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INTERNAL WAVES PRODUCED BY A SUBMERGED SLENDER BODY MOVING IN A STRATIFIED FLUID

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Contents

Nomenclature

F_n	Froude number $= \frac{U}{\sqrt{gL}}$
g	Gravity acceleration $= (0, 0, -g)$
L	Characteristic length, for example length of body
N	Brunt-Väisälä frequency $= \sqrt{-\frac{g}{\rho(z)}\frac{\partial\rho}{\partial z}}$
$\tilde{N}$	Non-dimensional Brunt-Väisälä frequency $= \frac{NL}{U}$
$Oxyz$	Moving reference coordinate system located at the body's centroid
$\boldsymbol{r}$	$= \sqrt{r_1^2 + r_2^2 + r_3^3}$
r_1	$= x - \xi$
r_2	$= y - \eta$
r_3	$= z - \zeta$
$-\boldsymbol{U}$	Translational velocity of body $= -\{U, V, W\}$
$\boldsymbol{u}(x, y, z)$	Parametric disturbance velocity vector $= (u(x, y, z), v(x, y, z), w(x, y, z))$
$\boldsymbol{V}(x, y, z)$	Disturbance velocity vector
$\boldsymbol{x}$	Position vector of the field point (x, y, z)
$\beta^{\pm}$	$= -\frac{\tilde{N}(\zeta \pm z)}{\sqrt{r_2^2 + (\zeta \pm z)^2}}$
γ_3	$= \sqrt{1 - \frac{\tilde{N}^2}{\lambda_1(\lambda_1 - i\epsilon)}}$
$\gamma^{\pm}$	$= \lambda_1^2[(y - \eta)^2 + (\zeta \pm z)^2] - \tilde{N}^2(\zeta \pm z)^2$
ϵ	a small parameter introduced as an artificial damping mechanism,
	a radiation condition is obeyed when $\epsilon \to 0$,
$\rho(z)$	Density stratification of fluid medium
$\boldsymbol{\xi}$	Position vector of source point (ξ, η, ζ)
Σ	Boundary surface enclosing the fluid domain, Ω
Ω	Fluid domain, $z < 0$
$\boldsymbol{\nabla}$	$= (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$
$\boldsymbol{\nabla}_h$	$= (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0)$

Abstract

Solutions are obtained for the velocity components generated by a singularity moving horizontally in a fluid of constant Brunt-Väisälä frequency. A radiation condition is enforced using an artificial damping mechanism. Two solutions are combined to produce a Rankine ovoid and a continuous line distribution is used to model a prolate spheroid. The disturbance velocity field generated by the body displays the characteristics associated with the propagation of internal waves. The disturbance velocities calculated on the fluid's surface are compared with those obtained from a three layer model. The patterns produced display significant differences.

1 Introduction

This report describes an investigation of the disturbance created by a slender body moving in an inviscid fluid of constant Brunt-Väisälä frequency. A complementary report detailing a panel method which models an arbitrarily shaped body has preceded this investigation, see Price and Westlake (1994). The mathematical model described in this previous report is adopted in this study.

2 Basis of the analysis

The non-dimensional coupled zero order equations describing the disturbance in a fluid of constant Brunt-Väisälä frequency are of the form (see equations 71 and 72 in Price and Westlake, 1994)

$$\nabla_h^2 \phi + \frac{\partial w}{\partial z} = 0 \tag{1}$$

$$\boldsymbol{U} \cdot \nabla(\boldsymbol{U} \cdot \nabla + \epsilon)\left(w - \frac{\partial \phi}{\partial z}\right) + \tilde{N}^2 w = 0 \tag{2}$$

where

$$\tfrac{\partial \phi}{\partial x} = u \quad \tfrac{\partial \phi}{\partial y} = v.$$

The introduction of a body force takes the form of a dirac delta function, this is introduced on the right hand side of equation 1. This equation is derived from the incompressibility condition, i.e. fluid is neither created or destroyed. The delta function singularity introduces fluid into the domain and therefore equation 1 must be modified.

Without any loss of generality we can set $\boldsymbol{U} \cdot \nabla = \frac{\partial}{\partial x}$ and together with the introduction of a variable ψ, related to the vertical velocity component w in the form

$$w = \frac{\partial \psi}{\partial z}$$

equations 1 and 2 become

$$\nabla_h^2 \phi + \frac{\partial^2 \psi}{\partial z^2} = \delta(\boldsymbol{r}) \tag{3}$$

$$\frac{\partial}{\partial x}(\frac{\partial}{\partial x} + \epsilon)\left(\frac{\partial \psi}{\partial z} - \frac{\partial \phi}{\partial z}\right) + \tilde{N}^2 \frac{\partial \psi}{\partial z} = 0. \tag{4}$$

Integrating equation 4 with respect to z we obtain

$$\frac{\partial}{\partial x}(\frac{\partial}{\partial x} + \epsilon)(\psi - \phi) + \tilde{N}^2 \psi = 0 \tag{5}$$

and the adjoint of this equation is given by

$$\frac{\partial}{\partial x}(\frac{\partial}{\partial x} - \epsilon)(\psi - \phi) + \tilde{N}^2 \psi = 0 \tag{6}$$

Equations 3 and 6 form the basis equations describing the fluid disturbance caused by the moving slender body.

3 Derivation of disturbance parameters

Initially the disturbance parameters are derived for a single source of unit strength. At this stage it is appropriate to ensure the radiation condition is satisfied. Any error in the expressions for the velocities or strains will violate this condition which can be observed graphically. The magnitudes of the velocities can be calculated manually for a potential field, these results can then be compared with the velocities obtained when $N \to 0$.

Secondly, two singularities can be employed to produce a body shape. The separation and magnitude of the singularities can be obtained from the potential flow theory which describes a Rankine ovoid.

Lastly, singularities are distributed continuously along the axis of revolution of the body, The magnitudes of the distribution obtained from slender body theory.

3.1 Derivation of the functions ϕ and ψ

The velocity components u, v and w are directly related to the functions ϕ and ψ. The derivation of these functions will then allow the velocity components to be obtained.

Eliminating ϕ using $\frac{\partial}{\partial x}(\frac{\partial}{\partial x} - \epsilon)$ [3] and ∇_h^2 [6] gives a partial differential equation in ψ only,

$$\frac{\partial}{\partial x}(\frac{\partial}{\partial x} - \epsilon)\nabla^2\psi + \tilde{N}^2\nabla_h^2\psi = \frac{\partial}{\partial x}(\frac{\partial}{\partial x} - \epsilon)\delta(\boldsymbol{r}) \tag{7}$$

and eliminating ψ using $\frac{\partial^2}{\partial z^2}$ [6] gives

$$\frac{\partial}{\partial x}(\frac{\partial}{\partial x} - \epsilon)\nabla^2\phi + \tilde{N}^2\nabla_h^2\phi = \frac{\partial}{\partial x}(\frac{\partial}{\partial x} - \epsilon)\delta(\boldsymbol{r}) + \tilde{N}^2\delta(\boldsymbol{r}) \tag{8}$$

Equations 7 and 8 may be written in the matrix form

$$\mathcal{L}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \begin{pmatrix}\frac{\partial}{\partial x}(\frac{\partial}{\partial x} - \epsilon) + \tilde{N}^2\\ \frac{\partial}{\partial x}(\frac{\partial}{\partial x} - \epsilon)\end{pmatrix}\delta(\boldsymbol{r}) \tag{9}$$

where

$$\mathcal{L}(\,) = \left\{\frac{\partial}{\partial x}(\frac{\partial}{\partial x} - \epsilon)\nabla^2 + \tilde{N}^2\nabla_h^2\right\}(\,)$$

By the application of Fourier transforms we can rewrite equation 9 into the form

$$L\begin{pmatrix}\Phi\\ \Psi\end{pmatrix} = \begin{pmatrix}1 - \frac{\tilde{N}^2}{\lambda_1[\lambda_1 - i\epsilon]}\\ 1\end{pmatrix}\delta(z-\zeta)\frac{e^{i\boldsymbol{\xi}\cdot\boldsymbol{\lambda}}}{2\pi} \tag{10}$$

where

$$L(\,) = \left\{\frac{\partial^2}{\partial z^2} + \left[\frac{\tilde{N}^2}{\lambda_1[\lambda_1 - i\epsilon]} - 1\right]\lambda^2\right\}(\,)$$

Let us now examine the equation

$$\left\{\frac{\partial^2}{\partial z^2} + \left[\frac{\tilde{N}^2}{\lambda_1[\lambda_1 - i\epsilon]} - 1\right]\lambda^2\right\}G(\lambda_1, \lambda_2, z; \xi) = \delta(z-\zeta) \tag{11}$$

subject to the boundary conditions

$$w(x, y, 0) = 0 \quad \text{and} \quad \Psi(\lambda_1, \lambda_2, -\infty; \boldsymbol{\xi}) \quad \text{bounded} \tag{12}$$

This equation has the following solution when considering Ψ

$$G_\Psi(\lambda_1,\lambda_2,z;\xi) = -\frac{1}{2\lambda\sqrt{1-\frac{\tilde{N}^2}{\lambda_1(\lambda_1-i\epsilon)}}}\begin{cases} e^{\lambda(z+\zeta)\sqrt{1-\frac{\tilde{N}^2}{\lambda_1(\lambda_1-i\epsilon)}}}+e^{\lambda(\zeta-z)\sqrt{1-\frac{\tilde{N}^2}{\lambda_1(\lambda_1-i\epsilon)}}} & 0>z>\zeta \\ e^{\lambda(\zeta+z)\sqrt{1-\frac{\tilde{N}^2}{\lambda_1(\lambda_1-i\epsilon)}}}+e^{\lambda(z-\zeta)\sqrt{1-\frac{\tilde{N}^2}{\lambda_1(\lambda_1-i\epsilon)}}} & 0>\zeta>z \end{cases} \tag{13}$$

and $G_\Phi(\lambda_1,\lambda_2,z;\xi)$ can be obtained through an examination of equation 10, that is

$$G_\Phi = \gamma_3^2 G_\Psi \quad \gamma_3 = \sqrt{1-\frac{\tilde{N}^2}{\lambda_1(\lambda_1-i\epsilon)}} \tag{14}$$

The functions $G_\Phi(\lambda_1,\lambda_2,z;\xi)$ and $G_\Psi(\lambda_1,\lambda_2,z;\xi)$ can now be used to determine the functions ϕ and ψ by the application of inverse Fourier transforms,

$$\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{pmatrix}1-\frac{\tilde{N}^2}{\lambda_1[\lambda_1-i\epsilon]}\\ 1\end{pmatrix} G(\lambda_1,\lambda_2,z;\xi)e^{-i\boldsymbol{\xi}\cdot\boldsymbol{\lambda}}d\lambda_1 d\lambda_2 \tag{15}$$

The function ψ is now given by

$$\psi = -\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{e^{\lambda(z+\zeta)\gamma_3}+e^{\lambda(\zeta-z)\gamma_3}}{2\lambda\gamma_3}\right)e^{-i[(x-\xi)\lambda_1+(y-\eta)\lambda_2]}d\lambda_1 d\lambda_2$$

$$= -\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{e^{-i(x-\xi)\lambda_1}}{\gamma_3}\int_0^{\infty}\left(\frac{e^{\lambda(z+\zeta)\gamma_3}+e^{\lambda(\zeta-z)\gamma_3}}{\lambda}\right)\cos r_2\lambda_2 d\lambda_2 d\lambda_1$$

$$= -\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{e^{-i(x-\xi)\lambda_1}}{\gamma_3}\left[K_0\left(|\lambda_1|\sqrt{r_2^2+(z+\zeta)^2\gamma_3^2}\right)+K_0\left(|\lambda_1|\sqrt{r_2^2+(z-\zeta)^2\gamma_3^2}\right)\right]d\lambda_1 \tag{16}$$

Now let us examine the behaviour of the constituent parts of the integrand as a function of λ_1 as $\epsilon \to 0$.

λ_1	$-\infty \to -\tilde{N}$	$-\tilde{N} \to -\beta^\pm$	$-\beta^\pm \to 0$	$0 \to \beta^\pm$	$\beta^\pm \to \tilde{N}$	$\tilde{N} \to \infty$
$\lvert\lambda_1\rvert\gamma_3$	$\sqrt{\lambda_1^2-\tilde{N}^2}$	$i\sqrt{\tilde{N}^2-\lambda_1^2}$	$i\sqrt{\tilde{N}^2-\lambda_1^2}$	$-i\sqrt{\tilde{N}^2-\lambda_1^2}$	$-i\sqrt{\tilde{N}^2-\lambda_1^2}$	$\sqrt{\lambda_1^2-\tilde{N}^2}$
$\lvert\lambda_1\rvert\sqrt{r_2^2+(z\pm\zeta)^2\gamma_3^2}$	$\sqrt{\gamma^\pm}$	$\sqrt{\gamma^\pm}$	$i\sqrt{-\gamma^\pm}$	$-i\sqrt{-\gamma^\pm}$	$\sqrt{\gamma^\pm}$	$\sqrt{\gamma^\pm}$

Table 1: Integrand properties of ψ

The introduction of $|\lambda_1|$ into the numerator and denominator allows the application of the integrand properties expressed in table 1 and using the property

$$K_0(\pm ix) = -\tfrac{\pi}{2}\left[Y_0(x) \pm iJ_0(x)\right] \quad x>0$$

(see Abramowitz and Stegun(1970), 9.6.4, $\nu = 0$) ψ becomes

$$-(2\pi)^2\psi = \int_{-\infty}^{-\tilde{N}}\frac{|\lambda_1|}{\sqrt{\lambda_1^2-\tilde{N}^2}}e^{-i(x-\xi)\lambda_1}\left[K_0\left(\sqrt{\gamma^+}\right)+K_0\left(\sqrt{\gamma^-}\right)\right]d\lambda_1$$

$$+\int_{-\tilde{N}}^{-\beta^+} \frac{|\lambda_1|}{i\sqrt{\tilde{N}^2-\lambda_1^2}} e^{-i(x-\xi)\lambda_1} K_0\left(\sqrt{\gamma^+}\right) d\lambda_1$$

$$+\int_{-\tilde{N}}^{-\beta^-} \frac{|\lambda_1|}{i\sqrt{\tilde{N}^2-\lambda_1^2}} e^{-i(x-\xi)\lambda_1} K_0\left(\sqrt{\gamma^-}\right) d\lambda_1$$

$$+\int_{-\beta^+}^{0} \frac{|\lambda_1|}{i\sqrt{\tilde{N}^2-\lambda_1^2}} e^{-i(x-\xi)\lambda_1} \cdot -\frac{\pi}{2}\left[Y_0\left(\sqrt{-\gamma^+}\right)+iJ_0\left(\sqrt{-\gamma^+}\right)\right] d\lambda_1$$

$$+\int_{-\beta^-}^{0} \frac{|\lambda_1|}{i\sqrt{\tilde{N}^2-\lambda_1^2}} e^{-i(x-\xi)\lambda_1} \cdot -\frac{\pi}{2}\left[Y_0\left(\sqrt{-\gamma^-}\right)+iJ_0\left(\sqrt{-\gamma^-}\right)\right] d\lambda_1$$

$$+\int_{0}^{\beta^+} \frac{|\lambda_1|}{-i\sqrt{\tilde{N}^2-\lambda_1^2}} e^{-i(x-\xi)\lambda_1} \cdot -\frac{\pi}{2}\left[Y_0\left(\sqrt{-\gamma^+}\right)-iJ_0\left(\sqrt{-\gamma^+}\right)\right] d\lambda_1$$

$$+\int_{0}^{\beta^-} \frac{|\lambda_1|}{-i\sqrt{\tilde{N}^2-\lambda_1^2}} e^{-i(x-\xi)\lambda_1} \cdot -\frac{\pi}{2}\left[Y_0\left(\sqrt{-\gamma^-}\right)-iJ_0\left(\sqrt{-\gamma^-}\right)\right] d\lambda_1$$

$$+\int_{\beta^+}^{\tilde{N}} \frac{|\lambda_1|}{-i\sqrt{\tilde{N}^2-\lambda_1^2}} e^{-i(x-\xi)\lambda_1} K_0\left(\sqrt{\gamma^+}\right) d\lambda_1$$

$$+\int_{\beta^-}^{\tilde{N}} \frac{|\lambda_1|}{-i\sqrt{\tilde{N}^2-\lambda_1^2}} e^{-i(x-\xi)\lambda_1} K_0\left(\sqrt{\gamma^-}\right) d\lambda_1$$

$$+\int_{\tilde{N}}^{\infty} \frac{|\lambda_1|}{\sqrt{\lambda_1^2-\tilde{N}^2}} e^{-i(x-\xi)\lambda_1} \left[K_0\left(\sqrt{\gamma^+}\right)+K_0\left(\sqrt{\gamma^-}\right)\right] d\lambda_1$$

Rearrangement of the terms gives

$$-2\pi^2\psi = \int_{\tilde{N}}^{\infty} \frac{\lambda_1}{\sqrt{\lambda_1^2-\tilde{N}^2}} \left[K_0\left(\sqrt{\gamma^+}\right)+K_0\left(\sqrt{\gamma^-}\right)\right] \cos r_1\lambda_1 d\lambda_1$$

$$+\int_{\beta^+}^{\tilde{N}} \frac{\lambda_1}{\sqrt{\tilde{N}^2-\lambda_1^2}} K_0\left(\sqrt{\gamma^+}\right) \sin r_1\lambda_1 d\lambda_1$$

$$+\int_{\beta^-}^{\tilde{N}} \frac{\lambda_1}{\sqrt{\tilde{N}^2-\lambda_1^2}} K_0\left(\sqrt{\gamma^-}\right) \sin r_1\lambda_1 d\lambda_1$$

$$-\frac{\pi}{2}\int_0^{\beta^+}\frac{\lambda_1}{\sqrt{\tilde{N}^2-\lambda_1^2}}Y_0\left(\sqrt{-\gamma^+}\right)\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{\pi}{2}\int_0^{\beta^+}\frac{\lambda_1}{\sqrt{\tilde{N}^2-\lambda_1^2}}J_0\left(\sqrt{-\gamma^+}\right)\cos r_1\lambda_1 d\lambda_1$$

$$-\frac{\pi}{2}\int_0^{\beta^-}\frac{\lambda_1}{\sqrt{\tilde{N}^2-\lambda_1^2}}Y_0\left(\sqrt{-\gamma^-}\right)\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{\pi}{2}\int_0^{\beta^-}\frac{\lambda_1}{\sqrt{\tilde{N}^2-\lambda_1^2}}J_0\left(\sqrt{-\gamma^-}\right)\cos r_1\lambda_1 d\lambda_1 \tag{17}$$

The form of this equation can be compared with the asymmetric form of $\bar{h}_1$ derived in Price and Westlake(1994), page 39. The relationship between the two solutions can be determined through an examination of the integrands before the inverse Fourier transforms are carried out. In fact,

$$\frac{\partial\psi}{\partial\xi} = -\bar{h}_1$$

differentiating ψ with respect to ξ and comparing with $\bar{h}_1$ confirms the correct form of ψ has been obtained.

Similarly ϕ can be derived in the form

$$-2\pi^2\phi = \int_{\tilde{N}}^{\infty}\frac{\sqrt{\lambda_1^2-\tilde{N}^2}}{\lambda_1}\left[K_0\left(\sqrt{\gamma^+}\right)+K_0\left(\sqrt{\gamma^-}\right)\right]\cos r_1\lambda_1 d\lambda_1$$

$$-\int_{\beta^+}^{\tilde{N}}\frac{\sqrt{\tilde{N}^2-\lambda_1^2}}{\lambda_1}K_0\left(\sqrt{\gamma^+}\right)\sin r_1\lambda_1 d\lambda_1$$

$$-\int_{\beta^-}^{\tilde{N}}\frac{\sqrt{\tilde{N}^2-\lambda_1^2}}{\lambda_1}K_0\left(\sqrt{\gamma^-}\right)\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta^+}\frac{\sqrt{\tilde{N}^2-\lambda_1^2}}{\lambda_1}Y_0\left(\sqrt{-\gamma^+}\right)\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta^+}\frac{\sqrt{\tilde{N}^2-\lambda_1^2}}{\lambda_1}J_0\left(\sqrt{-\gamma^+}\right)\cos r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta^-}\frac{\sqrt{\tilde{N}^2-\lambda_1^2}}{\lambda_1}Y_0\left(\sqrt{-\gamma^-}\right)\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta^-}\frac{\sqrt{\tilde{N}^2-\lambda_1^2}}{\lambda_1}J_0\left(\sqrt{-\gamma^-}\right)\cos r_1\lambda_1 d\lambda_1 \tag{18}$$

When $\tilde{N}=0$, ψ and ϕ reduce to

$$-2\pi^2\left\{\begin{array}{c}\psi\\ \phi\end{array}\right\}=\int_0^\infty\left[K_0\left(\sqrt{\lambda^2[r_2^2+(\zeta+z)^2]}\right)+K_0\left(\sqrt{\lambda^2[r_2^2+(\zeta-z)^2]}\right)\right]\cos r_1\lambda_1 d\lambda_1$$

Using the result

$$\int_0^\infty K_0(ax)\cos bx dx=\frac{\pi}{2}\frac{1}{\sqrt{a^2+b^2}}\qquad \mathcal{R}e(a)>0\quad b>0$$

(see Abramowitz and Stegun(1980), $\nu=1$), we find that ψ and ϕ become

$$-\frac{1}{4\pi}\left[\frac{1}{\sqrt{r_1^2+r_2^2+(z-\zeta)^2}}+\frac{1}{\sqrt{r_1^2+r_2^2+(z+\zeta)^2}}\right],$$

which is the correct result.

3.2 Disturbance velocities, u, v and w

Expressions for the disturbance velocities can now be obtained from the functions ϕ and ψ. That is,

$$u=\frac{\partial\phi}{\partial x}$$

such that it follows from equation 18 after differentiation,

$$-2\pi^2 u=-\int_{\tilde{N}}^\infty\sqrt{\lambda_1^2-\tilde{N}^2}\left[K_0\left(\sqrt{\gamma^+}\right)+K_0\left(\sqrt{\gamma^-}\right)\right]\sin r_1\lambda_1 d\lambda_1$$

$$-\int_{\beta^+}^{\tilde{N}}\sqrt{\tilde{N}^2-\lambda_1^2}K_0\left(\sqrt{\gamma^+}\right)\cos r_1\lambda_1 d\lambda_1$$

$$-\int_{\beta^-}^{\tilde{N}}\sqrt{\tilde{N}^2-\lambda_1^2}K_0\left(\sqrt{\gamma^-}\right)\cos r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta^+}\sqrt{\tilde{N}^2-\lambda_1^2}Y_0\left(\sqrt{-\gamma^+}\right)\cos r_1\lambda_1 d\lambda_1$$

$$-\frac{\pi}{2}\int_0^{\beta^+}\sqrt{\tilde{N}^2-\lambda_1^2}J_0\left(\sqrt{-\gamma^+}\right)\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta^-}\sqrt{\tilde{N}^2-\lambda_1^2}Y_0\left(\sqrt{-\gamma^-}\right)\cos r_1\lambda_1 d\lambda_1$$

$$-\frac{\pi}{2}\int_0^{\beta^-}\sqrt{\tilde{N}^2-\lambda_1^2}J_0\left(\sqrt{-\gamma^-}\right)\sin r_1\lambda_1 d\lambda_1 \tag{19}$$

Again,

$$v = \frac{\partial \phi}{\partial y},$$

and by using the results

$$\frac{\partial}{\partial y}\left[J_0(\sqrt{-\gamma^\pm})\right] = \frac{r_2\lambda_1^2}{\sqrt{-\gamma^\pm}} J_1(\sqrt{-\gamma^\pm})$$

$$\frac{\partial}{\partial y}\left[Y_0(\sqrt{-\gamma^\pm})\right] = \frac{r_2\lambda_1^2}{\sqrt{-\gamma^\pm}} Y_1(\sqrt{-\gamma^\pm})$$

$$\frac{\partial}{\partial y}\left[K_0(\sqrt{\gamma^\pm})\right] = -\frac{r_2\lambda_1^2}{\sqrt{\gamma^\pm}} K_1(\sqrt{\gamma^\pm})$$

$$\frac{\partial}{\partial y}(\beta^\pm) = \frac{\tilde{N} r_2(\zeta \pm z)}{[r_2^2 + (\zeta \pm z)^2]^{\frac{3}{2}}}$$

v becomes

$$-2\pi^2 v = -\int_{\tilde{N}}^{\infty} r_2\lambda_1\sqrt{\lambda_1^2 - \tilde{N}^2}\left[\frac{K_1\left(\sqrt{\gamma^+}\right)}{\sqrt{\gamma^+}} + \frac{K_1\left(\sqrt{\gamma^-}\right)}{\sqrt{\gamma^-}}\right]\cos r_1\lambda_1 d\lambda_1$$

$$+\frac{\tilde{N} r_2(\zeta + z)}{[r_2^2 + (\zeta + z)^2]^{\frac{3}{2}}}\frac{\sqrt{\tilde{N}^2 - (\beta^+)^2}}{\beta^+} K_0\left(\sqrt{\gamma^+|_{\lambda_1=\beta^+}}\right)\sin r_1\beta^+$$

$$+\int_{\beta^+}^{\tilde{N}} \frac{r_2\lambda_1\sqrt{\tilde{N}^2 - \lambda_1^2}}{\sqrt{\gamma^+}} K_1\left(\sqrt{\gamma^+}\right)\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\tilde{N} r_2(\zeta - z)}{[r_2^2 + (\zeta - z)^2]^{\frac{3}{2}}}\frac{\sqrt{\tilde{N}^2 - (\beta^-)^2}}{\beta^-} K_0\left(\sqrt{\gamma^-|_{\lambda_1=\beta^-}}\right)\sin r_1\beta^-$$

$$+\int_{\beta^-}^{\tilde{N}} \frac{r_2\lambda_1\sqrt{\tilde{N}^2 - \lambda_1^2}}{\sqrt{\gamma^-}} K_1\left(\sqrt{\gamma^-}\right)\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi\tilde{N} r_2(\zeta + z)}{2[r_2^2 + (\zeta + z)^2]^{\frac{3}{2}}}\frac{\sqrt{\tilde{N}^2 - (\beta^+)^2}}{\beta^+} Y_0\left(\sqrt{-\gamma^+|_{\lambda_1=\beta^+}}\right)\sin r_1\beta^+$$

$$+\frac{\pi}{2}\int_0^{\beta^+} \frac{r_2\lambda_1\sqrt{\tilde{N}^2 - \lambda_1^2}}{\sqrt{-\gamma^+}} Y_1\left(\sqrt{-\gamma^+}\right)\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi\tilde{N}r_2(\zeta+z)}{2[r_2^2+(\zeta+z)^2]^{\frac{3}{2}}}\frac{\sqrt{\tilde{N}^2-(\beta^+)^2}}{\beta^+}J_0\left(\sqrt{-\gamma^+|_{\lambda_1=\beta^+}}\right)\cos r_1\beta^+$$

$$+\frac{\pi}{2}\int_0^{\beta^+}\frac{r_2\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^+}}J_1\left(\sqrt{-\gamma^+}\right)\cos r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi\tilde{N}r_2(\zeta-z)}{2[r_2^2+(\zeta-z)^2]^{\frac{3}{2}}}\frac{\sqrt{\tilde{N}^2-(\beta^-)^2}}{\beta^-}Y_0\left(\sqrt{-\gamma^-|_{\lambda_1=\beta^-}}\right)\sin r_1\beta^-$$

$$+\frac{\pi}{2}\int_0^{\beta^-}\frac{r_2\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^-}}Y_1\left(\sqrt{-\gamma^-}\right)\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi\tilde{N}r_2(\zeta-z)}{2[r_2^2+(\zeta-z)^2]^{\frac{3}{2}}}\frac{\sqrt{\tilde{N}^2-(\beta^-)^2}}{\beta^-}J_0\left(\sqrt{-\gamma^-|_{\lambda_1=\beta^-}}\right)\cos r_1\beta^-$$

$$+\frac{\pi}{2}\int_0^{\beta^-}\frac{r_2\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^-}}J_1\left(\sqrt{-\gamma^-}\right)\cos r_1\lambda_1 d\lambda_1 \tag{20}$$

The terms $K_0\left(\sqrt{\gamma^\pm|_{\lambda_1=\beta^\pm}}\right)$ and $Y_0\left(\sqrt{-\gamma^\pm|_{\lambda_1=\beta^\pm}}\right)$ are undefined as $\gamma^\pm|_{\lambda_1=\beta^\pm}=0$. However combining these two terms produces the finite result

$$K_0\left(\sqrt{\gamma^\pm|_{\lambda_1=\beta^\pm}}\right)+\frac{\pi}{2}Y_0\left(\sqrt{-\gamma^\pm|_{\lambda_1=\beta^\pm}}\right)=0.$$

Also, using the results

$$J_0\left(\sqrt{-\gamma^\pm|_{\lambda_1=\beta^\pm}}\right)=J_0(0)=1$$

$$\frac{\sqrt{\tilde{N}^2-(\beta^\pm)^2}}{\beta^\pm}=-\frac{|r_2|}{\zeta\pm z}$$

the velocity component v is given by the expression

$$-2\pi^2 v=-\int_{\tilde{N}}^{\infty}r_2\lambda_1\sqrt{\lambda_1^2-\tilde{N}^2}\left[\frac{K_1\left(\sqrt{\gamma^+}\right)}{\sqrt{\gamma^+}}+\frac{K_1\left(\sqrt{\gamma^-}\right)}{\sqrt{\gamma^-}}\right]\cos r_1\lambda_1 d\lambda_1$$

$$+\int_{\beta^+}^{\tilde{N}}\frac{r_2\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{\gamma^+}}K_1\left(\sqrt{\gamma^+}\right)\sin r_1\lambda_1 d\lambda_1$$

$$+\int_{\beta^-}^{\tilde{N}}\frac{r_2\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{\gamma^-}}K_1\left(\sqrt{\gamma^-}\right)\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta^+}\frac{r_2\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^+}}Y_1\left(\sqrt{-\gamma^+}\right)\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{\pi\tilde{N}r_2|r_2|}{2[r_2^2+(\zeta+z)^2]^{\frac{3}{2}}}\cos r_1\beta^+$$

$$+\frac{\pi}{2}\int_0^{\beta^+}\frac{r_2\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^+}}J_1\left(\sqrt{-\gamma^+}\right)\cos r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta^-}\frac{r_2\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^-}}Y_1\left(\sqrt{-\gamma^-}\right)\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{\pi\tilde{N}r_2|r_2|}{2[r_2^2+(\zeta-z)^2]^{\frac{3}{2}}}\cos r_1\beta^-$$

$$+\frac{\pi}{2}\int_0^{\beta^-}\frac{r_2\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^-}}J_1\left(\sqrt{-\gamma^-}\right)\cos r_1\lambda_1 d\lambda_1 \qquad (21)$$

In a similar manner the velocity component w can be obtained. This is defined by the expression

$$-2\pi^2 w = -\int_{\tilde{N}}^{\infty}\lambda_1\sqrt{\lambda_1^2-\tilde{N}^2}\left[\frac{(\zeta+z)K_1\left(\sqrt{\gamma^+}\right)}{\sqrt{\gamma^+}}-\frac{(\zeta-z)K_1\left(\sqrt{\gamma^-}\right)}{\sqrt{\gamma^-}}\right]\cos r_1\lambda_1 d\lambda_1$$

$$+\int_{\beta^+}^{\tilde{N}}\frac{\lambda_1(\zeta+z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{\gamma^+}}K_1\left(\sqrt{\gamma^+}\right)\sin r_1\lambda_1 d\lambda_1$$

$$-\int_{\beta^-}^{\tilde{N}}\frac{\lambda_1(\zeta-z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{\gamma^-}}K_1\left(\sqrt{\gamma^-}\right)\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta^+}\frac{\lambda_1(\zeta+z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^+}}Y_1\left(\sqrt{-\gamma^+}\right)\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{\pi\tilde{N}|r_2|(\zeta+z)}{2[r_2^2+(\zeta+z)^2]^{\frac{3}{2}}}\cos r_1\beta^+$$

$$+\frac{\pi}{2}\int_0^{\beta^+}\frac{\lambda_1(\zeta+z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^+}}J_1\left(\sqrt{-\gamma^+}\right)\cos r_1\lambda_1 d\lambda_1$$

$$-\frac{\pi}{2}\int_0^{\beta^-}\frac{\lambda_1(\zeta-z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^-}}Y_1\left(\sqrt{-\gamma^-}\right)\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi\tilde{N}|r_2|(\zeta-z)}{2[r_2^2+(\zeta-z)^2]^{\frac{3}{2}}}\cos r_1\beta^-$$

$$-\frac{\pi}{2}\int_0^{\beta^-}\frac{\lambda_1(\zeta-z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\sqrt{-\gamma^-}}J_1\left(\sqrt{-\gamma^-}\right)\cos r_1\lambda_1 d\lambda_1 \tag{22}$$

3.3 Introduction of $J_1^*(x)$, $Y_1^*(x)$ and $K_1^*(x)$

The functions derived in the previous section contain poles at $\lambda_1 = \beta^{\pm}$. These must be removed before the velocities can be evaluated. A method which will facilitate this is to introduce new functions based on the behaviour of the Bessel functions near the singularity point. For instance, the Bessel function $Y_1(x)$ is undefined at $x = 0$ and has behaviour

$$Y_1(x) \sim -\tfrac{2}{\pi x} \quad x \to 0$$

By defining the function $Y_1^*(x)$ as

$$Y_1^*(x) = \left(Y_1(x) + \frac{2}{\pi x}\right)\frac{1}{x}$$

we can remove the singularity with the introduction of additional terms into each expression. Also defining

$$K_1^*(x) = \left(K_1(x) - \frac{1}{x}\right)\frac{1}{x}$$

$$J_1^*(x) = \frac{J_1(x)}{x}$$

For large arguments these functions can be evaluated directly, however for small arguments polynomials approximations are employed. Using $Y_1^*(x)$ as an example, $Y_1(x)$ is defined in Abramowitz and Stegun (1970) page 370 as

$$xY_1(x) = \left(\tfrac{2}{\pi}\right)\log\left(\tfrac{x}{2}\right)J_1(x) - \tfrac{2}{\pi} + 0.2212091\left(\tfrac{x}{3}\right)^2 + 2.1682709\left(\tfrac{x}{3}\right)^4 - 1.3164827\left(\tfrac{x}{3}\right)^6 + \ldots \quad x < 3$$

where

$$x^{-1}J_1(x) = J_1^*(x) = 0.5 - 0.56249985\left(\tfrac{x}{3}\right)^2 + 0.21093573\left(\tfrac{x}{3}\right)^4 - 0.03954289\left(\tfrac{x}{3}\right)^6 + \ldots \quad x < 3$$

Substituting these two expressions into the definition of $Y_1^*(x)$ we have

$$Y_1^*(x) = \tfrac{2}{\pi}\log\left(\tfrac{x}{2}\right)J_1^*(x) + \tfrac{1}{9}\left[+0.2212091 + 2.1682709\left(\tfrac{x}{3}\right)^2 - 1.3164827\left(\tfrac{x}{3}\right)^4 + \ldots\right] \quad x < 3$$

which is integratable.

The formulation of u does not require such modifications and u can be evaluated directly. The velocity components v and w however must be modified. The component v becomes,

$$-2\pi^2 v = -\int_{\tilde{N}}^{\infty} r_2\lambda_1\sqrt{\lambda_1^2 - \tilde{N}^2}\left[\frac{K_1\left(\sqrt{\gamma^+}\right)}{\sqrt{\gamma^+}} + \frac{K_1\left(\sqrt{\gamma^-}\right)}{\sqrt{\gamma^-}}\right]\cos r_1\lambda_1 d\lambda_1$$

$$+\int_{\beta+}^{\tilde{N}} r_2\lambda_1\sqrt{\tilde{N}^2 - \lambda_1^2}K_1^*\left(\sqrt{\gamma^+}\right)\sin r_1\lambda_1 d\lambda_1$$

$$+\int_{\beta-}^{\tilde{N}} r_2\lambda_1\sqrt{\tilde{N}^2 - \lambda_1^2}K_1^*\left(\sqrt{\gamma^-}\right)\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta+} r_2\lambda_1\sqrt{\tilde{N}^2 - \lambda_1^2}Y_1^*\left(\sqrt{-\gamma^+}\right)\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{\pi\tilde{N}r_2|r_2|}{2[r_2^2+(\zeta+z)^2]^{\frac{3}{2}}}\cos r_1\beta^+$$

$$+\frac{\pi}{2}\int_0^{\beta^+} r_2\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}J_1^*\left(\sqrt{-\gamma^+}\right)\cos r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta^-} r_2\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}Y_1^*\left(\sqrt{-\gamma^-}\right)\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{\pi\tilde{N}r_2|r_2|}{2[r_2^2+(\zeta-z)^2]^{\frac{3}{2}}}\cos r_1\beta^-$$

$$+\frac{\pi}{2}\int_0^{\beta^-} r_2\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}J_1^*\left(\sqrt{-\gamma^-}\right)\cos r_1\lambda_1 d\lambda_1$$

$$\underline{+\int_0^{\tilde{N}}\frac{r_2\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}}{\gamma^+}\sin r_1\lambda_1 d\lambda_1}$$

$$\underline{+\int_0^{\tilde{N}}\frac{r_2\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}}{\gamma^-}\sin r_1\lambda_1 d\lambda_1} \qquad (23)$$

and the velocity component w takes the form

$$-2\pi^2 w = -\int_{\tilde{N}}^{\infty}\lambda_1\sqrt{\lambda_1^2-\tilde{N}^2}\left[\frac{(\zeta+z)K_1\left(\sqrt{\gamma^+}\right)}{\sqrt{\gamma^+}}-\frac{(\zeta-z)K_1\left(\sqrt{\gamma^-}\right)}{\sqrt{\gamma^-}}\right]\cos r_1\lambda_1 d\lambda_1$$

$$+\int_{\beta^+}^{\tilde{N}}\lambda_1(\zeta+z)\sqrt{\tilde{N}^2-\lambda_1^2}K_1^*\left(\sqrt{\gamma^+}\right)\sin r_1\lambda_1 d\lambda_1$$

$$-\int_{\beta^-}^{\tilde{N}}\lambda_1(\zeta-z)\sqrt{\tilde{N}^2-\lambda_1^2}K_1^*\left(\sqrt{\gamma^-}\right)d\lambda_1\sin r_1\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta^+}\lambda_1(\zeta+z)\sqrt{\tilde{N}^2-\lambda_1^2}Y_1^*\left(\sqrt{-\gamma^+}\right)\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{\pi\tilde{N}|r_2|(\zeta+z)}{2[r_2^2+(\zeta+z)^2]^{\frac{3}{2}}}\cos r_1\beta^+$$

$$+\frac{\pi}{2}\int_0^{\beta^+}\lambda_1(\zeta+z)\sqrt{\tilde{N}^2-\lambda_1^2}J_1^*\left(\sqrt{-\gamma^+}\right)\cos r_1\lambda_1 d\lambda_1$$

$$-\frac{\pi}{2}\int_0^{\beta^-}\lambda_1(\zeta-z)\sqrt{\tilde{N}^2-\lambda_1^2}Y_1^*\left(\sqrt{-\gamma^-}\right)\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi\tilde{N}|r_2|(\zeta-z)}{2[r_2^2+(\zeta-z)^2]^{\frac{3}{2}}}\cos r_1\beta^-$$

$$-\frac{\pi}{2}\int_0^{\beta^-}\lambda_1(\zeta-z)\sqrt{\tilde{N}^2-\lambda_1^2}J_1^*\left(\sqrt{-\gamma^-}\right)\cos r_1\lambda_1 d\lambda_1$$

$$\underline{+\int_0^{\tilde{N}}\frac{\lambda_1(\zeta+z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\gamma^+}\sin r_1\lambda_1 d\lambda_1}$$

$$\underline{-\int_0^{\tilde{N}}\frac{\lambda_1(\zeta-z)\sqrt{\tilde{N}^2-\lambda_1^2}}{\gamma^-}\sin r_1\lambda_1 d\lambda_1} \qquad (24)$$

The additional terms underlined still possess singularities at $\lambda_1=\beta^{\pm}$. The previous process removes these singularities from the integrands involving Bessel functions. The integration and removal of the singularities can now be achieved through the application of the integral equality derived in appendix A. Let

$$a=\sqrt{r_2^2+(\zeta\pm z)^2},$$

$$b=\tilde{N}|\zeta\pm z|=-\tilde{N}(\zeta\pm z),$$

$$c=r_1$$

and

$$\frac{b}{a}=\beta^{\pm}.$$

So that an integral of the form

$$\int_0^{\tilde{N}}\frac{\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}}{\lambda_1^2[r_2^2+(\zeta\pm z)^2]-\tilde{N}^2(\zeta\pm z)^2}\sin r_1\lambda_1 d\lambda_1$$

$$=-\frac{1}{r_2^2+(\zeta\pm z)^2}\int_0^{\tilde{N}}\frac{\lambda_1\sin r_1\lambda_1}{\sqrt{\tilde{N}^2-\lambda_1^2}+\sqrt{\tilde{N}^2-(\beta^{\pm})^2}}d\lambda_1$$

$$+\frac{\tilde{N}|r_2|}{2[r_2^2+(\zeta\pm z)^2]^{\frac{3}{2}}}\left\{Si\left[r_1(\tilde{N}+\beta^{\pm})\right]+Si\left[r_1(\tilde{N}-\beta^{\pm})\right]\right\}\cos r_1\beta^{\pm}$$

$$-\frac{\tilde{N}|r_2|}{2[r_2^2+(\zeta\pm z)^2]^{\frac{3}{2}}}\left\{Ci\left[|r_1|(\tilde{N}+\beta^{\pm})\right]-Ci\left[|r_1|(\tilde{N}-\beta^{\pm})\right]\right\}\sin r_1\beta^{\pm}$$

Substituting this result into equation 23 we find that the velocity component v becomes

$$-2\pi^2 v = -\int_{\tilde{N}}^{\infty} r_2\lambda_1\sqrt{\lambda_1^2-\tilde{N}^2}\left[\frac{K_1\left(\sqrt{\gamma^+}\right)}{\sqrt{\gamma^+}}+\frac{K_1\left(\sqrt{\gamma^-}\right)}{\sqrt{\gamma^-}}\right]\cos r_1\lambda_1 d\lambda_1$$

$$+\int_{\beta^+}^{\tilde{N}} r_2\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}K_1^*\left(\sqrt{\gamma^+}\right)\sin r_1\lambda_1 d\lambda_1$$

$$+\int_{\beta^-}^{\tilde{N}} r_2\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}K_1^*\left(\sqrt{\gamma^-}\right)d\lambda_1\sin r_1\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta^+} r_2\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}Y_1^*\left(\sqrt{-\gamma^+}\right)\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{\pi\tilde{N}r_2|r_2|}{2[r_2^2+(\zeta+z)^2]^{\frac{3}{2}}}\cos r_1\beta^+$$

$$+\frac{\pi}{2}\int_0^{\beta^+} r_2\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}J_1^*\left(\sqrt{-\gamma^+}\right)\cos r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta^-} r_2\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}Y_1^*\left(\sqrt{-\gamma^-}\right)\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{\pi\tilde{N}r_2|r_2|}{2[r_2^2+(\zeta-z)^2]^{\frac{3}{2}}}\cos r_1\beta^-$$

$$+\frac{\pi}{2}\int_0^{\beta^-} r_2\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}J_1^*\left(\sqrt{-\gamma^-}\right)\cos r_1\lambda_1 d\lambda_1$$

$$-\frac{r_2}{r_2^2+(\zeta+z)^2}\int_0^{\tilde{N}}\frac{\lambda_1\sin r_1\lambda_1}{\sqrt{\tilde{N}^2-\lambda_1^2}+\sqrt{\tilde{N}^2-(\beta^+)^2}}d\lambda_1$$

$$+\frac{\tilde{N}r_2|r_2|}{2[r_2^2+(\zeta+z)^2]^{\frac{3}{2}}}\left\{Si\left[r_1(\tilde{N}+\beta^+)\right]+Si\left[r_1(\tilde{N}-\beta^+)\right]\right\}\cos r_1\beta^+$$

$$-\frac{\tilde{N}r_2|r_2|}{2[r_2^2+(\zeta+z)^2]^{\frac{3}{2}}}\left\{Ci\left[|r_1|(\tilde{N}+\beta^+)\right]-Ci\left[|r_1|(\tilde{N}-\beta^+)\right]\right\}\sin r_1\beta^+$$

$$-\frac{r_2}{r_2^2+(\zeta-z)^2}\int_0^{\tilde{N}}\frac{\lambda_1\sin r_1\lambda_1}{\sqrt{\tilde{N}^2-\lambda_1^2}+\sqrt{\tilde{N}^2-(\beta^-)^2}}d\lambda_1$$

$$+\frac{\tilde{N}r_2|r_2|}{2[r_2^2+(\zeta-z)^2]^{\frac{3}{2}}}\left\{Si\left[r_1(\tilde{N}+\beta^-)\right]+Si\left[r_1(\tilde{N}-\beta^-)\right]\right\}\cos r_1\beta^-$$

$$-\frac{\tilde{N}r_2|r_2|}{2[r_2^2+(\zeta-z)^2]^{\frac{3}{2}}}\left\{Ci\left[|r_1|(\tilde{N}+\beta^-)\right]-Ci\left[|r_1|(\tilde{N}-\beta^-)\right]\right\}\sin r_1\beta^- \tag{25}$$

whereas its substitution into equation 24 allows the velocity component w to take the form

$$-2\pi^2 w=-\int_{\tilde{N}}^{\infty}\lambda_1\sqrt{\lambda_1^2-\tilde{N}^2}\left[\frac{(\zeta+z)K_1\left(\sqrt{\gamma^+}\right)}{\sqrt{\gamma^+}}-\frac{(\zeta-z)K_1\left(\sqrt{\gamma^-}\right)}{\sqrt{\gamma^-}}\right]\cos r_1\lambda_1 d\lambda_1$$

$$+\int_{\beta+}^{\tilde{N}}\lambda_1(\zeta+z)\sqrt{\tilde{N}^2-\lambda_1^2}K_1^*\left(\sqrt{\gamma^+}\right)\sin r_1\lambda_1 d\lambda_1$$

$$-\int_{\beta^-}^{\tilde{N}}\lambda_1(\zeta-z)\sqrt{\tilde{N}^2-\lambda_1^2}K_1^*\left(\sqrt{\gamma^-}\right)\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta^+}\lambda_1(\zeta+z)\sqrt{\tilde{N}^2-\lambda_1^2}Y_1^*\left(\sqrt{-\gamma^+}\right)\sin r_1\lambda_1 d\lambda_1$$

$$-\frac{\pi\tilde{N}|r_2|(\zeta+z)}{2[r_2^2+(\zeta+z)^2]^{\frac{3}{2}}}\cos r_1\beta^+$$

$$+\frac{\pi}{2}\int_0^{\beta^+}\lambda_1(\zeta+z)\sqrt{\tilde{N}^2-\lambda_1^2}J_1^*\left(\sqrt{-\gamma^+}\right)\cos r_1\lambda_1 d\lambda_1$$

$$-\frac{\pi}{2}\int_0^{\beta^-}\lambda_1(\zeta-z)\sqrt{\tilde{N}^2-\lambda_1^2}Y_1^*\left(\sqrt{-\gamma^-}\right)\sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi\tilde{N}|r_2|(\zeta-z)}{2[r_2^2+(\zeta-z)^2]^{\frac{3}{2}}}\cos r_1\beta^-$$

$$-\frac{\pi}{2}\int_0^{\beta^-}\lambda_1(\zeta-z)^2\sqrt{\tilde{N}^2-\lambda_1^2}J_1^*\left(\sqrt{-\gamma^-}\right)\cos r_1\lambda_1 d\lambda_1$$

$$-\frac{\zeta+z}{r_2^2+(\zeta+z)^2}\int_0^{\tilde{N}}\frac{\lambda_1\sin r_1\lambda_1}{\sqrt{\tilde{N}^2-\lambda_1^2}+\sqrt{\tilde{N}^2-(\beta^+)^2}}d\lambda_1$$

$$+\frac{\tilde{N}|r_2|(\zeta+z)}{2[r_2^2+(\zeta+z)^2]^{\frac{3}{2}}}\left\{Si\left[r_1(\tilde{N}+\beta^+)\right]+Si\left[r_1(\tilde{N}-\beta^+)\right]\right\}\cos r_1\beta^+$$

$$-\frac{\tilde{N}|r_2|(\zeta+z)}{2[r_2^2+(\zeta+z)^2]^{\frac{3}{2}}}\left\{Ci\left[|r_1|(\tilde{N}+\beta^+)\right]-Ci\left[|r_1|(\tilde{N}-\beta^+)\right]\right\}\sin r_1\beta^+$$

$$+\frac{\zeta-z}{r_2^2+(\zeta-z)^2}\int_0^{\tilde{N}}\frac{\lambda_1\sin r_1\lambda_1}{\sqrt{\tilde{N}^2-\lambda_1^2}+\sqrt{\tilde{N}^2-(\beta^-)^2}}d\lambda_1$$

$$-\frac{\tilde{N}|r_2|(\zeta-z)}{2[r_2^2+(\zeta-z)^2]^{\frac{3}{2}}}\left\{Si\left[r_1(\tilde{N}+\beta^-)\right]+Si\left[r_1(\tilde{N}-\beta^-)\right]\right\}\cos r_1\beta^-$$

$$+\frac{\tilde{N}|r_2|(\zeta-z)}{2[r_2^2+(\zeta-z)^2]^{\frac{3}{2}}}\left\{Ci\left[|r_1|(\tilde{N}+\beta^-)\right]-Ci\left[|r_1|(\tilde{N}-\beta^-)\right]\right\}\sin r_1\beta^- \qquad (26)$$

To avoid excessively large expressions only the source term will now be shown and the superscripts $\pm$ will be dropped. Thus β^- becomes β. To compute any of the disturbance parameters **BOTH** the source term and the image term must be included in the calculation and the total disturbance can be obtained from

$$u = u_{source} + u_{image}$$

$$v = v_{source} + v_{image}$$

$$w = w_{source} - w_{image}$$

The expressions for the strains obey the same rules.

3.4 Disturbance strains, u_x, v_y and v_x

The components of strain can be obtained through further differentiation. That is, the strain component u_x involves the simple differentiation of the terms $\sin r_1\lambda_1$ or $\cos r_1\lambda_1$. v_x involves the differentiation of the sine and cosine integrals, but the remaining strain v_y however is problematic as the term r_2 appears in the argument of the Bessel functions and the limits of the integral. The following list of formulae allows simplification of the differentiation process.

$$\frac{\partial}{\partial y}\left[J_1^*(\sqrt{-\gamma})\right]=\frac{r_2\lambda_1^2}{\gamma}\left[J_0(\sqrt{-\gamma})-2J_1^*(\sqrt{-\gamma})\right]$$

$$\frac{\partial}{\partial y}\left[Y_1^*(\sqrt{-\gamma})\right]=\frac{r_2\lambda_1^2}{\gamma}\left[Y_0(\sqrt{-\gamma})-2Y_1^*(\sqrt{-\gamma})\right]$$

$$\frac{\partial}{\partial y}\left[K_1^*(\sqrt{\gamma})\right]=-\frac{r_2\lambda_1^2}{\gamma}\left[K_0(\sqrt{\gamma})+2K_1^*(\sqrt{\gamma})\right]$$

$$\frac{\partial}{\partial y}\left\{Si\left[r_1(N\pm\beta)\right]\right\}=\pm\frac{r_2r_3}{(\sqrt{r_2^2+r_3^2}\mp r_3)(r_2^2+r_3^2)}\sin r_1(\tilde{N}\pm\beta)$$

$$\frac{\partial}{\partial y}\left\{Ci\left[r_1(N\pm\beta)\right]\right\}=\pm\frac{r_2r_3}{(\sqrt{r_2^2+r_3^2}\mp r_3)(r_2^2+r_3^2)}\cos r_1(\tilde{N}\pm\beta)$$

By differentiating equation 19, we find that the strain component u_x is given by the expression

$$-2\pi^2 u_x = -\int_{\tilde{N}}^{\infty} \lambda_1 \sqrt{\lambda_1^2 - \tilde{N}^2} K_0\left(\sqrt{\gamma}\right) \cos r_1 \lambda_1 d\lambda_1$$

$$+\int_{\beta}^{\tilde{N}} \lambda_1 \sqrt{\tilde{N}^2 - \lambda_1^2} K_0\left(\sqrt{\gamma}\right) \sin r_1 \lambda_1 d\lambda_1$$

$$-\frac{\pi}{2}\int_{0}^{\beta} \lambda_1 \sqrt{\tilde{N}^2 - \lambda_1^2} Y_0\left(\sqrt{-\gamma}\right) \sin r_1 \lambda_1 d\lambda_1$$

$$-\frac{\pi}{2}\int_{0}^{\beta} \lambda_1 \sqrt{\tilde{N}^2 - \lambda_1^2} J_0\left(\sqrt{-\gamma}\right) \cos r_1 \lambda_1 d\lambda_1 \qquad (27)$$

By differentiating equation 25, the strain component v_y is found to take the form

$$-2\pi^2 v_x = \int_{\tilde{N}}^{\infty} r_2 \lambda_1^2 \sqrt{\lambda_1^2 - \tilde{N}^2} \frac{K_1\left(\sqrt{\gamma}\right)}{\sqrt{\gamma}} \sin r_1 \lambda_1 d\lambda_1$$

$$+\int_{\beta}^{\tilde{N}} r_2 \lambda_1^2 \sqrt{\tilde{N}^2 - \lambda_1^2} K_1^*\left(\sqrt{\gamma}\right) \cos r_1 \lambda_1 d\lambda_1$$

$$+\frac{\pi}{2}\int_{0}^{\beta} r_2 \lambda_1^2 \sqrt{\tilde{N}^2 - \lambda_1^2} Y_1^*\left(\sqrt{-\gamma}\right) \cos r_1 \lambda_1 d\lambda_1$$

$$-\frac{\pi}{2}\int_{0}^{\beta} r_2 \lambda_1^2 \sqrt{\tilde{N}^2 - \lambda_1^2} J_1^*\left(\sqrt{-\gamma}\right) \sin r_1 \lambda_1 d\lambda_1$$

$$-\frac{r_2}{r_2^2 + r_3^2}\int_{0}^{\tilde{N}} \frac{\lambda_1^2 \cos r_1 \lambda_1}{\sqrt{\tilde{N}^2 - \lambda_1^2} + \sqrt{\tilde{N}^2 - \beta^2}} d\lambda_1$$

$$-\frac{\pi \tilde{N}^2 r_2 |r_2| r_3}{2[r_2^2 + r_3^2]^2} \sin r_1 \beta$$

$$+\frac{\tilde{N} r_2 |r_2|}{2 r_1 [r_2^2 + r_3^2]^{\frac{3}{2}}} \left[\sin r_1(\tilde{N} + \beta) + \sin r_1(\tilde{N} - \beta)\right] \cos r_1 \beta$$

$$+\frac{\tilde{N}^2 r_2 |r_2| r_3}{2[r_2^2 + r_3^2]^2} \left\{ Si\left[r_1(\tilde{N} + \beta)\right] + Si\left[r_1(\tilde{N} - \beta)\right] \right\} \sin r_1 \beta$$

$$-\frac{\tilde{N} r_2 |r_2|}{2 r_1 [r_2^2 + r_3^2]^{\frac{3}{2}}} \left[\cos r_1(\tilde{N} + \beta) - \cos r_1(\tilde{N} - \beta)\right] \sin r_1 \beta$$

$$+\frac{\tilde{N} r_2 |r_2| r_3}{2[r_2^2 + r_3^2]^2} \left\{ Ci\left[|r_1|(\tilde{N} + \beta)\right] - Ci\left[|r_1|(\tilde{N} - \beta)\right] \right\} \cos r_1 \beta \qquad (28)$$

By differentiating equation 25, the strain component v_y is given by the expression

$$-2\pi^2 v_y = -\int_{\tilde{N}}^{\infty} \lambda_1 \sqrt{\lambda_1^2 - \tilde{N}^2} \frac{K_1\left(\sqrt{\gamma}\right)}{\sqrt{\gamma}} \cos r_1\lambda_1 d\lambda_1$$

$$+\int_{\tilde{N}}^{\infty} \sqrt{\lambda_1^2 - \tilde{N}^2} \frac{r_2^2\lambda_1^3}{\sqrt{\gamma}} \left[K_0\left(\sqrt{\gamma}\right) + \frac{2K_1\left(\sqrt{\gamma}\right)}{\sqrt{\gamma}}\right] \cos r_1\lambda_1 d\lambda_1$$

$$+\int_{\beta}^{\tilde{N}} \lambda_1 \sqrt{\tilde{N}^2 - \lambda_1^2} K_1^*\left(\sqrt{\gamma}\right) \sin r_1\lambda_1 d\lambda_1$$

$$+\int_{\beta}^{\tilde{N}} r_2\lambda_1 \sqrt{\tilde{N}^2 - \lambda_1^2} K_{1M}^*\left(\sqrt{\gamma}\right) \sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta} r_2\lambda_1 \sqrt{\tilde{N}^2 - \lambda_1^2} Y_1^*\left(\sqrt{-\gamma}\right) \sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta} \lambda_1 \sqrt{\tilde{N}^2 - \lambda_1^2} Y_{1M}^*\left(\sqrt{-\gamma}\right) \sin r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta} \lambda_1 \sqrt{\tilde{N}^2 - \lambda_1^2} J_1^*\left(\sqrt{-\gamma}\right) \cos r_1\lambda_1 d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta} r_2^2\lambda_1^3 \sqrt{\tilde{N}^2 - \lambda_1^2} J_{1M}^*\left(\sqrt{-\gamma}\right) \cos r_1\lambda_1 d\lambda_1$$

$$-\frac{r_2}{\left(r_2^2+r_3^2\right)^2}\int_0^{\tilde{N}} \frac{\lambda_1\left(r_2\sqrt{\tilde{N}^2-\lambda_1^2} + \tilde{N}\operatorname{sgn}(r_2)\sqrt{r_2^2+r_3^2}\right)\sin r_1\lambda_1}{\left(\sqrt{\tilde{N}^2-\lambda_1^2} + \sqrt{\tilde{N}^2-\beta^2}\right)^2} d\lambda_1$$

$$-\frac{r_2^2\beta^2(r_2^2+r_3^2) + 2r_3^2}{2\left(r_2^2+r_3^2\right)^2}\int_0^{\tilde{N}} \frac{\lambda_1 \sin r_1\lambda_1}{\sqrt{\tilde{N}^2-\lambda_1^2} + \sqrt{\tilde{N}^2-\beta^2}} d\lambda_1$$

$$-\frac{\pi\tilde{N}^3 r_2^2|r_2|r_3^2}{4(r_2^2+r_3^2)^{\frac{5}{2}}} \cos r_1\beta$$

$$+\frac{\tilde{N} r_2^2|r_2|\beta^2}{4(r_2^2+r_3^2)^{\frac{3}{2}}} \left\{Si\left[r_1(\tilde{N}+\beta)\right] + Si\left[r_1(\tilde{N}-\beta)\right]\right\} \cos r_1\beta$$

$$-\frac{\tilde{N} r_2^2|r_2|\beta^2}{4(r_2^2+r_3^2)^{\frac{3}{2}}} \left\{Ci\left[|r_1|(\tilde{N}+\beta)\right] - Ci\left[|r_1|(\tilde{N}-\beta)\right]\right\} \sin r_1\beta$$

$$-\frac{\pi N|r_2|}{2(r_2^2+r_3^2)^{\frac{5}{2}}}\left[(r_3^2-r_2^2)\cos r_1\beta-\frac{\tilde{N}r_1r_2^2r_3}{\sqrt{r_2^2+r_3^2}}\right]\sin r_1\beta$$

$$+\frac{\tilde{N}r_2^2|r_2|r_3}{2(r_2^2+r_3^2)^{\frac{5}{2}}}\left[\frac{\sin r_1(\tilde{N}+\beta)}{\sqrt{r_2^2+r_3^2}-r_3}-\frac{\sin r_1(\tilde{N}-\beta)}{\sqrt{r_2^2+r_3^2}+r_3}\right]\cos r_1\beta$$

$$+\frac{\tilde{N}|r_2|}{2(r_2^2+r_3^2)^{\frac{5}{2}}}\left[(r_3^2-r_2^2)\cos r_1\beta-\frac{\tilde{N}r_1r_2^2r_3}{\sqrt{r_2^2+r_3^2}}\sin r_1\beta\right]\left\{Si\left[r_1(\tilde{N}+\beta)\right]+Si\left[r_1(\tilde{N}-\beta)\right]\right\}$$

$$-\frac{\tilde{N}r_2^2|r_2|r_3}{2(r_2^2+r_3^2)^{\frac{5}{2}}}\left[\frac{\cos r_1(\tilde{N}+\beta)}{\sqrt{r_2^2+r_3^2}-r_3}+\frac{\cos r_1(\tilde{N}-\beta)}{\sqrt{r_2^2+r_3^2}+r_3}\right]\sin r_1\beta$$

$$-\frac{\tilde{N}|r_2|}{2(r_2^2+r_3^2)^{\frac{5}{2}}}\left[(r_3^2-r_2^2)\sin r_1\beta+\frac{\tilde{N}r_1r_2^2r_3}{\sqrt{r_2^2+r_3^2}}\cos r_1\beta\right]\left\{Ci\left[|r_1|(\tilde{N}+\beta)\right]-Ci\left[|r_1|(\tilde{N}-\beta)\right]\right\} \qquad (29)$$

where

$$K_{1M}^*(x)=-\frac{r_2}{x^2}\left\{\lambda_1^2\left[K_0(x)+2K_1^*(x)\right]+\frac{\beta^2}{2}\right\}$$

$$Y_{1M}^*(x)=-\frac{r_2}{x^2}\left\{\lambda_1^2\left[Y_0(x)+2Y_1^*(x)\right]-\frac{\beta^2}{\pi}\right\}$$

$$J_{1M}^*(x)=-\frac{1}{x^2}\left[J_0(x)-2J_1^*(x)\right]$$

These functions can be defined for small arguments using the technique previously applied to $K_1^*(x)$, $Y_1^*(x)$ and $J_1^*(x)$.

3.5 Application of slender body theory

Slender body theory assumes that the summation of solutions distributed along the longitudinal axis of the body is equivalent to a solution when the body is taken as a whole. The source strength associated with each solution being a function of its location on that axis. The summation can be expressed as

$$\phi(x,\xi) = \int_L Q(\xi)\phi(x,\xi)d\xi$$

where the source strength is given by

$$Q(\xi) = -U\frac{dA(\xi)}{d\xi}$$

This integral over the length of the body can be applied to the expressions for the velocities and strains directly as they are merely derivatives of ϕ. A fully submerged prolate spheroid requires the source distribution

$$Q(\xi) = 2\pi U\left(\frac{d}{L}\right)^2\xi$$

Thus to complete the ξ integration on the velocity u the following integrals are required :

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \xi\cos\lambda_1(x-\xi)d\xi$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \xi\sin\lambda_1(x-\xi)d\xi$$

If these results are defined as $C_1(x,\lambda_1)$ and $S_1(x,\lambda_1)$ respectively then u becomes

$$\begin{aligned}
-2\pi^2 u = &-\int_{\tilde{N}}^{\infty}\sqrt{\lambda_1^2-\tilde{N}^2}K_0\left(\sqrt{\gamma}\right)S_1(x,\lambda_1)d\lambda_1 \\
&-\int_{\beta}^{\tilde{N}}\sqrt{\tilde{N}^2-\lambda_1^2}K_0\left(\sqrt{\gamma}\right)C_1(x,\lambda_1)d\lambda_1 \\
&+\frac{\pi}{2}\int_0^{\beta}\sqrt{\tilde{N}^2-\lambda_1^2}Y_0\left(\sqrt{-\gamma}\right)C_1(x,\lambda_1)d\lambda_1 \\
&-\frac{\pi}{2}\int_0^{\beta}\sqrt{\tilde{N}^2-\lambda_1^2}J_0\left(\sqrt{-\gamma}\right)S_1(x,\lambda_1)d\lambda_1
\end{aligned}\tag{30}$$

where

$$C_1(x,\lambda_1) = \frac{2\sin x\lambda_1}{\lambda_1^2}\left(\sin\frac{\lambda_1}{2}-\frac{\lambda_1}{2}\cos\frac{\lambda_1}{2}\right)$$

$$S_1(x,\lambda_1) = \frac{2\cos x\lambda_1}{\lambda_1^2}\left(\frac{\lambda_1}{2}\cos\frac{\lambda_1}{2}-\sin\frac{\lambda_1}{2}\right)$$

To complete the ξ integration on the velocities v and w the following integrals are required :

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \xi Ci[|x-\xi|(\tilde{N} \pm \beta)] \sin \lambda_1 (x-\xi) d\xi$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \xi Si[(x-\xi)(\tilde{N} \pm \beta)] \cos \lambda_1 (x-\xi) d\xi$$

Defining the results as $CiS_1(x,\beta)$ and $SiC_1(x,\beta)$ respectively v and w become

$$\begin{aligned}
-2\pi^2 v = &- \int_{\tilde{N}}^{\infty} r_2 \lambda_1 \sqrt{\lambda_1^2 - \tilde{N}^2} \frac{K_1(\sqrt{\gamma})}{\sqrt{\gamma}} C_1(x,\lambda_1) d\lambda_1 \\
&+ \int_{\beta}^{\tilde{N}} r_2 \lambda_1 \sqrt{\tilde{N}^2 - \lambda_1^2} K_1^* (\sqrt{\gamma}) d\lambda_1 S_1(x,\lambda_1) \\
&+ \frac{\pi}{2} \int_0^{\beta} r_2 \lambda_1 \sqrt{\tilde{N}^2 - \lambda_1^2} Y_1^* (\sqrt{-\gamma}) S_1(x,\lambda_1) d\lambda_1 \\
&+ \frac{\pi}{2} \int_0^{\beta} r_2 \lambda_1 \sqrt{\tilde{N}^2 - \lambda_1^2} J_1^* (\sqrt{-\gamma}) C_1(x,\lambda_1) d\lambda_1 \\
&- \frac{r_2}{r_2^2 + r_3^2} \int_0^{\tilde{N}} \frac{\lambda_1 S_1(x,\lambda_1)}{\sqrt{\tilde{N}^2 - \lambda_1^2} + \sqrt{\tilde{N}^2 - \beta^2}} d\lambda_1 \\
&- \frac{\pi \tilde{N} r_2 |r_2|}{2[r_2^2 + r_3^2]^{\frac{3}{2}}} C_1(x,\beta) \\
&+ \frac{\tilde{N} r_2 |r_2|}{2[r_2^2 + r_3^2]^{\frac{3}{2}}} [SiC_1(x,\beta) + SiC_1(x,-\beta)] \\
&- \frac{\tilde{N} r_2 |r_2|}{2[r_2^2 + r_3^2]^{\frac{3}{2}}} [CiS_1(x,\beta) - CiS_1(x,-\beta)]
\end{aligned} \tag{31}$$

$$\begin{aligned}
-2\pi^2 w = &\int_{\tilde{N}}^{\infty} r_3 \lambda_1 \sqrt{\lambda_1^2 - \tilde{N}^2} \frac{K_1(\sqrt{\gamma})}{\sqrt{\gamma}} C_1(x,\lambda_1) d\lambda_1 \\
&- \int_{\beta}^{\tilde{N}} r_3 \lambda_1 \sqrt{\tilde{N}^2 - \lambda_1^2} K_1^* (\sqrt{\gamma}) d\lambda_1 S_1(x,\lambda_1) \\
&- \frac{\pi}{2} \int_0^{\beta} r_3 \lambda_1 \sqrt{\tilde{N}^2 - \lambda_1^2} Y_1^* (\sqrt{-\gamma}) S_1(x,\lambda_1) d\lambda_1
\end{aligned}$$

$$-\frac{\pi}{2}\int_0^{\beta} r_3\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}J_1^*\left(\sqrt{-\gamma}\right)C_1(x,\lambda_1)d\lambda_1$$

$$+\frac{r_3}{r_2^2+r_3^2}\int_0^{\tilde{N}}\frac{\lambda_1 S_1(x,\lambda_1)}{\sqrt{\tilde{N}^2-\lambda_1^2}+\sqrt{\tilde{N}^2-\beta^2}}d\lambda_1$$

$$+\frac{\pi\tilde{N}|r_2|r_3}{2[r_2^2+r_3^2]^{\frac{3}{2}}}C_1(x,\beta)$$

$$-\frac{\tilde{N}|r_2|r_3}{2[r_2^2+r_3^2]^{\frac{3}{2}}}\left[SiC_1(x,\beta)+SiC_1(x,-\beta)\right]$$

$$+\frac{\tilde{N}|r_2|r_3}{2[r_2^2+r_3^2]^{\frac{3}{2}}}\left[CiS_1(x,\beta)-CiS_1(x,-\beta)\right] \tag{32}$$

where

$$-\beta\ CiS_1(x\pm\beta)=(x+\frac{1}{2})\cos\beta(x+\frac{1}{2})Ci[|x+\frac{1}{2}|(\tilde{N}\pm\beta)]-(x-\frac{1}{2})\cos\beta(x-\frac{1}{2})Ci[|x-\frac{1}{2}|(\tilde{N}\pm\beta)]$$

$$-\frac{x}{2}\left\{Ci[|(x+\frac{1}{2})(\tilde{N}\pm 2\beta)|]-Ci[|(x-\frac{1}{2})(\tilde{N}\pm 2\beta)|]+Ci[|x+\frac{1}{2}|\tilde{N}]-Ci[|x-\frac{1}{2}|\tilde{N}]\right\}$$

$$-Ci[|x+\frac{1}{2}|(\tilde{N}\pm\beta)]\left[(x+\frac{1}{2})\cos\beta(x+\frac{1}{2})-\frac{\sin\beta(x+\frac{1}{2})}{\beta}\right]+Ci[(x-\frac{1}{2})(\tilde{N}\pm\beta)]\left[(x-\frac{1}{2})\cos\beta(x-\frac{1}{2})-\frac{\sin\beta(x-\frac{1}{2})}{\beta}\right]$$

$$+\frac{\cos x(\tilde{N}\pm 2\beta)\sin\left(\frac{\tilde{N}}{2}\pm\beta\right)}{\tilde{N}\pm 2\beta}-\frac{\cos x\tilde{N}\sin\frac{\tilde{N}}{2}}{\tilde{N}}$$

$$\mp\frac{1}{2\beta}\left\{Si[(x+\frac{1}{2})(\tilde{N}\pm 2\beta)]-Si[(x-\frac{1}{2})(\tilde{N}\pm 2\beta)]-Si[(x+\frac{1}{2})\tilde{N}]+Si[(x-\frac{1}{2})\tilde{N}]\right\}$$

$$\beta\ SiC_1(x\pm\beta)=(x+\frac{1}{2})\sin\beta(x+\frac{1}{2})Si[(x+\frac{1}{2})(\tilde{N}\pm\beta)]-(x-\frac{1}{2})\sin\beta(x-\frac{1}{2})Si[(x-\frac{1}{2})(\tilde{N}\pm\beta)]$$

$$\mp\frac{x}{2}\left\{Ci[|(x-\frac{1}{2})(\tilde{N}\pm 2\beta)|]-Ci[|(x+\frac{1}{2})(\tilde{N}\pm 2\beta)|]+Ci[|x+\frac{1}{2}|\tilde{N}]-Ci[|x-\frac{1}{2}|\tilde{N}]\right\}$$

$$-Si[(x+\frac{1}{2})(\tilde{N}\pm\beta)]\left[(x+\frac{1}{2})\sin\beta(x+\frac{1}{2})+\frac{\cos\beta(x+\frac{1}{2})}{\beta}\right]+Si[(x-\frac{1}{2})(\tilde{N}\pm\beta)]\left[(x-\frac{1}{2})\sin\beta(x-\frac{1}{2})+\frac{\cos\beta(x-\frac{1}{2})}{\beta}\right]$$

$$\mp\frac{\cos x(\tilde{N}\pm 2\beta\sin\left(\frac{\tilde{N}}{2}\pm\beta\right)}{\tilde{N}\pm 2\beta}\pm\frac{\cos x\tilde{N}\sin\frac{\tilde{N}}{2}}{\tilde{N}}$$

$$+\frac{1}{2\beta}\left\{Si[(x+\frac{1}{2})(\tilde{N}\pm 2\beta)] - Si[(x-\frac{1}{2})(\tilde{N}\pm 2\beta)] + Si[(x+\frac{1}{2})\tilde{N}] - Si[(x-\frac{1}{2})\tilde{N}]\right\}$$

A similar process can be applied to the strains. The functions created from the ξ integration are listed in appendix B. By this means, the strain component u_x given in equation 27 becomes

$$-2\pi^2 u_x = -\int_{\tilde{N}}^{\infty} \lambda_1\sqrt{\lambda_1^2-\tilde{N}^2}K_0\left(\sqrt{\gamma}\right)C_1(x,\lambda_1)d\lambda_1$$

$$+\int_{\beta}^{\tilde{N}} \lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}K_0\left(\sqrt{\gamma}\right)S_1(x,\lambda_1)d\lambda_1$$

$$-\frac{\pi}{2}\int_0^{\beta} \lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}Y_0\left(\sqrt{-\gamma}\right)S_1(x,\lambda_1)d\lambda_1$$

$$-\frac{\pi}{2}\int_0^{\beta} \lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}J_0\left(\sqrt{-\gamma}\right)C_1(x,\lambda_1)d\lambda_1 \qquad (33)$$

From equation 28, the strain component v_y is given by the expression

$$-2\pi^2 v_x = \int_{\tilde{N}}^{\infty} r_2\lambda_1^2\sqrt{\lambda_1^2-\tilde{N}^2}\frac{K_1\left(\sqrt{\gamma}\right)}{\sqrt{\gamma}}S_1(x,\lambda_1)d\lambda_1$$

$$+\int_{\beta}^{\tilde{N}} r_2\lambda_1^2\sqrt{\tilde{N}^2-\lambda_1^2}K_1^*\left(\sqrt{\gamma}\right)C_1(x,\lambda_1)d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta} r_2\lambda_1^2\sqrt{\tilde{N}^2-\lambda_1^2}Y_1^*\left(\sqrt{-\gamma}\right)C_1(x,\lambda_1)d\lambda_1$$

$$-\frac{\pi}{2}\int_0^{\beta} r_2\lambda_1^2\sqrt{\tilde{N}^2-\lambda_1^2}J_1^*\left(\sqrt{-\gamma}\right)S_1(x,\lambda_1)d\lambda_1$$

$$-\frac{r_2}{r_2^2+r_3^2}\int_0^{\tilde{N}} \frac{\lambda_1^2 C_1(x,\lambda_1)}{\sqrt{\tilde{N}^2-\lambda_1^2}+\sqrt{\tilde{N}^2-\beta^2}}d\lambda_1$$

$$-\frac{\pi\tilde{N}^2 r_2|r_2|r_3}{2[r_2^2+r_3^2]^2}S_1(x,\beta)$$

$$+\frac{\tilde{N}r_2|r_2|}{2r_1[r_2^2+r_3^2]^{\frac{3}{2}}}\left[SNCS_2(x,\beta)+SNCS_2(x,-\beta)\right]$$

$$+\frac{\tilde{N}^2 r_2|r_2|r_3}{2[r_2^2+r_3^2]^2}\left[SiS_1(x,\beta)+SiS_1(x,-\beta)\right]$$

$$-\frac{\tilde{N}r_2|r_2|}{2r_1[r_2^2+r_3^2]^{\frac{3}{2}}}\left[CSSN_2(x,\beta)-CSSN_2(x,-\beta)\right]$$

$$+\frac{\tilde{N}r_2|r_2|r_3}{2[r_2^2+r_3^2]^2}\left[CiC_1(x,\beta)-CiC_1(x,-\beta)\right] \tag{34}$$

From equation 29, the strain component v_y is given by

$$-2\pi^2 v_y = -\int_{\tilde{N}}^{\infty}\lambda_1\sqrt{\lambda_1^2-\tilde{N}^2}\frac{K_1\left(\sqrt{\gamma}\right)}{\sqrt{\gamma}}C_1(x,\lambda_1)d\lambda_1$$

$$+\int_{\tilde{N}}^{\infty}\sqrt{\lambda_1^2-\tilde{N}^2}\frac{r_2^2\lambda_1^3}{\sqrt{\gamma}}\left[K_0\left(\sqrt{\gamma}\right)+\frac{2K_1\left(\sqrt{\gamma}\right)}{\sqrt{\gamma}}\right]C_1(x,\lambda_1)d\lambda_1$$

$$+\int_{\beta}^{\tilde{N}}\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}K_1^*\left(\sqrt{\gamma}\right)S_1(x,\lambda_1)d\lambda_1$$

$$+\int_{\beta}^{\tilde{N}}r_2\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}K_{1M}^*\left(\sqrt{\gamma}\right)S_1(x,\lambda_1)d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta}r_2\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}Y_1^*\left(\sqrt{-\gamma}\right)S_1(x,\lambda_1)d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta}\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}Y_{1M}^*\left(\sqrt{-\gamma}\right)S_1(x,\lambda_1)d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta}\lambda_1\sqrt{\tilde{N}^2-\lambda_1^2}J_1^*\left(\sqrt{-\gamma}\right)C_1(x,\lambda_1)d\lambda_1$$

$$+\frac{\pi}{2}\int_0^{\beta}r_2^2\lambda_1^3\sqrt{\tilde{N}^2-\lambda_1^2}J_{1M}^*\left(\sqrt{-\gamma}\right)C_1(x,\lambda_1)d\lambda_1$$

$$-\frac{r_2}{\left(r_2^2+r_3^2\right)^2}\int_0^{\tilde{N}}\frac{\lambda_1\left(r_2\sqrt{\tilde{N}^2-\lambda_1^2}+\tilde{N}\mathrm{sgn}(r_2)\sqrt{r_2^2+r_3^2}\right)S_1(x,\lambda_1)}{\left(\sqrt{\tilde{N}^2-\lambda_1^2}+\sqrt{\tilde{N}^2-\beta^2}\right)^2}d\lambda_1$$

$$-\frac{r_2^2\beta^2(r_2^2+r_3^2)+2r_3^2}{2\left(r_2^2+r_3^2\right)^2}\int_0^{\tilde{N}}\frac{\lambda_1 S_1(x,\lambda_1)}{\sqrt{\tilde{N}^2-\lambda_1^2}+\sqrt{\tilde{N}^2-\beta^2}}d\lambda_1$$

$$-\frac{\pi\tilde{N}^3 r_2^2|r_2|r_3^2}{4(r_2^2+r_3^2)^{\frac{5}{2}}}C_1(x,\beta)$$

$$+\frac{\tilde{N}r_2^2|r_2|\beta^2}{4(r_2^2+r_3^2)^{\frac{3}{2}}}\left[SiC_1(x,\beta)+SiC_1(x,-\beta)\right]$$

$$-\frac{\tilde{N}r_2^2|r_2|\beta^2}{4(r_2^2+r_3^2)^{\frac{3}{2}}}\left[CiS_1(x,\beta)-CiS_1(x,-\beta)\right]$$

$$-\frac{\pi N|r_2|}{2(r_2^2+r_3^2)^{\frac{5}{2}}}\left[(r_3^2-r_2^2)\cos r_1\beta-\frac{\tilde{N}r_1r_2^2r_3}{\sqrt{r_2^2+r_3^2}}\right]\sin r_1\beta$$

$$+\frac{\tilde{N}r_2^2|r_2|r_3}{2(r_2^2+r_3^2)^{\frac{5}{2}}}\left[\frac{\sin r_1(\tilde{N}+\beta)}{\sqrt{r_2^2+r_3^2}-r_3}-\frac{\sin r_1(\tilde{N}-\beta)}{\sqrt{r_2^2+r_3^2}+r_3}\right]\cos r_1\beta$$

$$+\frac{\tilde{N}|r_2|}{2(r_2^2+r_3^2)^{\frac{5}{2}}}\left[(r_3^2-r_2^2)\cos r_1\beta-\frac{\tilde{N}r_1r_2^2r_3}{\sqrt{r_2^2+r_3^2}}\sin r_1\beta\right]\left\{Si\left[r_1(\tilde{N}+\beta)\right]+Si\left[r_1(\tilde{N}-\beta)\right]\right\}$$

$$-\frac{\tilde{N}r_2^2|r_2|r_3}{2(r_2^2+r_3^2)^{\frac{5}{2}}}\left[\frac{\cos r_1(\tilde{N}+\beta)}{\sqrt{r_2^2+r_3^2}-r_3}+\frac{\cos r_1(\tilde{N}-\beta)}{\sqrt{r_2^2+r_3^2}+r_3}\right]\sin r_1\beta$$

$$-\frac{\tilde{N}|r_2|}{2(r_2^2+r_3^2)^{\frac{5}{2}}}\left[(r_3^2-r_2^2)\sin r_1\beta+\frac{\tilde{N}r_1r_2^2r_3}{\sqrt{r_2^2+r_3^2}}\cos r_1\beta\right]\left\{Ci\left[|r_1|(\tilde{N}+\beta)\right]-Ci\left[|r_1|(\tilde{N}-\beta)\right]\right\} \quad (35)$$

Equations 33 - 35 are free from singularities and their evaluations are more readily obtained.

4 Implementation of results

The combination of a source and sink can be used to create a Rankine ovoid. The strength and separation of the singularities can be obtained from potential flow theory, eg Newman (1978). For the non-dimensional analysis used here these equations become

$$\left(\frac{1}{4} - \xi^2\right)^2 = \frac{\xi Q}{2\pi}$$

$$Q = \pi b^2 \sqrt{1 + \frac{b^2}{\xi^2}}$$

where b is the non-dimensional half beam, Q is the non-dimensional strength and ξ is half the non-dimensional separation of the source and sink.

Q and ξ can be determined numerically for a specified b. The expressions describing the disturbance velocities, equations 19, 25 and 26 are evaluated twice, once with $r_1 = x - \xi$ (the source) then $r_1 = x + \xi$ (the sink). The resultant velocities are the difference between the two results. Finally the velocities are dimensionalised by multiplying by the free stream U. A prolate spheroid can be modelled using the slender body method described previously.

Figures 1 and 2 illustrate the velocities u and v generated on the fluid surface by a prolate spheroid moving at 2 m/s in a fluid possessing a constant Brunt-Väisälä frequency of 0.02 $rads/sec$. The spheroid is 100 m long and 10 m diameter with its centre line at a depth of 45 m. The area shown in these figures lies between 450 $-$ 1950 m behind the body and extends 400 m from the track. The body is moving towards the bottom of the page. The associated strains are shown in figures 3, 4 and 5. Figures 6 to 10 show similar results for a fluid possessing a constant Brunt-Väisälä frequency of 0.04 $rads/sec$. It can be observed that the angle of the "v" has increased, this is in agreement with results from layer models. Figures 11 to 15 illustrate results for a body speed of 5 m/s and a constant Brunt-Väisälä frequency of 0.02 $rads/sec$. In this case the angle of the "v" decreases, again as expected. Figure 16 and 17 show the velocities u and v generated on the fluid's surface by a prolate spheroid moving in a three layer fluid, the densities and thicknesses of each layer selected to represent as closely as possible a stratification with a maximum Brunt-Väisälä frequency of $N = 0.04$. The difference between the patterns is readily apparent.

The following figures are generated by a prolate spheroid moving at 2 m/s in a fluid possessing a constant Brunt-Väisälä frequency of 0.02 $rads/sec$. The spheroid is 100 m long and 10 m diameter with its centre line at a depth of 45 m. Figures 18 and 19 demonstrate the variation in the disturbance at various distances downstream of the body. The examples display the vertical velocity w from the fluid's surface to a depth of 400 m and extend to 400 m off track. These figures are readily identifiable with the images of internal waves generated by moving bodies obtained using the schlieren photography technique, see Lighthill(1979), page 314. Figures 20 and 21 illustrate how w appears at several distances away from the body's track. The velocity is shown from the fluid's surface to a depth of 400 m and a range of 450 $-$ 1950 m behind the body. The body is moving towards the right hand side of the page. Figures 22 and 23 show w calculated at a range of depths. The distances off track and behind the body are identical to those stated above. The body is moving towards the bottom of the page. Note how the disturbance moves off track as the disturbance rises through the fluid. These figures confirm the body is generating internal waves.

It can be seen from the figures 18 - 23 that the disturbance propagates deep into the fluid. For realistic stratification the Brunt-Väisälä frequency reduces with increasing depth i.e. the frequency of internal waves which can be supported in the fluid reduces with increasing depth. If the fluid possesses a maximum Brunt-Väisälä of $\hat{N}$ an internal wave of frequency ω ($\omega < \hat{N}$) created in that region can propagate upward or downward until the local value of N reduces to ω. At this depth the wave may behave in one of three ways :

1. It may experience total reflection upward into the fluid retaining its internal wave characteristics ;

2. It may be absorbed, degenerating into an interfacial wave of frequency ω at that depth, the kinetic energy associated with that frequency dispersing up and down in the fluid with an exponential decay ;

3. A combination of the above.

The surface disturbances generated by a body moving in a realistic stratification will almost certainly deviate from the constant Brunt-Väisälä frequency model due to the effects listed above.

5 Conclusions

The constant Brunt-Väisälä frequency model is the simpliest form of continuous stratification and a first model not to rely on constant density layers. Although this model does not represent a realistic stratification a descrepancy between the three layer model and this model has been demonstrated. This descrepancy is due to the generation of true internal waves and the interfacial waves generated by layered models. The successful development of this model indicates that a layered $N = N_1$, $N = N_2$, $N = N_3$ model is feasible and will produce results which approach the disturbances generated by a body moving in a realistic stratification.

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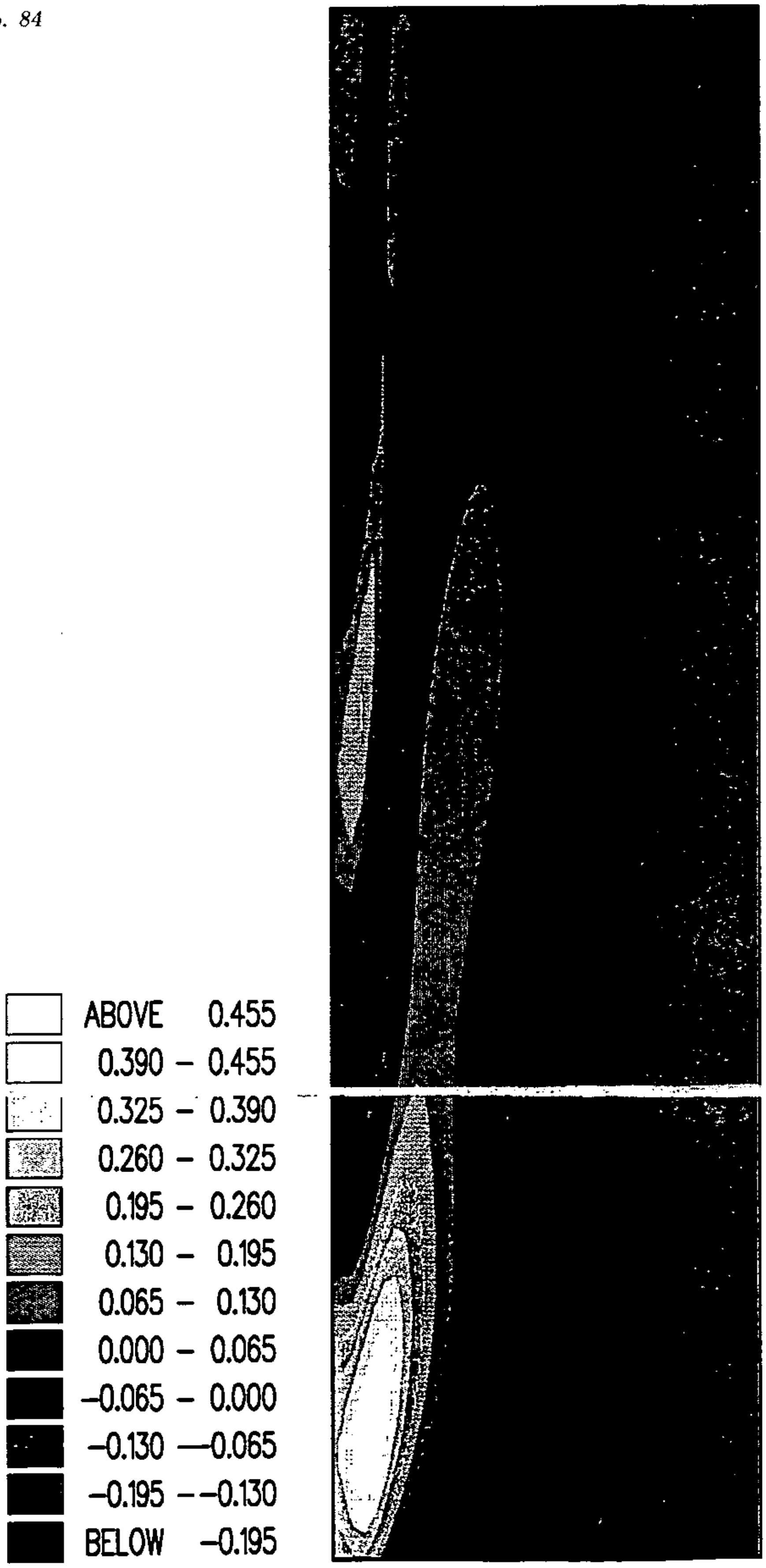


Figure 1: Contour plot of u (mm/s). $U = 2\ m/s$, $N = 0.02\ rads/s$.

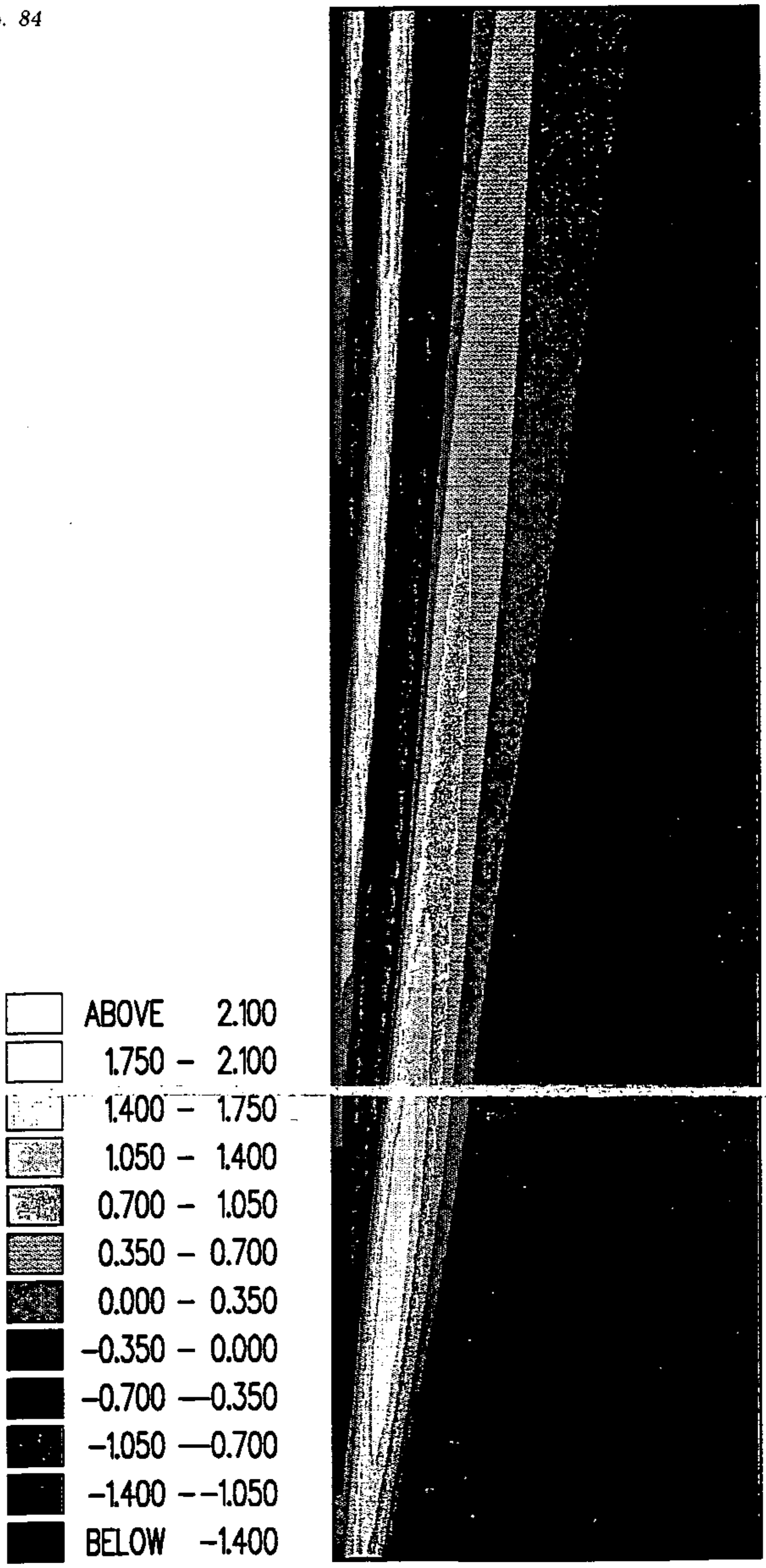


Figure 2: Contour plot of v (mm/s). $U = 2\ m/s$, $N = 0.02\ rads/s$.

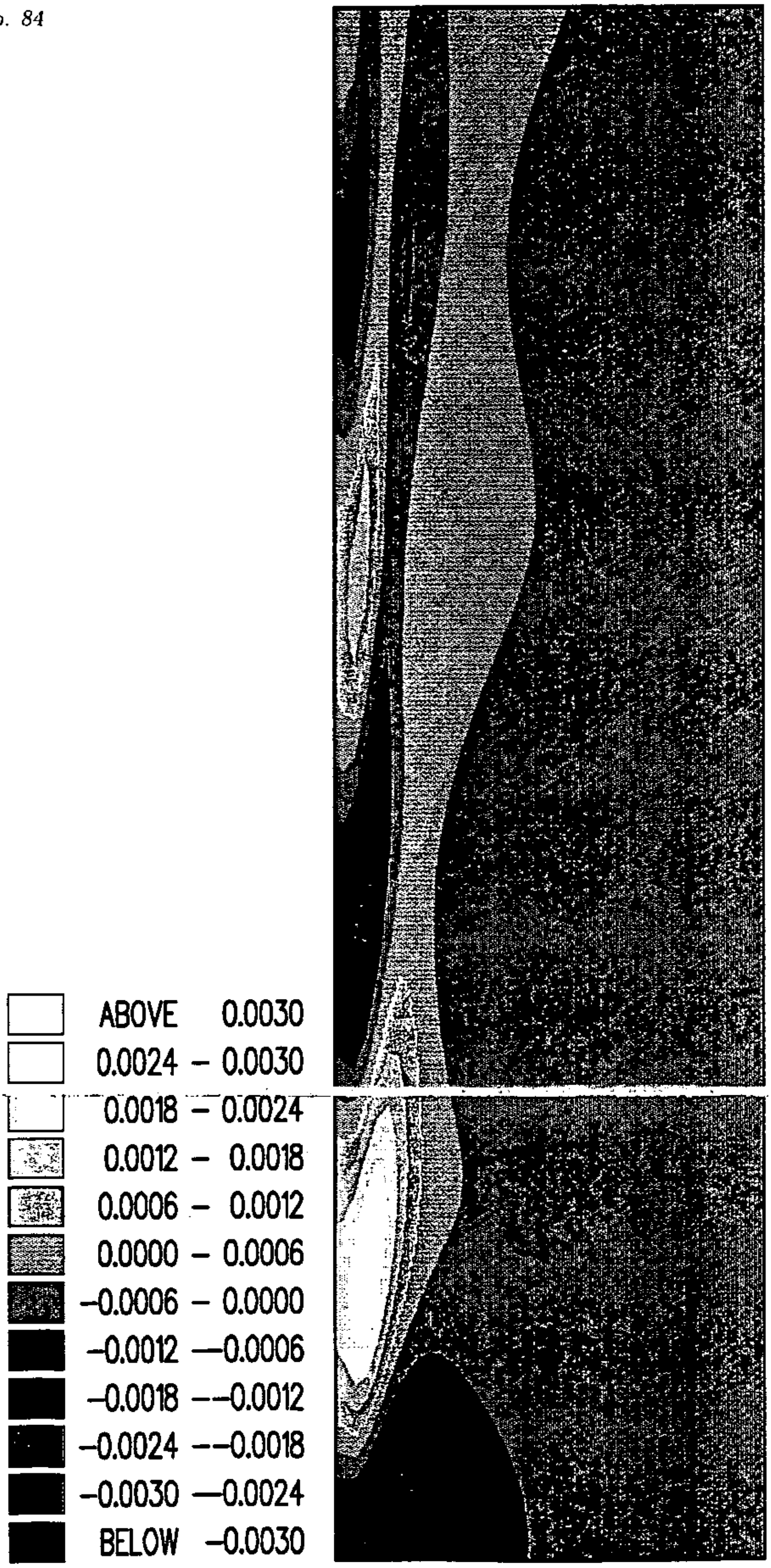
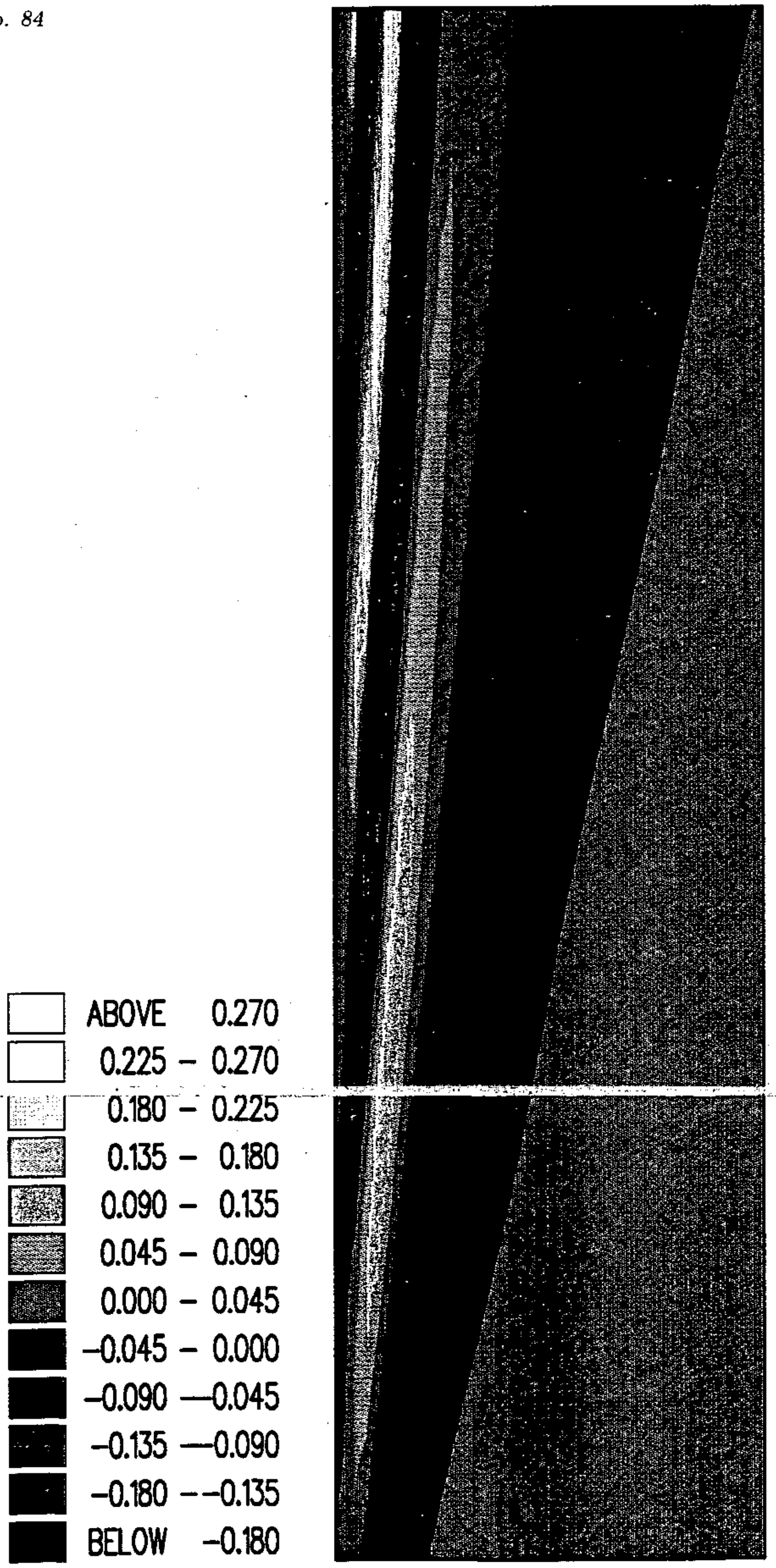


Figure 3: Contour plot of $10^3\ u_x$ $(1/s)$. $U = 2\ m/s$, $N = 0.02\ rads/s$.



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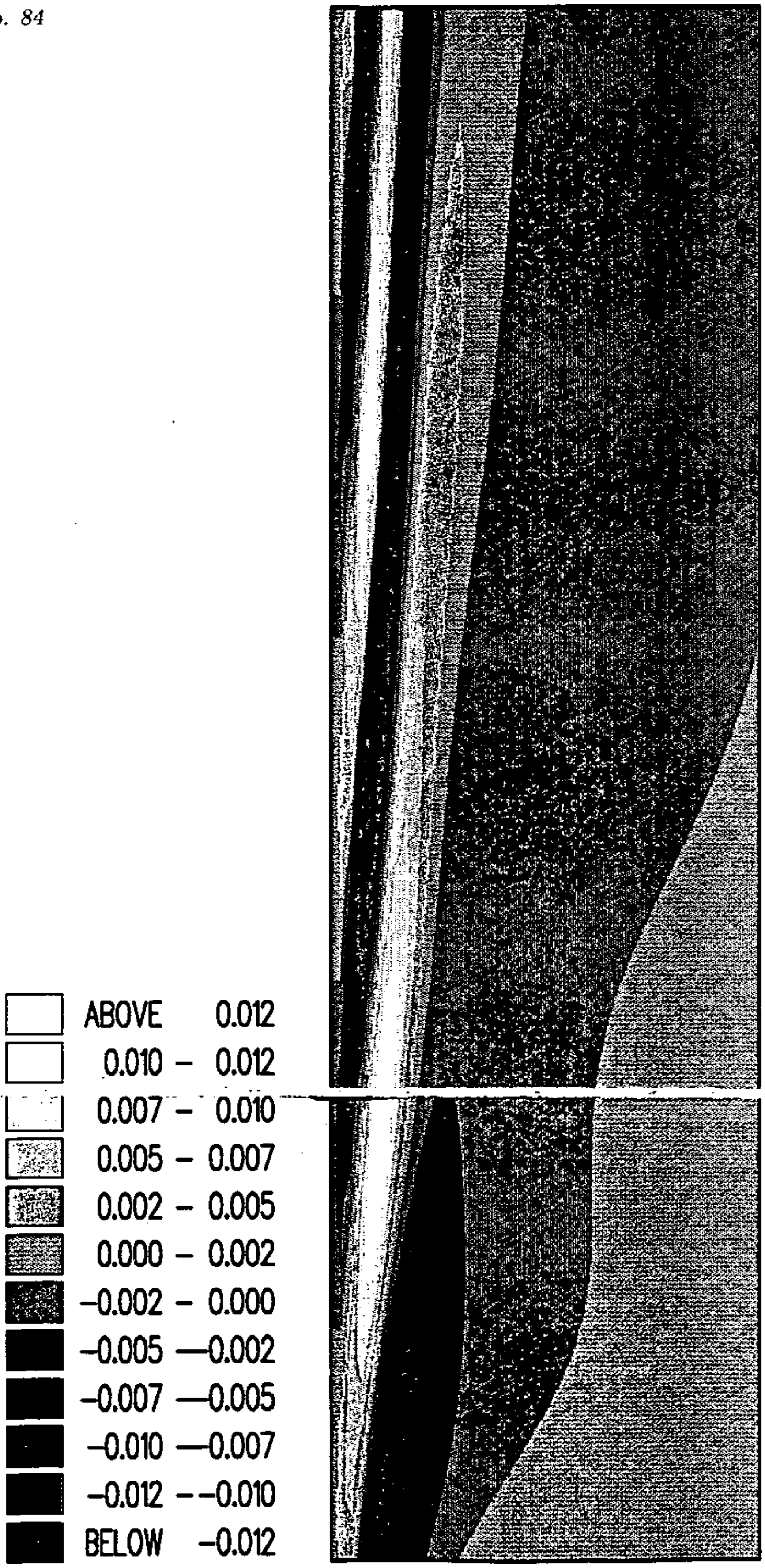

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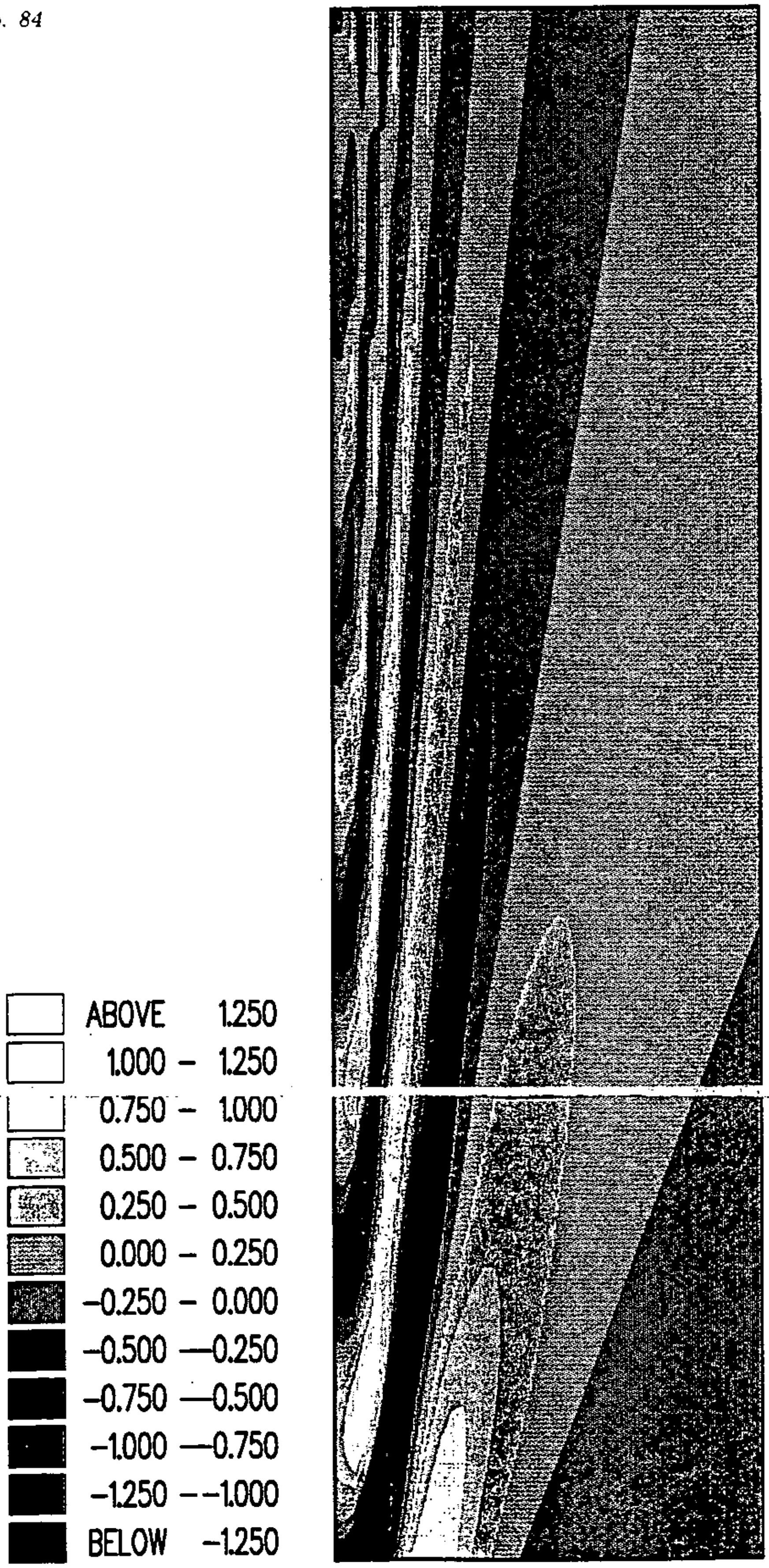

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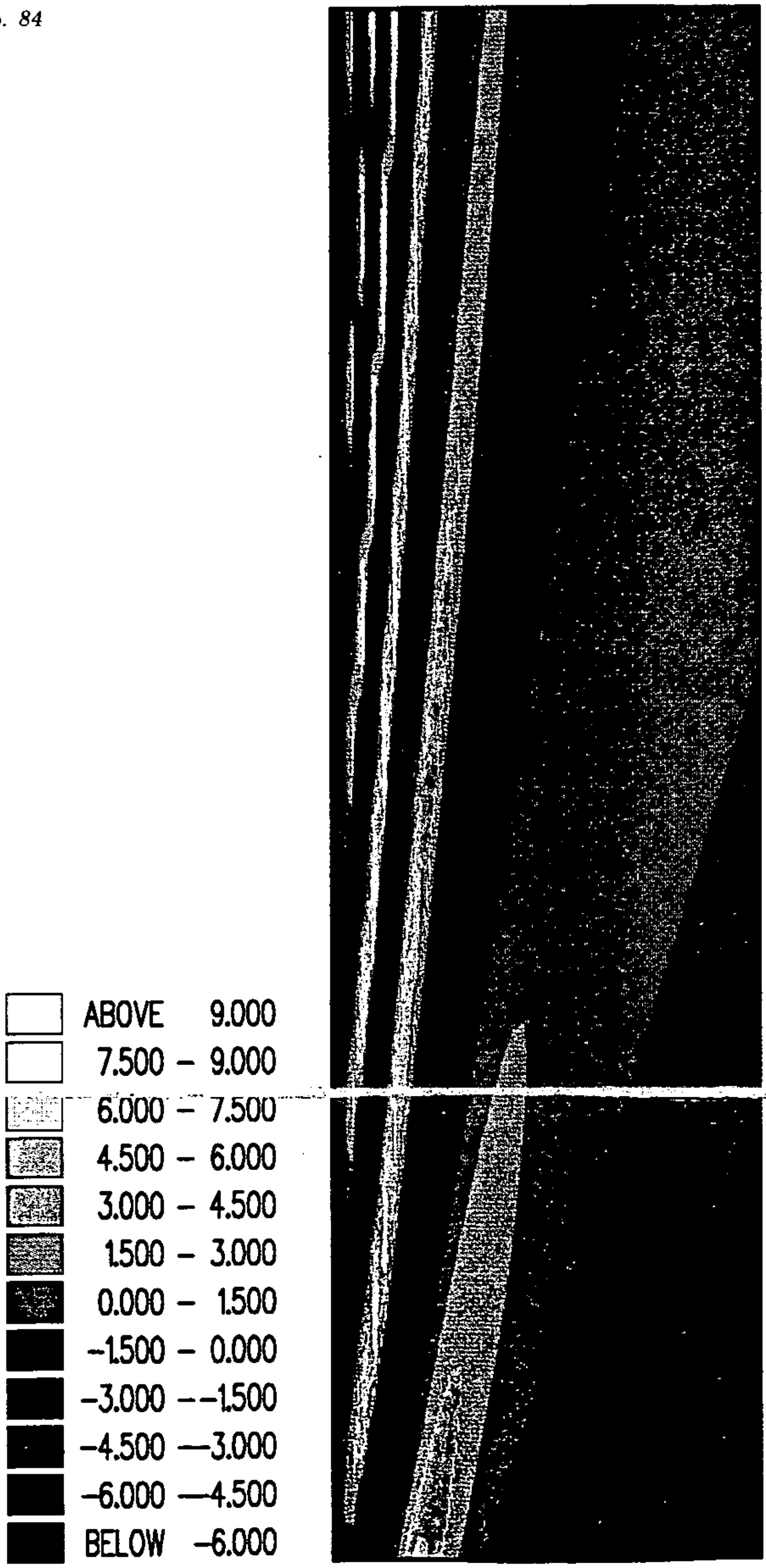

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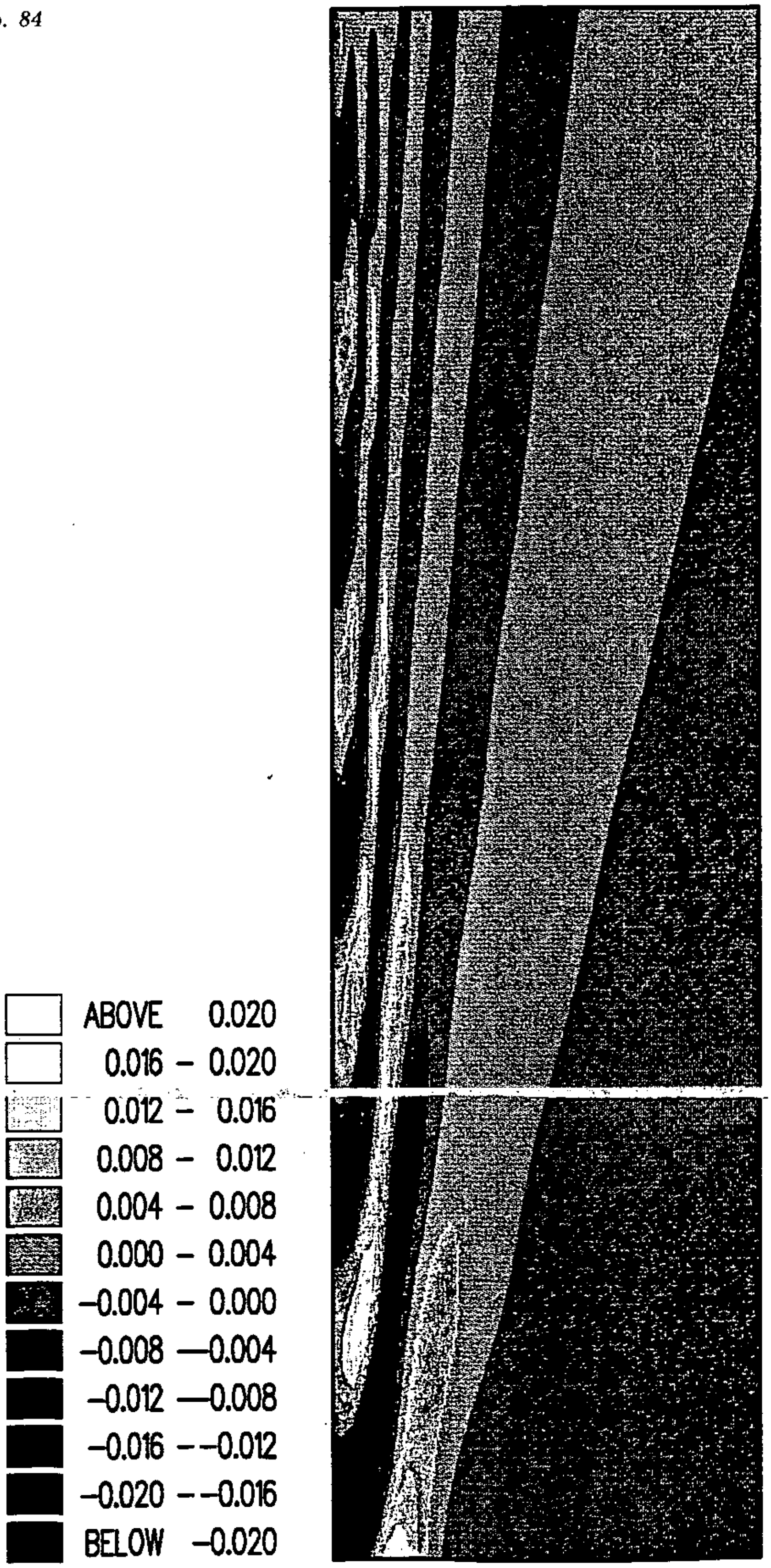

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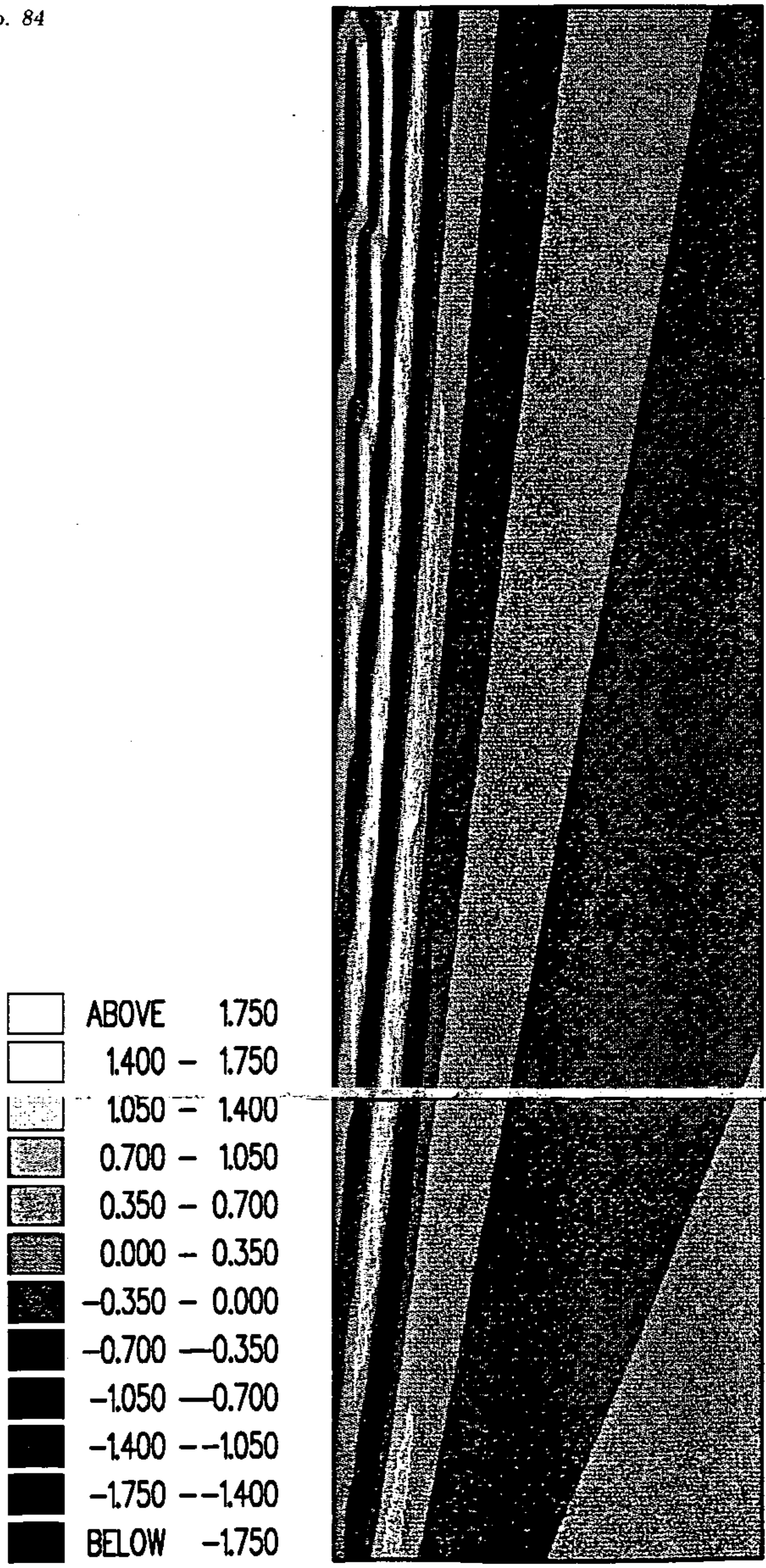

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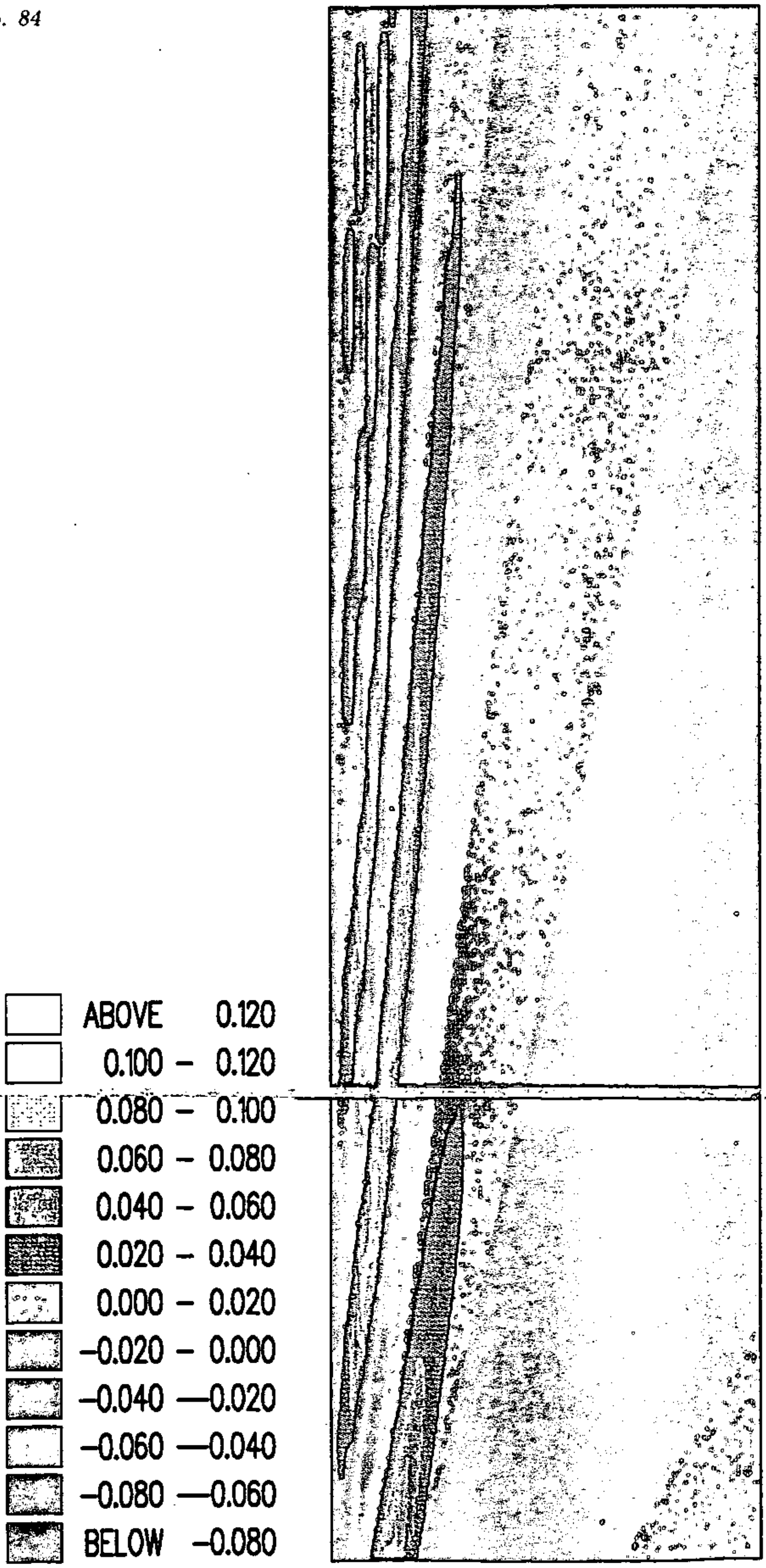

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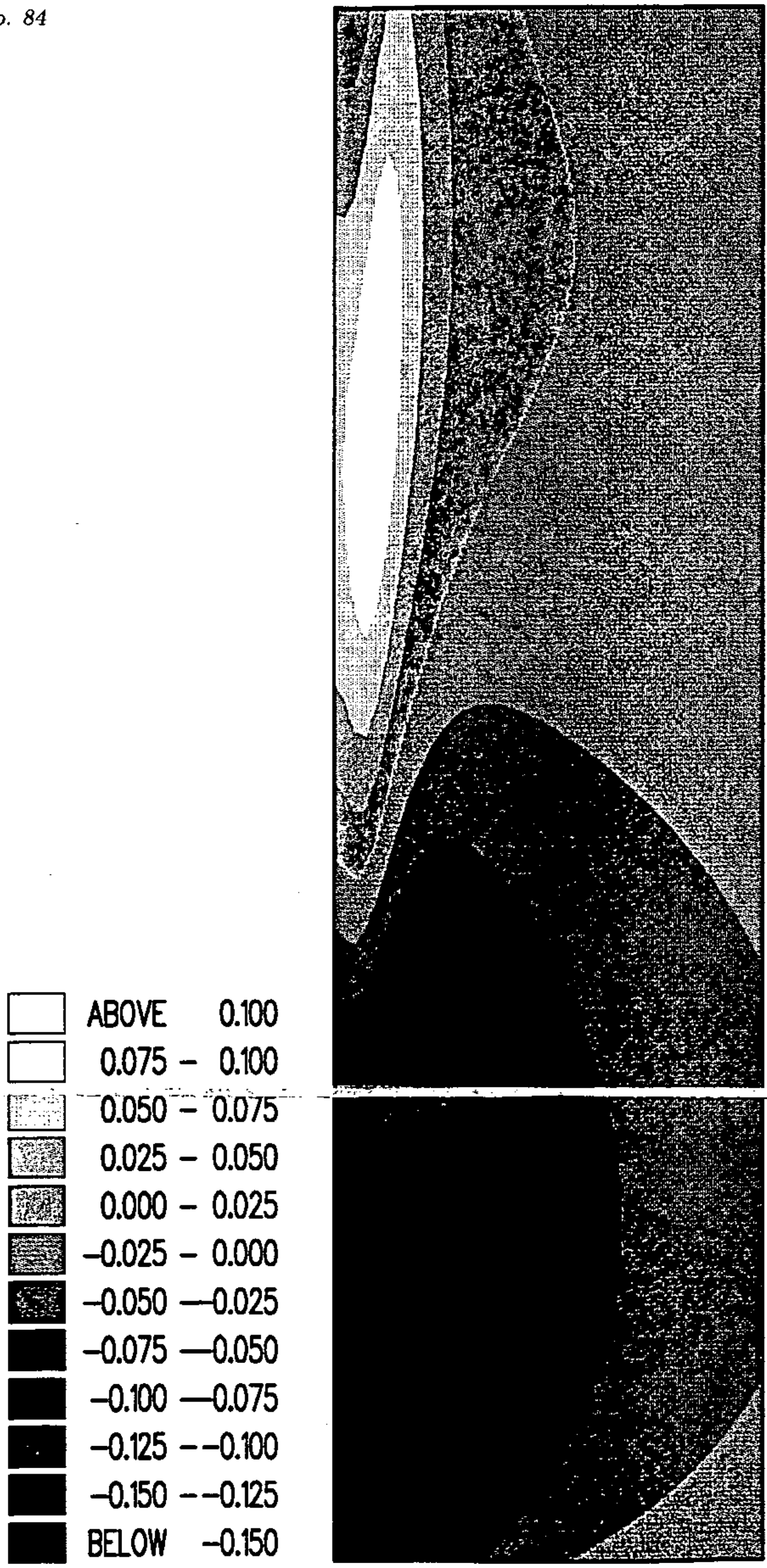

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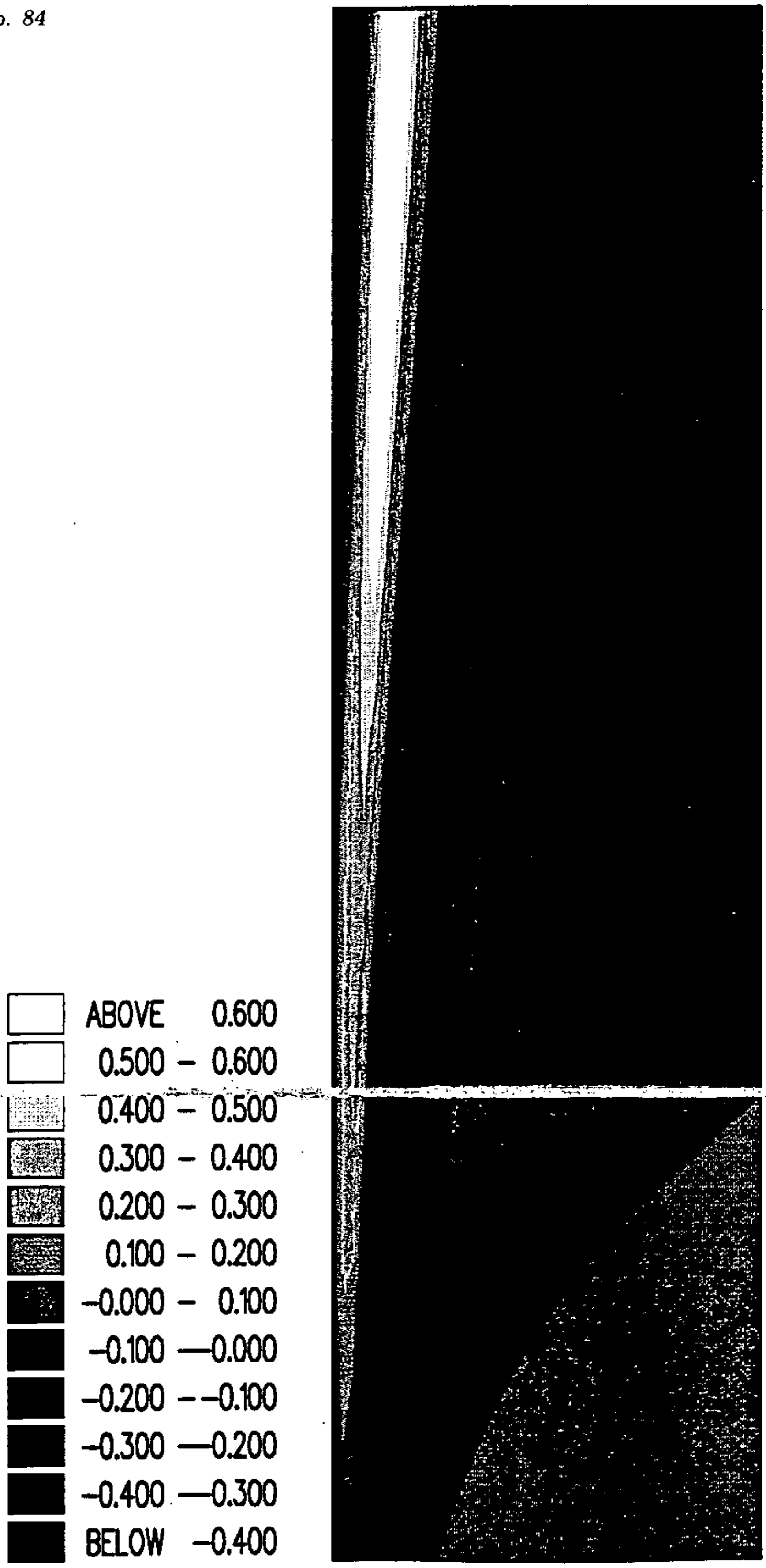

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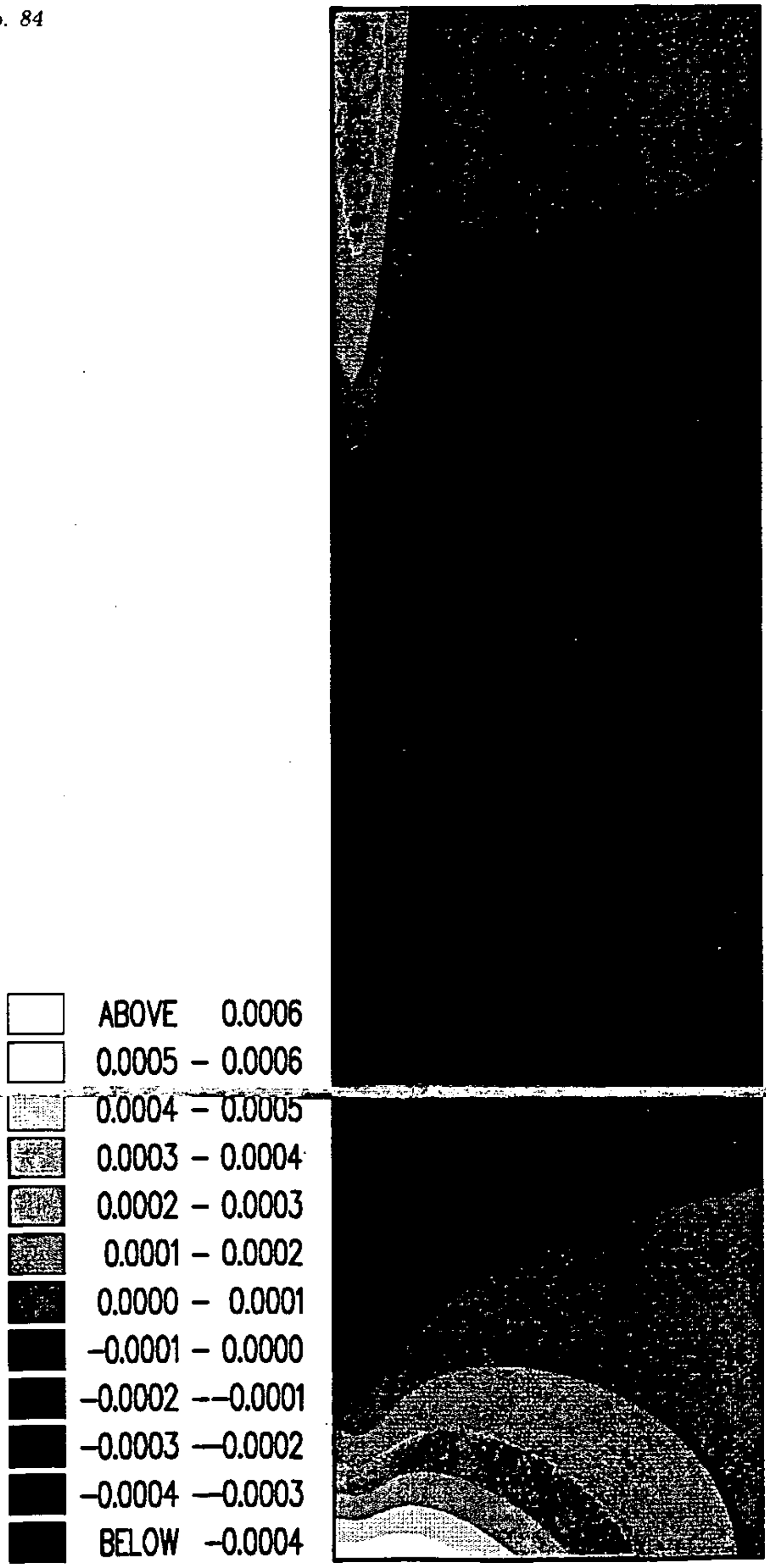

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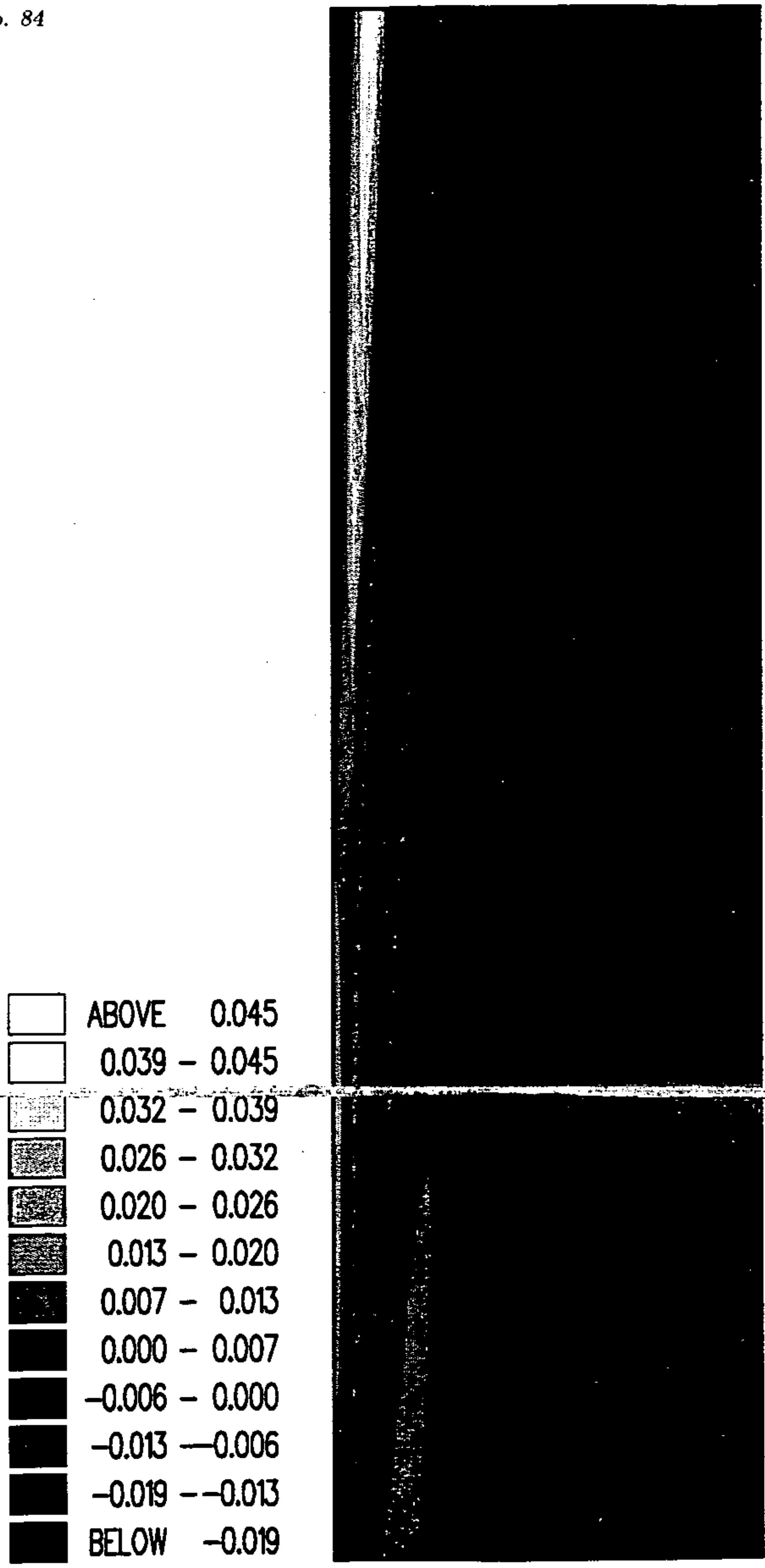

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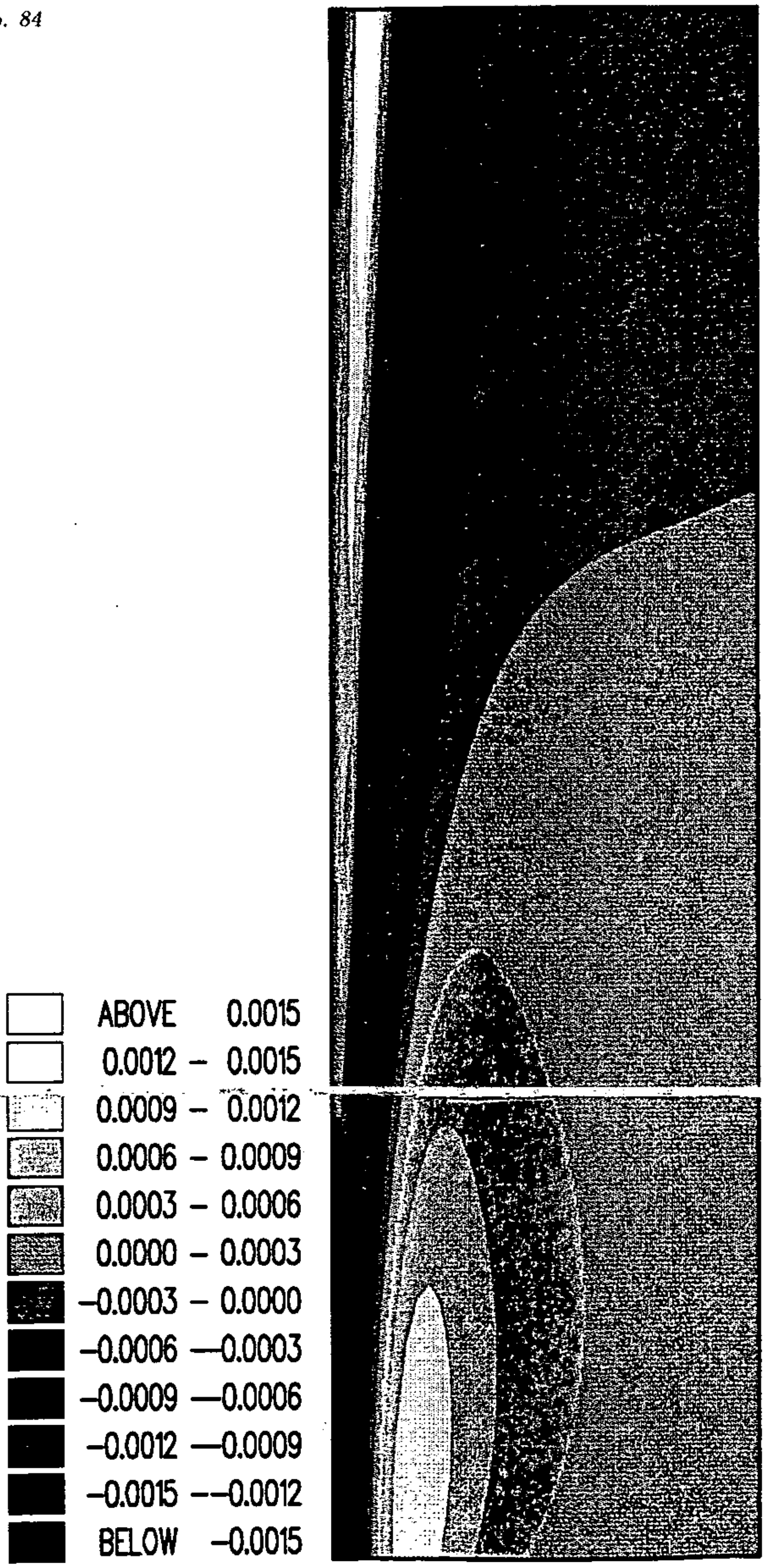

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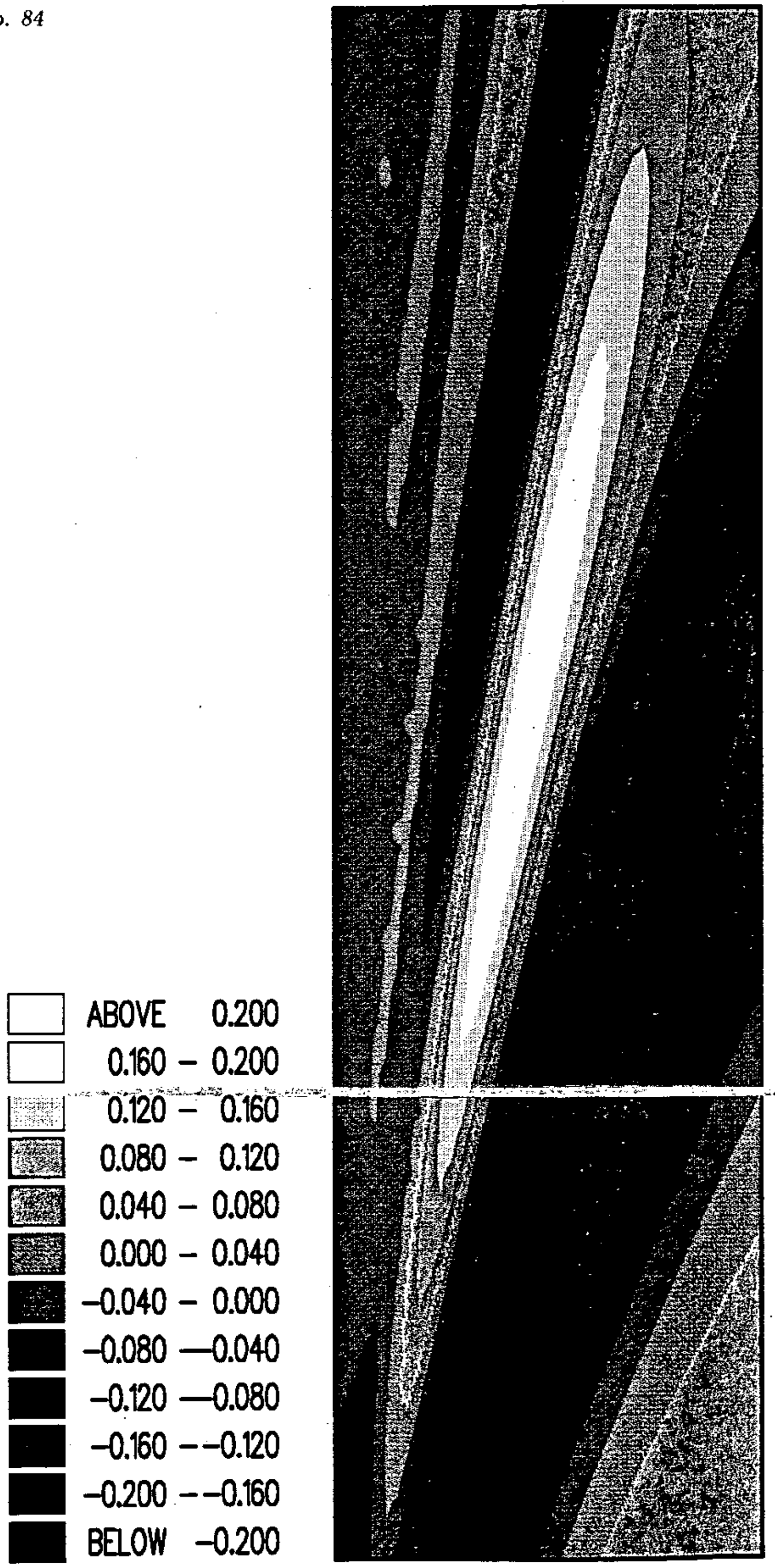

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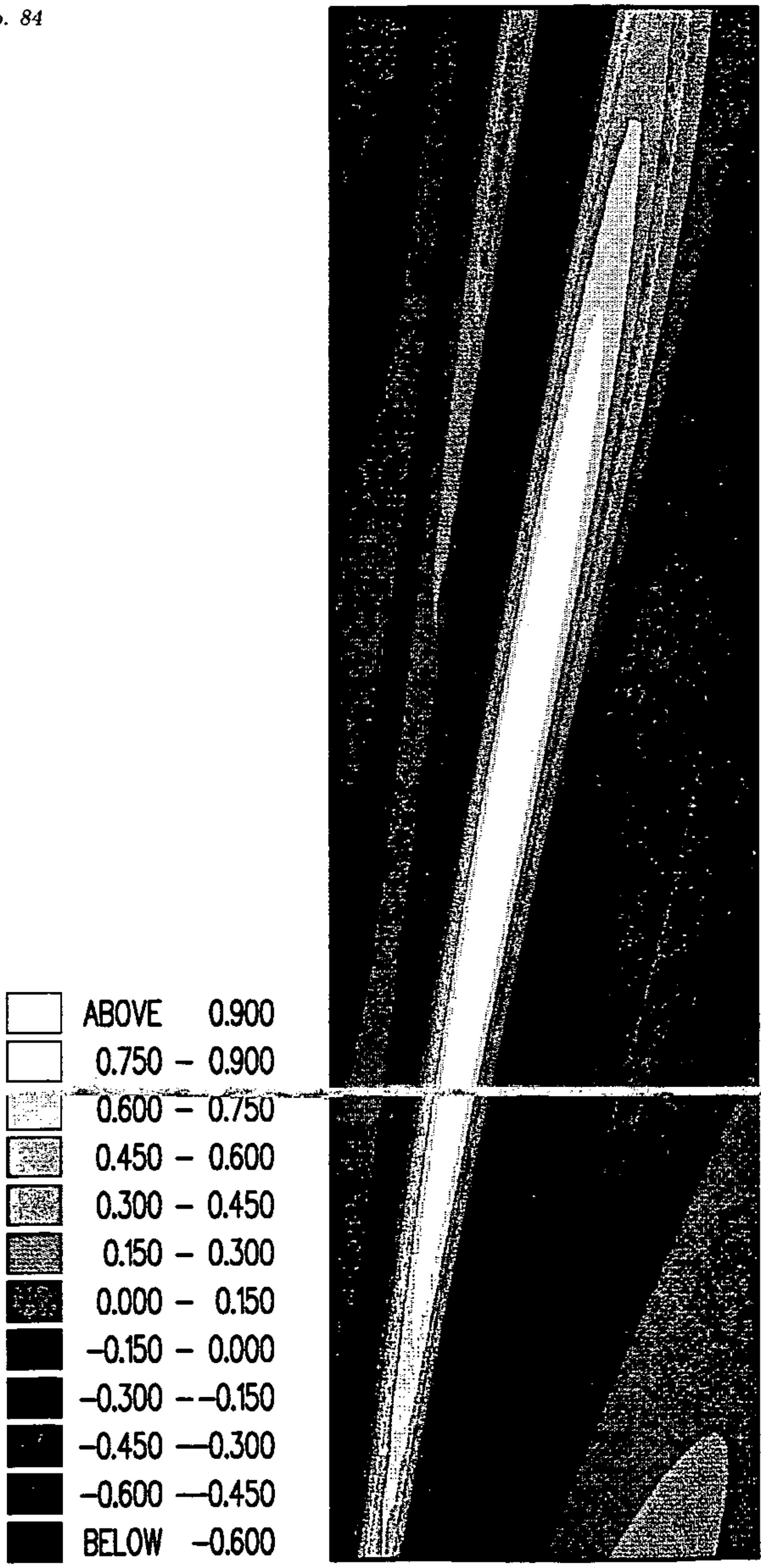
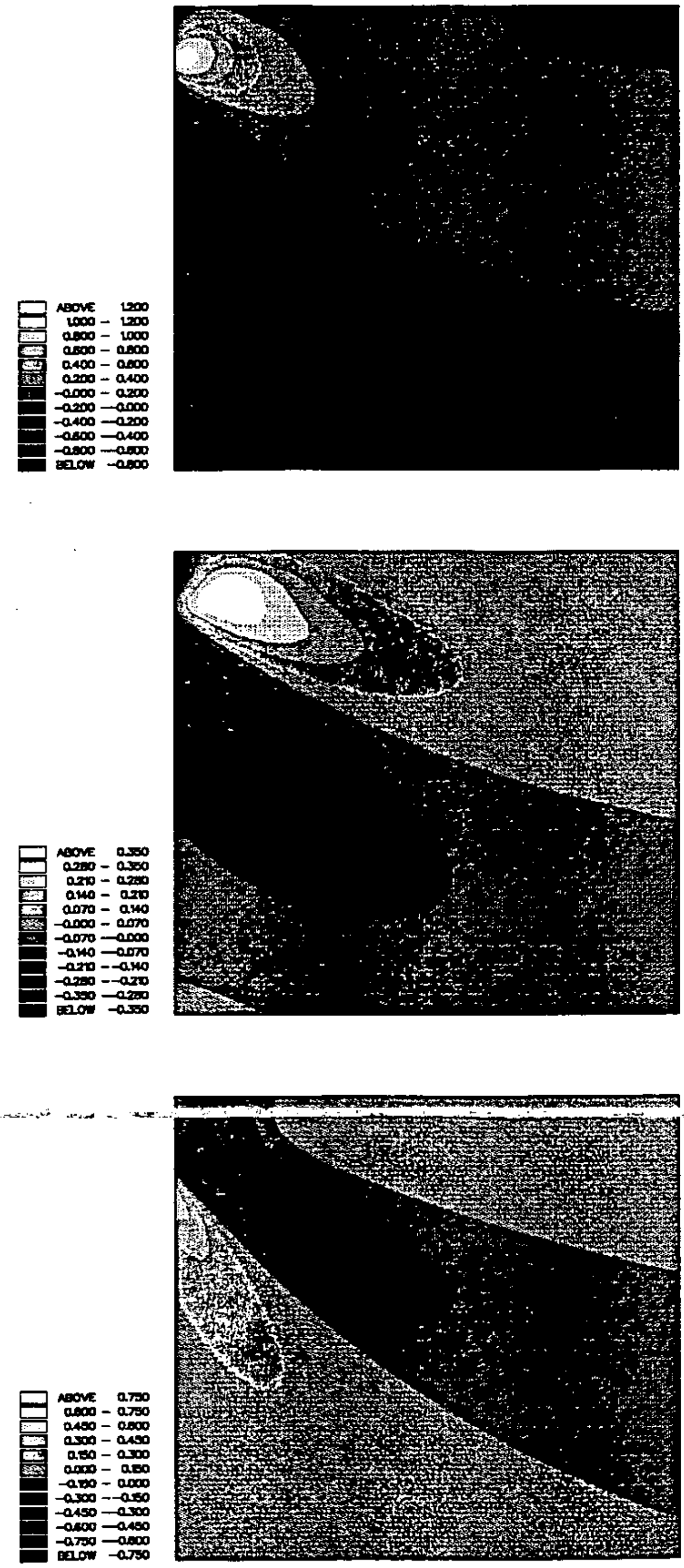


Figure 17: Contour plot of v (mm/s) using a three layer model. $U = 2\ m/s$, $\rho_1 = 1025\ Kg/m^3$, $\rho_2 = 1026.5\ Kg/m^3$, $\rho_3 = 1028\ Kg/m^3$, $t_1 = 30\ m$, $t_2 = 30\ m$.



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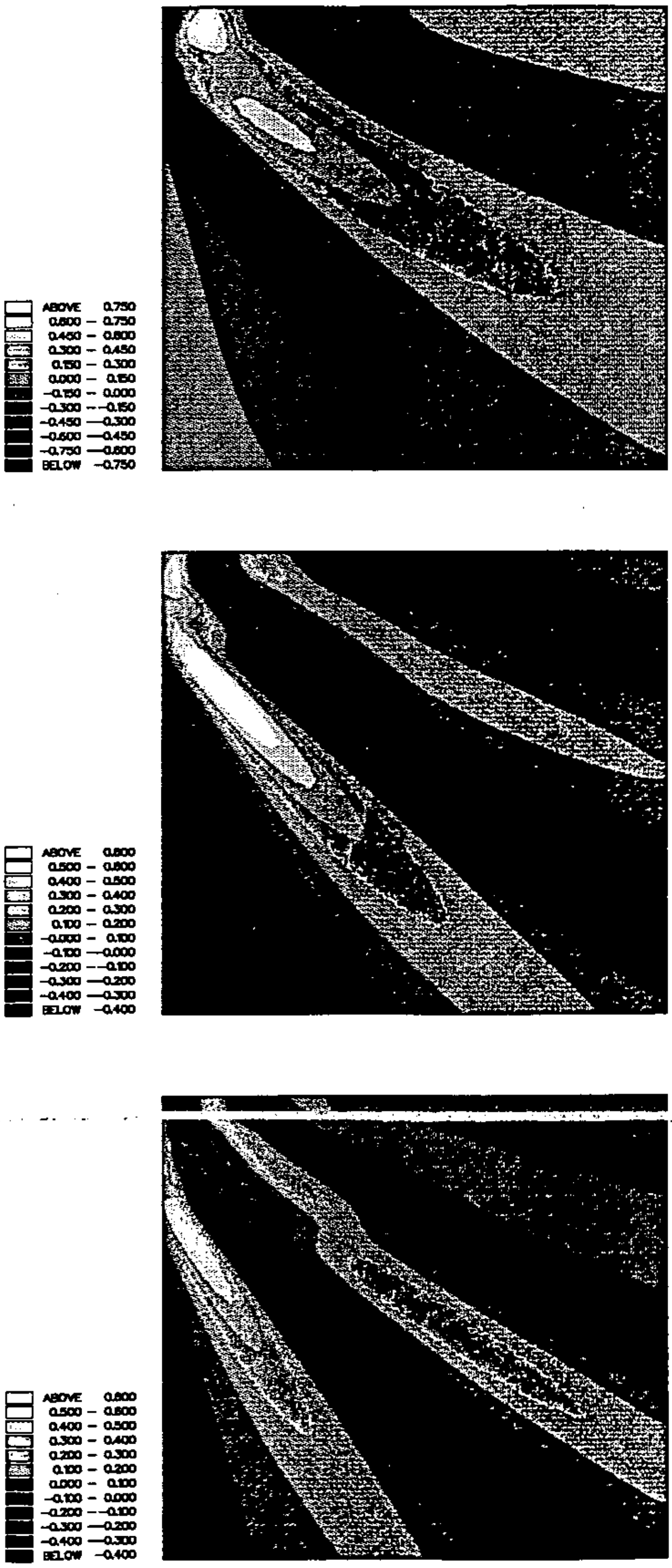

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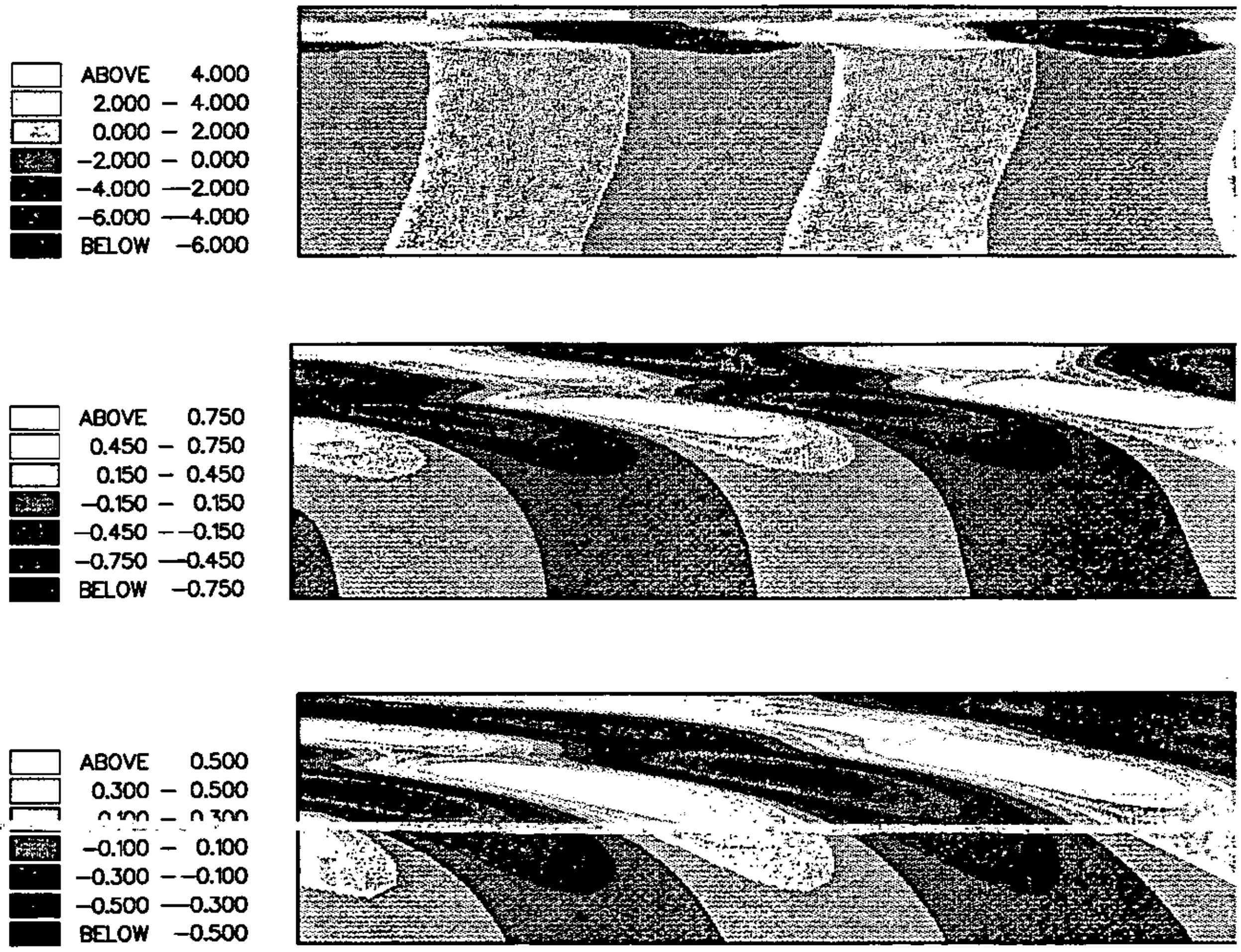

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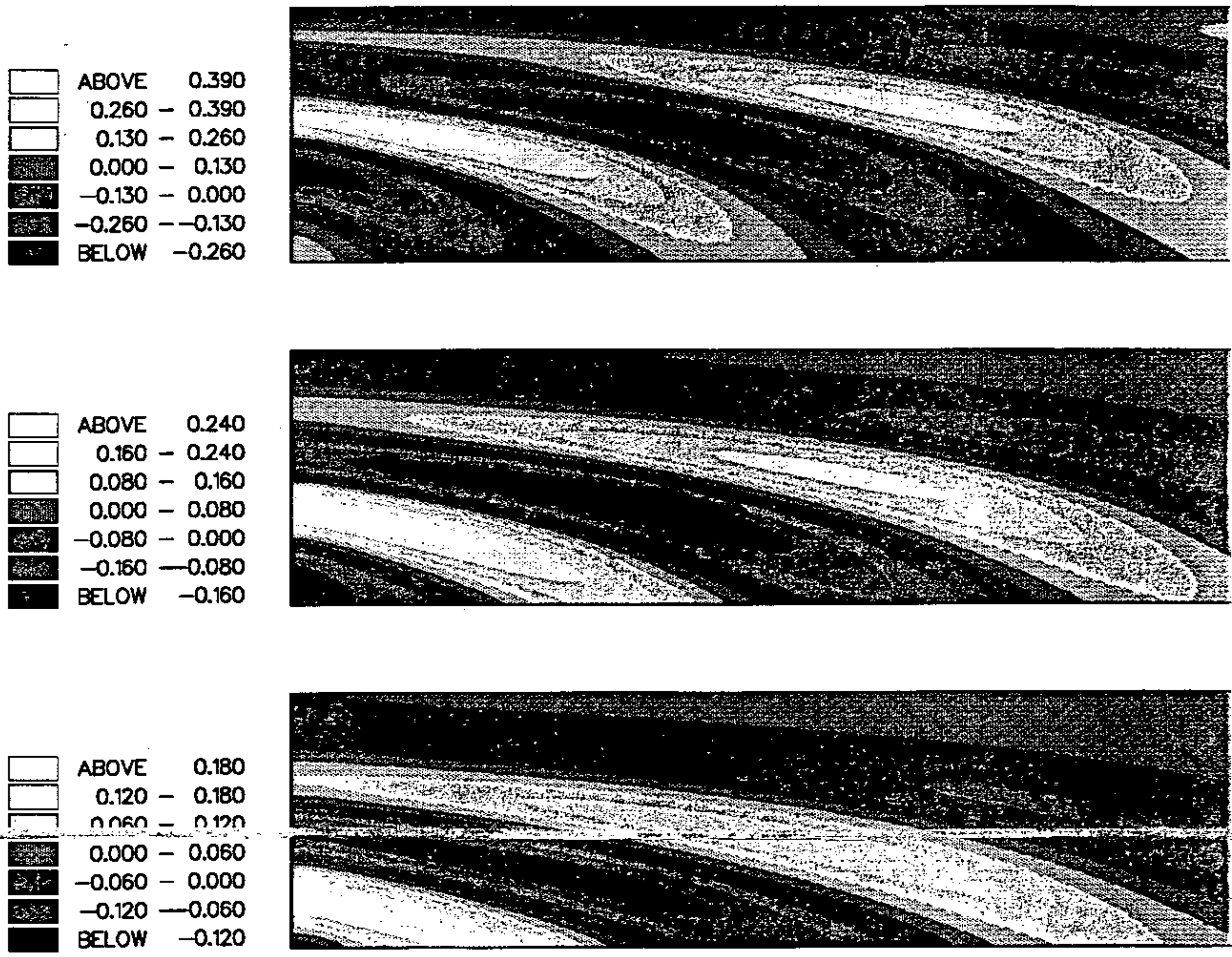

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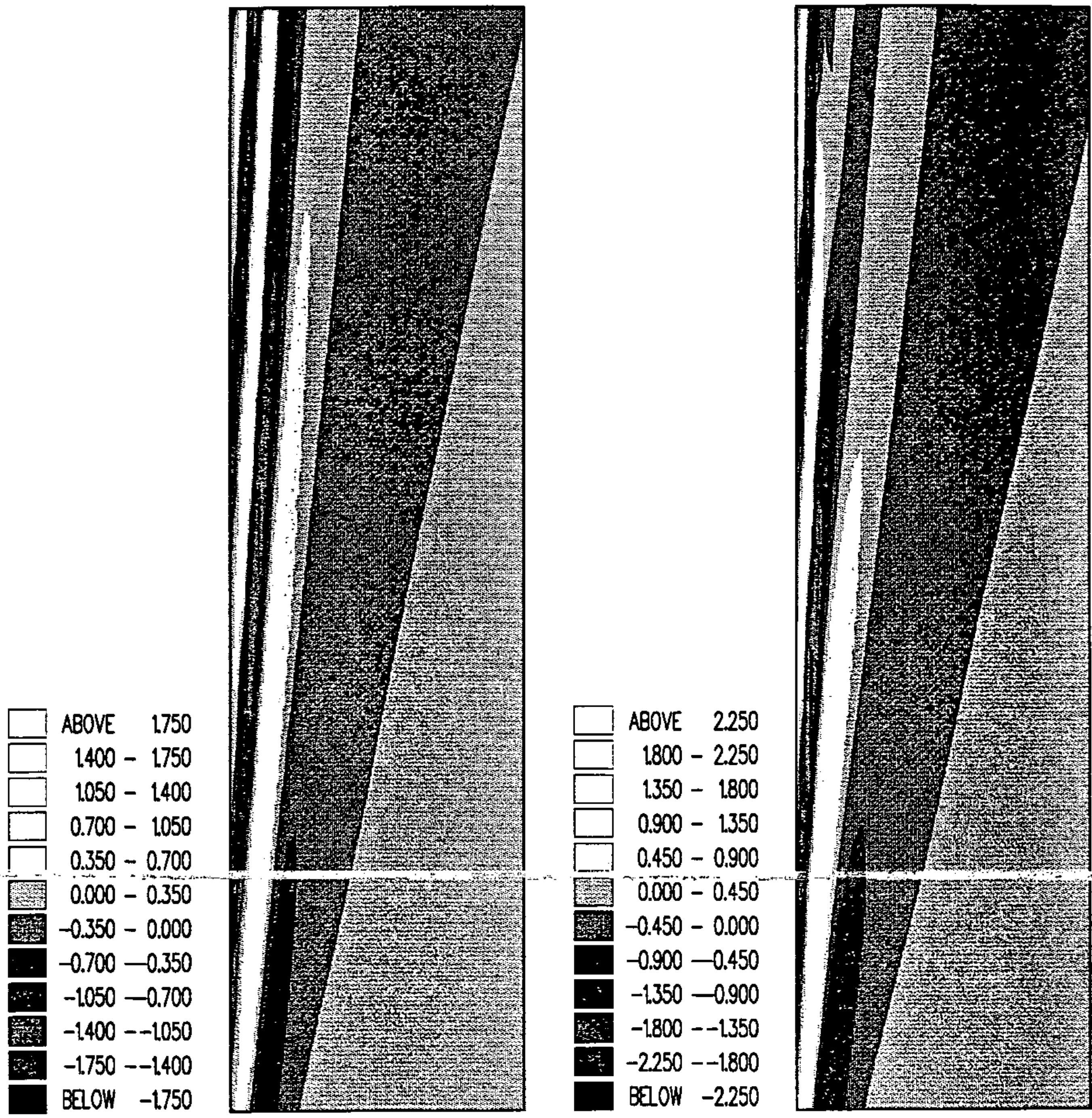

Figure 22: Vertical velocity w (mm/s) calculated on the horizontal planes $z = -10$ and -20 m

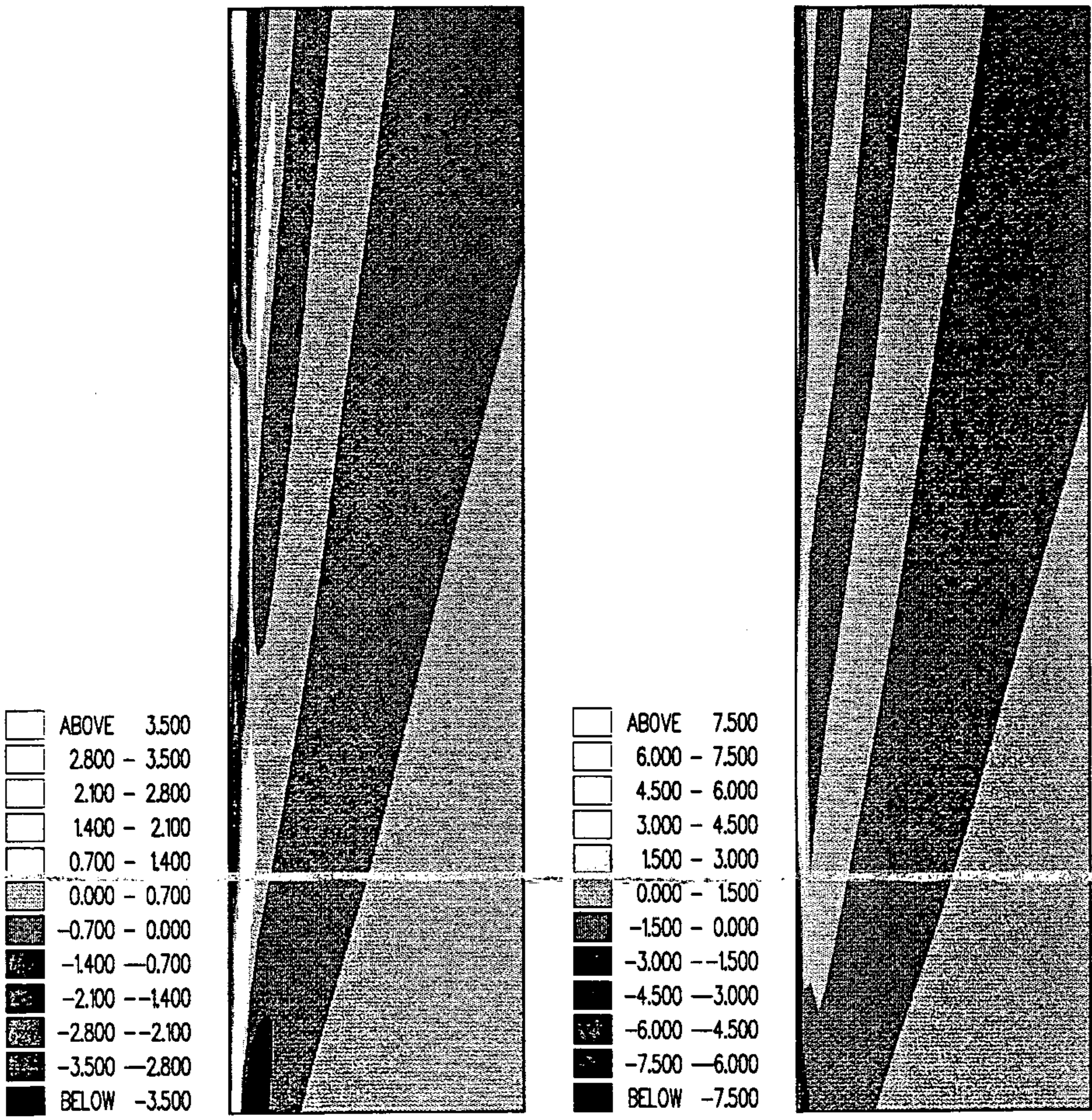

Figure 23: Vertical velocity w (mm/s) calculated on the horizontal planes $z = -30$ and -40 m

A : Solution of the singular integral

In order to evaluate the functions efficiently an integral of the form

$$\int_0^{\tilde{N}} \frac{x\sqrt{\tilde{N}^2 - x^2}}{(ax)^2 - b^2} \sin cx dx \tag{36}$$

appearing in the equations describing the fluid disturbance can be modified to remove the singularity at $x = \frac{b}{a}$. The singularity may be removed using the following method. Firstly let us rewrite the integral 36 as

$$\int_0^{\tilde{N}} \frac{\sqrt{\tilde{N}^2 - x^2} - \alpha}{(ax)^2 - b^2} x \sin cx dx + \int_0^{\tilde{N}} \frac{x\alpha \sin cx}{(ax)^2 - b^2} dx$$

The selection of an appropriate form of α will cause the integrand in the first integral to possess a finite limit at $x = \frac{b}{a}$, the integration can then be completed numerically using a NAG routine. The second integral has an analytic result. Expressed explicitly α must satisfy

$$\left[x(\sqrt{\tilde{N}^2 - x^2} - \alpha)\right]_{x=\frac{b}{a}} = 0$$

Therefore $\alpha = \sqrt{\tilde{N}^2 - (\frac{b}{a})^2}$. The integrand now has a finite limit at $x = \frac{b}{a}$

$$\lim_{x \to \frac{b}{a}} \left[\frac{\sqrt{\tilde{N}^2 - x^2} - \sqrt{\tilde{N}^2 - (\frac{b}{a})^2}}{(ax)^2 - b^2} \right]$$

$$= -\frac{1}{2a^2\sqrt{\tilde{N}^2 - (\frac{b}{a})^2}} \quad \text{for} \quad \tilde{N}^2 - (\tfrac{b}{a})^2 \neq 0$$

Integral 36 may now be written as

$$\int_0^{\tilde{N}} \frac{\sqrt{\tilde{N}^2 - x^2} - \sqrt{\tilde{N}^2 - (\frac{b}{a})^2}}{(ax)^2 - b^2} x \sin cx dx + \sqrt{\tilde{N}^2 - (\tfrac{b}{a})^2} \int_0^{\tilde{N}} \frac{x \sin cx}{(ax)^2 - b^2} dx$$

The first integrand can be rearranged to produce an analytic expression at $x = \frac{b}{a}$

$$\frac{\sqrt{\tilde{N}^2 - x^2} - \sqrt{\tilde{N}^2 - (\frac{b}{a})^2}}{(ax)^2 - b^2}$$

$$= -\frac{1}{a^2\left(\sqrt{\tilde{N}^2 - x^2} + \sqrt{\tilde{N}^2 - (\frac{b}{a})^2}\right)}$$

The second integral can be completed using the equalities

$$\int_0^{\tilde{N}} \frac{\sin cx}{ax \pm b} dx = \frac{1}{a}\left\{Si\left[c(\tilde{N} \pm \tfrac{b}{a})\right] \mp Si\left[\tfrac{cb}{a}\right]\right\} \cos\left(\tfrac{cb}{a}\right) \mp \frac{1}{a}\left\{Ci\left[|c|(\tilde{N} \pm \tfrac{b}{a})\right] - Ci\left[\tfrac{|c|b}{a}\right]\right\} \sin\left(\tfrac{cb}{a}\right) \quad a, b > 0$$

where $Ci(x)$ and $Si(x)$ are the cosine and sine integrals. Thus

$$\int_0^{\tilde{N}} \frac{x \sin cx}{(ax)^2 - b^2} dx = \frac{1}{2a}\left\{\int_0^{\tilde{N}} \frac{\sin cx}{ax + b} dx + \int_0^{\tilde{N}} \frac{\sin cx}{ax - b} dx\right\}$$

$$= \frac{1}{2a^2}\left\{Si\left[c(\tilde{N}+\frac{b}{a})\right]+Si\left[c(\tilde{N}-\frac{b}{c})\right]\right\}\cos\left(\frac{cb}{a}\right)-\frac{1}{2a^2}\left\{Ci\left[|c|(\tilde{N}+\frac{b}{a})\right]-Ci\left[|c|(\tilde{N}-\frac{b}{a})\right]\right\}\sin\left(\frac{cb}{a}\right)$$

Combining previous equalities, integral 36 becomes

$$\begin{aligned} \int_0^{\tilde{N}} \frac{x\sqrt{\tilde{N}^2-x^2}}{(ax)^2-b^2}\sin cx dx &= -\frac{1}{a^2}\int_0^{\tilde{N}} \frac{x\sin cx}{\sqrt{\tilde{N}^2-x^2}+\sqrt{\tilde{N}^2-(\frac{b}{a})^2}}dx \\ &+\frac{\sqrt{\tilde{N}^2-(\frac{b}{a})^2}}{2a^2}\left\{Si\left[c(\tilde{N}+\tfrac{b}{a})\right]+Si\left[c(\tilde{N}-\tfrac{b}{a})\right]\right\}\cos\left(\tfrac{cb}{a}\right) \\ &-\frac{\sqrt{\tilde{N}^2-(\frac{b}{a})^2}}{2a^2}\left\{Ci\left[|c|(\tilde{N}+\tfrac{b}{a})\right]+Ci\left[|c|(\tilde{N}-\tfrac{b}{a})\right]\right\}\sin\left(\tfrac{cb}{a}\right) \end{aligned} \quad (37)$$

which is free of singularities and can be evaluated easily.

B : Functions arising from the ξ integration

In section 3.5, for ease of representation it was convenient to denote integrals which remained undefined. This omission is corrected here and such terms are expressed as follows

$$C_1(x,\lambda_1) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \xi \cos\lambda_1(x-\xi)d\xi = \frac{2\sin x\lambda_1}{\lambda_1^2}\left(\sin\frac{\lambda_1}{2} - \frac{\lambda_1}{2}\cos\frac{\lambda_1}{2}\right)$$

$$\beta\ CiC_1(x,\pm\beta) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \xi Ci[|x-\xi|(\tilde{N}\pm\beta)]\cos\beta(x-\xi)d\xi$$

$$= (x+\frac{1}{2})\sin\beta(x+\frac{1}{2})Ci[|x+\frac{1}{2}|(\tilde{N}\pm\beta)] - (x-\frac{1}{2})\sin\beta(x-\frac{1}{2})Ci[|x-\frac{1}{2}|(\tilde{N}\pm\beta)]$$

$$-Ci[|x+\frac{1}{2}|(\tilde{N}\pm\beta)]\left[(x+\frac{1}{2})\sin\beta(x+\frac{1}{2}) + \frac{\cos\beta(x+\frac{1}{2})}{\beta}\right] + Ci[(x-\frac{1}{2})(\tilde{N}\pm\beta)]\left[(x-\frac{1}{2})\sin\beta(x-\frac{1}{2}) + \frac{\cos\beta(x-\frac{1}{2})}{\beta}\right]$$

$$+\frac{I_4}{\beta} + I_6 - xI_8$$

$$-\beta\ CiC_2(x,\pm\beta) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \xi(x-\xi)Ci[|x-\xi|(\tilde{N}\pm\beta)]\cos\beta(x-\xi)d\xi$$

$$= -xCi[|x+\frac{1}{2}|(\tilde{N}\pm\beta)]\left[(x+\frac{1}{2})\sin\beta(x+\frac{1}{2}) + \frac{\cos\beta(x+\frac{1}{2})}{\beta}\right]$$

$$+xCi[(x-\frac{1}{2})(\tilde{N}\pm\beta)]\left[(x-\frac{1}{2})\sin\beta(x-\frac{1}{2}) + \frac{\cos\beta(x-\frac{1}{2})}{\beta}\right]$$

$$+Ci[|x+\frac{1}{2}|(\tilde{N}\pm\beta)]\left[(x+\frac{1}{2})^2\sin\beta(x+\frac{1}{2}) + \frac{2(x+\frac{1}{2})}{\beta}\cos\beta(x+\frac{1}{2}) - \frac{2}{\beta^2}\sin\beta(x+\frac{1}{2})\right]$$

$$-Ci[|x-\frac{1}{2}|(\tilde{N}\pm\beta)]\left[(x-\frac{1}{2})^2\sin\beta(x-\frac{1}{2}) + \frac{2(x-\frac{1}{2})}{\beta}\cos\beta(x-\frac{1}{2}) - \frac{2}{\beta^2}\sin\beta(x-\frac{1}{2})\right]$$

$$+\frac{x}{\beta}I_4 - \frac{2}{\beta}I_5 + xI_6 + \frac{2}{\beta^2}I_8 - I_{10}$$

$$-\beta\ CiS_1(x,\pm\beta) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \xi Ci[|x-\xi|(\tilde{N}\pm\beta)]\sin\beta(x-\xi)d\xi$$

$$= (x+\frac{1}{2})\cos\beta(x+\frac{1}{2})Ci[|x+\frac{1}{2}|(\tilde{N}\pm\beta)] - (x-\frac{1}{2})\cos\beta(x-\frac{1}{2})Ci[|x-\frac{1}{2}|(\tilde{N}\pm\beta)]$$

$$-Ci[|x+\frac{1}{2}|(\tilde{N}\pm\beta)]\left[(x+\frac{1}{2})\cos\beta(x+\frac{1}{2})-\frac{\sin\beta(x+\frac{1}{2})}{\beta}\right]+Ci[(x-\frac{1}{2})(\tilde{N}\pm\beta)]\left[(x-\frac{1}{2})\cos\beta(x-\frac{1}{2})-\frac{\sin\beta(x-\frac{1}{2})}{\beta}\right]$$

$$-xI_4+I_5-\frac{I_8}{\beta}$$

$$CSSN_1(x,\pm\beta)=\int_{-\frac{1}{2}}^{\frac{1}{2}}\xi\cos[(x-\xi)(\tilde{N}\pm\beta)]\sin\beta(x-\xi)d\xi=xI_6-I_{10}$$

$$CSSN_2(x,\pm\beta)=\int_{-\frac{1}{2}}^{\frac{1}{2}}\xi(x-\xi)\cos[(x-\xi)(\tilde{N}\pm\beta)]\sin\beta(x-\xi)d\xi=xI_8-I_6$$

$$S_1(x,\lambda_1)=\int_{-\frac{1}{2}}^{\frac{1}{2}}\xi\sin\lambda_1(x-\xi)d\xi=\frac{2\cos x\lambda_1}{\lambda_1^2}\left(\frac{\lambda_1}{2}\cos\frac{\lambda_1}{2}-\sin\frac{\lambda_1}{2}\right)$$

$$\beta\ S_2(x,\pm\beta)=\int_{-\frac{1}{2}}^{\frac{1}{2}}\xi(x-\xi)\sin\beta(x-\xi)d\xi$$

$$=-\frac{(x+1)}{\beta}\sin\beta(x+\frac{1}{2})+\frac{(x-1)}{\beta}\sin\beta(x-\frac{1}{2})+\left(\frac{1}{4}+\frac{x}{2}-\frac{2}{\beta^2}\right)\cos\beta(x+\frac{1}{2})-\left(\frac{1}{4}-\frac{x}{2}-\frac{2}{\beta^2}\right)\cos\beta(x-\frac{1}{2})$$

$$\beta\ SiC_1(x,\pm\beta)=\int_{-\frac{1}{2}}^{\frac{1}{2}}\xi Si[(x-\xi)(\tilde{N}\pm\beta)]\cos\beta(x-\xi)d\xi$$

$$=(x+\frac{1}{2})\sin\beta(x+\frac{1}{2})Si[(x+\frac{1}{2})(\tilde{N}\pm\beta)]-(x-\frac{1}{2})\sin\beta(x-\frac{1}{2})Si[(x-\frac{1}{2})(\tilde{N}\pm\beta)]$$

$$-Si[(x+\frac{1}{2})(\tilde{N}\pm\beta)]\left[(x+\frac{1}{2})\sin\beta(x+\frac{1}{2})+\frac{\cos\beta(x+\frac{1}{2})}{\beta}\right]+Si[(x-\frac{1}{2})(\tilde{N}\pm\beta)]\left[(x-\frac{1}{2})\sin\beta(x-\frac{1}{2})+\frac{\cos\beta(x-\frac{1}{2})}{\beta}\right]$$

$$-xI_1+I_2+\frac{I_3}{\beta}$$

$$-\beta\ SiS_1(x,\pm\beta)=\int_{-\frac{1}{2}}^{\frac{1}{2}}\xi Si[(x-\xi)(\tilde{N}\pm\beta)]\sin\beta(x-\xi)d\xi$$

$$=(x+\frac{1}{2})\cos\beta(x+\frac{1}{2})Si[(x+\frac{1}{2})(\tilde{N}\pm\beta)]-(x-\frac{1}{2})\cos\beta(x-\frac{1}{2})Si[(x-\frac{1}{2})(\tilde{N}\pm\beta)]$$

$$-Si[(x+\frac{1}{2})(\tilde{N}\pm\beta)]\left[(x+\frac{1}{2})\cos\beta(x+\frac{1}{2})-\frac{\sin\beta(x+\frac{1}{2})}{\beta}\right]+Si[(x-\frac{1}{2})(\tilde{N}\pm\beta)]\left[(x-\frac{1}{2})\cos\beta(x-\frac{1}{2})-\frac{\sin\beta(x-\frac{1}{2})}{\beta}\right]$$

$$-\frac{I_1}{\beta} - xI_3 + I_7$$

$$-\beta\ SiS_2(x,\pm\beta) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \xi(x-\xi)Si[(x-\xi)(\tilde{N}\pm\beta)]\sin\beta(x-\xi)d\xi$$

$$= xSi[(x+\frac{1}{2})(\tilde{N}\pm\beta)]\left[(x+\frac{1}{2})\cos\beta(x+\frac{1}{2}) - \frac{\sin\beta(x+\frac{1}{2})}{\beta}\right]$$

$$-xSi[(x-\frac{1}{2})(\tilde{N}\pm\beta)]\left[(x-\frac{1}{2})\cos\beta(x-\frac{1}{2}) - \frac{\sin\beta(x-\frac{1}{2})}{\beta}\right]$$

$$-Si[(x+\frac{1}{2})(\tilde{N}\pm\beta)]\left[(x+\frac{1}{2})^2\cos\beta(x+\frac{1}{2}) - \frac{2(x+\frac{1}{2})}{\beta}\sin\beta(x+\frac{1}{2}) - \frac{2}{\beta^2}\cos\beta(x+\frac{1}{2})\right]$$

$$+Si[(x-\frac{1}{2})(\tilde{N}\pm\beta)]\left[(x-\frac{1}{2})^2\cos\beta(x-\frac{1}{2}) - \frac{2(x-\frac{1}{2})}{\beta}\sin\beta(x-\frac{1}{2}) - \frac{2}{\beta^2}\cos\beta(x-\frac{1}{2})\right]$$

$$+\frac{x}{\beta}I_1 - \frac{2}{\beta}I_2 - \frac{2}{\beta^2}I_3 - xI_7 + I_9$$

$$SNCS_1(x,\pm\beta) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \xi\sin[(x-\xi)(\tilde{N}\pm\beta)]\cos\beta(x-\xi)d\xi = xI_7 - I_9$$

$$SNCS_2(x,\pm\beta) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \xi(x-\xi)\sin[(x-\xi)(\tilde{N}\pm\beta)]\cos\beta(x-\xi)d\xi = xI_3 - I_7$$

where

$$I_1 = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \frac{\sin t\beta\sin t(\tilde{N}\pm\beta)}{t}dt$$

$$= \pm\frac{1}{2}\left\{-Ci[|(x+\frac{1}{2})(\tilde{N}\pm2\beta)|] + Ci[|(x-\frac{1}{2})(\tilde{N}\pm2\beta)|] + Ci[|x+\frac{1}{2}|\tilde{N}] - Ci[|x-\frac{1}{2}|\tilde{N}]\right\}$$

$$I_2 = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \sin t\beta\sin t(\tilde{N}\pm\beta)dt$$

$$= \mp\left[\frac{\cos x(\tilde{N}\pm2\beta)\sin(\frac{\tilde{N}}{2}\pm\beta)}{\tilde{N}\pm2\beta} - \frac{\cos x\tilde{N}\sin\frac{\tilde{N}}{2}}{\tilde{N}}\right]$$

$$I_3 = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \frac{\cos t\beta \sin t(\tilde{N} \pm \beta)}{t} dt$$

$$= \frac{1}{2}\left\{Si[(x+\frac{1}{2})(\tilde{N} \pm 2\beta)] - Si[(x-\frac{1}{2})(\tilde{N} \pm 2\beta)] + Si[(x+\frac{1}{2})\tilde{N}] - Si[(x-\frac{1}{2})\tilde{N}]\right\}$$

$$I_4 = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \frac{\cos t\beta \cos t(\tilde{N} \pm \beta)}{t} dt$$

$$= \frac{1}{2}\left\{Ci[|(x+\frac{1}{2})(\tilde{N} \pm 2\beta)|] - Ci[|(x-\frac{1}{2})(\tilde{N} \pm 2\beta)|] + Ci[|x+\frac{1}{2}|\tilde{N}] - Ci[|x-\frac{1}{2}|\tilde{N}]\right\}$$

$$I_5 = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \cos t\beta \cos t(\tilde{N} \pm \beta) dt$$

$$= \frac{\cos x(\tilde{N} \pm 2\beta)\sin(\frac{\tilde{N}}{2} \pm \beta)}{\tilde{N} \pm 2\beta} + \frac{\cos x\tilde{N}\sin\frac{\tilde{N}}{2}}{\tilde{N}}$$

$$I_6 = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \sin t\beta \cos t(\tilde{N} \pm \beta) dt$$

$$= \pm\left[\frac{\sin x(\tilde{N} \pm 2\beta)\sin(\frac{\tilde{N}}{2} \pm \beta)}{\tilde{N} \pm 2\beta} - \frac{\sin x\tilde{N}\sin\frac{\tilde{N}}{2}}{\tilde{N}}\right]$$

$$I_7 = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \cos t\beta \sin t(\tilde{N} \pm \beta) dt$$

$$= \left[\frac{\sin x(\tilde{N} \pm 2\beta)\sin(\frac{\tilde{N}}{2} \pm \beta)}{\tilde{N} \pm 2\beta} + \frac{\sin x\tilde{N}\sin\frac{\tilde{N}}{2}}{\tilde{N}}\right]$$

$$I_8 = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \frac{\sin t\beta \cos t(\tilde{N} \pm \beta)}{t} dt$$

$$= \pm\frac{1}{2}\left\{Si[(x+\frac{1}{2})(\tilde{N} \pm 2\beta)] - Si[(x-\frac{1}{2})(\tilde{N} \pm 2\beta)] - Si[(x+\frac{1}{2})\tilde{N}] + Si[(x-\frac{1}{2})\tilde{N}]\right\}$$

$$I_9 = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} t\cos t\beta \sin t(\tilde{N} \pm \beta) dt$$

$$= \frac{1}{2(\tilde{N} \pm 2\beta)} \left[\frac{2\cos x(\tilde{N} \pm 2\beta)\sin(\frac{\tilde{N}}{2} \pm \beta)}{\tilde{N} \pm 2\beta} + 2x\sin x(\tilde{N} \pm 2\beta)\sin(\frac{\tilde{N}}{2} \pm \beta) - \cos x(\tilde{N} \pm 2\beta)\cos(\frac{\tilde{N}}{2} \pm \beta) \right]$$

$$+ \frac{1}{2\tilde{N}} \left[\frac{2\cos x\tilde{N}\sin\frac{\tilde{N}}{2}}{\tilde{N}} + 2x\sin x\tilde{N}\sin\frac{\tilde{N}}{2} - \cos x\tilde{N}\cos\frac{\tilde{N}}{2} \right]$$

$$I_{10} = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} t\sin t\beta\cos t(\tilde{N} \pm \beta)dt$$

$$= \pm\frac{1}{2(\tilde{N} \pm 2\beta)} \left[\frac{2\cos x(\tilde{N} \pm 2\beta)\sin(\frac{\tilde{N}}{2} \pm \beta)}{\tilde{N} \pm 2\beta} + 2x\sin x(\tilde{N} \pm 2\beta)\sin(\frac{\tilde{N}}{2} \pm \beta) - \cos x(\tilde{N} \pm 2\beta)\cos(\frac{\tilde{N}}{2} \pm \beta) \right]$$

$$\mp \frac{1}{2\tilde{N}} \left[\frac{2\cos x\tilde{N}\sin\frac{\tilde{N}}{2}}{\tilde{N}} + 2x\sin x\tilde{N}\sin\frac{\tilde{N}}{2} - \cos x\tilde{N}\cos\frac{\tilde{N}}{2} \right]$$

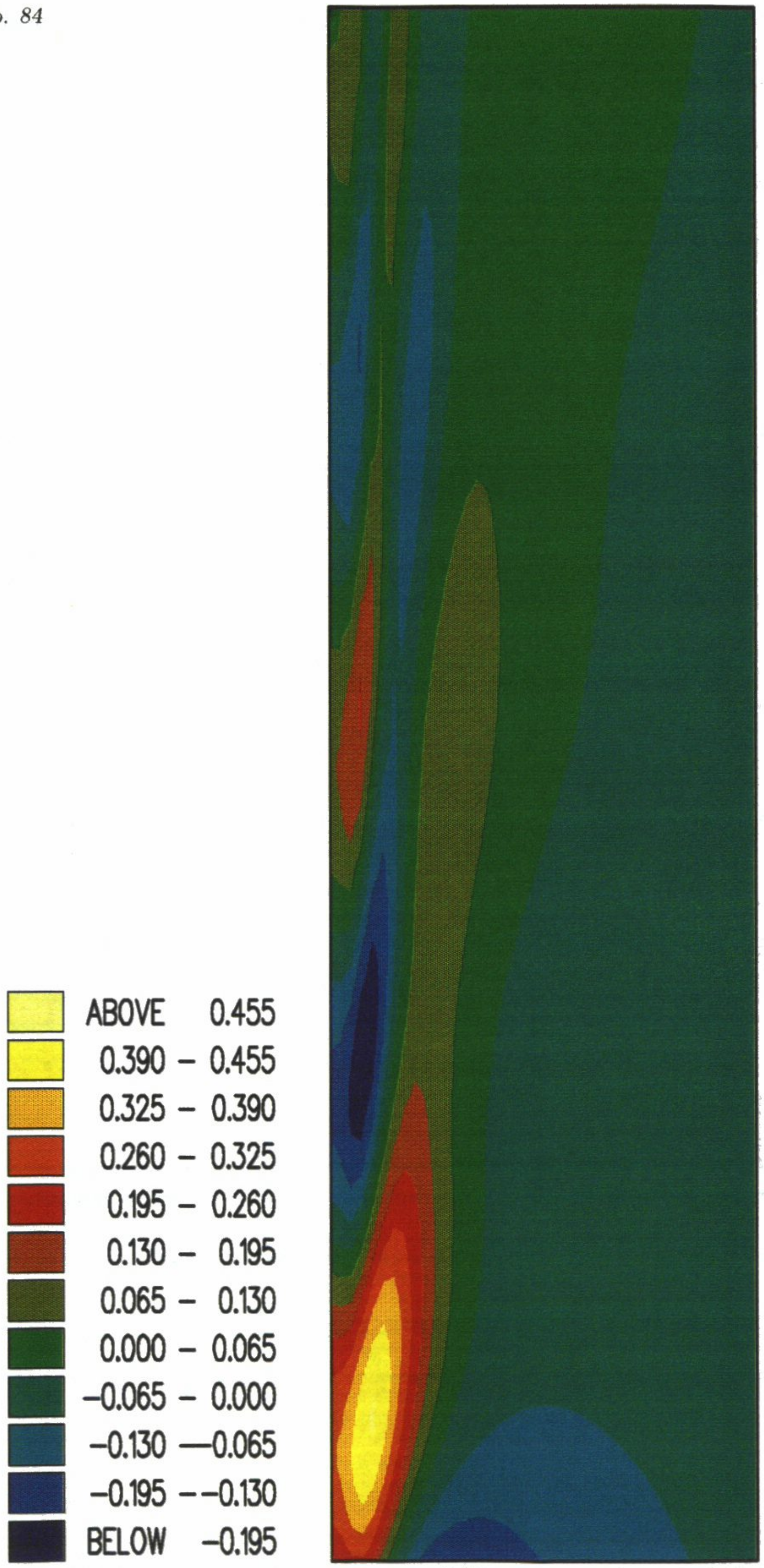

Figure 1: Contour plot of u (mm/s). $U = 2\ m/s$, $N = 0.02\ rads/s$.

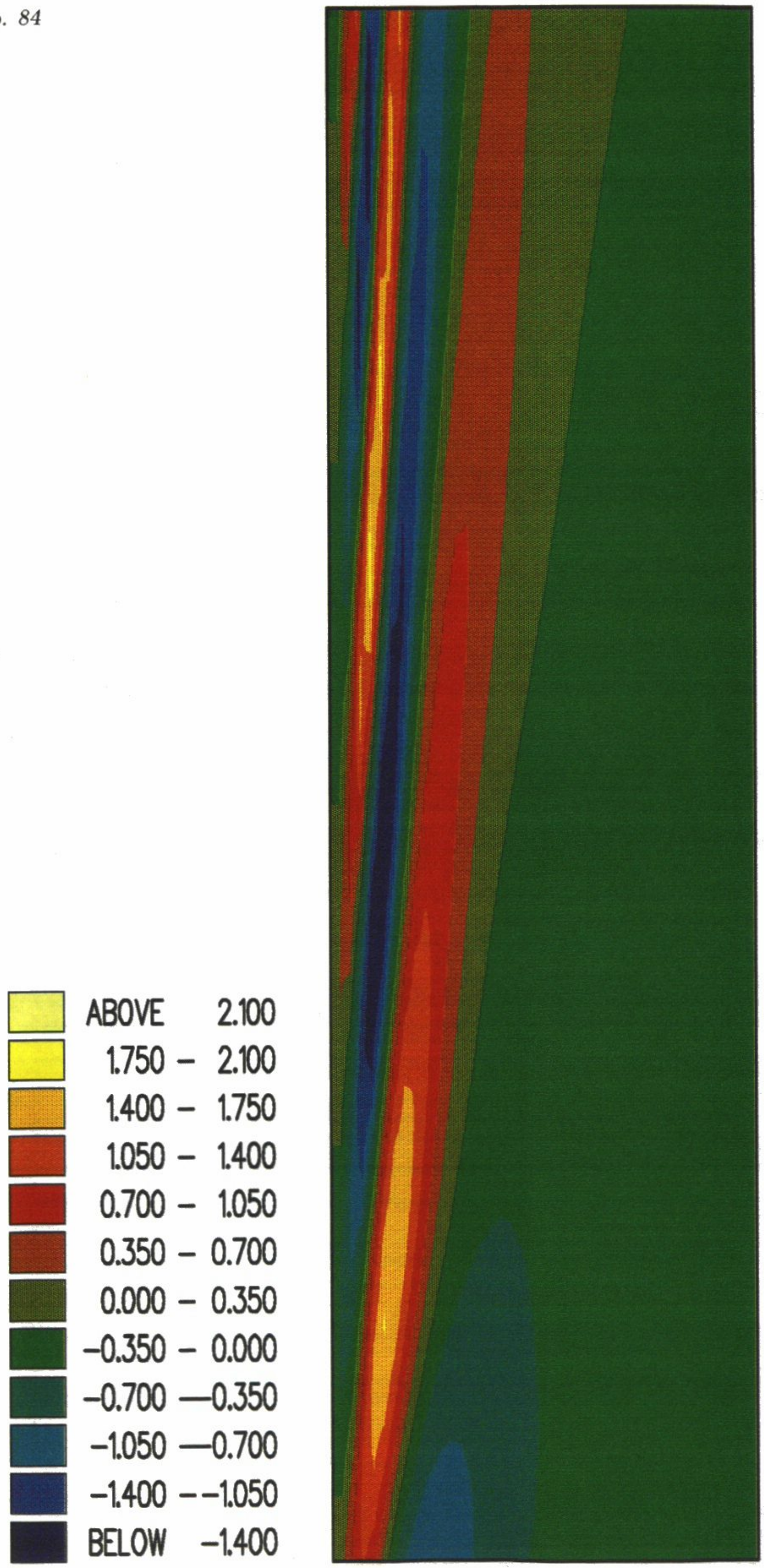

Figure 2: Contour plot of v (mm/s). $U = 2\ m/s$, $N = 0.02\ rads/s$.

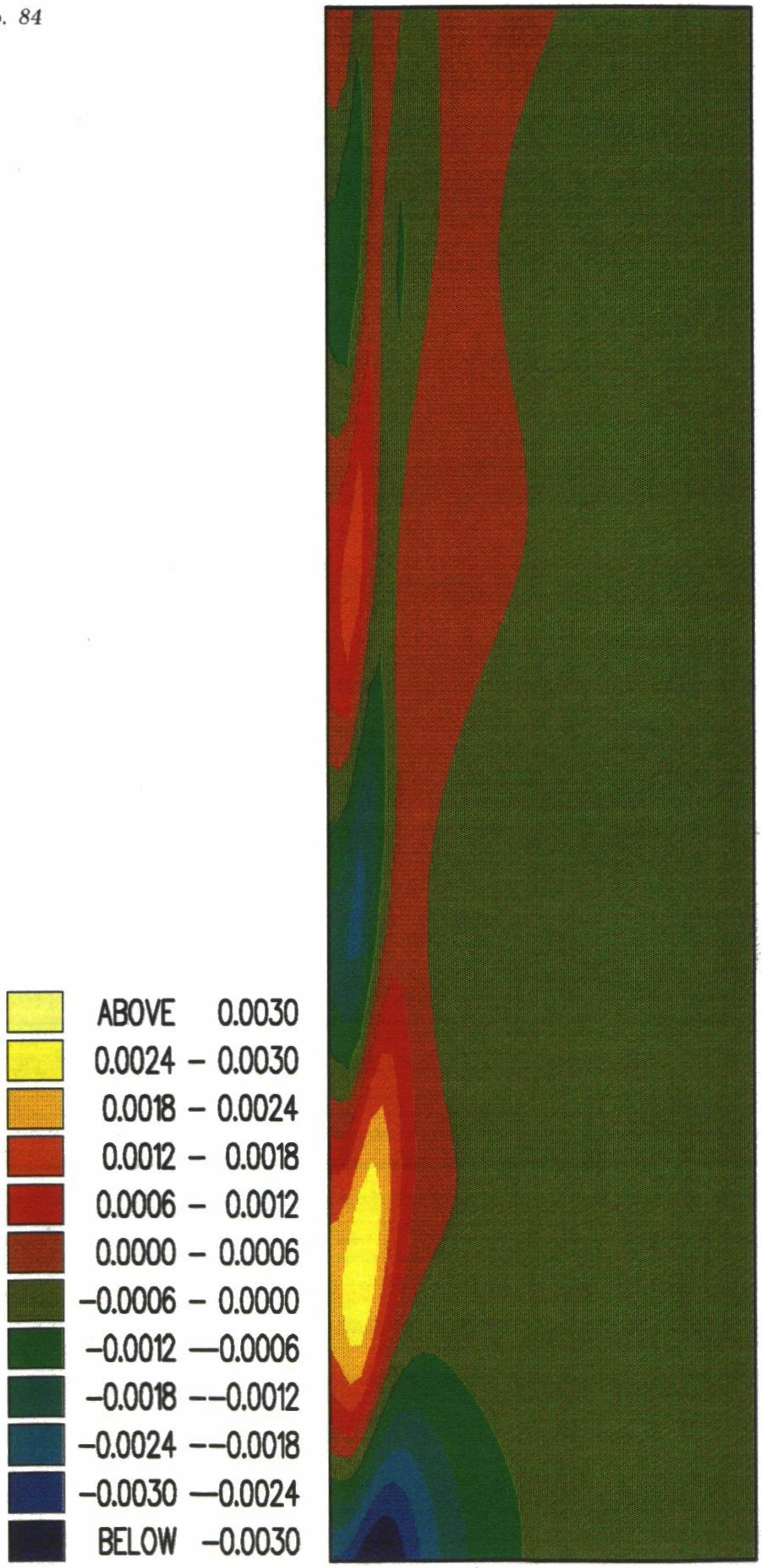

Figure 3: Contour plot of $10^3\ u_x$ $(1/s)$. $U\ =\ 2\ m/s$, $N\ =\ 0.02\ rads/s$.

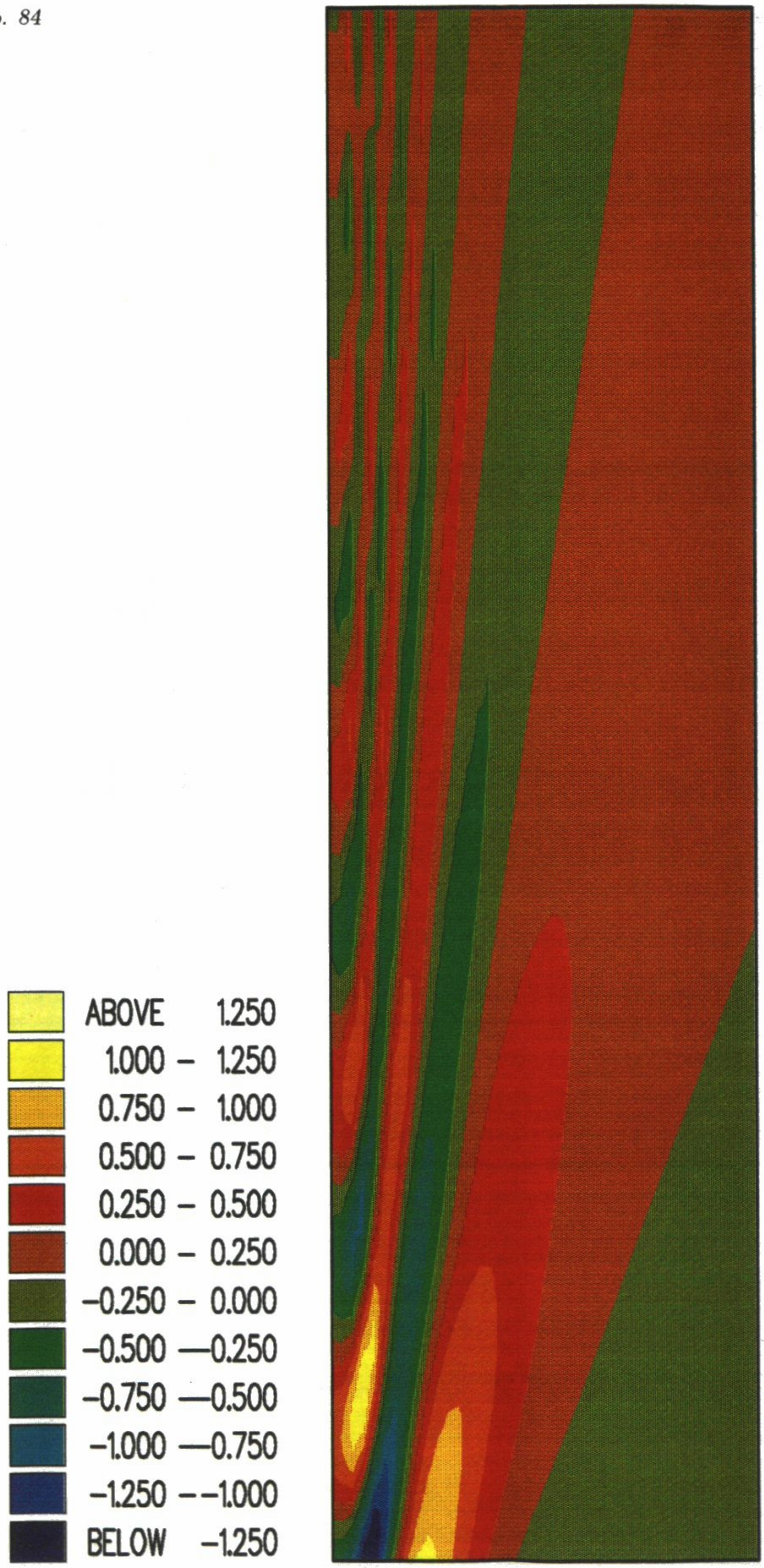

Figure 6: Contour plot of u (mm/s). $U = 2\ m/s$, $N = 0.04\ rads/s$.

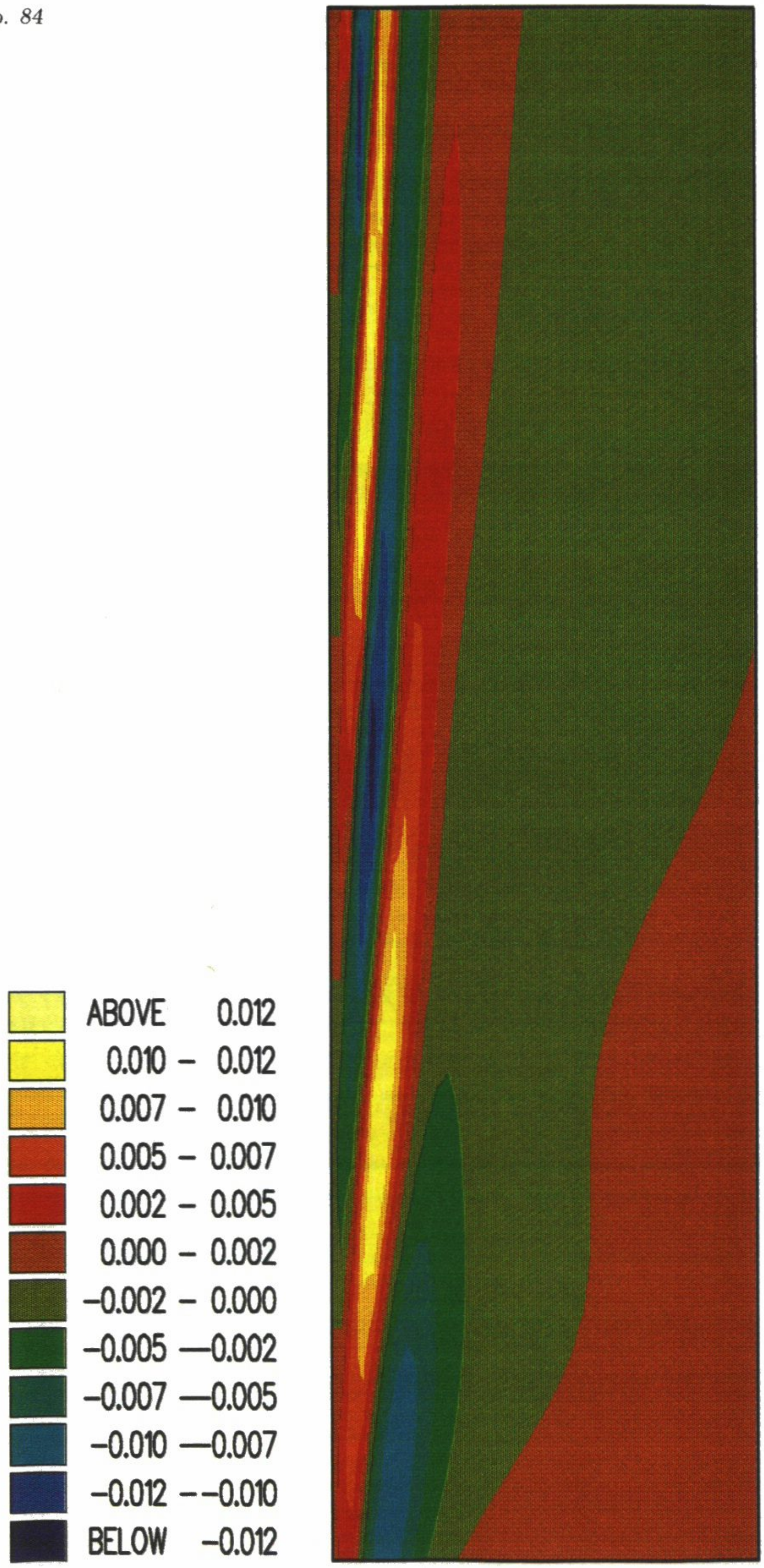

Figure 5: Contour plot of $10^3\ v_x\ (1/s)$. $U\ =\ 2\ m/s$, $N\ =\ 0.02\ rads/s$.

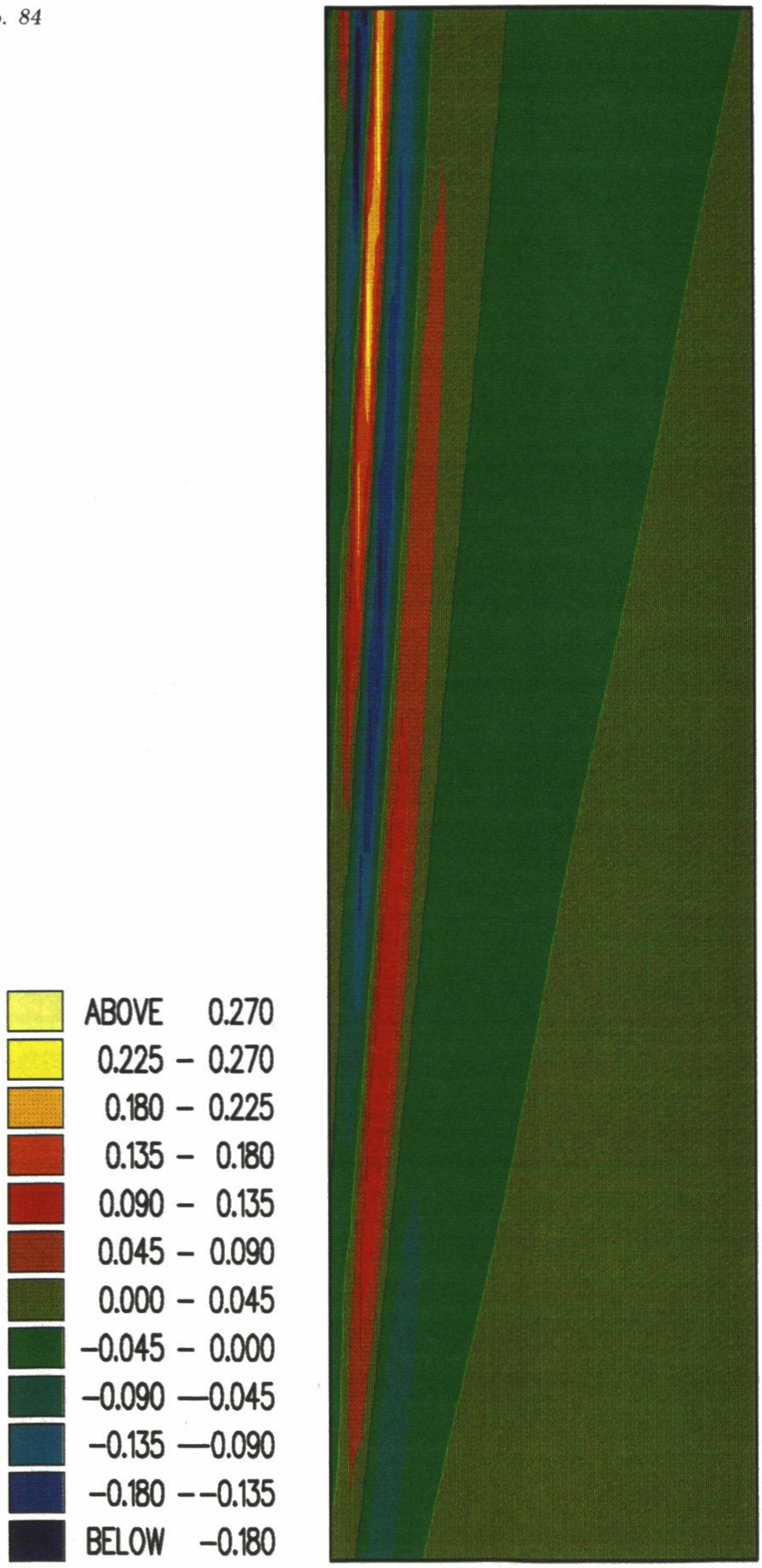

Figure 4: Contour plot of $10^3\ v_y\ (1/s)$. $U\ =\ 2\ m/s$, $N\ =\ 0.02\ rads/s$.

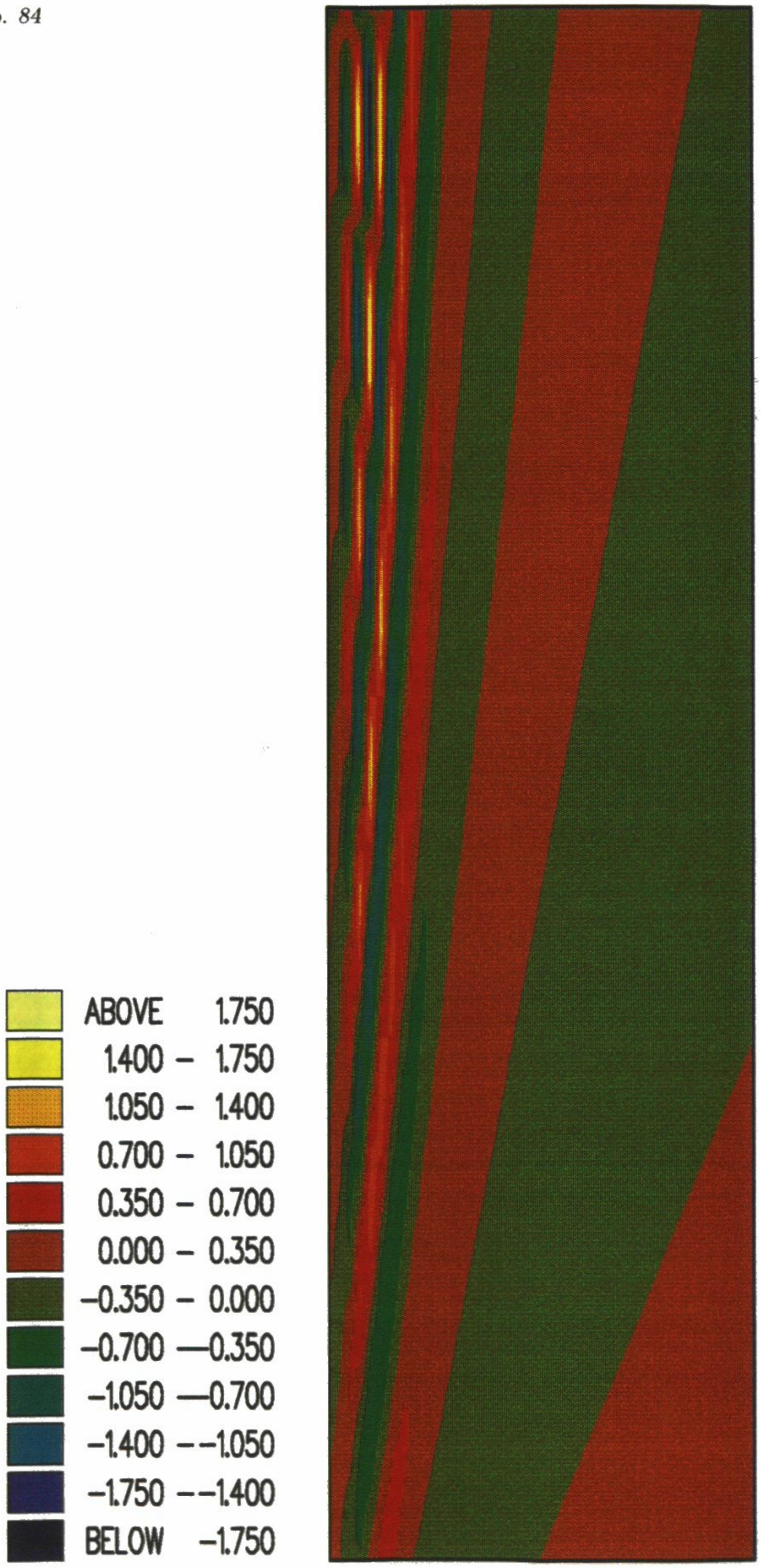

Figure 9: Contour plot of $10^3\ v_y$ $(1/s)$. $U = 2\ m/s$, $N = 0.04\ rads/s$.

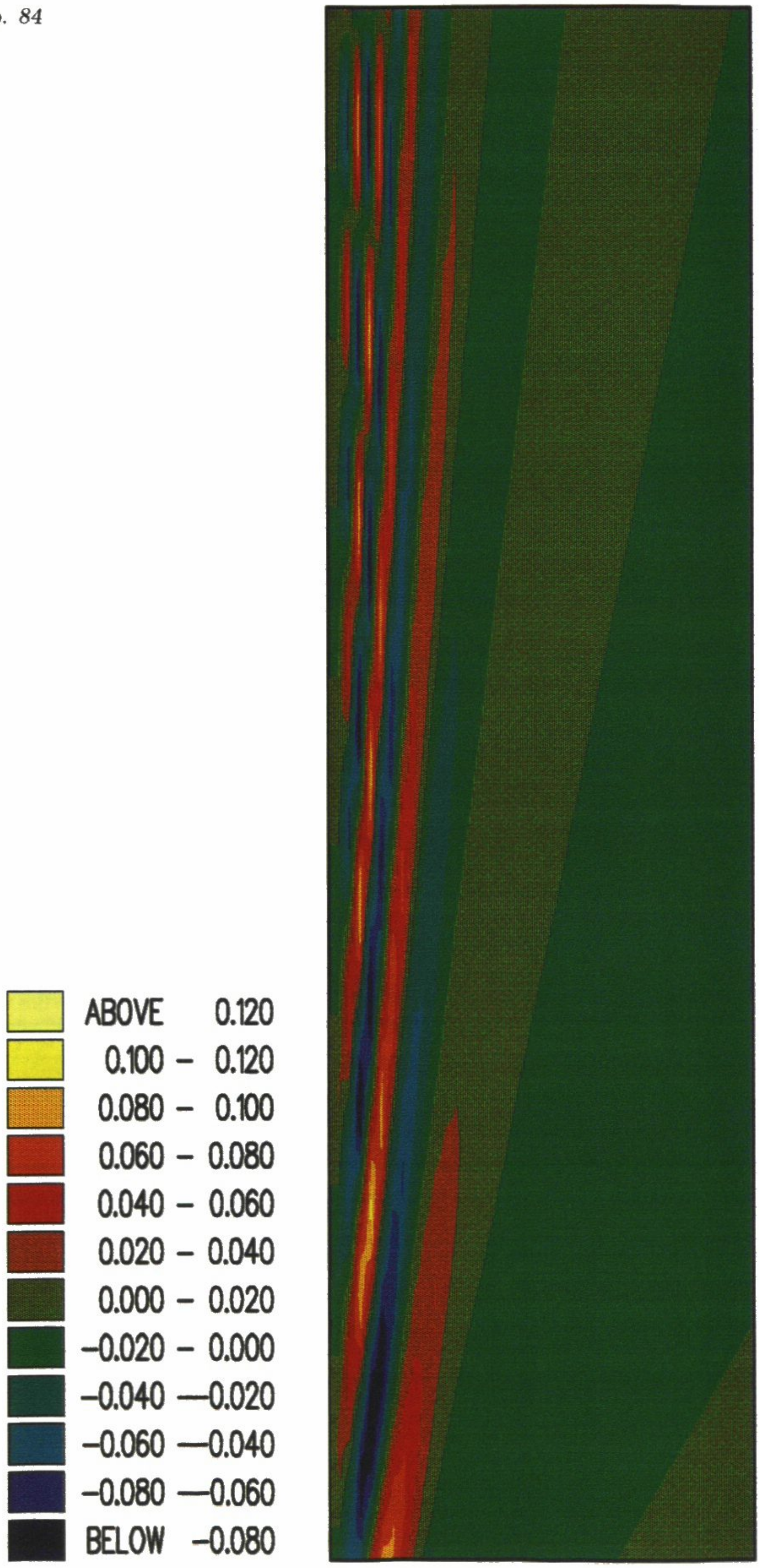

Figure 10: Contour plot of $10^3\ v_x\ (1/s)$. $U\ =\ 2\ m/s$, $N\ =\ 0.04\ rads/s$.

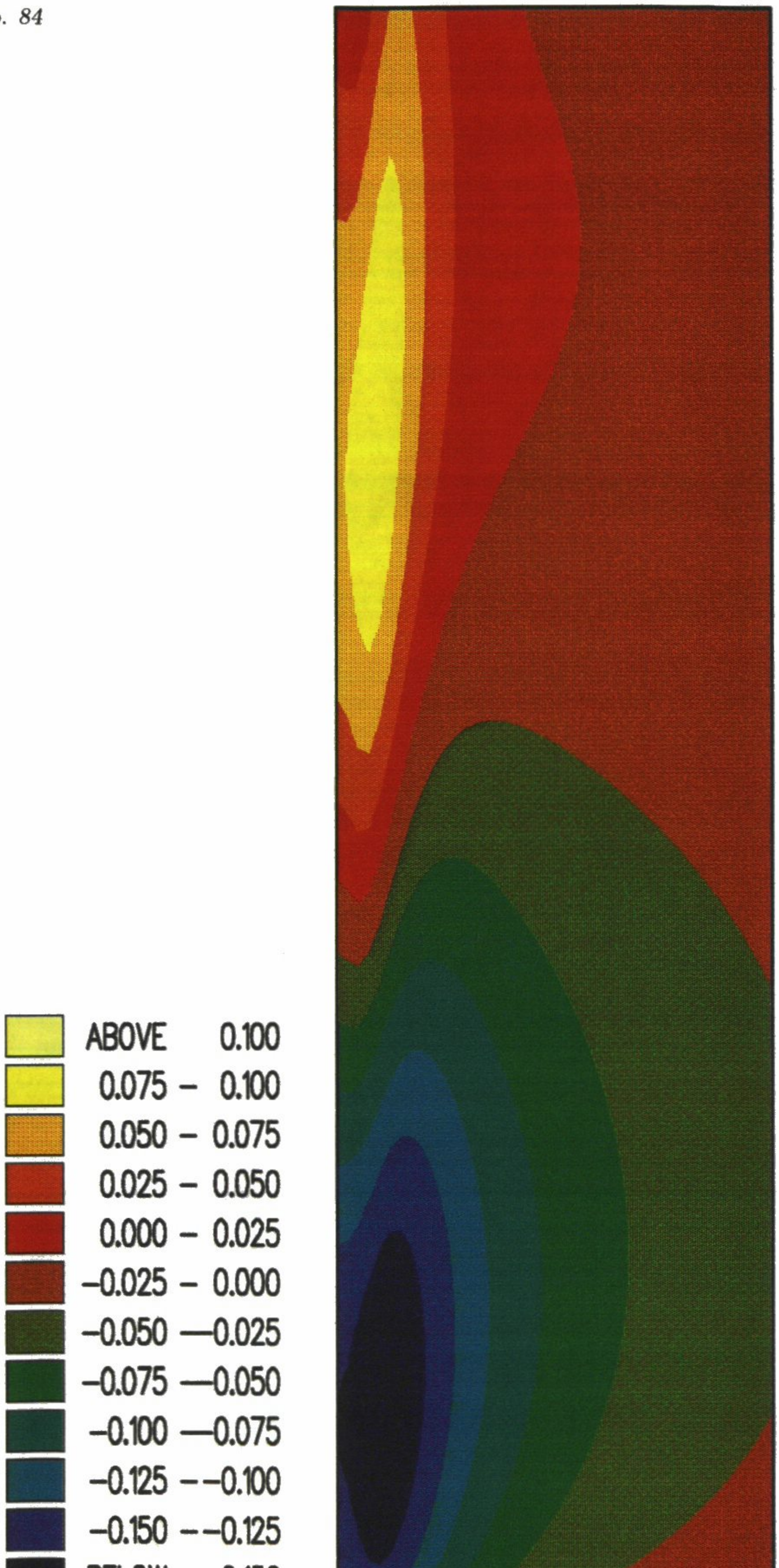

Figure 11: Contour plot of u (mm/s). $U = 5\ m/s$, $N = 0.02\ rads/s$.

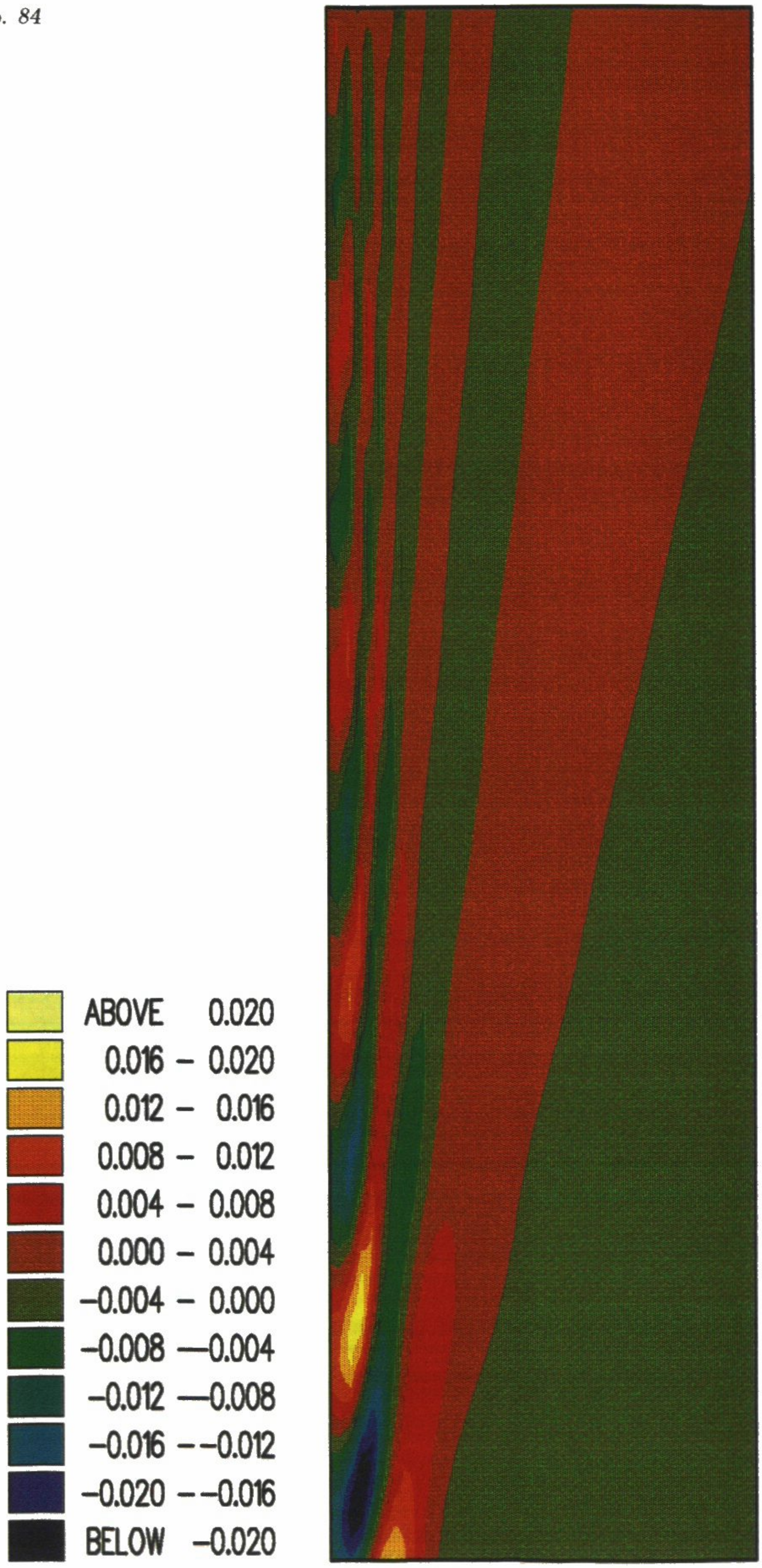

Figure 8: Contour plot of $10^3\ u_x\ (1/s)$. $U\ =\ 2\ m/s$, $N\ =\ 0.04\ rads/s$.

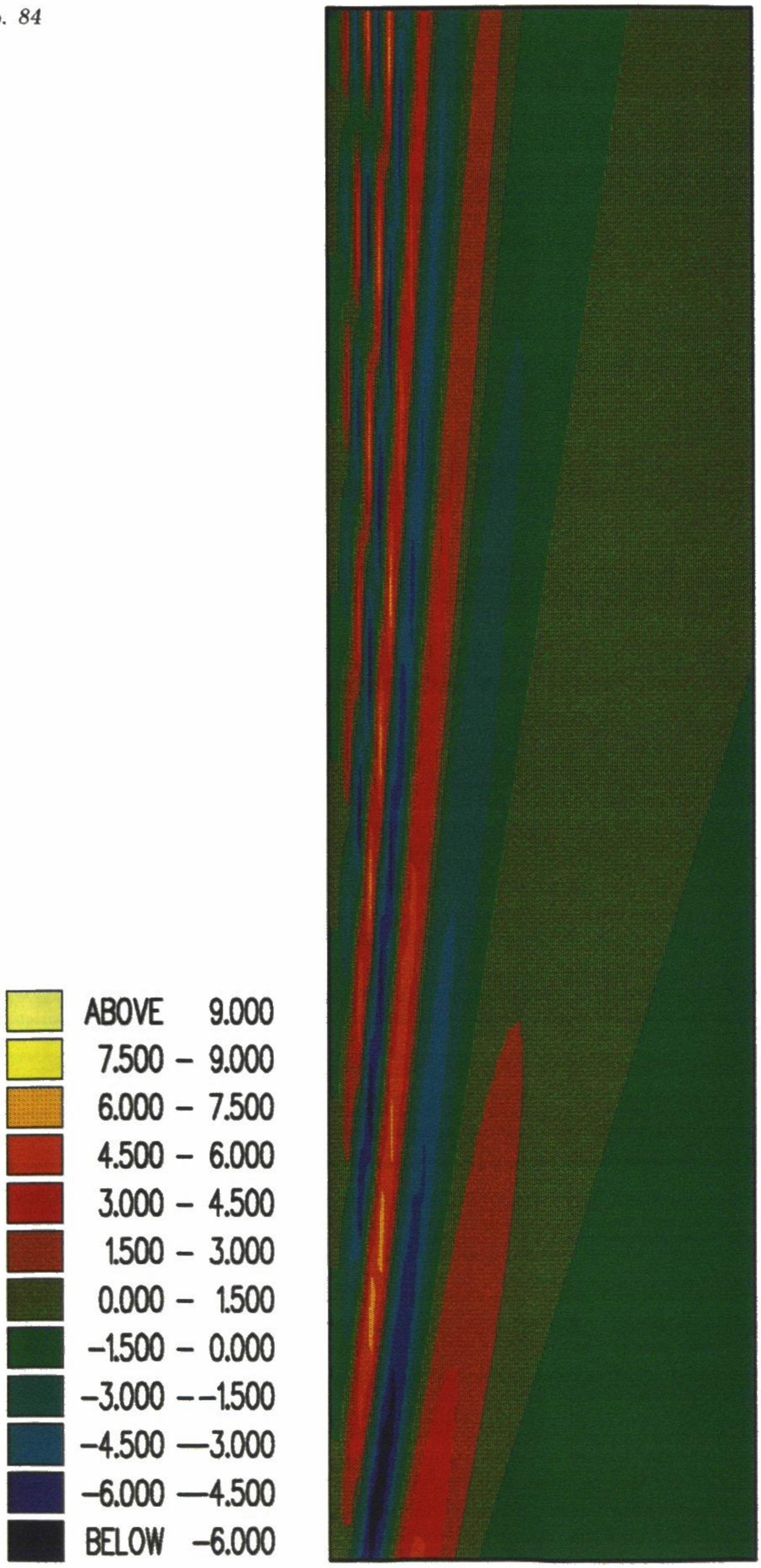

Figure 7: Contour plot of v (mm/s). $U\ =\ 2\ m/s$, $N\ =\ 0.04\ rads/s$.

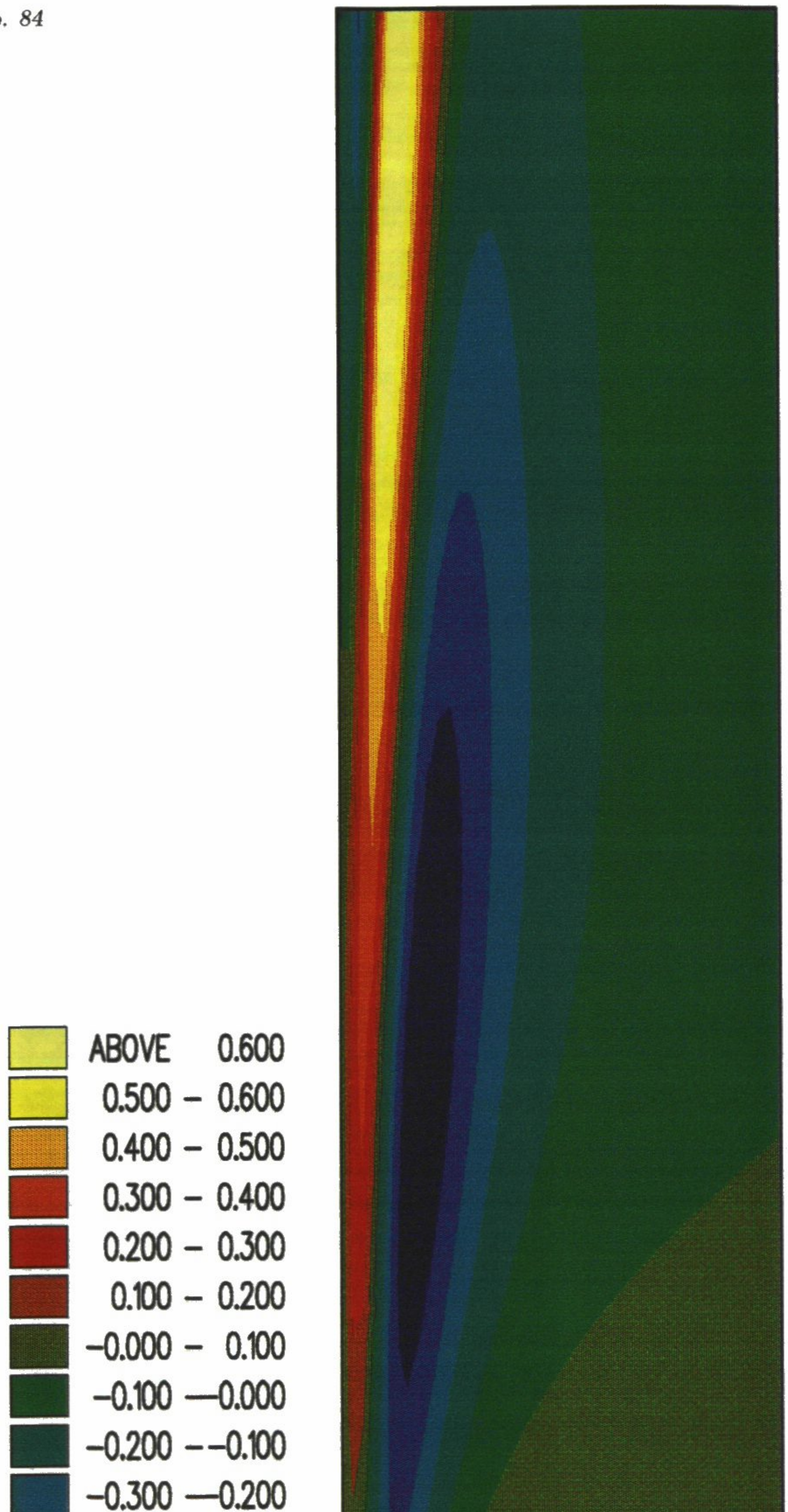

Figure 12: Contour plot of v (mm/s). $U = 5\ m/s$, $N = 0.02\ rads/s$.

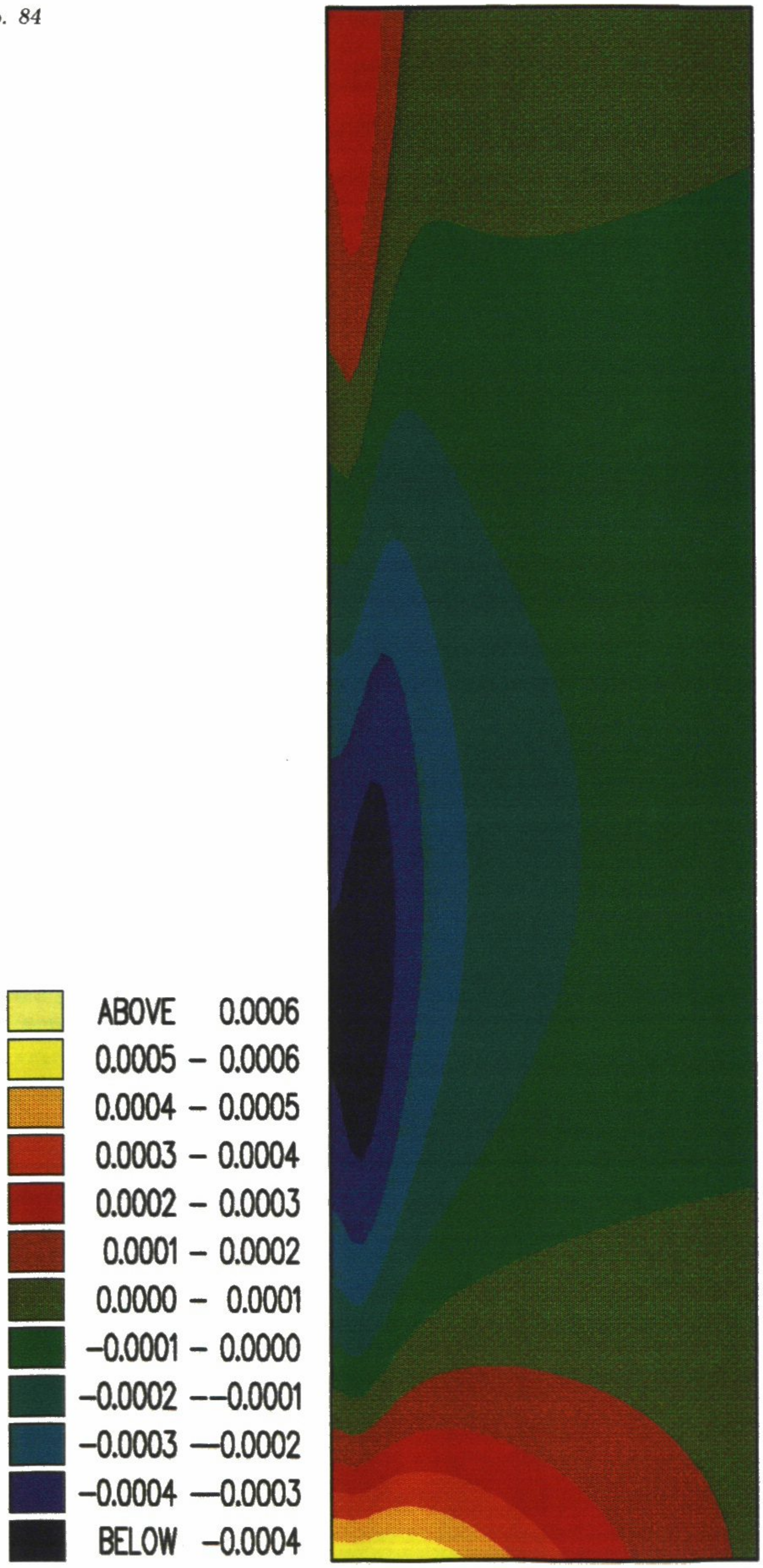

Figure 13: Contour plot of $10^3\ u_x$ $(1/s)$. $U\ =\ 5\ m/s$, $N\ =\ 0.02\ rads/s$.

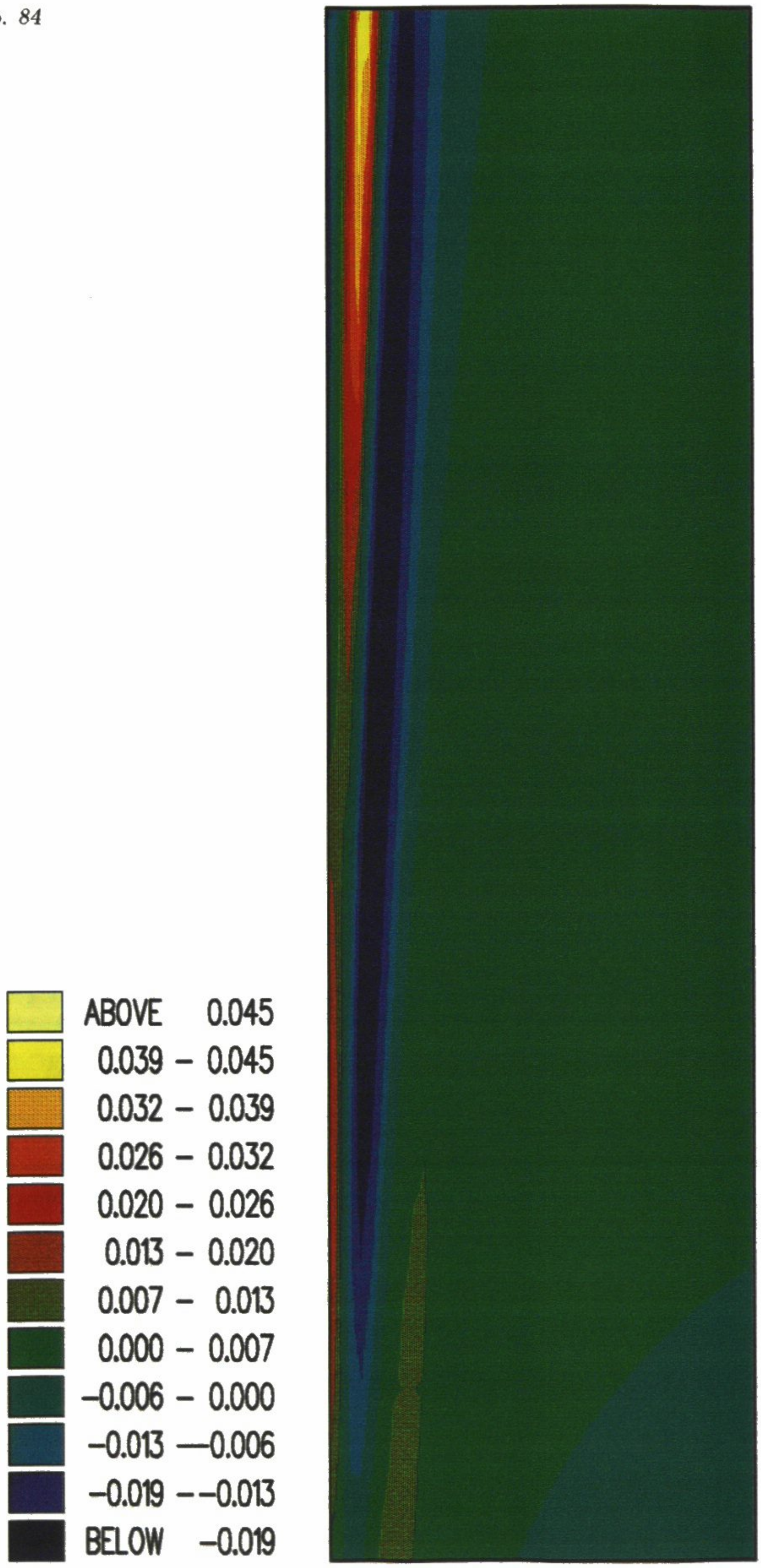

Figure 14: Contour plot of 10^3 v_y $(1/s)$. $U = 5$ m/s, $N = 0.02$ $rads/s$.

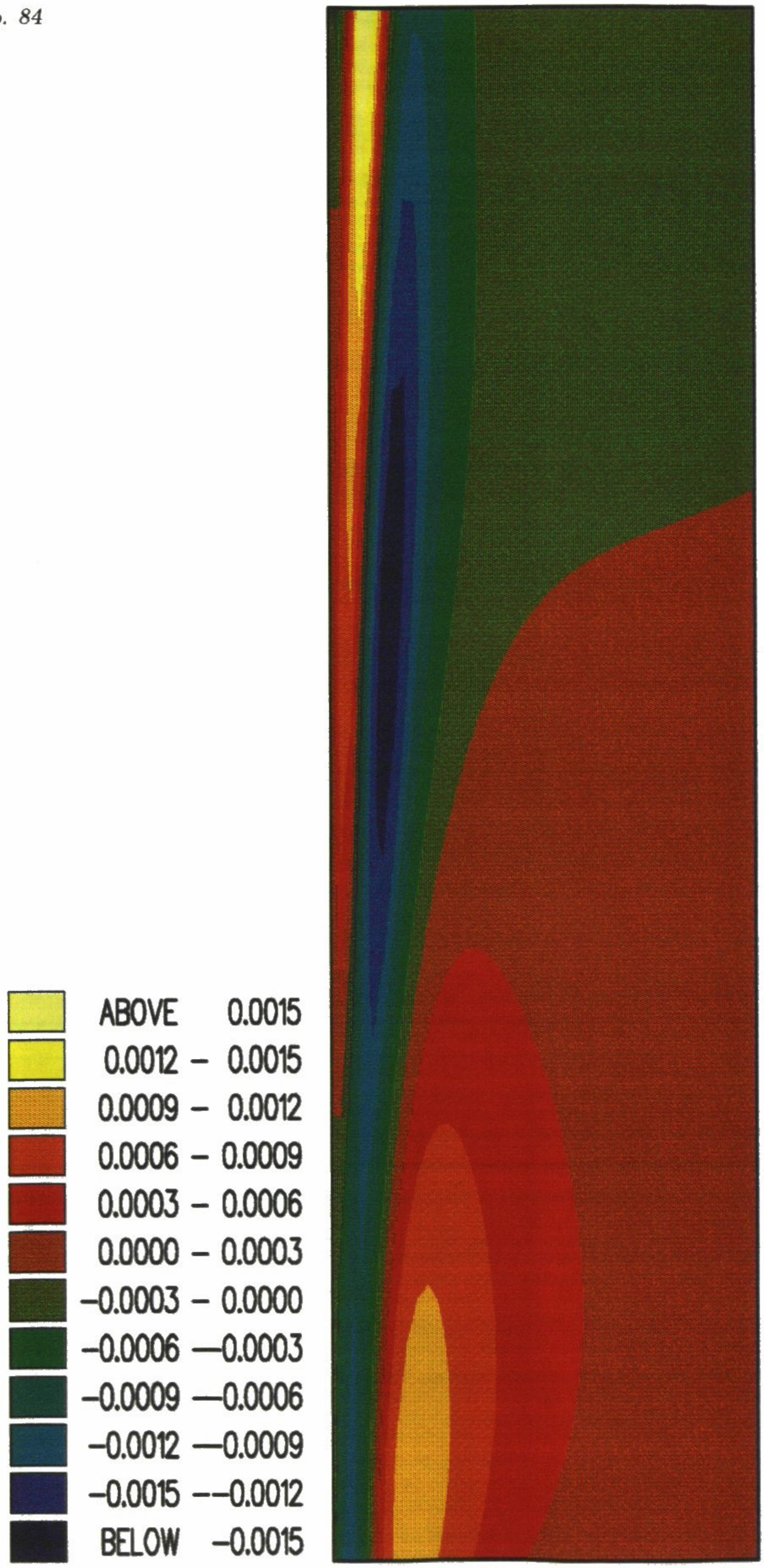

Figure 15: Contour plot of $10^3\ v_x\ (1/s)$. $U\ =\ 5\ m/s$, $N\ =\ 0.02\ rads/s$.

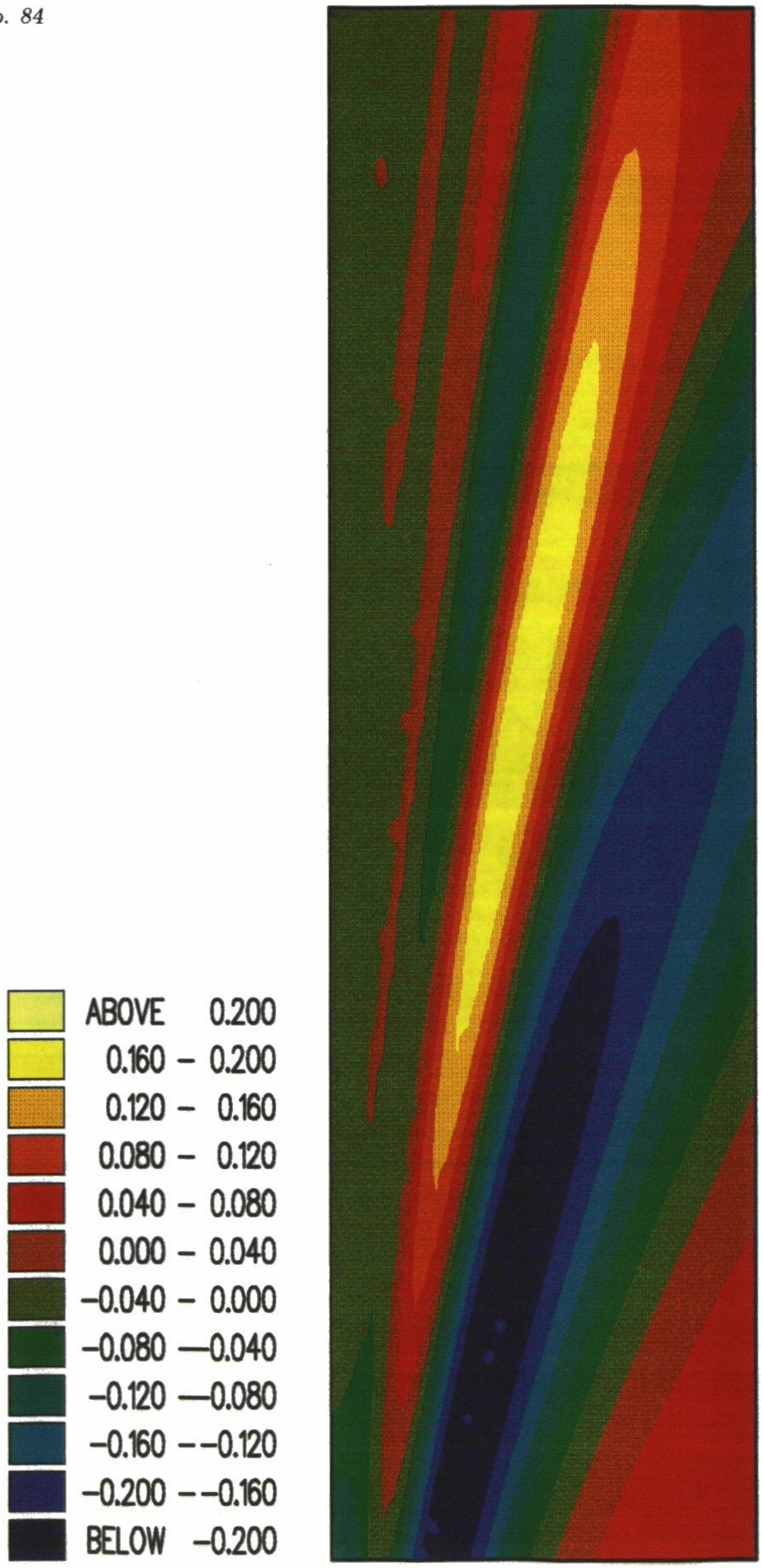

Figure 16: Contour plot of u (mm/s) using a three layer model. $U = 2\ m/s$, $\rho_1 = 1025\ Kg/m^3$, $\rho_2 = 1026.5\ Kg/m^3$, $\rho_3 = 1028\ Kg/m^3$, $t_1 = 30\ m$, $t_2 = 30\ m$.

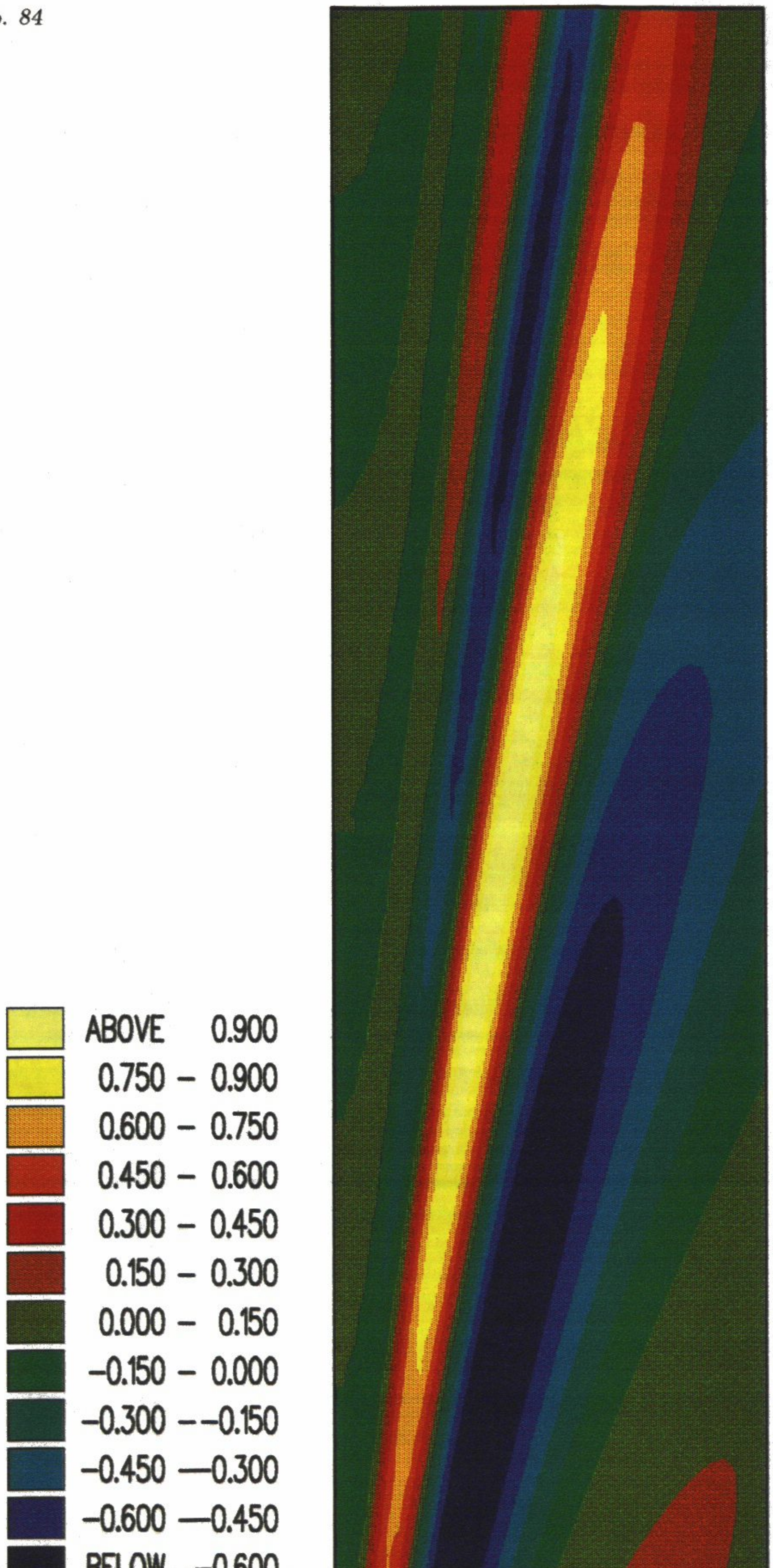

Figure 17: Contour plot of v (mm/s) using a three layer model. $U = 2\ m/s$, $\rho_1 = 1025\ Kg/m^3$, $\rho_2 = 1026.5\ Kg/m^3$, $\rho_3 = 1028\ Kg/m^3$, $t_1 = 30\ m$, $t_2 = 30\ m$.

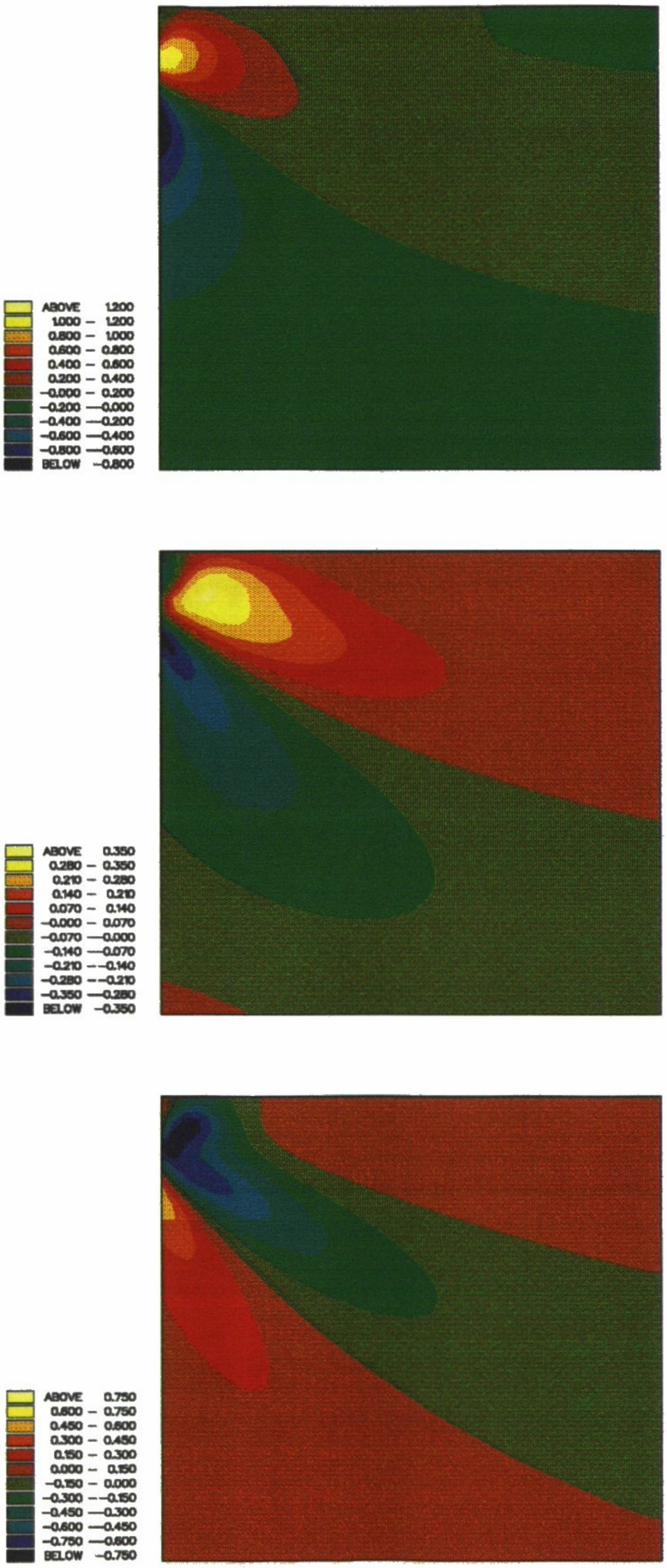

Figure 18: Vertical velocity w (mm/s) calculated on the vertical planes $x = -100$, -200 and -500 m

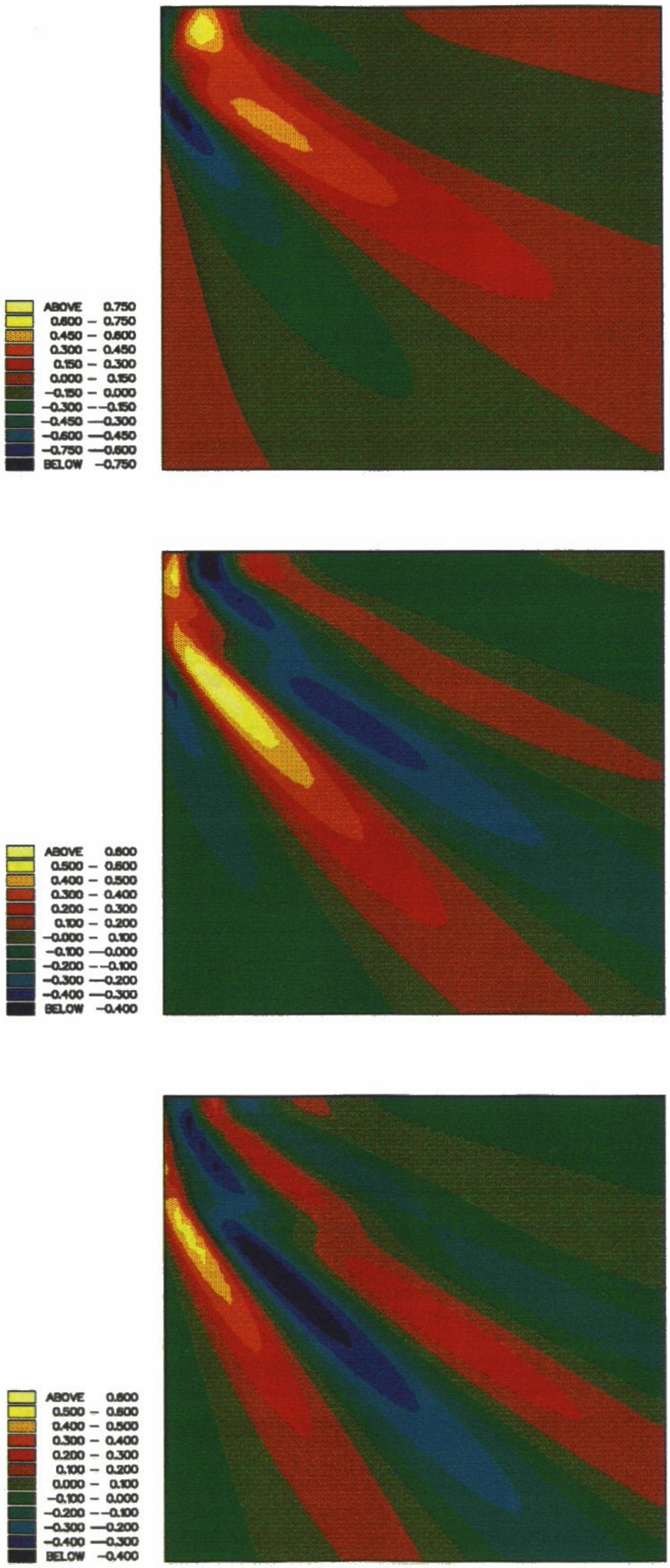

Figure 19: Vertical velocity w (mm/s) calculated on the vertical planes $x = -1000, -1500$ and -2000 m

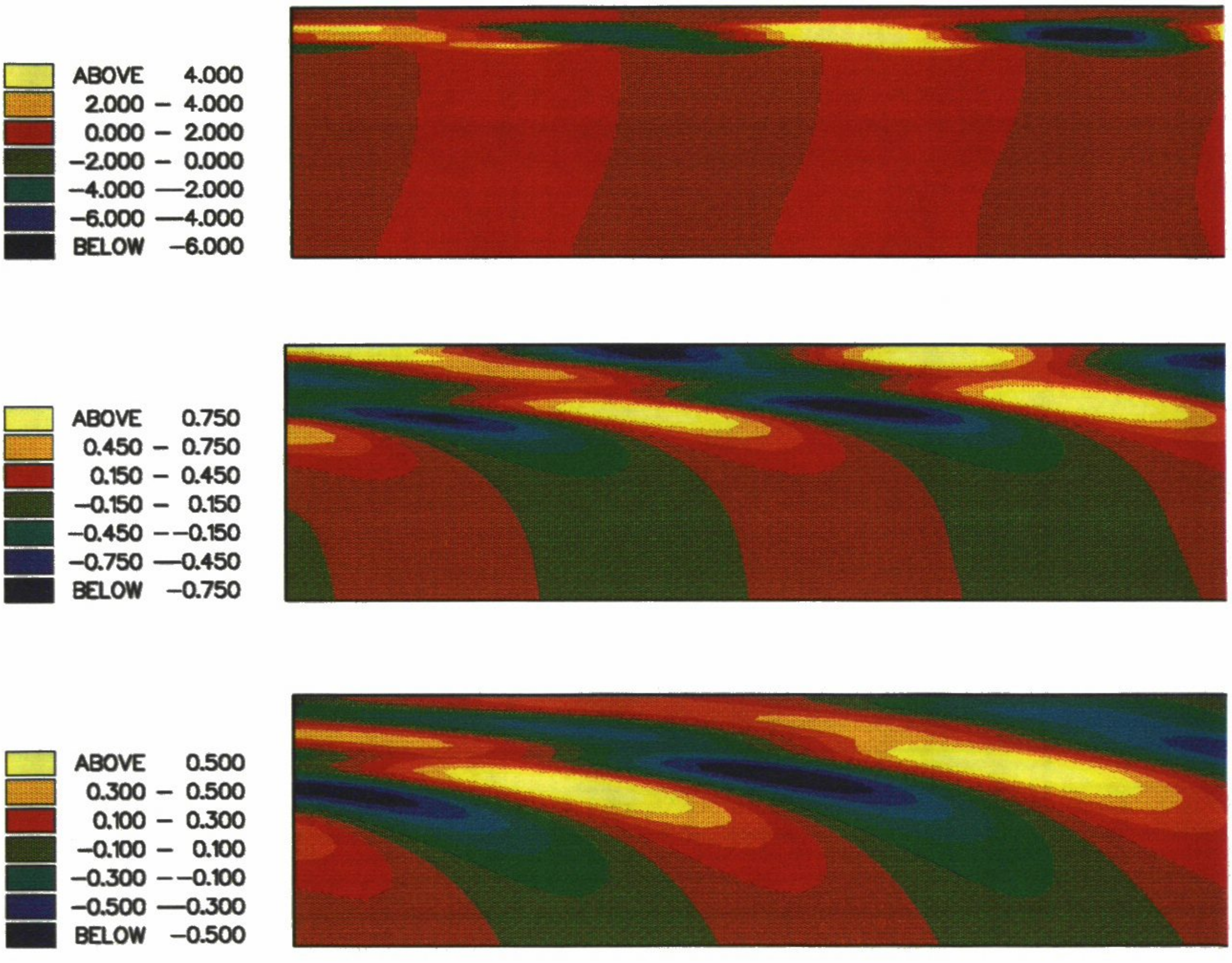

Figure 20: Vertical velocity w (mm/s) calculated on the vertical planes $y = 0$, 50 and 100 m

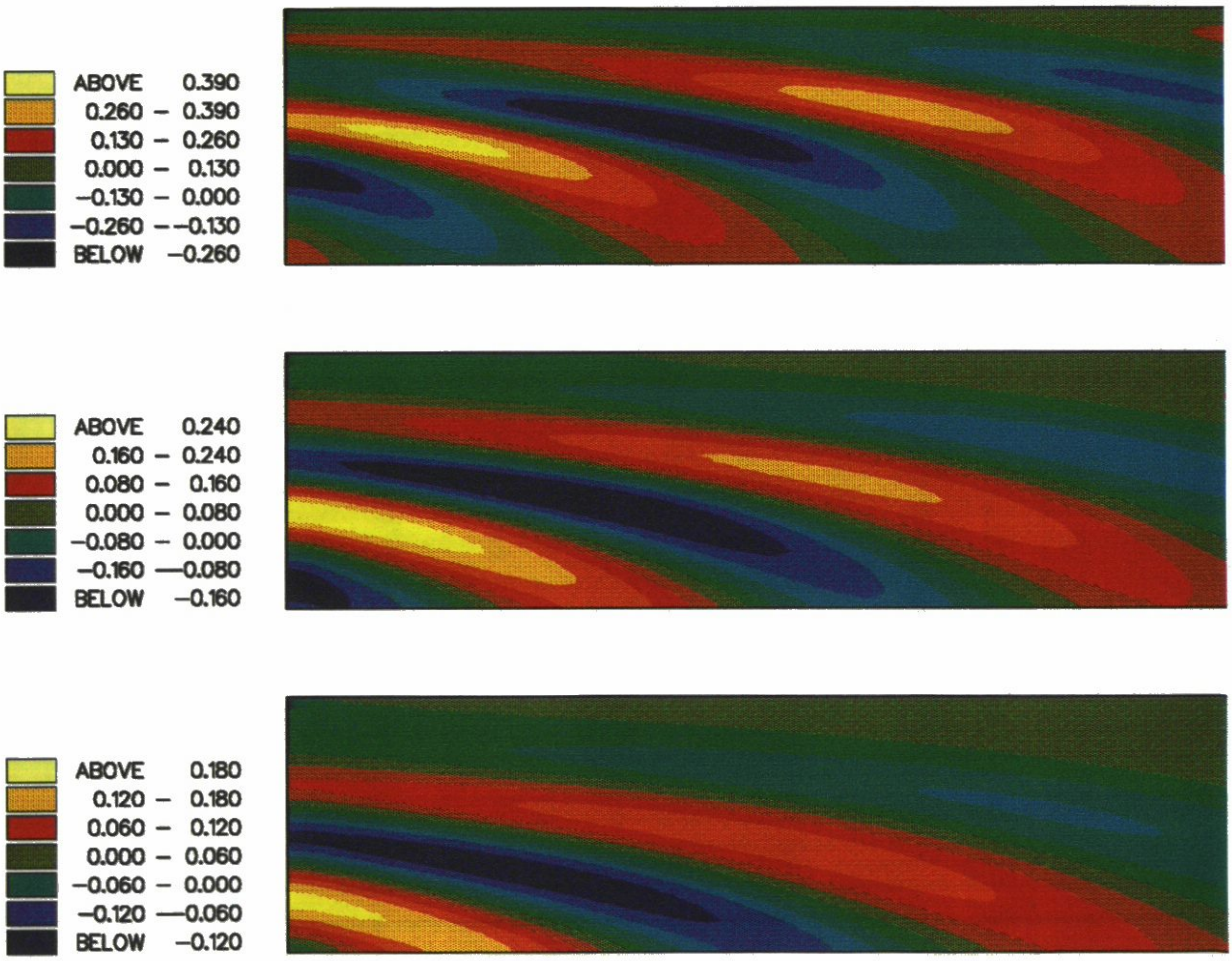

Figure 21: Vertical velocity w (mm/s) calculated on the vertical planes $y = 200$, 300 and 400 m

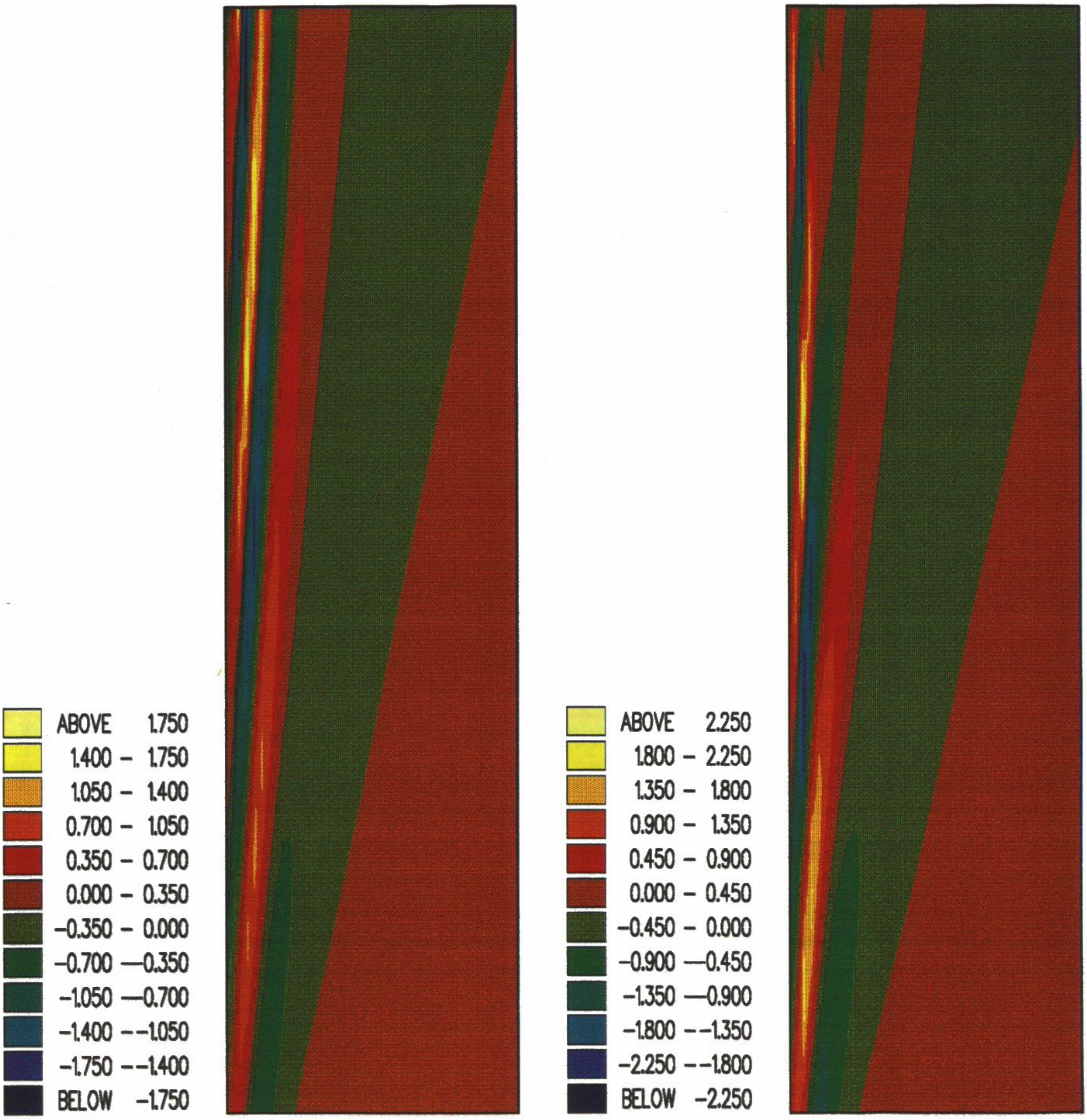

Figure 22: Vertical velocity w (mm/s) calculated on the horizontal planes $z = -10$ and -20 m

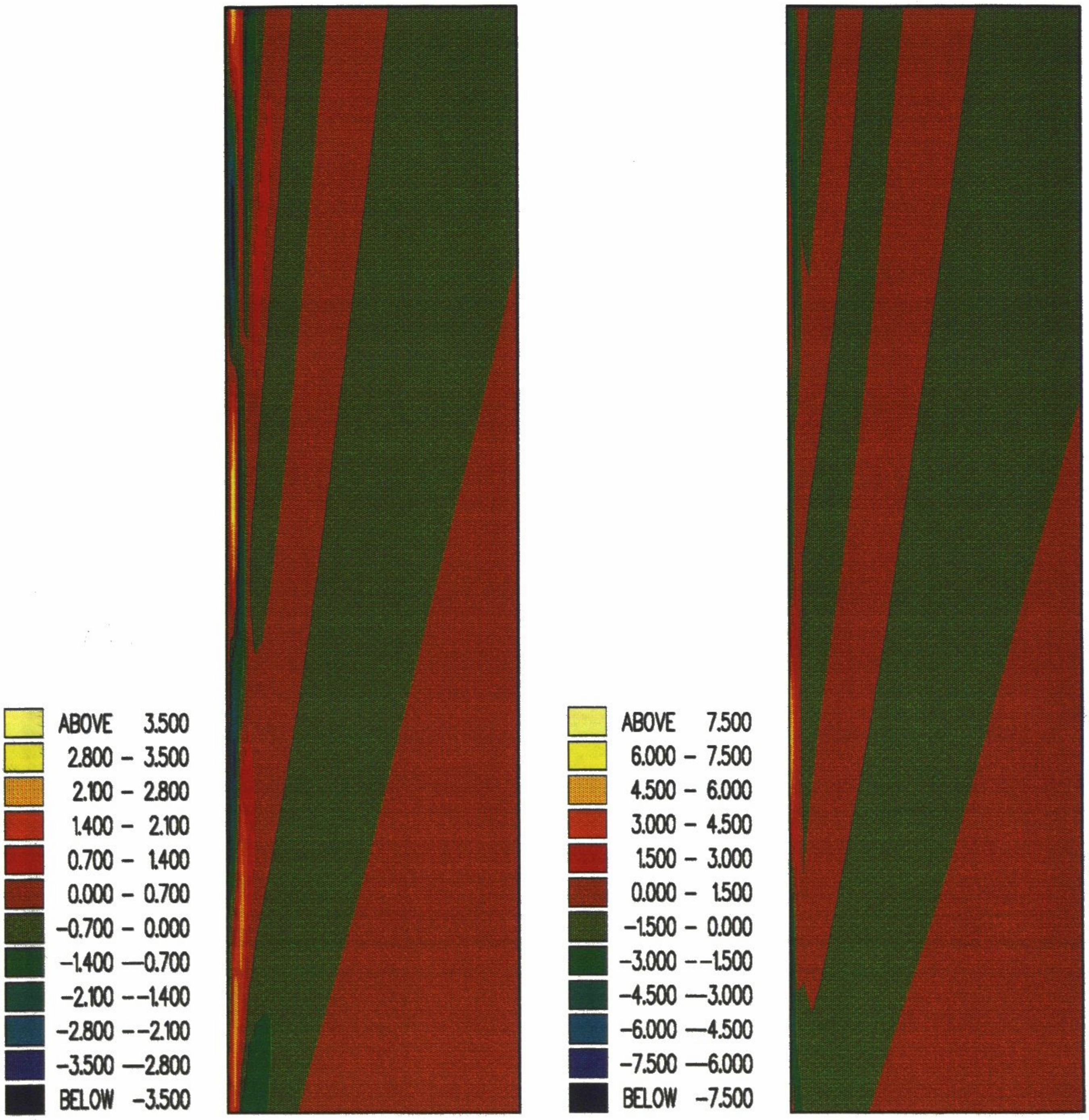

Figure 23: Vertical velocity w (mm/s) calculated on the horizontal planes $z = -30$ and -40 m