SSSU 70
ISSN 0140 3818

UNIVERSITY OF SOUTHAMPTON

DEPARTMENT OF SHIP SCIENCE

FACULTY OF ENGINEERING AND APPLIED SCIENCE

INTERNAL WAVES PRODUCED BY A SUBMERGED BODY MOVING IN A STRATIFIED FLUID

W. G. Price
P. C. Westlake

Ship Science Report No. 70

February 1994

SSSU 70
ISSN 0140 3818

UNIVERSITY OF SOUTHAMPTON

DEPARTMENT OF SHIP SCIENCE

FACULTY OF ENGINEERING AND APPLIED SCIENCE

INTERNAL WAVES PRODUCED BY A SUBMERGED BODY MOVING IN A STRATIFIED FLUID

W. G. Price
P. C. Westlake

Ship Science Report No. 70

February 1994

Internal waves produced by a submerged body moving in a stratified fluid

W.G.Price
P.C.Westlake

February 16, 1994

Contents

Nomenclature 3

1 Introduction 4

2 Equations of motion 4

3 Low speed approximation theory 7

4 High speed approximation theory 8

5 Integral equation 9

6 Fundamental solutions - $\tilde{N}^2$ constant 10

7 Summary of antisymmetric fundamental solutions 13

8 Application of a radiation condition 15

9 Summary of asymmetric fundamental solutions 18

10 Utilisation of the fundamental solutions 22

Figures 24

A : Derivation of antisymmetric fundamental solutions 28

B : Derivation of asymmetric fundamental solutions 36

Nomenclature

F_n	Froude number $= \frac{U}{\sqrt{gL}}$
g	Gravity acceleration $= (0,0,-g)$
L	Characteristic length, for example length of body
$N(z)$	Brunt-Väisälä frequency $= \sqrt{-\frac{g}{\rho(z)}\frac{\partial\rho}{\partial z}}$
$\tilde{N}(z)$	$= \frac{N(z)}{F_n}$
$Oxyz$	Moving reference coordinate system
r_1	$= x - \xi$
r_2	$= y - \eta$
r_3	$= z - \zeta$
-$\boldsymbol{U}(t)$	Translational velocity of body $= -\{U(t), V(t), W(t)\}$
$\boldsymbol{u}(x,y,z,t)$	Parametric disturbance velocity vector $= (u(x,y,z,t), v(x,y,z,t), w(x,y,z,t))$
$\boldsymbol{V}(x,y,z,t)$	Disturbance velocity vector
$\boldsymbol{x}$	Position vector of the field point (x,y,z)
$\beta^{\pm}$	$= -\frac{\tilde{N}(\zeta\pm z)}{\sqrt{r_2^2+(\zeta\pm z)^2}}$
γ_1	$= \sqrt{1-\left(\frac{\tilde{N}}{\lambda_1}\right)^2}$
γ_2	$= \sqrt{\left(\frac{\tilde{N}}{\lambda_1}\right)^2 - 1}$
γ_3	$= \sqrt{1-\frac{\tilde{N}^2}{\lambda_1(\lambda_1-i\epsilon)}}$
$\gamma^{\pm}$	$= \lambda_1^2[(y-\eta)^2+(\zeta\pm z)^2] - \tilde{N}^2(\zeta\pm z)^2$
$\mu(x,y,z,t)$	Viscosity of fluid medium
$\boldsymbol{n}$	Unit normal pointing outward from the fluid domain $= (n_1, n_2, n_3)$
$\boldsymbol{n}_h$	$= (n_1, n_2, 0)$
$\rho(x,y,z,t)$	Density stratification of fluid medium
$\boldsymbol{\xi}$	Position vector of source point (ξ,η,ζ)
Σ	Boundary surface enclosing the fluid domain, Ω
$\sigma^{\pm}$	$= \lambda_1^2[(y-\eta)^2-(\zeta\pm z)^2] + \tilde{N}^2(\zeta\pm z)^2$
Ω	Fluid domain, $z<0$
$\boldsymbol{\nabla}$	$= (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$
$\boldsymbol{\nabla}_h$	$= (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0)$

1 Introduction

In many maritime engineering applications, potential flow theory has proved successful in describing the behaviour of a body travelling or fixed in an irregular seaway, Faltinsen (1990). Furthermore, potential flow singularity distribution panel methods, developed from the original approach proposed by Hess and Smith (1964), have allowed descriptions of the behaviour and interaction mechanisms between arbitrary shaped bodies and fluid flows. This is achieved by discretising the wetted surface geometry of the body by panels over which appropriate singularity solutions (or fundamental solutions) of unknown strengths are distributed. The latter are determined subject to the imposed boundary conditions in the developed mathematical model.

This report describes a preliminary investigation into the development of a *boundary element method* to determine the fluid actions and velocity flow fields associated with an arbitrary shaped body moving in a fluid exhibiting a prescribed vertical density stratification. The approach is analogous to a potential flow singularity distribution panel method adopted in problems assuming an ideal fluid of constant density but now the singularity is replaced by a fundamental solution accounting for the density stratification and therefore the rotational characteristics of the fluid.

Commencing from the general equations of momentum and conservation of mass, we derive linearised equations of motion describing the fluid disturbance caused by a body moving with a constant translational velocity in the horizontal direction. The inclusion of density stratification into the mathematical model destroys the concept of an ideal fluid and hence irrotational fluid motion but in the present context this influence is conceived to be of far greater importance in the development of the mathematical model than viscous effects. For this reason, a simplified mathematical model is assumed which neglects the influence of viscosity but retains a prescriptive description of the vertical stratification of the fluid density.

Although a single equation of motion in the vertical velocity component of the fluid disturbance (say) can be deduced, it is found advantageous from a derivation point of view to treat the individual coupled equations describing the parameters defining the fluid disturbance separately. When these equations are expressed in non-dimensional forms, two approximate theories are deduced dependent on the relative magnitudes of forward speed and the density stratification of the fluid through the Froude number and a nondimensional Brunt-Väisälä frequency respectively. Namely a *high speed* and a *low speed* approximation. Because of practical considerations attention is focused on the low speed approximation although the high speed approximation and a theory void of any approximation are briefly examined.

A boundary integral identity equation is developed and this was found to simplify by grouping terms to create fundamental equations from which fundamental solutions are derived. It is these coupled solutions which replace the singularity solutions in an ideal fluid flow problem and they are functions of the prescribed forward motion of the body and density stratification. Fundamental solutions to the fundamental equations are sought in the presence of an implusive point action which permits significant simplification to the integral equation identity. Further refinement and development of this area of study are required but initial results show that the general proposed approach is feasible and the fundamental solutions show characteristics associated with internal waves.

This study concentrates on deriving suitable fundamental solutions which exhibit internal wave characteristics exerted by a body moving in a prescribed density stratified fluid. At present, we are interested in creating a mathematical model which demonstrates the physical fluid structure interaction processes in a qualitative sense rather than quantitatively. Later, when the proposed approach is developed sufficiently emphasis will be transferred to examinations of the quantitative aspects of a three fluid-structure interactive process between body, stratified fluid and free surface.

2 Equations of motion

It is assumed that the fluid structure interaction experienced by a rigid, arbitrary shaped body moving in a prescribed density stratified fluid can be described with reference to a body fixed coordinate system. For generality this moves with a translational velocity $-\boldsymbol{U}(t) = -\{U(t), V(t), W(t)\}$ and the equations of motion describing the velocity of the fluid disturbance $\boldsymbol{V}(t)$ in a stratified fluid with viscosity μ and density ρ are of the form (see Batchelor(1967))

Equation of momentum

$$\rho\frac{D\boldsymbol{V}}{Dt} = \rho\left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}\right] = -\boldsymbol{\nabla}p + \rho\boldsymbol{g} + \boldsymbol{\nabla}^2(\mu\boldsymbol{V})$$

$$-\boldsymbol{V}\boldsymbol{\nabla}^2\mu - (\boldsymbol{\nabla}\times\boldsymbol{V})\times\boldsymbol{\nabla}\mu + \boldsymbol{\nabla}(\beta\boldsymbol{\nabla}\cdot\boldsymbol{V}) + \frac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{V}) + \frac{2}{3}\boldsymbol{\nabla}\cdot\boldsymbol{V}\boldsymbol{\nabla}\mu + \rho\dot{\boldsymbol{U}} \tag{1}$$

Here p denotes the pressure in the fluid, $\boldsymbol{g} = (0, 0, -g)$ and the variables β and μ denote the kinematic viscosity coefficients. An overdot denotes an acceleration and $\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$.

Equation of Conservation of Mass or Continuity

$$\frac{D\rho}{Dt} + \rho\boldsymbol{\nabla}\cdot\boldsymbol{V} = \frac{\partial\rho}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\rho + \rho\boldsymbol{\nabla}\cdot\boldsymbol{V} = 0 \tag{2}$$

It is assumed that the fluid is incompressible and no heat transfer occurs, that is

$$\boldsymbol{\nabla}\cdot\boldsymbol{V} = 0 \tag{3}$$

The substitution of equation 3 into equation 1 gives

$$\rho\frac{D\boldsymbol{V}}{Dt} = -\boldsymbol{\nabla}p + \rho\boldsymbol{g} + \boldsymbol{\nabla}^2(\mu\boldsymbol{V}) - \boldsymbol{V}\boldsymbol{\nabla}^2\mu - (\boldsymbol{\nabla}\times\boldsymbol{V})\times\boldsymbol{\nabla}\mu + \rho\dot{\boldsymbol{U}} \tag{4}$$

and its substitution into equation 2 gives

$$\frac{\partial\rho}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\rho = 0 \tag{5}$$

Let us assume that the variables describing the fluid structure interaction can be expressed in the form

$$\begin{array}{rcccl} p(x,y,z,t) &=& p_0(z,t) &+& p_1(x,y,z,t) \\ \rho(x,y,z,t) &=& \rho_0(z,t) &+& \rho_1(x,y,z,t) \\ \mu(x,y,z,t) &=& \mu_0(z,t) &+& \mu_1(x,y,z,t) \\ \boldsymbol{V}(x,y,z,t) &=& \boldsymbol{U}(t) &+& \boldsymbol{u}(x,y,z,t) \end{array} \tag{6}$$

where p_1, ρ_1 and μ_1 and $|\boldsymbol{u}|$ are all small quantities compared to p_0, ρ_0 and μ_0 and $|\boldsymbol{U}|$ respectively.

Under these assumptions the equation of momentum, equation 4, describing the parametric disturbances becomes

$$(\rho_0+\rho_1)\frac{D(\boldsymbol{U}+\boldsymbol{u})}{Dt} = -\boldsymbol{\nabla}(p_0+p_1) + (\rho_0+\rho_1)\boldsymbol{g} + (\rho_0+\rho_1)\dot{\boldsymbol{U}}$$

$$+\boldsymbol{\nabla}^2\left[(\mu_0+\mu_1)(\boldsymbol{U}+\boldsymbol{u})\right] - (\boldsymbol{U}+\boldsymbol{u})\boldsymbol{\nabla}^2(\mu_0+\mu_1) - [\boldsymbol{\nabla}\times(\boldsymbol{U}+\boldsymbol{u})]\times\boldsymbol{\nabla}(\mu_0+\mu_1) \tag{7}$$

from which the first order terms produce

$$\rho_0\frac{D\boldsymbol{u}}{Dt} = \rho_0\left[\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{U}\cdot\boldsymbol{\nabla}\boldsymbol{u}\right] = -\boldsymbol{\nabla}p_1 + \rho_1\boldsymbol{g} + \boldsymbol{\nabla}^2(\mu_0\boldsymbol{u}) - \boldsymbol{u}\boldsymbol{\nabla}^2\mu_0 - (\boldsymbol{\nabla}\times\boldsymbol{u})\times\boldsymbol{\nabla}\mu_0 \tag{8}$$

It is interesting to note that to the chosen order of approximation this equation is not dependent on the parametric viscosity variation μ_1.

Similarly equation 5 becomes

$$\frac{\partial\rho_1}{\partial t} + \boldsymbol{U}\cdot\boldsymbol{\nabla}\rho_1 + \boldsymbol{u}\cdot\boldsymbol{\nabla}\rho_0 = 0 \tag{9}$$

and equation 3 becomes

$$\nabla \cdot \boldsymbol{u} = 0 \tag{10}$$

Furthermore, for a body travelling horizontally with a steady translational velocity $\boldsymbol{U}(t) = \boldsymbol{U} = (U, 0, 0)$ and assuming $\rho_0(z,t) = \rho_0(z)$, equation 8 in component form becomes

$$\rho_0 \left(\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} \right) = -\frac{\partial p_1}{\partial x} + \mu_0 \nabla^2 u + \frac{\partial \mu_0}{\partial z} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \tag{11}$$

$$\rho_0 \left(\frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} \right) = -\frac{\partial p_1}{\partial y} + \mu_0 \nabla^2 v + \frac{\partial \mu_0}{\partial z} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \tag{12}$$

$$\rho_0 \left(\frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} \right) = -\frac{\partial p_1}{\partial z} + \mu_0 \nabla^2 w - \rho_1 g + 2 \frac{\partial \mu_0}{\partial z} \frac{\partial w}{\partial z} \tag{13}$$

Equation 9 becomes

$$\frac{\partial \rho_1}{\partial t} + U \frac{\partial \rho_1}{\partial x} + w \frac{\partial \rho_0}{\partial z} = 0, \tag{14}$$

and equation 10,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{15}$$

By forming the combination $\frac{\partial}{\partial x}$ [11] + $\frac{\partial}{\partial y}$ [12], we obtain

$$\nabla_h^2 p_1 = -\rho_0 \left[\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 v}{\partial y \partial t} + U \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} \right) \right]$$

$$+ \mu_0 \nabla^2 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{\partial \mu_0}{\partial z} \left(\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 w}{\partial y^2} \right) \tag{16}$$

and applying equation 15, we find that

$$\nabla_h^2 p_1 = -\rho_0 \left(-\frac{\partial^2 w}{\partial z \partial t} - U \frac{\partial^2 w}{\partial x \partial z} \right) - \mu_0 \nabla^2 \left(\frac{\partial w}{\partial z} \right) + \frac{\partial \mu_0}{\partial z} \left(\nabla_h^2 w - \frac{\partial^2 w}{\partial z^2} \right)$$

$$= \rho_0 \frac{D}{Dt} \left(\frac{\partial w}{\partial z} \right) - \mu_0 \nabla^2 \left(\frac{\partial w}{\partial z} \right) + \frac{\partial \mu_0}{\partial z} \left(\nabla_h^2 w - \frac{\partial^2 w}{\partial z^2} \right), \tag{17}$$

whereas the combination $\frac{\partial}{\partial y}$ [11] - $\frac{\partial}{\partial x}$ [12] gives

$$\left[\rho_0 \frac{D}{Dt} - \mu_0 \nabla^2 - \frac{\partial \mu_0}{\partial z} \frac{\partial}{\partial z} \right] \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) = 0 \tag{18}$$

That is, there exists a function $\Phi(x, y, z)$ such that

$$u = \tfrac{\partial \Phi}{\partial x} \quad v = \tfrac{\partial \Phi}{\partial y}$$

is a solution of equation 18 and from equation 15 we obtain the result

$$\nabla_h^2 \Phi = -\frac{\partial w}{\partial z} \tag{19}$$

The substitution of these results into equation 17 gives

$$\nabla_h^2 \left[p_1 + \rho_0 \frac{D\Phi}{Dt} - \mu_0 \nabla^2 \Phi - \frac{\partial \mu_0}{\partial z} \left(w + \frac{\partial \Phi}{\partial z} \right) \right] = 0 \tag{20}$$

whereas the application of the joint operators $\frac{D}{Dt}$ and ∇_h^2 to equation 13 in conjunction with equations 14 and 17 gives an equation describing the vertical velocity disturbance, w, in the form

$$\rho_0 \frac{D^2}{Dt^2}\left(\nabla^2 w\right) + \frac{\partial \rho_0}{\partial z}\frac{D^2}{Dt^2}\left(\frac{\partial w}{\partial z}\right) - g\frac{\partial \rho_0}{\partial z}\nabla_h^2 w - \mu_0 \frac{D}{Dt}\left(\nabla^4 w\right)$$

$$-2\frac{\partial \mu_0}{\partial z}\frac{D}{Dt}\left(\nabla^2 \frac{\partial w}{\partial z}\right) + \frac{\partial^2 \mu_0}{\partial z^2}\frac{D}{Dt}\left(\nabla^2 w\right) - 2\frac{\partial \mu_0}{\partial z}\frac{D}{Dt}\frac{\partial^2 w}{\partial z^2} = 0 \quad (21)$$

The evaluation of w from this equation allows the remaining variables ρ_1, Φ and p_1 to be determined from equations 14, 19 and 20 respectively.

The inclusion of density stratification into the mathematical model destroys the concept of irrotational fluid motion and this influence is considered to be of far greater importance than viscous effects in this linearised theory. For this reason, we shall examine a simplified mathematical model adopting linear equations in the absence of viscosity, $\mu_0 = 0$, but with density stratification $\rho_0(z)$ and rotational fluid motion. In this case, the equations describing the fluid disturbance can be expressed as

$$\left.\begin{array}{ll} [\,11\,] \text{ and } [\,12\,] & \rho_0 \frac{D\Phi}{Dt} + p_1 = 0 \\ [\,13\,] & \rho_0 \frac{Dw}{Dt} + \frac{\partial p_1}{\partial z} + g\rho_1 = 0 \\ [\,15\,] & \frac{D\rho_1}{Dt} + w\frac{\partial \rho_1}{\partial z} = 0 \\ [\,19\,] & \nabla_h^2 \Phi + \frac{\partial w}{\partial z} = 0 \end{array}\right\} \quad (22)$$

Equation 22 can be written in non-dimensional form using the non-dimensionalising variables L, U, ρ_0 and g. That is

$$\begin{array}{lll} \Phi = UL\Phi' & p_1 = \rho_0 U^2 p_1' & w = Uw' \\ (x,y,z) = L(x',y',z') & t = \frac{L}{U}t' & \frac{D}{Dt} = \frac{U}{L}\frac{D}{Dt'} \\ \nabla_h^2 = \left(\frac{1}{L}\right)^2 \nabla_h'^2 & \rho_1 = \rho_0 \rho_1' & N^2(z) = \frac{g}{L}N'^2(z) \end{array}$$

and

$$\left.\begin{array}{c} \frac{D\Phi'}{Dt'} + p_1' = 0 \\ \frac{Dw'}{Dt'} + \frac{\partial p_1'}{\partial z'} - N'^2(z)p_1' + \frac{\rho_1'}{F_n^2} = 0 \\ \frac{D\rho_1'}{Dt'} - w'N'^2(z) = 0 \\ \nabla_h'^2 \Phi + \frac{\partial w'}{\partial z'} = 0 \end{array}\right\} \quad (23)$$

By eliminating p_1' and ρ_1' and dropping the superscripts in the subsequent analysis but retaining the understanding that the equations and variables are expressed in non-dimensional form, the following coupled equations are derived

$$\left.\begin{array}{c} \nabla_h^2 \Phi + \frac{\partial w}{\partial z} = 0 \\ \frac{D^2}{Dt^2}\left(w - \frac{\partial \Phi}{\partial z} + N^2(z)\Phi\right) + \left(\frac{N(z)}{F_n}\right)^2 w = 0 \end{array}\right\} \quad (24)$$

3 Low speed approximation theory

If we let N_m^2 to denote the maximum value of the non dimensional parameter $N^2(z)$ and restrict the analysis to the range $N_m^2 < 1$, $F_n < 1$ such that $\frac{N_m^2}{F_n^2} \sim O(1)$ then a low speed approximation can be introduced into the analysis by assuming that $\frac{N_m^2}{F_n^2} \gg N_m^2$. That is, the variables w and Φ can be written as

$$w = w_0 + N_m^2 w_1 + (N_m^2)^2 w_2 + ...$$
$$\Phi = \Phi_0 + N_m^2 \Phi_1 + (N_m^2)^2 \Phi_2 + ...$$

Substituting w and Φ into equation 24 a zero order theory is described by the equations

$$\left.\begin{aligned} \nabla_h^2 \Phi_0 + \frac{\partial w_0}{\partial z} = 0 \\ \frac{D^2}{Dt^2}\left(w_0 - \frac{\partial \Phi_0}{\partial z}\right) + \tilde{N}^2(z) w_0 = 0 \end{aligned}\right\} \tag{25}$$

whereas a first order theory is based on the equations

$$\left.\begin{aligned} \nabla_h^2 \Phi_1 + \frac{\partial w_1}{\partial z} = 0 \\ \frac{D^2}{Dt^2}\left(w_1 - \frac{\partial \Phi_1}{\partial z}\right) + \tilde{N}^2(z) w_1 = -N^2(z) \frac{D^2 \Phi_0}{Dt^2} \end{aligned}\right\} \tag{26}$$

with additional equations describing higher order contributions.

For the steady state case, $\frac{D}{Dt} = \boldsymbol{U} \cdot \nabla$, we find that the zeroth order approximation becomes

$$\left.\begin{aligned} \nabla_h^2 \Phi_0 + \frac{\partial w_0}{\partial z} = 0 \\ (\boldsymbol{U} \cdot \nabla)^2 \left(w_0 - \frac{\partial \Phi_0}{\partial z}\right) + \tilde{N}^2(z) w_0 = 0 \end{aligned}\right\} \tag{27}$$

and

$$\left.\begin{aligned} p_1 = -(\boldsymbol{U} \cdot \nabla)^2 \Phi_0 \\ \rho_1 = -F_n^2 (\boldsymbol{U} \cdot \nabla)\left(w_0 - \frac{\partial \Phi_0}{\partial z}\right) \end{aligned}\right\} \tag{28}$$

where $\tilde{N}^2(z) = \left(\frac{N(z)}{F_n}\right)^2$

4 High speed approximation theory

Alternatively, for $F_n \gg 1$, we can introduce a high speed approximation by assuming that $\frac{N_m^2}{F_n^2} \ll N_m^2$ and, in this case, the variables w and Φ can be written as

$$w = w_0 + \frac{1}{F_n^2} w_1 + \left(\frac{1}{F_n^2}\right)^2 w_2 + ...$$
$$\Phi = \Phi_0 + \frac{1}{F_n^2} \Phi_1 + \left(\frac{1}{F_n^2}\right)^2 \Phi_2 + ...$$

Substituting w and Φ into equation 24 a zero order theory is now described by the equations

$$\left.\begin{aligned} \nabla_h^2 \Phi_0 + \frac{\partial w_0}{\partial z} = 0 \\ \frac{D^2}{Dt^2}\left(w_0 - \frac{\partial \Phi_0}{\partial z} + N^2(z) \Phi_0\right) = 0 \end{aligned}\right\} \tag{29}$$

and the first order theory is based on the equations

$$\left.\begin{aligned} \nabla_h^2 \Phi_1 + \frac{\partial w_1}{\partial z} = 0 \\ \frac{D^2}{Dt^2}\left(w_1 - \frac{\partial \Phi_1}{\partial z} + N^2(z) \Phi\right) = -\tilde{N}^2(z) w_0 \end{aligned}\right\} \tag{30}$$

and additional equations can be derived to describe the higher order contributions.

For the steady state case, $\frac{D}{Dt} = \boldsymbol{U} \cdot \nabla$, we find that the zeroth approximation becomes

$$\left.\begin{array}{c}\nabla_h^2\Phi_0 + \frac{\partial w_0}{\partial z} = 0 \\ (\boldsymbol{U}\cdot\nabla)^2\left(w_0 - \frac{\partial \Phi_0}{\partial z} + N^2(z)\Phi_0\right) = 0\end{array}\right\} \tag{31}$$

and

$$\left.\begin{array}{c}p_1 = -(\boldsymbol{U}\cdot\nabla)^2\Phi_0 \\ (\boldsymbol{U}\cdot\nabla)\rho_1 = N^2(z)w_0\end{array}\right\} \tag{32}$$

A simple manipulation of this last equation, gives

$$\nabla^2 w_0 - N(z)\frac{\partial w_0}{\partial z} = 0. \tag{33}$$

If a solution is sought in the form

$$w_0(x, y, z) = W_0(z)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

where $\boldsymbol{k}\cdot\boldsymbol{x} = k_x x + k_y y$ and $k^2 = \sqrt{k_x^2 + k_y^2}$ we find that

$$\frac{d^2W_0}{dz^2} - N^2(z)\frac{dW_0}{dz} - k^2 W_0 = 0 \tag{34}$$

which has a Bessel type structure.

5 Integral equation

For the conditions imposed, the low speed approximation theory is applicable to studies describing the operation of submarines (i.e. $F_n \ll 1$) in a density stratified fluid whereas the high speed approximation theory is more appropriate to examine the behaviour of a torpedo (say) travelling in a similar medium. For this reason, in the main, we shall focus attention on the low speed approximation theory adopted and seek solutions to the coupled equations describing the zeroth order approximant. Namely, solutions to the equations

$$\nabla_h^2\Phi + \frac{\partial w}{\partial z} = 0 \tag{35}$$

$$(\boldsymbol{U}\cdot\nabla)^2\left(w - \frac{\partial\Phi}{\partial z}\right) + \tilde{N}^2(z)w = 0 \tag{36}$$

where, for convenience, the subscript " $_o$" present in equation 27 is omitted.

By using Gaussian integral formulae, we can transform equations 35 and 36 into an integral equation. To do so, we introduce two unknown auxillary functions $f(\boldsymbol{r})$ and $h(\boldsymbol{r})$, where $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{\xi}$ represents the position of the field point $\boldsymbol{x}$ relative to the source point $\boldsymbol{\xi}$. Thus, equation 36 when multiplied by f and integrated over the fluid domain Ω gives,

$$\int_\Omega f\left[(\boldsymbol{U}\cdot\nabla)^2\left(w - \frac{\partial\Phi}{\partial z}\right) + \tilde{N}^2(z)w\right]d\Omega = \int_\Omega\left\{w\left[(\boldsymbol{U}\cdot\nabla)^2 f + \tilde{N}^2(z)f\right] + \Phi\left[(\boldsymbol{U}\cdot\nabla)^2\frac{\partial f}{\partial z}\right]\right\}d\Omega$$

$$+\int_\Sigma\left\{(\boldsymbol{U}\cdot\boldsymbol{n})\,f\,(\boldsymbol{U}\cdot\nabla)\left(w - \frac{\partial\Phi}{\partial z}\right) - (\boldsymbol{U}\cdot\nabla)f\left[(\boldsymbol{U}\cdot\boldsymbol{n})w - n_3(\boldsymbol{U}\cdot\nabla)\Phi\right] - (\boldsymbol{U}\cdot\boldsymbol{n})\Phi(\boldsymbol{U}\cdot\nabla)\frac{\partial f}{\partial z}\right\}d\Sigma \tag{37}$$

whereas equation 35 when multiplied by h and integrated over Ω gives

$$\int_\Omega h\left[\nabla_h^2\Phi + \frac{\partial w}{\partial z}\right]d\Omega = \int_\Omega\left\{\Phi\nabla_h^2 h - w\frac{\partial h}{\partial z}\right\}d\Omega$$

$$+\int_{\Sigma}\{h\boldsymbol{n}_h\cdot\nabla_h\Phi-\Phi\boldsymbol{n}_h\cdot\nabla_h h+hwn_3\}\,d\Sigma \tag{38}$$

Here, Σ represents the boundary surface enclosing Ω, $\boldsymbol{n}=(n_1,n_2,n_3)$ is a unit normal pointing outward from the fluid domain, $\boldsymbol{n}_h=(n_1,n_2,0)$ and $\nabla_h=(\frac{\partial}{\partial x},\frac{\partial}{\partial y},0)$.

Since the left hand terms of equations 37 and 38 are zero according to equations 35 and 36, the combination of equations 37 and 38 gives the integral relation

$$\int_{\Omega}\left\{w\left[(\boldsymbol{U}\cdot\nabla)^2 f+\tilde{N}^2(z)f\frac{\partial h}{\partial z}\right]+\Phi\left[(\boldsymbol{U}\cdot\nabla)^2\frac{\partial f}{\partial z}-\nabla_h^2 h\right]\right\}d\Omega$$

$$=\int_{\Sigma}\left\{h\frac{\partial\Phi}{\partial n_h}-\Phi\left[\frac{\partial h}{\partial n_h}-(\boldsymbol{U}\cdot\boldsymbol{n})(\boldsymbol{U}\cdot\nabla)\frac{\partial f}{\partial z}\right]+hwn_3\right\}d\Sigma$$

$$+\int_{\Sigma}\left\{(\boldsymbol{U}\cdot\nabla)f\left[(\boldsymbol{U}\cdot\boldsymbol{n})w-n_3\boldsymbol{U}\cdot\nabla\Phi\right]-(\boldsymbol{U}\cdot\boldsymbol{n})f\boldsymbol{U}\cdot\nabla\left(w-\frac{\partial\Phi}{\partial z}\right)\right\}d\Sigma \tag{39}$$

So far, f and h are undefined arbitrary functions and we are at liberty to select their forms. In fact, examination of equation 39 shows that there would be significant simplification to this identity equation if we let these functions satisfy the following equations

$$(\boldsymbol{U}\cdot\nabla)^2 f_1+\tilde{N}^2(z)f_1+\frac{\partial h_1}{\partial z}=\alpha \tag{40}$$

$$(\boldsymbol{U}\cdot\nabla)^2\frac{\partial f_1}{\partial z}-\nabla_h^2 h_1=0 \tag{41}$$

and

$$(\boldsymbol{U}\cdot\nabla)^2 f_2+\tilde{N}^2(z)f_2+\frac{\partial h_2}{\partial z}=0 \tag{42}$$

$$(\boldsymbol{U}\cdot\nabla)^2\frac{\partial f_2}{\partial z}-\nabla_h^2 h_2=\beta \tag{43}$$

where α and β denote point singularity excitations of the form, $\alpha=\delta(\boldsymbol{x}-\boldsymbol{\xi})$ for example.

In effect, f_1, h_1, f_2 and h_2 are the fundamental solutions which cause the left hand side of the integral relation in equation 39 to simplify. For arbitrary vertical density stratification, $\tilde{N}^2(z)$ we could in principle solve for f_1, h_1, f_2 and h_2 numerically but to gain a fuller understanding of these solutions let us consider the simplified case when $\tilde{N}^2(z)$ is a constant, $\tilde{N}^2(z)=\tilde{N}$.

6 Fundamental solutions - $\tilde{N}^2$ constant

The selection of $h_1=\frac{\partial\bar{h}_1}{\partial z}$ allows the elimination of h_1 from equation 40 using ∇_h^2 [40] and $\frac{\partial}{\partial z}$ [41]. A partial differential equation involving f_1 only is deduced of the form

$$\nabla^2(\boldsymbol{U}\cdot\nabla)^2 f_1+\tilde{N}^2\nabla_h^2 f_1=\nabla_h^2\alpha$$

The elimination of f_1 using $(\boldsymbol{U}\cdot\nabla)^2\frac{\partial}{\partial z}$ [40] produces the following equation in $\bar{h}_1$ only,

$$\nabla^2(\boldsymbol{U}\cdot\nabla)^2\bar{h}_1+\tilde{N}^2\nabla_h^2\bar{h}_1=(\boldsymbol{U}\cdot\nabla)^2\alpha$$

and in a similar manner, equations for $\bar{f}_2$ and h_2 can be determined.

This process allows equations 40 - 43 to be expressed as

$$\mathcal{L}(f_1) = \nabla_h^2 \alpha \tag{44}$$

$$\mathcal{L}(\bar{h}_1) = (\boldsymbol{U} \cdot \boldsymbol{\nabla})^2 \alpha \tag{45}$$

$$\mathcal{L}(\bar{f}_2) = \beta \tag{46}$$

$$\mathcal{L}(h_2) = -\left[(\boldsymbol{U} \cdot \boldsymbol{\nabla})^2 + \tilde{N}^2\right] \beta \tag{47}$$

where the linear operator $\mathcal{L}(\,)$ is given by

$$\mathcal{L}(\,) = \left[(\boldsymbol{U} \cdot \boldsymbol{\nabla})^2 \boldsymbol{\nabla}^2 + \tilde{N}^2 \boldsymbol{\nabla}_h^2\right] (\,) \tag{48}$$

We are at liberty to take the doublet $\alpha = \beta = (\boldsymbol{U} \cdot \boldsymbol{\nabla})\gamma$, such that equations 44 - 47 can be written in the matrix form

$$\mathcal{L} \begin{bmatrix} f_1 & \bar{h}_1 \\ \bar{f}_2 & h_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\nabla}_h^2 (\boldsymbol{U} \cdot \boldsymbol{\nabla}) & (\boldsymbol{U} \cdot \boldsymbol{\nabla})^3 \\ \boldsymbol{U} \cdot \boldsymbol{\nabla} & -\left[(\boldsymbol{U} \cdot \boldsymbol{\nabla})^2 + \tilde{N}^2\right] (\boldsymbol{U} \cdot \boldsymbol{\nabla}) \end{bmatrix} \gamma \tag{49}$$

By the application of Fourier transforms, we can rewrite equation 49 into the form

$$L \begin{bmatrix} F_1 & \bar{H}_1 \\ \bar{F}_2 & H_2 \end{bmatrix} = \begin{bmatrix} -\frac{\lambda^2 i}{\boldsymbol{U} \cdot \boldsymbol{\lambda}} & -(\boldsymbol{U} \cdot \boldsymbol{\lambda}) i \\ \frac{i}{\boldsymbol{U} \cdot \boldsymbol{\lambda}} & -i \frac{\tilde{N}^2 - (\boldsymbol{U} \cdot \boldsymbol{\lambda})^2}{\boldsymbol{U} \cdot \boldsymbol{\lambda}} \end{bmatrix} \Gamma \tag{50}$$

where

$$L(\,) = \left\{ \frac{\partial^2}{\partial z^2} + \left[\left(\frac{\tilde{N}}{\boldsymbol{U} \cdot \boldsymbol{\lambda}} \right)^2 - 1 \right] \lambda^2 \right\} (\,) \tag{51}$$

and F_1, $\bar{H}_1$, $\bar{F}_2$, H_2 and Γ are the Fourier transforms of f_1, $\bar{h}_1$, $\bar{f}_2$, h_2 and γ respectively.

Furthermore, without any loss of generality, we can further assume $\boldsymbol{U}$ has a non-zero component in the x direction only such that $\boldsymbol{U} \cdot \boldsymbol{\lambda} = \lambda_1$. Thus equations 50 and 51 become

$$L \begin{bmatrix} F_1 & \bar{H}_1 \\ \bar{F}_2 & H_2 \end{bmatrix} = \begin{bmatrix} -\frac{\lambda^2 i}{\lambda_1} & -\lambda_1 i \\ \frac{i}{\lambda_1} & -i \frac{\tilde{N}^2 - \lambda_1^2}{\lambda_1} \end{bmatrix} \Gamma \tag{52}$$

where

$$L(\,) = \left\{ \frac{\partial^2}{\partial z^2} + \left[\left(\frac{\tilde{N}}{\lambda_1} \right)^2 - 1 \right] \lambda^2 \right\} (\,) \tag{53}$$

In particular, in our case, the singularity function γ can be chosen to have the form

$$\gamma = \delta(z - \zeta)\delta(r_h) \tag{54}$$

where $(z - \zeta)$ denotes the vertical separation between the singularity and the field point and r_h denotes the relative horizontal distance between the singularity and the field point. The Fourier transform of this singularity function is given by

$$\Gamma = \delta(z - \zeta) \frac{e^{i \boldsymbol{\xi} \cdot \boldsymbol{\lambda}}}{2\pi}.$$

Let us now examine the equation

$$\left\{\frac{\partial^2}{\partial z^2}+\left[\left(\frac{N}{\lambda_1}\right)^2-1\right]\lambda^2\right\}G(\lambda_1,\lambda_2,z;\boldsymbol{\xi})=\delta(z-\zeta) \tag{55}$$

subject to the boundary conditions

$$\left.\begin{array}{lllll} G(\lambda_1,\lambda_2,0;\boldsymbol{\xi})=0 & \text{and} & G(\lambda_1,\lambda_2,-\infty;\boldsymbol{\xi}) & \text{bounded} & \text{for } F_1 \text{ and } H_2 \\ \frac{\partial G(\lambda_1,\lambda_2,0;\boldsymbol{\xi})}{\partial z}=0 & \text{and} & G(\lambda_1,\lambda_2,-\infty;\boldsymbol{\xi}) & \text{bounded} & \text{for } \bar{F}_2 \text{ and } \bar{H}_1 \end{array}\right\} \tag{56}$$

This equation has the following solutions when considering F_1 and H_2,

$$G(\lambda_1,\lambda_2,z;\boldsymbol{\xi})=\begin{cases} \frac{e^{\gamma_1\lambda(\zeta+z)}-e^{\gamma_1\lambda(\zeta-z)}}{2\gamma_1\lambda} & 0>z>\zeta \\ \frac{e^{\gamma_1\lambda(z+\zeta)}-e^{\gamma_1\lambda(z-\zeta)}}{2\gamma_1\lambda} & 0>\zeta>z \end{cases}\Big\}\ \lambda_1^2>\tilde{N}^2 \\ \quad \begin{cases} z & 0>z>\zeta \\ \zeta & 0>\zeta>z \end{cases}\Big\}\ \lambda_1^2=\tilde{N}^2 \\ \quad \begin{cases} \frac{\sin\gamma_2\lambda(\zeta+z)-\sin\gamma_2\lambda(\zeta-z)}{2\gamma_2\lambda} & 0>z>\zeta \\ \frac{\sin\gamma_2\lambda(z+\zeta)-\sin\gamma_2\lambda(z-\zeta)}{2\gamma_2\lambda} & 0>\zeta>z \end{cases}\Big\}\ \lambda_1^2<\tilde{N}^2 \tag{57}$$

and for $\bar{F}_2$ and $\bar{H}_1$ we have

$$G(\lambda_1,\lambda_2,z;\boldsymbol{\xi})=\begin{cases} \frac{e^{\gamma_1\lambda(\zeta+z)}+e^{\gamma_1\lambda(\zeta-z)}}{-2\gamma_1\lambda} & 0>z>\zeta \\ \frac{e^{\gamma_1\lambda(z+\zeta)}+e^{\gamma_1\lambda(z-\zeta)}}{-2\gamma_1\lambda} & 0>\zeta>z \end{cases}\Big\}\ \lambda_1^2>\tilde{N}^2 \\ \quad \begin{cases} -\zeta & 0>z>\zeta \\ -z & 0>\zeta>z \end{cases}\Big\}\ \lambda_1^2=\tilde{N}^2 \\ \quad \begin{cases} \frac{\sin\gamma_2\lambda(\zeta+z)+\sin\gamma_2\lambda(\zeta-z)}{-2\gamma_2\lambda} & 0>z>\zeta \\ \frac{\sin\gamma_2\lambda(z+\zeta)+\sin\gamma_2\lambda(z-\zeta)}{-2\gamma_2\lambda} & 0>\zeta>z \end{cases}\Big\}\ \lambda_1^2<\tilde{N}^2 \tag{58}$$

where

$$\gamma_1=\sqrt{1-\left(\frac{\tilde{N}}{\lambda_1}\right)^2}$$

$$\gamma_2=\sqrt{\left(\frac{\tilde{N}}{\lambda_1}\right)^2-1}$$

Note that the function $G(\lambda_1,\lambda_2,z;\boldsymbol{\xi})$ can be obtained for the case $z<\zeta$ by exchanging z and ζ, this also confirms continuity in the vertical dimension across the singularity point.

The function $G(\lambda_1,\lambda_2,z;\boldsymbol{\xi})$ can now be used to determine the functions f_1, $\bar{h}_1$, $\bar{f}_2$ and h_2 by the application of inverse Fourier transforms. That is

$$\begin{pmatrix} f_1 & \bar{h}_1 \\ \bar{f}_2 & h_2 \end{pmatrix}=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{pmatrix} -i\frac{\lambda^2}{\lambda_1} & -i\lambda_1 \\ i\frac{1}{\lambda_1} & -i\frac{\tilde{N}^2-\lambda_1^2}{\lambda_1} \end{pmatrix}G(\lambda_1,\lambda_2,z;\boldsymbol{\xi})e^{-i\boldsymbol{\xi}\cdot\boldsymbol{\lambda}}d\lambda_1 d\lambda_2 \tag{59}$$

Detailed derivations of functions f_1, h_1, f_2 and h_2 are given in appendix A.

7 Summary of antisymmetric fundamental solutions

No radiation condition is included in the derivation of the functions, therefore the solutions display fore and aft antisymmetry. The structure of each function is organised into singularity and image expressions.

$$f_1 = \frac{1}{2\pi^2}\int_{\tilde{N}}^{\infty} \frac{\lambda_1^2}{\gamma^- \sqrt{\lambda_1^2 - \tilde{N}^2}} \left[(\lambda_1^2 - \tilde{N}^2)(\zeta - z)^2 K_0(\sqrt{\gamma^-}) - \frac{\sigma^-}{\sqrt{\gamma^-}} K_1(\sqrt{\gamma^-})\right] \sin r_1\lambda_1 d\lambda_1$$

$$+\frac{1}{4\pi}\int_0^{\beta^-} \frac{\lambda_1^2}{\gamma^- \sqrt{\tilde{N}^2 - \lambda_1^2}} \left[(\tilde{N}^2 - \lambda_1^2)(\zeta - z)^2 J_0(\sqrt{-\gamma^-}) - \frac{\sigma^-}{\sqrt{-\gamma^-}} J_1(\sqrt{-\gamma^-})\right] \sin r_1\lambda_1 d\lambda_1$$

$$-\frac{1}{2\pi^2}\int_{\tilde{N}}^{\infty} \frac{\lambda_1^2}{\gamma^+ \sqrt{\lambda_1^2 - \tilde{N}^2}} \left[(\lambda_1^2 - \tilde{N}^2)(\zeta + z)^2 K_0(\sqrt{\gamma^+}) - \frac{\sigma^+}{\sqrt{\gamma^+}} K_1(\sqrt{\gamma^+})\right] \sin r_1\lambda_1 d\lambda_1$$

$$-\frac{1}{4\pi}\int_0^{\beta^+} \frac{\lambda_1^2}{\gamma^+ \sqrt{\tilde{N}^2 - \lambda_1^2}} \left[(\tilde{N}^2 - \lambda_1^2)(\zeta + z)^2 J_0(\sqrt{-\gamma^+}) - \frac{\sigma^+}{\sqrt{-\gamma^+}} J_1(\sqrt{-\gamma^+})\right] \sin r_1\lambda_1 d\lambda_1 \quad (60)$$

where

$$\sigma^{\pm} = \lambda_1^2[(y - \eta)^2 - (\zeta \pm z)^2] + \tilde{N}^2(\zeta \pm z)^2$$

$$h_1 = \frac{1}{2\pi^2}\int_{\tilde{N}}^{\infty} \frac{\lambda_1^2(\zeta - z)\sqrt{\lambda_1^2 - \tilde{N}^2}}{\sqrt{\gamma^-}} K_1(\sqrt{\gamma^-}) \sin r_1\lambda_1 d\lambda_1$$

$$-\frac{1}{4\pi}\int_0^{\beta^-} \frac{\lambda_1^2(\zeta - z)\sqrt{\tilde{N}^2 - \lambda_1^2}}{\sqrt{-\gamma^-}} J_1(\sqrt{-\gamma^-}) \sin r_1\lambda_1 d\lambda_1$$

$$-\frac{\tilde{N}(\zeta - z)^2}{4\pi[r_2^2 + (\zeta - z)^2]^{\frac{3}{2}}} \sin r_1\beta^-$$

$$-\frac{1}{2\pi^2}\int_{\tilde{N}}^{\infty} \frac{\lambda_1^2(\zeta + z)\sqrt{\lambda_1^2 - \tilde{N}^2}}{\sqrt{\gamma^+}} K_1(\sqrt{\gamma^+}) \sin r_1\lambda_1 d\lambda_1$$

$$+\frac{1}{4\pi}\int_0^{\beta^+} \frac{\lambda_1^2(\zeta + z)\sqrt{\tilde{N}^2 - \lambda_1^2}}{\sqrt{-\gamma^+}} J_1(\sqrt{-\gamma^+}) \sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\tilde{N}(\zeta + z)^2}{4\pi[r_2^2 + (\zeta + z)^2]^{\frac{3}{2}}} \sin r_1\beta^+ \quad (61)$$

$$f_2 = -\frac{1}{2\pi^2}\int_{\tilde{N}}^{\infty}\frac{(\zeta-z)\sqrt{\lambda_1^2-\tilde{N}^2}}{\sqrt{\gamma^-}}K_1(\sqrt{\gamma^-})\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{1}{4\pi}\int_0^{\beta^-}\frac{(\zeta-z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^-}}J_1(\sqrt{-\gamma^-})\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{|r_2|}{4\pi[r_2^2+(\zeta-z)^2]}\sin r_1\beta^-$$

$$+\frac{1}{2\pi^2}\int_{\tilde{N}}^{\infty}\frac{(\zeta+z)\sqrt{\lambda_1^2-\tilde{N}^2}}{\sqrt{\gamma^+}}K_1(\sqrt{\gamma^+})\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{1}{4\pi}\int_0^{\beta^+}\frac{(\zeta+z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^+}}J_1(\sqrt{-\gamma^+})\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{|r_2|}{4\pi[r_2^2+(\zeta+z)^2]^{\frac{3}{2}}}\sin r_1\beta^+ \tag{62}$$

$$h_2 = -\frac{1}{2\pi^2}\int_{\tilde{N}}^{\infty}\sqrt{\lambda_1^2-\tilde{N}^2}K_0(\sqrt{\gamma^-})\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{1}{4\pi}\int_0^{\beta^-}\sqrt{\tilde{N}^2-\lambda_1^2}J_0(\sqrt{-\gamma^-})\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{1}{2\pi^2}\int_{\tilde{N}}^{\infty}\sqrt{\lambda_1^2-\tilde{N}^2}K_0(\sqrt{\gamma^+})\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{1}{4\pi}\int_0^{\beta^+}\sqrt{\tilde{N}^2-\lambda_1^2}J_0(\sqrt{-\gamma^+})\sin r_1\lambda_1 d\lambda_1 \tag{63}$$

8 Application of a radiation condition

The functions derived in the previous section obey the fundamental equation 49 subject to the singularity disturbance γ and the imposed boundary conditions. However, due to the symmetry of the operator $\mathcal{L}$ and the right hand side of the fundamental equation, the solutions are antisymmetric about the singularity point. As the body possesses forward motion, it is required that the functions display differential behaviour upstream and downstream. Observations indicate that the disturbance upstream decays far more rapidly than the disturbance downstream. Theoretical problems arise from the solution of the steady state case for the solution does not display uniqueness in the x or y dimensions and contains free wave solutions which destroy the physical meaning of the mathematical model. Three methods exist which allow an unique solution to be sought :

(1) Energy considerations. An unique solution is obtained by requiring that the energy is always radiating away from the disturbance. That is the Sommerfeld radiation condition is applied. This eliminates the upstream waves travelling away from the body ;

(2) Time dependent solution. An unique solution can always be found if the problem is treated as time dependent by starting the fluid motion from rest. The required solution is found by taking the limit of the time term to infinity ;

(2) Artificial damping mechanism. Artificial damping is introduced in the form of a body force, $\boldsymbol{F} = \epsilon \boldsymbol{V}$ and a solution sought as $\epsilon \to 0$. A simple example of this technique appears in the Encyclopaedia of Mathematical Sciences (Volume 13 , III §2.5, page 208, example 9) and is also widely used by Lighthill(1979 §3.9 page 267 and §4.9 page 363). This approach is the most readily applicable to the current problem and is now described briefly.

The introduction of damping into the equation of momentum (in the absence of viscosity) modifies equations 11,12 and 13 to the form

$$\rho_0 \frac{Du}{Dt} + \epsilon u + \frac{\partial p_1}{\partial x} = 0 \tag{64}$$

$$\rho_0 \frac{Dv}{Dt} + \epsilon v + \frac{\partial p_1}{\partial y} = 0 \tag{65}$$

$$\rho_0 \frac{Dw}{Dt} + \epsilon w + \frac{\partial p_1}{\partial z} + g\rho_1 = 0 \tag{66}$$

where the terms involving ϵ denote the contribution from the *artificial damping*.

Equations 14 and 15 remain unchanged. That is

$$\frac{\partial \rho_1}{\partial t} + U\frac{\partial \rho_1}{\partial x} + w\frac{\partial \rho_0}{\partial z} = 0 \tag{67}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{68}$$

Equation 22 becomes

$$\left.\begin{array}{c} \rho_0 \frac{D\Phi}{Dt} + \epsilon\Phi + p_1 = 0 \\ \rho_0 \frac{Dw}{Dt} + \epsilon w + \frac{\partial p_1}{\partial z} + g\rho_1 = 0 \\ \frac{D\rho_1}{Dt} + w\frac{\partial \rho_1}{\partial z} = 0 \\ \nabla_h^2 \Phi + \frac{\partial w}{\partial z} = 0 \end{array}\right\} \tag{69}$$

and after the non dimensionalising process we have

$$\left.\begin{aligned} \tfrac{D\Phi}{Dt} + \epsilon\Phi + p_1 = 0 \\ \tfrac{Dw}{Dt} + \epsilon w + \tfrac{\partial p_1}{\partial z} - N^2 p_1 + \tfrac{\rho_1}{F_n^2} = 0 \\ \tfrac{D\rho_1}{Dt} - wN^2 = 0 \\ \nabla_h^2\Phi + \tfrac{\partial w}{\partial z} = 0 \end{aligned}\right\} \tag{70}$$

By eliminating p_1 and ρ_1 and applying the steady state low speed approximation, the symmetric nature of the problem posed by equations 35 and 36 is destroyed and these equations now have the form

$$\nabla_h^2\Phi + \frac{\partial w}{\partial z} = 0 \tag{71}$$

$$\boldsymbol{U}\cdot\nabla(\boldsymbol{U}\cdot\nabla + \epsilon)\left(w - \frac{\partial \Phi}{\partial z}\right) + \tilde{N}^2 w = 0 \tag{72}$$

After the application of the integral formulation, equations 40 - 43 become

$$\left.\begin{aligned} \boldsymbol{U}\cdot\nabla(\boldsymbol{U}\cdot\nabla - \epsilon)f_1 + \tilde{N}^2 f_1 + \tfrac{\partial h_1}{\partial z} = \alpha \\ \boldsymbol{U}\cdot\nabla(\boldsymbol{U}\cdot\nabla - \epsilon)\tfrac{\partial f_1}{\partial z} - \nabla_h^2 h_1 = 0 \end{aligned}\right\} \tag{73}$$

and

$$\left.\begin{aligned} \boldsymbol{U}\cdot\nabla(\boldsymbol{U}\cdot\nabla - \epsilon)f_2 + \tilde{N}^2 f_2 + \tfrac{\partial h_2}{\partial z} = 0 \\ \boldsymbol{U}\cdot\nabla(\boldsymbol{U}\cdot\nabla - \epsilon)\tfrac{\partial f_2}{\partial z} - \nabla_h^2 h_2 = \beta \end{aligned}\right\} \tag{74}$$

from which the matrix equation 49 is modified to

$$\mathcal{L}\begin{bmatrix} f_1 & \bar{h}_1 \\ \bar{f}_2 & h_2 \end{bmatrix} = \begin{bmatrix} \nabla_h^2(\boldsymbol{U}\cdot\nabla) & (\boldsymbol{U}\cdot\nabla)^2(\boldsymbol{U}\cdot\nabla - \epsilon) \\ \boldsymbol{U}\cdot\nabla & -\left[(\boldsymbol{U}\cdot\nabla)(\boldsymbol{U}\cdot\nabla - \epsilon) + \tilde{N}^2\right](\boldsymbol{U}\cdot\nabla) \end{bmatrix}\gamma \tag{75}$$

where the linear operator $\mathcal{L}(\,)$ is now given by

$$\mathcal{L}(\,) = \left[\boldsymbol{U}\cdot\nabla(\boldsymbol{U}\cdot\nabla - \epsilon)\nabla^2 + \tilde{N}^2\nabla_h^2\right](\,) \tag{76}$$

Applying Fourier transforms and setting $\boldsymbol{U}\cdot\boldsymbol{\lambda} = \lambda_1$ we obtain

$$L\begin{bmatrix} F_1 & \bar{H}_1 \\ \bar{F}_2 & H_2 \end{bmatrix} = \begin{bmatrix} -\frac{i\lambda^2}{\lambda_1 - i\epsilon} & -i\lambda_1 \\ \frac{i}{\lambda_1 - i\epsilon} & i\left(\lambda_1 - \frac{\tilde{N}^2}{\lambda_1 - i\epsilon}\right) \end{bmatrix}\Gamma \tag{77}$$

where

$$L(\,) = \left\{\frac{\partial^2}{\partial z^2} + \left[\frac{\tilde{N}^2}{\lambda_1(\lambda_1 - i\epsilon)} - 1\right]\lambda^2\right\}(\,) \tag{78}$$

Let us now examine the equation

$$\left\{\frac{\partial^2}{\partial z^2} + \left[\frac{\tilde{N}^2}{\lambda_1(\lambda_1 - i\epsilon)} - 1\right]\lambda^2\right\} G(\lambda_1, \lambda_2, z; \boldsymbol{\xi}) = \delta(z - \zeta) \tag{79}$$

subject to the boundary conditions stated in equation 56. This equation has the following solutions when the boundary conditions applicable to F_1 and H_2 are applied

$$G(\lambda_1,\lambda_2,z;\boldsymbol{\xi}) = \frac{1}{2\lambda\sqrt{1-\frac{\bar{N}^2}{\lambda_1(\lambda_1-i\epsilon)}}}\begin{cases} e^{\lambda(z+\zeta)\sqrt{1-\frac{\bar{N}^2}{\lambda_1(\lambda_1-i\epsilon)}}} - e^{\lambda(\zeta-z)\sqrt{1-\frac{\bar{N}^2}{\lambda_1(\lambda_1-i\epsilon)}}} & 0>z>\zeta \\ e^{\lambda(\zeta+z)\sqrt{1-\frac{\bar{N}^2}{\lambda_1(\lambda_1-i\epsilon)}}} - e^{\lambda(z-\zeta)\sqrt{1-\frac{\bar{N}^2}{\lambda_1(\lambda_1-i\epsilon)}}} & 0>\zeta>z \end{cases} \tag{80}$$

and for $\bar{F}_2$ and $\bar{H}_1$ we have

$$G(\lambda_1,\lambda_2,z;\boldsymbol{\xi}) = -\frac{1}{2\lambda\sqrt{1-\frac{\bar{N}^2}{\lambda_1(\lambda_1-i\epsilon)}}}\begin{cases} e^{\lambda(z+\zeta)\sqrt{1-\frac{\bar{N}^2}{\lambda_1(\lambda_1-i\epsilon)}}} + e^{\lambda(\zeta-z)\sqrt{1-\frac{\bar{N}^2}{\lambda_1(\lambda_1-i\epsilon)}}} & 0>z>\zeta \\ e^{\lambda(\zeta+z)\sqrt{1-\frac{\bar{N}^2}{\lambda_1(\lambda_1-i\epsilon)}}} + e^{\lambda(z-\zeta)\sqrt{1-\frac{\bar{N}^2}{\lambda_1(\lambda_1-i\epsilon)}}} & 0>\zeta>z \end{cases} . \tag{81}$$

The function $G(\lambda_1,\lambda_2,z;\boldsymbol{\xi})$ can be obtained for the $z<\zeta$ case by exchanging z and ζ, exactly as described in the antisymmetric analysis.

$G(\lambda_1,\lambda_2,z;\boldsymbol{\xi})$ can now be used to determine the functions f_1, $\bar{h}_1$, $\bar{f}_2$ and h_2 by the application of inverse Fourier transforms. That is

$$\begin{pmatrix} f_1 & \bar{h}_1 \\ \bar{f}_2 & h_2 \end{pmatrix} = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{pmatrix} -i\frac{\lambda^2}{\lambda_1-i\epsilon} & -i\lambda_1 \\ -i\frac{1}{\lambda_1-i\epsilon} & i\left[\lambda_1-\frac{\bar{N}^2}{\lambda_1-i\epsilon}\right]\end{pmatrix} G(\lambda_1,\lambda_2,z;\boldsymbol{\xi})e^{-i\boldsymbol{\xi}\cdot\boldsymbol{\lambda}}d\lambda_1 d\lambda_2 \tag{82}$$

Detailed derivations of the functions f_1, h_1, f_2 and h_2 are described in appendix B.

9 Summary of asymmetric fundamental solutions

The functions below contain terms which do not appear in their counterparts derived in section 7. Their appearance is due to the radiation condition being satisfied. These additional terms destroy the antisymmetry of each function and are underlined.

$$f_1 = \frac{1}{2\pi^2}\int_{\tilde{N}}^{\infty} \frac{\lambda_1^2}{\gamma^-\sqrt{\lambda_1^2-\tilde{N}^2}}\left[(\lambda_1^2-\tilde{N}^2)(\zeta-z)^2K_0(\sqrt{\gamma^-}) - \frac{\sigma^-}{\sqrt{\gamma^-}}K_1(\sqrt{\gamma^-})\right]\sin r_1\lambda_1 d\lambda_1$$

$$\underline{-\frac{1}{2\pi^2}\int_{\beta^-}^{\tilde{N}} \frac{\lambda_1^2}{\gamma^-\sqrt{\tilde{N}^2-\lambda_1^2}}\left[(\lambda_1^2-\tilde{N}^2)(\zeta-z)^2K_0(\sqrt{\gamma^-}) - \frac{\sigma^-}{\sqrt{\gamma^-}}K_1(\sqrt{\gamma^-})\right]\cos r_1\lambda_1 d\lambda_1}$$

$$\underline{-\frac{1}{4\pi}\int_{0}^{\beta^-} \frac{\lambda_1^2}{\gamma^+\sqrt{\tilde{N}^2-\lambda_1^2}}\left[(\tilde{N}^2-\lambda_1^2)(\zeta-z)^2Y_0(\sqrt{-\gamma^-}) - \frac{\sigma^-}{\sqrt{-\gamma^-}}Y_1(\sqrt{-\gamma^-})\right]\cos r_1\lambda_1 d\lambda_1}$$

$$+\frac{1}{4\pi}\int_{0}^{\beta^-} \frac{\lambda_1^2}{\gamma^-\sqrt{\tilde{N}^2-\lambda_1^2}}\left[(\tilde{N}^2-\lambda_1^2)(\zeta-z)^2J_0(\sqrt{-\gamma^-}) - \frac{\sigma^-}{\sqrt{-\gamma^-}}J_1(\sqrt{-\gamma^-})\right]\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{1}{2\pi^2}\int_{\tilde{N}}^{\infty} \frac{\lambda_1^2}{\gamma^+\sqrt{\lambda_1^2-\tilde{N}^2}}\left[(\lambda_1^2-\tilde{N}^2)(\zeta+z)^2K_0(\sqrt{\gamma^+}) - \frac{\sigma^+}{\sqrt{\gamma^+}}K_1(\sqrt{\gamma^+})\right]\sin r_1\lambda_1 d\lambda_1$$

$$\underline{+\frac{1}{2\pi^2}\int_{\beta^+}^{\tilde{N}} \frac{\lambda_1^2}{\gamma^+\sqrt{\tilde{N}^2-\lambda_1^2}}\left[(\lambda_1^2-\tilde{N}^2)(\zeta+z)^2K_0(\sqrt{\gamma^+}) - \frac{\sigma^+}{\sqrt{\gamma^+}}K_1(\sqrt{\gamma^+})\right]\cos r_1\lambda_1 d\lambda_1}$$

$$\underline{+\frac{1}{4\pi}\int_{0}^{\beta^+} \frac{\lambda_1^2}{\gamma^+\sqrt{\tilde{N}^2-\lambda_1^2}}\left[(\tilde{N}^2-\lambda_1^2)(\zeta+z)^2Y_0(\sqrt{-\gamma^+}) - \frac{\sigma^+}{\sqrt{-\gamma^+}}Y_1(\sqrt{-\gamma^+})\right]\cos r_1\lambda_1 d\lambda_1}$$

$$-\frac{1}{4\pi}\int_{0}^{\beta^+} \frac{\lambda_1^2}{\gamma^+\sqrt{\tilde{N}^2-\lambda_1^2}}\left[(\tilde{N}^2-\lambda_1^2)(\zeta+z)^2J_0(\sqrt{-\gamma^+}) - \frac{\sigma^+}{\sqrt{-\gamma^+}}J_1(\sqrt{-\gamma^+})\right]\sin r_1\lambda_1 d\lambda_1 \quad (83)$$

$$h_1 = \frac{1}{2\pi^2}\int_{\tilde{N}}^{\infty} \frac{\lambda_1^2(\zeta - z)\sqrt{\lambda_1^2 - \tilde{N}^2}}{\sqrt{\gamma^-}} K_1\left(\sqrt{\gamma^-}\right) \sin r_1\lambda_1 d\lambda_1$$

$$+\frac{1}{2\pi^2}\int_{\beta^-}^{\tilde{N}} \frac{\lambda_1^2(\zeta - z)\sqrt{\tilde{N}^2 - \lambda_1^2}}{\sqrt{\gamma^-}} K_1\left(\sqrt{\gamma^-}\right) \cos r_1\lambda_1 d\lambda_1$$

$$+\frac{1}{4\pi}\int_{0}^{\beta^-} \frac{\lambda_1^2(\zeta - z)\sqrt{\tilde{N}^2 - \lambda_1^2}}{\sqrt{-\gamma^-}} Y_1\left(\sqrt{-\gamma^-}\right) \cos r_1\lambda_1 d\lambda_1$$

$$-\frac{1}{4\pi}\int_{0}^{\beta^-} \frac{\lambda_1^2(\zeta - z)\sqrt{\tilde{N}^2 - \lambda_1^2}}{\sqrt{-\gamma^-}} J_1\left(\sqrt{-\gamma^-}\right) \sin r_1\lambda_1 d\lambda_1$$

$$-\frac{\tilde{N}(\zeta - z)^2}{4\pi[r_2^2 + (\zeta - z)^2]^{\frac{3}{2}}} \sin r_1\beta^-$$

$$-\frac{1}{2\pi^2}\int_{\tilde{N}}^{\infty} \frac{\lambda_1^2(\zeta + z)\sqrt{\lambda_1^2 - \tilde{N}^2}}{\sqrt{\gamma^+}} K_1\left(\sqrt{\gamma^+}\right) \sin r_1\lambda_1 d\lambda_1$$

$$-\frac{1}{2\pi^2}\int_{\beta^+}^{\tilde{N}} \frac{\lambda_1^2(\zeta + z)\sqrt{\tilde{N}^2 - \lambda_1^2}}{\sqrt{\gamma^+}} K_1\left(\sqrt{\gamma^+}\right) \cos r_1\lambda_1 d\lambda_1$$

$$-\frac{1}{4\pi}\int_{0}^{\beta^+} \frac{\lambda_1^2(\zeta + z)\sqrt{\tilde{N}^2 - \lambda_1^2}}{\sqrt{-\gamma^+}} Y_1\left(\sqrt{-\gamma^+}\right) \cos r_1\lambda_1 d\lambda_1$$

$$+\frac{1}{4\pi}\int_{0}^{\beta^+} \frac{\lambda_1^2(\zeta + z)\sqrt{\tilde{N}^2 - \lambda_1^2}}{\sqrt{-\gamma^+}} J_1\left(\sqrt{-\gamma^+}\right) \sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\tilde{N}(\zeta + z)^2}{4\pi[r_2^2 + (\zeta + z)^2]^{\frac{3}{2}}} \sin r_1\beta^+ \tag{84}$$

$$
\begin{aligned}
f_2 = &-\frac{1}{2\pi^2}\int_{\tilde{N}}^{\infty}\frac{(\zeta - z)\sqrt{\lambda_1^2 - \tilde{N}^2}}{\sqrt{\gamma^-}}K_1(\sqrt{\gamma^-})\sin r_1\lambda_1 d\lambda_1 \\
&-\frac{1}{2\pi^2}\int_{\beta^-}^{\tilde{N}}\frac{(\zeta - z)\sqrt{\tilde{N}^2 - \lambda_1^2}}{\sqrt{\gamma^-}}K_1(\sqrt{\gamma^-})\cos r_1\lambda_1 d\lambda_1 \\
&-\frac{1}{4\pi}\int_{0}^{\beta^-}\frac{(\zeta - z)\sqrt{\tilde{N}^2 - \lambda_1^2}}{\sqrt{-\gamma^-}}Y_1(\sqrt{-\gamma^-})\cos r_1\lambda_1 d\lambda_1 \\
&+\frac{1}{4\pi}\int_{0}^{\beta^-}\frac{(\zeta - z)\sqrt{\tilde{N}^2 - \lambda_1^2}}{\sqrt{-\gamma^-}}J_1(\sqrt{-\gamma^-})\sin r_1\lambda_1 d\lambda_1 \\
&+\frac{|r_2|}{4\pi[r_2^2 + (\zeta - z)^2]}\sin r_1\beta^- \\
&+\frac{1}{2\pi^2}\int_{\tilde{N}}^{\infty}\frac{(\zeta + z)\sqrt{\lambda_1^2 - \tilde{N}^2}}{\sqrt{\gamma^+}}K_1(\sqrt{\gamma^+})\sin r_1\lambda_1 d\lambda_1 \\
&+\frac{1}{2\pi^2}\int_{\beta^+}^{\tilde{N}}\frac{(\zeta + z)\sqrt{\tilde{N}^2 - \lambda_1^2}}{\sqrt{\gamma^+}}K_1(\sqrt{\gamma^+})\cos r_1\lambda_1 d\lambda_1 \\
&+\frac{1}{4\pi}\int_{0}^{\beta^+}\frac{(\zeta + z)\sqrt{\tilde{N}^2 - \lambda_1^2}}{\sqrt{-\gamma^+}}Y_1(\sqrt{-\gamma^+})\cos r_1\lambda_1 d\lambda_1 \\
&-\frac{1}{4\pi}\int_{0}^{\beta^+}\frac{(\zeta + z)\sqrt{\tilde{N}^2 - \lambda_1^2}}{\sqrt{-\gamma^+}}J_1(\sqrt{-\gamma^+})\sin r_1\lambda_1 d\lambda_1 \\
&-\frac{|r_2|}{4\pi[r_2^2 + (\zeta + z)^2]}\sin r_1\beta^+
\end{aligned}
\tag{85}
$$

$$
\begin{aligned}
h_2 = &-\frac{1}{2\pi^2}\int_{\tilde{N}}^{\infty}\sqrt{\lambda_1^2-\tilde{N}^2}K_0(\sqrt{\gamma^-})\sin r_1\lambda_1 d\lambda_1 \\
&\underline{-\frac{1}{2\pi^2}\int_{\beta^-}^{\tilde{N}}\sqrt{\tilde{N}^2-\lambda_1^2}K_0(\sqrt{\gamma^-})\cos r_1\lambda_1 d\lambda_1} \\
&\underline{+\frac{1}{4\pi}\int_{0}^{\beta^-}\sqrt{\tilde{N}^2-\lambda_1^2}Y_0(\sqrt{-\gamma^-})\cos r_1\lambda_1 d\lambda_1} \\
&-\frac{1}{4\pi}\int_{0}^{\beta^-}\sqrt{\tilde{N}^2-\lambda_1^2}J_0(\sqrt{-\gamma^-})\sin r_1\lambda_1 d\lambda_1 \\
&+\frac{1}{2\pi^2}\int_{\tilde{N}}^{\infty}\sqrt{\lambda_1^2-\tilde{N}^2}K_0(\sqrt{\gamma^+})\sin r_1\lambda_1 d\lambda_1 \\
&\underline{+\frac{1}{2\pi^2}\int_{\beta^+}^{\tilde{N}}\sqrt{\tilde{N}^2-\lambda_1^2}K_0(\sqrt{\gamma^+})\cos r_1\lambda_1 d\lambda_1} \\
&\underline{-\frac{1}{4\pi}\int_{0}^{\beta^+}\sqrt{\tilde{N}^2-\lambda_1^2}Y_0(\sqrt{-\gamma^+})\cos r_1\lambda_1 d\lambda_1} \\
&+\frac{1}{4\pi}\int_{0}^{\beta^+}\sqrt{\tilde{N}^2-\lambda_1^2}J_0(\sqrt{-\gamma^+})\sin r_1\lambda_1 d\lambda_1
\end{aligned}
\tag{86}
$$

10 Utilisation of the fundamental solutions

The solutions obtained in section 9 may now be substituted into equation 39 together with the assumption previously stated that $\boldsymbol{U} = (1, 0, 0)$.

Substitution of the functions subscripted 1 gives

$$\frac{\partial w}{\partial x} = \int_{\Sigma} \left\{ h_1 \left[\frac{\partial \Phi}{\partial n_h} + w n_3 \right] - \Phi \left[\frac{\partial h_1}{\partial n_h} - n_1 \frac{\partial^2 f_1}{\partial x \partial z} \right] + \frac{\partial f_1}{\partial x} \left[n_1 w - n_3 \frac{\partial \Phi}{\partial x} \right] - n_1 f_1 \frac{\partial}{\partial x} \left(w - \frac{\partial \Phi}{\partial z} \right) \right\} d\Sigma \tag{87}$$

whilst the substitution of functions subscripted 2 gives

$$\frac{\partial \Phi}{\partial x} = \int_{\Sigma} \left\{ h_2 \left[\frac{\partial \Phi}{\partial n_h} + w n_3 \right] - \Phi \left[\frac{\partial h_2}{\partial n_h} - n_2 \frac{\partial^2 f_2}{\partial x \partial z} \right] + \frac{\partial f_2}{\partial x} \left[n_1 w - n_3 \frac{\partial \Phi}{\partial x} \right] - n_1 f_2 \frac{\partial}{\partial x} \left(w - \frac{\partial \Phi}{\partial z} \right) \right\} d\Sigma \tag{88}$$

The solution of this pair of simultaneous equations provides the velocity components at specified points, once the boundary conditions on the body's surface have been selected.

The appearance of the velocity field can be deduced from the characteristics displayed by the functions f_1, h_1, f_2 and h_2. The figures included in this report represent the disturbance function h_2 produced by a single translating singularity submerged in a fluid of constant Brunt-Väisälä frequency. The position of the singularity is indicated by an 'x' on each figure.

Figures 1 and 2 are the function calculated on a horizontal plane z = constant. Because the disturbance is symmetric about the plane $y = 0$ only half of the wave system is shown. The singularity is moving towards the bottom of the page.

Figure 1(a) illustrates the antisymmetric nature of the function h_2 before a radiation condition is introduced into the analysis. Figure 1(b) shows the equivalent asymmetric solution derived by introducing an artificial damping mechanism into the mathematical model and letting the artificial damping constant $\epsilon \to 0$. This solution exhibits the characteristic "v" shape pattern associated with a travelling pressure source.

The influence of increasing the forward speed of the source is seen when Figures 1(b) and 2(a) are compared. Namely, as the speed increases the "v" shaped pattern becomes sharper (i.e the angle in the apex decreases with increasing speed).

The strength of stratification also effects the magnitude of the apex angle. This is demonstrated by comparing figures 1(b) and 2(b). These figures show that reducing the stratification (i.e a lower value of $\bar{N}$) causes the apex angle to decrease.

The functions also possess wavelike characteristics in the vertical plane y = constant. Figure 3 is the function h_2 calculated on the plane $y = 0$. The motion of the singularity is towards the left hand side of the page. The upper horizontal surface of the figure is the plane $z = 0$, which is the fluid's surface.

The properties of the functions demonstrated in these figures indicate that a wavelike form of solution has been obtained and this behaviour will be reflected in the velocity field.

Conclusion

This report describes a non potential method to obtain the velocity field created by a uniformly translating, arbitrarily shaped body, fully submerged in a stratified fluid. In the case of a fluid of constant Brunt-Väisälä frequency fundamental solutions are obtained. These solutions demonstrate the properties which other investigations (constant density layer models) have indicated. The implementation of this model and comparison with layer models is now being undertaken.

References

Abramowitz, M. and Stegun, I. A. (1970). *Handbook of mathematical functions.* Dover Publications Inc, ninth edition. I.S.B.N. 0-486-61272-4.

Batchelor, G. K. (1967). *An introduction to fluid dynamics.* Cambridge University Press. I.S.B.N. 0-521-09817-3.

Faltinsen, O. M. (1990). *Sea loads on ships and offshore structures.* Cambridge University Press. I.S.B.N. 0-521-45870-6.

Gamkrelidze, R. V. (1989). *Encyclopaedia of mathematical sciences, Volume 13 - Analysis 1.* Springer - Verlag. I.S.B.N. 3-540-17008-1.

Gradshteyn, I. S. and Ryzhik, I. M. (1980). *Table of integrals, series and products.* Academic Press Inc (London) Ltd, corrected and enlarged edition. I.S.B.N. 0-12-294760-6.

Hess, J. L. and Smith, A. M. O. (1964). Calculation of nonlifting potential flow about arbitrary three dimensional bodies. *J. Ship Res.*, 8(2):22–44.

Lighthill, J. (1979). *Waves in fluids.* Cambridge University Press. I.S.B.N. 0-521-29233-6.

List of Figures

1 (a) antisymmetric and (b) asymmetric function h_2 - horizontal plane. $U = 2ms^{-1}$, $N = 0.10s^{-1}$ 25
2 Asymmetric function h_2 - horizontal plane. (a) $U = 5ms^{-1}$, $N = 0.10s^{-1}$ and (b) $U = 2ms^{-1}$, $N = 0.05s^{-1}$ 26
3 Asymmetric function h_2 - vertical plane. $U = 5ms^{-1}$, $N = 0.50s^{-1}$ 27

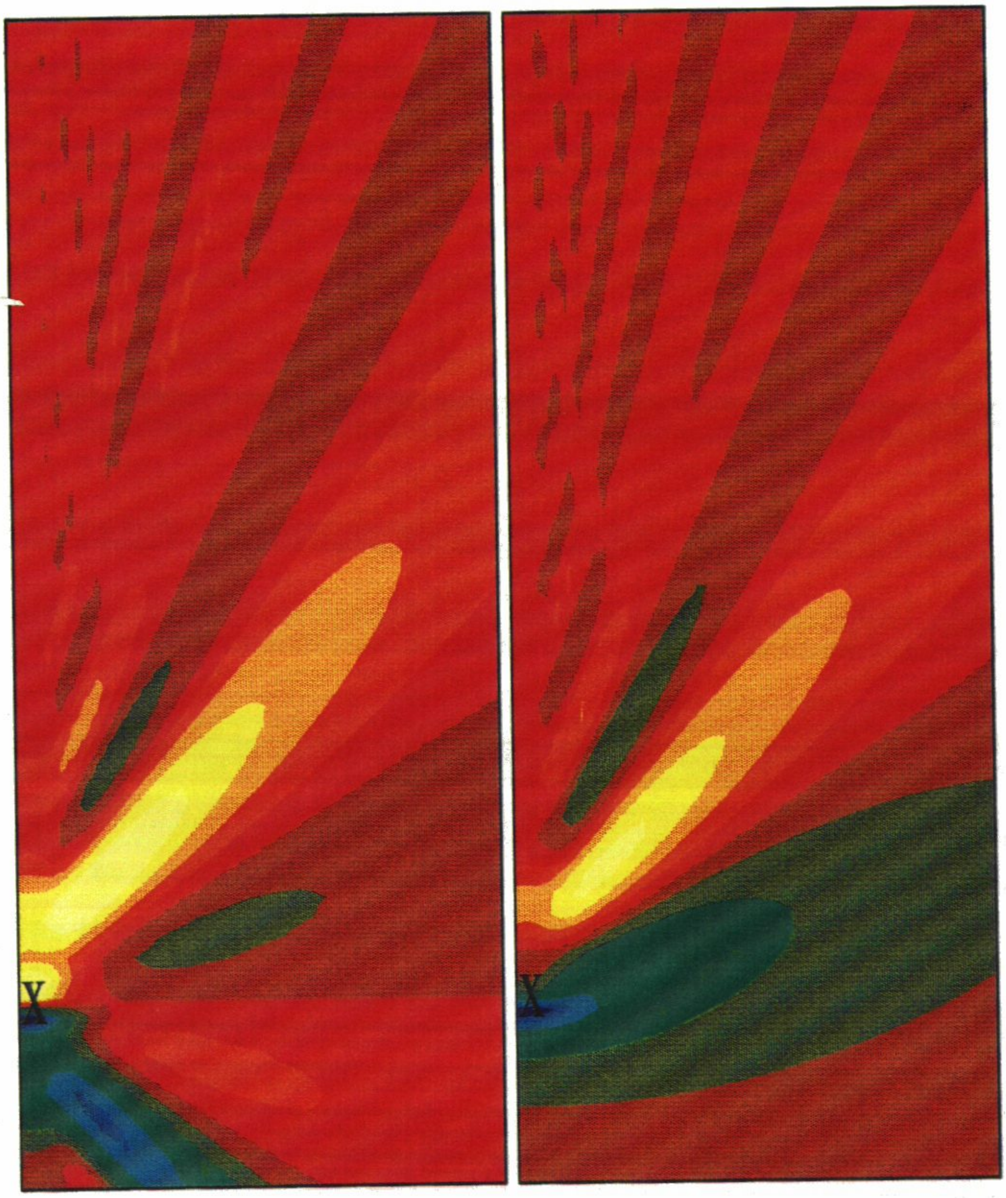

Figure 1: (a) antisymmetric and (b) asymmetric function h_2 - horizontal plane. $U = 2ms^{-1}$, $N = 0.10s^{-1}$

The disturbance function h_2 produced by a single translating singularity submerged in a fluid of constant Brunt-Väisälä frequency. The disturbance is symmetric about the plane $y = 0$ so only half (the port side) of the wave system is shown. The location of the singularity is indicated by an 'x' on each figure. The left hand boundary of each figure is the line $y = 0, z =$constant. The singularity is moving towards the bottom of the page.

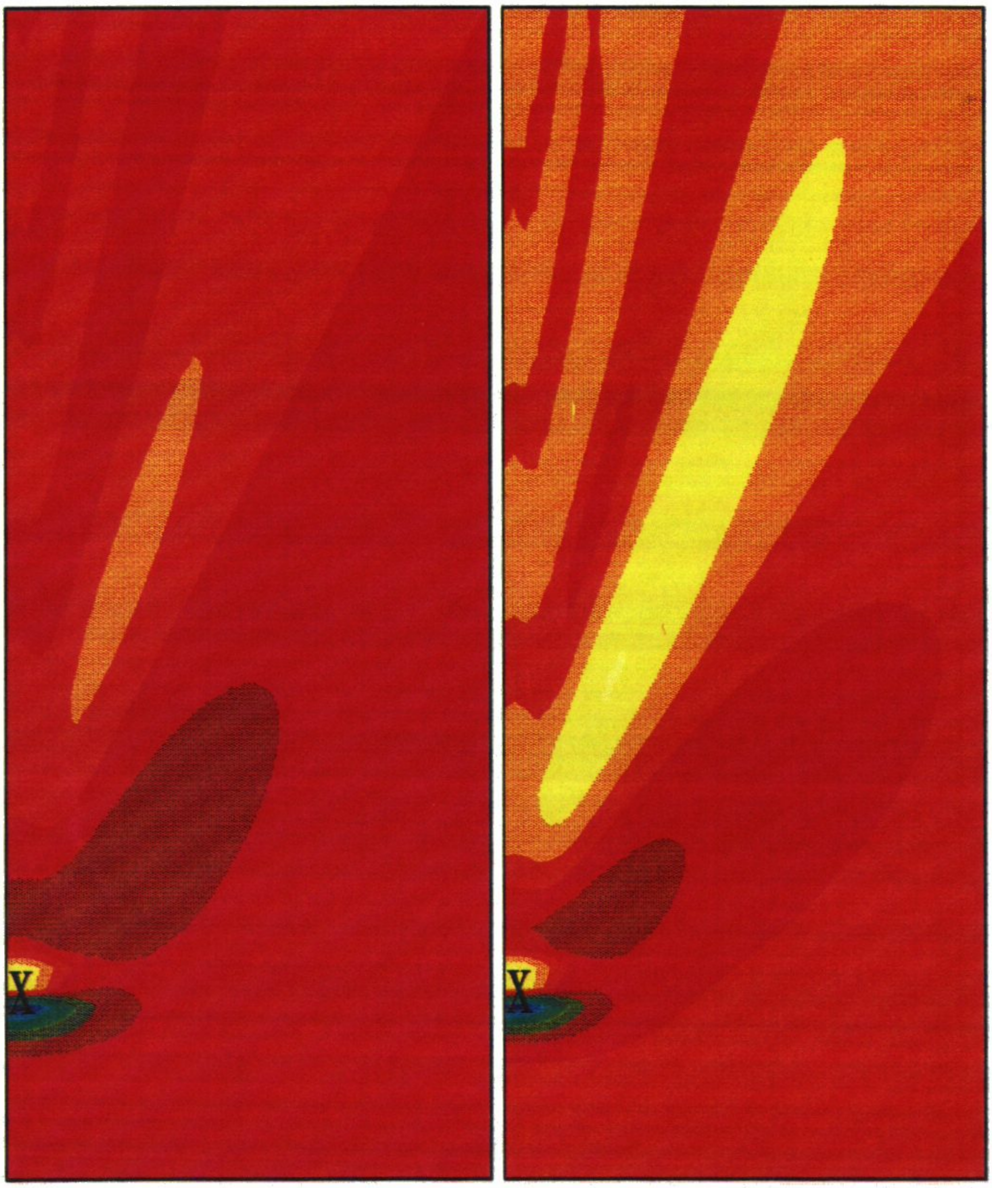

Figure 2: Asymmetric function h_2 - horizontal plane. (a) $U = 5ms^{-1}$, $N = 0.10s^{-1}$ and (b) $U = 2ms^{-1}$, $N = 0.05s^{-1}$

The disturbance function h_2 produced by a single translating singularity submerged in a fluid of constant Brunt-Väisälä frequency. The disturbance is symmetric about the plane $y = 0$ so only half (the port side) of the wave system is shown. The location of the singularity is indicated by an 'x' on each figure. The left hand boundary of each figure is the line $y = 0, z =$constant. The singularity is moving towards the bottom of the page.

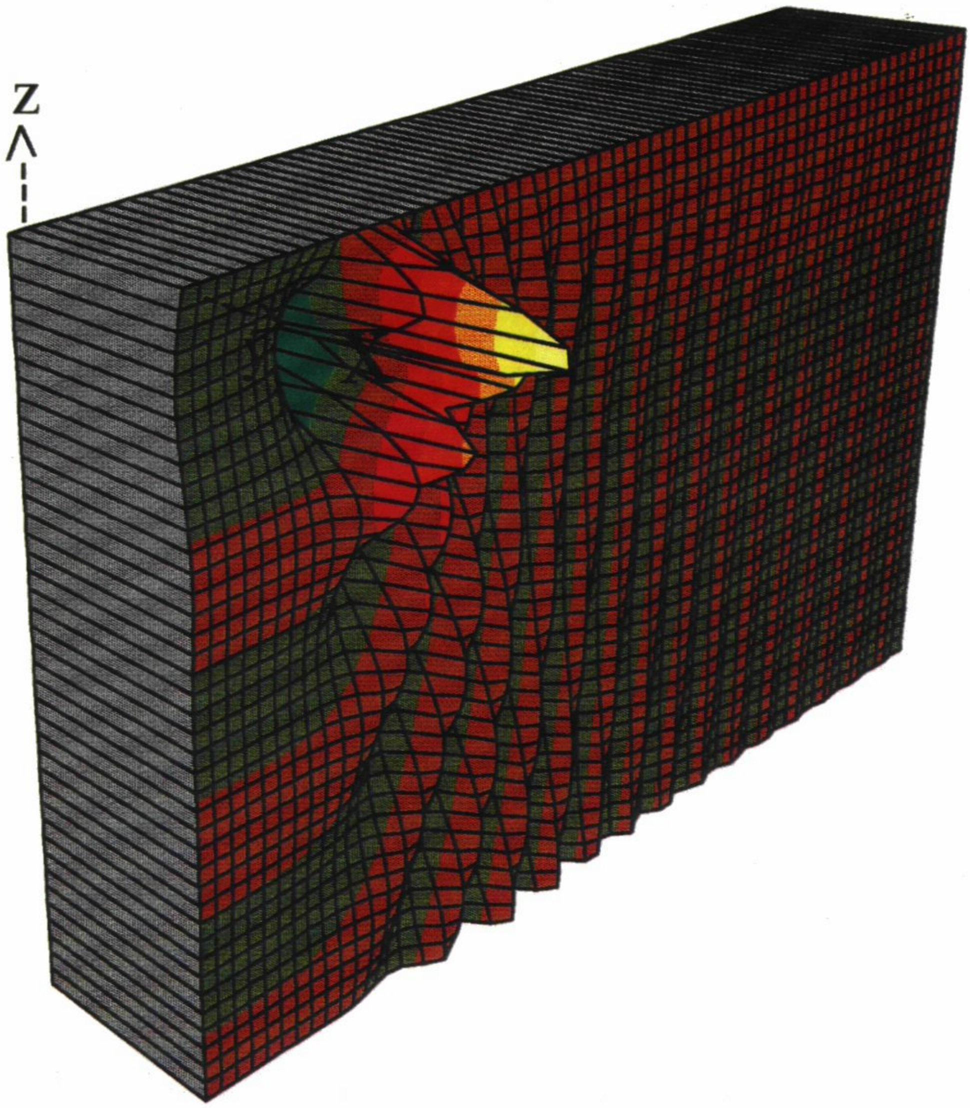

Figure 3: Asymmetric function h_2 - vertical plane. $U = 5ms^{-1}$, $N = 0.50s^{-1}$

X denotes the position the singularity which translates horizontally towards the left hand side of the page. The wavelike wake in the vertical plane is shown.

A : Derivation of antisymmetric fundamental solutions

$$(2\pi)^2 h_2 = \int_{\tilde N}^{\infty}\int_{-\infty}^{\infty} -i\frac{\tilde N^2-\lambda_1^2}{\lambda_1}\left(\frac{e^{\gamma_1\lambda(\zeta+z)}-e^{\gamma_1\lambda(\zeta-z)}}{2\gamma_1\lambda}\right)e^{-i[(x-\xi)\lambda_1+(y-\eta)\lambda_2]}d\lambda_2 d\lambda_1$$

$$+\int_{-\tilde N}^{\tilde N}\int_{-\infty}^{\infty} -i\frac{\tilde N^2-\lambda_1^2}{\lambda_1}\left(\frac{\sin\gamma_2\lambda(\zeta+z)-\sin\gamma_2\lambda(\zeta-z)}{2\gamma_2\lambda}\right)e^{-i[(x-\xi)\lambda_1+(y-\eta)\lambda_2]}d\lambda_2 d\lambda_1$$

$$+\int_{-\infty}^{-\tilde N}\int_{-\infty}^{\infty} -i\frac{\tilde N^2-\lambda_1^2}{\lambda_1}\left(\frac{e^{\gamma_1\lambda(\zeta+z)}-e^{\gamma_1\lambda(\zeta-z)}}{2\gamma_1\lambda}\right)e^{-i[(x-\xi)\lambda_1+(y-\eta)\lambda_2]}d\lambda_2 d\lambda_1$$

$$=\int_{\tilde N}^{\infty} -i\frac{\tilde N^2-\lambda_1^2}{\gamma_1\lambda_1}e^{-i(x-\xi)\lambda_1}\int_0^{\infty}\left(\frac{e^{\gamma_1\lambda(\zeta+z)}-e^{\gamma_1\lambda(\zeta-z)}}{\lambda}\right)\cos r_2\lambda_2 d\lambda_2 d\lambda_1$$

$$+\int_{-\tilde N}^{\tilde N} -i\frac{\tilde N^2-\lambda_1^2}{\gamma_2\lambda_1}e^{-i(x-\xi)\lambda_1}\int_0^{\infty}\left(\frac{\sin\gamma_2\lambda(\zeta+z)-\sin\gamma_2\lambda(\zeta-z)}{\lambda}\right)\cos r_2\lambda_2 d\lambda_2 d\lambda_1$$

$$+\int_{-\infty}^{-\tilde N} -i\frac{\tilde N^2-\lambda_1^2}{\gamma_1\lambda_1}e^{-i(x-\xi)\lambda_1}\int_0^{\infty}\left(\frac{e^{\gamma_1\lambda(\zeta+z)}-e^{\gamma_1\lambda(\zeta-z)}}{\lambda}\right)\cos r_2\lambda_2 d\lambda_2 d\lambda_1$$

where $r_2 = y-\eta$.

This simplification is possible as the exponential $e^{-ir_2\lambda_2} = \cos r_2\lambda_2 - i\sin r_2\lambda_2$. The remaining λ_2 integrand is an even function of λ_2 so the real part of the integration from $-\infty$ to ∞ becomes an integral with limits 0 and ∞ whilst the imaginary part of the integration is an odd function of λ_2 and is zero.

The integrals

$$\int_0^{\infty}\frac{e^{-\beta\sqrt{\gamma^2+x^2}}}{\sqrt{\gamma^2+x^2}}\cos ax\,dx = K_0\left(\gamma\sqrt{a^2+\beta^2}\right)$$

$$\mathcal{R}e(\beta)>0 \quad \mathcal{R}e(\gamma)>0 \quad a>0, \tag{89}$$

and

$$\int_0^{\infty}\frac{\sin\left(p\sqrt{x^2+a^2}\right)}{\sqrt{x^2+a^2}}\cos bx\,dx = \begin{cases}\frac{\pi}{2}J_0\left(a\sqrt{p^2-b^2}\right) & 0<b<p\\ 0 & b>p>0\end{cases} \tag{90}$$

(see Gradshteyn and Ryzhik (1980) 3.961(2) and 3.876(1) respectively) can be applied to the λ_2 integration if h_2 is rewritten as

$$(2\pi)^2 h_2 = \int_{\tilde N}^{\infty} -i\frac{\tilde N^2-\lambda_1^2}{\gamma_1\lambda_1}e^{-i(x-\xi)\lambda_1}\int_0^{\infty}\left(\frac{e^{-\gamma_1\lambda|\zeta+z|}-e^{\gamma_1\lambda|\zeta-z|}}{\lambda}\right)\cos|r_2|\lambda_2 d\lambda_2 d\lambda_1$$

$$-\int_{-\tilde N}^{\tilde N} -i\frac{\tilde N^2-\lambda_1^2}{\gamma_2\lambda_1}e^{-i(x-\xi)\lambda_1}\int_0^{\infty}\left(\frac{\sin\gamma_2\lambda|\zeta+z|-\sin\gamma_2\lambda|\zeta-z|}{\lambda}\right)\cos|r_2|\lambda_2 d\lambda_2 d\lambda_1$$

$$+\int_{-\infty}^{-\tilde N} -i\frac{\tilde N^2-\lambda_1^2}{\gamma_1\lambda_1}e^{-i(x-\xi)\lambda_1}\int_0^{\infty}\left(\frac{e^{\gamma_1\lambda|\zeta+z|}-e^{\gamma_1\lambda|\zeta-z|}}{\lambda}\right)\cos|r_2|\lambda_2 d\lambda_2 d\lambda_1$$

because

$$\gamma_{1,2} > 0 \quad \zeta \pm z < 0 \quad \lambda = \sqrt{\lambda_1^2 + \lambda_2^2},$$

thus

$$\gamma_{1,2}\lambda(\zeta \pm z) = -\gamma_{1,2}\lambda|\zeta \pm z|,$$

$$\cos r_2\lambda_2 = \cos|r_2|\lambda_2,$$

$$\sin\gamma_2\lambda(\zeta \pm z) = -\sin\gamma_2\lambda|\zeta \pm z|.$$

The inequality $b < p$ requires the modification of the integration limits for the λ_2 integral. That is

$$|r_2| = \gamma_2|\zeta \pm z|$$

which yields

$$\frac{\tilde{N}(\zeta \pm z)}{\sqrt{r_2^2 + (\zeta \pm z)^2}} < \lambda_1 < -\frac{\tilde{N}(\zeta \pm z)}{\sqrt{r_2^2 + (\zeta \pm z)^2}}$$

If we let

$$\beta^+ = -\frac{\tilde{N}(\zeta + z)}{\sqrt{r_2^2 + (\zeta + z)^2}} \quad 0 \leq \beta^+ \leq \tilde{N}$$

$$\beta^- = -\frac{\tilde{N}(\zeta - z)}{\sqrt{r_2^2 + (\zeta - z)^2}} \quad 0 \leq \beta^- \leq \tilde{N}$$

it follows that

$$(2\pi)^2 h_2 = \int_{\tilde{N}}^{\infty} -i\frac{\tilde{N}^2 - \lambda_1^2}{\lambda_1} e^{-i(x-\xi)\lambda_1} \left[K_0\left(\sqrt{\lambda_1^2[(y-\eta)^2 + (\zeta+z)^2] - \tilde{N}^2(\zeta+z)^2}\right)\right.$$
$$\left.-K_0\left(\sqrt{\lambda_1^2[(y-\eta)^2 + (\zeta-z)^2] - \tilde{N}^2(\zeta-z)^2}\right)\right] \frac{|\lambda_1|}{\sqrt{\lambda_1^2 - \tilde{N}^2}} d\lambda_1$$
$$-\frac{\pi}{2}\int_{-\beta^+}^{\beta^+} -i\frac{\tilde{N}^2 - \lambda_1^2}{\lambda_1} e^{-i(x-\xi)\lambda_1} J_0\left(\sqrt{\tilde{N}^2(\zeta+z)^2 - \lambda_1^2[(y-\eta)^2 + (\zeta+z)^2]}\right) \frac{|\lambda_1|}{\sqrt{\tilde{N}^2 - \lambda_1^2}} d\lambda_1$$
$$+\frac{\pi}{2}\int_{-\beta^-}^{\beta^-} -i\frac{\tilde{N}^2 - \lambda_1^2}{\lambda_1} e^{-i(x-\xi)\lambda_1} J_0\left(\sqrt{\tilde{N}^2(\zeta-z)^2 - \lambda_1^2[(y-\eta)^2 + (\zeta-z)^2]}\right) \frac{|\lambda_1|}{\sqrt{\tilde{N}^2 - \lambda_1^2}} d\lambda_1$$
$$+\int_{-\infty}^{-\tilde{N}} -i\frac{\tilde{N}^2 - \lambda_1^2}{\lambda_1} e^{-i(x-\xi)\lambda_1} \left[K_0\left(\sqrt{\lambda_1^2[(y-\eta)^2 + (\zeta+z)^2] - \tilde{N}^2(\zeta+z)^2}\right)\right.$$
$$\left.-K_0\left(\sqrt{\lambda_1^2[(y-\eta)^2 + (\zeta-z)^2] - \tilde{N}^2(\zeta-z)^2}\right)\right] \frac{|\lambda_1|}{\sqrt{\lambda_1^2 - \tilde{N}^2}} d\lambda_1 \tag{91}$$

Another simplification can be applied to h_2 through an examination of the even and odd behaviour of the integrands. That is

$$h_2 = \frac{1}{2\pi^2}\int_{\tilde{N}}^{\infty}\sqrt{\lambda_1^2-\tilde{N}^2}\left[K_0\left(\sqrt{\gamma^+}\right)-K_0\left(\sqrt{\gamma^-}\right)\right]\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{1}{4\pi}\int_0^{\beta^+}\sqrt{\tilde{N}^2-\lambda_1^2}J_0\left(\sqrt{-\gamma^+}\right)\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{1}{4\pi}\int_0^{\beta^-}\sqrt{\tilde{N}^2-\lambda_1^2}J_0\left(\sqrt{-\gamma^-}\right)\sin r_1\lambda_1 d\lambda_1 \qquad (92)$$

where

$$\gamma^{\pm} = \lambda_1^2[(y-\eta)^2+(\zeta\pm z)^2]-\tilde{N}^2(\zeta\pm z)^2$$

and

$$r_1 = x-\xi$$

Equation 91 can be modified using the appropriate formulation of $G(\lambda_1,\lambda_2,z;\ \boldsymbol{\xi}\ )$ and when applied into the derivation of $\bar{h}_1$ we obtain the result

$$-(2\pi)^2\bar{h}_1 = \int_{\tilde{N}}^{\infty}-i\lambda_1\left[K_0\left(\sqrt{\gamma^+}\right)+K_0\left(\sqrt{\gamma^-}\right)\right]\frac{|\lambda_1|}{\sqrt{\lambda_1^2-\tilde{N}^2}}e^{-i(x-\xi)\lambda_1}d\lambda_1$$

$$-\frac{\pi}{2}\int_{-\beta^+}^{\beta^+}-i\lambda_1 J_0\left(\sqrt{-\gamma^+}\right)\frac{|\lambda_1|}{\sqrt{\tilde{N}^2-\lambda_1^2}}e^{-i(x-\xi)\lambda_1}d\lambda_1$$

$$-\frac{\pi}{2}\int_{-\beta^-}^{\beta^-}-i\lambda_1 J_0\left(\sqrt{-\gamma^-}\right)\frac{|\lambda_1|}{\sqrt{\tilde{N}^2-\lambda_1^2}}e^{-i(x-\xi)\lambda_1}d\lambda_1$$

$$+\int_{-\infty}^{-\tilde{N}}-i\lambda_1\left[K_0\left(\sqrt{\gamma^+}\right)+K_0\left(\sqrt{\gamma^-}\right)\right]\frac{|\lambda_1|}{\sqrt{\lambda_1^2-\tilde{N}^2}}e^{-i(x-\xi)\lambda_1}d\lambda_1 \qquad (93)$$

$$-2\pi^2\bar{h}_1 = -\int_{\tilde{N}}^{\infty}\frac{\lambda_1^2}{\sqrt{\lambda_1^2-\tilde{N}^2}}\left[K_0\left(\sqrt{\gamma^+}\right)+K_0\left(\sqrt{\gamma^-}\right)\right]\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta^+}\frac{\lambda_1^2}{\sqrt{\tilde{N}^2-\lambda_1^2}}J_0\left(\sqrt{-\gamma^+}\right)\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta^-}\frac{\lambda_1^2}{\sqrt{\tilde{N}^2-\lambda_1^2}}J_0\left(\sqrt{-\gamma^-}\right)\sin r_1\lambda_1 d\lambda_1 \qquad (94)$$

The required function h_1 is the partial differential of $\bar{h}_1$ with respect to z. The integration limits are a function of z and the partial differential of an integral of this type has the general form

$$\frac{\partial}{\partial\alpha}\int_{\phi(\alpha)}^{\psi(\alpha)} f(x,\alpha)dx = \left(\frac{\partial\psi}{\partial\alpha}\right) f(\psi(\alpha),\alpha) - \left(\frac{\partial\phi}{\partial\alpha}\right) f(\phi(\alpha),\alpha) + \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial f(x,\alpha)}{\partial\alpha}dx$$

In particular,

$$\frac{\partial}{\partial z}(\beta^{\pm}) = \frac{\mp\tilde{N}r_2^2}{[r_2^2+(\zeta\pm z)^2]^{\frac{3}{2}}}$$

$$\frac{\partial}{\partial z}\left[K_0(\sqrt{\gamma^{\pm}})\right] = \frac{\mp(\lambda_1^2-\tilde{N}^2)(\zeta\pm z)}{\sqrt{\gamma^{\pm}}}K_1(\sqrt{\gamma^{\pm}})$$

$$\frac{\partial}{\partial z}\left[J_0(\sqrt{-\gamma^{\pm}})\right] = \frac{\mp(\tilde{N}^2-\lambda_1^2)(\zeta\pm z)}{\sqrt{-\gamma^{\pm}}}J_1(\sqrt{-\gamma^{\pm}})$$

and on completing the differentiation it follows that

$$-2\pi^2 h_1 = \int_{\tilde{N}}^{\infty} \frac{\lambda_1^2(\lambda_1^2-\tilde{N}^2)}{\sqrt{\lambda_1^2-\tilde{N}^2}}\left[\frac{(\zeta+z)}{\sqrt{\gamma^+}}K_1(\sqrt{\gamma^+}) - \frac{(\zeta-z)}{\sqrt{\gamma^-}}K_1(\sqrt{\gamma^-})\right]\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{\tilde{N}r_2^2\pi}{2[r_2^2+(\zeta+z)^2]^{\frac{3}{2}}}\frac{(\beta^+)^2}{\sqrt{\tilde{N}^2-(\beta^+)^2}}J_0\left(\sqrt{-\gamma^+|_{\lambda_1=\beta^+}}\right)\sin r_1\beta^+$$

$$-\frac{\pi}{2}\int_0^{\beta^+} \frac{\lambda_1^2(\zeta+z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^+}}J_1(\sqrt{-\gamma^+})\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\tilde{N}r_2^2\pi}{2[r_2^2+(\zeta-z)^2]^{\frac{3}{2}}}\frac{(\beta^-)^2}{\sqrt{\tilde{N}^2-(\beta^-)^2}}J_0\left(\sqrt{-\gamma^-|_{\lambda_1=\beta^-}}\right)\sin r_1\beta^-$$

$$+\frac{\pi}{2}\int_0^{\beta^-} \frac{\lambda_1^2(\zeta-z)\sqrt{\tilde{N}^2-\lambda_1}}{\sqrt{-\gamma^-}}J_1(\sqrt{-\gamma^-})\sin r_1\lambda_1 d\lambda_1$$

Using the results

$$\frac{(\beta^+)^2}{\sqrt{\tilde{N}^2-(\beta^+)^2}} = \frac{(\zeta\pm z)^2}{r_2^2}$$

$$J_0\left(\sqrt{-\gamma^{\pm}|_{\lambda_1=\beta^{\pm}}}\right) = J_0(0) = 1$$

simplifies the previous equation giving the expression

$$h_1 = -\frac{1}{2\pi^2}\int_{\tilde{N}}^{\infty} \lambda_1^2\sqrt{\lambda_1^2-\tilde{N}^2}\left[\frac{(\zeta+z)}{\sqrt{\gamma^+}}K_1(\sqrt{\gamma^+}) - \frac{(\zeta-z)}{\sqrt{\gamma^-}}K_1(\sqrt{\gamma^-})\right]\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\tilde{N}(\zeta+z)^2}{4\pi[r_2^2+(\zeta+z)^2]^{\frac{3}{2}}}\sin r_1\beta^+$$

$$+\frac{1}{4\pi}\int_0^{\beta^+}\frac{\lambda_1^2(\zeta+z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^+}}J_1(\sqrt{-\gamma^+})\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{\tilde{N}(\zeta-z)^2}{4\pi[r_2^2+(\zeta-z)^2]^{\frac{3}{2}}}\sin r_1\beta^-$$

$$-\frac{1}{4\pi}\int_0^{\beta^-}\frac{\lambda_1^2(\zeta-z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^-}}J_1(\sqrt{-\gamma^-})\sin r_1\lambda_1 d\lambda_1$$

Equation 93 can be utilised in the derivation of $\bar{f}_2$. This allows us to write

$$-(2\pi)^2\bar{f}_2=\int_{\tilde{N}}^{\infty}\frac{i}{\lambda_1}\left[K_0\left(\sqrt{\gamma^+}\right)+K_0\left(\sqrt{\gamma^-}\right)\right]\frac{|\lambda_1|}{\sqrt{\lambda_1^2-\tilde{N}^2}}e^{-i(x-\xi)\lambda_1}d\lambda_1$$

$$-\frac{\pi}{2}\int_{-\beta^+}^{\beta^+}\frac{i}{\lambda_1}J_0\left(\sqrt{-\gamma^+}\right)\frac{|\lambda_1|}{\sqrt{\tilde{N}^2-\lambda_1^2}}e^{-i(x-\xi)\lambda_1}d\lambda_1$$

$$-\frac{\pi}{2}\int_{-\beta^-}^{\beta^-}\frac{i}{\lambda_1}J_0\left(\sqrt{-\gamma^-}\right)\frac{|\lambda_1|}{\sqrt{\tilde{N}^2-\lambda_1^2}}e^{-i(x-\xi)\lambda_1}d\lambda_1$$

$$+\int_{-\infty}^{-\tilde{N}}\frac{i}{\lambda_1}\left[K_0\left(\sqrt{\gamma^+}\right)+K_0\left(\sqrt{\gamma^-}\right)\right]\frac{|\lambda_1|}{\sqrt{\lambda_1^2-\tilde{N}^2}}e^{-i(x-\xi)\lambda_1}d\lambda_1 \tag{95}$$

or, after rearrangement,

$$-2\pi^2\bar{f}_2=\int_{\tilde{N}}^{\infty}\frac{1}{\sqrt{\lambda_1^2-\tilde{N}^2}}\left[K_0\left(\sqrt{\gamma^+}\right)+K_0\left(\sqrt{\gamma^-}\right)\right]\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{\pi}{2}\int_0^{\beta^+}\frac{1}{\sqrt{\tilde{N}^2-\lambda_1^2}}J_0\left(\sqrt{-\gamma^+}\right)\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{\pi}{2}\int_0^{\beta^-}\frac{1}{\sqrt{\tilde{N}^2-\lambda_1^2}}J_0\left(\sqrt{-\gamma^-}\right)\sin r_1\lambda_1 d\lambda_1$$

Now differentiate $\bar{f}_2$ with respect to z to obtain f_2 gives

$$-2\pi^2 f_2=-\int_{\tilde{N}}^{\infty}\sqrt{\lambda_1^2-\tilde{N}^2}\left[\frac{(\zeta+z)}{\sqrt{\gamma^+}}K_1\left(\sqrt{\gamma^+}\right)-\frac{(\zeta-z)}{\sqrt{\gamma^-}}K_1\left(\sqrt{\gamma^-}\right)\right]\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\tilde{N}r_2^2\pi}{2[r_2^2+(\zeta+z)^2]^{\frac{3}{2}}}\frac{1}{\sqrt{\tilde{N}^2-(\beta^+)^2}}J_0\left(\sqrt{-\gamma^+|_{\lambda_1=\beta^+}}\right)\sin r_1\beta^+$$

$$+\frac{\pi}{2}\int_0^{\beta^+}\frac{\sqrt{\tilde{N}^2-\lambda_1^2}(\zeta+z)}{\sqrt{-\gamma^+}}J_1(\sqrt{-\gamma^+})\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{\tilde{N}r_2^2\pi}{2[r_2^2+(\zeta-z)^2]^{\frac{3}{2}}}\frac{1}{\sqrt{\tilde{N}^2-(\beta^-)^2}}J_0\left(\sqrt{-\gamma^-|_{\lambda_1=\beta^-}}\right)\sin r_1\beta^-$$

$$-\frac{\pi}{2}\int_0^{\beta^-}\frac{\sqrt{\tilde{N}^2-\lambda_1^2}(\zeta-z)}{\sqrt{-\gamma^-}}J_1(\sqrt{-\gamma^-})\sin r_1\lambda_1 d\lambda_1$$

and on using the result

$$\frac{1}{\sqrt{\tilde{N}^2-(\beta^\pm)^2}}=\frac{\sqrt{r_2^2+(\zeta-z)^2}}{\tilde{N}|r_2|}$$

we obtain the following expression,

$$f_2=\frac{1}{2\pi^2}\int_{\tilde{N}}^{\infty}\sqrt{\lambda_1^2-\tilde{N}^2}\left[\frac{(\zeta+z)}{\sqrt{\gamma^+}}K_1\left(\sqrt{\gamma^+}\right)-\frac{(\zeta-z)}{\sqrt{\gamma^-}}K_1\left(\sqrt{\gamma^-}\right)\right]\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{|r_2|}{4\pi[r_2^2+(\zeta+z)^2]}\sin r_1\beta^+$$

$$-\frac{1}{4\pi}\int_0^{\beta^+}\frac{\sqrt{\tilde{N}^2-\lambda_1^2}(\zeta+z)}{\sqrt{-\gamma^+}}J_1(\sqrt{-\gamma^+})\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{|r_2|}{4\pi[r_2^2+(\zeta-z)^2]}\sin r_1\beta^-$$

$$+\frac{1}{4\pi}\int_0^{\beta^-}\frac{\sqrt{\tilde{N}^2-\lambda_1^2}(\zeta-z)}{\sqrt{-\gamma^-}}J_1(\sqrt{-\gamma^-})\sin r_1\lambda_1 d\lambda_1$$

The function f_1 has to be dealt with slightly more carefully as the integrand has an additional λ_2 term, as can be seen from the following equation :

$$(2\pi)^2 f_1=\int_{\tilde{N}}^{\infty}\int_{-\infty}^{\infty}-i\frac{\lambda^2}{\lambda_1}\left(\frac{e^{\gamma_1\lambda(\zeta+z)}-e^{\gamma_1\lambda(\zeta-z)}}{2\gamma_1\lambda}\right)e^{-i[(x-\xi)\lambda_1+(y-\eta)\lambda_2]}d\lambda_2 d\lambda_1$$

$$+\int_{-\tilde{N}}^{\tilde{N}}\int_{-\infty}^{\infty}-i\frac{\lambda^2}{\lambda_1}\left(\frac{\sin\gamma_2\lambda(\zeta+z)-\sin\gamma_2\lambda(\zeta-z)}{2\gamma_2\lambda}\right)e^{-i[(x-\xi)\lambda_1+(y-\eta)\lambda_2]}d\lambda_2 d\lambda_1$$

$$+\int_{-\infty}^{-\tilde{N}}\int_{-\infty}^{\infty} -i\frac{\lambda^2}{\lambda_1}\left(\frac{e^{\gamma_1\lambda(\zeta+z)}-e^{\gamma_1\lambda(\zeta-z)}}{2\gamma_1\lambda}\right)e^{-i[(x-\xi)\lambda_1+(y-\eta)\lambda_2]}d\lambda_2 d\lambda_1$$

$$=\int_{\tilde{N}}^{\infty} -i\frac{1}{\gamma_1\lambda_1}e^{-i(x-\xi)\lambda_1}\int_0^{\infty}\lambda\left(e^{\gamma_1\lambda(\zeta+z)}-e^{\gamma_1\lambda(\zeta-z)}\right)\cos r_2\lambda_2 d\lambda_2 d\lambda_1$$

$$+\int_{-\tilde{N}}^{\tilde{N}} -i\frac{1}{\gamma_2\lambda_1}e^{-i(x-\xi)\lambda_1}\int_0^{\infty}\lambda\left(\sin\gamma_2\lambda(\zeta+z)-\sin\gamma_2\lambda(\zeta-z)\right)\cos r_2\lambda_2 d\lambda_2 d\lambda_1$$

$$+\int_{-\infty}^{-\tilde{N}} -i\frac{1}{\gamma_1\lambda_1}e^{-i(x-\xi)\lambda_1}\int_0^{\infty}\lambda\left(e^{\gamma_1\lambda(\zeta+z)}-e^{\gamma_1\lambda(\zeta-z)}\right)\cos r_2\lambda_2 d\lambda_2 d\lambda_1$$

Equation 89 may be applied to the λ_2 integration if it is differentiated twice with respect to β. This gives

$$\int_0^{\infty}\sqrt{\gamma^2+x^2}e^{-\beta\sqrt{\gamma^2+x^2}}\cos ax dx=\frac{\partial^2}{\partial\beta^2}\left[K_0\left(\gamma\sqrt{a^2+\beta^2}\right)\right]$$

Using the results

$$\frac{\partial}{\partial x}K_0(x)=-K_1(x),$$

$$\frac{\partial}{\partial x}\left(\frac{1}{x}K_1(x)\right)=-\frac{K_2(x)}{x},$$

$$K_2(x)=K_0(x)+\frac{2}{x}K_1(x)$$

(see Abramowitz and Stegun(1970) 9.6.27, 9.6.28, $k=1$, $\nu=1$ and 9.6.26, $\nu=1$), we derive the expression

$$\int_0^{\infty}\sqrt{\gamma^2+x^2}e^{-\beta\sqrt{\gamma^2+x^2}}\cos ax dx=\frac{\gamma}{a^2+\beta^2}\left[\gamma\beta^2K_0(\gamma\sqrt{a^2+\beta^2})-\frac{a^2-\beta^2}{\sqrt{a^2+\beta^2}}K_1(\gamma\sqrt{a^2+\beta^2})\right],$$

similarly,

$$\int_0^{\infty}\sqrt{x^2+a^2}\sin\left(p\sqrt{x^2+a^2}\right)\cos bx dx=\begin{cases}\frac{a\pi}{2(p^2-b^2)}\left[ap^2J_0(a\sqrt{p^2-b^2})-\frac{p^2+b^2}{\sqrt{p^2-b^2}}J_1(a\sqrt{p^2-b^2})\right] & 0<b<p\\ 0 & b>p>0.\end{cases}$$

Thus we find that

$$(2\pi)^2 f_1=\int_{\tilde{N}}^{\infty}-i\frac{|\lambda_1|\lambda_1}{\gamma^+\sqrt{\lambda_1^2-\tilde{N}^2}}e^{-i(x-\xi)\lambda_1}\left[(\lambda_1^2-\tilde{N}^2)(\zeta+z)^2K_0(\sqrt{\gamma^+})-\frac{\sigma^+}{\sqrt{\gamma^+}}K_1(\sqrt{\gamma^+})\right]d\lambda_1$$

$$-\int_{\tilde{N}}^{\infty}-i\frac{|\lambda_1|\lambda_1}{\gamma^-\sqrt{\lambda_1^2-\tilde{N}^2}}e^{-i(x-\xi)\lambda_1}\left[(\lambda_1^2-\tilde{N}^2)(\zeta-z)^2K_0(\sqrt{\gamma^-})-\frac{\sigma^-}{\sqrt{\gamma^-}}K_1(\sqrt{\gamma^-})\right]d\lambda_1$$

$$+\frac{\pi}{2}\int_{-\beta^+}^{\beta^+} -i\frac{|\lambda_1|\lambda_1}{\gamma^+\sqrt{\tilde{N}^2-\lambda_1^2}}e^{-i(x-\xi)\lambda_1}\left[(\tilde{N}^2-\lambda_1^2)(\zeta+z)^2 J_0(\sqrt{-\gamma^+}) - \frac{\sigma^+}{\sqrt{-\gamma^+}}J_1(\sqrt{-\gamma^+})\right]d\lambda_1$$

$$-\frac{\pi}{2}\int_{-\beta^-}^{\beta^-} -i\frac{|\lambda_1|\lambda_1}{\gamma^-\sqrt{\tilde{N}^2-\lambda_1^2}}e^{-i(x-\xi)\lambda_1}\left[(\tilde{N}^2-\lambda_1^2)(\zeta-z)^2 J_0(\sqrt{-\gamma^-}) - \frac{\sigma^-}{\sqrt{-\gamma^-}}J_1(\sqrt{-\gamma^-})\right]d\lambda_1$$

$$+\int_{-\infty}^{-\tilde{N}} -i\frac{|\lambda_1|\lambda_1}{\gamma^+\sqrt{\lambda_1^2-\tilde{N}^2}}e^{-i(x-\xi)\lambda_1}\left[(\lambda_1^2-\tilde{N}^2)(\zeta+z)^2 K_0(\sqrt{\gamma^+}) - \frac{\sigma^+}{\sqrt{\gamma^+}}K_1(\sqrt{\gamma^+})\right]d\lambda_1$$

$$-\int_{-\infty}^{-\tilde{N}} -i\frac{|\lambda_1|\lambda_1}{\gamma^-\sqrt{\lambda_1^2-\tilde{N}^2}}e^{-i(x-\xi)\lambda_1}\left[(\lambda_1^2-\tilde{N}^2)(\zeta-z)^2 K_0(\sqrt{\gamma^-}) - \frac{\sigma^-}{\sqrt{\gamma^-}}K_1(\sqrt{\gamma^-})\right]d\lambda_1$$

or, after rearrangement,

$$f_1 = -\frac{1}{2\pi^2}\int_{\tilde{N}}^{\infty} \frac{\lambda_1^2}{\gamma^+\sqrt{\lambda_1^2-\tilde{N}^2}}\left[(\lambda_1^2-\tilde{N}^2)(\zeta+z)^2 K_0(\sqrt{\gamma^+}) - \frac{\sigma^+}{\sqrt{\gamma^+}}K_1(\sqrt{\gamma^+})\right]\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{1}{2\pi^2}\int_{\tilde{N}}^{\infty} \frac{\lambda_1^2}{\gamma^-\sqrt{\lambda_1^2-\tilde{N}^2}}\left[(\lambda_1^2-\tilde{N}^2)(\zeta-z)^2 K_0(\sqrt{\gamma^-}) - \frac{\sigma^-}{\sqrt{\gamma^-}}K_1(\sqrt{\gamma^-})\right]\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{1}{4\pi}\int_{0}^{\beta^+} \frac{\lambda_1^2}{\gamma^+\sqrt{\tilde{N}^2-\lambda_1^2}}\left[(\tilde{N}^2-\lambda_1^2)(\zeta+z)^2 J_0(\sqrt{-\gamma^+}) - \frac{\sigma^+}{\sqrt{-\gamma^+}}J_1(\sqrt{-\gamma^+})\right]\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{1}{4\pi}\int_{0}^{\beta^-} \frac{\lambda_1^2}{\gamma^-\sqrt{\tilde{N}^2-\lambda_1^2}}\left[(\tilde{N}^2-\lambda_1^2)(\zeta-z)^2 J_0(\sqrt{-\gamma^-}) - \frac{\sigma^-}{\sqrt{-\gamma^-}}J_1(\sqrt{-\gamma^-})\right]\sin r_1\lambda_1 d\lambda_1$$

B : Derivation of asymmetric fundamental solutions

The derivation of the asymmetric fundamental solutions uses techniques and identities established in appendix A.

Let

$$\gamma_3 = \sqrt{1 - \frac{\tilde{N}^2}{\lambda_1(\lambda_1 - i\epsilon)}}$$

then

$$h_2 = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i\left(\lambda_1 - \frac{\tilde{N}^2}{\lambda_1 - i\epsilon}\right)\left(\frac{e^{\lambda(z+\zeta)\gamma_3} - e^{-\lambda(z-\zeta)\gamma_3}}{2\lambda\gamma_3}\right)e^{-i[(x-\xi)\lambda_1+(y-\eta)\lambda_2]}d\lambda_1 d\lambda_2$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{i(\lambda_1^2 - \tilde{N}^2 - i\epsilon\lambda_1)}{(\lambda_1 - i\epsilon)\gamma_3}e^{-i(x-\xi)\lambda_1}\int_0^{\infty}\left(\frac{e^{\lambda(z+\zeta)\gamma_3} - e^{-\lambda(z-\zeta)\gamma_3}}{\lambda}\right)\cos r_2\lambda_2 d\lambda_2 d\lambda_1$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{i(\lambda_1^2 - \tilde{N}^2 - i\epsilon\lambda_1)}{(\lambda_1 - i\epsilon)\gamma_3}e^{-i(x-\xi)\lambda_1}\left[K_0\left(|\lambda_1|\sqrt{r_2^2 + (z+\zeta)^2\gamma_3^2}\right) - K_0\left(|\lambda_1|\sqrt{r_2^2 + (z-\zeta)^2\gamma_3^2}\right)\right]d\lambda_1 \quad (96)$$

Now let us examine the behaviour of the constituent parts of the integrand as a function of λ_1 as $\epsilon \longrightarrow 0$.

λ_1	$-\infty \rightarrow -\tilde{N}$	$-\tilde{N} \rightarrow -\beta^{\pm}$	$-\beta^{\pm} \rightarrow 0$	$0 \rightarrow \beta^{\pm}$	$\beta^{\pm} \rightarrow \tilde{N}$	$\tilde{N} \rightarrow \infty$
$\lvert\lambda_1\rvert\gamma_3$	$\sqrt{\lambda_1^2 - \tilde{N}^2}$	$i\sqrt{\tilde{N}^2 - \lambda_1^2}$	$i\sqrt{\tilde{N}^2 - \lambda_1^2}$	$-i\sqrt{\tilde{N}^2 - \lambda_1^2}$	$-i\sqrt{\tilde{N}^2 - \lambda_1^2}$	$\sqrt{\lambda_1^2 - \tilde{N}^2}$
$\lvert\lambda_1\rvert\sqrt{r_2^2 + (z \pm \zeta)^2\gamma_3^2}$	$\sqrt{\gamma^{\pm}}$	$\sqrt{\gamma^{\pm}}$	$i\sqrt{-\gamma^{\pm}}$	$-i\sqrt{-\gamma^{\pm}}$	$\sqrt{\gamma^{\pm}}$	$\sqrt{\gamma^{\pm}}$

Table 1: Integrand properties of h_2

From the table above and using the property

$$K_0(\pm ix) = -\tfrac{\pi}{2}\left[Y_0(x) \pm iJ_0(x)\right] \quad x > 0$$

(see Abramowitz and Stegun(1970) 9.6.4, $\nu = 0$) h_2 becomes

$$(2\pi)^2 h_2 = \int_{-\infty}^{-\tilde{N}} -i\sqrt{\lambda_1^2 - \tilde{N}^2}\left[K_0(\sqrt{\gamma^+}) - K_0(\sqrt{\gamma^-})\right]e^{-i(x-\xi)\lambda_1}d\lambda_1$$

$$+\int_{-\tilde{N}}^{-\beta^+}\sqrt{\tilde{N}^2 - \lambda_1^2}K_0(\sqrt{\gamma^+})e^{-i(x-\xi)\lambda_1}d\lambda_1$$

$$-\int_{-\tilde{N}}^{-\beta^-}\sqrt{\tilde{N}^2 - \lambda_1^2}K_0(\sqrt{\gamma^-})e^{-i(x-\xi)\lambda_1}d\lambda_1$$

$$+\int_{-\beta^+}^{0}\sqrt{\tilde{N}^2 - \lambda_1^2}\cdot -\frac{\pi}{2}\left[Y_0(\sqrt{-\gamma^+}) + iJ_0(\sqrt{-\gamma^+})\right]e^{-i(x-\xi)\lambda_1}d\lambda_1$$

$$-\int_{-\beta^-}^{0} \sqrt{\tilde{N}^2-\lambda_1^2}\cdot -\frac{\pi}{2}\left[Y_0(\sqrt{-\gamma^-})+iJ_0(\sqrt{-\gamma^-})\right]e^{-i(x-\xi)\lambda_1}d\lambda_1$$

$$+\int_{0}^{\beta^+} \sqrt{\tilde{N}^2-\lambda_1^2}\cdot -\frac{\pi}{2}\left[Y_0(\sqrt{-\gamma^+})-iJ_0(\sqrt{-\gamma^+})\right]e^{-i(x-\xi)\lambda_1}d\lambda_1$$

$$-\int_{0}^{\beta^-} \sqrt{\tilde{N}^2-\lambda_1^2}\cdot -\frac{\pi}{2}\left[Y_0(\sqrt{-\gamma^-})-iJ_0(\sqrt{-\gamma^-})\right]e^{-i(x-\xi)\lambda_1}d\lambda_1$$

$$+\int_{\beta+}^{\tilde{N}} \sqrt{\tilde{N}^2-\lambda_1^2}K_0(\sqrt{\gamma^+})e^{-i(x-\xi)\lambda_1}d\lambda_1$$

$$-\int_{\beta-}^{\tilde{N}} \sqrt{\tilde{N}^2-\lambda_1^2}K_0(\sqrt{\gamma^-})e^{-i(x-\xi)\lambda_1}d\lambda_1$$

$$+\int_{\tilde{N}}^{\infty} i\sqrt{\lambda_1^2-\tilde{N}^2}\left[K_0(\sqrt{\gamma^+})-K_0(\sqrt{\gamma^-})\right]e^{-i(x-\xi)\lambda_1}d\lambda_1 \qquad (97)$$

Simplifications can be applied to h_2 through an examination of the odd and even behaviour of the integrands. That is

$$h_2=\frac{1}{2\pi^2}\int_{\tilde{N}}^{\infty} \sqrt{\lambda_1^2-\tilde{N}^2}\left[K_0(\sqrt{\gamma^+})-K_0(\sqrt{\gamma^-})\right]\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{1}{2\pi^2}\int_{\beta+}^{\tilde{N}} \sqrt{\tilde{N}^2-\lambda_1^2}K_0(\sqrt{\gamma^+})\cos r_1\lambda_1 d\lambda_1$$

$$-\frac{1}{2\pi^2}\int_{\beta-}^{\tilde{N}} \sqrt{\tilde{N}^2-\lambda_1^2}K_0(\sqrt{\gamma^-})\cos r_1\lambda_1 d\lambda_1$$

$$-\frac{1}{4\pi}\int_{0}^{\beta^+} \sqrt{\tilde{N}^2-\lambda_1^2}Y_0(\sqrt{-\gamma^+})\cos r_1\lambda_1 d\lambda_1$$

$$+\frac{1}{4\pi}\int_{0}^{\beta^+} \sqrt{\tilde{N}^2-\lambda_1^2}J_0(\sqrt{-\gamma^+})\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{1}{4\pi}\int_{0}^{\beta^-} \sqrt{\tilde{N}^2-\lambda_1^2}Y_0(\sqrt{-\gamma^-})\cos r_1\lambda_1 d\lambda_1$$

$$-\frac{1}{4\pi}\int_{0}^{\beta^-} \sqrt{\tilde{N}^2-\lambda_1^2}J_0(\sqrt{-\gamma^-})\sin r_1\lambda_1 d\lambda_1 \qquad (98)$$

Equation 96 can be modified using the appropriate formulation of $G(\lambda_1, \lambda_2, z;\ \boldsymbol{\xi}\ )$ and when applied into the derivation of $\bar{h}_1$ we obtain

$$-(2\pi)^2\bar{h}_1 = \int_{-\infty}^{\infty} -i\frac{\lambda_1}{\gamma_3}e^{-i(x-\xi)\lambda_1}\left[K_0\left(|\lambda_1|\sqrt{r_2^2+(z+\zeta)^2\gamma_3^2}\right)+K_0\left(|\lambda_1|\sqrt{r_2^2+(z-\zeta)^2\gamma_3^2}\right)\right]d\lambda_1,$$

The introduction of $|\lambda_1|$ into the numerator and denominator allows the application of the integrand properties expressed in table 1. That is

$$-(2\pi)^2\bar{h}_1 = \int_{-\infty}^{\infty} -i\frac{|\lambda_1|\lambda_1}{|\lambda_1|\gamma_3}e^{-i(x-\xi)\lambda_1}\left[K_0\left(|\lambda_1|\sqrt{r_2^2+(z+\zeta)^2\gamma_3^2}\right)+K_0\left(|\lambda_1|\sqrt{r_2^2+(z-\zeta)^2\gamma_3^2}\right)\right]d\lambda_1$$

$$= \int_{-\infty}^{-\tilde{N}} -i\frac{|\lambda_1|\lambda_1}{\sqrt{\lambda_1^2-\tilde{N}^2}}e^{-i(x-\xi)\lambda_1}\left[K_0\left(\sqrt{\gamma^+}\right)+K_0\left(\sqrt{\gamma^-}\right)\right]d\lambda_1$$

$$+\int_{-\tilde{N}}^{-\beta^+} -i\frac{|\lambda_1|\lambda_1}{i\sqrt{\tilde{N}^2-\lambda_1^2}}e^{-i(x-\xi)\lambda_1}K_0\left(\sqrt{\gamma^+}\right)d\lambda_1$$

$$+\int_{-\tilde{N}}^{-\beta^-} -i\frac{|\lambda_1|\lambda_1}{i\sqrt{\tilde{N}^2-\lambda_1^2}}e^{-i(x-\xi)\lambda_1}K_0\left(\sqrt{\gamma^-}\right)d\lambda_1$$

$$+\int_{-\beta^+}^{0} -i\frac{|\lambda_1|\lambda_1}{i\sqrt{\tilde{N}^2-\lambda_1^2}}e^{-i(x-\xi)\lambda_1}\cdot -\frac{\pi}{2}\left[Y_0\left(\sqrt{-\gamma^+}\right)+iJ_0\left(\sqrt{-\gamma^+}\right)\right]d\lambda_1$$

$$+\int_{-\beta^-}^{0} -i\frac{|\lambda_1|\lambda_1}{i\sqrt{\tilde{N}^2-\lambda_1^2}}e^{-i(x-\xi)\lambda_1}\cdot -\frac{\pi}{2}\left[Y_0\left(\sqrt{-\gamma^-}\right)+iJ_0\left(\sqrt{-\gamma^-}\right)\right]d\lambda_1$$

$$+\int_{0}^{\beta^+} -i\frac{|\lambda_1|\lambda_1}{-i\sqrt{\tilde{N}^2-\lambda_1^2}}e^{-i(x-\xi)\lambda_1}\cdot -\frac{\pi}{2}\left[Y_0\left(\sqrt{-\gamma^+}\right)-iJ_0\left(\sqrt{-\gamma^+}\right)\right]d\lambda_1$$

$$+\int_{0}^{\beta^-} -i\frac{|\lambda_1|\lambda_1}{-i\sqrt{\tilde{N}^2-\lambda_1^2}}e^{-i(x-\xi)\lambda_1}\cdot -\frac{\pi}{2}\left[Y_0\left(\sqrt{-\gamma^-}\right)-iJ_0\left(\sqrt{-\gamma^-}\right)\right]d\lambda_1$$

$$+\int_{\beta^+}^{\tilde{N}} -i\frac{|\lambda_1|\lambda_1}{-i\sqrt{\tilde{N}^2-\lambda_1^2}}e^{-i(x-\xi)\lambda_1}K_0\left(\sqrt{\gamma^+}\right)d\lambda_1$$

$$+\int_{\beta^-}^{\tilde{N}} -i\frac{|\lambda_1|\lambda_1}{-i\sqrt{\tilde{N}^2-\lambda_1^2}}e^{-i(x-\xi)\lambda_1}K_0\left(\sqrt{\gamma^-}\right)d\lambda_1$$

$$+\int_{\tilde{N}}^{\infty} -i\frac{|\lambda_1|\lambda_1}{\sqrt{\lambda_1^2-\tilde{N}^2}}e^{-i(x-\xi)\lambda_1}\left[K_0\left(\sqrt{\gamma^+}\right)+K_0\left(\sqrt{\gamma^-}\right)\right]d\lambda_1$$

Rearrangement of the terms gives

$$-2\pi^2\bar{h}_1 = -\int_{\tilde{N}}^{\infty} \frac{\lambda_1^2}{\sqrt{\lambda_1^2-\tilde{N}^2}}\left[K_0\left(\sqrt{\gamma^+}\right)+K_0\left(\sqrt{\gamma^-}\right)\right]\sin r_1\lambda_1 d\lambda_1$$

$$+\int_{\beta^+}^{\tilde{N}} \frac{\lambda_1^2}{\sqrt{\tilde{N}^2-\lambda_1^2}}K_0\left(\sqrt{\gamma^+}\right)\cos r_1\lambda_1 d\lambda_1$$

$$+\int_{\beta^-}^{\tilde{N}} \frac{\lambda_1^2}{\sqrt{\tilde{N}^2-\lambda_1^2}}K_0\left(\sqrt{\gamma^-}\right)d\lambda_1\cos r_1\lambda_1$$

$$-\frac{\pi}{2}\int_0^{\beta^+} \frac{\lambda_1^2}{\sqrt{\tilde{N}^2-\lambda_1^2}}Y_0\left(\sqrt{-\gamma^+}\right)\cos r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta^+} \frac{\lambda_1^2}{\sqrt{\tilde{N}^2-\lambda_1^2}}J_0\left(\sqrt{-\gamma^+}\right)\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{\pi}{2}\int_0^{\beta^-} \frac{\lambda_1^2}{\sqrt{\tilde{N}^2-\lambda_1^2}}Y_0\left(\sqrt{-\gamma^-}\right)\cos r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta^-} \frac{\lambda_1^2}{\sqrt{\tilde{N}^2-\lambda_1^2}}J_0\left(\sqrt{-\gamma^-}\right)\sin r_1\lambda_1 d\lambda_1$$

The function h_1 is the partial differential of $\bar{h}_1$ with respect to z. Noting,

$$\frac{\partial}{\partial z}\left[Y_0(\sqrt{-\gamma^\pm})\right] = \frac{\mp(\tilde{N}^2-\lambda_1^2)(\zeta\pm z)}{\sqrt{-\gamma^\pm}}Y_1(\sqrt{-\gamma^\pm})$$

it follows that

$$-2\pi^2 h_1 = \int_{\tilde{N}}^{\infty} \lambda_1^2\sqrt{\lambda_1^2-\tilde{N}^2}\left[\frac{(\zeta+z)}{\sqrt{\gamma^+}}K_1\left(\sqrt{\gamma^+}\right)-\frac{(\zeta-z)}{\sqrt{\gamma^-}}K_1\left(\sqrt{\gamma^-}\right)\right]\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\tilde{N}r_2^2}{[r_2^2+(\zeta+z)^2]^{\frac{3}{2}}}\frac{(\beta^+)^2}{\sqrt{\tilde{N}^2-(\beta^+)^2}}K_0\left(\sqrt{\gamma^+|_{\lambda_1=\beta^+}}\right)\cos r_1\beta^+$$

$$+\int_{\beta^+}^{\tilde{N}} \frac{\lambda_1^2(\zeta+z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{\gamma^+}}K_1\left(\sqrt{\gamma^+}\right)\cos r_1\lambda_1 d\lambda_1$$

$$-\frac{\tilde{N}r_2^2}{[r_2^2+(\zeta-z)^2]^{\frac{3}{2}}}\frac{(\beta^-)^2}{\sqrt{\tilde{N}^2-(\beta^-)^2}}K_0\left(\sqrt{\gamma^-|_{\lambda_1=\beta^-}}\right)\cos r_1\beta^-$$

$$-\int_{\beta^-}^{\tilde{N}} \frac{\lambda_1^2(\zeta - z)\sqrt{\tilde{N}^2 - \lambda_1^2}}{\sqrt{\gamma^-}} K_1\left(\sqrt{\gamma^-}\right) \cos r_1\lambda_1 d\lambda_1$$

$$+\frac{\tilde{N} r_2^2 \pi}{2[r_2^2 + (\zeta + z)^2]^{\frac{3}{2}}} \frac{(\beta^+)^2}{\sqrt{\tilde{N}^2 - (\beta^+)^2}} Y_0\left(\sqrt{-\gamma^+|_{\lambda_1=\beta^+}}\right) \cos r_1\beta^+$$

$$+\frac{\pi}{2}\int_0^{\beta^+} \frac{\lambda_1^2(\zeta + z)\sqrt{\tilde{N}^2 - \lambda_1^2}}{\sqrt{-\gamma^+}} Y_1\left(\sqrt{-\gamma^+}\right) \cos r_1\lambda_1 d\lambda_1$$

$$-\frac{\tilde{N} r_2^2 \pi}{2[r_2^2 + (\zeta - z)^2]^{\frac{3}{2}}} \frac{(\beta^-)^2}{\sqrt{\tilde{N}^2 - (\beta^-)^2}} Y_0\left(\sqrt{-\gamma^-|_{\lambda_1=\beta^-}}\right) \cos r_1\beta^-$$

$$-\frac{\pi}{2}\int_0^{\beta^-} \frac{\lambda_1^2(\zeta - z)\sqrt{\tilde{N}^2 - \lambda_1^2}}{\sqrt{-\gamma^-}} Y_1\left(\sqrt{-\gamma^-}\right) \cos r_1\lambda_1 d\lambda_1$$

$$-\frac{\tilde{N} r_2^2 \pi}{2[r_2^2 + (\zeta + z)^2]^{\frac{3}{2}}} \frac{(\beta^+)^2}{\sqrt{\tilde{N}^2 - (\beta^+)^2}} J_0\left(\sqrt{-\gamma^+|_{\lambda_1=\beta^+}}\right) \sin r_1\beta^+$$

$$-\frac{\pi}{2}\int_0^{\beta^+} \frac{\lambda_1^2(\zeta + z)\sqrt{\tilde{N}^2 - \lambda_1^2}}{\sqrt{-\gamma^+}} J_1\left(\sqrt{-\gamma^+}\right) \sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\tilde{N} r_2^2 \pi}{2[r_2^2 + (\zeta - z)^2]^{\frac{3}{2}}} \frac{(\beta^-)^2}{\sqrt{\tilde{N}^2 - (\beta^-)^2}} J_0\left(\sqrt{-\gamma^-|_{\lambda_1=\beta^-}}\right) \sin r_1\beta^+$$

$$+\frac{\pi}{2}\int_0^{\beta^-} \frac{\lambda_1^2(\zeta - z)\sqrt{\tilde{N}^2 - \lambda_1^2}}{\sqrt{-\gamma^-}} J_1\left(\sqrt{-\gamma^-}\right) \sin r_1\lambda_1 d\lambda_1$$

The terms $K_0\left(\sqrt{\gamma^\pm|_{\lambda_1=\beta^\pm}}\right)$ and $Y_0\left(\sqrt{-\gamma^\pm|_{\lambda_1=\beta^\pm}}\right)$ are undefined as $\gamma^\pm|_{\lambda_1=\beta^\pm} = 0$. However combining the two terms produces the finite result

$$K_0\left(\sqrt{\gamma^\pm|_{\lambda_1=\beta^\pm}}\right) + \frac{\pi}{2} Y_0\left(\sqrt{-\gamma^\pm|_{\lambda_1=\beta^\pm}}\right) = 0.$$

Utilising other previous identities, we find that h_1 becomes

$$h_1 = -\frac{1}{2\pi^2}\int_{\tilde{N}}^{\infty} \lambda_1^2\sqrt{\lambda_1^2 - \tilde{N}^2}\left[\frac{(\zeta + z)}{\sqrt{\gamma^+}} K_1\left(\sqrt{\gamma^+}\right) - \frac{(\zeta - z)}{\sqrt{\gamma^-}} K_1\left(\sqrt{\gamma^-}\right)\right] \sin r_1\lambda_1 d\lambda_1$$

$$-\frac{1}{2\pi^2}\int_{\beta^+}^{\tilde{N}} \frac{\lambda_1^2(\zeta + z)\sqrt{\tilde{N}^2 - \lambda_1^2}}{\sqrt{\gamma^+}} K_1\left(\sqrt{\gamma^+}\right) \cos r_1\lambda_1 d\lambda_1$$

$$+\frac{1}{2\pi^2}\int_{\beta^-}^{\tilde{N}}\frac{\lambda_1^2(\zeta-z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{\gamma^-}}K_1\left(\sqrt{\gamma^-}\right)\cos r_1\lambda_1 d\lambda_1$$

$$-\frac{1}{4\pi}\int_0^{\beta^+}\frac{\lambda_1^2(\zeta+z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^+}}Y_1\left(\sqrt{-\gamma^+}\right)\cos r_1\lambda_1 d\lambda_1$$

$$+\frac{1}{4\pi}\int_0^{\beta^-}\frac{\lambda_1^2(\zeta-z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^-}}Y_1\left(\sqrt{-\gamma^-}\right)\cos r_1\lambda_1 d\lambda_1$$

$$+\frac{\tilde{N}(\zeta+z)^2}{4\pi[r_2^2+(\zeta+z)^2]^{\frac{3}{2}}}\sin r_1\beta^+$$

$$+\frac{1}{4\pi}\int_0^{\beta^+}\frac{\lambda_1^2(\zeta+z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^+}}J_1\left(\sqrt{-\gamma^+}\right)\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{\tilde{N}(\zeta-z)^2}{4\pi[r_2^2+(\zeta-z)^2]^{\frac{3}{2}}}\sin r_1\beta^-$$

$$-\frac{1}{4\pi}\int_0^{\beta^-}\frac{\lambda_1^2(\zeta-z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^-}}J_1\left(\sqrt{-\gamma^-}\right)\sin r_1\lambda_1 d\lambda_1$$

In a similar manner f_2 can be derived. Namely,

$$-(2\pi)^2\bar{f}_2=\int_{-\infty}^{\infty}i\frac{1}{(\lambda_1-i\epsilon)\gamma_3}e^{-i(x-\xi)\lambda_1}\left[K_0\left(|\lambda_1|\sqrt{r_2^2+(z+\zeta)^2\gamma_3^2}\right)+K_0\left(|\lambda_1|\sqrt{r_2^2+(z-\zeta)^2\gamma_3^2}\right)\right]d\lambda_1$$

$$-2\pi^2\bar{f}_2=\int_{\tilde{N}}^{\infty}\frac{1}{\sqrt{\lambda_1^2-\tilde{N}^2}}\left[K_0(\sqrt{\gamma^+})+K_0(\sqrt{\gamma^-})\right]\sin r_1\lambda_1 d\lambda_1$$

$$-\int_{\beta^+}^{\tilde{N}}\frac{1}{\sqrt{\tilde{N}^2-\lambda_1^2}}K_0(\sqrt{\gamma^+})\cos r_1\lambda_1 d\lambda_1$$

$$-\int_{\beta^-}^{\tilde{N}}\frac{1}{\sqrt{\tilde{N}^2-\lambda_1^2}}K_0(\sqrt{\gamma^-})\cos r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta^+}\frac{1}{\sqrt{\tilde{N}^2-\lambda_1^2}}Y_0(\sqrt{-\gamma^+})\cos r_1\lambda_1 d\lambda_1$$

$$-\frac{\pi}{2}\int_0^{\beta^+}\frac{1}{\sqrt{\tilde{N}^2-\lambda_1^2}}J_0(\sqrt{-\gamma^+})\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta^-}\frac{1}{\sqrt{\tilde{N}^2-\lambda_1^2}}Y_0(\sqrt{-\gamma^-})\cos r_1\lambda_1 d\lambda_1$$

$$-\frac{\pi}{2}\int_0^{\beta^-}\frac{1}{\sqrt{\tilde{N}^2-\lambda_1^2}}J_0(\sqrt{-\gamma^-})\sin r_1\lambda_1 d\lambda_1$$

The differentiation of $\bar{f}_2$ with respect to z allows f_2 to be determined and this has the form

$$-2\pi^2 f_2 = -\int_{\tilde{N}}^{\infty}\sqrt{\lambda_1^2-\tilde{N}^2}\left[\frac{(\zeta+z)}{\sqrt{\gamma^+}}K_1(\sqrt{\gamma^+})-\frac{(\zeta-z)}{\sqrt{\gamma^-}}K_1(\sqrt{\gamma^-})\right]\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{\tilde{N}r_2^2}{[r_2^2+(\zeta+z)^2]^{\frac{3}{2}}}\frac{K_0(\sqrt{\gamma^+|_{\lambda_1=\beta^+}})}{\sqrt{\tilde{N}^2-(\beta^+)^2}}\cos r_1\beta^+$$

$$-\int_{\beta^+}^{\tilde{N}}\frac{(\zeta+z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{\gamma^+}}K_1(\sqrt{\gamma^+})\cos r_1\lambda_1 d\lambda_1$$

$$+\frac{\tilde{N}r_2^2}{[r_2^2+(\zeta-z)^2]^{\frac{3}{2}}}\frac{K_0(\sqrt{\gamma^-|_{\lambda_1=\beta^-}})}{\sqrt{\tilde{N}^2-(\beta^-)^2}}\cos r_1\beta^-$$

$$+\int_{\beta^-}^{\tilde{N}}\frac{(\zeta-z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{\gamma^-}}K_1(\sqrt{\gamma^-})\cos r_1\lambda_1 d\lambda_1$$

$$-\frac{\tilde{N}r_2^2\pi}{2[r_2^2+(\zeta+z)^2]^{\frac{3}{2}}}\frac{Y_0(\sqrt{-\gamma^+|_{\lambda_1=\beta^+}})}{\sqrt{\tilde{N}^2-(\beta^+)^2}}\cos r_1\beta^+$$

$$-\frac{\pi}{2}\int_0^{\beta^+}\frac{(\zeta+z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^+}}Y_1(\sqrt{-\gamma^+})\cos r_1\lambda_1 d\lambda_1$$

$$+\frac{\tilde{N}r_2^2\pi}{2[r_2^2+(\zeta+z)^2]^{\frac{3}{2}}}\frac{J_0(\sqrt{-\gamma^+|_{\lambda_1=\beta^+}})}{\sqrt{\tilde{N}^2-(\beta^+)^2}}\sin r_1\beta^+$$

$$+\frac{\pi}{2}\int_0^{\beta^+}\frac{(\zeta+z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^+}}J_1(\sqrt{-\gamma^+})\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\tilde{N}r_2^2\pi}{2[r_2^2+(\zeta-z)^2]^{\frac{3}{2}}}\frac{Y_0(\sqrt{-\gamma^-}|_{\lambda_1=\beta^-})}{\sqrt{\tilde{N}^2-(\beta^-)^2}}\cos r_1\beta^-$$

$$+\frac{\pi}{2}\int_0^{\beta^-}\frac{(\zeta-z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^-}}Y_1(\sqrt{-\gamma^-})\cos r_1\lambda_1 d\lambda_1$$

$$-\frac{\tilde{N}r_2^2\pi}{2[r_2^2+(\zeta-z)^2]^{\frac{3}{2}}}\frac{J_0(\sqrt{-\gamma^-}|_{\lambda_1=\beta^-})}{\sqrt{\tilde{N}^2-(\beta^-)^2}}\sin r_1\beta^-$$

$$-\frac{\pi}{2}\int_0^{\beta^-}\frac{(\zeta-z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^-}}J_1(\sqrt{-\gamma^-})\sin r_1\lambda_1 d\lambda_1$$

or, after rearranging like terms, we find

$$f_2=\frac{1}{2\pi^2}\int_{\tilde{N}}^{\infty}\sqrt{\lambda_1^2-\tilde{N}^2}\left[\frac{(\zeta+z)}{\sqrt{\gamma^+}}K_1(\sqrt{\gamma^+})-\frac{(\zeta-z)}{\sqrt{\gamma^-}}K_1(\sqrt{\gamma^-})\right]\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{1}{2\pi^2}\int_{\beta^+}^{\tilde{N}}\frac{(\zeta+z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{\gamma^+}}K_1(\sqrt{\gamma^+})\cos r_1\lambda_1 d\lambda_1$$

$$-\frac{1}{2\pi^2}\int_{\beta^-}^{\tilde{N}}\frac{(\zeta-z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{\gamma^-}}K_1(\sqrt{\gamma^-})\cos r_1\lambda_1 d\lambda_1$$

$$+\frac{1}{4\pi}\int_0^{\beta^+}\frac{(\zeta+z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^+}}Y_1(\sqrt{-\gamma^+})\cos r_1\lambda_1 d\lambda_1$$

$$-\frac{|r_2|}{4\pi[r_2^2+(\zeta+z)^2]}\sin r_1\beta^+$$

$$-\frac{1}{4\pi}\int_0^{\beta^+}\frac{(\zeta+z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^+}}J_1(\sqrt{-\gamma^+})\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{1}{4\pi}\int_0^{\beta^-}\frac{(\zeta-z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^-}}Y_1(\sqrt{-\gamma^-})\cos r_1\lambda_1 d\lambda_1$$

$$+\frac{|r_2|}{4\pi[r_2^2+(\zeta-z)^2]}\sin r_1\beta^-$$

$$+\frac{1}{4\pi}\int_0^{\beta^-}\frac{(\zeta-z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^-}}J_1(\sqrt{-\gamma^-})\sin r_1\lambda_1 d\lambda_1$$

Using the derivation of f_1 detailed in appendix A and the identity

$$K_1(\pm ix)=-\tfrac{\pi}{2}\left[J_1(x)\mp iY_1(x)\right]\quad x>0$$

(see Abramowitz and Stegun(1970) 9.6.4, $\nu=1$) the asymmetric form of f_1 can be written as

$$(2\pi)^2 f_1=\int_{-\infty}^{-\tilde{N}} i\frac{\lambda_1^2}{\gamma^+\sqrt{\lambda_1^2-\tilde{N}^2}}e^{-i(x-\xi)\lambda_1}\left[(\lambda_1^2-\tilde{N}^2)(\zeta+z)^2K_0(\sqrt{\gamma^+})-\frac{\sigma^+}{\sqrt{\gamma^+}}K_1(\sqrt{\gamma^+})\right]d\lambda_1$$

$$-\int_{-\infty}^{-\tilde{N}} i\frac{\lambda_1^2}{\gamma^-\sqrt{\lambda_1^2-\tilde{N}^2}}e^{-i(x-\xi)\lambda_1}\left[(\lambda_1^2-\tilde{N}^2)(\zeta-z)^2K_0(\sqrt{\gamma^-})-\frac{\sigma^-}{\sqrt{\gamma^-}}K_1(\sqrt{\gamma^-})\right]d\lambda_1$$

$$+\int_{-\tilde{N}}^{-\beta^+}\frac{\lambda_1^2}{\gamma^+\sqrt{\tilde{N}^2-\lambda_1^2}}e^{-i(x-\xi)\lambda_1}\left[(\lambda_1^2-\tilde{N}^2)(\zeta+z)^2K_0(\sqrt{\gamma^+})-\frac{\sigma^+}{\sqrt{\gamma^+}}K_1(\sqrt{\gamma^+})\right]d\lambda_1$$

$$-\int_{-\tilde{N}}^{-\beta^-}\frac{\lambda_1^2}{\gamma^-\sqrt{\tilde{N}^2-\lambda_1^2}}e^{-i(x-\xi)\lambda_1}\left[(\lambda_1^2-\tilde{N}^2)(\zeta-z)^2K_0(\sqrt{\gamma^-})-\frac{\sigma^-}{\sqrt{\gamma^-}}K_1(\sqrt{\gamma^-})\right]d\lambda_1$$

$$-\frac{\pi}{2}\int_{-\beta^+}^{0}\frac{\lambda_1^2e^{-i(x-\xi)\lambda_1}}{\gamma^+\sqrt{\tilde{N}^2-\lambda_1^2}}\left\{(\lambda_1^2-\tilde{N}^2)(\zeta+z)^2\left[Y_0(\sqrt{-\gamma^+})+iJ_0(\sqrt{-\gamma^+})\right]-\frac{\sigma^+}{\sqrt{-\gamma^+}}\left[-Y_1(\sqrt{-\gamma^+})-iJ_1(\sqrt{-\gamma^+})\right]\right\}d\lambda_1$$

$$+\frac{\pi}{2}\int_{-\beta^-}^{0}\frac{\lambda_1^2e^{-i(x-\xi)\lambda_1}}{\gamma^-\sqrt{\tilde{N}^2-\lambda_1^2}}\left\{(\lambda_1^2-\tilde{N}^2)(\zeta-z)^2\left[Y_0(\sqrt{-\gamma^-})+iJ_0(\sqrt{-\gamma^-})\right]-\frac{\sigma^-}{\sqrt{-\gamma^-}}\left[-Y_1(\sqrt{-\gamma^-})-iJ_1(\sqrt{-\gamma^-})\right]\right\}d\lambda_1$$

$$-\frac{\pi}{2}\int_{0}^{\beta^+}\frac{\lambda_1^2e^{-i(x-\xi)\lambda_1}}{\gamma^+\sqrt{\tilde{N}^2-\lambda_1^2}}\left\{(\lambda_1^2-\tilde{N}^2)(\zeta+z)^2\left[Y_0(\sqrt{-\gamma^+})-iJ_0(\sqrt{-\gamma^+})\right]-\frac{\sigma^+}{\sqrt{-\gamma^+}}\left[-Y_1(\sqrt{-\gamma^+})+iJ_1(\sqrt{-\gamma^+})\right]\right\}d\lambda_1$$

$$+\frac{\pi}{2}\int_{0}^{\beta^-}\frac{\lambda_1^2e^{-i(x-\xi)\lambda_1}}{\gamma^-\sqrt{\tilde{N}^2-\lambda_1^2}}\left\{(\lambda_1^2-\tilde{N}^2)(\zeta-z)^2\left[Y_0(\sqrt{-\gamma^-})-iJ_0(\sqrt{-\gamma^-})\right]-\frac{\sigma^-}{\sqrt{-\gamma^-}}\left[-Y_1(\sqrt{-\gamma^-})+iJ_1(\sqrt{-\gamma^-})\right]\right\}d\lambda_1$$

$$+\int_{\beta^+}^{\tilde{N}}\frac{\lambda_1^2}{\gamma^+\sqrt{\tilde{N}^2-\lambda_1^2}}e^{-i(x-\xi)\lambda_1}\left[(\lambda_1^2-\tilde{N}^2)(\zeta+z)^2K_0(\sqrt{\gamma^+})-\frac{\sigma^+}{\sqrt{\gamma^+}}K_1(\sqrt{\gamma^+})\right]d\lambda_1$$

$$-\int_{\beta^-}^{\tilde{N}}\frac{\lambda_1^2}{\gamma^-\sqrt{\tilde{N}^2-\lambda_1^2}}e^{-i(x-\xi)\lambda_1}\left[(\lambda_1^2-\tilde{N}^2)(\zeta-z)^2K_0(\sqrt{\gamma^-})-\frac{\sigma^-}{\sqrt{\gamma^-}}K_1(\sqrt{\gamma^-})\right]d\lambda_1$$

$$-\int_{\tilde{N}}^{\infty} i\frac{\lambda_1^2}{\gamma^+\sqrt{\lambda_1^2-\tilde{N}^2}}e^{-i(x-\xi)\lambda_1}\left[(\lambda_1^2-\tilde{N}^2)(\zeta+z)^2K_0(\sqrt{\gamma^+})-\frac{\sigma^+}{\sqrt{\gamma^+}}K_1(\sqrt{\gamma^+})\right]d\lambda_1$$

$$+\int_{\tilde{N}}^{\infty} i\frac{\lambda_1^2}{\gamma^-\sqrt{\lambda_1^2-\tilde{N}^2}}e^{-i(x-\xi)\lambda_1}\left[(\lambda_1^2-\tilde{N}^2)(\zeta-z)^2K_0(\sqrt{\gamma^-})-\frac{\sigma^-}{\sqrt{\gamma^-}}K_1(\sqrt{\gamma^-})\right]d\lambda_1$$

or, after rearranging, the expression can be written as

$$f_1=-\frac{1}{2\pi^2}\int_{\tilde{N}}^{\infty}\frac{\lambda_1^2}{\gamma^+\sqrt{\lambda_1^2-\tilde{N}^2}}\left[(\lambda_1^2-\tilde{N}^2)(\zeta+z)^2K_0(\sqrt{\gamma^+})-\frac{\sigma^+}{\sqrt{\gamma^+}}K_1(\sqrt{\gamma^+})\right]\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{1}{2\pi^2}\int_{\tilde{N}}^{\infty}\frac{\lambda_1^2}{\gamma^-\sqrt{\lambda_1^2-\tilde{N}^2}}\left[(\lambda_1^2-\tilde{N}^2)(\zeta-z)^2K_0(\sqrt{\gamma^-})-\frac{\sigma^-}{\sqrt{\gamma^-}}K_1(\sqrt{\gamma^-})\right]\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{1}{2\pi^2}\int_{\beta^+}^{\tilde{N}}\frac{\lambda_1^2}{\gamma^+\sqrt{\tilde{N}^2-\lambda_1^2}}\left[(\lambda_1^2-\tilde{N}^2)(\zeta+z)^2K_0(\sqrt{\gamma^+})-\frac{\sigma^+}{\sqrt{\gamma^+}}K_1(\sqrt{\gamma^+})\right]\cos r_1\lambda_1 d\lambda_1$$

$$-\frac{1}{2\pi^2}\int_{\beta^-}^{\tilde{N}}\frac{\lambda_1^2}{\gamma^-\sqrt{\tilde{N}^2-\lambda_1^2}}\left[(\lambda_1^2-\tilde{N}^2)(\zeta-z)^2K_0(\sqrt{\gamma^-})-\frac{\sigma^-}{\sqrt{\gamma^-}}K_1(\sqrt{\gamma^-})\right]\cos r_1\lambda_1 d\lambda_1$$

$$+\frac{1}{4\pi}\int_{0}^{\beta^+}\frac{\lambda_1^2}{\gamma^+\sqrt{\tilde{N}^2-\lambda_1^2}}\left[(\tilde{N}^2-\lambda_1^2)(\zeta+z)^2Y_0(\sqrt{-\gamma^+})-\frac{\sigma^+}{\sqrt{-\gamma^+}}Y_1(\sqrt{-\gamma^+})\right]\cos r_1\lambda_1 d\lambda_1$$

$$-\frac{1}{4\pi}\int_{0}^{\beta^+}\frac{\lambda_1^2}{\gamma^+\sqrt{\tilde{N}^2-\lambda_1^2}}\left[(\tilde{N}^2-\lambda_1^2)(\zeta+z)^2J_0(\sqrt{-\gamma^+})-\frac{\sigma^+}{\sqrt{-\gamma^+}}J_1(\sqrt{-\gamma^+})\right]\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{1}{4\pi}\int_{0}^{\beta^-}\frac{\lambda_1^2}{\gamma^+\sqrt{\tilde{N}^2-\lambda_1^2}}\left[(\tilde{N}^2-\lambda_1^2)(\zeta-z)^2Y_0(\sqrt{-\gamma^-})-\frac{\sigma^-}{\sqrt{-\gamma^-}}Y_1(\sqrt{-\gamma^-})\right]\cos r_1\lambda_1 d\lambda_1$$

$$+\frac{1}{4\pi}\int_{0}^{\beta^-}\frac{\lambda_1^2}{\gamma^-\sqrt{\tilde{N}^2-\lambda_1^2}}\left[(\tilde{N}^2-\lambda_1^2)(\zeta-z)^2J_0(\sqrt{-\gamma^-})-\frac{\sigma^-}{\sqrt{-\gamma^-}}J_1(\sqrt{-\gamma^-})\right]\sin r_1\lambda_1 d\lambda_1$$