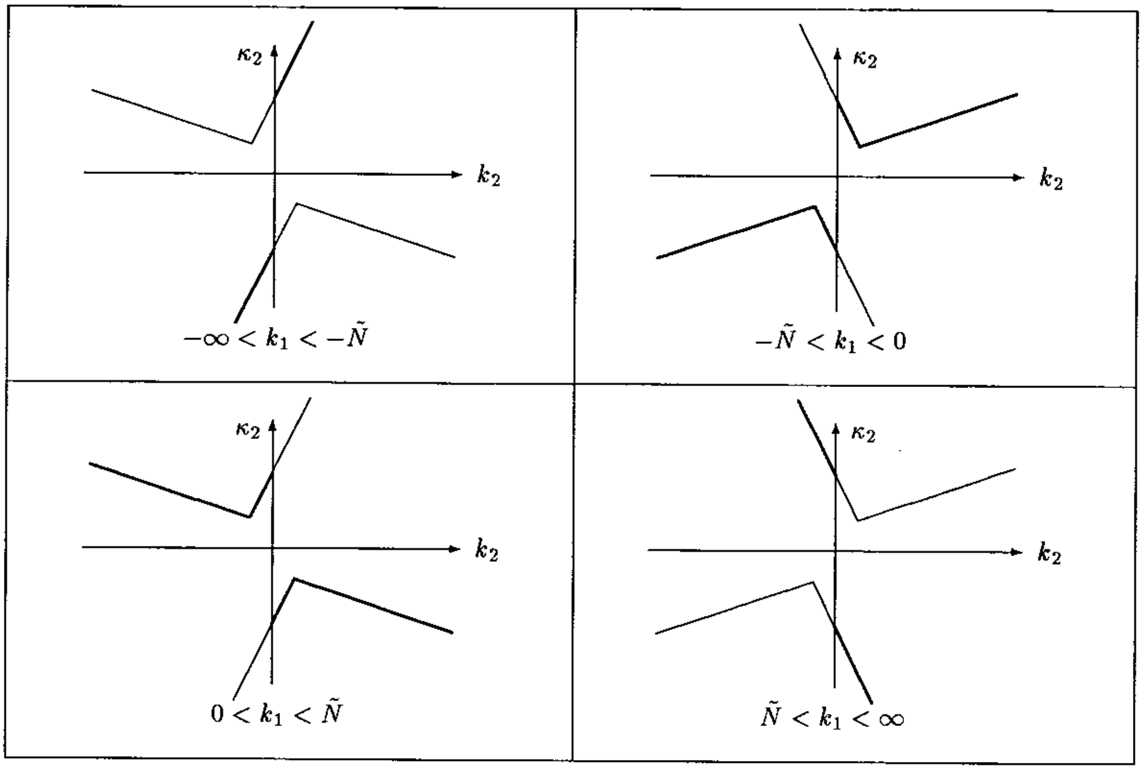

Figure 3: The location of the line discontinuites for $k\gamma$ for a non zero ϵ

Note: the actual appearance of the curves $\kappa_2 = function(k_2)$ are continuous curves here idealised by sets of straight lines. As $\epsilon \longrightarrow 0$ the curves rotate toward the axes and coincide with the axes when $\epsilon = 0$.

The contours required are therefore

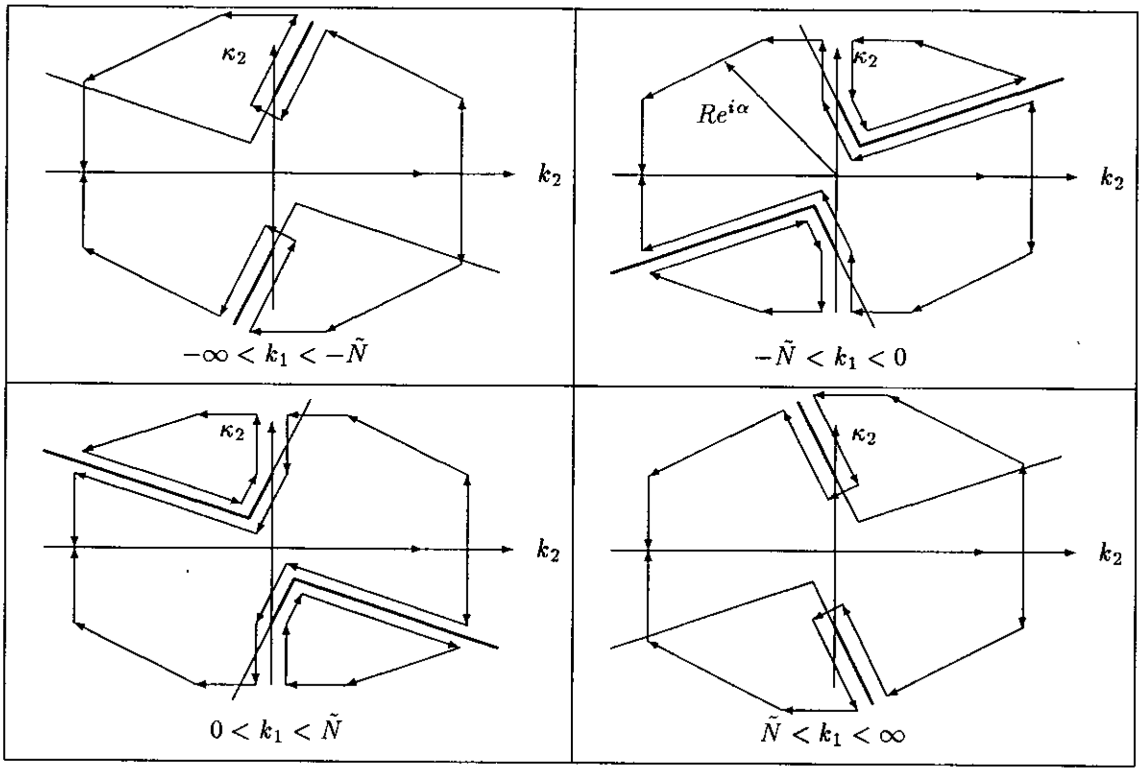

Figure 4: The path required for the contour integration of an integrand involving $k\gamma$ for a non zero ϵ.

The large radius path closing the contour is here represented by a series of straight lines. The behaviour of $k\gamma$ on each of these additional paths now has to be determined as $\epsilon \to 0$. For $r_2 < 0$ we use the upper half plane. Consider each interval individually.

1. $-\infty < k_1 < -\tilde{N}$

   As $\epsilon \to 0$ the cut and paths rotate anticlockwise and become parallel with the κ_2 axis when $\epsilon = 0$.

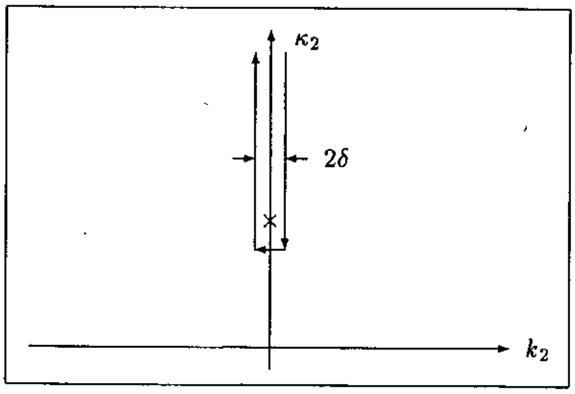

Figure 5: The path required for the contour integration of an integrand involving $k\gamma$ as $\epsilon \to 0$ for the interval $-\infty < k_1 < -\tilde{N}$.

On the right hand path $k\gamma$ becomes

$$k\gamma = \frac{1}{|k_1|}\left[(k_1^2+\delta^2-\kappa_2^2)(k_1^2-\tilde{N}^2)+i\delta\kappa_2(k_1^2-\tilde{N}^2)\right]^{\frac{1}{2}}$$

therefore as $\delta \to 0 \quad k\gamma = i\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(k_1^2-\tilde{N}^2)}$

whereas on the left hand path

$$k\gamma = \frac{1}{|k_1|}\left[(k_1^2+\delta^2-\kappa_2^2)(k_1^2-\tilde{N}^2)-i\delta\kappa_2(k_1^2-\tilde{N}^2)\right]^{\frac{1}{2}}$$

therefore $k\gamma = -i\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(k_1^2-\tilde{N}^2)}$ as $\epsilon \to 0$.

2. $-\tilde{N} < k_1 < 0$

   As $\epsilon \to 0$ the cut and paths rotate clockwise and become parallel with the k_2 and κ_2 axes when $\epsilon = 0$.

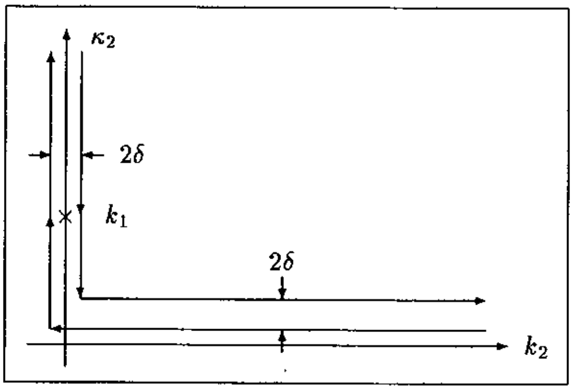

Figure 6: The path required for the contour integration of an integrand involving $k\gamma$ as $\epsilon \to 0$ for the interval $-\tilde{N} < k_1 < 0$.

Appling the same technique we find that on the upper right and left hand paths above $\kappa_2 = k_1$ as $\epsilon \to 0$ $k\gamma$ becomes $k\gamma = \sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(\tilde{N}^2-k_1^2)}$. On the lower right hand path $k\gamma$ becomes $k\gamma = -i\sqrt{(1-\frac{\kappa_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}$ and on the lower left hand path $k\gamma = i\sqrt{(1-\frac{\kappa_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}$.

The horizontal paths require slightly more care. On the upper horizontal path

$$k\gamma = \frac{1}{\sqrt{k_1^2+\epsilon^2}}\left\{\left[k_1^2+k_2^2-(\kappa_2+\delta)^2\right]\left[k_1^2+\epsilon^2-\tilde{N}^2\right]+\frac{2k_2(\kappa_2+\delta)\epsilon\tilde{N}^2}{k_1}\right.$$

$$\left.+i\left[2k_2(\kappa_2+\delta)(k_1^2+\epsilon^2-\tilde{N}^2)-\frac{\tilde{N}^2\epsilon}{k_1}\left[k_1^2+k_2^2-(\kappa_2+\delta)^2\right]\right]\right\}^{\frac{1}{2}}$$

As $\epsilon \to 0$ $\kappa_2 \to 0$ then as $\delta \to 0$ $k\gamma$ becomes $k\gamma = -i\sqrt{(1+\frac{k_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}$. On the lower horizontal path $k\gamma = i\sqrt{(1+\frac{k_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}$.

3. $0 < k_1 < \tilde{N}$

   As $\epsilon \to 0$ the cut and paths rotate anticlockwise and become parallel with the k_2 and κ_2 axes when $\epsilon = 0$.

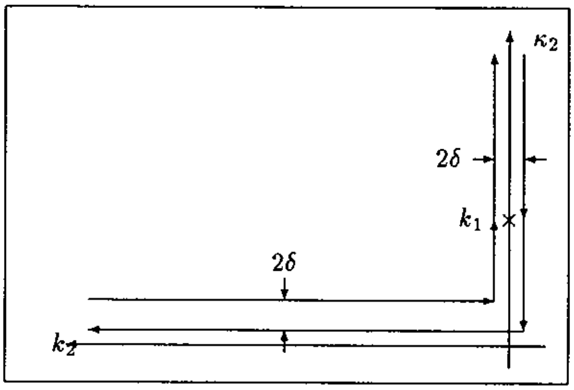

Figure 7: The path required for the contour integration of an integrand involving $k\gamma$ as $\epsilon \to 0$ for the interval $0 < k_1 < \tilde{N}$.

Here we find that on the upper right and left hand paths above $\kappa_2 = k_1$ as $\epsilon \to 0$ $k\gamma$ becomes $k\gamma = \sqrt{(\frac{\kappa_2^2}{k_1^2} - 1)(\tilde{N}^2 - k_1^2)}$. On the lower right hand path $k\gamma$ becomes $k\gamma = -i\sqrt{(1 - \frac{\kappa_2^2}{k_1^2})(\tilde{N}^2 - k_1^2)}$ and on the lower left hand path $k\gamma = i\sqrt{(1 - \frac{\kappa_2^2}{k_1^2})(\tilde{N}^2 - k_1^2)}$.

On the upper horizontal path $k\gamma$ becomes $k\gamma = i\sqrt{(1 + \frac{k_2^2}{k_1^2})(\tilde{N}^2 - k_1^2)}$ whilst on the lower horizontal path $k\gamma = -i\sqrt{(1 + \frac{k_2^2}{k_1^2})(\tilde{N}^2 - k_1^2)}$.

4. $\tilde{N} < k_1 < \infty$

   Here the process is identical to the first interval considered, on the right hand path $k\gamma = i\sqrt{(\frac{\kappa_2^2}{k_1^2} - 1)(k_1^2 - \tilde{N}^2)}$ whereas on the left hand path $k\gamma = -i\sqrt{(\frac{\kappa_2^2}{k_1^2} - 1)(k_1^2 - \tilde{N}^2)}$.

Combining all this information allows us now to complete the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{(z \mp \zeta)k\gamma}}{k\gamma} e^{-i(r_1 k_1 + r_2 k_2)} dk_1 dk_2 \qquad (z < \zeta < 0 \;\; , \;\; r_2 < 0)$$

if the k_2 variable is transformed into the complex variable $K_2 = k_2 + i\kappa_2$. Beginning with the large contours we have

$$\int_{-\infty}^{-\tilde{N}} e^{-ir_1 k_1} \int_{\infty}^{|k_1|} \frac{e^{(z \mp \zeta)i\sqrt{(\frac{\kappa_2^2}{k_1^2} - 1)(k_1^2 - \tilde{N}^2)}}}{i\sqrt{(\frac{\kappa_2^2}{k_1^2} - 1)(k_1^2 - \tilde{N}^2)}} e^{r_2 \kappa_2} i d\kappa_2 dk_1$$

$$+ \int_{-\infty}^{-\tilde{N}} e^{-ir_1 k_1} \int_{|k_1|}^{\infty} \frac{e^{-(z \mp \zeta)i\sqrt{(\frac{\kappa_2^2}{k_1^2} - 1)(k_1^2 - \tilde{N}^2)}}}{-i\sqrt{(\frac{\kappa_2^2}{k_1^2} - 1)(k_1^2 - \tilde{N}^2)}} e^{r_2 \kappa_2} i d\kappa_2 dk_1$$

$$+\int_{-\tilde{N}}^{0} e^{-ir_1k_1} \int_{\infty}^{0} \frac{e^{(z\mp\zeta)i\sqrt{(1+\frac{k_2^2}{k_2^2})(\tilde{N}^2-k_1^2)}}}{i\sqrt{(1+\frac{k_2^2}{k_2^2})(\tilde{N}^2-k_1^2)}} e^{-ir_2k_2} dk_2 dk_1$$

$$+\int_{-\tilde{N}}^{0} e^{-ir_1k_1} \int_{0}^{|k_1|} \frac{e^{(z\mp\zeta)i\sqrt{(1-\frac{\kappa_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}}}{i\sqrt{(1-\frac{\kappa_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}} e^{r_2\kappa_2} i d\kappa_2 dk_1$$

$$+\int_{-\tilde{N}}^{0} e^{-ir_1k_1} \int_{|k_1|}^{\infty} \frac{e^{(z\mp\zeta)\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(\tilde{N}^2-k_1^2)}}}{\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(\tilde{N}^2-k_1^2)}} e^{r_2\kappa_2} i d\kappa_2 dk_1$$

$$+\int_{0}^{\tilde{N}} e^{-ir_1k_1} \int_{\infty}^{|k_1|} \frac{e^{(z\mp\zeta)\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(\tilde{N}^2-k_1^2)}}}{\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(\tilde{N}^2-k_1^2)}} e^{r_2\kappa_2} i d\kappa_2 dk_1$$

$$+\int_{0}^{\tilde{N}} e^{-ir_1k_1} \int_{|k_1|}^{0} \frac{e^{-(z\mp\zeta)i\sqrt{(1-\frac{\kappa_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}}}{-i\sqrt{(1-\frac{\kappa_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}} e^{r_2\kappa_2} i d\kappa_2 dk_1$$

$$+\int_{0}^{\tilde{N}} e^{-ir_1k_1} \int_{0}^{-\infty} \frac{e^{-(z\mp\zeta)i\sqrt{(1+\frac{k_2^2}{k_2^2})(\tilde{N}^2-k_1^2)}}}{-i\sqrt{(1+\frac{k_2^2}{k_2^2})(\tilde{N}^2-k_1^2)}} e^{-ir_2k_2} dk_2 dk_1$$

$$+\int_{\tilde{N}}^{\infty} e^{-ir_1k_1} \int_{\infty}^{|k_1|} \frac{e^{(z\mp\zeta)i\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(k_1^2-\tilde{N}^2)}}}{i\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(k_1^2-\tilde{N}^2)}} e^{r_2\kappa_2} i d\kappa_2 dk_1$$

$$+\int_{\tilde{N}}^{\infty} e^{-ir_1k_1} \int_{|k_1|}^{\infty} \frac{e^{-(z\mp\zeta)i\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(k_1^2-\tilde{N}^2)}}}{-i\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(k_1^2-\tilde{N}^2)}} e^{r_2\kappa_2} i d\kappa_2 dk_1$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{(z\mp\zeta)k\gamma}}{k\gamma} e^{-i(r_1k_1+r_2k_2)} dk_1 dk_2 = 0$$

The integrals on the large radius path have been omitted as they tend to zero as $R \to \infty$. The smaller contours provide

$$\int_{-\tilde{N}}^{0} e^{-ir_1k_1} \int_{\infty}^{|k_1|} \frac{e^{(z\mp\zeta)\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(\tilde{N}^2-k_1^2)}}}{\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(\tilde{N}^2-k_1^2)}} e^{r_2\kappa_2} i d\kappa_2 dk_1$$

$$+\int_{-\tilde{N}}^{0} e^{-ir_1k_1} \int_{|k_1|}^{0} \frac{e^{-(z\mp\zeta)i\sqrt{(1-\frac{\kappa_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}}}{-i\sqrt{(1-\frac{\kappa_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}} e^{r_2\kappa_2} i d\kappa_2 dk_1$$

$$+\int_{-\tilde{N}}^{0} e^{-ir_1k_1} \int_{0}^{\infty} \frac{e^{-(z\mp\zeta)i\sqrt{(1+\frac{k_2^2}{k_2^2})(\tilde{N}^2-k_1^2)}}}{-i\sqrt{(1+\frac{k_2^2}{k_2^2})(\tilde{N}^2-k_1^2)}} e^{-ir_2k_2} dk_2 dk_1$$

$$+\int_{0}^{\tilde{N}} e^{-ir_1k_1} \int_{-\infty}^{0} \frac{e^{(z\mp\zeta)i\sqrt{(1+\frac{k_2^2}{k_2^2})(\tilde{N}^2-k_1^2)}}}{i\sqrt{(1+\frac{k_2^2}{k_2^2})(\tilde{N}^2-k_1^2)}} e^{-ir_2k_2} dk_2 dk_1$$

$$+\int_{0}^{\tilde{N}} e^{-ir_1k_1} \int_{0}^{|k_1|} \frac{e^{(z\mp\zeta)i\sqrt{(1-\frac{\kappa_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}}}{i\sqrt{(1-\frac{\kappa_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}} e^{r_2\kappa_2} i d\kappa_2 dk_1$$

$$+\int_{0}^{\tilde{N}} e^{-ir_1k_1} \int_{|k_1|}^{\infty} \frac{e^{(z\mp\zeta)\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(\tilde{N}^2-k_1^2)}}}{\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(\tilde{N}^2-k_1^2)}} e^{r_2\kappa_2} i d\kappa_2 dk_1 = 0$$

Adding these two expressions and simplifying gives

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{(z\mp\zeta)k\gamma}}{k\gamma} e^{-i(r_1k_1+r_2k_2)} dk_1 dk_2$$

$$= 4\int_{\tilde{N}}^{\infty} \cos r_1k_1 \int_{k_1}^{\infty} \frac{\cos(z\mp\zeta)\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(k_1^2-\tilde{N}^2)}}{\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(k_1^2-\tilde{N}^2)}} e^{r_2\kappa_2} d\kappa_2 dk_1$$

$$-4\int_{0}^{\tilde{N}} \cos r_1k_1 \int_{0}^{k_1} \frac{\cos(z\mp\zeta)\sqrt{(1-\frac{\kappa_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}}{\sqrt{(1-\frac{\kappa_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}} e^{r_2\kappa_2} d\kappa_2 dk_1$$

$$+4\int_0^{\tilde N}\int_0^\infty \frac{\cos(z\mp\zeta)\sqrt{(1+\frac{k_2^2}{k_1^2})(\tilde N^2-k_1^2)}}{\sqrt{(1+\frac{k_2^2}{k_1^2})(\tilde N^2-k_1^2)}}\left[\sin r_1k_1\cos r_2k_2-\cos r_1k_1\sin r_2k_2\right]dk_2dk_1 \qquad (z<\zeta<0\,,\ r_2<0)$$

Similarly for $r_2>0$ we have

$$\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{e^{(z\mp\zeta)k\gamma}}{k\gamma}e^{-i(r_1k_1+r_2k_2)}dk_1dk_2$$

$$=4\int_{\tilde N}^\infty \cos r_1k_1\int_{k_1}^\infty \frac{\cos(z\mp\zeta)\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(k_1^2-\tilde N^2)}}{\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(k_1^2-\tilde N^2)}}e^{-r_2\kappa_2}d\kappa_2dk_1$$

$$-4\int_0^{\tilde N}\cos r_1k_1\int_0^{k_1}\frac{\cos(z\mp\zeta)\sqrt{(1-\frac{\kappa_2^2}{k_1^2})(\tilde N^2-k_1^2)}}{\sqrt{(1-\frac{\kappa_2^2}{k_1^2})(\tilde N^2-k_1^2)}}e^{-r_2\kappa_2}d\kappa_2dk_1$$

$$+4\int_0^{\tilde N}\int_0^\infty \frac{\cos(z\mp\zeta)\sqrt{(1+\frac{k_2^2}{k_1^2})(\tilde N^2-k_1^2)}}{\sqrt{(1+\frac{k_2^2}{k_1^2})(\tilde N^2-k_1^2)}}\left[\sin r_1k_1\cos r_2k_2+\cos r_1k_1\sin r_2k_2\right]dk_2dk_1 \qquad (z<\zeta<0\,,\ r_2>0)$$

therefore the combined result for $-\infty<r_2<\infty$ is

$$\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{e^{(z\mp\zeta)k\gamma}}{k\gamma}e^{-i(r_1k_1+r_2k_2)}dk_1dk_2$$

$$=4\int_{\tilde N}^\infty \cos r_1k_1\int_{k_1}^\infty \frac{\cos(z\mp\zeta)\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(k_1^2-\tilde N^2)}}{\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(k_1^2-\tilde N^2)}}e^{-|r_2|\kappa_2}d\kappa_2dk_1$$

$$-4\int_0^{\tilde N}\cos r_1k_1\int_0^{k_1}\frac{\cos(z\mp\zeta)\sqrt{(1-\frac{\kappa_2^2}{k_1^2})(\tilde N^2-k_1^2)}}{\sqrt{(1-\frac{\kappa_2^2}{k_1^2})(\tilde N^2-k_1^2)}}e^{-|r_2|\kappa_2}d\kappa_2dk_1$$

$$+4\int_0^{\tilde N}\int_0^\infty \frac{\cos(z\mp\zeta)\sqrt{(1+\frac{k_2^2}{k_1^2})(\tilde N^2-k_1^2)}}{\sqrt{(1+\frac{k_2^2}{k_1^2})(\tilde N^2-k_1^2)}}\sin(r_1k_1+|r_2|k_2)dk_2dk_1 \qquad (z<\zeta<0)$$

Combining the source and image terms finally gives

$$-(2\pi)^2\psi(\boldsymbol{x},\boldsymbol{\xi}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{(z-\zeta)k\gamma}+e^{(z+\zeta)k\gamma}}{2k\gamma}e^{-i(r_1k_1+r_2k_2)}dk_1dk_2$$

$$= 4\int_{\tilde N}^{\infty}\cos r_1k_1\int_{k_1}^{\infty}\frac{\cos z\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(k_1^2-\tilde N^2)}\cos\zeta\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(k_1^2-\tilde N^2)}}{\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(k_1^2-\tilde N^2)}}e^{-|r_2|\kappa_2}d\kappa_2dk_1$$

$$-4\int_0^{\tilde N}\cos r_1k_1\int_0^{k_1}\frac{\cos z\sqrt{(1-\frac{\kappa_2^2}{k_1^2})(\tilde N^2-k_1^2)}\cos\zeta\sqrt{(1-\frac{\kappa_2^2}{k_1^2})(\tilde N^2-k_1^2)}}{\sqrt{(1-\frac{\kappa_2^2}{k_1^2})(\tilde N^2-k_1^2)}}e^{-|r_2|\kappa_2}d\kappa_2dk_1$$

$$+4\int_0^{\tilde N}\int_0^{\infty}\frac{\cos z\sqrt{(1+\frac{\kappa_2^2}{k_1^2})(\tilde N^2-k_1^2)}\cos\zeta\sqrt{(1+\frac{\kappa_2^2}{k_1^2})(\tilde N^2-k_1^2)}}{\sqrt{(1+\frac{\kappa_2^2}{k_1^2})(\tilde N^2-k_1^2)}}\sin(r_1k_1+|r_2|k_2)dk_2dk_1 \qquad (z<\zeta<0) \tag{40}$$

The $\zeta < z < 0$ case can be obtained through exchanging z and ζ.

The source or image term can be manipulated to obtain the expression describing ψ for a constant N fluid as detailed in Price and Westlake(1994b) (pages 7-8). Examine the first term on the right hand side.

$$\int_{\tilde N}^{\infty}\int_{k_1}^{\infty}\frac{\cos(z\mp\zeta)\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(k_1^2-\tilde N^2)}}{\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(k_1^2-\tilde N^2)}}e^{-|r_2|\kappa_2}d\kappa_2\cos r_1k_1dk_1$$

if we let $p=\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(k_1^2-\tilde N^2)}$ the integral is transformed into

$$\int_{\tilde N}^{\infty}\frac{k_1}{\sqrt{k_1^2-\tilde N^2}}\int_0^{\infty}\frac{e^{\frac{-|r_2|k_1}{\sqrt{k_1^2-\tilde N^2}}\sqrt{p^2+k_1^2-\tilde N^2}}\cos(z\mp\zeta)p}{\sqrt{p^2+k_1^2-\tilde N^2}}dp\cos r_1k_1dk_1$$

Applying the integral equality, Gradshteyn and Ryzhik(1980) 3.961(2)

$$\int_0^{\infty}\frac{e^{-\beta\sqrt{\gamma^2+x^2}}\cos ax}{\sqrt{\gamma^2+x^2}}dx = K_0\left(\gamma\sqrt{a^2+\beta^2}\right) \qquad \Re(\beta)>0\ ,\ \Re(\gamma)>0\ ,\ a>0$$

with $a=|z\mp\zeta|$, $\beta=\frac{|r_2|k_1}{\sqrt{k_1^2-\tilde N^2}}$ and $\gamma=\sqrt{k_1^2-\tilde N^2}$ we have

$$\int_{\tilde N}^{\infty}\int_{k_1}^{\infty}\frac{\cos(z\mp\zeta)\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(k_1^2-\tilde N^2)}}{\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(k_1^2-\tilde N^2)}}e^{-|r_2|\kappa_2}d\kappa_2\cos r_1k_1dk_1$$

$$= \int_{\tilde{N}}^{\infty} \frac{k_1}{\sqrt{k_1^2 - \tilde{N}^2}} K_0 \left\{ \sqrt{k_1^2 \left[r_2^2 + (z \mp \zeta)^2 \right] - \tilde{N}^2 (z \mp \zeta)^2} \right\} \cos r_1 k_1 dk_1$$

The second term and part of the third term can be combined

$$-4 \int_0^{\tilde{N}} \int_0^{k_1} \frac{\cos(z \mp \zeta) \sqrt{(1 - \frac{\kappa_2^2}{k_1^2})(\tilde{N}^2 - k_1^2)}}{\sqrt{(1 - \frac{\kappa_2^2}{k_1^2})(\tilde{N}^2 - k_1^2)}} e^{-|r_2|\kappa_2} d\kappa_2 \cos r_1 k_1 dk_1$$

$$+4 \int_0^{\tilde{N}} \int_0^{\infty} \frac{\cos(z \mp \zeta) \sqrt{(1 + \frac{k_2^2}{k_1^2})(\tilde{N}^2 - k_1^2)}}{\sqrt{(1 + \frac{k_2^2}{k_1^2})(\tilde{N}^2 - k_1^2)}} \sin |r_2| k_2 dk_2 \cos r_1 k_1 dk_1$$

giving

$$-\frac{\pi}{2} \int_0^{\beta} \frac{k_1}{\sqrt{\tilde{N}^2 - k_1^2}} J_0 \left\{ \sqrt{\tilde{N}^2 (z \mp \zeta)^2 - k_1^2 \left[r_2^2 + (z \mp zeta)^2 \right]} \right\} \cos r_1 k_1 dk_1$$

The remaining term on the right hand side is

$$\int_0^{\tilde{N}} \int_0^{\infty} \frac{\cos(z \mp \zeta) \sqrt{(1 + \frac{k_2^2}{k_1^2})(\tilde{N}^2 - k_1^2)}}{\sqrt{(1 + \frac{k_2^2}{k_1^2})(\tilde{N}^2 - k_1^2)}} \cos |r_2| k_2 dk_2 \sin r_1 k_1 dk_1$$

Applying an integral equality, Gradshteyn and Ryzhik(1980) 3.876(2)

$$\int_0^{\infty} \frac{\cos\left(p\sqrt{x^2 + a^2}\right)}{\sqrt{x^2 + a^2}} \cos bx dx = \begin{cases} -\frac{\pi}{2} Y_0 \left(a\sqrt{p^2 - b^2} \right) & (0 < b < p) \\ K_0 \left(a\sqrt{b^2 - p^2} \right) & (b < p < 0) \end{cases}$$

with $a = k_1$, $b = |r_2|$ and $p = |z \pm \zeta| \frac{\sqrt{k_1^2 - \tilde{N}^2}}{k_1}$ we have

$$\int_0^{\tilde{N}} \int_0^{\infty} \frac{\cos(z \mp \zeta) \sqrt{(1 + \frac{k_2^2}{k_1^2})(\tilde{N}^2 - k_1^2)}}{\sqrt{(1 + \frac{k_2^2}{k_1^2})(\tilde{N}^2 - k_1^2)}} \cos |r_2| k_2 dk_2 \sin r_1 k_1 dk_1$$

$$= \int_{\beta}^{\tilde{N}} \frac{k_1}{\sqrt{\tilde{N}^2 - k_1^2}} K_0 \left\{ \sqrt{k_1^2 \left[r_2^2 + (z \mp \zeta)^2 \right] - \tilde{N}^2 (z \mp \zeta)^2} \right\} \sin r_1 k_1 dk_1$$

$$-\frac{\pi}{2} \int_0^{\beta} \frac{k_1}{\sqrt{\tilde{N}^2 - k_1^2}} Y_0 \left\{ \sqrt{\tilde{N}^2 (z \mp \zeta)^2 - k_1^2 \left[r_2^2 + (z \mp zeta)^2 \right]} \right\} \sin r_1 k_1 dk_1$$

where β is the lower/upper limit of the integration arising from the inequality $b \gtrless p$, $\beta = -\frac{\tilde{N}(z \mp \zeta)}{\sqrt{r_2^2+(z\mp\zeta)^2}}$.

Collecting the underlined terms provides the the full expression for ψ. We have

$$-2\pi^2\psi(\boldsymbol{x},\boldsymbol{\xi}) = \int_{\tilde{N}}^{\infty} \frac{k_1}{\sqrt{k_1^2 - \tilde{N}^2}} K_0 \left\{ \sqrt{k_1^2\left[r_2^2 + (z\mp\zeta)^2\right] - \tilde{N}^2(z\mp\zeta)^2} \right\} \cos r_1 k_1 dk_1$$

$$\int_{\beta}^{\tilde{N}} \frac{k_1}{\sqrt{\tilde{N}^2 - k_1^2}} K_0 \left\{ \sqrt{k_1^2\left[r_2^2 + (z\mp\zeta)^2\right] - \tilde{N}^2(z\mp\zeta)^2} \right\} \sin r_1 k_1 dk_1$$

$$-\frac{\pi}{2}\int_0^{\beta} \frac{k_1}{\sqrt{\tilde{N}^2 - k_1^2}} Y_0 \left\{ \sqrt{\tilde{N}^2(z\mp\zeta)^2 - k_1^2\left[r_2^2 + (z\mp\zeta)^2\right]} \right\} \sin r_1 k_1 dk_1$$

$$-\frac{\pi}{2}\int_0^{\beta} \frac{k_1}{\sqrt{\tilde{N}^2 - k_1^2}} J_0 \left\{ \sqrt{\tilde{N}^2(z\mp\zeta)^2 - k_1^2\left[r_2^2 + (z\mp\zeta)^2\right]} \right\} \cos r_1 k_1 dk_1 \qquad (z < \zeta < 0)$$

which agrees with the previous formulation for the constant N model. This provides a means of validation of the proposed integration procedure though a verifiable process.

3.4 Extension to the three layer model

Now that the contour integration process has been established we can apply it to the integrals describing the three layer fluid system. The technique has to be extended as more intervals are present. Firstly an additional k has to be introduced on the numerator and denominator so that each γ_j only appears in the form $k\gamma_j$, The k on the numerator is absorbed into the $F_{ij}(k_1, k_2)$ function whilst the k on the denominator appears explicitly. We can split the required integral into eight parts -

$$-(2\pi)^2\phi_{ij}(\boldsymbol{x}, \boldsymbol{\xi}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{F_{ij}(k_1,k_2)}{k^2 D(k_1,k_2)} e^{-i(r_1k_1+r_2k_2)} dk_2 dk_1$$

$$= \int_{-\infty}^{-\tilde{N}_2}\int_{-\infty}^{\infty} \frac{F_{ij}(k_1,k_2)}{k^2 D(k_1,k_2)} e^{-i(r_1k_1+r_2k_2)} dk_2 dk_1$$

$$+ \int_{-\tilde{N}_2}^{-\tilde{N}_1}\int_{-\infty}^{\infty} \frac{F_{ij}(k_1,k_2)}{k^2 D(k_1,k_2)} e^{-i(r_1k_1+r_2k_2)} dk_2 dk_1$$

$$+ \int_{-\tilde{N}_1}^{-\tilde{N}_3}\int_{-\infty}^{\infty} \frac{F_{ij}(k_1,k_2)}{k^2 D(k_1,k_2)} e^{-i(r_1k_1+r_2k_2)} dk_2 dk_1$$

$$+ \int_{-\tilde{N}_3}^{0}\int_{-\infty}^{\infty} \frac{F_{ij}(k_1,k_2)}{k^2 D(k_1,k_2)} e^{-i(r_1k_1+r_2k_2)} dk_2 dk_1$$

$$+ \int_{0}^{\tilde{N}_3}\int_{-\infty}^{\infty} \frac{F_{ij}(k_1,k_2)}{k^2 D(k_1,k_2)} e^{-i(r_1k_1+r_2k_2)} dk_2 dk_1$$

$$+ \int_{\tilde{N}_3}^{\tilde{N}_1}\int_{-\infty}^{\infty} \frac{F_{ij}(k_1,k_2)}{k^2 D(k_1,k_2)} e^{-i(r_1k_1+r_2k_2)} dk_2 dk_1$$

$$+ \int_{\tilde{N}_1}^{\tilde{N}_2}\int_{-\infty}^{\infty} \frac{F_{ij}(k_1,k_2)}{k^2 D(k_1,k_2)} e^{-i(r_1k_1+r_2k_2)} dk_2 dk_1$$

$$+ \int_{\tilde{N}_2}^{\infty}\int_{-\infty}^{\infty} \frac{F_{ij}(k_1,k_2)}{k^2 D(k_1,k_2)} e^{-i(r_1k_1+r_2k_2)} dk_2 dk_1$$

Contour integration can be applied to all eight integrals, The first, fourth, fifth and eighth integrals have no poles in the integrand so the right hand side of the equality wil be zero. The remaining integrals have poles as described in section 3.1 and therefore the right hand side will be $2\pi i S$ where S is the sum of the residues within the contour. As there are now three BV frequencies there are now three forms of $k\gamma$ for each interval so the complexity of the problem increases. To apply the previously described approach additional paths are required to circumvent the discontinuities created by $k\gamma_j$. For $r_2 < 0$ we use the upper half plane. As $\epsilon \rightarrow 0$ the form of $k\gamma_j$ on each section of these additional paths become

1. $-\infty < k_1 < -\tilde{N}_2$

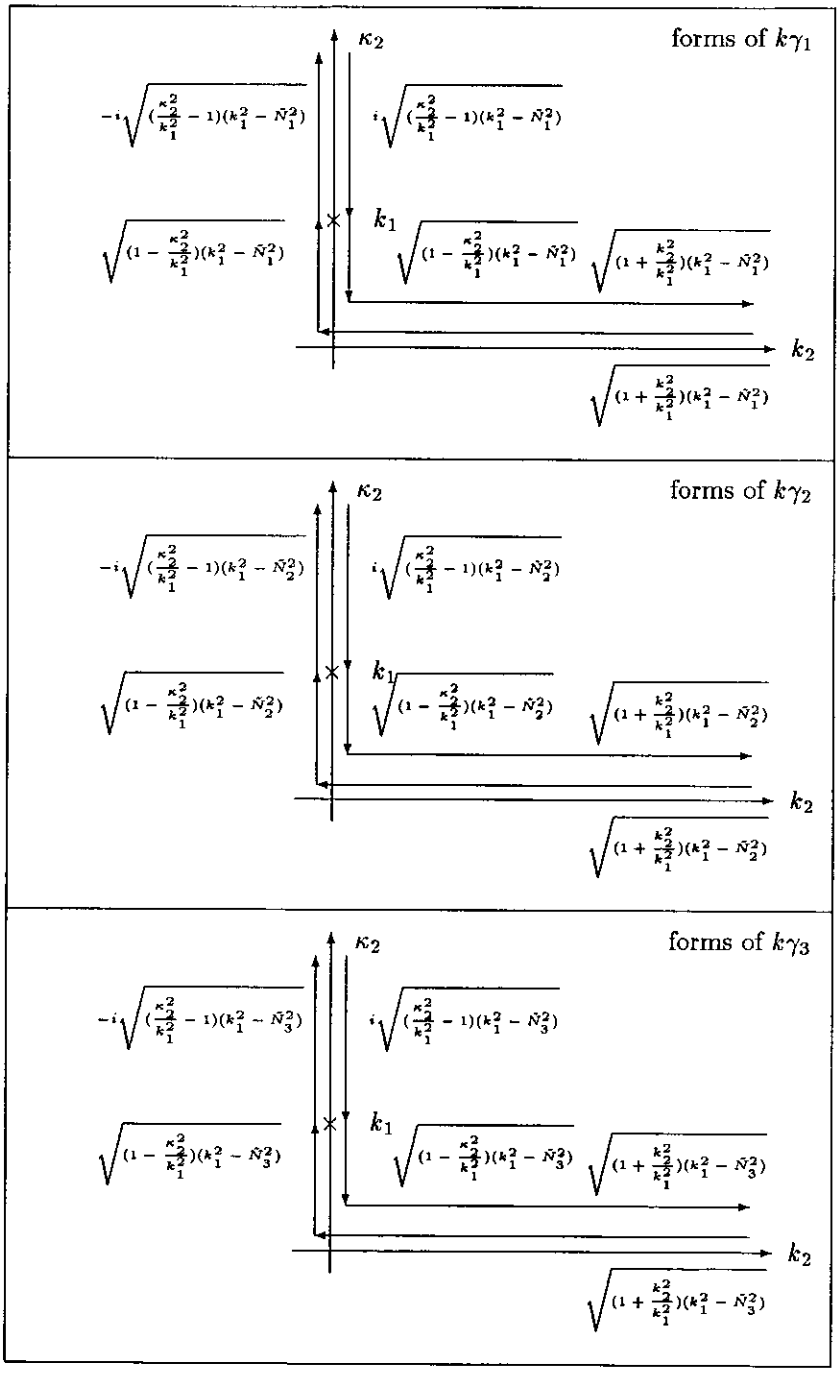

Figure 8: The path required and forms of $k\gamma_j$ as $\epsilon \to 0$ for the interval $-\infty < k_1 < -\tilde{N}_2$.

2. $-\tilde{N}_2 < k_1 < -\tilde{N}_1$

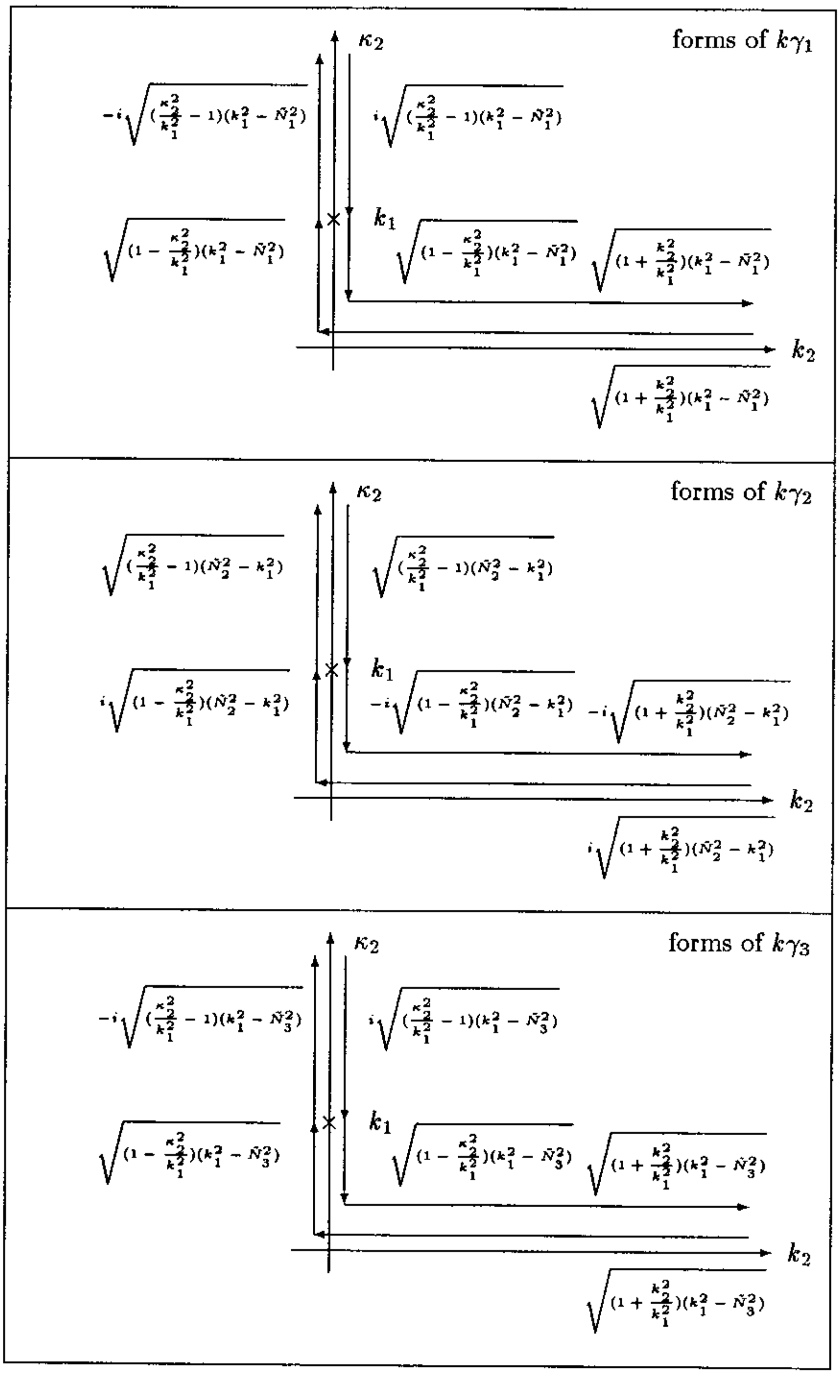

Figure 9: The path required and forms of $k\gamma_j$ as $\epsilon \to 0$ for the interval $-\tilde{N}_2 < k_1 < -\tilde{N}_1$.

3. $-\tilde{N}_1 < k_1 < -\tilde{N}_3$

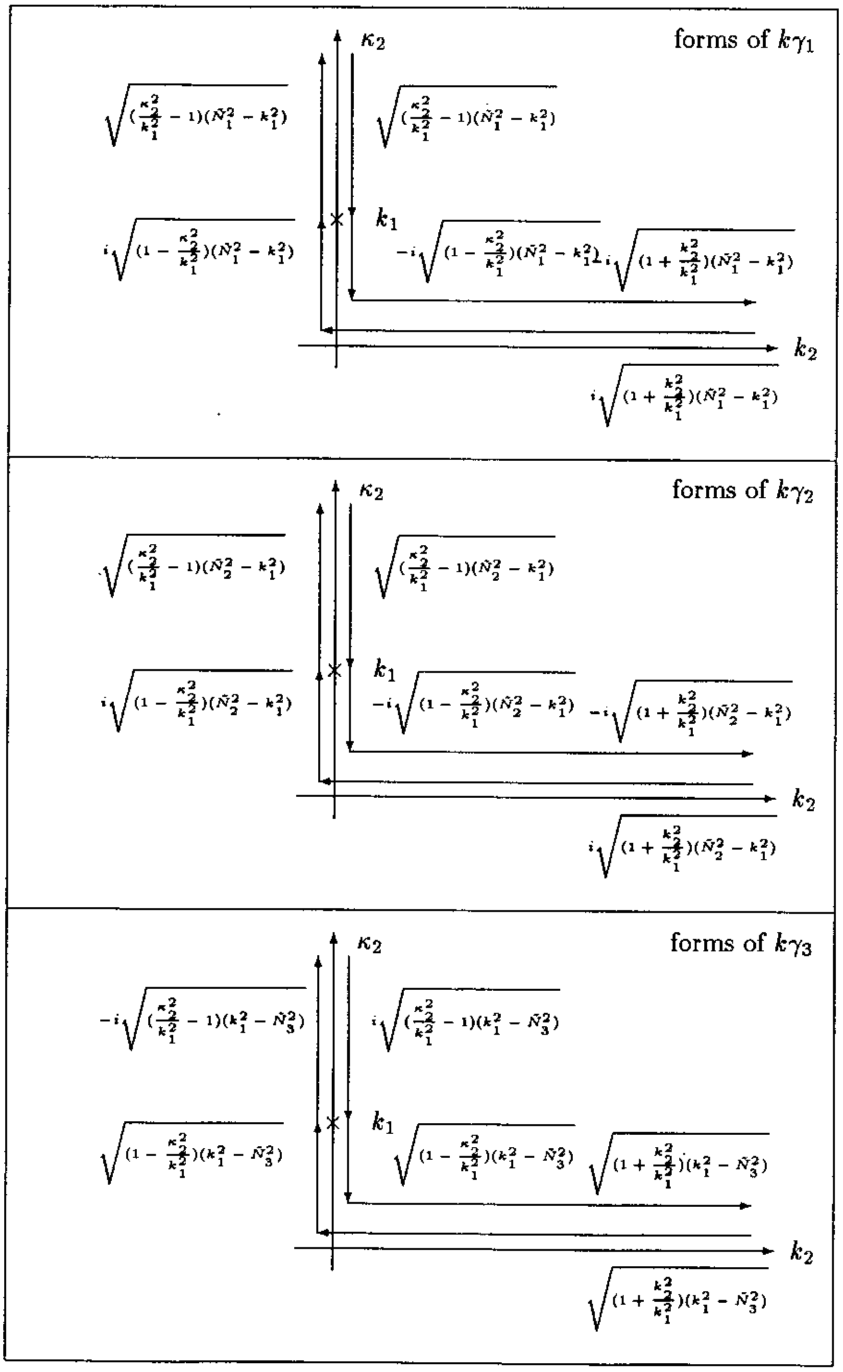

Figure 10: The path required and forms of $k\gamma_j$ as $\epsilon \to 0$ for the interval $-\tilde{N}_1 < k_1 < -\tilde{N}_3$.

4. $-\tilde{N}_3 < k_1 < 0$

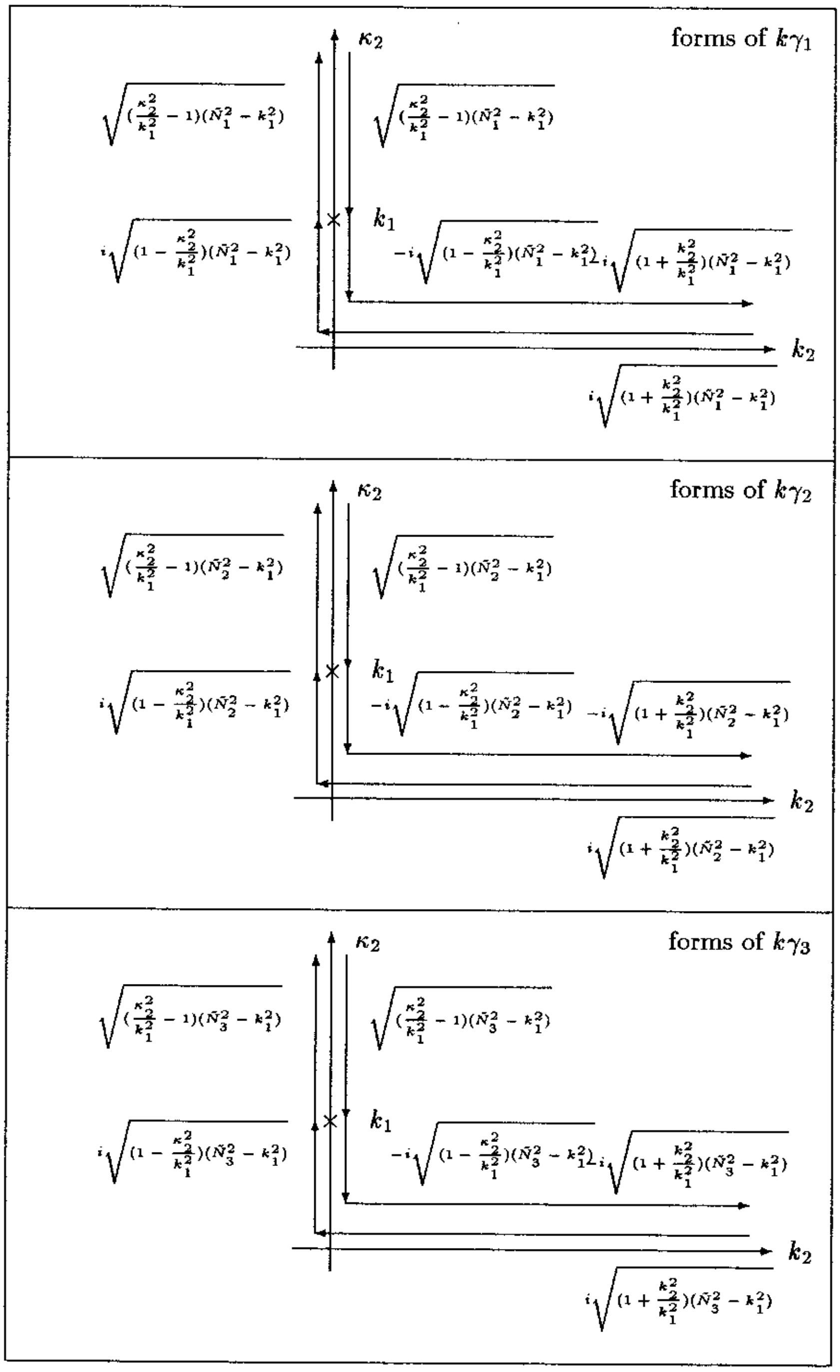

Figure 11: The path required and forms of $k\gamma_j$ as $\epsilon \to 0$ for the interval $-\tilde{N}_3 < k_1 < 0$.

5. $0 < k_1 < \tilde{N}_3$

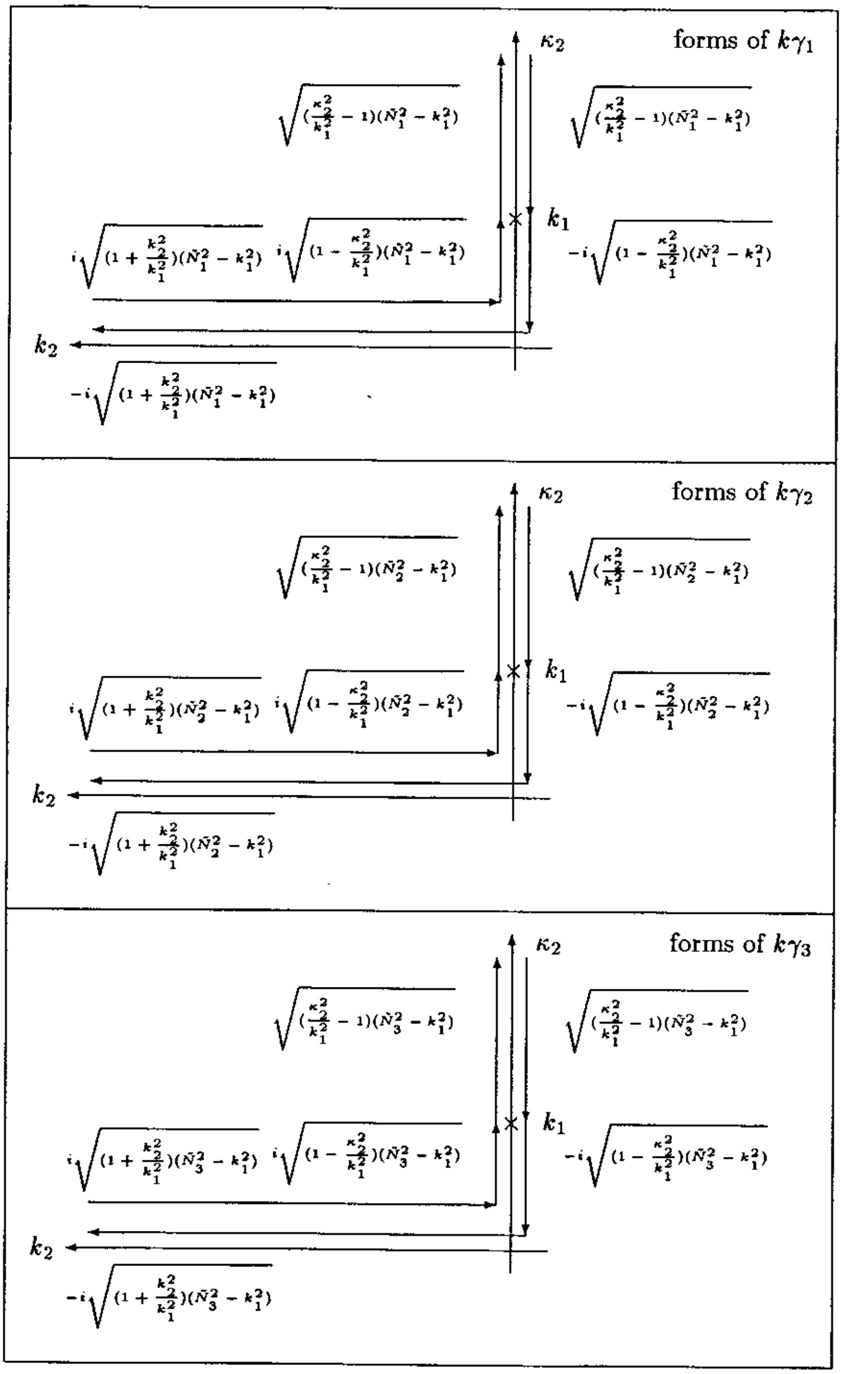

Figure 12: The path required and forms of $k\gamma_j$ as $\epsilon \to 0$ for the interval $0 < k_1 < \tilde{N}_3$.

6. $\tilde{N}_3 < k_1 < \tilde{N}_1$

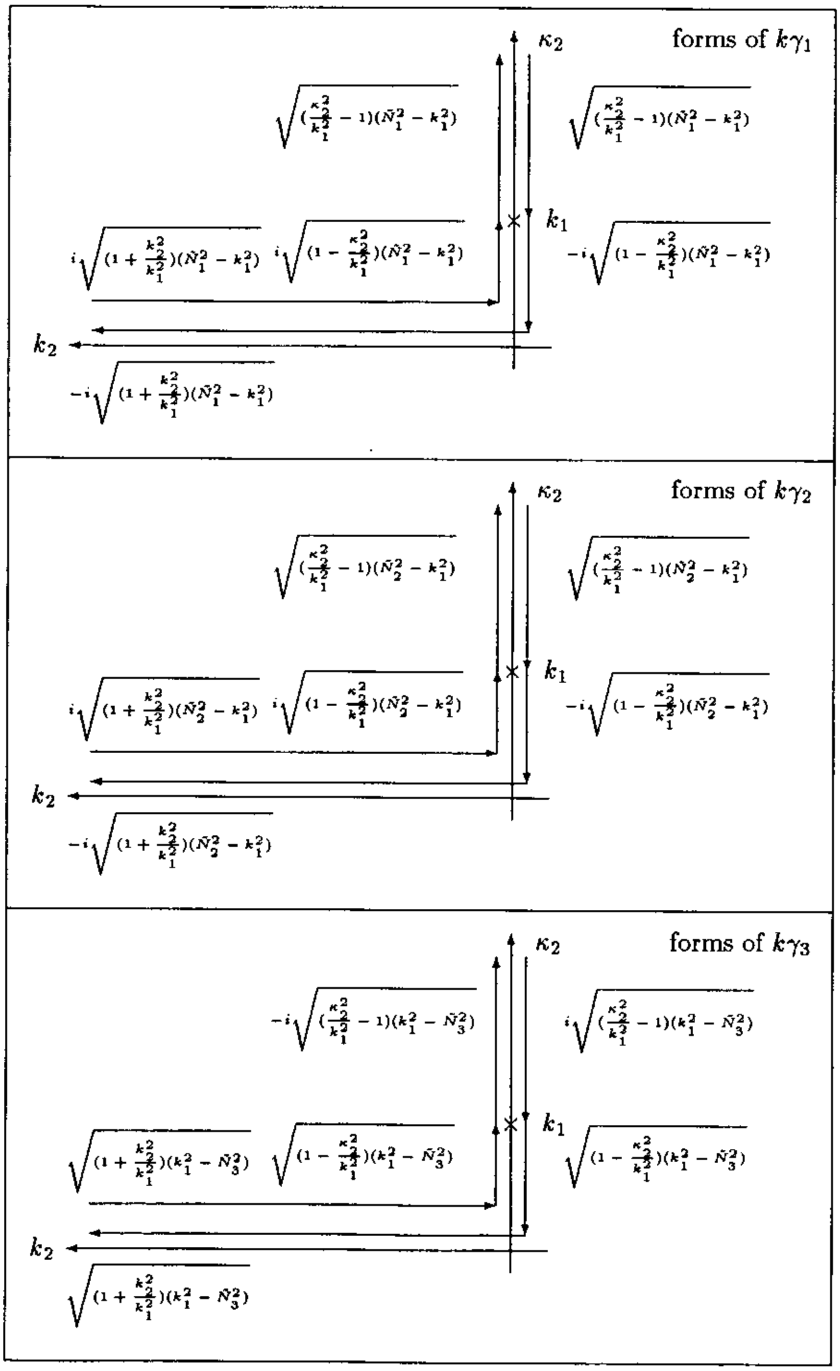

Figure 13: The path required and forms of $k\gamma_j$ as $\epsilon \rightarrow 0$ for the interval $\tilde{N}_3 < k_1 < \tilde{N}_1$.

7. $\tilde{N}_1 < k_1 < \tilde{N}_2$

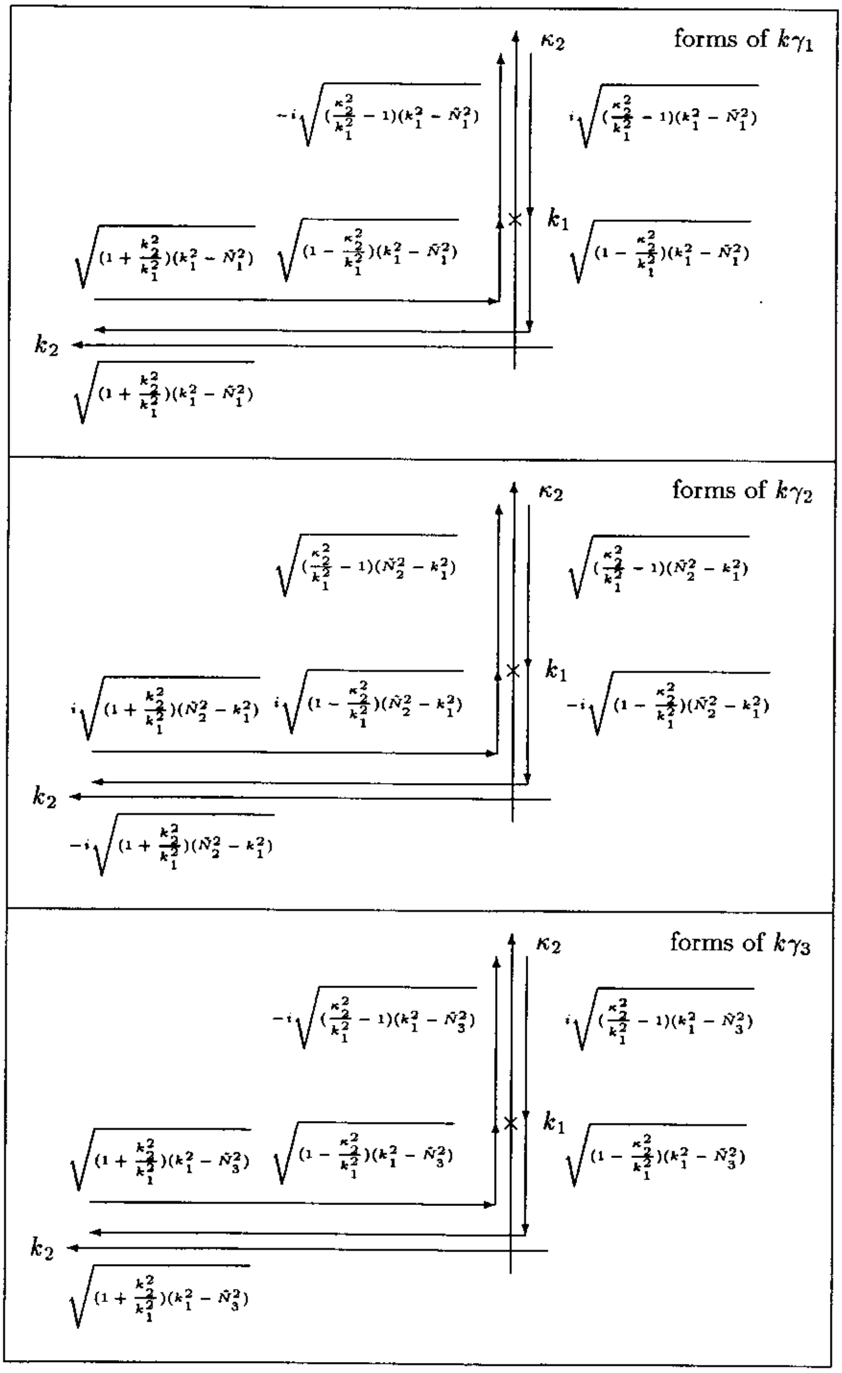

Figure 14: The path required and forms of $k\gamma_j$ as $\epsilon \to 0$ for the interval $\tilde{N}_1 < k_1 < \tilde{N}_2$.

8. $\tilde{N}_2 < k_1 < \infty$

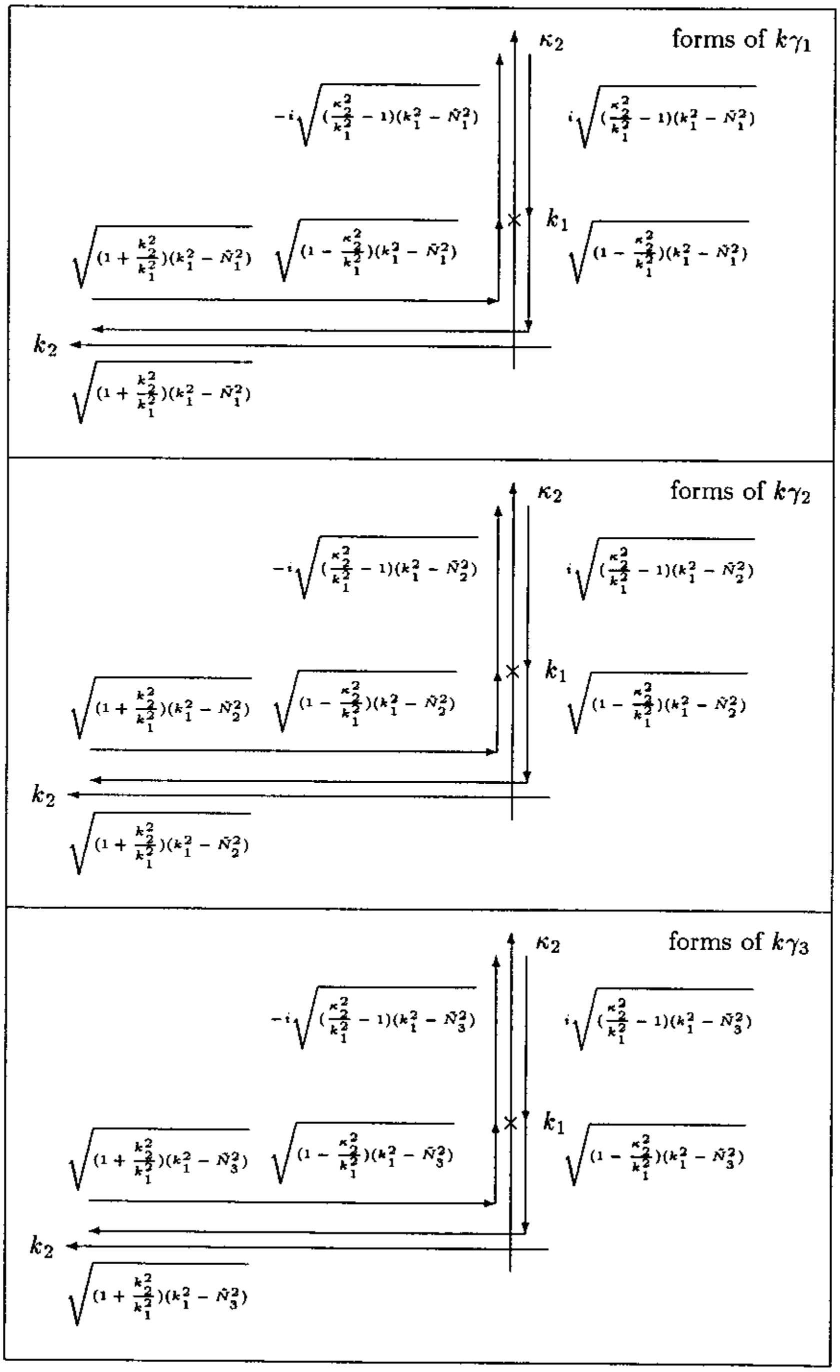

Figure 15: The path required and forms of $k\gamma_j$ as $\epsilon \to 0$ for the interval $\tilde{N}_2 < k_1 < \infty$.

To simplify the mathematics and reduce the length of the expressions each path is allocated a number, this will be used to denote the form of $k\gamma_j$ used in the expressions $F_{ij}(k_1, k_2)$ and $D(k_1, k_2)$ on that path. The path numbering is

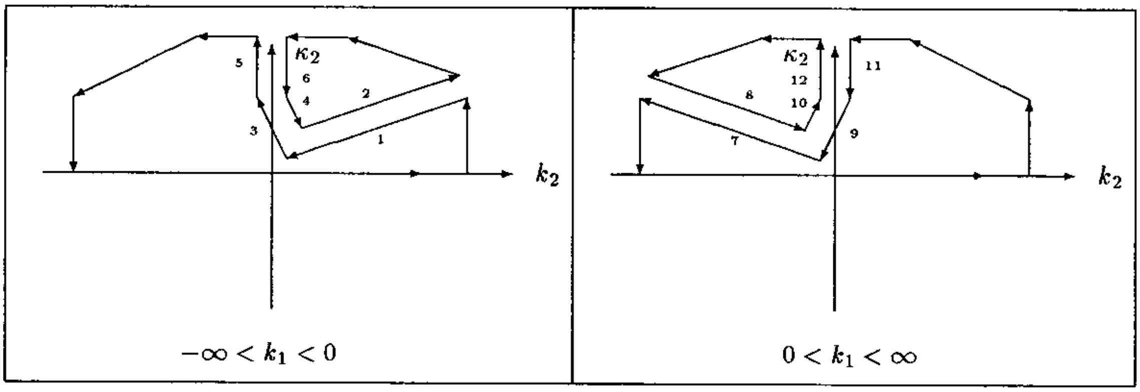

Figure 16: The path numbering for the contour integration.

The application of this numbering system allows the larger contour to provide for any of the pairs of intervals considered eg for the first and eighth intervals $a = \tilde{N}_2$ and $b = \tilde{N}_1$

$$\int_{-a}^{-b} \int_{-\infty}^{\infty} \frac{F_{ij}(k_1, k_2)}{k^2 D(k_1, k_2)} e^{-i(r_1 k_1 + r_2 k_2)} dk_2 dk_1$$

$$+ \int_{-a}^{-b} e^{-ir_1 k_1} \int_{\infty}^{0} \frac{F_{ij}[1](k_1, k_2)}{(k_1^2 + k_2^2) D[1](k_1, k_2)} e^{-ir_2 k_2} dk_2 dk_1$$

$$+ \int_{-a}^{-b} e^{-ir_1 k_1} \int_{0}^{|k_1|} \frac{F_{ij}[3](k_1, \kappa_2)}{(k_1^2 - \kappa_2^2) D[3](k_1, \kappa_2)} e^{r_2 \kappa_2} i \, d\kappa_2 dk_1$$

$$+ \int_{-a}^{-b} e^{-ir_1 k_1} \int_{|k_1|}^{\infty} \frac{F_{ij}[5](k_1, \kappa_2)}{(k_1^2 - \kappa_2^2) D[5](k_1, \kappa_2)} e^{r_2 \kappa_2} i \, d\kappa_2 dk_1$$

$$+ \int_{b}^{a} e^{-ir_1 k_1} \int_{\infty}^{|k_1|} \frac{F_{ij}[11](k_1, \kappa_2)}{(k_1^2 - \kappa_2^2) D[11](k_1, \kappa_2)} e^{r_2 \kappa_2} i \, d\kappa_2 dk_1$$

$$+ \int_{b}^{a} e^{-ir_1 k_1} \int_{|k_1|}^{0} \frac{F_{ij}[9](k_1, \kappa_2)}{(k_1^2 - \kappa_2^2) D[9](k_1, \kappa_2)} e^{r_2 \kappa_2} i \, d\kappa_2 dk_1$$

$$+ \int_{b}^{a} e^{-ir_1 k_1} \int_{0}^{-\infty} \frac{F_{ij}[7](k_1, k_2)}{(k_1^2 + k_2^2) D[7](k_1, k_2)} e^{-ir_2 k_2} dk_2 dk_1$$

$$+ \int_{b}^{a} \int_{-\infty}^{\infty} \frac{F_{ij}(k_1, k_2)}{k^2 D(k_1, k_2)} e^{-i(r_1 k_1 + r_2 k_2)} dk_2 dk_1 = 0$$

as the poles are contained in the smaller contour. This gives

$$\int_{-a}^{-b} e^{-ir_1k_1} \int_{\infty}^{|k_1|} \frac{F_{ij}[6](k_1,\kappa_2)}{(k_1^2-\kappa_2^2)D[6](k_1,\kappa_2)} e^{r_2\kappa_2} i\, d\kappa_2 dk_1$$

$$+\int_{-a}^{-b} e^{-ir_1k_1} \int_{|k_1|}^{0} \frac{F_{ij}[4](k_1,\kappa_2)}{(k_1^2-\kappa_2^2)D[4](k_1,\kappa_2)} e^{r_2\kappa_2} i\, d\kappa_2 dk_1$$

$$+\int_{-a}^{-b} e^{-ir_1k_1} \int_{0}^{\infty} \frac{F_{ij}[2](k_1,k_2)}{(k_1^2+k_2^2)D[2](k_1,k_2)} e^{-ir_2k_2} dk_2 dk_1$$

$$+\int_{b}^{a} e^{-ir_1k_1} \int_{-\infty}^{0} \frac{F_{ij}[8](k_1,k_2)}{(k_1^2+k_2^2)D[8](k_1,k_2)} e^{-ir_2k_2} dk_2 dk_1$$

$$+\int_{b}^{a} e^{-ir_1k_1} \int_{0}^{|k_1|} \frac{F_{ij}[10](^*)}{(k_1^2-\kappa_2^2)D[10](k_1,\kappa_2)} e^{r_2\kappa_2} i\, d\kappa_2 dk_1$$

$$+\int_{b}^{a} e^{-ir_1k_1} \int_{|k_1|}^{\infty} \frac{F_{ij}[12](k_1,\kappa_2)}{(k_1^2-\kappa_2^2)D[12](k_1,\kappa_2)} e^{r_2\kappa_2} i\, d\kappa_2 dk_1 = 2\pi i S$$

where S is the sum of the residues inside the smaller contours. For the intervals $-\infty < k_1 < -\tilde{N}_2$, $-\tilde{N}_3 < k_1 < 0$, $0 < k_1 < \tilde{N}_3$ and $\tilde{N}_2 < k_1 < \infty$ $S = 0$.

The equations may be added and simplified thus

$$\int_{-a}^{-b}\int_{-\infty}^{\infty} \frac{F_{ij}(k_1,k_2)}{k^2 D(k_1,k_2)} e^{-i(r_1k_1+r_2k_2)} dk_2 dk_1$$

$$+\int_{b}^{a}\int_{-\infty}^{\infty} \frac{F_{ij}(k_1,k_2)}{k^2 D(k_1,k_2)} e^{-i(r_1k_1+r_2k_2)} dk_2 dk_1$$

$$= \int_{b}^{a}\int_{0}^{\infty} \frac{1}{k_1^2+k_2^2} \left[\left(\frac{F_{ij}[1](k_1,k_2)}{D[1](k_1,k_2)} - \frac{F_{ij}[2](k_1,k_2)}{D[2](k_1,k_2)} \right) e^{i(r_1k_1-r_2k_2)} + \left(\frac{F_{ij}[7](k_1,k_2)}{D[7](k_1,k_2)} - \frac{F_{ij}[8](k_1,k_2)}{D[8](k_1,k_2)} \right) e^{-i(r_1k_1-r_2k_2)} \right] dk_2 dk_1$$

$$-i\int_{b}^{a}\int_{0}^{k_1} \frac{e^{r_2\kappa_2}}{k_1^2-\kappa_2^2} \left[\left(\frac{F_{ij}[3](k_1,\kappa_2)}{D[3](k_1,\kappa_2)} - \frac{F_{ij}[4](k_1,\kappa_2)}{D[4](k_1,\kappa_2)} \right) e^{ir_1k_1} - \left(\frac{F_{ij}[9](k_1,\kappa_2)}{D[9](k_1,\kappa_2)} - \frac{F_{ij}[10](k_1,\kappa_2)}{D[10](k_1,\kappa_2)} \right) e^{-ir_1k_1} \right] d\kappa_2 dk_1$$

$$-i\int_{b}^{a}\int_{k_1}^{\infty} \frac{e^{r_2\kappa_2}}{k_1^2-\kappa_2^2} \left[\left(\frac{F_{ij}[5](k_1,\kappa_2)}{D[5](k_1,\kappa_2)} - \frac{F_{ij}[6](k_1,\kappa_2)}{D[6](k_1,\kappa_2)} \right) e^{ir_1k_1} - \left(\frac{F_{ij}[11](k_1,\kappa_2)}{D[11](k_1,\kappa_2)} - \frac{F_{ij}[12](k_1,\kappa_2)}{D[12](k_1,\kappa_2)} \right) e^{-ir_1k_1} \right] d\kappa_2 dk_1$$

$$+ 2\pi i S \qquad (r_2 < 0) \tag{41}$$

For each pair of intervals the appropriate form of $k\gamma_j$ must be substituted into the above equation. The substitution may produce complex functions, let us denote the real part as F_{ijr} and the imaginary part as F_{iji}. In theory to complete the k_1 integration four forms of each of the expressions on the right hand side of equation 41 will exist corresponding to the four pairs of intervals giving twelve double integrals. However if the $k\gamma_j$ substituted forms of F_{ij} are examined great simplifications can be made.

1. $-\tilde{N}_3 < k_1 < 0$ and $0 < k_1 < \tilde{N}_3$

   Here the relationships between the various forms of F_{ij} are

$$F_{ij}[1](k_1,k_2) = F^*_{ij}[2](k_1,k_2) = F^*_{ij}[7](k_1,k_2) = F_{ij}[8](k_1,k_2) = F_{ijr} + iF_{iji}$$

$$F_{ij}[3](k_1,\kappa_2) = F^*_{ij}[4](k_1,\kappa_2) = F^*_{ij}[9](k_1,\kappa_2) = F_{ij}[10](k_1,\kappa_2) = F_{ijr} + iF_{iji}$$

$$F_{ij}[5](k_1,\kappa_2) = F_{ij}[6](k_1,\kappa_2) = F_{ij}[11](k_1,\kappa_2) = F_{ij}[12](k_1,\kappa_2)$$

2. $-\tilde{N}_1 < k_1 < -\tilde{N}_3$ and $\tilde{N}_3 < k_1 < \tilde{N}_1$

$$F_{ij}[1](k_1,k_2) = F_{ij}[2](k_1,k_2) = F_{ij}[7](k_1,k_2) = F_{ij}[8](k_1,k_2)$$

$$F_{ij}[3](k_1,\kappa_2) = F_{ij}[4](k_1,\kappa_2) = F_{ij}[9](k_1,\kappa_2) = F_{ij}[10](k_1,\kappa_2)$$

$$F^*_{ij}[5](k_1,\kappa_2) = F_{ij}[6](k_1,\kappa_2) = F_{ij}[11](k_1,\kappa_2) = F^*_{ij}[12](k_1,\kappa_2) = F_{ijr} - iF_{iji}$$

The forms of F_{ij} in the remaining two pairs of intervals are identical to (2) above.

The relationships between the various forms of D are identical to the relationships between F_{ij} in all cases. These relationships are possible as the $\sinh k\gamma_1$ and $\sinh k\gamma_2$ terms always appear with another $k\gamma_1$ and $k\gamma_2$ respectively thus rendering the $\pm$ sign on the imaginary forms of $k\gamma_j$ ineffective, the $\cosh k\gamma_1$ and $\cosh k\gamma_2$ terms have this property alone. Only the $k\gamma_3$ term has the effect of producing a conjugate function.

Using these properties the formulation reduces to

$$-(2\pi)^2\phi_{ij}(\boldsymbol{x}\ ,\boldsymbol{\xi}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{F_{ij}(k_1,k_2)}{k^2D(k_1,k_2)}e^{-i(r_1k_1+r_2k_2)}dk_2dk_1$$

$$= 4\int_0^{\tilde{N}_3}\int_0^{\infty}\frac{1}{k_1^2+k_2^2}\frac{F_{ijr}(k_1,k_2)D_i(k_1,k_2)-F_{iji}(k_1,k_2)D_r(k_1,k_2)}{D_r^2(k_1,k_2)+D_i^2(k_1,k_2)}\sin(r_1k_1-r_2k_2)dk_2dk_1$$

$$-4\int_0^{\tilde{N}_3}\cos r_1k_1\int_0^{k_1}\frac{e^{r_2\kappa_2}}{k_1^2-\kappa_2^2}\frac{F_{ijr}(k_1,\kappa_2)D_i(k_1,\kappa_2)-F_{iji}(k_1,\kappa_2)D_r(k_1,\kappa_2)}{D_r^2(k_1,\kappa_2)+D_i^2(k_1,\kappa_2)}d\kappa_2dk_1$$

$$+4\int_{\tilde{N}_3}^{\infty}\int_{k_1}^{\infty}\cos r_1k_1\frac{e^{r_2\kappa_2}}{k_1^2-\kappa_2^2}\frac{F_{ijr}(k_1,\kappa_2)D_i(k_1,\kappa_2)-F_{iji}(k_1,\kappa_2)D_r(k_1,\kappa_2)}{D_r^2(k_1,\kappa_2)+D_i^2(k_1,\kappa_2)}d\kappa_2dk_1$$

$$+2\pi iS \qquad (r_2 < 0)$$

The various forms of F_{ijr}, F_{iji}, D_r and D_i are listed in appendix B. The integrands no longer contain poles so the integration can be completed successfully. When $\tilde{N}_1 = \tilde{N}_2 = \tilde{N}_3 = \tilde{N}$ $\frac{F_{ijr}D_i - F_{iji}D_r}{D_r^2+D_i^2}$ becomes

1. $0 < k_1 < \tilde{N}$

$$-\sqrt{(1+\frac{k_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}\cos z\sqrt{(1+\frac{k_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}\cos\zeta\sqrt{(1+\frac{k_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}$$

2. $\tilde{N} < k_1 < \infty$

$$-\sqrt{(\frac{k_2^2}{k_1^2}-1)(k_1^2-\tilde{N}^2)}\cos z\sqrt{(\frac{k_2^2}{k_1^2}-1)(k_1^2-\tilde{N}^2)}\cos\zeta\sqrt{(\frac{k_2^2}{k_1^2}-1)(k_1^2-\tilde{N}^2)}$$

Thus

$$-(2\pi)^2\phi_{ij}(\boldsymbol{x}\ ,\boldsymbol{\xi}) =$$

$$-4\int_0^{\tilde{N}}\int_0^\infty \frac{1}{k_1^2+k_2^2}\sqrt{(1+\frac{k_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}\cos z\sqrt{(1+\frac{k_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}\cos\zeta\sqrt{(1+\frac{k_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}\sin(r_1k_1-r_2k_2)dk_2dk_1$$

$$4\int_0^{\tilde{N}}\cos r_1k_1\int_0^{k_1}\frac{e^{r_2\kappa_2}}{k_1^2-\kappa_2^2}\sqrt{(1-\frac{\kappa_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}\cos z\sqrt{(1-\frac{\kappa_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}\cos\zeta\sqrt{(1-\frac{\kappa_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}d\kappa_2dk_1$$

$$-4\int_{\tilde{N}}^\infty\cos r_1k_1\int_{k_1}^\infty\frac{e^{r_2\kappa_2}}{k_1^2-\kappa_2^2}\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(k_1^2-\tilde{N}^2)}\cos z\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(k_1^2-\tilde{N}^2)}\cos\zeta\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(k_1^2-\tilde{N}^2)}d\kappa_2dk_1 \quad (r_2<0)$$

which gives

$$-(2\pi)^2\psi_{ij}(\boldsymbol{x}\ ,\boldsymbol{\xi}) = 4\int_0^{\tilde{N}}\int_0^\infty\frac{\cos z\sqrt{(1+\frac{k_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}\cos\zeta\sqrt{(1+\frac{k_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}}{\sqrt{(1+\frac{k_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}}\sin(r_1k_1-r_2k_2)dk_2dk_1$$

$$-4\int_0^{\tilde{N}}\cos r_1k_1\int_0^{k_1}\frac{\cos z\sqrt{(1-\frac{\kappa_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}\cos\zeta\sqrt{(1-\frac{\kappa_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}}{\sqrt{(1-\frac{\kappa_2^2}{k_1^2})(\tilde{N}^2-k_1^2)}}e^{r_2\kappa_2}d\kappa_2dk_1$$

$$4\int_{\tilde{N}}^\infty\cos r_1k_1\int_{k_1}^\infty\frac{\cos z\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(k_1^2-\tilde{N}^2)}\cos\zeta\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(k_1^2-\tilde{N}^2)}}{\sqrt{(\frac{\kappa_2^2}{k_1^2}-1)(k_1^2-\tilde{N}^2)}}e^{r_2\kappa_2}d\kappa_2dk_1 \qquad (r_2<0)$$

which is identical to equation 40 derived in the previous section.

Extending the formulation for $r_2 > 0$ we have

$$-(2\pi)^2\phi_{ij}(\boldsymbol{x}\ ,\boldsymbol{\xi}) = \int_{-\infty}^\infty\int_{-\infty}^\infty\frac{F_{ij}(k_1,k_2)}{k^2D(k_1,k_2)}e^{-i(r_1k_1+r_2k_2)}dk_2dk_1$$

$$= 4\int_0^{\tilde{N}_3}\int_0^\infty\frac{1}{k_1^2+k_2^2}\frac{F_{ijr}(k_1,k_2)D_i(k_1,k_2)-F_{iji}(k_1,k_2)D_r(k_1,k_2)}{D_r^2(k_1,k_2)+D_i^2(k_1,k_2)}\sin(r_1k_1+|r_2|k_2)dk_2dk_1$$

$$-4\int_0^{\tilde{N}_3}\cos r_1k_1\int_0^{k_1}\frac{e^{-|r_2|\kappa_2}}{k_1^2-\kappa_2^2}\frac{F_{ijr}(k_1,\kappa_2)D_i(k_1,\kappa_2)-F_{iji}(k_1,\kappa_2)D_r(k_1,\kappa_2)}{D_r^2(k_1,\kappa_2)+D_i^2(k_1,\kappa_2)}d\kappa_2dk_1$$

$$+4\int_{\tilde{N}_3}^{\infty}\cos r_1k_1\int_{k_1}^{\infty}\frac{e^{-|r_2|\kappa_2}}{k_1^2-\kappa_2^2}\frac{F_{ijr}(k_1,\kappa_2)D_i(k_1,\kappa_2)-F_{iji}(k_1,\kappa_2)D_r(k_1,\kappa_2)}{D_r^2(k_1,\kappa_2)+D_i^2(k_1,\kappa_2)}d\kappa_2dk_1$$

$$-2\ \mathrm{sgn}(r_2)\pi iS \tag{42}$$

The $\mathrm{sgn}(r_2)$ function above arises due to the anticlockwise convention adopted when considering the sign of the residues.

3.5 The pole contribution

Finally it is necessary to include the pole contribution S in the intervals $-\tilde{N}_2 < k_1 < -\tilde{N}_3$ and $\tilde{N}_3 < k_1 < \tilde{N}_2$. The pole contribution is the summation of the residues at each pole location. The locations of the poles are detailed in figure 17.

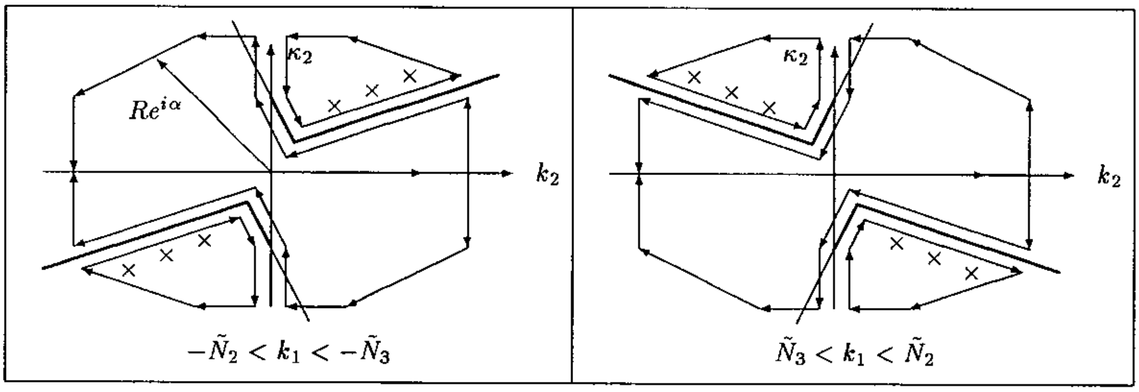

Figure 17: The locations of the poles for a non zero ϵ.

Let the location of a pole be α_n as $\epsilon \to 0$, this is the n^{th} root of $D(k_1, k_2) = 0$.For $r_2 < 0$ the contour integration uses the upper half plane so only the poles that exist there are required. S is given by

$$S = \int_{-\tilde{N}_2}^{-\tilde{N}_3} \sum_{n=1}^{n=\infty} \frac{1}{k_1^2 + \alpha_n^2} \frac{F_{ij}(k_1, \alpha_n)}{\frac{dD}{dk_2}(k_1, \alpha_n)} e^{-i(r_1 k_1 + r_2 \alpha_n)} dk_1$$

$$+ \int_{\tilde{N}_3}^{\tilde{N}_2} \sum_{n=1}^{n=\infty} \frac{1}{k_1^2 + \alpha_n^2} \frac{F_{ij}(k_1, -\alpha_n)}{\frac{dD}{dk_2}(k_1, -\alpha_n)} e^{-i(r_1 k_1 - r_2 \alpha_n)} dk_1 \qquad (r_2 < 0)$$

where $\frac{dD}{dk_2}(k_1, k_2)$ is

$$\frac{k}{k_2}\frac{dD}{dk_2}(k_1, k_2) = (t_1 + t_2)\cosh t_1 k\gamma_1 \left[\gamma_2 \sinh t_2 k\gamma_2 + \gamma_3 \cosh t_2 k\gamma_2\right] + (t_1 \frac{\gamma_1}{\gamma_2} + t_2 \frac{\gamma_2}{\gamma_1}) \sinh t_1 k\gamma_1 \left[\gamma_3 \sinh t_2 k\gamma_2 + \gamma_2 \cosh t_2 k\gamma_2\right.$$

S can be simplified using the results

$$F_{ij}(-k_1, k_2) = F_{ij}(k_1, k_2) \qquad F_{ij}(k_1, -k_2) = F_{ij}(k_1, k_2)$$

$$\tfrac{dD}{dk_2}(-k_1, k_2) = \tfrac{dD}{dk_2}(k_1, k_2) \quad \tfrac{dD}{dk_2}(k_1, -k_2) = -\tfrac{dD}{dk_2}(k_1, k_2)$$

we find that

$$S = 2i \int_{\tilde{N}_3}^{\tilde{N}_2} \sum_{n=1}^{n=\infty} \frac{1}{k_1^2 + \alpha_n^2} \frac{F_{ij}(k_1, \alpha_n)}{\frac{dD}{dk_2}(k_1, \alpha_n)} \sin(r_1 k_1 - r_2 \alpha_n) dk_1 \qquad (r_2 < 0)$$

and for $r_2 > 0$ we have

$$S = -2i \int_{\tilde{N}_3}^{\tilde{N}_2} \sum_{n=1}^{n=\infty} \frac{1}{k_1^2 + \alpha_n^2} \frac{F_{ij}(k_1, \alpha_n)}{\frac{dD}{dk_2}(k_1, \alpha_n)} \sin(r_1 k_1 + r_2 \alpha_n) dk_1 \qquad (r_2 > 0)$$

The combined result is

$$S = -2i \ \ \mathrm{sgn}(r_2) \int_{\tilde{N}_3}^{\tilde{N}_2} \sum_{n=1}^{n=\infty} \frac{1}{k_1^2+\alpha_n^2} \frac{F_{ij}(k_1,\alpha_n)}{\frac{dD}{dk_2}(k_1,\alpha_n)} \sin(r_1 k_1 + |r_2|\alpha_n) dk_1$$

Rearranging the denominator to obtain only $k\gamma_j$ terms we now have

$$S = -2i \ \ \mathrm{sgn}(r_2) \int_{\tilde{N}_3}^{\tilde{N}_2} \sum_{n=1}^{n=\infty} \frac{1}{\alpha_n} \frac{F_{ij}(k_1,\alpha_n)}{\frac{dD^*}{dk_2}(k_1,\alpha_n)} \sin(r_1 k_1 + |r_2|\alpha_n) dk_1 \tag{43}$$

where

$$\frac{dD^*}{dk_2}(k_1,k_2) = (t_1+t_2)\cosh t_1 k\gamma_1 \left[k\gamma_2 \sinh t_2 k\gamma_2 + k\gamma_3 \cosh t_2 k\gamma_2\right]$$

$$+(t_1\frac{k\gamma_1}{k\gamma_2} + t_2\frac{k\gamma_2}{k\gamma_1}) \sinh t_1 k\gamma_1 \left[k\gamma_3 \sinh t_2 k\gamma_2 + k\gamma_2 \cosh t_2 k\gamma_2\right]$$

The forms of $k\gamma_j$ to be used are those found on the horizontal paths.

3.6 The functions $\phi_{ij}(\boldsymbol{x},\boldsymbol{\xi})$ and $\psi_{ij}(\boldsymbol{x},\boldsymbol{\xi})$

Combining 42 and 43 we have

$$\phi_{ij}(\boldsymbol{x},\boldsymbol{\xi}) = -\frac{1}{\pi^2}\int_0^{\tilde{N}_3}\int_0^{\infty} \frac{1}{k_1^2+k_2^2} \frac{F_{ijr}(k_1,k_2)D_i(k_1,k_2) - F_{iji}(k_1,k_2)D_r(k_1,k_2)}{D_r^2(k_1,k_2)+D_i^2(k_1,k_2)} \sin(r_1k_1+|r_2|k_2)dk_2dk_1$$

$$+\frac{1}{\pi^2}\int_0^{\tilde{N}_3} \cos r_1k_1 \int_0^{k_1} \frac{e^{-|r_2|\kappa_2}}{k_1^2-\kappa_2^2} \frac{F_{ijr}(k_1,\kappa_2)D_i(k_1,\kappa_2) - F_{iji}(k_1,\kappa_2)D_r(k_1,\kappa_2)}{D_r^2(k_1,\kappa_2)+D_i^2(k_1,\kappa_2)} d\kappa_2 dk_1$$

$$-\frac{1}{\pi^2}\int_{\tilde{N}_3}^{\infty} \cos r_1k_1 \int_{k_1}^{\infty} \frac{e^{-|r_2|\kappa_2}}{k_1^2-\kappa_2^2} \frac{F_{ijr}(k_1,\kappa_2)D_i(k_1,\kappa_2) - F_{iji}(k_1,\kappa_2)D_r(k_1,\kappa_2)}{D_r^2(k_1,\kappa_2)+D_i^2(k_1,\kappa_2)} d\kappa_2 dk_1$$

$$+\frac{1}{\pi}\int_{\tilde{N}_3}^{\tilde{N}_2} \sum_{n=1}^{n=\infty} \frac{1}{\alpha_n} \frac{F_{ij}(k_1,\alpha_n)}{\frac{dD^*}{dk_2}(k_1,\alpha_n)} \sin(r_1k_1+|r_2|\alpha_n)dk_1 \tag{44}$$

and

$$\psi_{ij}(\boldsymbol{x},\boldsymbol{\xi}) = -\frac{1}{\pi^2}\int_0^{\tilde{N}_3} \frac{k_1^2}{k_1^2-\tilde{N}_j^2}\int_0^{\infty} \frac{1}{k_1^2+k_2^2} \frac{F_{ijr}(k_1,k_2)D_i(k_1,k_2) - F_{iji}(k_1,k_2)D_r(k_1,k_2)}{D_r^2(k_1,k_2)+D_i^2(k_1,k_2)} \sin(r_1k_1+|r_2|k_2)dk_2dk_1$$

$$+\frac{1}{\pi^2}\int_0^{\tilde{N}_3} \frac{k_1^2}{k_1^2-\tilde{N}_j^2} \cos r_1k_1 \int_0^{k_1} \frac{e^{-|r_2|\kappa_2}}{k_1^2-\kappa_2^2} \frac{F_{ijr}(k_1,\kappa_2)D_i(k_1,\kappa_2) - F_{iji}(k_1,\kappa_2)D_r(k_1,\kappa_2)}{D_r^2(k_1,\kappa_2)+D_i^2(k_1,\kappa_2)} d\kappa_2 dk_1$$

$$-\frac{1}{\pi^2}\int_{\tilde{N}_3}^{\infty} \frac{k_1^2}{k_1^2-\tilde{N}_j^2} \cos r_1k_1 \int_{k_1}^{\infty} \frac{e^{-|r_2|\kappa_2}}{k_1^2-\kappa_2^2} \frac{F_{ijr}(k_1,\kappa_2)D_i(k_1,\kappa_2) - F_{iji}(k_1,\kappa_2)D_r(k_1,\kappa_2)}{D_r^2(k_1,\kappa_2)+D_i^2(k_1,\kappa_2)} d\kappa_2 dk_1$$

$$+\frac{1}{\pi}\int_{\tilde{N}_3}^{\tilde{N}_2} \frac{k_1^2}{k_1^2-\tilde{N}_j^2} \sum_{n=1}^{n=\infty} \frac{1}{\alpha_n} \frac{F_{ij}(k_1,\alpha_n)}{\frac{dD^*}{dk_2}(k_1,\alpha_n)} \sin(r_1k_1+|r_2|\alpha_n)dk_1 \tag{45}$$

The distinct forms of the functions for each k_1 interval are contained in Appendix B.

These are the completed forms of $\phi_{ij}(\boldsymbol{x},\boldsymbol{\xi})$ and $\psi_{ij}(\boldsymbol{x},\boldsymbol{\xi})$ which describe the fluid disturbance created by a horizontally moving singularity of unit strength in the three layer stratified fluid system. The velocity components in the fluid can be determined using $u(\boldsymbol{x},\boldsymbol{\xi}) = \frac{\partial\phi}{\partial x}$, $v(\boldsymbol{x},\boldsymbol{\xi}) = \frac{\partial\phi}{\partial y}$ and $w(\boldsymbol{x},\boldsymbol{\xi}) = \frac{\partial\psi}{\partial z}$.

4 Application of slender body theory

The singularity solution may be extended to simulate a slender three dimensional body. Slender body theory assumes that the summation of singularity solutions distributed along the longitudinal axis of the body is equivalent to a solution when the body is taken as a whole. The source strength associated with each solution being a function of its location on that axis. The summation can be expressed as

$$\Phi_{ij}(x,\xi) = \int_L Q(\xi)\phi_{ij}(x,\xi)d\xi$$

where the source strength is given by

$$Q(\xi) = -U\frac{dA}{d\xi}(\xi)$$

A fully submerged prolate spheroid requires the source distribution

$$Q(\xi) = 2\pi U\left(\frac{d}{L}\right)^2 \xi$$

As ξ only appears in ϕ_{ij} and ψ_{ij} in the terms $\sin[(x-\xi)k_1 + |r_2|k_2]$ and $\cos(x-\xi)k_1$ the integration can be carried out analytically. Thus, it follows that

$$\Phi_{ij}(x,\xi) = \frac{4U}{\pi}\left(\frac{d}{L}\right)^2$$

$$*\int_0^{\tilde{N}_3} \frac{1}{k_1^2}\left(\sin\frac{k_1}{2} - \frac{k_1}{2}\cos\frac{k_1}{2}\right)\int_0^\infty \frac{1}{k_1^2+k_2^2}\frac{F_{ijr}(k_1,k_2)D_i(k_1,k_2) - F_{iji}(k_1,k_2)D_r(k_1,k_2)}{D_r^2(k_1,k_2)+D_i^2(k_1,k_2)}\cos(xk_1+|r_2|k_2)dk_2dk_1$$

$$+\frac{4U}{\pi}\left(\frac{d}{L}\right)^2\int_0^{\tilde{N}_3}\left(\sin\frac{k_1}{2} - \frac{k_1}{2}\cos\frac{k_1}{2}\right)\frac{\sin xk_1}{k_1^2}\int_0^{k_1}\frac{e^{-|r_2|\kappa_2}}{k_1^2-\kappa_2^2}\frac{F_{ijr}(k_1,\kappa_2)D_i(k_1,\kappa_2) - F_{iji}(k_1,\kappa_2)D_r(k_1,\kappa_2)}{D_r^2(k_1,\kappa_2)+D_i^2(k_1,\kappa_2)}d\kappa_2dk_1$$

$$-\frac{4U}{\pi}\left(\frac{d}{L}\right)^2\int_{\tilde{N}_3}^{\infty}\left(\sin\frac{k_1}{2} - \frac{k_1}{2}\cos\frac{k_1}{2}\right)\frac{\sin xk_1}{k_1^2}\int_{k_1}^{\infty}\frac{e^{-|r_2|\kappa_2}}{k_1^2-\kappa_2^2}\frac{F_{ijr}(k_1,\kappa_2)D_i(k_1,\kappa_2) - F_{iji}(k_1,\kappa_2)D_r(k_1,\kappa_2)}{D_r^2(k_1,\kappa_2)+D_i^2(k_1,\kappa_2)}d\kappa_2dk_1$$

$$-4U\left(\frac{d}{L}\right)^2\int_{\tilde{N}_3}^{\tilde{N}_2}\frac{1}{k_1^2}\left(\sin\frac{k_1}{2} - \frac{k_1}{2}\cos\frac{k_1}{2}\right)\sum_{n=1}^{n=\infty}\frac{1}{\alpha_n}\frac{F_{ij}(k_1,\alpha_n)}{\frac{dD^*}{dk_2}(k_1,\alpha_n)}\cos(xk_1+|r_2|\alpha_n)dk_1 \qquad (46)$$

and

$$\Psi_{ij}(x,\xi) = \frac{4U}{\pi}\left(\frac{d}{L}\right)^2$$

$$*\int_0^{\tilde{N}_3}\left(\sin\frac{k_1}{2} - \frac{k_1}{2}\cos\frac{k_1}{2}\right)\frac{1}{k_1^2-\tilde{N}_j^2}\int_0^\infty \frac{1}{k_1^2+k_2^2}\frac{F_{ijr}(k_1,k_2)D_i(k_1,k_2) - F_{iji}(k_1,k_2)D_r(k_1,k_2)}{D_r^2(k_1,k_2)+D_i^2(k_1,k_2)}\cos(xk_1+|r_2|k_2)dk_2dk_1$$

$$+\frac{4U}{\pi}\left(\frac{d}{L}\right)^2\int_0^{\tilde{N}_3}\left(\sin\frac{k_1}{2} - \frac{k_1}{2}\cos\frac{k_1}{2}\right)\frac{\sin xk_1}{k_1^2-\tilde{N}_j^2}\int_0^{k_1}\frac{e^{-|r_2|\kappa_2}}{k_1^2-\kappa_2^2}\frac{F_{ijr}(k_1,\kappa_2)D_i(k_1,\kappa_2) - F_{iji}(k_1,\kappa_2)D_r(k_1,\kappa_2)}{D_r^2(k_1,\kappa_2)+D_i^2(k_1,\kappa_2)}d\kappa_2dk_1$$

$$-\frac{4U}{\pi}\left(\frac{d}{L}\right)^2\int_{\tilde{N}_3}^{\infty}\left(\sin\frac{k_1}{2}-\frac{k_1}{2}\cos\frac{k_1}{2}\right)\frac{\sin xk_1}{k_1^2-\tilde{N}_j^2}\int_{k_1}^{\infty}\frac{e^{-|r_2|\kappa_2}}{k_1^2-\kappa_2^2}\frac{F_{ijr}(k_1,\kappa_2)D_i(k_1,\kappa_2)-F_{iji}(k_1,\kappa_2)D_r(k_1,\kappa_2)}{D_r^2(k_1,\kappa_2)+D_i^2(k_1,\kappa_2)}d\kappa_2 dk_1$$

$$-4U\left(\frac{d}{L}\right)^2\int_{\tilde{N}_3}^{\tilde{N}_2}\left(\sin\frac{k_1}{2}-\frac{k_1}{2}\cos\frac{k_1}{2}\right)\frac{1}{k_1^2-\tilde{N}_j^2}\sum_{n=1}^{n=\infty}\frac{1}{\alpha_n}\frac{F_{ij}(k_1,\alpha_n)}{\frac{dD^*}{dk_2}(k_1,\alpha_n)}\cos(xk_1+|r_2|\alpha_n)dk_1 \qquad (47)$$

Use has been made of the integrals

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\xi\cos(x-\xi)k_1]dk_1=\frac{2}{k_1^2}\left(\sin\frac{k_1}{2}-\frac{k_1}{2}\cos\frac{k_1}{2}\right)\sin xk_1$$

and

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\xi\sin[(x-\xi)k_1+|r_2|k_2]dk_1=-\frac{2}{k_1^2}\left(\sin\frac{k_1}{2}-\frac{k_1}{2}\cos\frac{k_1}{2}\right)\cos(xk_1+|r_2|k_2)$$

The velocity components u, v and w can now be determined from these expressions through differentiation.

These are the completed forms of the functions which describe the fluid disturbance created by a horizontally moving prolate spheroid in the three layer constant Brunt-Väisälä frequency stratified fluid system.

5 Interpretation and implementation of the theory

5.1 Explanation of the functions terms

The elements of the solution can be individually explained. Using ϕ_{ij} as an example we have

1.

$$\int_0^{\tilde{N}_3}\int_0^{\infty}\frac{1}{k_1^2+k_2^2}\frac{F_{ijr}(k_1,k_2)D_i(k_1,k_2)-F_{iji}(k_1,k_2)D_r(k_1,k_2)}{D_r^2(k_1,k_2)+D_i^2(k_1,k_2)}\sin(r_1k_1+|r_2|k_2)dk_2dk_1$$

This term originates from the $k\gamma$ discontinuity. It provides a very small wavelike contribution to the fluid disturbance when the three BV frequencies are not similar. When the BV frequencies approach the same value the amplitude of the wave disturbance increases and this term provides the only wavelike component.

2.

$$\int_0^{\tilde{N}_3}\cos r_1k_1\int_0^{k_1}\frac{e^{-|r_2|\kappa_2}}{k_1^2-\kappa_2^2}\frac{F_{ijr}(k_1,\kappa_2)D_i(k_1,\kappa_2)-F_{iji}(k_1,\kappa_2)D_r(k_1,\kappa_2)}{D_r^2(k_1,\kappa_2)+D_i^2(k_1,\kappa_2)}d\kappa_2dk_1$$

This term also originates from the $k\gamma$ discontinuity. It does not have any wavelike properties and only contributes to the fluid disturbance in the near field in the form of a localised peak and trough system.

3.

$$\int_{\tilde{N}_3}^{\infty}\cos r_1k_1\int_{k_1}^{\infty}\frac{e^{-|r_2|\kappa_2}}{k_1^2-\kappa_2^2}\frac{F_{ijr}(k_1,\kappa_2)D_i(k_1,\kappa_2)-F_{iji}(k_1,\kappa_2)D_r(k_1,\kappa_2)}{D_r^2(k_1,\kappa_2)+D_i^2(k_1,\kappa_2)}d\kappa_2dk_1$$

This is the final term originating from the $k\gamma$ discontinuity. Again it does not have any wavelike properties and only contributes to the fluid disturbance in the near field in the form of a localised peak and trough system.

4.

$$\int_{\tilde{N}_3}^{\tilde{N}_2}\sum_{n=1}^{n=\infty}\frac{1}{\alpha_n}\frac{F_{ij}(k_1,\alpha_n)}{\frac{dD^*}{dk_2}(k_1,\alpha_n)}\sin(r_1k_1+|r_2|\alpha_n)dk_1$$

This term exists due to the presence of poles in the integrand. It provides the major wavelike component when the three BV frequencies are distinct. As the BV frequencies approach a similar value the amplitude of the wavelike disturbance decreases and the term vanishes when $\tilde{N}_1=\tilde{N}_2=\tilde{N}_3$. The summation represents the addition of the individual contributions provided by the wave modes present in the fluid. Discrete modes exist due to reflections occurring between the rigid lid and the interfaces. The largest contribution is from the first mode, successive modes provide decreasing fractions of the total disturbance.

The nearfield disturbance should be calculated using all four terms, however the disturbance downstream may be calculated using the final term only.

5.2 Figures

There are two presentation types contained in this report:-

1. Three dimensional surface maps.

   These are three dimensional plots of the velocity components u and v calculated on the fluid's surface. They extend 5000 metres each side of the body's track, 5000 metres ahead of the body and 10000 metres behind the body.

2. One dimensional graphs.

   These also present the velocity components u and v calculated on the fluid's surface. The disturbance is calculated parallel to the body's track at 375, 875, 1375 and 1875 metres offset. The calculation extends 5000 metres ahead of the body and 10000 metres behind the body. The graphs provide a quantitive rather than qualitative presentation.

Figure 18 illustrates the appearance of the density and BV profiles used for the three layer BV model. The fluid parameters are $\tilde{N}_1 = 0.01$ rads/sec, $\tilde{N}_2 = 0.03$ rads/sec and $\tilde{N}_3 = 0.005$ rads/sec, the thickness of both layers is 30 metres. This profile is used in all the three layer BV model calculations contained in this report.

The fluid disturbance is generated using a prolate spheroid of length of 100 metres and a diameter of 10 metres.

Figure 19 is the surface velocity disturbance generated using a three layer constant density model. The body is located in the middle layer at a depth of 45 metres from the free surface (a free surface boundary condition was applied in this model). The density of the fluid in each layer is selected to approximate the profile in figure 18. That is $\rho_1 = 1025.0$ Kg/m^3, $\rho_2 = 1026.5$ Kg/m^3 and $\rho_3 = 1028.0$ Kg/m^3. The thickness of both layers is 30 metres. The near field disturbance for this model is not available as it was only developed for the far field.

Figure 20 are the surface velocities obtained from the constant BV model. The body is located at a depth of 45 metres from the rigid lid and the fluid possesses a BV frequency of 0.03 rads/sec. Figures 19 and 20 are included for comparative purposes.

Figure 21 shows the surface velocities for a body speed of 2 metres/sec when the body is in the lower layer. Figure 22 are individual lines taken from the maps. Similarly figures 23 and 24 are for the body in the middle layer. Figures 25 and 26 are for the body in the upper layer.

The effect of body speed may be observed using the final two sets of figures. Figure 27 is the surface disturbance when the body is in the lower layer for a body speed of 1 metres/sec, naturally the 'v' angle increases. Figure 28 is for the body in the same position moving at 5 metres/sec.

6 Conclusions

The solution obtained for the three layer Brunt-Väisälä frequency model not only provides a viable mechanism for the prediction of the disturbance created by the moving body, but also gives an insight into the structure of the fluid disturbance. The differences between the current model results and previous models demonstrate how predictions of the fluid disturbance are improved through using a constant BV layer model. Favourable comparisons with selected experimental measurements justify the additional complexity involved in producing this model. This is the most complex analytical model that can be attempted, further research involving a generalised density profile requires a numerical scheme which may not be as robust or enlightening.

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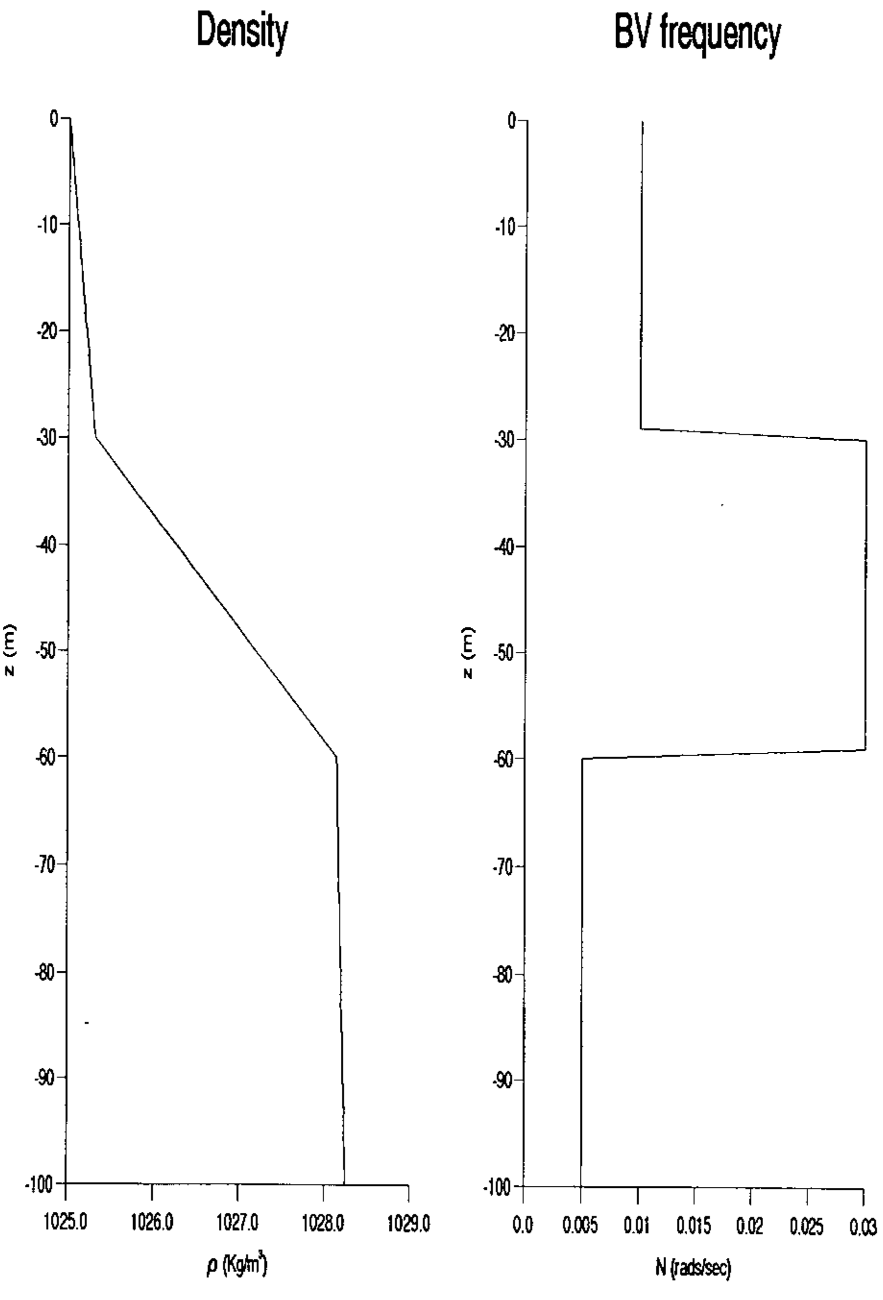


Figure 18: Density and Brunt-Väisälä frequency profiles.

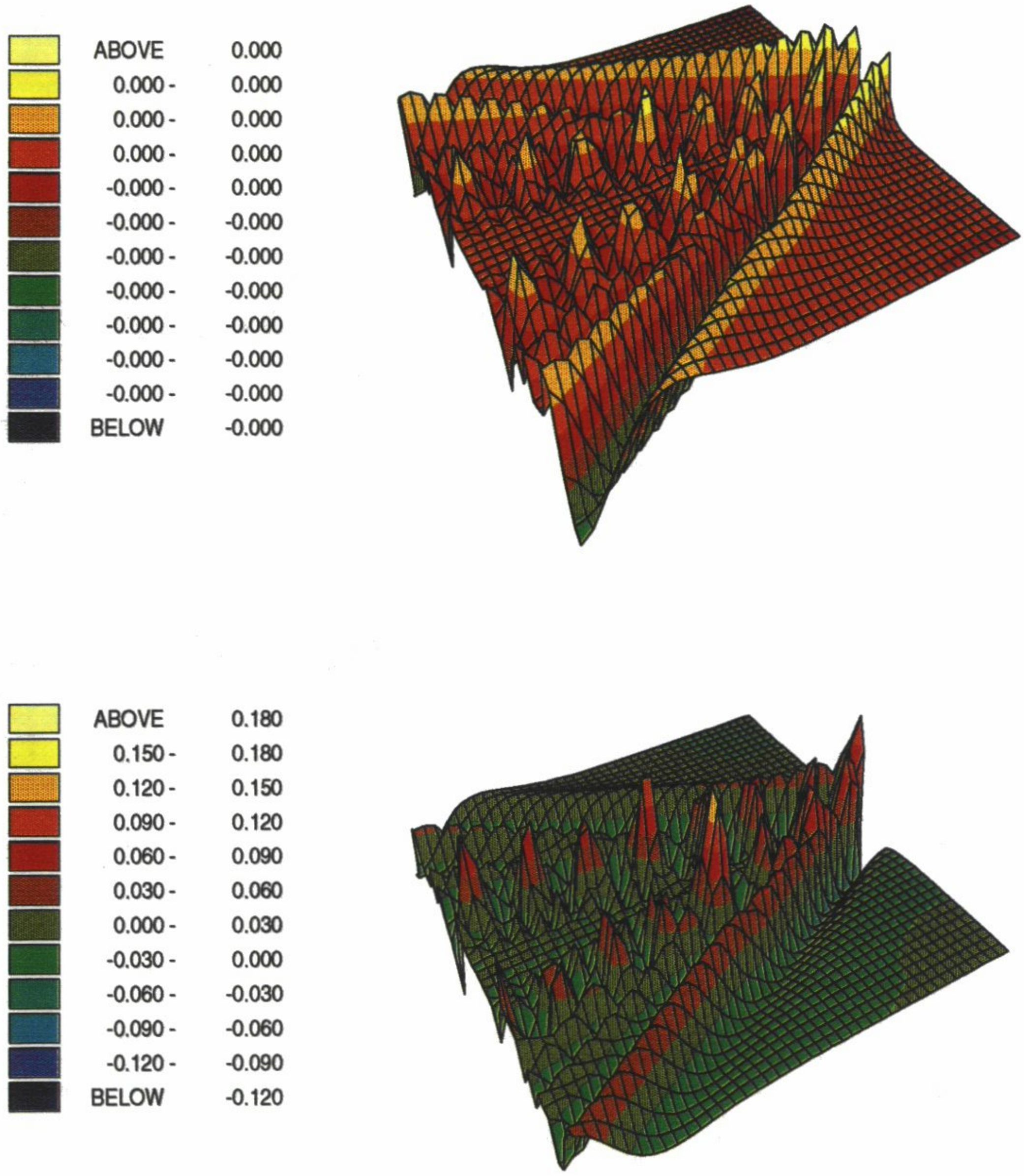


Figure 19: Velocities u (mm/s) and v (mm/s) calculated on the fluid's surface using the constant density three layer model. $U = 2$ m/s.

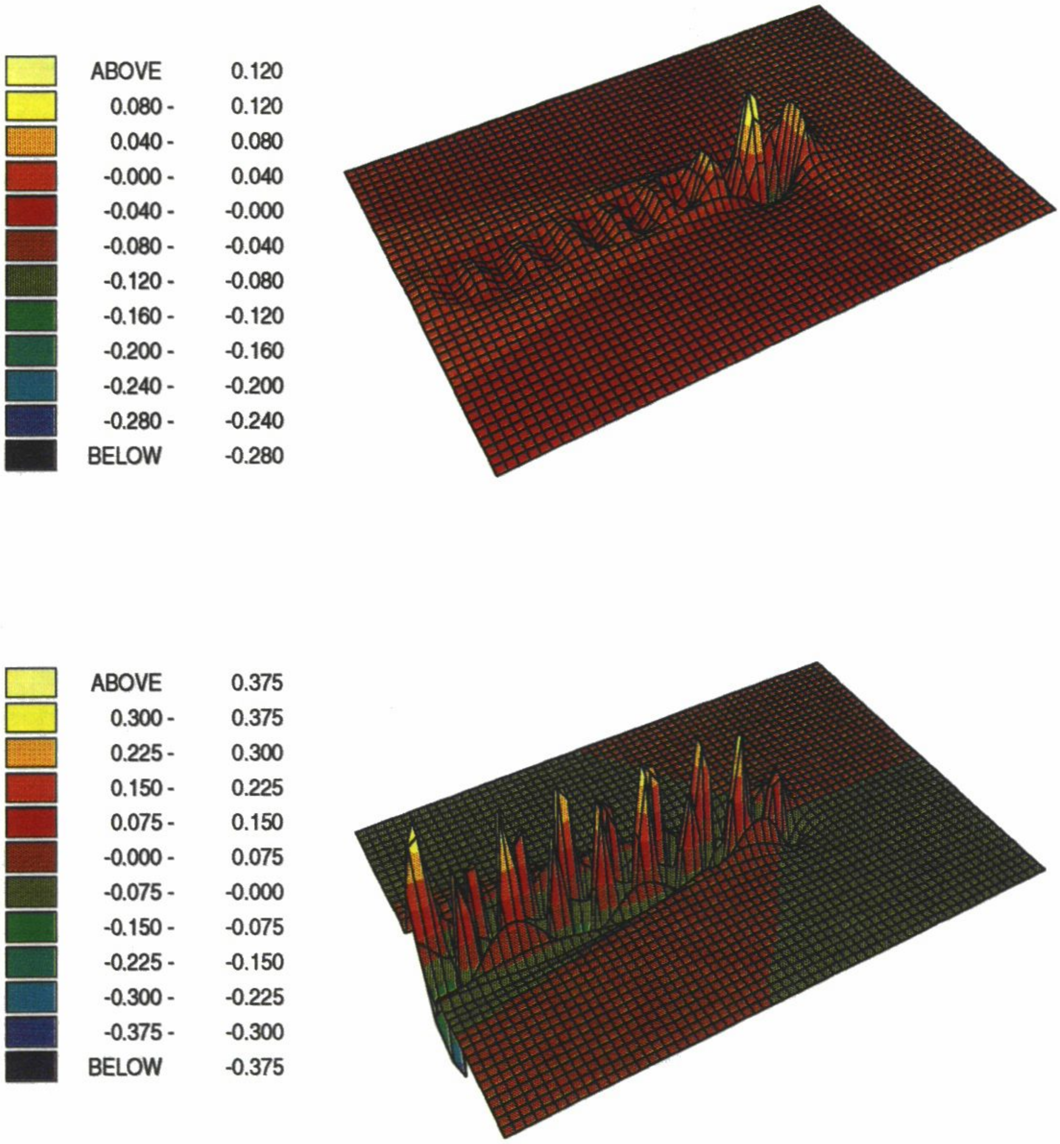
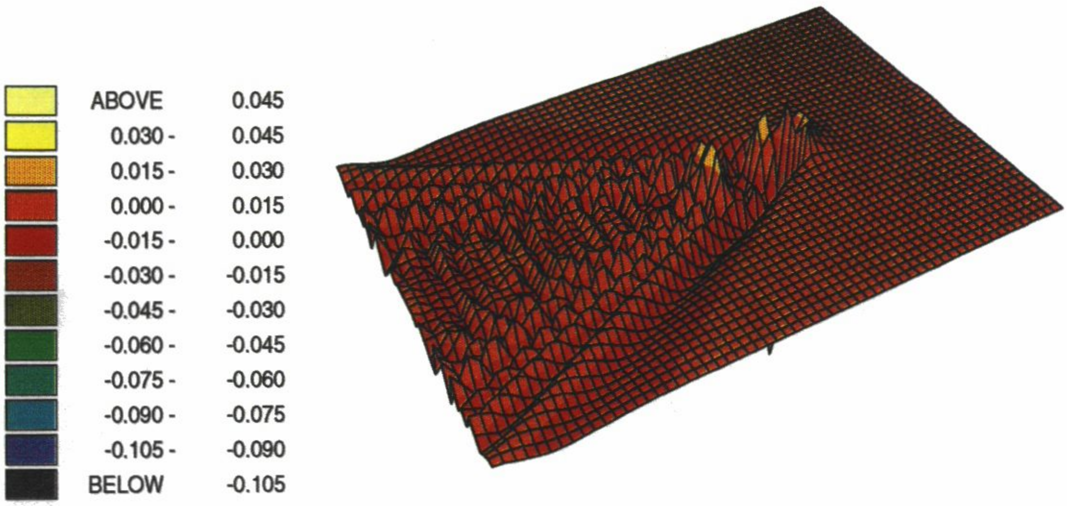


Figure 20: Velocities u (mm/s) and v (mm/s) calculated on the fluid's surface using the constant BV model. $U = 2$ m/s.



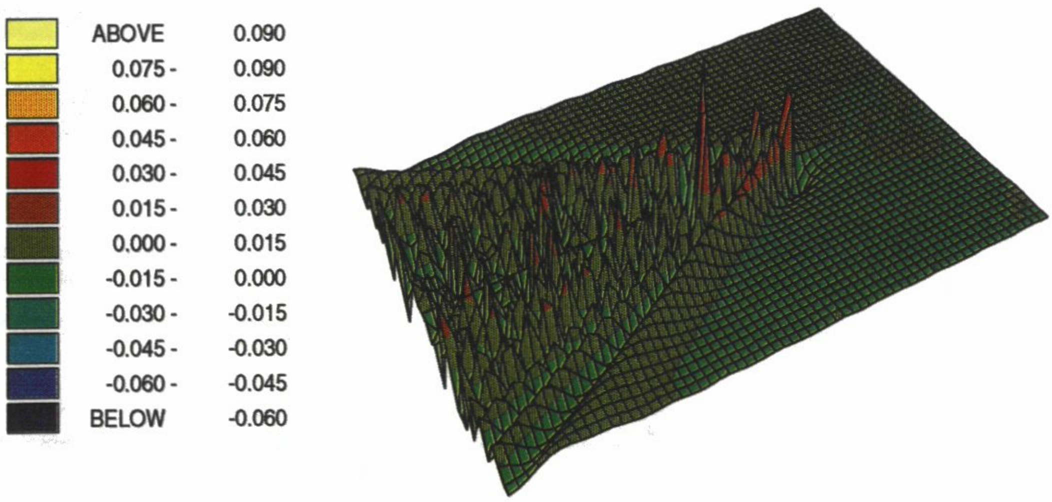


Figure 21: Velocities u (mm/s) and v (mm/s) calculated on the fluid's surface using the layered BV model when the body is in the lower layer. $U = 2$ m/s.

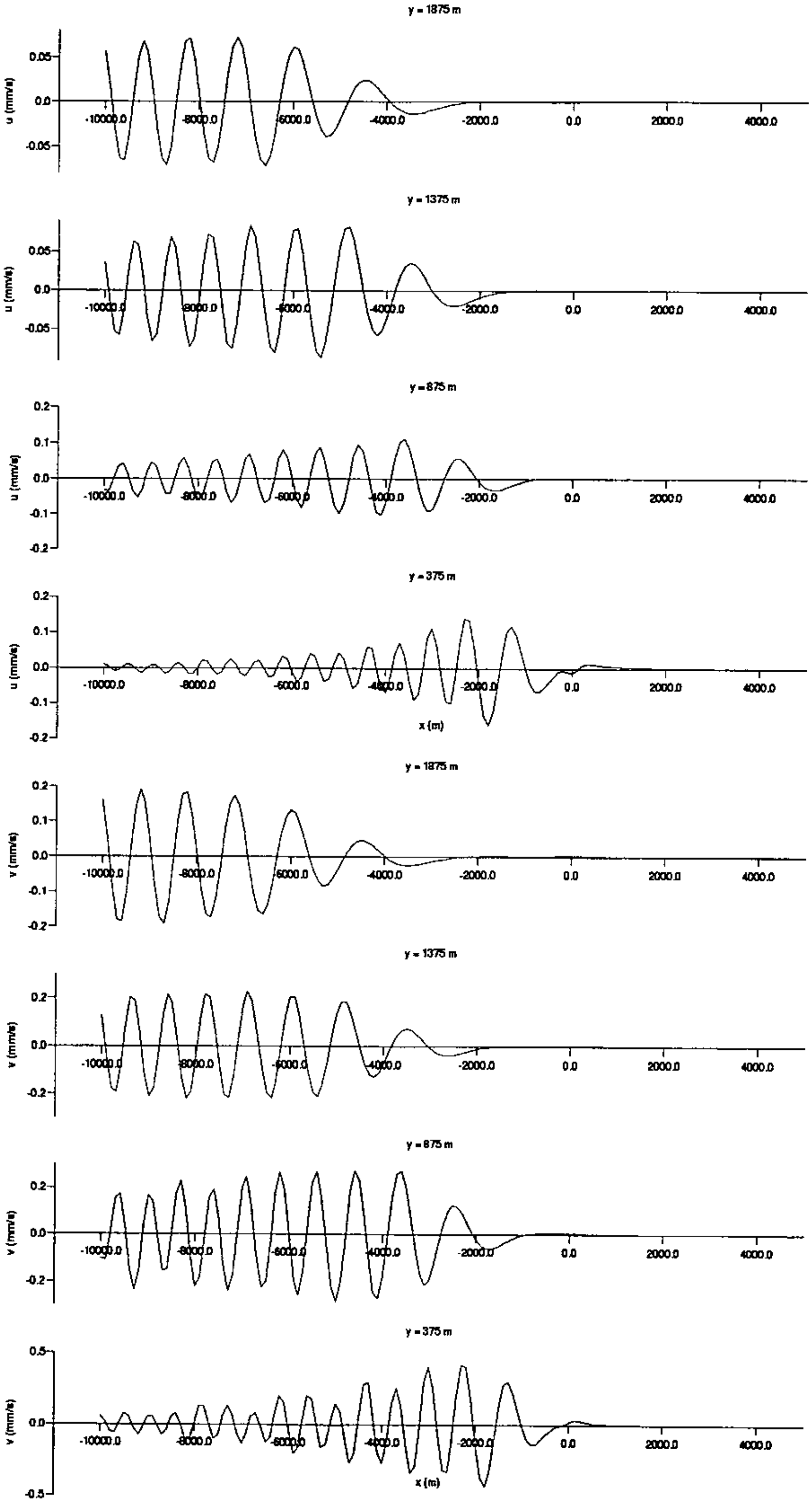


Figure 22: Velocities u (mm/s) and v (mm/s) calculated on the fluid's surface using the layered BV model when the body is in the lower layer. $U = 2$ m/s.

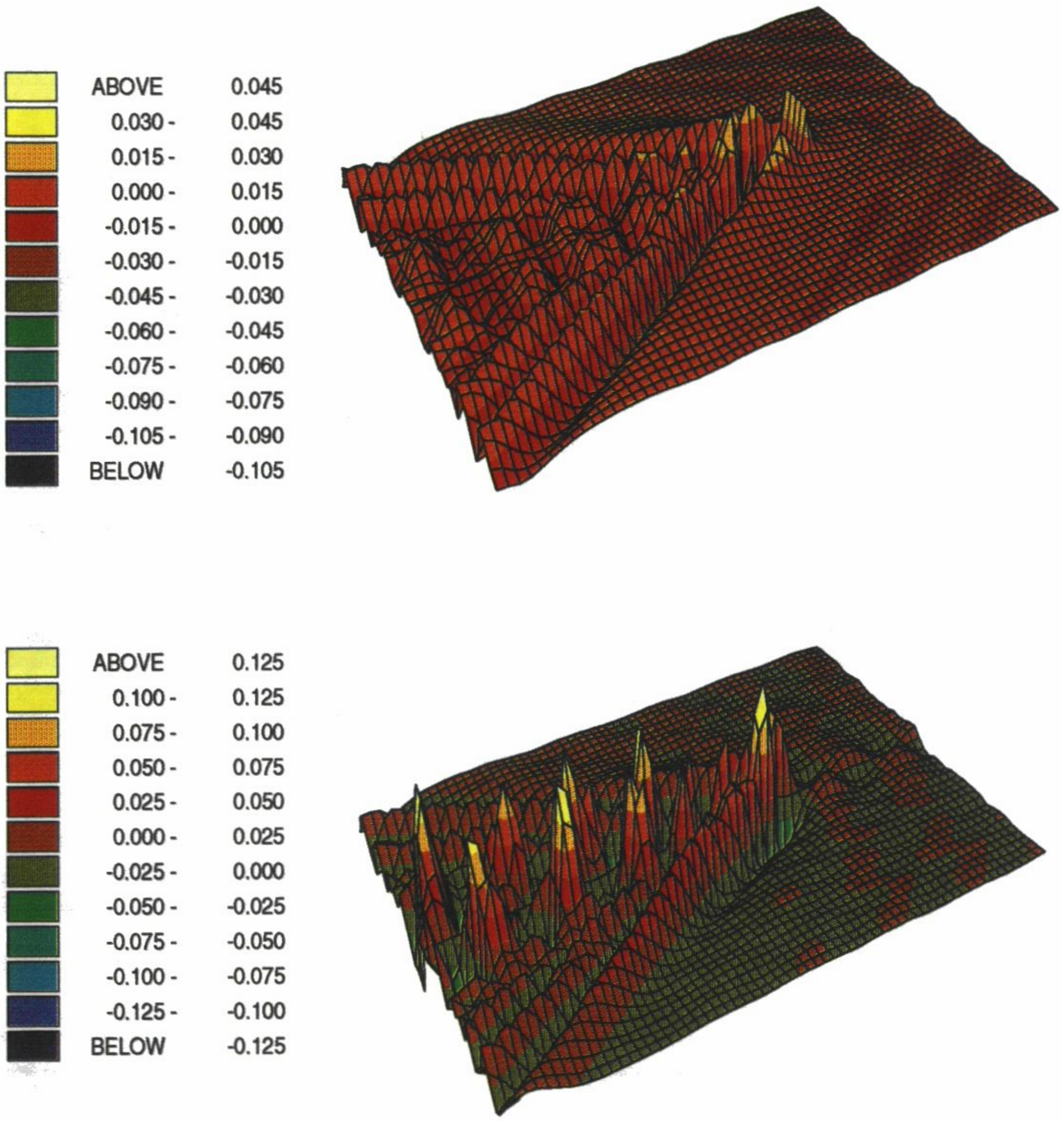


Figure 23: Velocities u (mm/s) and v (mm/s) calculated on the fluid's surface using the layered BV model when the body is in the middle layer. $U = 2$ m/s.

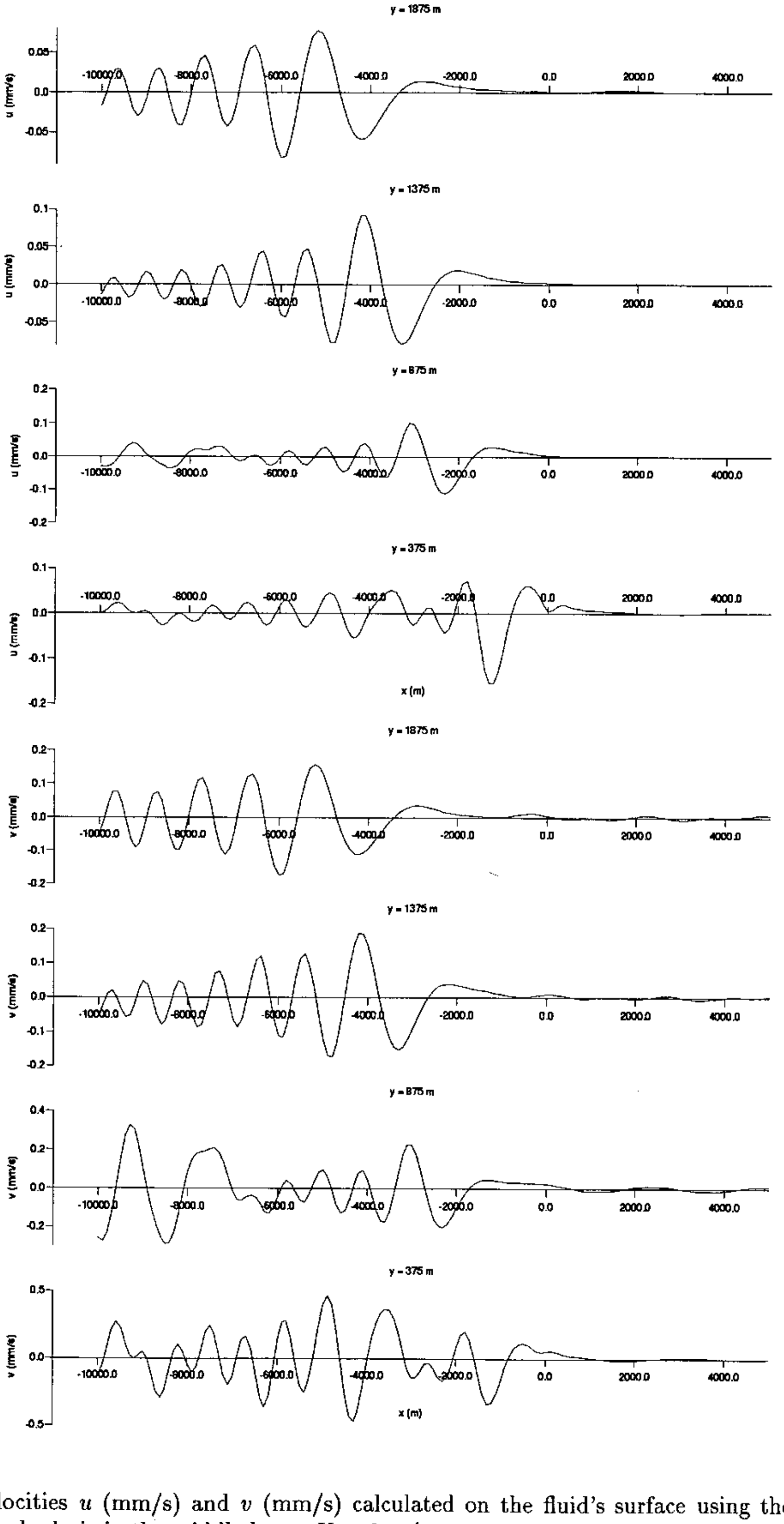


Figure 24: Velocities u (mm/s) and v (mm/s) calculated on the fluid's surface using the layered BV model when the body is in the middle layer. $U = 2$ m/s.

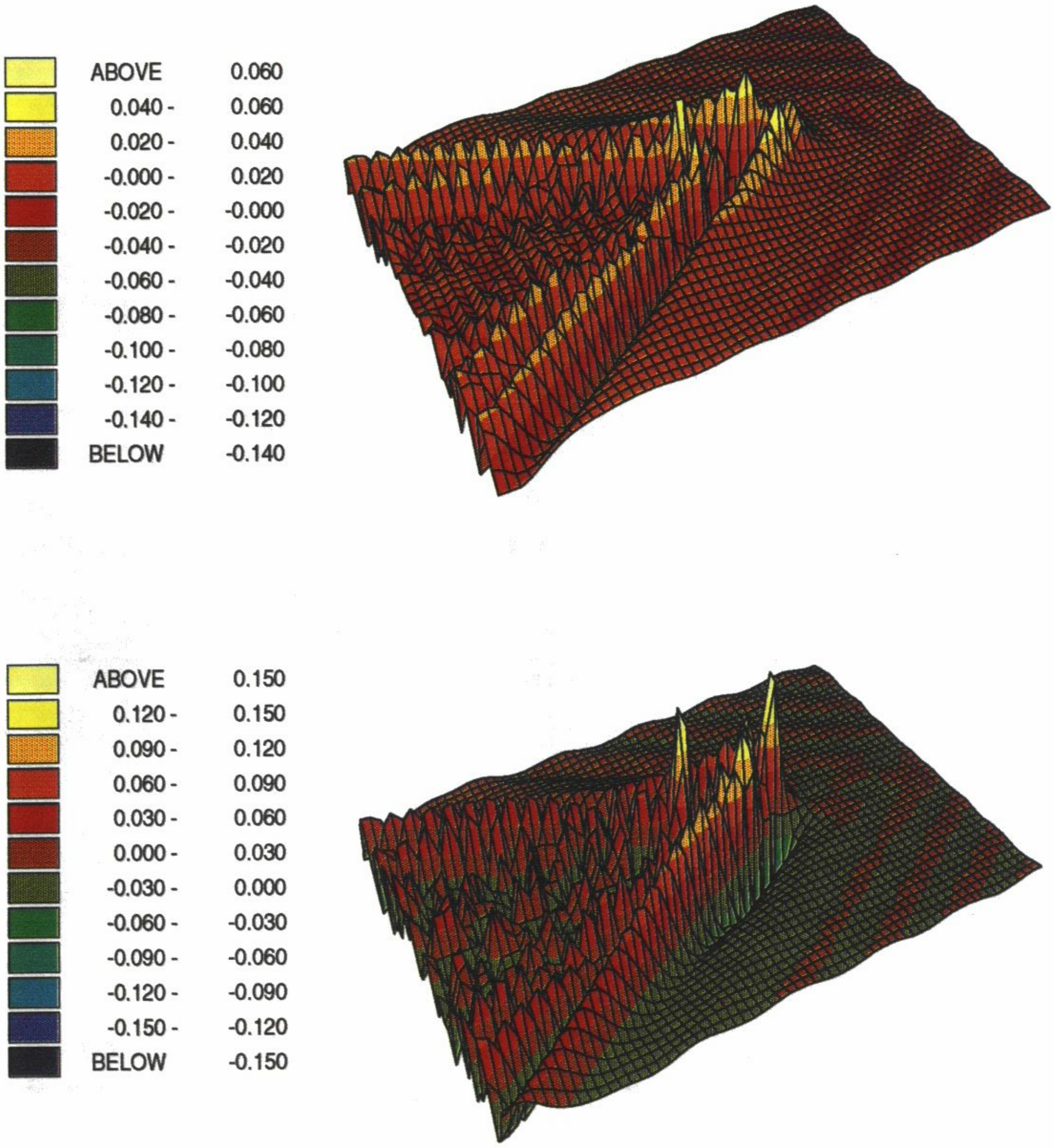


Figure 25: Velocities u (mm/s) and v (mm/s) calculated on the fluid's surface using the layered BV model when the body is in the upper layer. $U = 2$ m/s.

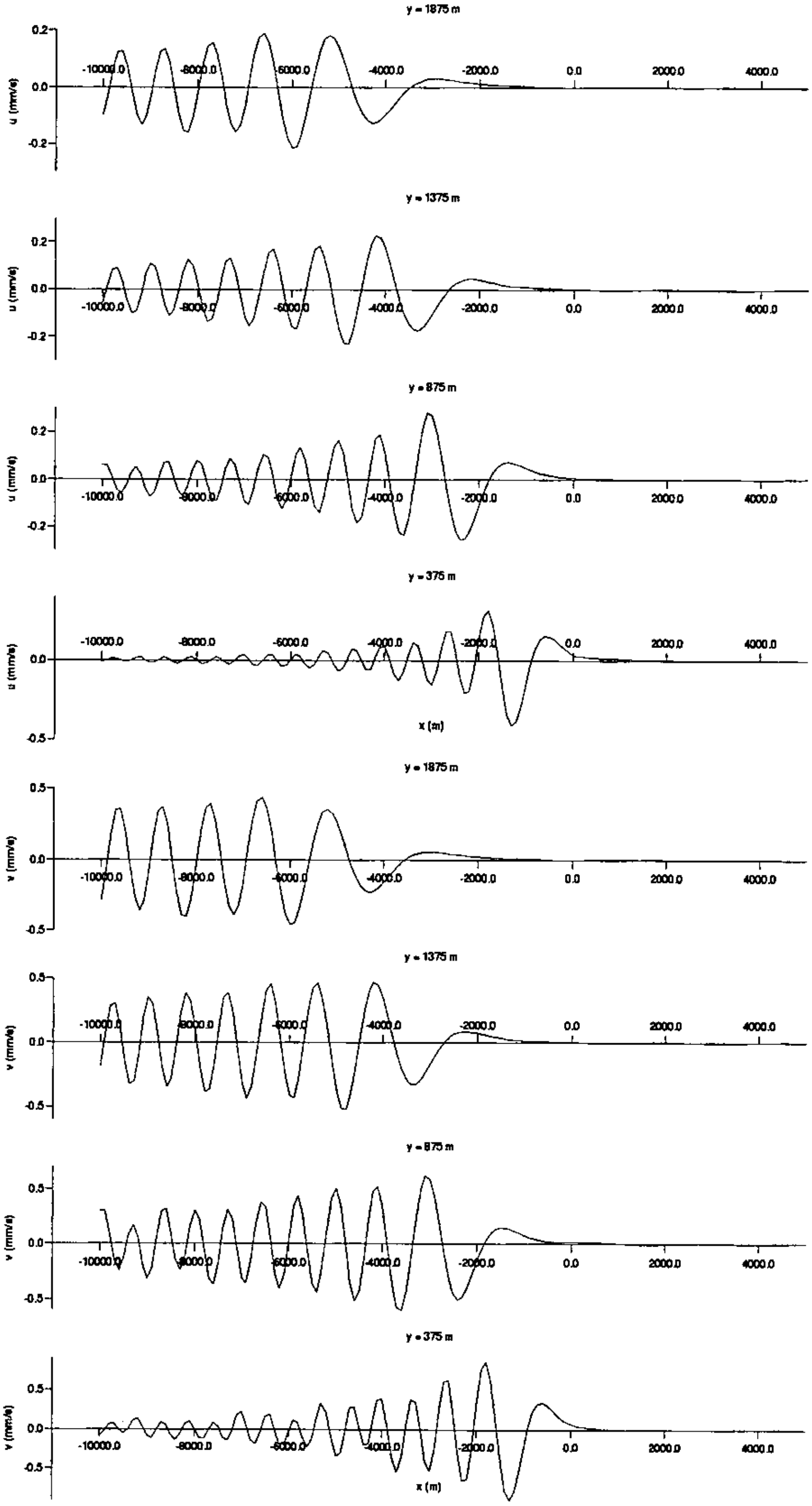


Figure 26: Velocities u(mm/s) and v (mm/s) calculated on the fluid's surface using the layered BV model when the body is in the upper layer. $U = 2$ m/s.

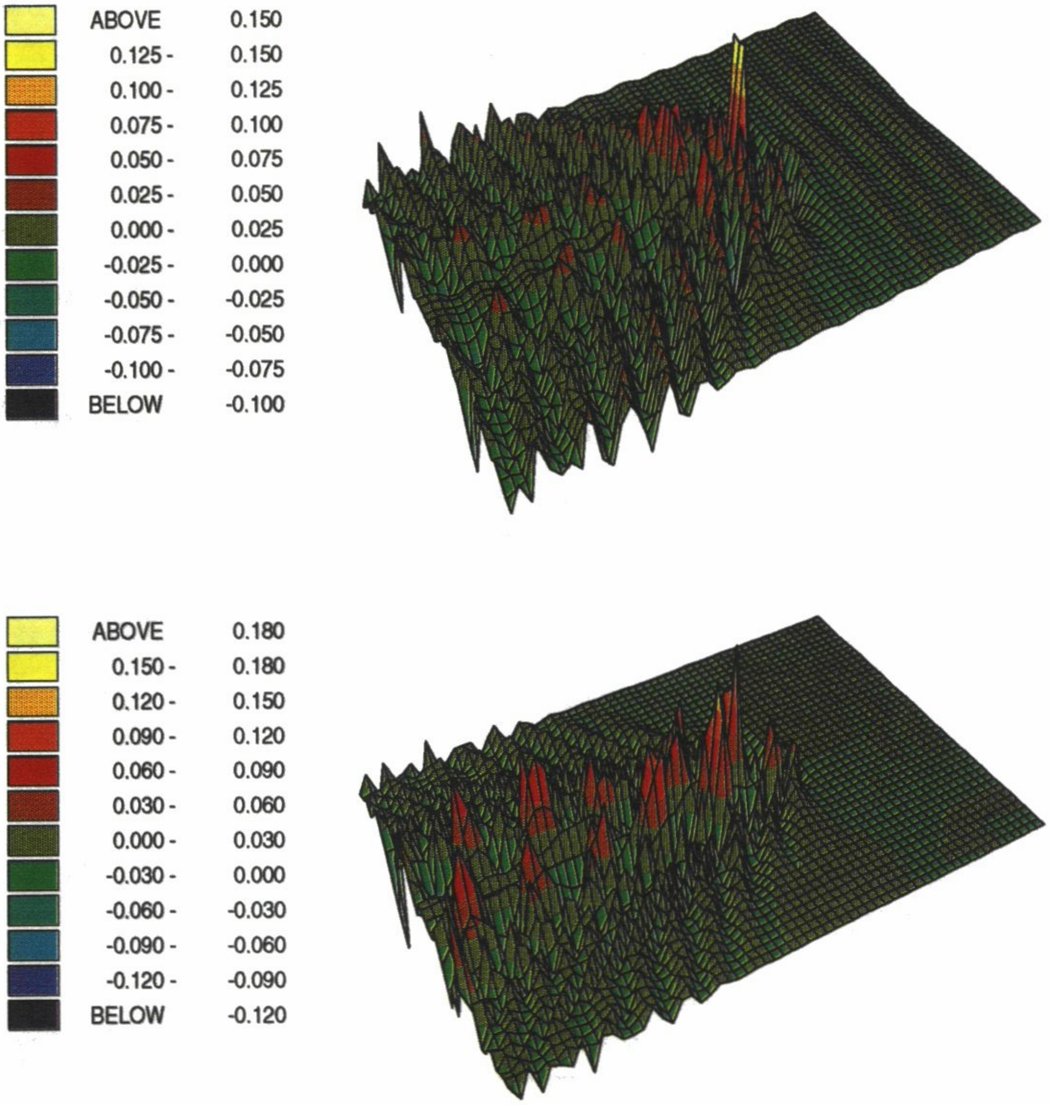


Figure 27: Velocities u (mm/s) and v (mm/s) calculated on the fluid's surface using the layered BV model when the body is in the lower layer. $U = 1$ m/s.

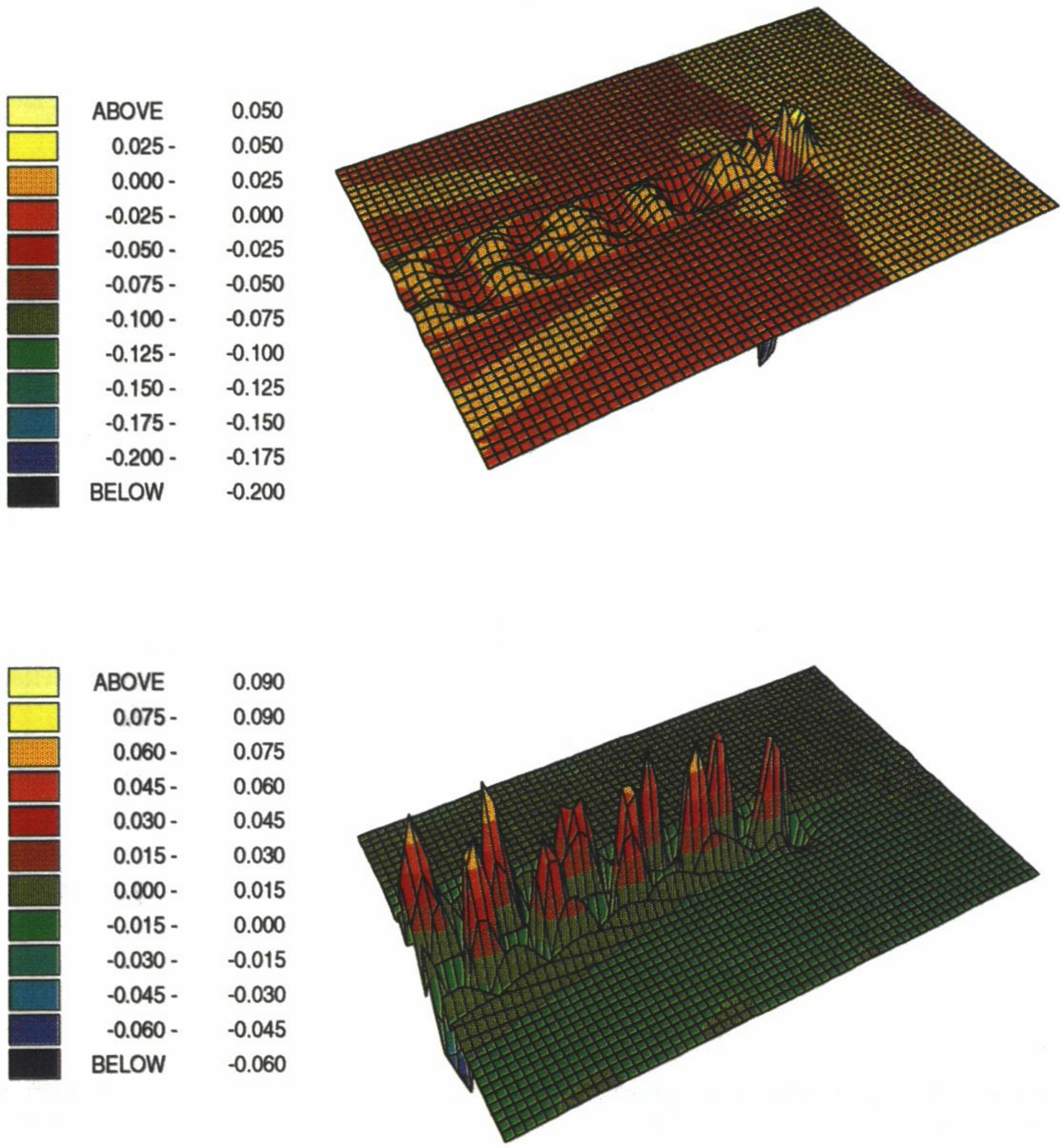


Figure 28: Velocities u (mm/s) and v (mm/s) calculated on the fluid's surface using the layered BV model when the body is in the lower layer. $U = 5$ m/s.

A : Example of contour integration

Consider the line integral

$$\int_0^\infty \frac{\cos bx}{\sqrt{a^2+x^2}}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{e^{ibx}}{\sqrt{a^2+x^2}}dx$$

Replacing the real variable x by the complex variable $z = x + iy$ we have a contour integral

$$\oint \frac{e^{ibz}}{\sqrt{a^2+z^2}}dz$$

The integrand has the complex square root $\sqrt{a^2+z^2} = \sqrt{a^2+x^2-y^2+i\,2xy}$ which requires additional paths to be added to a large radius contour in order to avoid the contour crossing the line discontinuity defined by

$$a^2 + x^2 - y^2 < 0 \quad \text{and} \quad xy = 0$$

so, it is possible that line discontinuities exist on $x = 0$ and $y = 0$. The inequality determines if a line discontinuity is present and the interval in which it exists.

1. $x = 0$ gives $a^2 - y^2 < 0$, thus the intervals on the line $x = 0$ are $a < y < \infty$ and $-\infty < y < -a$.
2. $y = 0$ gives $a^2 + x^2 < 0$, as x is real no intervals are present on the line $y = 0$ so no discontinuity exists.

The contour for $b > 0$ uses the upper half plane. This ensures the real part of the exponentials in the integrand are negative. Thus the contour becomes

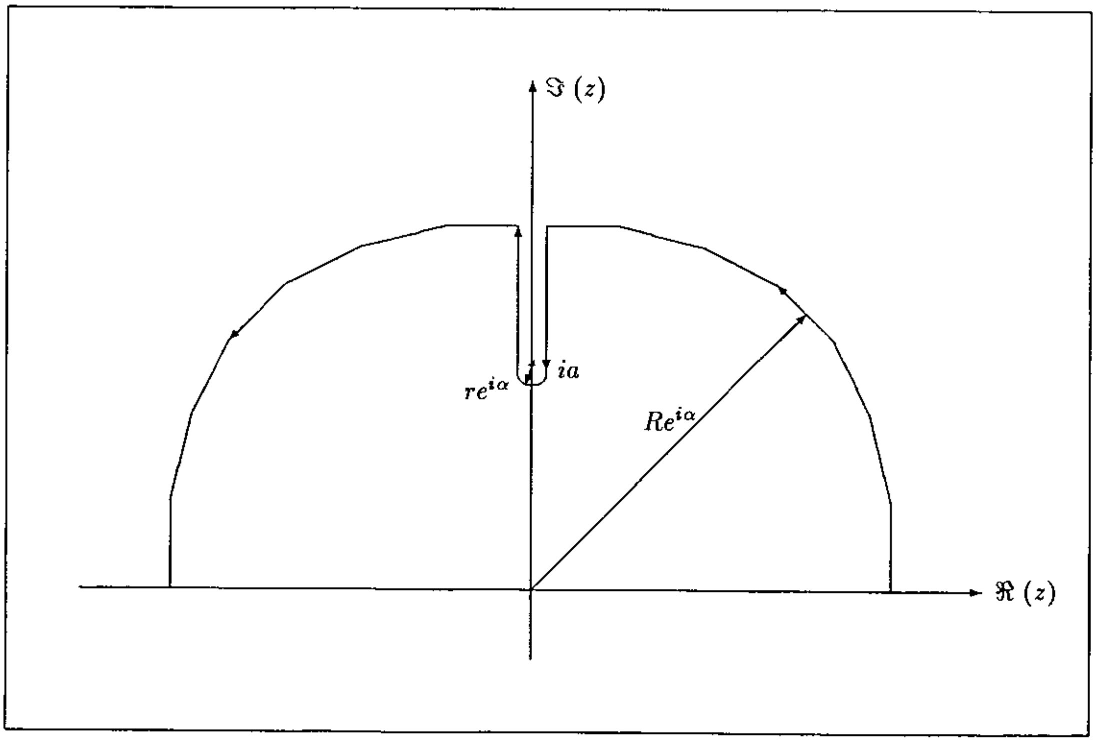


Figure 29: Contour path for the complex square root function

Note : The large radius arc $Re^{i\alpha}$, $0 < \alpha < \pi$ is represented in the figure by a series of straight lines. For $b < 0$ the lower half plane is utilised.

The integrals in the direction indicated around the contour are

$$\int_{-R}^{R} \frac{e^{ibx}}{\sqrt{a^2+x^2}}dx + \int_0^{\frac{\pi}{2}} \frac{e^{ibRe^{i\alpha}}}{\sqrt{a^2+(Re^{i\alpha})^2}} iRe^{i\alpha}d\alpha + \int_R^{|a|} \frac{e^{ib(r+iy)}}{\sqrt{a^2+(r+iy)^2}}idy$$

$$+\int_0^{-\pi} \frac{e^{ib(i|a|+re^{i\alpha})}}{\sqrt{a^2+(i|a|+re^{i\alpha})^2}} ire^{i\alpha}d\alpha + \int_{|a|}^{R} \frac{e^{ib(-r+iy)}}{\sqrt{a^2+(-r+iy)^2}}idy + \int_{\frac{\pi}{2}}^{\pi} \frac{e^{ibRe^{i\alpha}}}{\sqrt{a^2+(Re^{i\alpha})^2}} iRe^{i\alpha}d\alpha = 0$$

Now the examine behaviour of each integral as $r \to 0$ and $R \to \infty$. The second and sixth integrals will tend to zero as $R \to \infty$ as the real part of the exponential is $-bR\sin\alpha$ and is always negative. The fourth integral also tends to zero as $r \to 0$ because

$$\lim_{r\to 0}\left[\frac{r}{\sqrt{a^2+(i|a|+re^{i\alpha})^2}}\right] = 0$$

The third integral requires the application of equation 39 as $r \to 0$, the square root becomes $i\sqrt{y^2-a^2}$ and the integral becomes

$$-\int_{|a|}^{\infty} \frac{e^{-by}}{\sqrt{y^2-a^2}}dy$$

the fifth integral gives an identical result. Thus

$$\int_{-\infty}^{\infty} \frac{e^{ibx}}{\sqrt{a^2+x^2}}dx = 2\int_{|a|}^{\infty} \frac{e^{-by}}{\sqrt{y^2-a^2}} = 2\mathrm{K}_0(|a|b)$$

the last step utilises the equality given in Prudnikov et al(1986) 2.3.5(4).

$$\int_a^{\infty}(x^2-a^2)^{\beta-1}e^{-px}dx = \frac{1}{\sqrt{\pi}}\Gamma(\beta)\left(\frac{2a}{p}\right)^{\beta-\frac{1}{2}} K_{\beta-\frac{1}{2}}(ap) \quad (a>0 \quad , \quad \Re(\beta)>0 \quad , \quad \Re(p)>0)$$

with $p = b$, $x = y$ and $\beta = \frac{1}{2}$. Finally

$$\int_0^{\infty} \frac{\cos bx}{\sqrt{a^2+x^2}}dx = \mathrm{K}_0(|a|b)$$

This result is confirmed in the same volume at 2.5.6(4)

$$\int_0^{\infty} \frac{\cos bx}{(x^2+z^2)^{\rho}}dx = \left(\frac{2z}{b}\right)^{\frac{1}{2}-\rho} \frac{\sqrt{\pi}}{\Gamma(\rho)} K_{\frac{1}{2}-\rho}(bz) \qquad (b>0 \quad , \quad \Re(\rho)>0 \quad , \quad \Re(z)>0)$$

with $z = |a|$ and $\rho = \frac{1}{2}$.

B : Forms of the functions for different k_1 intervals

From the general forms of the functions derived in section 2.10 distinct variations are required for the four k_1 intervals. ϕ_{ij} is used as an example to illustrate where these forms are appropriate. A common function $C_{ij}(k_1, k_2)$ is removed from the numerator in all cases. Note when $i = j$ the functions are defined for $z < \zeta$, for $z > \zeta$ the functions can be obtained by exchanging z and ζ.

1. $0 < k_1 < \tilde{N}_3$. There are two integrals in this interval, both arising from the discontinuity. No cancellation of positive exponentials on the numerator and denominator is required in either case. The first is

$$\int_0^{\tilde{N}_3} \int_0^{\infty} \frac{C_{ij}(k_1,k_2)}{k_1^2+k_2^2} \frac{F_{ijr}(k_1,k_2)D_i(k_1,k_2) - F_{iji}(k_1,k_2)D_r(k_1,k_2)}{D_r^2(k_1,k_2) + D_i^2(k_1,k_2)} \sin(r_1 k_1 + |r_2| k_2) dk_2 dk_1$$

For this integral the horizontal forms of $k\gamma_j$ are required. Let

$$X_1 = \sqrt{(1+\frac{k_2^2}{k_1^2})(\tilde{N}_1^2 - k_1^2)}$$

$$X_2 = \sqrt{(1+\frac{k_2^2}{k_1^2})(\tilde{N}_2^2 - k_1^2)}$$

$$X_3 = \sqrt{(1+\frac{k_2^2}{k_1^2})(\tilde{N}_3^2 - k_1^2)}$$

then the functions become

$$D_r(k_1,k_2) = \cos t_1 X_1 \cos t_2 X_2 - X_2 \frac{\sin t_1 X_1}{X_1} \sin t_2 X_2$$

$$D_i(k_1,k_2) = X_3 \left[\frac{\sin t_1 X_1}{X_1} \cos t_2 X_2 + \cos t_1 X_1 \frac{\sin t_2 X_2}{X_2} \right]$$

$$F_{11r}(k_1,k_2) = X_1 \sin(z+t_1) X_1 \cos t_2 X_2 + X_2 \cos(z+t_1) X_1 \sin t_2 X_2$$

$$F_{11i}(k_1,k_2) = X_3 \left[X_1 \sin(z+t_1) X_1 \frac{\sin t_2 X_2}{X_2} - \cos(z+t_1) X_1 \cos t_2 X_2 \right]$$

$$F_{12r}(k_1,k_2) = -X_2 \sin(z+t_1+t_2) X_2$$

$$F_{12i}(k_1,k_2) = X_3 \cos(z+t_1+t_2) X_2$$

$$F_{13r}(k_1,k_2) = -\sin z X_3$$

$$F_{13i}(k_1, k_2) = \cos z X_3$$

$$F_{21r}(k_1, k_2) = -X_2 \sin(\zeta + t_2) X_2$$

$$F_{21i}(k_1, k_2) = X_3 \cos(\zeta + t_2) X_2$$

$$F_{22r}(k_1, k_2) = -X_2 \sin(z + t_2) X_2$$

$$F_{22i}(k_1, k_2) = X_3 \cos(z + t_2) X_2$$

$$F_{23r}(k_1, k_2) = -\sin(z + t_2) X_3$$

$$F_{23i}(k_1, k_2) = \cos(z + t_2) X_3$$

$$F_{31r}(k_1, k_2) = -\sin \zeta X_3$$

$$F_{31i}(k_1, k_2) = \cos \zeta X_3$$

$$F_{32r}(k_1, k_2) = -\sin \zeta X_3$$

$$F_{32i}(k_1, k_2) = \cos \zeta X_3$$

$$F_{33r}(k_1, k_2) = \sin z X_3$$

$$F_{33i}(k_1, k_2) = -\cos z X_3$$

$$C_{11}(k_1, k_2) = -\cos \zeta X_1$$

$$C_{12}(k_1, k_2) = \cos \zeta X_1$$

$$C_{13}(k_1, k_2) = X_3 \cos \zeta X_1$$

$$C_{21}(k_1, k_2) = \cos(z - t_1) X_1$$

$$C_{22}(k_1,k_2) = \cos t_1X_1 \cos\zeta X_2 + X_2\frac{\sin t_1X_1}{X_1}\sin\zeta X_2$$

$$C_{23}(k_1,k_2) = X_3\left[\cos t_1X_1 \cos\zeta X_2 + X_2\frac{\sin t_1X_1}{X_1}\sin\zeta X_2\right]$$

$$C_{31}(k_1,k_2) = X_3\cos(z-t_1-t_2)X_1$$

$$C_{32}(k_1,k_2) = X_3\left[\cos t_1X_1 \cos(z-t_2)X_2 + X_2\frac{\sin t_1X_1}{X_1}\sin(z-t_2)X_2\right]$$

$$C_{33}(k_1,k_2) = X_3\left\{\cos\zeta X_3\left[X_2\frac{\sin t_1X_1}{X_1}\sin t_2X_2 - \cos t_1X_1\cos t_2X_2\right]\right.$$

$$\left.-X_3\sin\zeta X_3\left[\frac{\sin t_1X_1}{X_1}\cos t_2X_2 + \cos t_1X_1\frac{\sin t_2X_2}{X_2}\right]\right\}$$

The second integral is

$$\int_0^{\tilde N_3}\cos r_1k_1\int_0^{k_1}C_{ij}(k_1,\kappa_2)\frac{e^{-|r_2|\kappa_2}}{k_1^2-\kappa_2^2}\frac{F_{ijr}(k_1,\kappa_2)D_i(k_1,\kappa_2)-F_{iji}(k_1,\kappa_2)D_r(k_1,\kappa_2)}{D_r^2(k_1,\kappa_2)+D_i^2(k_1,\kappa_2)}d\kappa_2dk_1$$

To obtain the functions for this integral the lower vertical forms of $k\gamma_j$ are required,

$$X_1 = \sqrt{(1-\frac{k_2^2}{k_1^2})(\tilde N_1^2-k_1^2)}$$

$$X_2 = \sqrt{(1-\frac{k_2^2}{k_1^2})(\tilde N_2^2-k_1^2)}$$

$$X_3 = \sqrt{(1-\frac{k_2^2}{k_1^2})(\tilde N_3^2-k_1^2)}$$

thus it is necessary to reassign the $'X'$ variables and use the previously defined functions.

2. $\tilde N_3 < k_1 < \tilde N_1$. There are two integrals in this interval. The first integral arising from the discontinuity is

$$\int_{\tilde N_3}^{\tilde N_1}\cos r_1k_1\int_{k_1}^{\infty}C_{ij}(k_1,\kappa_2)\frac{e^{-|r_2|\kappa_2}}{k_1^2-\kappa_2^2}\frac{F_{ijr}(k_1,\kappa_2)D_i(k_1,\kappa_2)-F_{iji}(k_1,\kappa_2)D_r(k_1,\kappa_2)}{D_r^2(k_1,\kappa_2)+D_i^2(k_1,\kappa_2)}d\kappa_2dk_1$$

The functions in this interval after substitution of the upper vertical forms of $k\gamma_j$,

$$X_1 = \sqrt{(\frac{k_2^2}{k_1^2} - 1)(\tilde{N}_1^2 - k_1^2)}$$

$$X_2 = \sqrt{(\frac{k_2^2}{k_1^2} - 1)(\tilde{N}_2^2 - k_1^2)}$$

$$X_3 = \sqrt{(\frac{k_2^2}{k_1^2} - 1)(k_1^2 - \tilde{N}_3^2)},$$

cancellation of positive exponentials on the numerator and denominator and removal of common factors become

$$D_r(k_1,k_2) = \frac{X_2}{X_1}\left(1 - e^{-2t_1X_1}\right)\left(1 - e^{-2t_2X_2}\right) + \left(1 + e^{-2t_1X_1}\right)\left(1 + e^{-2t_2X_2}\right)$$

$$D_i(k_1,k_2) = \frac{X_3}{X_1}\left(1 - e^{-2t_1X_1}\right)\left(1 + e^{-2t_2X_2}\right) + \frac{X_3}{X_2}\left(1 + e^{-2t_1X_1}\right)\left(1 - e^{-2t_2X_2}\right)$$

$$F_{11r}(k_1,k_2) = X_1\left(1 - e^{-2(z+t_1)X_1}\right)\left(1 + e^{-2t_2X_2}\right) + X_2\left(1 + e^{-2(z+t_1)X_1}\right)\left(1 - e^{-2t_2X_2}\right)$$

$$F_{11i}(k_1,k_2) = X_3\left[\frac{X_1}{X_2}\left(1 - e^{-2(z+t_1)X_1}\right)\left(1 - e^{-2t_2X_2}\right) + \left(1 + e^{-2(z+t_1)X_1}\right)\left(1 + e^{-2t_2X_2}\right)\right]$$

$$F_{12r}(k_1,k_2) = X_2\left(1 - e^{-2(z+t_1+t_2)X_2}\right)$$

$$F_{12i}(k_1,k_2) = X_3\left(1 + e^{-2(z+t_1+t_2)X_2}\right)$$

$$F_{13r}(k_1,k_2) = -\sin zX_3$$

$$F_{13i}(k_1,k_2) = \cos zX_3$$

$$F_{21r}(k_1,k_2) = X_2\left(1 - e^{-2(\zeta+t_2)X_2}\right)$$

$$F_{21i}(k_1,k_2) = X_3\left(1 + e^{-2(\zeta+t_2)X_2}\right)$$

$$F_{22r}(k_1,k_2) = X_2\left(1 - e^{-2(z+t_2)X_2}\right)$$

$$F_{22i}(k_1,k_2) = X_3\left(1+e^{-2(z+t_2)X_2}\right)$$

$$F_{23r}(k_1,k_2) = -\sin(z+t_2)X_3$$

$$F_{23i}(k_1,k_2) = \cos(z+t_2)X_3$$

$$F_{31r}(k_1,k_2) = -\sin\zeta X_3$$

$$F_{31i}(k_1,k_2) = \cos\zeta X_3$$

$$F_{32r}(k_1,k_2) = -\sin\zeta X_3$$

$$F_{32i}(k_1,k_2) = \cos\zeta X_3$$

$$F_{33r}(k_1,k_2) = -\sin zX_3$$

$$F_{33i}(k_1,k_2) = \cos zX_3$$

$$C_{11}(k_1,k_2) = \frac{1}{2}e^{(z-\zeta)X_1}\left(e^{2\zeta X_1}+1\right)$$

$$C_{12}(k_1,k_2) = e^{-(\zeta+t_1)X_1}e^{(z+t_1)X_2}\left(e^{2\zeta X_1}+1\right)$$

$$C_{13}(k_1,k_2) = 2X_3e^{-(\zeta+t_1)X_1}e^{-t_2X_2}\left(e^{2\zeta X_1}+1\right)$$

$$C_{21}(k_1,k_2) = e^{-zX_1}e^{\zeta X_2}\left(e^{2(z-t_1)X_1}+1\right)$$

$$C_{22}(k_1,k_2) = \frac{1}{2}e^{(z-\zeta)X_2}\left[\left(1+e^{-2t_1X_1}\right)\left(e^{2\zeta X_2}+1\right)-\frac{X_2}{X_1}\left(1-e^{-2t_1X_1}\right)\left(e^{2\zeta X_2}-1\right)\right]$$

$$C_{23}(k_1,k_2) = X_3e^{-(\zeta+t_2)X_2}\left[\left(1+e^{-2t_1X_1}\right)\left(e^{2\zeta X_2}+1\right)-\frac{X_2}{X_1}\left(1-e^{-2t_1X_1}\right)\left(e^{2\zeta X_2}-1\right)\right]$$

$$C_{31}(k_1,k_2) = 2X_3e^{-(z-t_2)X_1}e^{-t_2X_2}\left(e^{2(z-t_1-t_2)X_1}+1\right)$$

$$C_{32}(k_1,k_2) = X_3 e^{-zX_2}\left[\left(1+e^{-2t_1X_1}\right)\left(e^{2(z-t_2)X_2}+1\right) - \frac{X_2}{X_1}\left(1-e^{-2t_1X_1}\right)\left(e^{2(z-t_2)X_2}-1\right)\right]$$

$$C_{33}(k_1,k_2) = X_3\left\{\cos\zeta X_3\left[\frac{X_2}{X_1}\left(1-e^{-2t_1X_1}\right)\left(1-e^{-2t_2X_2}\right) + \left(1+e^{-2t_1X_1}\right)\left(1+e^{-2t_2X_2}\right)\right]\right.$$

$$\left. + X_3\sin\zeta X_3\left[\frac{1}{X_1}\left(1-e^{-2t_1X_1}\right)\left(1+e^{-2t_2X_2}\right) + \frac{1}{X_2}\left(1+e^{-2t_1X_1}\right)\left(1-e^{-2t_2X_2}\right)\right]\right\}$$

The second integral is from the pole contribution

$$\int_{\tilde{N}_3}^{\tilde{N}_1}\sum_{n=1}^{n=\infty}\frac{1}{\alpha_n}\frac{F_{ijr}(k_1,\alpha_n)}{\frac{dD^*}{dk_2}(k_1,\alpha_n)}\sin(r_1k_1+|r_2|\alpha_n)dk_1$$

The horizontal forms of $k\gamma_j$ are applicable here.

$$X_1 = \sqrt{(1+\frac{k_2^2}{k_1^2})(\tilde{N}_1^2-k_1^2)}$$

$$X_2 = \sqrt{(1+\frac{k_2^2}{k_1^2})(\tilde{N}_2^2-k_1^2)}$$

$$X_3 = \sqrt{(1+\frac{k_2^2}{k_1^2})(k_1^2-\tilde{N}_3^2)}$$

$$F_{11r}(k_1,k_2) = \cos\zeta X_1\left\{\cos(z+t_1)X_1\left[X_3\cos t_2X_2 - X_2\sin t_2X_2\right] - X_1\sin(z+t_1)X_1\left[X_3\frac{\sin t_2X_2}{X_2}+\cos t_2X_2\right]\right\}$$

$$F_{12r}(k_1,k_2) = \cos\zeta X_1\left[X_3\cos(z+t_1+t_2)X_2 - X_2\sin(z+t_1+t_2)X_2\right]$$

$$F_{13r}(k_1,k_2) = X_3e^{zX_3}\cos\zeta X_1$$

$$F_{21r}(k_1,k_2) = \cos(z-t_1)X_1\left[X_3\cos(\zeta+t_2)X_2 - X_2\sin(\zeta+t_2)X_2\right]$$

$$F_{22r}(k_1,k_2) = \left[X_3\cos(z+t_2)X_2 - X_2\sin(z+t_2)X_2\right]\left[\cos t_1X_1\cos\zeta X_2 + X_2\frac{\sin t_1X_1}{X_1}\sin\zeta X_2\right]$$

$$F_{23r}(k_1,k_2) = X_3e^{(z+t_2)X_3}\left[\cos t_1X_1\cos\zeta X_2 + X_2\frac{\sin t_1X_1}{X_1}\sin\zeta X_2\right]$$

$$F_{31r}(k_1,k_2) = X_3 e^{\zeta X_3} \cos(z-t_1-t_2)X_1$$

$$F_{32r}(k_1,k_2) = X_3 e^{\zeta X_3}\left[\cos t_1X_1 \cos(z-t_2)X_2 + X_2\frac{\sin t_1X_1}{X_1}\sin(z-t_2)X_2\right]$$

$$F_{33r}(k_1,k_2) = X_3 e^{zX_3}\left\{\cos\zeta X_3\left[\cos t_1X_1\cos t_2X_2 - X_2\frac{\sin t_1X_1}{X_1}\sin t_2X_2\right]\right.$$

$$\left.-X_3\sin\zeta X_3\left[\frac{\sin t_1X_1}{X_1}\cos t_2X_2 + \cos t_1X_1\frac{\sin t_2X_2}{X_2}\right]\right\}$$

$$\frac{dD^*}{dk_2}(k_1,k_2) = (t_1+t_2)\cos t_1X_1\left[X_3\cos t_2X_2 - X_2\sin t_2X_2\right] - \left(t_1\frac{X_1}{X_2}+t_2\frac{X_2}{X_1}\right)\sin t_1X_1\left[X_3\sin t_2X_2 + X_2\cos t_2X_2\right]$$

No imaginary functions exist on the horizontal paths in this interval.

3. $\tilde{N}_1 < k_1 < \tilde{N}_2$.

The analysis here is identical to the previous interval, only the functions themselves have a slightly different form due to the change in $k\gamma_1$. For the integral arising from the discontinuity we have

$$X_1 = \sqrt{\left(\frac{k_2^2}{k_1^2}-1\right)(k_1^2-\tilde{N}_1^2)}$$

$$X_2 = \sqrt{\left(\frac{k_2^2}{k_1^2}-1\right)(\tilde{N}_2^2-k_1^2)}$$

$$X_3 = \sqrt{\left(\frac{k_2^2}{k_1^2}-1\right)(k_1^2-\tilde{N}_3^2)}$$

$$D_r(k_1,k_2) = \frac{X_2}{X_1}\sin t_1X_1\left(1-e^{-2t_2X_2}\right) + \cos t_1X_1\left(1+e^{-2t_2X_2}\right)$$

$$D_i(k_1,k_2) = \frac{X_3}{X_1}\sin t_1X_1\left(1+e^{-2t_2X_2}\right) + \frac{X_3}{X_2}\cos t_1X_1\left(1-e^{-2t_2X_2}\right)$$

$$F_{11r}(k_1,k_2) = X_2\cos(z+t_1)X_1\left(1-e^{-2t_2X_2}\right) - X_1\sin(z+t_1)X_1\left(1+e^{-2t_2X_2}\right)$$

$$F_{11i}(k_1,k_2) = X_3\left[\cos(z+t_1)X_1\left(1+e^{-2t_2X_2}\right) - \frac{X_1}{X_2}\sin(z+t_1)X_1\left(1-e^{-2t_2X_2}\right)\right]$$

$$F_{12r}(k_1, k_2) = X_2 \left(1 - e^{-2(z+t_1+t_2)X_2}\right)$$

$$F_{12i}(k_1, k_2) = X_3 \left(1 + e^{-2(z+t_1+t_2)X_2}\right)$$

$$F_{13r}(k_1, k_2) = -\sin zX_3$$

$$F_{13i}(k_1, k_2) = \cos zX_3$$

$$F_{21r}(k_1, k_2) = X_2 \left(1 - e^{-2(\zeta+t_2)X_2}\right)$$

$$F_{21i}(k_1, k_2) = X_3 \left(1 + e^{-2(\zeta+t_2)X_2}\right)$$

$$F_{22r}(k_1, k_2) = X_2 \left(1 - e^{-2(z+t_2)X_2}\right)$$

$$F_{22i}(k_1, k_2) = X_3 \left(1 + e^{-2(z+t_2)X_2}\right)$$

$$F_{23r}(k_1, k_2) = -\sin(z + t_2)X_3$$

$$F_{23i}(k_1, k_2) = \cos(z + t_2)X_3$$

$$F_{31r}(k_1, k_2) = -\sin \zeta X_3$$

$$F_{31i}(k_1, k_2) = \cos \zeta X_3$$

$$F_{32r}(k_1, k_2) = -\sin \zeta X_3$$

$$F_{32i}(k_1, k_2) = \cos \zeta X_3$$

$$F_{33r}(k_1, k_2) = -\sin zX_3$$

$$F_{33i}(k_1, k_2) = \cos zX_3$$

$$C_{11}(k_1,k_2) = \cos\zeta X_1$$

$$C_{12}(k_1,k_2) = e^{(z+t_1)X_2}\cos\zeta X_1$$

$$C_{13}(k_1,k_2) = 2X_3 e^{-t_2 X_1}\cos\zeta X_1$$

$$C_{21}(k_1,k_2) = e^{\zeta X_2}\cos(z-t_1)X_1$$

$$C_{22}(k_1,k_2) = \frac{1}{2}e^{(z-\zeta)X_2}\left[\cos t_1 X_1\left(e^{2\zeta X_2}+1\right) - \frac{X_2}{X_1}\sin t_1 X_1\left(e^{2\zeta X_2}-1\right)\right]$$

$$C_{23}(k_1,\kappa_2) = X_3 e^{-(\zeta+t_2)X_2}\left[\cos t_1 X_1\left(e^{2\zeta X_2}+1\right) - \frac{X_2}{X_1}\sin t_1 X_1\left(e^{2\zeta X_2}-1\right)\right]$$

$$C_{31}(k_1,k_2) = 2X_3 e^{-t_2 X_2}\cos(z-t_1-t_2)X_1$$

$$C_{32}(k_1,k_2) = X_3 e^{-zX_2}\left[\cos t_1 X_1\left(e^{2(z-t_2)X_2}+1\right) - \frac{X_2}{X_1}\sin t_1 X_1\left(e^{2(z-t_2)X_2}-1\right)\right]$$

$$C_{33}(k_1,k_2) = X_3\left\{\cos\zeta X_3\left[\frac{X_2}{X_1}\sin t_1 X_1\left(1-e^{-2t_2X_2}\right) + \cos t_1 X_1\left(1+e^{-2t_2X_2}\right)\right]\right.$$

$$\left. + X_3\sin\zeta X_3\left[\frac{1}{X_1}\sin t_1 X_1\left(1+e^{-2t_2X_2}\right) + \frac{1}{X_2}\cos t_1 X_1\left(1-e^{-2t_2X_2}\right)\right]\right\}$$

whilst the functions corresponding to the pole contribution are

$$X_1 = \sqrt{(1+\frac{k_2^2}{k_1^2})(k_1^2-\tilde{N}_1^2)}$$

$$X_2 = \sqrt{(1+\frac{k_2^2}{k_1^2})(\tilde{N}_2^2-k_1^2)}$$

$$X_3 = \sqrt{(1+\frac{k_2^2}{k_1^2})(k_1^2-\tilde{N}_3^2)}$$

$$F_{11r}(k_1,k_2) = \frac{1}{2}e^{(z-\zeta)X_1}\left(e^{2\zeta X_1}+1\right)\left\{X_1\left(1-e^{-2(z+t_1)X_1}\right)\left[X_3\frac{\sin t_2 X_2}{X_2} + \cos t_2 X_2\right]\right.$$

$$+\left(1+e^{-2(z+t_1)X_1}\right)\left[X_3\cos t_2X_2 - X_2\sin t_2X_2\right]\Big\}$$

$$F_{12r}(k_1,k_2) = e^{-(\zeta+t_1)X_1}\left(e^{2\zeta X_1}+1\right)\left[X_3\cos(z+t_1+t_2)X_2 - X_2\sin(z+t_1+t_2)X_2\right]$$

$$F_{13r}(k_1,k_2) = X_3e^{-(\zeta+t_1)X_1}e^{zX_3}\left(e^{2\zeta X_1}+1\right)$$

$$F_{21r}(k_1,k_2) = e^{-zX_1}\left(e^{2(z-t_1)X_1}+1\right)\left[X_3\cos(\zeta+t_2)X_2 - X_2\sin(\zeta+t_2)X_2\right]$$

$$F_{22r}(k_1,k_2) = \left[X_3\cos(z+t_2)X_2 - X_2\sin(z+t_2)X_2\right]\left[\left(1+e^{-2t_1X_1}\right)\cos\zeta X_2 + X_2\frac{\left(1-e^{-2t_1X_1}\right)}{X_1}\sin\zeta X_2\right]$$

$$F_{23r}(k_1,k_2) = X_3e^{(z+t_2)X_3}\left[\left(1+e^{-2t_1X_1}\right)\cos\zeta X_2 + X_2\frac{\left(1-e^{-2t_1X_1}\right)}{X_1}\sin\zeta X_2\right]$$

$$F_{31r}(k_1,k_2) = X_3e^{-(z-t_2)X_1}e^{\zeta X_3}\left(e^{2(z-t_1-t_2)X_1}+1\right)$$

$$F_{32r}(k_1,k_2) = X_3e^{\zeta X_3}\left[\left(1+e^{-2t_1X_1}\right)\cos(z-t_2)X_2 + X_2\frac{\left(1-e^{-2t_1X_1}\right)}{X_1}\sin(z-t_2)X_2\right]$$

$$F_{33r}(k_1,k_2) = X_3e^{zX_3}\left\{\cos\zeta X_3\left[\left(1+e^{-2t_1X_1}\right)\cos t_2X_2 - X_2\frac{\left(1-e^{-2t_1X_1}\right)}{X_1}\sin t_2X_2\right]\right.$$

$$\left.-X_3\sin\zeta X_3\left[\frac{\left(1-e^{-2t_1X_1}\right)}{X_1}\cos t_2X_2 + \left(1+e^{-2t_1X_1}\right)\frac{\sin t_2X_2}{X_2}\right]\right\}$$

$$\frac{dD^*}{dk_2}(k_1,k_2) = (t_1+t_2)\left(1+e^{-2t_1X_1}\right)\left[X_3\cos t_2X_2 - X_2\sin t_2X_2\right]$$

$$+\left(t_1\frac{X_1}{X_2} - t_2\frac{X_2}{X_1}\right)\left(1-e^{-2t_1X_1}\right)\left[X_3\sin t_2X_2 + X_2\cos t_2X_2\right]$$

Again no imaginary functions exist on the horizontal paths in this interval.

4. $\tilde{N}_2 < k_1 < \infty$. Only the integral arising from the discontinuity exists in this interval

$$\int_{\tilde{N}_2}^{\infty} \cos r_1 k_1 \int_{k_1}^{\infty} C_{ij}(k_1, \kappa_2) \frac{e^{-|r_2|\kappa_2}}{k_1^2 - \kappa_2^2} \frac{F_{ijr}(k_1, \kappa_2) D_i(k_1, \kappa_2) - F_{iji}(k_1, \kappa_2) D_r(k_1, \kappa_2)}{D_r^2(k_1, \kappa_2) + D_i^2(k_1, \kappa_2)} d\kappa_2 dk_1$$

The functions here can be obtained by substituting

$$X_1 = \sqrt{(\frac{k_2^2}{k_1^2} - 1)(k_1^2 - \tilde{N}_1^2)}$$

$$X_2 = \sqrt{(\frac{k_2^2}{k_1^2} - 1)(k_1^2 - \tilde{N}_2^2)}$$

$$X_3 = \sqrt{(\frac{k_2^2}{k_1^2} - 1)(k_1^2 - \tilde{N}_3^2)}$$

in the functions defined for the first integral in the interval $0 < k_1 < \tilde{N}_3$.