UNIVERSITY OF SOUTHAMPTON

The Geometry of Lie Algebras and Broken SO(6) Symmetries

by

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Dedicated to my family and friends

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ABSTRACT

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Non-linear realisations of the groups SU(2) and SO(1,4) are analysed, described by the coset spaces SU(2)/U(1) and SO(1,4)/SO(1,3). The analysis consists of determining the transformation properties of the Goldstone bosons, constructing the most general possible Lagrangian for the realisations and finding the metric of the coset space. The Lie algebras of special unitary groups are studied and their projection operators are determined, leading to a general method for constructing the Lagrangian for a non-linear realisation of a special unitary group. The Lie algebra of SU(4) is looked at in depth and its homomorphism with SO(6) allows a full specification of the most general Lagrangian for the coset space SO(6)/SO(4)$\otimes$SO(2).

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Preface

Original work starts at the very end of Chapter 4. The contents of Chapter 5 have been presented by me in an internal group seminar but never published. The majority of the work in the sections of Chapter 6 concerning SU(4) and SO(6) is original and has been accepted for publication to the Journal of Physics A as a paper entitled 'How orbits of SU(N) can describe rotations in SO(6)' , (authors K J Barnes, J Hamilton-Charlton and T R Lawrence) as well as being presented by me in an internal group seminar. Likewise, the work presented in Chapters 7, 8 and 9 is original (except where otherwise stated).

Acknowledgements

I would first of all like to thank my supervisor, Prof. K. J. Barnes, to whom I am grateful for suggesting such an enjoyable area of research. Much of the early content of this thesis and its whole approach are due largely to him, as will be seen from the references. His unerring intuition has also proved a valuable guide.

I would also like to thank Mr. J. Hamilton-Charlton with whom I have collaborated on this work, for many stimulating discussions which have enhanced both our understandings. He has a knack of finding clear ways of presenting arguments, which I have occasionally drawn on in this thesis, of identifying points I have failed to consider and of identifying the most appropriate sign conventions, etc., although I still do not agree that the name SO(4,2) is more intuitive than SO(2,4)! Also, my thanks to him for providing the axodraw code for the SU(4) root diagram.

In general, I would like give my thanks to the past and present members of the Southampton High Energy Physics group for the sociable working environment and for helping to make my time in Southampton so enjoyable. I should give particular thanks to Drs. K. D. Anderson and C. Harvey-Fros for providing a template for the format of this thesis, compatible with regulations.

My thanks also goes to various members of the Southampton University Mathematics Department who have given me support in this work. In particular, I am very grateful for a most enlightening conversation with Dr. J. A. Vickers on various topics of differential geometry.

Finally, I give my sincere and warm thanks to my parents and to the rest of my family, as well as to all of those friends I have subjected to various ramblings about particle physics, and the people of Southampton for putting up with my wandering around musing out loud about what are now the contents of this thesis.

Note added after examination: I would also like to thank my examiners for drawing to my attention references [31], [42] and [48].

Chapter 1

Introduction

The importance of Lie group symmetries in particle physics has long been understood, with Wigner's classification of the unitary representations of the Poincaré group underpinning much of modern particle physics[1] and a local U(1) symmetry motivating the introduction of the electromagnetic gauge field described by quantum electrodynamics. The idea of introducing a gauge field to in order to maintain a local symmetry was extended to the (non-Abelian) isotopic spin group in 1955 [2] and in the following year to the Poincaré group [3], where it was shown to be the gravitational field as described by general relativity.

The roots of the research presented in this thesis, however, lie in the 1960s, in two topics which were studied in such different ways that it was not proved until after nearly a decade of research that they were two sides of the same coin. Interestingly, these were both initiated largely by papers submitted for publication in Nuovo Cimento in 1960. Goldstone's paper [4] looked at the possible interpretations of a situation in which a Lagrangian contains a scalar field or scalar multiplet with an imaginary mass. If such a Lagrangian has a discrete symmetry, the minima of the potential part of this Lagrangian must be discrete, whereas if it has a continuous symmetry the potential has a continuous set of degenerate minima. For example, the Lagrangian

$$\mathcal{L} = \frac{\partial \phi^a}{\partial x_\mu}\frac{\partial \phi^a}{\partial x^\mu} - \mu^2 \phi^a \phi^a - \frac{\lambda_0}{6}(\phi^a \phi^a)^2 \tag{1.1}$$

is invariant under an O(2) transformation of the doublet ϕ^a and with μ^2 negative, the potential looks like:

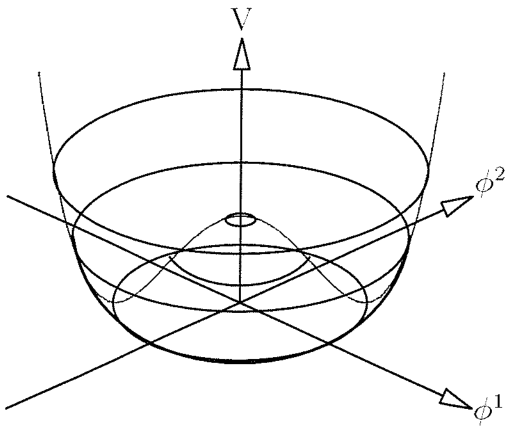

Figure 1.1: O(2) invariant potential with degenerate minima

To obtain physical fields, one must redefine the scalar fields such that one of the minima represents the vacuum state of the system - picking a minimum in this way breaks the symmetry (or at least part of it). Furthermore, following the field redefinition, some of the fields are massless (a flavour of why this occurs is given in Section 2.3).

This paper was followed by another [5] in which these conclusions were restated in a more general form: whenever a Lagrangian is invariant under a continuous symmetry group but its vacuum is not, there will be spinless fields of zero mass present. These are known as Goldstone bosons. It was shown that this theorem is generally valid for *global* symmetries in Lorentz covariant theories [5, 6]; the extension of Goldstone's method of symmetry breaking to the case of a *gauged* symmetry is the famous Higgs mechanism [7]. All of these papers were concerned with a way of going from a Lagrangian which is explicitly invariant under a given group of symmetries to a vacuum which is not invariant under all of them (though it may be invariant under a subgroup of these symmetries), which can be applied regardless of the particular group chosen.

Gell-Mann and Lévy's paper [8], by contrast, was classic phenomenology, concerning pion decays in a system of pions and nucleons. In such a system, the current which transforms neutron into proton could be split up into a vector part and an axial part. This paper described and considered three different models in which the axial vector current satisfied a certain condition, which it was shown led to a particular form for the decay rate which agreed with experiment (or would do if an unknown form factor behaved as expected).

In the second of these models, the nucleons transform as a representation of SO(4) but the pions only transform as a representation of its SO(3) (vector) subgroup. (We will look more closely at what this means in Chapter 2.) A fourth scalar (meson) field called σ' is introduced; under the remaining - axial - part of SO(4), $\boldsymbol{\pi}$ and σ' transform into each other, that is to say the pions and the σ' together form a multiplet of SO(4).

In the third model the σ' field is eliminated from the Lagrangian by constraining the modulus (the 'length') of this field:

$$\boldsymbol{\pi}^2 + \sigma'^2 = C^2 \tag{1.2}$$

where C is a constant, so that

$$\sigma' = -\sqrt{C^2 - \pi^2} \tag{1.3}$$

Wherever σ' previously appeared in the Lagrangian, then, it is now replaced by this non-linear function of $\boldsymbol{\pi}$. This model therefore became known as the 'non-linear sigma model'.

This idea of involving a scalar field non-linearly in the Lagrangian so that the full symmetries of the system (typically those of the 'chiral groups' $SU(2)\otimes SU(2) \approx SO(4)$ or $SU(3)\otimes SU(3)$) were not explicit, became increasingly popular through the 1960s. Much of the research was essentially phenomenological, considering one particular realisation of one particular group [9]. A notable exception was the work of Callan, Coleman, Wess and Zumino in 1969 [10, 11]. This demonstrated how, given any Lie group[1] and any Lie subgroup, it was possible in theory to derive the most general Lagrangian in which the subgroup was linearly represented but the rest of the symmetries were realised non-linearly. These papers form the starting point for this thesis and will be reviewed in detail in Chapters 2 and 3. They differ from their predecessors both by virtue of their geometric approach and by the generality of their application.

The geometry of these non-linear realisations was examined further by Isham [12], who introduced the concepts of Killing vectors and of a metric (prompted by Meetz [13]), and later by Boulware and Brown [14].

Salam and Strathdee [15] drew attention to the fact that in any such 'phenomenological Lagrangian' there are always terms involving a set of scalar fields, but none of these terms are pure powers of the fields - in particular there is no term quadratic in the fields, that is, they are massless. They proved that if a non-linear realisation of a particular group were obtained from a linear one, the vacuum could not be invariant under the full group. Goldstone's theorem then implied that there were massless scalars present in the non-linear realisation, which were exactly the fields identified by Callan, Coleman, Wess and Zumino. Also, if one

[1]Strictly speaking, this should be a linear Lie group (see, for example, Vol. 1 of [16]), but we will follow the convention of particle physics and use the phrase 'Lie group' in the understanding that it is a linear Lie group we are talking about.

were to specify that a system of fields is invariant under a Lie group G while the system's vacuum state is invariant under a subgroup H, one could ask whether there is a general method of determining the couplings between the Goldstone bosons and the other fields - Salam and Strathdee showed that in any such case the methods of Callan, Coleman, Wess and Zumino would do just this. In this way of looking at things, a non-linear realisation was just the effective theory resulting from the spontaneous breaking of a symmetry. (A more explicit comparison of the Lagrangians obtained from the methods of [11] and those from a spontaneous symmetry breaking scheme was carried out by Honerkamp [17] for the case of the chiral groups.)

Aided in their understanding of non-linear realisations by these papers, researchers in the area spent the next three years applying the methods of Callan, Coleman, Wess and Zumino and Isham to the chiral groups, culminating in the paper of Barnes, Dondi and Sarkar [18].

In this thesis we shall analyse various non-linear realisations. The first of these analyses, SU(2)/U(1), has been done before [19] - we shall just reproduce this work as a simple example of how to apply the above theory. The second, SO(1,4)/SO(1,3), has not been done before, but the formal manipulations are almost identical to those of SU(2)/U(1). For both of these non-linear realisations, we will determine the transformation properties of the fields involved (using the Killing vector method) and construct the most general possible Lagrangian. From this Lagrangian, we will obtain a metric, where the coordinates are the Goldstone fields themselves[2].

In the latter part of the thesis, we concentrate on SU(N) groups, in particular

[2]Whenever there is an even number of such real fields, they may be combined into complex fields; for certain non-linear realisations the resulting complex metric is particularly useful to anyone wishing to supersymmetrise the realisation [23, 24, 25, 26]

SU(4), and by way of a homomorphism, on SO(6). We make a general study of the geometry and algebra associated with the Lie algebras of these groups and then turn to the problem of specifying the Lagrangian and finding the metric for non-linear realisations of the groups. By using the machinery of projection operators, we are able to find very general forms of the required quantities for a large class of realisations, those with 'automorphism conjugate u-vectors'. However, these expressions assume one can find the projection operators appropriate to the realisation. This we do for particular realisations of SU(4), or equivalently of SO(6), including SO(6)/SO(4)⊗SO(2). These we shall see have 'automorphism conjugate u-vectors' so for SO(6)/SO(4)⊗SO(2) we will completely specify the Lagrangian.

Chapter 2 will begin by introducing the key concepts of a coset and a coset space which underpin the whole of this work. It will be seen that there is an initmate connection between the description of the coset space and the nature of the Goldstone fields. The bulk of this chapter will be spent looking at the transformation properties of the coset space and hence the Goldstone fields, using the coset space SU(2)/U(1) as an example. This analysis will be exactly that of [19], in which the field transformations are described in terms of the Killing vectors.

Chapter 3 introduces the 'standard field' description of the other particles that may be involved in the non-linear realisation. After looking at how these fields transform, we will look at how to construct the most general possible Lagrangian from the Goldstone fields and the standard fields, following the prescription given by [11]. Most of the chapter is involved with defining 'covariant derivatives' for the fields which transform in the same way as the fields themselves. Again, we shall see how to do all this for SU(2)/U(1) - we will obtain each term in the Lagrangian involving these covariant derivatives, and from the term involving the covariant derivatives of the Goldstone fields we will extract the metric of the coset space.

In Chapter 4 we start to look in detail at the properties of specific Lie algebras. We start off with the simplest (non-Abelian) example of all, SU(2). We describe the effects of similarity transformations on the vectors of the defining representation and see how this can be used to define the elements of the adjoint representation (of both the group and the algebra). We also identify a set of projection operators for each of these representations, which are so important in the non-linear realisations of higher-dimensional SU(N) groups. We go on to look at the special orthogonal groups SO(3), SO(4), SO(5), SO(1,3) and SO(1,4), making full use of homomorphisms between the groups. We consider the γ-matrices of the spinor representations and their products, as well as identifying projection operators for the spinor representation of SO(3) and the Weyl representation of SO(4).

This understanding of the Lie algebras of SO(1,3) and SO(1,4) is put to use in Chapter 5, which deals with the non-linear realisation SO(1,4)/SO(1,3). For this realisation we find the transformation properties of the Goldstone fields, the covariant derivatives and the metric, just as for SU(2)/U(1) in Chapters 2 and 3. Indeed, we also double-check this metric by introducing a method for obtaining it from the Killing vectors.

In Chapter 6 we go back to studying the intrinsic properties of Lie algebras. We see that SU(2) is something of a special case among the special unitary groups - the algebras of the higher-dimensional SU(N) all have additional features. We take a geometric approach to studying these features, based on the work of Michel and Radicati [27]. Although the bulk of this paper specifically concerns SU(3), they also outline a general way of describing the Lie algebra of any SU(N) group; after reviewing this theory we then apply it to SU(4). We see how the elements of the Lie algebra fall into four distinct classes or 'strata'. We also look at two bilinear operators on the Lie algebra which are related to the symmetric and antisymmetric structure constants. The final section notes that this Lie algebra contains the same elements as the Lie algebras of the two spinor representations of SO(6) and the space of matrices spanned by the products of SO(4) γ-matrices. This means that the strata of SU(4) can be seen as strata of SO(6). In this section we focus particularly on the SU(2), SO(3) and SO(4) subsets of SO(6) rotations, which gives us a deeper understanding of the geometry. Seen in this way the symmetric structure constants take on an unexpectedly simple form.

Chapter 7 starts by extending the definition of the elements of the adjoint representation of the Lie algebra of SU(2) to higher-dimensional SU(N). These elements act as tensor operators on the elements of the defining representation of the algebra and are constructed using the antisymmetric structure constants. We define a similar set of tensors based on the symmetric structure constants. We use the geometric properties of the projection operators of the defining representation to derive explicit forms for particular combinations of the projection operators of the *adjoint* representation. These are all the tensors we will employ in constructing a general Lagrangian for a non-linear realisation of SU(N). We close the chapter by finding explicit forms for these tensors for SO(6), by using the homomorphism with SU(4).

In Chapter 8 we set about trying to derive the covariant derivatives of an arbitrary non-linear realisation of SU(N) - these constitute a full specification of the most general possible Lagrangian. Given a set of (defining representation) projection operators relating to an arbitrary element of the coset space, we derive an expression for a key quantity, known as $L^{-1}\partial_\mu L$, in terms of the tensor operators and the traceless parts of the defining representation projection operators . For a large class of realisations (those with 'automorphism conjugate u-vectors') we are able to extract the vital information, resulting in general expressions for the covariant derivatives at the end of the chapter.

Having identified the covariant derivatives for a general SU(N) coset space, assuming the projection operators for that space to be known, in Chapter 9 we turn to a particular class of coset spaces of SU(4), for which we can determine the projection operators. We note some of the properties of this class and use these properties to find the projection operators for an arbitrary element of the coset space. We are able to provide a check on our result by using a method relating to the eigenvalues of the element. Each of these spaces has 'automorphism conjugate u-vectors', so at this point we have completely specified the projection operators *and* the covariant derivatives for the realisations. However, we further find that we are able to express one of the terms in the covariant derivative of the Goldstone fields in a more convenient form, giving us derivatives which look very much like those of the chiral non-linear realisations in [18].

Again, we can use the homomorphism to express all these results in SO(6) terms. Using the simple form of the symmetric structure constant we find simple expressions for the various vectors and invariants needed to construct the projection operators, explicitly in this basis. We end the chapter by demonstrating that the coset space SO(6)/SO(4)⊗SO(2) belongs to the class for which we have identified the covariant derivatives.

Finally, a note on ranges of indices. Throughout this thesis, we will stick to the following conventions. Whenever the indices a, b, c appear they run over the values 1,2. Similarly, the indices i, j, k, l, m will always run over 1,2,3. Whenever we are considering compact groups, so the metric of the Lie algebra is Euclidean (positive definite), greek indices will run 1,2,3,4, but when we come to consider the SO(t,s) group, we shall follow convention and use the index 0 to represent a timelike direction; in such cases the greek indices will run 0,1,2,3 (this will be clarified in Section 4.5). (We assume the fields to be functions of normal four-dimensional Minkowski spacetime, so whenever the index μ represents spacetime, for example in $\partial_\mu M^A$, it is assumed to run 0,1,2,3.)

The variety of Lie algebras and their subspaces we shall be working with means that it is not practical to rigidly define ranges of the upper case indices. We shall endeavour to define them as and when they are introduced, chapter by chapter. However, wherever possible, the following guidelines are observed. The indices X, Y are only used in the final chapter where they range over 5,6. When we work with a d-dimensional representation of a group, the indices S, T, U, V run $1, \ldots, d$ (these are largely used for projection operators). For SU(N) groups and their coset spaces, an A, B, C, D-index on a generator indicates it is a generator associated with the coset space, while a P, Q, R-index indicates it is a generator of the subgroup. The indices I, J, K, L in such cases run over all the group indices, from 1 to the dimension of the group. For special orthogonal groups, an arbitrary group index may be denoted by any capital letter upto and including P.

Chapter 2

Coset Spaces and Goldstone Bosons

2.1 Lie Algebras as Vector Spaces

Any element of an n-dimensional Lie group may be written $e^{-i\mathbf{x}}$ where $\mathbf{x}$ is a vector in an n-dimensional vector space. For example, an element of SU(2) (in the defining representation) may be written as $e^{-i\mathbf{x}}$ where $\mathbf{x} = \theta^j T_j$ is a vector in a 3-dimensional real vector space whose elements are 2×2 matrices. We can choose a basis set (of matrices) for this space - in the case of SU(2) the most common choice of basis is the set of generators given by half the Pauli matrices:

$$T_i = \frac{1}{2}\sigma_i$$

$$\text{where } \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.1}$$

For all SU(N) and SO(t,s) groups (the ones we shall be looking at), if we take the commutator of any two vectors in this vector space we get another vector in the space, or at least a vector in the space multiplied by a complex constant. In the case of unitary groups (all the vectors in the space are hermitian), this constant is purely imaginary, so following the notation and normalisation of Michel and Radicati [27], we can define the operator $_\wedge$:

$$\mathbf{x}_\wedge\mathbf{y} \equiv -\frac{i}{2}\left[\mathbf{x},\mathbf{y}\right] \tag{2.2}$$

For all SU(N) and SO(t,s) groups the matrix $\mathbf{x}_\wedge\mathbf{y}$ is an element of the vector space (it 'lies in the algebra', or the operator $_\wedge$ is an 'algebra' of the vector space). The $_\wedge$-algebra is, of course, a linear algebra, in that

$$(c_1\mathbf{x}+c_2\mathbf{y})_\wedge\,\mathbf{z} = -\frac{i}{2}\left[(c_1\mathbf{x}+c_2\mathbf{y}),\mathbf{z}\right] = -\frac{i}{2}c_1\left[\mathbf{x},\mathbf{z}\right] - \frac{i}{2}c_2\left[\mathbf{y},\mathbf{z}\right] = c_1\mathbf{x}_\wedge\mathbf{z} + c_2\mathbf{y}_\wedge\mathbf{z} \tag{2.3}$$

where c_1, c_2 are numerical coefficients

so we can write the commutator of any two vectors as a linear sum of commutators of basis vectors. The vector space is known as the Lie algebra, although the set of all commutation relations between the generators is often referred to also as the Lie algebra - it is always clear from the context which is meant.

The 'components' of the vector $\mathbf{x} = \theta^I T_I$ are the parameters θ^I. These can be thought of as coordinates on the space. Indeed, if we replace the basis vectors T_I by the normal Cartesian basis vectors, our vector space becomes the familiar n-dimensional space of real numbers, with the θ^I being the Cartesian coordinates of a vector in this space. This is defining a mapping from the (n-dimensional) Lie algebra to the (n-dimensional) space of real numbers:

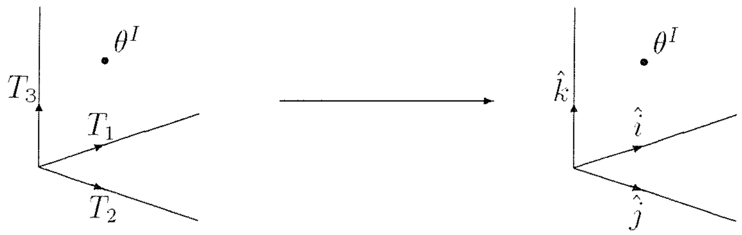

Figure 2.1: Mapping from the Lie algebra to $\mathbb{R}^n$

2.2 Cosets and Coset Spaces

In the basis of the generators, then, we can write an element g of a Lie group G as

$$g = \mathrm{e}^{-\mathrm{i}\omega^I T_I} \qquad I = 1, 2, \ldots, n \tag{2.4}$$

where n is the dimension of the group.

Let H be a subgroup of G. Then

$$h \in H = \mathrm{e}^{-\mathrm{i}\omega^P T_P} \tag{2.5}$$

$$\text{where } P = 1, \ldots, m \quad \text{where } m \text{ is the dimension of } H$$

We now define a left coset gH to be the set of elements $gH = \{gh \;\; \forall\, h \in H\}$; note that each g is in its own coset gH. (Right cosets can be similarly defined but they will not be used in this work; therefore whenever the word 'coset' is used in the remainder of this thesis I will always be referring to a left coset.)

(2.4) is not the only way of writing g; to construct gH we instead decompose it into two factors. For example, the elements of SU(2) may be written

$$g = e^{-\frac{i}{2}\omega^i\sigma_i} \qquad i = 1, 2, 3 \tag{2.6}$$

or alternatively we may express them as follows:

$$g = e^{-\frac{i}{2}\theta^a\sigma_a}e^{-\frac{i}{2}\theta^3\sigma_3} \qquad a = 1, 2 \tag{2.7}$$

(We will be working a lot with elements of SU(2). As the metric for the algebra is positive definite there is no distinction between, for example, θ^a and θ_a - we are free to raise and lower indices as we wish. However, for clarity, and to make things easier when we come to look at groups like SO(1,4) for which this is not the case, we shall ensure throughout this thesis that we keep the notation covariant, that is we keep our indices balanced.)

The elements of the U(1) subgroup generated by $T_3 = \sigma_3$ are

$$h = e^{-\frac{i}{2}\theta'^3\sigma_3} \tag{2.8}$$

The elements of gH are then given by

$$gH = e^{-\frac{i}{2}\theta^a\sigma_a}e^{-\frac{i}{2}\theta^3\sigma_3}\{e^{-\frac{i}{2}\theta'^3\sigma_3} \quad \forall\, \theta'^3\} \tag{2.9}$$

$$= e^{-\frac{i}{2}\theta^a\sigma_a}\{e^{-\frac{i}{2}\theta^3\sigma_3} \quad \forall\, \theta^3\} \tag{2.10}$$

In the context of broken symmetry, the part of the symmetry under G that is associated with the σ_a is broken, while the part associated with σ_3 survives. The σ_a are thus known as 'broken generators'.

Note in the above that if two groups elements have the same values of θ^a but different values of θ^3 they lie in the same coset. It is thus only the θ^a that distinguish between different cosets. We therefore define the element L as the part which distinguishes between cosets - in this particular case, it is

$$L = e^{-\frac{i}{2}\theta^a \sigma_a} \tag{2.11}$$

or in general for a Lie group

$$L \equiv e^{-i\omega^A T_A} \tag{2.12}$$

$$\text{where} \quad A = m+1, \ldots, n$$

so each coset is represented by one value of L - it is much simpler to work with L than with the entire coset. (Note that the L in (2.11) is a representative of the coset gH in (2.10), as indeed is g.)

We now define the coset space G/H to be the space of all of the cosets, so each point in the space is a coset, represented by a particular value of $L(\omega^A)$; that is to say, there is a one-to-one mapping from the coset space to the space of all $L(\omega^A)$. The space clearly has $n - m$ dimensions - one for each of the ω^A which distinguish the cosets.

2.3 Goldstone Bosons

Now consider a situation where we have a Lagrangian which is invariant under transformations in the group G but the vacuum states are only invariant under H. Let us look at what happens if we apply group elements to one such vacuum state, ϕ_0. If two elements of G have the same values of ω^A but different values of ω^P (i.e. they lie in the same coset) they map ϕ_0 to the same vacuum state. Conversely, if they have the same values of ω^P but different values of ω^A, they map ϕ_0 to different vacuum states. The coset space G/H thus represents the set of transformations which map one vacuum state into a different vacuum state.

The connection between these transformations and the Goldstone bosons that occur in spontaneous symmetry breaking can be seen by the following simple, rather heuristic argument. (A rigorous discussion of these details of spontaneous symmetry breaking lies outside the scope of this thesis, but may be found in [5] or any textbook dealing with the subject, such as [28].)

Consider a potential V of a scalar multiplet ϕ^S which transforms as an d-dimensional representation of SU(2) (i.e. $S = 1, 2, \ldots, d$). The minima of the vacuum occur when

$$|\phi| \equiv ((\phi^1)^2 + (\phi^2)^2 + \ldots + (\phi^d)^2)^{\frac{1}{2}} = a \tag{2.13}$$

- we take these minima to be invariant under the U(1) subgroup in (2.8). (This is like an d-dimensional version of the potential in Figure 1.) Under the transformation $\phi^S \rightarrow L\phi^S$, the vacuum state ϕ_0^S is transformed into a different vacuum state, as discussed above. Taylor expanding $V(\phi^S)$ under this transformation and evaluating at $\phi^S = \phi_0^S$, we get

$$V(L\phi_0^S) = V(\phi_0^S) + \left.\frac{\partial V}{\partial \phi^S}\right|_{\phi^S=\phi_0^S} \delta L\phi_0^S + \frac{1}{2}\left.\frac{\partial^2 V}{\partial \phi^S \partial \phi^T}\right|_{\phi^S=\phi_0^S} \delta L\phi_0^S \delta L\phi_0^T + \ldots \tag{2.14}$$

The second order term looks like a mass term:

$$\left.\frac{\partial^2 V}{\partial \phi^S \partial \phi^T}\right|_{\phi^S=\phi_0^S} \delta L\phi_0^S \delta L\phi_0^T \sim M_{ST}\chi^S\chi^T \tag{2.15}$$

The ϕ_0^S is just a set of numbers and contains no variables; rather the variables in

$\delta L\phi_0^S$ are the coset space parameters:

$$\delta L\phi_0^S = -\mathrm{i}\omega^A T_A \phi_0^S \tag{2.16}$$

so regardless of the representation, in this SU(2)/U(1) case $\delta L\phi_0^S$ is always a function of two fields, one for each of the coset space parameters.

Now ϕ_0^S is a minimum of V, so $\frac{\partial V}{\partial \phi^S}$ at this point is zero. Also, the Lagrangian is invariant under SU(2), so the potential is invariant under the action of L on ϕ^S. Thus to second order from (2.14) we obtain

$$M_{ST}\chi^S\chi^T \sim \left.\frac{\partial^2 V}{\partial\phi^S\partial\phi^T}\right|_{\phi^S=\phi_0^S} \delta L\phi_0^S \delta L\phi_0^T = 0 \tag{2.17}$$

i.e. our two fields are massless.

It should be clear from this argument that in general we have one massless field for each coset space parameter - these are the Goldstone bosons. Note that by assigning particular values to each of these parameters, we assign particular amplitudes to each of the Goldstone fields. In particular, if we set all the coset space parameters to zero, each of the Goldstone boson amplitudes are zero. There is thus a one-to-one mapping between the space of Goldstone fields and the space of coset space parameters, which maps the origin of one space into the origin of the other; alternatively we can think of this as changing coordinates on a space from coset space parameters to Goldstone fields.

2.4 Goldstone field transformations

We would like to determine the transformation properties of the Goldstone bosons - that is, to determine the transformation properties of the vector space described above. We can now see the advantage of decomposing g into a coset space part and a subgroup part: L is a simple function of the vector space whose coordinates are the ω^A, or alternatively the Goldstone fields. In order to find the transformation properties of the vector space, we therefore start by looking at the (rather simple) transformation properties of L.

Under the action of $g \in G$, it is clear that the coset $g'H$ will transform into another coset:

$$g(g'H) = (gg'H) = g''H \tag{2.18}$$

$$\text{where} \quad g'' \in G$$

However, we may write a coset $g'H$ as a product of a particular L with the subgroup (as in equation (2.10)), so this is equivalent to saying:

$$g(LH) = L'H \tag{2.19}$$

which implies for L that

$$gL = L'h \tag{2.20}$$

$$\text{where} \quad h \in H.$$

By L' here we mean a new 'point' in the coset space:

$$L' = \mathrm{e}^{-\mathrm{i}\omega'^A T_A} \tag{2.21}$$

2.4.1 A Simple Example

To see how this helps us find the transformation properties of the vector space (as parametrised by the coset space parameters), it is once again easiest to use an example. The example we shall use is the simplest possible example; we look at the transformation of the L considered above (that of SU(2)/U(1)) under the U(1) subgroup:

$$gL = h(\phi^3)L \tag{2.22}$$

with $h(\phi^3)$ given by (2.8) (with θ'^3 replaced by ϕ^3). We can always multiply by the identity in the form $h^{-1}h$, which makes the right-hand side look more like the right-hand side of (2.20):

$$gL = hL(h^{-1}h) = (hLh^{-1})h \tag{2.23}$$

We might therefore expect to have $L' = hLh^{-1}$. Let us try calculating hLh^{-1} - we start by expanding the L in (2.11) as a power series.

$$\begin{aligned}
hLh^{-1} &= h\mathrm{e}^{-\frac{\mathrm{i}}{2}\theta^a\sigma_a}h^{-1} && (2.24)\\
&= h[\mathbf{1} - \frac{\mathrm{i}}{2}\theta^a\sigma_a + \frac{1}{2}(-\frac{\mathrm{i}}{2}\theta^a\sigma_a)(-\frac{\mathrm{i}}{2}\theta^b\sigma_b) \\
&\quad + \frac{1}{3!}(-\frac{\mathrm{i}}{2}\theta^a\sigma_a)(-\frac{\mathrm{i}}{2}\theta^b\sigma_b)(-\frac{\mathrm{i}}{2}\theta^c\sigma_c) + \ldots]h^{-1} && (2.25)\\
&= \mathbf{1} - h\frac{\mathrm{i}}{2}\theta^a\sigma_a h^{-1} + \frac{1}{2}h(-\frac{\mathrm{i}}{2}\theta^a\sigma_a)h^{-1}h(-\frac{\mathrm{i}}{2}\theta^b\sigma_b)h^{-1} \\
&\quad + \frac{1}{3!}h(-\frac{\mathrm{i}}{2}\theta^a\sigma_a)h^{-1}h(-\frac{\mathrm{i}}{2}\theta^b\sigma_b)h^{-1}h(-\frac{\mathrm{i}}{2}\theta^c\sigma_c)h^{-1} + \ldots && (2.26)\\
&= \mathbf{1} - \frac{\mathrm{i}}{2}h\theta^a\sigma_a h^{-1} + \frac{1}{2}(-\frac{\mathrm{i}}{2}h\theta^a\sigma_a h^{-1})(-\frac{\mathrm{i}}{2}h\theta^b\sigma_b h^{-1}) \\
&\quad + \frac{1}{3!}(-\frac{\mathrm{i}}{2}h\theta^a\sigma_a h^{-1})(-\frac{\mathrm{i}}{2}h\theta^b\sigma_b h^{-1})(-\frac{\mathrm{i}}{2}h\theta^c\sigma_c h^{-1}) + \ldots && (2.27)\\
&= e^{-\frac{\mathrm{i}}{2}h\theta^a\sigma_a h^{-1}} && (2.28)
\end{aligned}$$

Now we just need to determine $h\theta^a\sigma_a h^{-1}$. For this it is easiest to use $h(\phi^3)$ not in the form (2.8) but in an equivalent trigonometric form:

$$\begin{aligned} e^{-i\frac{\phi^3}{2}\sigma_3} &= \mathbf{1} - i\frac{\phi^3}{2}\sigma_3 - \frac{1}{2}(\frac{\phi^3}{2})^2\mathbf{1} + \frac{i}{3!}(\frac{\phi^3}{2})^3\sigma_3 + \ldots && (2.29) \\ &= \mathbf{1}\cos\frac{\phi^3}{2} - i\sigma_3\sin\frac{\phi^3}{2} && (2.30) \end{aligned}$$

and similarly $h^{-1}(\phi^3) = \mathbf{1}\cos\frac{\phi^3}{2} + i\sigma_3\sin\frac{\phi^3}{2}$. Substituting these into $h\theta^a\sigma_a h^{-1}$ and using the product rule for the σ's

$$\sigma_i\sigma_j = \mathbf{1}\delta_{ij} + i\epsilon_{ij}{}^k\sigma_k \qquad (2.31)$$

and the trigonometric identities

$$2\sin\frac{\phi}{2}\cos\frac{\phi}{2} = \sin\phi \quad \text{and} \quad 2\sin^2\frac{\phi}{2} = 1 - \cos\phi$$

we find after a little calculation

$$h\theta^a\sigma_a h^{-1} = (\theta^1\cos\phi^3 - \theta^2\sin\phi^3)\sigma_1 + (\theta^1\sin\phi^3 + \theta^2\cos\phi^3)\sigma_2 \qquad (2.32)$$

Thus we can indeed write $hL(\theta^a)h^{-1}$ as $L' = L(\theta'^a)$ where θ'^a is given by

$$\begin{aligned} \theta'^1 &= \theta^1\cos\phi^3 - \theta^2\sin\phi^3 && (2.33) \\ \theta'^2 &= \theta^1\sin\phi^3 + \theta^2\cos\phi^3 && (2.34) \end{aligned}$$

or in matrix form

$$\begin{pmatrix} \theta'^1 \\ \theta'^2 2 \end{pmatrix} = \begin{pmatrix} \cos\phi^3 & -\sin\phi^3 \\ \sin\phi^3 & \cos\phi^3 \end{pmatrix} \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} \qquad (2.35)$$

i.e. θ^a is transformed as a doublet of U(1). This is a linear transformation: each θ'^a is just a linear sum of θ^as. This is actually the case for any such coset space G/H - if we act on L with the subgroup H we find that the coset space parameters transform as a representation of H.

Finding the transformation of the θ^as was particularly simple in this case. There are several reasons for this: the Abelian nature of H certainly helped, but also the fact that $L' = hLh^{-1}$ allowed us to use a helpful property of the similarity transformation of an exponential. This, of course, will not be possible if we look at transformations under $g \notin H$. Also, we have only found how the θ^as transform - we would like to see how the Goldstone fields themselves transform, though in general we only need to know the transformation properties under an infinitesimal transformation.

We shall address these two issues in turn, which will allow us to give a general prescription for finding the transformation properties of the Goldstone fields which will be valid for all the transformations we will be considering in this thesis.

2.4.2 The Outer Involutive Automorphism

When we consider the transformation of L under the action of elements of G which are not in the subgroup H, there is no simple technique for obtaining L' from (2.20) which is valid for all G/H. However, the three coset spaces we will be considering have a useful property which allows us to obtain an equation for L'^2 from (2.20). To get a feel for the meaning of this property, let us look at the commutators of the generators of G. In general, we may write the commutator of two generators as

$$[T_I, T_J] = \mathrm{i} f_{IJ}{}^K T_K \tag{2.36}$$

using the conventional normalisation, where ${f_{IJ}}^K$ are a set of totally antisymmetric structure constants (we will consider these in more detail in later chapters). We may decompose these relations into three sets (this is following an argument in [14]). Firstly, we have the commutator of two generators of H. As H is a group, the commutator must close onto generators of H. Therefore we have, for $P, Q, R = 1, \ldots, m$ where m is the dimension of H,

$$[T_P, T_Q] = \mathrm{i} {f_{PQ}}^R T_R \tag{2.37}$$

Note that all the ${f_{PQ}}^A$ for $A = m+1, \ldots, n$ (where n is the dimension of G) are zero. From the antisymmetry of the structure constants this then means that all of the ${f_{PA}}^Q$ are zero, so for the commutator of a subgroup generator with a coset space generator, we have

$$[T_P, T_A] = \mathrm{i} {f_{PA}}^B T_B \tag{2.38}$$

- i.e. this commutator closes onto the coset space generators.

Finally, we can look at the commutator of two coset space generators. In general, this is a linear sum of both subgroup and coset space generators:

$$[T_A, T_B] = \mathrm{i} {f_{AB}}^P T_P + \mathrm{i} {f_{AB}}^C T_C \tag{2.39}$$

However, the three coset spaces we will be considering belong to a class known as 'symmetric spaces' for which all of the ${f_{AB}}^C$ are zero. Thus for these, (2.39) reduces to the simple form

$$[T_A, T_B] = \mathrm{i} {f_{AB}}^P T_P \tag{2.40}$$

just closing onto the subgroup generators; thus the algebra has a '$\mathbb{Z}_2$ grading structure'. Note that if we map each of the coset space generators T_A into $-T_A$ (but do not alter the subgroup generators), the commutators (2.37), (2.38) and (2.40) are unaffected. However, this is not true for the commutator (2.39), so the algebra admits the 'outer involutive automorphism'

$$T_A \to \tilde{T}_A = -T_A \tag{2.41}$$

if and only if the coset space is symmetric. This property is particularly useful to us as we can use it to derive an expression for L'^2 from (2.20). First, note that under this automorphism, h, defined by (2.5) is unaffected, while using the forms of L and L' given in (2.12) and (2.21) we see that $L \to L^{-1}$ and $L' \to L'^{-1}$. Thus applying the automorphism to (2.20) we get

$$\tilde{g}L^{-1} = L'^{-1}h \tag{2.42}$$

where $\tilde{g} = g$ if g lies entirely in the subgroup or $\tilde{g} = -g$ if g lies entirely in the coset space. Now we invert this expression:

$$L\tilde{g}^{-1} = h^{-1}L' \tag{2.43}$$

and premultiply it by equation (2.20):

$$gL^2\tilde{g}^{-1} = L'^2 \tag{2.44}$$

We can thus use this expression to obtain L'^2 for any g in the same way as we used $L' = hLh^{-1}$.

2.4.3 Killing vectors

We can use the above equation to find how the coset space parameters ω^A transform under any transformation $g(\phi^I)$ in G. We would, however, like to know how the Goldstone bosons transform, at least to first order, under such transformations.

Recall that the Goldstone fields can be thought of as an alternative coordinate basis for the coset space to the parameters ω^A, one which has the same origin - we are now thinking of the space as a 'field space'[1]. We have denoted the invariance group of the Lagrangian G and we shall see in Chapter 3 that the Lagrangian and the metric of the space are very closely related and share an invariance group, i.e. G is the isometry group of the field space.

We shall denote the Goldstone fields M^A. Under an isometry transformation of the field space, they transform as

$$M^A \rightarrow M'^A = M^A + \phi^I K_I^A + \mathcal{O}(\phi^I)^2 \tag{2.45}$$

where the K_I^A are quantities known as Killing vectors (satisfying Killing's equation $\nabla_{(A} K_{B)I} = 0$) which clearly fully specify the first order transformation of the field space coordinates under the isometry. We can also Taylor expand the L'^2 resulting from these transformations in powers of M^A, which in principle we know how to find:

$$L'^2 = L^2 + \frac{\partial L^2}{\partial M^A}\delta M^A + \mathcal{O}(\delta M^A)^2 \tag{2.46}$$

[1] For SU(2)/U(1), this space looks like the surface of a sphere[19]

but we see from above that $\delta M^A = \phi^I K_I^A$, so we therefore have

$$\delta(L^2) = \frac{\partial L^2}{\partial M^A}\phi^I K_I^A \tag{2.47}$$

We can use this to find the Killing vectors. To do so, we simply need to find the first order variation in L^2 and the derivative $\frac{\partial L^2}{\partial M^A}$. The first of these we can do easily from (2.44):

$$\delta(L^2) = \delta g L^2 + L^2 \delta\tilde{g}^{-1} \tag{2.48}$$

giving us the important result

$$\delta g L^2 + L^2 \delta\tilde{g}^{-1} = \frac{\partial L^2}{\partial M^A}\phi^I K_I^A \tag{2.49}$$

Let us now see how to use this to obtain the Killing vectors of SU(2)/U(1).

2.4.4 The Killing vectors of SU(2)/U(1)

We will search first for the Killing vectors relating to the linear transformation (2.22). We start by identifying δg and $\delta\tilde{g}^{-1}$:

$$g = \tilde{g} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\phi^3\sigma_3} = \mathbf{1} - \frac{\mathrm{i}}{2}\phi^3\sigma_3 + \mathcal{O}(\phi^3)^2 \tag{2.50}$$

$$\Rightarrow \delta g = -\frac{\mathrm{i}}{2}\phi^3\sigma_3 \tag{2.51}$$

and similarly $\delta\tilde{g}^{-1} = \frac{\mathrm{i}}{2}\phi^3\sigma_3$ so in this case

$$\delta g L^2 + L^2\delta\tilde{g}^{-1} = -\frac{\mathrm{i}}{2}\phi^3\sigma_3 L^2 + \frac{\mathrm{i}}{2}\phi^3 L^2\sigma_3 = \frac{\mathrm{i}}{2}\phi^3[L^2, \sigma_3] \tag{2.52}$$

To calculate this commutator, or for that matter the derivative on the right-hand side of (2.49), we will need an explicit expression for L^2 as a linear sum of generators. We obtain this in much the same way as we got the trigonometric expression for h. We start with L^2 given by the square of (2.11), which is an exponential of the vector $\boldsymbol{\theta} = \theta^a \sigma_a$. Writing L^2 as a power series will give us terms in increasing powers of this vector. By using the product rule for the σ's, we find that

$$(\theta^a \sigma_a)^2 = (\theta^1)^2 \mathbf{1} + (\theta^2)^2 \mathbf{1} \tag{2.53}$$

We now define

$$\theta = \sqrt{(\theta^1)^2 + (\theta^2)^2} \tag{2.54}$$

and

$$n^a = \frac{\theta^a}{\theta} \tag{2.55}$$

(We will see in Section 4.2.1 that what we are doing here is in a very precise sense defining the 'length' of the vector $\boldsymbol{\theta}$ and an associated unit vector. We will always use a bold typeface for vectors of the algebra and a normal typeface for their lengths.)

With these definitions

$$(n^a \sigma_a)^2 = \frac{(\theta^1)^2 + (\theta^2)^2}{\theta^2} \mathbf{1} = \mathbf{1} \tag{2.56}$$

Now calculating L^2 as a linear sum of generators is easy:

$$\begin{aligned} L^2 &= e^{-i\theta^a\sigma_a} && (2.57)\\ &= e^{-i\theta n^a\sigma_a} && (2.58)\\ &= \mathbf{1} - i\theta n^a\sigma_a - \frac{1}{2}\theta^2\mathbf{1} + \frac{i}{3!}\theta^3 n^a\sigma_a + \ldots && (2.59)\\ &= \mathbf{1}\cos\theta - in^a\sigma_a\sin\theta && (2.60) \end{aligned}$$

Once again using the product rule for σ's, the commutator in (2.52) is then easily found to be

$$[L^2, \sigma_3] = 2\epsilon_{a3}{}^b n^a \sigma_b \sin\theta \tag{2.61}$$

so using (2.49) and (2.52) we obtain

$$\epsilon_{a3}{}^b n^a \sigma_b \sin\theta = -i\frac{\partial L^2}{\partial M^a} K_3^a \tag{2.62}$$

To find a helpful form of $\frac{\partial L^2}{\partial M^a}$ we have to understand a little about the vectors in this situation. In the same way that the θ^a are components of a vector $\boldsymbol{\theta}$ in the coset space part of the algebra, the fields M^a are components of a vector $\mathbf{M}$ in the coset space part of the algebra. So far, we have considered a completely general form of the vector $\boldsymbol{\theta}$ and similarly the field amplitudes M^a are completely arbitrary. We are only working with the θ^a to help us get a handle on the field amplitudes and we can considerably simply things if we now choose to work with a $\boldsymbol{\theta}$ which lies in the same direction in the coset space as $\mathbf{M}$. This choice entails no loss of generality in the field amplitudes and any other choice corresponds simply to taking linear combinations of the fields, which we are always free to do. (It will turn out that leaving the magnitude of the vector $\boldsymbol{\theta}$ as an arbitrary function of the fields $M \equiv M^a M_a$ - other than being one-to-one and respecting the same origin - does not unnecessarily complicate the calculations.)

With $\boldsymbol{\theta}$ and $\mathbf{M}$ lying in the same direction, they clearly share a unit vector n^a; i.e.

$$\theta^a = \theta n^a; \qquad M^a = M n^a; \qquad n^a n_a = 1 \tag{2.63}$$

From these relations we can derive the useful identities

$$\frac{\partial M}{\partial M^b} = n^b; \qquad \frac{\partial \theta}{\partial M^b} = \frac{\mathrm{d}\theta}{\mathrm{d}M} n^b \tag{2.64}$$

and we can then use the fact that

$$\frac{\partial M n^a}{\partial M^b} = \frac{\partial M^a}{\partial M^b} = \delta^a_b \tag{2.65}$$

to derive

$$\frac{\partial n^a}{\partial M^b} = \frac{1}{M}(\delta^a_b - n^a n_b) \tag{2.66}$$

Differentiating L^2 in the form (2.60) and using these identities we then find

$$\frac{\partial L^2}{\partial M^a} = -\mathbf{1} n_a \sin\theta \frac{\mathrm{d}\theta}{\mathrm{d}M} - \mathrm{i}\sigma_a \frac{\sin\theta}{M} + \mathrm{i} n_a n^b \sigma_b \frac{\sin\theta}{M} - \mathrm{i} n_a n^b \sigma_b \cos\theta \frac{\mathrm{d}\theta}{\mathrm{d}M} \tag{2.67}$$

so substituting into (2.62) we get

$$\epsilon_{a3}{}^b n^a \sigma_b = \left(\mathrm{i}\mathbf{1} n_a \frac{\mathrm{d}\theta}{\mathrm{d}M} - \frac{1}{M}\sigma_a + n_a n^b \sigma_b \frac{1}{M} - n_a n^b \sigma_b \cot\theta \frac{\mathrm{d}\theta}{\mathrm{d}M} \right) K^a_3 \tag{2.68}$$

Now this looks very long and messy, but actually all but one of the terms on the right hand side are zero, as can be seen by taking the trace of both sides, which gives us

$$0 = 2\left(\mathrm{i} n_a \frac{\mathrm{d}\theta}{\mathrm{d}M} \right) K^a_3 \Rightarrow n_a K^a_3 = 0 \tag{2.69}$$

so all that survives of the above expression is

$$\epsilon_{a3}{}^{b} n^{a} \sigma_{b} = -\frac{1}{M} \sigma_{a} K_{3}^{a} \tag{2.70}$$

from which we see that $K_3^b \sigma_b$ is a vector with components

$$K_3^b = M^a \epsilon_a{}^b{}_3 \tag{2.71}$$

Thus we have found the Killing vector which describes - through (2.45) - the transformation of the Goldstone fields under the U(1) subgroup to first order:

$$M^b \rightarrow M'^b = M^b + M^a \epsilon_a{}^b{}_3 \phi^3 \tag{2.72}$$

- again, this is a linear transformation; it is, as could be expected, the transformation of a vector of SU(2) under the U(1) subgroup, as we will see in Section 4.2.2.

Let us now briefly recap what we have done so far in this section. We started by looking at the transformation properties not of the Goldstone fields themselves but of the coset space representative L. We showed that L transformed according to (2.20). We then noted that for symmetric coset spaces we could make use of the involutive automorphism and doing so we obtained an equation for L'^2, equation (2.44). However, our ultimate aim was an expression for the first order transformation of the Goldstone fields. To find this, we expanded the left-hand side of (2.44) as a power series in the transformation parameter ϕ^I and similarly Taylor expanded the right-hand side in this same parameter and equated the first order variations. Writing the first order variation in the fields as $\phi^I K_I^A$, we thus obtained (2.49).

We then focused on the particular example of how the Goldstone fields of SU(2)/

U(1) transform under elements of the U(1) subgroup. Once we had found suitable descriptions of L^2 and $\frac{\partial L^2}{\partial M^a}$, the Killing vectors were relatively easy to find.

Having used this method to find the subgroup Killing vectors, we now turn to the transformations of the fields under other elements of SU(2). In fact, as a general group element may be decomposed in the form (2.7), we can restrict our attention to transformations under elements of the form

$$g = \mathrm{e}^{-\frac{\mathrm{i}}{2}\phi^a \sigma_a}$$

In this case, $\tilde{g} = g$, so equating first order variations of the left and right sides of (2.49) gives us

$$-\frac{\mathrm{i}}{2}\phi^b \{L^2, \sigma_b\} = \frac{\partial L^2}{\partial M^a}\phi^b K_b^a \tag{2.73}$$

Inserting (2.60) and (2.67), using the product rule for the σ's and equating coefficients of ϕ^b, we get

$$\begin{aligned} -\mathrm{i}\sigma_b \cos\theta - \mathbf{1} n_b \sin\theta \;=\; & \left(-\mathbf{1} n_a \sin\theta \frac{\mathrm{d}\theta}{\mathrm{d}M} - \mathrm{i}\sigma_a \frac{\sin\theta}{M} + \mathrm{i} n_a n^c \sigma_c \frac{\sin\theta}{M} \right. \\ & \left. - \mathrm{i} n_a n^c \sigma_c \cos\theta \frac{\mathrm{d}\theta}{\mathrm{d}M} \right) K_b^a \end{aligned} \tag{2.74}$$

Again, we can take the trace of both sides, which gives us

$$n_a K_b^a = \frac{\mathrm{d}M}{\mathrm{d}\theta} n_b \tag{2.75}$$

Substituting this back in, then dividing by $\mathrm{i}\sin\theta/M$ and rearranging we get

$$\sigma_a K_b^a = \sigma_b M \cot\theta - n_b n^c \sigma_c M \cot\theta + n_b n^c \sigma_c \frac{\mathrm{d}M}{\mathrm{d}\theta} \tag{2.76}$$

The easiest way to equate coefficients of the generators (components of the vectors) is to multiply by another σ and take the trace. Doing this finally gives us the Killing vector:

$$K^a_b = M \cot\theta(\delta^a_b - n^a n_b) + n^a n_b \frac{\mathrm{d}M}{\mathrm{d}\theta} \tag{2.77}$$

We therefore see that the transformation of the Goldstone fields in this case is given by

$$M^a \rightarrow M'^a = M^a + M \cot\theta(\phi^a - \phi^b n_b n^a) + \phi^b n_b n^a \frac{\mathrm{d}M}{\mathrm{d}\theta} \tag{2.78}$$

to first order in ϕ^b, where of course $\theta = \theta(M)$. Unlike the transformation under the subgroup, this is a non-linear transformation - the first order variation involves a complicated function of the fields. We say that the Goldstone fields, rather than forming a (linear) representation of G, form a *non-linear realisation* of G.

Finally, it is worth noting that although we chose for this transformation not to calculate $gL = L'h$ and opted instead to look at the first order variations of $gL^2\tilde{g}^{-1} = L'^2$, we could have multiplied g directly into L. The calculation and the result are very messy indeed, but one does get an answer of the required form $L'h$; the only feature of any interest to us is that in general h is a function of both ϕ^b and θ^a, i.e. of the transformation parameters and the Goldstone fields. (We shall need to be aware of this in the next chapter.)

Chapter 3

Constructing a Lagrangian

3.1 Standard Fields

In the last chapter we saw how whenever a Lie group symmetry is broken, a set of massless Goldstone bosons occur. We considered these fields as though they were the only ones present in the system and indeed Goldstone's theorem does *not* require that other fields are present, either in the broken or the unbroken theory. A consequence of this is that the Goldstone bosons must transform as a realisation of G on their own: under the action of an element of G, each Goldstone boson is transformed into combination of Goldstone bosons without involving other fields; we saw in the last chapter that in general this is a *non-linear* combination.

However, from a particle physics perspective, we are generally interested in involving fermionic and/or other bosonic fields in the theory which interact with the Goldstone bosons. Furthermore, as we want our non-linear theory to arise as a consequence of spontaneous symmetry breaking in a linear theory, these other fields must result from fields in the linear theory, that is, in a theory in which they transform as a representation of G. In breaking the symmetry, we redefine these fields such that the new fields, known as 'standard fields', transform linearly only under H and transform non-linearly under the rest of G (just as the Goldstone bosons do).

Unlike the Goldstone bosons, it is not necessarily true that the standard fields form a realisation of G on their own. Indeed if we involve the Goldstone bosons in the redefinition, the transformation of the standard fields will involve the Goldstone bosons and consequently be non-linear as required and the Lagrangian will naturally include interactions between the standard fields and the Goldstone bosons.

What we want, therefore, is a way to redefine a field multiplet of G, say Φ^S, involving the Goldstone bosons in the definition in such a way that the redefined fields, say ψ^S, transform as a representation of H. They will then form a non-linear realisation of G together with the Goldstone bosons. Such a redefinition is given in [10]; it is easiest to follow in the case where Φ^S transforms as the defining representation of G (so $S = 1, 2, \ldots, d$ where d is the dimension of the defining representation):

$$\Phi^S \to \Phi'^S = g^S_T \Phi^T \tag{3.1}$$

In particular, for $h \in H$,

$$\Phi^S \to \Phi'^S = h^S_T \Phi^T \tag{3.2}$$

The standard fields ψ^S are then given in terms of the original multiplet Φ^S and the Goldstone bosons by

$$\psi^S = (L^{-1})^S_T \Phi^T \tag{3.3}$$

We can then easily see that under the action of $h \in H$ the standard fields transform as the defining representation:

$$\psi^S = (L^{-1})^S_T \Phi^T \rightarrow \psi'^S = (L'^{-1})^S_T \Phi'^T \tag{3.4}$$

$$= ((hLh^{-1})^{-1})^S_T h^T_U \Phi^U \tag{3.5}$$

$$= (hL^{-1}h^{-1})^S_T h^T_U \Phi^U \tag{3.6}$$

$$= h^S_V (L^{-1})^V_U \Phi^U \tag{3.7}$$

$$= h^S_V \psi^V \tag{3.8}$$

We also need to know how these fields transform under elements of G which are functions only of the coset space parameters. We can find this in much the same way:

$$\psi^S = (L^{-1})^S_T \Phi^T \rightarrow \psi'^S = (L'^{-1})^S_T \Phi'^T \tag{3.9}$$

$$= (L'^{-1})^S_T g^T_U \Phi^U \tag{3.10}$$

$$= (L'^{-1} g)^S_U \Phi^U \tag{3.11}$$

$$= (L'^{-1} g)^S_U L^U_V \psi^V \tag{3.12}$$

$$= (L'^{-1} g L)^S_V \psi^V \tag{3.13}$$

but from (2.20), $L'^{-1} g L = h$ where h is a function of the Goldstone bosons

$$= h^S_V \psi^V \tag{3.14}$$

If Φ^S transforms as some other representation of G:

$$\Phi^S \rightarrow \Phi'^S = \Gamma(g)^S_T \Phi^T \tag{3.15}$$

we simply note that, by definition (of a representation),

$$\Gamma(g)^S_T \Gamma(g')^T_U = \Gamma(gg')^S_U \tag{3.16}$$

for any two elements $g, g' \in G$. Thus all of the above calculations are equally valid for in this representation, so we end up with

$$\psi^S \to \psi'^S = \Gamma(h)^S_V \psi^V \tag{3.17}$$

3.2 Constructing a Lagrangian

Having seen what fields we may include in our non-linear realisation and how they transform, we would now like to construct a Lagrangian for the system. As a whole, the Lagrangian must be Lorentz invariant and invariant under G. These restrictions actually allow us to determine the form of each term in the Lagrangian, as we shall now see[1].

For a normal, linear theory, the Lagrangian is composed of kinetic terms for each field, mass terms for the massive fields and interaction terms. The mass terms have form

$$\frac{m^2}{2}\phi^S\phi_S$$

for scalars, and

$$m\bar{\psi}^S\psi_S$$

for fermions.

[1] Actually, if we only require that the action be invariant under G, we may include extra terms in the action which are not invariant under transformations of G, but change by a total derivative[29, 30]. D'Hoker and Weinberg have shown that these terms are in one-to-one correspondence with the generators of the fifth cohomology class of G/H[31]. For SU(2)/U(1), which we consider in this chapter, this class is zero so there are no extra terms.

For our non-linear realisation it is clear from the transformations derived in the last chapter that such a term for the Goldstone bosons is invariant under H, but not the whole of G, so we cannot include such a term in the Lagrangian. (That is, the Goldstone bosons are massless, as they should be.) Indeed, it is observed in [15] that M^A cannot appear without derivatives. For the standard fields, on the other hand, we see from the transformations in the last section that if a polynomial in Φ^S is allowed in the linear theory (is invariant under G) the same polynomial in Ψ^S is allowed in the non-linear realisation.

We now turn to terms involving derivatives. First we note that the Goldstone bosons do not transform as representations of G so neither do their derivatives. This means that the normal kinetic term for scalars in the linear theory is not invariant under the whole of G. Obviously we cannot just throw away this term if we want our Goldstone bosons to be real, dynamical fields, so we must add other terms to it such that the sum of the terms is invariant. This is analogous to the case of a gauged symmetry where a gauge field is added to a Lagrangian to make it invariant under local transformations - this is usually achieved by constructing covariant derivatives which involve the gauge fields and, following [11], we will adopt precisely the same approach. Our 'kinetic term' thus has the form

$$\frac{1}{2}\mathrm{D}_\mu M^A \mathrm{D}^\mu M_A$$

(with A once again running over the coset space indices and μ running over the spacetime indices)

where $\mathrm{D}_\mu M^A$ is a 'covariant derivative' of the form

$$\mathrm{D}_\mu M^A = \partial_\mu M^A + \text{ something}$$

yet to be determined.

For standard fields the problem is similar - again the normal spacetime derivatives are not invariant:

$$\partial_\mu \psi^S \to \partial_\mu \psi'^S = (\partial_\mu \Gamma(h)^S_V)\psi^V + \Gamma(h)^S_V \partial_\mu \psi^V \tag{3.18}$$

so we need to find a covariant derivative $\mathrm{D}_\mu \psi$ which transforms in the same way as ψ, i.e.

$$\mathrm{D}_\mu \psi^S \to (\mathrm{D}_\mu \psi^S)' = \Gamma(h)^S_V \mathrm{D}_\mu \psi^V \tag{3.19}$$

To find these covariant derivatives we again turn to the work of Coleman *et al* [10, 11]. They use a particularly complicated and subtle argument to justify a particular form for these covariant derivatives - we shall present a basic outline of this argument here and then show that the forms that are obtained have the correct properties.

We think of the M^A and the Φ^S as coordinates on a manifold. The action of a group element on this manifold falls into two parts: the action on the M^A is given by multiplying the element into $L(M^A)$, while the action on the Φ^S is given simply by acting on the fields with the appropriate representation of the element. Under the action of L^{-1}, the Goldstone bosons are therefore transformed away, while the Φ^S are transformed into standard fields. This transformation therefore takes us from a set of coordinates which have a very complicated behaviour under the action of elements of G which are not in H to a set of coordinates with a particularly simple behaviour under this action.

Now if we consider the coordinates on this manifold as functions of spacetime, we can also ask about the transformation properties of the gradients of the fields. The important quantity here is the difference between the coordinates evaluated at two neighbouring points in space time, $(M^A(x), \Phi^S(x))$ and

$(M^A(x+\delta x), \Phi^S(x+\delta x))$. (For the moment we shall suppress the spacetime indices and write x^μ as x.) We know that both of these have complicated transformation properties under the action of elements of G which are not in H, but at least the former can be simplified by multiplying by L^{-1} and the latter will vary infinitesimally from this.

Take as an example the action of L^{-1} on the gradient of the Goldstone fields. If we act on the fields themselves by multiplying $L(M^A)$ by L^{-1}, we must act on the gradient of the fields by multiplying

$$\lim_{\delta x \to 0} \frac{L(x+\delta x) - L(x)}{\delta x} \equiv \lim_{\delta x \to 0} \frac{\delta L}{\delta x}$$

by L^{-1}. However, as the transformation of $L(M^A)$ by L^{-1} is a special case of (2.20), we can use take a δ-variation of both sides of (2.20) to find $L^{-1}\delta L$ and, using exponential forms of L' and h, it turns out to be

$$L^{-1}(x)\delta L = -\mathrm{i}\omega^A(\delta x)T_A - \mathrm{i}\eta^P(\delta x)T_P \tag{3.20}$$

where $\eta^P T_P$ is the vector in the exponential of h[1]:

$$h = \mathrm{e}^{-\mathrm{i}\eta^P T_P}$$

which, as we remarked at the end of the last chapter, is a function of the fields

[1]It is a vector in the sense that it is an element of a vector space, the Lie algebra. We shall see in Chapter 4 that this means that it transforms as a vector if you act on it with a group element by conjugation. However, all the transformations we will be considering will be applied by acting with a group element on L and these induce transformations on $\eta^P(\delta x)T_P$; under such transformations $\eta^P(\delta x)T_P$ does not transform as a vector. The same goes for $\omega^A(\delta x)T_A$, as we shall see on the next page.

and therefore of x. Thus (reinstating the spacetime indices)

$$\lim_{\delta x^\mu \to 0} \left(L^{-1} \frac{\delta L}{\delta x^\mu} \right) = L^{-1} \left(\lim_{\delta x^\mu \to 0} \frac{\delta L}{\delta x^\mu} \right) = L^{-1} \partial_\mu L$$

is a sum of a vector of the coset space and a vector of the subgroup (see footnote), both of which are derivatives with respect to x, which, the argument goes, transform in a very straightforward manner. It would seem logical that the part in the coset space is at least related to the covariant derivative of the Goldstone fields, while the part in the subgroup comes from the $h(\eta(\delta x))$ which describes the transformation of the standard fields, so we might expect this to be connected with the covariant derivative of the standard fields.

Now, for a given coset space such as SU(2)/U(1), if we can put L in a trigonometric form, we can of course find $L^{-1}\partial_\mu L$ directly. We shall do precisely this in the next section, where we shall see that $L^{-1}\partial_\mu L$ does indeed naturally fall into two terms, one in the coset space part of the algebra and one in the subgroup part - there are no terms involving the identity matrix. The two terms are usually written (upto a factor) as $\mathbf{a}_\mu$ and $\mathbf{v}_\mu$ respectively:

$$L^{-1}\partial_\mu L = -\frac{\mathrm{i}}{2}(\mathbf{a}_\mu + \mathbf{v}_\mu) \tag{3.21}$$

(For example, for SU(2)/U(1) we may write these vectors $\mathbf{a}_\mu = a^a_\mu \sigma_a$ and $\mathbf{v}_\mu = v^3_\mu \sigma_3$.)

To see how these vectors are related to the covariant derivatives, we need to use

$$L' = gLh^{-1} \tag{3.22}$$

(from (2.20)) and invert it (to obtain the transformation law for L^{-1}) and differentiate it (to obtain the transformation law for $\partial_\mu L$). The inverse is

$$L'^{-1} = hL^{-1}g^{-1} \tag{3.23}$$

and the differential is

$$\partial_\mu L' = g((\partial_\mu L)h^{-1} + L\partial_\mu(h^{-1})) \tag{3.24}$$

so multiplying them together we get

$$L'^{-1}\partial_\mu L' = hL^{-1}(\partial_\mu L)h^{-1} + h\partial_\mu(h^{-1}) \tag{3.25}$$

This is the transformation law for $L^{-1}\partial_\mu L$, i.e.

$$L^{-1}\partial_\mu L \to (L^{-1}\partial_\mu L)' = hL^{-1}(\partial_\mu L)h^{-1} + h\partial_\mu(h^{-1}) \tag{3.26}$$

Now h does not contain any coset space generators so therefore neither does $h\partial_\mu(h^{-1})$. Therefore we see by comparing (3.26) and (3.21) that $\mathbf{a}_\mu$ and $\mathbf{v}_\mu$ transform according to

$$\mathbf{a}_\mu \to \mathbf{a}'_\mu = h\mathbf{a}_\mu h^{-1} \tag{3.27}$$

$$-\frac{\mathrm{i}}{2}\mathbf{v}_\mu \to -\frac{\mathrm{i}}{2}\mathbf{v}'_\mu = -\frac{\mathrm{i}}{2}h\mathbf{v}_\mu h^{-1} + h\partial_\mu(h^{-1}) \tag{3.28}$$

We see that $\mathbf{a}_\mu$ has the correct transformation properties to be a covariant derivative by noting that

$$\mathbf{a}^\mu\mathbf{a}_\mu \to h\mathbf{a}^\mu\mathbf{a}_\mu h^{-1} \tag{3.29}$$

so that $\mathrm{tr}(\mathbf{a}^\mu \mathbf{a}_\mu)$ is an invariant under the action of G. Taking the trace just contracts the components - again, we show this for SU(2)/U(1) (we shall not look at the general case here because the normalisation of vectors of special orthogonal groups is different from that of special unitary groups, as we will see in Chapter 4)

$$\mathrm{tr}(\mathbf{a}^\mu \mathbf{a}_\mu) = \mathrm{tr}(a^\mu_a \sigma^a a^b_\mu \sigma_b) = a^\mu_a a^b_\mu \,\mathrm{tr}(\sigma^a \sigma_b) = a^\mu_a a^b_\mu \delta^a_b = a^\mu_a a^a_\mu \tag{3.30}$$

Thus with

$$\mathrm{D}^\mu M_A = a^\mu_A \tag{3.31}$$

the kinetic term $\frac{1}{2}\mathrm{D}^\mu M_A \mathrm{D}_\mu M^A$ is invariant as required.

(We will calculate this for the case of SU(2)/U(1) in the next section and we will see that $\mathrm{D}_\mu M^A$ does take the expected form $\partial_\mu M^A +$ something and what normalisation this gives the kinetic term. Furthermore, note that if we were to add another quantity to a^μ_A which transforms in the same way, so the transformation property of $\mathrm{D}^\mu M_A$ as a whole remains the same, we would get unwanted terms in $\frac{1}{2}\mathrm{D}^\mu M_A \mathrm{D}_\mu M^A$ which would spoil its invariance. Thus the form of the covariant derivative we have found is the only one which leads to an invariant kinetic term.)

We now turn to the derivatives of the standard fields. The Φ^S transform linearly under G and therefore so do their derivatives $\partial_\mu \Phi^S$ (indeed these are the derivatives in the linear theory). It must be the case, then, that

$$L^{-1}\partial_\mu \Phi^S = L^{-1}\partial_\mu (L\psi^S) = \partial_\mu \psi^S + (L^{-1}\partial_\mu L)\psi^S$$

transforms in the same way as

$$L^{-1}\Phi^S = \psi^S$$

(this is following an argument in [15]), so it seems that the 'something' we need to add to the usual partial derivative of the standard fields to get the covariant derivative is just $L^{-1}\partial_\mu L$. Actually, as suggested on the previous page, we only need the $-\frac{i}{2}\mathbf{v}_\mu$ part of $L^{-1}\partial_\mu L$. This is because to construct a covariant derivative, we need to add on to the partial derivative a term which transforms in such a way as to cancel the inhomogeneous term in (3.18) - that is, its transformation must contain a derivative of $\Gamma(h)$. The term of this form in the transformation of $(L^{-1}\partial_\mu L)\psi^S$ can be seen from (3.28) to lie entirely in the $-\frac{i}{2}\mathbf{v}_\mu\psi^S$ part. Let us now see that this cancellation does occur - that the combination $\partial_\mu\psi^S - \frac{i}{2}\mathbf{v}_\mu\psi^S$ does transform in the same way as ψ^S (we will work with standard fields transforming as the defining representation of H for simplicity but the following is valid for any representation).

$$\partial_\mu\psi - \frac{i}{2}\mathbf{v}_\mu\psi \quad\rightarrow\quad (\partial_\mu h)\psi + h\partial_\mu\psi + \left(-\frac{i}{2}h\mathbf{v}_\mu h^{-1} + h\partial_\mu(h^{-1})\right)h\psi \quad (3.32)$$

$$= (\partial_\mu h)\psi + h\partial_\mu\psi - \frac{i}{2}h\mathbf{v}_\mu\psi + h\partial_\mu(h^{-1})h\psi \quad (3.33)$$

but

$$hh^{-1} = \mathbf{1} \Rightarrow h(\partial_\mu h^{-1}) + (\partial_\mu h)h^{-1} = 0 \Rightarrow h(\partial_\mu h^{-1}) = -(\partial_\mu h)h^{-1} \quad (3.34)$$

so the fourth term becomes

$$-(\partial_\mu h)h^{-1}h\psi = -(\partial_\mu h)\psi$$

which does indeed cancel the first term, leaving us with

$$\partial_\mu\psi - \frac{i}{2}\mathbf{v}_\mu\psi \rightarrow h\partial_\mu\psi - \frac{i}{2}h\mathbf{v}_\mu\psi = h\left(\partial_\mu\psi - \frac{i}{2}\mathbf{v}_\mu\psi\right) \quad (3.35)$$

This combination thus clearly does transform in the same way as the standard fields themselves and is therefore the covariant derivative we have been looking for.

We close this section by summarising the above theory for the example of a system which contains only fermionic standard fields which have no self-interaction terms (fourth order self-couplings and so forth). For such a system, the demands of invariance under G and Lorentz invariance impose the following form on the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\mathrm{D}^{\mu} M_A \mathrm{D}_{\mu} M^A + \mathrm{i}\bar{\psi}^S \gamma^{\mu} \mathrm{D}_{\mu} \psi_S + m\bar{\psi}^S \psi_S \tag{3.36}$$

where

$$\mathrm{D}_{\mu}\psi = \partial_{\mu}\psi - \frac{\mathrm{i}}{2}\mathbf{v}_{\mu}\psi \tag{3.37}$$

and $\mathrm{D}_{\mu} M^A$ is given by (3.31), with $\mathbf{a}_{\mu}$ and $\mathbf{v}_{\mu}$ defined by (3.21). The first term in the Lagrangian contains the normal kinetic term for the Goldstone bosons, $\frac{1}{2}\partial^{\mu} M_A \partial_{\mu} M^A$, and also self-interaction terms, while the interactions between the Goldstone bosons and the standard fields are contained in the second term, courtesy of the $\mathbf{v}_{\mu}$.

3.3 SU(2)/U(1)

Let us now find the covariant derivatives for our simple example of SU(2)/U(1).

The easiest way to find a useful form of $L^{-1}\partial_{\mu} L$ is to start with L in trigonometric form, which we get by replacing θ^a by $\frac{\theta^a}{2}$ in (2.57):

$$L = \mathrm{e}^{-\mathrm{i}\frac{\theta^a}{2}\sigma_a} = \mathbf{1}\cos\frac{\theta}{2} - \mathrm{i}n^a\sigma_a \sin\frac{\theta}{2} \tag{3.38}$$

The x-dependence of the θ^a has clearly gone into the θ and the n_a, so the differential of this is

$$\partial_\mu L = -\frac{1}{2}\mathbf{1}\sin\frac{\theta}{2}\partial_\mu\theta - \frac{\mathrm{i}}{2}n^a\sigma_a\cos\frac{\theta}{2}\partial_\mu\theta - \mathrm{i}\sigma_a\sin\frac{\theta}{2}\partial_\mu n^a \tag{3.39}$$

while L^{-1} is just

$$L^{-1} = \mathbf{1}\cos\frac{\theta}{2} + \mathrm{i}n^a\sigma_a\sin\frac{\theta}{2} \tag{3.40}$$

Multiplying these together and using (2.56) and the identities

$$2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \sin\theta \quad \text{and} \quad \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} = 1$$

we get

$$L^{-1}\partial_\mu L = -\frac{\mathrm{i}}{2}n^a\sigma_a\partial_\mu\theta - \frac{\mathrm{i}}{2}\sin\theta\sigma_a\partial_\mu n^a + \sin^2\frac{\theta}{2}n^a\partial_\mu n^b\sigma_a\sigma_b \tag{3.41}$$

We can simplify the last of these terms by using the product rule for the σ's and by noting that

$$n^a n_a = \mathbf{1} \Rightarrow n^a\partial_\mu n_a + n_a\partial_\mu n^a = 0 \Rightarrow n^a\partial_\mu n_a = 0 \tag{3.42}$$

which gives us

$$L^{-1}\partial_\mu L = -\frac{\mathrm{i}}{2}n^a\sigma_a\partial_\mu\theta - \frac{\mathrm{i}}{2}\sin\theta\sigma_a\partial_\mu n^a + \mathrm{i}\sin^2\frac{\theta}{2}n^a\partial_\mu n^b\epsilon_{ab}{}^3\sigma_3 \tag{3.43}$$

i.e.,

$$\mathbf{a}_\mu = n^a\sigma_a\partial_\mu\theta + \sin\theta\sigma_a\partial_\mu n^a \tag{3.44}$$

$$\Rightarrow \quad \mathrm{D}_\mu M^a = a^a_\mu = n^a\partial_\mu\theta + \sin\theta\partial_\mu n^a \tag{3.45}$$

and

$$-\frac{\mathrm{i}}{2}\mathbf{v}_\mu = \mathrm{i}\sin^2\frac{\theta}{2}n^a\partial_\mu n^b\epsilon_{ab}{}^3\sigma_3 \tag{3.46}$$

We have yet to put $\mathrm{D}_\mu M^a$ in the form $\partial_\mu M^a +$ something. As both θ and n^a are defined in terms of the M^a, this is not difficult to do. We will, however, need some identities. We start by differentiating M^b.

$$\partial_\mu M^b = \partial_\mu(Mn^b) = M\partial_\mu n^b + n^b\partial_\mu M \tag{3.47}$$

$$\Rightarrow n_b\partial_\mu M^b = Mn_b\partial_\mu n^b + n_b n^b\partial_\mu M = \partial_\mu M \tag{3.48}$$

where we have used (3.42) and the fact that n^b is a unit vector. Substituting (3.48) into (3.47) we get

$$\partial_\mu M^a = M\partial_\mu n^a + n^a n_b\partial_\mu M^b \tag{3.49}$$

$$\Rightarrow \partial_\mu n^a = \frac{1}{M}(\partial_\mu M^a - n^a n_b\partial_\mu M^b) \tag{3.50}$$

while

$$\partial_\mu\theta = \frac{\mathrm{d}\theta}{\mathrm{d}M}\partial_\mu M = \frac{\mathrm{d}\theta}{\mathrm{d}M}n_b\partial_\mu M^b \tag{3.51}$$

Thus the covariant derivative for the Goldstone bosons becomes

$$\mathrm{D}_\mu M^a = n^a\frac{\mathrm{d}\theta}{\mathrm{d}M}n_b\partial_\mu M^b + \frac{\sin\theta}{M}(\partial_\mu M^a - n^a n_b\partial_\mu M^b) \tag{3.52}$$

Similarly, we can now write the covariant derivative for the standard fields as

$$\mathrm{D}_\mu\psi = \partial_\mu\psi - \frac{\mathrm{i}}{2}\mathbf{v}_\mu\psi = \partial_\mu\psi + \frac{\mathrm{i}}{M}\sin^2\frac{\theta}{2}n^a(\partial_\mu M^b - n^b n_c\partial_\mu M^c)\epsilon_{ab}{}^3\sigma_3\psi \tag{3.53}$$

but $n^a n^b \epsilon_{ab}{}^3 = 0$, so

$$\mathrm{D}_\mu \psi = \partial_\mu \psi + \frac{\mathrm{i}}{M^2} \sin^2 \frac{\theta}{2} M^a \partial_\mu M^b \epsilon_{ab}{}^3 \sigma_3 \psi \tag{3.54}$$

(This is all derived, albeit in a slightly more involved way, in [19].)

Having found the covariant derivatives, it is worth a close look at the term containing $\mathrm{D}_\mu M^a$ in (3.36).

$$\begin{aligned}
\frac{1}{2}\mathrm{D}^\mu M_a \mathrm{D}_\mu M^a &= \frac{1}{2}\left(n_a \frac{\mathrm{d}\theta}{\mathrm{d}M} n^b \partial^\mu M_b + \frac{\sin\theta}{M}(\partial^\mu M_a - n_a n^b \partial^\mu M_b)\right) \\
&\times \left(n^a \frac{\mathrm{d}\theta}{\mathrm{d}M} n_c \partial_\mu M^c + \frac{\sin\theta}{M}(\partial_\mu M^a - n^a n_c \partial_\mu M^c)\right) \qquad (3.55) \\
&= \frac{1}{2}\partial^\mu M_b \left(\frac{\mathrm{d}\theta}{\mathrm{d}M} n_a n^b + \frac{\sin\theta}{M}(\delta_a^b - n_a n^b)\right) \\
&\times \partial_\mu M^c \left(\frac{\mathrm{d}\theta}{\mathrm{d}M} n^a n_c + \frac{\sin\theta}{M}(\delta_c^a - n^a n_c)\right) \qquad (3.56)
\end{aligned}$$

Now

$$(\delta_a^b - n_a n^b) n^a n_c = n^b n_c - n^b n_c = 0 \tag{3.57}$$

and similarly

$$n_a n^b (\delta_c^a - n^a n_c) = 0 \tag{3.58}$$

while

$$(\delta_a^b - n_a n^b)(\delta_c^a - n^a n_c) = \delta_c^b - n_c n^b - n^b n_c + n^b n_c = \delta_c^b - n^b n_c \tag{3.59}$$

so

$$\frac{1}{2}\mathrm{D}^\mu M_a \mathrm{D}_\mu M^a = \frac{1}{2}\partial^\mu M_b \partial_\mu M^c \left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}M}\right)^2 n^b n_c + \left(\frac{\sin\theta}{M}\right)^2 (\delta_c^b - n^b n_c)\right] \tag{3.60}$$

Note that if we consider the power series expansion of $\theta(M)$:

$$\theta(M) = c_1 M + c_2 M^2 + c_3 M^3 + \ldots \tag{3.61}$$

(there is no constant term as we know that if all the coset space parameters are zero the amplitudes of the Goldstone fields are all zero) then it is clear that in the limit of small M,

$$\frac{\mathrm{d}\theta}{\mathrm{d}M} \to \frac{\theta}{M} \tag{3.62}$$

Also, in this limit

$$\frac{\sin\theta}{M} \to \frac{\theta}{M} \tag{3.63}$$

so two of the terms in the above invariant quantity vanish:

$$\frac{1}{2}\mathrm{D}^\mu M_a \mathrm{D}_\mu M^a \to \frac{1}{2}\partial^\mu M_b \partial_\mu M^b \frac{\theta}{M} \tag{3.64}$$

Thus if we take the fields to be normalised such that $\theta \to M$ for small M, the first term in a power series expansion of $\frac{1}{2}\mathrm{D}^\mu M_a \mathrm{D}_\mu M^a$ is the normal kinetic term, as expected.

Finally, we promised in the introduction that we would identify a metric for our non-linear realisation. The first to identify a metric for a non-linear realisation was Meetz[13]. He was concerned with the realisation of SU(2)⊗SU(2) obtained from the constraint (1.2). In the linear sigma model, one can define an 'interval' for the (flat) field space:

$$\mathrm{d}s^2 = (\mathrm{d}\boldsymbol{\pi})^2 + (\mathrm{d}\sigma')^2 \tag{3.65}$$

This is clearly G=SU(2)⊗SU(2) invariant. On eliminating the σ' field, this becomes

$$ds^2 = g_{ij}\mathrm{d}\pi^i\mathrm{d}\pi^j \tag{3.66}$$

where g_{ij} is the (non-flat) metric for the coset space. $\mathrm{d}s^2$ is still G-invariant, and therefore so is

$$g_{ij}\frac{\mathrm{d}\pi^i}{\mathrm{d}x^\mu}\frac{\mathrm{d}\pi^j}{\mathrm{d}x_\mu} = \partial_\mu\pi^i\partial^\mu\pi^j g_{ij}$$

This is valid for any choice of coordinates on the field space; in general for the non-linear realisation G/H, the quantity

$$\partial_\mu M^A\partial^\mu M^B g_{AB}$$

is G-invariant[12]. However, we have already shown that the only invariant of this form is $\mathrm{tr}(\mathbf{a}_\mu\mathbf{a}^\mu) = \mathrm{D}_\mu M^A\mathrm{D}^\mu M_A$. For SU(2)/U(1) this is given by (3.60), so the metric is the quantity in square brackets:

$$g_{bc} = \left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}M}\right)^2 n_b n_c + \left(\frac{\sin\theta}{M}\right)^2(\delta_{bc} - n_b n_c)\right] \tag{3.67}$$

(note that it is symmetric).

Chapter 4

Introduction to Lie Algebras and Projection Operators

4.1 Projection Operators

We have seen in the last two chapters how important it is to be able to write $L = \mathrm{e}^{-\mathrm{i}\theta^A T_A}$, an arbitrary element of the coset space, as a linear sum of the broken generators T_A. For each coset space we tackle we will need to be able to calculate this exponential. The exponential of such a matrix is defined by its power series:

$$\mathrm{e}^{-\mathrm{i}\theta^A T_A} = \mathbf{1} - \mathrm{i}\theta^A T_A - \frac{1}{2}\left(\theta^A T_A\right)^2 + \frac{\mathrm{i}}{3!}\left(\theta^A T_A\right)^3 + \ldots \tag{4.1}$$

where $\mathbf{1}$ is the identity matrix of the same dimension as the generators and where

$$\begin{aligned}\left(\theta^A T_A\right)^2 &= \left(\theta^1 T_1\right)^2 + \left(\theta^2 T_2\right)^2 + \left(\theta^3 T_3\right)^2 + \ldots \\ &\quad + \left\{\theta^1 T_1, \theta^2 T_2\right\} + \left\{\theta^1 T_1, \theta^3 T_3\right\} + \ldots\end{aligned} \tag{4.2}$$

If the generators involved all anticommute with each other and the sum of the square terms is proportional to the identity, (as is the case with any element of SU(2), for example), calculating this exponential becomes trivial, as every term in the power series is proportional either to $\mathbf{1}$ or to $\theta^A T_A$ so the entire series can be written as a linear sum of $\mathbf{1}$ and $\theta^A T_A$.

If this is not the case, the easiest way to find the exponential is to introduce objects called projection operators. A set of projection operators is a set of matrices of the same dimension as the generators, denoted P^S, which by definition have the properties

$$P^T P^U = \begin{cases} 0 & \text{if } T \neq U \\ P^T & \text{if } T = U \end{cases} \tag{4.3}$$

The set is a complete set if

$$\sum_T P^T = \mathbf{1} \tag{4.4}$$

or in terms of components,

$$\sum_T \left(P^T\right)_{UV} = \delta_{UV} \tag{4.5}$$

These properties means that any polynomial in the projection operators reduces to a linear sum, so the only combinations of these matrices that can be formed are linear ones. If we only have two projection operators, say P^+ and P^-, every such linear sum can be written as a linear sum of $P^+ + P^- = \mathbf{1}$ and $P^+ - P^-$. i.e. every traceless matrix which can be written in terms of the projection operators is a multiple of $P^+ - P^-$. This combination has the property

$$\left(P^+ - P^-\right)^2 = P^+P^+ - P^+P^- - P^-P^+ + P^-P^- = P^+ + P^- = \mathbf{1} \tag{4.6}$$

This means that if (a scalar multiple of) the vector we want to exponentiate squares to the identity we can write it as the difference of two projection operators. We will shortly see how to do this for an arbitrary SU(2) vector. However, this is of limited use as if (a scalar multiple of) the vector does square to the identity, we may calculate its exponential directly as remarked above. If this is not the case we clearly need more than two projection operators.

The simplest projection operators are those with a single 1 somewhere along the leading diagonal and zeros everywhere else. For a n-dimensional group representation, we clearly need n of these to form a complete set:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

By applying these operators to a n-dimensional vector (a multiplet of the n-dimensional representation) we can project out the individual components (fields):

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \phi_4 \\ 0 \end{pmatrix}$$

(hence the name). Clearly any diagonal $n \times n$ matrix can be written as a linear sum of these matrices. If we want to exponentiate a matrix with off-diagonal components, we can think of a similarity transformation reducing it to a diagonal matrix:

$$S^{-1}xS = d = c_1 P^1 + c_2 P^2 + \ldots + c_n P^n \tag{4.7}$$

which we can then invert:

$$x = SdS^{-1} = c_1 SP^1 S^{-1} + c_2 SP^2 S^{-1} + \ldots + c_n SP^n S^{-1} \tag{4.8}$$

The set of operators $SP^T S^{-1}$ satisfies all the conditions ((4.3) and (4.4)) to be a complete set of projection operators:

$$\left(SP^T S^{-1}\right)\left(SP^U S^{-1}\right) = SP^T P^U S^{-1} = \begin{cases} 0 & \text{if } T \neq U \\ SP^T S^{-1} & \text{if } T = U \end{cases} \tag{4.9}$$

$$SP^1 S^{-1} + SP^2 S^{-1} + \ldots + SP^n S^{-1} = S\left(P^1 + P^2 + \ldots + P^n\right) S^{-1} = SS^{-1} = \mathbf{1} \tag{4.10}$$

so we can write any vector of a Lie algebra as a linear sum of projection operators, providing we can find the appropriate set.

This similarity transformation acting on the multiplet corresponds to a field redefinition and the new projection operators project out various superpositions of states.

Such similarity transformations do not give us every complete set of projection operators for the group representation, as we can always add together projection operators - the sum of two projection operators is always another projection operator:

$$(P^T + P^U)P^V = P^T P^V + P^U P^V = 0 \qquad \text{if } T, U, V \text{ all different} \tag{4.11}$$

$$(P^T + P^U)(P^T + P^U) = P^T P^T + P^U P^T + P^T P^U + P^U P^U = P^T + P^U \tag{4.12}$$

(the logical extreme of adding projection operators to get new ones is of course when you just have one projection operator which is the identity). However, doing this obviously reduces the number of vectors which can be expressed in terms of the set and in general if we want to exponentiate an $n \times n$ matrix we will want a set of n projection operators.

To calculate a given L, i.e. to exponentiate an arbitrary linear sum of the broken generators, we must find a set of projection operators which we can write the sum in terms of. Once we have found such a set and written our coset space vector as a linear sum of them, (say $\mathbf{x} = \phi^1 P^1 + \phi^2 P^2 + \ldots + \phi^n P^n$), exponentiating it is easy:

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}\mathbf{x}} &= \mathbf{1} - \mathrm{i}(\phi^1 P^1 + \phi^2 P^2 + \ldots + \phi^n P^n) - \frac{1}{2}(\phi^1 P^1 + \phi^2 P^2 + \ldots + \phi^n P^n)^2 \\ &\quad + \frac{\mathrm{i}}{3!}(\phi^1 P^1 + \phi^2 P^2 + \ldots + \phi^n P^n)^3 + \ldots \end{aligned}$$

$$
\begin{aligned}
&= (P^1 + P^2 + \ldots + P^n) - \mathrm{i}(\phi^1 P^1 + \phi^2 P^2 + \ldots + \phi^n P^n) \\
&\qquad -\frac{1}{2}\left((\phi^1)^2 P^1 + (\phi^2)^2 P^2 + \ldots + (\phi^n)^2 P^n\right) \\
&\qquad +\frac{\mathrm{i}}{3!}\left((\phi^1)^3 P^1 + (\phi^2)^3 P^2 + \ldots + (\phi^n)^3 P^n\right) + \ldots \\
&= \mathrm{e}^{-\mathrm{i}\phi^1} P^1 + \mathrm{e}^{-\mathrm{i}\phi^2} P^2 + \ldots + \mathrm{e}^{-\mathrm{i}\phi^n} P^n \qquad (4.13)
\end{aligned}
$$

The generators are now all contained in the projection operators, so the coset space element is expressed as a linear sum of the generators, with coefficients of the form $c_1\mathrm{e}^{-\mathrm{i}\phi^1} + c_2\mathrm{e}^{-\mathrm{i}\phi^2} + \ldots + c_n\mathrm{e}^{-\mathrm{i}\phi^n}$ (where $c_1, c_2, \ldots, c_n$ are purely numerical).

Note that if one of the set of n projection operators does not appear in the expression for $\mathbf{x}$, it still appears in the expression for the exponential with a coefficient of $\mathrm{e}^0 = 1$, due to the identity matrix appearing in the expansion being a sum of all n projection operators.

4.2 SU(2)

4.2.1 Defining representation

The fundamental representation of SU(2) is the doublet. In quantum mechanics, we can only ever determine one component of a doublet's angular momentum (or isospin) at any one time, the component in the z-direction. This is because the SU(2) angular momentum (or isospin) group only has one diagonal generator, which is taken to be the one associated with the z-component, $T_z = T_3 = \frac{1}{2}\sigma_3$. (This generates a U(1) Cartan subgroup.) If we wish to project out the two components of the doublet, we must construct two projection operators to do this:

$$P^+\chi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} \chi_1 \\ 0 \end{pmatrix} \quad (4.14)$$

$$P^-\chi = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \chi_2 \end{pmatrix} \quad (4.15)$$

and we may use the diagonal generator to do this. In this case, the expressions for the projection operators are obvious:

$$P^+ = \frac{1}{2}(\mathbf{1} + \sigma_3) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (4.16)$$

$$P^- = \frac{1}{2}(\mathbf{1} - \sigma_3) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (4.17)$$

(Note that for any SU(N) or SO(t,s) group, any traceless diagonal matrix in the group's algebra can be written as a linear sum of the diagonal generators, so any diagonal matrices, including the diagonal projection operators, can be constructed from the diagonal generators and the unit matrix **1**.)

We just have two projection operators, so every matrix in the algebra which can be written in terms of these two is a multiple of $P^+ - P^- = \sigma_3$, that is, any diagonal vector can be written as

$$\mathbf{x} = \theta^3 T_3 = \frac{1}{2}\theta^3\sigma_3 = \frac{1}{2}\theta^3(P^+ - P^-) \quad (4.18)$$

We can think of this vector as lying along the same direction as T_3 and σ_3, with θ_3 being a measure of the 'length' of the vector. Every vector space, by definition, has some definition of 'length' or 'distance' associated with it, so we would like a definition in this case which allows us to take the generators or the σ's as an orthonormal basis with the θ's as their coefficients or 'components'.

The definition of the square of the length of $\mathbf{x}$ that we will use (following the conventions of Michel and Radicati[27]) is

$$(\mathbf{x}, \mathbf{x}) = \frac{1}{2}\,\mathrm{tr}(\mathbf{x}^2) \tag{4.19}$$

The factor at the beginning is matter of convention, depending on what multiple of the generators you wish to use as your basis vectors. However, we will find that for SU(N) and SO(t,s) groups, it is usually most convenient to deal with quantities which are given by doubling the generators when constructing projection operators. (For SU(2) these are obviously the Pauli matrices.) The factor of $\frac{1}{2}$ in the above expression is then the appropriate one:

$$(\sigma_i, \sigma_i) = \frac{1}{2}\,\mathrm{tr}(\sigma_i^2) = \frac{1}{2}\,\mathrm{tr}(\mathbf{1}) = 1 \tag{4.20}$$

We can see this is a sensible definition by considering the length of a vector $\mathbf{x} = x^i \sigma_i$

$$\begin{aligned}
(\mathbf{x}, \mathbf{x}) &= \frac{1}{2}\,\mathrm{tr}(\mathbf{x}^2) && (4.21)\\
&= \frac{1}{2} x^i x^j\,\mathrm{tr}(\sigma_i \sigma_j) && (4.22)\\
&= \frac{1}{2} x^i x^j\,\mathrm{tr}(\mathbf{1}\delta_{ij} + i\epsilon_{ij}{}^k \sigma_k) && (4.23)\\
&= x^i x_i && (4.24)
\end{aligned}$$

(here I have used x^i for the coefficients of σ_i as I do not want to cause confusion with the coefficients of T_i - the usual group parameters - which are precisely double those used above: $x^i = \frac{1}{2}\theta^i$).

We can generalise this to a scalar product of two vectors:

$$(\mathbf{x}, \mathbf{y}) \equiv \frac{1}{2}\,\mathrm{tr}(\mathbf{x}\mathbf{y}) \tag{4.25}$$

- then if $\mathbf{y} = y^i \sigma_i$,

$$(\mathbf{x}, \mathbf{y}) = x^i y_i \tag{4.26}$$

These definitions allow us to describe a set of vectors as orthogonal if their scalar products are all zero and orthonormal if they also each have unit length. For example, the basis of the σ's is orthonormal:

$$(\sigma^i, \sigma_j) = \frac{1}{2}\,\mathrm{tr}(\sigma^i \sigma_j) = \frac{1}{2}\,\mathrm{tr}(\mathbf{1}\delta^i_j) = \delta^i_j \tag{4.27}$$

Having found the projection operators for the diagonal group elements, we would now like to find the projection operators for a matrix in the algebra with off-diagonal components. (Vectors associated with the coset space SU(2)/U(1) are clearly all of this type.) Given such a matrix, we can do this by finding a similarity transformation which diagonalises the matrix and then applying the inverse transformation to the diagonal projection operators.

Now it is a well-known fact that any hermitian matrix can be diagonalised by a unitary (and hence invertible) similarity transformation. As every vector $\mathbf{x}$ of the SU(2) algebra is hermitian, we can apply a unitary similarity transformation to it to get a diagonal vector $\mathbf{d}$ in the algebra:

$$S\mathbf{x}S^{-1} = \mathbf{d} \tag{4.28}$$

Conversely, any such vector can be obtained by applying a unitary similarity transformation (the inverse transformation) to the appropriate diagonal vector:

$$S^{-1}\mathbf{d}S = \mathbf{x} \tag{4.29}$$

The S here is a 2×2 unitary matrix. Furthermore, it is easy to see that this transformation does not alter the length of the vector - any overall scaling from the S is cancelled by an inverse scaling from S^{-1} (the transformation as a whole is 'special' or 'unimodular'). It is not surprising, then, that this transformation corresponds to an SU(2) rotation of the vector - we will study this in depth in the next section.

For SU(2) we have seen that every diagonal vector is proportional to the diagonal generator T_3. This means that every vector $\mathbf{x}$ is proportional to $S^{-1}\sigma_3 S$ - that is, it can be obtained by applying an SU(2) transformation to σ_3 (which will give you another unit vector) followed by a simple scaling. This is obvious if you bear in mind the homomorphism between SU(2) and the group of rotations in 3-dimensional space (see Section 4.3.1). By applying an arbitrary SU(2) transformation to σ_3, then, we obtain the arbitrary unit vector $n^i\sigma_i$, where n^i are the components of the unit vector in the basis given by the Pauli matrices:

$$n^i n_i = n^1 n_1 + n^2 n_2 + n^3 n_3 = 1 \tag{4.30}$$

Any matrix in the SU(2) algebra can then be written as one of these unit vectors multiplied by a scaling factor, by factoring out its length:

$$\mathbf{x} = x^i\sigma_i = xn^i\sigma_i \qquad \text{where } x \equiv \sqrt{(\mathbf{x},\mathbf{x})} = \sqrt{x^i x_i} \tag{4.31}$$

just as we did for $\boldsymbol{\theta}$ in Section 2.4.4. To find the projection operators for such a matrix, we just apply the same SU(2) (similarity) transformation to the diagonal projection operators. This gives us

$$P^+ = \frac{1}{2}\left(\mathbf{1} + n^i\sigma_i\right) \tag{4.32}$$

$$P^- = \frac{1}{2}\left(\mathbf{1} - n^i\sigma_i\right) \tag{4.33}$$

With this form of P^+ and P^-, every matrix in the algebra is a multiple of

$P^+ - P^- = n^i\sigma_i.$

While our previous projection operators were constructed with explicit reference to the diagonal U(1) subgroup, these new forms do not relate to any particular subgroup. However, they still satisfy the defining properties of projection operators:

$$P^\pm P^\pm = \frac{1}{4}(\mathbf{1} \pm n^i\sigma_i)(\mathbf{1} \pm n^j\sigma_j) = \frac{1}{4}(\mathbf{1} \pm 2n^i\sigma_i + n^i n^j \sigma_i\sigma_j) \tag{4.34}$$

but

$$n^i n^j \sigma_i \sigma_j = (P^+ - P^-)^2 = n^i n^j(\delta_{ij}\mathbf{1} + \mathrm{i}\epsilon_{ij}{}^k\sigma_k) = n^i n_i \mathbf{1} = \mathbf{1} \tag{4.35}$$

so $P^\pm P^\pm = P^\pm$ as required and

$$P^+P^- = P^-P^+ = \frac{1}{4}(\mathbf{1} + n^i\sigma_i)(\mathbf{1} - n^j\sigma_j) = \frac{1}{4}(\mathbf{1} - n^i n^j\sigma_i\sigma_j) = \frac{1}{4}(\mathbf{1} - \mathbf{1}) = 0 \tag{4.36}$$

We can now exponentiate an arbitrary vector in the algebra easily:

$$\mathrm{e}^{-\mathrm{i}\theta^i T_i} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta^i\sigma_i} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta n^i\sigma_i} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta(P^+ - P^-)} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta P^+ + \frac{\mathrm{i}}{2}\theta P^-} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta}P^+ + \mathrm{e}^{\frac{\mathrm{i}}{2}\theta}P^- \tag{4.37}$$

This is clearly equivalent to the trigonometric form of L we used in the last two chapters with $n^a\sigma_a$ replaced by $n^i\sigma_i$.

4.2.2 Adjoint representation

Definition of the adjoint representation

We usually think of the defining representation of SU(2) as acting on a doublet and transforming it into another doublet. We can also, however, consider an element of the defining representation acting on a vector in the algebra by conjugation:

$$\mathbf{x} \to \mathbf{x}' = g\mathbf{x}g^{-1} \tag{4.38}$$

This is a similarity transformation of the vector using a 2×2 special unitary matrix and corresponds simply to an SU(2) rotation of the vector. Such a transformation preserves all scalar products and lengths:

$$\mathbf{xy} \to g\mathbf{xy}g^{-1} \Rightarrow \frac{1}{2}\,\mathrm{tr}(\mathbf{xy}) \to \frac{1}{2}\,\mathrm{tr}(g\mathbf{xy}g^{-1}) = \frac{1}{2}\,\mathrm{tr}(\mathbf{xy}) \tag{4.39}$$

In the basis of the Pauli matrices we can think of the SU(2) similarity transformation as rotating the components of the vector into each other:

$$x^i\sigma_i \to x'^i\sigma_i = gx^i\sigma_i g^{-1} = x^i g\sigma_i g^{-1} \tag{4.40}$$

To find out what these components of the transformed vector are, we take the scalar product of both sides with σ^j:

$$\begin{aligned}
(x'^i\sigma_i, \sigma^j) &= (x^i g\sigma_i g^{-1}, \sigma^j) & (4.41)\\
\Rightarrow x'^i(\sigma_i, \sigma^j) &= \frac{1}{2}\,\mathrm{tr}(x^i g\sigma_i g^{-1}\sigma^j) & (4.42)\\
\Rightarrow x^j \to x'^j &= \frac{1}{2}\,\mathrm{tr}(g\sigma_i g^{-1}\sigma^j)x^i & (4.43)
\end{aligned}$$

or writing the right-hand side as the action of a rotation matrix $R(g)$,

$$x^j \to x'^j = R(g)^j{}_i x^i \tag{4.44}$$

This is the definition of the adjoint representation[1] of SU(2) - the adjoint representation of such a group element is often written

$$(\mathbf{Ad}(g))_i{}^j = \frac{1}{2}\operatorname{tr}(g\sigma^j g^{-1}\sigma_i) \tag{4.45}$$

It is 3-dimensional (the matrix indices i, j run over 1,2,3) and we can show that this mapping is indeed homomorphic - the rule for combining elements, and therefore the commutator and anticommutator structure, is preserved - as follows.

If under the action of an SU(2) element g_1,

$$\mathbf{x} \to \mathbf{x}' = g_1 \mathbf{x} g_1^{-1}$$

we can then apply a second transformation:

$$\mathbf{x}'' = g_2 \mathbf{x}' g_2^{-1} = g_2 g_1 \mathbf{x} g_1^{-1} g_2^{-1} = (g_2 g_1)\mathbf{x}(g_2 g_1)^{-1} \tag{4.46}$$

so

$$(\mathbf{Ad}(g_2))^k{}_j x'^j = (\mathbf{Ad}(g_2))^k{}_j (\mathbf{Ad}(g_1))^j{}_i x^i = (\mathbf{Ad}(g_2 g_1))^k{}_i x^i \tag{4.47}$$

or

$$\mathbf{Ad}(g_2)\,\mathbf{Ad}(g_1) = \mathbf{Ad}(g_2 g_1) \tag{4.48}$$

By considering infinitesimal SU(2) transformations we can find the generators of

[1]See, for example, Vol. 2 of [16]

the adjoint representation. To first order an element g of SU(2) looks like

$$g \approx \mathbf{1} + \delta g = \mathbf{1} - \frac{\mathrm{i}}{2}\delta\theta^k \sigma_k \tag{4.49}$$

so, using (2.31) and the resulting commutator, as well as the tracelessness of the σ's,

$$\begin{aligned}
(\mathbf{Ad}(\mathbf{1} + \delta g))_i{}^j &= \frac{1}{2}\operatorname{tr}\left[(\mathbf{1} - \frac{\mathrm{i}}{2}\delta\theta^k\sigma_k)\sigma^j(\mathbf{1} + \frac{\mathrm{i}}{2}\delta\theta^l\sigma_l)\sigma_i\right] & (4.50)\\
&= \frac{1}{2}\operatorname{tr}\left(\sigma^j\sigma_i + \frac{\mathrm{i}}{2}[\sigma^j, \delta\theta^k\sigma_k]\sigma_i + \mathcal{O}(\delta\theta)^2)\right) & (4.51)\\
&= (\sigma^j, \sigma_i) + \frac{\mathrm{i}}{4}\delta\theta^k \operatorname{tr}(2\mathrm{i}\epsilon^j{}_{kl}\sigma_l\sigma_i) + \mathcal{O}(\delta\theta)^2 &\\
&= \delta_i^j - \frac{1}{2}\delta\theta^k \operatorname{tr}(\epsilon^j{}_{kl}\delta_{li}\mathbf{1} + \mathrm{i}\epsilon^j{}_{kl}\epsilon_{lim}\sigma_m) + \mathcal{O}(\delta\theta)^2 &\\
&= \delta_i^j - \delta\theta^k\epsilon_i{}^j{}_k + \mathcal{O}(\delta\theta)^2 & (4.52)
\end{aligned}$$

(hence the form of the linear transformation of the Goldstone fields in (2.72)).

We observe that the identity of the defining representation maps to

$$\delta_i^j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

the identity matrix in the adjoint representation, while σ_k maps to

$$(\mathbf{ad}(\sigma_k))_i{}^j = -2\mathrm{i}\epsilon_i{}^j{}_k \tag{4.53}$$

Note that we use the lower-case **ad** to denote the adjoint representation of the Lie algebra - an arbitrary element of the algebra can then be written

$$(\mathbf{ad}(\frac{1}{2}\theta^k\sigma_k))_i{}^j = -\mathrm{i}\theta^k\epsilon_i{}^j{}_k = \mathrm{i}\theta^k\epsilon_{ik}{}^j \tag{4.54}$$

in this representation. (We will look at this for a general SU(N) group in Section 7.1, where we will introduce a different, more appropriate notation for the adjoint representation of an element of the Lie algebra.)

Projection operators of the adjoint representation

In the defining representation we had two projection operators, P^+ and P^-, and we could write any element of the algebra as a multiple of $P^+ - P^-$. This allowed us to express an arbitrary group element as a linear sum of these projection operators (see (4.37)). In the adjoint representation we have an expression for an arbitrary group element, $\mathbf{Ad}(g)$, in terms of the corresponding element of the defining representation, g. We can write this too in terms of P^+ and P^- by substituting (4.37) into (4.45).

$$\begin{aligned}
(\mathbf{Ad}(g))_i{}^j &= \frac{1}{2}\operatorname{tr}(\mathrm{e}^{-\frac{\mathrm{i}}{2}\theta^k\sigma_k}\sigma^j\mathrm{e}^{\frac{\mathrm{i}}{2}\theta^l\sigma_l}\sigma_i) \\
&= \frac{1}{2}\operatorname{tr}\left[(\mathrm{e}^{-\frac{\mathrm{i}}{2}\theta}P^+ + \mathrm{e}^{\frac{\mathrm{i}}{2}\theta}P^-)\sigma^j(\mathrm{e}^{\frac{\mathrm{i}}{2}\theta}P^+ + \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta}P^-)\sigma_i\right] \\
&= \frac{1}{2}\operatorname{tr}(P^+\sigma^jP^+\sigma_i) + \frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}\operatorname{tr}(P^-\sigma^jP^+\sigma_i) + \frac{1}{2}\mathrm{e}^{-\mathrm{i}\theta}\operatorname{tr}(P^+\sigma^jP^-\sigma_i) \\
&\qquad + \frac{1}{2}\operatorname{tr}(P^-\sigma^jP^-\sigma_i)
\end{aligned} \tag{4.55}$$

It is worth comparing and contrasting this with (4.37). Both expressions have one term involving an exponential (of a multiple of θ) and another term involving its inverse. In (4.37), the coefficients of these exponentials are the two projection operators, P^+ and P^-, the difference of which is the vector we are exponentiating, upto a numerical factor.

For the adjoint representation, which we know is 3-dimensional, we should be able to construct three projection operators[2]. Let us suggest that an arbitrary

[2]The techniques for constructing projection operators for the adjoint representation of a

element of the algebra can once again be written, upto a factor, as a difference of two of these projection operators:

$$(\mathbf{ad}(\mathbf{x}))_i{}^j = -\mathrm{i}\theta^k \epsilon_i{}^j{}_k = -\mathrm{i}\theta n^k \epsilon_i{}^j{}_k \propto \theta(P^1 - P^2)_i{}^j \tag{4.56}$$

Now $(\mathbf{Ad}(g))_i{}^j$ can be written as an exponential power series in $(\mathbf{ad}(\mathbf{x}))_i{}^j$; the above expression would then allow us to write it in terms of P^1, P^2, P^3. The first term in the power series will be $\mathbf{1} = P^1 + P^2 + P^3$. It is easy to see that, just as for the defining representation,

$$(P^1 - P^2)^2 = P^1P^1 - P^1P^2 - P^2P^1 + P^2P^2 = P^1 + P^2 \tag{4.57}$$

although this is no longer equal to the identity, and

$$(P^1 - P^2)(P^1 + P^2) = P^1P^1 - P^2P^1 + P^1P^2 - P^2P^2 = P^1 - P^2 \tag{4.58}$$

so that once again the odd powers are proportional to $P^1 - P^2$ and the even powers are proportional to $P^1 + P^2$. Calculating this exponential will then give us coefficients for P^1 and P^2 of the same form as we got for P^+ and P^- in (4.37).

As noted above, we already have precisely these coefficients in (4.55). This would seem to indicate that (4.55) is in reality, an expression for $(\mathbf{Ad}(g))_i{}^j$ as a linear sum of P^1, P^2, P^3. To demonstrate this, we must show that the tensors in (4.55) have projection operator qualities and that the element of the algebra we are exponentiating is proportional to the difference of the appropriate two projectors.

general SU(N) group were established by Barnes and Delbourgo[21], based on earlier work on SU(3) by Rosen[32] and Barnes[22] - we will come back to this more general theory in Section 7.2.

For example, by substituting in (4.32) and (4.33) we can write

$$\frac{1}{2}\,\mathrm{tr}(P^+\sigma^j P^-\sigma_i) = \frac{1}{8}\,\mathrm{tr}\left(\sigma^j\sigma_i + n^k\sigma_k\sigma^j\sigma_i - n^l\sigma^j\sigma_l\sigma_i - n^k n^l\sigma_k\sigma^j\sigma_l\sigma_i\right) \tag{4.59}$$

then, by repeatedly using (2.31), we get

$$\frac{1}{2}\,\mathrm{tr}(P^+\sigma^j P^-\sigma_i) = \frac{1}{2}(\delta_i^j - n_i n^j - \mathrm{i}n^k\epsilon_i{}^j{}_k) \tag{4.60}$$

If this is to be a projection operator, say P^1, it must have the property

$$(P^1)_i{}^j(P^1)_j{}^k = (P^1)_i{}^k \tag{4.61}$$

To show this we note that

$$n_i n^j n_j n^k = n_i n^k \tag{4.62}$$

(from the fact that these are unit vectors), that

$$n^i n^j \epsilon_{ij}{}^k = 0 \tag{4.63}$$

(as one factor is symmetric on i and j and the other is antisymmetric) and that

$$\epsilon_{ij}{}^m\epsilon^{kl}{}_m = \delta_i^k\delta_j^l - \delta_i^l\delta_j^k \tag{4.64}$$

We then have

$$\left[\frac{1}{2}(\delta_i^j - n_i n^j - \mathrm{i}n^l \epsilon_i{}^j{}_l)\right]\left[\frac{1}{2}(\delta_j^k - n_j n^k - \mathrm{i}n^m \epsilon_j{}^k{}_m)\right]$$

$$= \frac{1}{4}(\delta_i^k - n_i n^k - \mathrm{i}n^l \epsilon_i{}^k{}_l - n_i n^k + n_i n^k + \mathrm{i}n^k n_j n^l \epsilon_i{}^j{}_l$$
$$-\mathrm{i}n^m \epsilon_i{}^k{}_m + \mathrm{i}n_i n^j n_m \epsilon_j{}^k{}_m - n_l n_m \epsilon_{lij}\epsilon_{kmj}) \quad (4.65)$$
$$= \frac{1}{4}(\delta_i^k - n_i n^k - \mathrm{i}n^l \epsilon_i{}^k{}_l - \mathrm{i}n^m \epsilon_i{}^k{}_m - (n_i n^k - \delta_i^k)) \quad (4.66)$$
$$= \frac{1}{2}(\delta_i^k - n_i n^k - \mathrm{i}n^l \epsilon_i{}^k{}_l) \quad (4.67)$$

as required.

Similarly, we find that

$$\frac{1}{2}\,\mathrm{tr}(P^- \sigma^j P^+ \sigma_i) = \frac{1}{2}(\delta_i^j - n_i n^j + \mathrm{i}n^k \epsilon_i{}^j{}_k) \quad (4.68)$$

which has the properties

$$\left[\frac{1}{2}(\delta_i^j - n_i n^j + \mathrm{i}n^l \epsilon_i{}^j{}_l)\right]\left[\frac{1}{2}(\delta_j^k - n_j n^k + \mathrm{i}n^m \epsilon_j{}^k{}_m)\right] = \frac{1}{2}(\delta_i^k - n_i n^k + \mathrm{i}n^l \epsilon_i{}^k{}_l) \quad (4.69)$$

and

$$\left[\frac{1}{2}(\delta_i^j - n_i n^j - \mathrm{i}n^l \epsilon_i{}^j{}_l)\right]\left[\frac{1}{2}(\delta_j^k - n_j n^k + \mathrm{i}n^m \epsilon_j{}^k{}_m)\right] = 0 \quad (4.70)$$

We therefore denote

$$P^1 = \frac{1}{2}\,\mathrm{tr}(P^+ \sigma^j P^- \sigma_i) = \frac{1}{2}(\delta_i^j - n_i n^j - \mathrm{i}n^k \epsilon_i{}^j{}_k) \quad (4.71)$$

and

$$P^2 = \frac{1}{2}\,\mathrm{tr}(P^-\sigma^j P^+\sigma_i) = \frac{1}{2}(\delta_i^j - n_i n^j + \mathrm{i} n^k \epsilon_i{}^j{}_k) \tag{4.72}$$

noting that this choice satisfies (4.56):

$$(\mathbf{ad}(\mathbf{x}))_i{}^j = -\mathrm{i}\theta n^k \epsilon_i{}^j{}_k = \theta(P^1 - P^2)_i{}^j \tag{4.73}$$

(Note also that all of these tensors commute, so that $P^1 P^2 = 0$ also tells us that $P^2 P^1 = 0$.)

Finally, the other two tensors turn out to be equal:

$$\frac{1}{2}\,\mathrm{tr}(P^+\sigma^j P^+\sigma_i) = \frac{1}{2}\,\mathrm{tr}(P^-\sigma^j P^-\sigma_i) = \frac{1}{2} n_i n^j \tag{4.74}$$

These appear added together in (4.55):

$$\frac{1}{2}\,\mathrm{tr}(P^+\sigma^j P^+\sigma_i) + \frac{1}{2}\,\mathrm{tr}(P^-\sigma^j P^-\sigma_i) = n_i n^j \tag{4.75}$$

and we know from (4.62) that this quantity squares to itself. If we think of writing $(\mathbf{Ad}(g))_i{}^j$ as an exponential power series in $(\mathbf{ad}(\mathbf{x}))_i{}^j$, P^3 only appears in the identity term, so the part of $(\mathbf{Ad}(g))_i{}^j$ which does not have a θ-dependent coefficient must be P^3. We therefore suspect that (4.75) is P^3. To show this, we only need to show that $P^1 P^3 = P^2 P^3 = 0$. This is quite trivial:

$$\frac{1}{2}(\delta_i^j - n_i n^j \pm \mathrm{i} n^l \epsilon_i{}^j{}_l) n_j n^k = \frac{1}{2}(n_i n^k - n_i n^k \pm n^l n_j n^k \epsilon_i{}^j{}_l) = 0 \tag{4.76}$$

We therefore have our third projection operator:

$$P^3 = \mathrm{tr}(P^+\sigma^j P^+\sigma_i) = \mathrm{tr}(P^-\sigma^j P^-\sigma_i) = n_i n^j \tag{4.77}$$

Note that the three form a complete set:

$$P^1 + P^2 + P^3 = \delta_{ij} - n_i n_j + P^3 = \delta_{ij} \tag{4.78}$$

We briefly recap the main results of this section. For the adjoint representation of SU(2), defined by the homomorphic mapping (4.45), there are three projection operators, given by (4.71), (4.72) and (4.77). Any element of the algebra may be written as

$$(\mathbf{ad}(\mathbf{x}))_i{}^j = -\mathrm{i}\theta^k \epsilon_i{}^j{}_k$$

for an appropriate θ^k and is thus proportional to $P^1 - P^2$. Exponentiating the element of the algebra then proceeds exactly as it did for the defining representation in (4.37), except for the fact that we now have a P^3 which occurs in the expression for the group element with a coefficient of $\mathrm{e}^0 = 1$. The expression for the group element ends up as

$$(\mathbf{Ad}(g))_i{}^j = P^1 \mathrm{e}^{-\mathrm{i}\theta} + P^2 \mathrm{e}^{\mathrm{i}\theta} + P^3 \tag{4.79}$$

in agreement with (4.55), or

$$(\mathbf{Ad}(g))_i{}^j = [\frac{1}{2}(\delta_i^j - n_i n^j - \mathrm{i}n^k \epsilon_i{}^j{}_k)]\mathrm{e}^{-\mathrm{i}\theta} + [\frac{1}{2}(\delta_i^j - n_i n^j + \mathrm{i}n^k \epsilon_i{}^j{}_k)]\mathrm{e}^{\mathrm{i}\theta} + n_i n^j \tag{4.80}$$

(of course this can be rearranged to give trigonometric coefficients).

4.3 SO(3)

4.3.1 Spinors

The elements of any special orthogonal group may be written

$$e^{-i\omega^{AB}T_{AB}} \tag{4.81}$$

where the generators T_{AB} are traceless matrices which are antisymmetric on the indices A and B, as are the parameters ω^{AB} (the ranges of A, B are explained below.) If the group is compact, the generators are also hermitian and obey the following commutation relations

$$[T_{AB}, T_{CD}] = -i(\delta_{BC}T_{AD} - \delta_{AC}T_{BD} - \delta_{BD}T_{AC} + \delta_{AD}T_{BC}) \tag{4.82}$$

The (compact) group of $N \times N$ special orthogonal matrices is known as SO(N) and is isomorphic to the group of rotations in N dimensions.

Unlike the SU(N) groups, the lowest dimensional representations of special orthogonal groups are not always the defining representations, as these groups have spinor representations. SO(2N+1) has one 2^N-dimensional spinor representation for which we can construct $2N+1$ γ-matrices which obey the Clifford algebra

$$\{\gamma_A, \gamma_B\} = 2\delta_{AB}\mathbf{1} \tag{4.83}$$

If we then take the generators for this rep. to be given by

$$T_{AB} = -\frac{i}{4}[\gamma_A, \gamma_B] \tag{4.84}$$

the Lie algebra of the group is automatically satisfied. (We will use this rule for all SO(s) groups, to keep the metric positive, but when we come to look at SO(t,s) groups we will change the sign - see Section 4.5.1.)

For SO(3) this means that we have one two-dimensional spinor representation with three γ-matrices. The generators for this two-dimensional representation are

$$T_{12} = -T_{21} \qquad T_{23} = -T_{32} \qquad T_{31} = -T_{13}$$

each of which generates an SO(2) subgroup isomorphic to a group of rotations in a plane. From (4.82), they satisfy the Lie algebra

$$[T_{12}, T_{23}] = \mathrm{i}T_{31}$$

$$[T_{23}, T_{31}] = \mathrm{i}T_{12}$$

$$[T_{31}, T_{12}] = \mathrm{i}T_{23}$$

Observe that if we make the replacement $T_{ij} \rightarrow \epsilon_{ij}{}^{k}T_{k}$ we get the algebra of SU(2), so these groups are homomorphic. This homomorphism tells us that SU(2) forms a representation of SO(3) (though not in this case a faithful one) which we know is two-dimensional with generators $T_k = \frac{1}{2}\sigma_k$. Thus the generators of the two-dimensional representation of SO(3) - the spinor representation - are given by

$$T_{ij} = \frac{1}{2}\epsilon_{ij}{}^{k}\sigma_k \qquad (4.85)$$

i.e.

$$T_{12} = \frac{1}{2}\sigma_3 \qquad T_{23} = \frac{1}{2}\sigma_1 \qquad T_{31} = \frac{1}{2}\sigma_2$$

We can now ask what the γ-matrices look like. We are looking for three matrices which satisfy (4.83) and, from (4.84) and (4.85),

$$[\gamma_i, \gamma_j] = 2\mathrm{i}\epsilon_{ijk}\sigma_k \qquad (4.86)$$

These conditions are obviously satisfied by

$$\gamma_i = \sigma_i \tag{4.87}$$

For special orthogonal groups, as for special unitary groups, it is often easier to work with matrices which are given by doubling the generators - we will call them σ_{AB}. These are products of gammas for the spinor representation:

$$\sigma_{AB} = -\frac{i}{2}[\gamma_A, \gamma_B] \tag{4.88}$$

$$\Rightarrow \sigma_{AB} = -i\gamma_A\gamma_B \qquad \text{if } A \neq B \tag{4.89}$$

but are also well defined for other representations. From (4.82) it is clear they have the commutation relations

$$[\sigma_{AB}, \sigma_{CD}] = -2i(\delta_{BC}\sigma_{AD} - \delta_{AC}\sigma_{BD} - \delta_{BD}\sigma_{AC} + \delta_{AD}\sigma_{BC}) \tag{4.90}$$

For the spinor representation of SO(3), it is clear that each of these σ's is a Pauli matrix, so they form an orthonormal basis for the space of all traceless, hermitian 2×2 matrices. (In Section 4.4.3 we will look at the analogous situation for SO(4) and SO(5).)

Projection operators of the spinor representation

We have seen that the fundamental spinor of SO(3) is nothing other than the doublet of SU(2). This means that the projection operators for the spinor representation of SO(3) are simply those of the defining representation of SU(2). However, we would like to be able to express them in terms of the vectors of the SO(3) algebra. To do this, we must find what a unit vector in this algebra looks like. We start by noting that

$$(\sigma_{ij}, \sigma^{lm}) = \epsilon_{ij}{}^k \epsilon^{lm}{}_n (\sigma_k, \sigma^n) = \epsilon_{ij}{}^k \epsilon^{lm}{}_k = \delta^l_i \delta^m_j - \delta^l_j \delta^m_i \tag{4.91}$$

so

$$(\omega^{ij}\sigma_{ij}, \omega^{lm}\sigma_{lm}) = \omega^{ij}\omega_{lm}(\sigma_{ij}, \sigma^{lm}) = \omega^{ij}\omega_{ij} - \omega^{ij}\omega_{ji} = 2\omega^{ij}\omega_{ij} \tag{4.92}$$

so the square of the length of $\omega^{ij}\sigma_{ij}$ is

$$2\omega^{ij}\omega_{ij} \equiv 2\omega^2 \tag{4.93}$$

which implies that a unit vector has the form

$$\frac{\omega^{ij}}{\sqrt{2\omega^2}}\sigma_{ij} = \frac{1}{\sqrt{2}} n^{ij}\sigma_{ij} \tag{4.94}$$

where

$$n_{ij} \equiv \frac{\omega_{ij}}{\omega} \tag{4.95}$$

Therefore the projection operators for the spinor representation of SO(3) are:

$$P^+ = \frac{1}{2}\left(\mathbf{1} + \frac{1}{\sqrt{2}} n^{ij}\sigma_{ij}\right) \tag{4.96}$$

$$P^- = \frac{1}{2}\left(\mathbf{1} - \frac{1}{\sqrt{2}} n^{ij}\sigma_{ij}\right) \tag{4.97}$$

4.3.2 Defining representation

Just for the sake of completeness, we note here that in the same way that SU(2) forms a representation of SO(3), (the defining representation of) SO(3) forms a representation of SU(2). We know that it is three-dimensional. However, we have already studied the three-dimensional representation of SU(2) (there is only one upto equivalence) - it is the adjoint representation. The homomorphic mapping (4.45) can therefore also be seen as the mapping from the spinor representation to the defining representation of SO(3).

4.4 SO(4) and SO(5)

4.4.1 Spinor Representations

SO(4) is an SO(2N) group so we know that its elements may be written

$$e^{-i\omega^{\mu\nu}T_{\mu\nu}} = e^{-\frac{i}{2}\omega^{\mu\nu}\sigma_{\mu\nu}} \tag{4.98}$$

where $\mu, \nu = 1, \ldots, 4$, with the generators and σ's again satisfying the commutation relations (4.82) and (4.90).

Now SO(2N) has two 2^{N-1}-dimensional spinor representations. For the direct sum of these (known as the Weyl representation) we can construct $2N$ γ-matrices which again obey the Clifford algebra (4.83) and the generators and σ's for this representation are given by (4.84) and (4.89).

To find the gammas for SO(4) we note that if a set of γ-matrices for a group SO(2N-1), labelled $\gamma_H^{(1)}$ (with, obviously, $H = 1, \ldots, 2N-1$) anticommute amongst themselves and all square to the unit matrix, so do the matrices

$$\gamma_H^{(2)} = \begin{pmatrix} 0 & i\gamma_H^{(1)} \\ -i\gamma_H^{(1)} & 0 \end{pmatrix} \tag{4.99}$$

Furthermore, the matrices

$$\gamma_{2N} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \quad \text{and} \quad \gamma_{2N+1} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{4.100}$$

also square to one and anticommute with each of the $\gamma_H^{(2)}$. Clearly the set composed of the $\gamma_H^{(2)}$ and γ_{2N} have the correct Clifford algebra to be the gammas for the group SO(2N) and if we add γ_{2N+1} to these we get a valid set of gammas for SO(2N+1). So if we start with the gamma matrices of SO(3) as given in (4.87), the gamma matrices of SO(5) from this method are:

$$\gamma_i = \begin{pmatrix} 0 & \mathrm{i}\sigma_i \\ -\mathrm{i}\sigma_i & 0 \end{pmatrix} \qquad \gamma_4 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \qquad \gamma_5 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{4.101}$$

where each entry is 2×2. (Incidentally, we could multiply any of these gammas by -1 if we wanted to, as it would still square to one and $\{\gamma_A, \gamma_B\} \to -\{\gamma_A, \gamma_B\} = 0$ so the anticommutations would still hold.) The first four of these are the gammas of SO(4), so by using (4.89) the σ's for the Weyl representation are:

$$\sigma_{ij} = \epsilon_{ij}{}^{k} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \qquad \sigma_{k4} = \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix} \tag{4.102}$$

The fifth is used to construct the projection operators

$$P^{\mathrm{R}} = \frac{1}{2}(\mathbf{1} + \gamma_5) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \tag{4.103}$$

and

$$P^{\mathrm{L}} = \frac{1}{2}(\mathbf{1} - \gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix} \tag{4.104}$$

which project out the right-handed and left-handed spinors:

$$P^{\rm R}\begin{pmatrix}\chi_1\\ \chi_2\\ \chi_3\\ \chi_4\end{pmatrix} = \begin{pmatrix}\chi_1\\ \chi_2\\ 0\\ 0\end{pmatrix} \equiv \chi^{\rm R} \qquad P^{\rm R}\sigma_{ij} = \epsilon_{ij}{}^{k}\begin{pmatrix}\sigma_k & 0\\ 0 & 0\end{pmatrix} \equiv \sigma^{\rm R}_{ij} \quad (4.105)$$

$$P^{\rm R}\sigma_{k4} = \begin{pmatrix}\sigma_k & 0\\ 0 & 0\end{pmatrix} \equiv \sigma^{\rm R}_{k4} \quad (4.106)$$

$$P^{\rm L}\begin{pmatrix}\chi_1\\ \chi_2\\ \chi_3\\ \chi_4\end{pmatrix} = \begin{pmatrix}0\\ 0\\ \chi_3\\ \chi_4\end{pmatrix} \equiv \chi^{\rm L} \qquad P^{\rm L}\sigma_{ij} = \epsilon_{ij}{}^{k}\begin{pmatrix}0 & 0\\ 0 & \sigma_k\end{pmatrix} \equiv \sigma^{\rm L}_{ij} \quad (4.107)$$

$$P^{\rm L}\sigma_{k4} = \begin{pmatrix}0 & 0\\ 0 & -\sigma_k\end{pmatrix} \equiv \sigma^{\rm L}_{k4} \quad (4.108)$$

(I will be using the phrase 'spinor' to refer to the multiplet which transforms under a spinor representation.)

4.4.2 The Homomorphism with SU(2) $\otimes$ SU(2)

Now certain linear combinations of the generators generate an SU(2) subgroup which acts only on the right-handed spinor:

$$\sigma^{\rm R}_k \equiv \frac{1}{2}\left(\frac{1}{2}\epsilon^{ij}{}_k\sigma_{ij} + \sigma_{k4}\right) = \begin{pmatrix}\sigma_k & 0\\ 0 & 0\end{pmatrix} \quad (4.109)$$

- we shall denote this subgroup $SU(2)_{\rm R}$. Similarly, the orthogonal combinations generate an $SU(2)_{\rm L}$ subgroup which acts solely on $\chi^{\rm L}$ and therefore commutes with $SU(2)_{\mathrm{R}}$:

$$\sigma_k^{\mathrm{L}} \equiv \frac{1}{2}\left(\frac{1}{2}\epsilon^{ij}{}_k\sigma_{ij} - \sigma_{k4}\right) = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_k \end{pmatrix} \tag{4.110}$$

(for example, $\sigma_3^{\mathrm{R}} = \frac{1}{2}(\sigma_{12} + \sigma_{34})$ and $\sigma_3^{\mathrm{L}} = \frac{1}{2}(\sigma_{12} - \sigma_{34})$).

Remembering that the generators form a basis for the vector space of all $\omega^{\mu\nu}\sigma_{\mu\nu}$, taking linear combinations in this way corresponds to changing basis in this space, from an SO(4) basis to an $SU(2)_{\mathrm{R}} \otimes SU(2)_{\mathrm{L}}$ basis. We can therefore rewrite an element of the SO(4) Lie algebra as an element of the $SU(2)_{\mathrm{R}} \otimes SU(2)_{\mathrm{L}}$ algebra.

$$\omega^{\mu\nu}\sigma_{\mu\nu} = \omega^{ij}(\epsilon_{ij}{}^k\sigma_k^{\mathrm{R}} + \epsilon_{ij}{}^k\sigma_k^{\mathrm{L}}) + 2\omega^{k4}(\sigma_k^{\mathrm{R}} - \sigma_k^{\mathrm{L}}) \tag{4.111}$$

$$= (\omega^{ij}\epsilon_{ij}{}^k + 2\omega^{k4})\sigma_k^{\mathrm{R}} + (\omega^{ij}\epsilon_{ij}{}^k - 2\omega^{k4})\sigma_k^{\mathrm{L}} \tag{4.112}$$

$$= \begin{pmatrix} \omega^{\mathrm{R}\,k}\sigma_k & 0 \\ 0 & \omega^{\mathrm{L}\,k}\sigma_k \end{pmatrix} \tag{4.113}$$

in the Weyl representation, where

$$\omega^{\mathrm{R}\,k} \equiv \omega^{ij}\epsilon_{ij}{}^k + 2\omega^{k4} \tag{4.114}$$

and

$$\omega^{\mathrm{L}\,k} \equiv \omega^{ij}\epsilon_{ij}{}^k - 2\omega^{k4} \tag{4.115}$$

What we are doing here is to think of σ_{ij} and σ_{k4} as generating 'vector' and 'axial' subsets of the SO(4) transformations (or 'rotations' and 'boosts' in Euclidean four-space, $\mathbb{R}^4$). We then write the 'vector' generators as $\epsilon_{ij}{}^k(\sigma_k^{\mathrm{R}} + \sigma_k^{\mathrm{L}})$ and the 'axial' generators as $(\sigma_k^{\mathrm{R}} - \sigma_k^{\mathrm{L}})$, that is we take combinations of them which generate a pair of mutually commuting SU(2) subgroups, $SU(2)_{\mathrm{L}}$ and $SU(2)_{\mathrm{R}}$. By manipulating the entire SO(4) vector, we can thus break it into an $SU(2)_{\mathrm{R}}$

3-vector (the quantity $\frac{1}{2}\omega^{ij}\epsilon_{ij}{}^{k} + \omega^{k4}$) and an $SU(2)_{\rm L}$ 3-vector ($\frac{1}{2}\omega^{ij}\epsilon_{ij}{}^{k} - \omega^{k4}$).

Projection Operators

Let us see how to use this homomorphism to find projection operators for SO(4). Now $\omega^{{\rm R}\,k}\sigma_k^{\rm R}$ is an $SU(2)_{\rm R}$ spinor. The projection operators for this SU(2) subgroup are obviously

$$P^1 = \frac{1}{2}\left(\mathbf{1}^{\rm R} + n^{{\rm R}\,k}\sigma_k^{\rm R}\right) \tag{4.116}$$

and

$$P^2 = \frac{1}{2}\left(\mathbf{1}^{\rm R} - n^{{\rm R}\,k}\sigma_k^{\rm R}\right) \tag{4.117}$$

where $\mathbf{1}^{\rm R} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}$ in 2×2 block notation (this is clearly just $P^{\rm R}$) and $n^{{\rm R}\,k}$ is a unit vector:

$$n^{{\rm R}\,k}n_k^{\rm R} = 1 \tag{4.118}$$

We can, of course, obtain such a unit vector in the usual way from $\omega^{{\rm R}\,k}\sigma_k^{\rm R}$, by dividing by its length, $\omega^{\rm R}$.

Similarly $\omega^{{\rm L}\,k}\sigma_k^{\rm L}$ is an $SU(2)_{\rm L}$ spinor and the projection operators for $SU(2)_{\rm L}$ are

$$P^3 = \frac{1}{2}\left(\mathbf{1}^{\rm L} + n^{{\rm L}\,k}\sigma_k^{\rm L}\right) \tag{4.119}$$

and

$$P^4 = \frac{1}{2}\left(\mathbf{1}^{\rm L} - n^{{\rm L}\,k}\sigma_k^{\rm L}\right) \tag{4.120}$$

where $\mathbf{1}^{\mathrm{L}} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}$ in 2×2 block notation and $n^{\mathrm{L}\,k}$ is a unit vector:

$$n^{\mathrm{L}\,k} n^{\mathrm{L}}_k = 1 \tag{4.121}$$

We can, of course, rewrite these in SO(4) terms:

$$\begin{aligned}
P^{\mathrm{R}\,+} &= \frac{1}{4}\left(\mathbf{1} + \gamma_5 + n^{\mathrm{R}\,k}(\frac{1}{2}\epsilon^{ij}{}_k\sigma_{ij} + \sigma_{k4})\right) && (4.122)\\
P^{\mathrm{R}\,-} &= \frac{1}{4}\left(\mathbf{1} + \gamma_5 - n^{\mathrm{R}\,k}(\frac{1}{2}\epsilon^{ij}{}_k\sigma_{ij} + \sigma_{k4})\right) && (4.123)\\
P^{\mathrm{L}\,+} &= \frac{1}{4}\left(\mathbf{1} - \gamma_5 + n^{\mathrm{L}\,k}(\frac{1}{2}\epsilon^{ij}{}_k\sigma_{ij} - \sigma_{k4})\right) && (4.124)\\
P^{\mathrm{L}\,-} &= \frac{1}{4}\left(\mathbf{1} - \gamma_5 - n^{\mathrm{L}\,k}(\frac{1}{2}\epsilon^{ij}{}_k\sigma_{ij} - \sigma_{k4})\right) && (4.125)
\end{aligned}$$

but in practice when exponentiating an SO(4) vector it can be easier to rewrite the vector in $SU(2) \otimes SU(2)$ terms, as in (4.112).

4.4.3 Clifford Algebra Structures of SO(4) and SO(5)

In the same way that the Pauli matrices form a basis for the space of all 2×2 traceless, hermitian matrices, the γ-matrices of SO(4) and their products form a basis for the space of all 4×4 traceless, hermitian matrices. As this is a 15-dimensional space, we require 11 such products as well as the four γ-matrices. From the Clifford algebra, the square of any γ-matrix is just the identity, while the product of two different γ's is proportional to a σ, for example:

$$\sigma_{13} = -\frac{\mathrm{i}}{2}[\gamma_1, \gamma_3] = -\mathrm{i}\gamma_1\gamma_3 \tag{4.126}$$

and we have seen there are six of these. Similarly, the product of all four γ's is just $\pm\gamma_5$, as can be seen by multiplying them in the Weyl representation using (4.101). (The order in which they are multiplied can only make the difference of a sign due to the Clifford algebra.) The remaining four matrices we can get as products of three different γ's, or equivalently they are products of $\gamma_5 = -\gamma_1\gamma_2\gamma_3\gamma_4$ with one of the γ_μ, e.g.

$$\mathrm{i}\gamma_2\gamma_3\gamma_4 = -\mathrm{i}\gamma_1\gamma_5 \tag{4.127}$$

where the factor of i ensures hermiticity. However, we know that together with the γ_μ, γ_5 is one of the γ-matrices of SO(5), so this quantity is one of the σ's of SO(5). In general, we have

$$\sigma_{\mu 5} \equiv -\frac{\mathrm{i}}{2}[\gamma_\mu, \gamma_5] = \frac{\mathrm{i}}{3!}\epsilon_\mu{}^{\nu\rho\lambda}\gamma_\nu\gamma_\rho\gamma_\lambda \tag{4.128}$$

4.5 SO(1,3) and SO(1,4)

4.5.1 Weyl Representation

SO(1,3) is isomorphic to the group of rotations in Minkowski spacetime. Elements of the group still have the form (4.98) where μ, ν now run $0, 1, 2, 3$, with the 0 representing the timelike direction. The generators are still traceless and are still antisymmetric on μ and ν, but are no longer all hermitian and they now satisfy the commutation relations

$$[T_{\mu\nu}, T_{\rho\lambda}] = \mathrm{i}(\eta_{\nu\rho}T_{\mu\lambda} - \eta_{\mu\rho}T_{\nu\lambda} - \eta_{\nu\lambda}T_{\mu\rho} + \eta_{\mu\lambda}T_{\nu\rho}) \tag{4.129}$$

where $\eta_{\mu\nu}$ is the Minkowski metric:

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{4.130}$$

Note that the overall sign has changed from the Euclidean case; this is to ensure that the subset of generators T_{ij} generate an SO(3) subgroup with the commutation relations we expect for SO(3). In terms of the σ's, the commutation relations look like:

$$[\sigma_{\mu\nu}, \sigma_{\rho\lambda}] = 2\mathrm{i}(\eta_{\nu\rho}\sigma_{\mu\lambda} - \eta_{\mu\rho}\sigma_{\nu\lambda} - \eta_{\nu\lambda}\sigma_{\mu\rho} + \eta_{\mu\lambda}\sigma_{\nu\rho}) \tag{4.131}$$

As for SO(4), SO(1,3) has two 2-dimensional spinor representations, with the γ-matrices for the direct sum (the Weyl representation) obeying a Clifford algebra:

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\mathbf{1} \tag{4.132}$$

This time, to ensure the correct commutation relations for the generators, we take

$$T_{\mu\nu} = \frac{\mathrm{i}}{4}[\gamma_\mu, \gamma_\nu] \tag{4.133}$$

$$\Rightarrow \sigma_{\mu\nu} = \mathrm{i}\gamma_\mu\gamma_\nu \qquad \text{if } \mu \neq \nu \tag{4.134}$$

To find the γ-matrices for SO(1,3) we first note that the anticommutator of any two different γ-matrices of SO(4) is zero and this is unaffected if we multiply any of them by a numerical factor. Next we note that the γ_4 of SO(4) squares to the identity, as does the γ_0 of SO(1,3). The remaining three γ's square to $\mathbf{1}$ for SO(4), but to $-\mathbf{1}$ for SO(1,3). Thus we can obtain a valid set of γ's for SO(1,3) from those for SO(4) by multiplying the γ_i by i and taking γ_0 of SO(1,3) to be equal to the γ_4 of SO(4), giving us:

$$\gamma_0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \qquad \gamma_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{4.135}$$

Using (4.134) we thus find:

$$\sigma_{0i} = \begin{pmatrix} \mathrm{i}\sigma_i & 0 \\ 0 & -\mathrm{i}\sigma_i \end{pmatrix} \qquad \sigma_{ij} = \epsilon_{ij}{}^{k} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \tag{4.136}$$

The extension to SO(1,4) is the obvious one. The metric becomes 5-dimensional with an extra -1 on the diagonal. This means that the fifth γ-matrix must square to $-\mathbf{1}$ and is of course taken to be i times the γ_5 of SO(5).

4.5.2 Clifford Algebra Structures of SO(1,3) and SO(1,4)

We found for SO(4) and SO(5) that we could construct 15 products of γ's, or 16 if we include the identity. The same can be done for SO(1,3) and SO(1,4). We have already seen the γ_μ and how to take products of two different ones to get $\sigma_{\mu\nu}$'s. The product of all four gives

$$\gamma_0\gamma_1\gamma_2\gamma_3 = -\gamma_5 \tag{4.137}$$

The remaining four can again either be written as products of three of the γ_μ or as products or commutators of one of the γ_μ with γ_5. However, in this case, we should be a little careful with our use of $\epsilon_{\mu\nu\rho\lambda}$. As we are now using an indefinite metric, the value of $\epsilon_{\mu\nu\rho\lambda}$ changes as we raise and lower indices. In particular,

$$\epsilon_{0123} = -\epsilon^{0123} \tag{4.138}$$

so if we are to use this tensor we should be careful to point out what sign convention we are using. If we adopt the convention that $\epsilon_{0123} = 1$, we have as the final four matrices (the remaining σ's of SO(1,4)):

$$\sigma_{\mu 5} \equiv \frac{\mathrm{i}}{2}[\gamma_\mu, \gamma_5] = \frac{\mathrm{i}}{3!}\epsilon_\mu{}^{\nu\rho\lambda}\gamma_\nu\gamma_\rho\gamma_\lambda \tag{4.139}$$

Finally, in the next chapter we will be considering a coset space for which the broken generators are the $\sigma_{\mu 5}$'s. In the same way that for SU(2)/U(1) it proved useful to have an expression for $\sigma_a\sigma_b$, it would be to our benefit in the next chapter if we can now derive an expression for $\sigma_{\mu 5}\sigma_{\nu 5}$:

$$\sigma_{\mu 5}\sigma_{\nu 5} = (\mathrm{i}\gamma_\mu\gamma_5)(\mathrm{i}\gamma_\nu\gamma_5) = -\gamma_\mu\gamma_5\gamma_\nu\gamma_5 = \eta_{55}\gamma_\mu\gamma_\nu \tag{4.140}$$

using the Clifford algebra (we keep the η_{55} rather than replacing it with -1 because we want to maintain covariance on all our indices). We can write the product of the γ's as half the sum of the commutator and the anticommutator:

$$\sigma_{\mu 5}\sigma_{\nu 5} = \frac{1}{2}\eta_{55}\{\gamma_\mu, \gamma_\nu\} + \frac{1}{2}\eta_{55}[\gamma_\mu, \gamma_\nu] = \eta_{55}\eta_{\mu\nu}\mathbf{1} - \mathrm{i}\eta_{55}\sigma_{\mu\nu} \tag{4.141}$$

From this we can see that

$$(\sigma_{\mu 5}, \sigma_{\nu 5}) = \frac{1}{2}\,\mathrm{tr}(\sigma_{\mu 5}\sigma_{\nu 5}) = 2\eta_{55}\eta_{\mu\nu} \tag{4.142}$$

and

$$\{\sigma_{\mu 5}, \sigma_{\nu 5}\} = 2\eta_{55}\eta_{\mu\nu}\mathbf{1} \tag{4.143}$$

Clifford Algebra Structures of Other SO(t,s) Groups

For each SO(2N) group, we can construct a set of products of the γ's (including the identity matrix and the γ's themselves) and by multiplying by i where appropriate we may make them all hermitian. For SO(4), as we have seen, there were 16 (including **1**), while for SO(6) there are 64 - in general, there will be enough to form a complete basis for the set of all hermitian matrices of the dimension of the Weyl representation. (For example, the 16 matrices in the Clifford algebra structure of SO(4) form a basis for the set of all hermitian 4×4 matrices, while for SO(6) the Weyl representation is 8-dimensional - as we will see in Section 6.3 - and the 64 products form a basis for the set of all hermitian 8×8 matrices.) The procedure is much the same: you start with **1** and the γ's and take products of increasing numbers of (different) γ's, or equivalently take commutators and anticommutators alternately. For matrices with more than N indices, the number of indices may be reduced (as for $\sigma_{\mu 5}$ above) by contracting with an appropriate ϵ-tensor. The last of these is always proportional to γ_{2N+1}.

For SO(t,s) groups with an indefinite metric, the process is just the same; the only difference is that the γ's with spatial indices are antihermitian rather than hermitian.

Chapter 5

SO(1,4)/SO(1,3)

5.1 Obtaining the Killing Vectors

5.1.1 Preliminaries

We have now established more than enough background to tackle our second coset space, SO(1,4)/SO(1,3). We will see that this coset space shares many of the features of SU(2)/U(1) and we will need very little of the machinery we developed in the last chapter.

An element of $G = \mathrm{SO}(1,4)$ is usually, as we have already seen, written

$$g = \mathrm{e}^{-\mathrm{i}\omega^{AB}T_{AB}} = \mathrm{e}^{-\frac{\mathrm{i}}{2}(\omega^{\mu\nu}\sigma_{\mu\nu}+\omega^{\mu 5}\sigma_{\mu 5}+\omega^{5\mu}\sigma_{5\mu})} = \mathrm{e}^{-\frac{\mathrm{i}}{2}(\omega^{\mu\nu}\sigma_{\mu\nu}+2\omega^{\mu 5}\sigma_{\mu 5})}$$

with an element of $H = \mathrm{SO}(1,3)$ written

$$h = \mathrm{e}^{-\frac{\mathrm{i}}{2}\omega^{\mu\nu}\sigma_{\mu\nu}} \tag{5.1}$$

but in order to identify L, we need to write g in a form equivalent to (2.7). In fact, we will subsume the factor of 2 resulting from the antisymmetry into the coset space parameters:

$$g = \mathrm{e}^{-\frac{\mathrm{i}}{2}\omega^{\mu 5}\sigma_{\mu 5}}\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega^{\mu\nu}\sigma_{\mu\nu}} \tag{5.2}$$

so that L, defined by (2.12), is in this case

$$L = \mathrm{e}^{-\frac{\mathrm{i}}{2}\omega^{\mu 5}\sigma_{\mu 5}} \tag{5.3}$$

Our first task is to find the Killing vectors of the Goldstone fields $M^{\mu 5}$ (remember, there is one for each coset space parameter). We know how to do this if our coset space admits the automorphism (2.41), so let us show that it does by decomposing the commutation relations for the generators in the form (2.37)-(2.39):

$$[T_{\mu\nu}, T_{\rho\lambda}] = \mathrm{i}(\eta_{\nu\rho}T_{\mu\lambda} - \eta_{\mu\rho}T_{\nu\lambda} - \eta_{\nu\lambda}T_{\mu\rho} + \eta_{\mu\lambda}T_{\nu\rho}) \tag{5.4}$$

$$[T_{\mu\nu}, T_{\rho 5}] = \mathrm{i}(\eta_{\nu\rho}T_{\mu 5} - \eta_{\mu\rho}T_{\nu 5}) \tag{5.5}$$

$$[T_{\mu 5}, T_{\rho 5}] = -\mathrm{i}\eta_{55}T_{\mu\rho} \tag{5.6}$$

These clearly have the required $\mathbb{Z}_2$ grading structure. This means that we can use equation (2.49), or rather its equivalent for this situation:

$$\delta g L^2 + L^2\delta\tilde{g}^{-1} = \frac{\partial L^2}{\partial M^{\mu 5}}\lambda^{AB}K^{\mu 5}_{AB} \tag{5.7}$$

to determine the Killing vectors $K^{\mu 5}_{AB}$. (Here the λ^{AB}'s are the parameters of the transformation - the equivalent of the ϕ^i for SU(2)/U(1).) To do this, we must of course find each of the other quantities in this equation. If we concentrate on the action under the subgroup H to start with, we see that the nature of δg is obvious from (5.1):

$$\delta g = -\frac{\mathrm{i}}{2}\lambda^{\mu\nu}\sigma_{\mu\nu} \tag{5.8}$$

and, as $\tilde{h} = h$,

$$\delta\tilde{g}^{-1} = \frac{\mathrm{i}}{2}\lambda^{\mu\nu}\sigma_{\mu\nu} \tag{5.9}$$

We now want to find L^2 as a linear sum of the generators, as we did in Section 2.4.4. We start with L^2 given by the square of (5.3), which is an exponential of $\omega^{\mu 5}\sigma_{\mu 5}$. By using (4.143), (assuming the generators to be in the spinor representation), we see that

$$(\omega^{\mu 5}\sigma_{\mu 5})^2 = \frac{1}{2}\{\omega^{\mu 5}\sigma_{\mu 5}, \omega^{\nu 5}\sigma_{\nu 5}\} = \omega^{\mu 5}\omega^{\nu 5}\eta_{55}\eta_{\mu\nu}\mathbf{1} = \omega^{\mu 5}\omega_{\mu 5}\mathbf{1} \tag{5.10}$$

As remarked in Section 4.1, this feature ensures that we will not need projection operators. Indeed, it is remarkably similar to the case for SU(2)/U(1) and we would expect the rest of the analysis to be along the same lines with an eventual expression for L^2 looking very like (2.60). We therefore propose to split $\omega^{\mu 5}$ into a magnitude and a direction.

This is where the one extra subtlety of this coset space comes in. For SU(2)/U(1), $\boldsymbol{\theta} = \theta^a\sigma_a$ lay in a 2-dimensional subspace of the Lie algebra of SU(2), which has positive definite metric. In this case, $\omega^{\mu 5}$ is a vector-like part of the antisymmetric 5-tensor ω^{AB}, that is, it is a tensor which is constrained such that its second index lies in the x^5 direction. Because the 5-dimensional space it lives in has an indefinite metric, we must be careful raising and lowering its indices, so we would be unwise to treat it as a vector $\omega^{\mu(5)}$. Indeed, in (5.10) we have kept the 5's in as (fixed) tensor indices which are raised and lowered with η_{55}'s. (Note that this practice means that, unlike indices which are summed over, we may have more than one covariant and one contravariant 5 in a term.) Thus rather than dealing with a unit vector, we will introduce a unit tensor, $n^{\mu 5}$, so that

$$\omega^{\mu 5} = \omega n^{\mu 5} \tag{5.11}$$

As the metric is indefinite, the tensor $\omega^{\mu 5}$ is timelike for some cosets and spacelike for others. We therefore have a two-way choice: we may adopt a timelike unit tensor, in which case for spacelike cosets ω and the components of $n^{\mu 5}$ will be imaginery, or we may adopt a spacelike unit tensor, in which case for timelike cosets ω and the components of $n^{\mu 5}$ will be imaginery. (Null cosets are a special case which cannot be dealt with in this way. We clearly cannot write a timelike or spacelike $\omega^{\mu 5}$ as $\omega n^{\mu 5}$ with $n^{\mu 5}$ null, as then ω^2 would have to be infinite; conversely we cannot use a timelike or spacelike $n^{\mu 5}$ for $\omega^{\mu 5}$ null.) For now, we will opt for a timelike unit tensor:

$$n^{\mu 5} n_{\mu 5} = 1 \tag{5.12}$$

as this gives us the results which look most like those of SU(2)/U(1), though we will describe in the final section of this chapter how the results would differ if we had chosen a spacelike unit tensor.

With this definition

$$(n^{\mu 5} \sigma_{\mu 5})^2 = n^{\mu 5} n_{\mu 5} \mathbf{1} = \mathbf{1} \tag{5.13}$$

and then

$$L^2 = e^{-i\omega n^{\mu 5}\sigma_{\mu 5}} \tag{5.14}$$

$$= \mathbf{1} - i\omega n^{\mu 5}\sigma_{\mu 5} - \frac{1}{2}\omega^2\mathbf{1} + \frac{i}{3!}\omega^3 n^{\mu 5}\sigma_{\mu 5} + \dots \tag{5.15}$$

$$= \mathbf{1}\cos\omega - i n^{\mu 5}\sigma_{\mu 5}\sin\omega \tag{5.16}$$

We now just need $\frac{\partial L^2}{\partial M^{\mu 5}}$. Once again we let $\omega^{\mu 5}$ and $M^{\mu 5}$ lie in the same direction, so they share a unit vector:

$$M^{\mu 5} = M n^{\mu 5} \tag{5.17}$$

This allows us to derive two identities analogous to those of SU(2)/U(1):

$$\frac{\partial\omega}{\partial M^{\mu 5}} = \frac{d\omega}{dM} n_{\mu 5} \tag{5.18}$$

$$\frac{\partial n^{\nu 5}}{\partial M^{\mu 5}} = \frac{1}{M}(\delta^\nu_\mu - n^{\nu 5} n_{\mu 5}) \tag{5.19}$$

Therefore

$$\frac{\partial L^2}{\partial M^{\mu 5}} = -\mathbf{1} n_{\mu 5}\sin\omega\frac{d\omega}{dM} - i\sigma_{\mu 5}\frac{\sin\omega}{M} + i n_{\mu 5} n^{\nu 5}\sigma_{\nu 5}\frac{\sin\omega}{M} - i n_{\mu 5} n^{\nu 5}\sigma_{\nu 5}\cos\omega\frac{d\omega}{dM} \tag{5.20}$$

5.1.2 An Aside: the unexpected projection operators of SO(1,4)/SO(1,3)

Let us quickly note a property of this coset space which is at odds with what we might have expected at this stage. An arbitrary unit vector of the coset space may be written $n^{\mu 5}\sigma_{\mu 5}$ and squares to the identity. In analogy with SU(2), this may be written as the difference of two projection operators:

$$P^{+} = \frac{1}{2}\left(\mathbf{1} + n^{\mu 5}\sigma_{\mu 5}\right) \quad (5.21)$$

$$P^{-} = \frac{1}{2}\left(\mathbf{1} - n^{\mu 5}\sigma_{\mu 5}\right) \quad (5.22)$$

so a vector of arbitrary length ω is

$$\omega^{\mu 5}\sigma_{\mu 5} = \omega P^{+} - \omega P^{-} \quad (5.23)$$

However, we noted in Section 4.1 that in general, to express an $n \times n$ matrix as a linear sum of projection operators, we will want a set of n projection operators, but there are no two-dimensional representations of SO(1,4). Indeed, the spinor representation of SO(1,4) is four-dimensional, so we would expect to need four projection operators if we were dealing with $\omega^{\mu 5}\sigma_{\mu 5}$ in this representation. In general, these would all have different coefficients in the linear sum.

The reason that we only need two projection operators in this case, with one arbitrary invariant ω, is that an arbitrary vector of SO(1,4)/SO(1,3) is not an arbitrary 4×4 traceless, hermitian matrix. All the vectors of the coset space belong to a special class of vectors. We shall look at such classes, or 'strata', for the case of SU(N) algebras in Chapter 6, where we shall see examples of subgroups of SO(6) which are entirely composed of vectors of one stratum. We will return to this particular coset space in Section 9.6.2, where we will show how all the vectors belong to the same special class.

5.1.3 The linear Killing vectors

We are now ready to substitute all the quantities we have found into (5.7). Using (5.8), (5.9) and (5.16), we find that the left-hand side is a commutator, which we can evaluate using (5.5) to get

$$\delta g L^2 + L^2 \delta \tilde{g}^{-1} = -\mathrm{i}\lambda^{\mu\nu} n^{\rho 5} \sin\omega (\eta_{\nu\rho}\sigma_{\mu 5} - \eta_{\mu\rho}\sigma_{\nu 5}) \tag{5.24}$$

Using (5.20) for the right-hand side and equating coefficients of $\lambda^{\mu\nu}$ (which are, of course, independent variables), we therefore find

$$\begin{aligned} n^{\rho 5} \sin\omega(\eta_{\nu\rho}\sigma_{\mu 5} - \eta_{\mu\rho}\sigma_{\nu 5}) &= \Bigg[-\mathrm{i}\mathbf{1} n_{\rho 5} \sin\omega \frac{\mathrm{d}\omega}{\mathrm{d}M} + \sigma_{\rho 5} \frac{\sin\omega}{M} \\ &\quad - n_{\rho 5} n^{\lambda 5} \sigma_{\lambda 5} \frac{\sin\omega}{M} + n_{\rho 5} n^{\lambda 5} \sigma_{\lambda 5} \cos\omega \frac{\mathrm{d}\omega}{\mathrm{d}M} \Bigg] K^{\rho 5}_{\mu\nu} \end{aligned} \tag{5.25}$$

Once again, we can take the trace of both sides to find that

$$n_{\rho 5} K^{\rho 5}_{\mu\nu} = 0 \tag{5.26}$$

which we can substitute back in to get

$$n^{\rho 5}(\eta_{\nu\rho}\sigma_{\mu 5} - \eta_{\mu\rho}\sigma_{\nu 5}) = \frac{1}{M} \sigma_{\rho 5} K^{\rho 5}_{\mu\nu} \tag{5.27}$$

Finally, we take the scalar product with $\sigma^{\lambda 5}$ which gives us

$$K^{\lambda 5}_{\mu\nu} = M n^{\rho 5}(\eta_{\nu\rho}\delta^5_5\delta^\lambda_\mu - \eta_{\mu\rho}\delta^5_5\delta^\lambda_\nu) = M^{\rho 5}(\eta_{\nu\rho}\delta^\lambda_\mu - \eta_{\mu\rho}\delta^\lambda_\nu) \tag{5.28}$$

Note that this is, as should be expected, antisymmetric under $\mu \leftrightarrow \nu$.

The transformation law for the Goldstone bosons is then

$$\begin{aligned} M^{\lambda 5} \to M'^{\lambda 5} &= M^{\lambda 5} + \lambda^{\mu\nu} M^{\rho 5}(\eta_{\nu\rho}\delta^\lambda_\mu - \eta_{\mu\rho}\delta^\lambda_\nu) && (5.29) \\ &= M^{\lambda 5} + \lambda^\lambda{}_\rho M^{\rho 5} - \lambda_\rho{}^\lambda M^{\rho 5} && (5.30) \\ &= M^{\lambda 5} + 2\lambda^\lambda{}_\rho M^{\rho 5} && (5.31) \end{aligned}$$

5.1.4 The non-linear Killing vectors

Now we turn to the transformations of the Goldstone fields under elements of the coset space. From (5.3) we see that in this case,

$$\delta g = \delta\tilde{g}^{-1} = -\frac{\mathrm{i}}{2}\lambda^{\mu 5}\sigma_{\mu 5} \tag{5.32}$$

thus, using (4.143) and (5.16),

$$\delta g L^2 + L^2 \delta\tilde{g}^{-1} = -\mathrm{i}\lambda^{\mu 5}\sigma_{\mu 5}\cos\omega - \mathbf{1}\lambda^{\mu 5} n_{\mu 5}\sin\omega \tag{5.33}$$

Substituting this and (5.20) into (5.7), we obtain (by equating coefficients of $\lambda^{\mu 5}$)

$$\begin{aligned}\sigma_{\mu 5}\cos\omega - \mathrm{i}\mathbf{1}n_{\mu 5}\sin\omega \quad = \quad & \left[-\mathrm{i}\mathbf{1}n_{\rho 5}\sin\omega\frac{\mathrm{d}\omega}{\mathrm{d}M} + \sigma_{\rho 5}\frac{\sin\omega}{M}\right. \\ & \left. -n_{\rho 5}n^{\lambda 5}\sigma_{\lambda 5}\frac{\sin\omega}{M} + n_{\rho 5}n^{\lambda 5}\sigma_{\lambda 5}\cos\omega\frac{\mathrm{d}\omega}{\mathrm{d}M}\right] K^{\rho 5}_{\mu 5}\end{aligned} \tag{5.34}$$

Taking traces (and rearranging) gives us

$$\frac{\mathrm{d}M}{\mathrm{d}\omega}n_{\mu 5} = n_{\rho 5}K^{\rho 5}_{\mu 5} \tag{5.35}$$

which we can substitute back in to get

$$\sigma_{\mu 5}\cos\omega \quad = \quad \sigma_{\rho 5}\frac{\sin\omega}{M}K^{\rho 5}_{\mu 5} - \frac{\mathrm{d}M}{\mathrm{d}\omega}n_{\mu 5}n^{\lambda 5}\sigma_{\lambda 5}\frac{\sin\omega}{M} + n_{\mu 5}n^{\lambda 5}\sigma_{\lambda 5}\cos\omega \tag{5.36}$$

$$\Rightarrow \sigma_{\rho 5}K^{\rho 5}_{\mu 5} \quad = \quad M\cot\omega(\sigma_{\mu 5} - n_{\mu 5}n^{\lambda 5}\sigma_{\lambda 5}) + \frac{\mathrm{d}M}{\mathrm{d}\omega}n_{\mu 5}n^{\lambda 5}\sigma_{\lambda 5} \tag{5.37}$$

Again, we can take a scalar product with $\sigma^{\nu 5}$ to give us the Killing vector:

$$K^{\cdot\nu 5}_{\mu 5} = M \cot\omega(\delta^\nu_\mu - n_{\mu 5}n^{\nu 5}) + \frac{\mathrm{d}M}{\mathrm{d}\omega} n_{\mu 5}n^{\nu 5} \tag{5.38}$$

- remarkably similar to (2.77) for SU(2)/U(1). Note that we could have used $\lambda^{5\mu}$ instead of $\lambda^{\mu 5}$ in (5.32). We would then be finding the Killing vector $K^{\cdot\nu 5}_{5\mu}$. This would interchange $\mu 5 \leftrightarrow 5\mu$ in (5.34) and hence in (5.37). Contracting with $\sigma^{\nu 5}$ would then give us

$$K^{\cdot\nu 5}_{5\mu} = M \cot\omega(-\delta^\nu_\mu - n_{5\mu}n^{\nu 5}) + \frac{\mathrm{d}M}{\mathrm{d}\omega} n_{5\mu}n^{\nu 5} \tag{5.39}$$

$$= -M \cot\omega(\delta^\nu_\mu - n_{\mu 5}n^{\nu 5}) - \frac{\mathrm{d}M}{\mathrm{d}\omega} n_{\mu 5}n^{\nu 5} \tag{5.40}$$

as could be expected. (Knowing that the Killing vector has this antisymmetry will be important at the end of this chapter.)

5.2 Finding the Covariant Derivatives

Our next task is to construct a Lagrangian for SO(1,4)/SO(1,3), which, following the prescription of Chapter 3, means calculating $L^{-1}\partial_\mu L$. We start with L in the trigonometric form

$$L = \mathrm{e}^{-\mathrm{i}\frac{\omega}{2} n^{\mu 5}\sigma_{\mu 5}} = \mathbf{1}\cos\frac{\omega}{2} - \mathrm{i}n^{\mu 5}\sigma_{\mu 5}\sin\frac{\omega}{2} \tag{5.41}$$

The x-dependence is in the ω and the $n_{\mu 5}$, so the differential of this is

$$\partial_\mu L = -\frac{1}{2}\mathbf{1}\sin\frac{\omega}{2}\partial_\mu\omega - \mathrm{i}\sigma_{\nu 5}\sin\frac{\omega}{2}\partial_\mu n^{\nu 5} - \frac{\mathrm{i}}{2}n^{\nu 5}\sigma_{\nu 5}\cos\frac{\omega}{2}\partial_\mu\omega \tag{5.42}$$

We now want to multiply by the inverse of (5.41) to get $L^{-1}\partial_\mu L$ and simplify the resulting expression; the stages of this calculation are precisely equivalent to those in equations (3.40)-(3.43), with the same trigonometric identities, but with the product rule for the σ's now being (4.141) and with $n^a\partial_\mu n_a = 0$ replaced by $n^{\nu 5}\partial_\mu n_{\nu 5} = 0$. The result is

$$
\begin{aligned}
L^{-1}\partial_\mu L = -\frac{\mathrm{i}}{2}n^{\nu 5}\sigma_{\nu 5}\partial_\mu\omega \quad &- \quad \frac{\mathrm{i}}{2}\sin\omega\sigma_{\nu 5}\partial_\mu n^{\nu 5} - \mathrm{i}\sin^2\frac{\omega}{2}n^{\rho 5}\partial_\mu n^{\nu}{}_5\sigma_{\rho\nu} \\
&- \quad \frac{\mathrm{i}}{4}\sin^2\omega\eta_{55}n^{\rho 5}n^{\nu 5}\sigma_{\rho\nu}\partial_\mu\omega
\end{aligned}
\tag{5.43}
$$

We now note that in the final term, $n^{\rho 5}n^{\nu 5}$ is symmetric under the interchange of μ and ν while $\sigma_{\rho\nu}$ is antisymmetric, so the last term is zero. Thus we split up $L^{-1}\partial_\mu L$ into

$$\mathbf{a}_\mu = (n^{\nu 5}\partial_\mu\omega + \sin\omega\partial_\mu n^{\nu 5})\sigma_{\nu 5} \tag{5.44}$$

$$\Rightarrow a_\mu^{\nu 5} = n^{\nu 5}\partial_\mu\omega + \sin\omega\partial_\mu n^{\nu 5} \tag{5.45}$$

and

$$-\frac{\mathrm{i}}{2}\mathbf{v}_\mu = -\mathrm{i}\sin^2\frac{\omega}{2}\eta_{55}n^{\rho 5}\partial_\mu n^{\nu 5}\sigma_{\rho\nu} \tag{5.46}$$

We saw in Section 3.2 that $\mathrm{tr}(\mathbf{a}^\mu\mathbf{a}_\mu) \propto a^\mu_{\nu 5}a^{\nu 5}_\mu$ is an invariant, from which we deduced that $a^{\nu 5}_\mu$ has the right properties to be a covariant derivative. We also remarked that the normalisation of vectors of special orthogonal algebras is different from the normalisation of vectors of SU(2)/U(1), for example - in this case, the normalisation is given by (4.142):

$$\mathrm{tr}(\mathbf{a}^\mu\mathbf{a}_\mu) = 2a^\mu_{\nu 5}a^{\rho 5}_\mu(\sigma^{\nu 5}, \sigma_{\rho 5}) = 4a^\mu_{\rho 5}a^{\rho 5}_\mu \tag{5.47}$$

We now want to put the covariant derivative $D_\mu M^{\nu 5} \propto a^{\nu 5}_\mu$ into the form $\partial_\mu M^{\nu 5}+$ something. Again, we use the same techniques as for SU(2)/U(1) to obtain the replacements

$$\partial_\mu \omega = \frac{\mathrm{d}\omega}{\mathrm{d}M}\partial_\mu M = \frac{\mathrm{d}\omega}{\mathrm{d}M} n_{\rho 5}\partial_\mu M^{\rho 5} \tag{5.48}$$

and

$$\partial_\mu n^{\nu 5} = \frac{1}{M}(\partial_\mu M^{\nu 5} - n^{\nu 5}\partial_\mu M) = \frac{1}{M}(\partial_\mu M^{\nu 5} - n^{\nu 5} n_{\rho 5}\partial_\mu M^{\rho 5}) \tag{5.49}$$

so that $a_\mu^{\nu 5}$ becomes

$$a_\mu^{\nu 5} = \frac{\mathrm{d}\omega}{\mathrm{d}M} n^{\nu 5} n_{\rho 5}\partial_\mu M^{\rho 5} + \frac{\sin\omega}{M}(\partial_\mu M^{\nu 5} - n^{\nu 5} n_{\rho 5}\partial_\mu M^{\rho 5}) \tag{5.50}$$

$$= \partial_\mu M^{\rho 5}\left(\frac{\mathrm{d}\omega}{\mathrm{d}M} n^{\nu 5} n_{\rho 5} + \frac{\sin\omega}{M}(\delta^\nu_\rho - n^{\nu 5} n_{\rho 5})\right) \tag{5.51}$$

Similarly, we can now write the covariant derivative for the standard fields as

$$\mathrm{D}_\mu \psi = \partial_\mu \psi - \frac{\mathrm{i}}{2}\mathbf{v}_\mu \psi = \partial_\mu \psi - \frac{\mathrm{i}}{M}\sin^2\frac{\omega}{2}\eta_{55} n^{\rho 5}(\partial_\mu M^{\nu 5} - n^{\nu 5}\partial_\mu M)\sigma_{\rho\nu}\psi \tag{5.52}$$

- again, the last term is zero because of the symmetry of $n^{\rho 5} n^{\nu 5}$ and the antisymmetry of $\sigma_{\rho\nu}$, so

$$\mathrm{D}_\mu \psi = \partial_\mu \psi - \frac{\mathrm{i}}{M^2}\sin^2\frac{\omega}{2} M^{\rho 5}\partial_\mu M^\nu{}_5 \sigma_{\rho\nu}\psi \tag{5.53}$$

Once again, we will have a closer look at the term in the Lagrangian involving the covariant derivative of the Goldstone fields. The quantities $n^{\nu 5} n_{\rho 5}$ and $\delta^\nu_\rho - n^{\nu 5} n_{\rho 5}$ which occur in $a_\mu^{\nu 5}$ have the same properties as $n^a n_b$ and $\delta^a_b - n^a n_b$, which we now recognise as the properties of projection operators[1].

[1] In Chapter 8 we will get a feel for why the projection operators of the adjoint representation emerge naturally in the expressions for the covariant derivatives and - for the case of SU(N) groups - we will see how sin and cos coefficients come to be associated with particular combinations of these operators

We thus find that

$$a^{\mu}_{\nu 5} a^{\nu 5}_{\mu} = \partial^{\mu} M_{\rho 5} \partial_{\mu} M^{\lambda 5} \left[\left(\frac{\mathrm{d}\omega}{\mathrm{d}M} \right)^2 n_{\rho 5} n^{\lambda 5} + \left(\frac{\sin \omega}{M} \right)^2 (\delta^{\lambda}_{\rho} - n_{\rho 5} n^{\lambda 5}) \right] \tag{5.54}$$

It makes sense to use the same normalisation for the fields as for SU(2)/U(1) and to take

$$D_{\mu} M^{\nu 5} = a^{\nu 5}_{\mu} \Rightarrow \frac{1}{2} D^{\mu} M_{\nu 5} D_{\mu} M^{\nu 5} = \frac{1}{2} a^{\mu}_{\nu 5} a^{\nu 5}_{\mu} = \frac{1}{8} \mathrm{tr}(\mathbf{a}^{\mu} \mathbf{a}_{\mu}) \tag{5.55}$$

5.3 A useful double-check on the metric

From (5.54) it is clear that the metric for this coset space is given by

$$g_{\rho 5\, \lambda 5} = \left(\frac{\mathrm{d}\omega}{\mathrm{d}M} \right)^2 n_{\rho 5} n_{\lambda 5} + \left(\frac{\sin \omega}{M} \right)^2 (\eta_{55} \eta_{\rho\lambda} - n_{\rho 5} n_{\lambda 5}) \tag{5.56}$$

However, we can check this using the following expression:

$$(g^{-1})^{BC} \propto K^{B}_{I} K^{C\, I} \tag{5.57}$$

where B, C are coset space indices and I ranges over all of the group's indices and the inverse metric is defined by

$$g_{AB} (g^{-1})^{BC} = \delta^{C}_{A} \tag{5.58}$$

Boulware and Brown[14] cite a similar expression, but we use the notation of an inverse metric rather than a contravariant metric because in this thesis indices are raised and lowered using the Minkowski (group) metric η_{IJ} rather than the coset space metric. The construction of the metric from Killing vectors was first performed by Isham[12] for the case of chiral groups and his expression for the metric was used by Barnes, Dondi and Sarkar[18] in a similar manner to the following.

For the coset space SO(1,4)/SO(1,3), (5.57) becomes

$$(g^{-1})^{\nu 5\,\sigma 5} \propto K^{\cdot\nu 5}_{AB} K^{\cdot\sigma 5\,AB} = K^{\cdot\nu 5}_{\rho 5} K^{\cdot\sigma 5\,\rho 5} + K^{\cdot\nu 5}_{5\rho} K^{\cdot\sigma 5\,5\rho} + K^{\cdot\nu 5}_{\rho\lambda} K^{\cdot\sigma 5\,\rho\lambda} \tag{5.59}$$

As the non-linear Killing vectors are linear sums of the projection operators $n^{\nu 5} n_{\rho 5}$ and $\delta^{\nu}_{\rho} - n^{\nu 5} n_{\rho 5}$, squaring them in this way is trivial:

$$K^{\cdot\nu 5}_{\rho 5} K^{\cdot\sigma 5\,\rho 5} + K^{\cdot\nu 5}_{5\rho} K^{\cdot\sigma 5\,5\rho} = 2\eta^{\sigma\mu}\eta^{55}\left[M^2 \cot^2\omega(\delta^{\nu}_{\mu} - n_{\mu 5}n^{\nu 5}) + \left(\frac{\mathrm{d}M}{\mathrm{d}\omega}\right)^2 n_{\mu 5}n^{\nu 5}\right] \tag{5.60}$$

$K^{\cdot\nu 5}_{\rho\lambda} K^{\cdot\sigma 5\,\rho\lambda}$ is not difficult to calculate either; with a little work one finds

$$K^{\cdot\nu 5}_{\rho\lambda} K^{\cdot\sigma 5\,\rho\lambda} = 2M^2(\eta^{55}\eta^{\sigma\nu} - n^{\sigma 5}n^{\nu 5}) = 2M^2\eta^{\sigma\mu}\eta^{55}(\delta^{\nu}_{\mu} - n_{\mu 5}n^{\nu 5}) \tag{5.61}$$

Adding these two together, we thus have

$$\begin{aligned} K^{\cdot\nu 5}_{AB} K^{\cdot\sigma 5\,AB} &= 2\eta^{\sigma\mu}\eta^{55}\left[M^2(\cot^2\omega + 1)(\delta^{\nu}_{\mu} - n_{\mu 5}n^{\nu 5}) + \left(\frac{\mathrm{d}M}{\mathrm{d}\omega}\right)^2 n_{\mu 5}n^{\nu 5}\right] \\ &= 2\eta^{\sigma\mu}\eta^{55}\left[M^2 \operatorname{cosec}^2\omega(\delta^{\nu}_{\mu} - n_{\mu 5}n^{\nu 5}) + \left(\frac{\mathrm{d}M}{\mathrm{d}\omega}\right)^2 n_{\mu 5}n^{\nu 5}\right] \end{aligned} \tag{5.62}$$

The fact that this sum is still written in terms of projection operators also makes inversion easy. If a vector can be written

$$\phi^1 P^1 + \phi^2 P^2 + \ldots$$

its inverse is simply

$$\frac{1}{\phi^1}P^1 + \frac{1}{\phi^2}P^2 + \ldots$$

as can easily be seen:

$$(\phi^1 P^1 + \phi^2 P^2 + \ldots)\left(\frac{1}{\phi^1}P^1 + \frac{1}{\phi^2}P^2 + \ldots\right) = P^1 + P^2 + \ldots = \mathbf{1} \tag{5.63}$$

The inverse of $K^{\nu 5}_{AB} K^{\sigma 5\, AB}$, which is proportional to the metric, is therefore

$$\begin{aligned} & 2\eta_{\sigma\mu}\eta_{55}\left[\frac{\sin^2\omega}{M^2}(\delta^\mu_\nu - n^{\mu 5}n_{\nu 5}) + \left(\frac{\mathrm{d}\omega}{\mathrm{d}M}\right)^2 n^{\mu 5}n_{\nu 5}\right] \\ & = 2\left[\frac{\sin^2\omega}{M^2}(\eta_{55}\eta_{\nu\sigma} - n_{\sigma 5}n_{\nu 5}) + \left(\frac{\mathrm{d}\omega}{\mathrm{d}M}\right)^2 n_{\sigma 5}n_{\nu 5}\right] \end{aligned} \tag{5.64}$$

in agreement with (5.56).

5.4 Results with a spacelike unit tensor

Finally, we remark on how the calculations and results differ if we take $n^{\mu 5}$ to be spacelike rather than timelike. We start with the expression for L^2, (5.16), which changes in a very simple manner - the sin and cos are replaced by sinh and cosh. The identity (5.18) just changes sign, while (5.19) becomes

$$\frac{\partial n^{\nu 5}}{\partial M^{\mu 5}} = \frac{1}{M}(\delta^\nu_\mu + n^{\nu 5}n_{\mu 5}) \tag{5.65}$$

Perhaps unsurprising, these changes have no overall impact on the linear Killing vectors. However, the non-linear Killing vector becomes

$$K^{\nu 5}_{\mu 5} = M\coth\omega(\delta^\nu_\mu + n_{\mu 5}n^{\nu 5}) - \frac{\mathrm{d}M}{\mathrm{d}\omega}n_{\mu 5}n^{\nu 5} \tag{5.66}$$

The effect on the expression for $L^{-1}\partial_\mu L$ (5.43) is just the same as for L^2, the sin and cos are replaced by sinh and cosh, while the identities (5.48) and (5.49) become

$$\partial_\mu \omega = -\frac{\mathrm{d}\omega}{\mathrm{d}M} n_{\rho 5} \partial_\mu M^{\rho 5} \tag{5.67}$$

and

$$\partial_\mu n^{\nu 5} = \frac{1}{M}(\partial_\mu M^{\nu 5} + n^{\nu 5} n_{\rho 5} \partial_\mu M^{\rho 5}) \tag{5.68}$$

The final results are

$$\mathrm{D}_\mu M^{\nu 5} = \partial_\mu M^{\rho 5} \left(-\frac{\mathrm{d}\omega}{\mathrm{d}M} n^{\nu 5} n_{\rho 5} + \frac{\sinh \omega}{M} (\delta^\nu_\rho + n^{\nu 5} n_{\rho 5}) \right) \tag{5.69}$$

$$\mathrm{D}_\mu \psi = \partial_\mu \psi - \frac{\mathrm{i}}{M^2} \sinh^2 \frac{\omega}{2} M^{\rho 5} \partial_\mu M^\nu{}_5 \sigma_{\rho\nu} \psi \tag{5.70}$$

and

$$\begin{aligned} \frac{1}{2} D^\mu M_{\nu 5} D_\mu M^{\nu 5} &= \frac{1}{2} \partial^\mu M_{\rho 5} \partial_\mu M^{\lambda 5} \left[-\left(\frac{\mathrm{d}\omega}{\mathrm{d}M} \right)^2 n_{\rho 5} n^{\lambda 5} \right. \\ &\qquad \left. + \left(\frac{\sinh \omega}{M} \right)^2 (\delta^\lambda_\rho + n_{\rho 5} n^{\lambda 5}) \right] \end{aligned} \tag{5.71}$$

Chapter 6

More Lie Algebras

So far we have studied two non-linear realisations based on the coset spaces SU(2)/U(1) and (SO1,4)/SO(1,3). These both had the properties that the broken generators all anticommute and the square of an arbitrary vector of the coset space is proportional to **1**. As remarked in Section 4.1 this vastly simplifies calculations.

In the rest of this thesis we concentrate on SU(N) groups, in particular SU(4), and by way of a homomorphism, on SO(6). The coset spaces of these groups do not in general have these properties and we will see that trying to determine the properties of the associated non-linear realisations is a different proposition entirely. For a start, the Lie algebras we will be using have features which are totally absent in those we have studied so far. Fortunately, these new features are common to all of these coset spaces (when dressed in the appropriate language) and this allows us to develop a standard set of techniques.

In this chapter, we will be looking at features of the Lie algebras of the groups we intend to use, from a geometrical point of view. This study is very much self contained and has been submitted as a paper entitled 'How orbits of SU(N) can describe rotations in SO(6)' to the Journal of Physics A, (authors K. J. Barnes, J. Hamilton-Charlton and T. R. Lawrence).

In the next chapter, we will go on to look at how to define tensor operators for a general SU(N) group and their algebraic properties, which will allow us to determine the covariant derivatives and metrics for the non-linear realisations of SU(N) as described in Chapter 1; however, for now we limit our study to the vectors of Lie algebras. We start by looking at the common features of the Lie algebras of special unitary groups.

6.1 General SU(N)

The elements of any special unitary group may be written

$$g = \mathrm{e}^{-\frac{i}{2}\theta^I \lambda_I} \tag{6.1}$$

where the λ_I are a set of $N^2 - 1$ traceless, hermitian $N \times N$ matrices which are twice the generators T_I. As we did in Section 4.2.1, we denote an arbitrary vector of the Lie algebra

$$\mathrm{x} = \theta^I T_I = \frac{1}{2}\theta^I \lambda_I = x^I \lambda_I \tag{6.2}$$

with the scalar product of two such vectors given by (4.25). Like the σ's of SU(2), the λ's form an orthonormal basis:

$$(\lambda^I, \lambda_J) = \delta^I_J \tag{6.3}$$

and have the product rule

$$\lambda_I \lambda_J = \frac{2}{N}\delta_{IJ}\mathbf{1} + d_{IJ}{}^K \lambda_K + \mathrm{i} f_{IJ}{}^K \lambda_K \tag{6.4}$$

where $d_{IJ}{}^K$ and $f_{IJ}{}^K$ are respectively totally symmetric and totally antisymmetric under rearrangements of I, J, K. (Note that for SU(2), the λ_I are the Pauli matrices and the $d_{IJ}{}^K$ are all zero. For higher-dimensional SU(N), they are not all zero.)

The group SU(N) has $N-1$ diagonal λ's, which are usually labelled $\lambda_3, \lambda_8, \lambda_{15}, \ldots, \lambda_{N^2-1}$. Any diagonal vector of the algebra can then be written as a linear sum of these. For example, for SU(3), the diagonal λ's are

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{6.5}$$

Now if we act on these by conjugation by a group element g:

$$\mathbf{x} \to \mathbf{x}' = g\mathbf{x}g^{-1} \tag{6.6}$$

we get new matrices with the same eigenvalues (this is once again a unitary similarity transformation on Hermitian matrices). Because the eigenvalues of λ_3 and λ_8 are different, it is not possible to use a group element in this way to transform one into the other.

We can effect all the possible unitary similarity transformations on a vector (say λ_3) by acting on it with all the group elements by conjugation, which will give us all the vectors in the algebra with the same eigenvalues (1, -1, 0). Thus under this action of the group on its own algebra - which, as we have seen in SU(2) is the action of the adjoint representation - the algebra falls into (is partitioned into) distinct orbits.

Two matrices have the same eigenvalues if they have the same characteristic equation. The general characteristic equation for an $N \times N$ traceless, hermitian matrix is

$$\mathbf{x}^N - \gamma_2(\mathbf{x})\mathbf{x}^{N-2} - \gamma_3(\mathbf{x})\mathbf{x}^{N-3} - \ldots - \gamma_N(\mathbf{x})\mathbf{1} = 0 \qquad (6.7)$$

where $\gamma_k(\mathbf{x})$ is invariant under the action of the group and is defined by

$$\gamma_k(\mathbf{x}) = \frac{1}{k}\,\mathrm{tr}\left(\mathbf{x}^k - \sum_{l=2}^{k-2} \gamma_l(\mathbf{x})\mathbf{x}^{k-l}\right) = 0 \qquad (6.8)$$

For SU(2) this is a quadratic equation

$$\mathbf{x}^2 - \gamma_2(\mathbf{x})\mathbf{1} = 0 \qquad (6.9)$$

with one invariant, the square of the length of $\mathbf{x}$

$$\gamma_2(\mathbf{x}) = (\mathbf{x}, \mathbf{x}) = \frac{1}{2}\,\mathrm{tr}\,\mathbf{x}^2 \qquad (6.10)$$

while for SU(3), for example, it is a cubic equation with the two invariants

$$\gamma_2(\mathbf{x}) = (\mathbf{x}, \mathbf{x}) \quad \text{and} \quad \gamma_3(\mathbf{x}) = \frac{1}{3}\,\mathrm{tr}\,\mathbf{x}^3 \qquad (6.11)$$

To proceed with the study of these Lie algebras, we turn to the work of Michel and Radicati[27]. This work is based on the notion that when the symmetric structure constants are non-zero, vectors in the space in general have non-trivial anticommutators. Besides the $_\wedge$-algebra (2.2), one can then define another (linearly independent) algebra on the vector space based on the anticommutator:

$$\mathbf{x}_\vee\mathbf{y} \equiv \frac{\sqrt{N}}{2}\{\mathbf{x},\mathbf{y}\} - \frac{1}{\sqrt{N}}\mathbf{1}\,\mathrm{tr}(\mathbf{x}\mathbf{y}) \tag{6.12}$$

This definition ensures that $\mathbf{x}_\vee\mathbf{y}$ is both hermitian and traceless and that this relation is preserved under the group action:

$$\begin{aligned}\mathbf{x}_\vee\mathbf{y} \to (g\mathbf{x}g^{-1})_\vee(g\mathbf{y}g^{-1}) &= \frac{\sqrt{N}}{2}(g\mathbf{x}\mathbf{y}g^{-1} + g\mathbf{y}\mathbf{x}g^{-1}) - \frac{1}{\sqrt{N}}\mathbf{1}\,\mathrm{tr}(g\mathbf{x}\mathbf{y}g^{-1}) \\ &= \frac{\sqrt{N}}{2}g(\mathbf{x}\mathbf{y}+\mathbf{y}\mathbf{x})g^{-1} - \frac{1}{\sqrt{N}}gg^{-1}\mathbf{1}\,\mathrm{tr}(\mathbf{x}\mathbf{y}) \\ &= g(\mathbf{x}_\vee\mathbf{y})g^{-1}\end{aligned} \tag{6.13}$$

(this is obviously true of the $_\wedge$-algebra as well). Furthermore, these are the only linearly independent algebras on the space which are invariant under the action of the group. Another way of saying this is that under the automorphism of the algebra generated by the adjoint representation of SU(N), only operators of the form

$$\mathbf{x}_\mathrm{T}\mathbf{y} = \alpha\mathbf{x}_\vee\mathbf{y} + \beta\mathbf{x}_\wedge\mathbf{y} \qquad \alpha,\beta\in\mathbb{R} \tag{6.14}$$

are preserved and give a vector in the space for $\mathbf{x}$ and $\mathbf{y}$. (For SU(2), $\mathbf{x}_\vee\mathbf{y} \equiv 0$.)

6.1.1 r-vectors and q-vectors

For $N > 2$, (the cases for which there is a non-trivial $_\vee$-algebra), there exist sets of vectors with particular values of the $N-1$ invariants which lead to a simpler characteristic equation than the general case (6.7). One such set is the set of unit r-vectors, defined by

$$\gamma_2(\mathbf{r}) = 1 \qquad \gamma_3(\mathbf{r}) = \gamma_4(\mathbf{r}) = \ldots = 0 \tag{6.15}$$

For every r-vector, there is a corresponding q-vector:

$$\mathbf{q}_r = \frac{1}{\sqrt{N-2}}\mathbf{r}_\vee\mathbf{r} \tag{6.16}$$

which has a quadratic characteristic equation:

$$\mathbf{q}_\vee\mathbf{q} = \frac{N-4}{\sqrt{N-2}}\mathbf{q} \tag{6.17}$$

and, from (6.12), clearly commutes with $\mathbf{r}$.

The beauty of this approach is that it is invariant under the group's action on the vector space - that is, it is independent of basis (as transforming from one set of basis vectors to another corresponds to a similarity transformation). However, many people are more at home working with components of vectors rather than the index-free style we are using. We therefore look in Appendix 1 at what the above relations imply for the components of the q-vectors and r-vectors of SU(3) if we explicitly choose the basis of the Gell-Mann λ-matrices.

Now SU(3) is a rank 2 group (this can be seen from the fact that there are two diagonal generators). This means that if we take a vector $\mathbf{x}$ we can always find another vector which commutes with it and - assuming them to be linearly independent - we can then use these two vectors to construct a plane of mutually commuting vectors (an Abelian subalgebra). No other vectors in the SU(3) algebra then commute with the entirety of this plane. Section III.4 of [27] is concerned with showing that in any such Cartan plane there are three unit positive q-vectors and six unit r-vectors, which are the roots of SU(3) for the plane. (We also show in Appendix 1 that $(\mathbf{r}, \mathbf{q}_r) = 0$ so that for any Cartan plane, a unit r-vector and its corresponding q-vector in that plane form an orthonormal basis.)

In general for SU(N), as Michel and Radicati state in Appendix 3 of [27], the r-vectors of any Cartan (maximal Abelian) subspace of SU(N) are the roots of that space. For the diagonal Cartan subspace, which we denote $\mathcal{C}_\mathrm{d}$, one way in which these can be found is to construct the weights using the eigenvalues of the diagonal generators and take the differences of them - we will see this for SU(4) in Section 6.2. For SU(3), this procedure yields the information that one of the diagonal λ's, λ_3, is a unit r-vector, with the other one, λ_8, its associated q-vector.

It is important to note that as the r-vectors are defined in terms of the invariants, under the group action an r-vector is transformed into another r-vector. Furthermore, all lengths, scalar products, $_\wedge$- and $_\vee$-relations are preserved - in particular, q-vectors are transformed into other q-vectors and an orthonormal basis is transformed into another orthonormal basis.

6.1.2 Orbits and Strata

It should be noted that for any vector $\mathbf{x}$ not every group element acting on it transforms it into another vector. If a group element g commutes with it,

$$\mathbf{x} \to \mathbf{x}' = g\mathbf{x}g^{-1} = \mathbf{x}gg^{-1} = \mathbf{x} \tag{6.18}$$

Such elements form a group called the little group or isotropy group of $\mathbf{x}$, or in terms of the group action on the vector space they are the stabiliser of $\mathbf{x}$ under this action. We can always express such an element as an exponential of a second vector,

$$g = \mathrm{e}^{-\mathrm{i}\mathbf{y}} \tag{6.19}$$

and by considering the power expansion of this it is clear that g commutes with $\mathbf{x}$ if and only if $\mathbf{y}$ commutes with $\mathbf{x}$. So the isotropy group of $\mathbf{x}$ is just a subgroup of SU(N) generated by the centraliser of $\mathbf{x}$ in the algebra (the set of vectors which commute with it).

The centraliser of a vector depends on its eigenvalues. We can see this by looking at diagonal vectors of SU(3). For example, if we take a diagonal vector whose eigenvalues are all different (such as λ_3), it will obviously commute with any other diagonal matrix. It will not, however, commute with any generator of an SU(2) subgroup with off-diagonal components, such as

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

because (due to the dimension of the fundamental representation of SU(2)) any 'identity' element for such a subgroup must have at least two 1's along the leading diagonal, or at any rate two eigenvalues of the same value. So the largest subalgebra such a vector commutes with is the (Cartan) subalgebra of all diagonal vectors, the algebra of U(1) $\odot$ U(1). Hence the stabiliser of any vector with all eigenvalues different is U(1) $\odot$ U(1).

λ_8, however, has a repeated eigenvalue. This means that it acts as an identity for the SU(2) group generated by

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

as well as with the U(1) it generates itself. The isotropy group for such a vector with a repeated eigenvalue is therefore SU(2)$\otimes$U(1)$\approx$U(2). Clearly the centraliser of two equivalent matrices is the same (upto equivalence) as the $\wedge$-algebra is preserved under similarity transformations, so all the vectors in one orbit necessarily have the same stabiliser. The orbits thus fall into two distinct sets - those with a U(1) $\otimes$ U(1) stabiliser and those with a U(2) stabiliser. These sets are known as 'strata'.

It is worth noting that for any SU(N) there is always one stratum which has as its isotropy group SU(N-1)$\otimes$U(1)$\approx$U(N-1) and one stratum which has as its isotropy group the Cartan subgroup U(1)$\otimes$U(1)...U(1), known as the 'generic' stratum (as discussed in [33]).

6.2 SU(4)

We now want to apply all of the above theory for SU(4), which has a 15-dimensional Lie algebra. The λ's which form a basis for this space have the product rule

$$\lambda_I \lambda_J = \frac{1}{2}\delta_{IJ}\mathbf{1} + d_{IJ}{}^K \lambda_K + \mathrm{i} f_{IJ}{}^K \lambda_K \tag{6.20}$$

so the anticommutators are non-zero:

$$\{\lambda_I, \lambda_J\} = \delta_{IJ}\mathbf{1} + 2d_{IJ}{}^K\lambda_K \tag{6.21}$$

Three of these λ's are diagonal:

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_{15} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

(for an explicit matrix representation of all fifteen based on the Gell-Mann λ's of SU(3), see, for example, [34]).

For an arbitrary vector in the Lie algebra, the characteristic equation is

$$\mathbf{x}^4 - \gamma_2(\mathbf{x})\mathbf{x}^2 - \gamma_3(\mathbf{x})\mathbf{x} - \gamma_4(\mathbf{x})\mathbf{1} = 0 \tag{6.22}$$

with $\gamma_2(\mathbf{x})$ and $\gamma_3(\mathbf{x})$ given by (6.11) and

$$\gamma_4(\mathbf{x}) = \frac{1}{4}\operatorname{tr}\left(\mathbf{x}^4 - \gamma_2(\mathbf{x})\mathbf{x}^2\right) = \frac{1}{4}\operatorname{tr}\mathbf{x}^4 - \frac{1}{8}(\operatorname{tr}\mathbf{x}^2)^2 \tag{6.23}$$

As the anticommutators are non-zero, there is a $\vee$-product given by (6.12) with

N=4:

$$\mathbf{x}_\vee\mathbf{y} = \{\mathbf{x},\mathbf{y}\} - (\mathbf{x},\mathbf{y})\mathbf{1} \tag{6.24}$$

- in particular,

$$\mathbf{x}_\vee\mathbf{x} = 2\mathbf{x}^2 - \gamma_2(\mathbf{x})\mathbf{1} \tag{6.25}$$

6.2.1 r-vectors and q-vectors of $\mathcal{C}_\mathrm{d}$

We have a set of unit r-vectors defined by (6.15), so their characteristic equation becomes

$$\mathbf{x}^2(\mathbf{x}^2 - 1) = 0 \tag{6.26}$$

- their eigenvalues are thus 1, -1, 0, 0.

For the diagonal Cartan subspace $\mathcal{C}_\mathrm{d}$, which in this case is 3-dimensional, we can show that this is in agreement with the statement that the r-vectors are the roots of the subspace by using the method outlined in Section 6.1.1.

The weights of $\mathcal{C}_\mathrm{d}$ are constructed from the eigenvalues of λ_3, λ_8 and λ_{15}:

$$\nu^1 = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{6}}\right) \qquad \nu^2 = \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{6}}\right) \tag{6.27}$$

$$\nu^3 = \left(0, -\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{6}}\right) \qquad \nu^4 = \left(0, 0, -\frac{3}{2\sqrt{6}}\right) \tag{6.28}$$

The roots are then just the differences of these:

$$\pm\beta^{12} = \pm(1,0,0) \qquad \pm\beta^{13} = \pm\left(\frac{1}{2},\frac{\sqrt{3}}{2},0\right) \tag{6.29}$$

$$\pm\beta^{14} = \pm\left(\frac{1}{2},\frac{1}{2\sqrt{3}},\sqrt{\frac{2}{3}}\right) \qquad \pm\beta^{23} = \pm\left(-\frac{1}{2},\frac{\sqrt{3}}{2},0\right) \tag{6.30}$$

$$\pm\beta^{24} = \pm\left(-\frac{1}{2},\frac{1}{2\sqrt{3}},\sqrt{\frac{2}{3}}\right) \qquad \pm\beta^{34} = \pm\left(0,-\frac{1}{\sqrt{3}},\sqrt{\frac{2}{3}}\right) \tag{6.31}$$

written as row vectors. More explicitly, the r-vectors are, for example,

$$\mathbf{r}_1 \equiv (\beta^{23})_3\lambda_3 + (\beta^{23})_8\lambda_8 + (\beta^{23})_{15}\lambda_{15} = -\frac{1}{2}\lambda_3 + \frac{\sqrt{3}}{2}\lambda_8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.32}$$

Similarly,

$$\mathbf{r}_2 = \frac{1}{2}\lambda_3 + \frac{\sqrt{3}}{2}\lambda_8 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{r}_3 = \lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.33}$$

(these are the three diagonal unit r-vectors of the SU(3) subgroup generated by $\lambda_1, \lambda_2, \ldots, \lambda_8$)

$$\mathbf{r}_4 = \frac{1}{2}\lambda_3 + \frac{1}{2\sqrt{3}}\lambda_8 + \sqrt{\frac{2}{3}}\lambda_{15} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{6.34}$$

$$\mathbf{r}_5 = -\frac{1}{2}\lambda_3 + \frac{1}{2\sqrt{3}}\lambda_8 + \sqrt{\frac{2}{3}}\lambda_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{6.35}$$

$$\mathbf{r}_6 = -\frac{1}{\sqrt{3}}\lambda_8 + \sqrt{\frac{2}{3}}\lambda_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{6.36}$$

In a diagram of $\mathcal{C}_\text{d}$, these roots form the familiar polyhedral root lattice:

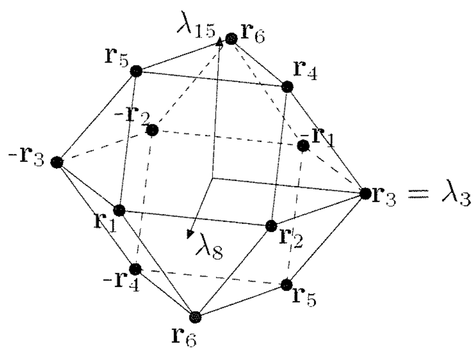

Figure 6.1: The SU(4) root lattice

To obtain the q-vectors of $\mathcal{C}_\text{d}$, we just use (6.16) with N=4:

$$\mathbf{q}_1 = \frac{1}{\sqrt{2}}\mathbf{r}_{1\vee}\mathbf{r}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{6.37}$$

$$\mathbf{q}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \mathbf{q}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{6.38}$$

$$\mathbf{q}_4 = -\mathbf{q}_1 \qquad \mathbf{q}_5 = -\mathbf{q}_2 \qquad \mathbf{q}_6 = -\mathbf{q}_3 \tag{6.39}$$

Note that for each of these,

$$\mathbf{q}^2 = \frac{1}{2}\mathbf{1} \Rightarrow (\mathbf{q}, \mathbf{q}) = 1 \tag{6.40}$$

but also, from (6.25),

$$\mathbf{q}_\vee \mathbf{q} = 0 \tag{6.41}$$

in agreement with (6.17). Note that each q-vector acts as an identity for its r-vector (upto a factor of $1/\sqrt{2}$):

$$\mathbf{q}_r \mathbf{r} = \frac{1}{\sqrt{2}}\mathbf{r} \tag{6.42}$$

so each r-vector is orthogonal to its associated q-vector. (Indeed, a q-vector such as $\mathbf{q}_3$ must be orthogonal to both $\mathbf{r}_3$ and $\mathbf{r}_6$, as $\mathbf{q}_6 = -\mathbf{q}_3$.)

An r-vector and its associated q-vector thus form an orthonormal basis for a plane, but to form a complete basis for $\mathcal{C}_{\rm d}$ we clearly need three linearly independent vectors. The most obvious set that comes to mind is the set of three independent diagonal q-vectors, $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$. Indeed,

$$(\mathbf{q}_1, \mathbf{q}_2) = \frac{1}{2}\,\mathrm{tr}\left(\frac{1}{2}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\right) \tag{6.43}$$

$$= \frac{1}{4}\,\mathrm{tr}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 0 \tag{6.44}$$

and similarly for $(\mathbf{q}_1, \mathbf{q}_3)$ and $(\mathbf{q}_2, \mathbf{q}_3)$, so they form an orthonormal set. We can therefore express any vectors of the subspace as linear combinations of these three q-vectors. The vectors we have considered so far are

$$\lambda_3 = \frac{1}{\sqrt{2}}(\mathbf{q}_2 - \mathbf{q}_1) \tag{6.45}$$

$$\lambda_8 = \frac{1}{\sqrt{6}}(2\mathbf{q}_3 - \mathbf{q}_1 - \mathbf{q}_2) \tag{6.46}$$

$$\lambda_{15} = \frac{1}{\sqrt{3}}(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3) \tag{6.47}$$

$$\mathbf{r}_1 = \frac{1}{\sqrt{2}}(\mathbf{q}_3 - \mathbf{q}_2) \qquad \mathbf{r}_4 = \frac{1}{\sqrt{2}}(\mathbf{q}_3 + \mathbf{q}_2) \tag{6.48}$$

$$\mathbf{r}_2 = \frac{1}{\sqrt{2}}(\mathbf{q}_3 - \mathbf{q}_1) \qquad \mathbf{r}_5 = \frac{1}{\sqrt{2}}(\mathbf{q}_3 + \mathbf{q}_1) \tag{6.49}$$

$$\mathbf{r}_3 = \frac{1}{\sqrt{2}}(\mathbf{q}_2 - \mathbf{q}_1) \qquad \mathbf{r}_6 = \frac{1}{\sqrt{2}}(\mathbf{q}_2 + \mathbf{q}_1) \tag{6.50}$$

6.2.2 Non-diagonal Cartan Subspaces

We can consider non-diagonal Cartan subspaces by looking at what happens to $\mathcal{C}_\mathrm{d}$ under the group action. Recall that all $_\wedge$-relations are preserved under the group action, so the Cartan subalgebra is preserved. This means that $\mathcal{C}_\mathrm{d}$ is transformed into another Cartan subspace, and as stated in Section 6.1.1, any orthonormal basis we take for $\mathcal{C}_\mathrm{d}$ is transformed into an orthonormal basis for the new Cartan subspace.

Furthermore, as any vector $\mathbf{x}$ in the algebra can be diagonalised to one lying in $\mathcal{C}_\mathrm{d}$ by the action of the appropriate group element g, it follows that by applying the inverse transformation to $\mathcal{C}_\mathrm{d}$ we get the Cartan subspace containing $\mathbf{x}$. Thus we can obtain any Cartan subspace by acting on $\mathcal{C}_\mathrm{d}$ with the appropriate group element.

If, as above, we take the set $\mathbf{q}_i = \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ to be our orthonormal basis for $\mathcal{C}_\mathrm{d}$, under the group action this is transformed into another set of orthonormal q-vectors (see Figure 6.2).

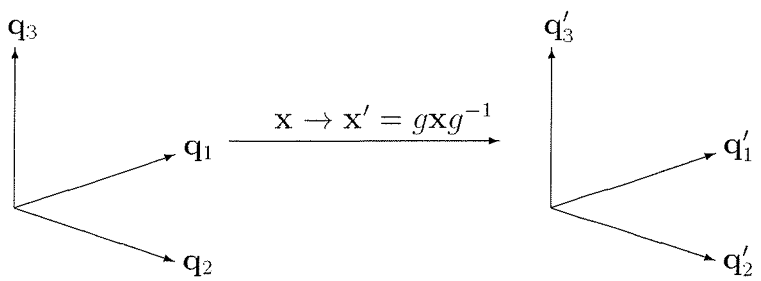

Figure 6.2: Transformation of $\mathbf{q}_i$ under the group action

Hence we see that any vector can be written as a linear sum of three q-vectors:

$$\mathbf{q}'_1 = g\mathbf{q}_1 g^{-1} \tag{6.51}$$

$$\mathbf{q}'_2 = g\mathbf{q}_2 g^{-1} \tag{6.52}$$

$$\mathbf{q}'_3 = g\mathbf{q}_3 g^{-1} \tag{6.53}$$

with the appropriate g.

The group action also preserves the $_\vee$-relations of the vectors. As the $_\vee$-algebra is linear, we only need to consider the $_\vee$-relations of the q-vectors we are using as a basis. For commuting, orthogonal vectors, (6.24) becomes

$$\mathbf{x}_\vee\mathbf{y} = 2\mathbf{x}\mathbf{y} \tag{6.54}$$

then using (6.37)-(6.38) we find

$$\mathbf{q}_{1\vee}\mathbf{q}_2 = -\sqrt{2}\mathbf{q}_3 \tag{6.55}$$

$$\mathbf{q}_{1\vee}\mathbf{q}_3 = -\sqrt{2}\mathbf{q}_2 \tag{6.56}$$

$$\mathbf{q}_{2\vee}\mathbf{q}_3 = -\sqrt{2}\mathbf{q}_1 \tag{6.57}$$

or using the tensor η_{ijk} introduced in [27],

$$\mathbf{q}_{i\vee}\mathbf{q}_j = -\sqrt{2}\eta_{ij}{}^{k}\mathbf{q}_k \tag{6.58}$$

(this tensor acts like the modulus of ϵ_{ijk}: it takes the value 1 if i, j, k are all different, otherwise it takes the value 0).

6.2.3 Orbits and Strata

As pointed out in [33], in SU(4) there are four strata. We shall label them the q-stratum (in analogy with [27]), the r- s- and t-strata.

i) q-stratum

This stratum is composed of vectors with two distinct eigenvalues, both with a multiplicity of 2. Remembering they must be traceless, this means that they must diagonalise to the form

$$\mathbf{d} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & -a \end{pmatrix} \tag{6.59}$$

which is the general form of a q-vector. Hence every vector in this stratum is a q-vector. $\mathbf{d}$ commutes with the SU(2) group generated by

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and with the SU(2) group generated by

$$\lambda_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \lambda_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix}, \quad \mathbf{r}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

as well as with the U(1) it generates itself. The isotropy group of the q-stratum is therefore $SU(2) \otimes SU(2) \otimes U(1)$.

As all q-vectors of a given length are related by similarity transformations, the q-stratum contains one orbit for each length of q-vector. An alternative way to see this is by looking at the values of the three invariants $\gamma_2(\mathbf{q})$, $\gamma_3(\mathbf{q})$ and $\gamma_4(\mathbf{q})$. Using the fact that each q-vector squares to $\frac{1}{2}\mathbf{1}\gamma_2(\mathbf{q})$ (which can be obtained from (6.25) and (6.41)), we see that

$$\gamma_3(\mathbf{q}) = 0 \tag{6.60}$$

Similarly, from the characteristic equation and the square of $\mathbf{q}$, we obtain

$$\gamma_4(\mathbf{q}) = -\frac{1}{4}(\gamma_2(\mathbf{q}))^2 \tag{6.61}$$

so two q-vectors with the same length have the same characteristic equation and therefore lie in the same orbit. (For example, we can act upon $\mathbf{q}_2$ by conjugation with the special unitary - and orthogonal - matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

to get $\mathbf{q}_3$.)

ii) r-stratum

This stratum contains r-vectors such as

$$\mathbf{r}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

but it also contains other vectors with the same multiplicities of eigenvalues, such as

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

All three of these commute with the SU(2) group generated by $\lambda_1, \lambda_2, \lambda_3$. They also commute with the U(1) they generate themselves as well as with a U(1) group generated by one other linearly independent vector. For example, $\mathbf{r}_6$ commutes with the SU(2) group, its own U(1) group and the U(1) group generated by its associated q-vector, $\mathbf{q}_3 = -\mathbf{q}_6$. The isotropy group of the r-stratum is therefore $SU(2) \otimes U(1) \otimes U(1)$.

The above three matrices have different eigenvalues and therefore different characteristic equations, or equivalently different values of γ_2, γ_3 and γ_4, as can easily be verified. In particular, unit r-vectors by definition have $\gamma_3 = \gamma_4 = 0$. We may ask what the consequences of these conditions are (individually) for the eigenvalues. Firstly, if $\gamma_4 = 0$, the characteristic equation becomes

$$\mathbf{x}(\mathbf{x}^3 - \gamma_2(\mathbf{x})\mathbf{x} - \gamma_3(\mathbf{x})) = 0 \qquad (6.62)$$

so one eigenvalue is zero. Secondly, if $\gamma_3 = 0$, the characteristic equation becomes

$$\mathbf{x}^4 - \gamma_2(\mathbf{x})\mathbf{x}^2 - \gamma_4(\mathbf{x})\mathbf{1} = 0 \tag{6.63}$$

- a quadratic equation in $\mathbf{x}^2$. This only has two roots, so $\mathbf{x}^2$ can have at most two eigenvalues, for example

$$\mathbf{x}^2 = \begin{pmatrix} a^2 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & b^2 & 0 \\ 0 & 0 & 0 & b^2 \end{pmatrix}$$

By removing the trace from this, we see that $\mathbf{x}_\vee\mathbf{x}$ is a q-vector. For $\mathbf{x}$ in the r-stratum, this implies that $\mathbf{x}$ has the form

$$\mathbf{x} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & -b \end{pmatrix}$$

where, to ensure the tracelessness of $\mathbf{x}$,

$$a + a + b - b = 2a = 0 \Rightarrow a = 0 \tag{6.64}$$

hence $\mathbf{x}$ is an r-vector. (This is obviously true for non-diagonal vectors as well as diagonal ones.)

Finally, if $\gamma_4 = -\gamma_3 \neq 0$, which is the case for the last of the above three matrices, the characteristic equation becomes

$$(\mathbf{x} - 1)(\mathbf{x}^3 + \mathbf{x}^2 - \gamma_3(\mathbf{x})\mathbf{1}) = 0 \tag{6.65}$$

so one of the eigenvalues is 1.

iii) s-stratum

This stratum is composed of vectors with a triple eigenvalue. From the tracelessness condition, we find that there are only eight diagonal unit vectors in this stratum:

$$\mathbf{s}_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{3}}(-\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3) \tag{6.66}$$

$$\mathbf{s}_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{3}}(\mathbf{q}_1 - \mathbf{q}_2 + \mathbf{q}_3) \tag{6.67}$$

$$\mathbf{s}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{3}}(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{q}_3) \tag{6.68}$$

$$\mathbf{s}_4 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} = \frac{1}{\sqrt{3}}(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) \qquad (6.69)$$

as well as $-\mathbf{s}_1, -\mathbf{s}_2, -\mathbf{s}_3$ and $-\mathbf{s}_4 = \lambda_{15}$

For any vector in this stratum, there is a similarity transformation which diagonalises it to

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & -3a \end{pmatrix}$$

where a is a real number. Calculating the invariants for this vector, we find they are

$$\gamma_2 = 6a^2, \qquad \gamma_3 = -8a^3, \qquad \gamma_4 = 3a^4$$

Clearly, vectors with a triple eigenvalue a and those with a triple eigenvalue $-a$ have the same value of γ_2 (the same length) and the same value of γ_4, but their values of γ_3 have opposite signs. Thus for a given length of vector there are two distinct orbits in this stratum and γ_3 distinguishes between them. This is much the same as the situation for q-vectors in SU(3), as discussed in [27].

Finally, using the same arguments as for the previous two strata, the isotropy group of this stratum is SU(3) $\otimes$ U(1) (recall it was noted in Section 6.1.2 that there is always such a stratum).

iv) t-stratum

This is the generic stratum: it is composed of vectors with all eigenvalues different, for example

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

For the second of these, $\gamma_3 = 0$. This is true for any vector which diagonalises to the form

$$\mathbf{x} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & -b \end{pmatrix}$$

so for any such vector $\mathbf{x}_\vee\mathbf{x}$ is a q-vector.

Clearly vectors in this stratum only commute with the Cartan subgroup, i.e. the isotropy group is $\mathrm{U}(1) \otimes \mathrm{U}(1) \otimes \mathrm{U}(1)$.

6.3 SO(6)

6.3.1 Spinor representations

The elements of SO(6) take the form (4.81), where the parameters ω^{AB} and the generators T_{AB} are, as always, antisymmetric under the interchange of A and B, which run $1, \ldots, 6$. The generators and σ's again satisfy the commutation relations (4.82) and (4.90).

SO(6) has two $2^{3-1} = 4$-dimensional spinor representations. For the direct sum of these we can construct six γ-matrices which again obey the Clifford algebra (4.83). To find these, we take the γ's of SO(5) to be the $\gamma_H^{(1)}$ of the method described in Section 4.4.1 and use this inductive method to obtain the gammas of SO(7):

$$\gamma_i = \begin{pmatrix} 0 & 0 & 0 & -\sigma_i \\ 0 & 0 & \sigma_i & 0 \\ 0 & \sigma_i & 0 & 0 \\ -\sigma_i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & 0 & 0 & \mathrm{i}\mathbf{1} \\ 0 & 0 & \mathrm{i}\mathbf{1} & 0 \\ 0 & -\mathrm{i}\mathbf{1} & 0 & 0 \\ -\mathrm{i}\mathbf{1} & 0 & 0 & 0 \end{pmatrix} \tag{6.70}$$

$$\gamma_5 = \begin{pmatrix} 0 & 0 & \mathrm{i}\mathbf{1} & 0 \\ 0 & 0 & 0 & -\mathrm{i}\mathbf{1} \\ -\mathrm{i}\mathbf{1} & 0 & 0 & 0 \\ 0 & \mathrm{i}\mathbf{1} & 0 & 0 \end{pmatrix}, \quad \gamma_6 = \begin{pmatrix} 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \end{pmatrix} \tag{6.71}$$

$$\gamma_7 = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & -\mathbf{1} & 0 \\ 0 & 0 & 0 & -\mathbf{1} \end{pmatrix} \tag{6.72}$$

Again we can take products of the first six of these to obtain the σ's for the Weyl representation and use the last to construct the projection operators

$$P_{\mathrm{R}} = \frac{1}{2}(\mathbf{1} + \gamma_7) = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.73}$$

$$P_{\rm L} = \frac{1}{2}(\mathbf{1} - \gamma_7) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix} \tag{6.74}$$

with which we can project out the σ's for the two spinor representations. We find that the σ's for the right-handed spinor are

$$\sigma^{\rm R}_{ij} = \epsilon_{ij}{}^{k} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \qquad \sigma^{\rm R}_{i4} = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} \tag{6.75}$$

$$\sigma^{\rm R}_{i5} = \begin{pmatrix} 0 & -\sigma_i \\ -\sigma_i & 0 \end{pmatrix} \qquad \sigma^{\rm R}_{45} = \begin{pmatrix} 0 & {\rm i}\mathbf{1} \\ -{\rm i}\mathbf{1} & 0 \end{pmatrix} \tag{6.76}$$

$$\sigma^{\rm R}_{i6} = \begin{pmatrix} 0 & {\rm i}\sigma_i \\ -{\rm i}\sigma_i & 0 \end{pmatrix} \quad \sigma^{\rm R}_{46} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \quad \sigma^{\rm R}_{56} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{6.77}$$

while the σ's for the left-handed spinor are

$$\sigma^{\rm L}_{ij} = \epsilon_{ij}{}^{k} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \qquad \sigma^{\rm L}_{i4} = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} \tag{6.78}$$

$$\sigma^{\rm L}_{i5} = \begin{pmatrix} 0 & -\sigma_i \\ -\sigma_i & 0 \end{pmatrix} \qquad \sigma^{\rm L}_{45} = \begin{pmatrix} 0 & {\rm i}\mathbf{1} \\ -{\rm i}\mathbf{1} & 0 \end{pmatrix} \tag{6.79}$$

$$\sigma^{\rm L}_{i6} = \begin{pmatrix} 0 & -{\rm i}\sigma_i \\ {\rm i}\sigma_i & 0 \end{pmatrix} \quad \sigma^{\rm L}_{46} = \begin{pmatrix} 0 & -\mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \quad \sigma^{\rm L}_{56} = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} \tag{6.80}$$

6.3.2 Connections with SO(4)

All of the above matrices are 4×4 traceless hermitian matrices, so they must lie in the Clifford algebra structure of SO(4), or alternatively, in the algebra of SU(4). In this subsection, we identify each of the σ's of the two spinor representations with the matrices of the Clifford algebra of SO(4).

We start by noting that the $\sigma_{\mu\nu}$ in the above two spinor representations which generate the subgroup of rotations in the first four dimensions - we will call this subgroup $\mathcal{H}$ - have precisely the same form as they do in the Weyl representation of SO(4) (see (4.102)). Indeed, together with the $\sigma_{\mu 5}$ they form an SO(5) subgroup, so we might expect the above $\sigma_{\mu 5}$ to be the $\sigma_{\mu 5}$ discussed in Section 4.4.3; if we commute the appropriate γ's in (4.101) we see this is correct.

This means by elimination that $\sigma_{\mu 6}$ and σ_{56} must be linear combinations of other matrices of the Clifford algebra structure, i.e. the γ's of SO(5). In fact, by inspection, we see that for the right-handed spinor, the $\sigma_{\mu 6}$ are just the γ_μ of SO(4) and similarly σ_{56} is the γ_5 of SO(4), with signs reversed for the left-handed spinor.

6.3.3 The geometry of the algebra of SU(4) in an SO(6) basis

We have shown that the σ's of the spinor representations of SO(6) form a basis for the space of all 4×4 traceless Hermitian matrices. This means that there is a one-to-one mapping between these and the λ's of SU(4) - this can be thought of as a change of axes in the algebra. (Incidentally, this mapping preserves the commutation relations - the algebras of SU(4) and SO(6) are then said to be isomorphic[1], which in this case gives a homomorphism between the two groups.) We should therefore be able to couch all of the results we obtained for SU(4) in the language of SO(6). It turns out that this is remarkably easy and in many ways this is the more natural description.

[1]See, for example, Vol. 2 of [16]

The obvious place to start is with the diagonal Cartan subspace. For SU(4), we found three q-vectors in this space - these are, upto a factor, precisely the diagonal generators of SO(6):

$$\mathbf{q}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = \frac{1}{\sqrt{2}}\sigma_{12} \tag{6.81}$$

$$\mathbf{q}_1 = -\frac{1}{\sqrt{2}}\begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} = -\frac{1}{\sqrt{2}}\sigma_{34} \tag{6.82}$$

$$\mathbf{q}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} = \frac{1}{\sqrt{2}}\sigma^{\mathrm{R}}_{56} \tag{6.83}$$

Using (6.45)-(6.47) then gives us the mapping between the diagonal SO(6) generators and the diagonal SU(4) generators:

$$\lambda_3 = \frac{1}{2}(\sigma_{12} + \sigma_{34}) \tag{6.84}$$

$$\lambda_8 = \frac{1}{2\sqrt{3}}(-\sigma_{12} + \sigma_{34}) + \frac{1}{\sqrt{3}}\sigma^{(3)}_{56} \tag{6.85}$$

$$\lambda_{15} = \frac{1}{\sqrt{6}}(\sigma_{12} - \sigma_{34} + \sigma^{(3)}_{56}) \tag{6.86}$$

We then have the r-vectors. Two of these, $\mathbf{r}_3$ and $\mathbf{r}_6$, can be written in terms of $\mathbf{q}_1$ and $\mathbf{q}_2$, i.e. can be written in terms of the diagonal generators of $\mathcal{H}$:

$$\mathbf{r}_3 = \frac{1}{\sqrt{2}}(\mathbf{q}_2 - \mathbf{q}_1) = \frac{1}{2}(\sigma_{12} + \sigma_{34}) \tag{6.87}$$

$$\mathbf{r}_6 = \frac{1}{\sqrt{2}}(\mathbf{q}_2 + \mathbf{q}_1) = \frac{1}{2}(\sigma_{12} - \sigma_{34}) \tag{6.88}$$

- these are what we called σ^{R}_3 and σ^{L}_3 in Section 4.4.2. Similarly, the other diagonal r-vectors are sums and differences of the diagonal SO(6) generators, for example

$$\mathbf{r}_2 = \frac{1}{\sqrt{2}}(\mathbf{q}_3 - \mathbf{q}_1) = \frac{1}{2}(\sigma_{34} + \sigma^{\mathrm{R}}_{56})$$

- if we consider other SO(4) subgroups, e.g. in the (x_3, x_4, x_5, x_6)-space, these are then the corresponding $SU(2)_{\mathrm{R}}$ and $SU(2)_{\mathrm{L}}$ diagonal generators. $\mathbf{q}_i$ and $\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_6$ are shown in this basis in Figure 6.3 (the r-vectors given by the negatives of these are omitted).

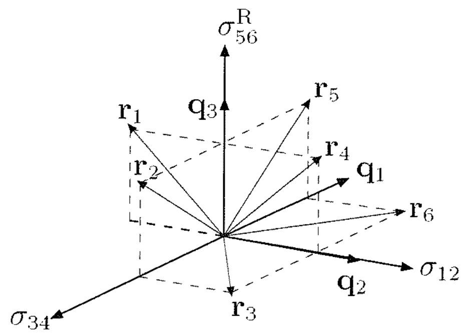

Figure 6.3: Diagonal q-vectors and r-vectors in the SO(6) basis

$_{\mathrm{V}}$-relations between σ's

We now turn to non-diagonal vectors. A key to this is looking at SU(2) subgroups of SO(6). Start by noting that for each generator in the Weyl representation, as the γ's anticommute and square to the identity,

$$(\sigma_{IJ})^2 = (-\mathrm{i}\gamma_I\gamma_J)^2 = -\gamma_I\gamma_J\gamma_I\gamma_J \tag{6.89}$$

$$= \gamma_I\gamma_I\gamma_J\gamma_J = \mathbf{1} \tag{6.90}$$

(with no sums) so for the two spinor representations, it is also true that the generators square to the identity:

$$(\sigma^{\mathrm{R}}_{IJ})^2 = (\sigma^{\mathrm{L}}_{IJ})^2 = \mathbf{1} \tag{6.91}$$

This has an interesting consequence:

$$\sigma_{IJ\vee}\sigma_{IJ} = 2(\sigma_{IJ})^2 - \frac{1}{2}\,\mathrm{tr}(\sigma_{IJ})^2\mathbf{1} = 2\mathbf{1} - \frac{1}{2} \times 4\mathbf{1} = 0 \tag{6.92}$$

i.e. all SO(6) generators are q-vectors[2].

We then note that sets of generators such as $\{\sigma_{24}, \sigma_{25}, \sigma_{45}\}$ form SO(3) subgroups of SO(6) (rotations in the (x_2, x_4, x_5)-subspace). Now SO(3) $\approx$ SU(2), and we know that upto an overall length, all the vectors in an SU(2) algebra have the same eigenvalues (lie in the same orbit). This means that in any SO(3) subgroup, all the vectors are q-vectors, and for any pair of generators which share an index, say σ_{AB} and σ_{AC}, (e.g. σ_{24} and σ_{25}), their $_\vee$-product is zero as we can always find an SO(3) subgroup they fall in, $\{\sigma_{AB}, \sigma_{AC}, \sigma_{BC}\}$:

$$\sigma_{IJ\vee}\sigma_{IK} = 0 \tag{6.93}$$

(no sum)

Note that this implies that for any SO(3) subgroup, $\gamma_3 = 0$ and $\gamma_4 = -\frac{1}{4}\gamma_2$, so there is only one independent invariant which varies from vector to vector. This is one of many cases we will see in which a subspace of the algebra has fewer arbitrary invariants than the algebra as a whole (this is, in fact, why we only needed one invariant for SO(1,4)/SO(1,3), as we shall see in Section 9.6.2).

Now consider two generators with all indices different, for example σ_{14} and σ_{35}. They are two mutually commuting generators of an SO(4) subgroup, $<\sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{34}, \sigma_{35}, \sigma_{45}>$. They satisfy $\sigma_{IJ\vee}\sigma_{IJ} = 0$ (no sum) and $\sigma_{IJ\wedge}\sigma_{KL} = 0$, so a unitary transformation can be used to transform them into another pair of mutually commuting q-vectors. We could, for example, diagonalise them, to

[2] we only explicitly use the R and L superscripts when they are required for an expression to make sense - in general, it is clear which representation is being used from the context

get two of the three diagonal q-vectors (of the same length). If we do this to obtain, for example, σ_{12} and σ_{34}, we can see from the algebra that applying this transformation to the entire SO(4) subgroup will give us the SO(4) subgroup $\mathcal{H}$.

Whichever generators we get after diagonalising σ_{14} and σ_{35}, we know from (6.55)-(6.57) that taking their $_\vee$-product we get the other diagonal generator which generates the SO(2) subgroup orthogonal to the SO(4) the first two lie in (upto a factor). All this is obviously preserved under the unitary transformation, so the $_\vee$-product of σ_{14} and σ_{35} generates an SO(2) subgroup orthogonal to $< \sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{34}, \sigma_{35}, \sigma_{45} >$, i.e. $\sigma_{14\vee}\sigma_{35} \propto \sigma_{26}$.

So we know that if σ_{IJ} and σ_{KL} have any indices in common, $\sigma_{IJ\vee}\sigma_{KL} = 0$, but if they do not, $\sigma_{IJ\vee}\sigma_{KL} \propto \sigma_{MN}$, where $M, N \neq I, J, K, L$. To find the proportionality, we go back to the 8-dimensional Weyl representation. It can be shown using the Clifford algebra and the orthogonality of the σ's that for I, J, K, L all different,

$$\{\sigma_{IJ}, \sigma_{KL}\} - \frac{1}{2}\,\mathrm{tr}(\sigma_{IJ}, \sigma_{KL})\mathbf{1} = -2\gamma_I\gamma_J\gamma_K\gamma_L \tag{6.94}$$

For example,

$$\{\sigma_{12}, \sigma_{35}\} - \frac{1}{2}\,\mathrm{tr}(\sigma_{12}, \sigma_{35})\mathbf{1} = -2\gamma_1\gamma_2\gamma_3\gamma_5 \tag{6.95}$$

However, in the same way that for SO(4) we had $\gamma_5 = -\gamma_1\gamma_2\gamma_3\gamma_4$, for SO(6) we have

$$\gamma_7 = \mathrm{i}\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6 = \frac{\mathrm{i}}{6!}\epsilon^{IJKLMN}\gamma_I\gamma_J\gamma_K\gamma_L\gamma_M\gamma_N = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{6.96}$$

and we know from the Clifford algebra that the γ's anticommute with each other and square to the identity. We can use this information to obtain any string of four γ's such as (6.95) as a product of γ_7 and a σ-matrix:

$$-\gamma_7\sigma_{46} = -\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6\gamma_4\gamma_6 = -\gamma_1\gamma_2\gamma_3\gamma_5 \tag{6.97}$$

Observe that every anticommutation introduces a minus sign, so in general we have

$$\{\sigma_{IJ}, \sigma_{KL}\} - \frac{1}{2}\mathrm{tr}(\sigma_{IJ}, \sigma_{KL})\mathbf{1} = \epsilon_{IJKL}{}^{MN}\gamma_7\sigma_{MN} \tag{6.98}$$

$$= \epsilon_{IJKL}{}^{MN}\begin{pmatrix} \sigma^{\mathrm{R}}_{MN} & 0 \\ 0 & -\sigma^{\mathrm{L}}_{MN} \end{pmatrix} \tag{6.99}$$

for I, J, K, L all different. (The factor of 2 in the right-hand side of (6.95) results from the fact that $\gamma_7\sigma_{64}$ is also included in the sum.) This implies for the individual spinor representations, together with (6.93), that

$$\sigma^{\mathrm{R}}_{IJ\vee}\sigma^{\mathrm{R}}_{KL} = \epsilon_{IJKL}{}^{MN}\sigma^{\mathrm{R}}_{MN} \tag{6.100}$$

and

$$\sigma^{\mathrm{L}}_{IJ\vee}\sigma^{\mathrm{L}}_{KL} = -\epsilon_{IJKL}{}^{MN}\sigma^{\mathrm{L}}_{MN} \tag{6.101}$$

Thus on changing from a basis where the group's parameters have a single vector index, such as the basis of the λ's, to the SO(6) basis where they have an antisymmetric pair of indices, we are replacing the totally symmetric structure constant d_{IJK} with the totally antisymmetric tensor of rank six, ϵ_{IJKLMN}.

Note that among other things, these equations contain the information that if we apply the algebra (6.24) to the vectors of the Weyl representation of SO(4) it does not close. Indeed, the Clifford algebra structure of SO(4) can be defined as the minimal extension of this vector space such that the algebra does close.

r-vectors and q-vectors in SO(4) subgroups

We have begun to see the power of studying the geometry of this Lie algebra in the SO(6) basis - so far we have seen that each 'basis vector' is a q-vector and the symmetric structure constants take a remarkably simple form. We can gain further insight into how the structure of this algebra can be described by rotations by looking at SO(4) subgroups of SO(6).

Every SO(4) subgroup is homomorphic to SU(2) $\otimes$ SU(2) and we know from $\mathcal{H}$ how to take orthogonal combinations of commuting SO(4) generators (q-vectors) to get the $SU(2)_{\rm R} \otimes SU(2)_{\rm L}$ generators (r-vectors). We also know that all vectors of a given length in one of these SU(2) subgroups lie in the same orbit, so by rotating $\mathbf{r}_3$ we see that all vectors of the form $n^{{\rm R}i}\sigma_i^{\rm R}$ (in the $SU(2)_{\rm R}$ subgroup) are r-vectors and similarly all $SU(2)_{\rm L}$ vectors are r-vectors[3]. This is obviously true of the $SU(2)_{\rm R}$ and $SU(2)_{\rm L}$ vectors of *any* SO(4) subgroup of SO(6). Furthermore, by applying these $SU(2)_{\rm R}$ and $SU(2)_{\rm L}$ rotations to $\mathbf{q}_2 = \frac{1}{\sqrt{2}}(\sigma_3^{\rm R} + \sigma_3^{\rm L})$ *independently*, we see that any vector that has 'equal parts' in $SU(2)_{\rm R}$ and $SU(2)_{\rm L}$:

$$\mathbf{q} = \theta(n^{{\rm R}i}\sigma_i^{\rm R} + n^{{\rm L}i}\sigma_i^{\rm L})$$

(i.e. its $SU(2)_{\rm R}$ and $SU(2)_{\rm L}$ components have equal magnitude) is a q-vector.

Let us write this expression explicitly for a q-vector in $\mathcal{H}$ in terms of σ_{IJ}'s:

$$\begin{aligned}
\mathbf{q} &= \frac{\theta}{2}\,(n^{\rm R1}(\sigma_{23}+\sigma_{14}) + n^{\rm R2}(\sigma_{31}+\sigma_{24}) + n^{\rm R3}(\sigma_{12}+\sigma_{34}) \\
&\quad + n^{\rm L1}(\sigma_{23}-\sigma_{14}) + n^{\rm L2}(\sigma_{31}-\sigma_{24}) + n^{\rm L3}(\sigma_{12}-\sigma_{34})) \qquad (6.102) \\
&= \frac{\theta}{2}\,((n^{\rm R1}+n^{\rm L1})\sigma_{23} + (n^{\rm R2}+n^{\rm L2})\sigma_{31} + (n^{\rm R3}+n^{\rm L3})\sigma_{12} \\
&\quad + (n^{\rm R1}-n^{\rm L1})\sigma_{14} + (n^{\rm R1}-n^{\rm L1})\sigma_{24} + (n^{\rm R1}-n^{\rm L1})\sigma_{34}) \qquad (6.103)
\end{aligned}$$

[3]These subalgebras therefore also only have one arbitrary invariant, γ_2

There are a number of interesting examples of such q-vectors obtained by equating components of $n_i^{\rm R}$ and $n_i^{\rm L}$:

$$n_i^{\rm R} = n_i^{\rm L} = n_i : \qquad \mathbf{q} = \theta\left(n^1\sigma_{23} + n^2\sigma_{31} + n^3\sigma_{12}\right)$$

- any element of the vector part of $\mathcal{H}$

$$n_i^{\rm R} = -n_i^{\rm L} = n_i : \qquad \mathbf{q} = \theta\left(n^1\sigma_{14} + n^2\sigma_{24} + n^3\sigma_{34}\right)$$

- any element of the axial part of $\mathcal{H}$

$$n_1^{\rm R} = n_1^{\rm L}, n_2^{\rm R} = -n_2^{\rm L}, n_3^{\rm R} = -n_3^{\rm L} : \qquad \mathbf{q} = \theta\left(n^{\rm R1}\sigma_{23} + n^{\rm R2}\sigma_{24} + n^{\rm R3}\sigma_{34}\right)$$

- any element of SO(3) in (x_2, x_3, x_4)-space

$$n_1^{\rm R} = -n_1^{\rm L}, n_2^{\rm R} = n_2^{\rm L}, n_3^{\rm R} = n_3^{\rm L} : \qquad \mathbf{q} = \theta\left(n^{\rm R1}\sigma_{14} + n^{\rm R2}\sigma_{31} + n^{\rm R3}\sigma_{12}\right)$$

- any element of $\mathcal{H}$/above SO(3)

$$n_1^{\rm R} = -n_1^{\rm L}, n_2^{\rm R} = n_2^{\rm L}, n_3^{\rm R} = -n_3^{\rm L} : \qquad \mathbf{q} = \theta\left(n^{\rm R1}\sigma_{14} + n^{\rm R2}\sigma_{31} + n^{\rm R3}\sigma_{34}\right)$$

- any element of SO(3) in (x_1, x_3, x_4)-space

$$n_1^{\rm R} = n_1^{\rm L}, n_2^{\rm R} = -n_2^{\rm L}, n_3^{\rm R} = n_3^{\rm L} : \qquad \mathbf{q} = \theta\left(n^{\rm R1}\sigma_{23} + n^{\rm R2}\sigma_{24} + n^{\rm R3}\sigma_{12}\right)$$

- any element of $\mathcal{H}$/above SO(3)

$$n_1^{\rm R} = -n_1^{\rm L}, n_2^{\rm R} = -n_2^{\rm L}, n_3^{\rm R} = n_3^{\rm L} : \qquad \mathbf{q} = \theta\left(n^{\rm R1}\sigma_{14} + n^{\rm R2}\sigma_{24} + n^{\rm R3}\sigma_{12}\right)$$

- any element of SO(3) in (x_1, x_2, x_4)-space

$$n_1^{\rm R} = n_1^{\rm L}, n_2^{\rm R} = n_2^{\rm L}, n_3^{\rm R} = -n_3^{\rm L} : \qquad \mathbf{q} = \theta\left(n^{\rm R1}\sigma_{23} + n^{\rm R2}\sigma_{31} + n^{\rm R3}\sigma_{34}\right)$$

- any element of $\mathcal{H}$/above SO(3)

Note that unitary transformations in (the adjoint representation of) the $SU(2)_{\rm R}$ subgroup of SO(6) transform one $SU(2)_{\rm R}$ vector into another, but there exist other unitary transformations in SO(6) which transform an $SU(2)_{\rm R}$ vector into a vector in the corresponding $SU(2)_{\rm L}$ subgroup or even a vector in a completely different SU(2) subgroup, as all the unit r-vectors in SO(6) lie in the same orbit (as their eigenvalues are the same). Similarly, acting on a q-vector in $\mathcal{H}$ with a unitary transformation in $\mathcal{H} \approx SU(2)_{\rm R} \otimes SU(2)_{\rm L}$ will transform it into another q-vector in $\mathcal{H}$, but there exist SO(6) transformations which will transform it into a q-vector in a completely different SO(4) subgroup.

Commuting sets of vectors

Finally, if one wishes to make use of all of this theory, it is usually important to have a clear and thorough understanding of which vectors of the algebra commute. It is all too easy, when considering all the subgroups of rotations in SO(6), to get confused about which vectors commute, so in this subsection we present this problem and its solutions as clearly and precisely as possible. We have already found the centralisers of the various strata in Section 6.2.3, but it would be helpful to review this in the language we have used in this section.

Firstly we took as an example of a q-vector the matrix

$$\mathbf{d} = a \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

which is clearly just a multiple of σ_{56}. We observed that, besides the $U(1) \approx SO(2)$ group it generates, it commutes with two SU(2) groups, which we are now calling $SU(2)_{\mathrm{R}}$ and $SU(2)_{\mathrm{L}}$; it should be noted that these two make up $\mathcal{H}$, the SO(4) orthogonal to this subgroup. So the isotropy group is $SU(2) \otimes SU(2) \otimes U(1)$ or $SO(4) \otimes SO(2)$.

For the r-vectors, we looked most closely at $\mathbf{r}_6$. We noted that it commutes with the whole of $SU(2)_{\mathrm{R}}$, as well as with the U(1) group it generates itself, which is the diagonal part of $SU(2)_{\mathrm{L}}$. Finally, it commutes with its associated q-vector, which is, upto a factor, σ_{56} - this clearly acts as an identity for both SU(2) groups, to which it is orthogonal.

We can now think of picking a vector in the algebra, choosing an orthogonal vector it commutes with and then picking a third vector which is orthogonal to both of the first two and commutes with them. For example, we can start with a q-vector. We have already found that any orthogonal vector which commutes with it must lie in the algebra of the SO(4) which commutes with the SO(2) it generates. However, we have the whole SO(4) space to choose from, which contains q-vectors, r-vectors and vectors which are neither. If we pick one of the q-vectors in this SO(4) as our second vector, the third must be a vector in the SO(4) which commutes with it and is orthogonal to it. This uniquely defines a third q-vector (upto a change of length) - we can see this from σ_{12}, whose centraliser in $\mathcal{H}$ is the U(1) $\otimes$ U(1) generated by σ_{12} and σ_{34}. However, if we pick an r-vector as our second vector, this is a vector in a right (or left) SU(2) subgroup of the SO(4) and commutes with the whole of the left (or right) SU(2) subgroup, so although we know our third vector must be another r-vector, we have the whole of an SU(2) subgroup to choose from.

Now start with an r-vector. Its isotropy group is SU(2) $\otimes$ U(1) $\otimes$ U(1), where the algebra of the SU(2) is composed entirely of r-vectors and the U(1) orthogonal to the r-vector is generated by a single q-vector. We could take a q-vector from the U(1) and an r-vector from the SU(2), in which case the q-vector is uniquely defined (upto a change of length), whereas we are free to choose any r-vector from the SU(2). We can also ask whether there are any r-vectors or q-vectors in the algebra of SU(2) $\otimes$ U(1) other than these. Take the example of $\mathbf{r}_3$ which lies in $SU(2)_{\mathrm{R}}$. We know that

$$\mathbf{r}_{3_\vee}\mathbf{r}_3 = \sqrt{2}\mathbf{q}_3 \qquad \text{and} \qquad \mathbf{r}_{6_\vee}\mathbf{r}_6 = -\sqrt{2}\mathbf{q}_3$$

We are asking whether there are any r-vectors or q-vectors which are linear sums of $\mathbf{q}_3$ and an $SU(2)_\mathrm{L}$ vector. These equations are preserved under $SU(2)_\mathrm{R}$ and $SU(2)_\mathrm{L}$ transformations. By applying these transformations to the r-vectors on the left-hand sides of these equations we can get any $SU(2)_\mathrm{R}$ or $SU(2)_\mathrm{L}$ vector. However, these transformations are in the stabiliser of $\mathbf{q}_3$, so the right-hand sides are unaffected. Hence for any $SU(2)_\mathrm{R}$ or $SU(2)_\mathrm{L}$ r-vector,

$$\mathbf{r}_\vee \mathbf{r} = \pm\sqrt{2}\mathbf{q}_3$$

So the vectors we are interested in are linear sums of an $SU(2)_\mathrm{L}$ r-vector and its *corresponding* q-vector, that is, they lie in a plane spanned by the $SU(2)_\mathrm{L}$ r-vector and its corresponding q-vector. If we look at the plane containing $\mathbf{r}_6$ and $\mathbf{q}_3$ in Figure 6.3, we see that $\pm\mathbf{r}_6$ and $\pm\mathbf{q}_3$ are the only r-vectors and q-vectors in this plane. It is not possible to take a linear sum of an r-vector and its corresponding q-vector to get another r-vector or q-vector, therefore the only r-vectors and q-vectors in the algebra of an SU(2) ⊗ U(1) isotropy group are the r-vectors in the SU(2) algebra and the q-vector which generates the U(1).

Finally, it is worth noting that if we take the first vector to be in the t-stratum, we have a choice for our second vector of any vector in a plane (generating a U(1) ⊗ U(1) which commutes with the U(1) generated by the first vector). Having chosen this, the third vector is uniquely defined (upto a change of length).

Chapter 7

Tensor operators of SU(N)

In order to find the covariant derviatives of a non-linear realisation of SU(N), we need to work with L for the realisation, as a linear sum of broken generators. This requires, as mentioned before, the use of projection operators of (the defining representation of) SU(N). These may be constructed from a set of 'u-vectors', which are a generalisation of the s-vectors defined in Section 6.2.3. However, in the next chapter, we will endeavour to construct a general form for $L^{-1}\partial_\mu L$ which will clearly involve the derivatives of these projection operators. In doing this, we will be following in the footsteps of Barnes *et al*[18], who construct just such an expression for SU(N)$\otimes$SU(N)/SU(N) - this paper demonstrates the intimate connection between these derivatives and the projection operators of the *adjoint* representation of SU(N).

In this chapter we therefore carry out an analysis of the adjoint representation of SU(N), much as we did for SU(2) in Section 4.2.2. This will necessarily be more involved for SU(N) as there are new features in the Lie algebra. In Section 7.1 we look at the much-studied f- and d-tensors of SU(N) and note some useful identities. (The former are related to **ad**(**x**).) In Section 7.2 we define the u-vectors and look at some of their properties. In Section 7.3 we turn to the adjoint representation projection operators. The complete set of these was identified by Barnes and Delbourgo[21]; however, we shall only be concerned with a subset, which are analogous to (4.71) and (4.72). By using the identities of Section 7.1 and the u-vector properties of Section 7.2, we are able to obtain simple forms for the symmetric and antisymmetric combinations of these operators. We close the chapter by using the homomorphism between SU(4) and SO(6) explored in Section 6.3 to investigate the form of the f- and d-tensors and the adjoint representation projection operators in the SO(6) description.

7.1 Adjoint representation of su(N)

In Section 4.2.2 we saw how the adjoint representation of an element of SU(2) is defined: acting with the element on a vector of the algebra by conjugation is equivalent to acting on the components with a 'rotation' matrix - this matrix is the adjoint representation of the group element. Using just the orthonormality of the Pauli σ-matrices, we showed that the general form of the matrix for a group element g is

$$R(g)_i{}^j = (\mathbf{Ad}(g))_i{}^j = \frac{1}{2}\,\mathrm{tr}(g\sigma^j g^{-1}\sigma_i)$$

As the λ's of SU(N) are also orthonormal, this definition extends trivially to SU(N):

$$(\mathbf{Ad}(g))_I{}^J \equiv \frac{1}{2}\,\mathrm{tr}(g\lambda^J g^{-1}\lambda_I) \tag{7.1}$$

Again, this mapping is homomorphic.

To find the adjoint representation of the algebra we once again consider infinitesimal transformations. We now have

$$
\begin{aligned}
(\mathbf{Ad}(\mathbf{1}+\delta g))_I{}^J &= \frac{1}{2}\operatorname{tr}\left[(\mathbf{1}-\frac{\mathrm{i}}{2}\delta\theta^K\lambda_K)\lambda^J(\mathbf{1}+\frac{\mathrm{i}}{2}\delta\theta^L\lambda_L)\lambda_I\right] && (7.2)\\
&= (\lambda^J,\lambda_I)+\frac{\mathrm{i}}{4}\delta\theta^K\operatorname{tr}([\lambda^J,\lambda_K]\lambda_I)+\mathcal{O}(\delta\theta)^2 && (7.3)\\
&= (\lambda^J,\lambda_I)-\frac{1}{2}\delta\theta^K f^J{}_K{}^L\operatorname{tr}(\lambda_L\lambda_I)+\mathcal{O}(\delta\theta)^2 \\
&= \delta_I^J-\delta\theta^K f_I{}^J{}_K+\mathcal{O}(\delta\theta)^2 && (7.4)
\end{aligned}
$$

so

$$
\left(\mathbf{ad}(x^K\lambda_K)\right)_I{}^J = 2\mathrm{i}x^K f_{IK}{}^J \tag{7.5}
$$

Let us look more closely at this quantity. Like the rotation matrix $R(g)$, this acts as an operator on the components of a vector:

$$
y^I \to y'^I = 2\mathrm{i}x^K f^I{}_{KJ}y^J \tag{7.6}
$$

i.e. it transforms the vector $\mathbf{y}$ thus:

$$
\mathbf{y} = y^I\lambda_I \to \mathbf{y}' = y'^I\lambda_I = (2\mathrm{i}x^K f^I{}_{KJ}y^J)\lambda_I \tag{7.7}
$$

We know that the structure constant $f_{IK}{}^J$ arises from the commutator of two λ's, so it is no surprise to learn that this transformation itself is a commutation:

$$
[\mathbf{x},\mathbf{y}] = [x^K\lambda_K, y^J\lambda_J] = x^K y^J[\lambda_K,\lambda_J] = 2\mathrm{i}x^K y^J f_{KJ}{}^I\lambda_I = (2\mathrm{i}x^K f^I{}_{KJ}y^J)\lambda_I \tag{7.8}
$$

To get $\mathbf{x}_\wedge\mathbf{y}$ we just multiply both sides by $-\mathrm{i}/2$

$$
\mathbf{x}_\wedge\mathbf{y} = (x^K f^I{}_{KJ}y^J)\lambda_I \tag{7.9}
$$

but the left-hand side is now a vector of the Lie algebra; the components of this vector are then

$$(\mathbf{x}_\wedge \mathbf{y})^I = x^K f^I{}_{KJ} y^J \tag{7.10}$$

Clearly, the operator $\mathbf{x}_\wedge$ acting on the vector $\mathbf{y}$ is equivalent to the rank-2 tensor operator $x^K f^I{}_{KJ}$ acting on its components. We shall adopt the notation of [27] and denote this operator f_x; that is

$$(f_x)^I{}_J \equiv x^K f^I{}_{KJ} \tag{7.11}$$

(for SU(2), this is obviously just the familiar $x^k \epsilon^i{}_{kj}$). This is clearly linear on the x argument:

$$(f_{\alpha x+\beta y})^I{}_J = \alpha(f_x)^I{}_J + \beta(f_y)^I{}_J \qquad \alpha, \beta \in \mathbb{R} \tag{7.12}$$

We can, in the same manner, define an operator d_x using the $_\vee$-algebra -

$$\mathbf{x}_\vee \mathbf{y} = x^K y^J \lambda_{K\vee} \lambda_J = x^K y^J (\sqrt{N} d_{KJ}{}^I \lambda_I) = (\sqrt{N} x^K d^I{}_{KJ} y^J)\lambda_I \tag{7.13}$$

naturally gives us the definition

$$(d_x)^I{}_J \equiv \sqrt{N} x^K d^I{}_{KJ} \tag{7.14}$$

(in this case we do not technically need to stagger the indices, as this operator is clearly symmetric on its free indices).

In [35] many identities are derived for the symmetric and antisymmetric structure constants and these are rephrased in terms of the f_x and d_x in [27]; for example, the Jacobi identity may be written in terms of the structure constants as

$$f_{IL}{}^{M} f_{MK}{}^{J} - f_{IK}{}^{M} f_{ML}{}^{J} = f^{M}{}_{LK} f_{IM}{}^{J} \tag{7.15}$$

or in terms of the f_x operators as

$$[f_x, f_y] = f_{x \wedge y} \tag{7.16}$$

There is a similar relation for the commutator of f_x and d_y:

$$[f_x, d_y] = d_{x \wedge y} \tag{7.17}$$

Three more of these relations will be useful to us:

$$f_x d_y + f_y d_x = f_{x \vee y} \tag{7.18}$$

$$f_x d_x = d_x f_x = \frac{1}{2} f_{x \vee x} \tag{7.19}$$

and

$$d_x d_y - N f_y f_x = d_{x \vee y} - 2x >< y + 2(\mathbf{x}, \mathbf{y})\mathbf{1} \tag{7.20}$$

where the tensor $x >< y$ is the outer product of $\mathbf{x}$ and $\mathbf{y}$:

$$(x >< y)_I{}^J \equiv x_I y^J \tag{7.21}$$

7.2 The u-vectors of SU(N)

7.2.1 Defining the u-vectors

To look at the connections between the projection operators of the defining representation P^S, those of the adjoint representation and the f- and d-tensors, we split the P^S into two parts. Each of the P^S has a trace of 1 and therefore is not a vector of the Lie algebra, but we may write it as a vector of the algebra plus a trace term:

$$P^S = \frac{1}{N}\mathbf{1} + \mathbf{u}^S \tag{7.22}$$

This clearly has a trace of 1; it also has the property

$$(\lambda_I, P^S) = \frac{1}{2N}\,\mathrm{tr}(\lambda_I) + (\lambda_I, \mathbf{u}^S) = u^{SJ}(\lambda_I, \lambda_J) = u_I^S \tag{7.23}$$

(these components are called P_{Ai} in [21]).

For the diagonal projection operators (those shown explicitly in Section 4.1 with a single 1 and all other elements 0), the 'u-vectors' take a particularly simple matrix form:

$$\mathbf{u}_{\mathrm{d}}^{1} = \frac{1}{N}\begin{pmatrix} N-1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$
\mathbf{u}_{\mathrm{d}}^{2} = \frac{1}{N}\begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & N-1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}
$$

etc.

For SU(4), by comparison with (6.66)-(6.69), we see that

$$
\mathbf{u}_{\mathrm{d}}^{T} = \frac{\sqrt{6}}{4}\mathbf{s}_{T} \tag{7.24}
$$

In general, these are vectors of the stratum with isotropy group SU(N-1)$\otimes$U(1) $\approx$U(N-1), mentioned at the end of Section 6.1.2.

7.2.2 Properties of u-vectors

Recall that we get non-diagonal projection operators from the diagonal ones by applying unitary similarity transformations. This means that we are acting with a group element on the u-vectors (the trace term is obviously unaffected by this action). This, we know, transforms the set of u-vectors into another set of vectors in the same stratum[1] with the same $_\vee$-products, $_\wedge$-products and scalar products. These products can be found from the basic properties of the projection operators, (4.3).

Firstly, we see from the diagonal case that the u-vectors all commute. This means

[1]As the u-vectors are defined by the properties of the associated projection operators, which are preserved under the transformation, a set of u-vectors is always transformed into another set of u-vectors

that

$$\mathbf{u}^S\mathbf{u}^T = \frac{1}{2}\{\mathbf{u}^S, \mathbf{u}^T\} = \frac{1}{\sqrt{N}}\mathbf{u}^S{}_\vee\mathbf{u}^T + \frac{2}{N}(\mathbf{u}^S, \mathbf{u}^T)\mathbf{1} \tag{7.25}$$

(using (6.12)), so

$$P^S P^T = \left(\frac{1}{N}\mathbf{1} + \mathbf{u}^S\right)\left(\frac{1}{N}\mathbf{1} + \mathbf{u}^T\right) \tag{7.26}$$

$$= \frac{1}{N^2}\mathbf{1} + \frac{1}{N}(\mathbf{u}^S + \mathbf{u}^T) + \frac{1}{\sqrt{N}}\mathbf{u}^S{}_\vee\mathbf{u}^T + \frac{2}{N}(\mathbf{u}^S, \mathbf{u}^T)\mathbf{1} \tag{7.27}$$

If $S \neq T$ we know that this is zero, so that

$$(\mathbf{u}^S, \mathbf{u}^T) = -\frac{1}{2N} \tag{7.28}$$

(we already knew they could not be orthogonal as there are N of them and they commute, while the Cartan subspace is only $(N-1)$-dimensional)

and

$$\mathbf{u}^S{}_\vee\mathbf{u}^T = -\frac{1}{\sqrt{N}}(\mathbf{u}^S + \mathbf{u}^T) \tag{7.29}$$

Similarly, if $S = T$, the property $P^S P^S = P^S$ implies that

$$(\mathbf{u}^S, \mathbf{u}^S) = \frac{1}{2} - \frac{1}{2N} \tag{7.30}$$

and

$$\mathbf{u}^S{}_\vee\mathbf{u}^S = \left(\sqrt{N} - \frac{2}{\sqrt{N}}\right)\mathbf{u}^S \tag{7.31}$$

(remember that we never have any implied sum on the S, T, U-indices).

7.3 Adjoint representation projection operators

Recall that the adjoint representation of SU(2) is 3-dimensional and for each vector of the algebra $\mathbf{x} = xn^i\sigma_i$ a set of three projection operators may be defined:

$$P^1 = \frac{1}{2}\operatorname{tr}(P^+\sigma^j P^-\sigma_i) = \frac{1}{2}(\delta_i^j - n_i n^j - in^k\epsilon_i{}^j{}_k)$$

$$P^2 = \frac{1}{2}\operatorname{tr}(P^-\sigma^j P^+\sigma_i) = \frac{1}{2}(\delta_i^j - n_i n^j + in^k\epsilon_i{}^j{}_k)$$

$$P^3 = \operatorname{tr}(P^+\sigma^j P^+\sigma_i) = \operatorname{tr}(P^-\sigma^j P^-\sigma_i) = n_i n^j$$

The adjoint representation of SU(N) is (N^2-1)-dimensional and therefore (for each vector of the algebra) there are N^2-1 projection operators[21]. In this section, we will be concerned with $N(N-1)$ of these, given by

$$(P^{TU})_I{}^J \equiv \frac{1}{2}\operatorname{tr}(P^T\lambda_I P^U\lambda^J) \qquad T \neq U \tag{7.32}$$

or, more precisely, their symmetric combinations

$$(P^{TU} + P^{UT})_I^J = \frac{1}{2}\operatorname{tr}(P^T\lambda_I P^U\lambda^J) + \frac{1}{2}\operatorname{tr}(P^U\lambda_I P^T\lambda^J) \qquad T \neq U \tag{7.33}$$

(these are themselves projection operators and are symmetric under the interchange of I and J due to the cyclicity of the trace) and their antisymmetric combinations

$$(P^{TU} - P^{UT})_I{}^J = \frac{1}{2}\operatorname{tr}(P^T\lambda_I P^U\lambda^J) - \frac{1}{2}\operatorname{tr}(P^U\lambda_I P^T\lambda^J) \qquad T \neq U \tag{7.34}$$

(which do not have projection operator properties).

We know that for any vector of the Lie algebra $\mathbf{x}$, there are N projection operators of the defining representation, of the form (7.22). We now want to substitute this expression into (7.32) to find a general expression for the P^{ST}. This is analogous to the way we substituted $P^{\pm} = \frac{1}{2}(\mathbf{1} \pm n^i\sigma_i)$ into (7.32) for SU(2), except here the $\mathbf{u}^S$ are as yet unknown functions of $\mathbf{x}$:

$$\mathbf{u}^S = \mathbf{u}^S(\mathbf{x}) \tag{7.35}$$

(We shall see how to determine the $\mathbf{u}^S$ as functions of $\mathbf{x}$ for certain coset spaces of SU(4) in Chapter 9.)

The tensor forms we obtain for the P^{ST} themselves are not particularly elegant, but by judicious use of the identities in Section 7.1 we can find simple forms for their symmetric and antisymmetric combinations. For example, for the symmetric combinations, simply substituting (7.22) and using the orthonormality of the λ's gives us

$$\begin{aligned}(P^{ST} + P^{TS})_I^J \quad = \quad & \frac{2}{N^2}\delta_I^J + \frac{1}{2N}\operatorname{tr}[\lambda_I(\mathbf{u}^S + \mathbf{u}^T)\lambda^J] + \frac{1}{2N}\operatorname{tr}[\lambda^J(\mathbf{u}^S + \mathbf{u}^T)\lambda_I] \\ & + \frac{1}{2}\operatorname{tr}(\mathbf{u}^S\lambda_I\mathbf{u}^T\lambda^J) + \frac{1}{2}\operatorname{tr}(\mathbf{u}^T\lambda_I\mathbf{u}^U\lambda^J) \qquad S \neq T \end{aligned} \tag{7.36}$$

To simplify this, we need a couple of identities which can be derived from those given in Section 7.1 (the derivations are given in Appendix 2):

$$\operatorname{tr}(\lambda_I\mathbf{x}\lambda^J) + \operatorname{tr}(\lambda^J\mathbf{x}\lambda_I) = \frac{4}{\sqrt{N}}(d_x)_I^J \tag{7.37}$$

and

$$\operatorname{tr}(\mathbf{x}\lambda_I\mathbf{y}\lambda^J) + \operatorname{tr}(\mathbf{y}\lambda_I\mathbf{x}\lambda^J) = \frac{8}{N}(\mathbf{x},\mathbf{y})\delta_I^J + \frac{4}{N}(d_{x\vee y})_I^J + 4\{f_x, f_y\}_I^J \tag{7.38}$$

Using these gives us

$$(P^{ST}+P^{TS})_I^J = \left(\frac{2}{N^2}+\frac{4}{N}(\mathbf{u}^S,\mathbf{u}^T)\right)\delta_I^J+\frac{2}{N\sqrt{N}}(d_{u^S+u^T})_I^J$$
$$+\frac{2}{N}(d_{u^S\vee u^T})_I^J+2\{f_{u^S},f_{u^T}\}_I^J \qquad S\neq T \qquad (7.39)$$

Now we know that $\mathbf{u}^S$ and $\mathbf{u}^T$ commute, so by (7.16) f_{u^S} and f_{u^T} commute. The anticommutator in the last term then reduces to a product. Similarly, we can use the scalar product and the $\vee$-product of $\mathbf{u}^S$ and $\mathbf{u}^T$, (7.28) and (7.29), to cancel the other terms, leaving us with

$$(P^{ST}+P^{TS})_I^J = 4(f_{u^S}f_{u^T})_I^J \qquad S\neq T \qquad (7.40)$$

Finding the antisymmetric combinations is much the same.

$$(P^{ST}-P^{TS})_I{}^J = \frac{1}{2N}\,\mathrm{tr}[\lambda_I(\mathbf{u}^T-\mathbf{u}^S)\lambda^J]-\frac{1}{2N}\,\mathrm{tr}[\lambda^J(\mathbf{u}^T-\mathbf{u}^S)\lambda_I]$$
$$+\frac{1}{2}\,\mathrm{tr}(\mathbf{u}^S\lambda_I\mathbf{u}^T\lambda^J)-\frac{1}{2}\,\mathrm{tr}(\mathbf{u}^T\lambda_I\mathbf{u}^U\lambda^J) \qquad S\neq T \ (7.41)$$

We can now use

$$\mathrm{tr}(\lambda_I\mathbf{x}\lambda^J)-\mathrm{tr}(\lambda^J\mathbf{x}\lambda_I)=4\mathrm{i}(f_x)_I{}^J \qquad (7.42)$$

and

$$\mathrm{tr}(\mathbf{x}\lambda_I\mathbf{y}\lambda^J)-\mathrm{tr}(\mathbf{y}\lambda_I\mathbf{x}\lambda^J)=4[f_x,f_y]_I{}^J-\frac{4\mathrm{i}}{\sqrt{N}}(f_xd_y-f_yd_x)_I{}^J+\mathrm{tr}(\lambda_I[\mathbf{x},\mathbf{y}]\lambda^J) \qquad (7.43)$$

(also derived in Appendix 2) and the fact that the u-vectors commute to get

$$(P^{ST} - P^{TS})_I{}^J = \frac{2\mathrm{i}}{N}(f_{u^T} - f_{u^S})_I{}^J + \frac{2\mathrm{i}}{\sqrt{N}}(f_{u^T} d_{u^S} - f_{u^S} d_{u^T})_I{}^J \qquad S \neq T \tag{7.44}$$

It is actually possible to use the identities in Section 7.1 to eliminate the d_u's from this last equation. We shall go through the technique for doing this in some detail, as it will involve some concepts which will be of use to us later. These concern the properties of the f_u operators and the $u >< u$ operators.

First consider the operator

$$(u^S >< u^T)_I{}^J \equiv u_I^S u^{TJ} \tag{7.45}$$

(these operators are called $P_{Ai} P_{Bj}$ in [21] and they are related to the other $N-1$ projection operators of the adjoint representation which we have not looked at). We act on an arbitrary vector of the algebra, $\mathbf{x}$, with this operator:

$$(u^S >< u^T)_I{}^J x_J = u_I^S u^{TJ} x_J = u_I^S (\mathbf{u}^T, \mathbf{x}) \tag{7.46}$$

or in coordinate-free notation,

$$u^S >< u^T \mathbf{x} = (\mathbf{u}^T, \mathbf{x}) \mathbf{u}^S \tag{7.47}$$

(this is how the operator $x >< y$ is originally defined in [27]). So we end up with a scaled version of the vector $\mathbf{u}^S$, provided $\mathbf{x}$ is not orthogonal to $\mathbf{u}^T$. Contrast this with the action of f_{u^S} on $\mathbf{x}$:

$$f_{u^S} \mathbf{x} = \mathbf{u}^S {}_\wedge \mathbf{x} = -\frac{\mathrm{i}}{2}(\mathbf{u}^S \mathbf{x} - \mathbf{x} \mathbf{u}^S) \tag{7.48}$$

This is actually orthogonal to all of the u-vectors, including $\mathbf{u}^T$:

$$
\begin{aligned}
(\mathbf{u}^T, \mathbf{u}^S{}_\wedge \mathbf{x}) &= -\frac{i}{4}\left(\mathrm{tr}(\mathbf{u}^T\mathbf{u}^S\mathbf{x}) - \mathrm{tr}(\mathbf{u}^T\mathbf{x}\mathbf{u}^S)\right) && (7.49)\\
&= -\frac{i}{4}\left(\mathrm{tr}(\mathbf{u}^S\mathbf{u}^T\mathbf{x}) - \mathrm{tr}(\mathbf{u}^S\mathbf{u}^T\mathbf{x})\right) && (7.50)\\
&= 0 && (7.51)
\end{aligned}
$$

where we have cycled the second trace and used the fact that the u-vectors commute in the first. This means that

$$u^U >< u^T f_{u^S}\mathbf{x} = (\mathbf{u}^T, \mathbf{u}^S{}_\wedge \mathbf{x})\mathbf{u}^U \equiv 0 \tag{7.52}$$

for all $\mathbf{x}$. Similarly,

$$f_{u^S} u^U >< u^T \mathbf{x} = (\mathbf{u}^T, \mathbf{x})\mathbf{u}^S{}_\wedge \mathbf{u}^U \equiv 0 \tag{7.53}$$

for all $\mathbf{x}$, i.e.

$$u^U >< u^T f_{u^S} = f_{u^S} u^U >< u^T = 0 \tag{7.54}$$

for any three u-vectors $\mathbf{u}^S, \mathbf{u}^T, \mathbf{u}^U$.

We can now make use of this property in adapting equation (7.20) to our purpose. Again, this is done in Appendix 2, with the final result

$$f_{u^T} d_{u^S} = 2\sqrt{N} f_{u^S} f_{u^T}^2 - \frac{1}{\sqrt{N}} f_{u^T} \tag{7.55}$$

Substituting this into (7.44) we finally find

$$P^{ST} - P^{TS} = 4i f_{u^S} f_{u^T} (f_{u^T} - f_{u^S}) \tag{7.56}$$

7.4 Tensor operators of SO(6)

We saw in section 6.3 that all of the properties of the su(4) Lie algebra can been seen in an so(6) 'description'. This can obviously be extended to the tensor operators. In particular, the f- and d-tensors have a particularly simple form in this description. The operator f_x is an element of the adjoint representation of the so(6) Lie algebra and may be written

$$(f_a)_{IJ}{}^{KL} = a^{MN} f_{IJMN}{}^{KL} \tag{7.57}$$

where the $f_{IJMN}{}^{KL}$ are the structure constants of SO(6). From the Lie algebra, we have

$$f_{IJKL}{}^{OP}\sigma_{OP} = -\delta_{JK}\sigma_{IL} + \delta_{IK}\sigma_{JL} + \delta_{JL}\sigma_{IK} - \delta_{IL}\sigma_{JK} \tag{7.58}$$

we can then take a scalar product with σ^{MN} to give us

$$\begin{aligned} f_{IJKL}{}^{MN} &= \frac{1}{2}[\delta_{IK}(\delta_J^M\delta_L^N - \delta_J^N\delta_L^M) - \delta_{JK}(\delta_I^M\delta_L^N - \delta_I^N\delta_L^M) \\ &\quad -\delta_{IL}(\delta_J^M\delta_K^N - \delta_J^N\delta_K^M) + \delta_{JL}(\delta_I^M\delta_K^N - \delta_I^N\delta_K^M)] \end{aligned} \tag{7.59}$$

From this we find that

$$(f_a)_{IJ}{}^{MN} = a_I{}^N\delta_J^M - a_I{}^M\delta_J^N - a_J{}^N\delta_I^M + a_J{}^M\delta_I^N \tag{7.60}$$

- we note that this has all the symmetries we expect of it. Similarly, $(d_a)_{IJ}{}^{MN}$ has the simple form

$$(d_a)_{IJ}{}^{MN} = \sqrt{4}a^{MN}d_{IJMN}{}^{KL} = \pm 2a^{MN}\epsilon_{IJMN}{}^{KL} \tag{7.61}$$

We can now turn to the projection operators P^{ST}. We know that these are projection operators constructed from the P^S and the generators. We would therefore expect to have

$$(P^{ST})_{KL}{}^{IJ} \propto \mathrm{tr}(P^S \sigma_{KL} P^T \sigma^{IJ})$$

such that these still have projection operator qualities. To find the normalisation appropriate for P^{ST} to be a projection operator, we look at the paper in which it was first shown that

$$(P^{ST})_J{}^I = \frac{1}{2}\,\mathrm{tr}(P^S \lambda_J P^T \lambda^I)$$

is a projection operator for SU(N), [21], and see that we require an identity equivalent to (4.3) of that paper:

$$\frac{1}{2}\,\mathrm{tr}(\lambda^I X)\,\mathrm{tr}(\lambda_I Y) = \mathrm{tr}(XY) - \frac{1}{N}\,\mathrm{tr}\,X\;\mathrm{tr}\,Y$$

where X and Y are arbitrary hermitian matrices (not necessarily traceless), using SO(6) σ's in place of λ's. In Appendix 3 we find such an identity:

$$\mathrm{tr}(\sigma^{IJ} X)\,\mathrm{tr}(\sigma_{IJ} Y) = 8\,\mathrm{tr}(XY) - 2\,\mathrm{tr}\,X\;\mathrm{tr}\,Y$$

With this, it is trivial to find the normalisation which ensures that

$$(P^{ST})_{KL}{}^{IJ}(P^{ST})_{IJ}{}^{MN} = (P^{ST})_{KL}{}^{MN}$$

for $S \neq T$:

$$\begin{aligned}(P^{ST})_{KL}{}^{IJ}(P^{ST})_{IJ}{}^{MN} &= k\,\mathrm{tr}(P^S \sigma_{KL} P^T \sigma^{IJ})k\,\mathrm{tr}(P^S \sigma_{IJ} P^T \sigma^{MN}) &(7.62)\\ &= k^2\,\mathrm{tr}(\sigma^{IJ} P^S \sigma_{KL} P^T)\,\mathrm{tr}(\sigma_{IJ} P^T \sigma^{MN} P^S) &(7.63)\end{aligned}$$

$$
\begin{aligned}
&= 8k^2 \operatorname{tr}(P^S\sigma_{KL}P^TP^T\sigma^{MN}P^S) - 2k^2 \operatorname{tr}(P^S\sigma_{KL}P^T)\operatorname{tr}(P^T\sigma^{MN}P^S) && (7.64)\\
&= 8k^2 \operatorname{tr}(P^S\sigma_{KL}P^T\sigma^{MN}P^S) - 2k^2 \operatorname{tr}(P^S\sigma_{KL}P^T)\operatorname{tr}(P^T\sigma^{MN}P^S) && (7.65)\\
&= 8k^2 \operatorname{tr}(P^SP^S\sigma_{KL}P^T\sigma^{MN}) - 2k^2 \operatorname{tr}(P^TP^S\sigma_{KL})\operatorname{tr}(P^SP^T\sigma^{MN}) && (7.66)\\
&= 8k^2 \operatorname{tr}(P^S\sigma_{KL}P^T\sigma^{MN}) && (7.67)
\end{aligned}
$$

The correct normalisation is clearly

$$8k^2 = k \Rightarrow k = \frac{1}{8}$$

i.e.

$$(P^{ST})_{KL}{}^{IJ} \equiv \frac{1}{8}\operatorname{tr}(P^S\sigma_{KL}P^T\sigma^{IJ}) \qquad (7.68)$$

Chapter 8

Lagrangians of SU(N) sigma models

8.1 The Content of $\partial_\mu L$

The purpose of this chapter is to determine the general form of the Lagrangian of a non-linear realisation of SU(N), otherwise known as an SU(N) sigma model. We know from Chapter 3 that the Lagrangian for a non-linear realisation is uniquely specified for a given set of standard fields by the covariant derivatives $D_\mu M^A$ and $D_\mu \psi$, and that these may be found by calculating $L^{-1}\partial_\mu L$ and splitting it up into $\mathbf{a}_\mu$ and $\mathbf{v}_\mu$ parts. Our main task, then, is to find an expression for $L^{-1}\partial_\mu L$ for a general SU(N) sigma model. As we are working with Lie algebras in which the generators do not mutually anticommute (there is a non-trivial $_\vee$-algebra), this will require the use of projection operators.

To guide us in this task, we may make use of the work of Barnes *et al* [18], who found the equivalent expression for the chiral sigma models, SU(N)$\otimes$SU(N)/SU(N). It will turn out that the basic approach of this paper is one we can emulate, though some of the methods used in the paper are not directly applicable to the coset spaces we are looking at. The starting point is to consider L as a function of $\boldsymbol{\theta}$, the arbitrary vector for the coset space, which may be written as a linear sum of projection operators:

$$\boldsymbol{\theta} = \theta^A \lambda_A = \sum_S \theta'_S P^S \tag{8.1}$$

(we always write sums over the projection operators indices, $S = 1, 2, \ldots, N$, explicitly; so if no summation is shown, none is implied). L then has the general form

$$L = e^{-\frac{i}{2}\boldsymbol{\theta}} = \sum_S e^{-\frac{i}{2}\theta'_S} P^S \tag{8.2}$$

as can be seen from (4.13). (We will look at how to carry out the decomposition (8.1) in the next chapter, and will find it for certain coset spaces of SU(4)). The derivative $\partial_\mu L$ obviously contains derivatives of both the θ'_S and the P^S:

$$\partial_\mu L = \sum_S \left(-\frac{i}{2}(e^{-\frac{i}{2}\theta'_S}) P^S \partial_\mu \theta'_S + e^{-\frac{i}{2}\theta'_S} \partial_\mu P^S \right) \tag{8.3}$$

The derivatives $\partial_\mu P^S$ are related to the projection operators of the adjoint representation studied in the last chapter. We shall see precisely what the connection between them is in Section 8.1.1, drawing on arguments in [18]. The coefficients θ'_S will be studied in Section 8.1.2, where we will gain a much clearer understanding of their significance. In both of these sections we will obtain expressions which will be very useful in simplifying $L^{-1}\partial_\mu L$ when we calculate it in Section 8.2. In the form of $L^{-1}\partial_\mu L$ we end up with at the end of this section, the terms involving $\partial_\mu P^S$ look very much like their equivalents for the chiral sigma models. It is also possible to bring the terms involving the $\partial_\mu \theta'_S$ into a form like their chiral equivalent, but we are principally interested in coset spaces of SU(4), for which an alternative form is more appropriate, as we shall see in the next chapter.

The final section looks at how to split $L^{-1}\partial_\mu L$ into its $\mathbf{a}_\mu$ and $\mathbf{v}_\mu$ parts for certain symmetric spaces.

8.1.1 Derivatives of P^S

Recall that the $\mathbf{u}^S$ are functions of $\mathbf{x}$ - or in this case $\boldsymbol{\theta}$, the arbitrary element of the coset space - which is in turn a function of the Goldstone fields M^A. We can therefore write the derivatives in the following way:

$$\partial_\mu P^S = \partial_\mu \left(\frac{1}{N}\mathbf{1} + \mathbf{u}^S(M^A) \right) = \frac{\partial \mathbf{u}^S}{\partial M^A}\partial_\mu M^A = \frac{\partial u^{SI}}{\partial M^A}\partial_\mu M^A \lambda_I \qquad (8.4)$$

The quantity

$$\frac{\partial u^{SI}}{\partial M^A}$$

looks like a tensor (with I and A tensor indices and an S 'label'), so we might expect to be able to write it as a linear sum of the projection operators of the adjoint representation. Again, [18] points the way to doing so. In general, $\boldsymbol{\theta}$ can be expressed as a power series in $\mathbf{M}$. This means that $\mathbf{M}$ commutes with $\boldsymbol{\theta}$, i.e. its lies in the Cartan subspace of $\boldsymbol{\theta}$. Therefore $\mathbf{M} = M^B\lambda_B$ may be written as a linear sum of the P^S:

$$M^B\lambda_B = \sum_S M'_S P^S \qquad (8.5)$$

(remembering that we explicitly write sums on the S, T, U-indices rather than implying them. In [18], the coefficients M'_S are written as lower-case m's.)

Now act on this from the left with one of the projection operators, say P^1:

$$P^1 M^B \lambda_B = M'_1 P^1 \tag{8.6}$$

then differentiate with respect to M^A and rearrange:

$$P^1 \lambda_A - \frac{\partial M'_1}{\partial M^A} P^1 = M'_1 \frac{\partial P^1}{\partial M^A} - \frac{\partial P^1}{\partial M^A} M^B \lambda_B \tag{8.7}$$

Substitute (8.5) into the right-hand side:

$$P^1 \lambda_A - \frac{\partial M'_1}{\partial M^A} P^1 = M'_1 \frac{\partial P^1}{\partial M^A} - \frac{\partial P^1}{\partial M^A} \sum_S M'_S P^S \tag{8.8}$$

and multiply from the right by a second projection operator, say P^2. Dividing both sides by $M'_1 - M'_2$ then gives

$$\frac{1}{M'_1 - M'_2} P^1 \lambda_A P^2 = \frac{\partial P^1}{\partial M^A} P^2 \tag{8.9}$$

More generally, for two different projection operators P^S and P^T,

$$\frac{1}{M'_S - M'_T} P^S \lambda_A P^T = \frac{\partial P^S}{\partial M^A} P^T \tag{8.10}$$

(All of the above is covered on page 401 of [18].)

To find an expression for

$$\frac{\partial P^S}{\partial M^A} = \frac{\partial u^S}{\partial M^A}$$

from this, we make use of the properties of projection operators. First we note that as the P^S form a complete set,

$$\frac{\partial P^1}{\partial M^A} = \frac{\partial P^1}{\partial M^A}(P^1 + P^2 + \ldots + P^N) \tag{8.11}$$

and

$$\frac{\partial P^1}{\partial M^A} = (P^1 + P^2 + \ldots + P^N)\frac{\partial P^1}{\partial M^A} \tag{8.12}$$

Adding these together gives us

$$2\frac{\partial P^1}{\partial M^A} = (P^1 + P^2 + \ldots + P^N)\frac{\partial P^1}{\partial M^A} + \frac{\partial P^1}{\partial M^A}(P^1 + P^2 + \ldots + P^N) \tag{8.13}$$

However,

$$P^1 P^1 = P^1 \Rightarrow P^1\frac{\partial P^1}{\partial M^A} + \frac{\partial P^1}{\partial M^A}P^1 = \frac{\partial P^1}{\partial M^A} \tag{8.14}$$

Therefore

$$\frac{\partial P^1}{\partial M^A} = \frac{\partial P^1}{\partial M^A}P^2 + P^2\frac{\partial P^1}{\partial M^A} + \frac{\partial P^1}{\partial M^A}P^3 + P^3\frac{\partial P^1}{\partial M^A} + \ldots + \frac{\partial P^1}{\partial M^A}P^N + P^N\frac{\partial P^1}{\partial M^A} \tag{8.15}$$

Also,

$$P^T P^1 = 0 \Rightarrow P^T\frac{\partial P^1}{\partial M^A} + \frac{\partial P^T}{\partial M^A}P^1 = 0 \tag{8.16}$$

for $T \neq 1$, which we can use to obtain

$$\frac{\partial P^1}{\partial M^A} = \frac{\partial P^1}{\partial M^A}P^2 - \frac{\partial P^2}{\partial M^A}P^1 + \frac{\partial P^1}{\partial M^A}P^3 - \frac{\partial P^3}{\partial M^A}P^1 + \ldots + \frac{\partial P^1}{\partial M^A}P^N - \frac{\partial P^N}{\partial M^A}P^1 \tag{8.17}$$

We can now substitute (8.10) in to this to get

$$\begin{aligned}\frac{\partial P^1}{\partial M^A} \quad = \quad & \frac{1}{M'_1 - M'_2}(P^1\lambda_A P^2 + P^2\lambda_A P^1) + \frac{1}{M'_1 - M'_3}(P^1\lambda_A P^3 + P^3\lambda_A P^1) \\ & + \ldots + \frac{1}{M'_1 - M'_N}(P^1\lambda_A P^N + P^N\lambda_A P^1)\end{aligned} \tag{8.18}$$

This is getting very close to what we were after; we have quantities of the form $P^S\lambda_A P^T$ on the right-hand side which we know are involved in some of the adjoint representation projection operators. To get the final expression, we use the following procedure:

$$\frac{1}{2}\,\mathrm{tr}\left(\frac{\partial P^1}{\partial M^A}\lambda^J\right)\lambda_J = \frac{1}{2}\,\mathrm{tr}\left(\frac{\partial u^{1I}}{\partial M^A}\lambda_I\lambda^J\right)\lambda_J = \frac{\partial u^{1I}}{\partial M^A}(\lambda_I,\lambda^J)\lambda_J = \frac{\partial u^{1I}}{\partial M^A}\lambda_I$$
$$= \frac{\partial P^1}{\partial M^A} \qquad (8.19)$$

Substituting (8.18) into the left-hand side of this, we finally get

$$\begin{aligned}\frac{\partial P^1}{\partial M^A} &= \frac{1}{M_1'-M_2'}(P^{12}+P^{21})^J_A\lambda_J + \frac{1}{M_1'-M_3'}(P^{13}+P^{31})^J_A\lambda_J + \dots \\ &\qquad + \frac{1}{M_1'-M_N'}(P^{1N}+P^{N1})^J_A\lambda_J \qquad (8.20)\\ &= \sum_{T\neq 1}\frac{1}{M_1'-M_T'}(P^{1T}+P^{T1})^J_A\lambda_J \qquad (8.21)\end{aligned}$$

or in general

$$\frac{\partial P^S}{\partial M^A} = \sum_{T\neq S}\frac{1}{M_S'-M_T'}(P^{ST}+P^{TS})^J_A\lambda_J = \sum_{T\neq S}\frac{4}{M_S'-M_T'}(f_{u^S}f_{u^T})^J_A\lambda_J \qquad (8.22)$$

so that

$$\begin{aligned}\partial_\mu P^S &= \sum_{T\neq S}\frac{1}{M_S'-M_T'}(P^{ST}+P^{TS})^J_A\lambda_J\partial_\mu M^A \qquad (8.23)\\ &= \sum_{T\neq S}\frac{4}{M_S'-M_T'}(f_{u^S}f_{u^T})^J_A\lambda_J\partial_\mu M^A \qquad (8.24)\end{aligned}$$

8.1.2 u-vectors and eigenvalues

To see the deeper significance of the θ'_S, consider diagonalising the coset space vector $\boldsymbol{\theta}$ using a group element g:

$$g\boldsymbol{\theta}g^{-1} = \begin{pmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_N \end{pmatrix} = \sum_S \mu_S P_{\mathrm{d}}^S \tag{8.25}$$

where the μ_S are the eigenvalues of $\boldsymbol{\theta}$ and P_{d}^S are the diagonal projection operators

$$P_{\mathrm{d}}^1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \frac{1}{N}\mathbf{1} + \mathbf{u}_{\mathrm{d}}^1$$

$$P_{\mathrm{d}}^2 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \frac{1}{N}\mathbf{1} + \mathbf{u}_{\mathrm{d}}^2$$

etc. Applying the inverse transformation we get

$$\boldsymbol{\theta} = \sum_S \mu_S g^{-1} P_{\mathrm{d}}^S g = \sum_S \mu_S (\frac{1}{N}\mathbf{1} + g^{-1}\mathbf{u}_{\mathrm{d}}^S g) \tag{8.26}$$

but applying such a transformation to the diagonal u-vectors gives another valid set of u-vectors (with all the same properties), so

$$\frac{1}{N}\mathbf{1} + g^{-1}\mathbf{u}_{\mathrm{d}}^S g$$

are a valid set of projection operators. Thus we have found the decomposition of $\boldsymbol{\theta}$ into projection operators:

$$\boldsymbol{\theta} = \sum_S \mu_S P^S \tag{8.27}$$

i.e. the θ'_S in equation (8.1) are just the eigenvalues of $\boldsymbol{\theta}$, which we can find by solving the characteristic equation of $\boldsymbol{\theta}$.[1]

This clarifies the nature of the decomposition - writing the P^S as u-vectors plus trace terms

$$\theta^A \lambda_A = \sum_S \theta'_S \left(\frac{1}{N}\mathbf{1} + \mathbf{u}^S\right) \tag{8.28}$$

$$= \frac{1}{N}\mathbf{1} \sum_S \theta'_S + \sum_S \theta'_S \mathbf{u}^S \tag{8.29}$$

we see that the left-hand side has no trace, so that

$$\theta^A \lambda_A = \sum_S \theta'_S \mathbf{u}^S \tag{8.30}$$

and

$$\sum_S \theta'_S = 0 \tag{8.31}$$

which is just the condition that the sum of the eigenvalues is zero.

For a completely arbitrary vector of SU(N), this is the only restriction on the eigenvalues. However, some coset spaces have algebras consisting entirely of vectors in a stratum with repeated eigenvalues. For example, in Chapter 9 we shall identify a coset space of SU(4) whose algebra consists entirely of vectors

[1]Many papers on non-linear realisations and finite transformations of SU(N) are based on this approach, such as those by Rosen[32] and Bincer[36]

in the r-stratum and another whose algebra consists entirely of vectors in the q-stratum. Indeed, all of the coset spaces we will study in detail in Chapter 9 have the property $\gamma_3(\theta) = 0$ for every vector of the space, which, as we saw in Section 6.2.3, means that we can only identify two independent eigenvalues (matching the two arbitrary independent invariants $\gamma_2(\theta)$ and $\gamma_4(\theta)$).

8.2 $L^{-1}\partial_\mu L$

We now move on to the core of this chapter: calculating $L^{-1}\partial_\mu L$. We start with the expression (8.2) for L, which has an inverse

$$L^{-1} = \sum_S e^{\frac{i}{2}\theta'_S} P^S \tag{8.32}$$

and a derivative given by (8.3). Multiplying these together, we get

$$L^{-1}\partial_\mu L = -\frac{i}{2}\sum_S P^S \partial_\mu \theta'_S + \sum_{S,T} e^{\frac{i}{2}(\theta'_S - \theta'_T)} P^S \partial_\mu P^T \tag{8.33}$$

The P^S in the first term can be replaced by a $\mathbf{u}^S$ by using (7.22):

$$-\frac{i}{2}\sum_S P^S \partial_\mu \theta'_S = -\frac{i}{2N}\mathbf{1}\sum_S \partial_\mu \theta'_S - \frac{i}{2}\sum_S \mathbf{u}^S \partial_\mu \theta'_S \tag{8.34}$$

and then noting that by differentiating (8.31) the first of these terms is zero; thus (8.33) becomes

$$L^{-1}\partial_\mu L = -\frac{i}{2}\sum_S \mathbf{u}^S \partial_\mu \theta'_S + \sum_{S,T} e^{\frac{i}{2}(\theta'_S - \theta'_T)} P^S \partial_\mu P^T \tag{8.35}$$

To deal with the second term, we substitute in (8.23):

$$\sum_{S,T} e^{\frac{\mathrm{i}}{2}(\theta'_S-\theta'_T)} P^S \partial_\mu P^T = \sum_{S,T} e^{\frac{\mathrm{i}}{2}(\theta'_S-\theta'_T)} P^S \sum_{U\neq T} \frac{1}{M'_T - M'_U}(P^{TU} + P^{UT})^J_A \lambda_J \partial_\mu M^A \tag{8.36}$$

Once again, we can use (7.22) to eliminate the P^S in favour of the $\mathbf{u}^S$, giving us

$$\begin{aligned} L^{-1}\partial_\mu L &= -\frac{\mathrm{i}}{2}\sum_S \mathbf{u}^S \partial_\mu \theta'_S \\ &\quad + \sum_{S,T} e^{\frac{\mathrm{i}}{2}(\theta'_S-\theta'_T)} (\frac{1}{N}\mathbf{1} + \mathbf{u}^S) \sum_{U\neq T} \frac{1}{M'_T - M'_U}(P^{TU} + P^{UT})^J_A \lambda_J \partial_\mu M^A \end{aligned} \tag{8.37}$$

To see how this helps, let us consider the product $\mathbf{u}^S\lambda_J$. By using (6.4) we see that

$$\mathbf{u}^S\lambda_J = u^{SI}\lambda_I\lambda_J = \frac{2}{N}u^S_J\mathbf{1} + \frac{1}{\sqrt{N}}(d_{u^S})_J{}^K\lambda_K - \mathrm{i}(f_{u^S})_J{}^K\lambda_K \tag{8.38}$$

but from (10.24),

$$\mathrm{tr}(\mathbf{u}^S\lambda_J\lambda^K) = u^{SI}\,\mathrm{tr}(\lambda_I\lambda_J\lambda^K) = \frac{2}{\sqrt{N}}(d_{u^S})_J{}^K - 2\mathrm{i}(f_{u^S})_J{}^K \tag{8.39}$$

so

$$\mathbf{u}^S\lambda_J = \frac{2}{N}u^S_J\mathbf{1} + \frac{1}{2}\,\mathrm{tr}(\mathbf{u}^S\lambda_J\lambda^K)\lambda_K \tag{8.40}$$

Look at the second term on the right-hand side. Using the fact that $\mathbf{u}^S = P^S - \frac{1}{N}\mathbf{1}$

and inserting $\mathbf{1} = \sum_T P^T$,

$$\frac{1}{2}\,\mathrm{tr}(\mathbf{u}^S\lambda_J\lambda^K)\lambda_K = \frac{1}{2}\,\mathrm{tr}\left((P^S - \frac{1}{N}\mathbf{1})\lambda_J\lambda^K\right)\lambda_K \tag{8.41}$$

$$= \frac{1}{2}\,\mathrm{tr}\left(P^S\lambda_J\sum_T P^T\lambda^K\right)\lambda_K - \frac{1}{2N}\,\mathrm{tr}(\lambda_J\lambda^K)\lambda_K \tag{8.42}$$

$$= \sum_T (P^{ST})_J{}^K\lambda_K - \frac{1}{N}\lambda_J \tag{8.43}$$

Thus

$$\mathbf{u}^S\lambda_J = \frac{2}{N}u_J^S\mathbf{1} + \sum_T (P^{ST})_J{}^K\lambda_K - \frac{1}{N}\lambda_J \tag{8.44}$$

We can then substitute this into the second term on the right-hand side of (8.37) to get

$$\begin{aligned}\sum_{S,T} e^{\frac{i}{2}(\theta'_S-\theta'_T)}P^S\sum_{U\neq T}\frac{1}{M'_T - M'_U}(P^{TU}+P^{UT})_A^J\lambda_J\partial_\mu M^A \\ = \sum_{S,T} e^{\frac{i}{2}(\theta'_S-\theta'_T)}\sum_{U\neq T}\frac{1}{M'_T - M'_U}(P^{TU}+P^{UT})_A^J \\ \times\left(\frac{2}{N}u_J^S\mathbf{1} + \sum_V (P^{SV})_J{}^K\lambda_K\right)\partial_\mu M^A\end{aligned} \tag{8.45}$$

However, we have in here the product

$$(P^{TU}+P^{UT})_A^J u_J^S = 4(f_{u^T}f_{u^U})_A^J u_J^S = 4(f_{u^T}f_{u^U}\mathbf{u}^S)_A$$

which we know to be zero. Therefore

$$\begin{aligned}\sum_{S,T} e^{\frac{i}{2}(\theta'_S-\theta'_T)}P^S\sum_{U\neq T}\frac{1}{M'_T - M'_U}(P^{TU}+P^{UT})_A^J\lambda_J\partial_\mu M^A \\ = \sum_{S,T} e^{\frac{i}{2}(\theta'_S-\theta'_T)}\sum_{U\neq T}\frac{1}{M'_T - M'_U}(P^{TU}+P^{UT})_A^J\sum_V (P^{SV})_J{}^K\lambda_K\partial_\mu M^A\end{aligned} \tag{8.46}$$

The terms where $T = S$ sum to

$$\sum_{U \neq S} \frac{1}{M'_S - M'_U} (P^{SU})_A{}^K \lambda_K \partial_\mu M^A$$

but we would like to bring out the symmetric and antisymmetric combinations; we can do this by splitting this in half and relabelling on one half:

$$\begin{aligned}
&\sum_{U \neq S} \frac{1}{M'_S - M'_U} (P^{SU})_A{}^K \lambda_K \partial_\mu M^A \\
&= \frac{1}{2} \sum_{U \neq S} \frac{1}{M'_S - M'_U} (P^{SU})_A{}^K \lambda_K \partial_\mu M^A + \frac{1}{2} \sum_{U \neq S} \frac{1}{M'_U - M'_S} (P^{US})_A{}^K \lambda_K \partial_\mu M^A \\
&= \frac{1}{2} \sum_{U \neq S} \frac{1}{M'_S - M'_U} (P^{SU})_A{}^K \lambda_K \partial_\mu M^A - \frac{1}{2} \sum_{U \neq S} \frac{1}{M'_S - M'_U} (P^{US})_A{}^K \lambda_K \partial_\mu M^A \\
&= \frac{1}{2} \sum_{U \neq S} \frac{1}{M'_S - M'_U} (P^{SU} - P^{US})_A{}^K \lambda_K \partial_\mu M^A \qquad (8.47) \\
&= \sum_{S < U} \frac{1}{M'_S - M'_U} (P^{SU} - P^{US})_A{}^K \lambda_K \partial_\mu M^A \qquad (8.48)
\end{aligned}$$

Similarly, the terms where $S \neq T$ but $S = U$ sum to

$$\begin{aligned}
&\sum_{S \neq T} e^{\frac{i}{2}(\theta'_S - \theta'_T)} \frac{1}{M'_T - M'_S} (P^{ST})^K_A \lambda_K \partial_\mu M^A \\
&= \sum_{S \neq T} e^{\frac{i}{2}(\theta'_S - \theta'_T)} \frac{1}{M'_T - M'_S} (P^{ST})^K_A \lambda_K \partial_\mu M^A \\
&\quad + \sum_{S \neq T} e^{\frac{i}{2}(\theta'_T - \theta'_S)} \frac{1}{M'_S - M'_T} (P^{TS})^K_A \lambda_K \partial_\mu M^A \qquad (8.49) \\
&= \sum_{S \neq T} e^{\frac{i}{2}(\theta'_S - \theta'_T)} \frac{1}{M'_T - M'_S} (P^{ST})^K_A \lambda_K \partial_\mu M^A \\
&\quad - \sum_{S \neq T} e^{-\frac{i}{2}(\theta'_S - \theta'_T)} \frac{1}{M'_T - M'_S} (P^{TS})^K_A \lambda_K \partial_\mu M^A \qquad (8.50)
\end{aligned}$$

which naturally fall into symmetric and antisymmetric combinations if we break the exponential upto into a sine and a cosine:

$$= \sum_{S<T} \frac{1}{M'_T - M'_S} \left(\mathrm{i} \sin\left(\frac{\theta'_S - \theta'_T}{2} \right) (P^{ST} + P^{TS}) + \cos\left(\frac{\theta'_S - \theta'_T}{2} \right) (P^{ST} - P^{TS}) \right)_A{}^K \lambda_K \partial_\mu M^A \tag{8.51}$$

and all other terms are zero. Therefore

$$\begin{aligned}
&\sum_{S,T} e^{\frac{\mathrm{i}}{2}(\theta'_S - \theta'_T)} P^S \sum_{U \neq T} \frac{1}{M'_T - M'_U} (P^{TU} + P^{UT})^J_A \lambda_J \partial_\mu M^A \\
&= \sum_{S<T} \frac{1}{M'_S - M'_T} (P^{ST} - P^{TS})_A{}^K \lambda_K \partial_\mu M^A \\
&- \sum_{S<T} \frac{1}{M'_S - M'_T} \left(\mathrm{i} \sin\left(\frac{\theta'_S - \theta'_T}{2} \right) (P^{ST} + P^{TS}) + \cos\left(\frac{\theta'_S - \theta'_T}{2} \right) (P^{ST} - P^{TS}) \right)_A{}^K \lambda_K \partial_\mu M^A \\
&= \sum_{S<T} \frac{1}{M'_S - M'_T} \left(-\mathrm{i} \sin\left(\frac{\theta'_S - \theta'_T}{2} \right) (P^{ST} + P^{TS}) + 2 \sin^2\left(\frac{\theta'_S - \theta'_T}{4} \right) (P^{ST} - P^{TS}) \right)_A{}^K \lambda_K \partial_\mu M^A
\end{aligned} \tag{8.52}$$

Having performed this manipulation, (8.37) becomes

$$\begin{aligned}
L^{-1} \partial_\mu L \;=\; & -\frac{\mathrm{i}}{2} \sum_S \mathbf{u}^S \partial_\mu \theta'_S \\
& -\mathrm{i} \sum_{S<T} \left(\frac{\sin\left(\frac{\theta'_S - \theta'_T}{2} \right)}{M'_S - M'_T} \right) (P^{ST} + P^{TS})^I_A \lambda_I \partial_\mu M^A \\
& + \sum_{S<T} \frac{2}{M'_S - M'_T} \sin^2\left(\frac{\theta'_S - \theta'_T}{4} \right) (P^{ST} - P^{TS})_A{}^I \lambda_I \partial_\mu M^A
\end{aligned} \tag{8.53}$$

where $P^{ST} + P^{TS}$ and $P^{ST} - P^{TS}$ are given by (7.40) and (7.56).

The second and third terms of this expression, are, as promised, very similar to those in equations (4.7) and (4.8) of [18]. We could in principle make the first term take the same form as the first term of (4.7) by introducing the remaining $N-1$ projection operators of the adjoint representation, called $p_{\alpha\beta}$ by Barnes *et al*[18, 21]. (These are related to the $u^S >< u^T$ operators.) In actual fact, for SU(4), as we shall see in the next chapter, it makes more sense to rewrite this term in terms of the operators $q_i >< q_j$. For this reason, we shall for this chapter leave this term as it stands.

8.3 $\mathbf{a}_\mu$ and $\mathbf{v}_\mu$

8.3.1 Automorphism conjugate u-vectors

We now specialise to the case of SU(N) coset spaces whose commutation relations have the $\mathbb{Z}_2$-grading structure of a symmetric space, as described in Section 2.4.2.

If the outer involutive automorphism can be effected on each u-vector by the action of a group element, it is possible to show that the first two terms of (8.53) form the $\mathbf{a}_\mu$ part of $L^{-1}\partial_\mu L$, while the third term forms the $\mathbf{v}_\mu$ part. This is easiest to see for a particular Lie algebra - we will take su(4) as an example.

For su(4), we know that for a given Cartan subspace, there are four u-vectors. In general, these are composed of a part in the subalgebra and a part in the coset space part of the algebra:

$$\mathbf{u}^S = u^{SP}\lambda_P + u^{SA}\lambda_A \tag{8.54}$$

Written in this way, $(\mathbf{u}^S, \mathbf{u}^S)$ takes the form

$$(\mathbf{u}^S, \mathbf{u}^S) = (u^{SP}\lambda_P + u^{SA}\lambda_A, u^{SQ}\lambda_Q + u^{SB}\lambda_B) \tag{8.55}$$

$$= (u^{SP}\lambda_P, u^{SQ}\lambda_Q) + (u^{SA}\lambda_A, u^{SB}\lambda_B) \tag{8.56}$$

Under the outer involutive automorphism, the lengths are preserved:

$$(\mathbf{u}^S, \mathbf{u}^S) \to (\tilde{\mathbf{u}}^S, \tilde{\mathbf{u}}^S) = (u^{SP}\lambda_P - u^{SA}\lambda_A, u^{SQ}\lambda_Q - u^{SB}\lambda_B) \tag{8.57}$$

$$= (u^{SP}\lambda_P, u^{SQ}\lambda_Q) + (u^{SA}\lambda_A, u^{SB}\lambda_B) \tag{8.58}$$

Furthermore, if there is a group element which carries out this automorphism, the other invariants $\gamma_3(\mathbf{u}^S)$ and $\gamma_4(\mathbf{u}^S)$ are preserved and $\tilde{\mathbf{u}}^S$ lies in the same orbit as $\mathbf{u}^S$.

However, we know that each of the $\mathbf{u}^S$ are functions of $\boldsymbol{\theta}$ - in general, they may be expressed as a power series in $\boldsymbol{\theta}$ (we will find such expressions for the u-vectors of certain coset spaces of SU(4) in Section 9.2). As $\boldsymbol{\theta}$ is a vector of the coset space, under the automorphism

$$\boldsymbol{\theta} \to -\boldsymbol{\theta} \tag{8.59}$$

$$\boldsymbol{\theta}^2 \to \boldsymbol{\theta}^2 \tag{8.60}$$

$$\boldsymbol{\theta}^3 \to -\boldsymbol{\theta}^3 \tag{8.61}$$

etc., so $\tilde{\mathbf{u}}^S$ may also be expressed as a power series in $\boldsymbol{\theta}$. This means that both $\mathbf{u}^S$ and $\tilde{\mathbf{u}}^S$ lie in the Cartan subspace of $\boldsymbol{\theta}$, and we know from the diagonal case that for any Cartan subspace, there are only four vectors in the same orbit as $\mathbf{u}^S$ (see Section 6.2.3) - $\mathbf{u}^1$, $\mathbf{u}^2$, $\mathbf{u}^3$ and $\mathbf{u}^4$. Thus all of the $\tilde{\mathbf{u}}^S$ must be members of the set $< \mathbf{u}^S >$. This is the idea of 'automorphism conjugate' u-vectors: each of the $\mathbf{u}^S$ has an 'automorphism conjugate' $\tilde{\mathbf{u}}^S$ also in the set. Note that if any of the u-vectors lie in the subalgebra, they are self-conjugate.

What is more, we can use this to deduce further properties of the θ'_S. Let us take a different example. Say that for a given non-linear realisation of SU(3) we have three u-vectors, two of which form a conjugate pair:

$$\tilde{\mathbf{u}}^1 = \mathbf{u}^2, \qquad \tilde{\mathbf{u}}^2 = \mathbf{u}^1, \qquad \tilde{\mathbf{u}}^3 = \mathbf{u}^3$$

$\boldsymbol{\theta}$ may be written

$$\boldsymbol{\theta} = \theta'_1\mathbf{u}^1 + \theta'_2\mathbf{u}^2 + \theta'_3\mathbf{u}^3 \tag{8.62}$$

but if we take the automorphism conjugate of this equation we get

$$-\boldsymbol{\theta} = \theta'_1\tilde{\mathbf{u}}^1 + \theta'_2\tilde{\mathbf{u}}^2 + \theta'_3\tilde{\mathbf{u}}^3 = \theta'_1\mathbf{u}^2 + \theta'_2\mathbf{u}^1 + \theta'_3\mathbf{u}^3 \tag{8.63}$$

$$\Rightarrow \boldsymbol{\theta} = -\theta'_2\mathbf{u}^1 - \theta'_1\mathbf{u}^2 - \theta'_3\mathbf{u}^3 \tag{8.64}$$

by comparing these, we see that

$$\theta'_3 = 0$$

(this is not surprising because $\mathbf{u}^3$ is self-conjugate and so lies in the subgroup and therefore we would not expect it to appear in the expansion of the coset space vector $\boldsymbol{\theta}$) and

$$\theta'_1 = -\theta'_2$$

i.e. the eigenvalues associated with two conjugate u-vectors are opposite and equal.

This is all obviously also true for the M'_S; as we can write $\mathbf{M}$ as a linear sum of the same u-vectors (see equation (8.5)) and $\mathbf{M}$ also changes sign under the automorphism, the same reasoning tells us that $M'_S = -M'_T$ for $\tilde{\mathbf{u}}^S = \mathbf{u}^T$.

Once again, we will see in Section 9.2 that for a whole class of coset spaces of SU(4) the u-vectors fall into such automorphism conjugate pairs.

8.3.2 Using the automorphism to decompose $L^{-1}\partial_\mu L$

It should now be clear how to identify which terms in (8.53) lie in the subalgebra and which in the coset space of the algebra - we simply apply the automorphism to each term and determine whether or not it changes sign.

This is particularly easy for the first term, which is a linear sum of the u-vectors just as $\boldsymbol{\theta}$ is. Just like $\boldsymbol{\theta}$, the coefficients for two conjugate u-vectors are equal and opposite, while self-conjugate u-vectors have zero coefficient. If we number the u-vectors such that $\tilde{\mathbf{u}}^1 = \mathbf{u}^2$, $\tilde{\mathbf{u}}^3 = \mathbf{u}^4$, etc., the automorphism acts just as it would for $\boldsymbol{\theta}$:

$$\begin{aligned}
&-\frac{\mathrm{i}}{2}(\mathbf{u}^1\partial_\mu\theta_1' + \mathbf{u}^2\partial_\mu\theta_2' + \mathbf{u}^3\partial_\mu\theta_3' + \mathbf{u}^4\partial_\mu\theta_4') \\
&\qquad\qquad \to \quad -\frac{\mathrm{i}}{2}(\mathbf{u}^2\partial_\mu\theta_1' + \mathbf{u}^1\partial_\mu\theta_2' + \mathbf{u}^4\partial_\mu\theta_3' + \mathbf{u}^3\partial_\mu\theta_4') \\
&\qquad\qquad = \quad \frac{\mathrm{i}}{2}(\mathbf{u}^2\partial_\mu\theta_2' + \mathbf{u}^1\partial_\mu\theta_1' + \mathbf{u}^4\partial_\mu\theta_4' + \mathbf{u}^3\partial_\mu\theta_3')
\end{aligned} \tag{8.65}$$

so overall this quantity has changed sign. Therefore $\sum_S \mathbf{u}^S\partial_\mu\theta_S'$ lies in the coset space part of the algebra.

The other parts of (8.53) are not quite so straightforward. The second is perhaps easiest to deal with if we note that

$$(P^{ST} + P^{TS})^I_A\lambda_I\partial_\mu M^A = 4(f_{u^S}f_{u^T})^I_A\lambda_I\partial_\mu M^A \tag{8.66}$$

$$= 4(\mathbf{u}^S{}_\wedge(\mathbf{u}^T{}_\wedge\partial_\mu\mathbf{M}))^I\lambda_I \tag{8.67}$$

$$= 4\mathbf{u}^S{}_\wedge(\mathbf{u}^T{}_\wedge\partial_\mu\mathbf{M}) \tag{8.68}$$

Now numbering the u-vectors as above, we consider just one term in the sum:

$$-4\mathrm{i}\left(\frac{\sin\left(\frac{\theta_1'-\theta_3'}{2}\right)}{M_1'-M_3'}\right)\mathbf{u}^1{}_\wedge(\mathbf{u}^3{}_\wedge\partial_\mu\mathbf{M})$$

Under the automorphism, this transforms as

$$\begin{aligned}
-4\mathrm{i}\left(\frac{\sin\left(\frac{\theta_1'-\theta_3'}{2}\right)}{M_1'-M_3'}\right)\mathbf{u}^1{}_\wedge(\mathbf{u}^3{}_\wedge\partial_\mu\mathbf{M}) &\rightarrow -4\mathrm{i}\left(\frac{\sin\left(\frac{\theta_1'-\theta_3'}{2}\right)}{M_1'-M_3'}\right)\mathbf{u}^2{}_\wedge(\mathbf{u}^4{}_\wedge\partial_\mu\tilde{\mathbf{M}}) \\
&= -4\mathrm{i}\left(\frac{\sin\left(\frac{-\theta_2'+\theta_4'}{2}\right)}{-M_2'+M_4'}\right)\mathbf{u}^2{}_\wedge(\mathbf{u}^4{}_\wedge\partial_\mu\tilde{\mathbf{M}}) \\
&= -4\mathrm{i}\left(\frac{\sin\left(\frac{\theta_2'-\theta_4'}{2}\right)}{M_2'-M_4'}\right)\mathbf{u}^2{}_\wedge(\mathbf{u}^4{}_\wedge\partial_\mu\tilde{\mathbf{M}}) \\
&= 4\mathrm{i}\left(\frac{\sin\left(\frac{\theta_2'-\theta_4'}{2}\right)}{M_2'-M_4'}\right)\mathbf{u}^2{}_\wedge(\mathbf{u}^4{}_\wedge\partial_\mu\mathbf{M})
\end{aligned} \tag{8.69}$$

Similarly,

$$-4\mathrm{i}\left(\frac{\sin\left(\frac{\theta_2'-\theta_4'}{2}\right)}{M_2'-M_4'}\right)\mathbf{u}^2{}_\wedge(\mathbf{u}^4{}_\wedge\partial_\mu\mathbf{M}) \rightarrow 4\mathrm{i}\left(\frac{\sin\left(\frac{\theta_1'-\theta_3'}{2}\right)}{M_1'-M_3'}\right)\mathbf{u}^1{}_\wedge(\mathbf{u}^3{}_\wedge\partial_\mu\mathbf{M}) \tag{8.70}$$

By using this reasoning it is not hard to see that each term in the sum is transformed into the negative of another term, so the entire sum changes sign under the automorphism. Therefore

$$\sum_{S<T}\left(\frac{\sin\left(\frac{\theta_S'-\theta_T'}{2}\right)}{M_S'-M_T'}\right)(P^{ST}+P^{TS})^I_A\lambda_I\partial_\mu M^A$$

lies in the coset space part of the algebra.

We can deal with the third part of (8.53) in much the same way. Using (7.56) we can write

$$(P^{ST} - P^{TS})^I_A \lambda_I \partial_\mu M^A = 4\mathrm{i}\mathbf{u}^S{}_\wedge \mathbf{u}^T{}_\wedge (\mathbf{u}^S - \mathbf{u}^T)_\wedge \partial_\mu \mathbf{M} \tag{8.71}$$

Again, we consider the transformation of just one term of the sum:

$$\frac{8\mathrm{i}}{M_1' - M_3'} \sin^2\left(\frac{\theta_1' - \theta_3'}{4}\right) \mathbf{u}^1{}_\wedge \mathbf{u}^3{}_\wedge (\mathbf{u}^1 - \mathbf{u}^3)_\wedge \partial_\mu \mathbf{M} \tag{8.72}$$

$$\begin{aligned} &\rightarrow \frac{8\mathrm{i}}{M_1' - M_3'} \sin^2\left(\frac{\theta_1' - \theta_3'}{4}\right) \mathbf{u}^2{}_\wedge \mathbf{u}^4{}_\wedge (\mathbf{u}^2 - \mathbf{u}^4)_\wedge \partial_\mu \tilde{\mathbf{M}} \\ &= \frac{8\mathrm{i}}{-M_2' + M_4'} \sin^2\left(\frac{-\theta_2' + \theta_4'}{4}\right) \mathbf{u}^2{}_\wedge \mathbf{u}^4{}_\wedge (\mathbf{u}^2 - \mathbf{u}^4)_\wedge \partial_\mu \tilde{\mathbf{M}} \\ &= -\frac{8\mathrm{i}}{M_2' - M_4'} \sin^2\left(\frac{\theta_2' - \theta_4'}{4}\right) \mathbf{u}^2{}_\wedge \mathbf{u}^4{}_\wedge (\mathbf{u}^2 - \mathbf{u}^4)_\wedge \partial_\mu \tilde{\mathbf{M}} \\ &= \frac{8\mathrm{i}}{M_2' - M_4'} \sin^2\left(\frac{\theta_2' - \theta_4'}{4}\right) \mathbf{u}^2{}_\wedge \mathbf{u}^4{}_\wedge (\mathbf{u}^2 - \mathbf{u}^4)_\wedge \partial_\mu \mathbf{M} \end{aligned} \tag{8.73}$$

and similarly

$$\frac{8\mathrm{i}}{M_2' - M_4'} \sin^2\left(\frac{\theta_2' - \theta_4'}{4}\right) \mathbf{u}^2{}_\wedge \mathbf{u}^4{}_\wedge (\mathbf{u}^2 - \mathbf{u}^4)_\wedge \partial_\mu \mathbf{M} \tag{8.74}$$

$$\rightarrow \frac{8\mathrm{i}}{M_1' - M_3'} \sin^2\left(\frac{\theta_1' - \theta_3'}{4}\right) \mathbf{u}^1{}_\wedge \mathbf{u}^3{}_\wedge (\mathbf{u}^1 - \mathbf{u}^3)_\wedge \partial_\mu \mathbf{M} \tag{8.75}$$

In this case, each term is transformed exactly into another term in the sum, so the entire sum is invariant under the automorphism. Therefore

$$\mathrm{i}\sum_{S<T} \frac{2}{M_S' - M_T'} \sin^2\left(\frac{\theta_S' - \theta_T'}{4}\right) (P^{ST} - P^{TS})_A{}^I \lambda_I \partial_\mu M^A$$

lies in the subalgebra.

We have thus shown that $L^{-1}\partial_\mu L$ breaks into $\mathbf{a}_\mu$ and $\mathbf{v}_\mu$ parts as described at the start of this section. This means that for these particular symmetric spaces, the covariant derivatives are

$$\mathrm{D}_\mu M^B = a_\mu^B = \sum_S u^{SB}\partial_\mu\theta'_S + 2\sum_{S<T}\left(\frac{\sin\left(\frac{\theta'_S-\theta'_T}{2}\right)}{M'_S - M'_T}\right)(P^{ST}+P^{TS})_A^B\partial_\mu M^A \tag{8.76}$$

$$= \sum_S u^{SB}\partial_\mu\theta'_S + 8\sum_{S<T}\left(\frac{\sin\left(\frac{\theta'_S-\theta'_T}{2}\right)}{M'_S - M'_T}\right)(f_{u^S}f_{u^T})_A^B\partial_\mu M^A \tag{8.77}$$

and

$$\mathrm{D}_\mu\psi = \partial_\mu\psi - \frac{\mathrm{i}}{2}\mathbf{v}_\mu\psi$$

$$= \partial_\mu\psi + \sum_{S<T}\frac{2}{M'_S - M'_T}\sin^2\left(\frac{\theta'_S-\theta'_T}{4}\right)(P^{ST}-P^{TS})_A{}^I\lambda_I\partial_\mu M^A\psi \tag{8.78}$$

$$= \partial_\mu\psi + \sum_{S<T}\frac{8\mathrm{i}}{M'_S - M'_T}\sin^2\left(\frac{\theta'_S-\theta'_T}{4}\right)(f_{u^S}f_{u^T}(f_{u^T}-f_{u^S}))_A{}^P\lambda_P\partial_\mu M^A\psi \tag{8.79}$$

Chapter 9

Coset spaces of SU(4)≈SO(6)

9.1 Introduction

All su(N) algebras are composed entirely of traceless, hermitian matrices.

Consider the matrix

$$\begin{pmatrix} 0 & 0 & 2i & -i & 0 \\ 0 & 0 & 1 & 3-2i & -3i \\ -2i & 1 & 0 & 0 & 0 \\ i & 3+2i & 0 & 0 & 0 \\ 0 & 3i & 0 & 0 & 0 \end{pmatrix}$$

This has the interesting property that all of its odd powers are traceless. Indeed, any matrix that can be partitioned thus:

$$\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$$

has this property.

Some subspaces of su(4) are entirely composed of vectors of this type. We have already seen how so(3) subalgebras and $su(2)_R$ and $su(2)_L$ subalgebras have the property $\gamma_3(\mathbf{x}) = \frac{1}{3}\,\mathrm{tr}\,\mathbf{x}^3 = 0$ for every vector $\mathbf{x}$ in the subalgebra, but we also saw in 6.2.3 that there are *generic* vectors of SU(4) with this property. There exist coset spaces of SU(4), some (but not all) containing generic vectors, for which $\gamma_3(\mathbf{x}) = 0$ for every vector of the space. One example[1] is the coset space $SU(4)/SU(2)_R \otimes SU(2)_L \otimes U(1)$ generated by

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_9 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_{10} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \lambda_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

(which is clearly homomorphic to $SO(6)/SO(4) \otimes SO(2)$ - we shall examine it in the SO(6) formulation in Section 9.6.2), so all of the vectors in the coset space

[1] One example of a coset space for which $\gamma_3(\mathbf{x}) = 0$ for every vector of the space but does *not* contain generic vectors is the coset space $SU(4)/SU(3) \otimes U(1)$[37]

have the form

$$\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$$

Such coset spaces are always symmetric spaces, as the $_\wedge$-product of two vectors is always an element of the subalgebra:

$$\frac{i}{2}\left[\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}, \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}\right] = \frac{i}{2}\begin{pmatrix} AB^* - BA^* & 0 \\ 0 & A^*B - B^*A \end{pmatrix} \tag{9.1}$$

(see Section 2.4.2). These are the main subject of this chapter. Let us identify some basic properties of the generic vectors of these coset spaces.

Recall that the condition $\gamma_3(\mathbf{x}) = 0$ reduces the characteristic equation for a 4×4 traceless, hermitian matrix, (6.22), to a quadratic equation in $\mathbf{x}^2$, so $\mathbf{x}^2$ has at most two distinct eigenvalues, both appearing twice and that $\mathbf{x}_\vee\mathbf{x}$ is then a q-vector. However, we also know that for a *generic* vector, $\mathbf{x}$, $\mathbf{x}^2$ and $\mathbf{x}^3$ are all linearly independent, or equivalently, $\mathbf{x}$, $\mathbf{x}_\vee\mathbf{x}$ and $\mathbf{x}_\vee\mathbf{x}_\vee\mathbf{x}$ are all linearly independent (the fourth power is related to the others by the characteristic equation). Furthermore, they all commute; therefore $\mathbf{x}$, $\mathbf{x}_\vee\mathbf{x}$ and $\mathbf{x}_\vee\mathbf{x}_\vee\mathbf{x}$ - or $\mathbf{x}$, $\mathbf{x}^2$ and $\mathbf{x}^3$ minus their traces - must form a basis (though not necessarily an orthonormal one) for a Cartan subspace, for which we know there are three orthonormal unit q-vectors (upto a sign).

Thus for a generic vector of SU(4) with the property $\gamma_3(\mathbf{x}) = 0$ we can say the following: there are two linearly independent unit q-vectors which are linear sums of $\mathbf{x}$ and $\mathbf{x}_\vee\mathbf{x}_\vee\mathbf{x}$ (which we shall call $\mathbf{q}'_1$ and $\mathbf{q}'_2$), while $\mathbf{x}_\vee\mathbf{x}$ is proportional to a third ($\mathbf{q}'_3$), and all three commute.

In the last chapter, we found a general form of $L^{-1}\partial_\mu L$ which was valid for any SU(N) coset space, based on the decomposition of an arbitrary vector of the coset space in terms of projection operators, (8.1). As promised, we shall now look at how to do this decomposition - in two equivalent ways - for a coset space of the above type.

9.2 q-vectors for an arbitrary vector

9.2.1 Obtaining the q-vectors

By combining (7.24) with equations (6.66)-(6.69) we already have expressions for the diagonal u-vectors of SU(4) in the orthonormal basis of the diagonal unit q-vectors:

$$\mathbf{u}_{\mathrm{d}}^1 = \frac{1}{2\sqrt{2}}(-\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3) \tag{9.2}$$
$$\mathbf{u}_{\mathrm{d}}^2 = \frac{1}{2\sqrt{2}}(\mathbf{q}_1 - \mathbf{q}_2 + \mathbf{q}_3) \tag{9.3}$$
$$\mathbf{u}_{\mathrm{d}}^3 = \frac{1}{2\sqrt{2}}(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{q}_3) \tag{9.4}$$
$$\mathbf{u}_{\mathrm{d}}^4 = \frac{1}{2\sqrt{2}}(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) \tag{9.5}$$

However, we are interested in a non-diagonal Cartan subspace. We discussed in Section 8.1.2 how the properties of the projection operators P^S are preserved under the transformation $\mathbf{u}_{\mathrm{d}}^S \to g\mathbf{u}_{\mathrm{d}}^S g^{-1}$. Thus, under the change of basis $\mathbf{q}_i \to \mathbf{q}_i' = g\mathbf{q}_i g^{-1}$, the vectors

$$\mathbf{u}^1 = \frac{1}{2\sqrt{2}}(-\mathbf{q}_1' + \mathbf{q}_2' + \mathbf{q}_3') \tag{9.6}$$
$$\mathbf{u}^2 = \frac{1}{2\sqrt{2}}(\mathbf{q}_1' - \mathbf{q}_2' + \mathbf{q}_3') \tag{9.7}$$
$$\mathbf{u}^3 = \frac{1}{2\sqrt{2}}(\mathbf{q}_1' + \mathbf{q}_2' - \mathbf{q}_3') \tag{9.8}$$
$$\mathbf{u}^4 = \frac{1}{2\sqrt{2}}(-\mathbf{q}_1' - \mathbf{q}_2' - \mathbf{q}_3') \tag{9.9}$$

are a valid set of u-vectors. With this established, finding a set of u-vectors we can use in our analysis is equivalent to finding the q-vectors for the Cartan subspace of an arbitrary coset space vector. As the Cartan subspace of an arbitrary vector is the same as that for its associated unit vector, we shall work with the unit vector in order to find the relevant q-vectors; we can then easily generalise to a vector of arbitrary length at the end.

We now refine our notation for coset space vectors, denoting an arbitrary *unit* coset space vector $\mathbf{x}$, and its Cartan subspace $\mathcal{C}_x$. We further define the vector $\mathbf{y}$ to be $\mathbf{x}^2$ with its trace removed and $\mathbf{z}$ to be $\mathbf{x}^3$ with its trace removed:

$$\mathbf{y} \equiv \mathbf{x}^2 - \frac{1}{4}\mathbf{1}\,\mathrm{tr}\,\mathbf{x}^2 = \mathbf{x}^2 - \frac{1}{2}\mathbf{1} \tag{9.10}$$

$$\mathbf{z} \equiv \mathbf{x}^3 - \frac{1}{4}\mathbf{1}\,\mathrm{tr}\,\mathbf{x}^3 \tag{9.11}$$

and recall that these form a basis for $\mathcal{C}_x$. We can rearrange these definitions to find $\mathbf{x}^2$ and $\mathbf{xy}$:

$$\mathbf{x}^2 = \mathbf{y} + \frac{1}{2}\mathbf{1} \tag{9.12}$$

$$\mathbf{xy} = \mathbf{z} - \frac{1}{2}\mathbf{x} + \frac{1}{4}\mathbf{1}\,\mathrm{tr}\,\mathbf{x}^3 \tag{9.13}$$

Other products can be found using the characteristic equation:

$$\begin{aligned}
\mathbf{xz} &= \mathbf{y} + \frac{1}{12}\mathbf{x}\,\mathrm{tr}\,\mathbf{x}^3 + \frac{1}{4}\mathbf{1}\,\mathrm{tr}\,\mathbf{x}^4 && (9.14)\\
\mathbf{y}^2 &= \frac{1}{3}\mathbf{x}\,\mathrm{tr}\,\mathbf{x}^3 + \frac{1}{4}\mathbf{1}\,\mathrm{tr}\,\mathbf{x}^4 - \frac{1}{4}\mathbf{1} && (9.15)\\
\mathbf{yz} &= \frac{1}{2}\mathbf{z} + \frac{1}{12}\mathbf{y}\,\mathrm{tr}\,\mathbf{x}^3 + \frac{1}{4}\mathbf{x}\,\mathrm{tr}\,\mathbf{x}^4 - \frac{1}{2}\mathbf{x} + \frac{7}{24}\mathbf{1}\,\mathrm{tr}\,\mathbf{x}^3 && (9.16)
\end{aligned}$$

$$\mathbf{z}^2 = -\frac{1}{6}\mathbf{z}\,\mathrm{tr}\,\mathbf{x}^3 + \frac{1}{4}\mathbf{y}\,\mathrm{tr}\,\mathbf{x}^4 + \frac{1}{2}\mathbf{y} + \frac{1}{3}\mathbf{x}\,\mathrm{tr}\,\mathbf{x}^3 + \frac{3}{8}\mathbf{1}\,\mathrm{tr}\,\mathbf{x}^4 + \frac{1}{48}\mathbf{1}(\mathrm{tr}\,\mathbf{x}^3)^2 - \frac{1}{4}\mathbf{1} \tag{9.17}$$

Remembering that $\mathrm{tr}\,\mathbf{x} = \mathrm{tr}\,\mathbf{y} = \mathrm{tr}\,\mathbf{z} = 0$, we can easily find the trace of each of these, which allows us to calculate the $_\vee$-products:

$$\mathbf{x}_\vee\mathbf{x} = 2\mathbf{y} \tag{9.18}$$

$$\mathbf{x}_\vee\mathbf{y} = 2\mathbf{z} - \mathbf{x} \tag{9.19}$$

$$\mathbf{x}_\vee\mathbf{z} = 2\mathbf{y} + \frac{1}{6}\mathbf{x}\,\mathrm{tr}\,\mathbf{x}^3 \tag{9.20}$$

$$\mathbf{y}_\vee\mathbf{y} = \frac{2}{3}\mathbf{x}\,\mathrm{tr}\,\mathbf{x}^3 \tag{9.21}$$

$$\mathbf{y}_\vee\mathbf{z} = \mathbf{z} + \frac{1}{6}\mathbf{y}\,\mathrm{tr}\,\mathbf{x}^3 + \frac{1}{2}\mathbf{x}\,\mathrm{tr}\,\mathbf{x}^4 - \mathbf{x} \tag{9.22}$$

$$\mathbf{z}_\vee\mathbf{z} = -\frac{1}{3}\mathbf{z}\,\mathrm{tr}\,\mathbf{x}^3 + \frac{1}{2}\mathbf{y}\,\mathrm{tr}\,\mathbf{x}^4 + \mathbf{y} + \frac{2}{3}\mathbf{x}\,\mathrm{tr}\,\mathbf{x}^3 \tag{9.23}$$

Note that we can rearrange the first two of these to see how to change basis from $\mathbf{x}$, $\mathbf{x}^2$ and $\mathbf{x}^3$ minus their traces to $\mathbf{x}$, $\mathbf{x}_\vee\mathbf{x}$ and $\mathbf{x}_\vee\mathbf{x}_\vee\mathbf{x}$:

$$\mathbf{y} = \frac{1}{2}\mathbf{x}_\vee\mathbf{x} \tag{9.24}$$

and

$$\mathbf{z} = \frac{1}{2}\mathbf{x} + \frac{1}{4}\mathbf{x}_\vee\mathbf{x}_\vee\mathbf{x} \tag{9.25}$$

Due to the nature of the characteristic equation it is easier to determine the q-vectors in terms of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, but we shall re-express them in terms of $\mathbf{x}$, $\mathbf{x}_\vee\mathbf{x}$ and $\mathbf{x}_\vee\mathbf{x}_\vee\mathbf{x}$ once we have found them as they are easier to handle in this form and it will be more in keeping with the existing literature.

As the q-vectors we are seeking lie in the subspace spanned by $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, we can write

$$\mathbf{q}_i' = \alpha_i\mathbf{x} + \beta_i\mathbf{y} + \gamma_i\mathbf{z} \tag{9.26}$$

where $\alpha_i, \beta_i, \gamma_i$ are real numbers. We can determine the values of the coefficients by using the q-vector property $\mathbf{q}'\vee\mathbf{q}' = 0$ (this is equivalent to what Barnes *et al* do for SU(3) in [20]; the main differences being that their calculation is to obtain the angle α in the Cartan plane with basis $\mathbf{r}$ and $\mathbf{q}_r$ while we are identifying the coefficients $\alpha_i, \beta_i, \gamma_i$ in a 3-dimensional Cartan space where the basis is the set of $\mathbf{q}_i$). Employing the above identities then gives us

$$\begin{aligned}2\alpha_i^2\mathbf{y} + 4\alpha_i\beta_i\mathbf{z} - 2\alpha_i\beta_i\mathbf{x} + 4\alpha_i\gamma_i\mathbf{y} + \frac{1}{3}\alpha_i\gamma_i\mathbf{x}\,\mathrm{tr}\,\mathbf{x}^3 + \frac{2}{3}\beta_i^2\mathbf{x}\,\mathrm{tr}\,\mathbf{x}^3 \\ +2\beta_i\gamma_i\mathbf{z} + \frac{1}{3}\beta_i\gamma_i\mathbf{y}\,\mathrm{tr}\,\mathbf{x}^3 + \beta_i\gamma_i\mathbf{x}\,\mathrm{tr}\,\mathbf{x}^4 - 2\beta_i\gamma_i\mathbf{x} - \frac{1}{3}\gamma_i^2\mathbf{z}\,\mathrm{tr}\,\mathbf{x}^3 \\ +\frac{1}{2}\gamma_i^2\mathbf{y}\,\mathrm{tr}\,\mathbf{x}^4 + \gamma_i^2\mathbf{y} + \frac{2}{3}\gamma_i^2\mathbf{x}\,\mathrm{tr}\,\mathbf{x}^3 = 0\end{aligned} \tag{9.27}$$

However, $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are all linearly independent, so by equating coefficients

$$-2\alpha_i\beta_i + \frac{1}{3}\alpha_i\gamma_i\,\mathrm{tr}\,\mathbf{x}^3 + \frac{2}{3}\beta_i^2\,\mathrm{tr}\,\mathbf{x}^3 + \beta_i\gamma_i\,\mathrm{tr}\,\mathbf{x}^4 - 2\beta_i\gamma_i + \frac{2}{3}\gamma_i^2\,\mathrm{tr}\,\mathbf{x}^3 = 0 \tag{9.28}$$

$$2\alpha_i^2 + 4\alpha_i\gamma_i + \frac{1}{3}\beta_i\gamma_i\,\mathrm{tr}\,\mathbf{x}^3 + \frac{1}{2}\gamma_i^2\,\mathrm{tr}\,\mathbf{x}^4 + \gamma_i^2 = 0 \tag{9.29}$$

$$4\alpha_i\beta_i + 2\beta_i\gamma_i - \frac{1}{3}\gamma_i^2\,\mathrm{tr}\,\mathbf{x}^3 = 0 \tag{9.30}$$

To find the q-vectors we must solve these equations. When $\mathrm{tr}\,\mathbf{x}^3 = 0$, (9.30) reduces to

$$\beta_i(2\alpha_i + \gamma_i) = 0 \tag{9.31}$$

so either

$$\gamma_i = -2\alpha_i \text{ or } \beta_i = 0$$

If $\gamma_i = -2\alpha_i$, (9.28) and (9.29) give us:

$$2\alpha_i\beta_i(1 - \operatorname{tr}\mathbf{x}^4) = 0 \tag{9.32}$$

and

$$2\alpha_i^2(1 - \operatorname{tr}\mathbf{x}^4) = 0 \tag{9.33}$$

so for an arbitrary value of $\operatorname{tr}\mathbf{x}^4$, we get

$$\alpha_i = 0 \Rightarrow \gamma_i = 0 \tag{9.34}$$

so that, as expected, we have a q-vector proportional to $\mathbf{x}_\vee\mathbf{x}$:

$$\mathbf{q}_3' = \beta_i\mathbf{y} = \frac{1}{2}\beta_i\mathbf{x}_\vee\mathbf{x} \tag{9.35}$$

To normalise this q-vector, we need $(\mathbf{y}, \mathbf{y})$, which is easily found to be

$$(\mathbf{y}, \mathbf{y}) = \frac{1}{2}\operatorname{tr}\mathbf{x}^4 - \frac{1}{2} = 2\gamma_4(\mathbf{x}) + \frac{1}{2} \tag{9.36}$$

So our normalised q-vector is

$$\mathbf{q}_3' = \pm(2\gamma_4(\mathbf{x}) + \frac{1}{2})^{-\frac{1}{2}}\mathbf{y} \tag{9.37}$$

If $\beta_i = 0$, both sides of (9.28) and (9.30) are zero, but (9.29) gives us a coupled equation in α_i and γ_i. We obviously cannot have $\gamma_i = 0$, as then α_i, β_i and γ_i would all be zero. However, any other choice of α_i and γ_i which satisfies this equation corresponds to a particular q-vector (of a particular length). We can get unnormalised q-vectors by, say, setting $\gamma_i = 1$; equation (9.29) then gives us

$$2\alpha_i^2 + 4\alpha_i + \frac{1}{2}\operatorname{tr}\mathbf{x}^4 + 1 = 0 \tag{9.38}$$

$$\Rightarrow \alpha_i = -1 \pm \sqrt{-\gamma_4(\mathbf{x})} \tag{9.39}$$

so two unnormalised q-vectors in the $(\mathbf{x}, \mathbf{z})$-space are

$$\mathbf{q}_1'' = \left(-1 + \sqrt{-\gamma_4(\mathbf{x})}\right)\mathbf{x} + \mathbf{z}$$

and

$$\mathbf{q}_2'' = \left(-1 - \sqrt{-\gamma_4(\mathbf{x})}\right)\mathbf{x} + \mathbf{z}$$

We now want to normalise these. The process is much the same as it was for $\mathbf{q}_3'$. However, (not surprisingly), the factor $\sqrt{-\gamma_4(\mathbf{x})}$ keeps making an appearance, so we define

$$\rho \equiv \sqrt{-\gamma_4(\mathbf{x})} = \sqrt{\frac{1}{2} - \frac{1}{4}\operatorname{tr}\mathbf{x}^4} \tag{9.40}$$

With this definition, we finally get for our three q-vectors:

$$\mathbf{q}_1' = \pm(2\rho^2 - 4\rho^3)^{-\frac{1}{2}}\left[(-1+\rho)\,\mathbf{x} + \mathbf{z}\right] \tag{9.41}$$

$$\mathbf{q}_2' = \pm(2\rho^2 + 4\rho^3)^{-\frac{1}{2}}\left[(-1-\rho)\,\mathbf{x} + \mathbf{z}\right] \tag{9.42}$$

$$\mathbf{q}_3' = \pm(\frac{1}{2} - 2\rho^2)^{-\frac{1}{2}}\mathbf{y} \tag{9.43}$$

or, using (9.24) and (9.25),

$$\mathbf{q}_1' = \pm\frac{1}{2\rho}\left[-(\frac{1}{2}-\rho)^{\frac{1}{2}}\mathbf{x} + \frac{1}{4}(\frac{1}{2}-\rho)^{-\frac{1}{2}}\mathbf{x}_\vee\mathbf{x}_\vee\mathbf{x}\right] \tag{9.44}$$

$$\mathbf{q}_2' = \pm\frac{1}{2\rho}\left[-(\frac{1}{2}+\rho)^{\frac{1}{2}}\mathbf{x} + \frac{1}{4}(\frac{1}{2}+\rho)^{-\frac{1}{2}}\mathbf{x}_\vee\mathbf{x}_\vee\mathbf{x}\right] \tag{9.45}$$

$$\mathbf{q}_3' = \pm(2 - 8\rho^2)^{-\frac{1}{2}}\mathbf{x}_\vee\mathbf{x} \tag{9.46}$$

Note that there is a pole in the normalisation factor of $\mathbf{q}'_1$ when $\gamma_4(\mathbf{x}) = 0$ or when $\gamma_4(\mathbf{x}) = -\frac{1}{4}$. However, we saw in Section 6.2 that a unit vector with $\gamma_3(\mathbf{x}) = \gamma_4(\mathbf{x}) = 0$ is (by definition) an r-vector, while a unit vector with $\gamma_3(\mathbf{x}) = 0, \gamma_4(\mathbf{x}) = -\frac{1}{4}$ is a q-vector. $\mathbf{q}'_2$ is similarly ill-defined for r-vectors and $\mathbf{q}'_3$ for q-vectors.

It can be shown that the scalar product of any (linearly independent) pair of these is zero as expected and that they satisfy the $\vee$-relations of $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ provided we pick an appropriate set of signs. For example, we could take + signs on all of the q-vectors in (9.41)-(9.43) to get the desired result

$$\mathbf{q}'_i{}_\vee \mathbf{q}'_j = -\sqrt{2}\eta_{ij}{}^k \mathbf{q}'_k \tag{9.47}$$

This is not a unique choice, though. If we change the sign of a single q-vector the three relations are not preserved, for example $\mathbf{q}'_1{}_\vee \mathbf{q}'_2 \neq -\sqrt{2}\mathbf{q}'_3$ if we make the replacement $\mathbf{q}'_3 \to -\mathbf{q}'_3$. Similarly, if we change the sign of all three the relations are not preserved. However, if we change the sign of two of them, the relations are all preserved (try, for example, $\mathbf{q}'_1 \to -\mathbf{q}'_1$, $\mathbf{q}'_2 \to -\mathbf{q}'_2$). Also, of course, we are also perfectly free to renumber them.

9.2.2 q-vectors in $\mathcal{H}$ and sign/numbering conventions

To develop a set of sign conventions/numbering conventions we recall from Section 6.3 how we viewed the vectors of SU(4) as vectors of SO(6), concentrating particularly on those in $\mathcal{H}$, the subgroup of rotations in the first four dimensions.

Consider an arbitrary vector in this subgroup,

$$\omega^{\mu\nu}\sigma_{\mu\nu} = \begin{pmatrix} \omega^{\mathrm{R}k}\sigma_k & 0 \\ 0 & \omega^{\mathrm{L}k}\sigma_k \end{pmatrix}$$

We now have two ways of finding the projection operators for this vector. The first is that described in Section 4.4.2; this provides a set of projection operators which we now number in the logical way:

$$P^1 = \frac{1}{2}(\mathbf{1}^{\mathrm{R}} + n^{\mathrm{R}\,k}\sigma_k^{\mathrm{R}}) \tag{9.48}$$

$$P^2 = \frac{1}{2}(\mathbf{1}^{\mathrm{R}} - n^{\mathrm{R}\,k}\sigma_k^{\mathrm{R}}) \tag{9.49}$$

$$P^3 = \frac{1}{2}(\mathbf{1}^{\mathrm{L}} + n^{\mathrm{L}\,k}\sigma_k^{\mathrm{L}}) \tag{9.50}$$

$$P^4 = \frac{1}{2}(\mathbf{1}^{\mathrm{L}} - n^{\mathrm{L}\,k}\sigma_k^{\mathrm{L}}) \tag{9.51}$$

Alternatively, we can use the method outlined in the last section. First we divide $\omega^{\mu\nu}\sigma_{\mu\nu}$ by its length to get the associated unit vector, $\mathbf{x}$. Now

$$(\omega^{\mu\nu}\sigma_{\mu\nu})^2 = \begin{pmatrix} (\omega^{\mathrm{R}})^2\mathbf{1} & 0 \\ 0 & (\omega^{\mathrm{L}})^2\mathbf{1} \end{pmatrix} \tag{9.52}$$

so the length is $\sqrt{(\omega^{\mathrm{R}})^2 + (\omega^{\mathrm{L}})^2}$. Therefore

$$\mathbf{x} = \frac{1}{\sqrt{(\omega^{\mathrm{R}})^2 + (\omega^{\mathrm{L}})^2}} \begin{pmatrix} \omega^{\mathrm{R}k}\sigma_k & 0 \\ 0 & \omega^{\mathrm{L}k}\sigma_k \end{pmatrix} \tag{9.53}$$

From this we can obtain $\mathbf{x}_{\vee}\mathbf{x}$:

$$\mathbf{x}_{\vee}\mathbf{x} = 2\mathbf{x}^2 - \mathbf{1} \tag{9.54}$$

$$= \frac{2}{(\omega^{\mathrm{R}})^2 + (\omega^{\mathrm{L}})^2} \begin{pmatrix} (\omega^{\mathrm{R}})^2\mathbf{1} & 0 \\ 0 & (\omega^{\mathrm{L}})^2\mathbf{1} \end{pmatrix} - \frac{(\omega^{\mathrm{R}})^2 + (\omega^{\mathrm{L}})^2}{(\omega^{\mathrm{R}})^2 + (\omega^{\mathrm{L}})^2} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}$$

$$= \frac{(\omega^{\mathrm{R}})^2 - (\omega^{\mathrm{L}})^2}{(\omega^{\mathrm{R}})^2 + (\omega^{\mathrm{L}})^2} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{9.55}$$

Note in this case $\mathbf{x}(\mathbf{x}_\vee\mathbf{x})$ is traceless, i.e.

$$(\mathbf{x}, \mathbf{x}_\vee\mathbf{x}) = 0 \tag{9.56}$$

This makes it particularly easy to find $\mathbf{x}_\vee\mathbf{x}_\vee\mathbf{x}$:

$$\mathbf{x}_\vee\mathbf{x}_\vee\mathbf{x} = 2\mathbf{x}(\mathbf{x}_\vee\mathbf{x}) = \frac{2[(\omega^{\mathrm{R}})^2 - (\omega^{\mathrm{L}})^2]}{[(\omega^{\mathrm{R}})^2 + (\omega^{\mathrm{L}})^2)]^2} \begin{pmatrix} \omega^{\mathrm{R}k}\sigma_k & 0 \\ 0 & -\omega^{\mathrm{L}k}\sigma_k \end{pmatrix} \tag{9.57}$$

Finally we have

$$\mathbf{x}^4 = \frac{1}{[(\omega^{\mathrm{R}})^2 + (\omega^{\mathrm{L}})^2)]^2} \begin{pmatrix} (\omega^{\mathrm{R}})^4\mathbf{1} & 0 \\ 0 & (\omega^{\mathrm{L}})^4\mathbf{1} \end{pmatrix} \tag{9.58}$$

from which we can obtain

$$\rho = \frac{\omega^{\mathrm{R}}\omega^{\mathrm{L}}}{(\omega^{\mathrm{R}})^2 + (\omega^{\mathrm{L}})^2} \tag{9.59}$$

Now, substituting our expressions for $\mathbf{x}$, $\mathbf{x}_\vee\mathbf{x}$, $\mathbf{x}_\vee\mathbf{x}_\vee\mathbf{x}$ and ρ into (9.44)-(9.46) we find our q-vectors; taking them to be numbered as in (9.44)-(9.46) with plus signs for each, we get after a little work:

$$\mathbf{q}'_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} n^{\mathrm{R}}_k\sigma_k & 0 \\ 0 & -n^{\mathrm{L}}_k\sigma_k \end{pmatrix} \qquad \mathbf{q}'_2 = -\frac{1}{\sqrt{2}} \begin{pmatrix} n^{\mathrm{R}}_k\sigma_k & 0 \\ 0 & n^{\mathrm{L}}_k\sigma_k \end{pmatrix}$$

$$\mathbf{q}'_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$$

We can then use (9.6)-(9.9) in conjunction with (7.22) to get the projection operators:

$$P'^1 = \frac{1}{2}\left(\mathbf{1}^{\mathrm{R}} - n^{\mathrm{R}}_k\sigma^{\mathrm{R}}_k\right)$$

$$P'^2 = \frac{1}{2}\left(\mathbf{1}^{\mathrm{R}} + n_k^{\mathrm{R}}\sigma_k^{\mathrm{R}}\right)$$
$$P'^3 = \frac{1}{2}\left(\mathbf{1}^{\mathrm{L}} - n_k^{\mathrm{L}}\sigma_k^{\mathrm{L}}\right)$$
$$P'^4 = \frac{1}{2}\left(\mathbf{1}^{\mathrm{L}} + n_k^{\mathrm{L}}\sigma_k^{\mathrm{L}}\right)$$

Clearly, this does not correspond with the projection operators we got from considering the $SU(2)_{\mathrm{R}}$ and $SU(2)_{\mathrm{L}}$ parts. If instead,

$$\mathbf{q}_1' = -\frac{1}{2\rho}\left[-(\frac{1}{2}-\rho)^{\frac{1}{2}}\mathbf{x} + \frac{1}{4}(\frac{1}{2}-\rho)^{-\frac{1}{2}}\mathbf{x}_\vee\mathbf{x}_\vee\mathbf{x}\right] \tag{9.60}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} -n_k^{\mathrm{R}}\sigma_k & 0 \\ 0 & n_k^{\mathrm{L}}\sigma_k \end{pmatrix} \tag{9.61}$$
$$\mathbf{q}_2' = -\frac{1}{2\rho}\left[-(\frac{1}{2}+\rho)^{\frac{1}{2}}\mathbf{x} + \frac{1}{4}(\frac{1}{2}+\rho)^{-\frac{1}{2}}\mathbf{x}_\vee\mathbf{x}_\vee\mathbf{x}\right] \tag{9.62}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} n_k^{\mathrm{R}}\sigma_k & 0 \\ 0 & n_k^{\mathrm{L}}\sigma_k \end{pmatrix} \tag{9.63}$$
$$\mathbf{q}_3' = (2-8\rho^2)^{-\frac{1}{2}}\mathbf{x}_\vee\mathbf{x} \tag{9.64}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{9.65}$$

we get the desired result. Indeed, in the special case where our vector of $\mathcal{H}$ is diagonal, $n_k^{\mathrm{R}}\sigma_k^{\mathrm{R}} \to \sigma_3^{\mathrm{R}}$, $n_k^{\mathrm{L}}\sigma_k^{\mathrm{L}} \to \sigma_3^{\mathrm{L}}$ and our q-vectors reduce to (6.81)-(6.83).

Recall that the reason we were finding these q-vectors was to be able to obtain a form for the u-vectors and projection operators of a generic vector as used in the last chapter. These u-vectors and projection operators are defined without reference to diagonal vectors, so we did not put primes on them. Similarly, we now have expressions for the q-vectors corresponding to a generic vector with the property $\operatorname{tr}\mathbf{x}^3 = 0$, which are defined without reference to diagonal vectors and which we can write our u-vectors and projection operators in terms of; therefore we shall drop the primes and simply write that for a generic $\mathbf{x}$ with $\mathrm{tr}\,\mathbf{x}^3 = 0$:

$$\mathbf{q}_1 = -\frac{1}{2\rho}\left[-(\frac{1}{2}-\rho)^{\frac{1}{2}}\mathbf{x} + \frac{1}{4}(\frac{1}{2}-\rho)^{-\frac{1}{2}}\mathbf{x}_\vee\mathbf{x}_\vee\mathbf{x}\right] \tag{9.66}$$

$$\mathbf{q}_2 = -\frac{1}{2\rho}\left[-(\frac{1}{2}+\rho)^{\frac{1}{2}}\mathbf{x} + \frac{1}{4}(\frac{1}{2}+\rho)^{-\frac{1}{2}}\mathbf{x}_\vee\mathbf{x}_\vee\mathbf{x}\right] \tag{9.67}$$

$$\mathbf{q}_3 = (2-8\rho^2)^{-\frac{1}{2}}\mathbf{x}_\vee\mathbf{x} \tag{9.68}$$

- note the single invariant ρ, as we have set $\gamma_3 = 0$ and factored out the length $\sqrt{\gamma_2}$.

The relevant u-vectors and projection operators are then given by

$$\mathbf{u}^1 = \frac{1}{2\sqrt{2}}(-\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3) \tag{9.69}$$

$$\mathbf{u}^2 = \frac{1}{2\sqrt{2}}(\mathbf{q}_1 - \mathbf{q}_2 + \mathbf{q}_3) \tag{9.70}$$

$$\mathbf{u}^3 = \frac{1}{2\sqrt{2}}(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{q}_3) \tag{9.71}$$

$$\mathbf{u}^4 = \frac{1}{2\sqrt{2}}(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) \tag{9.72}$$

and (7.22). Note that as $\mathbf{q}_1$ and $\mathbf{q}_2$ are in the coset space while $\mathbf{q}_3$ is in the subgroup, if the algebra admits the outer involutive automorphism - as it does for all coset spaces whose vectors are all of the form

$$\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$$

- these u-vectors fall into automorphism conjugate pairs:

$$\tilde{\mathbf{u}}^1 = \mathbf{u}^2 \qquad\qquad \tilde{\mathbf{u}}^3 = \mathbf{u}^4 \tag{9.73}$$

9.2.3 $\operatorname{tr}\mathbf{x}^3 \neq 0$

Let us now have a quick look at the case for vectors which do not have the property $\operatorname{tr}\mathbf{x}^3 = 0$.

We start again with the three equations (9.28)-(9.30). We note that the answers are nonsensical if $\gamma_i = 0$, as then we get from (9.29) that $\alpha_i = 0$ and consequently from (9.28) that $\beta_i = 0$. The plan therefore is to set $\gamma_i = 1$, obtain corresponding values for α_i and β_i, substitute these three into (9.26) and then normalise the resulting q-vectors.

We can start by noting that when $\gamma_i = 1$, (9.29) reduces to

$$2\alpha_i^2 + 4\alpha_i + \beta_i\gamma_3(\mathbf{x}) + 2\gamma_4(\mathbf{x}) + 2 = 0 \tag{9.74}$$

which we can rearrange to get an equation for β_i:

$$\beta_i = -\frac{1}{\gamma_3(\mathbf{x})}(2\alpha_i^2 + 4\alpha_i + 2\gamma_4(\mathbf{x}) + 2) \tag{9.75}$$

Similarly, (9.30) reduces to

$$\beta_i(4\alpha_i + 2) = \gamma_3(\mathbf{x}) \tag{9.76}$$

Combining these two equations we get a cubic equation for α_i:

$$\alpha_i^3 + \frac{5}{2}\alpha_i^2 + (2 - \rho^2)\alpha_i - \frac{1}{2}\rho^2 + \frac{1}{2} = -\frac{1}{8}\gamma_3(\mathbf{x})^2 \tag{9.77}$$

Unfortunately the solutions to this equation are long and messy. Obviously, the corresponding values for β_i obtained by substituting these into (9.75) or (9.76) are just as bad, and normalising the resulting q-vectors simply makes them worse.

However, it is worth noting that it is possible to factorise the left-hand side of this equation:

$$(\alpha_i + 1 + \rho)(\alpha_i + 1 - \rho)(\alpha_i + \frac{1}{2}) = -\frac{1}{8}\gamma_3(\mathbf{x})^2 \tag{9.78}$$

which is valid for any value of $\gamma_3(\mathbf{x})$ other than zero. So for sufficiently small $\gamma_3(\mathbf{x})$, we have the following approximate solutions:

$$\alpha_i = -1 - \rho, \quad -1 + \rho, \quad -\frac{1}{2} \tag{9.79}$$

and to obtain more accurate solutions one could use numerical methods.

9.2.4 x in the basis of the u-vectors

Before we go on to look at the other method for finding the u-vectors, it is instructive to find expressions for $\mathbf{x}$ for the case $\mathrm{tr}\,\mathbf{x}^3 = 0$ in terms of the q-vectors and also in terms of the u-vectors/projection operators.

From (9.66) and (9.67) we get

$$\mathbf{x} = -(\frac{1}{2} - \rho)^{\frac{1}{2}}\mathbf{q}_1 + (\frac{1}{2} + \rho)^{\frac{1}{2}}\mathbf{q}_2 \tag{9.80}$$

We can also use (9.2)-(9.5) to get

$$\mathbf{q}_1 = \frac{1}{\sqrt{2}}(-\mathbf{u}^1 + \mathbf{u}^2 + \mathbf{u}^3 - \mathbf{u}^4) \tag{9.81}$$

$$\mathbf{q}_2 = \frac{1}{\sqrt{2}}(\mathbf{u}^1 - \mathbf{u}^2 + \mathbf{u}^3 - \mathbf{u}^4) \tag{9.82}$$

Thus

$$
\begin{aligned}
\mathbf{x} &= \left((\frac{1}{4}-\frac{1}{2}\rho)^{\frac{1}{2}}+(\frac{1}{4}+\frac{1}{2}\rho)^{\frac{1}{2}}\right)\mathbf{u}^1+\left(-(\frac{1}{4}-\frac{1}{2}\rho)^{\frac{1}{2}}-(\frac{1}{4}+\frac{1}{2}\rho)^{\frac{1}{2}}\right)\mathbf{u}^2 \\
&\quad+\left(-(\frac{1}{4}-\frac{1}{2}\rho)^{\frac{1}{2}}+(\frac{1}{4}+\frac{1}{2}\rho)^{\frac{1}{2}}\right)\mathbf{u}^3+\left((\frac{1}{4}-\frac{1}{2}\rho)^{\frac{1}{2}}-(\frac{1}{4}+\frac{1}{2}\rho)^{\frac{1}{2}}\right)\mathbf{u}^4 \qquad (9.83) \\
&= \left((\frac{1}{4}-\frac{1}{2}\rho)^{\frac{1}{2}}+(\frac{1}{4}+\frac{1}{2}\rho)^{\frac{1}{2}}\right)P^1+\left(-(\frac{1}{4}-\frac{1}{2}\rho)^{\frac{1}{2}}-(\frac{1}{4}+\frac{1}{2}\rho)^{\frac{1}{2}}\right)P^2 \\
&\quad+\left(-(\frac{1}{4}-\frac{1}{2}\rho)^{\frac{1}{2}}+(\frac{1}{4}+\frac{1}{2}\rho)^{\frac{1}{2}}\right)P^3+\left((\frac{1}{4}-\frac{1}{2}\rho)^{\frac{1}{2}}-(\frac{1}{4}+\frac{1}{2}\rho)^{\frac{1}{2}}\right)P^4 \qquad (9.84)
\end{aligned}
$$

9.3 Consequences of the eigenvalue equation

We now turn to the second way to find the u-vectors for an arbitrary vector of the coset space. This is almost the above procedure in reverse: we find the decomposition of the vector in terms of the u-vectors, then similarly find the decomposition of its powers in terms of the u-vectors, then finally invert the relations. We discussed this decomposition of an arbitrary vector of the coset space in Section 8.1.2, and saw that if we write such a vector as a linear sum of projection operators or u-vectors, the coefficients are the eigenvalues of the vector. It was remarked that these can be found by solving the characteristic equation. In this case the characteristic equation is (6.63). Solving this for a unit vector gives us the eigenvalues:

$$
\begin{aligned}
\mu_S &= \left[\frac{1}{2}+\left(\gamma_4(\mathbf{x})+\frac{1}{4}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}, \quad \left[\frac{1}{2}-\left(\gamma_4(\mathbf{x})+\frac{1}{4}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}, \\
&\quad -\left[\frac{1}{2}+\left(\gamma_4(\mathbf{x})+\frac{1}{4}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}, \quad -\left[\frac{1}{2}-\left(\gamma_4(\mathbf{x})+\frac{1}{4}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} \qquad (9.85)
\end{aligned}
$$

According to the argument in Section 8.1.2, the coefficients in (9.83) and (9.84) should be these eigenvalues - this is easy to see by squaring the coefficients; for example:

$$\begin{aligned}\left((\frac{1}{4}-\frac{1}{2}\rho)^{\frac{1}{2}}+(\frac{1}{4}+\frac{1}{2}\rho)^{\frac{1}{2}}\right)^2 &= (\frac{1}{4}-\frac{1}{2}\rho)+2(\frac{1}{16}-\frac{1}{4}\rho^2)^{\frac{1}{2}}+(\frac{1}{4}+\frac{1}{2}\rho) \\ &= \frac{1}{2}+\left(\frac{1}{4}-\rho^2\right)^{\frac{1}{2}} \qquad (9.86) \\ &= \frac{1}{2}+\left(\gamma_4(\mathbf{x})+\frac{1}{4}\right)^{\frac{1}{2}} \qquad (9.87)\end{aligned}$$

So solving the eigenvalue equation leads us to precisely the expansion (9.83) for $\mathbf{x}$. This is, in principle, all we need to find the four u-vectors in terms of $\mathbf{x}$, $\mathbf{x}_\vee\mathbf{x}$ and $\mathbf{x}_\vee\mathbf{x}_\vee\mathbf{x}$: we can use (7.29) and (7.31) to find expressions for $\mathbf{x}_\vee\mathbf{x}$ and $\mathbf{x}_\vee\mathbf{x}_\vee\mathbf{x}$ as linear sums of the u-vectors; together with the fact that the u-vectors add up to zero (this is obvious as the projection operators form a complete set) this is sufficient to find the four u-vectors. (It is just solving four linear equations in four unknowns.) However, we will not do this here as we have already found our u-vectors in the last section and checked the form of the resulting expression for $\mathbf{x}$.

9.4 $L^{-1}\partial_\mu L$ for SU(4) coset spaces

Let us recap what we found in the last chapter and what we have done so far in this chapter. We showed that for any coset space of an SU(N) group, given the decomposition of an arbitrary vector of that space

$$\boldsymbol{\theta} \equiv \theta^A \lambda_A = \sum_S \theta'_S \mathbf{u}^S$$

$L^{-1}\partial_\mu L$ takes the form (8.53). If the u-vectors of the space form automorphism conjugate pairs, the first two terms in this expression form the $\mathbf{a}_\mu$ part while the third term forms the $\mathbf{v}_\mu$ part. For coset spaces of SU(4) for which $\mathrm{tr}\,\boldsymbol{\theta}^3 = 0$ for every vector in the space, we have found the above decomposition; furthermore we have shown that if the space is also a symmetric one (e.g. SU(4)/SU(2)⊗SU(2)⊗U(1)) the u-vectors do fall into these pairs. As remarked at the end of Section 8.2 it is possible to rewrite the first part of the right-hand side of (8.53) in terms of $q_i >< q_j$ (using techniques equivalent to those given in [18]) - this is what we shall do in this section.

We start by comparing and contrasting our two expressions for $\boldsymbol{\theta}$, in two different bases. In the case of $\boldsymbol{\theta} = \theta^A \lambda_A$, the θ^A are the components of the vector $\boldsymbol{\theta}$ in a coordinate system whose basis is the set of λ_A. The matrix nature of $\boldsymbol{\theta}$ is contained entirely in the λ's - the θ^A are just numerical coefficients. Furthermore, the λ_A form an orthonormal basis set. Similarly, the θ'_S are the components of $\boldsymbol{\theta}$ in a coordinate system whose basis is the set of $\mathbf{u}^S$; however, these are unlike the λ_I in several ways: they are mutually commuting and they are neither orthogonal nor normalised, as can be seen from (7.28) and (7.30).

The $\mathbf{u}^S$ are a set of four vectors, in the case of SU(4), lying in the 3-dimensional Cartan subspace containing $\boldsymbol{\theta}$. Also in this subspace is $\sum_S \mathbf{u}^S \partial_\mu \theta'_S$. Barnes *et al* construct from the $\mathbf{u}^S$ a set of orthonormal vectors in the Cartan subspace, which they call p^α and are defined by equation (3.8) of [18]. For the coset spaces of SU(4), we already have such an orthonormal set - the q-vectors; indeed, these are simpler functions of the $\mathbf{u}^S$ than the relevant p^α. We can rewrite $\boldsymbol{\theta}$ in the orthonormal basis of the q-vectors by defining a unit vector $\mathbf{x} = \boldsymbol{\theta}/\theta$ and using (9.80):

$$\boldsymbol{\theta} = \theta\mathbf{x} = \mathbf{q}_a \theta''^a \tag{9.88}$$

where $a = 1, 2$ and

$$\theta''^1 = -\theta(\frac{1}{2} - \rho)^{\frac{1}{2}}, \qquad \theta''^2 = \theta(\frac{1}{2} + \rho)^{\frac{1}{2}}, \tag{9.89}$$

(these are the equivalent of the ψ^α in [18]). We would also like to rewrite

$$\sum_S \mathbf{u}^S \partial_\mu \theta'_S = \mathbf{u}^1 \partial_\mu \theta'_1 + \mathbf{u}^2 \partial_\mu \theta'_2 + \mathbf{u}^3 \partial_\mu \theta'_3 + \mathbf{u}^4 \partial_\mu \theta'_4$$

in terms of the q-vectors. Now we already know the decomposition $\boldsymbol{\theta} = \theta \mathbf{x} = \sum_S \mathbf{u}^S \theta'_S$ from (9.83) - the θ'_S are just linear sums of the θ''^a:

$$\theta'_1 = \theta\left((\frac{1}{4} - \frac{1}{2}\rho)^{\frac{1}{2}} + (\frac{1}{4} + \frac{1}{2}\rho)^{\frac{1}{2}}\right) = \frac{1}{\sqrt{2}}(-\theta''^1 + \theta''^2) \tag{9.90}$$

$$\theta'_2 = \theta\left(-(\frac{1}{4} - \frac{1}{2}\rho)^{\frac{1}{2}} - (\frac{1}{4} + \frac{1}{2}\rho)^{\frac{1}{2}}\right) = \frac{1}{\sqrt{2}}(\theta''^1 - \theta''^2) \tag{9.91}$$

$$\theta'_3 = \theta\left(-(\frac{1}{4} - \frac{1}{2}\rho)^{\frac{1}{2}} + (\frac{1}{4} + \frac{1}{2}\rho)^{\frac{1}{2}}\right) = \frac{1}{\sqrt{2}}(\theta''^1 + \theta''^2) \tag{9.92}$$

$$\theta'_4 = \theta\left((\frac{1}{4} - \frac{1}{2}\rho)^{\frac{1}{2}} - (\frac{1}{4} + \frac{1}{2}\rho)^{\frac{1}{2}}\right) = \frac{1}{\sqrt{2}}(-\theta''^1 - \theta''^2) \tag{9.93}$$

We thus use this in conjunction with (9.69)-(9.72) - not surprisingly, this gives us

$$\sum_S \mathbf{u}^S \partial_\mu \theta'_S = \mathbf{q}_a \partial_\mu \theta''^a \tag{9.94}$$

Now, if we are to use $L^{-1}\partial_\mu L$ to find the metric for the realisation, we need to extract a $\partial_\mu M^A$ from this. We therefore write

$$\sum_S \mathbf{u}^S \partial_\mu \theta'_S = \mathbf{q}_a \partial_\mu \theta''^a = \mathbf{q}_a \frac{\partial \theta''^a}{\partial M^A} \partial_\mu M^A \tag{9.95}$$

To take this further, we recall that $\mathbf{M}$ has an analogous decomposition to $\boldsymbol{\theta}$,

$$\mathbf{M} \equiv M^A \lambda_A = \sum_S M'_S \mathbf{u}^S$$

where we have the same u-vectors (as they lie in the same Cartan subspace) and the M'_S have the same form as the θ'_S but with different values of the invariants:

$$M'_1 = M\left((\frac{1}{4}-\frac{1}{2}\lambda)^{\frac{1}{2}}+(\frac{1}{4}+\frac{1}{2}\lambda)^{\frac{1}{2}}\right) \quad (9.96)$$
$$M'_2 = M\left(-(\frac{1}{4}-\frac{1}{2}\lambda)^{\frac{1}{2}}-(\frac{1}{4}+\frac{1}{2}\lambda)^{\frac{1}{2}}\right) \quad (9.97)$$
$$M'_3 = M\left(-(\frac{1}{4}-\frac{1}{2}\lambda)^{\frac{1}{2}}+(\frac{1}{4}+\frac{1}{2}\lambda)^{\frac{1}{2}}\right) \quad (9.98)$$
$$M'_4 = M\left((\frac{1}{4}-\frac{1}{2}\lambda)^{\frac{1}{2}}-(\frac{1}{4}+\frac{1}{2}\lambda)^{\frac{1}{2}}\right) \quad (9.99)$$

M can therefore be expressed in the same way in the basis of the same q-vectors:

$$\mathbf{M} = \mathbf{q}_a M''^a \quad (9.100)$$

but with different values of the invariants:

$$M''^1 = -M(\frac{1}{2}-\lambda)^{\frac{1}{2}}, \qquad M''^2 = M(\frac{1}{2}+\lambda)^{\frac{1}{2}}, \quad (9.101)$$

We then view the θ''^a as functions of M''^b:

$$\sum_S \mathbf{u}^S \partial_\mu \theta'_S = \mathbf{q}_a \frac{\partial \theta''^a}{\partial M''^b}\frac{\partial M''^b}{\partial M^A}\partial_\mu M^A \quad (9.102)$$

In order to evaluate the differential

$$\frac{\partial M''^b}{\partial M^A}$$

(which is essentially a rotation matrix - a Jacobian matrix - describing the change of basis from the λ's to the **q**'s), we take a scalar product of (9.100) with a q-vector:

$$(\mathbf{M}, \mathbf{q}_b) = (\mathbf{q}_a, \mathbf{q}_b)M''^a = \delta_{ab}M''^a = M''_b \tag{9.103}$$

$$\Rightarrow M''^b = M^B q^b_B \tag{9.104}$$

(where covariant and contravariant a, b-indices have the same sign.) In differentiating this expression, we must be careful. Remember, what we have done is to take a vector $\mathbf{M} = M^A\lambda_A$ and construct *from it* three q-vectors, then decompose the vector into a linear sum of two of these. q^b_B is therefore a function of M^A. This transformation from the basis of the λ's to the basis of the q-vectors is *dependent* on $\mathbf{M}$ - if we pick a different $\mathbf{M}$, we must find new $\mathbf{q}$'s. Differentiating (9.104) with respect to M^A therefore gives us

$$\frac{\partial M''^b}{\partial M^A} = \delta^B_A q^b_B + M^B \frac{\partial q^b_B}{\partial M^A} = q^b_A + q^B_c M''^c \frac{\partial q^b_B}{\partial M^A} \tag{9.105}$$

where we have used the expression $M^B = q^B_c M''^c$ which comes from taking components of (9.100). This is where we differ from Barnes *et al*, who only admit the first of these terms in equation (3.35) of [18]. However, substituting this into (9.102), we get

$$\begin{aligned} \sum_S \mathbf{u}^S \partial_\mu \theta'_S &= \mathbf{q}_a q^b_A \frac{\partial \theta''^a}{\partial M''^b} \partial_\mu M^A + \mathbf{q}_a q^B_c M''^c \frac{\partial \theta''^a}{\partial M''^b} \frac{\partial q^b_B}{\partial M^A} \partial_\mu M^A & (9.106) \\ &= \left(q^K_a q^b_A \frac{\partial \theta''^a}{\partial M''^b} \partial_\mu M^A + q^K_a q^B_c M''^c \frac{\partial \theta''^a}{\partial M''^b} \frac{\partial q^b_B}{\partial M^A} \partial_\mu M^A \right) \lambda_K & (9.107) \end{aligned}$$

and we can show that the second of these terms is zero. To do this we note that if we write out the b's explicitly, we have a term involving

$$\frac{\partial q^1_B}{\partial M^A}$$

and a term involving

$$\frac{\partial q^2_B}{\partial M^A}$$

Using (9.81)-(9.82) and (7.22), we can rewrite these as linear sums of

$$\left(\frac{\partial P^S}{\partial M^A}\right)_B$$

which, from (8.22) we see are linear sums of

$$(f_{u^S} f_{u^T})_{AB}$$

Thus the second term of (9.107) contains lots of terms of the form

$$q^K_a q^B_c (f_{u^S} f_{u^T})_{AB} = (q_a >< q_c)^K{}_B (f_{u^S} f_{u^T})^B_A$$

which must be zero from (7.54).

Thus we are left with

$$\sum_S \mathbf{u}^S \partial_\mu \theta'_S = (q_a >< q^b)^K{}_A \frac{\partial \theta''^a}{\partial M''^b} \partial_\mu M^A \lambda_K \tag{9.108}$$

This is the form we were after. Substituting this into (8.53), we get

$$\begin{aligned} L^{-1}\partial_\mu L &= -\frac{\mathrm{i}}{2}\partial_\mu M^A \left[(q_a >< q^b)^I{}_A \frac{\partial \theta''^a}{\partial M''^b} \right. \\ &\qquad \left. + 2\sum_{S<T} \frac{\sin\left(\frac{\theta'_S - \theta'_T}{2}\right)}{M'_S - M'_T} (P^{ST} + P^{TS})^I_A \right] \lambda_I \end{aligned}$$

$$+\partial_\mu M^A \sum_{S<T} \frac{2}{M'_S - M'_T} \sin^2\left(\frac{\theta'_S - \theta'_T}{4}\right)(P^{ST} - P^{TS})_A{}^I \lambda_I \qquad (9.109)$$

We close this section by noting that we could also rewrite the other terms using $\mathbf{q}$'s and the θ''^a and M''^a, but the results are not pretty. For example, using (9.69)-(9.72) as well as (9.90)-(9.93) and similar relations between the M'_S and the M''^a, the second term in the square brackets becomes

$$2\sum_{S<T} \frac{\sin\left(\frac{\theta'_S-\theta'_T}{2}\right)}{M'_S - M'_T}(P^{ST} + P^{TS})^I_A$$

$$= \frac{1}{\sqrt{2}}\left(-\frac{\sin[\frac{\theta''^1-\theta''^2}{\sqrt{2}}]}{M''^1 - M''^2} - \frac{2\sin[\frac{\theta''^1}{\sqrt{2}}]}{M''^1} + \frac{2\sin[\frac{\theta''^2}{\sqrt{2}}]}{M''^2} - \frac{\sin[\frac{\theta''^1+\theta''^2}{\sqrt{2}}]}{M''^1 + M''^2}\right)(f^2_{q_1})^I_A$$

$$+\frac{1}{\sqrt{2}}\left(-\frac{\sin[\frac{\theta''^1-\theta''^2}{\sqrt{2}}]}{M''^1 - M''^2} + \frac{2\sin[\frac{\theta''^1}{\sqrt{2}}]}{M''^1} - \frac{2\sin[\frac{\theta''^2}{\sqrt{2}}]}{M''^2} - \frac{\sin[\frac{\theta''^1+\theta''^2}{\sqrt{2}}]}{M''^1 + M''^2}\right)(f^2_{q_2})^I_A$$

$$+\frac{1}{\sqrt{2}}\left(\frac{\sin[\frac{\theta''^1-\theta''^2}{\sqrt{2}}]}{M''^1 - M''^2} - \frac{2\sin[\frac{\theta''^1}{\sqrt{2}}]}{M''^1} - \frac{2\sin(\frac{\theta''^2}{\sqrt{2}}]}{M''^2} + \frac{\sin[\frac{\theta''^1+\theta''^2}{\sqrt{2}}]}{M''^1 + M''^2}\right)(f^2_{q_3})^I_A$$

$$+\sqrt{2}\left(\frac{\sin[\frac{\theta''^1-\theta''^2}{\sqrt{2}}]}{M''^1 - M''^2} - \frac{\sin[\frac{\theta''^1+\theta''^2}{\sqrt{2}}]}{M''^1 + M''^2}\right)(f_{q_1}f_{q_2})^I_A \qquad (9.110)$$

9.5 Covariant Derivatives and Metric

We are now in a situation where we can state the covariant derivatives for a whole class of coset spaces, with every quantity in the expressions being known. For any coset space of SU(4) in which $\mathrm{tr}\,\boldsymbol{\theta}^3 = 0$ for every vector of the coset space *and* the commutation relations have the $\mathbb{Z}_2$-grading structure,

$$\mathbf{a}_\mu = \partial_\mu M^A \left((q_a >< q^b)^B{}_A \frac{\partial \theta''^a}{\partial M''^b} + 2\sum_{S<T} \frac{\sin\left(\frac{\theta'_S-\theta'_T}{2}\right)}{M'_S - M'_T}(P^{ST} + P^{TS})^B_A\right)\lambda_B$$

(9.111)

and

$$-\frac{\mathrm{i}}{2}\mathbf{v}_\mu = \partial_\mu M^A \sum_{S<T} \frac{2}{M'_S - M'_T} \sin^2\left(\frac{\theta'_S - \theta'_T}{4}\right)(P^{ST} - P^{TS})_A{}^P \lambda_P \qquad (9.112)$$

so

$$D_\mu M^B = \partial_\mu M^A \left((q_a >< q^b)^B{}_A \frac{\partial \theta''^a}{\partial M''^b} + 8 \sum_{S<T} \frac{\sin\left(\frac{\theta'_S - \theta'_T}{2}\right)}{M'_S - M'_T} (f_{u^S} f_{u^T})^B_A \right) \qquad (9.113)$$

and

$$D_\mu \psi = \partial_\mu \psi + \partial_\mu M^A \sum_{S<T} \frac{8\mathrm{i}}{M'_S - M'_T} \sin^2\left(\frac{\theta'_S - \theta'_T}{4}\right)(f_{u^S} f_{u^T}(f_{u^T} - f_{u^S}))_A{}^P \lambda_P \psi \qquad (9.114)$$

The q-vectors and u-vectors in this are those of equations (9.66)-(9.72), which can be constructed either from the powers of $\mathbf{M}$ or those of $\boldsymbol{\theta}$, as both are generic vectors in the same Cartan subspace. The θ''^a, M''^a, θ'_S and M'_S are given by equations (9.89), (9.101), (9.90)-(9.93) and (9.96)-(9.99) respectively, where

$$\rho \equiv \sqrt{-\gamma_4\left(\frac{\boldsymbol{\theta}}{\theta}\right)} = \sqrt{\frac{1}{2} - \frac{1}{4}\mathrm{tr}\left(\frac{\boldsymbol{\theta}}{\theta}\right)} \qquad (9.115)$$

and

$$\chi \equiv \sqrt{-\gamma_4\left(\frac{\mathbf{M}}{M}\right)} = \sqrt{\frac{1}{2} - \frac{1}{4}\mathrm{tr}\left(\frac{\mathbf{M}}{M}\right)} \qquad (9.116)$$

Finally, let us determine the metric for this non-linear realisation. If we try to calculate $D_\mu M^B D^\mu M_B$, it is clear from the last section that the cross-terms will all vanish. The products of $P^{ST} + P^{TS}$'s are obvious, but we will need to know what the products of $(q_a >< q^b)^B{}_A$'s look like. Using the orthonormality of the

$\mathbf{q}$'s, the product is just

$$(q_a >< q^b)^B{}_A (q_c >< q^d)_B{}^C = q_a^B q_A^b q_{cB} q^{dC} = (\mathbf{q}_a, \mathbf{q}_c) q_A^b q^{dC} = \delta_{ac} (q^b >< q^d)_A{}^C \tag{9.117}$$

Therefore

$$\begin{aligned} D_\mu M^B D^\mu M_B &= \partial_\mu M^A \partial^\mu M_C \Bigg[(q^b >< q^c)_A{}^C \frac{\partial \theta''^a}{\partial M''^b} \frac{\partial \theta''_a}{\partial M''^c} \\ &\quad + 16 \sum_{S<T} \left(\frac{\sin\left(\frac{\theta'_S - \theta'_T}{2}\right)}{M'_S - M'_T} \right)^2 (f_{u^S} f_{u^T})_A^C \Bigg] \end{aligned} \tag{9.118}$$

with the metric being the quantity in square brackets:

$$g_{AC} = (q^b >< q^c)_{AC} \frac{\partial \theta''^a}{\partial M''^b} \frac{\partial \theta''_a}{\partial M''^c} + 16 \sum_{S<T} \left(\frac{\sin\left(\frac{\theta'_S - \theta'_T}{2}\right)}{M'_S - M'_T} \right)^2 (f_{u^S} f_{u^T})_{AC} \tag{9.119}$$

9.6 Non-linear realisations of SO(6)

Once again, we are able to use the homomorphism between SU(4) and SO(6) to rephrase our results, so that they express the properties of non-linear realisations of SO(6). Each term in (9.109) is a linear sum of λ's, so is a vector of SO(6) (upto a factor of i). We can therefore simply replace the λ's with σ's and change each I-index to an IJ-pair, with the appropriate normalisation:

$$\begin{aligned} L^{-1} \partial_\mu L &= -\frac{\mathrm{i}}{2} \partial_\mu M^{AB} \Bigg[(q_a >< q^b)^{IJ}{}_{AB} \frac{\partial \theta''^a}{\partial M''^b} \\ &\quad + 2 \sum_{S<T} \frac{\sin\left(\frac{\theta'_S - \theta'_T}{2}\right)}{M'_S - M'_T} (P^{ST} + P^{TS})^{IJ}_{AB} \Bigg] \sigma_{IJ} \end{aligned}$$

$$+\partial_\mu M^{AB} \sum_{S<T} \frac{2}{M'_S - M'_T} \sin^2\left(\frac{\theta'_S - \theta'_T}{4}\right) (P^{ST} - P^{TS})_{AB}{}^{IJ} \sigma_{IJ} \tag{9.120}$$

(where σ_{IJ} is an arbitrary generator of SO(6) while σ_{AB} is a broken generator). The operators $(q_a >< q^b)^{IJ}{}_{AB}$, $(P^{ST} + P^{TS})^{IJ}_{AB}$ and $(P^{ST} - P^{TS})_{AB}{}^{IJ}$ are then given by

$$(q_a >< q^b)^{IJ}{}_{AB} = q_a^{IJ} q^b_{AB} \tag{9.121}$$

$$(P^{ST} + P^{TS})^{IJ}_{AB} = \frac{1}{8}\,\mathrm{tr}(P^S \sigma_{AB} P^T \sigma^{IJ}) + \frac{1}{8}\,\mathrm{tr}(P^T \sigma_{AB} P^S \sigma^{IJ}) = 4(f_{u^S} f_{u^T})^{IJ}_{AB} \tag{9.122}$$

and

$$\begin{aligned} (P^{ST} - P^{TS})_{AB}{}^{IJ} &= \frac{1}{8}\,\mathrm{tr}(P^S \sigma_{AB} P^T \sigma^{IJ}) - \frac{1}{8}\,\mathrm{tr}(P^T \sigma_{AB} P^S \sigma^{IJ}) && (9.123) \\ &= 4\mathrm{i}[f_{u^S} f_{u^T}(f_{u^T} - f_{u^S})]_{AB}{}^{IJ} && (9.124) \end{aligned}$$

(see Section 7.4) with $(f_{u^S})_{AB}{}^{IJ}$ given by (7.60). For symmetric spaces in which $\mathrm{tr}\,\boldsymbol{\theta}^3 = 0$ for every vector of the coset space, (9.120) can then be split in the obvious way, giving covariant derivatives analogous to (9.113) and (9.114). The metric is

$$g_{AB\,CD} = (q^b >< q^c)_{AB\,CD} \frac{\partial \theta''^a}{\partial M''^b} \frac{\partial \theta''_a}{\partial M''^c} + 16 \sum_{S<T} \left(\frac{\sin\left(\frac{\theta'_S - \theta'_T}{2}\right)}{M'_S - M'_T} \right)^2 (f_{u^S} f_{u^T})_{AB\,CD} \tag{9.125}$$

Due to the simplicity of the $_\vee$-relations of the SO(6) spinor representations, we can also find relatively simple expressions for $\mathbf{x}$, $\mathbf{x}_\vee\mathbf{x}$ and $\mathbf{x}_\vee\mathbf{x}_\vee\mathbf{x}$, as we shall now see. We shall also find expressions for the invariants $\mathrm{tr}\,\mathbf{x}^3$ and ρ.

9.6.1 Vectors and invariants of SO(6)

First we want to define a unit vector. To do this we need to know that the scalar product of two σ's is

$$(\sigma_{AB}, \sigma_{CD}) = 2(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}) \tag{9.126}$$

It is easy to see that this is the correct expression by considering, for example, the scalar product of σ_{12} with σ_{12}, σ_{21} and σ_{23}. From (6.91) it can be seen that

$$(\sigma_{12}, \sigma_{12}) = 2$$

as given by the above expression; this also implies that

$$(\sigma_{12}, \sigma_{21}) = -2$$

- also in agreement. Finally, we know that the scalar product of two different σ's, e.g. σ_{12} and σ_{23}, is zero (they are orthogonal).

The square of the length of an arbitrary vector $\omega^{AB}\sigma_{AB}$ is then

$$\begin{aligned}(\omega^{AB}\sigma_{AB}, \omega^{CD}\sigma_{CD}) &= \omega^{AB}\omega^{CD}(\sigma_{AB}, \sigma_{CD}) && (9.127)\\ &= 2\omega^{AB}\omega^{CD}(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}) && (9.128)\\ &= 4\omega^{AB}\omega_{AB} && (9.129)\end{aligned}$$

Defining the scalar ω by

$$\omega \equiv \sqrt{\omega^{AB}\omega_{AB}} \tag{9.130}$$

the magnitude of $\omega^{AB}\sigma_{AB}$ is then 2ω, so we can define a unit vector $\mathbf{x}$:

$$\mathbf{x} = \frac{1}{2\omega}\omega^{AB}\sigma_{AB} = \frac{1}{2}n^{AB}\sigma_{AB} \tag{9.131}$$

where $n^{AB} \equiv \frac{\omega^{AB}}{\omega}$ is a 'unit tensor':

$$n^{AB}n_{AB} = \frac{\omega^{AB}}{\omega}\frac{\omega_{AB}}{\omega} = \frac{\omega^2}{\omega^2} = 1 \tag{9.132}$$

The next vector we need is $\mathbf{x}_\vee\mathbf{x}$; this is simply

$$\mathbf{x}_\vee\mathbf{x} = \frac{1}{4}n^{AB}n^{CD}\sigma_{AB\vee}\sigma_{CD} = \frac{1}{4}n^{AB}n^{CD}\epsilon_{ABCD}{}^{EF}\sigma_{EF} \tag{9.133}$$

Similarly

$$\begin{aligned} \mathbf{x}_\vee\mathbf{x}_\vee\mathbf{x} &= \frac{1}{8}n^{AB}n^{CD}n^{EF}\epsilon_{CDEF}{}^{GH}\sigma_{AB\vee}\sigma_{GH} && (9.134) \\ &= \frac{1}{8}n^{AB}n^{CD}n^{EF}\epsilon_{CDEF}{}^{GH}\epsilon_{ABGH}{}^{IJ}\sigma_{IJ} && (9.135) \\ &= 2n^{IJ}\sigma_{IJ} + 4n^{GJ}n^{HI}n_{GH}\sigma_{IJ} && (9.136) \end{aligned}$$

where we have used an identity for the contraction of two ϵ's.

We can get $\mathrm{tr}\,\mathbf{x}^3$ by writing

$$\mathbf{x}^2 = \frac{1}{2}\mathbf{x}_\vee\mathbf{x} + \frac{1}{2}\mathbf{1} \tag{9.137}$$

so that

$$\begin{aligned} \mathrm{tr}\,\mathbf{x}^3 &= (\mathbf{x}, \mathbf{x}_\vee\mathbf{x}) + \frac{1}{2}\,\mathrm{tr}\,\mathbf{x} && (9.138) \\ &= \frac{1}{8}n^{AB}n^{CD}n^{EF}\epsilon_{CDEF}{}^{GH}(\sigma_{AB}, \sigma_{GH}) && (9.139) \\ &= \frac{1}{4}n^{AB}n^{CD}n^{EF}\epsilon_{CDEF}{}^{GH}(\delta_{AG}\delta_{BH} - \delta_{AH}\delta_{BG}) && (9.140) \end{aligned}$$

$$= \frac{1}{4} n^{AB} n^{CD} n^{EF} (\epsilon_{CDEFAB} - \epsilon_{CDEFBA}) \tag{9.141}$$

$$= \frac{1}{2} n^{AB} n^{CD} n^{EF} \epsilon_{ABCDEF} \tag{9.142}$$

while we obtain ρ by squaring (9.137):

$$\mathbf{x}^4 = \frac{1}{4}(\mathbf{x}_\vee \mathbf{x})^2 + \frac{1}{2}\mathbf{x}_\vee \mathbf{x} + \frac{1}{4}\mathbf{1} \tag{9.143}$$

$$\Rightarrow \operatorname{tr} \mathbf{x}^4 = \frac{1}{2}(\mathbf{x}_\vee \mathbf{x}, \mathbf{x}_\vee \mathbf{x}) + 1 \tag{9.144}$$

$$= 1 + \frac{1}{32} n^{AB} n^{CD} n^{GH} n^{IJ} \epsilon_{ABCD}{}^{EF} \epsilon_{GHIJ}{}^{KL} (\sigma_{EF}, \sigma_{KL})$$

$$= 1 + \frac{1}{8} n^{AB} n^{CD} n^{GH} n^{IJ} \epsilon_{ABCD}{}^{EF} \epsilon_{GHIJEF} \tag{9.145}$$

$$= 3 + 4 n^{GH} n^{IJ} n_{GJ} n_{HI} \tag{9.146}$$

$$\Rightarrow \gamma_4(\mathbf{x}) = \frac{1}{4} \operatorname{tr} \mathbf{x}^4 - \frac{1}{2} = \frac{3}{4} + n^{GH} n^{IJ} n_{GJ} n_{HI} - \frac{1}{2} \tag{9.147}$$

$$= \frac{1}{4} + n^{GH} n^{IJ} n_{GJ} n_{HI} \tag{9.148}$$

$$\Rightarrow \rho = \sqrt{-\gamma_4(\mathbf{x})} = \sqrt{n^{GH} n^{IJ} n_{GJ} n_{IH} - \frac{1}{4}} \tag{9.149}$$

9.6.2 Properties of coset spaces of SO(6)

The coset spaces of SU(4)≈SO(6) we have found the q-vectors and u-vectors for are those with the property $\operatorname{tr} \mathbf{x}^3 = 0$, which from (9.142) we can see is equivalent to insisting that

$$n^{AB} n^{CD} n^{EF} \epsilon_{ABCDEF} = 0$$

It is possible to identify particular coset spaces contained in SO(6) for which this is guaranteed to be the case for all vectors. For example, for the SO(5) subgroup of rotations in the first five dimensions, it is clear that in the product $n^{AB} n^{CD} n^{EF}$ it is not possible for all six indices to be different, so any coset space of SO(5) has the above property. Similarly, if we look at the coset space SO(6)/SO(4), it is generated by $< T_{\mu 5}, T_{\mu 6}, T_{56} >$ so in the indices of the product $n^{AB} n^{CD} n^{EF}$

either a 5 or a 6 (or both) must be repeated. Hence the algebra of the coset space SO(6)/SO(4) is also composed entirely of vectors for which $\operatorname{tr}\mathbf{x}^3 = 0$, as are those of the two subspaces of this coset space, SO(6)/SO(5) and SO(6)/SO(4)⊗SO(2).

This can be seen from the spinor representations. It is clear from the form of the generators given in (6.75)-(6.80) that the coset space SO(6)/SO(4)⊗SO(2) is one of the type discussed in Section 9.1 where every vector of the coset space has the form

$$\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$$

If we include the generator σ_{56} in our coset space, the general form of a vector is

$$\begin{pmatrix} c\mathbf{1} & A \\ A^* & -c\mathbf{1} \end{pmatrix}$$

where c is a real number; it is not hard to show that the trace of the cube of this is also zero. Such a coset space, however, is not necessarily a symmetric space as the $\wedge$-product of two vectors is not necessarily a vector of the sublagebra.

This would seem to suggest that equation (9.109) is valid for each of the coset spaces SO(6)/SO(4), SO(6)/SO(5) and SO(6)/SO(4)⊗SO(2) and that the covariant derivatives are given by (9.113) and (9.114) for SO(6)/SO(4)⊗SO(2). However, we should explicitly check which of these are symmetric spaces, besides which there is a subtlety for SO(6)/SO(5).

SO(6)/SO(4)

This is homomorphic to SU(4)/SU(2)⊗SU(2).

For the spinor representations of SO(6), we can decompose the commutation relations (4.82) as follows:

$$[T_{\mu\nu}, T_{\rho\lambda}] = -\mathrm{i}(\delta_{\nu\rho}T_{\mu\lambda} - \delta_{\mu\rho}T^{\nu\lambda} - \delta_{\nu\lambda}T_{\mu\rho} + \delta_{\mu\lambda}T_{\nu\rho}) \quad (9.150)$$

$$[T_{\mu\nu}, T_{\rho 5}] = -\mathrm{i}(\delta_{\nu\rho}T_{\mu 5} - \delta_{\mu\rho}T_{\nu 5}) \quad (9.151)$$

$$[T_{\mu\nu}, T_{\rho 6}] = -\mathrm{i}(\delta^{\nu\rho}T_{\mu 6} - \delta_{\mu\rho}T_{\nu 6}) \quad (9.152)$$

$$[T_{\mu\nu}, T_{56}] = 0 \quad (9.153)$$

$$[T_{\mu 5}, T_{\nu 5}] = \mathrm{i}T_{\mu\nu} \quad (9.154)$$

$$[T_{\mu 6}, T_{\nu 6}] = \mathrm{i}T_{\mu\nu} \quad (9.155)$$

$$[T_{\mu 5}, T_{\nu 6}] = \mathrm{i}\delta_{\mu\nu}T_{56} \quad (9.156)$$

$$[T_{\mu 5}, T_{56}] = -\mathrm{i}T_{\mu 6} \quad (9.157)$$

$$[T_{\mu 6}, T_{56}] = \mathrm{i}T_{\mu 5} \quad (9.158)$$

The coset space SO(6)/SO(4) is generated by $T_{\mu 5}, T_{\mu 6}, T_{56}$; we can see from the last three commutators that the commutator of two of these generators does not always close onto the subalgebra. Hence SO(6)/SO(4) is not symmetric.

SO(6)/SO(5)

SO(6)/SO(5) is generated by $T_{\mu 6}, T_{56}$. It can be seen from the commutators above that this space is symmetric, but in this case it is also worth looking at the $_\vee$-algebra, (6.100)-(6.101). Note that the $_\vee$-product of any two broken generators is zero. This is equivalent to saying that they all square to the identity and anticommute with each other - precisely the conditions under which projection operators are not required to find the covariant derivatives. Indeed, the projection operators developed in this chapter are based on $\mathbf{M}$ and $\boldsymbol{\theta}$ being generic vectors of SU(4); however, we can see from these $_\vee$-relations that for a general vector $\boldsymbol{\theta} = \theta^{\mu 6}\sigma_{\mu 6} + \theta^{56}\sigma_{56}$ of SO(6)/SO(5) that $\boldsymbol{\theta}_\vee\boldsymbol{\theta} = 0$. Therefore this is a coset space for which there are no generic vectors - every vector of the space is a q-vector.

These two conditions are clearly equivalent:

1) All the coset space generators square to the identity and anticommute with each other

2) Every vector of the coset space is a q-vector

Other examples of subspaces of SO(6) for which this is true are SO(5)/SO(4), SO(4)/SO(3), ... - the fact that every vector of the subspace is a q-vector means that each vector only has two distinct eigenvalues and only one independent invariant. For sigma models based on these coset spaces, then, we would expect to get results akin to those of SU(2)/U(1) - this is precisely what we found for SO(1,4)/SO(1,3), the 'Minkowski version' of SO(5)/SO(4)[2].

SO(6)/SO(4)⊗SO(2)

SO(6)/SO(4)⊗SO(2) is homomorphic to SU(4)/SU(2)⊗SU(2)⊗U(1) and is generated by $T_{\mu 5}, T_{\mu 6}$. It is a symmetric space and the generators do not all anticommute. The algebra spanned by these generators does include generic vectors of SU(4) - for example, the vector

$$\sigma_{35} + \sigma_{36} + \sqrt{2}\sigma_{46} = \begin{pmatrix} 0 & 0 & \sqrt{2}-1+\mathrm{i} & 0 \\ 0 & 0 & 0 & \sqrt{2}+1-\mathrm{i} \\ \sqrt{2}-1-\mathrm{i} & 0 & 0 & 0 \\ 0 & \sqrt{2}+1+\mathrm{i} & 0 & 0 \end{pmatrix}$$

has the eigenvalues $\sqrt{4+2\sqrt{2}}, \sqrt{4-2\sqrt{2}}, -\sqrt{4+2\sqrt{2}}, -\sqrt{4-2\sqrt{2}}.$

[2]However, if we only require that the *action* is invariant under SO(6), we may add one extra term to the Lagrangian for SO(6)/SO(5) which changes by a total derivative[31]. This is not the case for SO(6)/SO(4) or SO(6)/SO(4)⊗SO(2), for which there are no extra terms possible.

For a generic $\boldsymbol{\theta}$, we can define the vectors $\mathbf{x}$, $\mathbf{x}_\vee\mathbf{x}$ and $\mathbf{x}_\vee\mathbf{x}_\vee\mathbf{x}$, as well as the invariant ρ, as in Section 9.6.1; in this case these reduce to the following (with $X, Y = 5, 6$):

$$\mathbf{x} = n^{\mu X}\sigma_{\mu X} \tag{9.159}$$

$$\mathbf{x}_\vee\mathbf{x} = n^{\mu X}n^{\nu Y}\epsilon_{\mu\nu}{}^{\rho\lambda}{}_{XY}\sigma_{\rho\lambda} \tag{9.160}$$

$$\mathbf{x}_\vee\mathbf{x}_\vee\mathbf{x} = 4\mathbf{x} - 8n^{\mu X}n^{\nu Y}n_{\mu Y}\sigma_{\nu X} \tag{9.161}$$

$$\rho = \sqrt{2n^{\mu X}n^{\nu Y}n_{\nu X}n_{\mu Y} - \frac{1}{4}} \tag{9.162}$$

The covariant derivatives are then (from (9.120))

$$\begin{aligned} D_\mu M^{\nu X} \quad &= \quad 2\partial_\mu M^{\rho Y}\Bigg[(q_a >< q^b)^{\nu X}{}_{\rho Y}\frac{\partial\theta''^a}{\partial M''^b} \\ &\qquad\qquad + 2\sum_{S<T}\frac{\sin\left(\frac{\theta'_S-\theta'_T}{2}\right)}{M'_S - M'_T}(P^{ST} + P^{TS})^{\nu X}_{\rho Y}\Bigg] \end{aligned} \tag{9.163}$$

and

$$D_\mu\psi \quad = \quad \partial_\mu\psi + \partial_\mu M^{\nu X}\sum_{S<T}\frac{4}{M'_S - M'_T}\sin^2\left(\frac{\theta'_S - \theta'_T}{4}\right)(P^{ST} - P^{TS})_{\nu X}{}^{\rho\lambda}\sigma_{\rho\lambda} \tag{9.164}$$

Chapter 10

Conclusions

In this chapter we summarise our main findings and take a brief look at potential avenues for further research.

To start with we saw how the transformations of Goldstone bosons could be described by Killing vectors and how the Lagrangian for a non-linear realisation of a Lie group is composed of a mass term for standard fields and terms involving covariant derivatives of standard fields and Goldstone bosons. We saw how the Killing vectors and covariant derivatives are related to the 'coset space representative' L, particularly for 'symmetric spaces'. A trigonometric/hyperbolic form of L proved easy to obtain for the coset spaces SU(2)/U(1) and SO(1,4)/SO(1,3). We were then able to go on and find the Killing vectors and covariant derivatives. For each of these coset spaces, we were also able to rephrase the term involving covariant derivatives of the Goldstone bosons as the contraction of two normal partial derivatives with a metric for the space.

For a general coset space of SU(N), we found that such a form of L was not easy to obtain. Due to the non-trivial nature of the $_V$-algebra, the Lie algebra was partitioned into 'strata' and to obtain L as a linear sum of generators one had to use projection operators based on vectors of one particular stratum (u-vectors). In seeking the covariant derivatives L had to be differentiated, which introduced tensor projection operators belonging to the adjoint representation. By using certain relations for the vectors and tensor operators, we were able to find a general form for $L^{-1}\partial_\mu L$, valid for any SU(N) coset space. For symmetric spaces with 'automorphism conjugate' u-vectors, it was possible to break this into $\mathbf{a}_\mu$ and $\mathbf{v}_\mu$ parts and thus to find the covariant derivatives.

For SU(4) we were able to identify four strata. One of these, the q-stratum, had three orthonormal members in every Cartan subalgebra. This allowed us to rewrite the first term of the expression for $L^{-1}\partial_\mu L$ in a form (9.109) more in keeping with the known results for chiral SU(N) $\otimes$ SU(N)[18]. We were also able to use the homomorphism with SO(6) to write the tensor operators in this expression explicitly as tensor operators of SO(6).

However, for a general coset space of SU(4), although we understood the meaning of the various invariants and tensors in this expression, we did not find expressions for each of them in terms of our original coset space vector $\boldsymbol{\theta}$. It was only when we limited ourselves to coset spaces for which $\gamma_3(\boldsymbol{\theta}) = 0$ for every $\boldsymbol{\theta}$ that we were able to construct u-vectors, and hence projection operators, from $\boldsymbol{\theta}$, as well as finding explicit forms for the invariants. Again, we could use the homomorphism with SO(6) to write the vectors and invariants explicitly as vectors and invariants of SO(6). We saw that the coset space SO(6)/SO(4)$\otimes$SO(2) has only vectors of the form

$$\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$$

and is therefore a symmetric space with $\gamma_3(\boldsymbol{\theta}) = 0$ for every vector. Such coset spaces were seen to have automorphism conjugate u-vectors, so we could split the expression for $L^{-1}\partial_\mu L$ into $\mathbf{a}_\mu$ and $\mathbf{v}_\mu$ parts and thus find the covariant derivatives and the metric.

What we have blatantly not addressed is the Killing vectors of SO(6)/SO(4) $\otimes$SO(2). Had we not gone through all the mechanics of Chapter 8, we might naively assume that once we have found L^2 as a linear sum of the generators, we could simply substitute it into an equation of the form of (2.49) and from there obtaining the Killing vectors would be much the same as for SU(2)/U(1) and SO(1,4)/SO(1,3). However, as we have seen, these coset spaces are much simpler than SO(6)/SO(4)$\otimes$SO(2). This time, the feature that makes the difference is that for SU(2)/U(1) and SO(1,4)/SO(1,3), only the first power of $\mathbf{x}$ appears in the expression for L^2, whereas for SO(6)/SO(4)$\otimes$SO(2), the projection operators in the expansion of L^2 contain $\mathbf{x}$ *and* $\mathbf{x}_\vee\mathbf{x}_\vee\mathbf{x}$. On differentiating $\mathbf{x} = n^a\sigma_a$ for SU(2)/U(1), for example, one gets

$$\frac{\partial \mathbf{x}}{\partial M^b} = \frac{1}{M}(\delta^a_b - n^a n_b)\sigma_a \tag{10.1}$$

so

$$\frac{\partial \mathbf{x}}{\partial M^b} K_3^{\cdot b} = \frac{1}{M}(\sigma_b - n^a\sigma_a n_b) K_3^{\cdot b} \tag{10.2}$$

and the second of these terms vanishes when we take the trace to find $n_b K_3^{\cdot b} = 0$. For SO(6)/SO(4)$\otimes$SO(2), on the other hand, we also have to deal with

$$\frac{\partial(\mathbf{x}_\vee\mathbf{x}_\vee\mathbf{x})}{\partial M^{\mu X}} K^{\cdot\mu X}_{AB}$$

which produces (non-vanishing) contractions of varying numbers of n's with $K^{\cdot\mu X}_{AB}$. Without resorting to projection operators, we know of no way of separating these. Equally, we cannot use the techniques of Barnes, Dondi and Sarkar[18], as their methods of finding the Killing vectors are based on the chiral nature of the coset spaces they are considering. The key to finding these Killing vectors once again appears to be expressing the derivative of L, or in this case L^2, in terms of the projection operators of the adjoint representation. Any results obtained using this technique may, of course, be checked against the metric using the method of Section 5.3.

For a realisation with an even number of Goldstone fields, of course, it is always possible to take pairs of these real fields and combine them into complex ones - create 'complex coordinates' for the field space. For a two-dimensional coset space there is only one way of doing this, while for a higher-dimensional coset space there are many such possible pairings. The partial derivatives of the Goldstone fields can then be expressed in terms of the partial derivatives of the complex fields, with an associated complex metric. For a set of complex fields, A_I, (with I running from 1 to half the dimension of the coset space) the term involving the metric will then fall into three parts, one with $\partial_\mu A^I \partial^\mu A_I$, one with $\partial_\mu A^I \partial^\mu \bar{A}_I$ and one with $\partial_\mu \bar{A}^I \partial^\mu \bar{A}_I$. By placing restrictions on the functions of the invariants of the fields, it is possible to chose a coordinate basis - 'stereographic coordinates' - such that the first and last of these terms vanish, leaving just the $\partial_\mu A^I \partial^\mu \bar{A}_I$ term.

For those interested in supersymmetric sigma models, the resulting complex metric (denoted $h_{I\bar{J}}$) can be very useful. A two-form, known as the Kähler form, can be constructed from it, and Zumino has shown[23] that if the exterior derivative of this two-form is zero (in which case the complex space is said to be a Kähler manifold) a supersymmetric version of this realisation is given simply by replacing the complex fields by chiral superfields. The Lagrangian density in superspace is given by the Kähler potential V, defined by

$$h_{I\bar{J}} = \frac{\partial^2 V}{\partial A^I \partial \bar{A}^J} \tag{10.3}$$

If the original coset space is two-dimensional - for example SU(2)/U(1) - the associated complex space has only one complex dimension and the exterior derivative of any two-form of such a space is trivially zero. It can be shown that for SO(1,4)/SO(1,3) the complex manifold is not a Kähler manifold by explicit calculation. We know that for SU(4)/SU(2)⊗SU(2)⊗U(1)≈SO(6)/SO(4)⊗SO(2), on the other hand, the complex manifold *is* Kähler, not by explicit calculation, but from a theorem of Borel[38]. This theorem states that for a coset space G/H, if H is the centraliser of a direct product of U(1) groups in G, then the complex version of G/H is Kähler. In this case, H is the centraliser of the final U(1), generated by σ_{56}. (This is not the same as saying that the centraliser of H in G is a direct product of U(1) groups, as can be seen with the example $SU(4)/SU(2)_R \otimes SU(2)_L$.) It would, however, be useful to show explicitly that this complex space is Kähler. (One problem that may arise is choosing which fields to combine into complex coordinates. The natural choices in SU(4) notation are somewhat different from the natural choices in SO(6) notation.)

If one can obtain the Kähler form for this complex space and show that it satisfies the Kähler condition, one can then test whether it satisfies the more stringent 'hyperKähler' conditions - if a realisation is hyperKähler, it will admit an extended (N=2) supersymmetry[24, 25, 26][1].

It is worth noting that the group SO(6) naturally occurs in supersymmetric field theories: it is the R-symmetry group of N=4 conformally-invariant field theories[39]. This fact has been made use of by Maldacena[40], who has conjectured that Type IIB string theory on an $AdS_5 \times S^5$ background (which has isometry group SO(2,4)⊗SO(6)) is dual to an N=4 conformally-invariant field

[1]All of this assumes that the fields inhabit normal four-dimensional space-time. If the theory is based on two-dimensional space-time, a Kähler manifold will admit N=2 supersymmetry, while a hyperKähler manifold will admit N=4 supersymmetry. There has been a lot of interest over the years in two-dimensional sigma models and the field of research has close connections with string theory. This provides yet another avenue for further research into the realisations described in this thesis.

theory in Minkowski spacetime. From the point of view of the symmetries of the two theories, this is quite plausible, as the field theory, as well as being invariant under the SO(6) R-symmetry is invariant under the conformal group in Minkowski space, which is homomorphic to the group SO(2,4). Neither of these symmetries are observed in nature, so for this conjecture to have any physical relevance, these symmetries must be broken at low energies[2]. We have dealt with broken SO(6) symmetries in this thesis. Is it possible to adapt the methods we have used to the case of SO(2,4)? Judging by the similarity of SO(1,4)/SO(1,3) to SO(5)/SO(4) one would think so. However, in this case we have to extend the methods for SO(6) to non-unitary matrices, which is considerably more difficult. For SO(6), the action of the group partitioned the algebra into orbits, with, for example, all the generators in the same orbit. For SO(2,4) this does not appear to be the case, as can be seen even looking at the action under a non-compact one-parameter subgroup. (Elements of such a subgroup involve hyperbolic rather than trigonometric functions of the group parameter.) However, this did not prevent us from obtaining results for SO(1,4)/SO(1,3). It remains to be seen whether this is a pathological case because of the algebra being composed entirely of (non-hermitian versions of) q-vectors, or whether our method of analysing this coset space contains hints as to how to deal with a generic case such as SO(2,4)/SO(4)⊗SO(2) or SO(2,4)/SO(1,3)⊗SO(1,1).

Even if we could adapt our methods to such cases, the realisations we are analysing would not represent broken global conformal symmetry. This is because the symmetries we are considering are internal symmetries of a set of fields which take values on a completely separate and unchanging spacetime, whereas

[2]Non-linear realisations have also been considered in the context of supergravity. A solution of the 11-dimensional supergravity equations is a space of the form $AdS_5 \times M^7$ where M^7 is a 7-dimensional Einstein space. The non-linear realisations forming 7-dimensional Einstein spaces have been catalogued[41] and the geometric properties of some of them have been studied in depth[42]. These also have connections with the AdS/CFT correspondence[43].

conformal symmetry is a symmetry of the underlying spacetime itself. Non-linear realisations of this spacetime symmetry have been studied, though it is more common to consider the breaking to the Poincaré group for obvious reasons[44, 45]. (The conformal group is composed of boosts, rotations, translations, special conformal transformations and dilatations. SO(2,4)/SO(1,3)$\otimes$SO(1,1) therefore contains translations and special conformal transformations while SO(2,4)/$\mathcal{P}$, where $\mathcal{P}$ is the Poincaré group, comprises special conformal transformations and dilatations.) If it were possible to adapt our methods to the case of SO(2,4), it would be interesting to compare our results with those from the methods of these papers.

Finally, while this thesis has focused on breaking global symmetries, it is in principle possible to consider gauged versions of these symmetries being broken. Again, non-linear realisations of local symmetries were first considered by Coleman, Callan, Wess and Zumino[11] and their work was expanded upon by Salam and Strathdee[15]. It may also be helpful to note in such a study that unbroken gauged SU(4) symmetry would look like a four-colour version of QCD, while linear representations of gauged conformal symmetry have been analysed by Kaku, Townsend and Van Nieuwenhuizen[46].

However, for groups with non-zero symmetric structure constants, such as SU(4), one must be aware of what happens at the quantum level. Just as for global symmetries of this kind there may be extra terms in the Lagrangian (see footnote in 9.6.2), for the gauge theories processes involving the gauge fields which break G-invariance are possible[47]. These anomalies have been studied recently for particular classes of realisation which include the realisations of SO(6) we focussed on (SO(6)/SO(5) and SO(6)/SO(4) as 'anomaly free embeddings' of H and SO(6)/SO(4)$\otimes$SO(2) as a symmetric space)[48]. For these realisations, it is possible to preserve local H-invariance by adding extra terms to the action and to obtain the true effective action from the expression for the anomaly.

Appendix 1

Consider a unit r-vector of SU(3):

$$(\mathbf{r}, \mathbf{r}) = 1 \tag{10.4}$$

In the basis of the Gell-Mann λ's, it may be written

$$\mathbf{r} = r^I \lambda_I \tag{10.5}$$

so that the normalisation condition is

$$(\mathbf{r}, \mathbf{r}) = r^I r^J (\lambda_I, \lambda_J) = r^I r_I = 1 \tag{10.6}$$

It has an associated q-vector:

$$\begin{aligned}
\mathbf{q}_r &= \mathbf{r}_\vee \mathbf{r} && (10.7)\\
&= \sqrt{3}\mathbf{r}^2 - \frac{1}{\sqrt{3}}\mathbf{1}\,\mathrm{tr}\,\mathbf{r}^2 && (10.8)\\
&= \sqrt{3} r^I r^J \lambda_I \lambda_J - \frac{2}{\sqrt{3}}\mathbf{1}(\mathbf{r}, \mathbf{r}) && (10.9)\\
&= \sqrt{3} r^I r^J (\frac{2}{3}\delta_{IJ}\mathbf{1} + d_{IJ}{}^K \lambda_K) - \frac{2}{\sqrt{3}}\mathbf{1} && (10.10)\\
&= \sqrt{3} r^I r^J d_{IJ}{}^K \lambda_K && (10.11)
\end{aligned}$$

If we also write $\mathbf{q}_r$ in terms of components:

$$\mathbf{q} = q^K \lambda_K \tag{10.12}$$

we see that we get the result

$$q^K = \sqrt{3} r^I r^J d_{IJ}{}^K \tag{10.13}$$

The relation

$$\mathbf{q}_\vee \mathbf{q} = -\mathbf{q} \tag{10.14}$$

in terms of components looks like

$$\begin{aligned} \sqrt{3} q^I q^J (\frac{2}{3}\delta_{IJ}\mathbf{1} + d_{IJ}{}^K \lambda_K) - \frac{1}{\sqrt{3}} q^I q^J \mathbf{1}\,\mathrm{tr}(\frac{2}{3}\delta_{IJ}\mathbf{1} + d_{IJ}{}^K \lambda_K) &= -q^K \lambda_K \\ \Rightarrow q^K = -\sqrt{3} q^I q^J d_{IJ}{}^K & \qquad (10.15) \end{aligned}$$

Finally, Michel and Radicati give a third relation between an r-vector and its associated q-vector which we have not put in the main body of the thesis:

$$\mathbf{q}_{r\vee} \mathbf{r} = \mathbf{r}_\vee \mathbf{q}_r = \mathbf{r} \tag{10.16}$$

In terms of components, this looks like

$$\begin{aligned} \sqrt{3} q^I r^J (\frac{2}{3}\delta_{IJ}\mathbf{1} + d_{IJ}{}^K \lambda_K) - \frac{1}{\sqrt{3}} q^I r^J \mathbf{1}\,\mathrm{tr}(\frac{2}{3}\delta_{IJ}\mathbf{1} + d_{IJ}{}^K \lambda_K) &= r^K \lambda_K \qquad (10.17) \\ \Rightarrow \sqrt{3} q^I r^J d_{IJ}{}^K \lambda_K &= r^K \lambda_K \qquad (10.18) \\ \Rightarrow r^K = \sqrt{3} q^I r^J d_{IJ}{}^K & \qquad (10.19) \end{aligned}$$

Note that from (10.8) we have

$$(\mathbf{r}, \mathbf{q}_r) = \frac{1}{2}\,\mathrm{tr}(\mathbf{r}\mathbf{q}_r) = \frac{1}{2}\,\mathrm{tr}\left(\sqrt{3}\mathbf{r}^3 - \frac{1}{\sqrt{3}}\mathbf{r}\,\mathrm{tr}\,\mathbf{r}^2\right) \tag{10.20}$$

but $\mathrm{tr}\,\mathbf{r} = 0$ and $\mathrm{tr}\,\mathbf{r}^3 = 3\gamma_3(\mathbf{r}) = 0$

so

$$(\mathbf{r}, \mathbf{q}_r) = 0 \tag{10.21}$$

Appendix 2

10.1 Symmetric combinations

The first thing we want to derive in this appendix is an expression for the trace of a product of three λ's. We start by writing this product as

$$\lambda_I \lambda_K \lambda^J = \frac{1}{2}\{\lambda_I \lambda_K, \lambda^J\} + \frac{1}{2}[\lambda_I \lambda_K, \lambda^J] \qquad (10.22)$$

Noting that the trace of a commutator is zero, we therefore have

$$\begin{aligned} \mathrm{tr}(\lambda_I \lambda_K \lambda^J) &= \frac{1}{2}\,\mathrm{tr}\{\lambda_I \lambda_K, \lambda^J\} && (10.23) \\ &= \frac{1}{N}\delta_{IK}\,\mathrm{tr}\{\mathbf{1}, \lambda^J\} + \frac{1}{2} d_{IK}{}^L\,\mathrm{tr}\{\lambda_L, \lambda^J\} + \frac{\mathrm{i}}{2} f_{IK}{}^L\,\mathrm{tr}\{\lambda_L, \lambda^J\} \\ &= 2d_{IK}{}^J + 2\mathrm{i} f_{IK}{}^J && (10.24) \end{aligned}$$

where we have used the product of two λ's, (6.4).

This has several consequences which are of use to us. The first is that if we contract with a vector x^K and symmetrise, using the definition of d_x we have

$$\mathrm{tr}(\lambda_I \mathbf{x} \lambda^J) + \mathrm{tr}(\lambda^J \mathbf{x} \lambda_I) = \frac{4}{\sqrt{N}}(d_x)_I^J \qquad (10.25)$$

The next is that we can find $\mathrm{tr}(\lambda_I\{\mathbf{x},\mathbf{y}\}\lambda^J)$:

$$\begin{aligned}
\{\mathbf{x},\mathbf{y}\} &= \frac{4}{N}\mathbf{1}(\mathbf{x},\mathbf{y}) + \frac{2}{\sqrt{N}}\mathbf{x}_\vee\mathbf{y} && (10.26)\\
\Rightarrow \mathrm{tr}(\lambda_I\{\mathbf{x},\mathbf{y}\}\lambda^J) &= \frac{4}{N}(\mathbf{x},\mathbf{y})\,\mathrm{tr}(\lambda_I\lambda^J) + \frac{2}{\sqrt{N}}(\mathbf{x}_\vee\mathbf{y})^K\,\mathrm{tr}(\lambda_I\lambda_K\lambda^J) && (10.27)\\
&= \frac{8}{N}(\mathbf{x},\mathbf{y})\delta_I^J + \frac{4}{N}(d_{x\vee y})_I^J + \frac{4\mathrm{i}}{\sqrt{N}}(f_{x\vee y})_I{}^J && (10.28)
\end{aligned}$$

Thirdly,

$$\begin{aligned}
[\lambda_L,\lambda_I] &= 2\mathrm{i}f_{LI}{}^M\lambda_M && (10.29)\\
\Rightarrow [\mathbf{x},\lambda_I] &= 2\mathrm{i}x^L f_{LI}{}^M\lambda_M && (10.30)\\
\Rightarrow \mathrm{tr}([\mathbf{x},\lambda_I]\mathbf{y}\lambda^J) &= 2\mathrm{i}x^L y^K f_{LI}{}^M\,\mathrm{tr}(\lambda_M\lambda_K\lambda^J) && (10.31)\\
&= 2\mathrm{i}x^L y^K f_{LI}{}^M(2d_{MK}{}^J + 2\mathrm{i}f_{MK}{}^J) && (10.32)\\
&= 4(f_x f_y)_I{}^J - \frac{4\mathrm{i}}{\sqrt{N}}(f_x d_y)_I{}^J && (10.33)
\end{aligned}$$

We can use these results to find $\mathrm{tr}(\mathbf{x}\lambda_I\mathbf{y}\lambda^J)+\mathrm{tr}(\mathbf{y}\lambda_I\mathbf{x}\lambda^J)$ which is vital in finding a general form for symmetric combinations of adjoint representation projection operators (see Section 7.3):

$$\begin{aligned}
\mathrm{tr}(\mathbf{x}\lambda_I\mathbf{y}\lambda^J)+\mathrm{tr}(\mathbf{y}\lambda_I\mathbf{x}\lambda^J) &= \mathrm{tr}([\mathbf{x},\lambda_I]\mathbf{y}\lambda^J) + \mathrm{tr}(\lambda_I\mathbf{x}\mathbf{y}\lambda^J) + \mathrm{tr}([\mathbf{y},\lambda_I]\mathbf{x}\lambda^J)\\
&\quad + \mathrm{tr}(\lambda_I\mathbf{y}\mathbf{x}\lambda^J) && (10.34)\\
&= \mathrm{tr}([\mathbf{x},\lambda_I]\mathbf{y}\lambda^J) + \mathrm{tr}([\mathbf{y},\lambda_I]\mathbf{x}\lambda^J) + \mathrm{tr}(\lambda_I\{\mathbf{x},\mathbf{y}\}\lambda^J)\\
&= \frac{8}{N}(\mathbf{x},\mathbf{y})\delta_I^J + \frac{4}{N}(d_{x\vee y})_I^J + \frac{4\mathrm{i}}{\sqrt{N}}(f_{x\vee y})_I{}^J\\
&\quad +4\{f_x,f_y\}_I{}^J - \frac{4\mathrm{i}}{\sqrt{N}}(f_x d_y + f_y d_x)_I{}^J && (10.35)
\end{aligned}$$

Substituting in the identity (7.18) this reduces to

$$\mathrm{tr}(\mathbf{x}\lambda_I\mathbf{y}\lambda^J)+\mathrm{tr}(\mathbf{y}\lambda_I\mathbf{x}\lambda^J) = \frac{8}{N}(\mathbf{x},\mathbf{y})\delta_I^J + \frac{4}{N}(d_{x\vee y})_I^J + 4\{f_x,f_y\}_I{}^J \qquad (10.36)$$

Note that the left-hand side is symmetric under the interchange of I and J, therefore the right-hand side must also be symmetric on these indices; this means that $\{f_x, f_y\}_I{}^J = \{f_x, f_y\}^J_I$.

10.2 Antisymmetric combinations

Similarly,

$$\begin{aligned}
\mathrm{tr}(\mathbf{x}\lambda_I\mathbf{y}\lambda^J) - \mathrm{tr}(\mathbf{y}\lambda_I\mathbf{x}\lambda^J) &= \mathrm{tr}([\mathbf{x},\lambda_I]\mathbf{y}\lambda^J) + \mathrm{tr}(\lambda_I\mathbf{x}\mathbf{y}\lambda^J) - \mathrm{tr}([\mathbf{y},\lambda_I]\mathbf{x}\lambda^J) \\
&\qquad - \mathrm{tr}(\lambda_I\mathbf{y}\mathbf{x}\lambda^J) \quad (10.37) \\
&= \mathrm{tr}([\mathbf{x},\lambda_I]\mathbf{y}\lambda^J) - \mathrm{tr}([\mathbf{y},\lambda_I]\mathbf{x}\lambda^J) + \mathrm{tr}(\lambda_I[\mathbf{x},\mathbf{y}]\lambda^J) \\
&= 4[f_x, f_y]_I{}^J - \frac{4\mathrm{i}}{\sqrt{N}}(f_x d_y - f_y d_x)_I{}^J + \mathrm{tr}(\lambda_I[\mathbf{x},\mathbf{y}]\lambda^J) \\
&\qquad (10.38)
\end{aligned}$$

Finally, if we contract (10.24) with a vector x^K and antisymmetrise, we get

$$\mathrm{tr}(\lambda_I\mathbf{x}\lambda^J) - \mathrm{tr}(\lambda^J\mathbf{x}\lambda_I) = 4\mathrm{i}(f_x)_I{}^J \qquad (10.39)$$

10.3 Simplifying P^{ST} - P^{TS}

Taking $\mathbf{x} = \mathbf{u}^S, \mathbf{y} = \mathbf{u}^T$ in (7.20) and postmultiplying by f_{u^T} we get

$$d_{u^S} d_{u^T} f_{u^T} - N f_{u^T} f_{u^S} f_{u^T} = d_{u^S \vee u^T} f_{u^T} + 2(\mathbf{u}^S, \mathbf{u}^T) f_{u^T} \qquad (10.40)$$

Now use (7.19) on the first term and the commutativity of the f_u's on the second

$$\frac{1}{2}d_{u^S}f_{u^T\vee u^T} - Nf_{u^S}f_{u^T}^2 = d_{u^S\vee u^T}f_{u^T} + 2(\mathbf{u}^S, \mathbf{u}^T)f_{u^T} \tag{10.41}$$

and then the u-vector properties (7.28), (7.29) and (7.31)

$$(\frac{\sqrt{N}}{2} - \frac{1}{\sqrt{N}})d_{u^S}f_{u^T} - Nf_{u^S}f_{u^T}^2 = -\frac{1}{\sqrt{N}}(d_{u^S}f_{u^T} + d_{u^T}f_{u^T}) - \frac{1}{N}f_{u^T} \tag{10.42}$$

Repeating the process for the $d_{u^T}f_{u^T}$ on the right gives us

$$\frac{\sqrt{N}}{2}d_{u^S}f_{u^T} = Nf_{u^S}f_{u^T}^2 - \frac{1}{2}f_{u^T} \tag{10.43}$$

Finally, noting from (7.17) that $d_{u^S}f_{u^T} = f_{u^T}d_{u^S}$, we obtain

$$f_{u^T}d_{u^S} = 2\sqrt{N}f_{u^S}f_{u^T}^2 - \frac{1}{\sqrt{N}}f_{u^T} \tag{10.44}$$

Appendix 3

Let X and Y be two 4×4 hermitian matrices:

$$X = \alpha \mathbf{1} + a^{KL}\sigma_{KL} \qquad Y = \beta \mathbf{1} + b^{MN}\sigma_{MN} \tag{10.45}$$

Then, using our scalar product of two σ's,

$$\begin{aligned}
\operatorname{tr}(\sigma^{IJ}X)\operatorname{tr}(\sigma_{IJ}Y) &= (\alpha \operatorname{tr}\sigma^{IJ} + 2a^{KL}(\sigma^{IJ},\sigma_{KL}))(\beta \operatorname{tr}\sigma_{IJ} + 2b^{MN}(\sigma_{IJ},\sigma_{MN})) \\
&= (0 + 8a^{IJ})(0 + 8b_{IJ}) \qquad (10.46) \\
&= 64a^{IJ}b_{IJ} \qquad (10.47)
\end{aligned}$$

However, we also have

$$\begin{aligned}
\operatorname{tr}(XY) &= \alpha\beta \operatorname{tr}\mathbf{1} + a^{IJ}b^{KL}\operatorname{tr}(\sigma_{IJ}\sigma_{KL}) \qquad (10.48) \\
&= 4\alpha\beta + 2a^{IJ}b^{KL}(\sigma_{IJ},\sigma_{KL}) \qquad (10.49) \\
&= 4\alpha\beta + 8a^{IJ}b_{IJ} \qquad (10.50)
\end{aligned}$$

and

$$\operatorname{tr}X\operatorname{tr}Y = \alpha \operatorname{tr}\mathbf{1}\beta \operatorname{tr}\mathbf{1} = 16\alpha\beta \tag{10.51}$$

Thus for any hermitian X and Y,

$$\operatorname{tr}(\sigma^{IJ}X)\operatorname{tr}(\sigma_{IJ}Y) = 8\operatorname{tr}(XY) - 2\operatorname{tr}X\operatorname{tr}Y \tag{10.52}$$

References

[1] E. Wigner, Ann. Math. **40** (1939) 149

[2] C. N. Yang and R. L. Mills, Phys. Rev. **96** (1954) 191

[3] Ryoyu Utiyama, Phys. Rev. **101** (1955) 1597

[4] J. Goldstone, Nuovo Cimento **Vol XIX** (1961)154

[5] Jeffrey Goldstone, Abdus Salam and Steven Weinberg, Phys. Rev. **127** (1962) 965

[6] Walter Gilbert, Phys. Rev. Lett **12** (1964) 713

[7] P. W. Higgs, Phys. Lett. **12** (1964) 132; Phys. Rev. Lett. **13** (1964) 508; Phys. Rev. **145** (1966) 1156

[8] M. Gell-Mann and M. Lévy, Nuov. Cim. **Vol XVI** (1960) 705

[9] See, for example, J. Schwinger, Phys. Lett. **24B** (1967) 473; Jeremiah A. Cronin, Phys. Rev. **161** (1967) 1483; Benjamin W. Lee and H. T. Nieh, Phys. Rev. **166** (1968), 1507; Steven Weinberg Phys. Rev. **166** (1968) 1507

[10] S. Coleman, J. Wess and Bruno Zumino, Phys. Rev. **177** (1969) 2239

[11] C. G. Callan, S. Coleman, J. Wess and Bruno Zumino, Phys. Rev. **177** (1969) 2247

[12] C. J. Isham, Nuov. Cim. A **61** (1969) 188

[13] Kurt Meetz, J. Math. Phys. **10** (1969) 589

[14] David G. Boulware and Lowell S. Brown, Ann. Phys. **138** (1982) 392

[15] Abdus Salam and J. Strathdee, Phys. Rev. **184** (1969) 1750

[16] J. F. Cornwell, *Group Theory in Physics*, Academic Press (1984)

[17] J. Honerkamp, Nucl. Phys. B **12** (1969) 227

[18] K. J. Barnes, P. H. Dondi and S.C. Sarkar, Proc. R. Soc. A **330** (1972) 389

[19] K. J. Barnes, J. M. Generowicz and P. J. Grimshare J. Phys. A **29** (1996) 4457

[20] K. J. Barnes, P. H. Dondi and S.C. Sarkar, J. Phys. A **5** (1972) 555

[21] K. J. Barnes and R. Delbourgo, J. Phys. A **5** (1972) 1043

[22] K. J. Barnes, J. Phys. A **5** (1972) 830

[23] B. Zumino, Phys. Lett. **87B** (1979) 203

[24] Luis Alvarez-Gaumé and Daniel Freedman, Commun. Math. Phys. **80** (1981) 443

[25] U. Lindström and M. Roček, Nucl. Phys. B **222** (1983) 285

[26] Martin Roček, Physica D **15** (1985) 75

[27] Louis Michel and Luigi A. Radicati, Ann. Inst. Henri Poincaré **Vol XVIII** (1973) 185

[28] Lewis H. Ryder, *Quantum Field Theory*, Cambridge University Press (1996)

[29] Y.-S. Wu, Phys. Lett. **153B** (1985) 70

[30] C. M. Hull and B. Spence, Nucl. Phys. **B353** (1991) 379

[31] Eric D'Hoker and Steven Weinberg, Preprint hep-ph/9409402

[32] S. P. Rosen, J. Math. Phys. **12** (1971) 673

[33] L. O'Raifeartaigh, Rep. Pr. Phys. **42** (1979) 159

[34] D. Amati, H. Bacry, J. Nuyts and J. Prentki, Nuovo Cimento 34, (1964), 1732-1750.

[35] A. J. MacFarlane, A. Sudbery and P. H. Weisz, Comm. Math. Phys. **11** (1968) 77

[36] Adam M. Bincer, J. Math. Phys. **31** (1990) 563

[37] J. D. Hamilton-Charlton, PhD Thesis, in preparation

[38] A. Borel, Proc. Natl. Acad. Sci. USA **40** (1954) 1147

[39] Rudolf Haag, Jan T. Łopuszański and Martin Sohnius, Nucl. Phys. B **88** (1975) 257

[40] Juan Maldacena, Adv. Theor. Math. Phys. **2** (1998) 231

[41] L. Castellani, L. J. Romans and N. P. Warner, Nucl. Phys. **B241** (1984) 429

[42] L. Castellani, Preprint hep-th/9912277

[43] L. Castellani, A. Ceresole, R. D'Auria, S. Ferrara, P. Fre' and M. Trigiante, Nucl. Phys. **B527** (1998) 142

[44] Abdus Salam and J. Strathdee, Phys. Rev. **184** (1969) 1760

[45] Alex Hankey, Phys. Rev. D **3** (1970) 2543

[46] M. Kaku, P. K. Townsend and P. Van Nieuwenhuizen, Phys. Lett. **69B** (1977) 304

[47] J. Wess and B. Zumino, Phys. Lett. **37B** (1971) 95

[48] Chong-Sun Chu, Pei-Ming Ho and Bruno Zumino, Preprint hep-th/9602093