SSSU 97
ISSN 0140 3818

UNIVERSITY OF SOUTHAMPTON

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FACULTY OF ENGINEERING AND APPLIED SCIENCE

MICROMECHANICS APPROACHES TO PREDICT DAMAGE IN FRP COMPOSITE LAMINATES

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Ship Science Report No. 97
June 1996

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1. INTRODUCTION

Widespread acceptance of composite structures requires confidence in their load-carrying capacity. For this reason it is necessary to be able to predict accurately the strength of a particular composite. Most available strength information is based on uniaxial stress states, though practical applications involve at least biaxial loading. Unlike isotropic materials, the strength of composites is directionally dependent. Furthermore, failure in some of the plies (constituting a laminated composite) need not necessarily mean rupture of the total laminate, since the multiplicity of layers provide alternate load paths.

This report briefly reviews the more commonly employed failure criteria. Sufficient theory is included to make the report self-contained. Because strength theories are conceived primarily to predict onset (and not mode) of failure, the macroscopic viewpoint is predominant. Finally, in view of large structures requiring to be analysed for progressive failure, an outline is given of an approach which could be adapted for practical analysis.

2. ANALYTICAL BACKGROUND

2.1 Orthotropic Elasticity

While several of the strength criteria do not in themselves address whether the material is elastic or inelastic, composite lamination theory does involve constitutive response. Usually this is linear elastic, orthotropic behaviour. The generalised Hooke's law can be written as (SHENOI and WELLICOME (1993)):

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{1}$$

where σ_{ij} is the stress tensor, ε_{ij} is the strain tensor and C_{ijkl} is the 4th order stiffness or material tensor. Symmetry reduces the number of independent stiffness components to 21 for even the most general, anisotropic material in three dimensions. If the response is orthotropic such that 1-, 2- and 3-directions are the three axes of material symmetry, then there are only nine independent material constants.

The plies of most structural components are subjected to plane stress. The constitutive relationship for an orthotropic lamina in a state of plane stress ($\sigma_3 = \tau_{13} = \tau_{23} = 0$) may be written as:

$$\left\{\sigma_{ij}\right\} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}_k = \begin{bmatrix} Q_{11} & .. & .. \\ Q_{12} & Q_{22} & .. \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}_k \tag{2}$$

where the components of the reduced stiffness matrix Q_{ij} are

$$\begin{aligned} Q_{11} &= E_{11}/(1-\nu_{12}\nu_{21}) \\ Q_{22} &= E_{22}/(1-\nu_{12}\nu_{21}) \\ Q_{12} &= \nu_{12}E_{11}/(1-\nu_{12}\nu_{21}) \\ Q_{66} &= G_{12} \\ Q_{16} &= Q_{26} = 0 \\ \nu_{12}E_{22} &= \nu_{21}E_{11} \end{aligned} \tag{3}$$

E_{11}, E_{22}, ν_{12} and G_{12} are the four independent elastic constants of the lamina with respect to the axes (1-2) of material symmetry. Transforming Eqn. 2 into the orthotropic laminate axes system results in

$$\left\{\sigma_{xy}\right\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k = \begin{bmatrix} \overline{Q}_{11} & .. & .. \\ \overline{Q}_{12} & \overline{Q}_{22} & .. \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_k \tag{4}$$

where $\overline{Q}_{ij}$ are the transformed reduced stiffnesses given by CHRISTENSEN (1979) as

$$\begin{aligned}
\overline{Q}_{11} &= U_1 + U_2\cos(2\theta) + U_3\cos(4\theta) \\
\overline{Q}_{22} &= U_1 - U_2\cos(2\theta) + U_3\cos(4\theta) \\
\overline{Q}_{12} &= U_4 - U_3\cos(4\theta) \\
\overline{Q}_{66} &= U_5 - U_3\cos(4\theta) \\
\overline{Q}_{16} &= -\tfrac{1}{2}U_2\sin(2\theta) - U_3\sin(4\theta) \\
\overline{Q}_{26} &= -\tfrac{1}{2}U_2\sin(2\theta) + U_3\sin(4\theta)
\end{aligned} \tag{5}$$

and the elastic constants are

$$\begin{aligned}
U_1 &= \tfrac{1}{8}(3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66} \\
U_2 &= \tfrac{1}{2}(Q_{11} - Q_{22}) \\
U_3 &= \tfrac{1}{8}(Q_{11} + Q_{22} - 2Q_{12} - 4Q_{16}) \\
U_4 &= \tfrac{1}{8}(Q_{11} + 2Q_{22} + 6Q_{12} - 4Q_{16}) \\
U_5 &= \tfrac{1}{8}(Q_{11} + Q_{22} - 2Q_{12} + 4Q_{66})
\end{aligned} \tag{6}$$

Small k represents the kth layer of the laminate and θ is the angle measured counter-clockwise from the laminate positive x-axis of the lamina. Eqn. 2 can be inverted to yield the strains ε as a function of material compliance S and the stress σ.

$$\{\varepsilon_{ij}\} = \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}_k = \begin{bmatrix} S_{11} & .. & .. \\ S_{12} & S_{22} & .. \\ S_{16} & S_{26} & S_{66} \end{bmatrix}_k \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}_k \tag{7}$$

where the lamina compliance matrix is S_{ij} is given by

$$[S] = [Q]^{-1} \tag{8}$$

and

$$\begin{aligned}
S_{11} &= 1/E_{11} \\
S_{22} &= 1/E_{22} \\
S_{12} &= -\nu_{12}/E_{11} = -\nu_{21}/E_{22} \\
S_{66} &= 1/G_{12}
\end{aligned} \tag{9}$$

Similarly, Eqn. 7 can be transformed to the principal material axes x-y of the laminate to become

$$\{\varepsilon_{ij}\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_k = \begin{bmatrix} \bar{S}_{11} & .. & .. \\ \bar{S}_{12} & \bar{S}_{22} & .. \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix}_k \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k \qquad (10)$$

where, like the transformed stiffnesses, the transformed compliances $\overline{S}_{ij}$ depend only on E_{11}, E_{22}, G_{12}, ν_{12} and θ. Stresses and strains can be converted from lamina (local) 1-2 axes to the laminate (global) x-y axes through the use of the transformation matrix T as below

$$\{\sigma_{12}\} = [T]\{\sigma_{xy}\} and \{\varepsilon_{12}\} = [T]\{\varepsilon_{xy}\} \qquad (11)$$

and

$$\{\sigma_{xy}\} = [T]^{-1}\{\sigma_{12}\} and \{\varepsilon_{xy}\} = [T]^{-1}\{\varepsilon_{12}\} \qquad (12)$$

where

$$[T] = \begin{bmatrix} \cos^2\theta & .. & .. \\ \sin^2\theta & \cos^2\theta & .. \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \qquad (13)$$

2.2 Lamination Theory

From classical plate theory, the laminate strains ε_{xy} at a distance z from the mid-plane are given by

$$\{\varepsilon\} = \{\varepsilon^0\} + z\{\kappa\} \qquad (14)$$

where ε^0 represents the mid-plane (or membrane) strains and κ represents the plate curvatures. Substituting Eqn. 14 into Eqn. 4 and integrating through the thickness h of the plate yields the following expression

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix} \qquad (15)$$

where the extensional, coupling and bending stiffnesses matrices are given by

$$\left(A_{ij}, B_{ij}, D_{ij}\right) = \int_{-h/2}^{h/2} \overline{Q}_{ij}\left(1, z, z^2\right) dz \tag{16}$$

N and M are the resultant force and moment vectors respectively, acting per unit length of the laminate, i.e.

$$\left(N_x, N_y, N_{xy}; M_x, M_y, M_{xy}\right) = \int \left(\sigma_x^k, \sigma_y^k, \tau_{xy}^k; z\sigma_x^k, z\sigma_y^k, z\tau_{xy}^k\right) dz \tag{17}$$

In general, the matrices A_{ij}, B_{ij}, D_{ij} are all complete and symmetric as defined in Eqn. 16. If the laminate lay-up is symmetrical in geometry and the material properties about z = 0, then $B_{ij} = 0$ and the bend-stretch coupling matrix vanishes. Stiffness components A_{11}, A_{22}, A_{12}, A_{66}, D_{11}, D_{22}, D_{12}, and D_{66} are positive definite. A_{16}, A_{26}, D_{16} and D_{26} can be made equal to zero for laminates made up entirely of plies at 0- or 90-degrees to the laminate axes. For angle ply laminates ($\pm\theta$) fabricated from a large number of alternating lamina, D_{16}, D_{26}, A_{16} and A_{26} are quite small compared to the other stiffness components.

Eqns. 22 can be inverted to yield the following

$$\begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{Bmatrix} N \\ M \end{Bmatrix} \tag{18}$$

where

$$\begin{aligned} A' &= A^* - B^*.D^{*-1}.C^* \\ B' &= B^*.D^{*-1} \\ C' &= D^{*-1}.C^* \\ D' &= D^{*-1} \end{aligned} \tag{19}$$

and

$$\begin{aligned} A^* &= A^{-1} \\ B^* &= A^{-1}.B \\ C^* &= B.A^{-1} \\ D^* &= D - B.A^{-1}.B \end{aligned} \tag{20}$$

For any laminate of known lamina elastic properties (E_{11}, E_{22}, ν_{12} and G_{12}) and subjected to forces N and moments M, the strains in any ply relative to the laminate axes can be calculated from Eqns. 18. From the strains and knowing the stiffness properties of the laminate, the stresses too can be determined. Thus having evaluated the stress and strain history in any ply of the laminate, imminence of lamina failure can be determined from one of several possible criteria. Some of these are described in the next section.

3. FAILURE CRITERIA

3.1 General Comments

With macroscopically homogeneous but orthotropic materials, development of a strength theory has often involved extending one of the isotropic analyses to account for orthotropy. It may be reasonable with homogeneous materials to base structure strength on the initial combination of loads which causes the postulated failure strength envelope or criterion to be reached. For laminates, a criterion is typically applied on a ply-by-ply basis and the load carrying capability of the entire composite is predicted using lamination theory outlined in Section 2. A laminate is sometimes assumed to fail analytically when the strength criterion of any one of its laminae is reached. While load distribution usually occurs within a laminate when one of the plies fails, this need not imply total failure of the laminate.

Several of the more commonly used composite strength theories are outlined below. Although some of the failure theories were originally postulated in three-dimensions, most laminates are subjected to two-dimensional stress; so only those forms of criteria are presented here. In addition to the specific references mentioned below, various anisotropic failure theories are reviewed in SENDECKYJ (1972), VICARO and TOLAND (1975) and SIH and SKUDRA (1985).

3.2 Maximum Strain Theory

This criterion states that a ply of a laminate has failed when either its longitudinal, transverse or shear strain reaches a limiting value determined from simple one-dimensional, uniaxial stress experiments. The minimum common envelope of the superposition of the interaction failure diagrams of all the individual plies related to the principal material axes of the laminate becomes the failure diagram of the laminate. From Eqn. .... and with the strains equalling experimentally determined limiting values ε_{1U} and ε_{2U}, the maximum strain criterion becomes

$$\begin{Bmatrix} \varepsilon_{1U} \\ \varepsilon_{2U} \end{Bmatrix} = \begin{bmatrix} S_{11} & .. \\ S_{12} & S_{22} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \end{Bmatrix} \tag{21}$$

for the case of $\tau_{12} = 0$. Upon re-arranging,

$$\sigma_2 = \frac{\varepsilon_{1U}}{S_{12}} - \frac{S_{11}}{S_{12}}\sigma_1 \tag{22}$$

$$\sigma_2 = \frac{\varepsilon_{2U}}{S_{22}} - \frac{S_{12}}{S_{22}}\sigma_1$$

Eqns. 22 represent two straight lines in the σ_1-σ_2 co-ordinate space system which define the failure of an orthotropic lamina. Utilisation of the limiting strains in tension and compression results in this failure envelope consisting of two straight lines in each quadrant. Superposition of the laminate failure envelope for varying values of τ_{xy} produces the complete laminate failure envelope.

3.3 Maximum Stress Theory

JENKINS (1920) extended the concept of the maximum normal or principal stress theory to predict the strength of planar orthotropic materials such as wood. With this theory, it is postulated that failure in a lamina will occur when any one of the stresses σ_1, σ_2 or τ_{12} attains a respective maximum value σ_{1U}, σ_{2U} or τ_{12U}.

If a unidirectionally reinforced laminate subjected to uniaxial tension σ at some angle θ to the fibres, then the maximum allowable loading according to this theory is the smallest of the following three equations

$$\sigma = \frac{\sigma_{1U}}{\cos^2\theta} \qquad (23)$$

$$\sigma = \frac{\sigma_{2U}}{\sin^2\theta}$$

$$\sigma = \frac{\tau_{12U}}{\sin\theta\cos\theta}$$

For comparison, if the strength were to be predicted according to the maximum strain criterion of Section 3.2, then the corresponding expressions become (JONES (1975))

$$\sigma = \frac{\sigma_{1U}}{\cos^2\theta - \nu_{12}\sin^2\theta} \qquad (24)$$

$$\sigma = \frac{\sigma_{2U}}{\sin^2\theta - \nu_{21}\cos\ \theta}$$

$$\sigma = \frac{\tau_{12U}}{\sin\theta\cos\theta}$$

The only difference between these maximum stress and strain predictions is the inclusion of the Poisson's ratio terms.

3.4 Hill's Criterion

Under plane stress, HILL (1950) proposed that failure would initiate when the magnitude of the stresses reach the following condition

$$\left(\frac{\sigma_1}{\sigma_{1U}}\right)^2 + \left(\frac{\sigma_2}{\sigma_{2U}}\right)^2 - \left(\frac{1}{\sigma_{1U}^2} + \frac{1}{\sigma_{2U}^2} - \frac{1}{\tau_{12U}^2}\right)\sigma_1\sigma_2 + \left(\frac{\tau_{12}}{\tau_{12U}}\right)^2 = 1 \qquad (25)$$

Hill assumed that the yield (failure) stresses are the same in tension and compression., i.e. there is no Bauschinger effect. Unlike the two previous criteria, the Hill theory contains interaction among the stress components and therefore involves combined modes of failure.

3.5 Tsai-Hill Theory

This postulates the imminence of failure in a ply is evident when

$$\left(\frac{\sigma}{\sigma_{1U}}\right)^2 - \left(\frac{\sigma_1\sigma_2}{\sigma_{1U}^2}\right) + \left(\frac{\sigma_2}{\sigma_{2U}}\right)^2 + \left(\frac{\tau_{12}}{\tau_{12U}}\right)^2 = 1 \tag{26}$$

This equation is obtained from the Hill criterion of Eqn. 25 by assuming $\sigma_{2U} = \tau_{12U}$ for fibre reinforced composites (AZZI and TSAI (1965)). This is very similar to the theory proposed by NORRIS (1950) to examine failures in wood. This too provides interaction of the stresses and is therefore a criterion that can be employed in mixed modes of failure. The one drawback is that the interaction is fixed and that it does not distinguish between compressive and tensile strengths; this is overcome by the next criterion to be examined.

3.6 Tsai-Wu Theory

In an effort to predict experimental results more accurately, TSAI and WU (1971) proposed a failure surface of the form

$$f(\sigma) = F_i\sigma_i + F_{ij}\sigma_{ij} = 1; .....with....i,j = 1,2,,,,6 \tag{27}$$

where F_i and F_{ij} are second and fourth order lamina strength tensors. The linear stress terms account for the possible differences in the tensile and compressive strengths. The quadratic stress terms are similar to those in the Tsai-Hill formulation and describe the ellipsoid stress space. The F_{ij} ($i \neq j$) terms are new. Off-diagonal terms of the strength tensor provide independent interactions among the stress components. Under plane stress conditions this failure criterion becomes

$$F_1\sigma_1 + F_2\sigma_2 + F_6\sigma_6 + F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + 2F_{12}\sigma_1\sigma_2 + F_{66}\sigma_6^2 = 1 \tag{28}$$

where

$$F_1 = \left(\sigma_{1U}^{T}\right)^{-1} - \left(\sigma_{1U}^{C}\right)^{-1} \tag{29}$$
$$F_2 = \left(\sigma_{2U}^{T}\right)^{-1} - \left(\sigma_{2U}^{C}\right)^{-1}$$
$$F_6 = \left(\tau_{12U}^{T}\right)^{-1} - \left(\tau_{12U}^{C}\right)^{-1}$$
$$F_{11} = \left(\sigma_{1U}^{T}\sigma_{1U}^{C}\right)^{-1}$$
$$F_{22} = \left(\sigma_{2U}^{T}\sigma_{2U}^{C}\right)^{-1}$$
$$f_{66} = \left(\tau_{12U}^{T}\tau_{12U}^{C}\right)^{-1}$$

with the notations as before and the superscript T/C in the failure strength values denoting tensile/compressive modes. It must be noted that uniaxial strength values, by themselves, are not adequate. To determine F_{12}, biaxial tests are required (WU (1972)).

3.7 Discussion

The four failure theories (noting that the Hill theory leads on to the Tsai-Hill criterion) discussed in the previous sub-sections are representative and most widely used ones. The validity and applicability of a specific theory depends on the convenience of application and agreement with experimental results. Furthermore, failure modes, and thus the failure criteria, depend greatly on material properties and type of loading. A comparison of the four theories is given in Table 1 on the next page.

The maximum stress/strain theories, for example, are more applicable when brittle behaviour is predominant, typically in the first quadrant of the failure envelope with $\sigma_{1,2} \geq 0$. Of the two, only the maximum strain theory allows for a small degree of interaction through Poisson's ratio effect. These theories are conceptually easy to use; however, they do contain three sub-criteria each. The necessary experimental parameters can be obtained from standard uniaxial tests.

The interactive theories, such as the Tsai-Hill and Tsai-Wu, may be more applicable when ductile behaviour under shear or compression loading is applicable. They also cater for mixed-mode failure scenarios. The Tsai-Wu theory is mathematically consistent and relatively simple to use. The additional coefficients in this theory allow for distinction between compression and tension strengths. A comprehensive materials testing programme, including some biaxial tests, are required to accurately determine the many materials parameters that are required.

Most of the experimental data available for comparison with the theoretical predictions is in the first quadrant. These type of data are easily gathered by uniaxial tensile testing. Given the usual scatter of data, all four major theories give a satisfactory comparison with test results. More substantial differences among the approaches emerge in other quadrants where compressive failure stresses and modes are present. Here, it appears that the Tsai-Wu theory fits the data best (DANIEL and ISHAI (1994)).

In terms of applicability, it is best to consider all theories because the materials in all cases need not be the same and hence may exhibit brittle or ductile behaviour depending on the dominant loading mode and material make-up.

Table 1: Comparison of failure theories

Theory	Physical Basis	Operational Convenience	Required Experimental Characterisation
Maximum stress	Tensile behaviour of brittle material; No stress interaction	Inconvenient	Few parameters by simple testing
Maximum strain	Tensile behaviour of brittle material; Some stress interaction	Inconvenient	Few parameters by simple testing
Tsai-Hill	Ductile behaviour of anisotropic materials; Curve fitting for heterogeneous materials	Can be programmed; Different functions required for tensile and compressive strengths	Biaxial testing is needed in addition to uniaxial test cases
Tsai-Wu	Mathematically consistent; Reliable "curve fitting"	General and comprehensive; Operationally simple	Numerous parameters; Comprehensive experimental programme needed

4. FAILURE ANALYSIS OF MULTIDIRECTIONAL LAMINATES

4.1 Types of Failure

Two main types or definitions of failure need to be considered: (a) initial or First Ply Failure (or FPF) and (b) Ultimate Laminate Failure (ULF). In the first case, the laminate is deemed to have failed when the first ply or lamina reaches the failure loads/stress levels. In the latter, failure is assumed to have occurred when the ultimate or final level is reached, when the laminate can take up no further load at all. The FPF approach is conservative but it can be used with relatively low safety factors. The ULF approach is more advanced and requires a more precise definition of the loading conditions and stress distributions. In aircraft and some boat applications, the practice is to use FPF approach. For instance, in the aircraft industry, the general practice is to limit operational strains in carbon-epoxy to 0.4%. In the boat industry, where sandwich configuration is used, the practice is to limit skin strains in glass-epoxy construction to about 0.6%. Both the FPF and ULF approaches are considered below.

4.2 FPF of Symmetric Laminates; In-Plane Loading

Given a symmetric laminate under general in-plane loading, the average laminate stresses are given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{1}{h} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} \qquad (30)$$

where h is the laminate thickness. The laminate strains are equal to the reference plane strains and are related to the forces by the Eqn. 18. These strains, in global x-y axes context, can be converted to the principal axis system for an individual lamina by using the relationship given in Eqn. 11. From these ply strains, the corresponding stresses can be calculated from Eqn. 2.

For the FPF approach, the selected failure criterion is applied to the state of stress in each layer separately. Thus, for a state of stress $(\sigma_1, \sigma_2, \tau_{12})_k$ in layer k, the state of stress at failure is $S_{fk}(\sigma_1, \sigma_2, \tau_{12})_k$, where S_{fk} is safety factor for layer k. Substitution of the critical or failure stress in the Tsai-Wu criterion of Eqn. 28, for instance, leads to

$$F_1 S_{fk}\sigma_1 + F_2 S_{fk}\sigma_2 + F_6 S_{fk}\tau_{12} + F_{11} S_{fk}\sigma_1^2 + \ldots\ldots = 1 \qquad (31)$$

or

$$aS_{fk}^2 + bS_{fk} - 1 = 0 \qquad (32)$$

where

$$a = F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 + 2F_{12}\sigma_1\sigma_2$$
$$b = F_1\sigma_1 + F_2\sigma_2 \tag{33}$$

The solutions to the quadratic in Eqns. 32 and 33 are

$$S_{fka} = \left|\frac{-b + \sqrt{b^2 + 4a}}{2a}\right|$$
$$S_{fkr} = \left|\frac{-b - \sqrt{b^2 + 4a}}{2a}\right| \tag{34}$$

where the S_{fka} is the safety factor for layer k with the actual state of stress (σ_1, σ_2, τ_{12}) and S_{fkr} is the safety factor when the state of stress is negative, i.e. ($-\sigma_1$, $-\sigma_2$, $-\tau_{12}$). The procedure above is carried out for all layers of the laminate to find the minimum values of S_{fka} and S_{fkr}. These minimums are the safety factors of the laminate based on the FPF approach for the actual and reversed loadings.

4.3 Computational Procedure for FPF Analysis

The procedure for the determination of safety factors consists of the following steps.

Step 1	Enter basic lamina properties (E_{11}, E_{22}, G_{12}, ν_{12})
Step 2	Compute ply stiffnesses, using Eqn. 3
Step 3	Enter orientation of principal material axes for layer k, θ_k
Step 4	Calculate transformed layer stiffnesses using Eqns. 5, 6
Step 5	Enter through-the-thickness co-ordinates for layer k, h_k and h_{k-1}
Step 6	Calculate laminate stiffness matrices A, B, D using Eqn. 16
Step 7	Calculate laminate compliance matrices A', B', D' using Eqns. 19, 20
Step 8	Enter mechanical loading, i.e. forces N, moments M
Step 9	Calculate reference plane strains $\varepsilon°$, and curvatures κ from Eqn. 18
Step 10	Calculate layer strains ε_{x-y} with respect to the structural global axes, using Eqn. 14
Step 11	Calculate layer strains ε_{1-2} with respect to principal material axes, using Eqn. 11
Step 12	Calculate layer stresses σ_{1-2} with respect to principal material axes, using Eqn. 2.
Step 13	Enter lamina strengths σ_{1U} etc. and calculate Tsai-Wu coefficients using Eqn. 29
Step 14	Calculate layer safety factors S_{fka}/S_{fkr} from Eqn. 34 and thence the laminate safety factor
Step 15	Determine laminate strength components $\bar{F}$ by applying unit stress in each direction and using Eqn. 35.

$$\overline{F}_{xt} = (S_{fks})_{min.}$$
$$\overline{F}_{xc} = (S_{fks})_{min.} \quad (35)$$

4.4 Progressive and Ultimate Laminate Failure

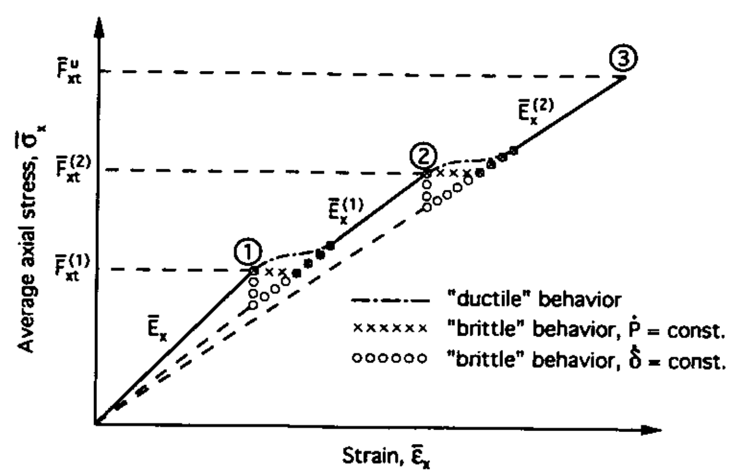
Progressive failure of a lamina within the laminate consists of cracking of the lamina upto a characteristic limiting crack density. Following this FPF, failure process continues upto ULF, which is usually higher than FPF.

The process can be explained by examining the stress-strain response of a multi-directional laminate under tensile loading, see Figure 1 below. Initially, the laminate behaves linearly, with the slope of the stress-strain curve equalling the initial modulus, $\overline{E}_x$, upto a point (1), where the first ply fails. After this ply reaches its maximum crack density (i.e. the characteristics damage state), its effective transverse modulus drops to E_2' and the laminate modulus drops to a value $\overline{E}_x^{(1)}$. If the material behaves in a brittle manner, then the modulus will drop suddenly. It will be manifested by a horizontal or vertical shift in the stress-strain curve, depending on whether the test is conducted under load or strain control respectively. Under increasing load, the specimen will respond linearly with a stress-strain slope equal to the reduced modulus $\overline{E}_x^{(1)}$ upto the point (2), where the next ply or plies will fail. Again, if the ply or plies fail suddenly in a brittle manner, there will be a further drop in modulus to a value, $\overline{E}_x^{(2)}$. This value corresponds to the laminate with all the failed plies to date discounted or reduced in stiffness. The progressive failure continues to a point, say, (3) where ultimate failure takes place.

At each stage of failure there is a corresponding strength. The ratio of the FPF and ULF values, ϕ, is an indicator of ply efficiency; it depends on the material system and laminate lay-up.

$$\phi_L = \frac{\overline{F}_{FPF}}{\overline{F}_{ULF}} \quad (36)$$

Figure 1 : Stress-Strain Response of a Multidirectional Laminate



4.5 Analysis of Progressive Laminate Failure

The determination of the ultimate strength of a laminate requires an interactive procedure taking into account the damage progression in the various plies. The computational scheme involved comprises the following steps.

Steps 1-15	Same as in Section 4.3
Step 16	Identify failed layer k_i, under i^{th} loading cycle from $S_{fki} = (S_{fk})_{min.}$
Step 17	Determine laminate strength components $[\bar{F}_{x,y}]^i$ for the i^{th} loading cycle
Step 18	Check if strength for the i^{th} load cycle, $\bar{F}^i$, is higher than the strength at the previous load cycle $\bar{F}^{i-1}$
Step 19	If the answer is "yes", the damaged lamina, k_i, is replaced by one having the following properties

$$E_1^{k_i} = r_1 E_1 \qquad (37)$$
$$E_2^{k_i} = r_2 E_2$$
$$G_{12}^{k_i} = r_{12} G_{12}$$
$$\nu_{12}^{k_i} = r_1 \nu_{12}$$

where r_1, r_2, r_{12} are the stiffness reduction factors, obtained previously from experimentation or analysis. Conservatively, $r_1 = 1$, $r_2 = r_{12} = 0$, i.e. complete ply discount.

Step 20	Go to i+1 load cycle and recalculate modified laminate stiffnesses, A, B, D and compliances, A', B', D'. Repeat all steps upto 18 above. In step 13, the lamina strength of the failed layer is made artificially high to avoid repeated failure indication of the failed layer
Step 21	If the answer to the question in Step 18 is "no", then ultimate failure occurs at the i-1 load cycle, i.e. $\bar{F}^u = \bar{F}^{i-1}$.

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