

UNIVERSITY OF SOUTHAMPTON

**DYNAMIC ANALYSIS OF FRP LAMINATED AND
SANDWICH PLATES**

by

Marion Meunier

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ABSTRACT

FACULTY OF ENGINEERING AND APPLIED SCIENCE
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The work presented in this thesis document is concerned with the investigation of the dynamic behaviour of Fibre Reinforced Plastic (FRP) laminated and sandwich plates composed of a PolyVinylChloride (PVC) foam core material. An extensive review, covering existing modelling techniques for FRP composite plates and for viscoelastic materials as well as published viscoelastic material data, has highlighted many areas of limitations. A comprehensive theoretical study was therefore carried out in order to develop new analytical models. It has been demonstrated that Reddy's refined HSDT appears to be the most versatile and accurate modelling method for FRP laminated and FRP sandwich plates for which dynamic responses need to be predicted. Since FRP laminated and FRP sandwich plates are made of, amongst other things, PVC foam material and/or resin system which exhibit viscoelastic damping, the method has been enhanced to offer the possibility to predict the forced response and the steady state response of such panels, while taking into account their material viscoelastic properties. The new models have been developed by combining the HSDT with viscoelastic material modelling tools namely the elastic viscoelastic correspondence principal or the GHM mathematical method offering the feasibility to carry out either steady state or transient analysis. In parallel to the development of novel analytical models, experimental testing has been carried out in order to gain knowledge and data on the dynamic properties and behaviour of virgin and damaged PVC foam materials. Finally results from the experimental and the theoretical aspects of the research have been combined and the dynamic analysis of specific FRP sandwich plates has been realised. From this research work, a better understanding of the dynamic behaviour of undamaged and damaged FRP composite panels with designated viscoelastic material properties has been acquired.

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NOMENCLATURE

ENGLISH ALPHABET

LOWER CASE SYMBOLS

a	Length of the plate
b	Width of the plate
f	Frequency in Hertz
h	Thickness of the studied panel
h_c	Thickness of the core for sandwich panel
h_d	Thickness of the damaged zone
h_f	Thickness of the faces for sandwich panel
m	Number of vibration half waves in the x direction
n	Number of vibration half waves in the y direction
n_2	Number of layers of each face of a sandwich plate
q	Transverse applied load
q_0	Maximum amplitude of the transverse applied load
s	Laplace domain operator
t	Time variable
Δt	Time step
u, v, w	Displacement components along the x, y, z co-ordinate directions
u_0, v_0, w_0	Displacement components along the x, y, z co-ordinate directions of a point on the midplane
w_b	Transverse displacement due to bending
w_s	Transverse displacement due to shear
$\bar{w}$	Dimensionless transverse displacement
(x, y, z)	Plate co-ordinate system
(x_1, x_2, x_3)	Material co-ordinate system
z, z_i	Thickness co-ordinate measured from the mid-plane of the plate

UPPER CASE SYMBOLS

A_{ij}, B_{ij}, D_{ij}	In-plane, coupled bending-stretching and bending plate stiffnesses
$A_{ij}^*, B_{ij}^*, D_{ij}^*$	Complex plate stiffnesses
$C(t)$	Relaxation modulus for isotropic material
$C_{ij}(t)$	Relaxation modulus for anisotropic material
C_{ij}	Material stiffnesses in the lamina co-ordinate system
$\bar{C}_{ij}, \bar{C}_{ij}^*$	Elastic and complex material stiffnesses expressed in the plate co-ordinate system

$\overline{C}_{ij}^{(k)} ; \overline{C}_{ij}^{*(k)}$	k th layer elastic and complex material stiffnesses expressed in the plate co-ordinate system
$\overline{C}_{ij}^{(core)} , \overline{C}_{ij}^{*(core)}$	Core elastic and complex material stiffnesses expressed in the plate co-ordinate system
$C_{ij}^*(\omega) = C_{ij}'(\omega) + iC_{ij}''(\omega)$	Frequency dependent complex material stiffnesses expressed in the lamina co-ordinate
E	Elastic Young's modulus
E' , E''	Storage and loss Young's modulus
E^*	Complex Young's modulus
E_c	Elastic modulus of the core for sandwich plate
E_f	Elastic modulus of the face for sandwich plate
E_1	Elastic modulus of FRP material in the direction along fibres
E_2 , E_3	Elastic moduli of FRP material in direction x_2 and x_3
E_1' , E_1''	Storage and loss modulus of FRP material in the direction along fibres
E_2' , E_2''	Storage and loss modulus of FRP material in direction normal to fibres
$E_1^* = E_1' + iE_1''$	Complex elastic modulus of FRP material in the direction along fibres
$E_2^* = E_2' + iE_2''$	Complex elastic modulus of FRP material in direction normal to fibres
$s\tilde{E}_c(s)$	Laplace transform function of the core Young's modulus
E_{ij} , F_{ij} , H_{ij}	High-order laminate stiffnesses
$E_{ij}^* , F_{ij}^* , H_{ij}^*$	Complex high-order laminate stiffnesses
F	Applied force
G	Elastic shear modulus
G_c	Elastic shear modulus of the core
G' , G''	Storage and loss shear modulus
G^*	Complex shear modulus
$s\tilde{G}_c(s)$	Laplace transform function of the core shear modulus
G_{12} , G_{13} , G_{23}	FRP in-plane shear modulus and transverse shear moduli
$G'_{12} , G'_{13} , G'_{23}$	FRP in-plane storage shear modulus and transverse storage shear moduli
$G''_{12} , G''_{13} , G''_{23}$	FRP in-plane loss shear modulus and transverse loss shear moduli
$G_{12}^* , G_{13}^* , G_{23}^*$	FRP in-plane complex shear modulus and transverse complex shear moduli
G^o	Equilibrium value of modulus
$\hat{G}^o$	GHM approximation to G^o
$s\hat{G}(s)$	Laplace transform function of the material modulus
I_1 , I_2 , I_3	Normal, coupled normal-rotary and rotary inertia

I_4, I_5, I_7	High-order inertia terms
J_{ijkl}	Creep functions
K	Shear correction factor
K_{ij}	Coefficients of the stiffness matrix $[K]$
M_i	Moment resultants
N_i	Force resultants
P_i	Higher stress resultants
Q_i	Transverse force resultants
Q_{mn}	Transverse load coefficient
R_i	Higher stress resultants
$S(t)$	Creep compliance for isotropic material
$S_{ij}(t)$	Relaxation modulus for anisotropic material
$T = 2\pi/\omega$	Period of vibration
$U_{mn}, V_{mn}, W_{mn}, X_{mn}, Y_{mn}$	Generalised spatial variation amplitudes

GREEK ALPHABET

LOWER CASE SYMBOLS

α, β	$= m\pi/a, n\pi/b$
$\hat{\alpha}$	GHM mini-oscillator parameter
χ, γ	Newmark integration technique parameters
δ	Phase angle
$\varepsilon, \varepsilon^o$	Strain and strain amplitude
ε_{ij}	Strain components
ε_i	Contracted notation of the strain components
κ, κ_1	Spring stiffnesses
ϕ_x, ϕ_y	Rotations of a transverse normal about the y- and x- axes, respectively
φ_i	Phase angle components
η	Modal loss factor
η_E, η_G	Tan delta values associated respectively with Young's and shear complex modulus
η_1, η_2	Tan delta values associated respectively with Young's complex modulus E_1^* and E_2^*
$\eta_{12}, \eta_{13}, \eta_{23}$	Tan delta values associated respectively with Young's complex modulus G_{12}^*, G_{13}^* and G_{23}^*
$\lambda^* = (\omega^*)^2 = \lambda' + i\lambda''$	Complex eigenvalues
θ	Fibre orientation relative to the x -axis in a lamina
θ_i	Fibre orientation relative to the x -axis in the i^{th} layer of a laminate

μ	Viscosity or dashpot parameter
ν, ν_{ij}	Poisson ratios
ν_c	Poisson ratio of the core material
ν_f	Poisson ratio of sandwich faces isotropic material
ρ	Density
ρ_c	Density of the core material in sandwich panel
ρ_f	Density of the face material in sandwich panel
$\rho^{(k)}$	Material density of the k th layer
σ	Stress
σ_{ij}	Stress components
σ_i	Contracted notation of the stress components
τ	Input time
ω	Frequency in rad/sec
$\bar{\omega}$	Dimensionless natural frequency
ω^*	Complex frequency
$\hat{\omega}$	GHM mini-oscillator parameter
$\hat{\zeta}$	GHM mini-oscillator parameter

VECTOR NOTATIONS

$\{\bar{z}\}, \{\bar{\bar{z}}\}$	Vector of the dissipative co-ordinates
$\{F\}$	Vector of the applied forces
$\{F_0\}$	Vector of the amplitudes of harmonic applied forces
$\{F_v\}$	Viscoelastic vector of the applied forces
$\{\delta^*\}$	Complex eigenvectors
$\{\epsilon\}$	Vector of the strain components
$\{\sigma\}$	Vector of the stress components
$\{\Delta\}$	Vector of the generalised spatial variation amplitudes
$\{\Delta^*\} = \{\delta^*\} \exp(i\omega t)$	Vector of the generalised periodic solutions
$\{\Delta_0\}$	Vector of the amplitudes of the generalised periodic solutions.
$\{\Delta_v\}$	Viscoelastic vector of the generalised spatial variation amplitudes

MATRIX NOTATIONS

$[a_p], [b_r]$	Constant matrices of the differential form of the stress-strain relations
$[C]$	Material stiffness matrix
$[D]$	Damping matrix
$[D_v]$	Viscoelastic damping matrix

$[D_z]$	Sub matrix of $[D_v]$
$[D_{\bar{z}}]$	Sub matrix of the viscoelastic damping matrix before decomposition
$[K]$	Stiffness matrix
$[K']$	Real part of the complex stiffness matrix
$[K'']$	Imaginary part of the complex stiffness matrix
$[K^*] = [K'] + i[K'']$	Complex stiffness matrix
$[K_v]$	Viscoelastic stiffness matrix
$[K_q], [K_{qz}], [K_z]$	Sub matrices of $[K_v]$
$[K_{q\bar{z}}], [K_{\bar{z}}]$	Sub matrices of the viscoelastic stiffness matrix before decomposition
$[s\tilde{K}(s)]$	Matrix of the Laplace transform function of the complex stiffness coefficients
$[\bar{K}]$	Modulus factored out stiffness matrices
$[K]_s$	Stiffness matrix due to the faces of the sandwich plate
$[\bar{K}]_{G_c}$	Modulus factored out stiffness matrix associated with the core shear modulus
$[\bar{K}]_{E_c}$	Modulus factored out stiffness matrix associated with the core Young's modulus
M_{ij}	Coefficients of the mass matrix $[M]$
$[M]$	Mass matrix
$[M_v]$	Viscoelastic mass matrix
$[M_z]$	Sub-matrix of $[M_v]$
$[M_{\bar{z}}]$	Sub matrix of the viscoelastic matrix before decomposition
$[\bar{R}]$	Matrix of the mode eigenvectors of the stiffness matrix $[K]$
$[R]$	$= [\bar{R}][\Lambda]$
$[S]$	Compliance matrix
$[T]$	Transformed matrix
$[\bar{\Lambda}]$	Diagonal matrices of the positive eigenvalues of $[\bar{K}]$
$[\Lambda]$	$= \hat{G}^o[\bar{\Lambda}]$

OTHER SYMBOLS

$(\sim)$	Laplace transform
$(\dot{})$	First time derivative
$(\ddot{})$	Second time derivative
$[\]^T$	Matrix transform

PART A

Chapter 1: Introduction.

Chapter 2: Introduction to FRP laminated and sandwich structures.

1. INTRODUCTION

1.1 BACKGROUND TO THE SUBJECT

Fibre reinforced plastic (FRP) materials have been used for a relatively long time, in a variety of structural engineering applications and are now finding an increased usage in the marine industry through the development of high speed light craft. This is mainly because of the high strength to weight ratio of these materials, which is ideal for the construction of ship hulls and aiding cost efficiency. Furthermore, FRP materials are corrosion resistant, have a low maintenance cost and have low magnetic characteristics which makes them well suited for smaller naval ships assigned to special tasks, such as mine hunting. A structurally efficient manner in which these materials can be used is in a sandwich configuration. This configuration has certain advantages over a more convenient single skin laminated structure, namely improvement in the flexural rigidity at a reduced weight. For marine applications sandwich structures are often employed in weight critical fast hull forms where high stiffness is important in order to retain structural integrity and hydrodynamic efficiency. Additional benefits gained are thermal and sound insulation and damping of vibrations. In the marine context, the FRP sandwich technique was first applied in the late 1960s Royal Swedish Navy minesweepers. However it is only since the 1980s with major developments in high-speed light craft such as high-speed passenger ferries, lifeboats, special military vessels and high-performance sailing and power boats, that it has become a common construction material for hulls, decks and superstructures. This popularity is linked with the emergence of a new relatively rigid core material, PVC foam, which is nowadays the most commonly used sandwich core material in the construction of high speed vessels.



Figure 1.1: R.N.L.I. Trent Class lifeboat (the hull is made of glass/kevlar epoxy skins and a closed cell polymer foam core).

In parallel with the increased use of FRP laminated and FRP sandwich panels in high performance vessels, the necessity for developing new and more accurate structural design methods has become apparent. This issue is mainly linked with the principal objectives of today's ship designers; namely to reduce the overall structural weight and to make increased cost savings without lengthening the design time or altering the integrity and safe behaviour of ship structures in the conditions and lifetime expected. So, as high speed craft are likely to be subjected to extreme and/or repeated dynamic loads, that have in the past been the cause of damage in FRP sandwich hulls of some rescue vessels, one of the new key design issues is related to the dynamic performance of FRP laminated and FRP sandwich structures. That is to say, it is essential, from the design stage, not only to carry out static analysis but also to predict with accuracy the dynamic responses of such structures. However, up until now, less work has been dictated to the development of dynamic analytical models and so existing models for FRP laminated and FRP sandwich panels are based on assumptions that lead to accurate results when static analyses are carried out but not always when dynamic analyses are performed. There is consequently a requirement for further research in order to develop analytical models that allow accurate prediction of the dynamic behaviour of both FRP laminated and FRP sandwich panels. Such models should account for all the fundamental design elements such as loads, material properties, and geometric characteristics since each one of them forms a base for the accuracy of the whole process. Indeed, the lack of information on for example material properties or an unsuitable method for the case studied will result in an incorrect design. For the specific case of FRP laminated and FRP sandwich structures, which are composed of, among other things, a resin system and/or a PVC foam core material, it is essential to account for the material viscoelastic properties when dynamic analyses are carried out. Indeed, energy dissipation mechanisms in such structures play an important role in vibration control and fatigue endurance. However, it is not a common practice in actual modelling techniques especially when transient analyses are carried out to take into consideration these properties. This opens up the work to new development in order to overcome this important drawback and to offer dynamic analytical models that account for the dynamic properties of the constitutive materials. Concurrently, the lack of published data on the viscoelastic properties of common constitutive materials of FRP laminated and FRP sandwich panels such as PVC foam materials will need to be overcome. Indeed, it is not standard for manufacturers to provide information on their material dynamic properties.

The principal objective of this research work is therefore to develop new analytical methods that allow the prediction and the analysis of the dynamic responses of FRP laminated and FRP

sandwich plates. The developed methods should offer the possibility to account for the viscoelastic properties of the constitutive materials and for the degradation of these properties during the service life of the panels. The development of such models is a complex problem that requires knowledge and understanding in various fields. In fact, results from both theoretical and experimental research works will need to be combined to satisfy this task. This thesis document sets out to present and to draw together these results in order to produce the desired new analytical model.

1.2 LAYOUT OF THE THESIS

The first part of this thesis document assesses the existing field of expertise with respect to FRP laminated and FRP sandwich plates modelling techniques and with respect to published viscoelastic material data and viscoelastic material mathematical modelling tools in order to establish direction towards further research. First of all, a critical review of the most commonly used FRP laminated and FRP sandwich plate modelling theories is presented in chapter 3. Their accuracy and their limitations, as far as the prediction of the dynamic behaviour of the panels is concerned, is discussed. In chapter 4, the existing analytical models available to describe linear viscoelastic material behaviour are introduced. The efficiency and the deficiency of these models to represent viscoelastic material behaviour are outlined and the feasibility of incorporating these models in the previously introduced plate theories is investigated. Besides, in this chapter the importance of the lack of published dynamic material data is highlighted.

In the light of the knowledge gained from the literary review the second part of this thesis document concentrates on the development of new dynamic analytical models and the analysis of experimental data on the dynamic properties of intact and damaged closed cell PVC foam material. Chapter 6 investigates the possibility of using analytical models based on Reddy's FSDT and HSDT theories to model and predict the dynamic responses of both FRP laminated and FRP sandwich plates, assuming elastic material properties. By comparing the predicted results with published data and finite element results conclusions are made on the accuracy and efficiency of both proposed methods. In chapter 7 existing experimentation rigs are first of all briefly reviewed in order to provide sufficient knowledge on available experiment techniques, which can be used when published data are insufficient. The experiments carried out on PVC foam material are then described and the results obtained for both intact and damaged samples are presented and discussed. Finally in chapter 8, new analytical models which account for

viscoelastic properties of the constitutive materials and which offer the feasibility to accurately predict the dynamic responses of FRP laminated and FRP sandwich plates are presented. Then, the developed models and the measured dynamic material properties are used to first of all predict the natural frequencies and modal loss factors and then the transient responses of simply supported sandwich panels made of three layers of Eglass/polyester skins and a PVC foam core. The dynamic responses, which are predicted for both intact and damaged PVC foam material, are discussed and analysed.

In the last part of this thesis document an overall discussion is given. Then, conclusions are drawn and recommendations for further work on the subject provided.

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2. INTRODUCTION TO FRP LAMINATED AND SANDWICH STRUCTURES

2.1 INTRODUCTION

As mentioned in the introductory chapter, the research work presented in this thesis document considers two specific structural configurations widely used in high speed and high performance vessels, namely FRP laminated plates and FRP sandwich plates. These two fundamental structural configurations, which are represented in figure 2.1, are introduced and described in more detail in this chapter.

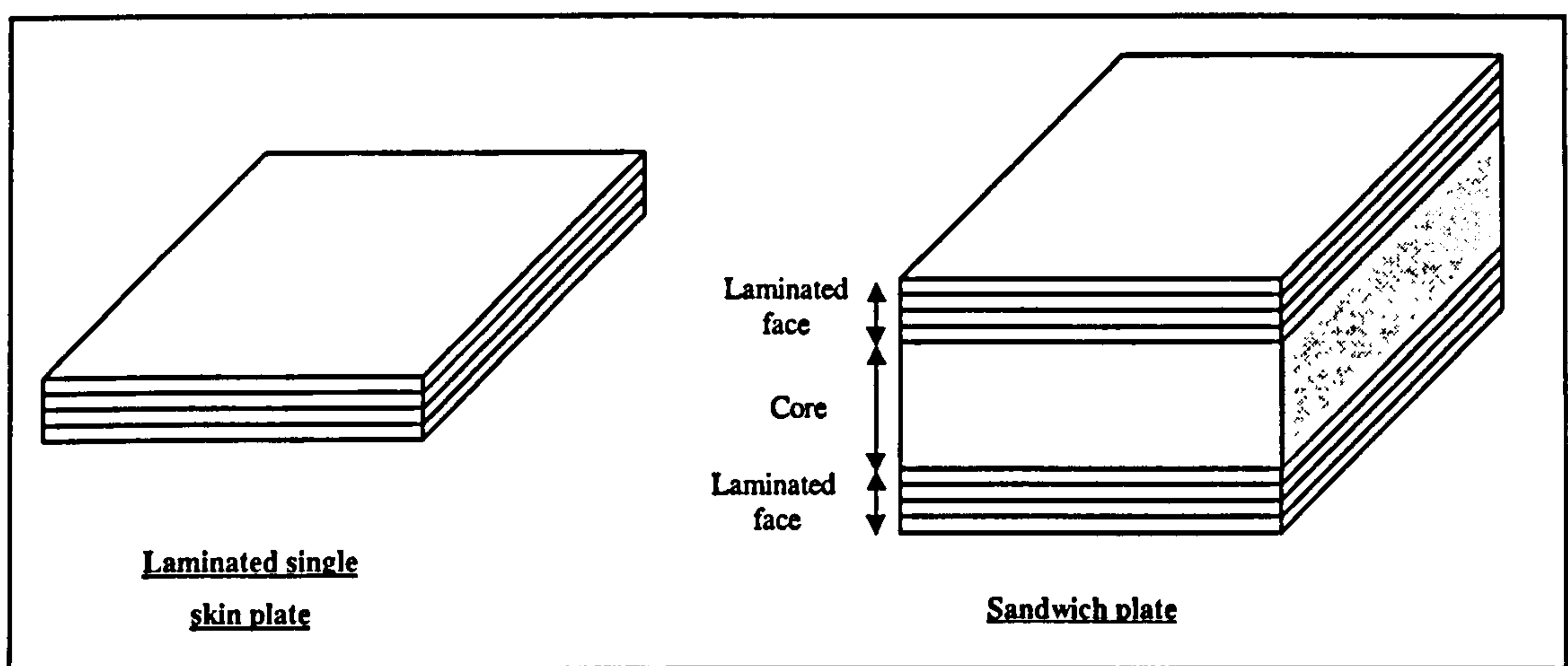


Figure 2.1: Schematic drawing of laminated and sandwich structures

2.2 FRP LAMINATED AND FRP SANDWICH PLATES

2.2.1 FRP LAMINATED PLATES

FRP laminated structures, also called laminates, are made of several unidirectional laminae (or plies or layers) stacked together at various orientations, as can be seen in figure 2.2, in order to achieve the desired stiffness and thickness. FRP laminate is, for example, the construction material of single skin structures, also called monolithic structures, as well as the construction material of the bottom and the top faces of sandwich structures.

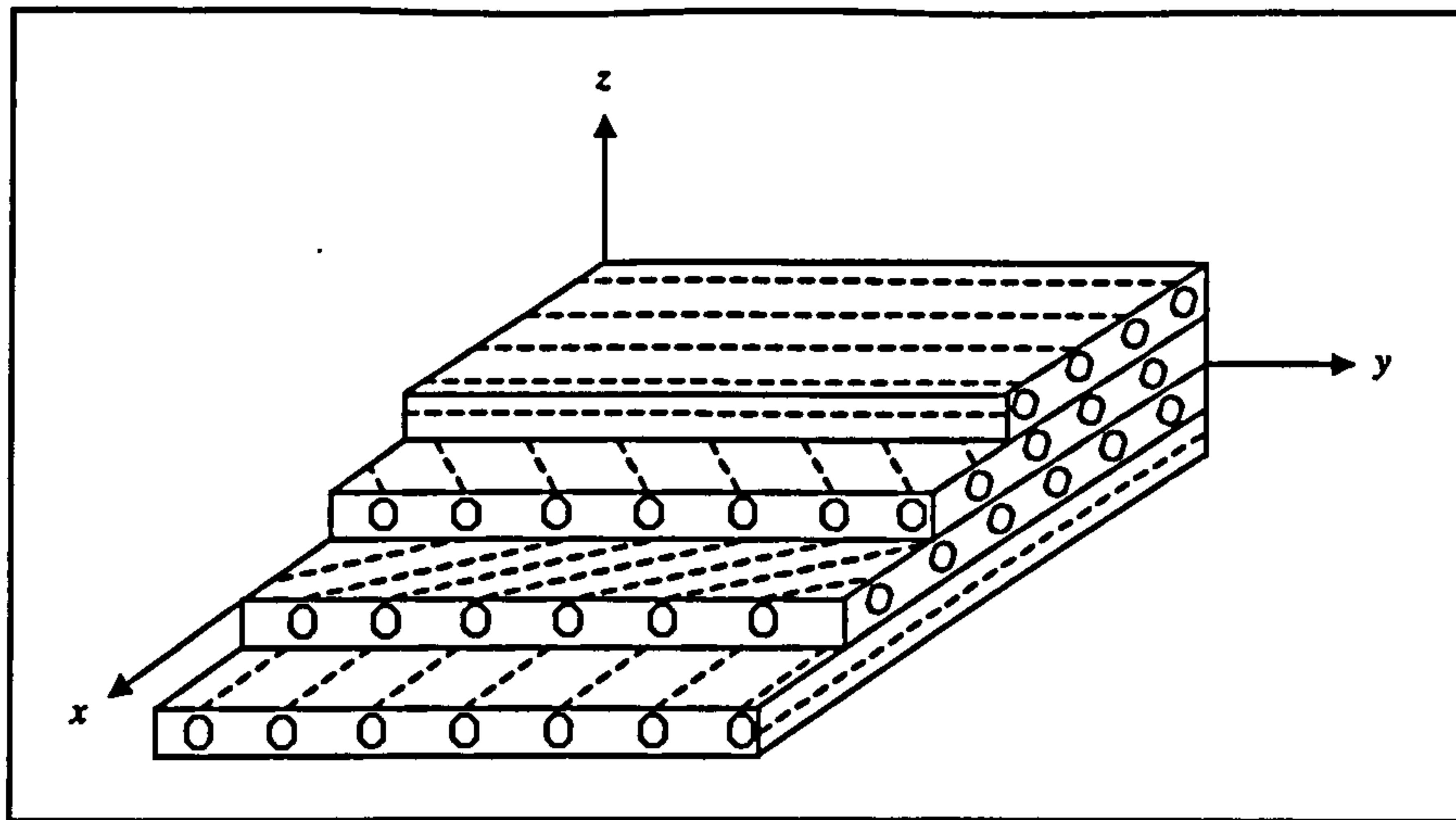


Figure 2.2: Sketch of a FRP laminated plate

The sequence of various orientations of a fibre reinforced composite layer in a laminate is termed the lamination scheme or stacking sequence. The lamination scheme of a laminate is denoted by $[\theta_1/\theta_2/\theta_3/\dots]$, where θ_1 is the orientation of the first ply, θ_2 is the orientation of the second ply, and so on. These angles are measured from the x -axis as shown in figure 2.3 and their values are given in degrees.

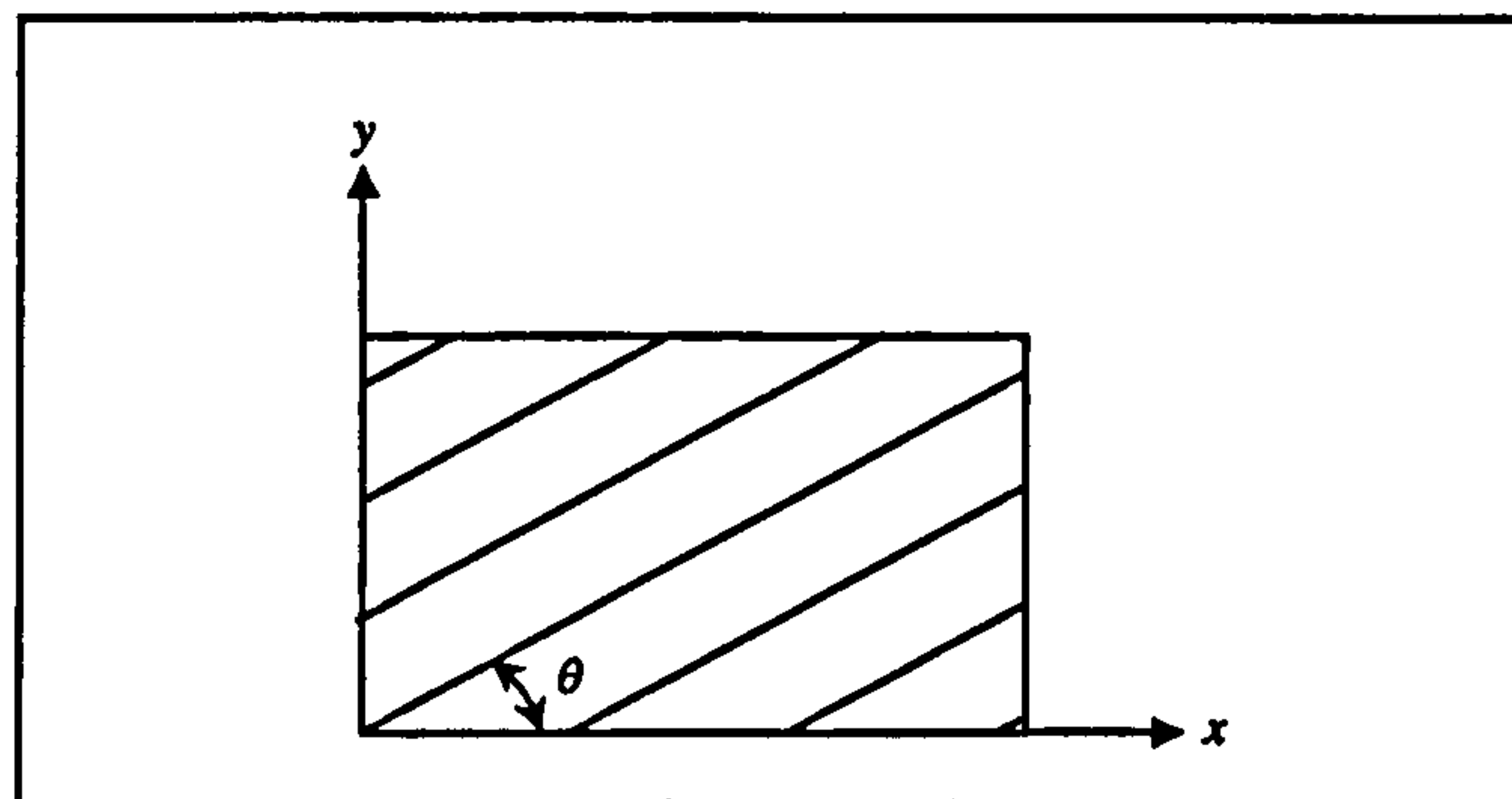


Figure 2.3: Plane view of a lamina with fibre orientated at an angle θ from the x -axis.

Laminates with ply orientations of θ and $-\theta$ where $0^\circ \leq \theta \leq 90^\circ$, and with at least one layer having an orientation other than 0° and 90° are called angle-ply laminates. An example of angle-ply is provided by $[15/30/0/90/45/-45]$. On the other hand, laminates with ply orientations of 0° and 90° are called cross-ply laminates. An example of a cross-ply laminate is $[0/90/0/90/0/90]$. When ply stacking sequence, material and geometry are symmetric about the midplane of the laminate, the laminate is called a symmetric laminate. For a symmetric laminate the upper half through the laminate thickness is a mirror image of the lower half. The laminate $[-45/45/45/-45]_s$, with all layers having the same thickness and material, is an example of a symmetric angle-ply laminate, alternatively $[0/90/90/0]_s = [0/90]_s$ is a symmetric cross-ply

laminate. An antisymmetric laminate is one whose lamination scheme is antisymmetric and material and thicknesses are symmetric about the midplane. Examples of antisymmetric angle-ply and cross-ply laminates are provided, respectively, by $[-45/45/-45/45]$ and $[0/90/0/90]$. It should be noticed that the feasibility of having different lamination scheme and different material properties for each individual lamina provide an added flexibility to designers to tailor the stiffness and strength of the laminate to match the structural requirements of specific applications.

2.2.2 FRP SANDWICH PANELS

FRP sandwich panels are special forms of laminated shell plates. They consist of three distinct layers (i.e.: the bottom face, the top face and the core) which are bonded together to form efficient load carrying assemblies. ASTM [2.1] defines the sandwich structure as follows:

A structural sandwich is a special form of a laminated composite comprising a combination of different materials that are bonded to each other so as to utilise the properties of each separate component to the structural advantage of the whole assembly.

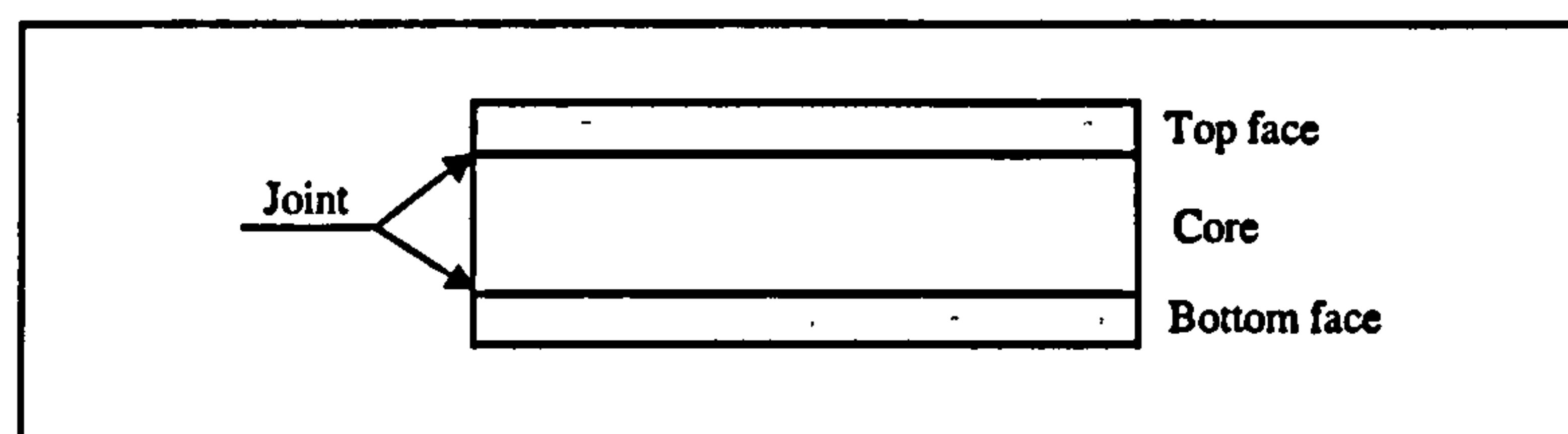


Figure 2.4: Schematic of a structural sandwich panels

The faces of FRP sandwich structures are a stack of arbitrary orientated laminae assembled to form a laminated plate or shell. The core is made of a lower density material to add as little as possible to the total weight of the structure. It can be made of balsa wood, honeycomb or cellular foam. The theory to model composite panels, which will be presented in chapter 6, offers the possibility to investigate the behaviour of sandwich structures with the faces made of FRP laminate and with the core made of any of the three previously mentioned materials. However all the applications presented in this thesis are realised only for the specific case of cellular foam core material. Cellular foam material is widely used as a construction material in the marine industry. Indeed, sandwich structures consisting of a foam core material combined with high strength skins are today's solution for demanding lightweight and stiff structures. The hull of the Visby corvette of the Swedish Navy is, for example, a sandwich construction comprising a

cellular foam core with carbon fibre and vinyl ester laminate. Another example is the hull of the R.N.L.I Trent class lifeboats that is made of glass/kevlar epoxy skins and a closed cell polymer foam core.

The greatest advantage with sandwich construction compared to solid laminates is that the strength and stiffness are increased without a corresponding increase in the weight. Indeed by increasing the thickness of the core and so increasing the total thickness of the laminates, the stiffness to weight ratio and the strength to weight ratio are significantly increased. Numerical example of the increase of flexural rigidity and bending strength of sandwich panels due to an increase in thickness can be found in Zenkert's book [2.2]. Sandwich design appears therefore as an attractive concept to use in FRP hull instead of a stiffened single skin structure since the built-in-flexural stiffness of the sandwich makes the stiffener system unnecessary. It should be mentioned that as well as offering a high stiffness and strength to weight ratios, the sandwich design gives further advantages such as thermal and acoustic insulation, high energy absorption capacity and buoyancy.

2.3 CONCLUSION

This part has been devoted to the presentation of the structural configurations, namely FRP laminated and FRP sandwich plates, under investigation in this thesis document. Indeed, the research work presented in the following parts of this document is dedicated to the development of analytical models that allow accurate predictions of the dynamic behaviour of such structures.

2.4 REFERENCES

- [2.1] ASTM. Annual book of ASTM standards. Technical report, American Society for Testing and Materials, Philadelphia, Pennsylvania, USA, 1991.
- [2.2] Zenkert D., *An Introduction to Sandwich Construction*, EMAS Publishing, 1995.

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PART B

Chapter 3: Laminated and sandwich plate theories.

Chapter 4: Linear viscoelastic damping in composite materials.

3. LAMINATED AND SANDWICH PLATE THEORIES

3.1. INTRODUCTION

The dynamic behaviour of a structure is determined by solving its equations of motion. The first step of the work, as for any structural vibration problem, is therefore the formulation of the equations of motion, also known as the mathematical model of the structure. This is an important part of the work since the accuracy of the results is dependent upon these equations being formulated correctly. In the case of composite panels, they are derived from the assumed displacement field or stress field using an energy principle, namely the principle of virtual displacements, which is the Hamilton principle expressed for the case of a continuous system [3.1]. The derivation of such mathematical models requires good background knowledge of the fundamental equations of anisotropic elasticity. These equations are provided for the case of composite materials in appendices A and B and their application to the case of multilayered composite panels is reviewed in appendix C. A good understanding of the existing structural theories is also required. Indeed, the accuracy of the formulation of the equations of motion is mainly determined by the choice of a proper structural theory for the problem. Different displacement based theories have been proposed in the literature to model FRP single-skin laminated and FRP sandwich panels. In this chapter, a critical review of the most commonly used ones is presented. Their validity, as far as the prediction of the dynamic behaviour of FRP laminated and FRP sandwich plates is concerned, is discussed. The advantages and disadvantages of each of them are outlined with the objective of establishing the most appropriate theory for the study of the dynamic behaviour of both FRP single-skin and FRP sandwich composite plates.

3.2 LAMINATED COMPOSITE PLATE THEORIES

3.2.1 EQUIVALENT SINGLE LAYER THEORIES

In the equivalent single layer laminate theories a heterogeneous laminated plate is treated as a statically equivalent single layer, having a complex constitutive behaviour, reducing the three-dimensional continuum problem to a two-dimensional problem. The equivalent single layer theories are derived from the three-dimensional elasticity theory by assuming the form of the displacement field or the stress field as a combination of unknown functions and the thickness co-ordinate.

3.2.1.1 Classical laminated plate theory

The simplest equivalent single layer laminate theory is the Classical Laminated Plate Theory (or CLPT) [3.2], which is an extension of the Kirchhoff (classical) plate theory to laminated composite plates. It is based on the following assumptions (see figure 3.1):

- Straight lines perpendicular to the midsurface (i.e., transverse normals) before deformation remain straight after deformation.
- The transverse normals do not experience elongation (i.e., they are inextensible).
- The transverse normals rotate such that they remain perpendicular to the midsurface after deformation.

The first two assumptions imply that the transverse displacement is independent of the transverse (or thickness) co-ordinate and the transverse normal strain is zero. The third assumption results in zero transverse shear strains. So, since the Kirchhoff amounts to neglecting both transverse shear and transverse normal effects, the deformation is due entirely to bending and to inplane stretching.

The classical plate theory is adequate for the analysis of thin plates, especially when the transverse deformation is negligible. However, this is no longer the case for moderately thick plates, in which the transverse shear and transverse normal deformation cannot be ignored. As far as fibre reinforced composite laminated plates are concerned, the shear deformation effects are more pronounced in comparison with the isotropic plates due to the high ratios of in-plane modulus to transverse shear modulus induced by the great difference in elastic properties between fibre filaments and matrix. These high ratios make CLPT inadequate for the analysis of fibre reinforced composite plates. Indeed, by not taking into account the transverse shear deformation in a laminate, the effective stiffnesses are overpredicted leading to the deflections being underpredicted and the natural frequencies being overpredicted [3.3]. In the case of the transient analysis, it has been found that the amplitude and period of oscillation are underpredicted [3.4, 3.5].

3.2.1.2 First-order shear deformation theory

The next theory in the hierarchy of equivalent single layer laminate theories is the First-order Shear Deformation Theory (FSDT), more commonly known as “the Mindlin plate theory”. In the FSDT, the Kirchhoff hypothesis is relaxed by assuming that the transverse normals do not remain perpendicular to the midsurface after deformation (see figure

3.1). This is achieved by including constant transverse shear strains in the theory, with respect to the thickness co-ordinate. Since the transverse shear strains are represented as constant through each layer thickness of the laminate, it follows that the transverse shear stresses will also be constant. However, it is well known that the transverse shear stress varies at least parabolically through the thickness. So, the theory requires shear correction factors to account for the discrepancy between the constant state of shear strains and stresses in the FSDT and the quadratic or higher-order distribution of shear strains and stresses in the elasticity theory. These coefficients are dimensionless quantities, which have been calculated for homogeneous and isotropic plates by various static and dynamic methods [3.6, 3.7, 3.8]. For composite laminates, they are difficult to determine since they depend on a number of different parameters such as the constituent ply properties, the lamination scheme and the type of structure (i.e., geometry and boundary conditions). Values of $5/6$ and $\pi^2/12$ are commonly used to model laminated plates. However, because the accuracy of the prediction of the response of composite plates is strongly dependent on these coefficients, they should be calculated for each specific case.

The FSDT has been used to study the static and dynamic (free vibration and transient analysis) response of various laminated composite plates [3.9, 3.10]. This theory, when used to model layered composite plates, yields acceptable solutions for the global response of plates (displacement, fundamental mode of vibration). Nevertheless, the theory, as mentioned by Reddy in his review [3.11], does not accurately predict higher-order frequencies. Indeed, Reddy and Kuppusamy [3.12] have found that the use of the FSDT with a shear correction factor value of $5/6$ leads to higher frequency than those predicted by the 3D theory. The degree of inaccuracy has been found to be function of the degree of material anisotropy as well as plate aspect ratio. Besides, Sun and Whitney found that if the out-of-plane shear moduli of the constituent layer are dissimilar, even the predicted fundamental modes of vibration are not accurate [3.13]. These results lead to the conclusion that a more accurate theory is required when the dynamic behaviour of complex laminated plates is likely to be investigated.

3.2.1.3 Higher-order shear deformation theories

Development of two-dimensional theories, which incorporate higher-order modes of transverse cross-sectional deformation, has been attempted in recent years. In principle it

is possible to expand the displacement field in terms of the thickness co-ordinate up to any desired degree. However due to the algebraic complexity and computational effort involved with higher-order theories, those higher than third order have not been attempted. The reason for expanding the displacements up to the cubic term in the thickness co-ordinate is to have quadratic variation of the transverse shear strains and transverse shear stresses through the thickness. This avoids the need for shear correction coefficients. There are many papers on third order theories. Although many of them seem to differ from each other on the surface, the displacement fields of these theories are related. Reddy's refined High-order Shear Deformation Theory (HSDT) [3.14, 3.15] is the most commonly used since it contains other lower-order laminate theories, including the CLPT and the FSDT, as special cases.

Reddy's refined HSDT is based on the same assumptions as the CLPT and FSDT, except that the assumption on the straightness and normality of a transverse normal after deformation is relaxed (see figure 3.1) by expanding the displacements as cubic functions of the thickness co-ordinate. The form of the displacement field is dictated by the satisfaction of the conditions that the transverse shear stresses vanish on the plate surfaces and are nonzero elsewhere. Any other choice of displacement field would either not satisfy the stress-free boundary conditions or lead to a theory that would involve more dependent unknowns than those in the FSDT. Indeed, Reddy's refined HSDT contains the same dependent unknowns as the FSDT [3.15].

HSDT, in general, gives more accurate results than the FSDT when compared to the three-dimensional elastic solution. Whereas FSDT is quite accurate when transverse deflections are concerned [3.14], HSDT greatly improves the accuracy of the computed natural frequencies and stresses [3.14, 3.15]. The difference between the computed results using HSDT and FSDT increases with decreasing the length to thickness ratio of the plate and with increasing mode numbers [3.14]. Therefore, since the effect of transverse shear moduli has been found to be more pronounced in thicker laminates, the use of the HSDT is recommended to investigate their behaviour. Besides, the fact that no shear correction coefficients are needed in the refined HSDT makes it easier to use.

It should be noticed that whereas Reddy's refined HSDT has been widely used to study the static behaviour and to carry out the free vibration analysis of laminated composite plates, only one paper has been found on transient analysis [3.5]. It has been observed

that for cross-ply laminated plates, the HSDT slightly improves the prediction of the deflections (amplitude and period) and stresses. Since only one specific case of laminated plates has been considered it should be interesting to carry on the analysis by investigating the prediction of the transient response of various laminated plates modelled using both FSDT and HSDT. Comments on this matter can be found in part 6.5 where some results are presented.

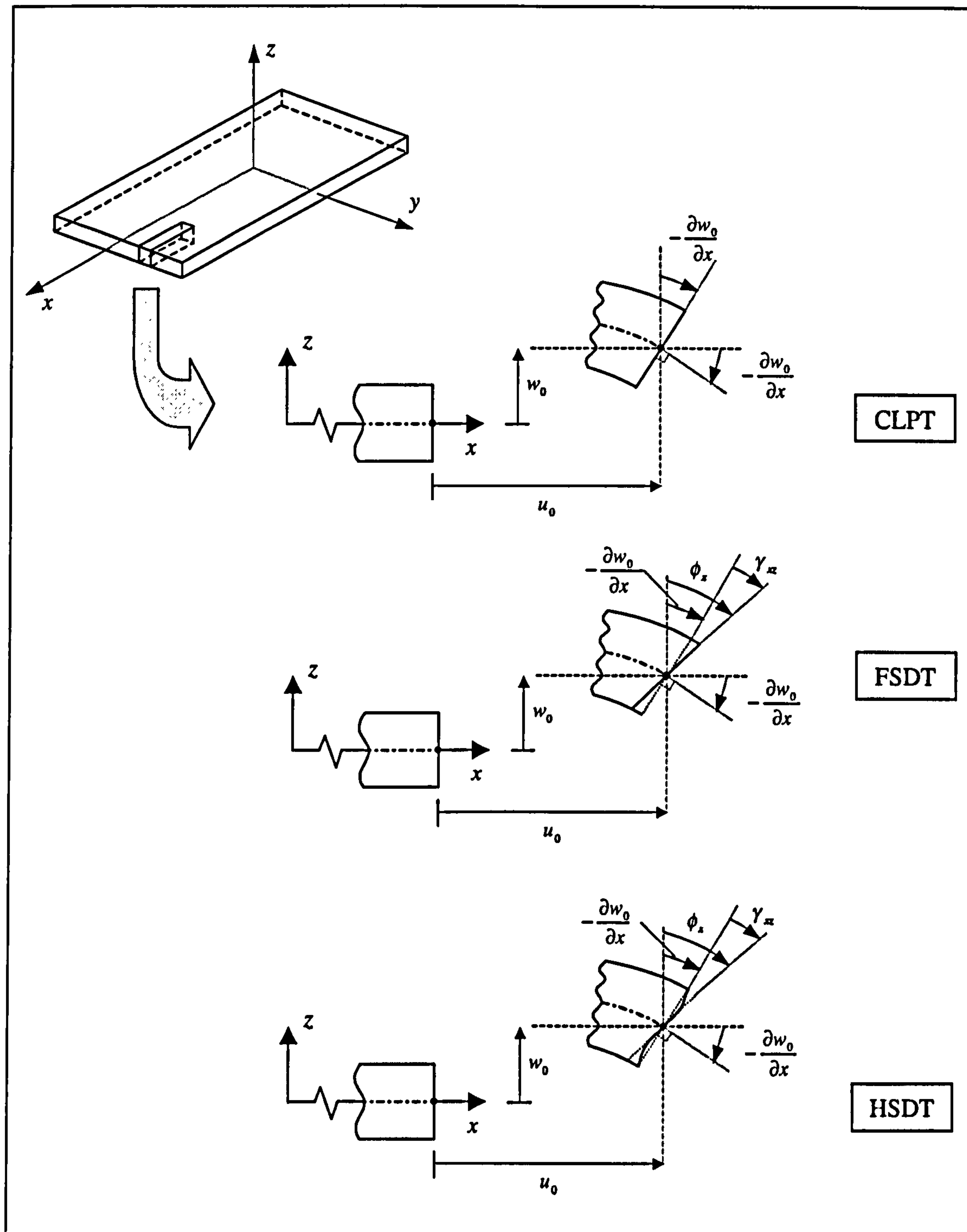


Figure 3.1: Undeformed and deformed geometries of an edge of a plate under the assumption of the equivalent single layer theories.

3.2.1.4 General remarks on equivalent single layer theories

The displacement fields associated to each of the equivalent single layer theories presented in this part are summarised in table 3.1.

Theory	Displacement field equations
CLPT [3.2]	$\begin{cases} u(x, y, z, t) = u_0(x, y, t) - z \frac{\partial w_0}{\partial x} \\ v(x, y, z, t) = v_0(x, y, t) - z \frac{\partial w_0}{\partial y} \\ w(x, y, z, t) = w_0(x, y, t) \end{cases}$
FSDT [3.9]	$\begin{cases} u(x, y, z, t) = u_0(x, y, t) + z \phi_x(x, y, t) \\ v(x, y, z, t) = v_0(x, y, t) + z \phi_y(x, y, t) \\ w(x, y, z, t) = w_0(x, y, t) \end{cases}$
Reddy's refined HSDT [3.14]	$\begin{cases} u(x, y, z, t) = u_0(x, y, t) + z \phi_x(x, y, t) + z^3 \left(-\frac{4}{3h^2} \right) \left(\phi_x + \frac{\partial w_0}{\partial x} \right) \\ v(x, y, z, t) = v_0(x, y, t) + z \phi_y(x, y, t) + z^3 \left(-\frac{4}{3h^2} \right) \left(\phi_y + \frac{\partial w_0}{\partial y} \right) \\ w(x, y, z, t) = w_0(x, y, t) \end{cases}$

Table 3.1. Displacement fields of laminated plate theories.

In addition to their inherent simplicity and low computational cost, the equivalent single layer models often provide a sufficiently accurate description of global response for thin to moderately thick laminates, e.g., gross deflections, critical buckling loads and fundamental vibration frequencies and associated mode shapes. However, the equivalent single layer models have limitations that prevent them from being used to solve the whole spectrum of composite laminate problems. The equivalent single layer models are often incapable of accurately describing the state of stress and strain at the ply level near geometry and material discontinuities or near regions of intense loading, where accurate stresses are needed the most. In such cases, three-dimensional theories or multiple model approaches are required.

Although several single-layer, high-order theories have been developed to improve stress distributions, it remains that none of these can give good interlaminar stresses. Further, they cannot be used to model delaminations in composite laminates.

3.2.2. OTHER THEORIES

The equivalent single layer theories are most often incapable of accurately determining the three-dimensional stress field at the ply level. Thus, the analysis of composite structural components may for specific cases require the use of three-dimensional elasticity theory or a layer-wise laminate theory that contains full three-dimensional kinematics and constitutive relations.

3.2.2.1. Three dimensional elasticity theories

Three-dimensional theories are required for the determination of three-dimensional stress states. Three-dimensional elasticity models have been applied to calculate the stress and to study free vibrations and buckling problems of laminated orthotropic rectangular plates [3.16, 3.17]. They give more accurate results than the equivalent single layer theories. However, the use of three-dimensional models for predicting the response characteristics of laminated anisotropic plates with complicated geometry is computationally expensive. The storage and computational costs, especially for geometrically non-linear problems or transient analysis preclude the use of such a theory.

3.2.2.2 Layer-wise theories

In contrast to the equivalent single layer theories, the layer-wise theories are developed by assuming that the displacement components are continuous through the laminate thickness but the derivatives of the displacements with respect to the thickness coordinate may be discontinuous at various points through the thickness. This allows the possibility of continuous transverse stresses at interfaces separating dissimilar materials. Consequently, the layer-wise displacement fields provide a much more kinematically correct representation of the moderate to severe cross sectional warping associated with the deformation of thick laminates.

Reddy proposed a layer-wise displacement plate theory [3.18] in which the three-dimensional displacement field is expanded as a linear combination of the thickness coordinate and undetermined functions of position within each layer.

Several other layer-wise theories have been proposed in the literature but many of these require numerous unknowns and become nontractable in the solution of practical engineering problems.

The layer-wise laminate theories prove to be the best alternative to three-dimensional theories. They allow the determination if needed of information at the ply level or between plies. They can therefore be used effectively to model local effects such as delamination. However, most of the time, they are computationally expensive and are used in combination with finite element analysis.

3.2.3 SUMMARY OF PUBLISHED WORK ON LAMINATED PLATE THEORIES

From this review on laminated composite plate theories, the following points should be emphasised:

- Equivalent single layer theories are adequate for global response prediction.
- HSDT and more specifically Reddy's refined HSDT, when static and free vibration analyses are carried out, provide more accurate results than FSDT and CLPT when compared to three-dimensional elastic solutions.
- As far as transient analyses are concerned, the lack of data obtained using HSDT needs to be overcome in order to be able to conclude on the improvement provided by this theory as compared to FSDT and CLPT.
- Three-dimensional theory and layer-wise theories are more computationally expensive than equivalent single layer theories.
- When the three-dimensional stress field at the ply level is required then a layer-wise theory should be used to model the laminated plate structures.

3.3 SANDWICH PLATE THEORIES

Research into the theoretical analysis of sandwich construction began following World War Two. The work was then devoted to the development of theories for conventional or classical sandwich structures in which the facings were thin, stiff and heavy as compared to the core. A detailed review of this work is given in books by Plantema [3.19] and Allen [3.20]. The pioneer worker Reissner [3.21] suggested a simple and useful model, known as the Classical Sandwich Theory (CST), to describe such plates. He took into account the transverse shear stiffness in the core and the in-plane or membrane stiffness of the faces. Up to the late eighties little further work had been conducted into the analysis of sandwich panels, principally due to the fact that the emphasis in theoretical research had shifted to laminates. In the recent years, efforts have been

made to improve performance of high-speed marine craft. This has led to a new beginning in research for composite sandwich panels. The main aim was then to improve the commonly used CST in order to be able to predict accurately the dynamic behaviour of such structures. Most of the theories, which have been developed in the last ten years, are based on the knowledge acquired from the CST and from the equivalent single layer theories for laminated plates. A critical review of the existing methods to model composite sandwich panels is presented in this part.

3.3.1 CLASSICAL SANDWICH PLATE THEORY

The CST used most often up to now to design sandwich plates is based on the following assumptions, which have been first described by Allen [3.20] and Plantema [3.19] and summarised later by Zenkert [3.22]:

- The faces are thin compared to the core, $h_f \ll h_c$;
- The Young's modulus of the core is much lower than the one of the faces, $E_c \ll E_f$.

These assumptions can be summarised by saying that in sandwich plates the faces carry bending moments as tensile and compressive stresses and the core carries transverse forces as shear stresses. The stress distribution due to these approximations is compared in figure 3.2 to the actual stress distribution, which occurs in the sandwich plate.

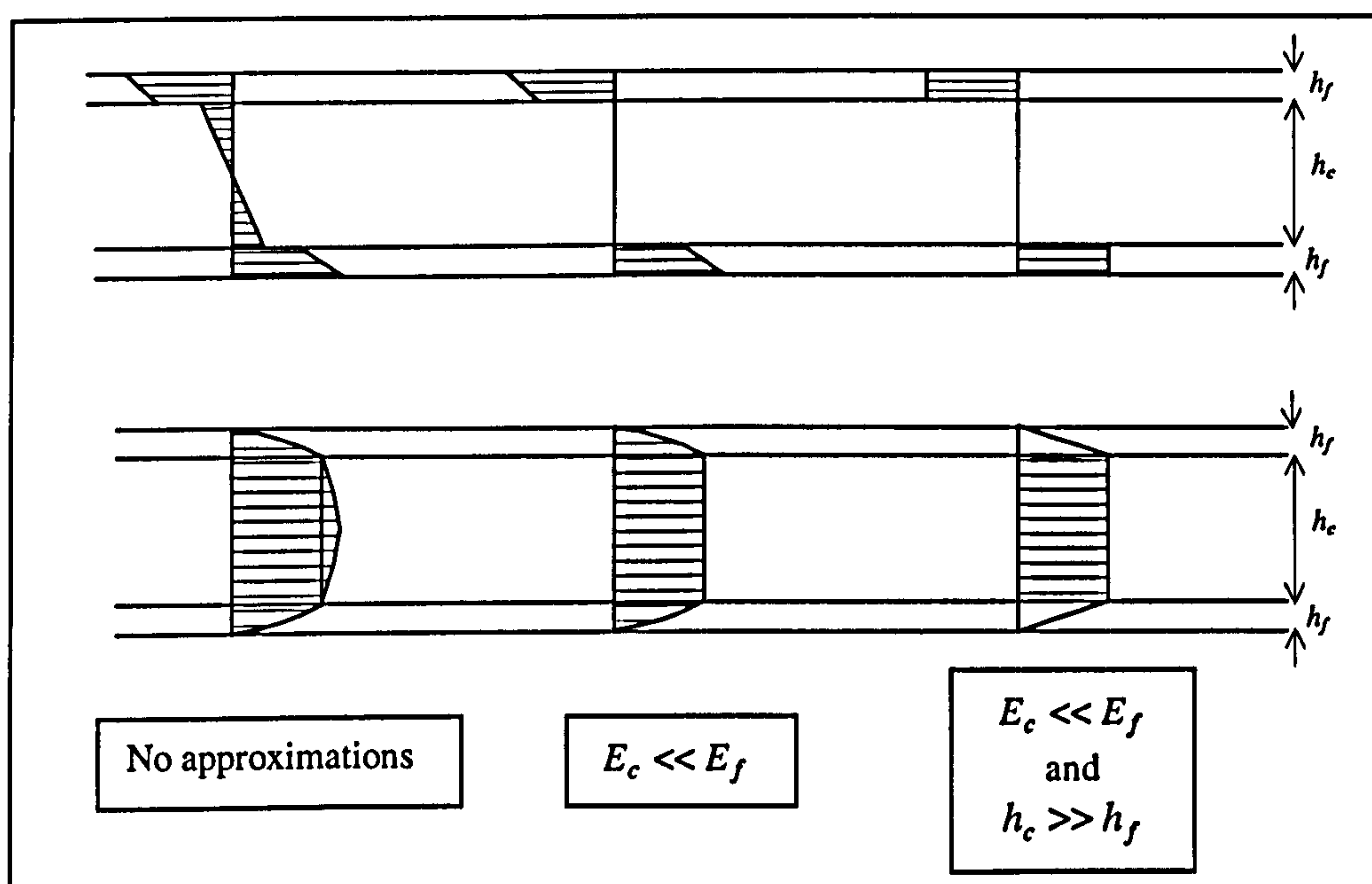


Figure 3.2: Direct and shear stresses for different level of approximations.

3.3.1.1 Analytical models based on CST

Based on the work of Allen [3.20] and Plantema [3.19], Zenkert [3.22] presented a method to study anisotropic sandwich plates in which it is assumed that the deformations consist of two different and independent parts:

- The deformations due to bending moments.
- The deformations due to transverse forces.

He demonstrated that in the particular case of sandwich plates with thin face approximation, it is possible to calculate these two deformations separately and then superimpose them. Figure 3.3 illustrates this assumption by showing the deformation due respectively to bending, to shear and to the combination of both. In this case the bending moments and the shearing forces are independent of each other. It is a correct assumption for panels with equal rigidities in both x - and y - directions or having the same neutral axis for both cross-sections. The main reason for introducing partial deflections is to uncouple the equilibrium equation and so to reduce the computational time.

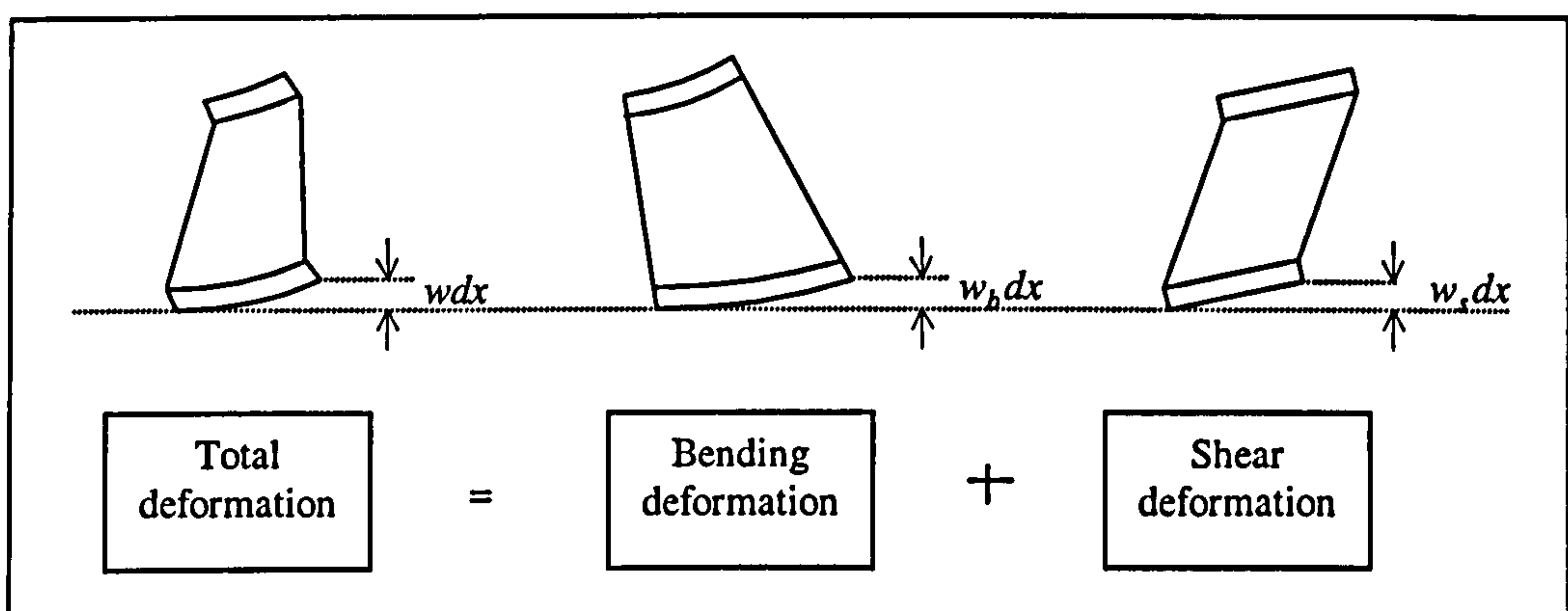


Figure 3.3: A deformed sandwich element.

Later on, based on Zenkert's linear sandwich plate theory, Riber [3.23] developed a non-linear analytical model for orthotropic sandwich plates. He applied his non-linear method to investigate the static (deflection and stresses) and dynamic (natural frequencies and transient response) behaviour of simply supported and clamped rectangular sandwich plates.

3.3.1.2 Finite element models based on CST

Chate and Mäkinen solved the problems of bending and vibration of sandwich plates by carrying out a finite element analysis. They developed a triangular finite element [3.24], which satisfies the assumptions of the CST that is to say only the membrane stresses are considered in the faces and only the transverse shear stresses in the core. This new finite element is therefore only applicable to sandwich construction with thin facings and with low core Young's modulus. In this method, the FSDT displacement field has been assumed to be valid through the whole plate. This finite element has been used to model simply supported and fully clamped rectangular sandwich plates with isotropic faces and isotropic or orthotropic cores and to determine their deflection, their bending moment and their natural frequencies. It has been found that the method gives accurate results as long as the plate thickness to face thickness ratio is greater than 25 and the ratio of Young's modulus of the faces to shear modulus of the core is greater than 200. This method could also be used to model composite sandwich plates with orthotropic faces as long as the previous ratios are respected.

3.3.1.3 Remarks on existing methods based on CST

In all the presented models, which are based on CST, it is assumed that the core resists the entire shear load. This is a pessimistic assumption, but one that is commonly used in the design of sandwich panels. However, in fact, the skins will take some of the shear load depending upon the relative thicknesses and the ratio of the core and skin shear modulus. Consequently, if a general accurate analytical model is required, with no restriction on the ratio of face to core stiffness or the ratio of face to core thickness, then the shear capacity of the skins should not be ignored. In addition to this drawback it should be noticed that all these methods do not allow the consideration of laminated faces. Indeed, they require the input of the properties of each face considered as one entity and not as made up of different laminae with different characteristics. Since most composite sandwich panels have laminated faces, it is necessary to undertake a preliminary calculation before being able to predict their static or dynamic behaviour with these methods. So, with the increasing use of FRP sandwich structures in high performance craft, the necessity of being able to predict the dynamic behaviour of such structures easily and accurately at the design stage has highlighted the need to develop new theories for FRP composite sandwich panels.

3.3.2 COMPOSITE SANDWICH PANELS MODELLED USING LAMINATED PLATE THEORIES

The recent research work has focused on improving the CST by including more “physical mechanics”. This was achieved by combining the knowledge acquired from both the CST and from the laminated plate theories. Analytical methods [3.25] and finite elements [3.26, 3.27] have been developed based on both FSDT and CST.

From the work carried out on composite laminated plates, it has been shown, that the shear deformation can have a significant influence on the dynamic response of a studied panel and more especially on its natural frequencies. This observation led to the conclusion that it is important to be able to account for the shear deformation in the faces of FRP sandwich plates. Therefore, in order to carry out the free vibration analysis of sandwich panels, Cupial and Niziol [3.25] proposed an analytical method and Lee and Fan [3.27] developed a finite element model, which make allowances for the shear effects in both the faces and the core. They used the FSDT to model the displacement field in the faces and they applied the continuity of displacement at the interfaces between the faces and the core to find the displacement in the core. By using the FSDT to model the faces, they offer the possibility of considering the laminated faces to be made of dissimilar laminae. However, in the same time, the use of FSDT leads to the necessity of including in the analytical model shear correction coefficients. Finally, these methods do offer the possibility to consider the constitutive material damping properties. Nevertheless, they only account for material damping due to shear in the core. These methods have shown some improvements as compared to CST but still some drawbacks need to be overcome.

Chate et al. [3.26], in the same time, developed a “broken line” model to study the dynamic behaviour of sandwich plates. In each individual layer the FSDT displacement field is used and the displacement continuity between the core and the faces is considered. This method has the advantage over the previous ones of taking into account both the shear and bending effects in both the faces and the core. Besides, it does not require shear correction factors. However this method does not account for damping material properties and the developed finite element model is rather complicated and takes much computational time and resources. This method is too computationally expensive to be used analytically.

So far, the reviewed methods, even though they are based on the FSDT displacement field, do not use this theory explicitly to model sandwich plates. Reddy and Chao [3.28] are the first and

to the knowledge of the author the only ones who have been applying the FSDT, as presented in section 3.2.2.2 for laminated plates, directly to the case of sandwich plates by assuming a thick middle layer. Using this technique, they have carried out static analyses of sandwich panels made with isotropic materials. However, they did not apply the method to study the behaviour of FRP composite sandwich plates. This is probably related to the difficulty in determining the value of the shear correction factors with accuracy. Indeed, their values are likely to be different from one configuration to another. Nevertheless, it should be noticed that this method offers the benefit of providing a similar theory for laminated and sandwich composite plates, which implies that only one computer program is required.

All the new methods reviewed in this part have been developed based on the FSDT displacement field. Compared to the CST they provide a great improvement since both the shear and bending effects are considered in the faces. However they show either the necessity to calculate shear correction coefficients or in the specific case of the broken line model of Chate et al. [3.26] they become too computationally expensive to be considered as feasible. Besides, since they are based on FSDT displacement field, the shear distribution is taken as constant through the layer. This assumption has been demonstrated as inaccurate for relatively thick laminated plates especially if their dynamic behaviour is investigated (see section 3.2.2.3). In this case, the use of higher-order theory is recommended. As composite sandwich plates can be considered as relatively thick panels it appears important to use a model based on a HSDT displacement field to predict their behaviour. Reddy's refined HSDT is the most general HSDT. This theory, presented in section 3.2.1.3 has not been used yet to model FRP composite sandwich plates and to investigate their dynamic behaviour. It seems that the feasibility of using this method, previously developed for laminated plates, to model composite sandwich plates should be investigated. If accurate, this new method should provide a better distribution of the shear stress through the thickness and avoid the difficult task of determining the shear correction factors for sandwich plates. The method would then be valid for both sandwich plates and laminated plates, which would allow the designer to predict the behaviour of various composite plates using the same analytical method.

3.4 SUMMARY

In this chapter a critical review of the theories available to model composite laminated plates and composite sandwich plates has been presented. The following main points should be emphasised:

- All the published models have been used for either FRP laminated panels or FRP sandwich panels. No analytical models have yet been proven to be valid for both types of composite panel.
- The FSDT, which is most commonly used to model FRP laminated panels, does not always provide accurate results when dynamic analyses are carried out.
- When global response prediction of laminated plates is required, Reddy's refined HSDT appears to be the more versatile and accurate method. However, when transient analysis is carried out, the accuracy of the predicted responses has not yet been clearly identified.
- As far as sandwich plates are concerned, models are still most often based on the CST.
- Attempts to develop new models for FRP sandwich panels based mainly on FSDT principles have been carried out. However, none have yet proven themselves to be both accurate and simple to use.
- To the knowledge of the author, no analytical models based on HSDT have yet been developed for modelling relatively thick FRP sandwich panels.
- The dynamic analysis of FRP sandwich panels has so far been mainly limited to free vibration study.

The literary review work, presented in this chapter, clearly outlines the fact that there is a lack in current knowledge with respect to modelling of composite plates. New works need therefore to be carried out in order to make up for the current weaknesses of the identified models. The following points should be kept in mind and should greatly influence the development of a more efficient analytical model.

- An ideal model should offer the possibility to predict accurately the dynamic responses of both FRP laminated and FRP sandwich panels when both free vibration and transient analyses are carried out.
- Reddy's refined HSDT has been presented as an attractive method. Even so it has mainly been used only for static and free vibration analysis of homogeneous and laminated panels. Further work should therefore be realised in order to be able to conclude on the accuracy of this method when transient analysis is carried out. Besides, the feasibility of developing a new analytical model for FRP sandwich panels based on this theory should be investigated.
- Few models account for material damping properties. Nevertheless these properties should not be neglected when dynamic analyses are carried out.

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4. LINEAR VISCOELASTIC DAMPING IN COMPOSITE PLATES

4.1. INTRODUCTION

FRP laminated and FRP sandwich structures when used in high-speed vessels are likely to be subjected to dynamic loading during their service life. So, if such structures are not properly designed, they may fail due to dynamic instability or fatigue. Damping, of which the physical characteristic is the result of energy dissipation, plays an important role in vibration control, noise monitoring and fatigue endurance by reducing the amplitude of resonant vibration response. So, the damping capacities of FRP composite panels should be considered from the design stage when predicting their dynamic responses. Damping can originate in materials such as viscoelastic materials that dissipate energy during each cycle of vibration or it can also be developed through the interfacial friction between two materials. Damping can also be introduced through design by using a constrained viscoelastic damping layer on the surface of the structure or by using damping joints in the structure. Composite plates contain one or more polymeric materials including for example a resin system and PVC foam which exhibit viscoelastic damping. This type of damping arises from relaxation and recovery of the polymer network, itself composed of long molecular chains, after it has been deformed. Viscoelastic energy loss is the principal source of damping in FRP single skin and FRP sandwich composite structures and therefore this form of damping must be considered when the dynamic behaviour of such panels is investigated. In this chapter, a critical review of the analytical models available to describe linear viscoelastic material behaviour is introduced. The efficiency of these models to represent material damping, which is one of the manifestations of viscoelastic behaviour, will be outlined and the feasibility of incorporating these models into the laminated and sandwich plates theories, presented in chapter 3, will be investigated. It should be emphasised that the main objective is to develop an analytical model (see chapter 8), which will allow an accurate prediction of the dynamic behaviour of single skin and sandwich composite panels made of viscoelastic materials.

4.2 VISCOELASTICITY

4.2.1 DEFINITION

The concept of viscoelasticity comes from the fact that most materials do not exhibit purely elastic (ideal solids) or purely viscous (ideal liquids) behaviour but a combination of both.

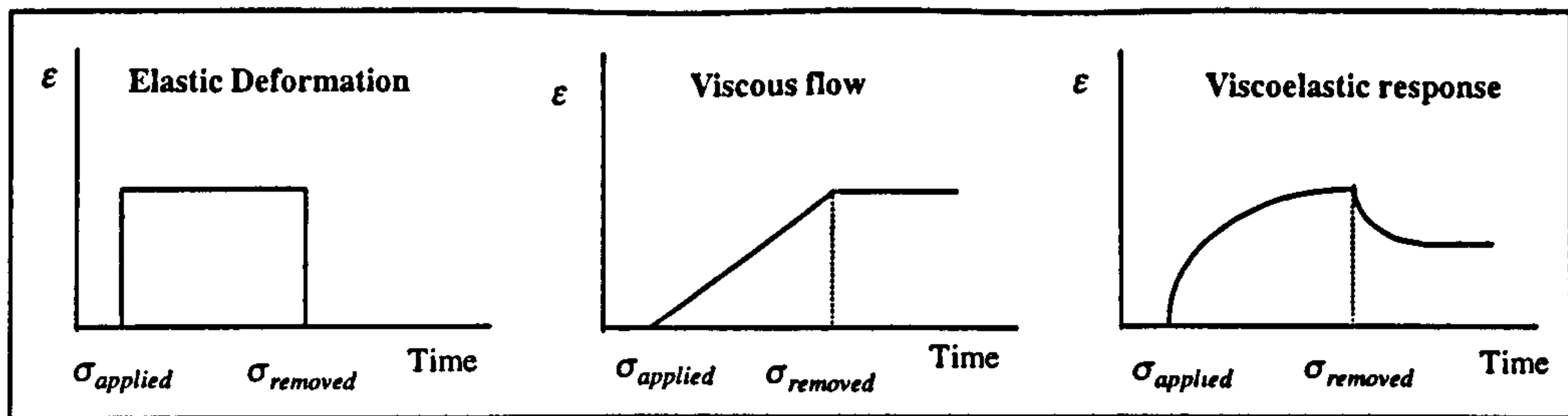


Figure 4.1: Response of elastic, viscous and viscoelastic materials to loading followed by unloading.

A purely elastic material, as can be seen in figure 4.1, changes its shape when deformed and regains its original shape when the deformation is removed. All the energy applied to the material is stored by itself during deformation. An ideal fluid, in contrast, is a material that has no elasticity and so stores no deformational energy. When it is deformed it changes its shape and when the deformation is removed it does not regain its original shape but remains in its deformed shape. All the energy is dissipated during deformation. Viscoelastic materials, however, possess a capacity to both store and dissipate energy under load. This dual viscous and elastic character of viscoelastic materials leads to a more complicated behaviour, which cannot be described by either an elasticity or viscosity theory but a combination of both. Due to this complicated dual behaviour the mechanical properties of viscoelastic material exhibit both a time-dependency and memory effects. Indeed, whereas the response of an elastic material depends only on the total stress level at every instant of time, the viscoelastic material response is not only determined by the current state of stress, but also determined by all past states of stress.

4.2.2 CHARACTERISTICS OF VISCOELASTIC BEHAVIOUR

In order to be able to choose an accurate theory to model viscoelastic material behaviour, it is first of all essential to understand well their different characteristics. There are four important physical manifestations of viscoelastic behaviour in structural materials. They are illustrated in figure 4.2 and are briefly described below.

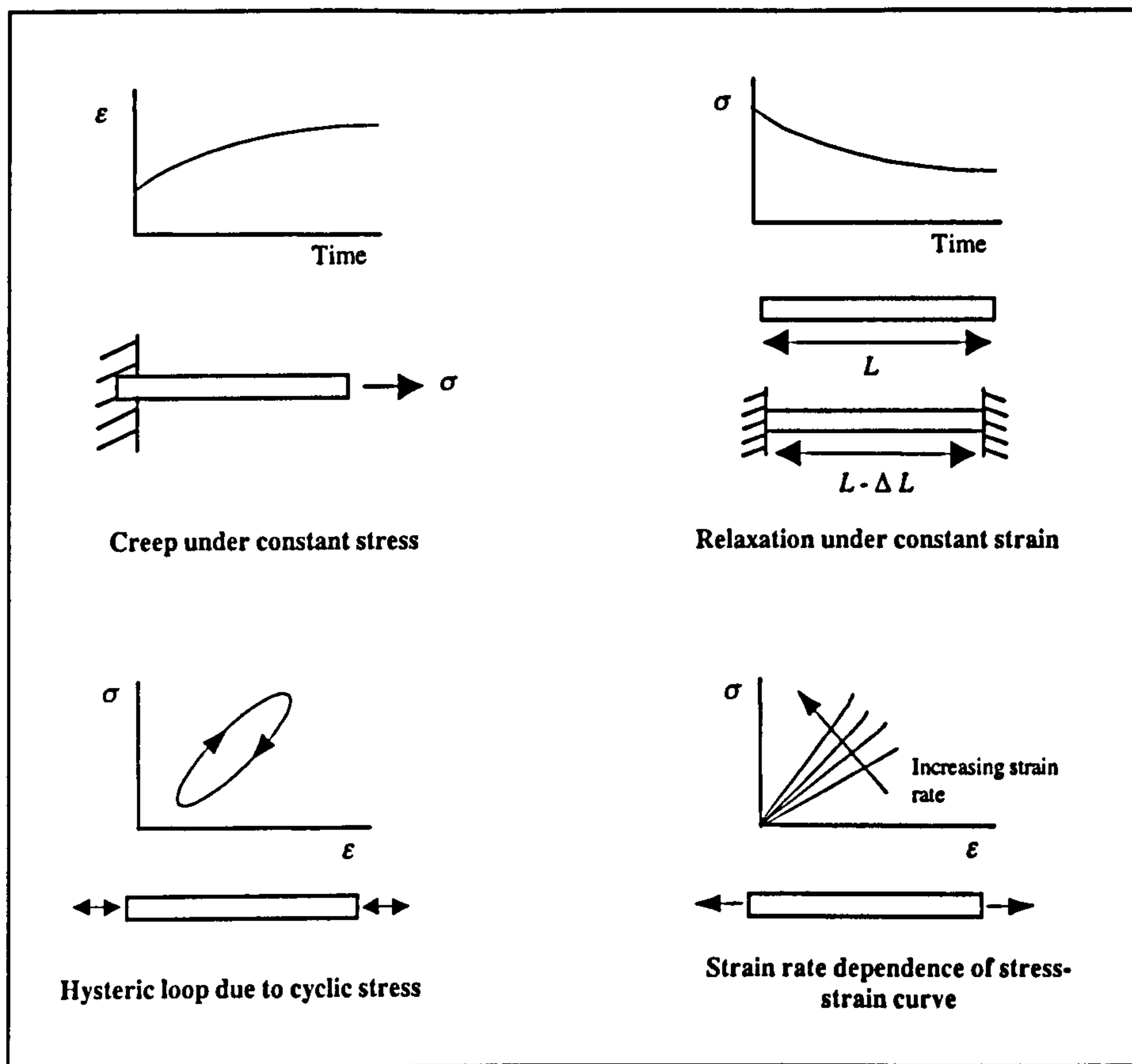


Figure 4.2: Physical manifestations of viscoelastic behaviour in structural materials as demonstrated by various type of loading applied to a viscoelastic rod [4.1].

These characteristics are as follows:

- If the viscoelastic structure is subjected to a constant stress, the resulting strain will exhibit creep. The time-dependent creep strains are superimposed on the initial elastic strains.
- If it is subjected to a constant strain or displacement, the resulting stress will exhibit relaxation. That is, the stress relaxes from the initial elastic stress.
- If it is subjected to oscillatory loading, the resulting stress-strain curve will describe a “hysteresis loop”. The area enclosed by the hysteresis loop is a measure of the damping, or dissipation of energy, in the material.
- If it is loaded at various strain rates, the stress-strain curves will exhibit a strain-rate dependence. That is, the stress corresponding to a given strain depends on the rate of straining.

It should be noticed that an ideal elastic material exhibits none of the above characteristics. Therefore, specific strain-stress constitutive relations should be used to model the complex behaviour of structural viscoelastic materials.

4.3 LINEAR VISCOELASTIC MATERIALS STRESS-STRAIN CONSTITUTIVE RELATIONS

In this part, the various stress-strain constitutive relations, available in the literature, for linear viscoelastic materials are reviewed. A viscoelastic material is said to be linear if it exhibits a linearity between stress and strain. That is to say, the ratio of stress to strain is a function of time (or frequency) alone, and not of stress magnitude.

4.3.1 DIFFERENTIAL FORM OF STRESS-STRAIN CONSTITUTIVE RELATIONS

Linear viscoelastic behaviour can be described by an ordinary differential equation as explained by Johnson in his paper [4.2]. In this case the constitutive equation for linear viscoelasticity is given in a differential form as shown in equation 4.1.

$$\left([a_p] \frac{\partial^p}{\partial t^p} + [a_{p-1}] \frac{\partial^{p-1}}{\partial t^{p-1}} + \dots + [a_0] \right) \{\sigma\} = \left([b_r] \frac{\partial^r}{\partial t^r} + [b_{r-1}] \frac{\partial^{r-1}}{\partial t^{r-1}} + \dots + [b_0] \right) \{\epsilon\} \quad (4.1)$$

Equation (4.1) is based on a mechanical model consisting of springs, elastic elements obeying Hooke's law (applied stress proportional to strain) and dashpots, viscous time dependent components obeying Newton's law (applied stress proportional to the strain rate). The coefficients of the matrices $[a_p]$ and $[b_r]$ are constants and must be constructed so that equation 4.1 represents a material that obeys the law of thermodynamics. In the late 1800s, Maxwell, Voigt, and Kelvin developed models of viscoelastic materials that can be cast in the form of equation 4.1 and for which it is easy to ensure that the thermodynamic restrictions are satisfied. These models are reviewed in figure 4.3 for viscoelastic materials. They are often used since they provide a relatively simple qualitative picture of the manner in which viscoelastic materials behave.

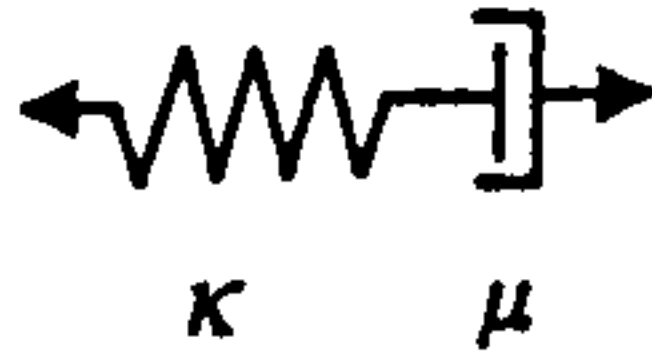
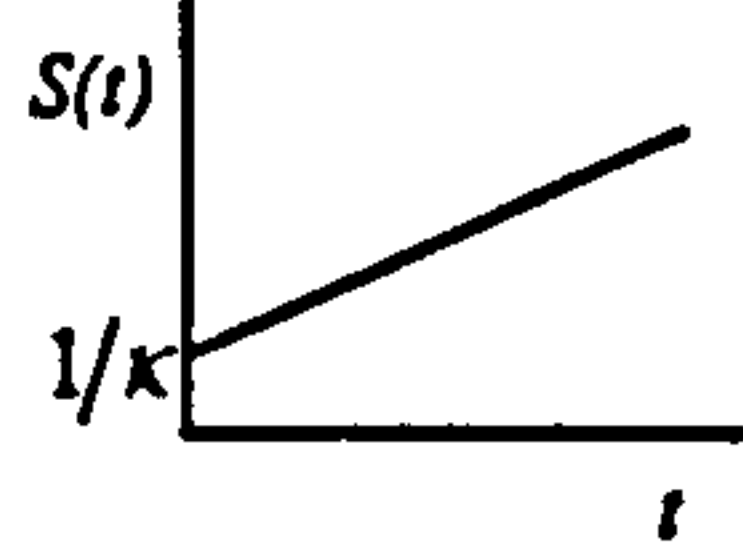
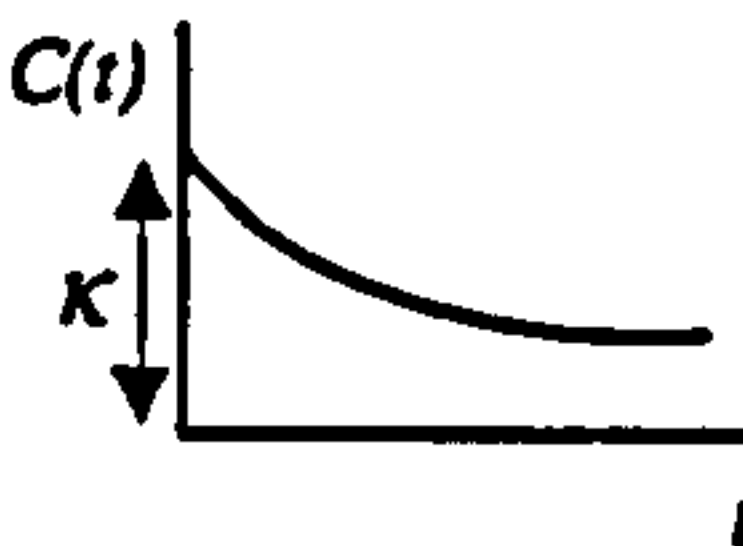
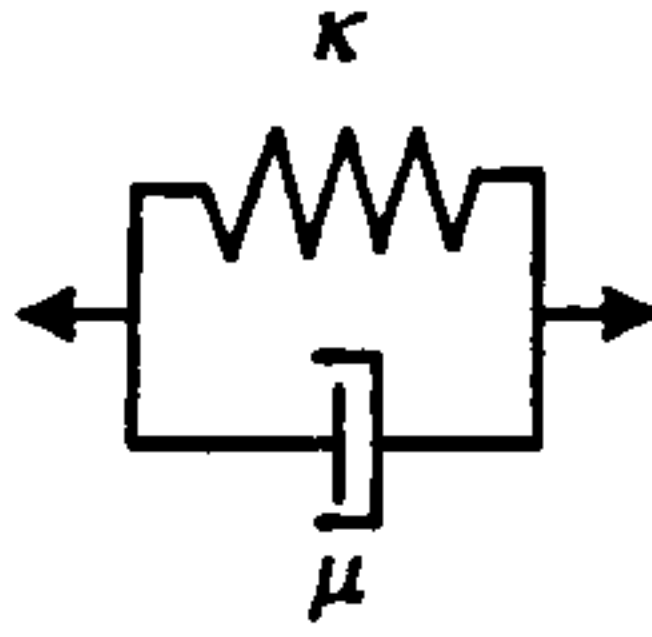
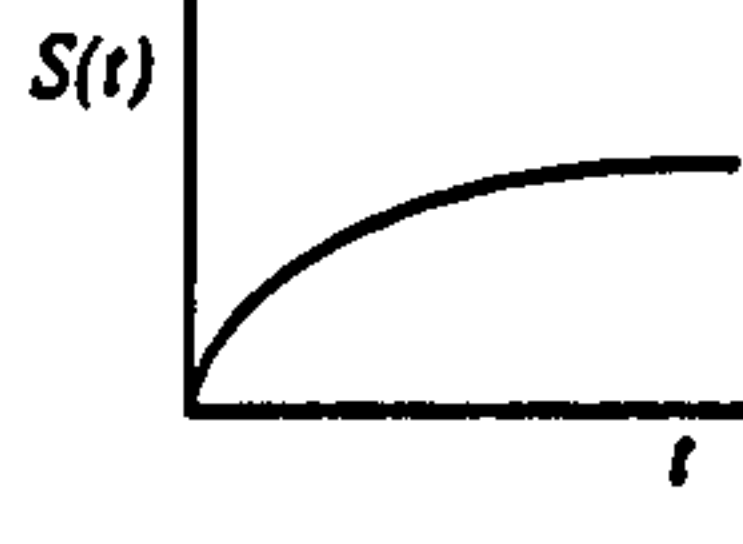
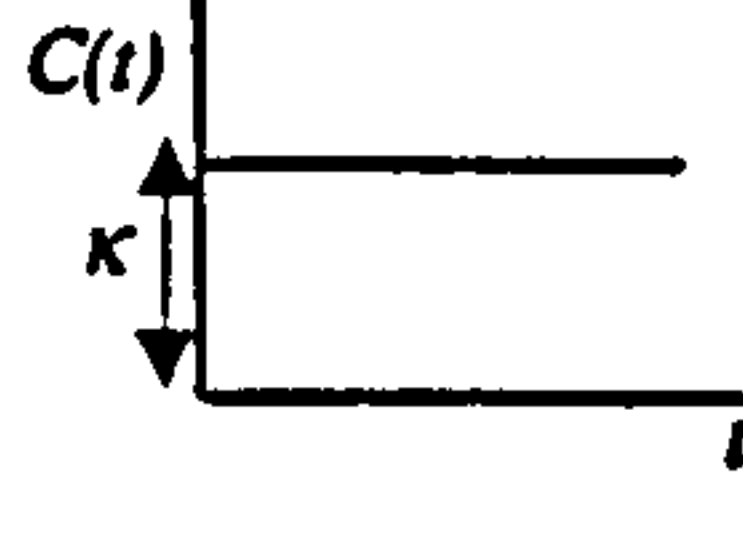
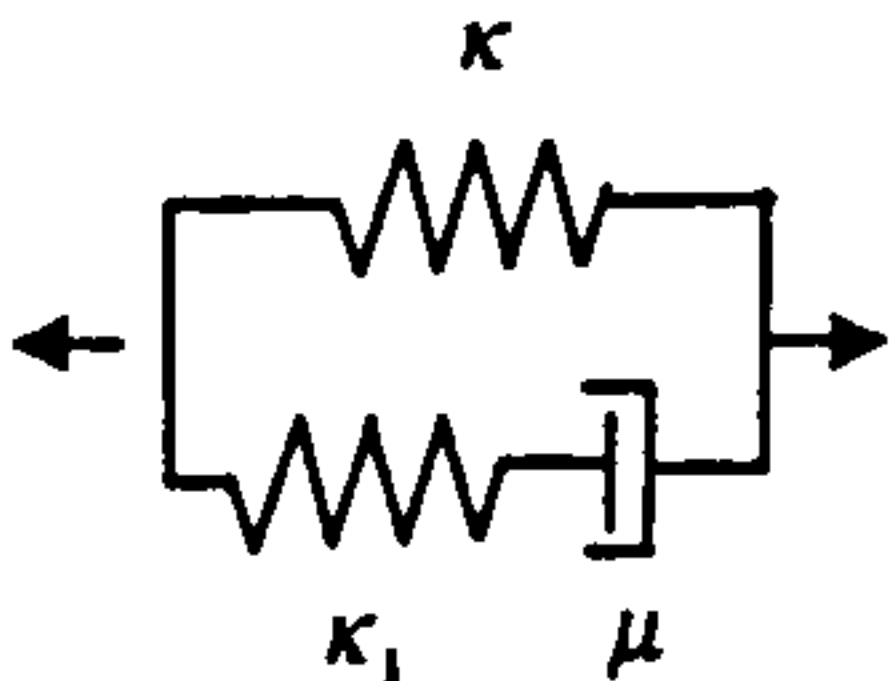
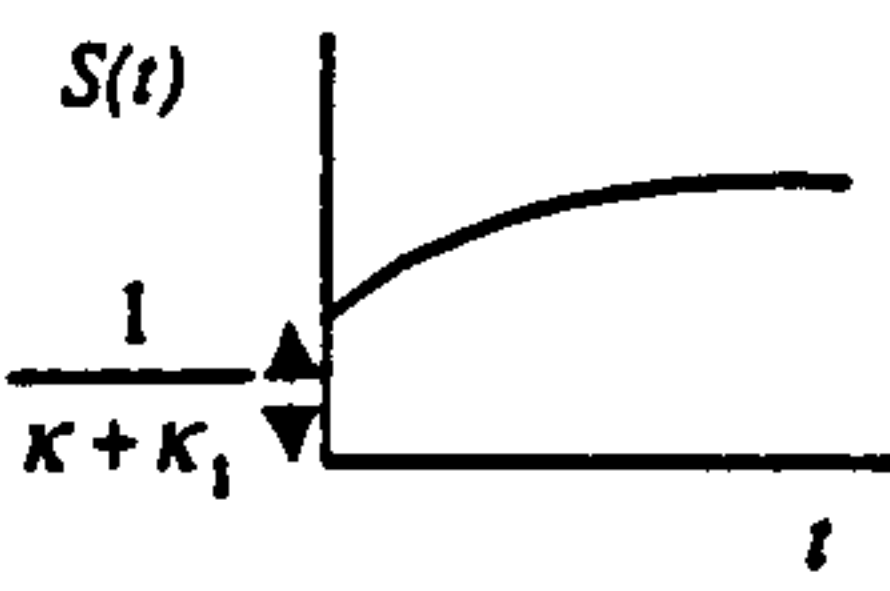
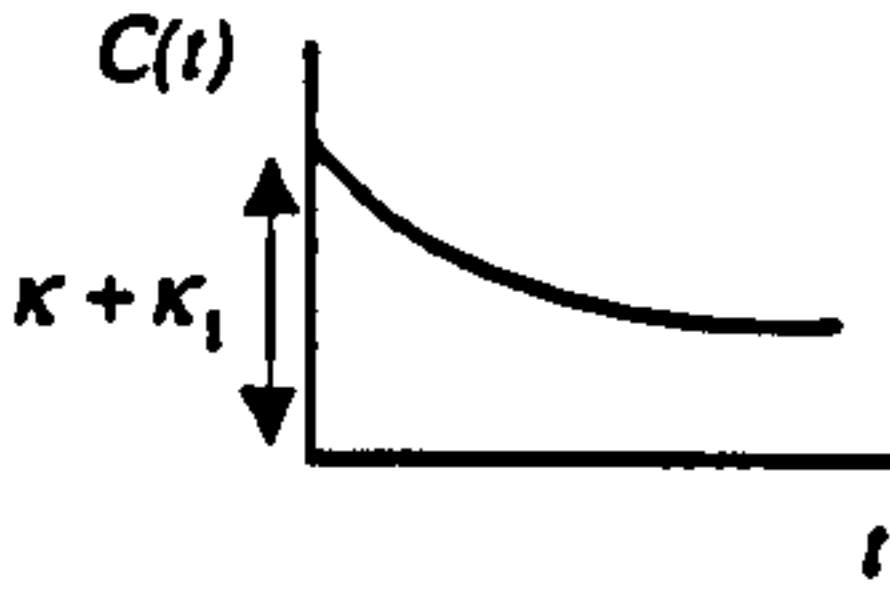
MODEL	Spring-dashpot arrangement	Creep compliance versus time	Relaxation modulus versus time	Stress-strain relations
Maxwell model				$\frac{d\varepsilon}{dt} = \frac{1}{\kappa} \frac{d\sigma}{dt} + \frac{\sigma}{\mu}$
Kelvin-Voigt model				$\sigma = \kappa\varepsilon + \mu \frac{d\varepsilon}{dt}$
Standard linear solid model				$\sigma + \frac{\mu}{\kappa_1} \frac{d\sigma}{dt} = \kappa\varepsilon + \frac{\mu}{\kappa_1} (\kappa + \kappa_1) \frac{d\varepsilon}{dt}$

Figure 4.3: Models for viscoelastic isotropic materials.

4.3.1.1 Maxwell model

This model was proposed by Maxwell to explain the time-dependent mechanical behaviour of viscous materials. It consists of a spring and dashpot in series as shown in figure 4.3. Its response to an applied load is to produce an instantaneous elastic deflection followed by a time dependent constant slow rate of increasing deflection, which is totally irrecoverable. The model can be used to predict the response of a polymer during stress relaxation as explained by Young and Lovell in their book [4.3]. However, it cannot predict accurately the creep behaviour of polymeric material.

4.3.1.2 Kelvin or Voigt model

The Kelvin or Voigt model consists of a spring and a dashpot in parallel as shown in figure 4.3. In the Kelvin or Voigt model viscous stresses are proportional to the strain rate. This model is particularly useful in describing the behaviour during creep where the stress is held constant. On the other hand, it fails to predict the stress relaxation behaviour of a polymer. The deficiency of the Kelvin-Voigt model under stress relaxation conditions can be mitigated by the addition of a spring in series with the basic element to absorb the effect of instantaneous deformation as explained in more detailed by McCullough [4.4].

4.3.1.3 Standard linear solid model

It has been mentioned that the Maxwell model describes the stress relaxation of a polymer to a first approximation whereas the Voigt or Kelvin model similarly describes creep. A logical step forward is therefore to find some combination of these two basic models, which can account for both phenomena. A simple model, which does this, is known as the standard linear solid model [4.3]. In this case a Maxwell element and a spring are in parallel. The presence of the second spring stops the tendency of the Maxwell element undergoing simple viscous flow during creep loading, but will still allow the stress relaxation to occur.

The models presented in this chapter are based on the differential form of the stress-strain constitutive relation. They give a mathematical representation of the mechanical behaviour of viscoelastic materials. These mechanical models, which have been established from studies of similar systems, are generally too simple to be able to capture all the features of viscoelastic behaviour. As mentioned by Christensen [4.5], these mechanical models provide a qualitative basis for the interpretation of viscoelastic behaviour, but they are usually restrictive and are not fundamental to the development of a consistent theory for viscoelastic structural materials. To solve this problem, there have been many attempts to develop more complex models based on equation 4.1, which can give a better representation of the viscoelastic behaviour of polymers. Bland [4.6] has reviewed some of these. However, these models, which are characterised by their large number of time derivatives (e.g.: large value of p and r in equation 4.1), are time consuming to manipulate and, when input into the equations of motion produce high-order differential equations. Hence, as the number of elements increases the mathematical models become too cumbersome to be easily solved when introduced in the equations of motion. Alternative models should therefore be considered.

4.3.2 INTEGRAL FORM OF STRESS-STRAIN CONSTITUTIVE RELATIONS

The stress-strain relationships for a linear viscoelastic material can also be determined, as explained by Christensen [4.5], using the Boltzmann Superposition Principle. It assumes that the response of the material at any time t due to an input at time $t = \tau$ is a function of the input and the elapsed time $(t - \tau)$ only. This implies that the strain response is linearly proportional to the input stress, but the proportional factor is a function of the elapsed time since the stress input.

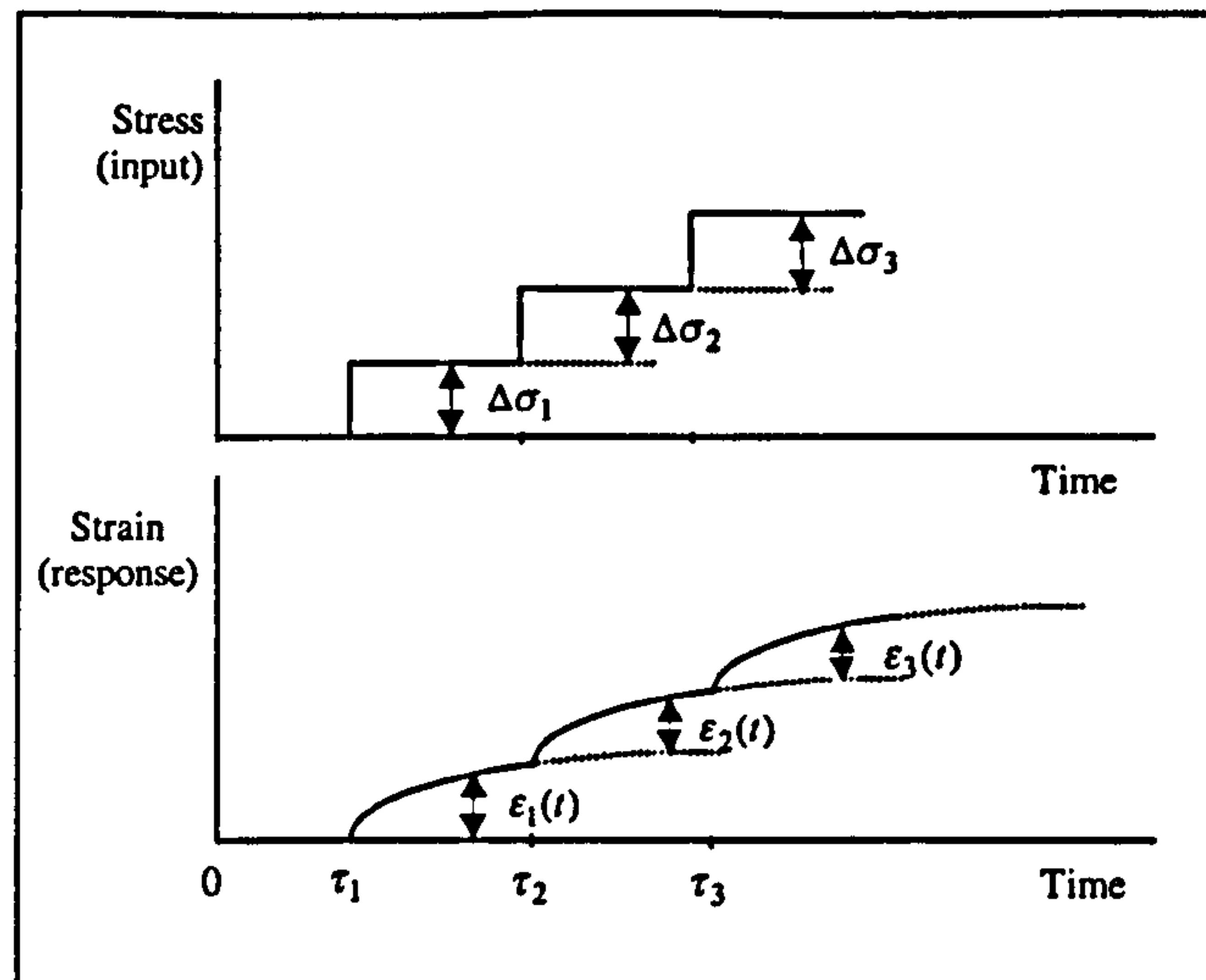


Figure 4.4: The response of a linear viscoelastic material to a series of loading steps [4.3].

Assuming, as shown in figure 4.4, that the effects of sequential changes in stress are additive the strain response, for input stresses having arbitrary time histories, is given by [4.5]:

$$\epsilon_i(t) = S_{ij}(t)\sigma_j(0) + \int_0^t S_{ij}(t-\tau) \frac{d\sigma_j(\tau)}{d\tau} d\tau \quad (4.2)$$

Alternatively the stress resulting from strain inputs may be given by:

$$\sigma_i(t) = C_{ij}(t)\epsilon_j(0) + \int_0^t C_{ij}(t-\tau) \frac{d\epsilon_j(\tau)}{d\tau} d\tau \quad (4.3)$$

In relations 4.2 and 4.3 $S_{ij}(t)$ and $C_{ij}(t)$ are the creep and relaxation functions of the anisotropic viscoelastic material. From equations 4.2 and 4.3 it can be noticed that the current stress is determined by the superposition of the responses to the complete spectrum of increments of strain and the current strain is determined by the current value and past history of stress.

Frequently, when the dynamic analysis of viscoelastic structures is carried out, the Boltzmann representation involved in the stress-strain relation is used to model the viscoelastic material behaviour. The use of the Boltzmann theory leads to an integro-differential equation of motion rather than an ordinary differential equation as in the elastic case. This implies further

complication in the analysis. Cederbaum and Aboudi [4.7] have been using the Boltzmann superposition principle in order to model the viscoelastic behaviour of fibre reinforced plastic laminated plates. To avoid the difficulty of having to solve an integro-differential equation of motion they considered the Boltzmann representation of the viscoelastic phases in the frequency domain by application of the Fourier transform. The use of the Fourier transform has enabled them to express the governing equations of the studied plate in the frequency domain using laminated plate theories. With this method the imposed dynamic load has to be expressed in its Fourier transform form. Finally, the predicted response is obtained by inversion of the response function into the time domain using the Fast Fourier transform algorithm. This method requires the knowledge of the elastic material properties of the fibres and the time dependent viscoelastic material properties of the matrix.

4.3.3 STEADY STATE STRESS-STRAIN CONSTITUTIVE RELATIONS

The general Boltzmann's representation, when used to model viscoelastic material behaviour, leads to complex time dependent equations of motion, which can be difficult to solve. In the case where the studied structure is subjected to steady state oscillatory conditions, the viscoelastic stress-strain relation can be brought to the same form as the elastic stress-strain equation using the so-called elastic viscoelastic correspondence principle, which is summarised in this section.

4.3.3.1 Elastic viscoelastic correspondence principle

When the studied viscoelastic structure is subjected to steady state oscillatory conditions, the strain can be specified as being a harmonic function of time according to:

$$\varepsilon_j(t) = \varepsilon_j^o \exp(i\omega t) \quad (4.4)$$

In this particular situation, equation 4.3 can be derived as explained by Christensen [4.5] and the stress can be re-written in the steady state form shown by relation 4.5.

$$\begin{aligned} \sigma_i(t) &= C_{ij}^*(i\omega) \varepsilon_j^o \exp(i\omega t) \\ \Leftrightarrow \sigma_i(t) &= C_{ij}^*(i\omega) \varepsilon_j(t) \end{aligned} \quad (4.5)$$

The complex moduli $C_{ij}^*(i\omega)$ can be represented using the frequency dependent storage moduli $C_{ij}'(\omega)$ and the frequency dependent loss moduli $C_{ij}''(\omega)$ as shown by equation 4.6.

$$C_{ij}^*(i\omega) = C_{ij}'(\omega) + iC_{ij}''(\omega) \quad (4.6)$$

So, when a structure is subjected to steady state oscillatory condition, its stress-strain constitutive relations can be expressed using equation 4.5. It should be noticed that this equation is similar to the stress-strain relations obtained for the elastic case. With this method the real moduli of classical elasticity in the stiffness matrix are simply replaced by their complex viscoelastic counterparts. This method is called the elastic viscoelastic correspondence principle.

4.3.3.2 Existing applications

The complex modulus approach is commonly used to model the material viscoelastic behaviour of constrained viscoelastic layers ([4.8], [4.9]). The use of constrained viscoelastic damping treatment is a common design solution for aerospace structures to improve their performance in dynamic load environments. The design of such treatment requires accurate and efficient methods in order to be able to analyse and optimise the overall damping ability of the structure. For this type of analysis the elastic viscoelastic principle is seen as the most appropriate technique. For example, it has been introduced in existing modelling techniques by Alam *et al.* [4.8] and by Rao *et al.* [4.9] to study the dynamic behaviour of structures having constrained viscoelastic treatment. Cupial and Niziol [4.10] used as well the complex modulus representation to model the constitutive material viscoelastic behaviour with the aim to predict the modal loss factor of sandwich laminated plates. Nevertheless, they only considered the viscoelastic damping due to shear deformation in the core and they did not account for the frequency dependency of the dynamic material properties. Later on, Barkanov, Rikards and Chate [4.11] introduced the frequency dependent complex modulus representation in their finite element model developed to describe the dynamic behaviour of sandwich panels made of a viscoelastic middle layer. As previous authors, they only considered the viscoelastic damping properties due to shear in the core.

By using the elastic viscoelastic principle, it has been demonstrated that it is possible to represent the equations of motion in the frequency domain, and so to directly solve the free vibration equations and harmonic vibration equations. However, the prediction of the behaviour of the structure in the time domain using this representation is not straightforward.

Finally it should be emphasised that published works have confirmed the possibility of using the complex modulus approach to predict modal loss factors and natural frequencies of panels made of frequency or non-frequency dependent linear viscoelastic materials. The success of these methods outlines the necessity to investigate the possibility using this complex modulus approach to carry out the steady state analysis of FRP laminated and FRP sandwich panels, made of anisotropic or orthotropic (e.g., FRP material) and isotropic (e.g., undamaged PVC foam) linear viscoelastic materials.

4.3.3.3 Application to orthotropic laminae

Fibre reinforced plastics laminated panels are composed of a stack of arbitrarily orientated orthotropic laminae. If the constitutive materials (e.g.: matrix) exhibit viscoelastic behaviour then using the elastic-viscoelastic principle the stress-strain relation in each lamina can be expressed using equation 4.7. This relation is valid for linear viscoelastic orthotropic materials.

$$\begin{Bmatrix} \sigma_1(t) \\ \sigma_2(t) \\ \sigma_3(t) \\ \sigma_4(t) \\ \sigma_5(t) \\ \sigma_6(t) \end{Bmatrix} = \begin{bmatrix} C_{11}^*(i\omega) & C_{12}^*(i\omega) & C_{13}^*(i\omega) & 0 & 0 & 0 \\ C_{12}^*(i\omega) & C_{22}^*(i\omega) & C_{23}^*(i\omega) & 0 & 0 & 0 \\ C_{13}^*(i\omega) & C_{23}^*(i\omega) & C_{33}^*(i\omega) & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44}^*(i\omega) & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55}^*(i\omega) & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66}^*(i\omega) \end{bmatrix} \begin{Bmatrix} \varepsilon_1(t) \\ \varepsilon_2(t) \\ \varepsilon_3(t) \\ \varepsilon_4(t) \\ \varepsilon_5(t) \\ \varepsilon_6(t) \end{Bmatrix} \quad (4.7)$$

As for the elastic coefficients, the viscoelastic coefficients are related to the engineering properties as follows:

$$\begin{aligned}
C_{11}^*(i\omega) &= E_1^*(i\omega) * \frac{(1 - \nu_{23}\nu_{32})}{\Delta(i\omega)}; & C_{22}^*(i\omega) &= E_2^*(i\omega) * \frac{(1 - \nu_{13}\nu_{31})}{\Delta(i\omega)}; \\
C_{33}^*(i\omega) &= E_3^*(i\omega) * \frac{(1 - \nu_{12}\nu_{21})}{\Delta(i\omega)}; & C_{12}^*(i\omega) &= E_1^*(i\omega) * \frac{\nu_{21} + \nu_{31}\nu_{23}}{\Delta(i\omega)}; \\
C_{23}^*(i\omega) &= E_2^*(i\omega) * \frac{\nu_{32} + \nu_{12}\nu_{31}}{\Delta(i\omega)}; & C_{13}^*(i\omega) &= E_3^*(i\omega) * \frac{\nu_{13} + \nu_{12}\nu_{23}}{\Delta(i\omega)} \\
C_{44}^*(i\omega) &= G_{23}^*(i\omega); & C_{55}^*(i\omega) &= G_{31}^*(i\omega); & C_{66}^*(i\omega) &= G_{12}^*(i\omega).
\end{aligned} \tag{4.8}$$

$$\text{where, } \Delta(i\omega) = \begin{vmatrix} 1 & -\nu_{12} & -\nu_{31} \\ -\nu_{12} & 1 & -\nu_{32} \\ -\nu_{13} & -\nu_{23} & 1 \end{vmatrix}.$$

The material presented in this section 4.3.3 leads to the conclusion that the use of the elastic viscoelastic correspondence principle is ideal to carry out steady state analysis. As soon as it is possible to determine the complex moduli of the viscoelastic material properties, the analysis is then as simple as in the elastic case. On the other hand, when the transient analysis needs to be undertaken, the use of the correspondence principle does not simplify the analysis any further. Indeed, if the viscoelastic material behaviour is modelled using the correspondence principle, then the equations of motion written in the frequency domain cannot be solved directly and therefore they need to be transformed into the time domain using, for example, the fast Fourier transform technique. Then, the accuracy of the method is dependent on various factors such as the number of samples, the sampling interval and the shape of the frequency response.

4.3.4 SUMMARY

From this review on stress-strain constitutive relations of linear viscoelastic materials, the following points should be emphasised:

- The differential forms of stress-strain constitutive relations are either too restrictive or mathematically too complex and therefore should not be used in the derivation of the equations of motion of viscoelastic structures. However they can be used as a tool to explain and interpret viscoelastic behaviour.
- The Boltzmann's representation on the other hand can be used to include the viscoelastic behaviour in the dynamic analysis of a given structure. It leads to integro-differential time

dependent equations of motion, which can be difficult to solve. Since with this representation the equations of motion are written in the time domain, the knowledge of the viscoelastic material functions in the time domain is required.

- By using the elastic-viscoelastic principle the viscoelastic equations of motion in the frequency domain can be easily derived from the elastic ones and the steady state response easily predicted. However, the obtained equations of motion cannot be solved directly with standard methods if a non-steady state case is considered. This method requires the knowledge of viscoelastic material functions in the frequency domain.

4.4 TRANSIENT ANALYSIS OF VISCOELASTIC STRUCTURES

4.4.1 DISCUSSION ON THE EFFICIENCY OF THE LINEAR VISCOELASTIC MODELS

As mentioned in the previous part, different models can be used to represent the behaviour of viscoelastic materials. Depending on the analysis required, certain methods appear to be more attractive than others. In the case where a steady state analysis needs to be undertaken, it is convenient to write the equations of motion in the frequency domain using the complex material functions. In this situation, the steady state analysis will be similar to the steady state analysis of an elastic structure but with the moduli of elasticity replaced by the viscoelastic complex moduli. This method offers the advantage of keeping the analysis relatively simple.

The transient analysis of viscoelastic structures is a lot more complicated. Indeed, if the forced vibration equations of a structure are expressed using the complex modulus approach, as in the steady state case, then the transient response of the system cannot be obtained effectively, as in the elastic case by applying direct integration methods or modal superposition methods. Consequently, more complex mathematical techniques have to be used to solve these equations. For example, the time domain behaviour of the structures may be obtained, as mentioned before, from the frequency domain response by applying the inverse Fourier transform technique. Another alternative would be to write the equations of motion in the time domain. However, as explained before both the derivative and the integral representation complicate the problem greatly and therefore increase the solution time. In order to improve and to simplify the transient analysis of viscoelastic structures work has recently been carried out on the development of new approaches. In this part 4.4 a critical review of these new methods is provided.

4.4.2 NEW APPROACHES

New methods, based on the Laplace transform models, have been developed. In the transform Laplace domain, the equations of motion of viscoelastic structures can be expressed as explained by Bagley and Torvik [4.12] or McTavish and Hughes [4.13] in the form of equation 4.9.

$$s^2 [M] \{\tilde{\Delta}(s)\} + s \tilde{G}(s) [K] \{\tilde{\Delta}(s)\} = \{\tilde{F}(s)\} \quad (4.9)$$

In equation 4.9 $s \tilde{G}(s)$ is referred to as the Laplace transform function of the material modulus. This function can be defined using different rational expressions that contain several scalar parameters. The unknown scalar parameters are determined by fitting the function to the experimental complex modulus data. Hence, these new methods start with measured material properties for the viscoelastic material in the form of frequency dependent complex moduli that are, then, curve fitted to obtain model parameters over a frequency range of interest. This can be realised because the modulus function, when evaluated along the imaginary axis $s = j\omega$, produces a more familiar quantity $G^*(\omega)$ known as the complex modulus as illustrated by the relation 4.10.

$$s \tilde{G}(s) = j\omega \tilde{G}(j\omega) = G^*(\omega) = G'(\omega) + iG''(\omega) \quad (4.10)$$

The choice of the rational expression for the Laplace transform of the material modulus is therefore the most important and the most critical point of these new methods. It should allow, by using another Laplace transform, the expression of the dynamic problem of linear second-order equations of motion in the time domain in the form of equation 4.11.

$$[M] \{\ddot{\Delta}\} + [C] \{\dot{\Delta}\} + [K] \{\Delta\} = \{F(t)\} \quad (4.11)$$

Based on this approach new viscoelastic material models have been developed. Different authors have proposed various expressions for the modulus function. The fractional calculus method and the Golla-Hughes-McTavish (GHM) method are two particular examples, which are reviewed in more detail below. All the developed expressions for the modulus function should satisfy the two following conditions:

- The model should fit the experimental data over a significant range of frequency,

- The model should facilitate the derivation of the time-dependent second order equations of motion.

4.4.2.1 Fractional calculus method

Bagley and Torvik [4.12] proposed the first alternative model that satisfies the conditions mentioned above. It is a fractional derivative model, which describes viscoelastic material behaviour. A four or five parameter Laplace domain expression is used to model complex modulus data. The modulus function is represented by an expression of the form:

$$s\tilde{G}(s) = \frac{G_0 + G_1 s^p}{1 + cs^r} \quad (4.12)$$

The model described by equation 4.12 suggests that the frequency dependent modulus is a function of fractional powers of frequency. The small number of empirical parameters needed with this model, compared for example with the differential models, facilitates the process of obtaining a least squares fit to the mechanical properties of the materials and reduces the effort required to solve the resulting equations of motion. This model has been shown to closely fit experimental data; that is to say, to accurately describe the mathematical properties over significant ranges of frequency. This method is therefore recognised for its efficiency in modelling viscoelastic material behaviour. However, this simplicity is superficial. Indeed, its major drawbacks include the awkward assembly of the global equations of motion and the large cumbersome system matrices produced. The use of this method leads to really complicated equations to solve when the transient analysis of viscoelastic structures is carried out.

4.4.2.2 Golla-Hughes-McTavish method (GHM)

The GHM approach models viscoelastic damping by adding additional “dissipation co-ordinates” to the system in order to achieve a linear model providing the required damping properties. McTavish has developed this method as part of his PhD research work [4.14] in order to fulfil the need of the aerospace industry to characterise energy dissipation in large structures in an accurate and convenient manner. The aim was to provide an alternative method which includes viscoelastic damping effects without the restriction of the steady-state motion by providing extra co-ordinates. The GHM

method represents the material modulus function as a series of damped mini-oscillator terms. That is to say, the complex modulus can be written in the Laplace domain using equation 4.13 [4.14]:

$$s\tilde{G}(s) = G^*(s) = \hat{G}^o(1 + h(s)) = \hat{G}^o \left(1 + \sum_{n=1}^k \hat{\alpha}_n \frac{s^2 + 2\hat{\zeta}_n \hat{\omega}_n s}{s^2 + 2\hat{\zeta}_n \hat{\omega}_n s + \hat{\omega}_n^2} \right) \quad (4.13)$$

In relation 4.13, $\hat{G}^o$ is the equilibrium value of the modulus, i.e. the final value of the relaxation function $G(t)$, and s is the Laplace domain operator. The hatted terms in relation 4.13, that is to say the scalar parameters, are obtained from the curve fit to the complex modulus data for a particular viscoelastic material. The expansion of $h(s)$ represents the material modulus as a series of mini-oscillator terms. The number of terms kept in the expansion is determined by the high or low frequency dependence of the complex modulus. In many cases only two to four terms are necessary [4.15].

The main advantage of this method, also called the mini-oscillator technique, is that the algebraic form of the mini-oscillator terms allows the development of a linear second order equations of motion in the time domain. The strength of this method as explained in appendix D lies in the fact that it has the characteristics of a well-known mechanical system. Developed as a tool for the analysis of damping in homogeneous large flexible space structures, it is believed that this method can be applicable to any structure that incorporates viscoelastic materials. Being able to enhance this mathematical model so that it offers the possibility to predict the transient response of FRP laminated and FRP sandwich panels taking into consideration their viscoelastic material properties would offer a great improvement to the current available analytical models.

4.4.3 GENERAL REMARKS

Accounting for material viscoelastic behaviour when transient analysis of a given structure is carried out is from a modelling point of view a really difficult task. From the presented review, the following points should be emphasised:

- The application of the elastic viscoelastic principle to introduce the constitutive viscoelastic material properties into a previously developed analytical model is not recommended when transient analysis needs to be carried out.
- When transient analysis needs to be carried out, the GHM approach which is defined as a Laplace transform method appears to be the best alternative modelling technique for viscoelastic material. Since the GHM method has only been applied to model isotropic viscoelastic material more research work needs to be carried out to investigate the feasibility of using this method to model anisotropic or orthotropic constitutive viscoelastic material of FRP laminated and FRP sandwich panels.

4.5 VISCOELASTIC PROPERTIES OF FRP AND CORE MATERIALS

In order to carry out the dynamic analysis of structures made of viscoelastic materials, first the knowledge of their complex dynamic material properties is required. Indeed, when either the constitutive viscoelastic material properties are introduced into an analytical model using the elastic viscoelastic principle or when using one of the new Laplace transform approaches, the input of the frequency dependent complex material moduli is required. For FRP laminated and FRP sandwich plates, since damping is mainly due to the viscoelastic behaviour of the core material and/or the viscoelastic behaviour of the matrix material in the fibre reinforced plastic laminae, the knowledge of the complex moduli of both the core material and the FRP material is required.

4.5.1 EXISTING VISCOELASTIC MATERIAL DATA

As far as FRP materials are concerned, experimental and analytical research work has been carried out, principally for the aerospace industry, in order to determine their complex material dynamic properties. However, since FRP materials are anisotropic, their damping characteristics are more complex and more difficult to measure than for isotropic materials and so more extensive analyses are required. Indeed, for example for unidirectional fibre-reinforced composites, the damping is different in longitudinal and in transverse directions. Sun and Lu [4.16], have proposed analytical methods to predict the damping characteristics of FRP material based on the knowledge of the dynamic properties of the fibre and the matrix and on the knowledge of the fibre volume fraction and the fibre aspect ratio. Whereas these approaches seem very practical, their accuracy is limited. Indeed, in addition to the material

constituent properties, their interaction and physical characteristics such as voids need also to be incorporated since, as mentioned by Crane and Gillespie [4.17], numerous experimental investigations have shown that these effects will influence the damping response. The best and most accurate way of determining the dynamic properties of FRP materials is, therefore, to carry out experiments. Schultz and Tsai [4.18] and then later on Ni and Adams [4.19] were the first to measure the dynamic material properties of unidirectional FRP materials namely glass and carbon fibres in epoxy resin. More recently, numerous researchers have carried out more experiments to determine the damping properties of FRP composite structures with the aim of improving and optimising their damping characteristics. For example, Benchechou *et al.*, [4.20] determined experimentally the material properties of carbon fibre reinforced plastic as a function of frequency. Grant *et al.* [4.21] measured the dynamic properties, as a function of temperature, of plain weave carbon fibre in highly damped epoxy resin systems. Ghosh *et al.* [4.22] monitored the change of the dynamic properties of various glass-reinforced composites as a function of temperature. Published complex modulus data for FRP materials can therefore be found in the literature. However, their amount is still limited and if the analysis has to be carried out for a given specific material it is likely that new experiments will need to be realised.

As far as PVC foam materials are concerned little work has been carried out. Danielson and Olsson [4.23] measured the complex shear modulus of a high resilient rigid foam, the P2044 foam material, using a Rheometrics instrument in torsion mode. Their objective was to investigate the damping properties of this foam. Lin [4.24] measured, using a DMA instrument, the dynamic Young's modulus of a Klegecell PVC commercial foam as a function of temperature and at three frequencies. The aim of his work was to investigate the mechanical characteristics of commercial foam. More experimental work such as the ones carried out by Danielsson and Olsson and Lin should be undertaken on PVC foam materials. Indeed, the lack of data and knowledge on the dynamic material properties of commonly used structural PVC foam materials needs to be overcome since these materials play an important role in vibration damping of FRP sandwich structures.

4.5.2 FATIGUE AND DAMAGE EFFECTS ON THE DYNAMIC MATERIAL PROPERTIES

It should be emphasised at this point that not only data for new or undamaged constitutive materials should be determined but as well data for materials which have been damaged or fatigued during the service life of FRP laminated or FRP sandwich panels. Indeed, being able

to predict the dynamic responses of fatigued or damaged structures is as important from a design point of view as predicting the dynamic responses of undamaged structures. Nevertheless this can only be achieved if sufficient knowledge on the degradation of the material properties during normal service life of a structure has been gathered. However up until now no information is available in the literature on the change of the complex viscoelastic moduli of composite structure materials due to fatigue or presence of damage such as cracks.

4.5.3 GENERAL REMARKS

It is apparent from this literary review on the viscoelastic properties of FRP and core materials that there is a lack in current knowledge. This outlines the necessity of carrying out more experimental work in order to improve the current knowledge on the dynamic properties of FRP and core materials and on their degradation due to presence of damage. Indeed, this knowledge is absolutely necessary in order to be able to obtain an accurate prediction of the dynamic responses of FRP laminated or FRP sandwich plates.

4.6 CONCLUSION

In this chapter a critical review of the different methods available to model the linear viscoelastic behaviour of materials has been presented with the aim of establishing the most suitable method that could be incorporated into the laminated and sandwich plate theories. The following conclusions from this chapter should be outlined.

- The elastic viscoelastic correspondence principle offers the possibility to easily and accurately model the frequency dependent dynamic material properties. However, this method should ideally only be applied under steady state conditions.
- Few attempts have been made to introduce this elastic viscoelastic principle into analytical models based on sandwich plate theories. However these methods only account for the viscoelastic properties due to shear in the core and most of the time do not consider the frequency dependency of the dynamic material properties.
- Modelling the viscoelastic material without restricting the analysis to steady state analysis is a complex problem. Indeed the use of traditional viscoelastic models leads to equations of motion that are cumbersome to solve.

- The lack of efficient current analytical methods to predict the transient response of FRP laminated and FRP sandwich panels having viscoelastic material properties outlines the necessity to find alternative new mathematical techniques.
- Methods based on Laplace transform models have recently been developed. To the knowledge of the author they have only been applied to study the transient response of homogeneous specific space structures made of an isotropic viscoelastic material.
- The use of viscoelastic material model is amongst other things restricted by the serious lack in published data on the dynamic material properties of FRP materials and PVC foam core materials.

The review presented in this chapter has clearly outlined the lack of analytical models accounting for viscoelastic material properties and allowing accurate prediction of the dynamic response of FRP laminated and FRP sandwich panels. Consequently, to overcome this deficiency alternative new analytical models need to be developed. This should be linked with the realisation of new experimental work in order to reduce the shortage of material data and increase the knowledge on the constitutive materials dynamic behaviour.

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PART C

Chapter 5: Investigation philosophy.

Chapter 6: Analytical models for predicting dynamic responses of FRP laminated and FRP sandwich plates.

Chapter 7: Experimental characterisation.

Chapter 8: Dynamic analysis of FRP composite plates with viscoelastic materials.

5. INVESTIGATION PHILOSOPHY

Nowadays design is more and more performance orientated. Indeed, the tendency is to produce lighter, faster and more economically efficient vessels without altering their structural integrity. During their operating life ship structures are subjected to various levels of vibration due to the natural environment in which they are operating and on board machinery and equipment. Consequently dynamic analysis as well as static analysis needs to be carried out at the design stage in order to accurately optimise the design of structural members, to avoid resonance and fatigue damage and to minimise damage to sensitive equipment and improve comfort for the crew and passengers.

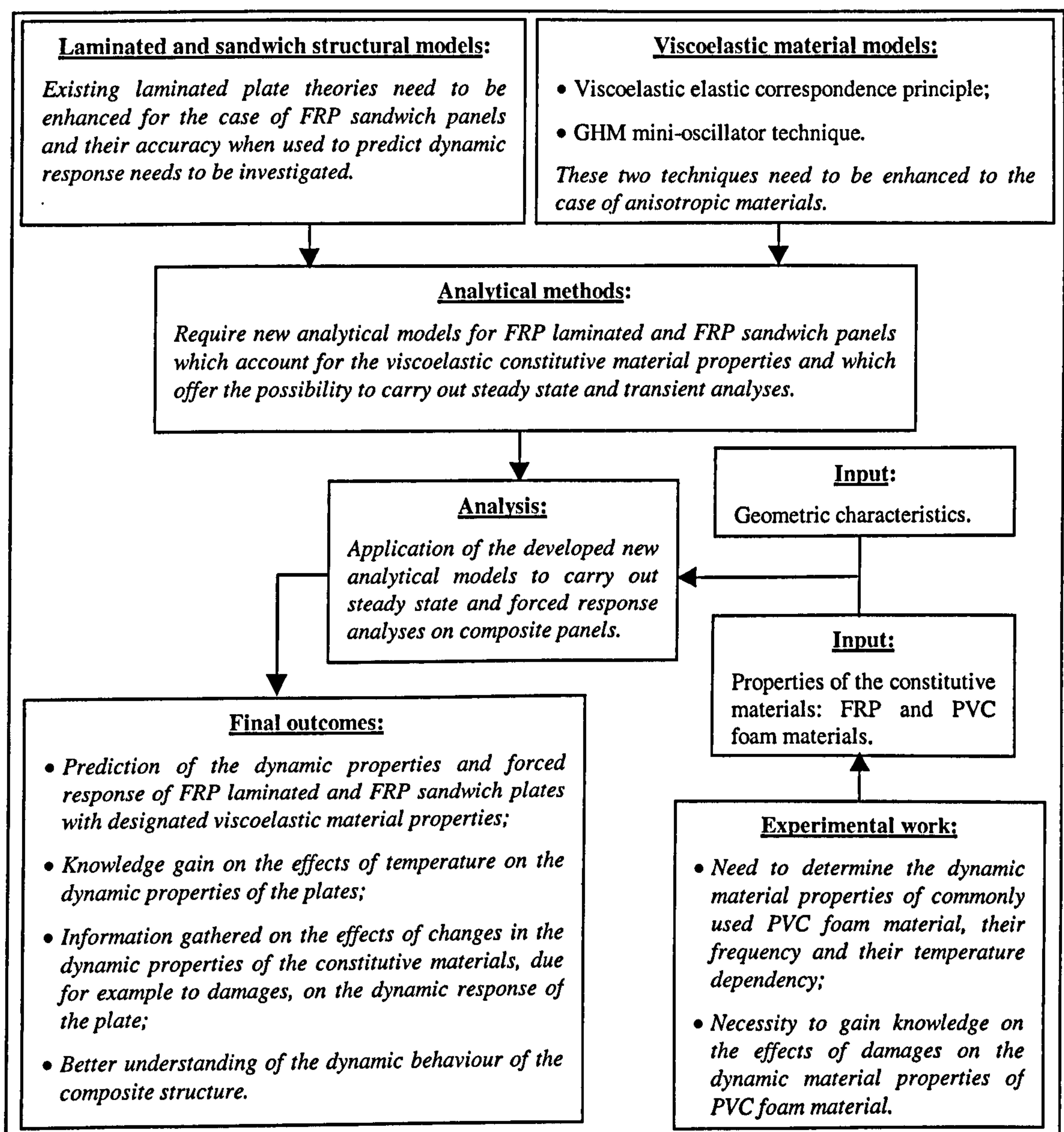


Figure 5.1: General overview of the proposed approach (novelty and contribution to knowledge highlighted in italics)

The main objective of this research work is to develop new analytical methods that allow the prediction and the analysis of the dynamic responses of FRP laminated and FRP sandwich plates while accounting for their viscoelastic material properties. Being able to fulfil this aim requires knowledge to be gained in various domain of research. Figure 5.1 summarises the different aspects that need to be considered and highlights in italics the novelty and contribution to knowledge. The general overview of the proposed approach provided in this figure is explained in more detail in the following.

As indicated in the literary review, techniques exist to model composite panels. Nevertheless, it has been identified that the commonly used methods are different for FRP laminated plates and for FRP sandwich plates and present several limitations when used to predict dynamic responses. Indeed, most of the existing methods are based on pessimistic assumptions and on simplifications that have proved to be valid for static cases but which lead to erroneous results under dynamic analysis. Consequently, dynamic analysis of FRP laminated and FRP sandwich plates requires the development of more accurate analytical models. These models, as pointed out in the review, need to account for the damping viscoelastic properties of the constitutive materials. Indeed, the shortage of analytical models which account for the dynamic material properties of the constitutive materials and their frequency dependency, especially when forced response analyses are carried out, needs to be overcome. Existing methods to model linear viscoelastic material behaviour have been presented in the literature review and suitable methods that could be incorporated into laminated or sandwich plate theories have been identified. However these methods need to be enhanced since their use, up until now, has mainly been restricted to isotropic viscoelastic material and/or to steady state condition. Finally it should be kept in mind that accounting for the viscoelastic behaviour of the constitutive materials requires the knowledge of their complex dynamic moduli. This last point emphasises the need to create a database on the dynamic material properties of the constitutive materials, such as FRP materials and PVC foam materials, and to increase the actual, but presently limited, knowledge of the constitutive materials dynamic behaviour.

The development of new design techniques, which enable the ship designer to carry out dynamic analyses with greater precision and confidence, are required. The necessity of developing ideal analytical models, that is to say models that are as accurate as possible and as simple as possible and that account for the effect of the viscoelastic behaviour of the constitutive materials, has been highlighted. The developed models should offer the possibility to carry out free vibration and transient analyses of either FRP laminated panels or FRP sandwich panels. Besides, they

should give the designer the opportunity to account for the dynamic material properties and their frequency dependency. Finally, the derived methods should provide to the designer the opportunity to have a better understanding of the dynamic behaviour of the ship structure. In parallel to the development of new analytical models, experimental works need to be carried out in order to provide information and data on the dynamic properties and behaviour of the constitutive materials. Indeed, being able to increase the knowledge at a material level is essential to the improvement in the understanding of the dynamic behaviour of composite ship structure.

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6. ANALYTICAL MODELS FOR PREDICTING DYNAMIC RESPONSES OF FRP LAMINATED AND FRP SANDWICH PLATES

6.1 INTRODUCTION

The first step of the research consists of investigating the feasibility of using FSDT and HSDT to model both FRP laminated and FRP sandwich panels. At this stage of the analysis, the constitutive materials are assumed to be elastic and so any damping due to viscoelasticity is neglected. The FSDT, as mentioned in the literary review, is computationally attractive and provides accurate results when the study of relatively thin panels is carried out. However, when the thickness of the studied panels is increased, the efficiency of the prediction of their dynamic response using analytical model based on FSDT is affected. On the other hand, Reddy's refined HSDT has been found to still provide accurate results when the thickness of the modelled laminated plates is increased. This is due to the fact that, as mentioned earlier in the literary review, Reddy's refined HSDT provides a more realistic parabolic distribution of the shear stresses through the thickness by which the condition of zero transverse shear stresses in the plate bounding planes are fulfilled and so the need for shear correction factors is removed. The use of higher order shear deformation theories have up until now been restricted mainly due to the fact that they lead to computationally expensive methods. However analytical methods based on higher order theories and more specifically Reddy's refined HSDT appear to be more attractive methods when it comes to modelling thicker composite panels such as FRP sandwich panels.

In this chapter, after a brief review of the theory related to the development of analytical methods based on either FSDT or HSDT to predict the behaviour of laminated plates, the feasibility of using these methods to model FRP sandwich panels is investigated. First of all a free vibration analysis is carried out and the results obtained using analytical models based on both FSDT and HSDT are compared with finite element results. From this analysis both the advantages and the disadvantages of the presented methods are outlined. Then, transient analyses for both FRP laminated and FRP sandwich plates modelled using HSDT are carried out. The validity of the results is discussed since, as highlighted in the literary review, the accuracy of the transient response of panels modelled using HSDT has not been yet clearly identified.

6.2 ANALYTICAL MODELS FOR LAMINATES

The analytical models, which are considered in this analysis, are based on either the FSDT or the refined HSDT. The development of the equations of motion based on these shear deformation theories for homogeneous and laminated rectangular plates has been widely described by Reddy in his various publications ([6.1], [6.2], [6.3]). The aim of this part is to outline the various assumptions and the main steps of the development of the equations of motion. This is realised with the objective of helping later with the understanding of the changes that will be applied to the theory in order to update it firstly to the case of sandwich panels and then subsequently to the case of structures made of viscoelastic materials.

6.2.1 EQUATIONS OF MOTION

The first step in the derivation of the equations of motion of displacement based analytical models is to determine the form of the displacement field that best models the studied continuous body. In the present case, the displacement fields associated with both the FSDT and the HSDT are considered. Their expression can be found in table 3.1 of chapter 3. In formulating these shear deformation theories the assumption that all the layers are perfectly bonded together is made. This implies that the displacements and the strains are continuous through the laminate thickness. Besides, with this assumption, some restrictions are placed as stated here:

- Each layer is of uniform thickness;
- The transverse shear stresses on the top and bottom surfaces of the laminate are zero;

For each shear deformation theory, knowing the displacement field, the strain in the studied body can be computed using the strain-displacement relations. Then the derivation of the equations of motion can be carried out using, for example, the energy principle also known as the principle of virtual displacements. After that, by using the laminated constitutive equations the Euler-Lagrange equations associated with each theory can be computed. The derivation of the Euler-Lagrange equations of motion for laminated plates modelled using FSDT and HSDT respectively can be found in appendix E and F respectively. Finally, the equations of motion can be expressed in terms of the generalised displacements by using the laminate constitutive relations. The obtained equations of motion for laminates modelled using FSDT can be found in Ochoa and Reddy's book [6.1]. For the case of HSDT, these equations can be found, for a simplified case such as symmetric cross-ply laminated plates, in Khdeir and Reddy's paper [6.4]. It should be noticed that the equations of motion written in terms of the displacements are large and therefore

are most of the times presented only for simplified cases (i.e., specific lamination scheme) where some of the laminate stiffness terms can be neglected.

6.2.2 LAMINATED STIFFNESSES FOR SELECTED LAMINATES

The laminate stiffnesses, which can be computed for any arbitrary laminate using the relations provided in appendices E or F, depend on the material stiffnesses, layer thicknesses and the lamination scheme. Symmetry and antisymmetry of the lamination scheme and symmetry of the material properties about the midplane of the laminate reduce some of the laminate stiffnesses to zero and so lead to simplification in the equations of motion.

6.2.2.1 Symmetric laminates

When symmetric laminates are modelled the governing equations can be simplified since there is no coupling between bending and extension (i.e., the coupling stiffnesses B_{ij} and the higher-order coupling stiffnesses E_{ij} are zero). If, as well as being symmetric, the studied laminate is composed of layers orientated at 0° and 90° , that is to say if it is a symmetric cross-ply laminate, then further simplification can be made. Indeed, when cross-ply laminates are considered in each layer the material stiffnesses $\bar{C}_{16}$, $\bar{C}_{26}$ and $\bar{C}_{45}$ written in the plate co-ordinate system are zero. Consequently as well as having no coupling stiffnesses B_{ij} and E_{ij} , symmetric cross-ply laminates have their laminate stiffnesses A_{16} , A_{26} , A_{45} , D_{16} , D_{26} , D_{45} , F_{16} , F_{26} , F_{45} , H_{16} and H_{26} equal to zero. It should be noticed that symmetric angle-ply laminates do not offer any further simplification than general symmetric laminates, that is to say only the coupling stiffnesses B_{ij} and E_{ij} are zero.

6.2.2.2 Antisymmetric laminates

Due to the antisymmetry of the lamination scheme but symmetry of the thicknesses of each pair of layers, antisymmetric laminates have the feature that A_{16} , A_{26} , D_{16} , D_{26} , F_{16} , F_{26} , H_{16} and H_{26} are zero. Further simplification can be considered for specific lamination schemes such as cross-ply or angle-ply laminates.

- **Antisymmetric cross-ply laminates**

For these laminates, the coupling stiffnesses B_{ij} and E_{ij} have the properties $B_{22} = -B_{11}$, $E_{22} = -E_{11}$ and all other B_{ij} and E_{ij} are zero. In addition, as for any cross-ply laminates, the stiffnesses A_{16} , A_{26} , A_{45} , D_{16} , D_{26} , D_{45} , F_{16} , F_{26} , F_{45} , H_{16} and H_{26} are equal to zero.

- **Antisymmetric angle-ply laminates**

For antisymmetric angle-ply laminates without 90° layers, some coupling stiffnesses can be simplified. Indeed, in this particular case B_{11} , B_{22} , B_{12} and B_{66} are zero as well as E_{11} , E_{22} , E_{12} and E_{66} . In addition, due to the specific geometric features of angle-ply laminates and due to the fact that $\bar{C}_{45}(\theta) = -\bar{C}_{45}(-\theta)$ the following stiffnesses A_{45} , D_{45} and F_{45} are also equal to zero.

6.2.3 CLOSED FORM SOLUTIONS

The equations of motion expressed in terms of the displacement co-ordinates are valid for any arbitrary laminated composite plate. As demonstrated by Reddy [6.2], [6.3] and Khdeir [6.4] these equations can be solved exactly using either a Navier type or a Lévy type solution. However, using these methods leads to some restrictions. Indeed, each of them is valid only under specific geometric parameters, boundary conditions and stacking layers. The restrictions associated with both these methods are presented here in this chapter.

6.2.3.1 Navier solution

In the Navier solution the generalised displacements are expanded in a double trigonometric series in terms of unknown parameters. The choice of the functions in the series is restricted to those which satisfy the boundary conditions of the problem. Substitution of the displacement expansions into the governing equations should result in a unique, invertible, set of algebraic equations among the parameters of the expansion. Otherwise the Navier solution cannot be developed for the problem. As demonstrated by Whitney and Pagano [6.5] and then as mentioned by Reddy and Chao [6.6] the Navier solutions can be developed for rectangular laminates with only two sets of simply supported boundary conditions which are summarised in table 6.1. It should be noticed that even for these boundary conditions, not all the lamination schemes permit such solutions.

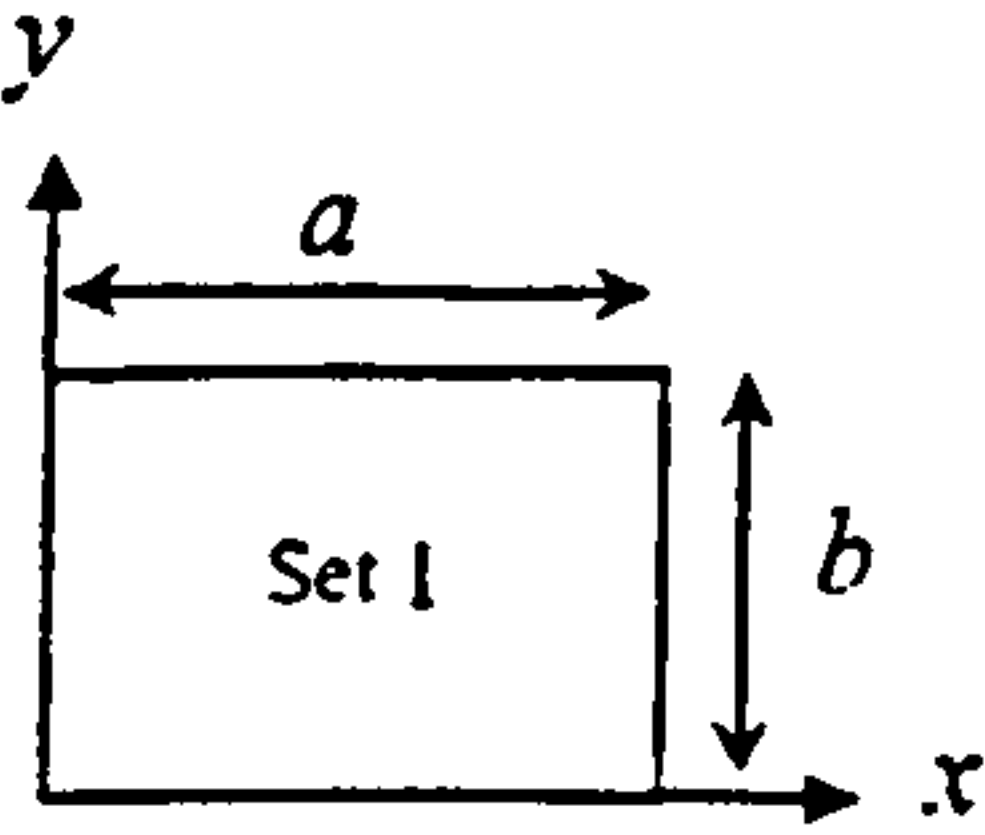
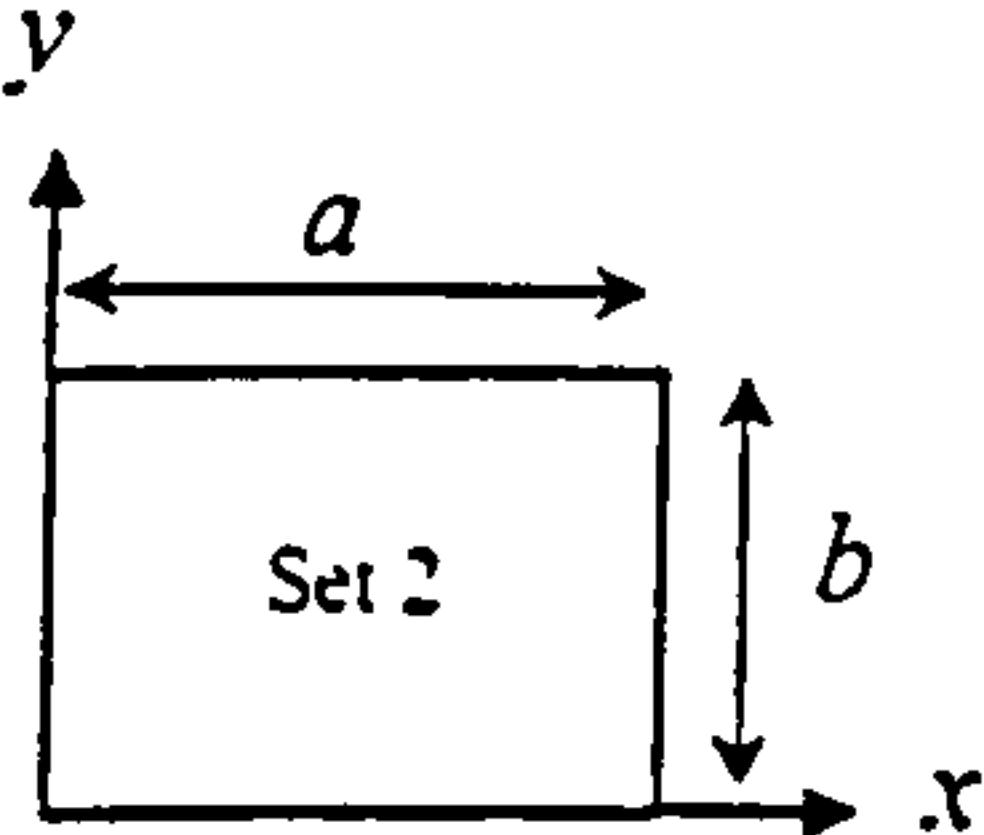
	FSDT	HSDT
SSSS-1 	At $x=0$ and $x=a$: $v_0 = w_0 = \phi_y = 0$ $N_1 = M_1 = 0$ At $y=0$ and $y=b$: $u_0 = w_0 = \phi_x = 0$ $N_2 = M_2 = 0$	At $x=0$ and $x=a$: $v_0 = w_0 = \phi_y = 0$ $N_1 = M_1 = P_1 = 0$ At $y=0$ and $y=b$: $u_0 = w_0 = \phi_x = 0$ $N_2 = M_2 = P_2 = 0$
SSSS-2 	At $x=0$ and $x=a$: $u_0 = w_0 = \phi_y = 0$ $N_6 = M_1 = 0$ At $y=0$ and $y=b$: $v_0 = w_0 = \phi_x = 0$ $N_6 = M_2 = 0$	At $x=0$ and $x=a$: $u_0 = w_0 = \phi_y = 0$ $N_6 = M_1 = P_1 = 0$ At $y=0$ and $y=b$: $v_0 = w_0 = \phi_x = 0$ $N_6 = M_2 = P_2 = 0$

Table 6.1: Two types of simply supported boundary conditions

It has been demonstrated by Reddy [6.3] that the Navier solution using the first set of boundary conditions, noted SSSS-1, can be obtained only for laminates whose stiffnesses A_{16} , A_{26} , B_{16} , B_{26} , D_{16} , D_{26} and A_{45} are zero and whose higher-order stiffnesses E_{16} , E_{26} , F_{16} , F_{26} , H_{16} , H_{26} , D_{45} and F_{45} are zero. Thus, the boundary conditions SSSS-1 can be developed only for laminates with a single orthotropic layer, symmetric cross-ply laminates and antisymmetric cross-ply laminates. Similarly, the Navier solutions using the second set of boundary conditions, noted as SSSS-2, can be obtained only for laminates whose stiffnesses A_{16} , A_{26} , B_{11} , B_{12} , B_{22} , B_{66} , D_{16} , D_{26} and A_{45} are zero and whose higher-order stiffnesses, F_{16} , F_{26} , H_{16} , H_{26} , D_{45} and F_{45} are zero. Consequently the boundary conditions SSSS-2 can be developed only for laminates with a single orthotropic layer, symmetric cross-ply laminated plates and antisymmetric angle-ply laminated plates.

6.2.3.2 Lévy solution

The Lévy method can be used to solve the governing equations of various plate theories for rectangular laminates for which two (parallel) opposite edges are simply supported

and the other two edges can have any boundary conditions (e.g., simply supported, clamped or free). As with the Navier solution, the Lévy solution can be used to model homogeneous, symmetric and antisymmetric cross-ply laminates and antisymmetric angle-ply laminates. The Lévy solution technique involves choosing a solution form that satisfies the first set of the simply supported boundary condition, SSSS-1, presented for the Navier solution on two parallel edges of a rectangular laminate. Then, the partial differential equations of equilibrium are reduced to ordinary differential equations in the co-ordinate parallel to the simply supported edges. The ordinary differential equations are then solved using the state space approach as shown in Khdeir's paper [6.4].

6.2.3.3 General remarks on closed form solutions

Closed form solutions can be used to investigate the dynamic behaviour of specific laminated panels. Nevertheless, by using closed form solutions, restriction in the lamination schemes and boundary conditions of the studied panels exist. The problem can be solved using either Lévy solution or Navier solution. The use of each of these methods leads to different feasible sets of boundary conditions. In the rest of the analysis only the Navier solution is considered. However it should not be forgotten that the work realised using analytical models developed assuming a Navier solution type could as well be easily carried out with analytical models developed assuming a Lévy solution type.

6.2.4 EQUATIONS OF MOTION

Assuming a set of Navier solution form that satisfies the boundary conditions, it is possible for homogeneous, symmetric and antisymmetric cross-ply and antisymmetric angle-ply laminates to solve the equations of motion. Indeed by substituting the Navier type form of the generalised displacements into the equations of motion, the set of ordinary differential equations in time is obtained as represented by equation 6.1. This derivation technique has been widely explained by Reddy [6.3] for laminated plates.

$$[M]\{\ddot{\Delta}\} + [K]\{\Delta\} = \{F\} \quad (6.1)$$

Here in relation 6.1 $\{\Delta\}$ is the column vector of the amplitude values of the generalised displacements and $\{F\}$ is the column vector of the applied forces. The coefficients of the

stiffness matrix $[K]$ and of the mass matrix $[M]$ are listed in appendix G and H respectively. The relations, used to compute the mass and stiffness coefficients, are different for each shear deformation theory used to model the plate and for each set of boundary conditions considered. Besides, these coefficients are a function of the laminate stiffnesses and inertia coefficients as well as of the mode considered and the geometric properties of the studied laminate.

6.3 DEVELOPMENT OF THE THEORY FOR SANDWICH PLATES

The studied rectangular sandwich plates are composed of two FRP composite laminated faces and a core of thickness h_c . The plate considered in this part is assumed to have length a , width b and total thickness h as shown in figure 6.1. The faces are considered to be made of thin orthotropic layers while the theory allows the core material to be either isotropic or orthotropic. Since a PVC foam core is being considered, the core is assumed to be isotropic for the rest of the study.

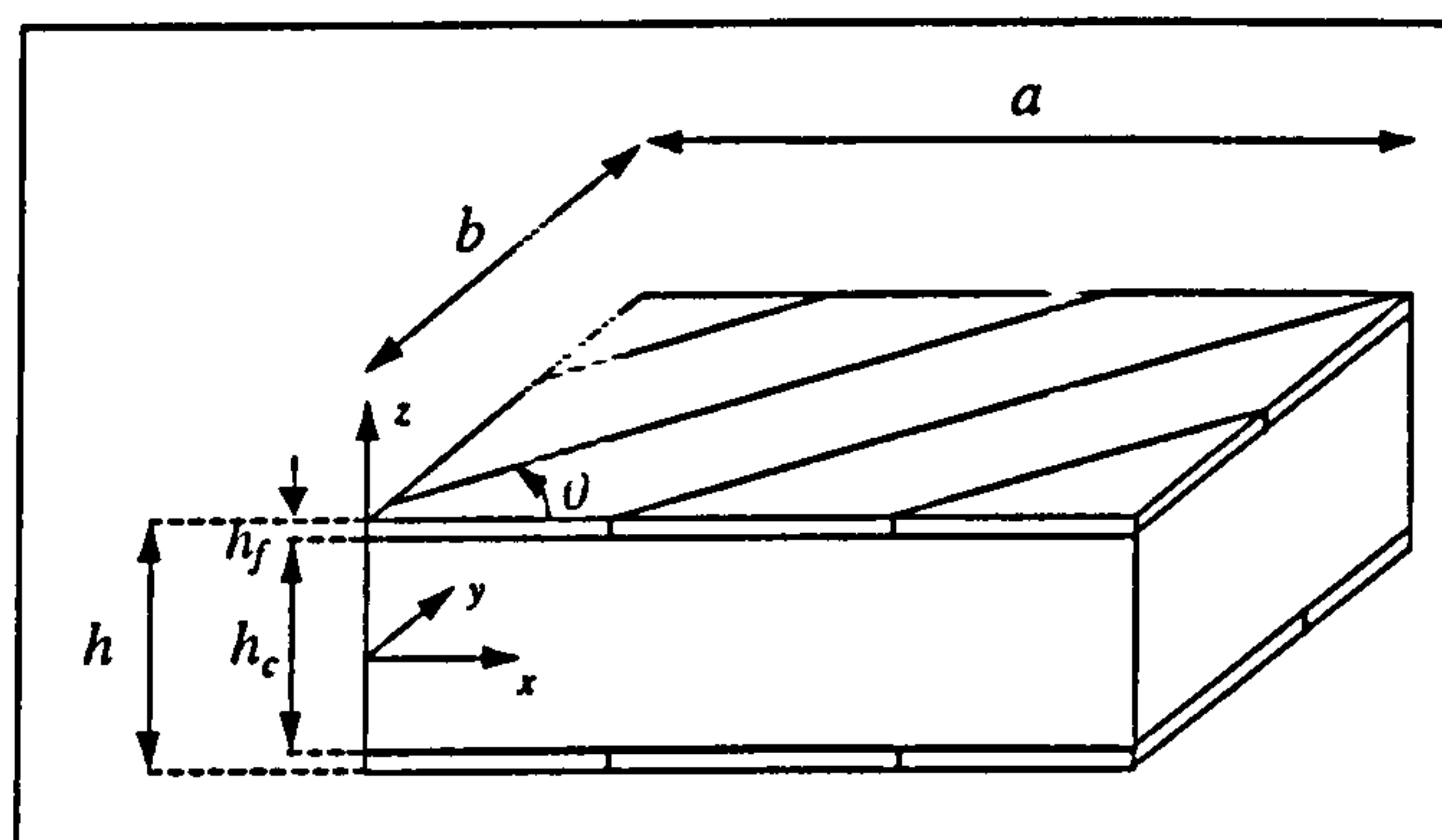


Figure 6.1: Sandwich plate geometry.

It is proposed in this part to enhance the existing analytical model developed for the case of laminated plates to the case of sandwich panels. As mentioned in the previous part, it is not possible to study the behaviour of laminated plates having an arbitrary geometry, constructed of arbitrarily orientated layers and which are subjected to arbitrary boundary conditions using closed-form analytical techniques. Nevertheless, analytical solutions can be obtained for simply supported rectangular laminates made of specific lamination schemes (i.e., symmetric cross-ply laminates and antisymmetric angle-ply and cross-ply laminates). The same restrictions exist when the study of sandwich plates is undertaken. Therefore, the analysis can be carried out for simply supported sandwich plates comparable with symmetric cross-ply rectangular plates or

antisymmetric angle-ply or cross-ply rectangular plates to which a thick core made of PVC foam material has been added in the middle as represented in figure 6.2.

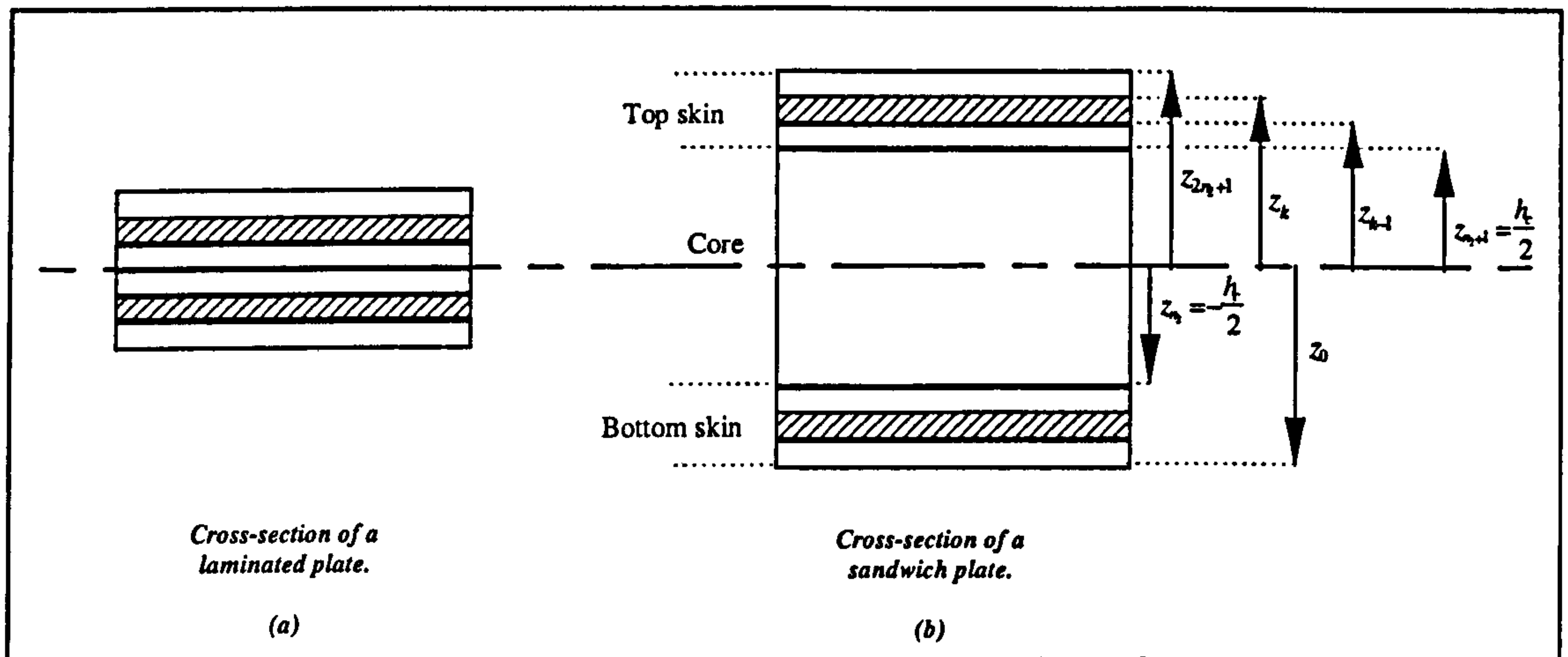


Figure 6.2: Comparison of the cross sections for the base laminate (a) and the studied sandwich panel (b).

The theory that is presented here concentrates only on the implementation of the analytical methods previously developed for laminates or sandwich plates. It should be noticed that the introduction of a thick middle layer, which can be made of orthotropic or isotropic material, does not affect drastically the analytical model. Indeed, the stiffness coefficients and the matrix coefficients of the equations of motion 6.1 can still be calculated using the relations provided in appendix G and H respectively. Nevertheless, the laminate stiffness terms and the inertia terms can be re-written for FSDT and HSDT respectively as explained in the author's paper [6.7] and in the following two parts.

6.3.1 FSDT

In the case of laminated composite plates the inplane stiffness A_{ij} , the coupled bending-stretching stiffness B_{ij} and the bending stiffness D_{ij} as well as the normal inertia I_1 , the coupled normal-rotary inertia I_2 and the rotary inertia I_3 can be found in appendix E. When the study of FRP rectangular sandwich plates composed of n_2 layers in each skin, as shown in figure 6.2, is carried out, the expressions for the laminate stiffness terms and for the inertia terms respectively should be re-written in the form shown by relations 6.2 and 6.3 respectively.

$$\begin{aligned}
 (A_{ij}, B_{ij}, D_{ij}) &= \sum_{k=1}^{n_2} \int_{z_{k-1}}^{z_k} \bar{C}_{ij}^{(k)}(1, z, z^2) dz + \int_{-h_c/2}^{h_c/2} \bar{C}_{ij}^{(core)}(1, z, z^2) dz + \sum_{k=n_2+2}^{2n_2+1} \int_{z_{k-1}}^{z_k} \bar{C}_{ij}^{(k)}(1, z, z^2) dz \\
 &\quad \text{for } i, j = 1, 2, 6 \\
 A_{ij} &= \sum_{k=1}^{n_2} \int_{z_{k-1}}^{z_k} K \bar{C}_{ij}^{(k)} dz + \int_{-h_c/2}^{h_c/2} K \bar{C}_{ij}^{(core)} dz + \sum_{k=n_2+2}^{2n_2+1} \int_{z_{k-1}}^{z_k} K \bar{C}_{ij}^{(k)} dz \quad \text{for } i, j = 4, 5
 \end{aligned} \tag{6.2}$$

and

$$(I_1, I_2, I_3) = \sum_{k=1}^{n_2} \int_{z_{k-1}}^{z_k} \rho^{(k)}(1, z, z^2) dz + \int_{-h_c/2}^{h_c/2} \rho^{(core)}(1, z, z^2) dz + \sum_{k=n_2+2}^{2n_2+1} \int_{z_{k-1}}^{z_k} \rho^{(k)}(1, z, z^2) dz \tag{6.3}$$

Generally, when the study of laminated plates is undertaken the shear correction factor K is assumed to be equal to $5/6$ or $\pi^2/12$ as mentioned in part 3.2.2 of the literary review. As explained by Reddy and Kuppasamy [6.8] the use of the FSDT, assuming a shear correction factor value of $5/6$, to model laminates leads to higher frequency values than those predicted by the 3-D elasticity theory. This discrepancy in the results has been seen to increase with a decrease in the length to thickness ratio. This is linked to the fact that the shear correction factor depends on the constituent ply properties, lamination schemes and type of the structure. Consequently, with the insertion of a thick PVC foam core material in the sandwich plates, new values for the shear correction factors might be required. The first numerical study presented later on in this chapter will therefore consist of investigating the accuracy of the results obtained with commonly used shear correction factor values. This will be realised by comparing the natural frequency values of given sandwich panels obtained by using closed form solution method based on FSDT with results obtained by carrying out finite element analysis. This analysis will allow us to conclude on the efficiency of using FSDT to model sandwich panels.

6.3.2 HSDT

In the case of laminated composite the stiffnesses A_{ij} , B_{ij} , D_{ij} , E_{ij} , F_{ij} and H_{ij} and the inertia terms I_1 , I_2 , I_3 , I_4 , I_5 and I_7 for the same laminated plate can be found in appendix F. When the study of FRP rectangular sandwich plates composed of n_2 layers in each skin, as shown in figure 6.2, is carried out, the expressions for the laminate stiffness terms and for the inertia terms respectively should be re-written in the form shown by relations 6.4 and 6.5 respectively.

$$\begin{aligned}
(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) &= \sum_{k=1}^{n_2} \int_{z_{k-1}}^{z_k} \bar{C}_{ij}^{(k)}(1, z, z^2, z^3, z^4, z^6) dz + \int_{-h_c/2}^{h_c/2} \bar{C}_{ij}^{(core)}(1, z, z^2, z^3, z^4, z^6) \\
&\quad + \sum_{k=n_2+2}^{2n_2+1} \int_{z_{k-1}}^{z_k} \bar{C}_{ij}^{(k)}(1, z, z^2, z^3, z^4, z^6) dz \quad (i, j = 1, 2, 6) \\
(A_{ij}, D_{ij}, F_{ij}) &= \sum_{k=1}^{n_2} \int_{z_{k-1}}^{z_k} \bar{C}_{ij}^{(k)}(1, z^2, z^4) dz + \int_{-h_c/2}^{h_c/2} \bar{C}_{ij}^{(core)}(1, z^2, z^4) \\
&\quad + \sum_{k=n_2+2}^{2n_2+1} \int_{z_{k-1}}^{z_k} \bar{C}_{ij}^{(k)}(1, z^2, z^4) dz \quad (i, j = 4, 5)
\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}
(I_1, I_2, I_3, I_4, I_5, I_7) &= \sum_{k=1}^{n_2} \int_{z_{k-1}}^{z_k} \rho^{(k)}(1, z, z^2, z^3, z^4, z^6) dz + \int_{-h_c/2}^{h_c/2} \rho_c(1, z, z^2, z^3, z^4, z^6) dz + \\
&\quad \sum_{k=n_2+2}^{2n_2+1} \int_{z_{k-1}}^{z_k} \rho^{(k)}(1, z, z^2, z^3, z^4, z^6) dz
\end{aligned} \tag{6.5}$$

6.3.3 SUMMARY

The use of the equations 6.2 and 6.3 for the FSDT or the equations 6.4 and 6.5 for the HSDT allows the determination of the laminate stiffnesses and the inertia terms that are required to compute the equations of motion of the studied FRP sandwich panels. To enhance the model previously developed for laminates it has been assumed that the thicker middle layer, which constitutes the core, can be computed as for any of the other laminae of the studied panels. The main advantage of this proposed technique is that it provides only one analytical model that can be used directly to study FRP laminates or FRP sandwich panels.

6.4 FREE VIBRATION ANALYSIS

The main objective of this part is to investigate and discuss the accuracy of the developed analytical models based on FSDT and HSDT when used to carry out free vibration analysis on FRP sandwich panels. First of all the feasibility of using either the FSDT or HSDT based analytical model is investigated. Results are compared with available experimental data for sandwich panels made of aluminium faces and with finite element results for FRP sandwich

panels. Finally, the influence of the material property parameters and plate geometry related variables on natural frequencies is investigated.

6.4.1 THEORY

The free vibration analysis is realised by setting all applied loads to zero, that is to say $\{F\} = \{0\}$ in relation 6.1, and by assuming a periodic solution form as expressed by equation 6.6.

$$\{\Delta\} = \{\Delta_0\} \exp(i\omega t) \quad (6.6)$$

where, $\{\Delta_0\} = \{U_{mn}^o, V_{mn}^o, W_{mn}^o, X_{mn}^o, Y_{mn}^o\}$, and ω is the frequency of natural vibration.

Equation 6.6 can then be reduced to the following eigenvalue problem:

$$([K] - \omega^2 [M])\{\Delta_0\} = \{0\} \quad (6.7)$$

This eigenvalue equation 6.7 can be solved using standard procedures, which yields the estimations for the natural frequencies and mode shapes of flexural vibration.

6.4.2 ANSYS MODEL FOR FRP SANDWICH PANELS

Due to the lack of published data on the dynamic response of simply supported FRP sandwich panels it has been decided to validate the results obtained using the presented analytical models with finite element results. The finite element model, which has been developed using the finite element package ANSYS, is presented here. It has been realised using the element SOLID 46. It is an 8-node solid element designed to model layered thick solids. The element is defined by eight nodal points, average thickness, direction angles of the layer material and orthotropic material properties. The boundary conditions that are applied to the developed model are identical to the ones defined in table 6.1. That is to say depending on the lamination scheme, the boundary condition SSSS-1 or the boundary conditions SSSS-2 are applied to the modelled FRP sandwich panel. Finally, the meshes of the finite element model were built as shown in figure 6.3 so that the solution process is optimised. This model has been used to determine the natural frequencies and modal shape of the studied structure. This has been realised by carrying out a modal analysis in the ANSYS solution phase.

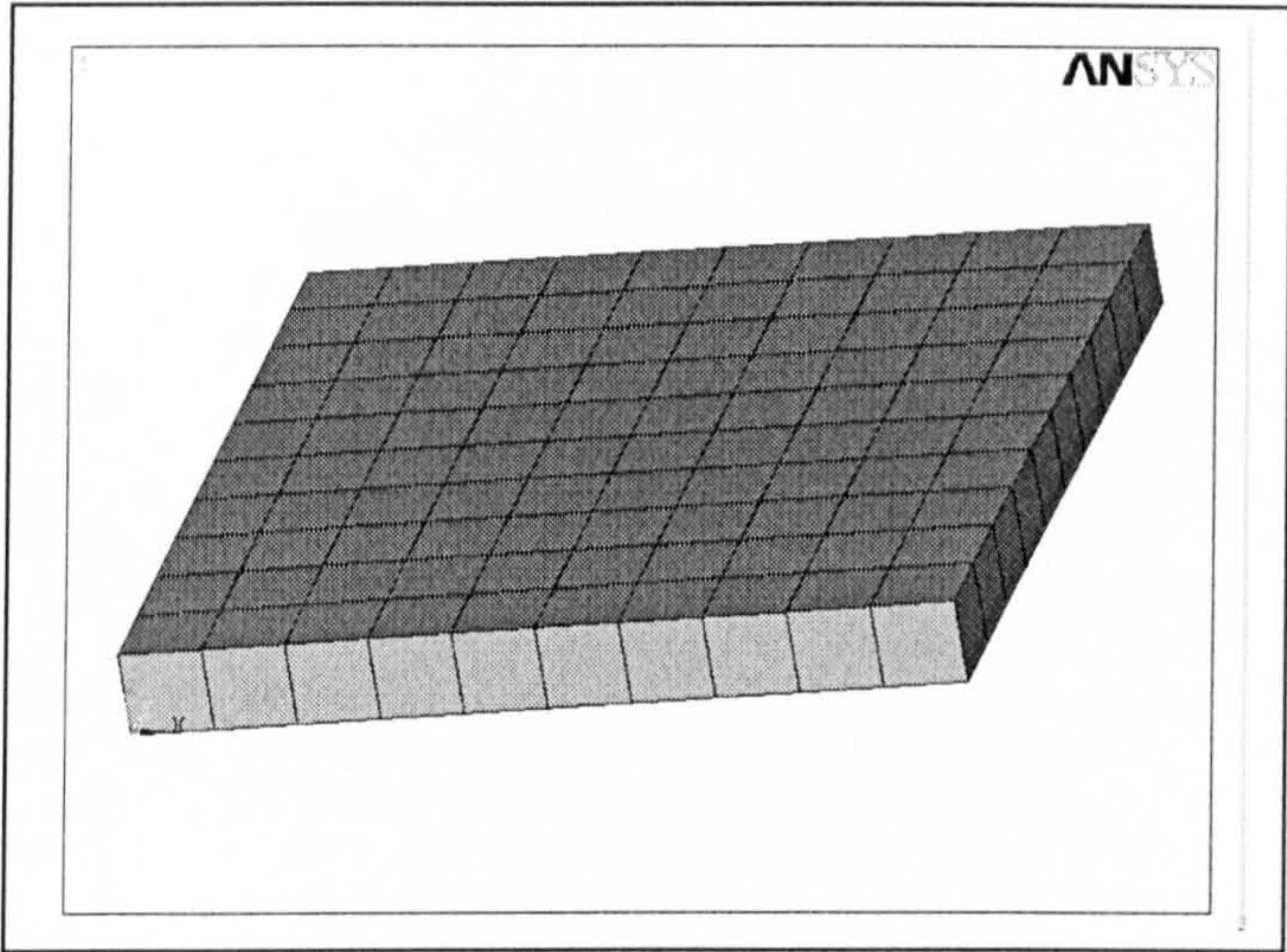


Figure 6.3: FRP sandwich plate finite element model.

6.4.3 DETERMINATION OF THE SHEAR CORRECTION FACTORS FOR THE FSDT

The range of validity of the first order shear deformation theory is known to be strongly dependent on the factor used in adjusting the transverse shear stiffness of the plate. The shear correction factor depends, as mentioned before, on the constituent ply properties, lamination scheme and type of the structure. Consequently, an appropriate shear correction factor needs to be used for each specific study. In this part, the accuracy of the shear correction factor values commonly used when laminated plates are modelled using FSDT is investigated and then the feasibility of using this method to model FRP sandwich plates is discussed. In order to realise this analysis, first of all the results obtained for different fixed values of K , the shear correction factor, are compared with results obtained using the finite element model presented in part 6.4.2. Square sandwich plates made of either HEREX C70.130 or HEREX C70.40 closed cell foam core of relative thickness $h_c/h = 0.94$ and of FRP laminated skins are considered. The bottom and top faces of the sandwich panels are made of three layers of glass in polyester resin orientated at 0° , 90° and 0° and of identical thickness. The length to thickness ratio a/h is assumed to be equal to 10. The material properties for the fibre reinforced plastic and for the two different PVC foams can be found in table 6.2.

Material Properties	E_1 (MPa)	E_2 (MPa)	G_{12} (MPa)	G_{13} (MPa)	G_{23} (MPa)	ρ (kg/m ³)	ν_{12}
HEREX C70.130	109	109	50	50	50	130	0.32
HEREX C70.40	28	28	14	14	14	40	0.32
Eglass/Polyester ($V_f=35\%$)	27860	8070	3170	3170	2070	1650	0.27

Table 6.2: Properties of two HEREX C70 foam materials and of an Eglass/polyester FRP material.

The results from the analysis are presented in tables 6.3 and table 6.4. In these tables, the dimensionless natural frequencies ($\bar{\omega} = \omega a / (100h)$) obtained from the ANSYS model are compared to data obtained using the presented analytical model based on FSDT with different values for the shear correction factors. The advantage of using dimensionless values is that by changing the length and thickness dimensions without changing the length to thickness ratio the dimensionless results are not affected. Therefore, results provided under this dimensionless form are easier to use at the design stage. In the analysis presented below only the first four modes of vibration are considered. The modal displacements in the z direction for the first four modes of the studied sandwich panel made of HEREX C70.130 core material have been represented in figure 6.4.

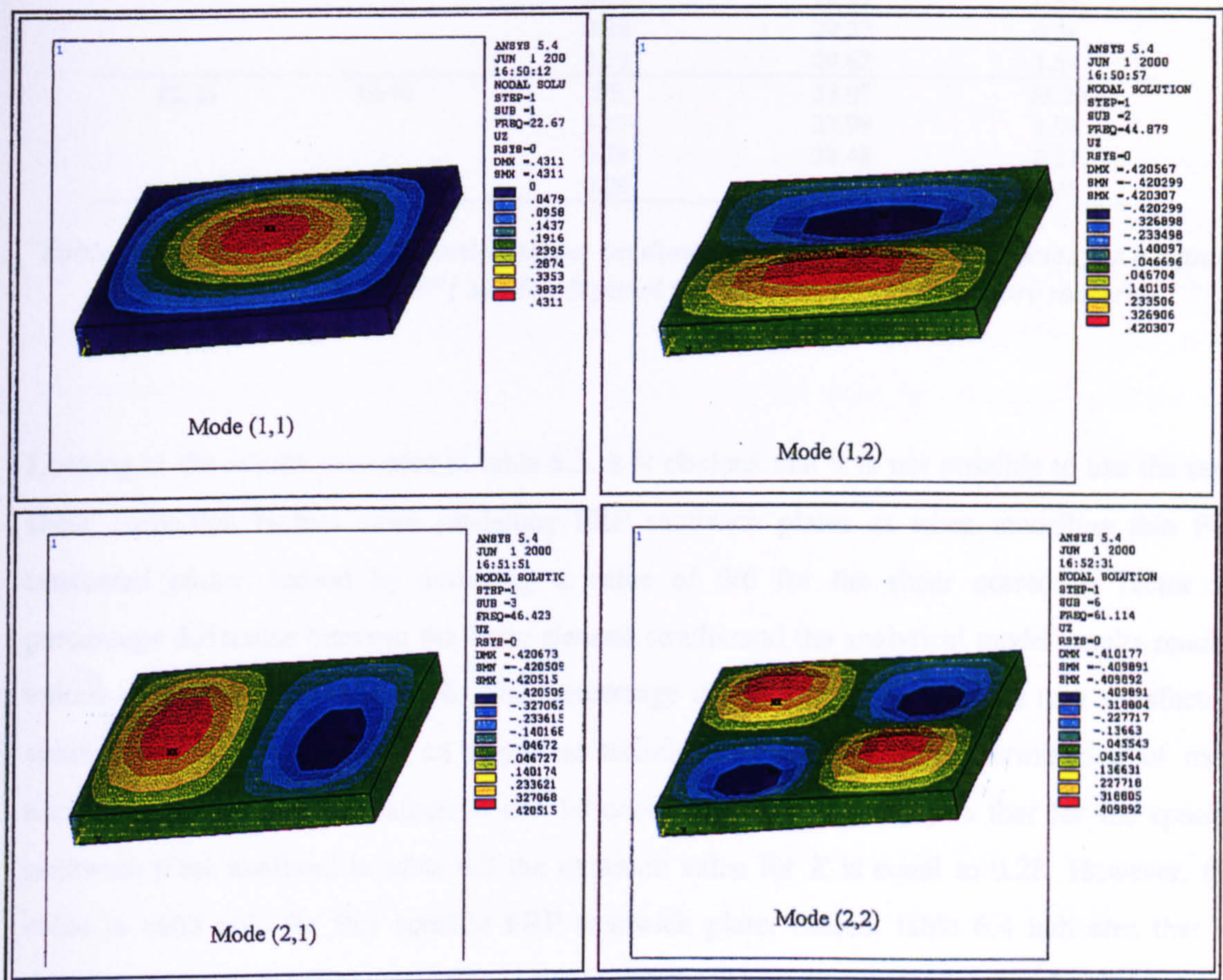


Figure 6.4: Displacement in the z direction for the first four normal modes of the studied square sandwich plate.

From this figure 6.4 the following remarks can be made. It can be noticed that the first mode of vibration also called mode (1, 1) is the first symmetric mode. It is characterised by half a sine wave in the x direction and half a sine wave in the y direction. The mode (1, 2) and (2, 1)

respectively are the first and the second symmetric-antisymmetric modes that have half a sine wave in the x direction and y direction respectively and two half sine waves in the y direction and x direction respectively. Finally, the mode (2, 2) is the first antisymmetric-antisymmetric mode with two half waves in each direction.

Mode (m, n)	$\bar{\omega}$ (ANSYS)	Shear correction factors (K)	$\bar{\omega}$ (FSDT)	Difference (%)
(1, 1)	14.24	5/6	16.70	17.27
		0.27	14.22	0.14
		0.28	14.32	0.56
		0.29	14.42	1.26
(1, 2)	28.20	5/6	36.45	29.26
		0.27	27.78	1.49
		0.28	28.10	0.35
		0.29	28.40	0.7
(2, 1)	29.17	5/6	39.33	34.83
		0.27	28.92	0.85
		0.28	29.27	0.34
		0.29	29.62	1.54
(2, 2)	38.40	5/6	53.07	38.20
		0.27	37.98	1.09
		0.28	38.48	0.21
		0.29	38.97	1.48

Table 6.3: Effect of shear correction factor on dimensionless natural frequencies for a square $[0^\circ/90^\circ/0^\circ/\text{core}/0^\circ/90^\circ/0^\circ]$ sandwich panel with a HEREX C70.130 core material.

Looking at the results presented in table 6.3, it is obvious that it is not possible to use the same shear correction factors when modelling FRP sandwich plates as when modelling thin FRP laminated plates. Indeed by assuming a value of 5/6 for the shear correction factor the percentage difference between the finite element results and the analytical model results reaches values of between 15% and 40%. This percentage difference can be reduced to a satisfactory value of around 0.5% using an empirical technique that allows the determination of more accurate shear coefficient values. It can be concluded from this analysis that for the specific sandwich plate analysed in table 6.3 the optimum value for K is equal to 0.28. However, this value is valid only for this specific FRP sandwich plate. Indeed, table 6.4 indicates that by changing the core material properties to the HEREX C70.40 the optimum value for the factor K is changed to 0.09.

Mode (m, n)	$\bar{\omega}$ (ANSYS)	Shear correction factors (K)	$\bar{\omega}$ (FSDT)	Difference (%)
(1, 1)	12.47	5/6	20.55	64.79
		0.08	12.01	3.68
		0.09	12.53	0.48
		0.1	12.99	4.17
(1, 2)	21.89	5/6	44.23	102
		0.08	20.67	5.57
		0.09	21.74	0.68
		0.1	22.74	3.88
(2, 1)	22.13	5/6	47.63	115
		0.08	21.00	5.1
		0.09	22.12	0.05
		0.1	23.16	4.65
(2, 2)	29.50	5/6	63.87	116
		0.08	26.89	8.85
		0.09	28.37	3.8
		0.1	29.74	0.81

Table 6.4: Effect of shear correction factor on dimensionless natural frequencies for a square [0°/90°/0°/core/0°/90°/0°] sandwich panel with a HEREX C70.40 core material.

First of all it has been demonstrated that the commonly used shear correction factors cannot be used when FRP sandwich panels are modelled. Secondly it has been demonstrated that a completely different value for the shear correction factor needs to be used when the material properties of the core are modified even though the geometric characteristics of the studied plates are the same. The shear correction factor value is therefore strongly dependent upon the core material properties and therefore needs to be determined for each specific study. This makes the FSDT inconvenient to use to model FRP sandwich plates.

6.4.4 VALIDATION OF THE HSDT BASED ANALYTICAL MODEL

It has been demonstrated that the FSDT, most commonly used to model FRP laminated plates, becomes difficult and cumbersome to use when modelling thicker panels such as FRP sandwich plates. It is therefore proposed in this part to investigate the accuracy of the results obtained for sandwich plates modelled using a higher-order shear deformation theory, namely Reddy’s refined HSDT.

6.4.4.1 Validation of the results using published experimental data

The analytical model developed using Reddy’s refined HSDT is first of all validated by comparing the results obtained using the proposed analytical model with existing numerical and experimental results ([6.9], [6.10], [6.11]). To the knowledge of the author, the only experimental data available were obtained by Raville and Ueng for a

three layered plate with isotropic faces and orthotropic core [6.10]. The plate is assumed to be simply supported on all edges (SSSS) and to have the following geometrical dimensions and material properties: $a = 1.828$ m, $b = 1.219$ m, $h_f = 0.4064$ mm, $h_c = 6.35$ mm, $E_f = 70.23$ GPa, $\nu_f = 0.33$, $G_{c_{13}} = 137$ MPa, $G_{c_{23}} = 52.7$ MPa, $\rho_f = 2820$ kg/m³, $\rho_c = 124.1$ kg/m³. The flexural natural frequencies of this sandwich plate can be easily calculated by using the proposed analytical model. In this instance, the studied plate can be compared to a symmetric cross-ply sandwich plate ($[0^\circ/\text{core}/0^\circ]$) where the single layer faces are made of an isotropic material (specific case for orthotropic material) and the bending effects of the orthotropic core are neglected ($E_c = 0$).

Mode (m, n)	f (Present analytical model)	f (Experimental [6.10])	f (FEM [6.11])	f (FEM [6.9])
(1, 1)	23		23	23
(2, 1)	45	45	44	45
(1, 2)	71	69	69	72
(3, 1)	80	78	78	82
(2, 2)	93	92	90	96
(3, 2)	128	129	123	134
(4, 1)	129	133	126	134
(1, 3)	151	152	143	152
(2, 3)	172	169	162	157

Table 6.5: Frequencies f (Hz) for SSSS sandwich plate.

In table 6.5, the natural frequencies obtained using the presented analytical model based on Reddy's refined HSDT have been compared with published experimental results as well as published finite element results. From table 6.5, it can be noticed that the data obtained using the present analysis compares well with the experimental results. Indeed the percentage difference between the computed results and the experimental results do not exceed 3% and do not increase with an increase in mode number. By looking now at the results obtained by Alam and Asnani [6.11], it can be noticed that as the mode number increases the accuracy decreases. Indeed, for the last four modes, the percentage difference between Alam and Asnani results and the experimental results exceeds 4% whereas it stays lower than 3% for the 5 first modes considered. In the same way, the percentage difference between the results obtained by Chate *et al.* [6.9] and the experimental results quite often exceeds 3% and reaches 7% for the last mode considered. So, it can be concluded, that the present analytical model in comparison with

previously published finite element methods leads to a slight increase in the accuracy of the predicted flexural natural frequencies, especially for higher mode numbers. It is believed that this can be explained by the use of a higher order shear deformation theory, which takes into account the parabolic distribution of the strain through the thickness. Indeed, the previously developed finite element methods assumed a linear distribution of the shear stress through the thickness of the plate.

6.4.4.2 Validation of the results obtained for FRP sandwich panels

It has been demonstrated that the present analytical model provides accurate results for the dynamic study of sandwich plates made of isotropic faces and orthotropic core. In this part, the specific case of FRP sandwich plates is considered. The possibility of using HSDT to model FRP sandwich plates and to predict their dynamic behaviour is investigated. The analysis is realised for simply supported FRP sandwich plates with the faces made of Eglass/polyester and the core of HEREX C70.130 closed cell foam, whose material properties are provided in table 6.2 of part 6.4.3. In order to determine the accuracy of the present analytical method, when it is used to model FRP sandwich plates, the results obtained from the free vibration analysis are compared with the data acquired using a finite element model. The results from the free vibration analysis are presented in table 6.6 for a symmetric cross-ply sandwich plate and in table 6.7 for an antisymmetric angle-ply. The results are in the form of dimensionless natural frequencies, $\bar{\omega} = \omega a / (100h)$, obtained from the finite element model as compared to the analytical model.

Mode (m, n)	$\bar{\omega}$ (Finite element model)	$\bar{\omega}$ (Analytical model)	% Difference
(1, 1)	14.24	14.32	0.56%
(1, 2)	28.20	28.15	0.18%
(2, 1)	29.17	29.31	0.48%
(2, 2)	38.40	38.55	0.40%
(1, 3)	45.28	45.93	1.43%
(3, 1)	46.33	47.30	2.09%
(2, 3)	52.33	51.88	0.86%
(3, 2)	52.77	52.54	0.44%

Table 6.6: Dimensionless natural frequencies for a square sandwich plate $[0^\circ/90^\circ/0^\circ/\text{core}/0^\circ/90^\circ/0^\circ]$ with $a/h = 10$ and $h_c/h = 0.94$.

Mode (m, n)	$\bar{\omega}$ (Finite element model)	$\bar{\omega}$ (Analytical model)	% Difference
(1, 1)	15.39	15.50	0.71%
(1, 2)	28.97	29.10	0.48%
(2, 1)	28.97	29.10	0.48%
(2, 2)	39.62	40.04	1.06%
(1, 3)	45.45	44.77	1.50%
(3, 1)	45.48	44.77	1.56%
(2, 3)	53.31	53.19	0.22%
(3, 2)	53.31	53.19	0.22%

Table 6.7: Dimensionless natural frequencies for a square sandwich plate [-45°/45°/-45°/core/45°/-45°/45°] with $a/h = 10$ and $h_c/h = 0.94$.

It is noted that the agreement between the finite element results and the analytical results are quite good. Indeed, the percentage difference between the results obtained with the analytical model and the results obtained with the finite element method is less than 2.1% for the symmetric cross-ply laminates and is less than 1.6% for the antisymmetric angle-ply laminates. Therefore, the percentage difference is small enough to conclude that the analytical method based on Reddy's refined HSDT can be used with accuracy to model FRP sandwich plates and to study their dynamic behaviour.

6.4.5 INFLUENCE OF THE MATERIAL AND GEOMETRIC PROPERTIES ON THE NATURAL FREQUENCIES

In order to study the influence of the plate dimensions on the natural frequencies, rectangular sandwich plates composed of glass/polyester faces and HEREX C70.130 foam core are considered. Two different types of stacking sequences are adopted for the faces of the sandwich plates, namely [0°/90°/0°] laid symmetrically about the midplane of the core and [45°/-45°/45°] laid antisymmetrically about the midplane of the core. They are respectively designated by S1 and S2. The ratio of the thickness of the core to the total thickness is assumed to be equal to 0.88. The effect of the aspect ratios a/h and a/b on the natural frequencies is examined first and the results obtained are shown in figure 6.5 and figure 6.6.

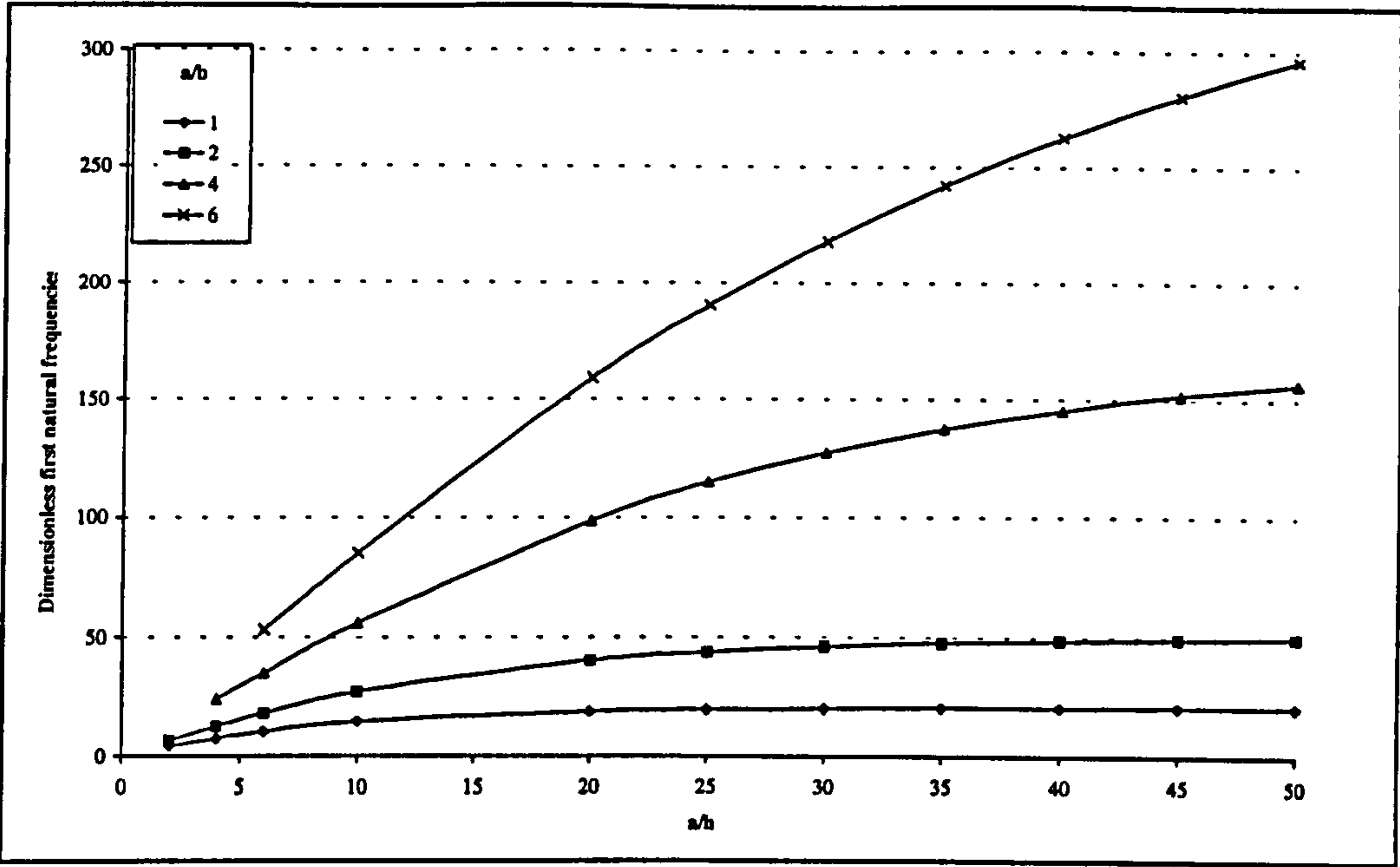


Figure 6.5: Effect of the plate aspect ratio, a/b , and the length to thickness ratio, a/h , on the dimensionless first natural frequencies of sandwich plates ($[0^\circ/90^\circ/0^\circ/\text{core}/0^\circ/90^\circ/0^\circ]$).

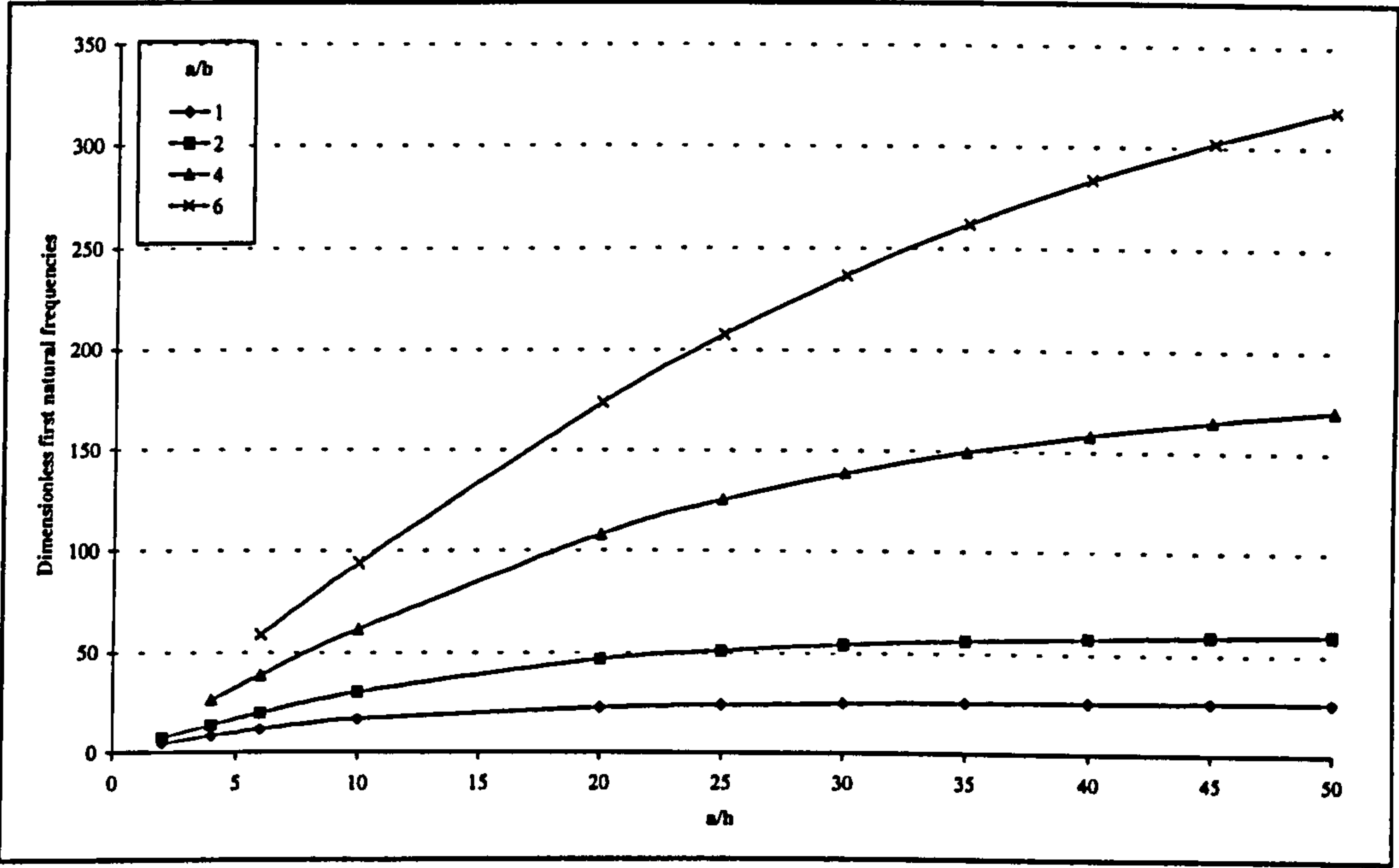


Figure 6.6: Effect of the plate aspect ratio, a/b , and the length to thickness ratio, a/h , on the dimensionless first natural frequencies of sandwich plates ($[45^\circ/-45^\circ/45^\circ/\text{core}/-45^\circ/45^\circ/-45^\circ]$).

By looking at the results presented in figures 6.5 and 6.6, it can be noticed that dimensionless first natural frequencies of both plate S1 and plate S2 are similarly affected by changes of the aspect ratios a/h and a/b . Indeed, the dimensionless first natural frequencies of both plates

increase with increasing a/h and with increasing a/b . Two different phenomena can be highlighted. First of all, the increase of the dimensionless first natural frequencies with increasing a/h is due to the reduction of the effects of shear deformation and rotary inertia. Indeed, these effects become less important when the thickness of the panels decreases which corresponds to an increase of the ratio a/h . Secondly, the increase of the dimensionless first natural frequencies with increasing a/b is related to the fact that panels become stiffer as their width is reduced. It should be noticed at this point that an increase in the ratio a/b for a constant ratio a/h leads to a decrease of the ratio b/h . A decrease of the ratio b/h implies that the effects of shear and rotary inertia are increased. However these increases, which should lead to a reduction in the natural frequencies, have a negligible effect as compared to the increase in stiffness of the panel due to reduction of the a/b ratio. Besides, it should be pointed out that as the length to width ratio increases, the effect of an increase of the ratio a/h on the fundamental frequencies is more significant. Finally, by comparing figure 6.5 and figure 6.6 it appears that for the same geometric characteristics, the dimensionless first natural frequency of the plate S2 is larger than that for the plate S1. This is due to the inherent higher through width stiffnesses of the angle-ply laminated faces as compared to the cross-ply laminated faces.

The effect of the relative thickness of the core, represented by the ratio h_c/h , on the first natural frequencies is investigated next. The results are presented in figures 6.7 and 6.8.

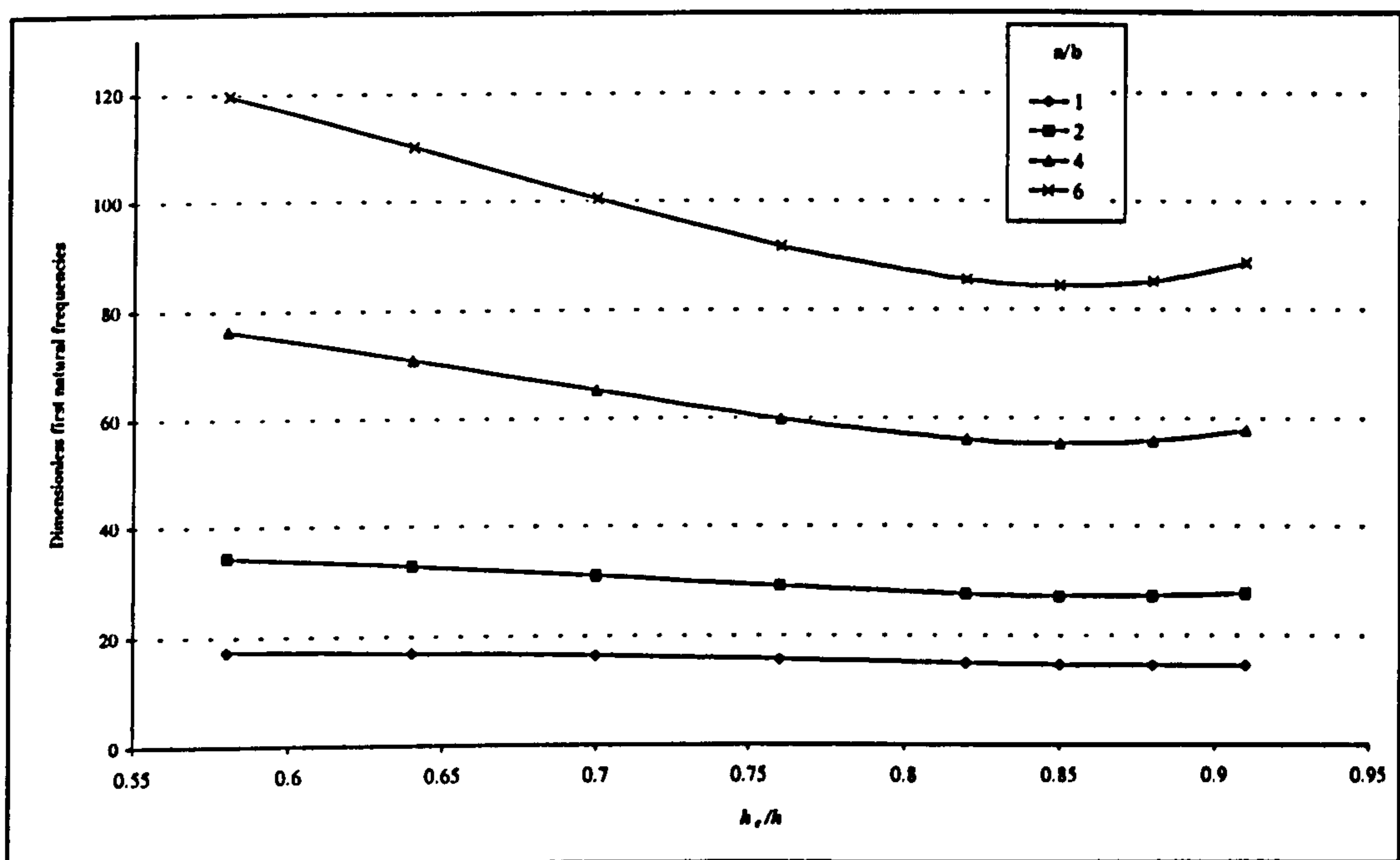


Figure 6.7: Effect of the plate thickness ratio, a/b , and core relative thickness ratio, h_c/h on the first dimensionless natural frequencies of sandwich plates ($[0^\circ/90^\circ/0^\circ/\text{core}/0^\circ/90^\circ/0^\circ]$).

Figure 6.7 shows that for a square plate as the relative thickness core ratio increases the first natural frequency decreases. However, for higher values of the length to width ratio, the first natural frequency reaches a minimum for a relative thickness core value of around 0.85. To understand the general trend of these curves, it is necessary to look at the effect of the stiffness on the behaviour of sandwich plates. It is known that an increase in the stiffness of a sandwich plate leads to an increase in its first natural frequency. A stiffness increase can be due to an increase in the length to width ratio or to a decrease in the relative thickness of the core for a constant plate thickness. When, as in figure 6.7, there is the combination of an increase in the length to width ratio and an increase in the relative thickness of the core, the interaction produces curves with distinct sections. For small values of a/b the effect of the decrease of the relative thickness of the core appears to be always dominant. This is why the first natural frequency keeps decreasing as h_c/h ratio increases. However for larger values of a/b ratio, three different stages can be identified. When h_c/h ratio is smaller than 0.85-0.88, the effect due to the relative thickness is the principal one whereas when h_c/h is greater than 0.85-0.88 it is the effect due of the length to width ratio which is greater. For the value of h_c/h which corresponds to the minimum natural frequency, these two parameters have an equal effect. In this case, the minimum overall sandwich plate stiffness is obtained, which corresponds to the minimum first natural frequency.

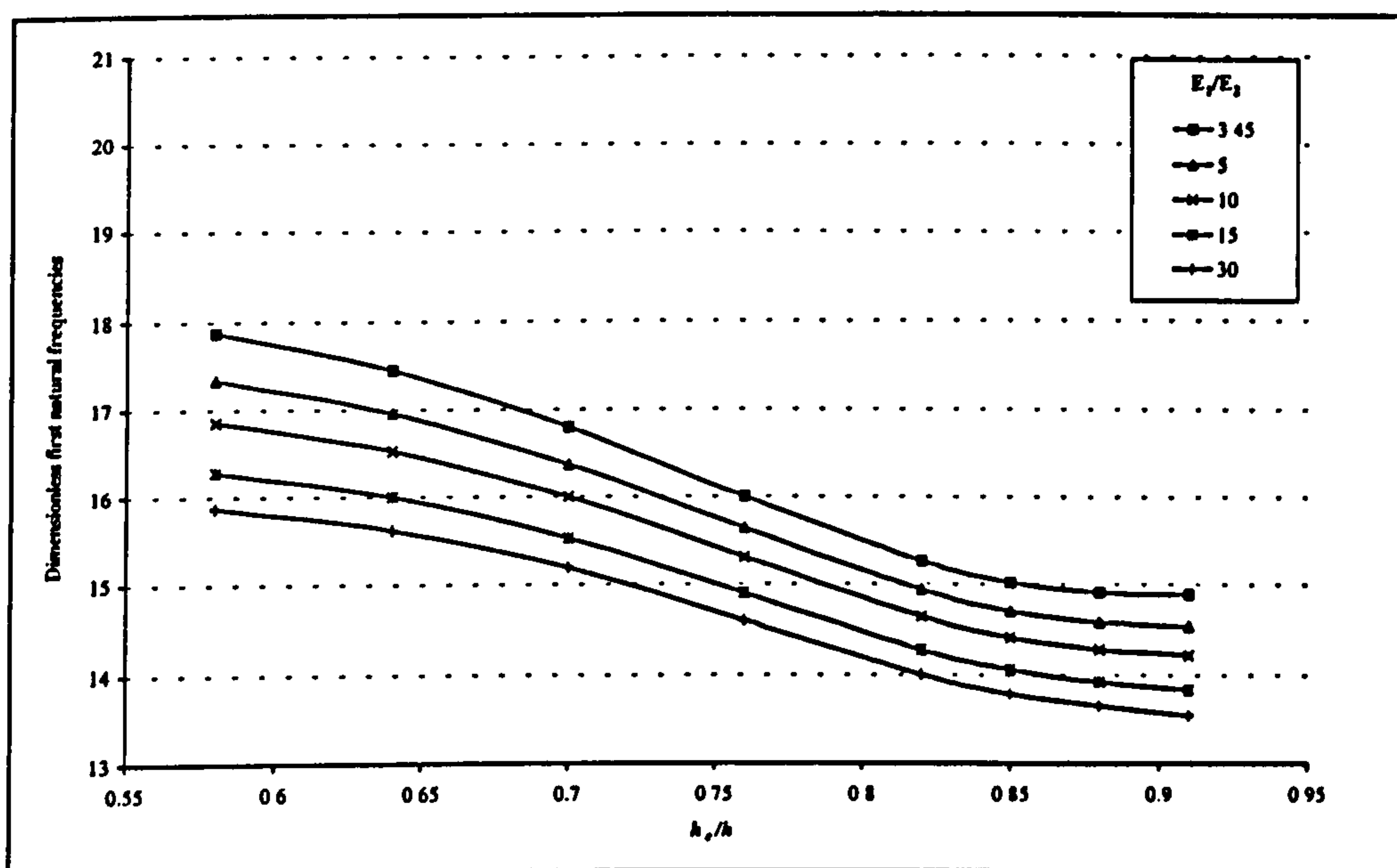


Figure 6.8: Effect of the degree of orthotropy of the skins, E_1/E_2 , and of the core relative thickness ratio h_c/h on the first dimensionless natural frequencies of square sandwich plates ($[0^\circ/90^\circ/0^\circ/\text{core}/0^\circ/90^\circ/0^\circ]$).

By looking at figure 6.8, it can be noticed that a decrease in the degree of orthotropy of the skins leads to an increase in the first natural frequency of the studied square sandwich plates. Indeed, as mentioned by Jones [6.12] a decrease in the orthotropy of the cross-ply laminated faces leads to an increase in stiffness.

6.4.6 SUMMARY

The analytical models based on FSDT and HSDT previously developed for thin laminated plates have been enhanced in order to be used to analyse the dynamic behaviour of FRP sandwich panels. The accuracy of the results obtained using FSDT is strongly dependent on the shear correction factor's value. It has been found that this factor is different for each specific sandwich panel and so standard values cannot be defined. This makes the use of analytical model based on FSDT not practical for modelling FRP sandwich panels. On the other hand the analytical model based on Reddy's refined HSDT does not require the use of a shear correction factor. It has been found to accurately predict the natural frequencies of sandwich panels. This method has been used to analyse the influence of the material properties and geometric properties on the first natural frequencies of specific simply supported FRP sandwich panels. From the results observed it can be concluded that the first natural frequency of simply supported composite sandwich plates made with PVC foam core is influenced by:

- The effect of shear deformation and rotary inertia, that is to say, the aspect ratios, a/h and b/h ;
- The general stiffness of the plate, which is a function of the relative thickness ratio of the core, defined by h_c/h , the length to width ratio of the plate, defined by a/b and by the degree of orthotropy of the faces.

6.5 TRANSIENT ANALYSIS

In this section, transient solutions to FRP laminated and FRP sandwich panels are developed. To achieve this, it is proposed to model the composite panels using Reddy's refined HSDT as this model has been shown to provide an accurate prediction of the natural frequencies of even relatively thick composite plates such as FRP sandwich panels. The first main aim of this part is to investigate the accuracy of the proposed method when used to predict the transient response of FRP laminated and FRP sandwich panels.

6.5.1 THEORY

The solution process, which leads to the determination of the transient response of FRP laminated or FRP sandwich plates, is divided in two major steps.

- First of all, assuming a spatial variation of the displacements, the governing partial equations can be reduced to a set of ordinary differential equations in time. This first step of the analysis has already been presented in section 6.2.5 assuming a Navier solution.
- Secondly, the derivation of the ordinary differential equations needs to be carried out using either exact or numerical methods. In the present case it has been decided to carry out a numerical analysis as explained in more detail in the following sections.

6.5.1.1 Representation of the applied forces

The studied panels are assumed to be subjected to transverse forces. That is to say only the third component of the column vector of the applied forces $\{F\}$ is not equal to zero.

As an example a uniformly distributed transverse load applied to a rectangular plate is represented in figure 6.9.

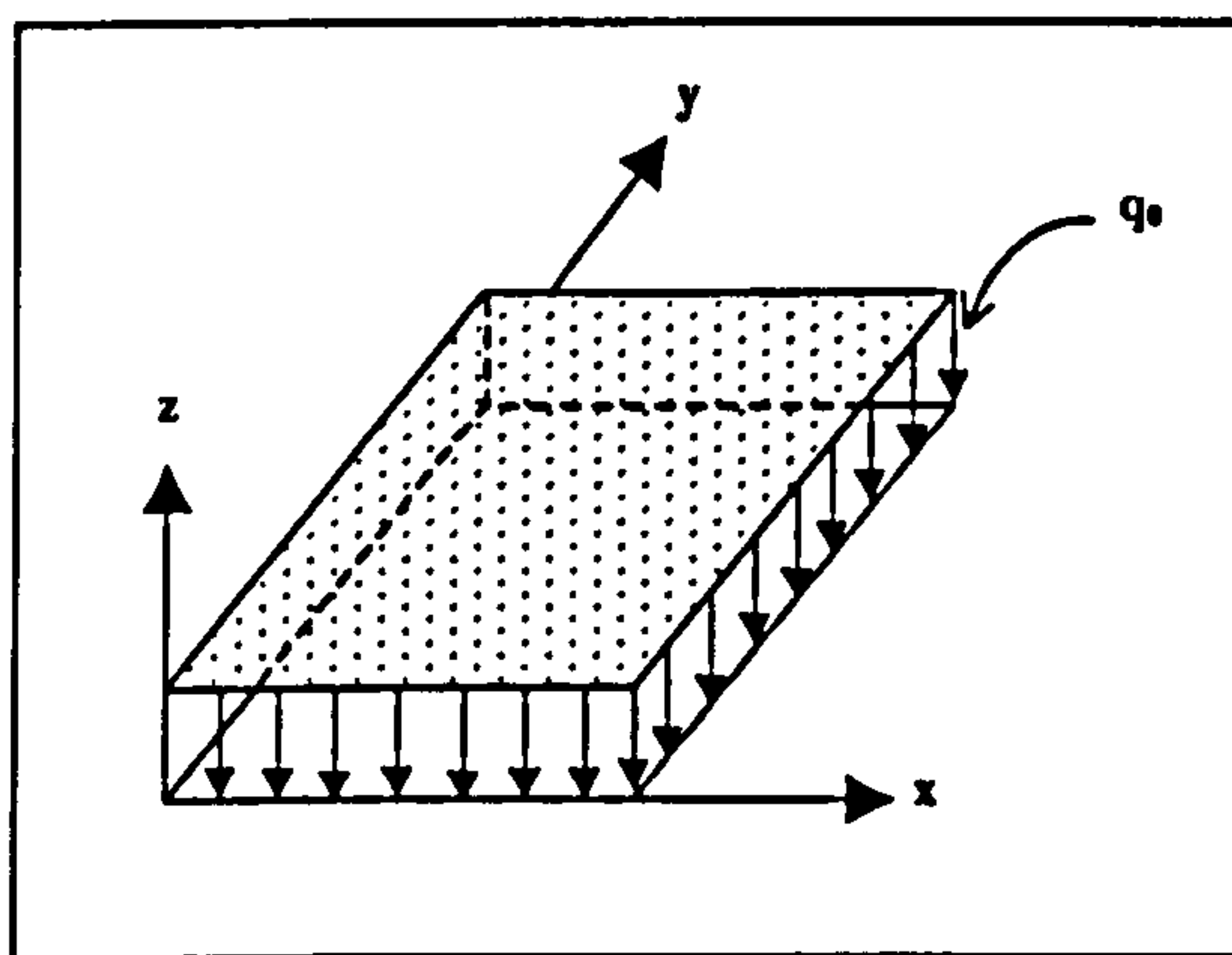


Figure 6.9: Uniformly distributed transverse load.

The ordinary equations of motion as explained earlier in this chapter are derived assuming a Navier solution type, that is to say the generalised displacements are expanded in a double trigonometric series. These series are defined depending on the boundary conditions of the studied panel. As explained in part 6.2.3.1 only the SSSS-1 type of boundary condition or SSSS-2 type of boundary condition can be analysed. On one hand if the set SSSS-1 is considered, the boundary conditions are satisfied by assuming the form of displacements as shown by relation 6.8.

$$\begin{aligned}
u_0(x, y, t) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} U_{mn}(t) \cos(\alpha x) \sin(\beta y) \\
v_0(x, y, t) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} V_{mn}(t) \sin(\alpha x) \cos(\beta y) \\
w_0(x, y, t) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} W_{mn}(t) \sin(\alpha x) \sin(\beta y) \\
\phi_x(x, y, t) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} X_{mn}(t) \cos(\alpha x) \sin(\beta y) \\
\phi_y(x, y, t) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Y_{mn}(t) \sin(\alpha x) \cos(\beta y)
\end{aligned} \tag{6.8}$$

On the other hand if the set SSSS-2 is applied, the boundary conditions are then satisfied by assuming the form of displacements as shown by relation 6.9.

$$\begin{aligned}
u_0(x, y, t) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} U_{mn}(t) \sin(\alpha x) \cos(\beta y) \\
v_0(x, y, t) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} V_{mn}(t) \cos(\alpha x) \sin(\beta y) \\
w_0(x, y, t) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} W_{mn}(t) \sin(\alpha x) \sin(\beta y) \\
\phi_x(x, y, t) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} X_{mn}(t) \cos(\alpha x) \sin(\beta y) \\
\phi_y(x, y, t) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Y_{mn}(t) \sin(\alpha x) \cos(\beta y)
\end{aligned} \tag{6.9}$$

Consequently, to be able to derive the solution the applied load must as well be expanded in the series form as expressed by equation 6.10.

$$q(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Q_{mn}(t) \sin(\alpha x) \sin(\beta y) \tag{6.10}$$

This implies that the column vector of the applied forces in the ordinary equations of motion 6.1 can be expressed as shown by relation 6.11.

$$\{F\} = \{0, 0, Q_{mn}(t), 0, 0\} \tag{6.11}$$

In this relation 6.11, $Q_{mn}(t)$ is the transverse load coefficient. This load coefficient will be different for various types of loading. The types of transverse loads which can be considered and their associated transverse load coefficient expressions are summarised in table 6.8.

Load $q(x, y)$	Coefficients Q_{mn}
Sinusoidal load: $q(x, y) = q_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$	$Q_{mn}(t) = q_0$ with $m=1$ and $n=1$.
Uniform load: $q(x, y) = -q_0$	$Q_{mn}(t) = -\frac{16q_0}{\pi^2 mn}$ with m and n odd numbers.
Hydrostatic load: $q(x, y) = -q_0 \frac{y}{b}$	$Q_{mn}(t) = -\frac{8q_0 \cos n\pi}{\pi^2 mn}$ with m and n odd numbers.
Point load: $q(x, y) = -q_0$ at (x_0, y_0)	$Q_{mn}(t) = -\frac{8q_0 \cos n\pi}{\pi^2 mn}$ with m and n odd numbers.
Line load: $q(x, y) = -q_0$ at $y = y_0$	$Q_{mn}(t) = -\frac{4q_0}{\pi b m} \sin \frac{n\pi y_0}{b}$ with m and n odd numbers.

Table 6.8: Coefficients in the double trigonometric series expansion of loads in the Navier method.

6.5.1.2 Derivation of the solution

Having defined the applied load, the set of equations 6.1, for any mode number (i.e., any m and n), can be solved exactly using Newmark's integration schemes [6.13]. In this numerical integration method, the time derivatives are approximated using difference approximations, and therefore solution is obtained only for discrete times and not as a continuous function of time. Different Newmark integration schemes exist [6.13]. They are a function of the parameters γ and β which indicate how much the acceleration affects the velocity and displacement equations at the end of the interval Δt . For the present analysis, it has been decided to use the constant acceleration method (i.e., $\gamma = 1/2$ and $\beta = 1/4$) since this particular scheme has numerically stable properties. The principle of the Newmark's direct integration method is reviewed for background information in appendix I. It should be noticed that in appendix I the method has been presented for the general linear case. Since in the present analysis the damping is

neglected, the damping matrix needs to be set to zero in order to be able to apply the relations presented in the appendix.

When the equations of motion 6.1 are solved numerically using Newmark's direct integration technique, the coefficients of the solution vector $\{\Delta\}$, that is to say the coefficients $U_{mn}(t)$, $V_{mn}(t)$, $W_{mn}(t)$, $X_{mn}(t)$ and $Y_{mn}(t)$ are determined for each time increment at a given mode number (i.e., given values for m and n). Since Navier solutions have been used to derive the equations of motion, the transverse deflection at a given point of the plate and at a given time t can be derived using equation 6.12. This last derivation can be realised assuming that the coefficients $W_{mn}(t)$ have been previously determined for each mode number.

$$w(x, y, z, t) = w_0(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn}(t) \sin(\alpha x) \sin(\beta y) \quad (6.12)$$

For most of the applied transverse loads the determination of the response required the derivation of the vector of the generalised displacements for a wide range of mode number. The solution is complete when the result of the summation calculation is no longer altered by an increase in mode number (i.e., an increase in m and/or n values) that is to say when the convergence is reached. The time required to reach convergence is therefore an important parameter that needs to be taken into account. It should be emphasised that, when a sinusoidal transverse load is applied, the problem of computing time is less critical since the calculation needs to be carried out only for one mode, the first mode (i.e., $m=n=1$). As well as being dependent on the number of modes required to reach convergence, the computing time is affected by the time step increment value. If the time step value is extremely small when compared to the period of time under which the analysis needs to be carried out the computing time is likely to become a critical parameter. The main objective is consequently to develop a program that can give an accurate prediction of the response of the structure in a relatively short time. The selection of the time step parameter is guided by the estimate of the time step for conditionally stable time integration schemes. The critical time step can be expressed in terms of the period of vibration, $T = 2\pi/\omega$, where ω is the natural frequency of the system. For good accuracy Petyt [6.13] recommends the use of $\Delta t = T/10$. For each specific analysis, which will be carried out, the optimum time step will be obtained by

using Petyt's criteria combined with a convergence study of the results predicted using different time steps.

Finally, it should be noticed that when using the same resolution method as the one presented here not only the transverse deflection but also all the generalised displacements $u_0(x, y, t)$, $v_0(x, y, t)$, $w_0(x, y, t)$, $\phi_x(x, y, t)$ and $\phi_y(x, y, t)$ can be determined at any time t for any position (x, y) in the centre of the plate. Then, using the displacement field associated to the HSDT (see table 3.1), these results can be used to calculate the displacement components $u(x, y, z, t)$, $v(x, y, z, t)$ and $w(x, y, z, t)$ as well as the stress components at any point in the plate.

6.5.1.3 Conclusion

The theory related to the transient analysis of laminated and FRP composite panels has been presented in this part. It has been shown that the responses of a plate subjected to a transverse applied load can be determined under a given period of time by solving the equations of motion for each mode number using the Newmark's integration method and then by deriving the responses using the double trigonometric displacement forms and the associated displacement field. A program has been implemented using the software MATLAB. It allows the transient response analysis of both simply supported FRP laminated and FRP sandwich panels subjected to transverse loading.

6.5.2 VALIDATION OF THE METHOD

6.5.2.1 FRP laminated plates

The program, which has been developed using MATLAB, has first of all been validated by comparing the data obtained with published results. Reddy [6.15] investigated the transient response of a two layer square cross-ply laminate ($[0^\circ/90^\circ]$), with a length $a=25\text{cm}$ and a plate thickness $h=5\text{cm}$, modelled using FSDT under suddenly applied transverse sinusoidal load. The properties of the constitutive material of the studied laminated plate can be found in table 6.9.

E_1 (N/cm ²)	E_2 (N/cm ²)	$G_{12}=G_{23}=G_{31}$ (N/cm ²)	ρ (Nsec ² /cm ⁴)	ν_{12}
52.5×10^6	2.1×10^6	1.05×10^6	8×10^{-6}	0.25

Table 6.9: Material properties [6.15].

The program that has been developed using the software MATLAB offers the possibility to model simply supported laminated plates using either FSDT or HSDT. The results obtained using both of these methods have been compared with Reddy’s published data. As expected, the results obtained by using the developed program based on the FSDT are identical to the results obtained by Reddy. However when the HSDT is selected a difference in the results is observed as can be noticed in table 6.10.

Time (μs)	10	20	40	60	80	100	120	160
$w \times 10^6$ (cm) FSDT	7.6	36.5	147.4	292.5	411.9	460.6	417.2	155.8
$w \times 10^6$ (cm) HSDT	7.6	36.4	146	288	402	442	390	129
Percentage difference	0%	0.3%	0.9%	1.5%	2.4%	4%	6.5%	17.2%

Table 6.10: Comparison of transverse deflection of the centre of a 2 layer square laminated plate obtained using FSDT with those obtained using HSDT.

This difference in the results, which is negligible at the beginning of the analysis, increases with an increase of time. This increase in difference in the results is explained by the fact that any error at a given time t will have an effect on the results at time $t + \Delta t$, that is to say the error has a knock on cumulative effect. It is why, when carrying out a transient analysis, it is essential to have an analytical model which predicts the response as accurately as possible. The observed difference in the results predicted using either FSDT or HSDT is dependent on the geometry of the studied laminate. Indeed the larger the length to thickness ratio is, that is to say the thinner the plate is, the smaller the error is. This is illustrated in figure 6.10 where the dimensionless transverse displacements ($\bar{w} = w \times 10^3 \times h^3 E_2 / (q_0 a^4)$) of the middle of square plates obtained using both FSDT and HSDT are represented for two different length to thickness plate ratios. The plates are made of the same material properties as before (see table 6.9) and are subjected from the first time step to a load, which is sinusoidal in the

spatial domain and constant with time. It can be noticed that after 0.001 seconds the difference in the results obtained using FSDT and HSDT is clearly noticeable for the thickest plate whereas they are still negligible for the thinnest plate.

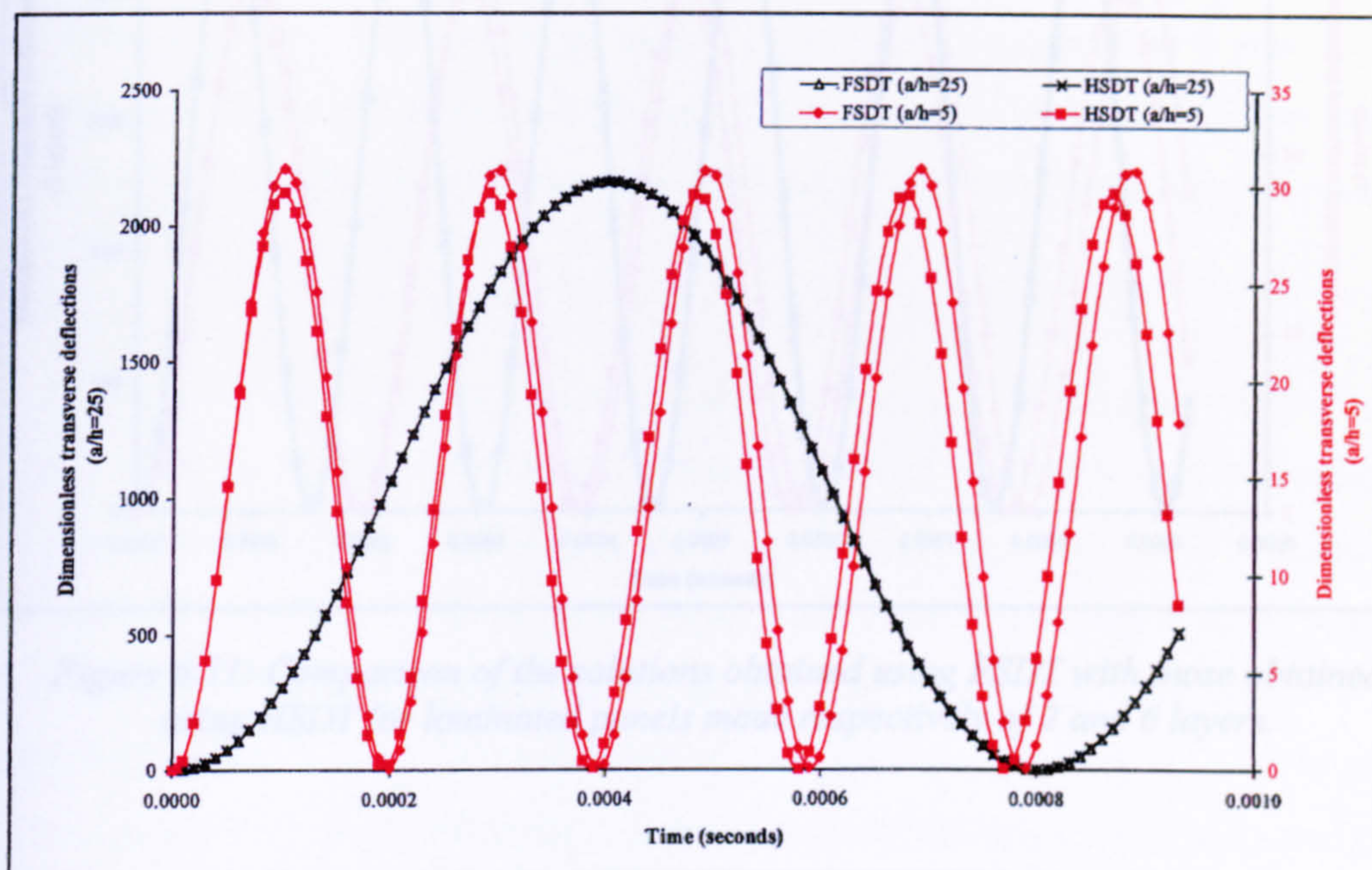


Figure 6.10: Comparison of the solutions obtained using FSDT with those obtained using HSDT for two different length to thickness plate ratios.

The influence of the number of layers at a constant length to thickness ratio, $a/h = 5$, has also been investigated. In figure 6.11 the transient response of cross-ply square laminated panels made of the same material as before and constituted of either 2 or 6 layers have been compared. The difference in the predicted dimensionless transverse displacements ($\bar{w} = w \times 10^3 \times h^3 E_2 / (q_0 a^4)$) of the middle of the laminates, in both cases, increases with an increase in time. However, it is observed that this increase is more pronounced for a 2 layers panel than for a 6 layers one. This is believed to be due to the bending-stretching coupling. Indeed as the number of layers increases the bending-stretching stiffness coefficients E_{ij} for anti-symmetric cross-ply laminates tend to zero. So these stiffness coefficients, which are neglected in the FSDT, affect the results of antisymmetric laminated plate made of a small number of layers.

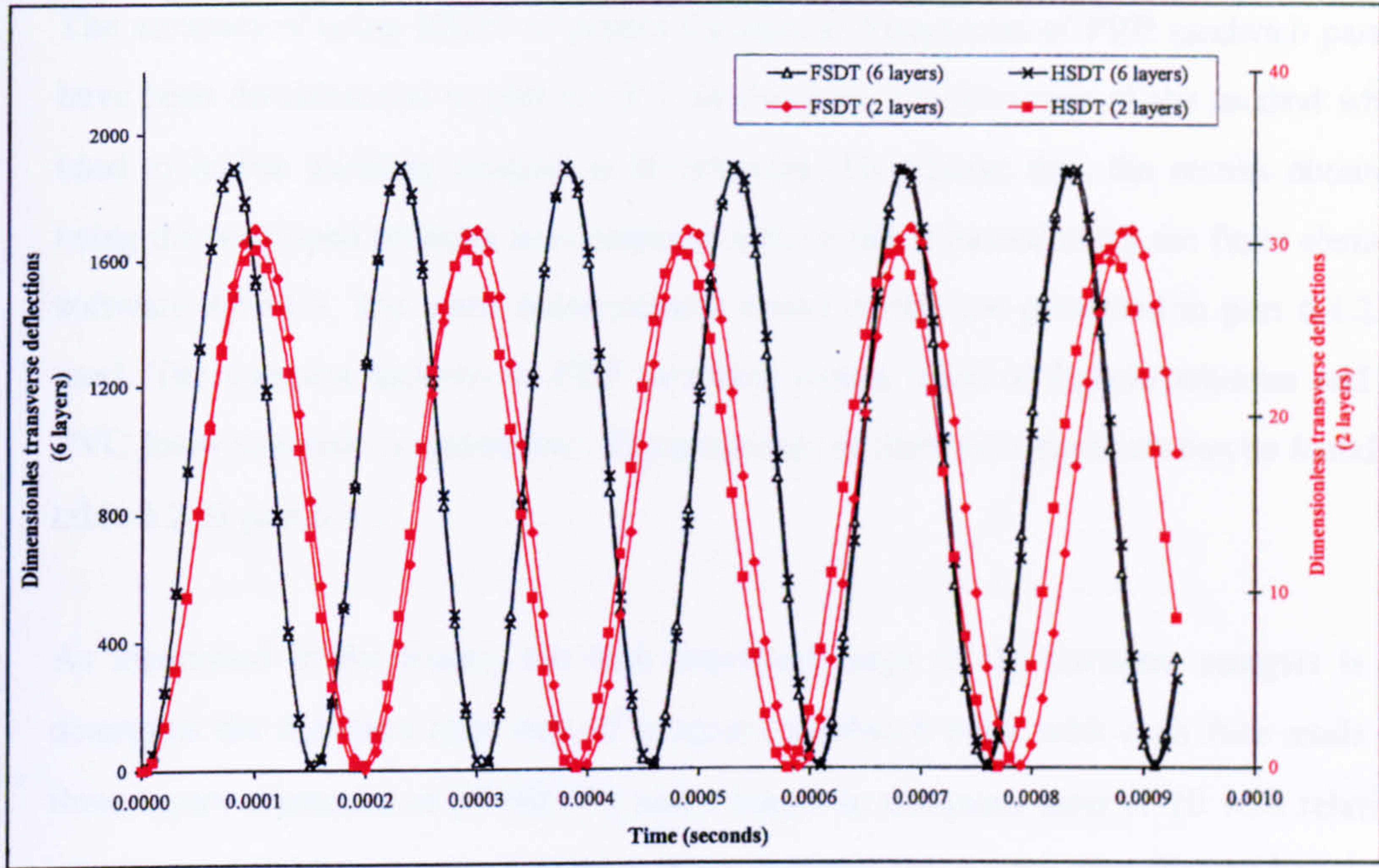


Figure 6.11: Comparison of the solutions obtained using FSDT with those obtained using HSDT for laminated panels made respectively of 2 and 6 layers.

The results presented in figures 6.10 and 6.11 agree with previously carried out static analysis. Indeed, it has been observed [6.3] during static analysis that the reduction in the accuracy of the results obtained using FSDT is much greater than for the results obtained using HSDT when the thickness of the studied plate is increased. Besides, the effect of the stiffness coefficients for antisymmetric cross-ply had been previously observed but considered as insignificant. This slight inaccuracy in the prediction of the response when using FSDT is most of the time considered as being small enough to be negligible when a static analysis is carried out. This assumption is no longer valid when a transient analysis or a long-term behaviour analysis is carried since after a given period of time, which can vary depending on the geometric characteristics of the studied laminates, the error becomes significant.

The analysis carried out in this part leads to the conclusion that the use of the HSDT to model FRP laminated panels leads to a better accuracy in the results especially for a long term analysis.

6.5.2.2 FRP sandwich panels

The accuracy of using HSDT to predict the natural frequencies of FRP sandwich panels have been demonstrated in part 6.4.4.2. In this part, the accuracy of the method when used to realise transient analysis is investigated. To achieve this, the results obtained using the developed program are compared with results obtained using the finite element software ANSYS. The same finite element model as the one presented in part 6.4.2 is used. The transient analysis of FRP sandwich panels made of Eglass/polyester and of PVC foam materials is undertaken. The properties of these two materials can be found in table 6.2 of part 6.4.3.

As mentioned in the theory, the first important stage of the transient analysis is to determine the optimum time step. If a square sandwich plate with each face made of three layers orientated at $[0^\circ/90^\circ/0^\circ]$ and a length to thickness ratio of 10 with relative core to total thickness ratio of 0.94 is being investigated, a time step of $4 \cdot 10^{-3}$ seconds should be used following Petyt's recommendation. Using this time step value, a preliminary analysis has been undertaken in order to determine how many mode numbers are required to reach convergence assuming that the studied plate is subjected to a uniformly distributed load. It has been observed that when m and n arrive at the value of 15 convergence is reached since a further increase in mode number does not affect the results. Knowing the mode number for which convergence is reached, further investigations have been undertaken in order to determine the validity of the time step value derived following Petyt's criteria. This has been achieved by running the transient analysis program, assuming a uniformly distributed transverse load, for different time step values from $4 \cdot 10^{-3}$ seconds up to $1 \cdot 10^{-4}$ seconds. The analysis has led to the conclusion that the accuracy of the results could be improved by using smaller time step values than $4 \cdot 10^{-3}$ seconds. Indeed, it can be noted by looking at figure 6.12, where the dimensionless displacement $\bar{w} = w \times 10^{10} h^3 / a^4$ has been plotted for three different time steps in the time range from 0.15 seconds to 0.19 seconds, that the predicted results using a time step of $4 \cdot 10^{-3}$ seconds are not exactly in phase with the results predicted using smaller time step values such as $5 \cdot 10^{-4}$ or $1 \cdot 10^{-4}$ seconds. The results obtained using either of the last two mentioned time steps are very similar to each other. The increase in the accuracy obtained using a time step of $1 \cdot 10^{-4}$ seconds instead of $5 \cdot 10^{-4}$ seconds does not justify the observed and important increase in the computing time. Therefore it has been decided to use a time step of $5 \cdot 10^{-4}$ seconds for the rest of the study.

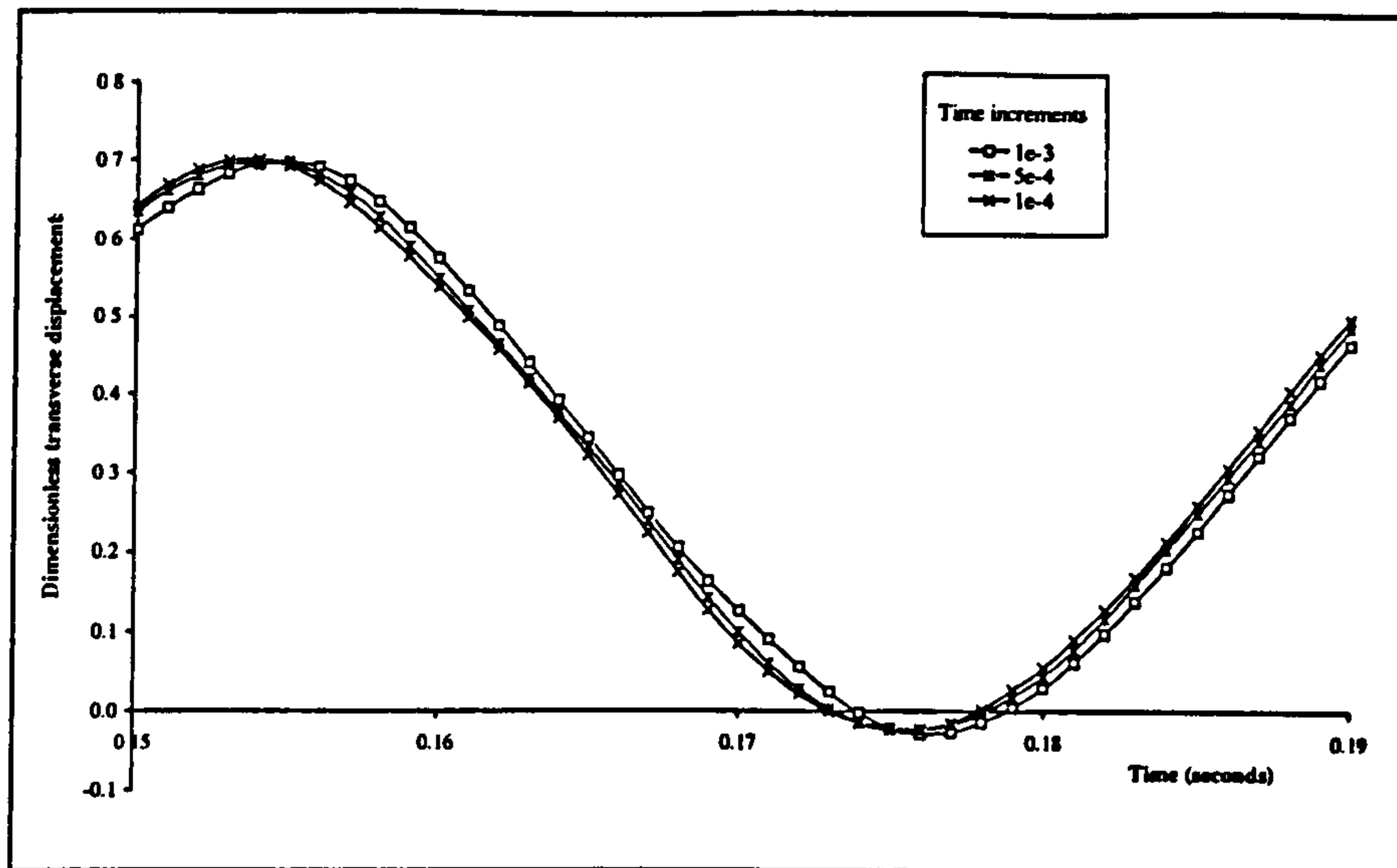


Figure 6.12: Dynamic response of a square sandwich plate $[0^\circ/90^\circ/0^\circ/\text{core}/0^\circ/90^\circ/0^\circ]$ with $a/h = 10$ and $h_c/h = 0.94$ obtained using various time step values.

In the following, the transient analysis of the defined sandwich panels is undertaken for various time histories of uniformly distributed transverse loads in order to validate the theory. In figures 6.13 and 6.14, the dimensionless mid-span plate deflection $\bar{w} = 10^2 w h^3 / a^4$, obtained from the finite element model as compared to the analytical model, has been plotted as a function of time. In both cases a uniformly distributed step load has been applied but for different period of time. In figure 6.13 the load is applied for 0.17 seconds whereas it is applied for 0.2 seconds in figure 6.14. From these two figures, it can be noticed that, as for the free vibration analysis, the response obtained using the two different methods shows a really good agreement between them.

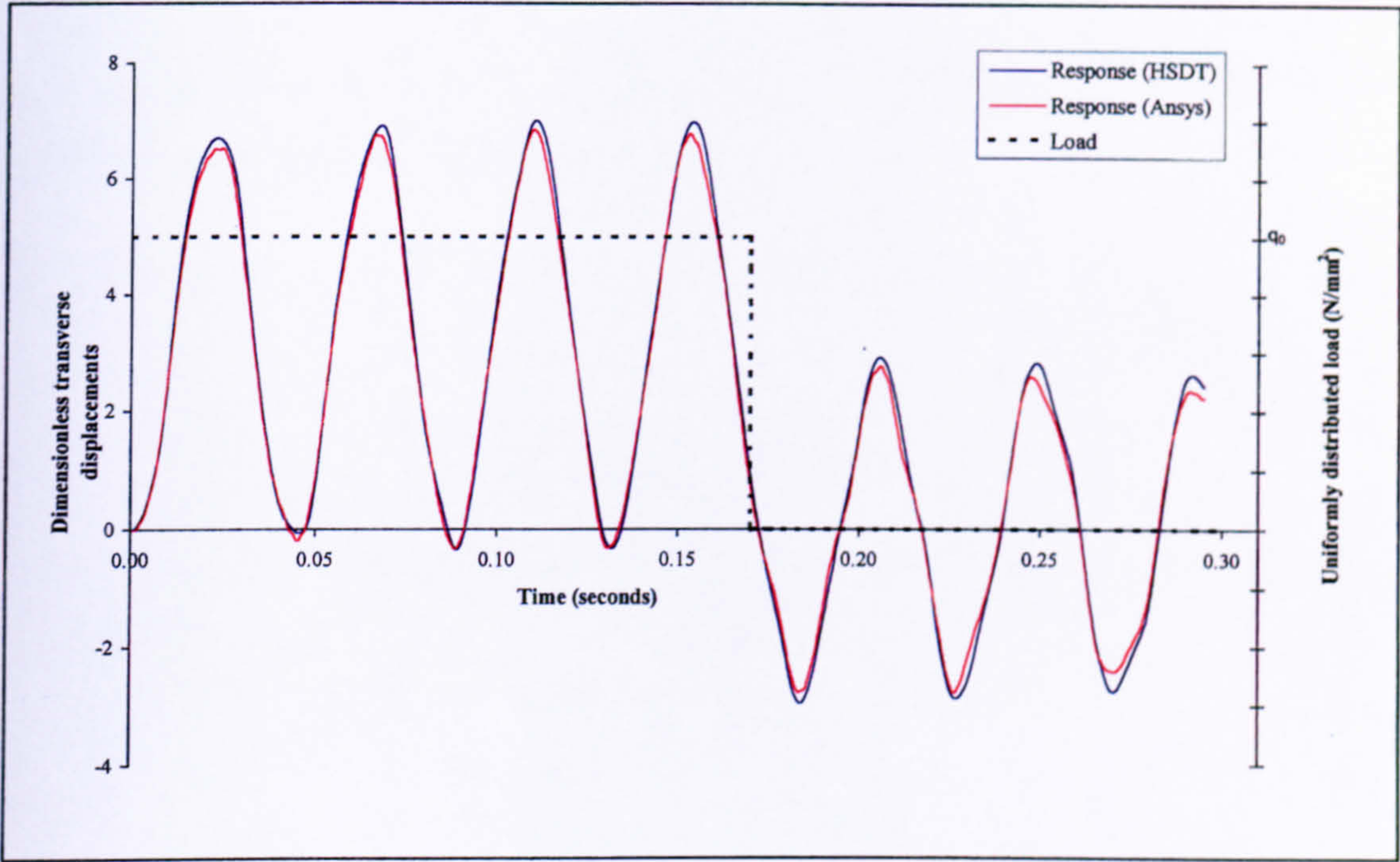


Figure 6.13: Dynamic response of a square sandwich plate $[0^\circ/90^\circ/0^\circ/\text{core}/0^\circ/90^\circ/0^\circ]$ with $a/h = 10$ and $h_c/h = 0.94$.

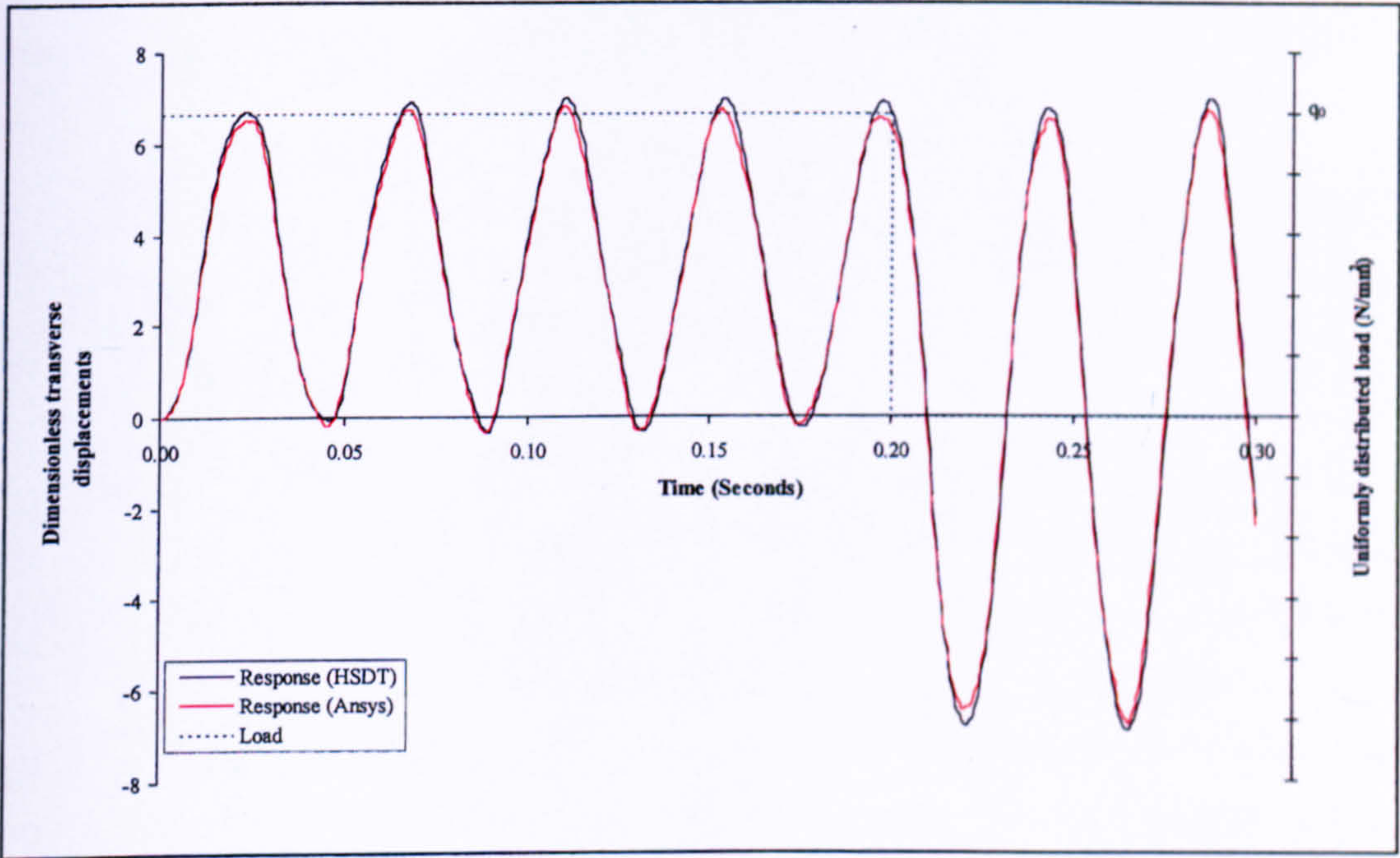


Figure 6.14: Dynamic response of a square sandwich plate $[0^\circ/90^\circ/0^\circ/\text{core}/0^\circ/90^\circ/0^\circ]$ with $a/h = 10$ and $h_c/h = 0.94$.

In figure 6.15, linearly varying load has been applied to the square sandwich panel. As for the two previous examples the difference between the results predicted using the software ANSYS and the developed program are small.

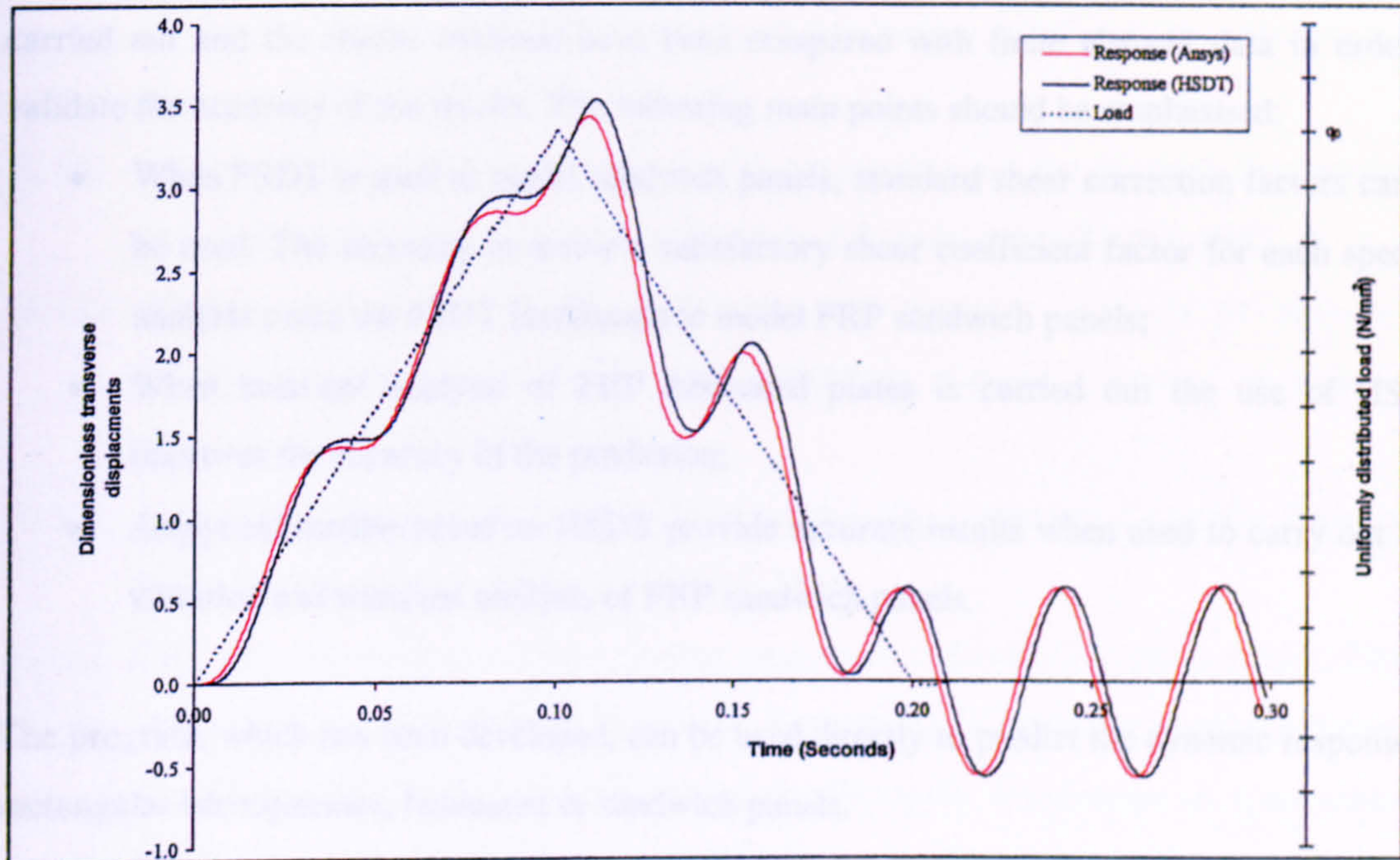


Figure 6.15: Dynamic response of a square sandwich plate $[0^\circ/90^\circ/0^\circ/\text{core}/0^\circ/90^\circ/0^\circ]$ with $a/h = 10$ and $h_c/h = 0.94$.

It should be noticed that in the three cases, which have been presented, the computing time required to predict the results over a period of 0.3 seconds was around 2 minutes using the MATLAB program whereas it took nearly a day to get the ANSYS results. Finally, from the results presented in the previous three graphs, it is possible to conclude that it is possible, by using the present analytical method based on Reddy's refined HSDT, to predict with accuracy the dynamic response of FRP sandwich structures subjected to various transverse loadings. However if a long-term behaviour study needs to be carried out, it is necessary to be careful to not increase dramatically the computing time. The presented developed program requires only the input of the geometric and material properties of the studied plate and therefore can be easily used to quickly predict the response of various possible structural and material configurations.

6.6 FINAL REMARKS

In this chapter analytical models based on FSDT and HSDT that had been previously developed to study the static and dynamic behaviour of rectangular composite laminates have been enhanced in order to be able to as well predict, with accuracy, the dynamic behaviour of FRP sandwich panels. Using these proposed models, free vibration and transient analyses have been carried out and the results obtained have been compared with finite element data in order to validate the accuracy of the results. The following main points should be emphasised:

- When FSDT is used to model sandwich panels, standard shear correction factors cannot be used. The necessity to derive a satisfactory shear coefficient factor for each specific analysis make the FSDT inadequate to model FRP sandwich panels;
- When transient analysis of FRP laminated plates is carried out the use of HSDT improves the accuracy of the prediction;
- Analytical models based on HSDT provide accurate results when used to carry out free vibration and transient analysis of FRP sandwich panels.

The program, which has been developed, can be used directly to predict the dynamic response of rectangular homogeneous, laminated or sandwich panels.

6.7 REFERENCES

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7. EXPERIMENTAL CHARACTERISATION

7.1 INTRODUCTION

An analytical method has been developed in chapter 6 in order to study the dynamic behaviour of undamped FRP single skin and FRP sandwich composite panels. The developed method, so far, does not account for the energy dissipation mechanisms in the structures. However, as mentioned in chapter 4, single skin and sandwich composite structures are made of polymeric materials that exhibit viscoelastic damping. Consequently, in order to predict with accuracy the dynamic behaviour of these structures, it is essential to take into account the viscoelastic damping properties of their constitutive materials. As seen in chapter 4, different models can be used to represent the viscoelastic behaviour of the constitutive materials. As far as this work is concerned, the methods, which have been chosen to be the most appropriate to model viscoelastic behaviour, required the knowledge of the complex dynamic material properties. These dynamic properties are frequently not provided by the manufacturer and therefore need to be determined using experimental techniques. In this chapter, firstly a short review of the existing vibrational methods is presented, then a more in depth description of the Dynamic Mechanical Analysis (DMA) equipment, which has been used to carry out the experimental work, is displayed. Due to the greater lack of knowledge on the dynamic material behaviour of PVC foam materials as compared to FRP materials, the experimental investigation, presented in the part 7.3 and part 7.4 of this chapter, is focussed on the study of the dynamic properties of HEREX C70.130 PVC foam material. The results of the tests on virgin and damaged samples of the foam are presented and discussed.

7.2 EXPERIMENTAL TECHNIQUES

The dynamic properties of materials can be determined by carrying out non-destructive tests using various vibrational methods. These experimental techniques deduce the dynamic material properties from the measurement of the deformation of a material in response to vibrational forces. Then from these measurements, the modulus, which gives an indication of the stiffness of the material, and the mechanical damping, which gives the amount of energy dissipated as heat during the deformation, are determined. The modulus may be a shear, a tensile or a flexural modulus, depending upon the experimental equipment.

7.2.1 REVIEW OF THE DIFFERENT VIBRATIONAL MEASUREMENT TECHNIQUES

Different classes of vibration measurement techniques (e.g.: free vibration, resonance vibration, wave propagation and sinusoidal excitation and response) are often used in order to carry out dynamic mechanical analysis. A brief summary of the various vibrational experimental techniques is presented as follow.

- The first class of vibration technique, the free vibration method, can be used to determine the mechanical properties of linear material. It consists of measuring, at a given frequency of oscillation, the period and the logarithmic decrement that defines the mechanical behaviour of the tested materials.
- Another method, the resonance vibration technique, consists of applying an oscillating force whose amplitude is fixed, but whose frequency may be varied, to a mechanical system. The amplitude of the resulting vibration passes through a maximum at a frequency, which is known as the resonant frequency of the system. The value of the resonant frequency depends on the elastic properties of the system, while the breadth of the resonance gives a measure of the dissipative forces. There are basically three kinds of vibration which may be employed in resonance studies, there are flexural, torsional and longitudinal. The resonance vibration technique is known to give good results when used to study the dynamic behaviour of samples of low energy loss and sufficiently high stiffness.
- The wave propagation technique can also be employed. Indeed, several types of elastic waves can propagate in solids. The velocity of propagation of all these waves depends on the elastic constants and the density of the material. Hence, dynamic elastic constants can be calculated from the velocity of the wave propagation in the given medium.
- Finally, it is possible to use the sinusoidal excitation and response method also called the Dynamic Mechanical Analysis (DMA) approach. It is an analytical technique that measures the modulus (stiffness) and damping (energy dissipation) properties of materials as they are deformed under periodic (oscillatory) stress.

7.2.2 VIBRATIONAL INSTRUMENTS USED TO TEST FRP AND FOAM MATERIALS

Many instruments based on these different classes of vibration have been developed to study the properties of various polymeric materials. Since polymeric materials exhibit a wide variety of shapes and moduli, each instrument is often restricted to narrow ranges of frequency, to given types of deformation and to one type of material. It is indeed very difficult to develop one instrument that has the capability of characterising all types of polymers over a wide range of frequencies and temperatures. An extended review of the available commercial instruments can be found in Murayama's book [7.1]. In this part only the experimental techniques, which have previously been used to measure the dynamic properties of FRP and foam material, are presented. It should be kept in mind that the objective is to identify an efficient and versatile technique which is suitable for the measurement, as a function of frequency, of the complex Young's modulus and complex shear modulus of FRP and foam materials.

7.2.2.1 Free vibration and resonance vibration analyses

Two experimental facilities have been developed by Wright [7.2] to measure the dynamic properties of FRP materials. These two apparatus have been used later on by White [7.3] and Abdin [7.4] to study the dynamic properties of CFRP materials. They proposed to measure the flexural stiffness and damping from the natural frequency and free decay respectively, of a free-free beam. As far as the shear properties are concerned they proposed to extract the shear stiffness and damping from the response around resonance of a rod, clamped at each end and excited by a central inertia bar. It should be noticed that other authors such as Adams and Bacon [7.5] have been using the same techniques to measure the dynamic properties of various FRP materials. These experiments have been shown to give accurate results when used to study the dynamic behaviour of FRP materials. However these methods present several limitations. The first drawback is linked to the fact that these apparatus do not provide the possibility to carry out the tests over a wide range of frequencies or temperatures. Indeed, each test has to be carried out at a given frequency and at a given temperature. Knowing that the natural frequencies are dependent on the geometry of the structure, Wright [7.2] proposed to investigate the effect of frequency on the dynamic material properties by carrying out experiments with beams of different length. Similarly to investigate the effect of frequency on the complex shear modulus, rods with different geometry need to be tested. These simple experimental techniques turn out to be time consuming especially when the effects of

frequency on the dynamic material properties are investigated. Besides, they are restricted to frequency ranges that can be practicably obtained using these apparatus. Indeed, only the natural frequencies of specimens, whose geometry can be feasibly tested, can be considered. On top of these restrictions, these methods have been causing some problems when tests have been carried out using more flexible materials such as resin systems. Indeed, Wright [7.2] explains in his thesis that if the samples are too flexible they cannot carry their own weight between supports during the free-free beam test. Besides, Wright showed [7.2] that with the two techniques presented here it is impossible to measure loss factors greater than 0.4. So, if the requirement is to carry out experiments over a wide range of frequencies or temperatures and to test both stiff and flexible material, a more versatile and sophisticated apparatus should be considered.

7.2.2.2 Wave propagation technique

Wave propagation technique has been used, as indicated before, to measure the dynamic properties of isotropic solid materials. Indeed, as mentioned by Murayama [7.1], experiments of this type have been largely conducted in solid blocks to obtain greater understanding of the dynamic mechanical properties of elastic and viscoelastic solids. In principle the method should enable the dynamic storage and loss moduli of anisotropic material to be determined through measurement of the velocity and attenuation, respectively, of ultrasonic waves travelling along known directions in a sample. In practice, as explained by Read and Dean in their book [7.6], the method is used to determine the dynamic properties of isotropic polymers but only the dynamic storage modulus of low-loss, anisotropic composites. This is due to the fact that for anisotropic materials some of the equations required to analyse the data are very complicated, and their solution is only practical for small loss factors. So as far as anisotropic materials are concerned, the method is suitable only for FRP materials with high modulus fibres. Sierakowski and Chaturvedi [7.7], for example, have used the wave propagation technique to determine the elastic properties of FRP materials. However they did not consider the dissipative forces. As far as closed cell PVC foam material is concerned, the use of the wave propagation technique is compromised by the presence of numerous voids inside the material. Indeed, the presence of voids contributes to the attenuation of the wave amplitude due to scattering at phase boundaries. So, false results of the loss moduli, which are

calculated from the measurement of the attenuation of the wave amplitude, are likely to be obtained. Finally, the wave propagation technique does not appear to be an ideal method to measure the dynamic properties of a wide range of FRP and PVC foam materials.

7.2.2.3 Dynamic Mechanical Analysis (DMA)

Based on the sinusoidal excitation and response vibration technique the dynamic mechanical analyser has been developed. DMA is a thermal analysis technique that measures the properties of materials as they are deformed under periodic stress. Specifically, during dynamic testing, an oscillatory (sinusoidal) strain (or stress) is applied to the materials and the resulting stress (or strain) developed in the material is derived. If the material being evaluated is purely elastic, the phase difference between the stress and the strain sine waves is 0° (i.e.: they are in phase). If the material is purely viscous, the phase difference is 90° . However, most real-world materials including polymers are viscoelastic and exhibit a phase angle that will lie somewhere between 0° and 90° . This phase difference, together with the amplitudes of the stress and strain waves, can be used to determine a variety of fundamental material parameters such as storage modulus, loss modulus, tan delta, transition temperature, creep and stress relaxation. The DMA technique is, therefore, particularly useful for evaluating polymeric materials properties which exhibit time, frequency and temperature dependency. Most of DMA measurements are made using a single frequency and constant deformation (strain) amplitude while varying temperature. This experimental method allows the determination over a wide range of temperatures of the storage modulus, which gives information on the stiffness of the tested material, and the tan delta values, which are used to assess the noise and vibration damping performance of the material. It should be noticed that dynamic mechanical analysers offer the possibility to measure both complex Young's and shear moduli. Complex Young's moduli can be obtained by measuring the deformation of rectangular samples fixed in single or double cantilever clamps. Complex shear moduli can be determined using two different techniques. One of them consists of testing two identical flat samples using the so-called shear sandwich clamp (see figure 7.3). The second possibility consists of measuring the deformation of a rod specimen subjected to a twisting torque. As far as FRP materials are concerned, it is recognised, as explained by Grant *et al.* [7.8], that the DMA technique is a

convenient method of testing large numbers of resin formulation and composite systems quickly and simply. For example it has been used by Benchechou *et al.* [7.9] to determine the complex Young's modulus and the glass transition temperature of CFRP materials with different resin systems. Moreover, DMA has not been used to measure the complex shear modulus of FRP materials. As far as PVC foam materials are concerned, the complex Young's modulus of PVC commercial foam has been measured and investigated with this technique [7.10]. Moreover, using rod specimens Danielsson and Olsson measured the complex shear modulus of high resilience "rigid" foam material [7.11]. It should be outlined at this point that there is no available example of measurements obtained using the shear sandwich clamp for either FRP or PVC foam materials. Finally it should be emphasised that DMA is a valuable thermal analysis technique that can be used to characterise the mechanical properties of the materials over a wide temperature range. The obtained measurements provide quantitative and qualitative information about the performance of materials. The mathematical equations, related to the calculation of the mechanical properties from the measurement of the amplitude of the stress and strain waves and their phase difference, can be found in appendix J.

7.2.2.4 General remarks on vibrational instruments

Various methods have been used to measure the properties of FRP materials. As far as foam materials are concerned little experimental work has been done and the only published results found have been obtained using the DMA technique. The DMA technique emerges as being more versatile than the other methods since it offers the possibility of determining the dynamic materials properties in bending and shear over a wide range of temperatures and frequencies.

7.2.3 DYNAMIC MECHANICAL ANALYSIS (DMA) FACILITIES

After having reviewed the different experimental methods available to measure the dynamic properties of viscoelastic materials, it has been decided to carry out the experimental work using the DMA technique. The experiments have been conducted using the TA DMA 2980 instrument at DERA Farnborough. A sketch of this instrument can be found in figure 7.1.

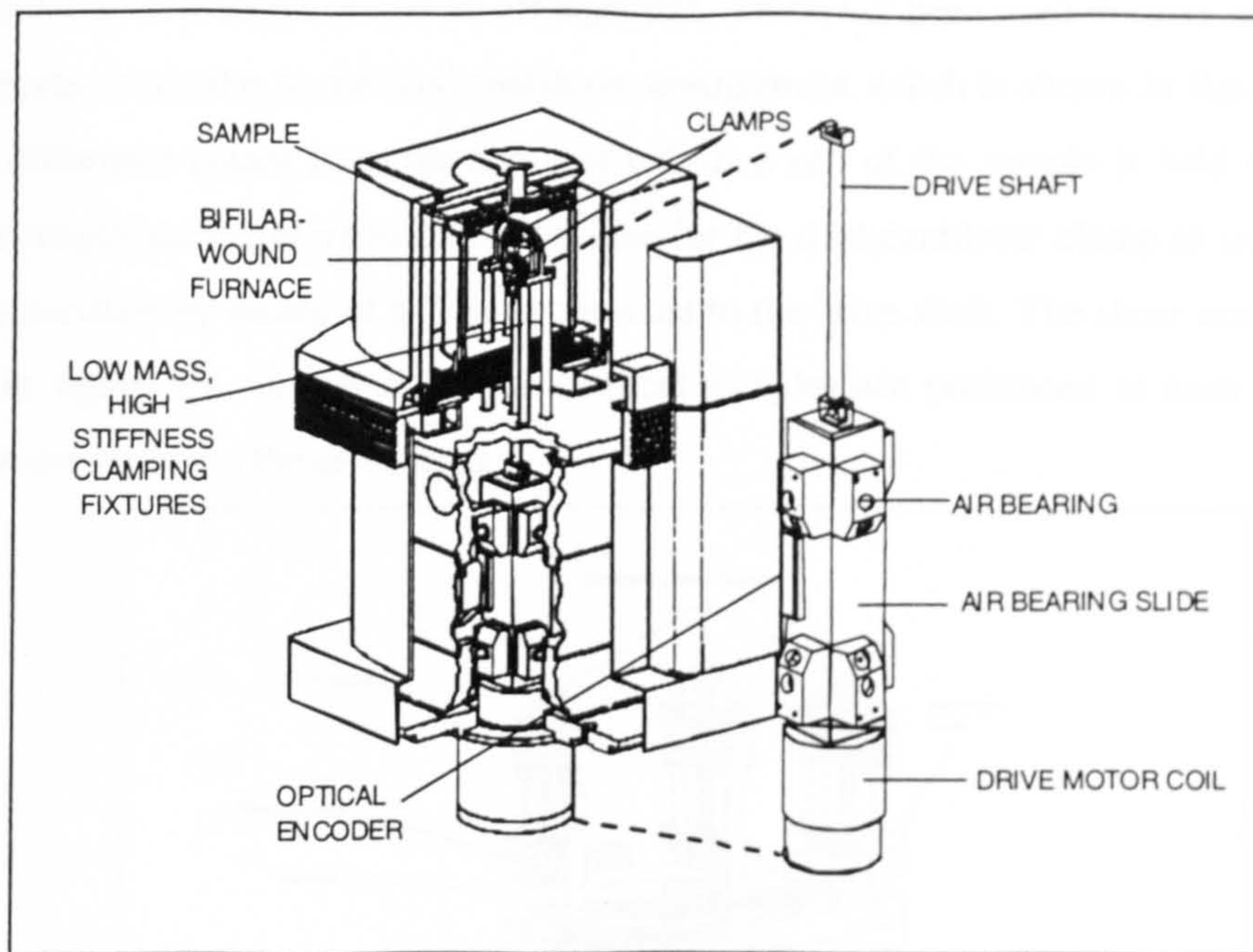


Figure 7.1: DMA 2980 design elements [7.12].

They are several components that are critical to the design and resultant performance of a dynamic mechanical analyser. These components are:

- the drive motor which applies the sinusoidal deformation force to the sample material,
- the drive shaft support and guidance system which transfers the force from the drive motor to the clamps which hold the sample,
- the displacement sensor which measures the sample deformation, that is to say the oscillation amplitude, that occurs under the applied force,
- the temperature control system (furnace), which associated with a gas cooling accessory can cover a broad range of temperature (-145° to 600°C) with excellent precision for transitions ($\pm 1^{\circ}\text{C}$),
- the sample clamps (a variety of sample clamping and deformation modes are available),
- the high resolution optical encoder (1nm over the whole displacement range of 25mm), which allows the precision (1%) of both the storage modulus and the loss modulus and the sensitivity (0.0001) of the tan delta to be excellent.

The DMA instrument determines the various moduli of the studied material by using specific clamp types available with the instrument along with the “Thermal Solutions” instrument control software. When the experiments are run using cantilever clamp the Young’s modulus and its associated damping factor can be determined. Concurrently, when the experiments are run using the shear sandwich clamp, the shear modulus and its associated damping factors

are calculated. The single cantilever arrangement, which has been used to carry out some the experiments, is similar to the dual cantilever arrangement which is shown in figure 7.2. The unique difference comes from the fact that only one end of the sample is held rigidly. The samples length has therefore to be shorter than for the dual cantilever clamp so that the other end can oscillate by means of a T-bar connected to the drive shaft. The shear arrangement is shown in figure 7.3. In this case two identical samples are positioned in each side of the setscrew connected to the drive shaft.

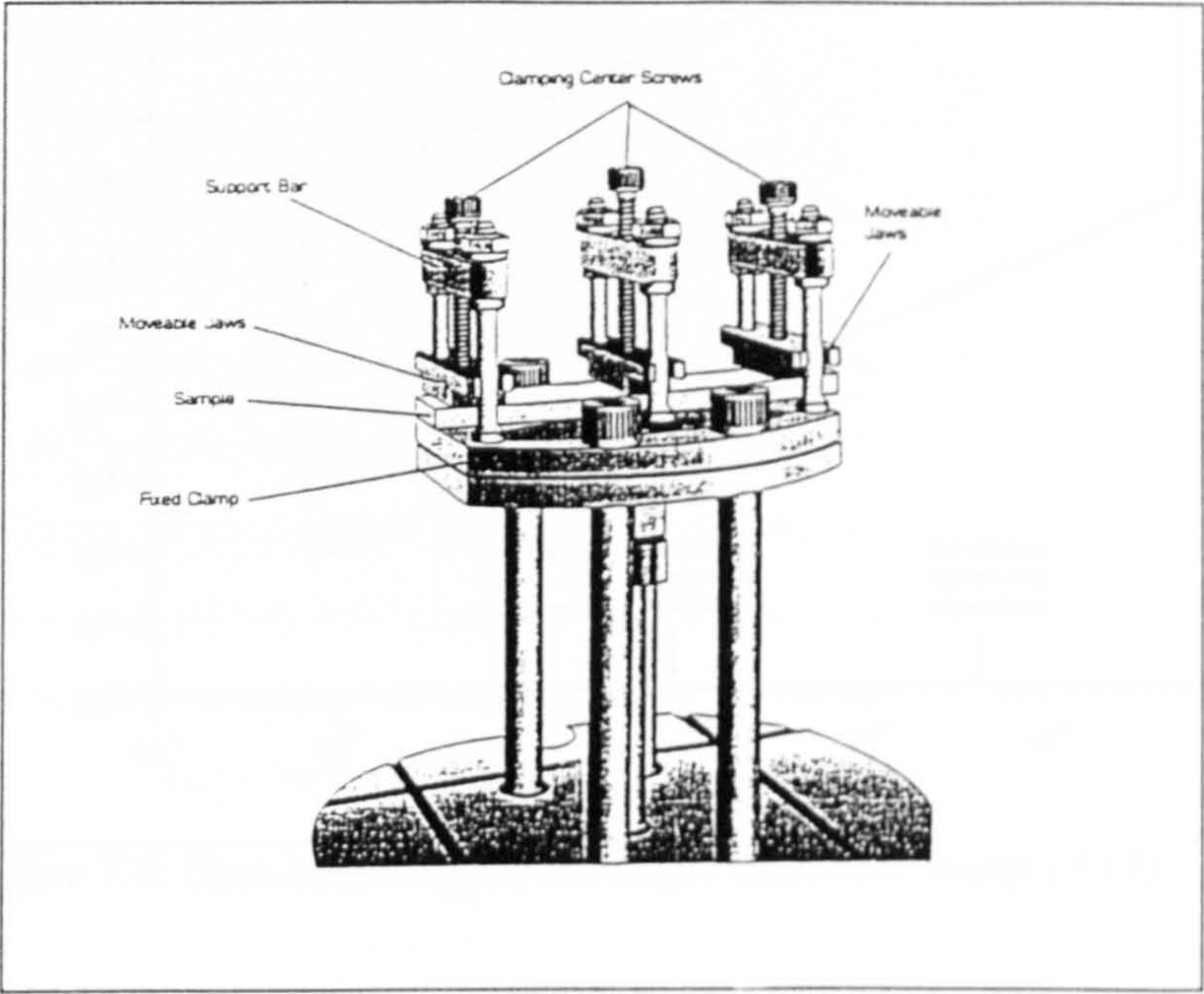


Figure 7.2: Cantilever clamp [7.13].

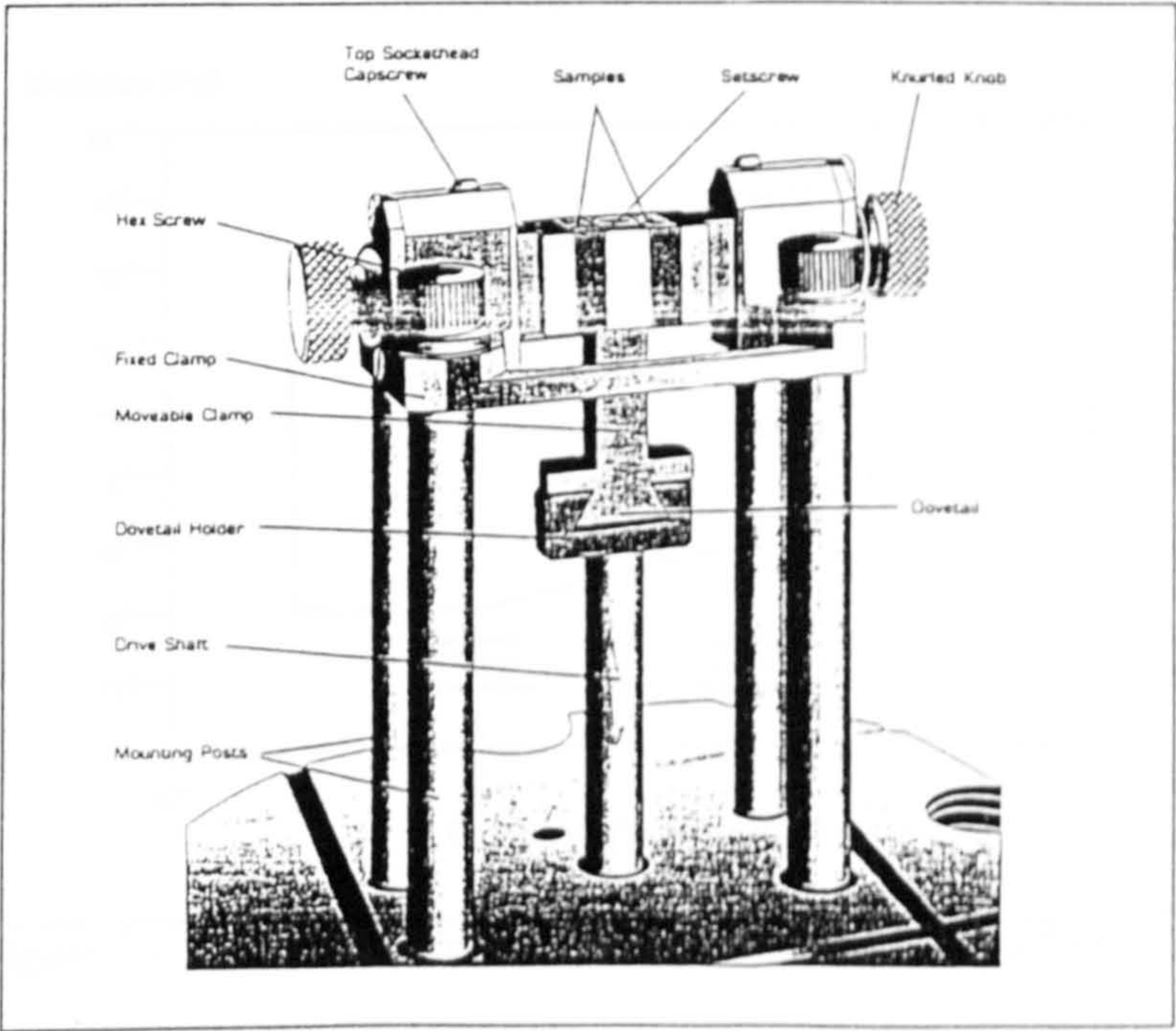


Figure 7.3: Shear sandwich clamp [7.13].

For each clamp the manufacturer defines a different operating range, based on the range of stiffness over which the DMA can operate. As far as DMA 2980 is concerned figure 7.4 and figure 7.5, respectively, illustrate the operating range for single cantilever clamp and shear clamp, respectively. The operating range is a function of the material moduli and the sample geometry.

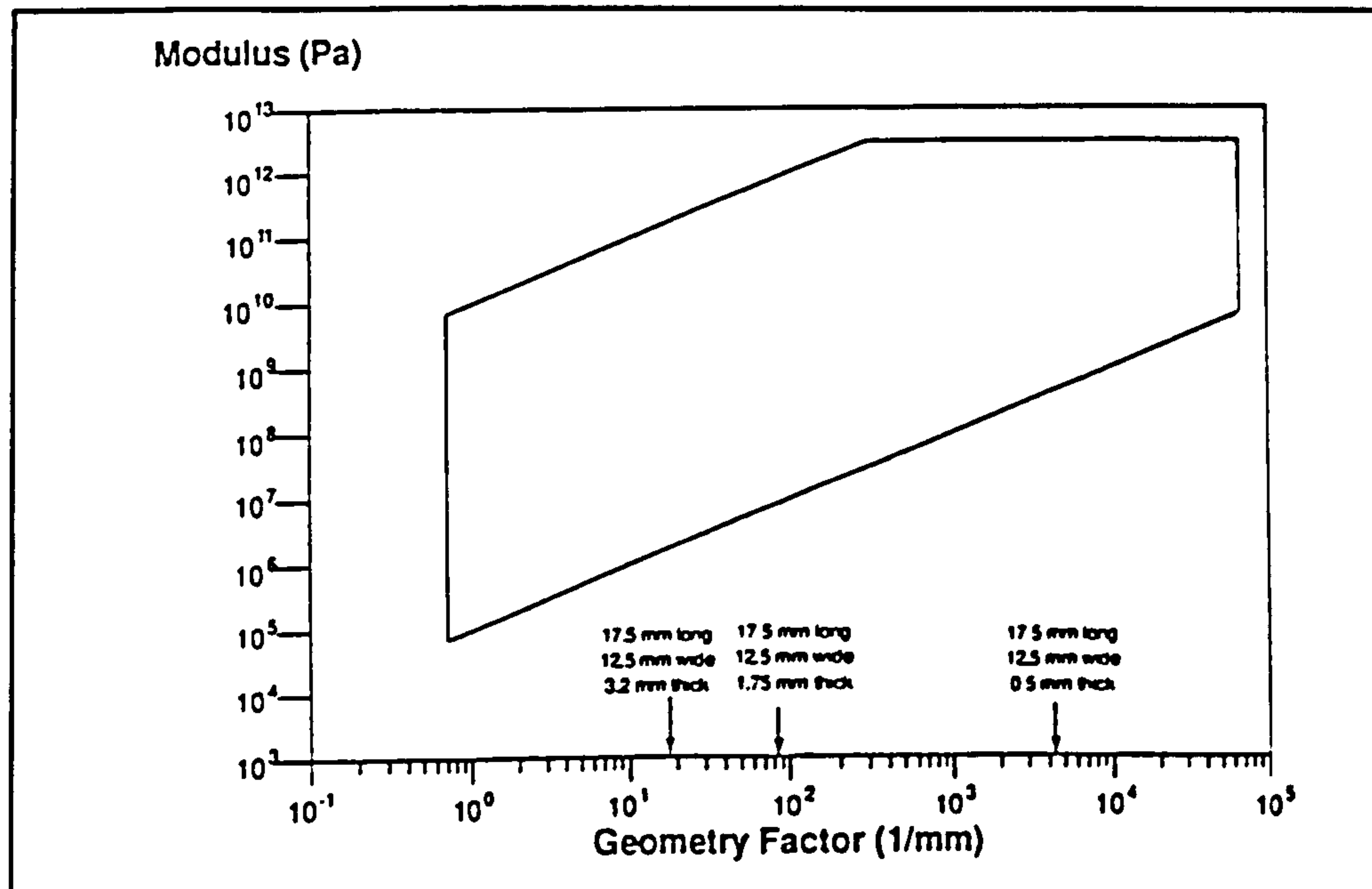


Figure 7.4: Operating range of the single cantilever clamp [7.13].

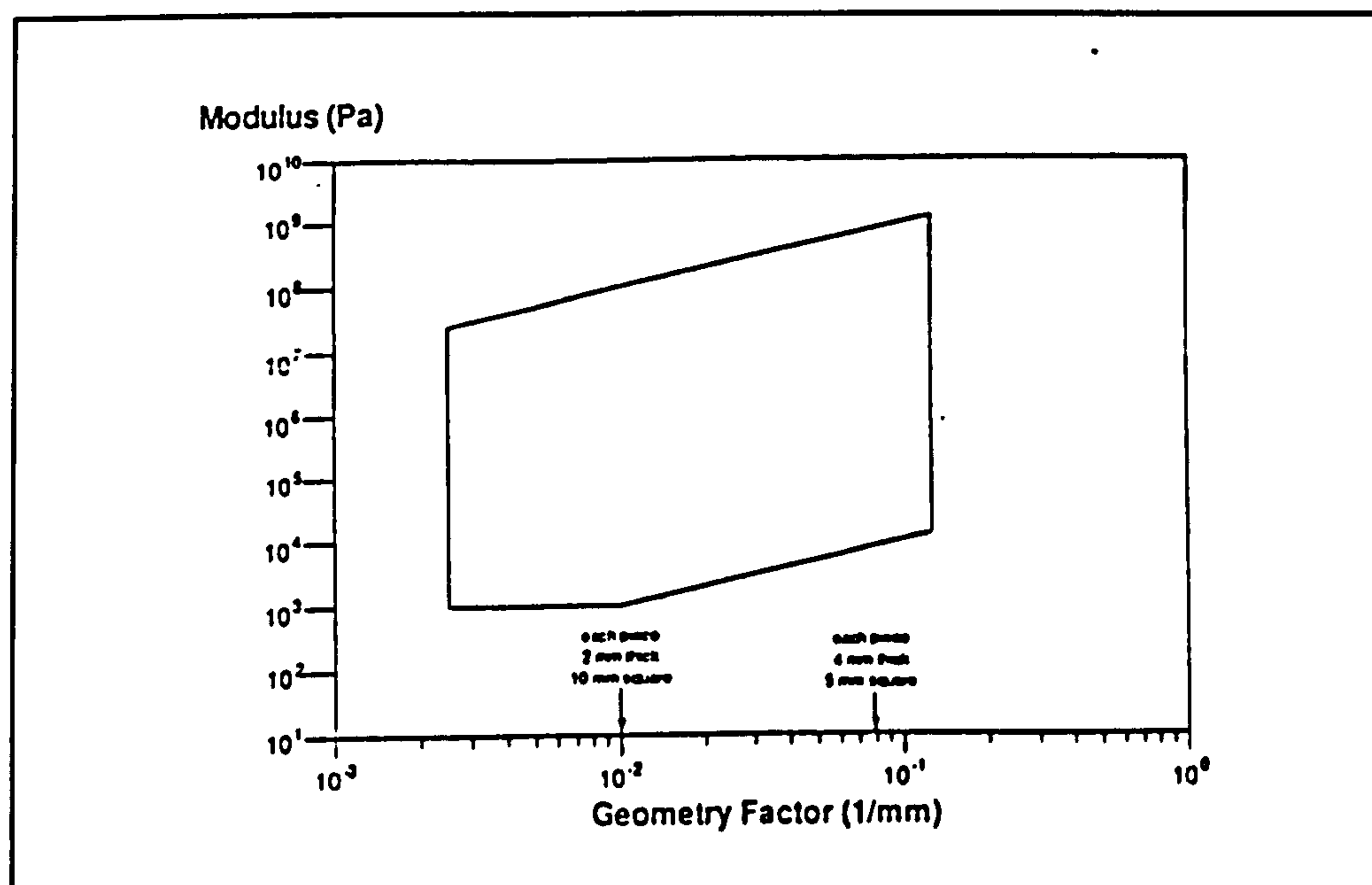


Figure 7.5: Operating range of the shear sandwich clamp [7.13].

By using the information provided on the figures 7.4 and 7.5 and knowing the elastic Young's and shear moduli of a given material (these values can be obtained from the manufacturer's data sheet) it is possible to investigate the feasibility of measuring the dynamic material properties of the given material using the DMA technique. That is to say, the possibility to obtain samples, which can be defined as being in the operating range, is investigated. From this analysis, a range of possible sizes of samples is defined for each clamp. Finally, when, for a given material, the possible sample dimensions are specified, the samples can be prepared and the test carried out.

7.2.4 CONCLUSION

In this part a review of the various vibrational methods available to measure the dynamic properties of viscoelastic materials has been conducted. The DMA technique appears as being more versatile since it offers the feasibility of measuring the complex Young's and shear moduli over a large range of temperature at various frequencies. The features of the TA DMA 2980 equipment, which has been used to carry out the experiments presented in the following parts, has been introduced. It should be noticed that the obtained results will offer not only the feasibility of gaining knowledge on the dynamic properties of the tested material but also the chance of commenting on the effectiveness of the DMA equipment.

7.3 EXPERIMENTAL PROGRAMME

7.3.1. CHOICE OF MATERIAL

It has been decided as mentioned in the introduction of this chapter that the experimental programme will be focussed on sandwich core materials and not on FRP materials. This has been determined due to the greater lack of knowledge on the dynamic material properties of commonly used PVC foam materials in FRP sandwich structures. Sandwich structures consisting of a foam core material combined with high strength skins are today's solution for demanding lightweight and stiff structures. The preferred choice for current sandwich cores, as explained by Alusuisse Airex AG manufacturer [7.14], are structural closed-cell foam due to their large surface bonding and their compatibility with contemporary industrial practices. Various closed cell commercial foams are available. HEREX C70 product, manufactured by Alusuisse Airex AG, is an example of closed cell foam characterised by its high stiffness and strength to weight ratio. It is a cross-linked thermoset PVC (polyisocyanate) foam. This foam

exists under different density from 40kg/m³ up to 200kg/m³. The HEREX C70.130 foam, characterised by a density of 130kg/m³ and by typical cell sizes range from approximately 0.2mm to approximately 0.4mm, has been used for example in the Trend class lifeboats. It is this specific foam which is the subject of our investigations. The properties of this foam are given for information in table 7.1.

Material properties	E (MPa)	G (MPa)	ρ (kg/m ³)	ν
HEREX C70.130	109	50	130	0.32

Table 7.1: Material properties of HEREX C70.130 [7.14].

A picture of a sample of HEREX C70.130 foam has been taken through a microscope. The picture shows the cell shape configuration in a virgin sample taken in the middle of the sheet of foam provided by the manufacturer.

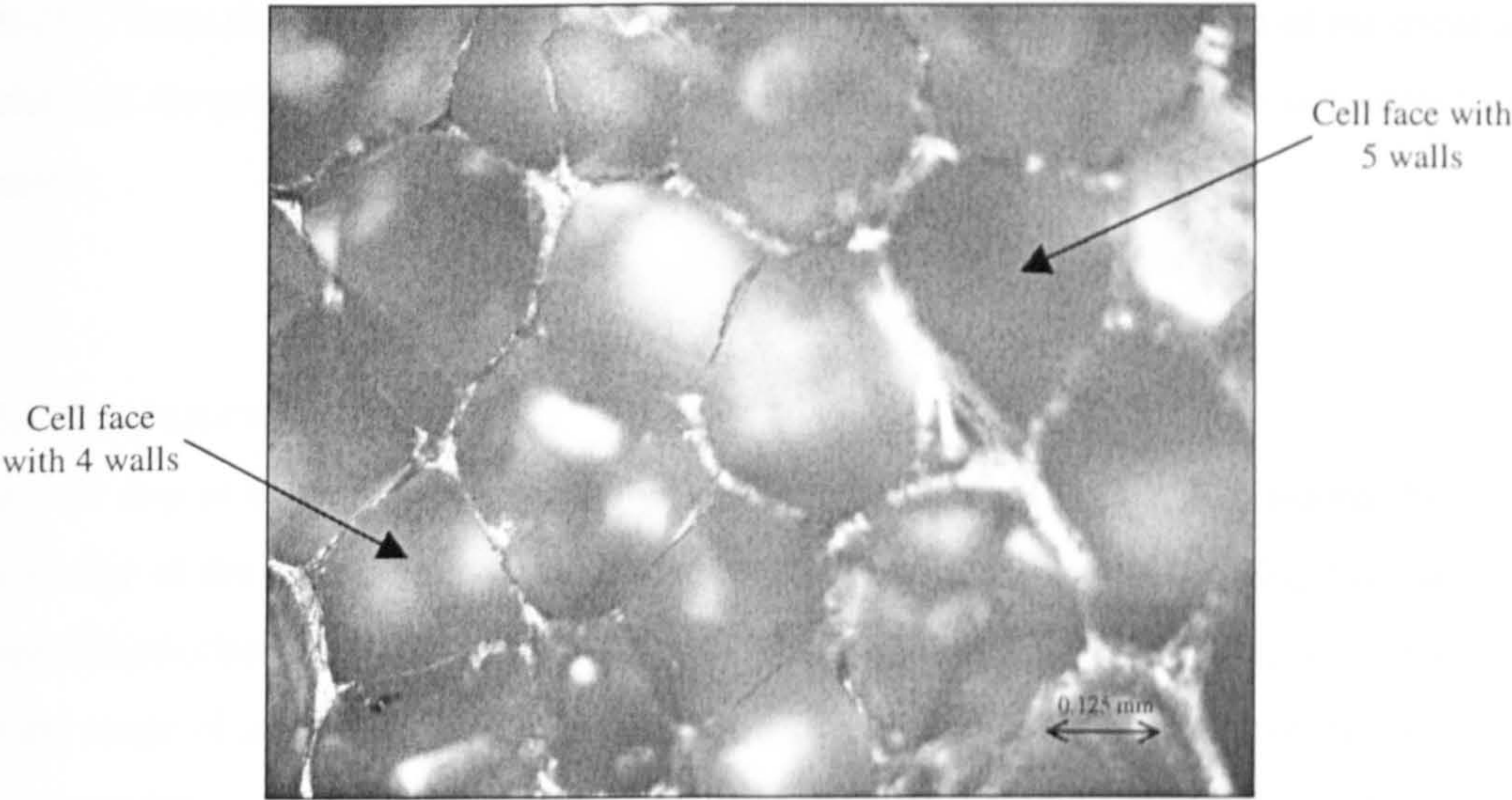


Figure 7.6: Image of the cell shape of the HEREX C70.130 foam.

It can be noted from this image that the cell faces are constituted of either five or four walls. This, if one refers to the Gibson and Asby study [7.15], indicates that the HEREX C70.130 PVC foam is likely to be made of cells having a tetrakaidecahedron shape as shown in figure 7.7. It should as well be noticed that even though the regularity of the structure is not perfect, all the cells tend to have the same shape.

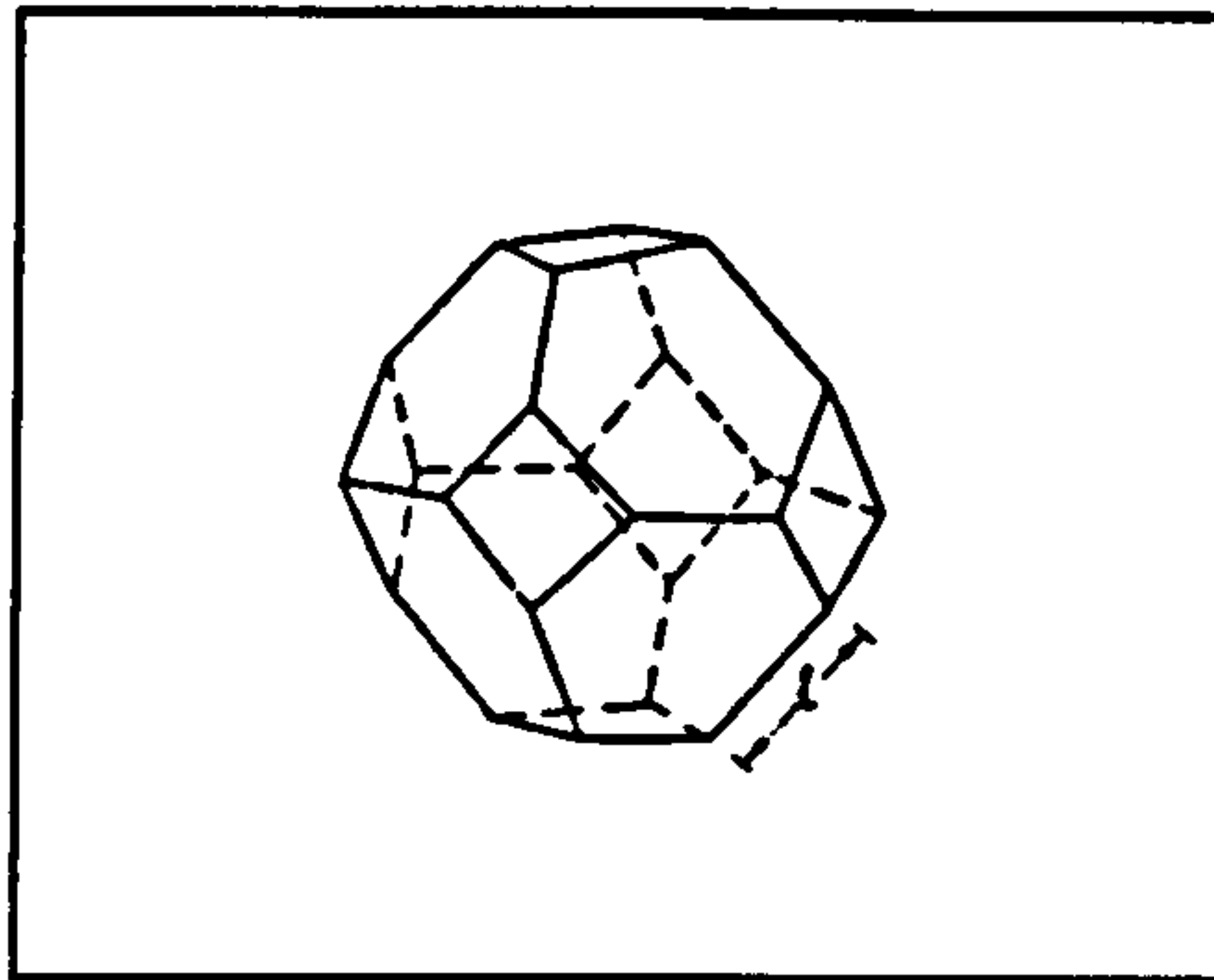


Figure 7.7: Sketch of a tetrakaidecahedron cell shape [7.15].

In most of the works carried out up to now the properties of the foam were assumed to be the same in all the directions, that is to say the foam was assumed to be isotropic. As explained by Gibson and Ashby [7.15], the properties of a cellular solid are mainly dependent on the shape and structure of the cells so that isotropic foam should be composed of ideal regular packing of identical cells. However, due to the manufacturing process used to make closed cell PVC foam sheet, the cells appear to be slightly elongated in the direction of the edges and corners of the panel. This implies that the assumption of isotropy should be considered with caution.

7.3.2 SAMPLE CHARACTERISTICS

The first step of the experimental work is to define the required sample dimensions. From the knowledge of the elastic material properties of the foam (see table 7.1), using the operating range of both clamps, given in figure 7.4 and 7.5, sample sizes are determined. It appears that a wide range of sample sizes can be considered in the case of single cantilever test since the elastic modulus of the HEREX C70.130 PVC foam is found to be in the middle of the operating range. However when the shear sandwich clamp is used more care is required to choose the samples size since the shear modulus of the considered foam is found to be at the top limit of the operating range. Indeed the shear clamp appears to not work very well with high modulus material and therefore should probably not be used for FRP materials. Finally, for the cantilever clamp sample dimensions of 25-30 mm $\times$ 12-13 mm $\times$ 1.8-2 mm are considered. With the shear clamp specimen dimensions of 8-9 mm $\times$ 8-9 mm $\times$ 2-2.2 mm are used. These samples are tested with the DMA equipment using a single frequency and constant deformation (strain) amplitude while varying temperature in the range from ambient to 120°C. The temperature increase rates are defined so as to obtain accurate results without

increasing drastically the experimentation time. In the case of the cantilever test, it is set to 2°C per minute whereas it is chosen to be 1°C per minute for the shear test. This choice of different values is explained by the fact that the thermocouple with the shear clamp cannot be as close to the sample as for the cantilever clamp. Finally, the measured storage modulus values and their associated loss factor values are directly plotted as a function of temperature. Then, using the frequency-temperature superposition principle that will be described in more detail in part 7.4, the results can be given as a function of frequency.

7.3.3 OUTLINE OF TESTING PROGRAMME

The overall aim of the experimental programme is to achieve a more complete knowledge of the dynamic behaviour of the PVC foam, HEREX C70.130 and to conclude, from the observed results, on the efficiency of the DMA technique when used to test this kind of material. The testing programme is split into two categories. The first step of the work consists of carrying out tests on non-damaged samples, that is to say on foam material directly provided by the manufacturer. The aim is then to determine the dynamic material properties of the foam. The second step of the work consists in investigating the changes in the dynamic material properties due to the formation of cracks under fatigue loading. To realise this, samples are cut from the crack area of previously tested specimen as explained in more detail in the next chapter.

7.4 EXPERIMENTAL RESULTS

7.4.1 NON-DAMAGED SAMPLES

First of all, the dynamic properties of the HEREX C70.130 PVC foam, provided by the manufacturer, have been determined as a function of temperature and frequency. To achieve this, samples have been cut from different areas of the provided sheet of foam and in different orientations. This has been achieved in order to be able to investigate the influence of a possible variation of density and cell shape on the dynamic properties of the foam. These variations can occur due to its process of fabrication. Three tests have been carried out using the single cantilever clamp and using the shear clamp, respectively.

7.4.1.1 Temperature dependency

- Single cantilever test

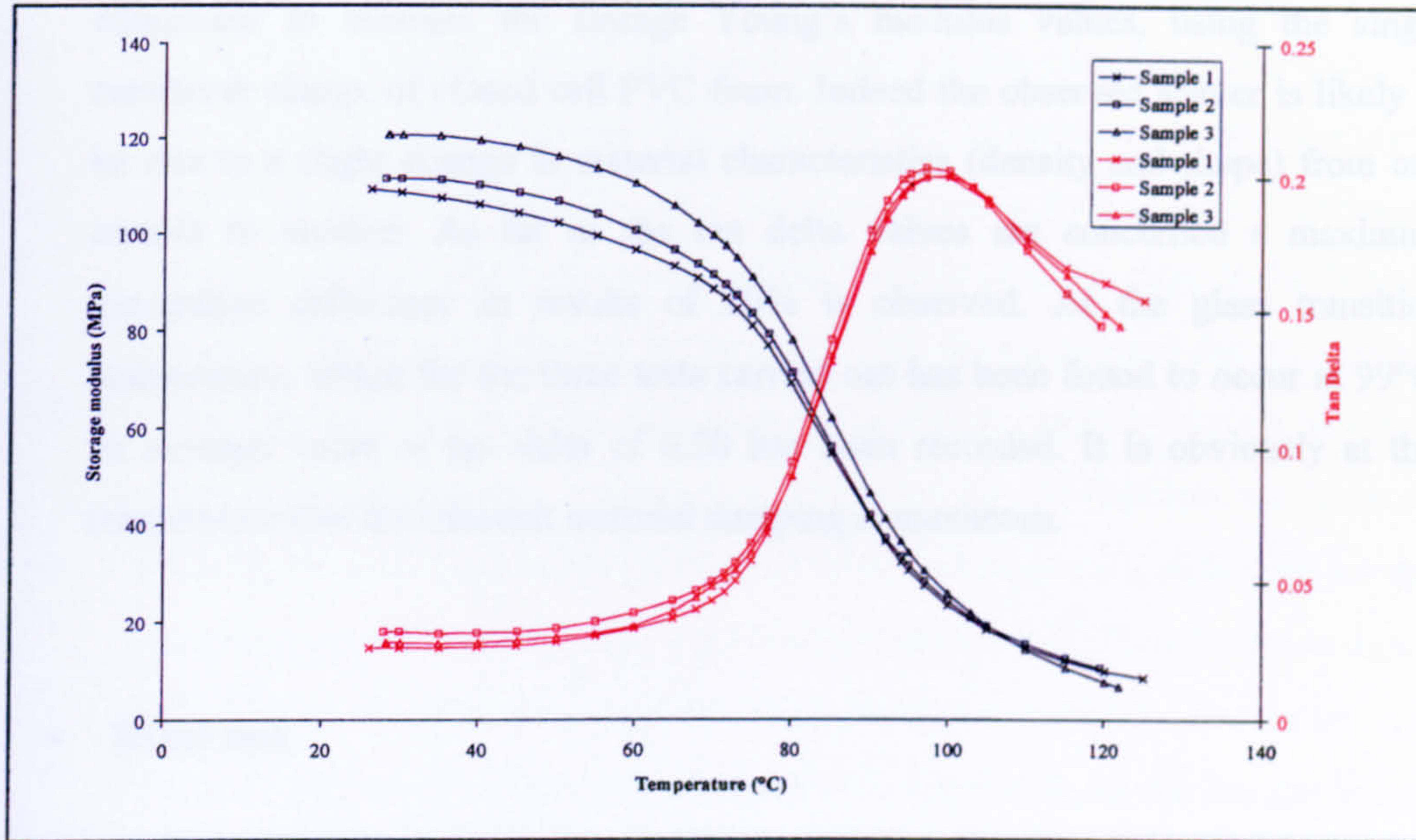


Figure 7.8: DMA single cantilever test results for PVC foam material (HEREX C70.130).

As for many polymeric materials, dynamic properties of the studied cross-linked PVC foam core depends on the temperature. This is illustrated in figure 7.8 where the effect of temperature on the measured dynamic storage Young's modulus and its associated loss factor is presented. From this figure distinct regions can be observed. The first, below 60°C, is the so-called glassy region, where the material takes on its maximum value for the storage modulus while having low values for the loss factor. The change in the dynamic properties with temperature increase is really small. The second region is characterised by having a storage modulus that decreases rapidly with increasing temperature, while the loss factor takes on its maximum value. This region is called the transition region. Near the end of the glass transition region, that is to say around 120°C, the experiments have been stopped. This explains why the rubbery region, which theoretically occurs after the transition temperature and which is characterised by constant values of storage modulus and tan delta is not observed in the figure 7.8. At ambient, the values of the measured storage modulus vary between 108.6MPa and 120.6MPa. These values are in accordance with the elastic Young's modulus value of 109MPa provided by the manufacturer. The manufacturer of the PVC foam considers that the density of the foam inside a sheet can show a

variation of 10% which leads to a variation of 20% in the storage modulus values. In the present case, the maximum scatter in the measured storage modulus values is observed at ambient and is of 10%. This result confirms the efficiency of the DMA equipment to measure the storage Young's modulus values, using the single cantilever clamp, of closed cell PVC foam. Indeed the observed scatter is likely to be due to a slight change in material characteristics (density and shape) from one sample to another. As far as the tan delta values are concerned a maximum percentage difference in results of 20% is observed. At the glass transition temperature, which for the three tests carried out has been found to occur at 99°C, an average value of tan delta of 0.20 has been recorded. It is obviously at this temperature that the inherent material damping is maximum.

• Shear test

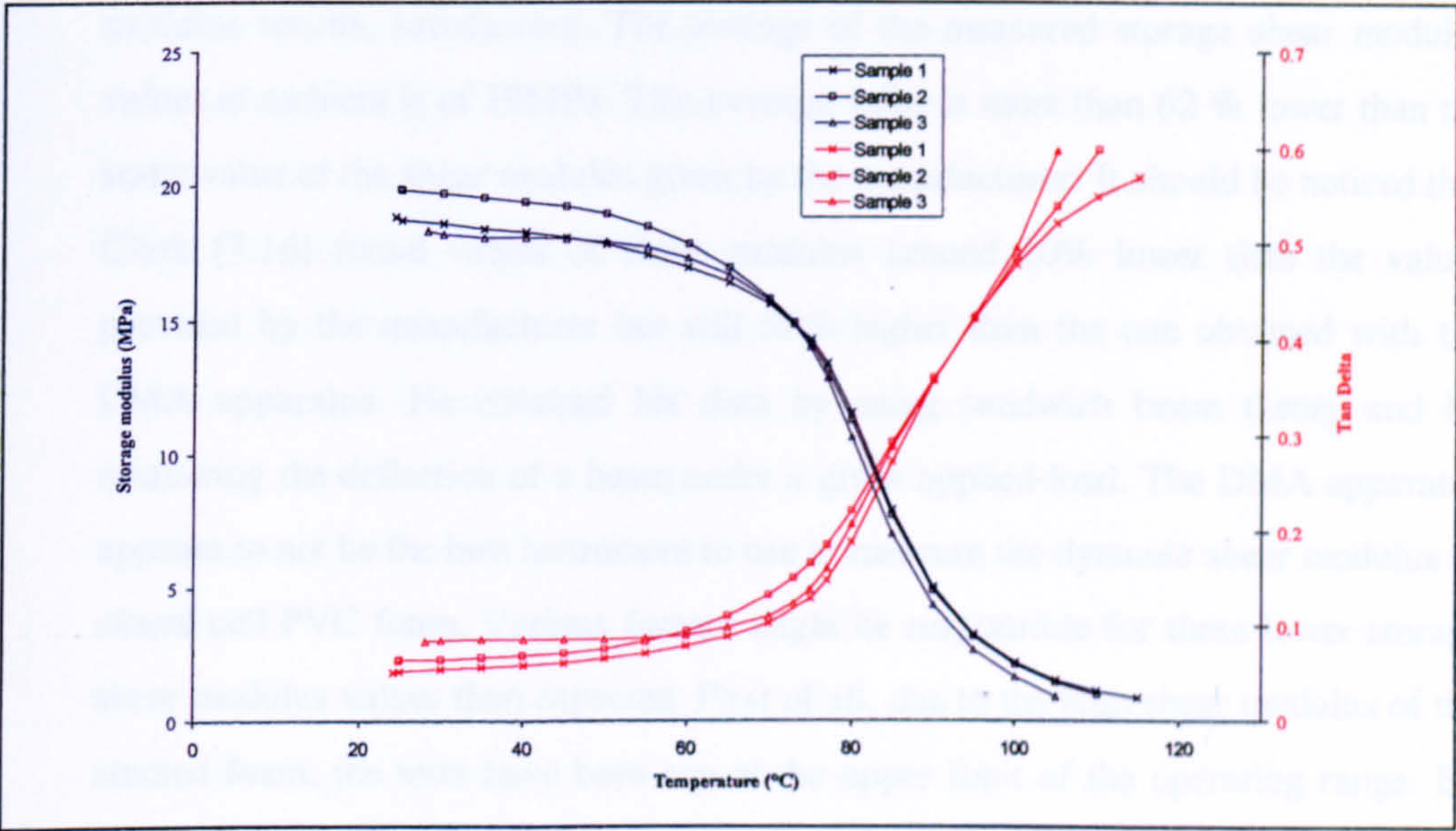


Figure 7.9: DMA shear test results for PVC foam material (HEREX C70.130).

In figure 7.9 the effect of temperature on the measured dynamic storage shear modulus and its associated loss factor is presented. Up to a temperature of 90°C the curves follow an expected trend. For temperatures greater then 90°C, the trend of the curves becomes unsatisfactory. First of all the glass transition temperature, which should be the same as the one observed during the single cantilever tests is never found. Instead, a continuous increase of the tan delta with increase in temperature is

observed. This can be due to the appearance of creep before reaching the glass transition temperature. Besides, during the experiments, around a temperature of 110°C, an increase in the amplitude of oscillation of the strain values has been noticed. This is believed to be the consequence of loosening of the clamp. Indeed, at the end of the test, the sample dimensions were smaller than the initial ones. This indicates that the samples shrunk during the test. The samples are initially maintained between the clamps by the action of a compressive force. If the samples shrink then they are no longer restrained properly in the clamp and might slide. If this happens, inaccurate results are likely to be obtained. The data measured for temperatures greater than 90°C should therefore not be considered as valid. By looking now at the data obtained at ambient, a percentage difference of 8% between the measured storage values and a percentage difference of 40% between the measured tan delta values are observed. The tan delta values measured with the shear test show a much greater scatter than the results obtained with the single cantilever test. As far as the shear storage modulus values are concerned the scatter is, as for the storage Young's modulus results, satisfactory. The average of the measured storage shear modulus values at ambient is of 19MPa. This average value is more than 62 % lower than the static value of the shear modulus given by the manufacturer. It should be noticed that Clark [7.16] found values of shear modulus around 20% lower than the values provided by the manufacturer but still 50% higher than the one obtained with the DMA apparatus. He obtained his data by using sandwich beam theory and by measuring the deflection of a beam under a given applied load. The DMA apparatus appears to not be the best instrument to use to measure the dynamic shear modulus of closed cell PVC foam. Various factors might be responsible for these lower storage shear modulus values than expected. First of all, due to the high shear modulus of the studied foam, the tests have been run at the upper limit of the operating range. By using the equipment at the limit of its operating range, the accuracy of the results might have been affected. Indeed, when high modulus materials are used, stiffness might be lost in the clamp which leads to lower modulus and higher damping values. Besides, in the theory used by the control software, which is associated to the testing equipment, the shear clamping correction factor is ignored in the calculation of the dynamic properties. In theory, a correction factor should always be used since the dilatation of the sample due to compression when the screws are tightened, causes the measured stiffness to differ from the model equation. However in the case of the shear clamp, the amount of dilatation is dependent upon the degree of compression of

the sample which cannot be quantified and therefore in the case of shear clamp the correction factor for this effect is ignored. This is likely to affect the accuracy of the results. Finally, Read [7.6] recognised that materials with shear modulus greater than 10MPa cannot conveniently be tested with the use of non-resonance method like the DMA technique. He reckons that the shear modulus and its associated loss factor can be more accurately measured using a torsion pendulum. It would therefore be interesting in the future to be able to use a DMA in torsion mode to measure the dynamic material properties of PVC foam materials and compared the results with the results obtained with the DMA apparatus set up with sandwich shear clamp.

7.4.1.2 Frequency dependency

The modulus of a viscoelastic material is a function of frequency as well as temperature. In fact, an increase in temperature has the same effect on the measured viscoelastic behaviour as an increase in time or a decrease in frequency. This inverse relationship between the temperature and frequency effects is very useful in establishing the temperature-frequency superposition principle [7.17]. This principle can be used to determine the frequency dependency of the dynamic material properties at a given temperature from the knowledge of the temperature dependency of these properties at a given frequency. Based on this theory, a shift frequency software associated with the DMA 2980 apparatus allows, using the material properties data measured over a given range of temperature, the determination of their frequency dependency. This software, which includes the William-Landel-Ferry equation [7.1], has been used to determine the frequency dependency of the dynamic properties of the HEREX C70.130 PVC foam. The dynamic material properties of the studied foam have therefore been determined as a function of both temperature and frequency. Figure 7.10 and figure 7.11 show the temperature and frequency dependency of the storage Young's modulus and its associated loss factor, respectively.

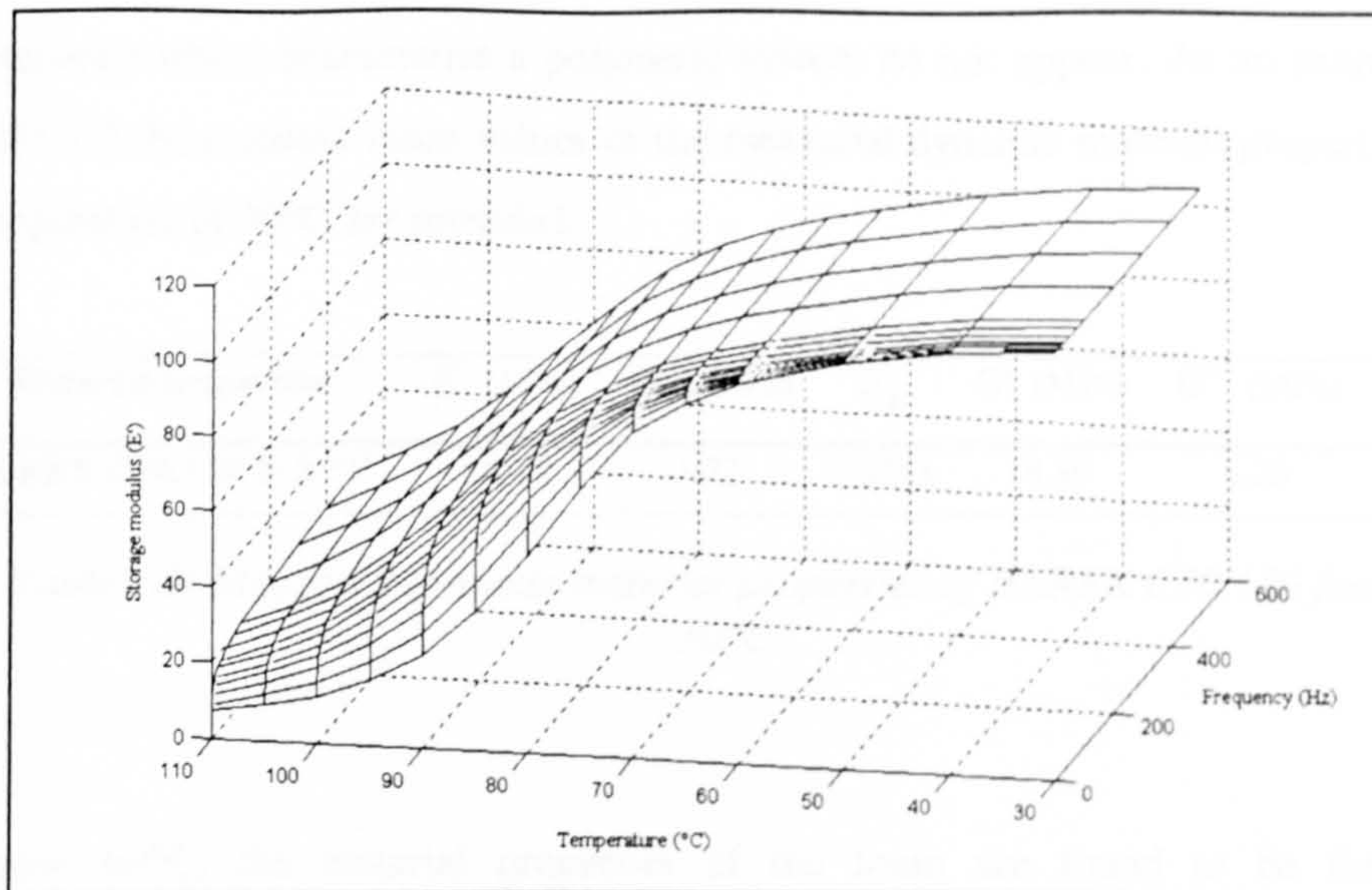


Figure 7.10: Temperature and frequency dependency of the storage Young's modulus data.

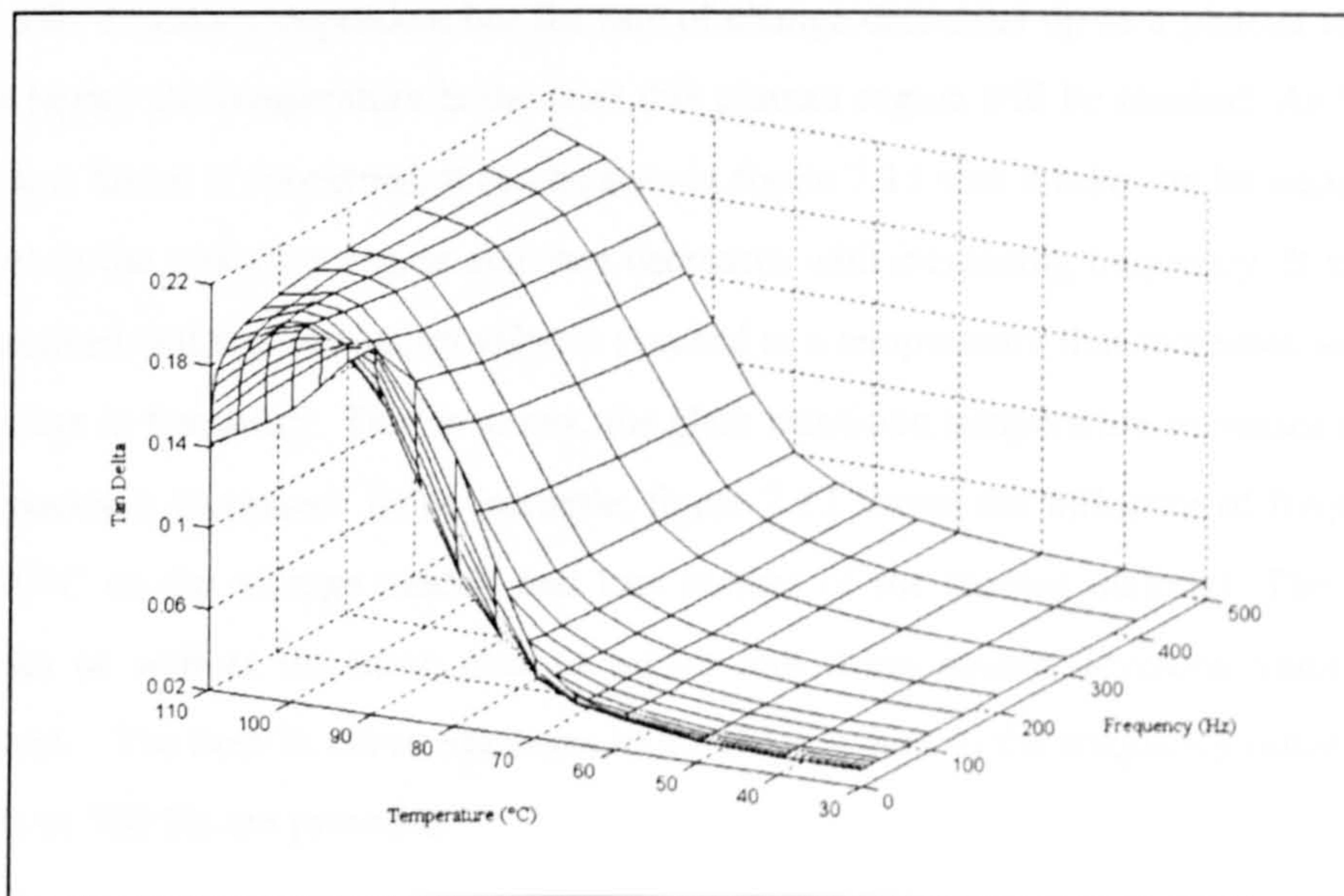


Figure 7.11: Temperature and frequency dependency of the tan delta values associated with the storage Young's modulus data.

From these figures, it can be noticed that the effect of frequency on the dynamic material properties of the HEREX C70.130 PVC foam is different for different temperatures. Between ambient and 60°C, which correspond to its glassy region, both the storage modulus and tan delta values are found to be independent of frequency. This is explained by the fact that in the glassy region the configurations of the polymer chain are largely immobilised and the changes in viscoelastic properties with

frequency which characterise a polymeric system do not appear. As an example, in table 7.2 the constant mean values of the measured dynamic material properties at a temperature of 30°C are provided.

Material properties	E' (MPa)	E'' (MPa)	η_E	G' (MPa)	G'' (MPa)	η_G
HEREX C70.130 @ 30°C	113.5	3.27	0.0288	18.86	1.26	0.067

Table 7.2: Measured dynamic material properties of HEREX C70.130 foam at 30°C.

Above 60°C, the material properties of the foam are found to be frequency dependent. One of the effects of frequency is that the modulus increases with increasing frequency. This increase takes on its greatest rate of change in the transition region and for frequencies below 100Hz. Above this value, the properties are still frequency dependent but the rate of change decreases up to a plateau region. The higher the temperature is the later this plateau region will be reached. As far as the loss factor is concerned, it can be seen in figure 7.11 that it takes on its maximum value in the transition region and then decreases with increasing frequency. It should be noticed that this maximum value is reached at a temperature that increases with an increase in frequency. That is to say, the glass transition temperature increases as the frequency is increased. As an example, figure 7.12 shows the influence of frequency at 90°C on the storage moduli and loss factors of the studied material. The mean values as well as the mean plus or minus root mean square deviation values are plotted. The best-fit curve equations to the mean values in the frequency range from 0 Hz to 500 Hz are provided.

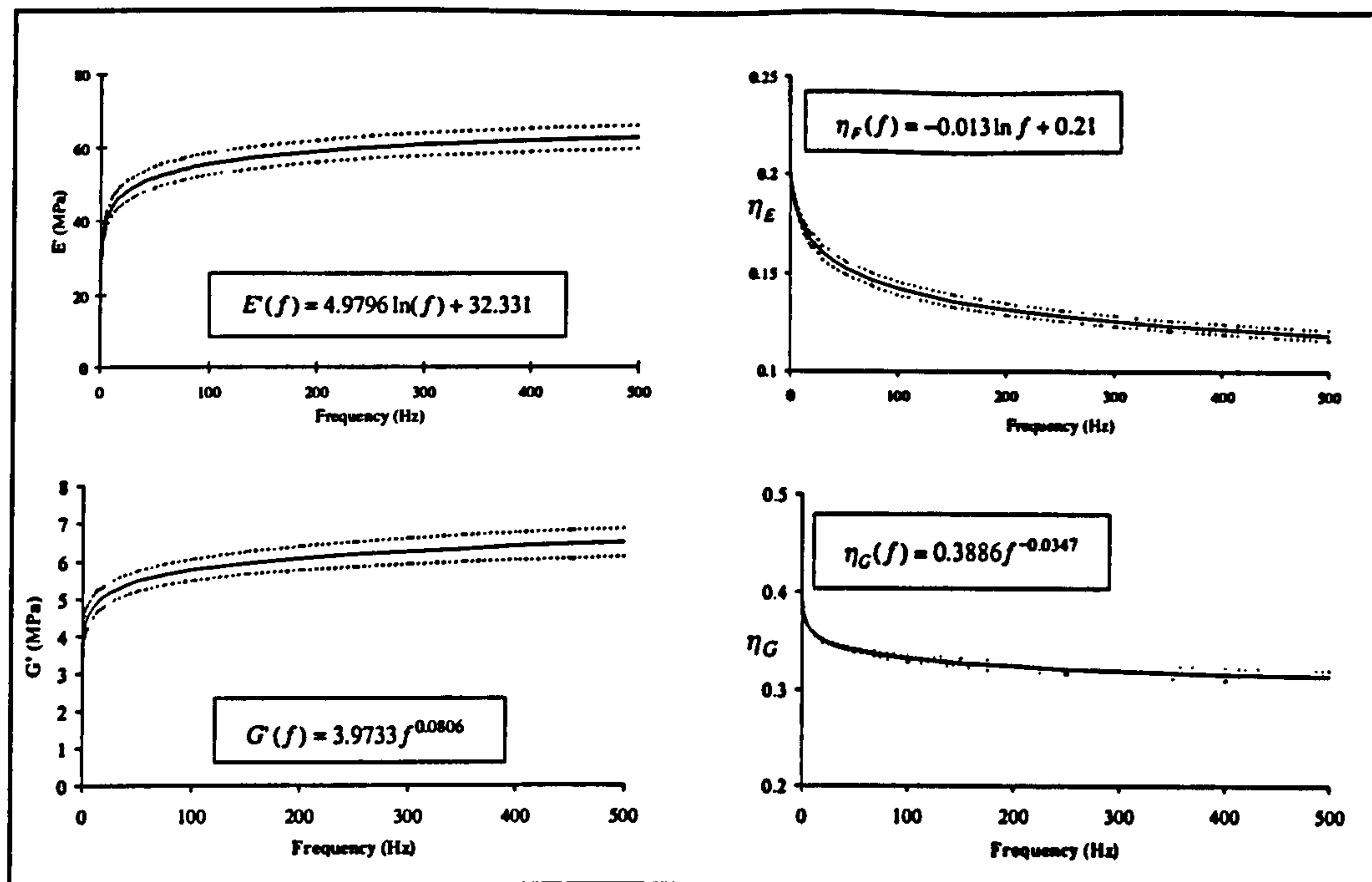


Figure 7.12: Frequency dependency of the dynamic properties of HEREX C70.130 PVC foam at 90°C.

The results obtained from the analysis leads to the conclusion that the dynamic properties of HEREX C70.130 PVC foam core are independent of frequency for temperatures lower than 60°C but are frequency dependent for greater temperatures. This implies that for greater temperature than 60°C the study of the dynamic response of FRP sandwich plates made of HEREX C70.130 PVC foam core requires allowance to be made for the frequency dependency of the material properties. The possibility of taking into account the frequency dependency of the constitutive materials of composite panels is investigated in chapter 8.

7.4.1.3 Remarks

The DMA apparatus has been used to determine the dynamic properties of the HEREX C70.130 PVC foam material. The results obtained using both sandwich shear clamp and single cantilever clamp have been presented and the effects of both temperature and frequency have been investigated. The analysis of the results has pointed out that the DMA apparatus, when employed to study the dynamic behaviour of closed cell PVC foam with relatively high modulus, should be used with caution. Indeed, whereas the results obtained with the single cantilever clamp are acceptable, the results obtained with the sandwich shear clamp should be considered with care.

Nevertheless, shear test can still be carried out along with single cantilever test to investigate qualitatively the influence of damage on the dynamic properties of the HEREX C70.130 PVC foam.

7.4.2 FATIGUED PVC FOAM SPECIMENS UNDER MIXED MODE LOADING

The main objective of this part of the work is to analyse the consequences of the presence of damage, such as a crack, on the dynamic properties of PVC foam. This is realised by measuring, with the DMA equipment, the dynamic material properties of samples that have been cut from the crack area of a specimen previously tested in fatigue. The fatigue tests have been conducted by Noury and Shenoï [7.18], in order to determine the fatigue crack growth under mixed loading mode conditions in cross-linked PVC foam materials. Using Noury's specimen made of HEREX C70.130, samples of 2mm thickness have been cut, as represented in figure 7.13, in two different orientations one parallel to the crack area and one perpendicular to it. The obtained samples have then been tested using the DMA apparatus in order to investigate the local effect of damage on the dynamic properties (i.e.: complex Young's modulus and complex shear modulus) of the PVC foam.

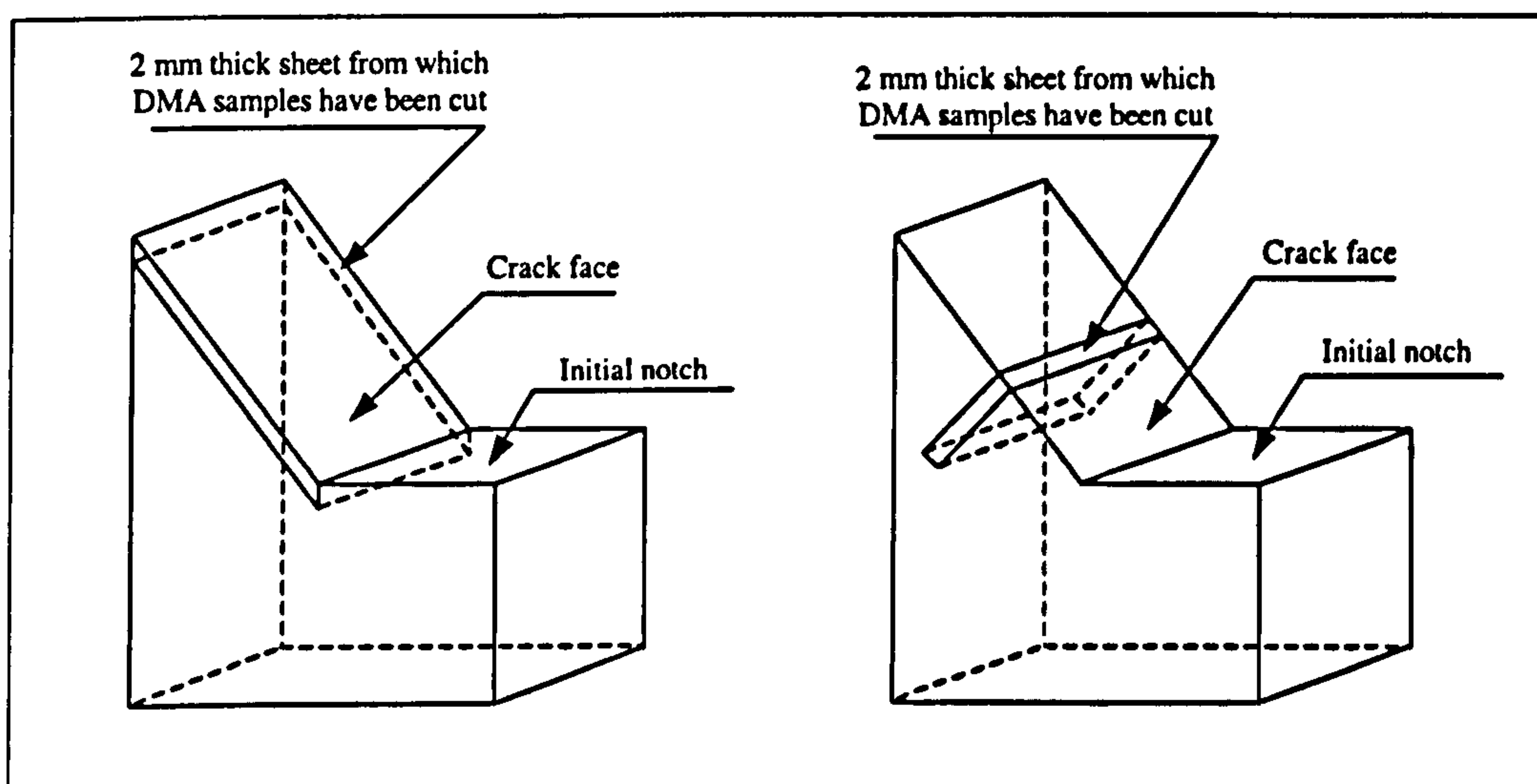


Figure 7.13: Sketch showing how the DMA samples were obtained from Noury damaged samples.

7.4.2.1. Complex Young's modulus

Rectangular samples with the crack face situated on one surface, on one short edge and on one long edge, respectively, have been tested. Two tests in each case have been carried out. For each condition, the two tests carried out gave similar results.

So, for better clarity the average of the measured storage Young's modulus and its associated loss factor values are represented in figure 7.14.

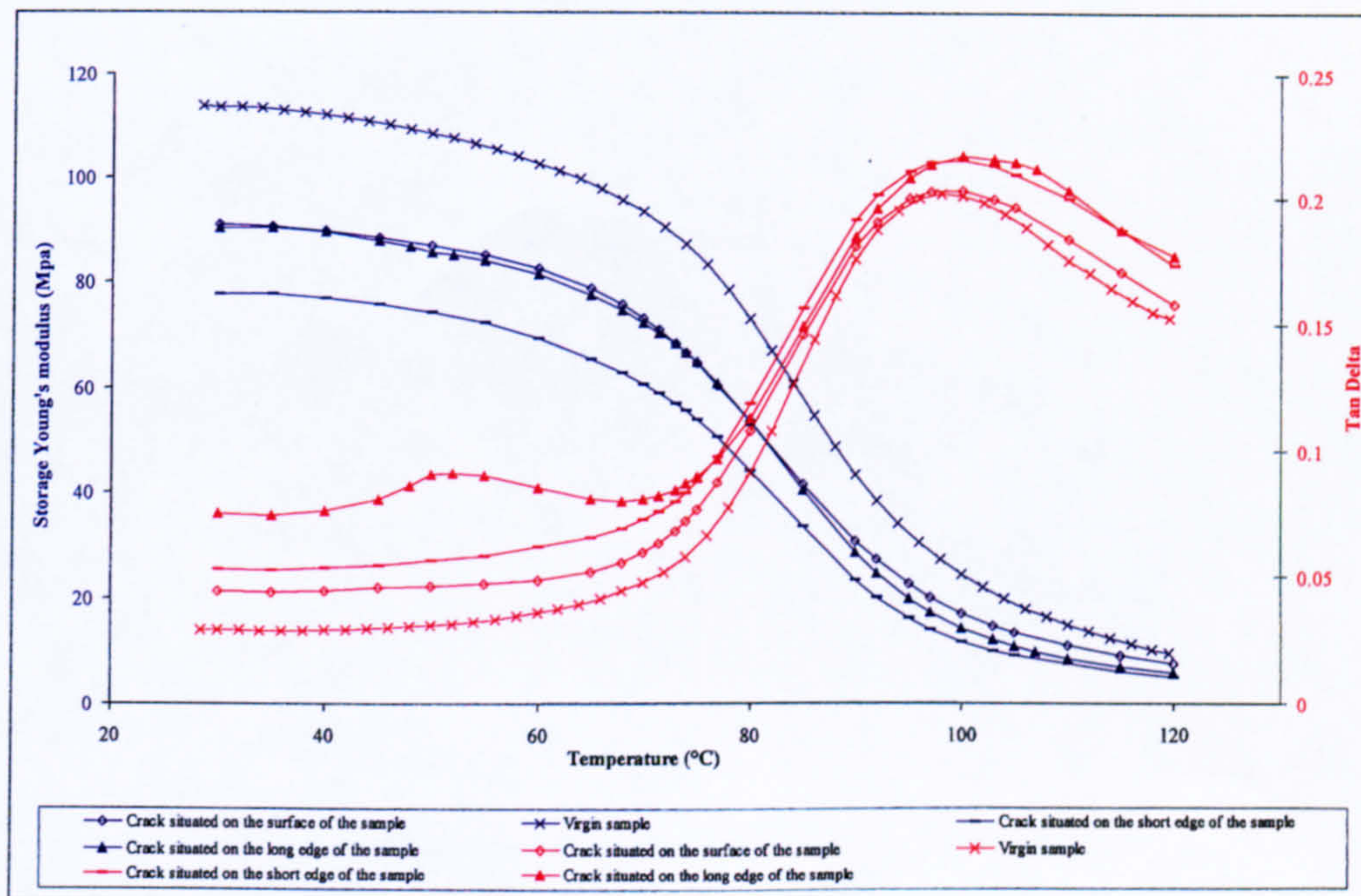
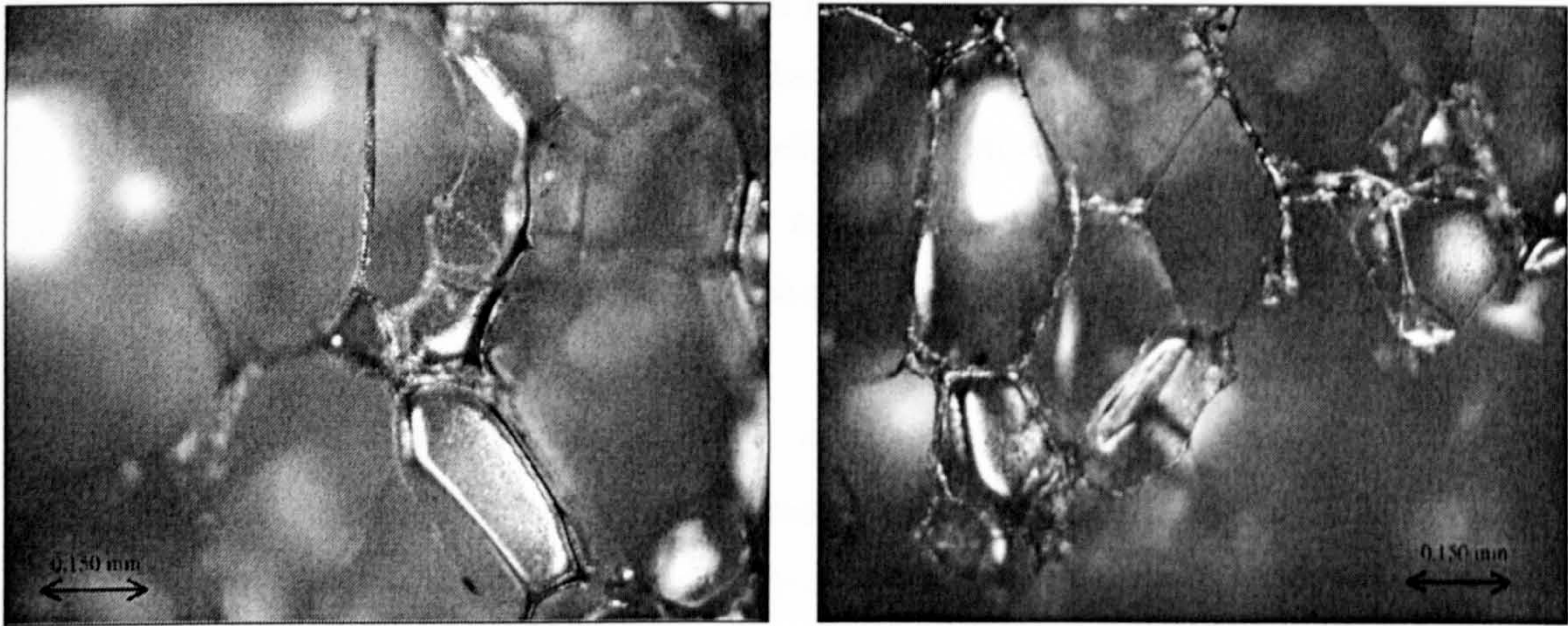


Figure 7.14: Effect of crack on the storage Young's modulus and its associated loss factor of HEREX C70.130 PVC foam material.

The results obtained, with the samples cut from the crack area, are compared with the average of the results presented in part 7.4.1.1 for virgin samples. The main issue highlighted by the analysis of the measured data concerned the behaviour of the dynamic properties of the HEREX C70.130 in the presence of cracks. Indeed, figure 7.14 shows that near the crack area the storage Young's modulus values are lower and the tan delta values are higher than for virgin samples. At ambient the drop in storage modulus is found to be between 20% and 32% of the initial value. At the same time an increase between 35% and 61% in the loss factor is observed. The presence of cracks appears to dramatically affect the dynamic material properties of the studied foam. The rate of decrease in the storage modulus values and the rate of increase in the loss factor values appear to be dependent on the orientation of the samples respective to the crack surface. Indeed, samples for which the crack surface is situated on a short edge show a greater loss in storage modulus values whereas samples for which the crack surface is localised on a long edge show a greater increase in loss factor values. These differences in the results, which are linked with different sample orientations, demonstrate that close to the crack surface the studied

foam material behaves as an anisotropic material. By looking at the figures 7.15, which have been taken at the crack surface, it can be observed that the cells are deformed. Indeed the shape of the cells is dramatically different to the shape of the cells observed in figure 7.6 for virgin material.



Figures 7.15: Images of the crack surface.

The formation of the crack has led to the elongation of the cells and to the rupture of the cell walls. From this observation it can be deduced that the anisotropy in the material properties found in the vicinity of the crack is likely to be linked to the deformation of the cell shape. Indeed, as explained by Gibson and Ashby [7.15], the anisotropy in the Young's modulus of closed cell foam materials varies at least as fast as the square value of the ratio of the largest cell dimension to the smallest one. This last ratio, which defines the degree of anisotropy in cell shape, has been found to reach values of 3.35 for a maximum measured cell length of 0.483 mm and a minimum measured cell width of 0.144 mm. The measurement of this ratio for 20 different cells situated on the crack surface gives an average shape-anisotropic ratio of 2.49 with a root mean square value of 0.55. The same measurements from cells of virgin samples provide an average ratio of 1.16 with a root mean square value of 0.16. It should be mentioned at this stage that no samples of virgin foam had been cut close to the edge of the manufacturing sheet. This last point leads to the conclusion that the value of 1.16 is likely to be a little bit lower than the one which would have been found if the ratio of cells situated at the edge of the manufacturing sheet had been taken into account. Consequently, this result for virgin material appears to be in good agreement with the typical value of 1.3 given by Gibson and Ashby for polymer foam [7.15]. These sets of measurements for both damaged and undamaged foam

confirmed the major increase in anisotropy of the cell shape during the formation of the crack. Since the shape-anisotropy ratio has been multiplied by about 2.14, the anisotropy in the complex Young's modulus is supposed, following Gibson and Ashby theory, to have been multiplied by about 4.6 in the specific area of the crack. It should be noticed that the ratio of anisotropy of the storage Young's modulus calculated using the DMA test results is not that high. This is believed to be due to the fact that the major deformation in the cell shape is localised at the crack surface and therefore does not affect all the cells of the tested samples. Finally, it can be concluded from this analysis that the complex Young's modulus of the foam is largely affected by a change in cell shape due to the presence of cracks.

It should be noticed at this stage that it was only possible to observe the surface of the crack area through the microscope. Indeed, in order to investigate the evolution of the deformation when moving away from the crack some samples had been cut perpendicular to it. Unfortunately taking pictures through the microscope of samples previously mechanically cut proved to be impossible. The equipment used to cut the samples leads to an imperfect surface finish that results in a fuzzy image when viewed under the microscope.

From the results presented in this part the following points should be emphasised:

- The complex Young's modulus of the PVC foam is affected by the presence of a crack.
- In the vicinity of the crack surface, the storage modulus values are reduced and the tan delta values are increased.
- The changes in the properties are linked with the deformation and elongation of the cell shape.
- It is believed that at the crack surface the anisotropy of the foam is widely dependent on the deformation, namely the elongation of the cells. This implies that in the presence of the crack the properties of the foam become widely direction dependent.

7.4.2.2 Complex shear modulus

In order to investigate the effects of the presence of a crack on the complex shear modulus of the HEREX C70.130 PVC foam, three shear DMA tests have been carried out using square samples with the crack surface situated on one face and then with the crack surface situated on one edge. In figure 7.16 for each specific case the average measured values of the storage shear modulus and its associated tan delta are plotted versus temperature. Before carrying out the analysis of the results it should be reminded that it has been previously found that the dynamic material properties measured using the shear clamp are not as accurate as the dynamic properties measured using the cantilever clamp. Consequently, only a qualitative analysis is realised here with the aim to determine if the presence of a crack has the same effects on the complex shear modulus as it has on the complex Young's modulus.

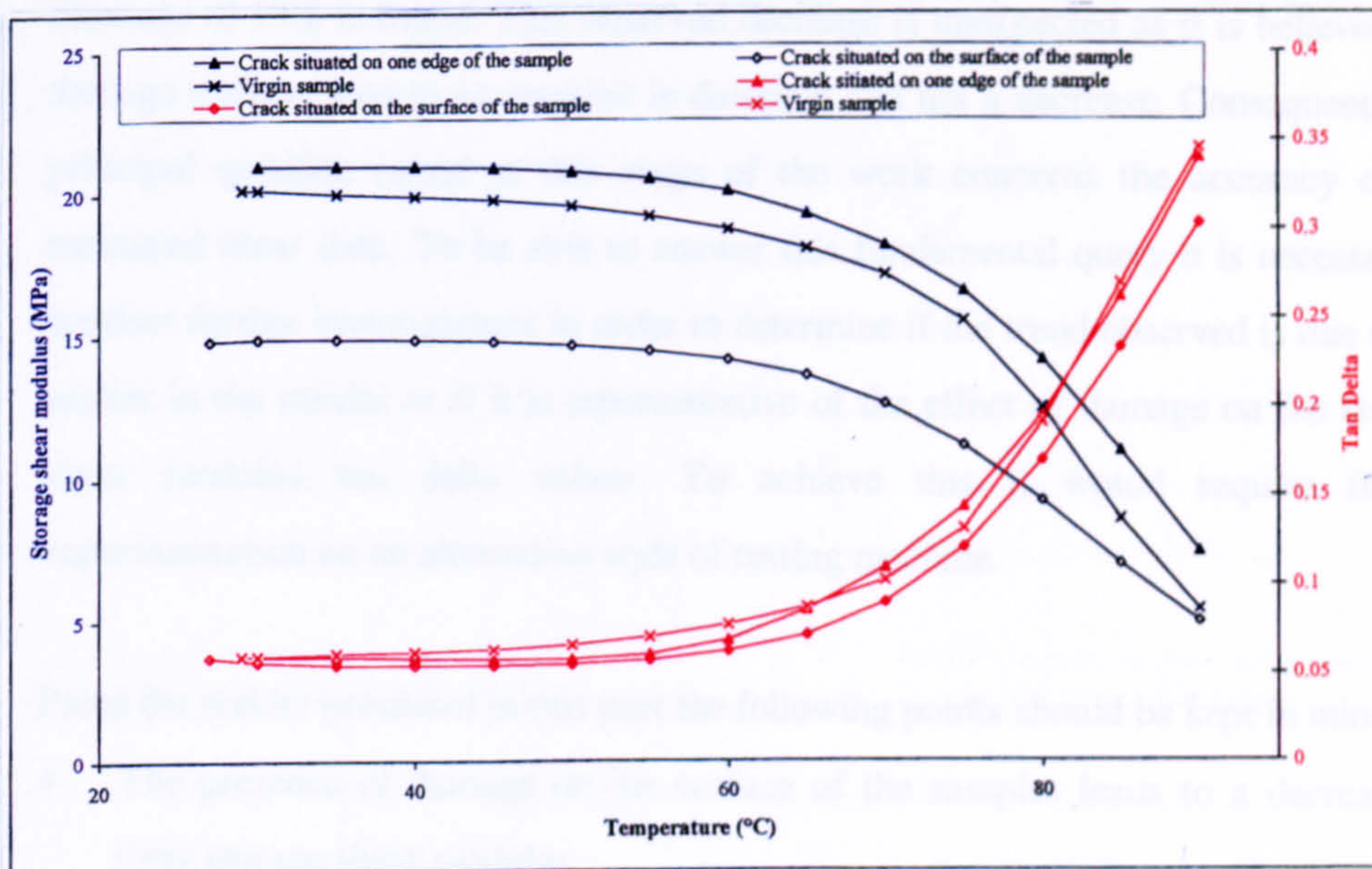


Figure 7.16: Effect of a crack on the storage shear modulus and its associated loss factor of HEREX C70.130 PVC foam material.

It is found from the measured data that the storage shear modulus values of samples with a crack situated on the surface are 26% lower than the storage shear modulus values of the virgin samples at 30°C. This decrease in the storage shear modulus due to the presence of crack on the surface of the sample is similar to the decrease observed for the storage Young's modulus. Besides this result confirms the results observed by Clark [7.16]. Indeed, just before failure, he observed a decrease in the

shear modulus value of several sandwich beams made of HEREX C70.130 PVC foam core material tested in fatigue.

The other measured results are more unexpected. The slight increase in the storage modulus of the sample with a crack on the side as compared to the virgin sample is believed to be due to the inaccuracy of the shear DMA experimental technique. As explained by Asby and Gibson [7.15], the shear modulus is a lot less sensitive to anisotropy in cell shape as compared to the Young's modulus. Besides, it is believed that the presence of cracks on one side of these samples is not significant enough to affect the results of the experiments. By looking now at the tan delta measured values for the three studied cases, it appears that the presence of cracks tends to lead to a slight decrease in the tan delta values. Indeed, at 30°C a decrease of 5.8% in the tan delta values due to the presence of cracks on the surface is observed and at 60°C a decrease of 19% is found. This observed decrease is unexpected as it is believed that damage tends to lead to an increase in damping and not a decrease. Consequently the principal question raised at this stage of the work concerns the accuracy of the measured shear data. To be able to answer this fundamental query it is necessary to conduct further investigations in order to determine if the trend observed is due to the scatter in the results or if it is representative of the effect of damage on the storage shear modulus tan delta values. To achieve this it would require further experimentation on an alternative style of testing machine.

From the results presented in this part the following points should be kept in mind.

- The presence of damage on the surface of the samples leads to a decrease in their storage shear modulus.
- The experiments undertaken using the DMA equipment with sandwich shear clamp does not allow a conclusion to be confirmed on the effect of damage on the tan delta values.

7.4.3 FOUR-POINT-BENDING FATIGUED PVC FOAM SPECIMENS

Four point bending fatigue experiments had been carried out by Noury [7.19] on beams made of HEREX C70.130 PVC foam. The specimens were subjected to 75% of the static failure load at a frequency of 2Hz and with a load ratio of 0.1. DMA cantilever samples have been cut following the dimension requirements given in part 7.3.2 from three different Noury beam

specimens as shown in figure 7.17. The first specimen considered had been tested until failure. The second and third specimens had been tested up to 2800 and 2700 cycles respectively. As can be seen on the figure 7.17 rectangular DMA samples have been cut from both the middle and the side of each specimen. Then the complex Young's modulus values of each of the prepared samples have been measured using the DMA equipment from ambient up to 120°C. The aim of this present analysis is to investigate if the dynamic properties of the foam are affected only by the presence of cracks or if they are as well sensitive to fatigue. Besides, by taking samples from different areas of Noury's specimens it is hoped to be able to identify if any change in the dynamic properties is constant through the thickness of the beam or not.

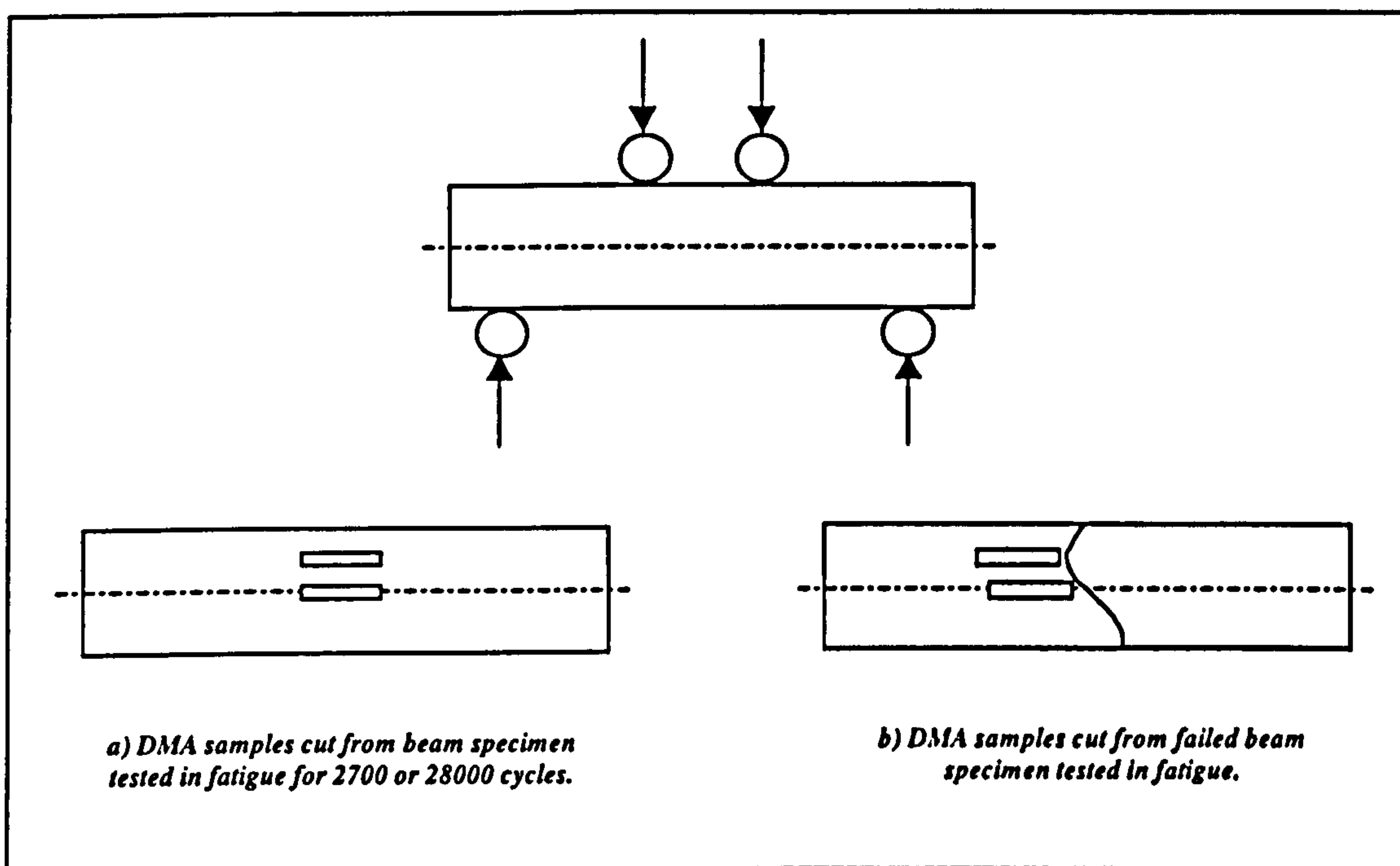


Figure 7.17: Four point bending PVC foam specimen.

It should be noted that in this part only the changes of the complex Young's modulus values due to fracture or failure are investigated. Indeed, due to the difficulties of obtaining satisfactory complex shear modulus data using the DMA equipment, it has been decided that no shear test will be carried out in this instance.

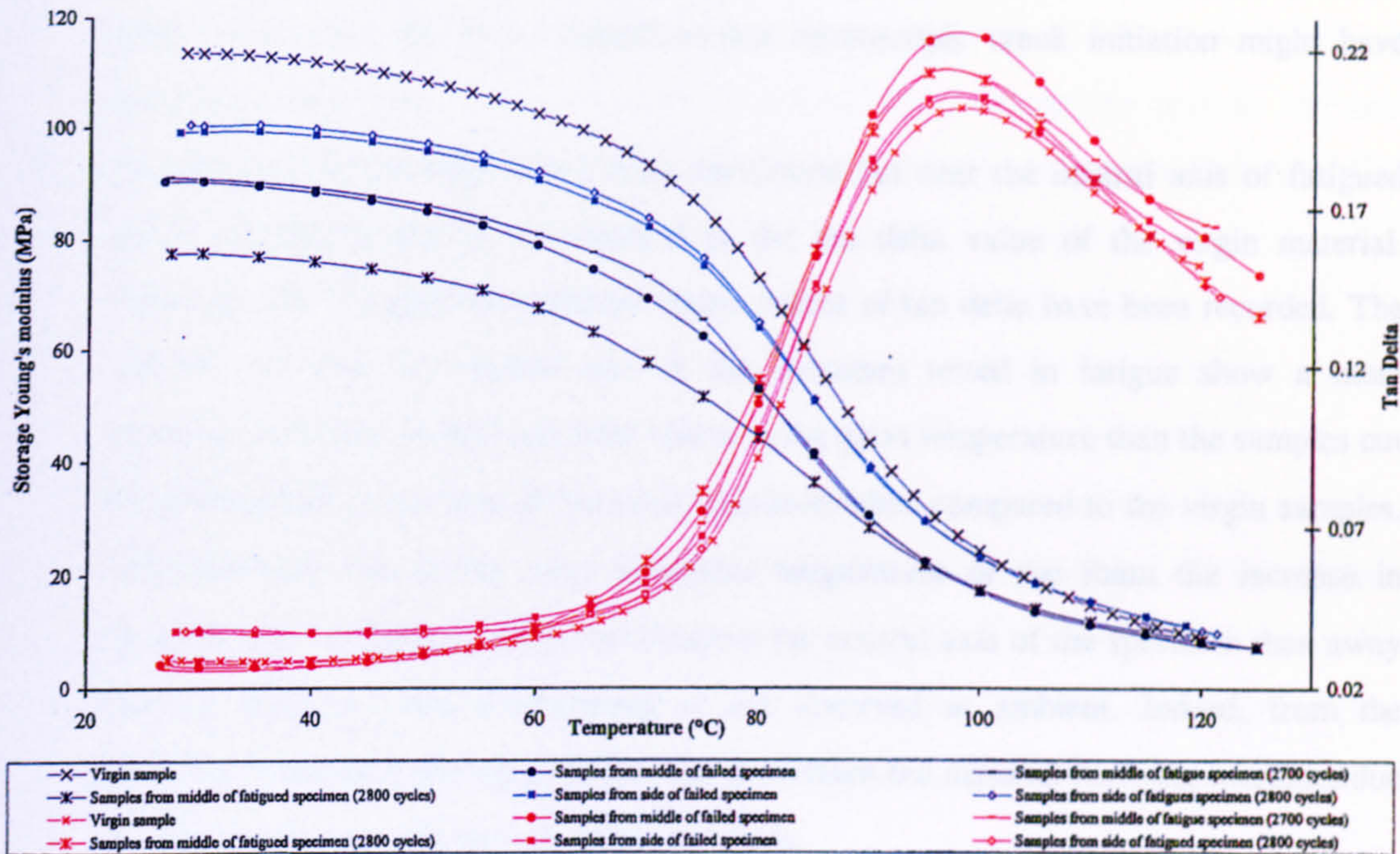


Figure 7.18: Effect of fatigue on the storage Young's modulus and its associated loss factor of HEREX C70.130 PVC foam material.

From the results presented in figure 7.18 various observations can be made.

- The decrease in the storage Young's modulus values appears to be more significant near the neutral axis of the beams tested in fatigue than away from the neutral axis of the beams. Indeed a decrease of between 20% and 32% have been found with the samples cut near the neutral axis of the beam whereas only a decrease of around 11.5% have been found with the samples cut away from the neutral axis of the specimen. These results correlate with Noury results [7.19]. Indeed, he observed that cracks initiate at the damage zone, which is located along the neutral axis of the core. Besides, he noticed that micro-deformations appear along the centre line of the fracture process zone before any damage could be identified visually. All these findings gathered together link to the conclusion that micro-deformation near the neutral axis of the core leads to a decrease in the storage Young's modulus of the foam.
- The decrease of the storage Young's modulus value is more significant near the neutral axis of the specimen fatigued for 2800 cycles than near the neutral axis of the failed specimen. This is believed to be due to the fact that the DMA samples for the fatigued specimen have been taken from the area where the failure would have been initiated, whereas for the failed beam the sample had been cut in the close vicinity of the crack but not exactly at the crack surface as represented in figure 7.17. It is observed that when the specimen reaches 2800 cycles the decrease in this storage modulus value is of 32%.

The same percentage of decrease has already been observed in the previous part for crack specimen. This is an indication that microscopic crack initiation might have occurred in this area.

- At ambient, the tan delta values from specimens cut near the neutral axis of fatigued tested samples appear to be identical to the tan delta value of the virgin material. However near the glass temperature higher values of tan delta have been recorded. The samples cut near the neutral axis of the specimen tested in fatigue show a more significant increase in their tan delta values at the glass temperature than the samples cut away from the neutral axis of the same specimen when compared to the virgin samples. This indicates that at the glass transition temperature of the foam the increase in damping due to fatigue is more significant at the neutral axis of the specimen than away from it. However this phenomenon is not observed at ambient. Indeed, from the experiment results it can be observed that at ambient the material damping increases due to fatigue only towards the side of the specimen.

The various pieces of information gathered during these experiments reveal a complex evolution of the behaviour of the HEREX C70.130 PVC foam when fatigued or damaged. More experiments will need to be carried out in the future to confirm the present observations. Besides, more in depth study will be required to fully understand the complex behaviour of the closed cell PVC foam.

7.5 FINAL REMARKS ON EXPERIMENTAL WORK

In this chapter, experimental methods, which can be used to measure the dynamic material properties of the composite panels constitutive materials, have been reviewed. Results of DMA experiments carried out on HEREX C70.130 PVC foam have been presented. The results obtained provide us with information on:

- The temperature and frequency dependence of the dynamic material properties of the HEREX C70.130 PVC foam;
- The degradation of these dynamic material properties due to the presence of damage.

The data gathered during this experimental work will allow us, as presented later on in chapter 8, to take into account the effect of the viscoelastic properties of the foam when investigating the dynamic behaviour of sandwich panels. It should be noted that the work

presented in this chapter should only be considered as a step forward that provides a better understanding of the complex behaviour of closed cell PVC foam and not as a full and complete analysis. Indeed, lots of questions have been raised during this analysis and more experimental works are required to be able to provide a more comprehensive picture of the complex behaviour of closed cell PVC foam.

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8. DYNAMIC ANALYSIS OF FRP COMPOSITE PLATES WITH VISCOELASTIC MATERIALS

8.1 INTRODUCTION

An analytical model based on Reddy's refined High order Shear Deformation Theory (HSDT), which allows the study of the dynamic behaviour of undamped FRP laminated or FRP sandwich panels, has been proposed in chapter 6. In this chapter, the method is enhanced in order to include the damping viscoelastic material properties and their frequency dependence. The aim is to create analytical models that will allow the prediction of the dynamic responses of FRP laminated and FRP sandwich panels, taking into account the viscoelastic properties of their constitutive materials. To achieve this, it is proposed to model the viscoelastic material properties using different mathematical tools depending on whether a steady state or a transient analysis needs to be carried out. The developed new methods are then used to predict the dynamic properties and the dynamic responses of specific FRP sandwich plates. Besides, an example of how these new techniques can be used to model damaged panels is presented.

8.2 THEORY

In this part both the theory related to the derivation of the new equations of motion, which take into consideration the viscoelastic behaviour of the constitutive materials, and the theory associated with the resolution of these equations are presented. It should be noticed that only the changes in the theory related to the introduction of viscoelastic models are outlined.

Viscoelasticity is the study of materials whose mechanical properties have both time-dependency and memory effects that is to say, the viscoelastic material response is not only determined by the current state of stress, but also by all past states of stress. Consequently, accounting for viscoelasticity can lead to difficulties in the derivation and the solution of the equations of motion of composite panels. Methods that account for the constitutive viscoelastic material properties have been reviewed in chapter 4. Different viscoelastic models have been suggested depending on the dynamic analysis that needs to be carried out. Under steady state condition, the use of the elastic-viscoelastic principle, which offers the possibility of easily and accurately modelling the frequency dependent dynamic properties, has been suggested. However, when transient analysis needs to be carried out an alternative solution should be considered. The feasibility of using the GHM Laplace transform method for the transient analysis case has been presented, in the

review, as the most suitable approach to consider for the investigation. In the following, the theory related to the implementation of the elastic viscoelastic principle into the steady state equations of motion and of the GHM mathematical model into the forced vibration equations of motion is presented.

8.2.1 STEADY STATE ANALYSIS

In the specific case where the structure is subjected to steady state oscillatory conditions, using the elastic viscoelastic principle, the stress-strain relation for each lamina with respect to the principal direction of the lamina can be expressed using equation 4.5. This relation for lamina made of orthotropic or isotropic viscoelastic materials can be re-written in the form of equation 8.1. It should be noticed that in relation 8.1 the through-thickness stress, σ_3 , is assumed to be negligible.

$$\begin{Bmatrix} \sigma_1(t) \\ \sigma_2(t) \\ \sigma_4(t) \\ \sigma_5(t) \\ \sigma_6(t) \end{Bmatrix} = \begin{bmatrix} C_{11}^*(i\omega) & C_{12}^*(i\omega) & 0 & 0 & 0 \\ C_{21}^*(i\omega) & C_{22}^*(i\omega) & 0 & 0 & 0 \\ 0 & 0 & C_{44}^*(i\omega) & 0 & 0 \\ 0 & 0 & 0 & C_{55}^*(i\omega) & 0 \\ 0 & 0 & 0 & 0 & C_{66}^*(i\omega) \end{bmatrix} \begin{Bmatrix} \varepsilon_1(t) \\ \varepsilon_2(t) \\ \varepsilon_4(t) \\ \varepsilon_5(t) \\ \varepsilon_6(t) \end{Bmatrix} \quad (8.1)$$

As for the elastic coefficients, the viscoelastic coefficients are related to the engineering properties as follows:

$$\begin{aligned} C_{11}^*(i\omega) &= \frac{E_1^*(i\omega)}{1 - \nu_{12}\nu_{21}}; & C_{22}^*(i\omega) &= \frac{E_2^*(i\omega)}{1 - \nu_{12}\nu_{21}}; & C_{12}^*(i\omega) &= \frac{\nu_{12} \times E_2^*(i\omega)}{1 - \nu_{12}\nu_{21}}; \\ C_{44}^*(i\omega) &= G_{23}^*(i\omega); & C_{55}^*(i\omega) &= G_{31}^*(i\omega); & C_{66}^*(i\omega) &= G_{12}^*(i\omega). \end{aligned} \quad (8.2)$$

The main advantage of using the elastic-viscoelastic principle to model viscoelastic structures subjected to steady state conditions is that the elastic constitutive equations can be easily updated to the viscoelastic cases. This is achieved simply by replacing the real elastic moduli by their frequency dependent complex viscoelastic counterparts. Therefore, as for the elastic case, using the Hamilton's principle the equations of motion appropriate for the displacement field associated with Reddy's refined HSDT (see table 3.1) and the constitutive equations 8.1 can be derived. The general vibration equations of motion can then be written in the form of relation 8.3.

$$[M]\{\ddot{\Delta}^*\} + [K^*(i\omega)]\{\Delta^*\} = \{F\} \quad (8.3)$$

The coefficients of the mass matrix are independent of the viscoelastic properties of the constitutive materials and so they can be derived as for the case of simply supported undamped FRP laminated and FRP sandwich plates using the relations provided in the appendix H. On the other hand, the coefficients of the stiffness matrix depend on the viscoelastic characteristics of the materials used. The coefficients of the stiffness matrix are therefore complex values and can be expressed in the following form:

$$[K^*(i\omega)] = [K'(\omega)] + i[K''(\omega)] \quad (8.4)$$

These complex coefficients for both simply supported rectangular FRP laminated or FRP sandwich plates are provided in appendix K and in the author's papers ([8.1], [8.2]). It should be noticed that they are a function of the complex laminate stiffnesses A_{ij}^* , B_{ij}^* , D_{ij}^* , E_{ij}^* , F_{ij}^* and H_{ij}^* which are calculated, knowing the constitutive material complex moduli in each layer, using the expressions provided in appendix L.

8.2.1.1 Free vibration analysis

The dynamic properties of structures made of amongst other things viscoelastic materials are characterised by their natural frequencies and their corresponding modal loss factor η for each frequency. One way to derive these properties is to carry out a free vibration analysis. The general free vibration equations for viscoelastic structures derived from relation 8.3 are given by:

$$[M]\{\ddot{\Delta}^*\} + [K^*(i\omega)]\{\Delta^*\} = \{0\} \quad (8.5)$$

To realise the free vibration analysis, a general periodic solution of the form $\{\Delta^*\} = \{\delta^*\} \exp(i\omega^* t)$ is considered where ω^* is the complex frequency and t is the time. By introducing this general periodic solution into relation 8.5, the free vibration equations 8.5 can be reduced to the eigenvalue equations 8.6.

$$([K'(\omega)] + i[K''(\omega)] - \lambda^*[M])\{\delta^*\} = \{0\} \quad (8.6)$$

where $\lambda^* = (\omega^*)^2$ are the complex eigenvalues and $\{\delta^*\}$ are the eigenvectors. By solving equations 8.6 the complex eigenvalues are obtained and are equal to $\lambda^* = \lambda' + i\lambda''$. After the complex eigenvalues have been obtained, the circular frequency ω and the modal loss factor η of the plate can be calculated for a given mode of vibration from the following formulae:

$$\omega = \sqrt{\lambda'} \text{ and } \eta = \frac{\lambda''}{\lambda'}. \quad (8.7)$$

It should be noticed that since the stiffness matrix coefficients are a function of the circular frequency, the free vibration equation 8.6 must therefore be solved iteratively. The procedure used is as described in figure 8.1.

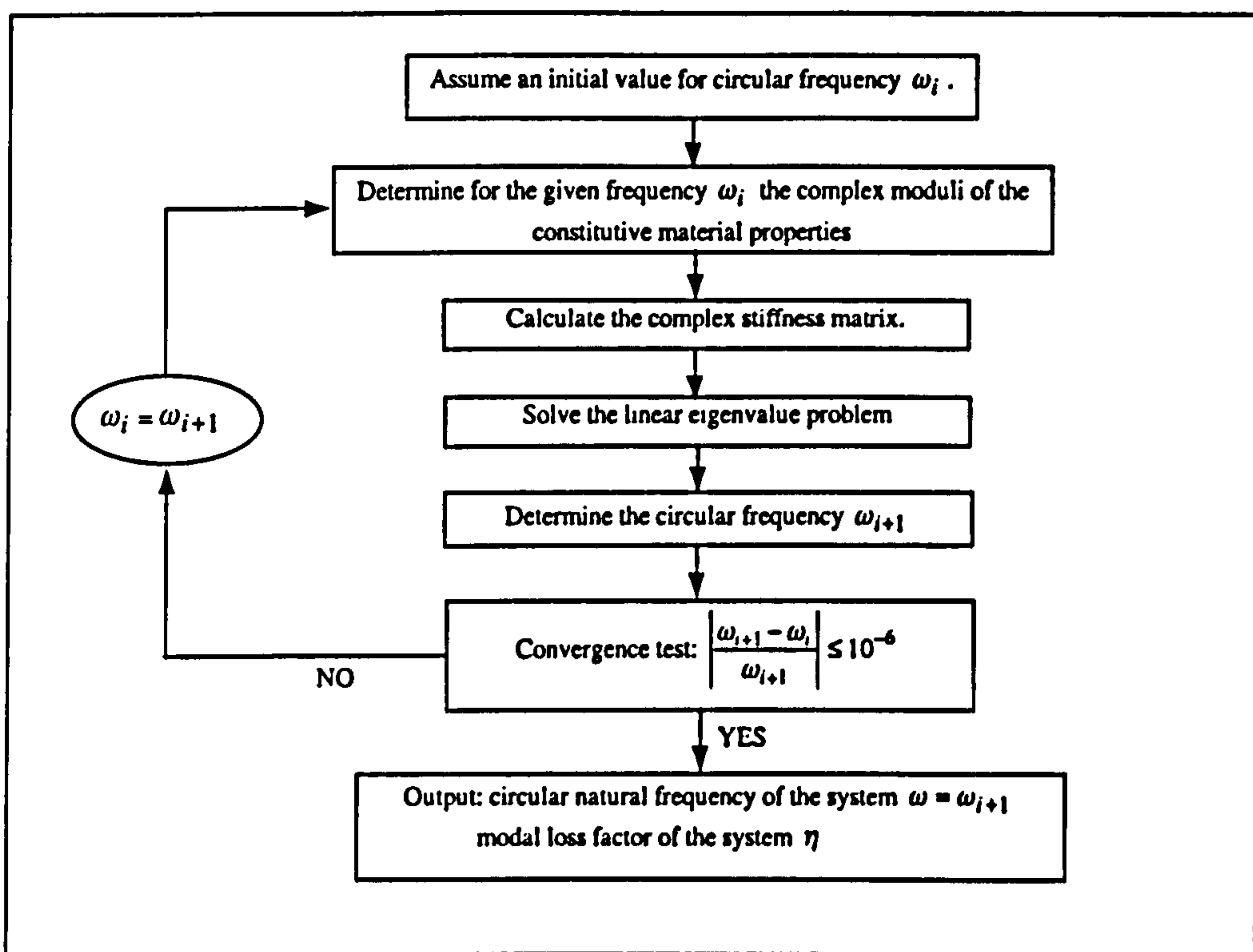


Figure 8.1: Solution flow diagram [8.1].

8.2.1.2 The direct frequency response method

The modal loss factor and the natural frequencies of the system can be determined as explained before by carrying out a free vibration analysis. They can as well be obtained using a different technique namely the direct frequency response method. In the direct frequency response method, a forced vibration over a range of frequencies is considered and a theoretical response spectrum is generated. The main objective of this task is to predict the damped steady state response of the studied structure. The studied structure

is therefore assumed to be subjected to forces that are harmonic in time. This implies that the column vector of the applied forces can be expressed using relation 8.8.

$$\{F\} = \{F_0 \exp(i\omega t)\} \quad (8.8)$$

In equation 8.8, ω is defined as the forcing frequency and t the time. The response induced by this type of applied force is assumed to be harmonic and to vibrate at the excitation frequency. Therefore, the relation 8.3, for the specific case where the studied structure is subjected to harmonic forces, reduces to relation 8.9.

$$([K'(\omega)] + i[K''(\omega)] - \omega^2[M])\{\delta^*\} = \{F_0\} \quad (8.9)$$

The matrix on the left-hand side of equation 8.9 is called the displacement impedance matrix. By solving this equation, for each frequency of excitation, the complex components of the column vector $\{\delta^*\}$ are calculated. Since these components are complex values their moduli are obtained by taking the square root of the sum of the square values of the real and imaginary parts and the phase angles are obtained by taking the ratio of the imaginary to the real parts as shown by equations 8.10 and 8.11 respectively.

$$|\delta_i| = \sqrt{(\text{Re}(\delta_i))^2 + (\text{Im}(\delta_i))^2} \quad (8.10)$$

$$\varphi_i = \tan^{-1}(\text{Im}(\delta_i)/\text{Re}(\delta_i)) \quad (8.11)$$

It should be noticed that both the modulus and the phase angle can only be calculated for one frequency at a time. In order to be able to obtain results over a wide range of frequency the same harmonic analysis needs to be carried out consecutively for several frequencies included in the range of interest. For each analysis it is first required to derive the frequency dependent coefficients of the complex stiffness matrix before being able to calculate the response of the structure using relations 8.10 and 8.11. The predicted response is then plotted as a function of frequency. The results can be used to determine the natural frequency of the system. Indeed, at resonance, which occurs when the predicted response reaches a maximum, the frequency of the exciting force is equal to the natural frequency. Besides, at resonance, the phase angle reaches the value of $\pm 90^\circ$ indicating that the exciting force and the predicted response are out of phase. The

loss factor of the studied structure can as well be calculated from the generated response by using the half power bandwidth technique. In this case the loss factor is given by relation 8.12.

$$\eta = \frac{(\omega_2 - \omega_1)}{2\omega_n} \quad (8.12)$$

In relation 8.12, ω_n is the frequency at resonance and ω_1 and ω_2 are the frequencies for which the value of $|\delta_i|$ falls to $|\delta_i|_{\max}/\sqrt{2}$.

8.2.1.3 Response to harmonic excitation

Two different methods can be used to determine the response of simply supported rectangular FRP laminated or FRP sandwich plates to a given harmonic excitation.

- Direct frequency response method

First of all the results, namely the modulus and phase values, obtained using the direct frequency response method can be directly used to derive, as a function of time t , the response of the simply supported rectangular FRP laminated or FRP sandwich panels subjected to an harmonic excitation, such as $\{F\} = \{F_0 \cos(\omega t)\}$. This is achieved by using relation 8.13.

$$\begin{Bmatrix} U_{mn}(t) \\ V_{mn}(t) \\ W_{mn}(t) \\ X_{mn}(t) \\ Y_{mn}(t) \end{Bmatrix} = \begin{Bmatrix} \operatorname{Re}(\delta_u^* \exp(i\omega t)) \\ \operatorname{Re}(\delta_v^* \exp(i\omega t)) \\ \operatorname{Re}(\delta_w^* \exp(i\omega t)) \\ \operatorname{Re}(\delta_x^* \exp(i\omega t)) \\ \operatorname{Re}(\delta_y^* \exp(i\omega t)) \end{Bmatrix} = \begin{Bmatrix} |\delta_u| \cos(\omega t + \varphi_u) \\ |\delta_v| \cos(\omega t + \varphi_v) \\ |\delta_w| \cos(\omega t + \varphi_w) \\ |\delta_x| \cos(\omega t + \varphi_x) \\ |\delta_y| \cos(\omega t + \varphi_y) \end{Bmatrix} \quad (8.13)$$

The transverse displacement at any point $M(x, y)$ in the plate can then be determined as a function of time and is given by equation 8.14.

$$w(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn}(t) \sin(\alpha x) \sin(\beta y) \quad (8.14)$$

- Direct time integration method

A second approach to solve this problem consists of using the well-known equations of motion 8.15 of a damped linear system subjected to harmonic forces of equal frequency.

$$[M]\{\ddot{\Delta}\} + [D]\{\dot{\Delta}\} + [K]\{\Delta\} = \{F_o \exp(i\omega t)\} \quad (8.15)$$

In relation 8.15 $[M]$, $[D]$ and $[K]$ are respectively the real mass, damping and stiffness matrices. Equation 8.15 may be solved by means of the trial solution $\{\Delta\} = \{\delta^* \exp(i\omega t)\}$ where the column vector $\{\delta^*\}$ is allowed to be complex. This leads to relation 8.16

$$([K] - \omega^2 [M] + i\omega [D])\{\delta^*\} = \{F_o\} \quad (8.16)$$

By comparing equations 8.9 and 8.16 it appears that for the excitation frequency and for this frequency only it is possible to calculate the damping matrix $[D]$ using relation 8.17 and the stiffness matrix $[K]$ using relation 8.18.

$$[D] = \frac{[K''(\omega)]}{\omega} \quad (8.17)$$

$$[K] = [K'(\omega)] \quad (8.18)$$

Consequently the first step of the analysis consists of deriving for the frequency of excitation the coefficients of the complex stiffness matrix. Then the real stiffness matrix coefficients and the real damping matrix coefficients of relation 8.15 can be determined as explained above using relation 8.17 and 8.18. Knowing all the real matrices coefficients, equations 8.15 can be solved using Newmark's integration technique. The results provide the coefficients of the column vector $\{\Delta\}$ for each time step. Finally the transverse displacement at any point within the plate can be determined, as for when the problem is solved using the direct frequency response method, using relation 8.13.

8.2.2 TRANSIENT RESPONSE

The main objective of this part is to develop a method, which will allow the study of the transient response of FRP laminated and FRP sandwich panels having viscoelastic materials. As far as

structures having viscoelastic material properties are concerned, it is not possible to write their equations of motion in the commonly used form 8.19.

$$[M]\{\ddot{\Delta}\} + [D]\{\dot{\Delta}\} + [K]\{\Delta\} = \{F\} \quad (8.19)$$

Indeed, for a structure to be described by 8.19, its materials, including any viscoelastic materials, must exhibit dynamic stress-strain behaviour of a certain type. Storage modulus must be constant and loss moduli must increase linearly with frequency. As can be seen in figures 7.10 and 7.11, the dynamic material properties of the HEREX C70.130 do not behave in such an accommodating way. This is in fact the case for most real viscoelastic materials. Consequently, due to the non-perfect hysteresis behaviour of most viscoelastic materials it is not feasible when transient analysis is carried out to simply replace the damping matrix $[D]$ by $[K'']/\omega$ in relation 8.19. It is not feasible either to suppress the damping term $[D]$ and to treat the stiffness matrix as complex. Indeed, both these approaches when used to model the frequency dependent dynamic material properties of the studied panels restrict the analysis to a single frequency, steady state analysis. Consequently, when the transient analysis of composite panels made of viscoelastic materials is carried out, a different viscoelastic model, which should be consistent across a broad range of frequencies, should be considered. The lack of such a model has limited up until now most of the analysis to steady state conditions. However, as reviewed in chapter 4, Golla, Hughes and McTavish have recently developed the so-called GHM method for the modelling of linear viscoelastic space structures. This approach provides an alternative method which includes viscoelastic damping effects without restriction of the steady state motion by providing extra co-ordinates. The complex modulus is in this case written in the Laplace domain as shown by relation 4.13. Being able to use this new approach to model the viscoelastic behaviour of FRP composite structures appears really attractive since it will offer the feasibility to predict their transient response. The main objective of this part 8.2.2 is to present the theory which has been developed to enhance the GHM method such that it can be used to model the viscoelastic behaviour of FRP laminated and FRP sandwich structures. This has been achieved, keeping in mind that to be useful this method needs be combined with the previously developed analytical model for FRP laminated and FRP sandwich panels (see chapter 6). By combining these two models, a new method is proposed which offers the possibility of accurately predicting the transient responses of FRP laminated and FRP sandwich panels accounting for their frequency dependent dynamic properties. In the following, the theory, which is related to the derivation of this new approach, is presented.

8.2.2.1 Equations of motion in the Laplace domain

The equations of motion for composite panels made of linear viscoelastic materials can be derived in the same way as the equations of motion for panels made of elastic materials simply by using the linear hereditary stress-strain law (see equation 4.3) instead of the generalised Hooke's law (see equation A.1). The obtained equations of motion are expressed in the form of relation 8.20.

$$[M]\{\ddot{\Delta}(t)\} + \int_0^t [K(t-\tau)]\{\dot{\Delta}(\tau)\}d\tau = \{F(t)\} \quad (8.20)$$

Equations 8.20 can then be transformed in the Laplace domain. The result of this transformation leads to relation 8.21.

$$s^2[M]\{\tilde{\Delta}(s)\} + [s\tilde{K}(s)]\{\tilde{\Delta}(s)\} = \{\tilde{F}(s)\} \quad (8.21)$$

In relations 8.20 and 8.21, the coefficients of the mass matrix $[M]$, which are independent of the viscoelastic material properties, can be calculated as for the steady state analysis using the relations provided in the appendix H. Having calculated the coefficients of the mass matrix, the main task results in deriving the coefficients of the stiffness matrix $[s\tilde{K}(s)]$ in the Laplace domain. This has to be achieved such that the transformation of relation 8.21 in the time domain leads to linear second order equations of motion. This task can be accomplished as explained in the part 8.2.2.2 by following the main outline of the approach developed by Golla Hughes and McTavish [8.3] for space structures made of isotropic viscoelastic material.

8.2.2.2 Determination of the stiffness coefficients functions

As mentioned before, an important step in the resolution of the problem lies in the derivation of expressions in the Laplace domain of the coefficients of the stiffness matrix $[s\tilde{K}(s)]$. The method developed to achieve this is based on the GHM approach presented in part 4.4.2.2 of the literary review. The GHM method requires the representation of the material modulus functions as a series of mini-oscillator terms in the Laplace domain as shown by relation 4.13.

In the particular case where the studied structure is made of a single material modulus the equation 8.21 can be simplified in the form of relation 8.22.

$$s^2 [M] \{\tilde{\Delta}(s)\} + s \tilde{G}(s) [K] \{\tilde{\Delta}(s)\} = \{\tilde{F}(s)\} \quad (8.22)$$

In this situation, only one material modulus function $s \tilde{G}(s)$ needs to be determined. This is achieved by modelling the material modulus function using a series of GHM mini-oscillator terms. Having determined the values of the independent parameters of the series of mini-oscillator terms using best-fit curves to the measured storage and loss modulus data, equations 8.22 can be solved following the procedure described in McTavish thesis [8.3]. The problem becomes much more complicated when the method is used to model FRP laminated or FRP sandwich panels. Indeed, in these cases equation 8.21 cannot be simplified and the stiffness matrix is composed of 15 different coefficients each of them being a function of at least one material modulus.

In order to apply the GHM method to this multi-modulus problem, a factorisation of the matrix $[s \tilde{K}(s)]$ is necessary. Indeed, the matrix has to be expressed as a series of component matrices $[s \tilde{K}_i(s)]$, each dependent upon a single modulus function $s \tilde{G}_i(s)$ as shown in relation 8.23.

$$[s \tilde{K}(s)] = \sum_i [s \tilde{K}_i(s)] = \sum_i s \tilde{G}_i(s) [\bar{K}] \quad (8.23)$$

Using relation 8.23 the equations of motion 8.21 in the Laplace domain can be re-written in the form of relation 8.24.

$$s^2 [M] \{\tilde{\Delta}(s)\} + \sum_i (s \tilde{G}_i(s) [\bar{K}]) \{\tilde{\Delta}(s)\} = \{\tilde{F}(s)\} \quad (8.24)$$

Each of the i^{th} modulus function in relation 8.24 can then be expressed in the mini-oscillator form 8.25.

$$s\tilde{G}_i(s) = \hat{G}_i^o \left(1 + \sum_k \alpha_i^k \frac{s^2 + 2\hat{\zeta}_i^k \hat{\omega}_i^k s}{s^2 + 2\hat{\zeta}_i^k \hat{\omega}_i^k s + (\hat{\omega}_i^k)^2} \right) \quad (8.25)$$

Then, by introducing a column vector of dissipation co-ordinates, as expressed by relation 8.26, the equations of motion in the Laplace domain 8.24 can be re-written in the form of relation 8.27.

$$\{\tilde{z}^k(s)\}_i = \frac{(\hat{\omega}_i^k)^2}{s^2 + 2\hat{\zeta}_i^k \hat{\omega}_i^k s + (\hat{\omega}_i^k)^2} \{\tilde{\Delta}(s)\} \quad (8.26)$$

$$\left\{ s^2 \begin{bmatrix} [M] & [0] \\ [0] & [M_{\bar{z}}] \end{bmatrix} + s \begin{bmatrix} [0] & [0] \\ [0] & [D_{\bar{z}}] \end{bmatrix} + \begin{bmatrix} [K_q] & [K_{q\bar{z}}] \\ [K_{q\bar{z}}]^T & [K_{\bar{z}}] \end{bmatrix} \right\} \begin{Bmatrix} \{\tilde{\Delta}\} \\ \{\tilde{z}\} \end{Bmatrix} = \begin{Bmatrix} \{\tilde{F}\} \\ \{0\} \end{Bmatrix} \quad (8.27)$$

The expression of the constitutive sub matrices of the mass, stiffness and damping matrices in equations 8.27 are provided in appendix M.

The coefficients of the sub matrix $[M_{\bar{z}}]$ are a function of the coefficients of the modulus factored out stiffness matrices $[\bar{K}]$ as can be seen by looking at the relations provided in appendix M. Since the matrices $[\bar{K}]$ admit rigid motion the mass matrix in relation 8.27 is not guaranteed to be a definite positive. To remedy this situation, spectral decomposition of the stiffness matrices $[\bar{K}]$ is achieved and it leads to the derivation, as shown in relation 8.28, of the diagonal matrices $[\bar{\Lambda}]$, of the nonzero eigenvalues of the matrices $[\bar{K}]$ and of the matrices $[\bar{R}]$ of which each column corresponds to the associated orthogonalised eigenvectors. Finally it should be noticed that this decomposition separates the eigenvalues into elastic (non-zero) and rigid-body (or zero) modes.

$$[\bar{K}] = [\bar{R}] [\bar{\Lambda}] [\bar{R}]^T \quad (8.28)$$

Finally to achieve the objective of having fewer dissipative coordinates and a positive definite viscoelastic matrix, firstly each diagonal matrix $[\bar{\Lambda}]$ is multiplied by the

associated equilibrium modulus $\hat{G}_i^o$, such that $[\Lambda]_i = \hat{G}_i^o [\bar{\Lambda}]_i$. Then the substitutions provided in relations 8.29 and 8.30 are carried out.

$$\{z\}_i = [\bar{R}]^T \{\bar{z}\}_i \quad (8.29)$$

$$[R]_i = [\bar{R}]_i [\Lambda]_i \quad (8.30)$$

Applying these substitutions to equations 8.27 leads to the final form of the viscoelastic equations of motion written in the Laplace domain as expressed by relation 8.31. This relation is then re-written in the simplified notation form in relation 8.32.

$$\left\{ s^2 \begin{bmatrix} [M] & [0] \\ [0] & [M_z] \end{bmatrix} + s \begin{bmatrix} [0] & [0] \\ [0] & [D_z] \end{bmatrix} + \begin{bmatrix} [K_q] & [K_{qz}] \\ [K_{qz}]^T & [K_z] \end{bmatrix} \right\} \begin{Bmatrix} \{\tilde{\Delta}\} \\ \{\tilde{z}\} \end{Bmatrix} = \begin{Bmatrix} \{\tilde{F}\} \\ \{0\} \end{Bmatrix} \quad (8.31)$$

$$(s^2 [M_v] + s [D_v] + [K_v]) \{\tilde{\Delta}_v\} = \{\tilde{F}_v\} \quad (8.32)$$

In relation 8.32, the matrices $[M_v]$, $[D_v]$ and $[K_v]$ are respectively the GHM viscoelastic mass, damping and stiffness matrices. Their coefficients, which are amongst other things dependent on the mini-oscillator parameters and on the coefficients of the matrices $[R]_i$ and $[\Lambda]_i$, are derived using the expressions provided in appendix N.

8.2.2.3 Equations of motion in the time-domain

In the Laplace domain the stiffness coefficients are expressed using the GHM analytical model involving a sum of rational functions of second degree. The algebraic form of the mini-oscillator terms allows one to write the equations of motion in the Laplace domain in the form of relation 8.32. This then enables one to derive linear second order equations of motion in the time domain. The derived second order equations of motion in the time domain are expressed as shown by relation 8.33.

$$[M_v] \{\ddot{\Delta}_v\} + [D_v] \{\dot{\Delta}_v\} + [K_v] \{\Delta_v\} = \{F_v\} \quad (8.33)$$

After deriving the equations of motion in the time domain, the response of the studied structure subjected to a given applied load can be obtained by solving equation 8.33 using Newmarks numerical integration technique.

8.3 NUMERICAL ANALYSIS

The presented new methods are used in this part to predict the dynamic response of rectangular sandwich plates composed of two FRP composite laminated faces and a rigid PVC core. To achieve this, programs based on the theories presented above have been developed using the software MATLAB. Even so only FRP sandwich plates are considered in this part, it should be noticed that the developed programs can as easily be used to predict the dynamic responses of FRP laminated panels.

In order to predict the dynamic responses of simply supported cross-ply and/or antisymmetric angle-ply rectangular FRP sandwich panels the knowledge of the viscoelastic constitutive material properties (e.g.: complex moduli) is required. As far as FRP materials are concerned, a reasonable amount of data can be found in the literature. In the following examples, it is assumed that the faces of the sandwich panels are made of Eglass in DX210 epoxy resin for which the material properties, taken from Lin, Ni and Adams paper [8.4], can be found in table 8.1. The core on the other hand is assumed to be made of closed cell polymer foam, namely the HEREX C70.130 PVC foam, of which the viscoelastic properties have been determined experimentally as explained in chapter 7. The measured material properties of the HEREX C70.130 PVC foam are summarised in table 8.2.

Material properties	E_1' (GPa)	E_2' (GPa)	$G_{12}' = G_{13}' = G_{23}'$ (GPa)	η_1	η_2	$\eta_{12} = \eta_{13} = \eta_{23}$	ν_{12}	ρ (kg/m ³)
Eglass/DX-210	37.78	10.90	4.91	$1.385 \cdot 10^{-3}$	$8.037 \cdot 10^{-3}$	$1.0998 \cdot 10^{-2}$	0.3	1813.9

Table 8.1: FRP laminated faces dynamic material properties [8.4].

Material properties	E' (MPa)	η_E	G' (MPa)	η_G	ν	ρ (kg/m ³)
HEREX C70.130 @ 30°C	113.5	0.0288	18.86	0.067	0.32	130
HEREX C70.130 @ 90°C	$7.980\ln(f) + 32.331$	$-0.013\ln(f) + 0.210$	$3.973f^{0.0806}$	$0.389f^{-0.0347}$	0.32	130

Table 8.2: Measured dynamic material properties of HEREX C70.130 PVC foam at two different temperatures.

8.3.1 DYNAMIC PROPERTIES OF FRP SANDWICH PLATES

In each of the following two examples the free vibration analysis of two different plates is carried out. For the first plate each face is assumed to be made of three layers orientated at 0°, 90° and 0°. In the second case the bottom face is composed of three layers orientated at -45°, 45° and -45° whereas for the top face the layers are orientated at 45°, -45° and 45°. All the layers in both the bottom and top faces are assumed to be of the same thickness.

8.3.1.1 Example 1

In this first example, simply supported sandwich plates made of HEREX C70.130 and of Eglass in epoxy resin are assumed to be at 30°C. At this temperature the core material properties have been found to be independent of frequency. This implies that no iterations are required to obtain the dimensionless natural frequencies, $\bar{\omega} = \omega a / (200\pi h)$, and modal loss factors, η , of the studied plates. The results obtained for the first mode of vibration as a function of different core thickness to total thickness ratios and different length to width ratios are provided in table 8.3 for the first plate and in table 8.4 for the second one.

a/b	h_c/h	0.58	0.64	0.70	0.76	0.82	0.88	0.91	0.94	0.97
1	$\bar{\omega}$	32.33	31.54	30.09	27.77	24.52	20.91	19.57	18.96	18.5
	η	0.009	0.009	0.010	0.012	0.017	0.029	0.038	0.044	0.040
2	$\bar{\omega}$	66.34	62.97	58.15	51.71	43.95	36.40	33.93	33.25	34.07
	η	0.009	0.010	0.012	0.014	0.020	0.034	0.043	0.051	0.051
4	$\bar{\omega}$	153.3	140.4	124.9	107.2	88.28	71.66	66.64	65.86	69.96
	η	0.011	0.012	0.013	0.016	0.022	0.036	0.047	0.056	0.059

Table 8.3: First dimensionless natural frequencies and modal loss factor of the simply supported sandwich plate ($a/h = 10$ and $[0^\circ/90^\circ/0^\circ/\text{core}/0^\circ/90^\circ/0^\circ]$) at 30°C.

a/b	h_c/h	0.58	0.64	0.70	0.76	0.82	0.88	0.91	0.94	0.97
1	$\bar{\omega}$	35.11	34	32.15	29.35	25.61	21.63	20.22	19.66	19.48
	η	0.008	0.009	0.010	0.013	0.018	0.030	0.039	0.046	0.043
2	$\bar{\omega}$	62.62	65.59	60.09	53.02	44.75	36.87	34.22	33.66	34.74
	η	0.010	0.010	0.012	0.015	0.021	0.034	0.044	0.052	0.053
4	$\bar{\omega}$	155.3	141.9	126.1	108.1	88.91	71.99	66.83	65.92	69.95
	η	0.011	0.012	0.013	0.016	0.022	0.036	0.047	0.056	0.059

Table 8.4: First dimensionless natural frequencies and modal loss factor of the simply supported sandwich plate ($a/h = 10$ and $[-45^\circ/45^\circ/-45^\circ/\text{core}/45^\circ/-45^\circ/45^\circ]$) at 30°C .

8.3.1.2 Example 2

In this second example the same plates are considered. The temperature is assumed to be higher (90°C), so the frequency dependence of the dynamic material properties of the foam should be taken into account. The dynamic properties of the FRP laminated faces are assumed to be unchanged. The first dimensionless natural frequencies, $\bar{\omega} = \omega a / (200\pi h)$, and modal loss factors, η , obtained after a few iterations are presented in table 8.5 and 8.6. In this case it was found that a maximum of 6 iterations was required to achieve convergence.

a/b	h_c/h	0.58	0.64	0.70	0.76	0.82	0.88	0.91	0.94	0.97
1	$\bar{\omega}$	32.23	31.37	29.77	27.13	23.11	17.68	14.81	12.56	11.78
	η	0.009	0.010	0.012	0.016	0.026	0.065	0.116	0.204	0.270
2	$\bar{\omega}$	66.05	62.48	57.28	50.13	40.87	30.05	24.86	21.03	20.07
	η	0.010	0.011	0.013	0.018	0.031	0.074	0.128	0.220	0.296
4	$\bar{\omega}$	152.35	138.9	122.52	103.26	81.39	58.46	48.18	41.01	39.83
	η	0.011	0.013	0.016	0.021	0.035	0.080	0.135	0.226	0.303

Table 8.5: First dimensionless natural frequencies and modal loss factor of the simply supported sandwich plate ($a/h = 10$ and $[0^\circ/90^\circ/0^\circ/\text{core}/0^\circ/90^\circ/0^\circ]$) at 90°C .

a/b	h_c/h	0.58	0.64	0.70	0.76	0.82	0.88	0.91	0.94	0.97
1	$\bar{\omega}$	34.99	33.80	31.77	28.61	24.03	18.13	15.10	12.77	12.02
	η	0.009	0.010	0.012	0.016	0.028	0.067	0.120	0.210	0.281
2	$\bar{\omega}$	69.29	65.04	59.15	51.34	41.54	30.35	25.04	21.15	20.20
	η	0.010	0.012	0.014	0.019	0.032	0.075	0.130	0.222	0.300
4	$\bar{\omega}$	153.3	140.4	123.8	104.2	82.06	58.86	48.44	41.13	39.84
	η	0.012	0.013	0.016	0.021	0.035	0.079	0.134	0.224	0.303

Table 8.6: First dimensionless natural frequencies and modal loss factor of the simply supported sandwich plate ($a/h = 10$ and $[-45^\circ/45^\circ/-45^\circ/\text{core}/45^\circ/-45^\circ/45^\circ]$) at 90°C .

8.3.1.3 Discussion

By comparing the results in table 8.3 and 8.5 respectively with the results in tables 8.4 and 8.6, it can be noticed that due to the inherent higher across the width stiffnesses of the angle ply FRP skins the natural frequencies for the angle ply sandwich panels are slightly larger than for the same cross ply sandwich panel. Nevertheless it can be observed that the difference in the layer orientation in the faces of the two sandwich panels does not affect the trend of the results. Consequently, both cases are analysed together. The aim of the following discussion is to identify from the obtained results the significance of the effects of the plate geometry, of the effects of the material properties and of the influence of external factors such as temperature on the dynamic response of simply supported rectangular FRP sandwich plates.

First of all, it can be observed that, for a constant plate thickness as the core thickness to plate thickness ratio decreases, the natural frequency increases and the modal loss factor decreases. This implies that the plate becomes stiffer and the damping effect principally due to the core material viscoelastic properties is reduced. Besides, as expected when the length to width ratio increases the natural frequency increases. This increase in the stiffness comes with a slight increase of the damping characteristics of the plate. Indeed a slight increase in the modal loss factor is observed. So as expected both the natural frequency and the modal loss factor are affected by a change in geometric characteristics.

As the temperature increases, the dynamic material properties of the core have been found to become frequency dependent. Besides, it has been observed that for a given frequency the storage moduli of the foam decreases as the temperature increases whereas the loss factor value attains a maximum value at the glass transition temperature. This frequency and temperature dependency of the dynamic material properties are factors that influence the dynamic response of the structure. As it can be observed in figure 8.2 a change in temperature noticeably affects the natural frequency and modal loss factor of the studied sandwich plates. Indeed an increase in temperature from 30°C up to 90°C leads to a decrease in the natural frequency and an increase in the modal loss factor. This variation in the dynamic material properties due to a change in temperature appears to be more pronounced for larger core thickness to total thickness ratio as can be noticed by comparing table 8.3 with table 8.5 and table 8.4 with table 8.6. This is principally due to the fact that the dynamic material properties of the faces are assumed to be unaltered.

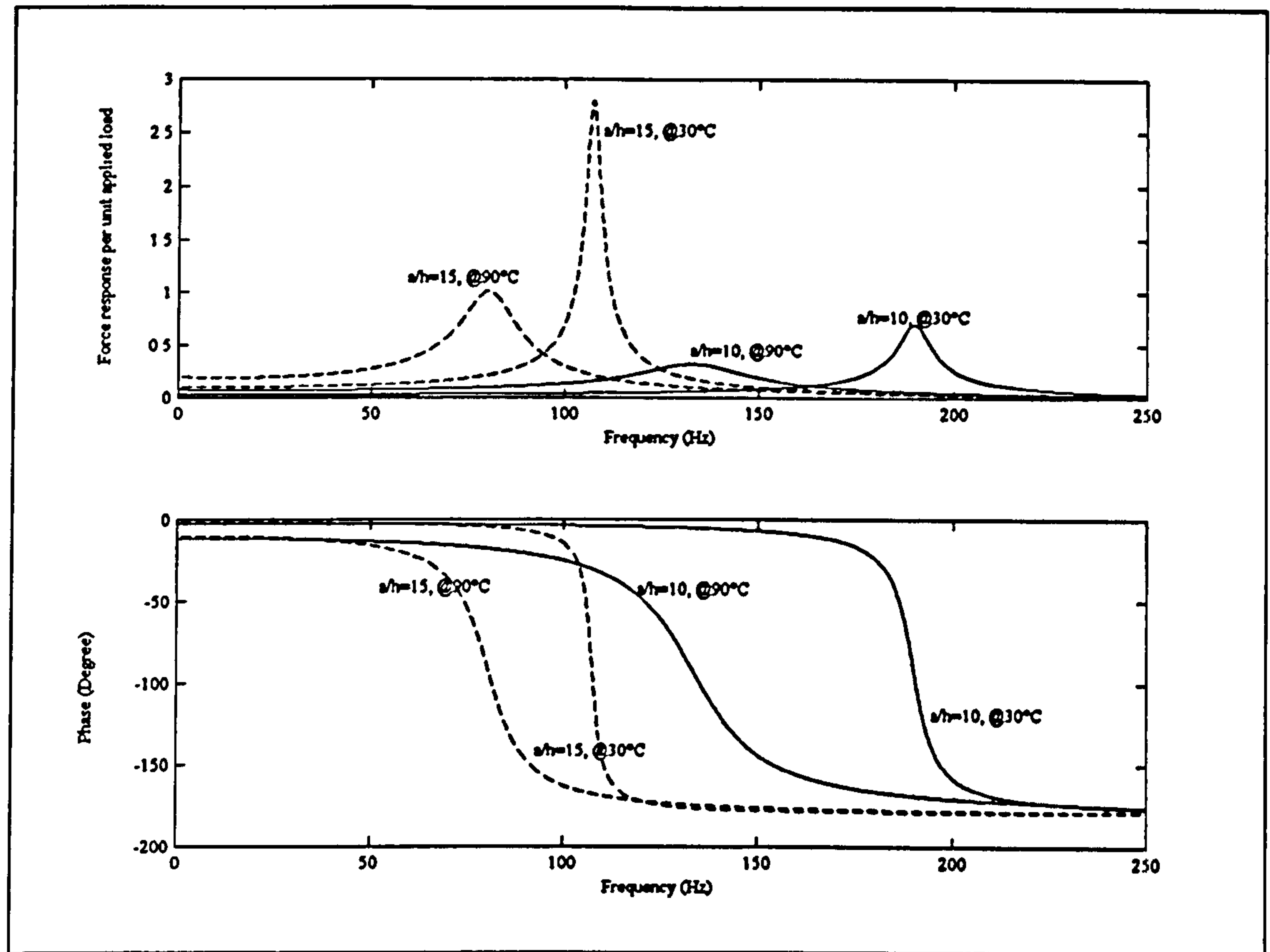


Figure 8.2: Force response, $|\delta_3| \cdot 10^6 / F_o$, and phase spectra, φ_3 , of square simply supported sandwich plates ($h=0.1m$, $h_c/h=0.94$ and $[0^\circ/90^\circ/0^\circ/\text{core}/0^\circ/90^\circ/0^\circ]$).

The results presented above leads to the conclusion that when the prediction of the dynamic response of FRP sandwich structures is carried out, it is essential to be able to include in the analysis the viscoelastic properties of their constitutive materials and it is necessary to be able to account for their temperature and frequency dependency.

8.3.2 HARMONIC ANALYSIS

In the following example the response of a cross-ply square sandwich panel subjected to a harmonic sinusoidally distributed force having an excitation frequency of 100 Hz is predicted. The panel is as described at the beginning of part 8.3. For this specific analysis it is assumed to be kept at a temperature of 30°C. Its core relative thickness, h_c/h , is assumed to be equal to 0.94 and the length to thickness ratio, a/h , to 10.

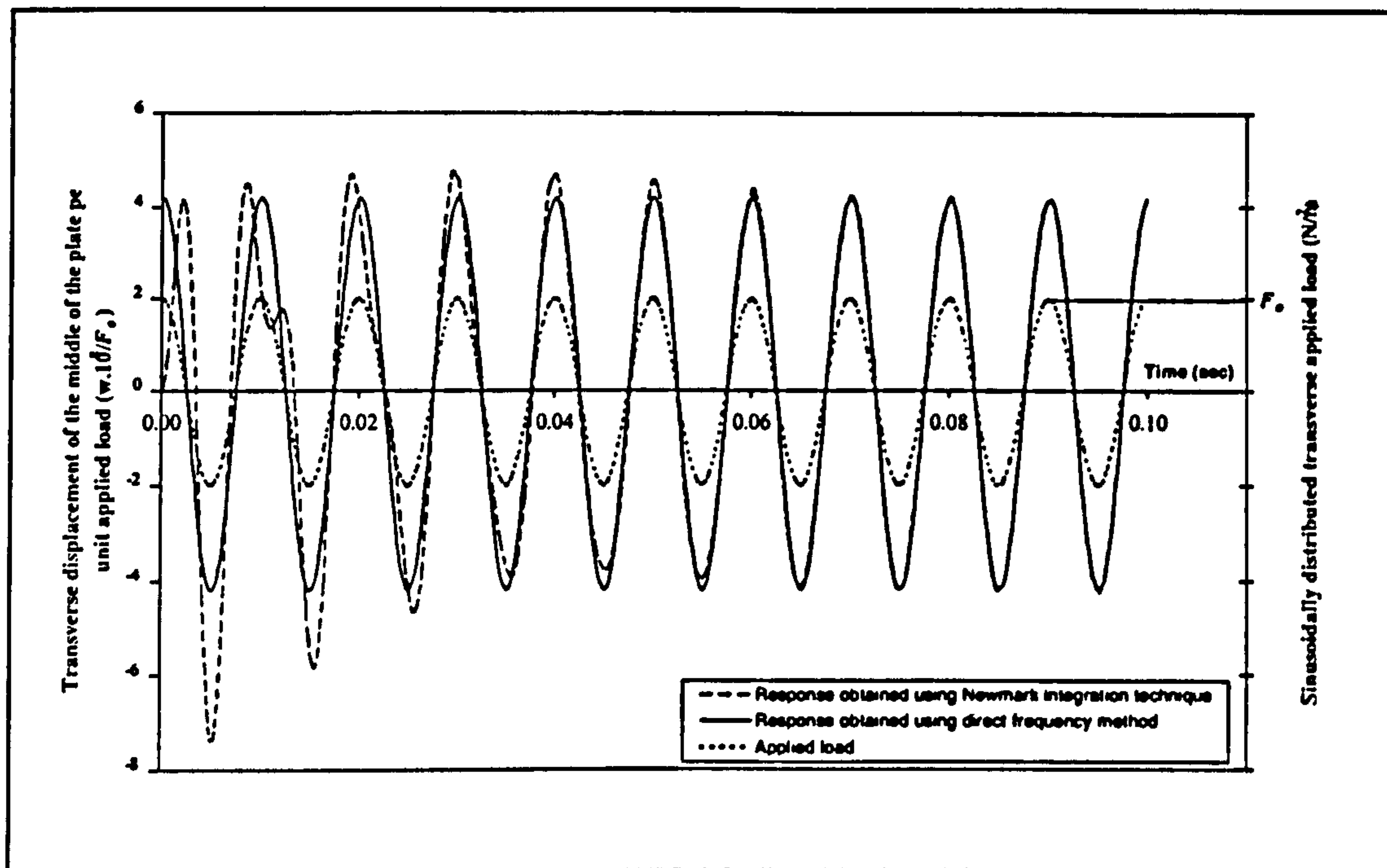


Figure 8.3: Response of FRP sandwich plate subjected to harmonic excitation ($f = 100\text{Hz}$).

In figure 8.3, the transverse displacement of the middle of the panel predicted using the two defined techniques, namely the direct frequency response method and the direct time integration method, is presented. At the excitation frequency of 100Hz , as can be observed in figure 8.2, the angle is nearly equal to 0 , which explains why the response of the studied panel appears to be in phase with the applied load in figure 8.3. The results presented in figure 8.3 show that when Newmark integration scheme is used to determine the harmonic response of the structure, the steady state conditions are only reached after a certain lapse of time. When, finally, the steady state conditions are reached, the response obtained with the Newmark method appears to be identical to the response obtained using the direct frequency method. Both methods can therefore be used to determine the response as a function of time of simply supported composite panel subjected to harmonic excitation. However, from a computational point of view, the time required to get the results over the same period of time using the direct frequency method is much faster than when the Newmark integration technique is used. Besides, there is no need with the first method to wait for the steady state condition to be reached. Consequently, the direct frequency method would be retained as the most appropriate method to carry out harmonic analysis.

8.3.3 TRANSIENT ANALYSIS

The transient analysis results presented in this part have been carried out for the same simply supported cross-ply sandwich panel as the one considered in part 8.3.2 kept at a temperature of 30°C. However in the present case the viscoelastic properties of the faces are neglected that is to say the tan delta values associated with the Eglass/DX-210 moduli are equal to 0. Since the faces are assumed to have elastic material properties only the complex Young's modulus and complex shear modulus of the core material will need to be modelled using the GHM mini-oscillator technique. The equations of motion in the Laplace domain of the studied FRP sandwich plate can be derived from relation 8.24 and so can be expressed in the form of relation 8.34.

$$s^2 [M] \{\tilde{\Delta}(s)\} + ([K]_f + s\tilde{G}_c(s)[\bar{K}]_{G_c} + s\tilde{E}_c(s)[\bar{K}]_{E_c}) \{\tilde{\Delta}(s)\} = \{\tilde{F}(s)\} \quad (8.34)$$

In this relation 8.34, $[K]_f$ is the stiffness due only to the faces of the sandwich panels, $s\tilde{G}_c(s)$ is the shear modulus function of the core material, $s\tilde{E}_c(s)$ is the Young's modulus function of the core material and $[\bar{K}]_{G_c}$ and $[\bar{K}]_{E_c}$ are the factorised out stiffness matrices respectively associated with the core shear and Young's modulus functions. The first step of the analysis consists of deriving the expressions for the modulus functions $s\tilde{G}_c(s)$ and $s\tilde{E}_c(s)$. This task is described in the following.

8.3.2.1 Derivation of the mini-oscillator parameters

As mentioned before, the first task consists of deriving the parameters of each mini-oscillator mathematical tool used to model each material complex modulus. Since in the present case only the core constitutive material is assumed to exhibit viscoelastic damping, only two series of mini-oscillator expression need to be determined namely one for the complex Young's modulus and one for the complex shear modulus of the HEREX C70.130 measured at 30°C. To achieve this, a least squares fit has been applied to both of the complex moduli spanning a defined range of frequencies. The real and imaginary parts of each complex modulus are considered simultaneously with equal weighting using the relations 8.35 and 8.36 which are derived from relation 8.25.

$$G_i'(\omega) = \hat{G}_i^o \left(1 + \omega^2 \sum_k \alpha_i^k \frac{(\omega^2 - (\hat{\omega}_i^k)^2) + (2\hat{\zeta}_i^k \hat{\omega}_i^k)^2}{(\omega^2 - (\hat{\omega}_i^k)^2)^2 + \omega^{21} (2\hat{\zeta}_i^k \hat{\omega}_i^k)^2} \right) \quad (8.35)$$

$$G_i''(\omega) = \hat{G}_i^o \left(\omega \sum_k \alpha_i^k \frac{(\hat{\omega}_i^k)^2 (2\hat{\zeta}_i^k \hat{\omega}_i^k)}{(\omega^2 - (\hat{\omega}_i^k)^2)^2 + \omega^{21} (2\hat{\zeta}_i^k \hat{\omega}_i^k)^2} \right) \quad (8.36)$$

The program used has been implemented using the software MATLAB. The function “lsqcurvefit” has been used to achieve the task [8.5]. “lsqcurvefit” function allows one to solve non-linear curve-fitting problems in the least-squares sense. This function requires:

- The input data; that is to say the frequencies for which GHM approximation of the material data is required;
- The output data; that is to say the real and imaginary values of the complex modulus for the corresponding input frequency;
- The equations containing the parameters that need to be derived.

It should be noticed that first of all the program has been run assuming a varying number k of mini-oscillator terms with the objective of finding the best approximation over the interval $f \in [1\text{Hz}, 100\text{Hz}]$. The computational time has been found to increase drastically when k increases. The optimum number of mini-oscillator terms for the studied examples has been found to be 3. However, depending on the frequency range of interest a different number of terms might be required. Indeed, if the frequency range of interest is increased in order to kept the same accuracy more mini-oscillator terms would be required for the modelling.

In figure 8.4 and in figure 8.5 respectively, the real part and imaginary part of the complex Young's modulus and the complex shear modulus respectively are presented in the frequency range from 1Hz to 100Hz. In these figures the approximated real and imaginary values of the complex moduli over the defined frequency range are compared with the calculated values using the equations provided in appendix K. The GHM mini-oscillator approximation form, of which coefficients have been determined by best fit curve, appears to accurately model, over the considered frequency range, the complex material moduli.

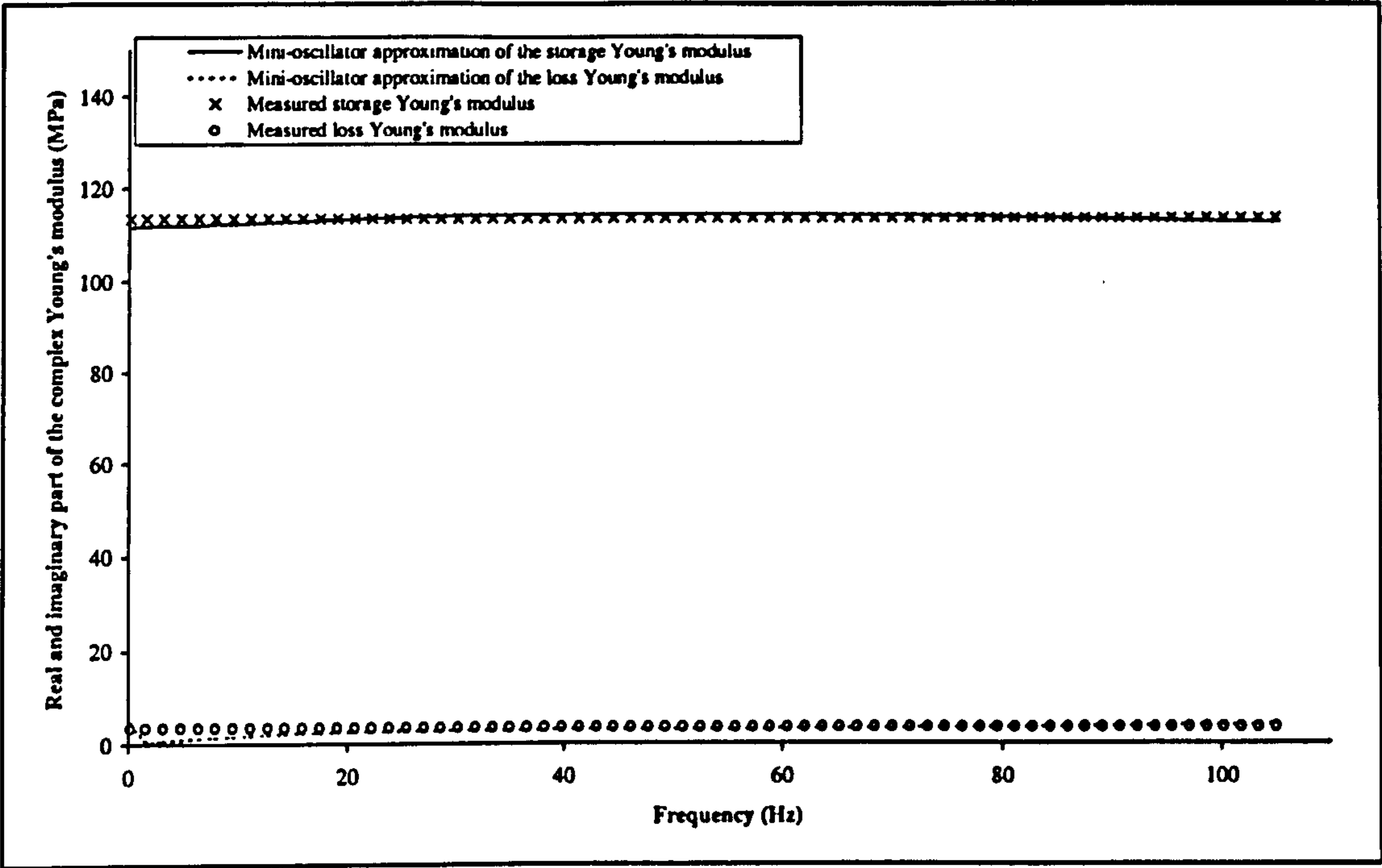


Figure 8.4: Comparison of the calculated values with the approximated values using the GHM approach for the real and imaginary parts of the complex Young's modulus.

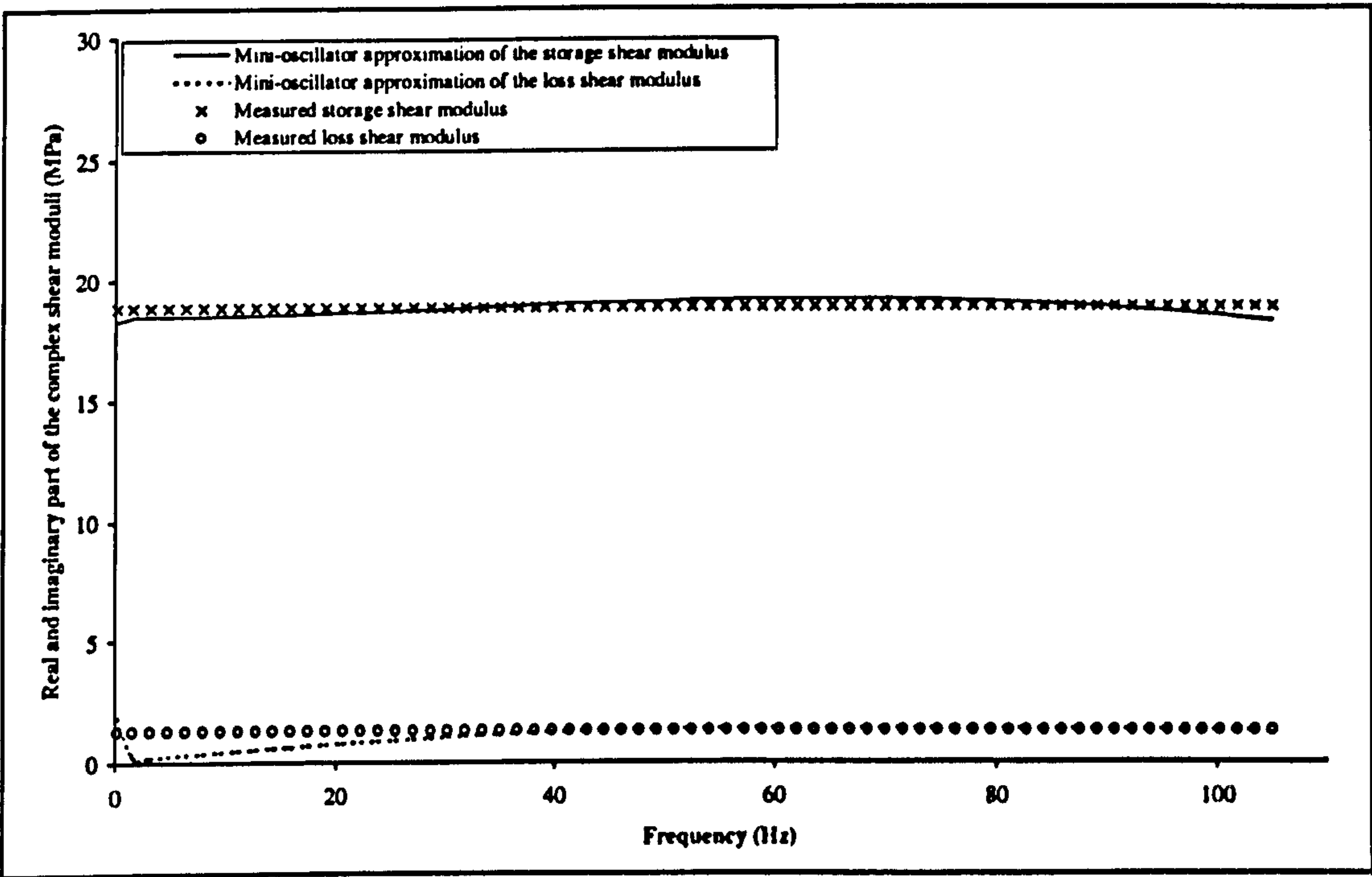


Figure 8.5: Comparison of the calculated values with the approximated values using the GHM approach for the real and imaginary parts of complex shear modulus.

8.3.2.2 Predicted transient response

Having determined for each stiffness coefficient the parameters of the GHM mini-oscillator representation, it is possible to derive the coefficients of the GHM viscoelastic mass, damping and stiffness matrices of the equations of motion 8.34 using the relations provided in appendix N. Having derived the coefficients of the viscoelastic matrices it is then possible to derive the resulting time domain second order equations as expressed by relation 8.33. These time domain second order equations of motion are solved using Newmark integration technique.

The example presented in figure 8.6 has been obtained assuming that the modelled sandwich plate is subjected to a transverse sinusoidally distributed load, $q = \sin(\pi x/a) \sin(\pi y/b)$, applied over a given period of time $t=0.1s$. Figure 8.6 shows the transverse displacement of the middle of the sandwich panel as a function of time when subjected to the previously defined load. It should be noticed from this example that the damping due to the viscoelastic behaviour of the PVC foam core material is not negligible. This emphasises the necessity of using the developed new method when the forced response of FRP composite plates needs to be predicted and analysed.

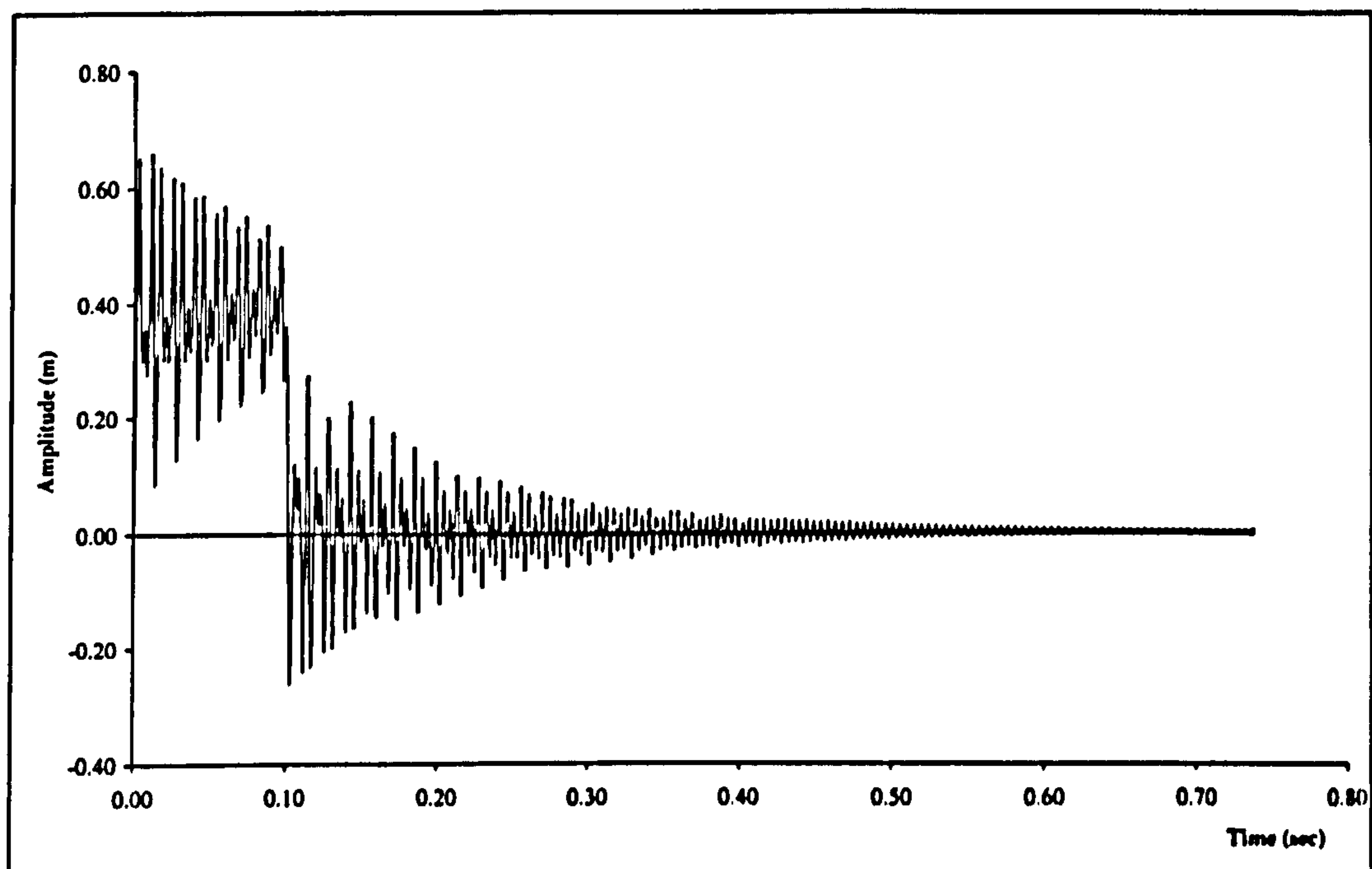


Figure 8.6: Dynamic response of a viscoelastically damped FRP square sandwich plate.

8.3.4 DYNAMIC PROPERTIES OF DAMAGED FRP SANDWICH PLATES

During their service life panels are likely to be subjected to damage. FRP sandwich panels with each face made of three layers orientated at 0° , 90° and 0° and the core made of HEREX C70.130 PVC foam is considered. The objective of this part is to predict the dynamic properties of such a panel assuming that cracks have appeared in the core. As noticed by Noury cracks start to develop in the middle of the core leading, as measured in chapter 7, to a local change in the dynamic material properties of the foam. Besides, it has been observed that cracks continuously grow away from the centreline towards the faces. The thickness of the damage zone, which is defined by h_d as shown in figure 8.7, consequently increases as the cracks propagate.

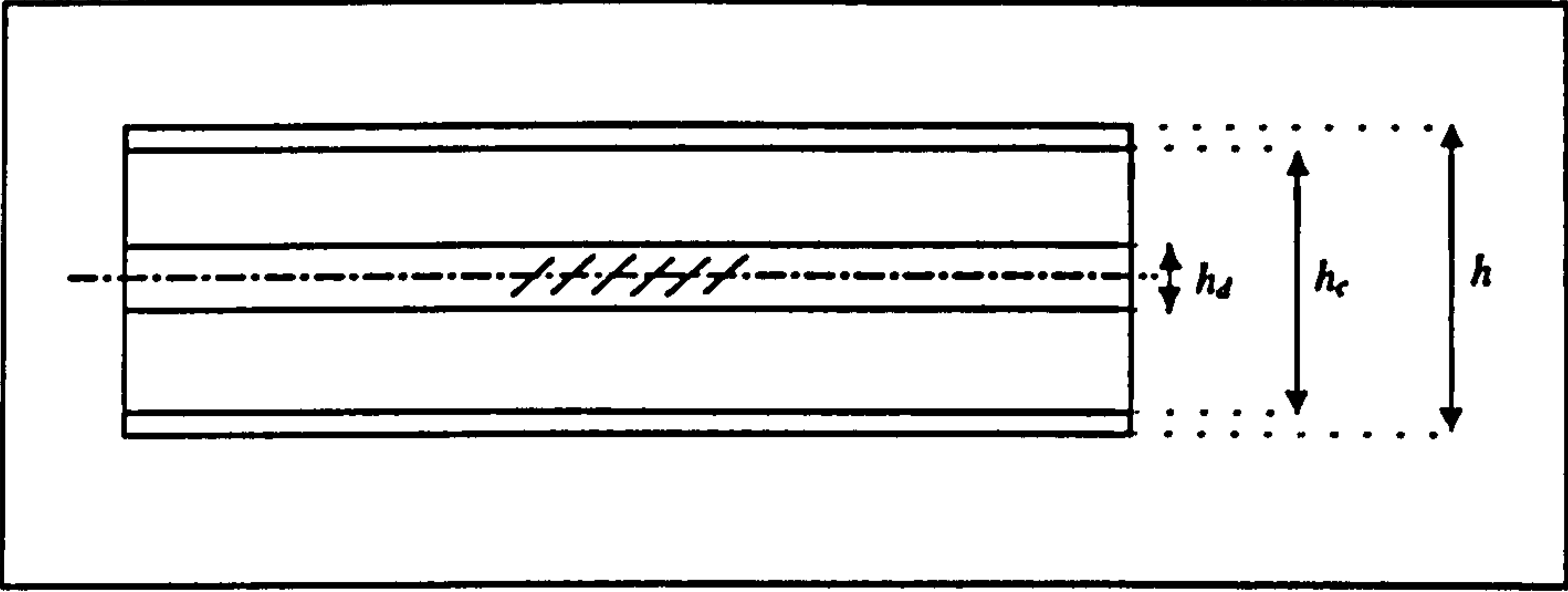


Figure 8.7: Cross section of a damaged sandwich panel

The dynamic properties of such a damaged panel can therefore be determined by running the program previously developed for an undamaged panel. This is achieved by assuming that a layer of thickness h_d of damaged PVC material has been added in the middle. That is to say the modelled damaged panel is made of a bottom layer of FRP material, a layer of undamaged PVC material, a middle layer of damaged PVC material, another layer of undamaged material and a top layer of FRP material. The material properties for the skins and the undamaged core respectively can be found as for the previous study of this chapter in table 8.1 and table 8.2 respectively. As far as the damaged PVC foam is concerned, it is assumed, from the analysis of the experimental results presented in chapter 7, that the dynamic material properties of the foam can be summarised as shown in table 8.7. It should be noticed that in this case it is assumed that the damaged foam material has isotropic viscoelastic material properties.

Material properties	E (Mpa)	η_E	G (Mpa)	η_G (not affected by damages)	η_G (30% increase)	ν	ρ (kg/m ³)
Damaged HEREX C70.130 @ 30°C	91	0.0388	14	0.067	0.087	0.37	130

Table 8.7: Dynamic material properties of damaged HEREX C70.130 PVC foam.

The dynamic properties of the damaged foam material presented in table 8.7 have been obtained by assuming a decrease of 20% in the storage Young's modulus, a decrease of 26 % in the storage shear modulus and an increase of 35 % in the tan delta value associated with the complex Young's modulus. Due, as explained in chapter 7, to the difficulty of obtaining good results with the shear test, no satisfactory tan delta value associated with the complex shear modulus has been measured. Consequently the analysis is undertaken for two different situations. In the first case, it is assumed that the tan delta value associated with the complex shear modulus is not affected by the presence of damage. In the second case it is considered that as for the tan delta value associated with the complex Young's modulus the tan delta value associated with the complex shear modulus is subjected to an increase. For this situation an increase of 30% in the tan delta value associated with the complex shear modulus is assumed.

In order to compare the dynamic properties of undamaged panels with the ones for damaged panels a free vibration analyses is carried out. For these analyses square cross ply sandwich panels of lengths equal to 1m, length to thickness ratio, a/h , equals 10 and relative core to thickness ratio, h_c/h , equals 0.94 are considered. In table 8.8 and 8.9 the dynamic properties of the previously defined square undamaged FRP sandwich panel are compared with the dynamic properties of the same panel with varying thicknesses of damaged zone.

h_d/h	0	0.04	0.08	0.12	0.16	0.20	0.28	0.36	0.44	0.52	0.60
$\bar{\omega}$	18.96	18.87	18.75	18.64	18.51	18.42	18.18	17.97	17.80	17.64	17.51
η	0.0437	0.0439	0.044	0.0441	0.0442	0.0443	0.0444	0.0446	0.0447	0.0448	0.0449

Table 8.8: First dimensionless natural frequencies ($\bar{\omega} = \omega a / (200 \pi h)$) and modal loss factor η of damaged simply supported sandwich plate ($\eta_G = 0.067$, $([0^\circ/90^\circ/0^\circ/\text{core}/0^\circ/90^\circ/0])$).

h_d/h	0	0.04	0.08	0.12	0.16	0.20	0.28	0.36	0.44	0.52	0.60
$\bar{\omega}$	18.96	18.85	18.74	18.62	18.51	18.40	18.18	17.98	17.80	17.64	17.51
η	0.0437	0.0445	0.0454	0.0462	0.0471	0.0479	0.0497	0.0513	0.0529	0.0543	0.0555

Table 8.9: First dimensionless natural frequencies $\bar{\omega} = \omega a / (200 \pi h)$ and modal loss factor η of damaged simply supported sandwich plate ($\eta_G = 0.087$, $([0^\circ/90^\circ/0^\circ/\text{core}/0^\circ/90^\circ/0])$).

As can be noticed by looking at table 8.8 and 8.9, the presence of damage in the core of FRP sandwich panels leads to a reduction in the first natural frequency and an increase in the modal loss factor. It should be noticed that if damage leads to an increase in the tan delta values associated with the complex shear modulus of the PVC foam then the modal loss factor of the sandwich panel becomes much more sensitive to the presence of damage. Indeed the increase in the modal loss factor with the increase in the damaged zone thickness is more pronounced for the second case than for the first case as can be noticed by comparing table 8.8 with table 8.9. Finally it should be emphasised that as far as the dynamic material properties of damaged foam are known it is possible to predict the effect of the damage and its evolution on the dynamic properties and hence the dynamic response of FRP sandwich plates. This type of analysis is valid as long as the complete failure of the structure has not occurred.

8.4 SUMMARY

New analytical models, which allow the prediction of the dynamic properties and the forced response of FRP laminated and FRP sandwich plates, have been developed. These models are based on Reddy's refined HSDT and offer the possibility, using the elastic-viscoelastic correspondence principle or the GHM mini-oscillator approximation respectively to carry out steady state analysis or transient analysis respectively. Analyses carried out using these new developed models have demonstrated that the dynamic response of FRP sandwich panels made of closed cell PVC foam core materials is influenced by the viscoelastic properties of the constitutive materials. That is to say a change in temperature or frequency which leads to a variation in the dynamic properties of the materials leads to a non-negligible change in the response of the structure. Therefore in order to predict with accuracy, under different environmental conditions, the dynamic response of FRP sandwich structures the viscoelastic properties of the constitutive materials as well as their temperature and frequency dependence should not be omitted from the analysis. Finally, it has been outlined that these new models can as well be used to predict the changes in the dynamic behaviour of FRP sandwich panels due to the appearance and the propagation of damage such as cracks in the core material.

8.5 REFERENCES

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PART D

Chapter 9: Discussion.

Chapter 10: Further work.

Chapter 11: Concluding remarks.

9. DISCUSSION

The research work, which has been presented in this thesis document, has been undertaken following the approach proposed in chapter 5. Extensive numerical and experimental investigations have been carried out. The new knowledge gained from these studies has been combined, as suggested in figure 5.1, to develop new analytical models that allow the designer to carry out the dynamic analysis of FRP laminated and FRP sandwich panels composed of viscoelastic constitutive materials. The developed models provide a new and a unique predictive tool to assess the overall dynamic behaviour of FRP laminated and FRP sandwich structures, commonly used in high performance vessels, at the initial design stage. A number of novel features have been highlighted in the relevant sections of the preceding chapters. They are summarised in the following.

9.1 ANALYSIS OF UNDAMPED FRP LAMINATED AND FRP SANDWICH PLATES

The dynamic analysis of FRP laminated and FRP sandwich plate is a subject that has received limited attention especially as far as forced response prediction is concerned. The feasibility of using analytical models based on either FSDT or HSDT to carry out dynamic analysis has been investigated. The first step of the work consisted of enhancing existing models based on FSDT and HSDT such that transient analysis as well as free vibration analysis of FRP sandwich plates and FRP laminated plates could be carried out. The work realised using the developed new analytical models has highlighted several limitations related to the use of FSDT. Indeed, it has been found that the accuracy of the results predicted using FSDT is strongly dependent on the shear correction factor value that needs to be determined for each specific sandwich panel. Besides, the use of the FSDT leads to inaccuracy in the prediction of forced response of relatively thick panels and of panels made of small numbers of layers. This work has therefore outlined the fact that the FSDT, while it is commonly used to carry out static analysis of FRP laminated plate, should not be used to model FRP sandwich plates or panels subjected to dynamic loading. On the other hand, the research has shown that the use of HSDT to model FRP laminated and FRP sandwich plates, leads to accurate prediction of the dynamic response when either free vibration or transient analysis is carried out. This HSDT modelling method, which has not been widely used up until now, has been revealed as the most appropriate technique to employ when dynamic analyses of FRP laminated and FRP sandwich plates need to be carry out.

9.2 DYNAMIC PROPERTIES AND BEHAVIOUR OF CLOSED CELL PVC FOAM

Due to the limited amount of previous work, there was a significant lack of data on the dynamic material properties of PVC foam material commonly used as core material in FRP sandwich structures. These data are the principal requirement for mathematical modelling methods applied to viscoelastic material. Their knowledge is therefore essential if dynamic analysis of FRP sandwich plate is carried out. Experimental work has provided new data and hence a better understanding on the dynamic material properties of closed cell PVC foam material and their temperature and frequency dependency. It has been highlighted that the frequency dependency of the dynamic material properties might need to be accounted for in the design stage depending on the temperature the constitutive PVC foam core material is likely to encounter during its service. Indeed, It has been found that dynamic material properties of PVC foam are significantly affected by an increase in temperature. Measurements have shown that at ambient the dynamic material properties are independent of frequency and that they become frequency dependent when the temperature increases. Besides, results have shown that for a given frequency, temperature increase leads to a reduction in storage modulus and an increase of tan delta up to the glass transition temperature.

In addition experimental work has provided information on the effect of damage such as cracks on the dynamic material properties of closed cell PVC foam material. This analysis has outlined the complex behaviour of this type of material. The presence of damage has been found to alter dramatically the shape and the structure of the cells, in the vicinity of the crack surface. This degradation was shown to lead to an important change in the dynamic material properties of the foam. The formation of cracks in a given FRP sandwich panel, due for example to fatigue, will lead to an important change in the dynamic material properties of the PVC foam core material in the cracks vicinity which cannot be neglected if dynamic analysis of such panels is carried out.

9.3 ANALYTICAL MODEL FOR VISCOELASTICALLY DAMPED FRP LAMINATED AND FRP SANDWICH PLATES

Forced response and steady state analysis of FRP laminated and FRP sandwich plates having viscoelastic material properties have been minimal due to the deficit of existing dynamic material data and due to the lack of analytical methods. The theoretical aspect of the research has led to new models that allow the prediction of the dynamic properties and forced responses of FRP

laminated and FRP sandwich plates having frequency dependent viscoelastic constitutive material properties. These new models have been developed by combining the HSDT with viscoelastic material modelling tools namely the elastic-viscoelastic correspondence principle or the GHM mini-oscillator mathematical method offering the feasibility to carry out either steady state analysis or transient analysis. This theoretical work has required the enhancement of the viscoelastic material mathematical tools in order to account for the anisotropy of the FRP laminated and FRP sandwich plates constitutive materials.

9.4 ANALYSIS OF VISCOELASTICALLY DAMPED FRP SANDWICH PLATES

Results from the experimental and the theoretical aspects of the research have been combined and the dynamic analysis of specific FRP sandwich plates has been carried out as an example. The analysis has highlighted the importance of being able to account for the dynamic material properties of the constitutive materials as well as their temperature and frequency dependency. Indeed it has been demonstrated that a change in temperature which affects the dynamic material properties has a non-negligible effect on the dynamic response of the structure. Besides, it has been demonstrated that the initiation of damage at the centre of the panel, leads to a change in the dynamic response of the structure caused by a degradation of the dynamic material properties of the core material in the damaged area. Finally, it has been observed that using the developed new analytical models, it was possible to gain a good understanding of the dynamic behaviour of the studied FRP sandwich panels. This can be of particular importance in the refining of safety factors presently applied even in the most carefully optimised design.

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10. FURTHER WORK

Several areas for further experimental and theoretical investigations have been identified. They are listed and described in the following.

10.1 EXPERIMENTAL WORK

The experimental work presented in this thesis document represents a fundamental first stage of understanding regarding the dynamic characteristics of closed cell PVC foam material. This current experimental work has highlighted some areas which require further work:

- The complex shear modulus of the HEREX C70.130 PVC foam should be measured using experimental tools other than the DMA equipment with sandwich shear clamp. It would be of interest to compare the obtained results with the results presented in this thesis document. The feasibility of using the DMA equipment in torsion mode to carry out these new experiments should be considered.
- Knowledge has been gathered concerning the dynamic behaviour and the dynamic properties of the HEREX C70.130 PVC foam material. Other structural foams are available to be used in FRP sandwich structure. Consequently, more experimental work should be carried out on a larger selection of foam materials in order to increase the database on the dynamic material properties of foam core materials.
- Published data concerning the dynamic properties of some FRP materials exist in the literature. However most of the time these properties have been obtained for FRP material commonly used in the aerospace industry. Carrying out experiments on FRP materials commonly used in the marine industry would be of great value.
- Presence of damage in PVC foam materials leads to the degradation of their dynamic material properties. Extensive experimental investigations should be carried out to try to improve the understanding of the complex behaviour of such materials. This would offer a more comprehensive picture of the complex behaviour of closed cell PVC foam. In parallel to the experimental work new mathematical models for the degradation of PVC foam material properties should be proposed. Such models implemented into the analytical models

presented in this thesis document will facilitate the prediction of the transformation of the dynamic response of FRP sandwich panels during their service life.

It would as well be of great interest to carry out experimentation at the plate level in order to get experimental data on the dynamic properties of FRP sandwich panels. The experimental results obtained for simply supported panels could then be compared with the theoretical results presented in this thesis document.

10.2 THEORETICAL WORK

New analytical models have been developed to study the dynamic behaviour of FRP laminated and FRP sandwich plates. These novel models open up the work to new areas of possible research. Suggestions for further theoretical works are summarised in the following.

- The new analytical models, which have been developed, restrict the analysis to simply supported plates. Based on the theory developed in this thesis, similar analytical models should be developed using Lévy solutions instead of Navier solutions. This will offer the possibility to investigate the behaviour of panels having two opposite edges simply supported with the other two being either free or clamped.
- It would be useful to combine the developed models with:
 - An optimisation algorithm in order to account for the dynamic behaviour and dynamic response of FRP laminated and FRP sandwich plates in design optimisation analysis;
 - Damage monitoring technique and/or damage detection techniques.
- Finally, a logical progression would be to implement this new technique into a design office/yard environment in order to fully evaluate the developed novel analytical model and to confirm the advantages and savings to be gained from being able to predict and understand the dynamic behaviour of composite structures more thoroughly.

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11. CONCLUDING REMARKS

The work presented in this thesis has provided an extensive examination on the dynamic analysis of FRP laminated and FRP sandwich plates. Theoretical investigations have been carried out leading to new analytical models. These new models are based on Reddy's refined HSDT. The HSDT has been demonstrated as being the most versatile and practical method to use, since it offers the feasibility to model accurately FRP laminated and FRP sandwich plates and does not require the derivation of extra parameters such as shear correction factors. The new approach based on HSDT has been formulated to account for the viscoelastic properties of the constitutive materials. It has been established that in order to predict with accuracy, under different environmental conditions, the dynamic response of FRP laminated and FRP sandwich structures, the viscoelastic properties of the constitutive materials as well as their temperature and frequency dependence should not be omitted in the analysis.

In addition, experimental results exposed in this thesis document have provided a comprehensive examination of the dynamic behaviour of the closed cell HEREX C70.130 PVC foam material commonly used as a core material in sandwich structures of high performance vessels. This experimental work has contributed to increasing the available knowledge on PVC foam materials as well as the database of their dynamic material properties. The gathered experimental data have provided essential information on the temperature and frequency dependency of the dynamic material properties of the studied foam. Furthermore, the experimental results on damaged samples have outlined the complex behaviour of closed cell PVC foam and have provided a firm basis of the understanding of the dynamic properties of damaged or fatigued foam.

The new developed analytical methods when combined with the knowledge of dynamic material properties provide a unique design tool which allows the prediction of the dynamic properties and the forced response of FRP laminated and FRP sandwich plates without neglecting the viscoelastic damping effects. Overall, a significant contribution to the understanding of how composite structures behave in a dynamic environment has been achieved. Such an understanding can in the long term lead to more precise optimisation of composite structural design.

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APPENDICES

A. CONSTITUTIVE EQUATIONS OF A LAMINA

A.1 GENERALISED HOOKE'S LAW FOR ANISOTROPIC MATERIALS

In this appendix, the material of the studied body is assumed to be linearly elastic. That is to say the body recovers its original form completely upon removal of the forces causing deformation and not on the history of that deformation. A body whose elastic properties are different in different directions is said to be anisotropic. For a homogeneous elastic body of general anisotropy the generalised Hooke's law is expressed by relation A.1.

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{21} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} & C_{1132} & C_{1131} & C_{1121} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} & C_{2232} & C_{2231} & C_{2221} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312} & C_{3332} & C_{3331} & C_{3321} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312} & C_{2332} & C_{2331} & C_{2321} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} & C_{1332} & C_{1331} & C_{1321} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} & C_{1232} & C_{1231} & C_{1221} \\ C_{3211} & C_{3222} & C_{3233} & C_{3223} & C_{3213} & C_{3212} & C_{3232} & C_{3231} & C_{3221} \\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3113} & C_{3112} & C_{3132} & C_{3131} & C_{3121} \\ C_{2111} & C_{2122} & C_{2133} & C_{2123} & C_{2113} & C_{2112} & C_{2132} & C_{2131} & C_{2121} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{13} \\ \epsilon_{12} \\ \epsilon_{32} \\ \epsilon_{31} \\ \epsilon_{21} \end{Bmatrix} \quad (\text{A.1})$$

This linear relation A.1 relates the nine components of stress, which are represented in a three dimensional Cartesian system in figure A.1, to the nine components of strain.

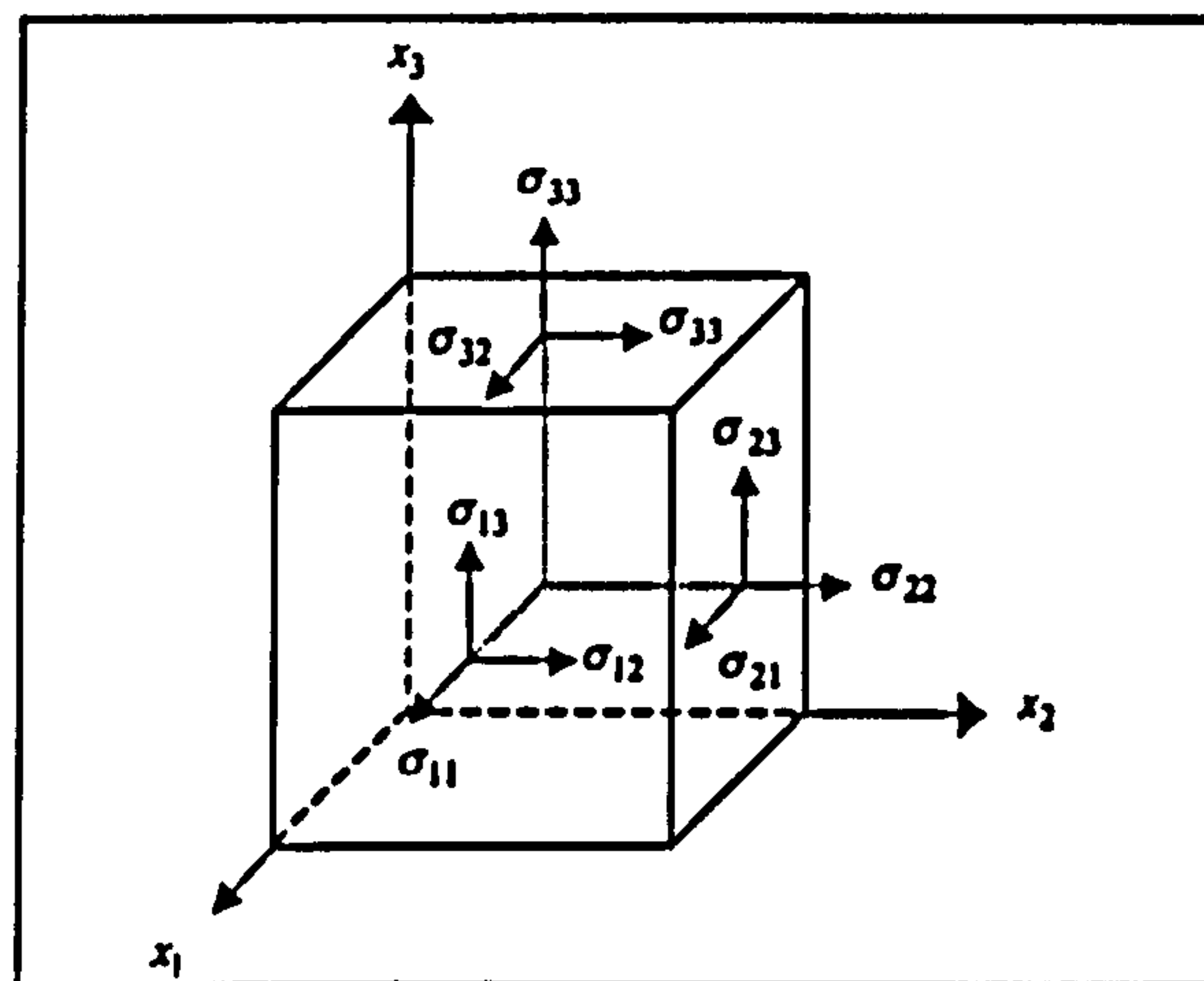


Figure A.1: Stress components of stress tensor on co-ordinate planes.

Due to the symmetry of both σ_{ij} and ε_{ij} , these nine components of stress and strain can be reduced to six components, namely three normal and three shear components. Consequently, equation A.1 can be re-written in the more convenient form of relation A.2.

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} \quad (\text{A.2})$$

Besides, due to the symmetry of the coefficients C_{ijkl} (i.e., $C_{ijkl} = C_{klij}$), the 6×6 matrix defined above is symmetric. Using the single-subscript notation, relation A.2 can be re-written in the form of relation A.3. It should be noticed that the number of independent coefficients is reduced to 21.

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} \quad (\text{A.3})$$

The single-subscript notation, also called the contracted notation for stress and strain, is based on the convention described by relations A.4.

$$\begin{aligned} \sigma_1 &= \sigma_{11}, \sigma_2 = \sigma_{22}, \sigma_3 = \sigma_{33}, \sigma_4 = \sigma_{23}, \sigma_5 = \sigma_{13}, \sigma_6 = \sigma_{12} \\ \varepsilon_1 &= \varepsilon_{11}, \varepsilon_2 = \varepsilon_{22}, \varepsilon_3 = \varepsilon_{33}, \varepsilon_4 = 2\varepsilon_{23}, \varepsilon_5 = 2\varepsilon_{13}, \varepsilon_6 = 2\varepsilon_{12} \end{aligned} \quad (\text{A.4})$$

Here $(\sigma_1, \sigma_2, \sigma_3)$ denote the normal stresses and $(\sigma_4, \sigma_5, \sigma_6)$ denote the shear stresses; $(\sigma_1, \sigma_2, \sigma_6)$ are the inplane stresses and $(\sigma_3, \sigma_4, \sigma_5)$ are the out-of-plane (i.e.: transverse) stresses. This terminology, which is similar for the strain components is used in the thesis document.

A.2 GENERALISED HOOKE'S LAW FOR ORTHOTROPIC MATERIALS

When three mutually orthogonal planes of material symmetry exist, the number of material stiffness coefficients C_{ij} is reduced to 9. Materials which satisfy this condition are called orthotropic. For orthotropic material the material stiffness coefficients can be expressed in terms of engineering constants, i.e. Young's moduli, Poisson's ratios and shear moduli, as is explained in more details in appendix B. They can therefore be easily calculated from the knowledge of the engineering constants of the considered material. These constants can be determined either experimentally or by using a theoretical approach such as a micromechanics approach. Often fibre-reinforced lamina are characterised as having three orthogonal planes of symmetry and therefore as being orthotropic. The stress-strain relations for an orthotropic material take the form shown in table A.1. Three important observations can be made with respect to the stress-strain relations:

- No interaction exists between normal stresses and shear strains; i.e., normal stresses acting along principal material directions produce only normal strains.
- No interaction exists between shear stresses and normal strains; i.e., shear stresses acting on principal material planes produce only shear strains.
- No interaction exists between shear stresses and shear strains on different planes; i.e., a shear stress acting on a principal plane produces a shear strain only on that plane.

A.3 GENERALISED HOOKE'S LAW FOR TRANSVERSELY ISOTROPIC MATERIALS

An orthotropic material is called transversely isotropic when one of its principal planes is a plane of isotropy. Transversely isotropic materials are of special importance in the study of fibre-reinforced composite materials. Indeed many unidirectional composites, as represented in figure A.2, can be considered transversely isotropic, with the x_2 - x_3 plane (normal to the fibres) as the plane of isotropy. The stress-strain relations for a transversely isotropic material, whose material coefficients are the same with respect to axes x_2 and x_3 , takes the form shown in table A.1.

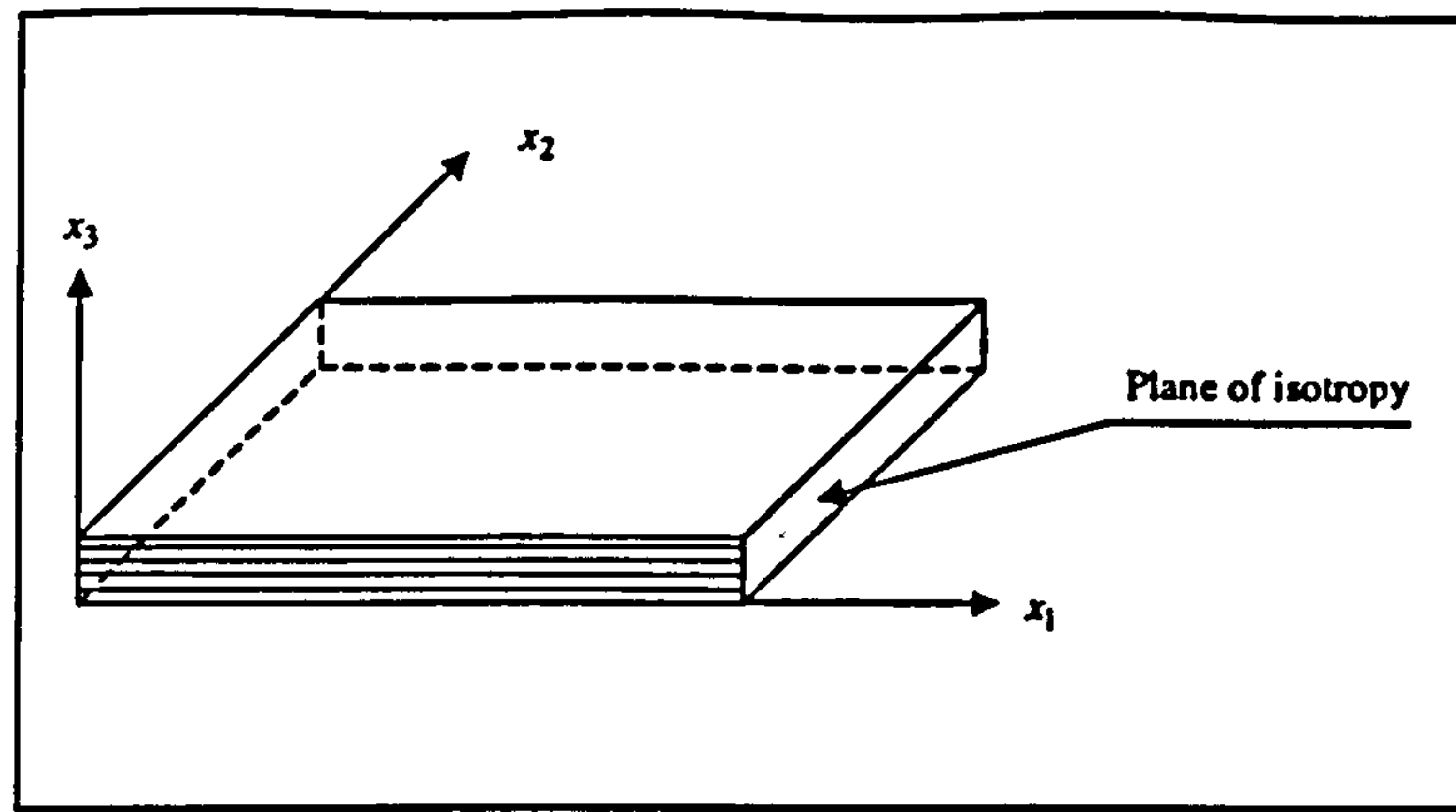


Figure A.2: Orthotropic material with transverse isotropy.

A.4 GENERALISED HOOKE'S LAW FOR ISOTROPIC MATERIALS

An isotropic material is characterised by an infinite number of planes of material symmetry through a point. So, the stress-strain relation can be reduced to the expression shown in table A.1. For example, undamaged cellular foam materials are commonly considered as being isotropic.

A.5 REMARKS

In this appendix, the Hooke's law has been presented for various material characteristics. It has been noticed that transversely isotropic materials and isotropic materials are specific cases of orthotropic materials. This implies that if a structure is modelled assuming its material properties are orthotropic, the developed model would be as well valid for transversely isotropic and isotropic materials. This remark is essential since it explains why in the thesis document the analytical method developed to investigate the behaviour of panels made of orthotropic materials will be as well valid for panels containing transversely isotropic or isotropic materials. To summarise, a review of the stress-strain relations for various material characteristics mentioned in this part is provided in table A.1. It should be noticed that these stress-strain relations are important relations of linear elasticity that are widely used and mentioned in the thesis document.

Material characteristics	Stress-strain relations
Anisotropic materials	$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix}$
Orthotropic materials	$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix}$
Transversely isotropic materials	$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_{22}-C_{23}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix}$
Isotropic materials	$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix}$

Table A.1: Stress-strain relations.

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B. ENGINEERING CONSTANTS OF ORTHOTROPIC MATERIALS

The constitutive equations for an orthotropic material are given as explained in chapter 2 by equation B.1.

$$\{\sigma\} = [C]\{\epsilon\} \quad (\text{B.1})$$

In relations B.1, the resulting stresses are related to the applied strains through the use of stiffness coefficients C_{ij} . However instead of stiffness constants, engineering constants are usually used to characterise the stiffness behaviour of orthotropic materials as explained below.

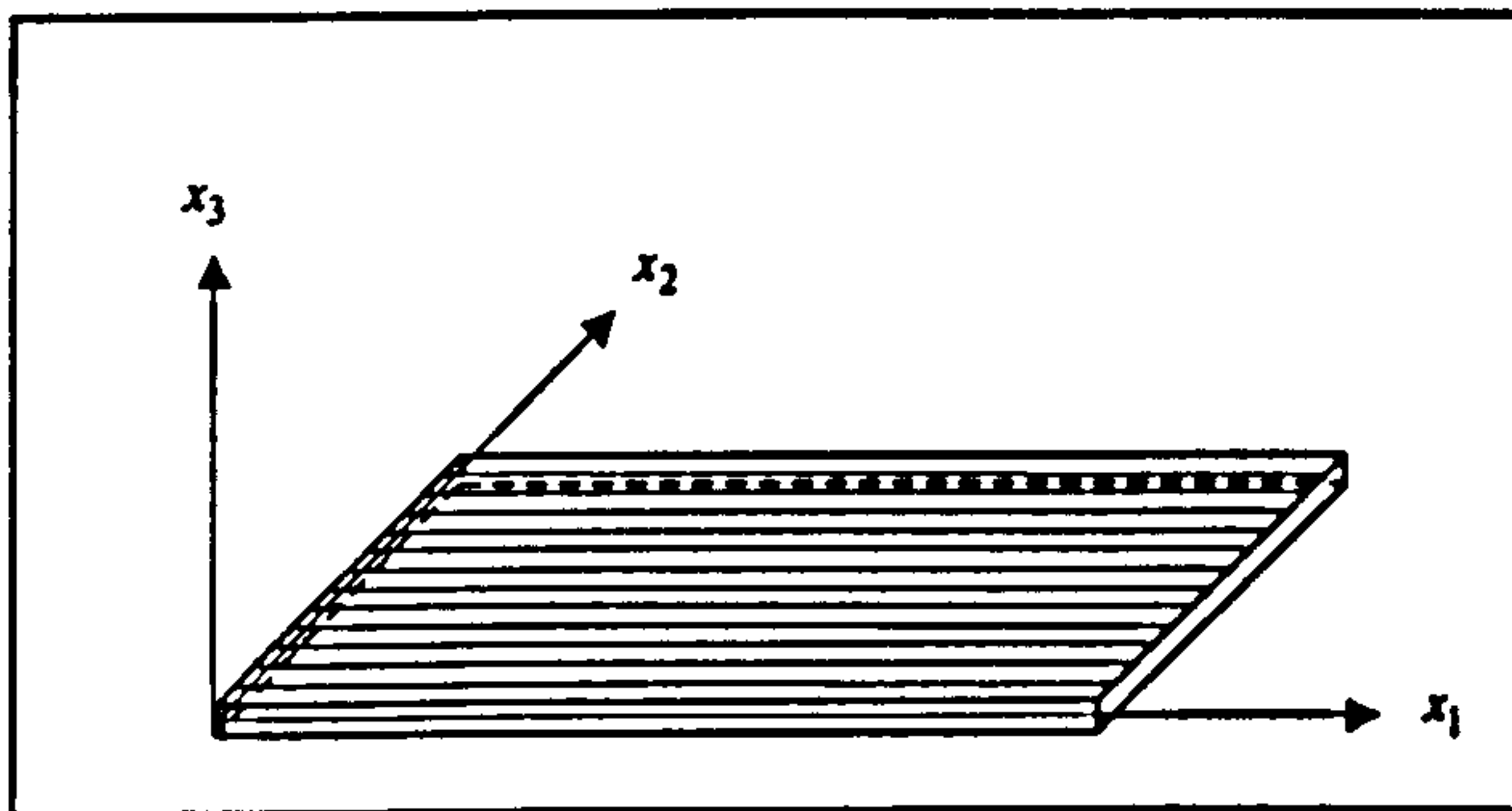


Figure B.1: A unidirectional fibre reinforced lamina

For linear elastic behaviour, the strains due to the application of individual uniaxial and shear stress components can be superposed. For example, the strain ϵ_1 in the principal material direction x_1 represented in figure B.1 due to the stress σ_1 in the same direction is σ_1/E_1 , where E_1 denotes the Young's modulus of the material in the principal direction. The strain ϵ_1 due to stress σ_2 in the direction x_2 is $-\nu_{21}\sigma_2/E_2$ where ν_{21} is a Poisson ratio and E_2 is the Young's modulus of the material in the direction x_2 . Similarly, σ_3 contributes to ϵ_1 by the amount $-\nu_{31}\sigma_3/E_3$. Hence, the total strain ϵ_1 due to simultaneous application of all three normal stress components is:

$$\epsilon_1 = \frac{\sigma_1}{E_1} - \frac{\sigma_2\nu_{21}}{E_2} - \frac{\sigma_3\nu_{31}}{E_3} \quad (\text{B.2})$$

Similarly, we can write

$$\epsilon_2 = \frac{\sigma_2}{E_2} - \frac{\sigma_1\nu_{12}}{E_1} - \frac{\sigma_3\nu_{32}}{E_3} \quad (\text{B.3})$$

$$\epsilon_3 = \frac{\sigma_3}{E_3} - \frac{\sigma_1\nu_{13}}{E_1} - \frac{\sigma_2\nu_{23}}{E_2} \quad (\text{B.4})$$

The simple shear tests with an orthotropic material give the results shown by relations B.5.

$$\varepsilon_6 = \frac{\sigma_6}{G_{12}}, \quad \varepsilon_5 = \frac{\sigma_5}{G_{13}}, \quad \varepsilon_4 = \frac{\sigma_4}{G_{23}} \quad (\text{B.5})$$

So using relations B.2 up to relations B.5, the strain-stress relations can be expressed in matrix form as shown in equation B.6.

$$\{\varepsilon\} = [S]\{\sigma\} \quad \text{or} \quad (\text{B.6})$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{12}/E_1 & -\nu_{13}/E_1 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{23}/E_2 & 0 & 0 & 0 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{12} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}$$

Since the compliance matrix $[S]$ is the inverse of the stiffness matrix $[C]$, the coefficients of the stiffness matrix can easily be obtained from the knowledge of the coefficients of the compliance matrix. Both the stiffness and the compliance matrix are symmetric. Finally the coefficients of the stiffness matrix are a function of the engineering parameters as shown by the relations B.7.

$$\begin{aligned} C_{11} &= \frac{1 - \nu_{23}\nu_{32}}{\Delta} E_1; \quad C_{22} = \frac{1 - \nu_{31}\nu_{13}}{\Delta} E_2; \quad C_{33} = \frac{1 - \nu_{12}\nu_{21}}{\Delta} E_3; \\ C_{12} = C_{21} &= \frac{\nu_{21} + \nu_{23}\nu_{31}}{\Delta} E_1; \quad C_{23} = C_{32} = \frac{\nu_{32} + \nu_{31}\nu_{12}}{\Delta} E_2; \quad C_{31} = C_{13} = \frac{\nu_{13} + \nu_{12}\nu_{23}}{\Delta} E_3; \\ C_{44} &= G_{23}; \quad C_{55} = G_{31}; \quad C_{66} = G_{12}; \\ \text{where } \Delta &= 1 - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - \nu_{12}\nu_{21} - \nu_{23}\nu_{31}\nu_{12} - \nu_{21}\nu_{32}\nu_{13} \end{aligned} \quad (\text{B.7})$$

Since isotropic and transversely isotropic materials are special cases of orthotropic material, the formulae presented for orthotropic materials can as well be used for transversely isotropic and isotropic materials. For example for transversely isotropic materials with the x_2x_3 -plane which coincides with the transverse plane of isotropy, the subscript 2 and 3 can be considered as being the same subscript.

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C. TRANSFORMATION OF THE STRESS-STRAIN RELATIONS

The composite plates, which are under investigation in this thesis, are composed of a stack of layers made of orthotropic, transversely isotropic or isotropic materials. In order to keep the theory to the most general case, each layer (i.e, lamina) is assumed to be made of an orthotropic material such as fibre reinforced material. The resulting stress due to a given applied strain in an orthotropic lamina can be modelled using the stress-strain relations presented in table A.1 of appendix A. These relations are given in the material co-ordinate system. In the specific case of a unidirectional fibre reinforced lamina, the material co-ordinate axis x_1 is taken to be parallel to the fibre and the x_2 -axis transverse to the fibre direction in the plane of the lamina, and the x_3 -axis perpendicular to the plane of the lamina. The material co-ordinate system can be visualised in figure C.1.

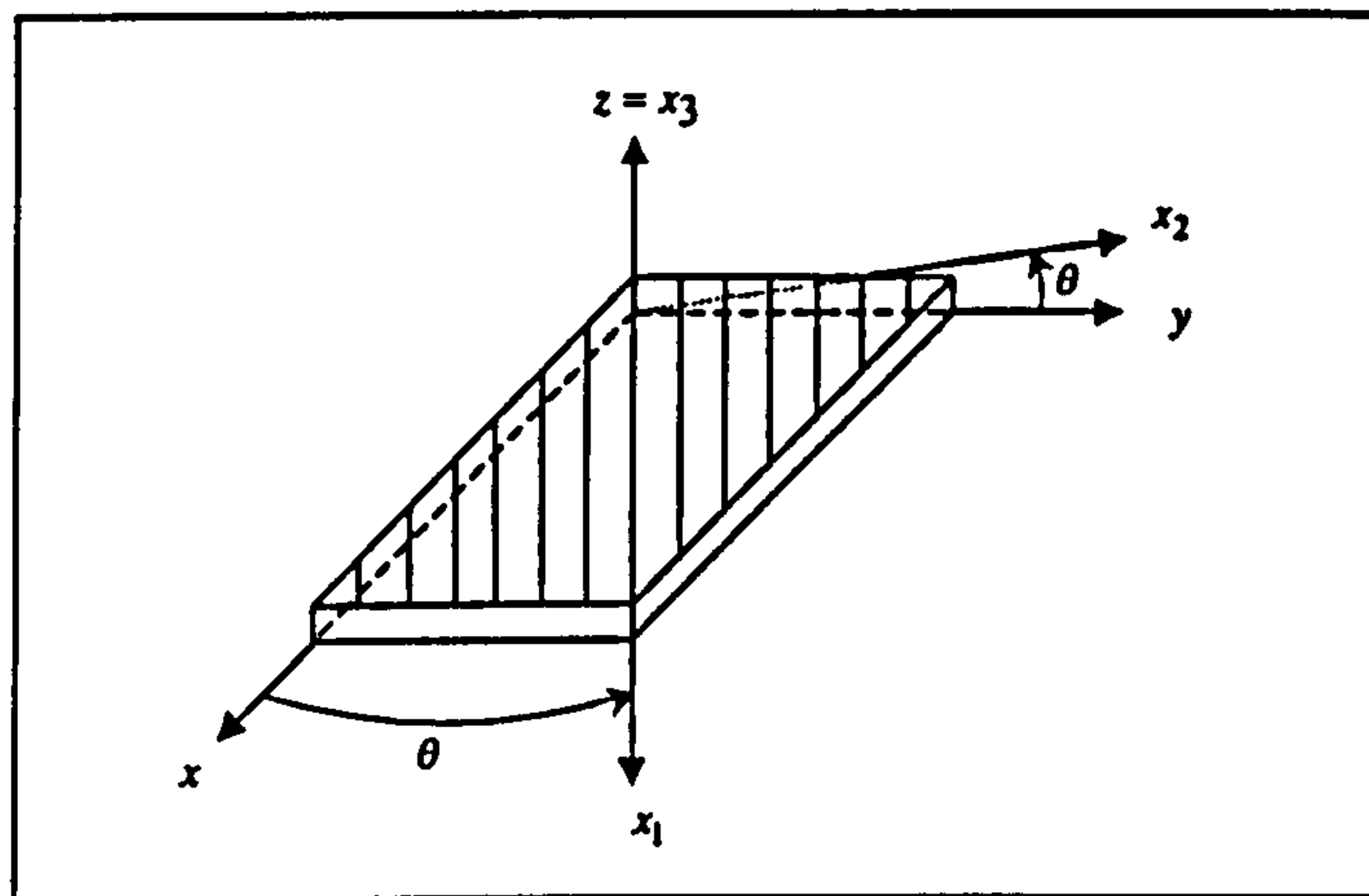


Figure C.1: A fibre reinforced lamina with global and material co-ordinate systems.

However, since composite panels are likely to be composed of several laminae having various orientations of their material co-ordinate systems, there is a need to establish transformation relations among stresses and strains in one co-ordinate system to the corresponding quantities in another co-ordinate system. These relations can be used to transform the constitutive equations of each lamina from the material co-ordinate systems to the co-ordinate system used in the solution problem. For the case of a laminate made of fibre reinforced lamina such as the one shown in figure C.1, it is a matter of transforming the constitutive relations from the material co-ordinate system (x_1, x_2, x_3) to the plate co-ordinate system (x, y, z) . It should be noticed that the material co-ordinate system (x_1, x_2, x_3) is obtained from the global co-ordinate system (x, y, z) by rotating the xy -plane by an angle θ about the z -axis.

The transformation of stress and strain components from the material co-ordinate system to the plate co-ordinate system is given respectively by equation C.1 and C.2.

$$\{\sigma\}_p = [T]\{\sigma\}_m \quad (\text{C.1})$$

$$\{\epsilon\}_p = [T]\{\epsilon\}_m \quad (\text{C.2})$$

The transformation matrix $[T]$, which is function of the angle θ , is given by relation C.3.

$$[T] = \begin{bmatrix} \cos^2 \theta & \sin^2 \theta & 0 & 0 & 0 & -\sin 2\theta \\ \sin^2 \theta & \cos^2 \theta & 0 & 0 & 0 & \sin 2\theta \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos \theta & \sin \theta & 0 \\ 0 & 0 & 0 & -\sin \theta & \cos \theta & 0 \\ \sin \theta \cos \theta & -\sin \theta \cos \theta & 0 & 0 & 0 & \cos^2 \theta - \sin^2 \theta \end{bmatrix} \quad (\text{C.3})$$

The constitutive equations for an orthotropic material, which are given by equation A.1 in appendix A, can be re-written using equations C.1 and C.2 on the form of relations C.4.

$$\{\sigma\}_p = [T][C][T]^T \{\epsilon\}_p = [\bar{C}]\{\epsilon\}_p \quad (\text{C.4})$$

The transformed elastic coefficients $\bar{C}_{ij}$ referred to the (x, y, z) co-ordinate system are related to the known elastic coefficients C_{ij} in the material co-ordinate system (x_1, x_2, x_3) as shown by the relations C.5.

$$\bar{C}_{11} = C_{11} \cos^4 \theta - 4C_{16} \cos^3 \theta \sin \theta + 2(C_{12} + 2C_{66}) \cos^2 \theta \sin^2 \theta - 4C_{26} \cos \theta \sin^3 \theta + C_{22} \sin^4 \theta$$

$$\bar{C}_{12} = C_{12} \cos^4 \theta + 2(C_{16} - C_{26}) \cos^3 \theta \sin \theta + (C_{11} + C_{22} - 4C_{66}) \cos^2 \theta \sin^2 \theta$$

$$+ 2(C_{26} - C_{16}) \cos \theta \sin^3 \theta + C_{12} \sin^4 \theta$$

$$\bar{C}_{13} = C_{13} \cos^2 \theta - 2C_{36} \cos \theta \sin \theta + C_{23} \sin^2 \theta$$

$$\bar{C}_{16} = C_{16} \cos^4 \theta + (C_{11} - C_{12} - 2C_{66}) \cos^3 \theta \sin \theta + 3(C_{26} - C_{16}) \cos^2 \theta \sin^2 \theta$$

$$+ (2C_{66} + C_{12} - C_{22}) \cos \theta \sin^3 \theta - C_{26} \sin^4 \theta$$

$$\bar{C}_{22} = C_{22} \cos^4 \theta + 4C_{26} \cos^3 \theta \sin \theta + 2(C_{12} + 2C_{66}) \cos^2 \theta \sin^2 \theta + 4C_{16} \cos \theta \sin^3 \theta + C_{11} \sin^4 \theta$$

$$\bar{C}_{23} = C_{23} \cos^2 \theta + 2C_{36} \cos \theta \sin \theta + C_{13} \sin^2 \theta$$

$$\begin{aligned}
\bar{C}_{26} &= C_{26} \cos^4 \theta + (C_{12} - C_{22} + 2C_{66}) \cos^3 \theta \sin \theta + 3(C_{16} - C_{26}) \cos^2 \theta \sin^2 \theta \\
&\quad + (C_{11} - C_{12} - 2C_{66}) \cos \theta \sin^3 \theta - C_{16} \sin^4 \theta \\
\bar{C}_{33} &= C_{33} \\
\bar{C}_{36} &= (C_{13} - C_{23}) \cos \theta \sin \theta + C_{36} (\cos^2 \theta - \sin^2 \theta) \\
\bar{C}_{66} &= 2(C_{16} - C_{26}) \cos^3 \theta \sin \theta + (C_{11} + C_{22} - 2C_{12} - 2C_{66}) \cos^2 \theta \sin^2 \theta + 2(C_{26} - C_{16}) \cos \theta \sin^3 \theta \\
&\quad + C_{66} (\cos^4 \theta + \sin^4 \theta) \\
\bar{C}_{44} &= C_{44} \cos^2 \theta + C_{55} \sin^2 \theta + 2C_{45} \cos \theta \sin \theta \\
\bar{C}_{45} &= C_{45} (\cos^2 \theta - \sin^2 \theta) + (C_{55} - C_{44}) \cos \theta \sin \theta \\
\bar{C}_{55} &= C_{55} \cos^2 \theta + C_{44} \sin^2 \theta - 2C_{45} \cos \theta \sin \theta \\
\bar{C}_{14} &= C_{14} \cos^3 \theta + (C_{15} - 2C_{46}) \cos^2 \theta \sin \theta + (C_{24} - 2C_{56}) \cos \theta \sin^2 \theta + C_{25} \sin^3 \theta \\
\bar{C}_{15} &= C_{15} \cos^3 \theta - (C_{14} + 2C_{56}) \cos^2 \theta \sin \theta + (C_{25} + 2C_{46}) \cos \theta \sin^2 \theta - C_{24} \sin^3 \theta \\
\bar{C}_{24} &= C_{24} \cos^3 \theta + (C_{25} + 2C_{46}) \cos^2 \theta \sin \theta + (C_{14} + 2C_{56}) \cos \theta \sin^2 \theta + C_{15} \sin^3 \theta \\
\bar{C}_{25} &= C_{25} \cos^3 \theta + (2C_{56} - C_{24}) \cos^2 \theta \sin \theta + (C_{15} - 2C_{46}) \cos \theta \sin^2 \theta - C_{14} \sin^3 \theta \\
\bar{C}_{34} &= C_{34} \cos \theta + C_{35} \sin \theta \\
\bar{C}_{35} &= C_{35} \cos \theta - C_{34} \sin \theta \\
\bar{C}_{46} &= C_{46} \cos^3 \theta + (C_{56} + C_{14} - C_{24}) \cos^2 \theta \sin \theta + (C_{15} - C_{25} - C_{46}) \cos \theta \sin^2 \theta - C_{56} \sin^3 \theta \\
\bar{C}_{56} &= C_{56} \cos^3 \theta + (C_{15} - C_{25} - C_{46}) \cos^2 \theta \sin \theta + (C_{24} - C_{14} - C_{56}) \cos \theta \sin^2 \theta + C_{46} \sin^3 \theta
\end{aligned}
\tag{C.5}$$

This transformation process is really important since it will be used each time the constitutive equations of an orthotropic lamina need to be written in the global co-ordinate system in order to be able to model and then investigate the behaviour of composite panels.

* *

*

D. A MECHANICAL ANALOGY FOR THE MINI-OSCILLATOR

The GHM (Golla-Hughes-McTavish) method requires the representation of the material modulus function as series of mini-oscillator terms in the Laplace domain:

$$s\tilde{G}(s) \equiv \hat{G}^o \left[1 + \sum_{r=1}^n \hat{\alpha}_r \frac{s^2 + 2\hat{\zeta}_r \hat{\omega}_r s}{s^2 + 2\hat{\zeta}_r \hat{\omega}_r s + \hat{\omega}_r^2} \right] \quad (D.1)$$

In relation D.1 the factor $\hat{G}^o$ corresponds to the GHM approximation to G^o the equilibrium value of the modulus, that is to say the final value of the relaxation function $G(t)$. Each mini-oscillator term is a second-order rational function involving three positive constants $\hat{\alpha}_r$, $\hat{\omega}_r$ and $\hat{\zeta}_r$. These constants govern the shape of the modulus function over the complex s -plane. Depending on the nature of a material modulus function and the range of s over which it is to be modelled, any number of mini-oscillator terms may be used in the GHM expression. The strength of this new approach lies in the fact that by using a model that has the characteristics of well known mechanical systems the derived equations of motion can easily be solved with existing techniques commonly used in dynamic analyses.

In order to explain the mechanical analogy for the mini-oscillator first of all an elementary mass spring system with an applied force is considered. With an elastic spring the motion of the system is described by the second-order equation of motion D.2.

$$M\ddot{\Delta}(t) + \kappa\Delta(t) = F(t) \quad (D.2)$$

If the spring is viscoelastic it is then described by a relaxation function and the equation of motion can be re-written in the form of relation D.3.

$$M\Delta(t) + \int_0^t \kappa(t-\tau)\dot{\Delta}(\tau)d\tau = F(t) \quad (D.3)$$

In relation D.3 it is assumed for convenience that the initial conditions are zero. Using a single-term GHM expression to model the material modulus function leads to the equation of motion D.4.

$$s^2 M \tilde{\Delta}(s) + \kappa \left[\hat{\alpha} \frac{s^2 + 2\hat{\zeta}\hat{\omega}s}{s^2 + 2\hat{\zeta}\hat{\omega}s + \hat{\omega}^2} \right] \tilde{\Delta}(s) = \tilde{F}(s) \quad (D.4)$$

By introducing a new auxiliary co-ordinate $\tilde{z}(s)$ defined by relation D.5, the Laplace transformed equation of motion may be re-written as two-coupled second order equations D.6 and D.7.

$$\tilde{z}(s) = \frac{\hat{\omega}^2}{s^2 + 2\hat{\zeta}\hat{\omega}s + \hat{\omega}^2} \tilde{\Delta}(s) \quad (D.5)$$

$$s^2 M \tilde{\Delta}(s) + (\kappa + \hat{\alpha}\kappa) \tilde{\Delta}(s) - \hat{\alpha}\kappa \tilde{z}(s) = \tilde{F} \quad (D.6)$$

$$s^2 \tilde{z}(s) + 2\hat{\zeta}\hat{\omega}s \tilde{z}(s) - \hat{\omega}^2 \tilde{\Delta}(s) + \hat{\omega}^2 \tilde{z}(s) = 0 \quad (D.7)$$

After multiplying the second equation D.7 by $\hat{\alpha}\kappa/\hat{\omega}^2$, the resulting system of equations D.8 has a symmetric matrix-second-order time-domain realisation.

$$\begin{bmatrix} M & 0 \\ 0 & \hat{\alpha} \frac{1}{\hat{\omega}^2} \kappa \end{bmatrix} \begin{Bmatrix} \ddot{\Delta} \\ \ddot{z} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \hat{\alpha} \frac{2\hat{\zeta}}{\hat{\omega}} \end{bmatrix} \begin{Bmatrix} \dot{\Delta} \\ \dot{z} \end{Bmatrix} + \begin{bmatrix} \kappa + \hat{\alpha}\kappa & -\hat{\alpha}\kappa \\ -\hat{\alpha}\kappa & \hat{\alpha}\kappa \end{bmatrix} \begin{Bmatrix} \Delta \\ z \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix} \quad (D.8)$$

This system has the mechanical representation shown in figure D.1.

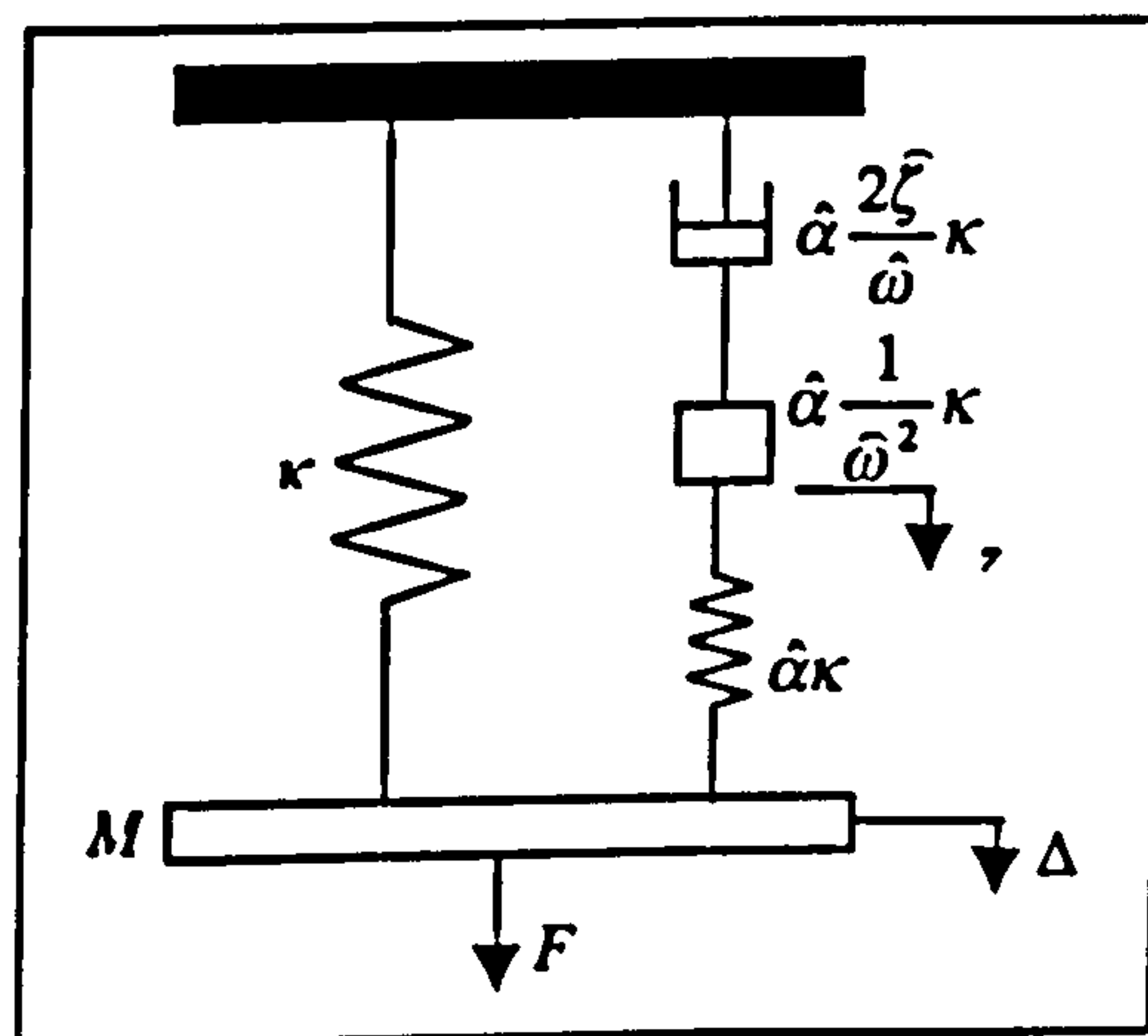


Figure D.1: The mini-oscillator mechanical analogy

With the spatial co-ordinate Δ held fixed, the motion of the fictitious mass is described by the familiar damped second-order equation of motion: $\ddot{z} + 2\hat{\zeta}\hat{\omega}\dot{z} + \hat{\omega}^2 z = 0$. This mini-oscillator mechanical analogy is not in itself a model for the viscoelastic material behaviour but an equivalent system with respect to the dynamics of the spatial co-ordinate Δ .



E. DERIVATION OF THE EQUATIONS OF MOTION FOR LAMINATED PLATES MODELLED USING FSDT

In this appendix, the different stages of the derivation of the equations of motion for laminated plates modelled using FSDT are presented. The laminated plates under consideration, are made of n orthotropic layers as shown in figure E.1.

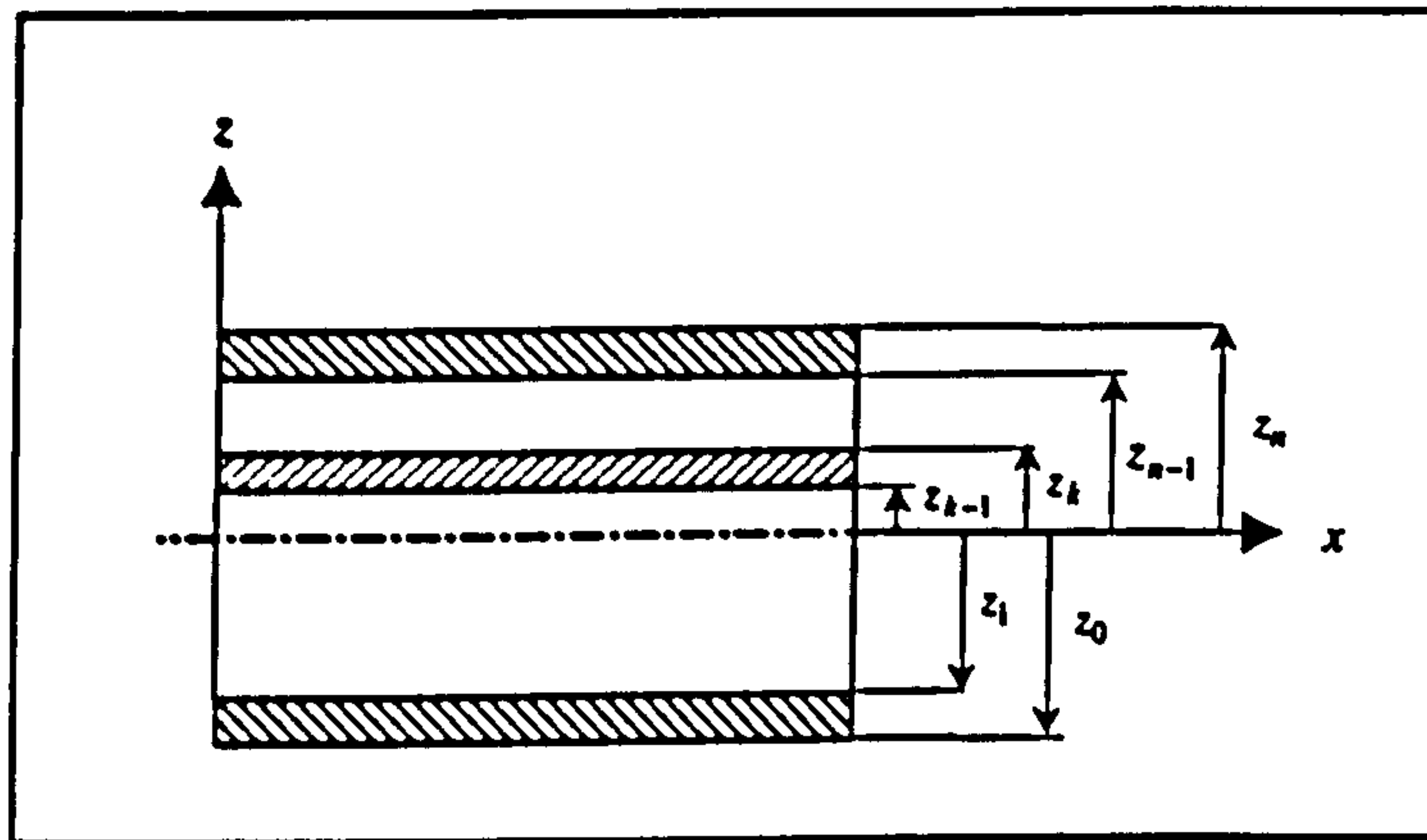


Figure E.1: Cross-section of a typical laminated plate.

The assumptions associated with the use of the FSDT to model the laminated plate have been presented in the literary review (see chapter 3). The displacement field associated with the theory has also been introduced in chapter 3. However, for a better understanding of the derivation techniques, which lead to the determination of the equations of motion of laminated plates modelled using FSDT, the displacement field associated with this theory is re-written here in equation E.1.

$$\begin{cases} u(x, y, z, t) = u_0(x, y, t) + z\phi_x(x, y, t) \\ v(x, y, z, t) = v_0(x, y, t) + z\phi_y(x, y, t) \\ w(x, y, z, t) = w_0(x, y, t) \end{cases} \quad (\text{E.1})$$

The first step of the derivation process consists in computing the strains associated with the displacement field using the strain-displacement relations E.2

$$\begin{aligned} \epsilon_1 &= \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2, & \epsilon_2 &= \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2, & \epsilon_3 &= \frac{\partial w}{\partial z} \\ \epsilon_4 &= \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right), & \epsilon_5 &= \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right), & \epsilon_6 &= \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \end{aligned} \quad (\text{E.2})$$

This relation E.2 can then be re-written in the form of equation E.3 by substituting the displacement field relations E.1 into the equations E.2.

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{Bmatrix} \partial u_0 / \partial x \\ \partial v_0 / \partial y \\ 0 \\ \phi_y + \partial w_0 / \partial y \\ \phi_x + \partial w_0 / \partial x \\ \partial v_0 / \partial x + \partial u_0 / \partial y \end{Bmatrix} + z \begin{Bmatrix} \partial \phi_x / \partial x \\ \partial \phi_y / \partial y \\ 0 \\ 0 \\ 0 \\ \partial \phi_x / \partial y + \partial \phi_y / \partial x \end{Bmatrix} \quad (\text{E.3})$$

Relation E.3 can also be written in the form shown by relation E.4.

$$\{\varepsilon\} = \{\varepsilon^0\} + z\{\kappa^0\} \quad (\text{E.4})$$

The stress-strain relations in each layer of a lamina can be expressed in the plate co-ordinate system using the transformed Hooke's law relation given by the equation 2.4 in chapter 2.

The substitution of equation E.4 into 2.4 gives the strain-displacement relation E.5.

$$\{\sigma\} = [\bar{C}]^T \{\varepsilon^0\} + z[\bar{C}]^T \{\kappa^0\} \quad (\text{E.5})$$

This relation E.5 is valid only in each lamina. The constitutive relations in the studied laminate are obtained by relating the force resultants N_i , the moment resultants M_i and the transverse force resultants Q_i to the strains of the laminate. These relations are expressed as shown by relations E.6.

$$\begin{aligned} \begin{Bmatrix} N_1 \\ N_2 \\ N_6 \end{Bmatrix} &= \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} dz = \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \left(\begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{26} \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \end{Bmatrix} + z \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{26} \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \kappa_1^0 \\ \kappa_2^0 \\ \kappa_6^0 \end{Bmatrix} \right) dz \\ &= \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \kappa_1^0 \\ \kappa_2^0 \\ \kappa_6^0 \end{Bmatrix} \end{aligned}$$

$$\begin{aligned}
 \begin{Bmatrix} M_1 \\ M_2 \\ M_6 \end{Bmatrix} &= \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} z dz = \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{26} \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \end{Bmatrix} + z \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16} \\ \bar{C}_{21} & \bar{C}_{22} & \bar{C}_{26} \\ \bar{C}_{61} & \bar{C}_{62} & \bar{C}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \kappa_1^0 \\ \kappa_2^0 \\ \kappa_6^0 \end{Bmatrix} dz \\
 &= \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \kappa_1^0 \\ \kappa_2^0 \\ \kappa_6^0 \end{Bmatrix} \\
 \begin{Bmatrix} Q_4 \\ Q_5 \end{Bmatrix} &= \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_4 \\ \sigma_5 \end{Bmatrix} dz = K \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \begin{bmatrix} \bar{C}_{44} & \bar{C}_{45} \\ \bar{C}_{45} & \bar{C}_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_4^0 \\ \varepsilon_5^0 \end{Bmatrix} + z \begin{bmatrix} \bar{C}_{44} & \bar{C}_{45} \\ \bar{C}_{45} & \bar{C}_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \kappa_4^0 \\ \kappa_5^0 \end{Bmatrix} dz \\
 &= \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_4^0 \\ \varepsilon_5^0 \end{Bmatrix} + \begin{bmatrix} B_{44} & B_{45} \\ B_{45} & B_{55} \end{bmatrix} \begin{Bmatrix} \kappa_4^0 \\ \kappa_5^0 \end{Bmatrix}
 \end{aligned} \tag{E.6}$$

It should be noticed that the laminate stiffness coefficients A_{ij} , B_{ij} and D_{ij} are functions of the lamination scheme, the material properties and the orientation of each layers.

Finally, the governing equations are derived using the principle of virtual displacements. The dynamic version of the principle of virtual work is given by relation (E.7).

$$0 = \int_0^T (\delta U + \delta V - \delta K) dt \tag{E.7}$$

In this relation (E.7) δU is the virtual strain energy, δV is the virtual work done by the applied forces and δK is the virtual kinetic energy. By solving this relation (E.7) the Euler-Lagrange equations of motion are obtained. For the case where the plate is modelled using the FSDT these equations are given by relation (E.8).

$$\begin{aligned}
 \delta u_0 : \frac{\partial N_1}{\partial x} + \frac{\partial N_6}{\partial y} &= I_0 \ddot{u}_0 + I_1 \ddot{\phi}_x \\
 \delta v_0 : \frac{\partial N_6}{\partial x} + \frac{\partial N_2}{\partial y} &= I_0 \ddot{v}_0 + I_1 \ddot{\phi}_y \\
 \delta w_0 : \frac{\partial Q_5}{\partial x} + \frac{\partial Q_4}{\partial y} + N(w_0) + q &= I_0 \ddot{w}_0 \\
 \delta \phi_x : \frac{\partial M_1}{\partial x} + \frac{\partial M_6}{\partial y} - Q_5 &= I_2 \ddot{\phi}_x + I_1 \ddot{u}_0
 \end{aligned} \tag{E.8}$$

$$\delta\phi_y : \frac{\partial M_6}{\partial x} + \frac{\partial M_2}{\partial y} - Q_4 = I_2 \ddot{\phi}_y + I_1 \ddot{v}_0$$

$$\text{where, } N(w_0) = \frac{\partial}{\partial x} \left(N_1 \frac{\partial w_0}{\partial x} + N_6 \frac{\partial w_0}{\partial y} \right) + \frac{\partial}{\partial y} \left(N_6 \frac{\partial w_0}{\partial x} + N_2 \frac{\partial w_0}{\partial y} \right)$$

In relation E.8 I_i are the mass moments of inertia that can be expressed by relation E.9.

$$(I_1, I_2, I_3) = \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \rho^{(k)} (1, z, z^2) dz \quad (\text{E.9})$$

Finally, using this relation E.8 and using the laminate constitutive relations E.6, which relate the stress resultants N_i , the moments M_i and the force resultants Q_i to the generalised displacements $(u_0, v_0, w_0, \phi_x, \phi_y)$, the equations of motion in terms of displacements can be determined as required. It should be noticed that these relations are normally provided for each specific studied case. Indeed, depending on the lamination scheme of the studied laminated plate different simplifications can be realised as explained in more detail in chapter 6. These simplifications facilitate the derivation of the equations of motion as a function of the generalised displacement coefficients and allow us to solve these equations using a Navier or a Lévy solution as explained in chapter 6.

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F. DERIVATION OF THE EQUATIONS OF MOTION FOR LAMINATED PLATES MODELLED USING REDDY'S REFINED HSDT

In this appendix, the different stages of the derivation of the equations of motion for laminated plates modelled using HSDT are presented. The laminated plates under consideration are made of n orthotropic layers as shown in figure F.1.

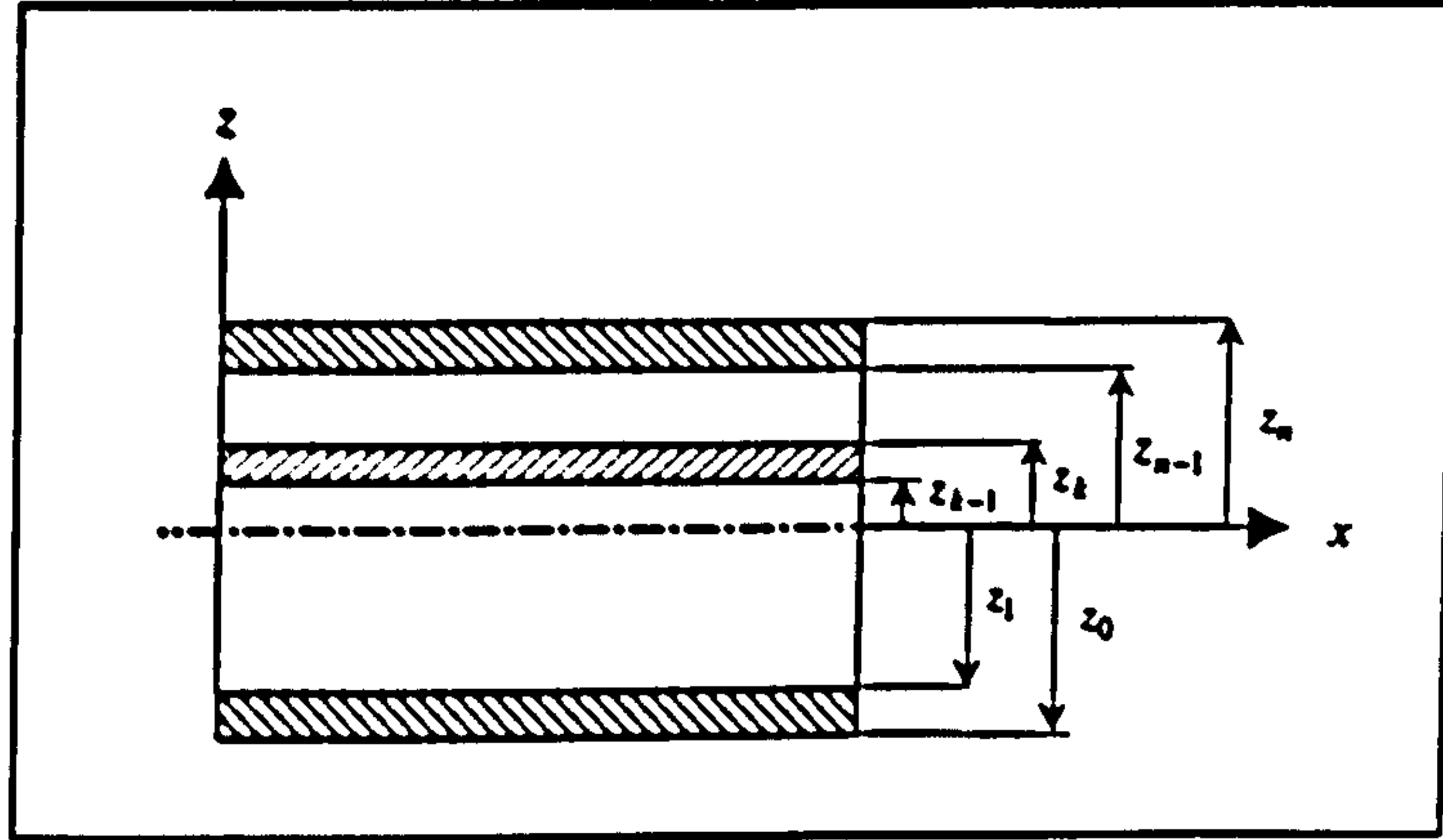


Figure F.1: Cross-section of a typical laminated plate.

The assumptions associated with the use of the HSDT to model the laminated plate have been presented in the literary review (see chapter 3). The displacement field associated with the theory has also been introduced in chapter 3. However, for a better understanding of the derivation techniques, which lead to the determination of the equations of motion of laminated plates modelled using HSDT, the displacement field associated with this theory is re-written here in equation F.1.

$$\begin{cases} u(x, y, z, t) = u_0(x, y, t) + z\phi_x(x, y, t) + z^3\left(-\frac{4}{3h^2}\right)\left(\phi_x + \frac{\partial w_0}{\partial x}\right) \\ v(x, y, z, t) = v_0(x, y, t) + z\phi_y(x, y, t) + z^3\left(-\frac{4}{3h^2}\right)\left(\phi_y + \frac{\partial w_0}{\partial y}\right) \\ w(x, y, z, t) = w_0(x, y, t) \end{cases} \quad (F.1)$$

The first step of the derivation process consists in computing the strains associated with the displacement field using the strain-displacement relations F.2.

$$\begin{aligned} \epsilon_1 &= \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2, & \epsilon_2 &= \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2, & \epsilon_3 &= \frac{\partial w}{\partial z} \\ \epsilon_4 &= \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right), & \epsilon_5 &= \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right), & \epsilon_6 &= \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \end{aligned} \quad (F.2)$$

This relation F.2 can then be re-written in the form of equation F.3 by substituting the displacement field relations F.1 into the equations F.2.

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} = \begin{Bmatrix} \partial u_0 / \partial x \\ \partial v_0 / \partial y \\ 0 \\ \phi_y + \partial w_0 / \partial y \\ \phi_x + \partial w_0 / \partial x \\ \partial v_0 / \partial x + \partial u_0 / \partial y \end{Bmatrix} + z \begin{Bmatrix} \partial \phi_x / \partial x \\ \partial \phi_y / \partial y \\ 0 \\ 0 \\ 0 \\ \partial \phi_x / \partial y + \partial \phi_y / \partial x \end{Bmatrix} + z^2 \begin{Bmatrix} -\frac{4z}{3h^2} (\partial \phi_x / \partial x + \partial^2 w / \partial x^2) \\ -\frac{4z}{3h^2} (\partial \phi_y / \partial y + \partial^2 w / \partial y^2) \\ 0 \\ -\frac{4}{h^2} (\phi_y + \partial w / \partial y) \\ -\frac{4}{h^2} (\phi_x + \partial w / \partial x) \\ -\frac{4z}{3h^2} (\partial \phi_x / \partial y + \partial \phi_y / \partial x + \partial^2 w / \partial x \partial y) \end{Bmatrix} \quad (F.3)$$

Relation F.3 can also be written in the form shown by relation F.4.

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} = \begin{Bmatrix} \epsilon_1^0 \\ \epsilon_2^0 \\ \epsilon_4^0 \\ \epsilon_5^0 \\ \epsilon_6^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_1^0 \\ \kappa_2^0 \\ 0 \\ 0 \\ \kappa_6^0 \end{Bmatrix} + z^2 \begin{Bmatrix} 0 \\ 0 \\ \kappa_4^1 \\ \kappa_5^1 \\ 0 \end{Bmatrix} + z^3 \begin{Bmatrix} \kappa_1^2 \\ \kappa_2^2 \\ 0 \\ 0 \\ \kappa_5^2 \end{Bmatrix} \quad (F.4)$$

The stress-strain relations in each layers of a lamina can be expressed in the plate co-ordinate system using the transformed Hooke's law relation given by the equation 2.4 in chapter 2.

The substitution of equation F.4 into 2.4 gives the strain-displacement relation F.5.

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & 0 & 0 & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & 0 & 0 & \bar{C}_{26} \\ 0 & 0 & \bar{C}_{44} & \bar{C}_{45} & 0 \\ 0 & 0 & \bar{C}_{45} & \bar{C}_{55} & 0 \\ \bar{C}_{16} & \bar{C}_{26} & 0 & 0 & \bar{C}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1^0 \\ \epsilon_2^0 \\ \epsilon_4^0 \\ \epsilon_5^0 \\ \epsilon_6^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_1^0 \\ \kappa_2^0 \\ 0 \\ 0 \\ \kappa_6^0 \end{Bmatrix} + z^2 \begin{Bmatrix} 0 \\ 0 \\ \kappa_4^1 \\ \kappa_5^1 \\ 0 \end{Bmatrix} + z^3 \begin{Bmatrix} \kappa_1^2 \\ \kappa_2^2 \\ 0 \\ 0 \\ \kappa_5^2 \end{Bmatrix} \quad (F.5)$$

This relation F.5 is valid only in each lamina. The constitutive relations in the studied laminate are obtained by relating the force resultants N_i , the moment resultants M_i , the transverse force resultants Q_i and the higher stress resultants P_i and R_i to the strains of the laminate. These relations are expressed as shown by relations F.6.

$$\begin{aligned}
 \begin{Bmatrix} N_1 \\ N_2 \\ N_6 \end{Bmatrix} &= \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} dz \\
 &= \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \left(\begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{26} \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \end{Bmatrix} + z \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{26} \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \kappa_1^0 \\ \kappa_2^0 \\ \kappa_6^0 \end{Bmatrix} + z^2 \begin{Bmatrix} \kappa_1^2 \\ \kappa_2^2 \\ \kappa_6^2 \end{Bmatrix} \right) dz \\
 &= \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \kappa_1^0 \\ \kappa_2^0 \\ \kappa_6^0 \end{Bmatrix} + \begin{bmatrix} E_{11} & E_{12} & E_{16} \\ E_{12} & E_{22} & E_{26} \\ E_{16} & E_{26} & E_{66} \end{bmatrix} \begin{Bmatrix} \kappa_1^2 \\ \kappa_2^2 \\ \kappa_6^2 \end{Bmatrix} \\
 \\
 \begin{Bmatrix} M_1 \\ M_2 \\ M_6 \end{Bmatrix} &= \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} z dz \\
 &= \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \left(\begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{26} \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \end{Bmatrix} + z \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{26} \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \kappa_1^0 \\ \kappa_2^0 \\ \kappa_6^0 \end{Bmatrix} + z^2 \begin{Bmatrix} \kappa_1^2 \\ \kappa_2^2 \\ \kappa_6^2 \end{Bmatrix} \right) z dz \\
 &= \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \kappa_1^0 \\ \kappa_2^0 \\ \kappa_6^0 \end{Bmatrix} + \begin{bmatrix} F_{11} & F_{12} & F_{16} \\ F_{12} & F_{22} & F_{26} \\ F_{16} & F_{26} & F_{66} \end{bmatrix} \begin{Bmatrix} \kappa_1^2 \\ \kappa_2^2 \\ \kappa_6^2 \end{Bmatrix} \\
 \\
 \begin{Bmatrix} P_1 \\ P_2 \\ P_6 \end{Bmatrix} &= \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} z^3 dz \\
 &= \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \left(\begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{26} \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \end{Bmatrix} + z \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{26} \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \kappa_1^0 \\ \kappa_2^0 \\ \kappa_6^0 \end{Bmatrix} + z^2 \begin{Bmatrix} \kappa_1^2 \\ \kappa_2^2 \\ \kappa_6^2 \end{Bmatrix} \right) z^3 dz \\
 &= \begin{bmatrix} E_{11} & E_{12} & E_{16} \\ E_{12} & E_{22} & E_{26} \\ E_{16} & E_{26} & E_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \end{Bmatrix} + \begin{bmatrix} F_{11} & F_{12} & F_{16} \\ F_{12} & F_{22} & F_{26} \\ F_{16} & F_{26} & F_{66} \end{bmatrix} \begin{Bmatrix} \kappa_1^0 \\ \kappa_2^0 \\ \kappa_6^0 \end{Bmatrix} + \begin{bmatrix} H_{11} & H_{12} & H_{16} \\ H_{12} & H_{22} & H_{26} \\ H_{16} & H_{26} & H_{66} \end{bmatrix} \begin{Bmatrix} \kappa_1^2 \\ \kappa_2^2 \\ \kappa_6^2 \end{Bmatrix}
 \end{aligned}$$

$$\begin{aligned} \begin{Bmatrix} Q_4 \\ Q_5 \end{Bmatrix} &= \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_4 \\ \sigma_5 \end{Bmatrix} dz \\ &= \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \left(\begin{bmatrix} \bar{C}_{44} & \bar{C}_{45} \\ \bar{C}_{45} & \bar{C}_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_4^0 \\ \varepsilon_5^0 \end{Bmatrix} + z^2 \begin{bmatrix} \bar{C}_{44} & \bar{C}_{45} \\ \bar{C}_{45} & \bar{C}_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \kappa_4^1 \\ \kappa_5^1 \end{Bmatrix} \right) dz \\ &= \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_4^0 \\ \varepsilon_5^0 \end{Bmatrix} + \begin{bmatrix} D_{44} & D_{45} \\ D_{45} & D_{55} \end{bmatrix} \begin{Bmatrix} \kappa_4^1 \\ \kappa_5^1 \end{Bmatrix} \end{aligned}$$

$$\begin{aligned} \begin{Bmatrix} R_4 \\ R_5 \end{Bmatrix} &= \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_4 \\ \sigma_5 \end{Bmatrix} z^2 dz \\ &= \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \left(\begin{bmatrix} \bar{C}_{44} & \bar{C}_{45} \\ \bar{C}_{45} & \bar{C}_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_4^0 \\ \varepsilon_5^0 \end{Bmatrix} + z^2 \begin{bmatrix} \bar{C}_{44} & \bar{C}_{45} \\ \bar{C}_{45} & \bar{C}_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \kappa_4^1 \\ \kappa_5^1 \end{Bmatrix} \right) z^2 dz \\ &= \begin{bmatrix} D_{44} & D_{45} \\ D_{45} & D_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_4^0 \\ \varepsilon_5^0 \end{Bmatrix} + \begin{bmatrix} F_{44} & F_{45} \\ F_{45} & F_{55} \end{bmatrix} \begin{Bmatrix} \kappa_4^1 \\ \kappa_5^1 \end{Bmatrix} \end{aligned}$$

(F.6)

It should be noticed that the laminate stiffness coefficients A_{ij} , B_{ij} , D_{ij} , E_{ij} and F_{ij} are functions of the lamination scheme, the material properties and the orientation of each layers.

Finally, the governing equations are derived using the principle of virtual displacements. The dynamic version of the principle of virtual work is given by relation F.7.

$$0 = \int_0^T (\delta U + \delta V - \delta K) dt \quad (F.7)$$

In this relation F.7 δU is the virtual strain energy, δV is the virtual work done by the applied forces and δK is the virtual kinetic energy. By solving this relation F.7 the Euler-Lagrange equations of motion are obtained. For the case where the plate is modelled using the HSDT these equations are given by relation F.8.

$$\begin{aligned} \delta u_0: \frac{\partial V_1}{\partial x} + \frac{\partial V_6}{\partial y} &= I_0 \ddot{u}_0 + \left(I_1 - \frac{4}{3h^2} I_3 \right) \ddot{\phi}_x - \frac{4}{3h^2} I_3 \frac{\partial \ddot{w}_0}{\partial x} \\ \delta v_0: \frac{\partial V_6}{\partial x} + \frac{\partial V_2}{\partial y} &= I_0 \ddot{v}_0 + \left(I_1 - \frac{4}{3h^2} I_3 \right) \ddot{\phi}_y - \frac{4}{3h^2} I_3 \frac{\partial \ddot{w}_0}{\partial y} \end{aligned}$$

$$\begin{aligned}
 \delta w_0 : & \frac{\partial Q_3}{\partial x} - \frac{4}{h^2} \frac{\partial R_3}{\partial x} + \frac{\partial Q_4}{\partial y} - \frac{4}{h^2} \frac{\partial R_4}{\partial y} + N(w_0) \\
 & + \frac{4}{3h^2} \left(\frac{\partial^2 P_1}{\partial x^2} + 2 \frac{\partial^2 P_6}{\partial x \partial y} + \frac{\partial^2 P_2}{\partial y^2} \right) + q = I_0 \ddot{w}_0 - \left(\frac{4}{3h^2} \right)^2 I_6 \left(\frac{\partial^2 \ddot{w}_0}{\partial x^2} + \frac{\partial^2 \ddot{w}_0}{\partial y^2} \right) \\
 & + \left(\frac{4}{3h^2} \left(I_3 \left(\frac{\partial \ddot{u}_0}{\partial x} + \frac{\partial \ddot{v}_0}{\partial y} \right) + \left(I_4 - \left(\frac{4}{3h^2} I_6 \right) \right) \left(\frac{\partial \ddot{\phi}_x}{\partial x} + \frac{\partial \ddot{\phi}_y}{\partial y} \right) \right) \right) \\
 \delta \phi_x : & \frac{\partial M_1}{\partial x} - \left(\frac{4}{3h^2} \right) \frac{\partial P_1}{\partial x} + \frac{\partial M_6}{\partial y} - \left(\frac{4}{3h^2} \right) \frac{\partial P_6}{\partial y} - Q_3 + \frac{4}{h^2} R_3 = \\
 & \left(I_2 - \frac{8}{3h^2} I_4 + \left(\frac{4}{3h^2} \right)^2 I_6 \right) \ddot{\phi}_x + \left(I_1 - \left(\frac{4}{3h^2} \right) I_3 \right) \ddot{u}_0 - \left(\frac{4}{3h^2} \right) \left(I_4 - \left(\frac{4}{3h^2} I_6 \right) \right) \frac{\partial \ddot{w}_0}{\partial x} \\
 \delta \phi_y : & \frac{\partial M_6}{\partial x} - \left(\frac{4}{3h^2} \right) \frac{\partial P_6}{\partial x} + \frac{\partial M_2}{\partial y} - \left(\frac{4}{3h^2} \right) \frac{\partial P_2}{\partial y} - Q_4 - \left(\frac{4}{h^2} \right) R_4 = \\
 & \left(I_2 - \frac{8}{3h^2} I_4 + \left(\frac{4}{3h^2} \right)^2 I_6 \right) \ddot{\phi}_y + \left(I_1 - \frac{4}{3h^2} I_3 \right) \ddot{v}_0
 \end{aligned} \tag{F.8}$$

$$\text{where, } N(w_0) = \frac{\partial}{\partial x} \left(N_1 \frac{\partial w_0}{\partial x} + N_6 \frac{\partial w_0}{\partial y} \right) + \frac{\partial}{\partial y} \left(N_6 \frac{\partial w_0}{\partial x} + N_2 \frac{\partial w_0}{\partial y} \right)$$

In relation F.8 I_i are the mass moments of inertia that can be expressed by relations F.9.

$$(I_1, I_2, I_3, I_4, I_5, I_7) = \sum_{k=1}^N \int_{z_{k-1}}^{z_k} \rho^{(k)} (1, z, z^2, z^3, z^4, z^5, z^7) dz \tag{F.9}$$

Finally, using this relation F.8 and using the laminate constitutive relations F.6, which relate the stress resultants N_i , the moments M_i , the force resultants Q_i and the high order stress resultants R_i and P_i to the generalised displacements $(u_0, v_0, w_0, \phi_x, \phi_y)$, the equations of motion in terms of displacements can be determined as required. It should be noticed that these relations are normally provided for each specific studied case. Indeed, depending on the lamination scheme of the studied laminated plate different simplifications can be realised as explained in more detail in chapter 6. These simplifications facilitate the derivation of the equations of motion as a function of the generalised displacement coefficients and allow us to solve these equations using a Navier or a Lévy solution as explained in chapter 6.

* * *

G. STIFFNESS MATRIX COEFFICIENTS

G.1 STIFFNESS MATRIX COEFFICIENTS FOR SIMPLY SUPPORTED (SSSS-1) RECTANGULAR LAMINATED OR SANDWICH PLATES MODELLED USING FSDT

$$K_{11} = \alpha^2 A_{11} + \beta^2 A_{66}$$

$$K_{12} = \alpha\beta(A_{12} + A_{66})$$

$$K_{13} = 0$$

$$K_{14} = \alpha^2 B_{11} + \beta^2 B_{66}$$

$$K_{15} = \alpha\beta(B_{12} + B_{66})$$

$$K_{22} = \beta^2 A_{22} + \alpha^2 A_{66}$$

$$K_{23} = 0$$

$$K_{24} = K_{15}$$

$$K_{25} = \beta^2 B_{22} + \alpha^2 B_{66}$$

$$K_{33} = \alpha^2 A_{55} + \beta^2 A_{44}$$

$$K_{34} = \alpha A_{55}$$

$$K_{35} = \beta A_{44}$$

$$K_{44} = A_{55} + \alpha^2 D_{11} + \beta^2 D_{66}$$

$$K_{45} = \alpha\beta(D_{12} + D_{66})$$

$$K_{55} = A_{44} + \alpha^2 D_{66} + \beta^2 D_{22}$$

G.2 STIFFNESS MATRIX COEFFICIENTS FOR SIMPLY SUPPORTED (SSSS-2) RECTANGULAR LAMINATED OR SANDWICH PLATES MODELLED USING FSDT

$$K_{11} = \alpha^2 A_{11} + \beta^2 A_{66}$$

$$K_{12} = \alpha\beta(A_{12} + A_{66})$$

$$K_{13} = 0$$

$$K_{14} = 2\beta\alpha B_{16}$$

$$K_{15} = \alpha^2 B_{16} + \beta^2 B_{26}$$

$$K_{22} = \beta^2 A_{22} + \alpha^2 A_{66}$$

$$K_{23} = 0$$

$$K_{24} = K_{15}$$

$$K_{25} = 2\alpha\beta B_{26}$$

$$K_{33} = \alpha^2 A_{55} + \beta^2 A_{44}$$

$$K_{34} = \alpha A_{55}$$

$$K_{35} = \beta A_{44}$$

$$K_{44} = A_{55} + \alpha^2 D_{11} + \beta^2 D_{66}$$

$$K_{45} = \alpha\beta(D_{12} + D_{66})$$

$$K_{55} = A_{44} + \alpha^2 D_{66} + \beta^2 D_{22}$$

G.3 STIFFNESS MATRIX COEFFICIENTS FOR SIMPLY SUPPORTED (SSSS-1) RECTANGULAR LAMINATED OR SANDWICH PLATES MODELLED USING HSDT

$$K_{11} = \alpha^2 A_{11} + \beta^2 A_{66}$$

$$K_{12} = \alpha\beta(A_{12} + A_{66})$$

$$K_{13} = -\left(\frac{4}{3h^2}\right)\alpha^3 E_{11} - \left(\frac{4}{3h^2}\right)\alpha\beta^2(E_{12} + 2E_{66})$$

$$K_{14} = \alpha^2 B_{11} - \alpha^2\left(\frac{4}{3h^2}\right)E_{11} + \beta^2\left[B_{66} - \left(\frac{4}{3h^2}\right)E_{66}\right]$$

$$K_{15} = \alpha\beta\left[(B_{12} + B_{66}) - \left(\frac{4}{3h^2}\right)(E_{12} + E_{66})\right]$$

$$K_{22} = \beta^2 A_{22} + \alpha^2 A_{66}$$

$$K_{23} = -\beta^3\left(\frac{4}{3h^2}\right)E_{22} - \left(\frac{4}{3h^2}\right)\alpha^2\beta(E_{12} + 2E_{66})$$

$$K_{24} = \alpha\beta\left[(B_{12} + B_{66}) - \left(\frac{4}{3h^2}\right)(E_{12} + E_{66})\right]$$

$$K_{25} = \beta^2 B_{22} - \beta^2\left(\frac{4}{3h^2}\right)E_{22} + \alpha^2\left[B_{66} - \left(\frac{4}{3h^2}\right)E_{66}\right]$$

$$K_{33} = \alpha^2 A_{55} + \beta^2 A_{44} - \frac{8}{h^2}(\alpha^2 D_{55} + \beta^2 D_{44}) + \left(\frac{4}{h^2}\right)^2(\alpha^2 F_{55} + \beta^2 F_{44})$$

$$+ \left(\frac{4}{3h^2}\right)^2[\alpha^4 H_{11} + 2(H_{12} + 2H_{66})\alpha^2\beta^2 + \beta^4 H_{22}]$$

$$K_{34} = \alpha A_{55} - \frac{8}{h^2}\alpha D_{55} + \left(\frac{4}{h^2}\right)^2\alpha F_{55} - \left(\frac{4}{3h^2}\right)[\alpha^3 F_{11} + \alpha\beta^2(F_{12} + 2F_{66})]$$

$$+ \left(\frac{4}{3h^2}\right)^2[\alpha^3 H_{11} + \alpha\beta^2(H_{12} + 2H_{66})]$$

$$K_{35} = \beta A_{44} - \frac{8}{h^2} \beta D_{44} + \left(\frac{4}{h^2}\right)^2 \beta F_{44} - \left(\frac{4}{3h^2}\right) [\alpha^2 \beta (F_{12} + 2F_{66}) + \beta^3 F_{22}] \\ + \left(\frac{4}{3h^2}\right)^2 [\alpha^2 \beta (H_{12} + 2H_{66}) + \beta^3 H_{22}]$$

$$K_{44} = A_{55} + \alpha^2 D_{11} + \beta^2 D_{66} - \frac{8}{h^2} D_{55} + \left(\frac{4}{h^2}\right)^2 F_{55} - \frac{8}{3h^2} (\alpha^2 F_{11} + \beta^2 F_{66}) \\ + \left(\frac{4}{3h^2}\right)^2 (\alpha^2 H_{11} + \beta^2 H_{66})$$

$$K_{45} = \alpha\beta \left[D_{12} + D_{66} - \frac{8}{3h^2} (F_{12} + F_{66}) + \left(\frac{4}{3h^2}\right)^2 (H_{12} + H_{16}) \right]$$

$$K_{55} = A_{44} + \alpha^2 D_{66} + \beta^2 D_{22} - \frac{8}{h^2} D_{44} + \left(\frac{4}{h^2}\right)^2 F_{44} - \left(\frac{8}{3h^2}\right) (\alpha^2 F_{66} + \beta^2 F_{22}) \\ + \left(\frac{4}{3h^2}\right)^2 (\beta^2 H_{22} + \alpha^2 H_{66})$$

G.4 STIFFNESS MATRIX COEFFICIENTS FOR SIMPLY SUPPORTED (SSSS-2) RECTANGULAR LAMINATED OR SANDWICH PLATES MODELLED USING HSDT

$$K_{11} = \alpha^2 A_{11} + \beta^2 A_{66}$$

$$K_{12} = \alpha\beta (A_{12} + A_{66})$$

$$K_{13} = -\left(\frac{4}{3h^2}\right) (3\alpha^2 \beta E_{16} + \beta^3 E_{26})$$

$$K_{14} = 2\beta\alpha B_{16} - \left(\frac{8}{3h^2}\right) \alpha\beta E_{16}$$

$$K_{15} = \alpha^2 B_{16} + \beta^2 B_{26} - \left(\frac{4}{3h^2}\right) (\alpha^2 E_{16} + \beta^2 E_{26})$$

$$K_{22} = \beta^2 A_{22} + \alpha^2 A_{66}$$

$$K_{23} = -\left(\frac{4}{3h^2}\right)(\alpha^3 E_{16} + 3\beta^2 \alpha E_{26})$$

$$K_{24} = K_{15}$$

$$K_{25} = 2\alpha\beta B_{26} - \left(\frac{8}{3h^2}\right)\alpha\beta E_{26}$$

$$K_{33} = \alpha^2 A_{55} + \beta^2 A_{44} - \frac{8}{h^2}(\alpha^2 D_{55} + \beta^2 D_{44}) + \left(\frac{4}{h^2}\right)^2(\alpha^2 F_{55} + \beta^2 F_{44}) \\ + \left(\frac{4}{3h^2}\right)^2[\alpha^4 H_{11} + 2(H_{12} + 2H_{66})\alpha^2\beta^2 + \beta^4 H_{22}]$$

$$K_{34} = \alpha A_{55} - \frac{8}{h^2}\alpha D_{55} + \left(\frac{4}{h^2}\right)^2 \alpha F_{55} - \left(\frac{4}{3h^2}\right)[\alpha^3 F_{11} + \alpha\beta^2(F_{12} + 2F_{66})] \\ + \left(\frac{4}{3h^2}\right)^2[\alpha^3 H_{11} + \alpha\beta^2(H_{12} + 2H_{66})]$$

$$K_{35} = \beta A_{44} - \frac{8}{h^2}\beta D_{44} + \left(\frac{4}{h^2}\right)\beta F_{44} - \left(\frac{4}{3h^2}\right)[\alpha^2\beta(F_{12} + 2F_{66}) + \beta^3 F_{22}] \\ + \left(\frac{3}{4h^2}\right)^2[\alpha^2\beta(H_{12} + 2H_{66}) + \beta^3 H_{22}]$$

$$K_{44} = A_{55} + \alpha^2 D_{11} + \beta^2 D_{66} - \frac{8}{h^2} D_{55} + \left(\frac{4}{h^2}\right)^2 F_{55} - \frac{8}{3h^2}(\alpha^2 F_{11} + \beta^2 F_{66}) \\ + \left(\frac{4}{3h^2}\right)^2(\alpha^2 H_{11} + \beta^2 H_{66})$$

$$K_{45} = \alpha\beta\left[D_{12} + D_{66} - \frac{8}{3h^2}(F_{12} + F_{66}) + \left(\frac{4}{3h^2}\right)^2(H_{12} + H_{16})\right]$$

$$K_{55} = A_{44} + \alpha^2 D_{66} + \beta^2 D_{22} - \frac{8}{h^2} D_{44} + \left(\frac{4}{h^2}\right)^2 F_{44} - \left(\frac{8}{3h^2}\right)(\alpha^2 F_{66} + \beta^2 F_{22}) \\ + \left(\frac{4}{3h^2}\right)^2(\beta^2 H_{22} + \alpha^2 H_{66})$$

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H. MASS MATRIX COEFFICIENTS

H.1 MASS MATRIX COEFFICIENTS FOR SIMPLY SUPPORTED RECTANGULAR LAMINATED OR SANDWICH PLATES MODELLED USING FSDT

$$M_{11} = I_1$$

$$M_{12} = 0$$

$$M_{13} = 0$$

$$M_{14} = I_2$$

$$M_{15} = 0$$

$$M_{22} = I_1$$

$$M_{23} = 0$$

$$M_{24} = 0$$

$$M_{25} = I_2$$

$$M_{33} = I_1$$

$$M_{34} = 0$$

$$M_{35} = 0$$

$$M_{44} = I_3$$

$$M_{45} = 0$$

$$M_{55} = I_3$$

H.2 STIFFNESS MATRIX COEFFICIENTS FOR SIMPLY SUPPORTED RECTANGULAR LAMINATED OR SANDWICH PLATES MODELLED USING HSDT

$$M_{11} = I_1$$

$$M_{12} = 0$$

$$M_{13} = -\left(\frac{4}{3h^2}\right)\alpha I_4$$

$$M_{14} = I_2 - \left(\frac{4}{3h^2}\right)I_4$$

$$M_{15} = 0$$

$$M_{22} = I_1$$

$$M_{23} = -\left(\frac{4}{3h^2}\right)\beta I_4$$

$$M_{24} = 0$$

$$M_{25} = I_2 - \left(\frac{7}{3h^2}\right)I_4$$

$$M_{33} = I_1 + \left(\frac{4}{3h^2}\right)^2 (\alpha^2 + \beta^2) I_7$$

$$M_{34} = -\left(\frac{4}{3h^2}\right)\alpha I_5 + \left(\frac{4}{3h^2}\right)^2 \alpha I_7$$

$$M_{35} = -\left(\frac{4}{3h^2}\right)\beta I_5 + \left(\frac{4}{3h^2}\right)^2 \beta I_7$$

$$M_{44} = I_3 - \left(\frac{8}{3h^2}\right)I_5 + \left(\frac{4}{3h^2}\right)^2 I_7$$

$$M_{45} = 0$$

$$M_{55} = I_3 - \left(\frac{8}{3h^2}\right)I_5 + \left(\frac{4}{3h^2}\right)^2 I_7$$

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I. NEWMARK'S DIRECT INTEGRATION TECHNIQUE

Newmark's direct numerical integration technique or a step-by-step method is employed to integrate the equations of motion of the form I.1.

$$[M]\{\ddot{\Delta}\} + [D]\{\dot{\Delta}\} + [K]\{\Delta\} = \{F\} \quad (\text{I.1})$$

This method allows the determination of the state of the system at a time $t = t_{n+1} = t_n + \Delta t$ knowing the state of the system at time t_n . In the Newmark's integration method the first time derivative $\{\dot{\Delta}\}$ and the solution vector $\{\Delta\}$ of the generalised displacements are approximated at time t_{n+1} using relations I.2.

$$\begin{aligned} \{\dot{\Delta}\}_{n+1} &= \{\Delta\}_n + (1-\gamma)\Delta t\{\ddot{\Delta}\}_n + \gamma\Delta t\{\ddot{\Delta}\}_{n+1} \\ \{\Delta\}_{n+1} &= \{\Delta\}_n + \Delta t\{\dot{\Delta}\}_n + (\Delta t)^2\left(\frac{1}{2} - \chi\right)\{\ddot{\Delta}\}_{n+1} \end{aligned} \quad (\text{I.2})$$

In these equations I.2, γ and χ are parameters that control the stability and the accuracy of the scheme and the subscript n indicates that the solution is evaluated at the n th time step. For a linear problem, that is to say when the mass, damping and stiffness matrices are independent of the vector solution $\{\Delta\}$, the Newmark's method is unconditionally stable when the parameter values are set to $\gamma = 1/2$ and $\chi = 1/4$.

Finally by substituting relations I.2 into the equations of motion I.1, the unknown acceleration vector $\{\ddot{\Delta}\}_{n+1}$ at time t_{n+1} can be expressed (see relation I.3) as a function of the previously determined acceleration vector $\{\ddot{\Delta}\}_n$, first time derivative vector $\{\dot{\Delta}\}_n$ and vector solution $\{\Delta\}_n$ at time t_n .

$$\begin{aligned} ([M] + \gamma\Delta t[D] + \chi(\Delta t)^2[K])\{\ddot{\Delta}\}_{n+1} &= \{F\}_{n+1} - [D](\{\dot{\Delta}\}_n + (1-\gamma)\Delta t\{\ddot{\Delta}\}_n) \\ &\quad - [K]\left(\{\Delta\}_n + \Delta t\{\dot{\Delta}\}_n + \left(\frac{1}{2} - \chi\right)(\Delta t)^2\{\ddot{\Delta}\}_n\right) \end{aligned} \quad (\text{I.3})$$

So for a given set of initial condition $\{\dot{\Delta}\}_0$ and $\{\Delta\}_0$ the equations of motion can be solved step by step as summarised in the Newmark's direct integration technique algorithm as shown in figure I.1.

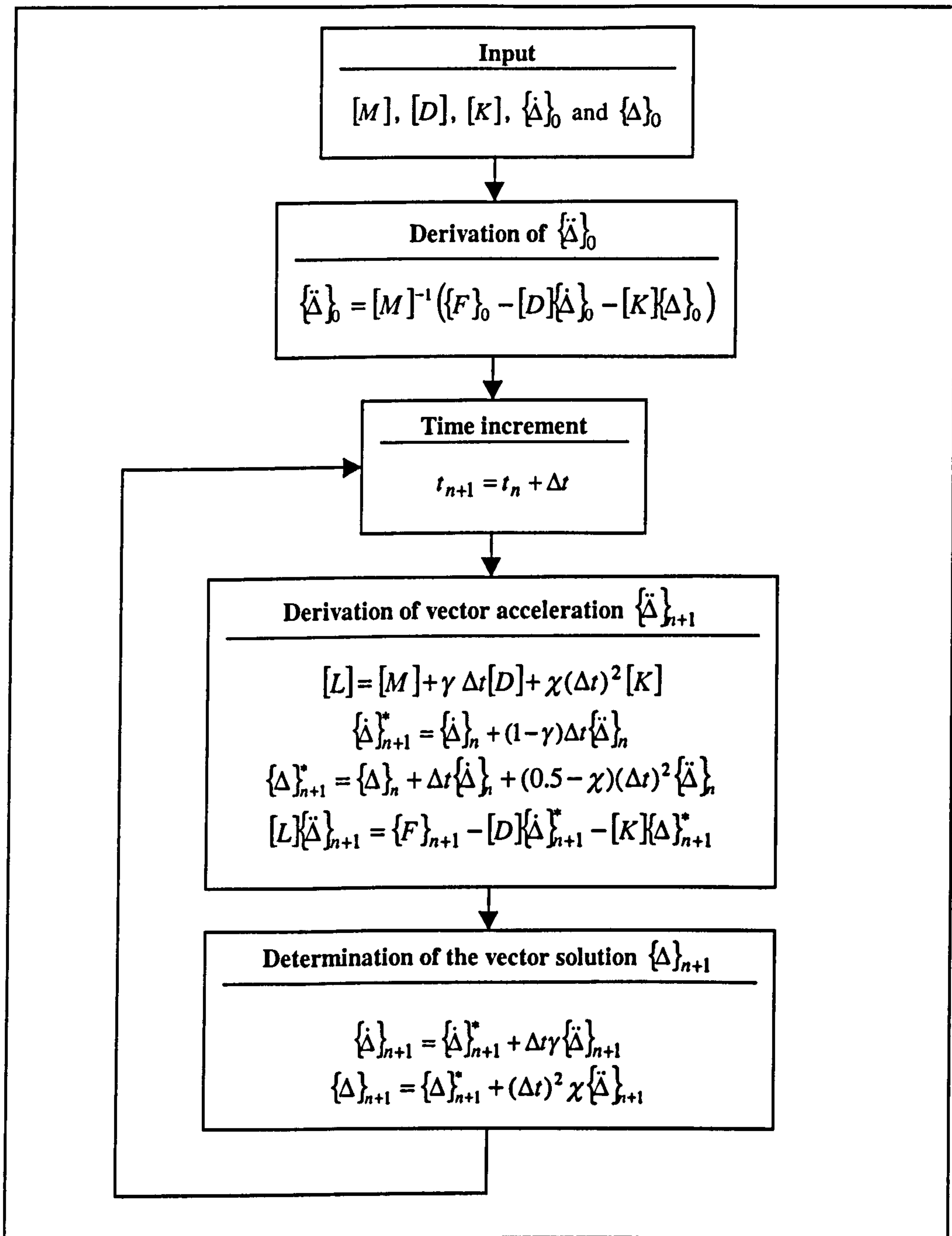


Figure I.1: Newmark's direct integration method algorithm.

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J. DMA: SINUSOIDAL EXCITATION AND RESPONSE

In viscoelastic studies of polymeric materials the method of sinusoidal excitation and response is very useful. During dynamic testing, a sinusoidal strain is applied to the material and the resulting stress developed in the material is measured. For linear viscoelastic behaviour, the strain will alternate sinusoidally but will be out of phase with the stress, as seen in figure J.1. This phase lag results from the time necessary for molecular rearrangements and is associated with relaxation phenomena.

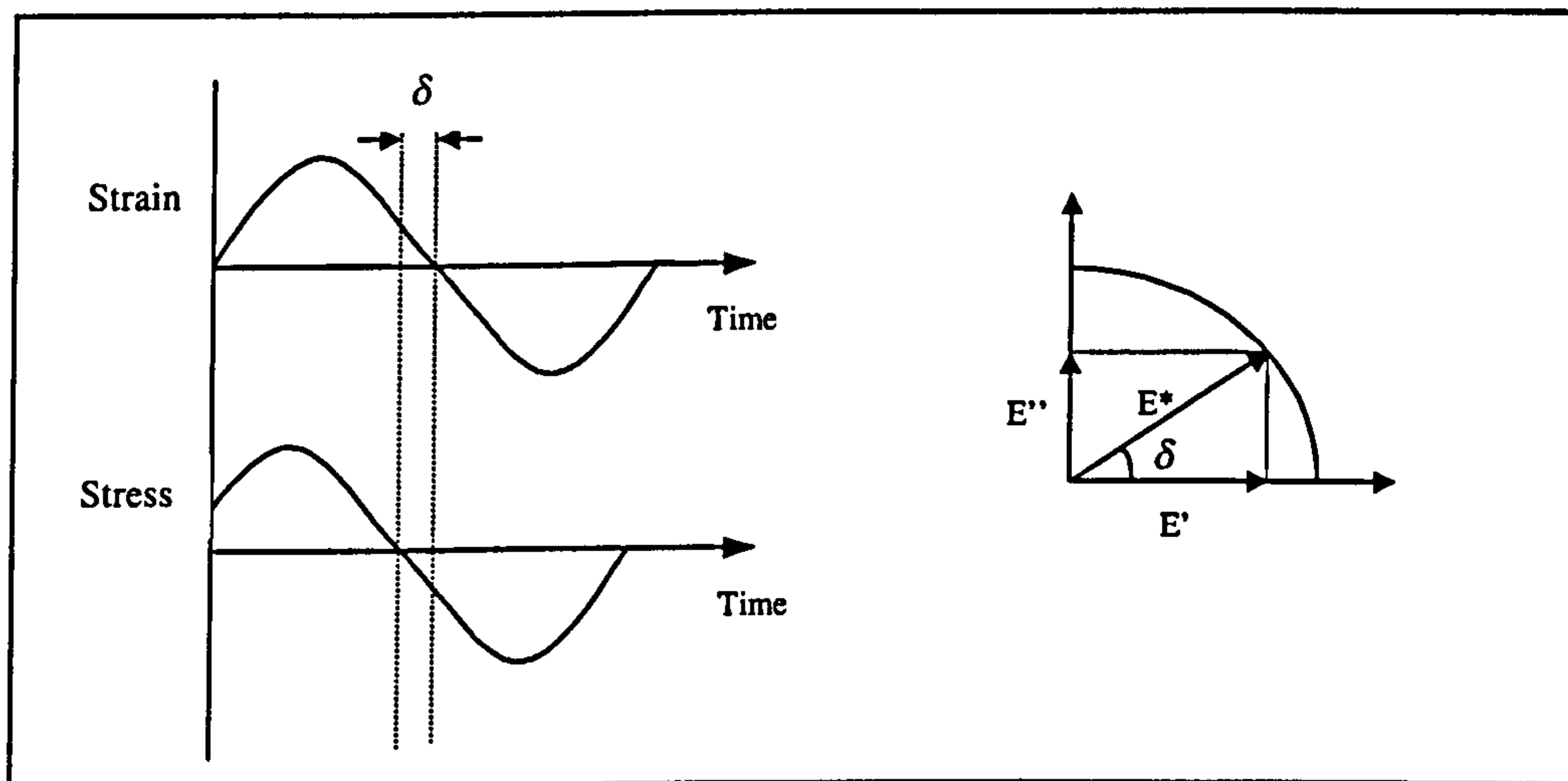


Figure J.1: Viscoelastic materials - DMA stress and strain.

The expression for the stress σ and the strains ε due to bending are provided in equations J.1 and J.2.

$$\sigma = \sigma_o \cos(\omega t + \delta) \quad (\text{J.1})$$

$$\varepsilon = \varepsilon_o \cos(\omega t) \quad (\text{J.2})$$

For an ideal solid material, which obeys Hooke's law, the resulting stress will be proportional to the amplitude of the applied strain and the stress and the strain waves will be in phase $\delta = 0^\circ$. For an ideal fluid, which obeys Newton's law, the stress will be proportional to the strain rate and the stress signal will lead the strain signal by 90° . For a viscoelastic material the phase angle will be between 0° and 90° . This phase lag results from the time necessary for molecular rearrangements and it is associated with relaxation phenomena.

Expression J.1 can be re-written in the following form:

$$\sigma = \sigma_o \cos \omega t \sin \delta - \sigma_o \sin \omega t \cos \delta \quad (\text{J.3})$$

From this equation, it can be noticed that the stress can be considered to consist of two components, an elastic stress $(\sigma_o \sin \delta)$ in phase with the strain and a viscous stress $(\sigma_o \cos \delta)$ 90° out of phase with the strain.

Since during DMA test, the studied structure is subjected to steady state oscillatory condition, the stress-strain relation 4.5 presented in chapter 4 can be used. This relation for a one-dimensional case can be written as follows:

$$\sigma^* = E^* \varepsilon_o \exp(i\omega t) \quad (\text{J.4})$$

The complex moduli E^* is defined as:

$$E^* = E' + iE'' \quad (\text{J.5})$$

Recognising that only the real quantities of the stress-strain equation have physical significance and using relation J.5, equation J.4 becomes:

$$\sigma = \text{Re}(\sigma^*) = \varepsilon_o [E' \cos \omega t - E'' \sin \omega t] \quad (\text{J.6})$$

By comparing equation J.3 with equation J.6, it appears that:

$$E' = \frac{\sigma_o}{\varepsilon_o} \sin \delta \quad (\text{J.7})$$

$$E'' = \frac{\sigma_o}{\varepsilon_o} \cos \delta \quad (\text{J.8})$$

In these equations J.7 and J.8, E' is the real part of the complex modulus and E'' is the imaginary part.

So, the complex Young's modulus E^* , defined as the ratio of stress to strain, can be calculated from the resultant stress generated in a viscoelastic material. It encompasses both elastic and viscous responses. The elastic modulus or storage modulus and the viscous modulus or loss modulus can be calculated directly from the elastic and viscous stress respectively. Therefore, by measuring during the experiments σ_0 , ε_0 and δ , it is then possible using equation J.7 and J.8 to calculate the complex Young's modulus of viscoelastic materials for a given temperature and a given frequency. It should be noticed that if the stress and strain due to shear are considered instead of the stress and strain due to bending then the complex shear modulus, G^* , can be determined in the same way as the complex Young's modulus.

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K. COMPLEX STIFFNESS MATRIX COEFFICIENTS

K.1 COMPLEX STIFFNESS MATRIX COEFFICIENTS FOR SIMPLY SUPPORTED (SSSS-1) RECTANGULAR LAMINATED OR SANDWICH PLATES MODELLED USING HSDT

$$K_{11}^* = \alpha^2 A_{11}^* + \beta^2 A_{66}^*$$

$$K_{12}^* = \alpha\beta(A_{12}^* + A_{66}^*)$$

$$K_{13}^* = -\left(\frac{4}{3h^2}\right)\alpha^3 E_{11}^* - \left(\frac{4}{3h^2}\right)\alpha\beta^2(E_{12}^* + 2E_{66}^*)$$

$$K_{14}^* = \alpha^2 B_{11}^* - \alpha^2\left(\frac{4}{3h^2}\right)E_{11}^* + \beta^2\left[B_{66}^* - \left(\frac{4}{3h^2}\right)E_{66}^*\right]$$

$$K_{15}^* = \alpha\beta\left[(B_{12}^* + B_{66}^*) - \left(\frac{4}{3h^2}\right)(E_{12}^* + E_{66}^*)\right]$$

$$K_{22}^* = \beta^2 A_{22}^* + \alpha^2 A_{66}^*$$

$$K_{23}^* = -\beta^3\left(\frac{4}{3h^2}\right)E_{22}^* - \left(\frac{4}{3h^2}\right)\alpha^2\beta(E_{12}^* + 2E_{66}^*)$$

$$K_{24}^* = \alpha\beta\left[(B_{12}^* + B_{66}^*) - \left(\frac{4}{3h^2}\right)(E_{12}^* + E_{66}^*)\right]$$

$$K_{25}^* = \beta^2 B_{22}^* - \beta^2\left(\frac{4}{3h^2}\right)E_{22}^* + \alpha^2\left[B_{66}^* - \left(\frac{4}{3h^2}\right)E_{66}^*\right]$$

$$K_{33}^* = \alpha^2 A_{55}^* + \beta^2 A_{44}^* - \frac{8}{h^2}(\alpha^2 D_{55}^* + \beta^2 D_{44}^*) + \left(\frac{4}{h^2}\right)^2(\alpha^2 F_{55}^* + \beta^2 F_{44}^*) \\ + \left(\frac{4}{3h^2}\right)^2[\alpha^4 H_{11}^* + 2(H_{12}^* + 2H_{66}^*)\alpha^2\beta^2 + \beta^4 H_{22}^*]$$

$$K_{34}^* = \alpha A_{55}^* - \frac{8}{h^2}\alpha D_{55}^* + \left(\frac{4}{h^2}\right)^2\alpha F_{55}^* - \left(\frac{4}{3h^2}\right)[\alpha^3 F_{11}^* + \alpha\beta^2(F_{12}^* + 2F_{66}^*)] \\ + \left(\frac{4}{3h^2}\right)^2[\alpha^3 H_{11}^* + \alpha\beta^2(H_{12}^* + 2H_{66}^*)]$$

$$K_{35}^* = \beta A_{44}^* - \frac{8}{h^2} \beta D_{44}^* + \left(\frac{4}{h^2} \right) \beta F_{44}^* - \left(\frac{4}{3h^2} \right) \left[\alpha^2 \beta (F_{12}^* + 2F_{66}^*) + \beta^3 F_{22}^* \right] \\ + \left(\frac{3}{4h^2} \right)^2 \left[\alpha^2 \beta (H_{12}^* + 2H_{66}^*) + \beta^3 H_{22}^* \right]$$

$$K_{44}^* = A_{55}^* + \alpha^2 D_{11}^* + \beta^2 D_{66}^* - \frac{8}{h^2} D_{55}^* + \left(\frac{4}{h^2} \right)^2 F_{55}^* - \frac{8}{3h^2} (\alpha^2 F_{11}^* + \beta^2 F_{66}^*) \\ + \left(\frac{4}{3h^2} \right)^2 (\alpha^2 H_{11}^* + \beta^2 H_{66}^*)$$

$$K_{45}^* = \alpha \beta \left[D_{12}^* + D_{66}^* - \frac{8}{3h^2} (F_{12}^* + F_{66}^*) + \left(\frac{4}{3h^2} \right)^2 (H_{12}^* + H_{16}^*) \right]$$

$$K_{55}^* = A_{44}^* + \alpha^2 D_{66}^* + \beta^2 D_{22}^* - \frac{8}{h^2} D_{44}^* + \left(\frac{4}{h^2} \right)^2 F_{44}^* - \left(\frac{8}{3h^2} \right) (\alpha^2 F_{66}^* + \beta^2 F_{22}^*) \\ + \left(\frac{4}{3h^2} \right)^2 (\beta^2 H_{22}^* + \alpha^2 H_{66}^*)$$

K.2 COMPLEX STIFFNESS MATRIX COEFFICIENTS FOR SIMPLY SUPPORTED (SSSS-2) RECTANGULAR LAMINATED OR SANDWICH PLATES MODELLED USING HSDT

$$K_{11}^* = \alpha^2 A_{11}^* + \beta^2 A_{66}^*$$

$$K_{12}^* = \alpha \beta (A_{12}^* + A_{66}^*)$$

$$K_{13}^* = - \left(\frac{4}{3h^2} \right) (3\alpha^2 \beta E_{16}^* + \beta^3 E_{26}^*)$$

$$K_{14}^* = 2\beta \alpha B_{16}^* - \left(\frac{8}{3h^2} \right) \alpha \beta E_{16}^*$$

$$K_{15}^* = \alpha^2 B_{16}^* + \beta^2 B_{26}^* - \left(\frac{4}{3h^2} \right) (\alpha^2 E_{16}^* + \beta^2 E_{26}^*)$$

$$K_{22}^* = \beta^2 A_{22}^* + \alpha^2 A_{66}^*$$

$$K_{23}^* = -\left(\frac{4}{3h^2}\right)(\alpha^3 E_{16}^* + 3\beta^2 \alpha E_{26}^*)$$

$$K_{24}^* = K_{15}^*$$

$$K_{25}^* = 2\alpha\beta B_{26}^* - \left(\frac{8}{3h^2}\right)\alpha\beta E_{26}^*$$

$$K_{33}^* = \alpha^2 A_{55}^* + \beta^2 A_{44}^* - \frac{8}{h^2}(\alpha^2 D_{55}^* + \beta^2 D_{44}^*) + \left(\frac{4}{h^2}\right)^2 (\alpha^2 F_{55}^* + \beta^2 F_{44}^*) \\ + \left(\frac{4}{3h^2}\right)^2 [\alpha^4 H_{11}^* + 2(H_{12}^* + 2H_{66}^*)\alpha^2 \beta^2 + \beta^4 H_{22}^*]$$

$$K_{34}^* = \alpha A_{55}^* - \frac{8}{h^2} \alpha D_{55}^* + \left(\frac{4}{h^2}\right)^2 \alpha F_{55}^* - \left(\frac{4}{3h^2}\right) [\alpha^3 F_{11}^* + \alpha\beta^2 (F_{12}^* + 2F_{66}^*)] \\ + \left(\frac{4}{3h^2}\right)^2 [\alpha^3 H_{11}^* + \alpha\beta^2 (H_{12}^* + 2H_{66}^*)]$$

$$K_{35}^* = \beta A_{44}^* - \frac{8}{h^2} \beta D_{44}^* + \left(\frac{4}{h^2}\right)^2 \beta F_{44}^* - \left(\frac{4}{3h^2}\right) [\alpha^2 \beta (F_{12}^* + 2F_{66}^*) + \beta^3 F_{22}^*] \\ + \left(\frac{3}{4h^2}\right)^2 [\alpha^2 \beta (H_{12}^* + 2H_{66}^*) + \beta^3 H_{22}^*]$$

$$K_{44}^* = A_{55}^* + \alpha^2 D_{11}^* + \beta^2 D_{66}^* - \frac{8}{h^2} D_{55}^* + \left(\frac{4}{h^2}\right)^2 F_{55}^* - \frac{8}{3h^2} (\alpha^2 F_{11}^* + \beta^2 F_{66}^*) \\ + \left(\frac{4}{3h^2}\right)^2 (\alpha^2 H_{11}^* + \beta^2 H_{66}^*)$$

$$K_{45}^* = \alpha\beta \left[D_{12}^* + D_{66}^* - \frac{8}{3h^2} (F_{12}^* + F_{66}^*) + \left(\frac{4}{3h^2}\right)^2 (H_{12}^* + H_{16}^*) \right]$$

$$K_{55}^* = A_{44}^* + \alpha^2 D_{66}^* + \beta^2 D_{22}^* - \frac{8}{h^2} D_{44}^* + \left(\frac{4}{h^2}\right)^2 F_{44}^* - \left(\frac{8}{3h^2}\right) (\alpha^2 F_{66}^* + \beta^2 F_{22}^*) \\ + \left(\frac{4}{3h^2}\right)^2 (\beta^2 H_{22}^* + \alpha^2 H_{66}^*)$$

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L. COMPLEX STIFFNESSES

L.1 COMPLEX STIFFNESSES FOR FRP LAMINATED PLATES MODELLED USING HSDT

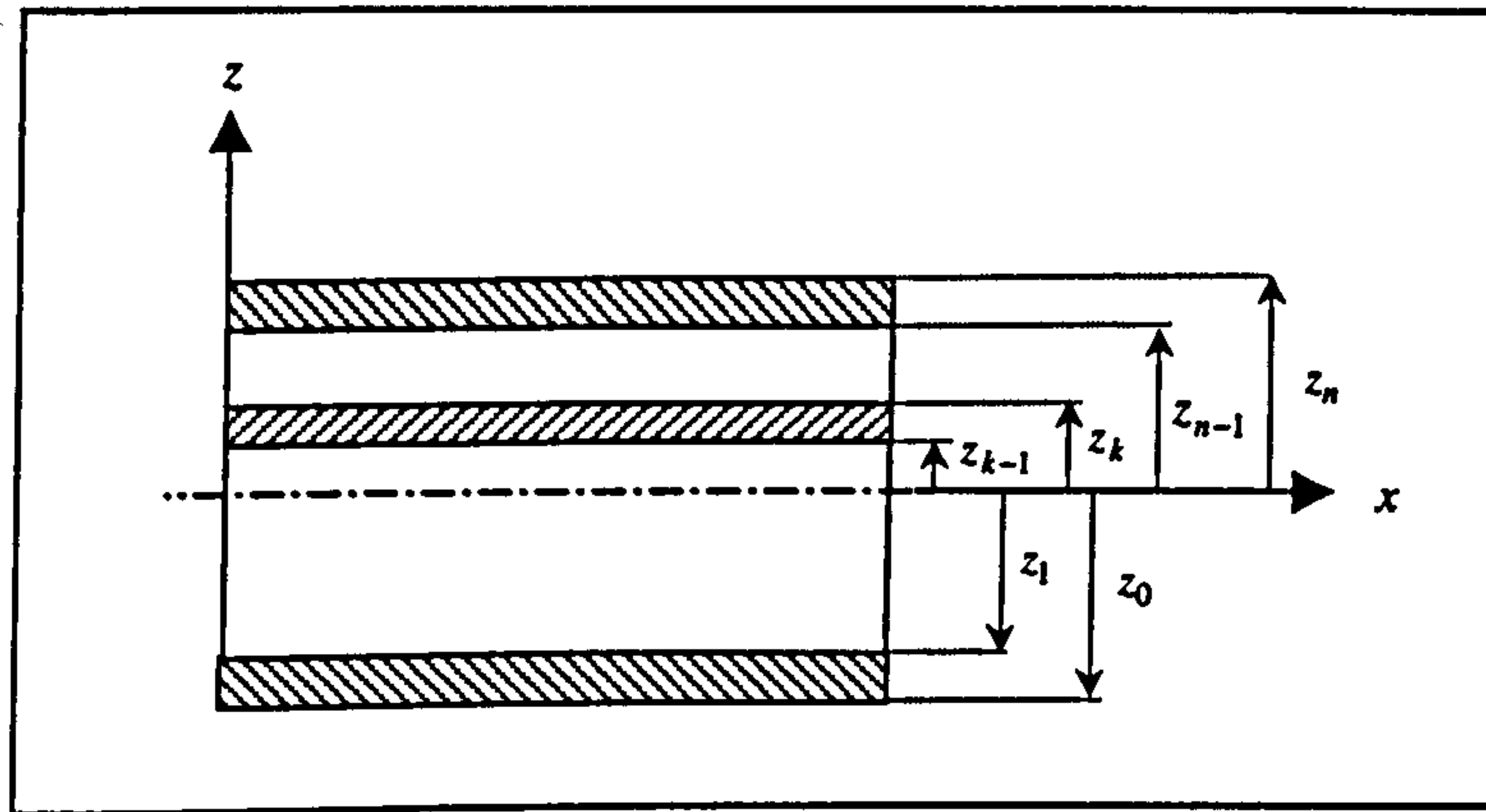


Figure L.1: Cross section of a typical laminated panel.

When FRP laminated panels are modelled using Reddy's refined HSDT, considering the notation shown in figure L.1, the complex laminate stiffnesses A_{ij}^* , B_{ij}^* , D_{ij}^* , E_{ij}^* , F_{ij}^* and H_{ij}^* can be determined using the following relations:

$$\left(A_{ij}^*, B_{ij}^*, D_{ij}^*, E_{ij}^*, F_{ij}^*, H_{ij}^* \right) = \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \bar{C}_{ij}^{*(k)} \left(1, z, z^2, z^3, z^4, z^6 \right) dz \quad (i, j = 1, 2, 6)$$

$$\left(A_{ij}^*, D_{ij}^*, F_{ij}^* \right) = \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \bar{C}_{ij}^{*(k)} \left(1, z^2, z^4 \right) dz \quad (i, j = 4, 5)$$

L.2 COMPLEX STIFFNESSES FOR FRP SANDWICH PLATES MODELLED USING HSDT

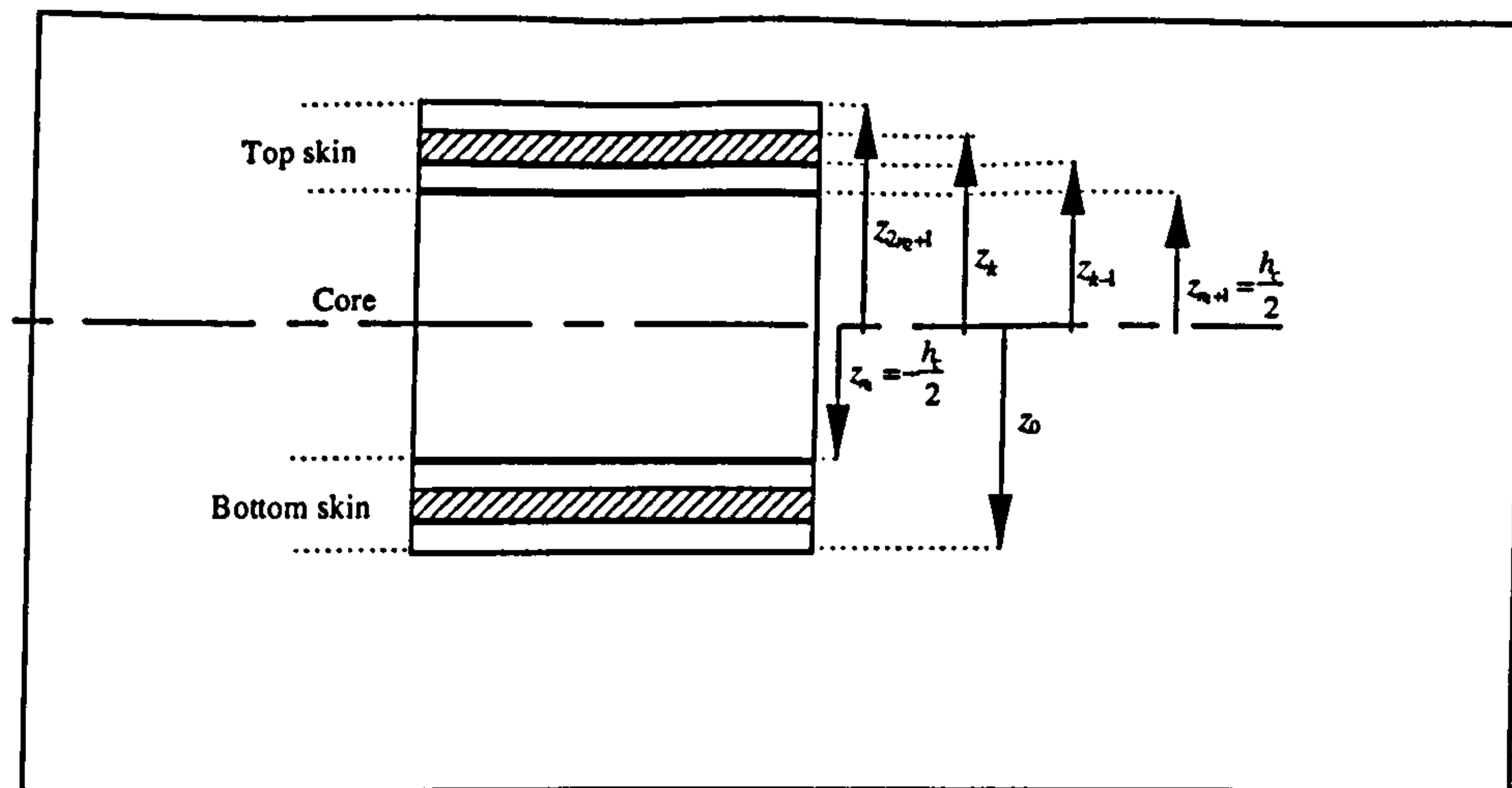


Figure L.2: Cross section of a sandwich plate.

When FRP sandwich panels are modelled using Reddy's refined HSDT, considering the notation shown in figure L.2, the complex stiffnesses A_{ij}^* , B_{ij}^* , D_{ij}^* , E_{ij}^* , F_{ij}^* and H_{ij}^* can be determined using the following relations:

$$\begin{aligned} (A_{ij}^*, B_{ij}^*, D_{ij}^*, E_{ij}^*, F_{ij}^*, H_{ij}^*) = & \sum_{k=1}^{n_2} \int_{z_{k-1}}^{z_k} \bar{C}_{ij}^{*(k)} (1, z, z^2, z^3, z^4, z^6) dz + \int_{-h_c/2}^{h_c/2} \bar{C}_{ij}^{*(core)} (1, z, z^2, z^3, z^4, z^6) dz \\ & + \sum_{k=n_2+2}^{2n_2+1} \int_{z_{k-1}}^{z_k} \bar{C}_{ij}^{*(k)} (1, z, z^2, z^3, z^4, z^6) dz \end{aligned} \quad (i, j = 1, 2, 6)$$

$$(A_{ij}^*, D_{ij}^*, F_{ij}^*) = \sum_{k=1}^{n_2} \int_{z_{k-1}}^{z_k} \bar{C}_{ij}^{*(k)} (1, z^2, z^4) dz + \int_{-h_c/2}^{h_c/2} \bar{C}_{ij}^{*(core)} (1, z^2, z^4) dz + \sum_{k=n_2+2}^{2n_2+1} \int_{z_{k-1}}^{z_k} \bar{C}_{ij}^{*(k)} (1, z^2, z^4) dz \quad (i, j = 4, 5)$$

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M. EXPRESSION FOR THE COEFFICIENTS OF THE VISCOELASTIC MATRICES BEFORE DECOMPOSITION.

The coefficients of the viscoelastic mass, damping and stiffness matrices of relation 8.27 can be derived using the relations presented in this appendix M.

M.1 THE GHM VISCOELASTIC MASS MATRIX COEFFICIENTS

The GHM viscoelastic mass matrix before decomposition is function of the mass matrix $[M]$, whose coefficient expressions can be found in the appendix G, and of the diagonal mass matrix $[M_{\bar{z}}]$, whose coefficients are a function of the parameters of the GHM mini-oscillator forms. The diagonal sub mass matrix $[M_{\bar{z}}]$ is defined as follows:

$$[M_{\bar{z}}] = \begin{bmatrix} [M_{\bar{z}}]_1 & [0] & \dots & [0] \\ [0] & \ddots & & \vdots \\ \vdots & & \ddots & [0] \\ [0] & \dots & [0] & [M_{\bar{z}}]_n \end{bmatrix} \quad (\text{M.1})$$

Then, each of the diagonal matrices $[M_{\bar{z}}]_i$ are defined by:

$$[M_{\bar{z}}]_i = \begin{bmatrix} \hat{\alpha}_i^1 \frac{1}{(\hat{\omega}_i^1)^2} [\bar{K}] & [0] & \dots & [0] \\ [0] & \ddots & & \vdots \\ \vdots & & \ddots & [0] \\ [0] & \dots & [0] & \hat{\alpha}_i^k \frac{1}{(\hat{\omega}_i^k)^2} [\bar{K}] \end{bmatrix} \quad (\text{M.2})$$

M.2 THE GHM VISCOELASTIC DAMPING MATRIX COEFFICIENTS

The GHM viscoelastic damping matrix before decomposition is function of the diagonal damping matrix $[D_{\bar{z}}]$, whose coefficients are a function of the parameters of the GHM mini-oscillator forms. This diagonal matrix is defined as follows:

$$[D_{\bar{z}}] = \begin{bmatrix} [D_{\bar{z}}]_1 & [0] & \dots & [0] \\ [0] & \ddots & & \vdots \\ \vdots & & \ddots & [0] \\ [0] & \dots & [0] & [D_{\bar{z}}]_n \end{bmatrix} \quad (\text{M.3})$$

Then each of the diagonal matrices $[D_{\bar{z}}]_i$ are defined by:

$$[D_{\bar{z}}]_i = \begin{bmatrix} \hat{\alpha}_i^1 \frac{2\hat{\zeta}_i^1}{\hat{\omega}_i^1} [\bar{K}]_1 & [0] & \dots & [0] \\ [0] & \ddots & & \vdots \\ \vdots & & \ddots & [0] \\ [0] & \dots & [0] & \hat{\alpha}_i^k \frac{2\hat{\zeta}_i^k}{\hat{\omega}_i^k} [\bar{K}]_k \end{bmatrix} \quad (\text{M.4})$$

M.3 THE GHM VISCOELASTIC STIFFNESS MATRIX COEFFICIENTS

The GHM viscoelastic stiffness matrix before decomposition is function of the stiffness matrix $[K_q]$, of the stiffness matrix $[K_{q\bar{z}}]$ and its matrix transpose and of the diagonal stiffness matrix $[K_{\bar{z}}]$. These sub matrices are defined as shown by relations M.5, M.6, M.7 and M.8.

$$[K_q] = [K] + \sum_{i=1}^n \sum_{j=1}^k \alpha_i^j [K]_i \quad (\text{M.5})$$

Where, in relation M.5, $[K] = \sum_{i=1}^n [K]_i = \sum_{i=1}^n \hat{G}_i^o [\bar{K}]_i$.

$$[K_{q\bar{z}}] = [-\hat{\alpha}_1^1 [\bar{K}]_1 \quad \dots \quad -\alpha_n^k [\bar{K}]_k] \quad (\text{M.6})$$

$$[K_{\bar{z}q}] = [K_{q\bar{z}}]^T \quad (\text{M.7})$$

$$[K_{\bar{z}}] = \begin{bmatrix} [K_{\bar{z}}]_1 & [0] & \dots & [0] \\ [0] & \ddots & & \vdots \\ \vdots & & \ddots & [0] \\ [0] & \dots & [0] & [K_{\bar{z}}]_n \end{bmatrix} \quad (\text{M.8})$$

Each of the diagonal matrices $[K_{\bar{z}}]_i$ in relation M.8 are given by:

$$[K_{\bar{z}}]_i = \begin{bmatrix} \alpha_i^1 [\bar{K}]_1 & [0] & \dots & [0] \\ [0] & \ddots & & \vdots \\ \vdots & & \ddots & [0] \\ [0] & \dots & [0] & \alpha_i^k [\bar{K}]_k \end{bmatrix} \quad (\text{M.9})$$

N. THE GHM VISCOELASTIC MATRICES

N.1 THE GHM VISCOELASTIC MASS MATRIX COEFFICIENTS

The GHM viscoelastic mass matrix $[M_v]$, as shown in relation N.1, is function of the mass matrix $[M]$, whose coefficients can be derived using the relations provided in the appendix G, and of the diagonal mass matrix $[M_z]$, whose coefficients are a function of the parameters of the GHM mini-oscillator forms.

$$[M_v] = \begin{bmatrix} [M] & [0] \\ [0] & [M_z] \end{bmatrix} \quad (N.1)$$

The diagonal mass matrix $[M_z]$ is defined as follows:

$$[M_z] = \begin{bmatrix} [M_z]_1 & [0] & \dots & [0] \\ [0] & \ddots & & \vdots \\ \vdots & & \ddots & [0] \\ [0] & \dots & [0] & [M_z]_n \end{bmatrix} \quad (N.2)$$

Each of the diagonal matrices $[M_z]_i$ in relation N.2 are defined by:

$$[M_z]_i = \begin{bmatrix} \hat{\alpha}_i^1 \frac{1}{(\hat{\omega}_i^1)^2} [\Lambda]_i & [0] & \dots & [0] \\ [0] & \ddots & & \vdots \\ \vdots & & \ddots & [0] \\ [0] & \dots & [0] & \hat{\alpha}_i^k \frac{1}{(\hat{\omega}_i^k)^2} [\Lambda]_i \end{bmatrix} \quad (N.3)$$

N.2 THE GHM VISCOELASTIC DAMPING MATRIX COEFFICIENTS

The GHM viscoelastic damping matrix $[D_v]$ is function of the diagonal damping matrix $[D_z]$, whose coefficients are a function of the parameters of the GHM mini-oscillator forms.

$$[D_v] = \begin{bmatrix} [0] & [0] \\ [0] & [D_z] \end{bmatrix} \quad (N.4)$$

The diagonal damping matrix $[D_z]$ is defined as follows:

$$[D_z] = \begin{bmatrix} [D_z]_1 & [0] & \dots & [0] \\ [0] & \ddots & & \vdots \\ \vdots & & \ddots & [0] \\ [0] & \dots & [0] & [D_z]_n \end{bmatrix} \quad (\text{N.5})$$

Each of the diagonal matrices $[D_z]_i$ in relation N.5 are defined by:

$$[D_z]_i = \begin{bmatrix} \hat{\alpha}_i^1 \frac{2\hat{\zeta}_i^1}{\hat{\omega}_i^1} [\Lambda]_i & [0] & \dots & [0] \\ [0] & \ddots & & \vdots \\ \vdots & & \ddots & [0] \\ [0] & \dots & [0] & \hat{\alpha}_i^k \frac{2\hat{\zeta}_i^k}{\hat{\omega}_i^k} [\Lambda]_i \end{bmatrix} \quad (\text{N.6})$$

The diagonal matrices $[\Lambda]_i$ are defined in chapter 8 (see relation 8.28).

N.3 THE GHM VISCOELASTIC STIFFNESS MATRIX COEFFICIENTS

The GHM viscoelastic stiffness matrix $[K_v]$ is function of the diagonal stiffness matrix $[K_z]$, of the stiffness matrix $[K_{qz}]$ and of the stiffness matrix $[K_q]$ whose coefficients are a function of the parameters of the GHM mini-oscillator forms.

$$[K_v] = \begin{bmatrix} [K_q] & [K_{qz}] \\ [K_{qz}]^T & [K_z] \end{bmatrix} \quad (\text{N.7})$$

The sub matrices of the viscoelastic stiffness matrix $[K_v]$ are defined as follows:

$$[K_q] = [K] + \sum_{i=1}^n \sum_{j=1}^k \alpha_i^j [K] \quad (\text{N.8})$$

Where, in relation N.8, $[K] = \sum_{i=1}^n [K]_i = \sum_{i=1}^n \hat{G}_i^o [\bar{K}]_i$.

$$[K_{qz}] = [-\hat{\alpha}_1^1 [R]_1 \quad \dots \quad -\alpha_n^k [R]_n] \quad (\text{N.9})$$

The matrices $[R]_i$ are defined in chapter 8 (see relation 8.30).

$$[K_{zq}] = [K_{qz}]^T \quad (\text{N.10})$$

$$[K_z] = \begin{bmatrix} [K_z]_1 & [0] & \dots & [0] \\ [0] & \ddots & & \vdots \\ \vdots & & \ddots & [0] \\ [0] & \dots & [0] & [K_z]_n \end{bmatrix} \quad (\text{N.11})$$

Each of the diagonal matrices $[K_z]_i$ in relation N.11 are defined by:

$$[K_z]_i = \begin{bmatrix} \alpha_i^1 [\Lambda]_i & [0] & \dots & [0] \\ [0] & \ddots & & \vdots \\ \vdots & & \ddots & [0] \\ [0] & \dots & [0] & \alpha_i^k [\Lambda]_i \end{bmatrix} \quad (\text{N.12})$$

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