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by

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CONTENTS

1 Introduction

Classical laminated plate theory(CLPT) does not consider the influence of transverse deformation. The theory assumes that shear strains $\gamma_{xz} = \gamma_{yz} = \mathbf{0}$. However, plates or shells made of FRP(Fibre Reinforced Plastics) composite materials are much weaker in their transverse directions. They are susceptible to failures through the thickness because their effective shear moduli $\boldsymbol{G}_{xz}$ and $\boldsymbol{G}_{yz}$ are significantly smaller than their Young's moduli $\boldsymbol{E}_x$ and $\boldsymbol{E}_y$. Thus, CLPT by itself is inadequate for FRP composites.

In the available finite element programs, for example, ANSYS and NISA, the most widely used displacement based theory is the first-order shear deformation theory [1] to account for the influence of the transverse deformation. In this theory shear deformation is constant through the thickness of the plate. This leads to an anomaly in that the top and bottom surfaces, although not subjected to shear, still are shown to have shear strain values. In order to correct this anomaly, various factors are introduced into the first-order theory leading to higher-order shear deformation theories. In this paper Reddy's higher-order shear deformation theory [2] [3] will be used to derive the stiffness matrix of an 8-node laminated plate element.

2 Stress-Strain Relation

Consider a laminated plate element of N layers with thickness h, length $2a$, and width $2b$, as shown in *Fig.1* . Each layer is taken to be macroscopically homogeneous and orthotropic.

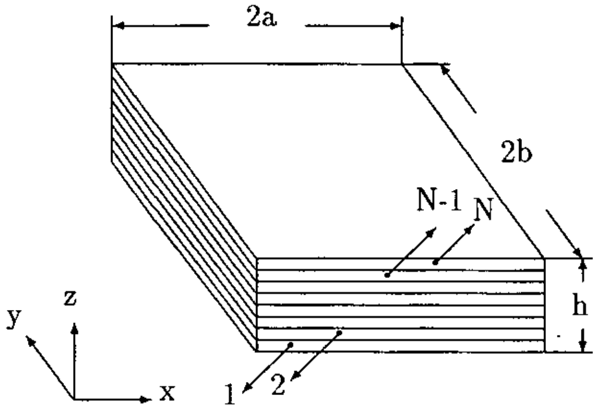

Figure 1

Based on the *Duhamel − Neumann* law, the stress-strain relation of the kth layer is

$$\left\{ \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{array} \right\} = \left[\begin{array}{cccccc} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_{66} \end{array} \right] \left\{ \begin{array}{c} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{array} \right\} \quad (1)$$

where

$$Q_{11} = E_1/(1 - \nu_{12}\nu_{21})$$

$$
\begin{aligned}
Q_{12} &= \nu_{12}E_2/(1-\nu_{12}\nu_{21}) = \nu_{21}E_1/(1-\nu_{12}\nu_{21}) = Q_{13} \\
Q_{22} &= E_2/(1-\nu_{12}\nu_{21}) \\
Q_{44} &= G_{23} \\
Q_{55} &= G_{12} = G_{13} = Q_{66} \\
Q_{23} &= \nu_{23}E_2/(1-\nu_{23}^2) = \nu_{32}E_3/(1-\nu_{32}^2)
\end{aligned} \tag{2}
$$

In the derivation of Eq. (1), the stresses and strains are defined in the principal material directions for that orthotropic lamina. However, in angle-ply laminated plates the principal directions of orthotropy of each individual lamina do not coincide with the geometrical coordinate frame. It is necessary to use the transformed reduced stiffness

$$
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{13} & 0 & 0 & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{23} & o & 0 & \overline{Q}_{26} \\ \overline{Q}_{13} & \overline{Q}_{23} & \overline{Q}_{33} & 0 & 0 & \overline{Q}_{36} \\ 0 & 0 & 0 & \overline{Q}_{44} & \overline{Q}_{45} & 0 \\ 0 & 0 & 0 & \overline{Q}_{45} & \overline{Q}_{55} & 0 \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{36} & 0 & 0 & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \tag{3}
$$

The thirteen constants $\overline{Q}_{ij}$ are related to the nine Q_{ij} through the following transformation formulae

$$
\begin{aligned}
\overline{Q}_{11} &= Q_{11}m^4 + 2(Q_{12}+2Q_{66})m^2n^2 + Q_{22}n^4 \\
\overline{Q}_{12} &= (Q_{11}+Q_{22}-4Q_{66})m^2n^2 + Q_{12}(m^4+n^4)
\end{aligned}
$$

$$
\begin{aligned}
\overline{Q}_{13} &= Q_{13}m^2 + Q_{23}n^2 \\
\overline{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66})m^3 n + (Q_{12} - Q_{22} + 2Q_{66})mn^3 \\
\overline{Q}_{22} &= Q_{11}n^4 + 2(Q_{12} + 2Q_{66})m^2 n^2 + Q_{22}m^4 \\
\overline{Q}_{23} &= Q_{13}n^2 + Q_{23}m^2 \\
\overline{Q}_{33} &= Q_{33} \\
\overline{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66})mn^3 + (Q_{12} - Q_{22} + 2Q_{66})m^3 n \\
\overline{Q}_{36} &= (Q_{13} - Q_{23})mn \\
\overline{Q}_{44} &= Q_{44}m^2 + Q_{55}n^2 \\
\overline{Q}_{45} &= (Q_{45} - Q_{44})mn \\
\overline{Q}_{55} &= Q_{55}m^2 + Q_{44}n^2 \\
\overline{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12})m^2 n^2 + Q_{66}(m^2 - n^2)^2
\end{aligned}
\tag{4}
$$

where

$$m = cos\theta_k \quad n = sin\theta_k \quad (see\ Fig.2) \tag{5}$$

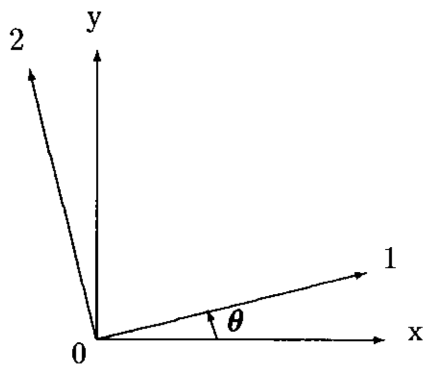

Figure 2

3 Reddy's Higher-Order Shear Deformation Theory

Generally speaking, the displacement components of a laminated plate are of the form

$$\begin{aligned} u(x,y,z) &= u_0(x,y) + \sum_i \phi_{1i} z^i \\ v(x,y,z) &= v_0(x,y) + \sum_i \phi_{2i} z^i \\ w(x,y,z) &= w_0(x,y) + \sum_i \phi_{3i} z^i \end{aligned} \tag{6}$$

If terms of higher order than z^3 are neglected

$$\begin{aligned} u(x,y,z) &= u_0(x,y) + z\phi_{11} + z^2\phi_{12} + z^3\phi_{13} \\ v(x,y,z) &= v_0(x,y) + z\phi_{21} + z^2\phi_{22} + z^3\phi_{23} \\ w(x,y,z) &= w_0(x,y) + z\phi_{31} + z^2\phi_{32} + z^3\phi_{33} \end{aligned} \tag{7}$$

Assuming that $\epsilon_z = 0$ then

$$\phi_{31} = \phi_{32} = \phi_{33} = 0 \tag{8}$$

For a symmetrical laminated plate, the assumption $\phi_{12} = \phi_{22} = 0$ is exactly satisfied. If the laminated plate is not symmetrical, we always approximately assume

$$\phi_{12} \approx \phi_{22} \approx 0 \tag{9}$$

If the shear stresses $\sigma_{xz}$and σ_{yz} in the top surface and bottom surface are zero, we have

$$\begin{aligned}\phi_{13} &= -\frac{4}{3h^2}(\phi_{11} + \frac{\partial w_0}{\partial x}) \\ \phi_{23} &= -\frac{4}{3h^2}(\phi_{21} + \frac{\partial w_0}{\partial y})\end{aligned} \tag{10}$$

Substituting Eqs. (8),(9),(10) into Eq.(7) we obtain the deformation components of Reddy's higher-order shear deformation theory

$$\begin{aligned}u(x,y,z) &= u_0(x,y) + z\phi_{11}(x,y) - \frac{4z^3}{3h^2}[\phi_{11}(x,y) + \frac{\partial w}{\partial x}] \\ v(x,y,z) &= v_0(x,y) + z\phi_{21}(x,y) - \frac{4z^3}{3h^2}[\phi_{21}(x,y) + \frac{\partial w}{\partial y}] \\ w(x,y,z) &= w_0(x,y)\end{aligned} \tag{11}$$

where u_0, v_0, w_0 are associated midplane displacements, and ϕ_{11} and ϕ_{21} are the rotations of the transverse normal in the xz and yz planes. The coordinate frame is chosen in such a way that the xy plane coincides with the midplane of plate.

Based on the strain-displacement equations of linear elasticity, we have

$$\begin{aligned}\epsilon_x &= \epsilon_x^0 + zk_x^1 - \frac{4z^3}{3h^2}\frac{\partial \gamma_{xz}^1}{\partial x} \\ \epsilon_y &= \epsilon_y^0 + zk_y^1 - \frac{4z^3}{3h^2}\frac{\partial \gamma_{yz}^1}{\partial y} \\ \epsilon_z &= 0 \\ \gamma_{xy} &= \gamma_{xy}^0 + zk_{xy}^1 - \frac{4z^3}{3h^2}(\frac{\partial \gamma_{yz}^1}{\partial x} + \frac{\partial \gamma_{xz}^1}{\partial y}) \\ \gamma_{xz} &= \gamma_{xz}^1(1 - \frac{4z^2}{h^2})\end{aligned} \tag{12}$$

$$\gamma_{yz} = \gamma_{yz}^1(1 - \frac{4z^2}{h^2})$$

where

$$\{\epsilon^0\} = \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix} \tag{13}$$

$$\{k^1\} = \begin{Bmatrix} k_x^1 \\ k_y^1 \\ k_{xy}^1 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \phi_{11}}{\partial x} \\ \frac{\partial \phi_{21}}{\partial y} \\ \frac{\partial \phi_{21}}{\partial x} + \frac{\partial \phi_{11}}{\partial y} \end{Bmatrix} \tag{14}$$

$$\{\gamma^1\} = \begin{Bmatrix} \gamma_{xz}^1 \\ \gamma_{yz}^1 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial w_0}{\partial x} + \phi_{11} \\ \frac{\partial w_0}{\partial y} + \phi_{21} \end{Bmatrix} \tag{15}$$

Substituting Eq.(12) into Eq.(3) ,one has the stresses expressed in terms of mid-plane strains $\{\epsilon^0\}$,and rotations of transverse normal $\{k^1\}$,and $\{\gamma^1\}$.

$$\begin{aligned}
\sigma_x &= \overline{Q}_{11}\epsilon_x^0 + \overline{Q}_{12}\epsilon_y^0 + \overline{Q}_{16}\gamma_{xy}^0 + \overline{Q}_{11}k_x^1 z + \overline{Q}_{12}k_y^1 z + \overline{Q}_{16}k_{xy}^1 z \\
&\quad -\frac{4}{3h^2}[(\overline{Q}_{11}\frac{\partial}{\partial x} + \overline{Q}_{16}\frac{\partial}{\partial y})\gamma_{xz}^1]z^3 - \frac{4}{3h^2}[(\overline{Q}_{12}\frac{\partial}{\partial y} + \overline{Q}_{16}\frac{\partial}{\partial x})\gamma_{yz}^1]z^3 \\
\sigma_y &= \overline{Q}_{12}\epsilon_x^0 + \overline{Q}_{22}\epsilon_y^0 + \overline{Q}_{26}\gamma_{xy}^0 + \overline{Q}_{12}k_x^1 z + \overline{Q}_{22}k_y^1 z + \overline{Q}_{26}k_{xy}^1 z \\
&\quad -\frac{4}{3h^2}[(\overline{Q}_{12}\frac{\partial}{\partial x} + \overline{Q}_{26}\frac{\partial}{\partial y})\gamma_{xz}^1]z^3 - \frac{4}{3h^2}[(\overline{Q}_{22}\frac{\partial}{\partial y} + \overline{Q}_{26}\frac{\partial}{\partial x})\gamma_{yz}^1]z^3 \\
\sigma_z &= \overline{Q}_{13}\epsilon_x^0 + \overline{Q}_{23}\epsilon_y^0 + \overline{Q}_{36}\gamma_{xy}^0 + \overline{Q}_{13}k_x^1 z + \overline{Q}_{23}k_y^1 z + \overline{Q}_{36}k_{xy}^1 z \\
&\quad -\frac{4}{3h^2}[(\overline{Q}_{13}\frac{\partial}{\partial x} + \overline{Q}_{36}\frac{\partial}{\partial y})\gamma_{xz}^1]z^3 - \frac{4}{3h^2}[(\overline{Q}_{23}\frac{\partial}{\partial y} + \overline{Q}_{36}\frac{\partial}{\partial x})\gamma_{yz}^1]z^3
\end{aligned}$$

$$
\begin{aligned}
\sigma_{yz} &= \overline{Q}_{44}\gamma^1_{yz}(1-\frac{4z^2}{h^2}) + \overline{Q}_{45}\gamma^1_{xz}(1-\frac{4z^2}{h^2}) \\
\sigma_{xz} &= \overline{Q}_{45}\gamma^1_{yz}(1-\frac{4z^2}{h^2}) + \overline{Q}_{55}\gamma^1_{xz}(1-\frac{4z^2}{h^2}) \\
\sigma_{xy} &= \overline{Q}_{16}\epsilon^0_x + \overline{Q}_{26}\epsilon^0_y + \overline{Q}_{66}\gamma^0_{xy} + \overline{Q}_{16}k^1_x z + \overline{Q}_{26}k^1_y z + \overline{Q}_{66}k^1_{xy} z \\
&\quad -\frac{4}{3h^2}[(\overline{Q}_{16}\frac{\partial}{\partial x} + \overline{Q}_{66}\frac{\partial}{\partial y})\gamma^1_{xz}]z^3 - \frac{4}{3h^2}[(\overline{Q}_{26}\frac{\partial}{\partial y} + \overline{Q}_{66}\frac{\partial}{\partial x})\gamma^1_{yz}]z^3
\end{aligned} \tag{16}
$$

The stress resultants and moment resultants are defined as

$$
\{N\} = \left\{ \begin{array}{c} N_x \\ N_y \\ N_{xy} \end{array} \right\} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{array} \right\} dz \tag{17}
$$

$$
\{Q\} = \left\{ \begin{array}{c} Q_x \\ Q_y \end{array} \right\} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \left\{ \begin{array}{c} \sigma_{xz} \\ \sigma_{yz} \end{array} \right\} dz \tag{18}
$$

$$
\{M\} = \left\{ \begin{array}{c} M_x \\ M_y \\ M_{xy} \end{array} \right\} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{array} \right\} z dz \tag{19}
$$

where the geometrical notation is as shown below:

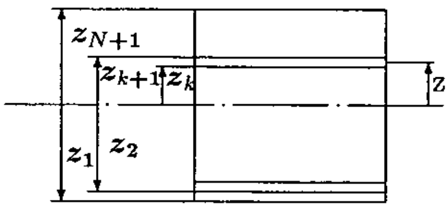

Figure 3

Substituting Eq. (16) into Eqs. (17) , (18) , (19) , we have

$$\left\{\begin{array}{c} \boldsymbol{N} \\ \boldsymbol{M} \end{array}\right\} = \left\{\begin{array}{c} \boldsymbol{N}^1 \\ \boldsymbol{M}^1 \end{array}\right\} + \left\{\begin{array}{c} \boldsymbol{N}^3 \\ \boldsymbol{M}^3 \end{array}\right\} \tag{20}$$

$$\left\{\begin{array}{c} \boldsymbol{Q}_x \\ \boldsymbol{Q}_y \end{array}\right\} = \left\{\begin{array}{c} \boldsymbol{Q}_x^1 \\ \boldsymbol{Q}_y^1 \end{array}\right\} + \left\{\begin{array}{c} \boldsymbol{Q}_x^3 \\ \boldsymbol{Q}_y^3 \end{array}\right\} \tag{21}$$

where

$$\left\{\begin{array}{c} \boldsymbol{N}^1 \\ \boldsymbol{M}^1 \end{array}\right\} = \left\{\begin{array}{c} N_x^1 \\ N_y^1 \\ N_{xy}^1 \\ M_x^1 \\ M_y^1 \\ M_{xy}^1 \end{array}\right\} = \left[\begin{array}{cccccc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{array}\right] \left\{\begin{array}{c} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \\ k_x^1 \\ k_y^1 \\ k_{xy}^1 \end{array}\right\} \tag{22}$$

$$\left\{\begin{array}{c} \boldsymbol{N}^3 \\ \boldsymbol{M}^3 \end{array}\right\} = \left\{\begin{array}{c} N_x^3 \\ N_y^3 \\ N_{xy}^3 \\ M_x^3 \\ M_y^3 \\ M_{xy}^3 \end{array}\right\} = \left[\begin{array}{cccc} E_{11} & E_{16} & E_{16} & E_{12} \\ E_{12} & E_{26} & E_{26} & E_{22} \\ E_{16} & E_{66} & E_{66} & E_{26} \\ F_{11} & F_{16} & F_{16} & F_{12} \\ F_{12} & F_{26} & F_{26} & F_{22} \\ F_{16} & F_{66} & F_{66} & F_{26} \end{array}\right] \left\{\begin{array}{c} \frac{\partial \gamma_{xz}^1}{\partial x} \\ \frac{\partial \gamma_{xz}^1}{\partial y} \\ \frac{\partial \gamma_{yz}^1}{\partial x} \\ \frac{\partial \gamma_{yz}^1}{\partial y} \end{array}\right\}$$

$$\left\{\begin{array}{c} Q_x^1 \\ Q_y^1 \end{array}\right\} = \left[\begin{array}{cc} A_{55} & A_{45} \\ A_{45} & A_{44} \end{array}\right] \left\{\begin{array}{c} \gamma_{xz}^1 \\ \gamma_{yz}^1 \end{array}\right\}$$

$$
\begin{Bmatrix} Q_x^3 \\ Q_y^3 \end{Bmatrix} = -\frac{4}{h^2} \begin{bmatrix} D_{55} & D_{45} \\ D_{45} & D_{44} \end{bmatrix} \begin{Bmatrix} \gamma_{xz}^1 \\ \gamma_{yz}^1 \end{Bmatrix} \tag{23}
$$

$$
\begin{aligned}
A_{ij} &= \sum_{k=1}^{N} \overline{Q}_{ij}^k (z_{k+1} - z_k) \\
B_{ij} &= \frac{1}{2} \sum_{k=1}^{N} \overline{Q}_{ij}^k (z_{k+1}^2 - z_k^2) \\
D_{ij} &= \frac{1}{3} \sum_{k=1}^{N} \overline{Q}_{ij}^k (z_{k+1}^3 - z_k^3) \\
E_{ij} &= -\frac{1}{3h^2} \sum_{k=1}^{N} \overline{Q}_{ij}^k (z_{k+1}^4 - z_k^4) \\
F_{ij} &= -\frac{4}{15h^2} \sum_{k=1}^{N} \overline{Q}_{ij}^k (z_{k+1}^5 - z_k^5)
\end{aligned} \tag{24}
$$

4 Interpolation Function

We adopt Serendipity 8-node element with five degrees of freedom for each node. The total number of degrees of freedom for the element is 40. The natural coordinate system (ξ, η) as shown in Fig.4 is taken to define the element geometry. The element has sides $\xi = \pm 1$ and $\eta = \pm 1$ (see Fig.4). For the element of side 2a by 2b

$$
\begin{aligned}
\xi &= (x - x_c)/a \\
\eta &= (y - y_c)/b
\end{aligned} \tag{25}
$$

where (x_c, y_c) are the coordinates at the centre of the element . Thus we have

$$
\frac{d\xi}{dx} = \frac{1}{a} \qquad \frac{d\eta}{dy} = \frac{1}{b} \tag{26}
$$

and the element area of the rectangular element is given as

$$dxdy = abd\xi d\eta \tag{27}$$

To integrate any function f(x,y) over the element we transform to the natural coordinate system, so that

$$\int\int_{\Omega^e} f(x,y)\, dxdy = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\, abd\xi d\eta \tag{28}$$

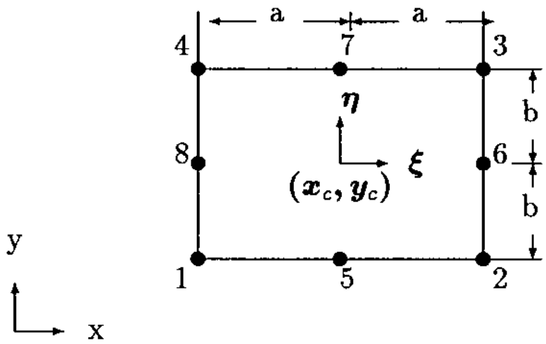

Figure 4

For the 8-node Serendipity element shown in Fig.4 the interpolation function has the following forms for the corner and midside nodes:

1. for the corner nodes

$$\psi_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)(\xi\xi_i+\eta\eta_i-1) \qquad i = 1,2,3,4 \tag{29}$$

2. for the midside nodes

$$\psi_i = \frac{\xi_i^2}{2}(1+\xi\xi_i)(1-\eta^2)+\frac{\eta_i^2}{2}(1+\eta\eta_i)(1-\xi^2) \qquad i = 5,6,7,8 \tag{30}$$

According to the values of 8 node coordinates we have from Eqs.(29) and (30) the interpolation functions for each node as follows :

$$
\begin{aligned}
\psi_1 &= -\frac{1}{4}(1-\xi)(1-\eta)(\xi+\eta+1) \\
\psi_2 &= \frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1) \\
\psi_3 &= \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1) \\
\psi_4 &= \frac{1}{4}(1-\xi)(1+\eta)(-\xi+\eta-1) \\
\psi_5 &= \frac{1}{2}(1-\eta)(1-\xi^2) \\
\psi_6 &= \frac{1}{2}(1+\xi)(1-\eta^2) \\
\psi_7 &= \frac{1}{2}(1+\eta)(1-\xi^2) \\
\psi_8 &= \frac{1}{2}(1-\xi)(1-\eta^2)
\end{aligned}
\tag{31}
$$

The derivatives of the interpolation functions are as follows :

1. for the corner nodes

$$
\begin{aligned}
\frac{\partial\psi_i}{\partial\xi} &= \frac{1}{4}\xi_i(1+\eta\eta_i)(2\xi\xi_i+\eta\eta_i) \\
\frac{\partial\psi_i}{\partial\eta} &= \frac{1}{4}\eta_i(1+\xi\xi_i)(2\eta\eta_i+\xi\xi_i) \\
\frac{\partial^2\psi_i}{\partial\xi^2} &= \frac{1}{2}\xi_i^2(1+\eta\eta_i) \\
\frac{\partial^2\psi_i}{\partial\eta^2} &= \frac{1}{2}\eta_i^2(1+\xi\xi_i) \\
\frac{\partial^2\psi_i}{\partial\xi\partial\eta} &= \frac{\partial^2\psi_i}{\partial\eta\partial\xi} = \frac{1}{4}\xi_i\eta_i(1+2\eta\eta_i+2\xi\xi_i)
\end{aligned}
\tag{32}
$$

2. for the midside nodes

$$\begin{aligned}
\frac{\partial \psi_i}{\partial \xi} &= \frac{1}{2}\xi_i^3(1-\eta^2) - \eta_i^2(1+\eta\eta_i)\xi \\
\frac{\partial \psi_i}{\partial \eta} &= \frac{1}{2}\eta_i^3(1-\xi^2) - \xi_i^2(1+\xi\xi_i)\eta \\
\frac{\partial^2 \psi_i}{\partial \xi^2} &= -\eta_i^2(1+\eta\eta_i) \\
\frac{\partial^2 \psi_i}{\partial \eta^2} &= -\xi_i^2(1+\xi\xi_i) \\
\frac{\partial^2 \psi_i}{\partial \xi \partial \eta} &= \frac{\partial^2 \psi_i}{\partial \eta \partial \xi} = -\xi_i^3\eta - \xi\eta_i^3
\end{aligned} \tag{33}$$

Note that the polynomial terms contained in this element are $1, x, y, x^2, xy, y^2, x^2y$, xy^2 . For this element, the interpolation functions have satisfied the conditions

$$\sum_i \psi_i(\xi\eta) = 1 \tag{34}$$

and

$$\psi_i(\xi_j\eta_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & i \neq j \end{cases} \tag{35}$$

The displacement components are approximated by the product of the interpolation function matrix $[\psi_i]$ and the nodal displacement vector $\{q_i^e\} = [u_{0i}\ v_{0i}\ w_{0i}\ \phi_{1i}\ \phi_{2i}]^T$, i.e.,

$$\{q\} = \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \phi_1 \\ \phi_2 \end{Bmatrix} = \sum_{i=1}^{8}[\psi_i]\{q_i^e\} \tag{36}$$

the superscript e of $\{q_i^e\}$ denotes these variables are defined on the element and need to be determined . Note that $\phi_1 = \phi_{11}, \phi_2 = \phi_{21}$.

5 The Stiffness Matrix of an 8-Node Laminated Plate Element

Finite element models developed for plate theory can be grouped into three major categories:

1. displacement models based on the principle of virtual displacements;
2. mixed and hybrid models based on the modified or mixed variational statements of the plate theories;
3. equilibrium models based on the principle of virtual forces.

Among the three types of models, the displacement finite element models are most natural and commonly used in commercial finite element programs.

In this paper we will use the displacement models to derive the stiffness matrix of an 8-node rectangular layered plate element.

By means of the principle of virtual displacement, we have

$$\begin{aligned}
\int_{\Omega^e} -\left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y}\right)\delta u_0 dx dy &= 0 \\
\int_{\Omega^e} -\left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y}\right)\delta v_0 dx dy &= 0 \\
\int_{\Omega^e} -\left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q\right)\delta w_0 dx dy &= 0
\end{aligned} \tag{37}$$

$$\int_{\Omega^e} -(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x)\delta\phi_1 dxdy = 0$$
$$\int_{\Omega^e} -(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y)\delta\phi_2 dxdy = 0$$

Recall that $N_x, N_{xy}, ..., M_{xy}$ are functions of the derivatives of the displacement $u_0, v_0, w_0, \phi_1, \phi_2$. To reduce the differentiability of the interpolation functions used in the finite element approximation of $u_0, v_0, w_0, \phi_1, \phi_2$ the differentiation on $N_x, N_{xy}, ..., M_{xy}$ is treated to weight function $\delta u_0, \delta v_0, \delta w_0, \delta\phi_1, \delta\phi_2$ by using integration-by-parts

$$\begin{aligned}
\int_{\Omega^e} (\frac{\partial \delta u_0}{\partial x} N_x + \frac{\partial \delta u_0}{\partial y} N_{xy}) dxdy - \int_{\Gamma^e} N_n \delta u_0 ds &= 0 \\
\int_{\Omega^e} (\frac{\partial \delta v_0}{\partial x} N_{xy} + \frac{\partial \delta v_0}{\partial y} N_y) dxdy - \int_{\Gamma^e} N_{ns} \delta v_0 ds &= 0 \\
\int_{\Omega^e} (\frac{\partial \delta w_0}{\partial x} Q_x + \frac{\partial \delta w_0}{\partial y} Q_y - q\delta w_0) dxdy - \int_{\Gamma^e} Q_n \delta w_0 ds &= 0 \\
\int_{\Omega^e} (\frac{\partial \delta\phi_1}{\partial x} M_x + \frac{\partial \delta\phi_1}{\partial y} M_{xy} + Q_x \delta\phi_1) dxdy - \int_{\Gamma^e} M_n \delta\phi_1 ds &= 0 \\
\int_{\Omega^e} (\frac{\partial \delta\phi_2}{\partial x} M_{xy} + \frac{\partial \delta\phi_2}{\partial y} M_y + Q_y \delta\phi_2) dxdy - \int_{\Gamma^e} M_{ns} \delta\phi_2 ds &= 0
\end{aligned} \tag{38}$$

where

Γ^e ——— the boundary edge of the element domain Ω^e;

Ω^e ——— the element domain;

N_n, N_{ns} – the stress resultants at the boundary edge of the element;

Q_n ——— the transverse shear force at the boundary edge of the element;

M_n, M_{ns} – the moment resultants at the boundary edge of the element.

Substituting $\delta u_0 = \psi_i$, $\delta v_0 = \psi_i$, $\delta w_0 = \psi_i$, $\delta\phi_1 = \psi_i$ and $\delta\phi_2 = \psi_i$ into Eqs. (38) , we have

$$
\begin{aligned}
\int_{\Omega^e} \left(\frac{\partial \psi_i}{\partial x} N_x + \frac{\partial \psi_i}{\partial y} N_{xy}\right) dx dy - \int_{\Gamma^e} N_n \psi_i ds &= 0 \\
\int_{\Omega^e} \left(\frac{\partial \psi_i}{\partial x} N_{xy} + \frac{\partial \psi_i}{\partial y} N_y\right) dx dy - \int_{\Gamma^e} N_{ns} \psi_i ds &= 0 \\
\int_{\Omega^e} \left(\frac{\partial \psi_i}{\partial x} Q_x + \frac{\partial \psi_i}{\partial y} Q_y - q\psi_i\right) dx dy - \int_{\Gamma^e} Q_n \psi_i ds &= 0 \qquad (39) \\
\int_{\Omega^e} \left(\frac{\partial \psi_i}{\partial x} M_x + \frac{\partial \psi_i}{\partial y} M_{xy} + Q_x \psi_i\right) dx dy - \int_{\Gamma^e} M_n \psi_i ds &= 0 \\
\int_{\Omega^e} \left(\frac{\partial \psi_i}{\partial x} M_{xy} + \frac{\partial \psi_i}{\partial y} M_y + Q_y \psi_i\right) dx dy - \int_{\Gamma^e} M_{ns} \psi_i ds &= 0
\end{aligned}
$$

Substituting Eq.(36) for u_0, v_0, w_0, ϕ_1, and ϕ_2) into Eqs.(13), (14), (15), we obtain $\{\epsilon^0\}, \{k^1\}$ and $\{\gamma^1\}$ represented by interpolation functions and nodal displacements. Then substituting $\{\epsilon^0\}$, $\{k^1\}$ and $\{\gamma^1\}$ into Eqs. (20), (21) , we have stress resultants, moment resultants and transverse shear forces $\{N\}, \{M\}$ and $\{Q\}$ represented by interpolation functions and node displacements . After substituting $\{N\}, \{M\}$ and $\{Q\}$ represented by interpolation functions and node displacements into Eq.(39) we obtain the finite element model of the higher-order shear deformation theory

$$\sum_{j=1}^{8} \sum_{\beta=1}^{5} K_{ij}^{\alpha\beta} \Delta_j^\beta - F_i^\alpha = 0 \quad (\alpha = 1, 2, ..., 5) \qquad (40)$$

or

$$[K^e]\{\Delta^e\} = \{F^e\} \qquad (41)$$

where the variables Δ_j^β , the stiffness and force coefficients are defined by

$$
\begin{aligned}
\Delta_j^1 &= u_{0j} \qquad \Delta_j^2 = v_{0j} \qquad \Delta_j^3 = w_{0j} \\
\Delta_j^4 &= \phi_{1j} \qquad \Delta_j^5 = \phi_{2j}
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
K_{ij}^{1\alpha} &= \int_{\Omega^e} [\frac{\partial \psi_i}{\partial x}(N_{1j}^\alpha + \overline{N}_{1j}^\alpha) + \frac{\partial \psi_i}{\partial y}(N_{6j}^\alpha + \overline{N}_{6j}^\alpha)] dxdy \\
K_{ij}^{2\alpha} &= \int_{\Omega^e} [\frac{\partial \psi_i}{\partial x}(N_{6j}^\alpha + \overline{N}_{6j}^\alpha) + \frac{\partial \psi_i}{\partial y}(N_{2j}^\alpha + \overline{N}_{2j}^\alpha)] dxdy \\
K_{ij}^{3\alpha} &= \int_{\Omega^e} [\frac{\partial \psi_i}{\partial x}(Q_{1j}^\alpha + \overline{Q}_{1j}^\alpha) + \frac{\partial \psi_i}{\partial y}(Q_{2j}^\alpha + \overline{Q}_{2j}^\alpha)] dxdy \\
K_{ij}^{4\alpha} &= \int_{\Omega^e} [\frac{\partial \psi_i}{\partial x}(M_{1j}^\alpha + \overline{M}_{1j}^\alpha) + \frac{\partial \psi_i}{\partial y}(M_{6j}^\alpha + \overline{M}_{6j}^\alpha) + \psi_i(Q_{1j}^\alpha + \overline{Q}_{1j}^\alpha)] dxdy \\
K_{ij}^{5\alpha} &= \int_{\Omega^e} [\frac{\partial \psi_i}{\partial x}(M_{6j}^\alpha + \overline{M}_{6j}^\alpha) + \frac{\partial \psi_i}{\partial y}(M_{2j}^\alpha + \overline{M}_{2j}^\alpha) + \psi_i(Q_{2j}^\alpha + \overline{Q}_{2j}^\alpha)] dxdy
\end{aligned} \tag{43}
$$

The coefficients $N_{Ij}^\alpha, \overline{N}_{Ij}^\alpha, M_{Ij}^\alpha, \overline{M}_{Ij}^\alpha, Q_{Ij}^\alpha$ and $\overline{Q}_{Ij}^\alpha$ for $\alpha = 1, 2, ..., 5$ and $I = 1, 2, 6$ are given by

$$
\begin{aligned}
N_{1j}^1 &= A_{11}\frac{\partial \psi_j}{\partial x} + A_{16}\frac{\partial \psi_j}{\partial y} \qquad N_{1j}^2 = A_{12}\frac{\partial \psi_j}{\partial y} + A_{16}\frac{\partial \psi_j}{\partial x} \\
N_{1j}^4 &= B_{11}\frac{\partial \psi_j}{\partial x} + B_{16}\frac{\partial \psi_j}{\partial y} \qquad N_{1j}^5 = B_{12}\frac{\partial \psi_j}{\partial y} + B_{16}\frac{\partial \psi_j}{\partial x} \\
N_{2j}^1 &= A_{12}\frac{\partial \psi_j}{\partial x} + A_{26}\frac{\partial \psi_j}{\partial y} \qquad N_{2j}^2 = A_{22}\frac{\partial \psi_j}{\partial y} + A_{26}\frac{\partial \psi_j}{\partial x} \\
N_{2j}^4 &= B_{12}\frac{\partial \psi_j}{\partial x} + B_{26}\frac{\partial \psi_j}{\partial y} \qquad N_{2j}^5 = B_{22}\frac{\partial \psi_j}{\partial y} + B_{26}\frac{\partial \psi_j}{\partial x} \\
N_{6j}^1 &= A_{16}\frac{\partial \psi_j}{\partial x} + A_{66}\frac{\partial \psi_j}{\partial y} \qquad N_{6j}^2 = A_{26}\frac{\partial \psi_j}{\partial y} + A_{66}\frac{\partial \psi_j}{\partial x} \\
N_{6j}^4 &= B_{16}\frac{\partial \psi_j}{\partial x} + B_{66}\frac{\partial \psi_j}{\partial y} \qquad N_{6j}^5 = B_{26}\frac{\partial \psi_j}{\partial y} + B_{66}\frac{\partial \psi_j}{\partial x}
\end{aligned} \tag{44}
$$

$$
\overline{N}_{1j}^3 = E_{11}\frac{\partial^2 \psi_j}{\partial x^2} + 2E_{16}\frac{\partial^2 \psi_j}{\partial x \partial y} + E_{12}\frac{\partial^2 \psi_j}{\partial y^2}
$$

$$
\begin{aligned}
\overline{N}^4_{1j} &= E_{11}\frac{\partial\psi_j}{\partial x} + E_{16}\frac{\partial\psi_j}{\partial y} \qquad \overline{N}^5_{1j} = E_{12}\frac{\partial\psi_j}{\partial y} + E_{16}\frac{\partial\psi_j}{\partial x} \\
\overline{N}^3_{2j} &= E_{12}\frac{\partial^2\psi_j}{\partial x^2} + 2E_{26}\frac{\partial^2\psi_j}{\partial x\partial y} + E_{22}\frac{\partial^2\psi_j}{\partial y^2} \\
\overline{N}^4_{2j} &= E_{12}\frac{\partial\psi_j}{\partial x} + E_{26}\frac{\partial\psi_j}{\partial y} \qquad \overline{N}^5_{2j} = E_{22}\frac{\partial\psi_j}{\partial y} + E_{26}\frac{\partial\psi_j}{\partial x} \\
\overline{N}^3_{6j} &= E_{16}\frac{\partial^2\psi_j}{\partial x^2} + 2E_{66}\frac{\partial^2\psi_j}{\partial x\partial y} + E_{26}\frac{\partial^2\psi_j}{\partial y^2} \\
\overline{N}^4_{6j} &= E_{16}\frac{\partial\psi_j}{\partial x} + E_{66}\frac{\partial\psi_j}{\partial y} \qquad \overline{N}^5_{6j} = E_{26}\frac{\partial\psi_j}{\partial y} + E_{66}\frac{\partial\psi_j}{\partial x}
\end{aligned} \tag{45}
$$

$$
\begin{aligned}
Q^3_{1j} &= A_{55}\frac{\partial\psi_j}{\partial x} + A_{45}\frac{\partial\psi_j}{\partial y} \quad Q^4_{1j} = A_{55}\psi_j \quad Q^5_{1j} = A_{45}\psi_j \\
Q^3_{2j} &= A_{45}\frac{\partial\psi_j}{\partial x} + A_{44}\frac{\partial\psi_j}{\partial y} \quad Q^4_{2j} = A_{45}\psi_j \quad Q^5_{2j} = A_{44}\psi_j
\end{aligned} \tag{46}
$$

$$
\begin{aligned}
\overline{Q}^3_{1j} &= -\frac{4}{h^2}(D_{55}\frac{\partial\psi_j}{\partial x} + D_{45}\frac{\partial\psi_j}{\partial y}) \quad \overline{Q}^4_{1j} = -\frac{4}{h^2}D_{55}\psi_j \\
\overline{Q}^5_{1j} &= -\frac{4}{h^2}D_{45}\psi_j \\
\overline{Q}^3_{2j} &= -\frac{4}{h^2}(D_{45}\frac{\partial\psi_j}{\partial x} + D_{44}\frac{\partial\psi_j}{\partial y}) \quad \overline{Q}^4_{2j} = -\frac{4}{h^2}D_{45}\psi_j \\
\overline{Q}^5_{2j} &= -\frac{4}{h^2}D_{44}\psi_j
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
M^1_{1j} &= B_{11}\frac{\partial\psi_j}{\partial x} + B_{16}\frac{\partial\psi_j}{\partial y} \qquad M^2_{1j} = B_{12}\frac{\partial\psi_j}{\partial y} + B_{16}\frac{\partial\psi_j}{\partial x} \\
M^4_{1j} &= D_{11}\frac{\partial\psi_j}{\partial x} + D_{16}\frac{\partial\psi_j}{\partial y} \qquad M^5_{1j} = D_{12}\frac{\partial\psi_j}{\partial y} + D_{16}\frac{\partial\psi_j}{\partial x} \\
M^1_{2j} &= B_{12}\frac{\partial\psi_j}{\partial x} + B_{26}\frac{\partial\psi_j}{\partial y} \qquad M^2_{2j} = B_{22}\frac{\partial\psi_j}{\partial y} + B_{26}\frac{\partial\psi_j}{\partial x} \\
M^4_{2j} &= D_{12}\frac{\partial\psi_j}{\partial x} + D_{26}\frac{\partial\psi_j}{\partial y} \qquad M^5_{2j} = D_{22}\frac{\partial\psi_j}{\partial y} + D_{26}\frac{\partial\psi_j}{\partial x} \\
M^1_{6j} &= B_{16}\frac{\partial\psi_j}{\partial x} + B_{66}\frac{\partial\psi_j}{\partial y} \qquad M^2_{6j} = B_{26}\frac{\partial\psi_j}{\partial y} + B_{66}\frac{\partial\psi_j}{\partial x} \\
M^4_{6j} &= D_{16}\frac{\partial\psi_j}{\partial x} + D_{66}\frac{\partial\psi_j}{\partial y} \qquad M^5_{6j} = D_{26}\frac{\partial\psi_j}{\partial y} + D_{66}\frac{\partial\psi_j}{\partial x}
\end{aligned} \tag{48}
$$

$$
\begin{aligned}
\overline{M}^3_{1j} &= F_{11}\frac{\partial^2\psi_j}{\partial x^2} + 2F_{16}\frac{\partial^2\psi_j}{\partial x\partial y} + F_{12}\frac{\partial^2\psi_j}{\partial y^2} \\
\overline{M}^4_{1j} &= F_{11}\frac{\partial\psi_j}{\partial x} + F_{16}\frac{\partial\psi_j}{\partial y} \qquad \overline{M}^5_{1j} = F_{12}\frac{\partial\psi_j}{\partial y} + F_{16}\frac{\partial\psi_j}{\partial x} \\
\overline{M}^3_{2j} &= F_{12}\frac{\partial^2\psi_j}{\partial x^2} + 2F_{26}\frac{\partial^2\psi_j}{\partial x\partial y} + F_{22}\frac{\partial^2\psi_j}{\partial y^2} \\
\overline{M}^4_{2j} &= F_{12}\frac{\partial\psi_j}{\partial x} + F_{26}\frac{\partial\psi_j}{\partial y} \qquad \overline{M}^5_{2j} = F_{22}\frac{\partial\psi_j}{\partial y} + F_{26}\frac{\partial\psi_j}{\partial x} \\
\overline{M}^3_{6j} &= F_{16}\frac{\partial^2\psi_j}{\partial x^2} + 2F_{66}\frac{\partial^2\psi_j}{\partial x\partial y} + F_{26}\frac{\partial^2\psi_j}{\partial y^2} \\
\overline{M}^4_{6j} &= F_{16}\frac{\partial\psi_j}{\partial x} + F_{66}\frac{\partial\psi_j}{\partial y} \qquad \overline{M}^5_{6j} = F_{26}\frac{\partial\psi_j}{\partial y} + F_{66}\frac{\partial\psi_j}{\partial x}
\end{aligned} \tag{49}
$$

$$
\begin{aligned}
F_i^1 &= \int_{\Gamma^e} \psi_i N_n ds \\
F_i^2 &= \int_{\Gamma^e} \psi_i N_{ns} ds \\
F_i^3 &= \int_{\Omega^e} q\psi_i dxdy + \int_{\Gamma^e} \psi_i Q_n ds \\
F_i^4 &= \int_{\Gamma^e} \psi_i M_n ds \\
F_i^5 &= \int_{\Gamma^e} \psi_i M_{ns} ds
\end{aligned} \tag{50}
$$

All other coefficients are zero.

Equation (43) represents the stiffness factors of an 8-node laminated plate element based on Reddy's higher-order shear deformation theory.If we set Eqs. (45), (47), (49) equal to zero , we obtain the stiffness factors of an 8-node laminated plate element based on first shear deformation theory[4] from Eq.(43).

6 References

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