ISSN 0140-3818
SSSU 85

# UNIVERSITY OF SOUTHAMPTON

DEPARTMENT OF SHIP SCIENCE

FACULTY OF ENGINEERING AND APPLIED SCIENCE

**STIFFNESS MATRIX OF AN ANISOTROPIC LAYERED PLATE ELEMENT BASED ON EQUILIBRIUM EQUATIONS DERIVED DIRECTLY FROM THE PRINCIPLE OF VIRTUAL DISPLACEMENTS**

**Pei Junhou and R.A. Shenoi**

**Ship Science Report 85**

**September 1994**

# Stiffness Matrix of an Anisotropic Layered Plate Element Based on Equilibrium Equations Derived Directly from the Principle of Virtual Displacements

by

**Pei Junhou** **R. A. Shenoi**

**Department of Ship Science**

**University of Southampton**

**September 1994**

CONTENTS

# 1 Introduction

In reference [1] [2], authors have derived the stiffness matrix of an 8-node laminated plate taking into account higher-order shear deformation, linear strain-displacement relations and nonlinear strain-displacement relations . In order to derive the stiffness matrix, authors adopted the equilibrium equations derived by using an element equilibrium conditions. However, the equilibrium equations, which authors used to derive the stiffness matrix, are variationally inconsistent, for the displacement field used, with those derived from the principle of virtual displacements [3] [4]. In general shell theory content, it is known that different methods to derive equilibrium equations result in marginally different results. Although most of the terms in those equilibrium equations are the same, some are different. Generally speaking, the differences are small and relate to higher order terms. These can be neglected in most cases.

In order to check the above conclusion, in this paper, the equilibrium equations, which are derived from the principle of virtual displacements, will be used to obtain the stiffness matrix of an 8-node laminated plate element taking into account higher-order shear deformation, linear strain-displacement relations and nonlinear displacement relations.

At the end of the paper, a unified stiffness matrix is presented used not only for linear solution but also for geometric nonlinear problems.

## 2 Stress-Strain Relation

Consider a laminated plate element of $N$ layers with thickness $h$, length $2a$, and width $2b$, as shown in *Fig.*1 . Each layer is taken to be macroscopically homogeneous and orthotropic.

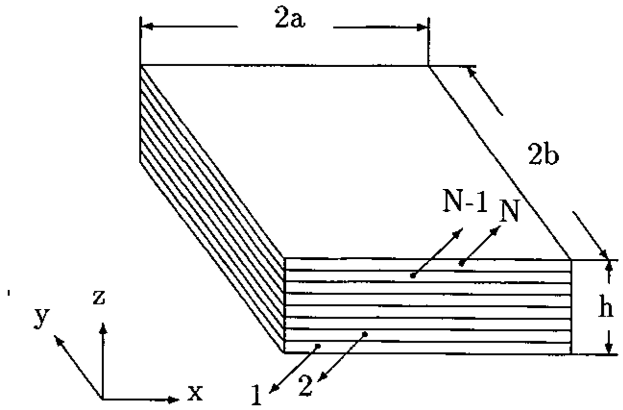

Figure 1

Based on the $Duhamel - Neumann$ law, the stress-strain relation of the $k$th layer is

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} \tag{1}$$

where

$$Q_{11} = E_1/(1 - \nu_{12}\nu_{21})$$

$$
\begin{aligned}
Q_{12} &= \nu_{12}E_2/(1-\nu_{12}\nu_{21}) = \nu_{21}E_1/(1-\nu_{12}\nu_{21}) = Q_{13} \\
Q_{22} &= E_2/(1-\nu_{12}\nu_{21}) \\
Q_{44} &= G_{23} \\
Q_{55} &= G_{12} = G_{13} = Q_{66} \\
Q_{23} &= \nu_{23}E_2/(1-\nu_{23}^2) = \nu_{32}E_3/(1-\nu_{32}^2)
\end{aligned} \tag{2}
$$

In the derivation of Eq. (1), the stresses and strains are defined in the principal material directions for that orthotropic lamina. However, in angle-ply laminated plates the principal directions of orthotropy of each individual lamina do not coincide with the geometrical coordinate frame. It is necessary to use the transformed reduced stiffness

$$
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{13} & 0 & 0 & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{23} & o & 0 & \overline{Q}_{26} \\ \overline{Q}_{13} & \overline{Q}_{23} & \overline{Q}_{33} & 0 & 0 & \overline{Q}_{36} \\ 0 & 0 & 0 & \overline{Q}_{44} & \overline{Q}_{45} & 0 \\ 0 & 0 & 0 & \overline{Q}_{45} & \overline{Q}_{55} & 0 \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{36} & 0 & 0 & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \tag{3}
$$

The thirteen constants $\overline{Q}_{ij}$ are related to the nine $Q_{ij}$ through the following transformation formulae

$$
\begin{aligned}
\overline{Q}_{11} &= Q_{11}m^4 + 2(Q_{12}+2Q_{66})m^2n^2 + Q_{22}n^4 \\
\overline{Q}_{12} &= (Q_{11}+Q_{22}-4Q_{66})m^2n^2 + Q_{12}(m^4+n^4)
\end{aligned}
$$

$$
\begin{aligned}
\overline{Q}_{13} &= Q_{13}m^2 + Q_{23}n^2 \\
\overline{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66})m^3n + (Q_{12} - Q_{22} + 2Q_{66})mn^3 \\
\overline{Q}_{22} &= Q_{11}n^4 + 2(Q_{12} + 2Q_{66})m^2n^2 + Q_{22}m^4 \\
\overline{Q}_{23} &= Q_{13}n^2 + Q_{23}m^2 \\
\overline{Q}_{33} &= Q_{33} \qquad (4) \\
\overline{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66})mn^3 + (Q_{12} - Q_{22} + 2Q_{66})m^3n \\
\overline{Q}_{36} &= (Q_{13} - Q_{23})mn \\
\overline{Q}_{44} &= Q_{44}m^2 + Q_{55}n^2 \\
\overline{Q}_{45} &= (Q_{45} - Q_{44})mn \\
\overline{Q}_{55} &= Q_{55}m^2 + Q_{44}n^2 \\
\overline{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12})m^2n^2 + Q_{66}(m^2 - n^2)^2
\end{aligned}
$$

where

$$m = cos\theta_k \quad n = sin\theta_k \quad (see\ \ Fig.2) \qquad (5)$$

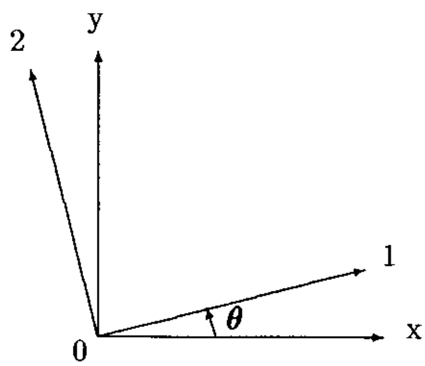


Figure 2

## 3 Reddy's Higher-Order Shear Deformation Theory

The displacement field of Reddy's higher-order shear deformation theory has been given in reference [1] [2] [5] as following

$$
\begin{aligned}
u(x,y,z) &= u_0(x,y) + z\phi_1(x,y) - \frac{4z^3}{3h^2}[\phi_1(x,y) + \frac{\partial w}{\partial x}] \\
v(x,y,z) &= v_0(x,y) + z\phi_2(x,y) - \frac{4z^3}{3h^2}[\phi_2(x,y) + \frac{\partial w}{\partial y}] \\
w(x,y,z) &= w_0(x,y)
\end{aligned}
\tag{6}
$$

where $u_0, v_0, w_0$ are associated midplane displacements, and $\phi_1$ and $\phi_2$ are the rotations of the transverse normal in the $xz$ and $yz$ planes. The coordinate frame is chosen in such a way that the $xy$ plane coincides with the midplane of plate.

Based on the linear strain-displacement equations and nonlinear strain-displacement equations, we have

$$
\begin{aligned}
\epsilon_x &= \epsilon_x^0 + z(k_x^1 + z^2 k_x^2) + L_n \epsilon_x^n \\
\epsilon_y &= \epsilon_y^0 + z(k_y^1 + z^2 k_x^2) + L_n \epsilon_y^n \\
\epsilon_z &= 0 \\
\gamma_{xy} &= \gamma_{xy}^0 + z(k_{xy}^1 + z^2 k_{xy}^2) + L_n \gamma_{xy}^n \\
\gamma_{xz} &= \gamma_{xz}^1 + z^2 \gamma_{xz}^2 \\
\gamma_{yz} &= \gamma_{yz}^1 + z^2 \gamma_{yz}^2
\end{aligned}
\tag{7}
$$

where

$$\{\epsilon^0\} = \left\{ \begin{array}{c} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{array} \right\} = \left\{ \begin{array}{l} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{array} \right\} \tag{8}$$

$$\{\epsilon^n\} = \left\{ \begin{array}{c} \epsilon_x^n \\ \epsilon_y^n \\ \gamma_{xy}^n \end{array} \right\} = \left\{ \begin{array}{l} \frac{1}{2}(\frac{\partial w_0}{\partial x})^2 \\ \frac{1}{2}(\frac{\partial w_0}{\partial y})^2 \\ \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} \end{array} \right\} \tag{9}$$

$$\{k^1\} = \left\{ \begin{array}{c} k_x^1 \\ k_y^1 \\ k_{xy}^1 \end{array} \right\} = \left\{ \begin{array}{l} \frac{\partial \phi_1}{\partial x} \\ \frac{\partial \phi_2}{\partial y} \\ \frac{\partial \phi_2}{\partial x} + \frac{\partial \phi_1}{\partial y} \end{array} \right\} \tag{10}$$

$$\{k^2\} = \left\{ \begin{array}{c} k_x^2 \\ k_y^2 \\ k_{xy}^2 \end{array} \right\} = -\frac{4}{3h^2} \left\{ \begin{array}{l} \frac{\partial \phi_1}{\partial x} + \frac{\partial^2 w_0}{\partial x^2} \\ \frac{\partial \phi_2}{\partial y} + \frac{\partial^2 w_0}{\partial y^2} \\ \frac{\partial \phi_2}{\partial x} + \frac{\partial \phi_1}{\partial y} + 2\frac{\partial^2 w_0}{\partial x \partial y} \end{array} \right\} \tag{11}$$

$$\{\gamma^1\} = \left\{ \begin{array}{c} \gamma_{xz}^1 \\ \gamma_{yz}^1 \end{array} \right\} = \left\{ \begin{array}{l} \frac{\partial w_0}{\partial x} + \phi_1 \\ \frac{\partial w_0}{\partial y} + \phi_2 \end{array} \right\} \tag{12}$$

$$\{\gamma^2\} = \left\{ \begin{array}{c} \gamma_{xz}^2 \\ \gamma_{yz}^2 \end{array} \right\} = -\frac{4}{h^2} \left\{ \begin{array}{l} \frac{\partial w_0}{\partial x} + \phi_1 \\ \frac{\partial w_0}{\partial y} + \phi_2 \end{array} \right\} \tag{13}$$

Where, $L_n$ in equation (7) is solution character coefficient. $L_n = 0$ means linear solution, and $L_n = 1$ means nonlinear solution.

Substituting Eq.(7) into Eq.(3) one has the stresses expressed in terms of midplane strains $\{\epsilon^0\}$, and rotations of transverse normal $\{k^1\}$ ,and $\{k^2\}$ , $\{\gamma^1\}$ , $\{\gamma^2\}$ and $\{\epsilon^n\}$ .

$$\begin{aligned}
\sigma_x &= \overline{Q}_{11}\epsilon_x^0 + \overline{Q}_{12}\epsilon_y^0 + \overline{Q}_{16}\gamma_{xy}^0 + (\overline{Q}_{11}k_x^1 + \overline{Q}_{12}k_y^1 + \overline{Q}_{16}k_{xy}^1)z \\
&\quad + (\overline{Q}_{11}k_x^2 + \overline{Q}_{12}k_y^2 + \overline{Q}_{16}k_{xy}^2)z^3 + L_n(\overline{Q}_{11}\epsilon_x^n + \overline{Q}_{12}\epsilon_y^n + \overline{Q}_{16}\gamma_{xy}^n) \\
\sigma_y &= \overline{Q}_{12}\epsilon_x^0 + \overline{Q}_{22}\epsilon_y^0 + \overline{Q}_{26}\gamma_{xy}^0 + (\overline{Q}_{12}k_x^1 + \overline{Q}_{22}k_y^1 + \overline{Q}_{26}k_{xy}^1)z \\
&\quad + (\overline{Q}_{12}k_x^2 + \overline{Q}_{22}k_y^2 + \overline{Q}_{26}k_{xy}^2)z^3 + L_n(\overline{Q}_{12}\epsilon_x^n + \overline{Q}_{22}\epsilon_y^n + \overline{Q}_{26}\gamma_{xy}^n) \\
\sigma_{yz} &= \overline{Q}_{44}(\gamma_{yz}^1 + z^2\gamma_{yz}^2) + \overline{Q}_{45}(\gamma_{xz}^1 + z^2\gamma_{xz}^2) \qquad (14) \\
\sigma_{xz} &= \overline{Q}_{45}(\gamma_{yz}^1 + z^2\gamma_{yz}^2) + \overline{Q}_{55}(\gamma_{xz}^1 + z^2\gamma_{xz}^2) \\
\sigma_{xy} &= \overline{Q}_{16}\epsilon_x^0 + \overline{Q}_{26}\epsilon_y^0 + \overline{Q}_{66}\gamma_{xy}^0 + (\overline{Q}_{16}k_x^1 + \overline{Q}_{26}k_y^1 + \overline{Q}_{66}k_{xy}^1)z \\
&\quad + (\overline{Q}_{16}k_x^2 + \overline{Q}_{26}k_y^2 + \overline{Q}_{66}k_{xy}^2)z^3 + L_n(\overline{Q}_{16}\epsilon_x^n + \overline{Q}_{26}\epsilon_y^n + \overline{Q}_{66}\gamma_{xy}^n)
\end{aligned}$$

The stress resultants and moment resultants are defined as

$$\{N\} = \left\{ \begin{array}{c} N_x \\ N_y \\ N_{xy} \end{array} \right\} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{array} \right\} dz \qquad (15)$$

$$\{M\} = \left\{ \begin{array}{c} M_x \\ M_y \\ M_{xy} \end{array} \right\} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{array} \right\} zdz \qquad (16)$$

$$\{P\} = \begin{Bmatrix} P_x \\ P_y \\ P_{xy} \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} z^3 dz \tag{17}$$

$$\{Q\} = \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} dz \tag{18}$$

$$\{R\} = \begin{Bmatrix} R_x \\ R_y \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} z^2 dz \tag{19}$$

where the geometrical notation is as shown below:

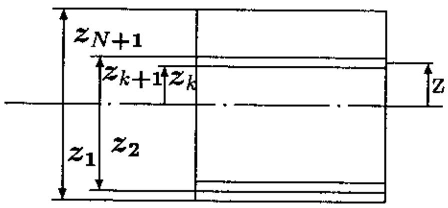


Figure 3

Substituting Eq. (14) into Eqs. (15), (16), (17), (18) and (19), we have

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \\ P_x \\ P_y \\ P_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & E_{11} & E_{12} & E_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & E_{12} & E_{22} & E_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & E_{16} & E_{26} & E_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & F_{11} & F_{12} & F_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & F_{12} & F_{22} & F_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & F_{16} & F_{26} & F_{66} \\ E_{11} & E_{12} & E_{16} & F_{11} & F_{12} & F_{16} & H_{11} & H_{12} & H_{16} \\ E_{12} & E_{22} & E_{26} & F_{12} & F_{22} & F_{26} & H_{12} & H_{22} & H_{26} \\ E_{16} & E_{26} & E_{66} & F_{16} & F_{26} & F_{66} & H_{16} & H_{26} & H_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^0 + L_n \epsilon_x^n \\ \epsilon_y^0 + L_n \epsilon_y^n \\ \gamma_{xy}^0 + L_n \gamma_{xy}^n \\ k_x^1 \\ k_y^1 \\ k_{xy}^1 \\ k_x^2 \\ k_y^2 \\ k_{xy}^2 \end{Bmatrix} \tag{20}$$

$$\begin{Bmatrix} Q_y \\ Q_x \\ R_y \\ R_x \end{Bmatrix} = \begin{bmatrix} A_{44} & A_{45} & D_{44} & D_{45} \\ A_{45} & A_{55} & D_{45} & D_{55} \\ D_{44} & D_{45} & F_{44} & F_{45} \\ D_{45} & D_{55} & F_{45} & F_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^1 \\ \gamma_{xz}^1 \\ \gamma_{yz}^2 \\ \gamma_{xz}^2 \end{Bmatrix} \tag{21}$$

$$\begin{aligned} A_{ij} &= \sum_{k=1}^{N} \overline{Q}_{ij}^k (z_{k+1} - z_k) \\ B_{ij} &= \frac{1}{2} \sum_{k=1}^{N} \overline{Q}_{ij}^k (z_{k+1}^2 - z_k^2) \\ D_{ij} &= \frac{1}{3} \sum_{k=1}^{N} \overline{Q}_{ij}^k (z_{k+1}^3 - z_k^3) \\ E_{ij} &= \frac{1}{4} \sum_{k=1}^{N} \overline{Q}_{ij}^k (z_{k+1}^4 - z_k^4) \end{aligned} \tag{22}$$

$$
\begin{aligned}
F_{ij} &= \frac{1}{5}\sum_{k=1}^{N}\overline{Q}_{ij}^{k}(z_{k+1}^{5}-z_{k}^{5}) \\
H_{ij} &= \frac{1}{7}\sum_{k=1}^{N}\overline{Q}_{ij}^{k}(z_{k+1}^{7}-z_{k}^{7})
\end{aligned}
$$

where i,j =1,2,6.

$$
\begin{aligned}
A_{ij} &= \sum_{k=1}^{N}\overline{Q}_{ij}^{k}(z_{k+1}-z_{k}) \\
D_{ij} &= \frac{1}{3}\sum_{k=1}^{N}\overline{Q}_{ij}^{k}(z_{k+1}^{3}-z_{k}^{3}) \\
F_{ij} &= \frac{1}{5}\sum_{k=1}^{N}\overline{Q}_{ij}^{k}(z_{k+1}^{5}-z_{k}^{5})
\end{aligned}
\tag{23}
$$

where i,j =4,5.

# 4 Interpolation Function

We adopt Serendipity 8-node element with five degrees of freedom for each node. The total number of degrees of freedom for the element is 40. The natural coordinate system $(\xi, \eta)$ as shown in Fig.4 is taken to define the element geometry. The element has sides $\xi = \pm 1$ and $\eta = \pm 1$ (see Fig.4). For the element of side 2a by 2b

$$
\begin{aligned}
\xi &= (x - x_c)/a \\
\eta &= (y - y_c)/b
\end{aligned}
\tag{24}
$$

where $(x_c, y_c)$ are the coordinates at the centre of the element . Thus we have

$$
\frac{d\xi}{dx} = \frac{1}{a} \qquad \frac{d\eta}{dy} = \frac{1}{b} \tag{25}
$$

and the element area of the rectangular element is given as

$$dxdy = abd\xi d\eta \tag{26}$$

To integrate any function f(x,y) over the element we transform to the natural coordinate system, so that

$$\int\int_{\Omega^e} f(x,y)\, dxdy = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\, abd\xi d\eta \tag{27}$$

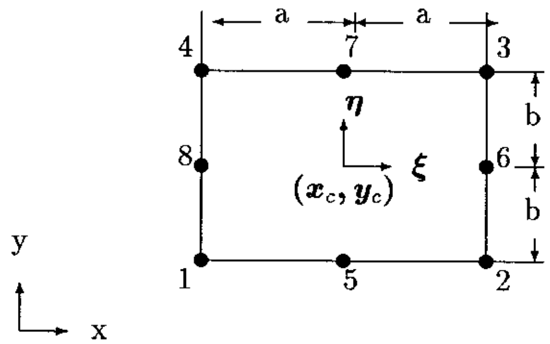


Figure 4

For the 8-node Serendipity element shown in Fig.4 the interpolation function has the following forms for the corner and midside nodes:

1. for the corner nodes

$$\psi_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)(\xi\xi_i+\eta\eta_i-1) \quad i = 1,2,3,4 \tag{28}$$

2. for the midside nodes

$$\psi_i = \frac{\xi_i^2}{2}(1+\xi\xi_i)(1-\eta^2)+\frac{\eta_i^2}{2}(1+\eta\eta_i)(1-\xi^2) \quad i = 5,6,7,8 \tag{29}$$

According to the values of 8 node coordinates we have from Eqs.(28) and (29) the interpolation functions for each node as follows :

$$
\begin{aligned}
\psi_1 &= -\frac{1}{4}(1-\xi)(1-\eta)(\xi+\eta+1) \\
\psi_2 &= \frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1) \\
\psi_3 &= \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1) \\
\psi_4 &= \frac{1}{4}(1-\xi)(1+\eta)(-\xi+\eta-1) \\
\psi_5 &= \frac{1}{2}(1-\eta)(1-\xi^2) \\
\psi_6 &= \frac{1}{2}(1+\xi)(1-\eta^2) \\
\psi_7 &= \frac{1}{2}(1+\eta)(1-\xi^2) \\
\psi_8 &= \frac{1}{2}(1-\xi)(1-\eta^2)
\end{aligned}
\tag{30}
$$

The derivatives of the interpolation functions are as follows :

1. for the corner nodes

$$
\begin{aligned}
\frac{\partial\psi_i}{\partial\xi} &= \frac{1}{4}\xi_i(1+\eta\eta_i)(2\xi\xi_i+\eta\eta_i) \\
\frac{\partial\psi_i}{\partial\eta} &= \frac{1}{4}\eta_i(1+\xi\xi_i)(2\eta\eta_i+\xi\xi_i) \\
\frac{\partial^2\psi_i}{\partial\xi^2} &= \frac{1}{2}\xi_i^2(1+\eta\eta_i) \\
\frac{\partial^2\psi_i}{\partial\eta^2} &= \frac{1}{2}\eta_i^2(1+\xi\xi_i) \\
\frac{\partial^2\psi_i}{\partial\xi\partial\eta} &= \frac{\partial^2\psi_i}{\partial\eta\partial\xi} = \frac{1}{4}\xi_i\eta_i(1+2\eta\eta_i+2\xi\xi_i)
\end{aligned}
\tag{31}
$$

2. for the midside nodes

$$\begin{aligned}
\frac{\partial \psi_i}{\partial \xi} &= \frac{1}{2}\xi_i^3(1-\eta^2) - \eta_i^2(1+\eta\eta_i)\xi \\
\frac{\partial \psi_i}{\partial \eta} &= \frac{1}{2}\eta_i^3(1-\xi^2) - \xi_i^2(1+\xi\xi_i)\eta \\
\frac{\partial^2 \psi_i}{\partial \xi^2} &= -\eta_i^2(1+\eta\eta_i) \\
\frac{\partial^2 \psi_i}{\partial \eta^2} &= -\xi_i^2(1+\xi\xi_i) \\
\frac{\partial^2 \psi_i}{\partial \xi \partial \eta} &= \frac{\partial^2 \psi_i}{\partial \eta \partial \xi} = -\xi_i^3\eta - \xi\eta_i^3
\end{aligned} \tag{32}$$

Note that the polynomial terms contained in this element are $1, x, y, x^2, xy, y^2, x^2y, xy^2$ . For this element, the interpolation functions have satisfied the conditions

$$\sum_i \psi_i(\xi\eta) = 1 \tag{33}$$

and

$$\psi_i(\xi_j\eta_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & i \neq j \end{cases} \tag{34}$$

The displacement components are approximated by the product of the interpolation function matrix $[\psi_i]$ and the nodal displacement vector $\{q_i^e\} = [u_{0i}\ v_{0i}\ w_{0i}\ \phi_{1i}\ \phi_{2i}]^T$, i.e.,

$$\{q\} = \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \phi_1 \\ \phi_2 \end{Bmatrix} = \sum_{i=1}^{8} [\psi_i]\{q_i^e\} \tag{35}$$

the superscript e of $\{q_i^e\}$ denotes these variables are defined on the element and need to be determined .

# 5 The Stiffness Matrix of an 8-Node Laminated Plate Element

Finite element models developed for plate theory can be grouped into three major categories:

1. displacement models based on the principle of virtual displacements;
2. mixed and hybrid models based on the modified or mixed variational statements of the plate theories;
3. equilibrium models based on the principle of virtual forces.

Among the three types of models, the displacement finite element models are most natural and commonly used in commercial finite element programs.

In this paper we will use the displacement models to derive the stiffness matrix of an 8-node rectangular layered plate element.

By means of the principle of virtual displacement, we have

$$\begin{aligned}
\int_{\Omega^e}\left(\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}\right)\delta u_0 dxdy &= 0\\
\int_{\Omega^e}\left(\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}\right)\delta v_0 dxdy &= 0\\
\int_{\Omega^e}\{\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+q-\frac{4}{h^2}\left(\frac{\partial R_x}{\partial x}+\frac{\partial R_y}{\partial y}\right)+\frac{4}{3h^2}(\frac{\partial^2 P_x}{\partial x^2}
\end{aligned}$$

$$
\begin{aligned}
&+2\frac{\partial^2 P_{xy}}{\partial x\partial y}+\frac{\partial^2 P_y}{\partial y^2})+L_n[\frac{\partial}{\partial x}(N_x\frac{\partial w_0}{\partial x}+N_{xy}\frac{\partial w_0}{\partial y}) \\
&+\frac{\partial}{\partial y}(N_{xy}\frac{\partial w_0}{\partial x}+N_y\frac{\partial w_0}{\partial y})]\}\delta w_0 dxdy \;=\; 0 \qquad (36) \\
&\int_{\Omega^e}[\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x+\frac{4}{h^2}R_x-\frac{4}{3h^2}(\frac{\partial P_x}{\partial x}+\frac{\partial P_{xy}}{\partial y})]\delta\phi_1 dxdy \;=\; 0 \\
&\int_{\Omega^e}[\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-Q_y+\frac{4}{h^2}R_y-\frac{4}{3h^2}(\frac{\partial P_{xy}}{\partial x}+\frac{\partial P_y}{\partial y})]\delta\phi_2 dxdy \;=\; 0
\end{aligned}
$$

Recall that $N_x, N_{xy}, ..., M_{xy}, R_x, ..., P_{xy}$ are functions of the derivatives of the displacement $u_0, v_0, w_0, \phi_1, \phi_2$. To reduce the differentiability of the interpolation functions used in the finite element approximation of $u_0, v_0, w_0, \phi_1, \phi_2$ the differentiation on $N_x, N_{xy}, ..., M_{xy}, R_x, ..., P_{xy}$ is treated to weight functions $\delta u_0, \delta v_0, \delta w_0$, $\delta\phi_1, \delta\phi_2$ by using integration-by-parts

$$
\begin{aligned}
&\int_{\Omega^e}(\frac{\partial\delta u_0}{\partial x}N_x+\frac{\partial\delta u_0}{\partial y}N_{xy})dxdy-\int_{\Gamma^e}N_n\delta u_0 ds \;=\; 0 \\
&\int_{\Omega^e}(\frac{\partial\delta v_0}{\partial x}N_{xy}+\frac{\partial\delta v_0}{\partial y}N_y)dxdy-\int_{\Gamma^e}N_{ns}\delta v_0 ds \;=\; 0 \\
&\int_{\Omega^e}\{\frac{\partial\delta w_0}{\partial x}Q_x+\frac{\partial\delta w_0}{\partial y}Q_y-\frac{4}{h^2}(\frac{\partial\delta w_0}{\partial x}R_x+\frac{\partial\delta w_0}{\partial y}R_y)-\frac{4}{3h^2}(P_x\frac{\partial^2\delta w_0}{\partial x^2} \\
&+2P_{xy}\frac{\partial^2\delta w_0}{\partial x\partial y}+P_y\frac{\partial^2\delta w_0}{\partial y^2})+L_n[\frac{\partial\delta w_0}{\partial x}(N_x\frac{\partial w_0}{\partial x}+N_{xy}\frac{\partial w_0}{\partial y}) \\
&+\frac{\partial\delta w_0}{\partial y}(N_{xy}\frac{\partial w_0}{\partial x}+N_y\frac{\partial w_0}{\partial y})]\}dxdy-\int_{\Omega^e}q\delta w_0 dxdy \qquad (37) \\
&-\int_{\Gamma^e}(Q_n\delta w_0-\frac{4}{3h^2}P_n\frac{\partial\delta w_0}{\partial x}-\frac{4}{3h^2}P_{ns}\frac{\partial\delta w_0}{\partial y})ds \;=\; 0 \\
&\int_{\Omega^e}[\frac{\partial\delta\phi_1}{\partial x}M_x+\frac{\partial\delta\phi_1}{\partial y}M_{xy}+Q_x\delta\phi_1-\frac{4}{3h^2}(\frac{\partial\delta\phi_1}{\partial x}P_x+\frac{\partial\delta\phi_1}{\partial y}P_{xy}) \\
&-\frac{4}{h^2}R_x\delta\phi_1]dxdy-\int_{\Gamma^e}M_n\delta\phi_1 ds \;=\; 0
\end{aligned}
$$

$$\int_{\Omega^e}[\frac{\partial\delta\phi_2}{\partial x}M_{xy} + \frac{\partial\delta\phi_2}{\partial y}M_y + Q_y\delta\phi_2 - \frac{4}{3h^2}(\frac{\partial\delta\phi_2}{\partial x}P_{xy} + \frac{\partial\delta\phi_2}{\partial y}P_y)$$
$$-\frac{4}{h^2}R_y\delta\phi_2]dxdy - \int_{\Gamma^e} M_{ns}\delta\phi_2 ds = 0$$

where

$\Gamma^e$ ——— the boundary edge of the element domain $\Omega^e$;

$\Omega^e$ ——— the element domain;

$$N_n = N_x n_x + N_{xy} n_y$$
$$N_{ns} = N_{xy} n_x + N_y n_y$$
$$Q_n = Q_x n_x + Q_y n_y - \frac{4}{h^2}(R_x n_x + R_y n_y) + \frac{4}{3h^2}[(\frac{\partial P_x}{\partial x} + \frac{\partial P_{xy}}{\partial y})n_x + (\frac{\partial P_{xy}}{\partial x}$$
$$+\frac{\partial P_y}{\partial y})n_y] + L_n[(N_x\frac{\partial w_0}{\partial x} + N_{xy}\frac{\partial w_0}{\partial y})n_x + (N_{xy}\frac{\partial w_0}{\partial x} + N_y\frac{\partial w_0}{\partial y})n_y]$$
$$P_n = P_x n_x + P_{xy} n_y$$
$$P_{ns} = P_{xy} n_x + P_y n_y$$
$$M_n = M_x n_x + M_{xy} n_y - \frac{4}{3h^2}(P_x n_x + P_{xy} n_y)$$
$$M_{ns} = M_{xy} n_x + M_y n_y - \frac{4}{3h^2}(P_{xy} n_x + P_y n_y)$$
$$n_x = \frac{dy}{ds} \qquad n_y = -\frac{dx}{ds}$$

Substituting $\delta u_0 = \psi_i$, $\delta v_0 = \psi_i$, $\delta w_0 = \psi_i$, $\delta\phi_1 = \psi_i$ and $\delta\phi_2 = \psi_i$ into Eqs. (37) , we have

$$
\begin{aligned}
\int_{\Omega^e}(\frac{\partial\psi_i}{\partial x}N_x + \frac{\partial\psi_i}{\partial y}N_{xy})dxdy - \int_{\Gamma^e} N_n\psi_i ds &= 0 \\
\int_{\Omega^e}(\frac{\partial\psi_i}{\partial x}N_{xy} + \frac{\partial\psi_i}{\partial y}N_y)dxdy - \int_{\Gamma^e} N_{ns}\psi_i ds &= 0 \\
\int_{\Omega^e}\{\frac{\partial\psi_i}{\partial x}Q_x + \frac{\partial\psi_i}{\partial y}Q_y - \frac{4}{h^2}(\frac{\partial\psi_i}{\partial x}R_x + \frac{\partial\psi_i}{\partial y}R_y) - \frac{4}{3h^2}(P_x\frac{\partial^2\psi_i}{\partial x^2} \\
+2P_{xy}\frac{\partial^2\psi_i}{\partial x\partial y} + P_y\frac{\partial^2\psi_i}{\partial y^2}) + L_n[\frac{\partial\psi_i}{\partial x}(N_x\frac{\partial w_0}{\partial x} + N_{xy}\frac{\partial w_0}{\partial y}) \\
+\frac{\partial\psi_i}{\partial y}(N_{xy}\frac{\partial w_0}{\partial x} + N_y\frac{\partial w_0}{\partial y})]\}dxdy - \int_{\Omega^e} -q\psi_i dxdy \\
- \int_{\Gamma^e}(Q_n\psi_i - \frac{4}{3h^2}P_n\frac{\partial\psi_i}{\partial x} - \frac{4}{3h^2}P_{ns}\frac{\partial\psi_i}{\partial y})ds &= 0 \\
\int_{\Omega^e}[\frac{\partial\psi_i}{\partial x}M_x + \frac{\partial\psi_i}{\partial y}M_{xy} + Q_x\psi_i - \frac{4}{3h^2}(P_x\frac{\partial\psi_i}{\partial x} + P_{xy}\frac{\partial\psi_i}{\partial y}) \\
-\frac{4}{h^2}R_x\psi_i]dxdy - \int_{\Gamma^e} M_n\psi_i ds &= 0 \\
\int_{\Omega^e}[\frac{\partial\psi_i}{\partial x}M_{xy} + \frac{\partial\psi_i}{\partial y}M_y + Q_y\psi_i - \frac{4}{3h^2}(P_{xy}\frac{\partial\psi_i}{\partial x} + P_y\frac{\partial\psi_i}{\partial y}) \\
-\frac{4}{h^2}R_y\psi_i]dxdy - \int_{\Gamma^e} M_{ns}\psi_i ds &= 0
\end{aligned} \qquad (38)
$$

Substituting Eq.(35) for $u_0, v_0, w_0, \phi_1$, and $\phi_2$ into Eqs. (8), (9), (10), (11), (12), (13) we obtain $\{\epsilon^0\}$, $\{\epsilon^n\}, \{k^1\}, \{k^2\}, \{\gamma^1\}$ and $\{\gamma^2\}$ represented by interpolation functions and nodal displacements. Then substituting $\{\epsilon^0\}$, $\{\epsilon^n\}$, $\{k^1\}$, $\{k^2\}$, $\{\gamma^1\}$ and $\{\gamma^2\}$ into Eqs. (20), (21) we have stress resultants, moment resultants and transverse shear forces $\{N\}, \{M\}, \{Q\}, \{P\}$ and $\{R\}$ represented by interpolation functions and node displacements . After substituting $\{N\}, \{M\}, \{Q\}, \{P\}$ and $\{R\}$ represented by interpolation functions and node displacements into Eq.(38)

we obtain the finite element model of the higher-order shear deformation theory

$$\sum_{j=1}^{8}\sum_{\alpha=1}^{5} K_{ij}^{\beta\alpha}\Delta_j^{\alpha} - F_i^{\beta} = 0 \quad (\beta = 1,2,...,5; i = 1,2,...,8.) \tag{39}$$

or

$$[K^e]\{\Delta^e\} \;=\; \{F^e\} \tag{40}$$

where the variables $\Delta_j^{\alpha}$ , the stiffness and force coefficients are defined by

$$\begin{aligned} \Delta_j^1 &= u_{0j} \qquad \Delta_j^2 = v_{0j} \qquad \Delta_j^3 = w_{0j} \\ \Delta_j^4 &= \phi_{1j} \qquad \Delta_j^5 = \phi_{2j} \end{aligned} \tag{41}$$

$$\begin{aligned}
K_{ij}^{1\alpha} &= \int_{\Omega^e}[\frac{\partial\psi_i}{\partial x}(N_{1j}^{\alpha} + \overline{N}_{1j}^{\alpha} + L_n N_{1j}^{n\alpha}) + \frac{\partial\psi_i}{\partial y}(N_{6j}^{\alpha} + \overline{N}_{6j}^{\alpha} + L_n N_{6j}^{n\alpha})]dxdy \\
K_{ij}^{2\alpha} &= \int_{\Omega^e}[\frac{\partial\psi_i}{\partial x}(N_{6j}^{\alpha} + \overline{N}_{6j}^{\alpha} + L_n N_{6j}^{n\alpha}) + \frac{\partial\psi_i}{\partial y}(N_{2j}^{\alpha} + \overline{N}_{2j}^{\alpha} + L_n N_{2j}^{n\alpha})]dxdy \\
K_{ij}^{3\alpha} &= \int_{\Omega^e}\{\frac{\partial\psi_i}{\partial x}(Q_{1j}^{\alpha} + Q_{3j}^{\alpha}) + \frac{\partial\psi_i}{\partial y}(Q_{2j}^{\alpha} + Q_{4j}^{\alpha}) + \frac{\partial^2\psi_i}{\partial x^2}(Q_{5j}^{\alpha} + L_n Q_{5j}^{n\alpha}) \\
&\quad +2\frac{\partial^2\psi_i}{\partial x\partial y}(Q_{6j}^{\alpha} + L_n Q_{6j}^{n\alpha}) + \frac{\partial^2\psi_i}{\partial y^2}(Q_{7j}^{\alpha} + L_n Q_{7j}^{n\alpha}) \\
&\quad +L_n[\frac{\partial\psi_i}{\partial x}\frac{\partial w_0}{\partial x}(N_{1j}^{\alpha} + \overline{N}_{1j}^{\alpha}) + \frac{\partial\psi_i}{\partial x}\frac{\partial w_0}{\partial y}(N_{6j}^{\alpha} + \overline{N}_{6j}^{\alpha}) \\
&\quad +\frac{\partial\psi_i}{\partial y}\frac{\partial w_0}{\partial x}(N_{6j}^{\alpha} + \overline{N}_{6j}^{\alpha}) + \frac{\partial\psi_i}{\partial y}\frac{\partial w_0}{\partial y}(N_{2j}^{\alpha} + \overline{N}_{2j}^{\alpha}) \\
&\quad +\frac{\partial\psi_i}{\partial x}(\frac{\partial\psi_j}{\partial x}Q_1^{\alpha} + \frac{\partial\psi_j}{\partial y}Q_6^{\alpha}) + \frac{\partial\psi_i}{\partial y}(\frac{\partial\psi_j}{\partial x}Q_6^{\alpha} + \frac{\partial\psi_j}{\partial y}Q_2^{\alpha})]\}dxdy \\
K_{ij}^{4\alpha} &= \int_{\Omega^e}\{\frac{\partial\psi_i}{\partial x}[M_{1j}^{\alpha} + M_{3j}^{\alpha} + L_n(M_{1j}^{n\alpha} + Q_{5j}^{n\alpha})] + \frac{\partial\psi_i}{\partial y}[M_{2j}^{\alpha} + M_{4j}^{\alpha} \\
&\quad +L_n(M_{2j}^{n\alpha} + Q_{6j}^{n\alpha})] + \psi_i(Q_{1j}^{\alpha} + Q_{3j}^{\alpha})\}dxdy \\
K_{ij}^{5\alpha} &= \int_{\Omega^e}\{\frac{\partial\psi_i}{\partial x}[M_{2j}^{\alpha} + M_{4j}^{\alpha} + L_n(M_{2j}^{n\alpha} + Q_{6j}^{n\alpha})] + \frac{\partial\psi_i}{\partial y}[M_{5j}^{\alpha} + M_{6j}^{\alpha} \\
&\quad +L_n(M_{5j}^{n\alpha} + Q_{7j}^{n\alpha})] + \psi_i(Q_{2j}^{\alpha} + Q_{4j}^{\alpha})\}dxdy
\end{aligned} \tag{42}$$

The coefficients $N_{Ij}^{\alpha}, \overline{N}_{Ij}^{\alpha}, M_{Ij}^{\alpha}$ and $Q_{Ij}^{\alpha}$ for $\alpha = 1, 2, ..., 5$ and $I = 1, 2, 6$ are given by

$$\begin{aligned}
N_{1j}^1 &= A_{11}\frac{\partial\psi_j}{\partial x} + A_{16}\frac{\partial\psi_j}{\partial y} & N_{1j}^2 &= A_{12}\frac{\partial\psi_j}{\partial y} + A_{16}\frac{\partial\psi_j}{\partial x} \\
N_{1j}^4 &= B_{11}\frac{\partial\psi_j}{\partial x} + B_{16}\frac{\partial\psi_j}{\partial y} & N_{1j}^5 &= B_{12}\frac{\partial\psi_j}{\partial y} + B_{16}\frac{\partial\psi_j}{\partial x} \\
N_{2j}^1 &= A_{12}\frac{\partial\psi_j}{\partial x} + A_{26}\frac{\partial\psi_j}{\partial y} & N_{2j}^2 &= A_{22}\frac{\partial\psi_j}{\partial y} + A_{26}\frac{\partial\psi_j}{\partial x} \\
N_{2j}^4 &= B_{12}\frac{\partial\psi_j}{\partial x} + B_{26}\frac{\partial\psi_j}{\partial y} & N_{2j}^5 &= B_{22}\frac{\partial\psi_j}{\partial y} + B_{26}\frac{\partial\psi_j}{\partial x} \\
N_{6j}^1 &= A_{16}\frac{\partial\psi_j}{\partial x} + A_{66}\frac{\partial\psi_j}{\partial y} & N_{6j}^2 &= A_{26}\frac{\partial\psi_j}{\partial y} + A_{66}\frac{\partial\psi_j}{\partial x} \\
N_{6j}^4 &= B_{16}\frac{\partial\psi_j}{\partial x} + B_{66}\frac{\partial\psi_j}{\partial y} & N_{6j}^5 &= B_{26}\frac{\partial\psi_j}{\partial y} + B_{66}\frac{\partial\psi_j}{\partial x}
\end{aligned} \tag{43}$$

$$\begin{aligned}
\overline{N}_{1j}^3 &= -\frac{4}{3h^2}(E_{11}\frac{\partial^2\psi_j}{\partial x^2} + E_{12}\frac{\partial^2\psi_j}{\partial y^2} + 2E_{16}\frac{\partial^2\psi_j}{\partial x\partial y}) \\
\overline{N}_{1j}^4 &= -\frac{4}{3h^2}(E_{11}\frac{\partial\psi_j}{\partial x} + E_{16}\frac{\partial\psi_j}{\partial y}) \qquad \overline{N}_{1j}^5 = -\frac{4}{3h^2}(E_{12}\frac{\partial\psi_j}{\partial y} + E_{16}\frac{\partial\psi_j}{\partial x}) \\
\overline{N}_{2j}^3 &= -\frac{4}{3h^2}(E_{12}\frac{\partial^2\psi_j}{\partial x^2} + E_{22}\frac{\partial^2\psi_j}{\partial y^2} + 2E_{26}\frac{\partial^2\psi_j}{\partial x\partial y}) \\
\overline{N}_{2j}^4 &= -\frac{4}{3h^2}(E_{12}\frac{\partial\psi_j}{\partial x} + E_{26}\frac{\partial\psi_j}{\partial y}) \qquad \overline{N}_{2j}^5 = -\frac{4}{3h^2}(E_{22}\frac{\partial\psi_j}{\partial y} + E_{26}\frac{\partial\psi_j}{\partial x}) \\
\overline{N}_{6j}^3 &= -\frac{4}{3h^2}(E_{16}\frac{\partial^2\psi_j}{\partial x^2} + E_{26}\frac{\partial^2\psi_j}{\partial y^2} + 2E_{66}\frac{\partial^2\psi_j}{\partial x\partial y}) \\
\overline{N}_{6j}^4 &= -\frac{4}{3h^2}(E_{16}\frac{\partial\psi_j}{\partial x} + E_{66}\frac{\partial\psi_j}{\partial y}) \qquad \overline{N}_{6j}^5 = -\frac{4}{3h^2}(E_{26}\frac{\partial\psi_j}{\partial y} + E_{66}\frac{\partial\psi_j}{\partial x})
\end{aligned} \tag{44}$$

$$\begin{aligned}
Q_{1j}^3 &= A_{45}\frac{\partial\psi_j}{\partial y} + A_{55}\frac{\partial\psi_j}{\partial x} - \frac{4}{h^2}D_{45}\frac{\partial\psi_i}{\partial y} - \frac{4}{h^2}D_{55}\frac{\partial\psi_i}{\partial x} \\
Q_{1j}^4 &= A_{55}\psi_j - \frac{4}{h^2}D_{55}\psi_j \qquad Q_{1j}^5 = A_{45}\psi_j - \frac{4}{h^2}D_{45}\psi_j \\
Q_{2j}^3 &= A_{44}\frac{\partial\psi_j}{\partial y} + A_{45}\frac{\partial\psi_j}{\partial x} - \frac{4}{h^2}D_{44}\frac{\partial\psi_i}{\partial y} - \frac{4}{h^2}D_{45}\frac{\partial\psi_i}{\partial x}
\end{aligned}$$

$$
\begin{aligned}
Q_{2j}^4 &= A_{45}\psi_j - \frac{4}{h^2}D_{45}\psi_j \qquad Q_{2j}^5 = A_{44}\psi_j - \frac{4}{h^2}D_{44}\psi_j \\
Q_{3j}^3 &= -\frac{4}{h^2}(D_{45}\frac{\partial\psi_j}{\partial y} + D_{55}\frac{\partial\psi_j}{\partial x} - \frac{4}{h^2}F_{45}\frac{\partial\psi_i}{\partial y} - \frac{4}{h^2}F_{55}\frac{\partial\psi_i}{\partial x}) \\
Q_{3j}^4 &= -\frac{4}{h^2}(D_{55}\psi_j - \frac{4}{h^2}F_{55}\psi_j) \qquad Q_{3j}^5 = -\frac{4}{h^2}(D_{45}\psi_j - \frac{4}{h^2}F_{45}\psi_j) \\
Q_{4j}^3 &= -\frac{4}{h^2}(D_{44}\frac{\partial\psi_j}{\partial y} + D_{45}\frac{\partial\psi_j}{\partial x} - \frac{4}{h^2}F_{44}\frac{\partial\psi_i}{\partial y} - \frac{4}{h^2}F_{45}\frac{\partial\psi_i}{\partial x}) \\
Q_{4j}^4 &= -\frac{4}{h^2}(D_{45}\psi_j - \frac{4}{h^2}F_{45}\psi_j) \qquad Q_{4j}^5 = -\frac{4}{h^2}(D_{44}\psi_j - \frac{4}{h^2}F_{44}\psi_j) \\
Q_{5j}^1 &= -\frac{4}{3h^2}(E_{11}\frac{\partial\psi_j}{\partial x} + E_{16}\frac{\partial\psi_j}{\partial y}) \qquad Q_{5j}^2 = -\frac{4}{3h^2}(E_{12}\frac{\partial\psi_j}{\partial y} + E_{16}\frac{\partial\psi_j}{\partial x}) \\
Q_{5j}^3 &= \frac{16}{9h^4}(H_{11}\frac{\partial^2\psi_j}{\partial x^2} + H_{12}\frac{\partial^2\psi_j}{\partial y^2} + 2H_{16}\frac{\partial^2\psi_j}{\partial x\partial y}) \qquad (45) \\
Q_{5j}^4 &= -\frac{4}{3h^2}(F_{11}\frac{\partial\psi_j}{\partial x} + F_{16}\frac{\partial\psi_j}{\partial y} - \frac{4}{3h^2}H_{11}\frac{\partial\psi_j}{\partial x} - \frac{4}{3h^2}H_{16}\frac{\partial\psi_j}{\partial y}) \\
Q_{5j}^5 &= -\frac{4}{3h^2}(F_{12}\frac{\partial\psi_j}{\partial y} + F_{16}\frac{\partial\psi_j}{\partial x} - \frac{4}{3h^2}H_{12}\frac{\partial\psi_j}{\partial y} - \frac{4}{3h^2}H_{16}\frac{\partial\psi_j}{\partial x}) \\
Q_{6j}^1 &= -\frac{4}{3h^2}(E_{16}\frac{\partial\psi_j}{\partial x} + E_{66}\frac{\partial\psi_j}{\partial y}) \qquad Q_{6j}^2 = -\frac{4}{3h^2}(E_{26}\frac{\partial\psi_j}{\partial y} + E_{66}\frac{\partial\psi_j}{\partial x}) \\
Q_{6j}^3 &= \frac{16}{9h^4}(H_{16}\frac{\partial^2\psi_j}{\partial x^2} + H_{26}\frac{\partial^2\psi_j}{\partial y^2} + 2H_{66}\frac{\partial^2\psi_j}{\partial x\partial y}) \\
Q_{6j}^4 &= -\frac{4}{3h^2}(F_{16}\frac{\partial\psi_j}{\partial x} + F_{66}\frac{\partial\psi_j}{\partial y} - \frac{4}{3h^2}H_{16}\frac{\partial\psi_j}{\partial x} - \frac{4}{3h^2}H_{66}\frac{\partial\psi_j}{\partial y}) \\
Q_{6j}^5 &= -\frac{4}{3h^2}(F_{26}\frac{\partial\psi_j}{\partial y} + F_{66}\frac{\partial\psi_j}{\partial x} - \frac{4}{3h^2}H_{26}\frac{\partial\psi_j}{\partial y} - \frac{4}{3h^2}H_{66}\frac{\partial\psi_j}{\partial x}) \\
Q_{7j}^1 &= -\frac{4}{3h^2}(E_{12}\frac{\partial\psi_j}{\partial x} + E_{26}\frac{\partial\psi_j}{\partial y}) \qquad Q_{7j}^2 = -\frac{4}{3h^2}(E_{22}\frac{\partial\psi_j}{\partial y} + E_{26}\frac{\partial\psi_j}{\partial x}) \\
Q_{7j}^3 &= \frac{16}{9h^4}(H_{12}\frac{\partial^2\psi_j}{\partial x^2} + H_{22}\frac{\partial^2\psi_j}{\partial y^2} + 2H_{26}\frac{\partial^2\psi_j}{\partial x\partial y}) \\
Q_{7j}^4 &= -\frac{4}{3h^2}(F_{12}\frac{\partial\psi_j}{\partial x} + F_{26}\frac{\partial\psi_j}{\partial y} - \frac{4}{3h^2}H_{12}\frac{\partial\psi_j}{\partial x} - \frac{4}{3h^2}H_{26}\frac{\partial\psi_j}{\partial y}) \\
Q_{7j}^5 &= -\frac{4}{3h^2}(F_{22}\frac{\partial\psi_j}{\partial y} + F_{26}\frac{\partial\psi_j}{\partial x} - \frac{4}{3h^2}H_{22}\frac{\partial\psi_j}{\partial y} - \frac{4}{3h^2}H_{26}\frac{\partial\psi_j}{\partial x})
\end{aligned}
$$

$$
\begin{aligned}
M_{1j}^1 &= B_{11}\frac{\partial\psi_j}{\partial x} + B_{16}\frac{\partial\psi_j}{\partial y} \qquad M_{1j}^2 = B_{12}\frac{\partial\psi_j}{\partial y} + B_{16}\frac{\partial\psi_j}{\partial x} \\
M_{1j}^3 &= -\frac{4}{3h^2}(F_{11}\frac{\partial^2\psi_j}{\partial x^2} + F_{12}\frac{\partial^2\psi_j}{\partial y^2} + 2F_{16}\frac{\partial^2\psi_j}{\partial x\partial y}) \\
M_{1j}^4 &= D_{11}\frac{\partial\psi_j}{\partial x} + D_{16}\frac{\partial\psi_j}{\partial y} - \frac{4}{3h^2}F_{11}\frac{\partial\psi_j}{\partial x} - \frac{4}{3h^2}F_{16}\frac{\partial\psi_j}{\partial y} \\
M_{1j}^5 &= D_{12}\frac{\partial\psi_j}{\partial y} + D_{16}\frac{\partial\psi_j}{\partial x} - \frac{4}{3h^2}F_{12}\frac{\partial\psi_j}{\partial y} - \frac{4}{3h^2}F_{16}\frac{\partial\psi_j}{\partial x} \\
M_{2j}^1 &= B_{16}\frac{\partial\psi_j}{\partial x} + B_{66}\frac{\partial\psi_j}{\partial y} \qquad M_{2j}^2 = B_{26}\frac{\partial\psi_j}{\partial y} + B_{66}\frac{\partial\psi_j}{\partial x} \\
M_{2j}^3 &= -\frac{4}{3h^2}(F_{16}\frac{\partial^2\psi_j}{\partial x^2} + F_{26}\frac{\partial^2\psi_j}{\partial y^2} + 2F_{66}\frac{\partial^2\psi_j}{\partial x\partial y}) \\
M_{2j}^4 &= D_{16}\frac{\partial\psi_j}{\partial x} + D_{66}\frac{\partial\psi_j}{\partial y} - \frac{4}{3h^2}F_{16}\frac{\partial\psi_j}{\partial x} - \frac{4}{3h^2}F_{66}\frac{\partial\psi_j}{\partial y} \\
M_{2j}^5 &= D_{26}\frac{\partial\psi_j}{\partial y} + D_{66}\frac{\partial\psi_j}{\partial x} - \frac{4}{3h^2}F_{26}\frac{\partial\psi_j}{\partial y} - \frac{4}{3h^2}F_{66}\frac{\partial\psi_j}{\partial x} \\
M_{3j}^1 &= -\frac{4}{3h^2}(E_{11}\frac{\partial\psi_j}{\partial x} + E_{16}\frac{\partial\psi_j}{\partial y}) \qquad M_{3j}^2 = -\frac{4}{3h^2}(E_{12}\frac{\partial\psi_j}{\partial y} + E_{16}\frac{\partial\psi_j}{\partial x}) \\
M_{3j}^3 &= \frac{16}{9h^4}(H_{11}\frac{\partial^2\psi_j}{\partial x^2} + H_{12}\frac{\partial^2\psi_j}{\partial y^2} + 2H_{16}\frac{\partial^2\psi_j}{\partial x\partial y}) \\
M_{3j}^4 &= -\frac{4}{3h^2}(F_{11}\frac{\partial\psi_j}{\partial x} + F_{16}\frac{\partial\psi_j}{\partial y} - \frac{4}{3h^2}H_{11}\frac{\partial\psi_j}{\partial x} - \frac{4}{3h^2}H_{16}\frac{\partial\psi_j}{\partial y}) \\
M_{3j}^5 &= -\frac{4}{3h^2}(F_{12}\frac{\partial\psi_j}{\partial y} + F_{16}\frac{\partial\psi_j}{\partial x} - \frac{4}{3h^2}H_{12}\frac{\partial\psi_j}{\partial y} - \frac{4}{3h^2}H_{16}\frac{\partial\psi_j}{\partial x}) \qquad (46) \\
M_{4j}^1 &= -\frac{4}{3h^2}(E_{16}\frac{\partial\psi_j}{\partial x} + E_{66}\frac{\partial\psi_j}{\partial y}) \qquad M_{4j}^2 = -\frac{4}{3h^2}(E_{26}\frac{\partial\psi_j}{\partial y} + E_{66}\frac{\partial\psi_j}{\partial x}) \\
M_{4j}^3 &= \frac{16}{9h^4}(H_{16}\frac{\partial^2\psi_j}{\partial x^2} + H_{26}\frac{\partial^2\psi_j}{\partial y^2} + 2H_{66}\frac{\partial^2\psi_j}{\partial x\partial y}) \\
M_{4j}^4 &= -\frac{4}{3h^2}(F_{16}\frac{\partial\psi_j}{\partial x} + F_{66}\frac{\partial\psi_j}{\partial y} - \frac{4}{3h^2}H_{16}\frac{\partial\psi_j}{\partial x} - \frac{4}{3h^2}H_{66}\frac{\partial\psi_j}{\partial y}) \\
M_{4j}^5 &= -\frac{4}{3h^2}(F_{26}\frac{\partial\psi_j}{\partial y} + F_{66}\frac{\partial\psi_j}{\partial x} - \frac{4}{3h^2}H_{26}\frac{\partial\psi_j}{\partial y} - \frac{4}{3h^2}H_{66}\frac{\partial\psi_j}{\partial x})
\end{aligned}
$$

$$
\begin{aligned}
M_{5j}^{1} &= B_{12}\frac{\partial\psi_j}{\partial x} + B_{26}\frac{\partial\psi_j}{\partial y} \qquad M_{5j}^{2} = B_{22}\frac{\partial\psi_j}{\partial y} + B_{26}\frac{\partial\psi_j}{\partial x} \\
M_{5j}^{3} &= -\frac{4}{3h^2}(F_{12}\frac{\partial^2\psi_j}{\partial x^2} + F_{22}\frac{\partial^2\psi_j}{\partial y^2} + 2F_{26}\frac{\partial^2\psi_j}{\partial x\partial y}) \\
M_{5j}^{4} &= D_{12}\frac{\partial\psi_j}{\partial x} + D_{26}\frac{\partial\psi_j}{\partial y} - \frac{4}{3h^2}F_{12}\frac{\partial\psi_j}{\partial x} - \frac{4}{3h^2}F_{26}\frac{\partial\psi_j}{\partial y} \\
M_{5j}^{5} &= D_{22}\frac{\partial\psi_j}{\partial y} + D_{26}\frac{\partial\psi_j}{\partial x} - \frac{4}{3h^2}F_{22}\frac{\partial\psi_j}{\partial y} - \frac{4}{3h^2}F_{26}\frac{\partial\psi_j}{\partial x}) \\
M_{6j}^{1} &= -\frac{4}{3h^2}(E_{12}\frac{\partial\psi_j}{\partial x} + E_{26}\frac{\partial\psi_j}{\partial y}) \qquad M_{6j}^{2} = -\frac{4}{3h^2}(E_{22}\frac{\partial\psi_j}{\partial y} + E_{26}\frac{\partial\psi_j}{\partial x}) \\
M_{6j}^{3} &= \frac{16}{9h^4}(H_{12}\frac{\partial^2\psi_j}{\partial x^2} + H_{22}\frac{\partial^2\psi_j}{\partial y^2} + 2H_{26}\frac{\partial^2\psi_j}{\partial x\partial y}) \\
M_{6j}^{4} &= -\frac{4}{3h^2}(F_{12}\frac{\partial\psi_j}{\partial x} + F_{26}\frac{\partial\psi_j}{\partial y} - \frac{4}{3h^2}H_{12}\frac{\partial\psi_j}{\partial x} - \frac{4}{3h^2}H_{26}\frac{\partial\psi_j}{\partial y}) \\
M_{6j}^{5} &= -\frac{4}{3h^2}(F_{22}\frac{\partial\psi_j}{\partial y} + F_{26}\frac{\partial\psi_j}{\partial x} - \frac{4}{3h^2}H_{22}\frac{\partial\psi_j}{\partial y} - \frac{4}{3h^2}H_{26}\frac{\partial\psi_j}{\partial x})
\end{aligned}
$$

$$
\begin{aligned}
N_{1j}^{n3} &= \frac{1}{2}(A_{11}\frac{\partial w_0}{\partial x}\frac{\partial\psi_j}{\partial x} + A_{12}\frac{\partial w_0}{\partial y}\frac{\partial\psi_j}{\partial y} + A_{16}\frac{\partial w_0}{\partial x}\frac{\partial\psi_j}{\partial y} + A_{16}\frac{\partial w_0}{\partial y}\frac{\partial\psi_j}{\partial x}) \\
N_{2j}^{n3} &= \frac{1}{2}(A_{12}\frac{\partial w_0}{\partial x}\frac{\partial\psi_j}{\partial x} + A_{22}\frac{\partial w_0}{\partial y}\frac{\partial\psi_j}{\partial y} + A_{26}\frac{\partial w_0}{\partial x}\frac{\partial\psi_j}{\partial y} + A_{26}\frac{\partial w_0}{\partial y}\frac{\partial\psi_j}{\partial x}) \\
N_{6j}^{n3} &= \frac{1}{2}(A_{16}\frac{\partial w_0}{\partial x}\frac{\partial\psi_j}{\partial x} + A_{26}\frac{\partial w_0}{\partial y}\frac{\partial\psi_j}{\partial y} + A_{66}\frac{\partial w_0}{\partial x}\frac{\partial\psi_j}{\partial y} + A_{66}\frac{\partial w_0}{\partial y}\frac{\partial\psi_j}{\partial x})
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
Q_1^3 &= \frac{1}{2}A_{11}(\frac{\partial w_0}{\partial x})^2 + \frac{1}{2}A_{12}(\frac{\partial w_0}{\partial y})^2 + A_{16}\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} \\
Q_2^3 &= \frac{1}{2}A_{12}(\frac{\partial w_0}{\partial x})^2 + \frac{1}{2}A_{22}(\frac{\partial w_0}{\partial y})^2 + A_{26}\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} \\
Q_6^3 &= \frac{1}{2}A_{16}(\frac{\partial w_0}{\partial x})^2 + \frac{1}{2}A_{26}(\frac{\partial w_0}{\partial y})^2 + A_{66}\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}
\end{aligned} \tag{48}
$$

$$
\begin{aligned}
M_{1j}^{n3} &= \frac{1}{2}(B_{11}\frac{\partial w_0}{\partial x}\frac{\partial\psi_j}{\partial x} + B_{12}\frac{\partial w_0}{\partial y}\frac{\partial\psi_j}{\partial y} + B_{16}\frac{\partial w_0}{\partial x}\frac{\partial\psi_j}{\partial y} + B_{16}\frac{\partial w_0}{\partial y}\frac{\partial\psi_j}{\partial x}) \\
M_{5j}^{n3} &= \frac{1}{2}(B_{12}\frac{\partial w_0}{\partial x}\frac{\partial\psi_j}{\partial x} + B_{22}\frac{\partial w_0}{\partial y}\frac{\partial\psi_j}{\partial y} + B_{26}\frac{\partial w_0}{\partial x}\frac{\partial\psi_j}{\partial y} + B_{26}\frac{\partial w_0}{\partial y}\frac{\partial\psi_j}{\partial x}) \\
M_{2j}^{n3} &= \frac{1}{2}(B_{16}\frac{\partial w_0}{\partial x}\frac{\partial\psi_j}{\partial x} + B_{26}\frac{\partial w_0}{\partial y}\frac{\partial\psi_j}{\partial y} + B_{66}\frac{\partial w_0}{\partial x}\frac{\partial\psi_j}{\partial y} + B_{66}\frac{\partial w_0}{\partial y}\frac{\partial\psi_j}{\partial x})
\end{aligned} \tag{49}
$$

$$Q^{n3}_{5j} = -\frac{2}{3h^2}(E_{11}\frac{\partial w_0}{\partial x}\frac{\partial \psi_j}{\partial x} + E_{12}\frac{\partial w_0}{\partial y}\frac{\partial \psi_j}{\partial y} + E_{16}\frac{\partial w_0}{\partial x}\frac{\partial \psi_j}{\partial y} + E_{16}\frac{\partial w_0}{\partial y}\frac{\partial \psi_j}{\partial x})$$
$$Q^{n3}_{6j} = -\frac{2}{3h^2}(E_{16}\frac{\partial w_0}{\partial x}\frac{\partial \psi_j}{\partial x} + E_{26}\frac{\partial w_0}{\partial y}\frac{\partial \psi_j}{\partial y} + E_{66}\frac{\partial w_0}{\partial x}\frac{\partial \psi_j}{\partial y} + E_{66}\frac{\partial w_0}{\partial y}\frac{\partial \psi_j}{\partial x}) \quad (50)$$
$$Q^{n3}_{7j} = -\frac{2}{3h^2}(E_{12}\frac{\partial w_0}{\partial x}\frac{\partial \psi_j}{\partial x} + E_{22}\frac{\partial w_0}{\partial y}\frac{\partial \psi_j}{\partial y} + E_{26}\frac{\partial w_0}{\partial x}\frac{\partial \psi_j}{\partial y} + E_{26}\frac{\partial w_0}{\partial y}\frac{\partial \psi_j}{\partial x})$$

$$\begin{aligned} F^1_i &= \int_{\Gamma^e} \psi_i N_n ds \\ F^2_i &= \int_{\Gamma^e} \psi_i N_{ns} ds \\ F^3_i &= \int_{\Omega^e} q\psi_i dxdy + \int_{\Gamma^e} [\psi_i Q_n - \frac{4}{3h^2}(\frac{\partial \psi_i}{\partial x}P_n + \frac{\partial \psi_i}{\partial y}P_{ns})]ds \\ F^4_i &= \int_{\Gamma^e} \psi_i M_n ds \\ F^5_i &= \int_{\Gamma^e} \psi_i M_{ns} ds \end{aligned} \quad (51)$$

All other coefficients are zero in Eqs. (43), (44), (45), (46), (47), (48), (49), (50).

Equation (42) represents the stiffness factors of an 8-node laminated plate element based on the equilibrium equations derived from the principle of virtual displacements. It is applicable not only to linear solution but also to nonlinear solution.

## 6 References

[1] J.H.Pei, and R.A.Shenoi, 'The Stiffness Matrix of 8-Node Laminated Plate Element Based on Reddy's Higher-Order Shear Deformation Theory', Research Report. Southampton University, January 1994.

[2] J.H.Pei, and R.A.Shenoi, 'The Stiffness Matrix of an 8-Node Laminated Plate

Element Taking into Account Higher-Order Shear Deformation and Geometric Nonlinearity Considerations', Research Report. Southampton University, February 1994.

[3] Reddy, J. N., 'A Simple Higher-Order Theory for Laminated Composite Plate', Journal of Applied Mechanics, 51, 1984, pp. 745-752.

[4] Reddy, J. N., 'A Refined Nonlinear Theory of Plates with Transverse Shear Deformation', Int.J. Solids Structures, Vol. 20,No. 9/10, 1984, pp.881-896.

[5] O.O. Ochoa and J.N. Reddy, 'Finite Element Analysis of Composite Laminates', Kluwer Academic Publishers, 1992.