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The Effect of Missing Values on Designed Experiments

by

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ABSTRACT

FACULTY OF MATHEMATICAL STUDIES

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Loss of observations in a designed experiment is a common occurrence. It is prudent to identify designs that are resistant in a statistical sense to the loss of data. The main aim of the designs in this thesis is to compare of a set of treatments. The precision, or variance, of the individual pairwise treatment comparisons increases when data become unavailable. These alterations can be evaluated through the information matrix for treatment effects for block and row-column designs.

In this thesis, the effect of different configurations of missing values on block designs, row-column designs, and diallel cross designs is investigated. The average variance of all pairwise treatment comparisons has been used as a measure of robustness by the majority of researchers. The maximum variance of comparisons is computed numerically, or developed theoretically, in this thesis for most patterns of missing data. The reduced normal equations can be solved with a suitable choice of generalised inverse, and formulae for the individual variances of pairwise treatment differences can also be derived.

The effect of missing values on block designs, in particular randomised and balanced incomplete block designs is studied. It is shown that designs with a small number of treatments and a small number of blocks are severely affected by the loss of one, two, or three observations. Larger designs are not as seriously affected when the average variance is considered, but there are a small number of pairwise treatment comparisons that suffer a large loss of efficiency.

Row-column designs have also been investigated for similar patterns of missing data. The lack of orthogonality introduced by the loss of data in many situations complicates the analysis and derivation of general expressions for the variances. The loss of efficiency for small Latin square designs is substantial after the removal of only one or two units. Constructing a design with multiple squares is shown to reduce the impact of the missing data. Youden square designs also suffer a similar loss of information after the loss of a few observations, and it is also shown that the structure of the design affects the distributions of efficiencies for a given number of missing values.

The last class of designs considered in this thesis are diallel cross designs, where each experimental unit is a combination of two of the treatments. Diallel cross designs suffer a large reduction in efficiency, in general, for the loss of only one cross. This loss of efficiency is more serious when two or three crosses become unavailable.

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Chapter 1

Introduction

The methodology of Design and Analysis of Experiments has been developed to maximise the statistical information gathered from a planned experiment. In many situations, it is possible that data may become unavailable for reasons unconnected with the starting design, which may result in the ruination of the experiment. Designs which are optimal, in a statistical sense, when complete may become inefficient after the loss of only a small number of observations. It would therefore appear sensible to identify designs that are robust in some sense to the loss of data.

The aim of many experiments is to compare a set of treatments. The precision of the estimates of differences between pairs of treatments decreases when data become unavailable, and the overall effect of drop-out is summarised by computing the average variance of all pairwise treatment comparisons. The majority of recent research has concentrated on calculating this average for various distinct configurations of missing observations. It is also useful to consider the maximum of the individual variances, because a small number of badly affected comparisons may be concealed if only the average is computed.

1.1 Block Designs and Missing Data

In designed experiments, the loss of data is a frequent occurrence. In a medical situation, patients may leave a clinical trial for reasons unconnected with the treatment received. For example, if a patient moves to another part of the country, then it may not be possible to measure the efficacy of the treatment. The aim of many experiments is to compare a set of treatments, e.g. in a clinical trial new drugs versus the current standard.

There is frequently non-response in sample surveys. Sensitive personal information such as the income of the household or political affiliation may be undisclosed by the respondent. Another mechanism for missing data is censoring, where the event of interest does not occur before the completion of the experiment. The various drop-out processes can be classified as follows:

1. Completely random drop-out, where the drop-out and measurement processes are independent.

2. Random drop-out, in which the drop-out process is dependent on the observed measurements, i.e. those preceding drop-out.

3. Informative drop-out. In this case the drop-out process depends on the unobserved measurements, i.e. those that would have been observed if the unit had not dropped out.

It is unlikely in practical applications that the experimenter will have any control over the particular observations that become unavailable. It is, therefore, useful to have an overall measure of how badly a design is affected by the loss of one or more observations scattered throughout the starting design.

There are a variety of techniques for handling missing data, which are summarised by Little and Rubin (1987). Three main approaches to analysing an experiment with missing observations are

1. Data are accepted as they stand and the analysis proceeds using the general normal equations.

2. Missing plot values are estimated, and the data are then analysed as if there was no drop-out.

3. Replace missing values with approximate values and make adjustments using the analysis of covariance.

Early work concentrated on the loss of complete treatments from balanced incomplete block (BIB) designs. Hedayat and John (1974) classified these BIB designs into three categories - susceptible, locally resistant, and globally resistant. Variance balance of the resulting design was used as the criterion of robustness. The degree of resistance of a given design identifies the number of complete treatments that can be removed while the resulting design remains variance balanced. Conditions for these classifications were derived by Hedayat and John (1974), who also showed that the structure of the starting design as well as the design parameters influenced the robustness of a given BIB design. Most (1975) extended this work from one missing treatment, and showed that for a design to be resistant to the loss of all observations relating to more than one treatment, all the subdesigns formed using the missing treatments must be BIB designs.

Shah and Gujarathi (1977) extended one of the results of Hedayat and John (1974) to produce a theorem outlining the construction of locally resistant BIB designs of degree one with respect to two of the treatments in the starting design. Shah and Gujarathi (1983) gave necessary conditions for BIB designs to be globally resistant of degree two, identified by considering the form of successive sub-designs generated by the sequential removal of treatments.

John (1976) derived a lower bound on the efficiency after the removal of one treatment from a BIB design, and showed that symmetric BIB designs attained this bound. Dey and Dhall (1988) studied the robustness of augmented BIB designs, where the new treatment is added to every block of a BIB design, in terms of the A-efficiency of the resulting design. They derived upper and lower bounds for the average variance of pairwise treatment comparisons.

The loss of complete blocks of observations has been investigated by Bhaumik and Whittinghill (1991), Gupta and Srivastava (1992), and Das and Kageyama (1992). These studies used the average variance of pairwise treatment differences as the measure of robustness. Bhaumik and Whittinghill (1991) derived the eigenvalues of the information matrix for treatment effects, and used majorisation on the vectors of these eigenvalues to compare the different ways of losing complete blocks of observations. They showed that the best case was the loss of blocks with disjoint sets of treatments, and the worst was the removal of identical blocks. Bhaumik and Whittinghill (1991) also illustrated the loss of two blocks from a BIB design. The loss of a single block from binary balanced block designs and augmented BIB designs was investigated by Gupta and Srivastava (1992), and it was shown that the reduction in efficiency for the average variance was in general small. Das and Kageyama (1992) used the efficiency of the residual design to consider the loss of a block from BIB, extended BIB, and Youden square designs.

Many investigations have considered the removal of specific patterns of observations, for example the loss of complete treatments or blocks of observations. In practice, it is likely that the missing data will be scattered throughout the starting design. Whittinghill (1995) considered the effects of missing data on block designs, and showed that the loss of observations from the same block of the starting design, or single observations from blocks with disjoint treatment sets, are the least severely affected cases. The worst case occurred when it is possible to lose the same treatment from identical blocks. The eigenvalues of the information matrix for treatment effects were used to calculate the average variance of pairwise treatment comparisons. When more than one observation becomes unavailable, there are different configurations of missing values with different properties that need to be considered separately. Whittinghill (1995) ranked all cases of two missing observations using majorisation theory on the vectors of eigenvalues of the information matrices for treatment effects. It will be shown in Section 3.5.3 that this technique cannot be extended to resulting designs with more than two missing observations. The maximum variance of pairwise treatment comparisons is also computed for the configurations of missing values for the different designs studied in the thesis. The loss of efficiency for the average variance is frequently small, but the maximum variance may suffer a large reduction in efficiency for only a small number of observations.

Duan and Kageyama (1995) considered augmented BIB designs, and computed the efficiency of designs resulting from the loss of any number of observations in a single block or any configuration of two missing values. They showed that these designs are fairly robust to the loss of data when the average variance is the measure of robustness. Lal et al. (2001) considered the robustness of block designs against the loss of up to three observations. The A-efficiency of the resulting designs was evaluated for different patterns of missing data, and a lower bound was derived for the loss of t observations. Srivastava et al. (1996) investigated the loss of a single observation from block designs used to make treatment-control comparisons by calculating the efficiency of the average variance.

The effect of missing observations in the final period of cross-over designs based on Latin squares, has been studied by Low et al. (1999). In the study, it was shown that it is sensible to construct designs from different Latin squares rather than replicates of the same square, when there is the possibility of missing data. When the use of multiple

Latin squares is considered in Chapter 5 it can be seen that the choice of square does not influence the overall properties of the starting design, because carry-over is not included in the model.

Another class of designs considered for the effect of missing observations are diallel cross designs. The effect of the loss of a block from complete diallel cross experiments has been investigated by Ghosh and Biswas (2000), and the average variance of pairwise treatment comparisons is considered. These designs were shown to be robust to the unavailability of one block. Ghosh and Desai (1999) considered the loss of a single block from complete diallel cross designs where there was unequal replication of the crosses, and concluded that these designs were fairly robust to this pattern of missing values.

1.2 Aims and Outline of the Thesis

The aim of this thesis is to assess the impact of missing observations on experimental designs with one or two blocking factors. Some specific objectives are to

1. investigate different patterns of missing data on a variety of block, row-column, and diallel cross designs.

2. derive theoretical results for variances of all pairwise treatment differences for all configurations of t missing values and to compare these cases of resulting designs.

3. use the maximum variance as well as the average to assess the robustness of these designs to missing data.

The structure of the thesis is as follows.

Chapter 2 covers the underlying theory used to compare a set of treatments in designs with blocking factors. The reduced normal equations for treatment effects are derived for complete block and row-column designs, and are also given more generally for situations where data are unavailable.

In Chapter 3, the influence of missing observations on block designs, and in particular randomised block and BIB designs, is studied in detail. The loss of all observations in one or more blocks, and up to three observations scattered throughout the starting design is considered theoretically. These formulae are also evaluated for a large range of design parameters. It is shown that, for the majority of designs, the increase in average variance is minimal representing a small loss of efficiency, but when the maximum is calculated a few of the treatment comparisons are severely affected by the missing observations.

The effect of drop-out on row-column designs, and in particular single replicate Latin squares, is the focus of Chapter 4. There are three patterns of missing data considered in this Chapter, and it is shown that the structure of the starting design does not influence the distribution of configurations of two and three missing values. Small squares suffer a substantial loss of efficiency when a small number of values become unavailable, but the loss of efficiency is reduced as the size of the square increases.

Chapter 5 considers the use of more than one Latin square to reduce the effect of missing observations. It is seen that the choice of squares does not affect the overall properties of the design when data become unavailable, which is different from the cross-over situation covered by Low et al. (1999). The results suggest that the use of multiple squares reduces the impact of drop-out.

In Chapter 6, Youden square designs are investigated to determine the effect of missing observations on the initial design. Loss of complete treatments is considered briefly at the start of the Chapter and it appears that Youden squares constructed by removing a single column from a Latin square design are variance balanced after the removal of any treatment from the starting design. The loss of a block of observations from the BIB component of the Youden square is also studied and the variances of pairwise treatment comparisons are derived theoretically and evaluated for a range of Youden square designs. The unavailability of one or two missing observations has also been considered to assess the impact on the starting design.

Diallel cross designs are introduced in Chapter 7 and the effect of missing values is also investigated. There are differences from conventional block designs because each plot corresponds to a cross, which is composed of two treatments, allocated to it at the design stage. The loss of blocks of crosses and one, two, or three missing crosses are studied numerically for a range of different diallel cross designs. It is shown that the efficiencies of some pairwise treatment differences are substantially reduced by the loss of only a small number of crosses.

The conclusions of this thesis are given in Chapter 8, and some recommendations for future work are suggested.

Chapter 2

Background to Designed Experiments

In many experiments, where the aim is to compare a set of treatments, there are one or two sources of variation that can be accounted for at the design stage. Blocking is a technique used in these situations, and it can reduce the variances of the estimated pairwise treatment differences. When there is a single blocking factor, a randomised block design (RBD) can be used if all treatments can be accommodated in the blocks of the starting design. An incomplete block design is required when there are insufficient plots in the blocks, and the balanced incomplete block (BIB) design is the optimal design in this situation, but such designs exist for only a small number of design parameter combinations.

There are two popular designs employed when the experiment has two blocking factors, often referred to as row-column designs. These designs are constructed using Latin squares and Youden squares respectively. The rows and columns can be considered individually as block designs and these designs are covered in Section 2.2.

2.1 The General Block Experiment

Consider an experiment where a set of v treatments is allocated to n units (or plots) in b blocks. Suppose that the jth block of the design contains k_j plots, and let n_{ij} denote the number of times the ith treatment is applied to a plot in the jth block. An additive model is assumed for the responses y_{ijk}, which corresponds to the application of the ith

treatment to the *jth* block for the *kth* time. The model is described by

$$y_{ijk} = \mu + \tau_i + \beta_j + \epsilon_{ijk} \tag{2.1}$$
$$(i = 1, \cdots, v; j = 1, \cdots, b; k = 1, \cdots, n_{ij})$$

where μ represents the overall mean, the effect of the *ith* treatment is given by τ_i, β_j is the effect of block j, and ϵ_{ijk} is random error. It is assumed that the errors are uncorrelated normal random variables, with $E(\epsilon_{ijk}) = 0$ and $Var(\epsilon_{ijk}) = \sigma^2$. This linear model can be expressed in matrix form as

$$\mathbf{y} = \mathbf{X}\mathbf{a} + \boldsymbol{\epsilon} \tag{2.2}$$

where $\mathbf{y}$ is a vector of the n observations, $\mathbf{a} = (\mu, \tau_1, \cdots, \tau_v, \beta_1, \cdots, \beta_b)'$ is the vector of $v + b + 1$ model parameters, and $\mathbf{X}$ is the design matrix for the experiment. In general, the matrix $\mathbf{X}$ is not of full rank so that $\mathbf{X}'\mathbf{X}$ is a singular matrix, and consequently the normal equations cannot be solved uniquely.

2.1.1 Derivation of the Normal Equations

To solve the normal equations and estimate the treatment effects and variances of pairwise treatment comparisons, a generalised inverse can be used. First, partition the vector of model parameters and design matrix for the experiment, such that $\mathbf{a} = (\mu|\boldsymbol{\tau}'|\boldsymbol{\beta}')'$ and $\mathbf{X} = (\mathbf{1}_n|\mathbf{X}_\tau|\mathbf{X}_\beta)$. Note that $\mathbf{1}_n$ is an n dimensional vector of ones, and $\mathbf{X}_\tau$ and $\mathbf{X}_\beta$ are design matrices for treatments and blocks in the design with dimensions $n \times v$ and $n \times b$ respectively. Equation (2.2) can be rewritten in the form

$$\mathbf{y} = \mathbf{1}_n\mu + \mathbf{X}_\tau\boldsymbol{\tau} + \mathbf{X}_\beta\boldsymbol{\beta} + \boldsymbol{\epsilon} \tag{2.3}$$

The model parameters are estimated using ordinary least squares, where the error sum of squares, $\boldsymbol{\epsilon}'\boldsymbol{\epsilon}$, is minimised with respect to the vector of model parameters. The error sum of squares can be expressed as

$$\boldsymbol{\epsilon}'\boldsymbol{\epsilon} = (\mathbf{y} - \mathbf{X}\mathbf{a})'(\mathbf{y} - \mathbf{X}\mathbf{a}) = \mathbf{y}'\mathbf{y} - 2\mathbf{y}'\mathbf{X}\mathbf{a} + \mathbf{a}'\mathbf{X}'\mathbf{X}\mathbf{a} \tag{2.4}$$

To minimise this expression, the full normal equations $(\mathbf{X}'\mathbf{X})\hat{\mathbf{a}} = \mathbf{X}'\mathbf{y}$ need to be solved for $\hat{\mathbf{a}}$. If $\mathcal{G}$ is the grand total of the observations, $\mathcal{T}$ the vector of treatment totals, and $\mathcal{B}$ a vector of block totals, then, for the general block experiment, the matrices involved in the full normal equations can now be written as

$$(\mathbf{X}'\mathbf{X}) = \begin{bmatrix} \mathbf{1}'_n \\ \mathbf{X}'_\tau \\ \mathbf{X}'_\beta \end{bmatrix} \begin{bmatrix} \mathbf{1}_n & \mathbf{X}_\tau & \mathbf{X}_\beta \end{bmatrix} = \begin{bmatrix} \mathbf{1}'_n\mathbf{1}_n & \mathbf{1}'_n\mathbf{X}_\tau & \mathbf{1}'_n\mathbf{X}_\beta \\ \mathbf{X}'_\tau\mathbf{1}_n & \mathbf{X}'_\tau\mathbf{X}_\tau & \mathbf{X}'_\tau\mathbf{X}_\beta \\ \mathbf{X}'_\beta\mathbf{1}_n & \mathbf{X}'_\beta\mathbf{X}_\tau & \mathbf{X}'_\beta\mathbf{X}_\beta \end{bmatrix} \tag{2.5}$$

and

$$\mathbf{X}'\mathbf{y} = \begin{bmatrix} \mathbf{1}'_n \\ \mathbf{X}'_\tau \\ \mathbf{X}'_\beta \end{bmatrix} \mathbf{y} = \begin{bmatrix} \mathcal{G} \\ \mathcal{T} \\ \mathcal{B} \end{bmatrix} \tag{2.6}$$

To simplify the solution of the normal equations, it is practical to define additional vectors and matrices. Let the number of plots in each block of a design d be stored in a vector $\mathbf{k} = (k_1, \cdots, k_b)'$, and the number of replicates of the v treatments be the elements of the vector $\mathbf{r} = (r_1, \cdots, r_v)'$. The incidence matrix of the design is a $v \times b$ matrix $\mathbf{N} = \{n_{ij}\}$, where the *(i,j)th* element is the number of times that the *ith* treatment occurs in the *jth* block of the design. Let $\mathbf{r}^\delta = diag(\mathbf{r})$ and $\mathbf{k}^\delta = diag(\mathbf{k})$ be diagonal matrices with elements obtained from the vectors $\mathbf{r}$ and $\mathbf{k}$ respectively. The matrix $\mathbf{X}'\mathbf{X}$ can now be expressed using these vectors and matrices as

$$(\mathbf{X}'\mathbf{X}) = \begin{bmatrix} n & \mathbf{r}' & \mathbf{k}' \\ \mathbf{r} & \mathbf{r}^\delta & \mathbf{N} \\ \mathbf{k} & \mathbf{N}' & \mathbf{k}^\delta \end{bmatrix} \tag{2.7}$$

This matrix is not of full rank in general, but non-unique solutions to the normal equations can be found using generalised inverses. Estimates of the treatment effects are of primary interest, so the nuisance block and mean parameters, μ and $\boldsymbol{\beta}$, can be eliminated from the normal equations initially. The full normal equations for a general block design are

$$n\hat{\mu} + \mathbf{r}'\hat{\boldsymbol{\tau}} + \mathbf{k}'\hat{\boldsymbol{\beta}} = \mathcal{G} \tag{2.8}$$

$$\mathbf{r}\hat{\mu} + \mathbf{r}^\delta\hat{\boldsymbol{\tau}} + \mathbf{N}\hat{\boldsymbol{\beta}} = \mathcal{T} \tag{2.9}$$

$$\mathbf{k}\hat{\mu} + \mathbf{N}'\hat{\boldsymbol{\tau}} + \mathbf{k}^\delta\hat{\boldsymbol{\beta}} = \mathcal{B} \tag{2.10}$$

To estimate the treatment effects, the block parameters and overall mean are eliminated from Equation (2.9). Subtract $\mathbf{Nk}^{-\delta}$ times Equation (2.10) from Equation (2.9) to leave

$$(\mathbf{r} - \mathbf{Nk}^{-\delta}\mathbf{k})\hat{\mu} + (\mathbf{r}^{\delta} - \mathbf{Nk}^{-\delta}\mathbf{N}')\hat{\tau} + (\mathbf{N} - \mathbf{Nk}^{-\delta}\mathbf{k}^{\delta})\hat{\beta} = \mathcal{T} - \mathbf{Nk}^{-\delta}\mathcal{B} \tag{2.11}$$

In Equation (2.11) the vector of block parameters is removed because $\mathbf{k}^{-\delta}\mathbf{k}^{\delta} = \mathbf{I}$, and the overall mean is eliminated because $\mathbf{k}^{-\delta}\mathbf{k} = \mathbf{1}_b$ and $\mathbf{N1}_b = \mathbf{r}$. The reduced normal equations for treatment effects simplify to

$$(\mathbf{r}^{\delta} - \mathbf{Nk}^{-\delta}\mathbf{N}')\hat{\tau} = (\mathcal{T} - \mathbf{Nk}^{-\delta}\mathcal{B}) \tag{2.12}$$

or, more concisely, to

$$\mathbf{C}\hat{\tau} = \mathcal{Q} \tag{2.13}$$

where $\mathbf{C} = (\mathbf{r}^{\delta} - \mathbf{Nk}^{-\delta}\mathbf{N}')$ is the information matrix for treatment effects, and $\mathcal{Q} = (\mathcal{T} - \mathbf{Nk}^{-\delta}\mathcal{B})$ is the vector of treatment totals adjusted for block effects. The information matrix is not of full rank, so a generalised inverse, $\mathbf{\Omega}$, of $\mathbf{C}$ can be used to solve the reduced normal equations, such that the estimates of treatment effects are given by $\hat{\tau} = \mathbf{\Omega}\mathcal{Q}$. The matrix $\mathbf{\Omega}$ has the property that $\mathbf{C\Omega C} = \mathbf{C}$. The Moore-Penrose generalised inverse is a common choice used to solve the reduced normal equations, and is defined in terms of the non-zero eigenvalues and their corresponding eigenvectors of the information matrix for treatment effects. Let $\mu_i, i = 1, \cdots, v-1$ be the non-zero eigenvalues and ξ_i be their corresponding orthonormal normalised eigenvectors of $\mathbf{C}$. The Moore-Penrose generalised inverse of a singular matrix is defined as

$$\Omega = \sum_{i=1}^{v-1} \frac{1}{\mu_i}\xi_i\xi_i' \tag{2.14}$$

An alternative generalised inverse for deriving general expressions for the variances of pairwise treatment comparisons will be discussed in Chapter 3.

2.1.2 Comparison of Treatments

The aim of an experiment is to compare the effects of a set of treatments. Treatments can be compared directly if they appear together in one or more blocks of the starting design, or indirectly if there is at least one treatment common to the set of blocks in which the two treatments occur. A design is connected if all pairs of the v treatments can be compared directly or indirectly, and the design is disconnected if this is not possible.

In the general block experiment the difference between the *ith* and *jth* treatments is estimated by $\hat{\tau}_i - \hat{\tau}_j$. There are $v(v-1)/2$ pairwise treatment comparisons which may be expressed in a contrast matrix $\mathbf{\Gamma}$. Note that $\mathbf{\Gamma}\boldsymbol{\tau}$ is estimable if and only if $\mathbf{\Gamma\Omega C} = \mathbf{\Gamma}$. In this case the variances of the pairwise treatment comparisons are the diagonal elements of $\mathbf{\Gamma\Omega\Gamma}'$. The average variance of all the pairwise treatment differences is now given by

$$\frac{2\sigma^2}{v(v-1)} trace(\mathbf{\Gamma\Omega\Gamma}') \tag{2.15}$$

To compare the merits of two competing designs, a set of efficiency factors are computed. These factors are the ratios of variances of estimated treatment contrasts for a completely randomised design to variances of the same contrasts for the block design. The variance of the treatment difference in a randomised block design with r replicates is $2\sigma^2/r$, so that pairwise efficiency factors e_{ij} for the *ith* and *jth* treatments in a general block design are given by

$$e_{ij} = \frac{2\sigma^2/r}{Var(\hat{\tau}_i - \hat{\tau}_j)} \tag{2.16}$$

Another important property associated with the estimation of treatment comparisons is balance, which has also been referred to as variance balance. A design is variance balanced if all elementary pairwise treatment comparisons are made with the same precision (i.e. if all variances of pairwise treatment differences are equal). In this case $e_{ij} = e$ for all i, j. When the design is variance balanced it may be shown that the information matrix for treatment effects of a balanced design can be written in the form $\mathbf{C} = c_1\mathbf{I} + c_2\mathbf{J}$, where c_1 and c_2 are constants.

2.1.3 Randomised Block Design

In a randomised block design the treatments in the experiment are replicated the same number of times within each block. Treatments are assigned to the plots within the blocks at random to avoid the introduction of systematic error. In the designs studied in later Chapters there are v treatments replicated r times in b equally sized blocks of k plots.

Therefore $\mathbf{r}^\delta = r\mathbf{I}_v$, $\mathbf{k}^\delta = k\mathbf{I}_b$, and $\mathbf{N} = \mathbf{J}_{v,b}$ because every treatment occurs once in each block of the design. The information matrix for treatment effects can be expressed as

$$\mathbf{C} = r\mathbf{I}_v - \mathbf{J}_{v,b}\frac{1}{k}\mathbf{I}_b\mathbf{J}_{b,v} = r\mathbf{I}_v - \frac{b}{k}\mathbf{J}_{v,v} \tag{2.17}$$

To solve these reduced normal equations and estimate the differences between any pair of treatments, a generalised inverse can be used. A simple generalised inverse is $\mathbf{I}_v/r$, and consequently all variances of pairwise treatment comparisons, $\hat{\tau}_i - \hat{\tau}_j$, are $2\sigma^2/r$.

2.1.4 Balanced Incomplete Block Design

When an incomplete block design is required, there is only a small set of design parameters where a balanced incomplete block design may be selected. There are three conditions that need to hold for an incomplete block design to be a BIB design; these are

1. every treatment label occurs at most once in each block,
2. every treatment occurs exactly r times, and
3. each pair of treatments occurs together in λ blocks of the starting design.

There are five parameters for a BIB design. The number of treatments v, the number of blocks b, treatment replication r, block size k, and the pairing parameter λ. There are three relationships based on the parameters, $vr = bk$, $\lambda(v-1) = r(k-1)$, and $k < v$. A BIB design is *symmetric* if the number of treatments is equal to the number of blocks in the starting design, i.e. $v = b$. Symmetric BIB designs are useful for constructing Youden square designs which are discussed later in this Chapter. Equal block sizes and treatment replication simplifies the information matrix for treatment effects to

$$\mathbf{C} = r\mathbf{I}_v - \frac{1}{k}\mathrm{NN}' \tag{2.18}$$

The matrix NN' is the concurrence matrix, which records the number of times that two treatments occur together in a block for a binary design. All treatments are replicated r times in the initial design, and every pair occurs together in λ blocks of the design. The concurrence matrix of a complete BIB design can be expressed as

$$\mathbf{NN}' = (r - \lambda)\mathbf{I}_v + \lambda\mathbf{J}_{v,v} \tag{2.19}$$

Substitute Equation (2.19) into Equation (2.18) and, after simplification, the information matrix for treatment effects is

$$\mathbf{C} = \frac{\lambda}{k}\{v\mathbf{I}_v - \mathbf{J}_{v,v}\} \tag{2.20}$$

To solve the normal equations a generalised inverse of the form

$$\boldsymbol{\Omega} = \frac{k}{v\lambda}\mathbf{I}_v \tag{2.21}$$

can be used. The variances of all pairwise treatment comparisons are $e = 2k\sigma^2/v\lambda$. The efficiency of a complete BIB design relative to a completely randomised block design is $v\lambda/kr$. The effect of missing data on BIB designs is studied in detail in Chapter 3.

2.2 Row-and-column Designs

When there are two orthogonal blocking factors, a row-column design can be used. The plots are arranged in a rectangular array. The two blocking systems are referred to as *rows* and *columns*, and a standard additive model for the data from a row-column design with v treatments with r rows and c columns is

$$y_{ij(l)} = \mu + \rho_i + \gamma_j + \tau_{(l)} + \epsilon_{ij(l)} \tag{2.22}$$
$$(i = 1, \cdots, r; j = 1, \cdots, c)$$

where the model parameters μ, ρ_i, γ_j, and $\tau_{(l)}$ are the overall mean, the effect of the *ith* row, the effect of the *jth* column, and the effect of the *lth* treatment applied to the plot in the *ith* row and *jth* column of the row-column design respectively. The experimental error is represented by $\epsilon_{ij(l)}$, and the model can be written in matrix form

$$\mathbf{y} = \mathbf{Xa} + \boldsymbol{\epsilon} \tag{2.23}$$

where $\mathbf{y}$ is a vector of the $n = r \times c$ observations, $\mathbf{a} = (\mu, \rho_1, \cdots, \rho_r, \gamma_1, \cdots, \gamma_c, \tau_1, \cdots, \tau_v)'$

is the vector of model parameters, and $\mathbf{X}$ is the design matrix.

2.2.1 Reduced Normal Equations

To estimate the treatment effects, the row and column parameters need to be eliminated from the full normal equations. The design matrix and vector of model parameters can be partitioned appropriately to simplify the elimination of the row and column parameters. The components of the full normal equations can therefore be written as

$$(\mathbf{X}'\mathbf{X}) = \begin{bmatrix} \mathbf{1}_n'\mathbf{1}_n & \mathbf{1}_n'\mathbf{X}_\rho & \mathbf{1}_n'\mathbf{X}_\gamma & \mathbf{1}_n'\mathbf{X}_\tau \\ \mathbf{X}_\rho'\mathbf{1}_n & \mathbf{X}_\rho'\mathbf{X}_\rho & \mathbf{X}_\rho'\mathbf{X}_\gamma & \mathbf{X}_\rho'\mathbf{X}_\tau \\ \mathbf{X}_\gamma'\mathbf{1}_n & \mathbf{X}_\gamma'\mathbf{X}_\rho & \mathbf{X}_\gamma'\mathbf{X}_\gamma & \mathbf{X}_\gamma'\mathbf{X}_\tau \\ \mathbf{X}_\tau'\mathbf{1}_n & \mathbf{X}_\tau'\mathbf{X}_\rho & \mathbf{X}_\tau'\mathbf{X}_\gamma & \mathbf{X}_\tau'\mathbf{X}_\tau \end{bmatrix} \tag{2.24}$$

and

$$\mathbf{X}'\mathbf{y} = \begin{bmatrix} \mathcal{G} & \mathcal{R}' & \mathcal{C}' & \mathcal{T}' \end{bmatrix}' \qquad \mathbf{a} = \begin{bmatrix} \mu & \rho' & \gamma' & \tau' \end{bmatrix}' \tag{2.25}$$

To estimate the treatment means, the row and column effects are eliminated from these normal equations. The number of observations in a given design is $\mathbf{1}_n'\mathbf{1}_n = n$, and the number of units in each row and column and treatment replication are given by the vectors $\mathbf{r}$, $\mathbf{c}$, and $\mathbf{t}$ respectively. These three vectors can be expressed using the treatment, row, and column design matrices, i.e. $\mathbf{r} = \mathbf{X}_\rho'\mathbf{1}_n$, $\mathbf{c} = \mathbf{X}_\gamma'\mathbf{1}_n$, and $\mathbf{t} = \mathbf{X}_\tau'\mathbf{1}_n$.

To simplify the normal equations, define three diagonal matrices $\mathbf{r}^\delta = \mathbf{X}_\rho'\mathbf{X}_\rho$, $\mathbf{c}^\delta = \mathbf{X}_\gamma'\mathbf{X}_\gamma$, and $\mathbf{t}^\delta = \mathbf{X}_\tau'\mathbf{X}_\tau$ whose elements are the number of units in each row, the number of units in each column, and the number of replicates of each treatment respectively. Let $\mathbf{N}_1 = \mathbf{X}_\tau'\mathbf{X}_\rho$ and $\mathbf{N}_2 = \mathbf{X}_\tau'\mathbf{X}_\gamma$ be the incidence matrices for the row and column components of the design. The *(i,j)th* element of the matrix $\mathbf{N}_1$ records the number of occurrences of the *ith* treatment in the *jth* row, and similarly, the *(i,j)th* element of the matrix $\mathbf{N}_2$ records the number of occurrences of the *ith* treatment in the *jth* column of the starting design. The *(i,j)th* element of $\mathbf{N}_3 = \mathbf{X}_\gamma'\mathbf{X}_\rho$ is equal to 1 if there is an observation in the *ith* column and *jth* row of the design and 0 otherwise. The full normal equations can therefore be written as

$$n\hat{\mu} + \mathbf{r}'\hat{\boldsymbol{\rho}} + \mathbf{c}'\hat{\boldsymbol{\gamma}} + \mathbf{t}'\hat{\boldsymbol{\tau}} = \mathcal{G} \tag{2.26}$$

$$\mathbf{r}\hat{\mu} + \mathbf{r}^{\delta}\hat{\boldsymbol{\rho}} + \mathrm{N}_3'\hat{\boldsymbol{\gamma}} + \mathrm{N}_1'\hat{\boldsymbol{\tau}} = \mathcal{R} \tag{2.27}$$

$$\mathbf{c}\hat{\mu} + \mathrm{N}_3\hat{\boldsymbol{\rho}} + \mathbf{c}^{\delta}\hat{\boldsymbol{\gamma}} + \mathrm{N}_2'\hat{\boldsymbol{\tau}} = \mathcal{C} \tag{2.28}$$

$$\mathbf{t}\hat{\mu} + \mathrm{N}_1\hat{\boldsymbol{\rho}} + \mathrm{N}_2\hat{\boldsymbol{\gamma}} + \mathbf{t}^{\delta}\hat{\boldsymbol{\tau}} = \mathcal{T} \tag{2.29}$$

To estimate treatment effects, the row parameters $\hat{\boldsymbol{\rho}}$ need to be eliminated from Equations (2.28) and (2.29). Premultiply Equation (2.27) by $\mathrm{N}_3\mathbf{r}^{-\delta}$ and by $\mathrm{N}_1\mathbf{r}^{-\delta}$ to get

$$\mathbf{c}\hat{\mu} + \mathrm{N}_3\hat{\boldsymbol{\rho}} + \mathrm{N}_3\mathbf{r}^{-\delta}\mathrm{N}_3'\hat{\boldsymbol{\gamma}} + \mathrm{N}_3\mathbf{r}^{-\delta}\mathrm{N}_1'\hat{\boldsymbol{\tau}} = \mathrm{N}_3\mathbf{r}^{-\delta}\mathcal{R} \tag{2.30}$$

$$\mathbf{t}\hat{\mu} + \mathrm{N}_1\hat{\boldsymbol{\rho}} + \mathrm{N}_1\mathbf{r}^{-\delta}\mathrm{N}_3'\hat{\boldsymbol{\gamma}} + \mathrm{N}_1\mathbf{r}^{-\delta}\mathrm{N}_1'\hat{\boldsymbol{\tau}} = \mathrm{N}_1\mathbf{r}^{-\delta}\mathcal{R} \tag{2.31}$$

where $\mathrm{N}_1\mathbf{r}^{-\delta}\mathbf{r} = \mathrm{N}_1\mathbf{1}_r = \mathbf{t}$ and $\mathrm{N}_3\mathbf{r}^{-\delta}\mathbf{r} = \mathrm{N}_3\mathbf{1}_c = \mathbf{c}$. Subtract Equation (2.31) from Equation (2.29) and Equation (2.30) from Equation (2.28) to leave a set of simultaneous equations in the column and treatment parameters $\hat{\boldsymbol{\gamma}}$ and $\hat{\boldsymbol{\tau}}$, given by

$$(\mathrm{N}_2 - \mathrm{N}_1\mathbf{r}^{-\delta}\mathrm{N}_3')\hat{\boldsymbol{\gamma}} + (\mathbf{t}^{\delta} - \mathrm{N}_1\mathbf{r}^{-\delta}\mathrm{N}_1')\hat{\boldsymbol{\tau}} = \mathcal{T} - \mathrm{N}_1\mathbf{r}^{-\delta}\mathcal{R} \tag{2.32}$$

$$(\mathbf{c}^{\delta} - \mathrm{N}_3\mathbf{r}^{-\delta}\mathrm{N}_3')\hat{\boldsymbol{\gamma}} + (\mathrm{N}_2' - \mathrm{N}_3\mathbf{r}^{-\delta}\mathrm{N}_1')\hat{\boldsymbol{\tau}} = \mathcal{C} - \mathrm{N}_3\mathbf{r}^{-\delta}\mathcal{R} \tag{2.33}$$

After the column parameters, $\hat{\boldsymbol{\gamma}}$, are removed from these equations, the information matrix for treatment effects is given by

$$\mathbf{C} = \mathbf{t}^{\delta} - \mathrm{N}_1\mathbf{r}^{-\delta}\mathrm{N}_1' - (\mathrm{N}_2 - \mathrm{N}_1\mathbf{r}^{-\delta}\mathrm{N}_3')(\mathbf{c}^{\delta} - \mathrm{N}_3\mathbf{r}^{-\delta}\mathrm{N}_3')^{-}(\mathrm{N}_2' - \mathrm{N}_3\mathbf{r}^{-\delta}\mathrm{N}_1') \tag{2.34}$$

where $\mathbf{D}^{-}$ represents a generalised inverse, and this expression can be used for situations where data are missing.

2.2.2 Latin Square Design

In a Latin square design, all r treatments occur exactly once in every row and column and are all replicated the same number of times, r. When this class of design is complete, the information matrix for treatment effects simplifies substantially to

$$\mathbf{C} = r\mathbf{I}_r - \mathbf{J}_{r,r} \tag{2.35}$$

where $\mathbf{I}_r$ is a $r \times r$ identity matrix and $\mathbf{J}_{r,r}$ is a $r \times r$ matrix of ones. A simple generalised inverse $\boldsymbol{\Omega}$ for a complete single replicate Latin square design is $\mathbf{I}_r/r$, and all pairwise treatments comparisons are measured with the same variance, $2\sigma^2/r$. The effect of missing data on Latin square designs is investigated in Chapter 4. It is also possible to use multiple squares to construct row-column designs. If there are k complete Latin squares, then the information matrix for treatment effects is given by

$$\mathbf{C} = kr\mathbf{I}_r - k\mathbf{J}_{r,r} \tag{2.36}$$

and the variances of all pairwise treatment comparisons are $2\sigma^2/kr$. The influence of missing data on these designs is considered in Chapter 5.

2.2.3 Youden Square Design

A Youden square is an amalgamation of a symmetric BIB design and a RBD and can be used to construct a row-column design when all treatments cannot be accommodated in the rows. The information matrix for treatment effects, when the design is complete, can be written as

$$\mathbf{C} = \frac{v\lambda}{k}\mathbf{I}_v - \frac{\lambda}{k}\mathbf{J}_{v,v} \tag{2.37}$$

Two common ways to construct Youden squares are (a) rearrangement of the treatments within the blocks of a symmetric BIB design, and (b) the removal of one column from a Latin square. Missing data in Youden square designs is investigated in Chapter 6.

Chapter 3

Robustness of Block Designs to Missing Observations

This Chapter considers the effect of missing data scattered throughout a binary variance balanced block design. Whittinghill (1989, 1995) and Prescott and Mansson (2001b) studied the situation where one, two, or three observations are removed from a BIB design in detail. Majorisation theory was used by Whittinghill (1995) to identify the best and worst configurations of a fixed number of missing values. A complete ordering of all potential configurations of two missing observations in a BIB design was given by Whittinghill (1995), using the average variance of pairwise treatment comparisons as the criterion. Prescott and Mansson (2001b) demonstrated that, although the majorisation approach gave the best and worst configurations for t missing values in general, a complete ordering of all cases was not always possible.

The different measures of robustness used to study block designs are covered in the next Section, and the loss of all observations in a block of a RB or BIB design is considered to extend the work of Bhaumik and Whittinghill (1991) and Das and Kageyama (1992). Expressions for the variances of pairwise treatment differences, as well as the average variance and relative efficiency of the resulting designs are derived. The normal equations are solved to compute these formulae, and a generalised inverse (g-inverse), denoted by $\boldsymbol{\Omega}$, is necessary because the information matrix for treatment effects is singular. Although the estimates of treatment effects, $\hat{\boldsymbol{\tau}}$, are not unique, the variances of elementary treatment contrasts are invariant to the choice of generalised inverse. Derivation of the variance formulae will be shown to be simplified by a sensible selection of g-inverse.

The theoretical results for the RBD are applied to starting designs with a range of block sizes and number of treatments. A detailed example will conclude the Chapter using a particular BIB design to illustrate the theoretical results of Prescott and Mansson (2001b) and the limitations of the majorisation approach for ordering different configurations of t missing values.

3.1 Measures of Robustness

In a block design, the aim is to compare a set of v treatments. The precision of any pairwise comparison, between treatments τ_i and τ_j say, is measured by the variance of the estimate of the treatment difference, given by $\hat{\tau}_i - \hat{\tau}_j$. To investigate the robustness of a particular starting design to missing observations, the alterations to the $(v \times v)$ information matrix for treatment effects need to be considered. For a general block experiment, this information matrix is given by

$$\mathbf{C} = \mathbf{r}^{\delta} - \mathbf{N}\mathbf{k}^{-\delta}\mathbf{N}' \tag{3.1}$$

The average variance of pairwise treatment differences is commonly used in assessing the A-efficiency of different designs. It can be defined using the non-zero eigenvalues of the information matrix for treatment effects. For a design d, the average variance is given by

$$\text{A.V.} = \frac{2}{v-1}\sum_{i=1}^{v-1}\frac{1}{\mu_i(d)} \tag{3.2}$$

where $\mu_i(d)$ are the $v-1$ non-zero eigenvalues of the information matrix for treatment effects. The sum of these eigenvalues is $v(v-1)\lambda/k - t$ for any design resulting from the removal of t observations. The configuration of missing values dictates the form of the affected eigenvalues, and consequently the average variance. Two designs, d_1 and d_2 say, can be compared by calculating the ratio of their average variances, which is known as the relative efficiency of the two designs. If the relative efficiency is expressed as

$$\text{R.E.}(d_1 \text{ to } d_2) = \frac{\sum_{t=1}^{v-1}\frac{1}{\mu_i(d_2)}}{\sum_{t=1}^{v-1}\frac{1}{\mu_i(d_1)}} \tag{3.3}$$

then d_1 is less (more) efficient than d_2 if the ratio is less (greater) than one. The universal optimality result used by Whittinghill (1995) implies that the relative efficiency of d_1 and

d_2 is greater than or equal to one if

$$\sum_{i=p}^{v-1} \mu_i(d_2) \leq \sum_{i=p}^{v-1} \mu_i(d_1) \tag{3.4}$$

$$(p = 1, \cdots, v-1)$$

If the design d_1 is less efficient then d_2, the vector of eigenvalues for d_1 majorises the vector of eigenvalues for d_2.

To study the effect of drop-out on a given design, it may be practical to express the information matrix as a sum of block information matrices, i.e.

$$\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2 + \cdots + \mathbf{C}_b = \sum_{j=1}^{b} \mathbf{C}_j \tag{3.5}$$

where $\mathbf{C}_j$ is the information matrix for the jth block of the starting design. Blocks are considered separately to assess the impact of the missing data, and are combined to derive the overall form of the information matrix. The different robustness criterion are calculated from the information matrix for treatment effects. In general, let t observations be removed from the initial design, with t_j removed from the jth block. Suppose that $\mathbf{C}_j^{(n)}$ is the information matrix for block j after the removal of the t_j missing observations. Let $\mathbf{A}_j$ be the difference between the matrices $\mathbf{C}_j$ and $\mathbf{C}_j^{(n)}$ representing the impact of losing t_j observations from the jth block. Therefore, the information matrix of the resulting design, $d(t)$, is given by

$$\mathbf{C}_{d(t)} = \mathbf{C} - \sum_{j=1}^{b}(\mathbf{C}_j - \mathbf{C}_j^{(n)}) = \mathbf{C} - \sum_{j=1}^{b} \mathbf{A}_j = \mathbf{C} - \mathbf{A}_d \tag{3.6}$$

where $\mathbf{A}_d = \sum_{j=1}^{b} \mathbf{A}_j$. Adjustments to the non-zero eigenvalues of the information matrix can be found by investigating the alterations to the eigenvalues of the adjustment matrices, $\mathbf{A}_j$. Whittinghill (1995, page 25) gave the next result for a single block adjustment matrix.

Lemma 3.1 *Let t scattered observations be removed from the first b_0 blocks of* d *, a connected, binary incomplete block design. Then*

1. *The mth eigenvalue of $\mathbf{A}_j$ is such that $\mu_m(\mathbf{A}_j) = 1$ for $1 \leq m \leq t_j$ and $\mu_m(\mathbf{A}_j) = 0$*

for $t_j + 1 \leq m \leq v$. *Therefore* $\mathbf{A}_j$ *is positive semidefinite, and trace*$(\mathbf{A}_j) = t_j$, $1 \leq j \leq b_0$.

2. $\mathbf{A}_d$ *is positive semidefinite, and trace*$(\mathbf{A}_d) = t$.

These results were used by Whittinghill (1995) to identify the best and worst cases of t missing values. For some designs, however, these configurations of missing values are not realisable. These cases are given in the next Lemma of Whittinghill (1995, page 26).

Lemma 3.2 *The bounding cases are*

1. *The worst situation is where one observation is removed from each of* t *identical blocks, and each missing observation corresponds to the same treatment. Here the sum of the adjustment matrices is denoted* $\mathbf{A}_d^{\circ}$*, and its eigenvalues are*

$$\mu_1(\mathbf{A}_d^{\circ}) = t \qquad \textit{and} \qquad \mu_m(\mathbf{A}_d^{\circ}) = 0 \qquad m = 2, \cdots, v$$

2. *One of the two best cases occurs when* t *observations are removed from the same block. Eigenvalues of the sum of the adjustment matrices, denoted* $\mathbf{A}_d^{+}$*, are*

$$\mu_m(\mathbf{A}_d^{+}) = 1 \qquad m = 1, \cdots, t \qquad \textit{and} \qquad \mu_m(\mathbf{A}_d^{+}) = 0 \qquad \textit{otherwise}$$

3. *The other best case is where one observation is removed from each of* t *blocks which have disjoint sets of treatments. The eigenvalues of* $\mathbf{A}_d^{\#}$*, the overall adjustment matrix, are also*

$$\mu_m(\mathbf{A}_d^{\#}) = 1 \qquad m = 1, \cdots, t \qquad \textit{and} \qquad \mu_m(\mathbf{A}_d^{\#}) = 0 \qquad \textit{otherwise}$$

The information matrices of all connected resulting designs have a common form, and their eigenvalues are given by a result from Das and Kageyama (1992).

Lemma 3.3 *Let* u, $c_1, \cdots, c_u$ *be positive integers,* $a_1, \cdots, a_u$, $b_1, \cdots, b_u$ *real constants,* *and consider the $(c \times c)$ partitioned matrix*

$$\mathbf{A} = \begin{bmatrix} a_1\mathbf{I}_{c_1} + b_{11}\mathbf{J}_{c_1,c_1} & b_{12}\mathbf{J}_{c_1,c_2} & \cdots & b_{1u}\mathbf{J}_{c_1,c_u} \\ b_{21}\mathbf{J}_{c_2,c_1} & a_2\mathbf{I}_{c_2} + b_{22}\mathbf{J}_{c_2,c_2} & \cdots & b_{2u}\mathbf{J}_{c_2,c_u} \\ \vdots & \vdots & \ddots & \vdots \\ b_{u1}\mathbf{J}_{c_u,c_1} & b_{u2}\mathbf{J}_{c_u,c_2} & \cdots & a_u\mathbf{I}_{c_u} + b_{uu}\mathbf{J}_{c_u,c_u} \end{bmatrix} \tag{3.7}$$

where $c = c_1 + \cdots + c_u$ and the $(u \times u)$ matrix $\mathbf{B} = (b_{ij})$ is symmetric. The eigenvalues of the matrix $\mathbf{A}$ are a_i with multiplicity $c_i - 1$ together with the eigenvalues of $\mathbf{\Delta} = \mathbf{D}_a + \mathbf{D}_c^{1/2}\mathbf{B}\mathbf{D}_c^{1/2}$, where $\mathbf{D}_a = diag(a_1, \cdots, a_u)$, $\mathbf{D}_c = diag(c_1, \cdots, c_u)$, and $\mathbf{D}_c^{1/2} = diag(c_1^{1/2}, \cdots, c_u^{1/2})$.

When the eigenvalues of the information matrix are known, it may often be possible to use majorisation to rank different configurations of t missing observations. Individual variances of pairwise treatment comparisons can also be computed theoretically. These will give the range of possible variances, and the maximum shows how badly the initial design is affected by a given configuration of t missing values. To derive these formulae, a generalised inverse, $\mathbf{\Omega}$, of the information matrix for treatment effects is required. A suitably defined contrast matrix $\mathbf{\Gamma}$ can then be used to select the appropriate elements of $\mathbf{\Omega}$ to generate the variances of all pairwise differences. A useful result for this approach is

Theorem 3.1 *Given a singular matrix $\mathbf{H}$, then for any constant $a \neq 0$,*

$$\mathbf{H}^{-} = (\mathbf{H} + a\mathbf{J})^{-1} \tag{3.8}$$

is a generalised inverse of $\mathbf{H}$, where $\mathbf{J}$ is a matrix of ones.

A sensible choice of a will simplify the form of the particular inverse, and the derivation of the variances of pairwise treatment comparisons. It is prudent to choose the constant a to eliminate a many of the elements of the information matrix. The following Theorems concerning inverses of patterned matrices, given by Graybill (1983), will be useful in determining a generalised inverse of the information matrix for different configurations of missing values.

Theorem 3.2 *Let* $\mathbf{D}$ *be a* $(m+n) \times (m+n)$ *matrix defined as*

$$\mathbf{D} = \begin{bmatrix} a_1 \mathbf{I}_m & a_2 \mathbf{J}_{m,n} \\ a_2 \mathbf{J}_{n,m} & a_3 \mathbf{I}_n \end{bmatrix} \tag{3.9}$$

where $a_3 \neq 0$*, and* $n > 0$*. The inverse exists if and only if* $a_1 \neq 0$ *and* $a_1 \neq mna_2^2/a_3$*. If* $\mathbf{D}^{-1}$ *exists, then it is given by*

$$\mathbf{D}^{-1} = \begin{bmatrix} \frac{1}{a_1}\mathbf{I}_m + b_1 \mathbf{J}_{m,m} & b_2 \mathbf{J}_{m,n} \\ b_2 \mathbf{J}_{n,m} & \frac{1}{a_3}\mathbf{I}_n + b_3 \mathbf{J}_{n,n} \end{bmatrix} \tag{3.10}$$

where

$$b_1 = -\frac{na_2^2}{a_1(mna_2^2 - a_1 a_3)}$$

$$b_2 = \frac{a_2}{(mna_2^2 - a_1 a_3)}$$

$$b_3 = -\frac{ma_2^2}{a_3(mna_2^2 - a_1 a_3)}$$

Theorem 3.3 *Given a non-singular matrix of the form*

$$\mathbf{D} = \begin{bmatrix} \alpha_1 \mathbf{I}_m + \beta_{11} \mathbf{J}_{m,m} & \beta_{12} \mathbf{J}_{m,n} \\ \beta_{21} \mathbf{J}_{n,m} & \alpha_2 \mathbf{I}_n + \beta_{22} \mathbf{J}_{n,n} \end{bmatrix} \tag{3.11}$$

then there exists an inverse $\mathbf{D}^{-1}$ *with a similar form*

$$\mathbf{D}^{-1} = \begin{bmatrix} a_1 \mathbf{I}_m + b_{11} \mathbf{J}_{m,m} & b_{12} \mathbf{J}_{m,n} \\ b_{21} \mathbf{J}_{n,m} & a_2 \mathbf{I}_n + b_{22} \mathbf{J}_{n,n} \end{bmatrix} \tag{3.12}$$

where $a_1 = \alpha_1^{-1}$*,* $a_2 = \alpha_2^{-1}$*, and*

$$b_{11} = -\frac{\alpha_2 \beta_{11} + n\beta_{11}\beta_{22} - n\beta_{12}\beta_{21}}{\alpha_1(\alpha_1\alpha_2 + n\alpha_1\beta_{22} + m\alpha_2\beta_{11} + mn\beta_{11}\beta_{22} - mn\beta_{12}\beta_{21})}$$

$$b_{12} = -\frac{\beta_{12}}{(\alpha_1\alpha_2 + n\alpha_1\beta_{22} + m\alpha_2\beta_{11} + mn\beta_{11}\beta_{22} - mn\beta_{12}\beta_{21})}$$

$$b_{21} = -\frac{\beta_{21}}{(\alpha_1\alpha_2 + n\alpha_1\beta_{22} + m\alpha_2\beta_{11} + mn\beta_{11}\beta_{22} - mn\beta_{12}\beta_{21})}$$

$$b_{22} = -\frac{\alpha_1\beta_{22} + m\beta_{11}\beta_{22} - m\beta_{12}\beta_{21}}{\alpha_2(\alpha_1\alpha_2 + n\alpha_1\beta_{22} + m\alpha_2\beta_{11} + mn\beta_{11}\beta_{22} - mn\beta_{12}\beta_{21})}$$

3.2 Loss of Complete Blocks

There are situations where all the observations within one or more blocks of the starting design are lost. Bhaumik and Whittinghill (1991) considered the removal of complete blocks from binary, variance balanced incomplete block designs, and used the vector of eigenvalues of the information matrix for treatment effects, $\mathbf{C}$, to compare the different situations. The average variance of a design d_1 is larger than that of another design d_2, if the vector of eigenvalues of d_1 is majorised by the corresponding vector of eigenvalues for the other design d_2.

When t blocks are removed from the starting design, there are $\binom{b}{t}$ possible resulting designs. Bhaumik and Whittinghill (1991) concentrated on the best and worst of these configurations, and showed that the loss of t identical blocks was the worst (largest average variance of treatment comparisons) case and that the best (smallest average variance) situation was where the treatment sets of the t blocks were mutually disjoint.

Das and Kageyama (1992) considered the loss of any number of observations from a single block of a BIB design and extended BIB designs. The measure of robustness was the relative efficiency of the resulting design. Their special case corresponded to the removal of one block from the starting design.

In this section, variances of the individual pairwise treatment comparisons are derived for some of the situations covered by Bhaumik and Whittinghill (1991) and Das and Kageyama (1992). These provide more detailed information about the effect of losing blocks, because the maximum increase in variances may be concealed when the average variance is used for comparison.

3.2.1 Loss of blocks from a RBD

A simple design is the RBD, introduced in Section 2.1.3, where there are v treatments allocated one to a plot in each of the b blocks. There are $r = b$ replicates of every treatment and $k = v$ units in all blocks of the starting design. When one block is removed, the

Table 3.1: Average variance and relative efficiency for RBDs after the loss of one or two blocks of observations.

	Replicates, $r = b$					
	3	4	5	6	7	8
$t = 1$ block						
A.V.	σ^2	$0.667\sigma^2$	$0.500\sigma^2$	$0.400\sigma^2$	$0.333\sigma^2$	$0.286\sigma^2$
R.E.	(0.667)	(0.750)	(0.800)	(0.833)	(0.857)	(0.875)
$t = 2$ blocks						
A.V.	$2\sigma^2$	σ^2	$0.667\sigma^2$	$0.500\sigma^2$	$0.400\sigma^2$	$0.333\sigma^2$
R.E.	(0.333)	(0.500)	(0.600)	(0.667)	(0.714)	(0.750)

information matrix for treatment effects is altered to

$$\mathbf{C}_d = (r-1)\mathbf{I}_v - \frac{(r-1)}{k}\mathbf{J}_{v,v} \tag{3.13}$$

and the average variance of pairwise treatment differences increases to $2\sigma^2/(r-1)$, because all of the $v(v-1)/2$ individual variances are $2\sigma^2/(r-1)$. The average is dependent only on the number of replicates of each treatment in the design. Table 3.1 shows these values for different replication in RBDs. More generally, when t blocks, which are all necessarily identical, are removed, the information matrix for the resulting designs corresponding to every realisable combination of lost blocks is given by

$$\mathbf{C}_d = (r-t)\mathbf{I}_v - \frac{(r-t)}{k}\mathbf{J}_{v,v} \tag{3.14}$$

The average variance increases from $2\sigma^2/r$ in the complete design to $2\sigma^2/(r-t)$, and the efficiency relative to the complete design is $(r-t)/r$. Although the resulting designs when t blocks are lost are always variance balanced, the loss of complete blocks correspond to the most serious way to lose observations, as shown by Bhaumik and Whittinghill (1991).

3.2.2 Loss of blocks from a BIB design

The adjustments to the information matrix when a single block is lost from a BIB design were derived by Bhaumik and Whittinghill (1991) and Das and Kageyama (1992). If we assume that the missing block contained the first k treatments, then the information

Table 3.2: Formulae for the variances of pairwise treatment comparisons in a BIB design when a single block, assumed to contain the first k treatments, is removed from the initial design.

	Individual Pairwise Variances			
	Treatment i	Treatment j	Variance	Number
(i)	$1, \cdots, k$	$1, \cdots, k$	$\frac{2k}{\lambda v - k}\sigma^2$	$k(k-1)/2$
(ii)	$1, \cdots, k$	$k+1, \cdots, v$	$\frac{k(k+1-2\lambda v)}{\lambda v(k-\lambda v)}\sigma^2$	$k(v-k)$
(iii)	$k+1, \cdots, v$	$k+1, \cdots, v$	$\frac{2k}{\lambda v}\sigma^2$	$(v-k)(v-k-1)/2$
			Average Variance	
			$\frac{2k(k^2 - vk + \lambda v^2 - \lambda v)}{\lambda v(\lambda v - k)(v-1)}\sigma^2$	
			Relative Efficiency	
			$1 - \frac{(k-1)}{b(k-1)-v+k}$	

matrix is given by

$$\mathbf{C}_d = \begin{bmatrix} \frac{(\lambda v - k)}{k}\mathbf{I}_k - \frac{(\lambda-1)}{k}\mathbf{J}_{k,k} & -\frac{\lambda}{k}\mathbf{J}_{k,v-k} \\ -\frac{\lambda}{k}\mathbf{J}_{v-k,k} & \frac{\lambda v}{k}\mathbf{I}_{v-k} - \frac{\lambda}{k}\mathbf{J}_{v-k,v-k} \end{bmatrix} \tag{3.15}$$

The non-zero eigenvalues of $\mathbf{C}_d$ are $\lambda v/k$ and $\lambda v/k - 1$ with multiplicities $v-k$ and $k-1$ respectively. To find a generalised inverse, $\boldsymbol{\Omega}$, of $\mathbf{C}_d$, add $\frac{\lambda}{k}\mathbf{J}_{v,v}$ to the information matrix $\mathbf{C}_d$. The inverse of the resulting non-singular matrix is a generalised inverse of $\mathbf{C}_d$. This particular generalised inverse is given by

$$\boldsymbol{\Omega} = \begin{bmatrix} \frac{k}{(\lambda v - k)}\mathbf{I}_k - \frac{k}{\lambda v(\lambda v - k)}\mathbf{J}_{k,k} & \mathbf{0}_{k,v-k} \\ \mathbf{0}_{v-k,k} & \frac{k}{\lambda v}\mathbf{I}_{v-k} \end{bmatrix} \tag{3.16}$$

The variances of the individual pairwise treatment comparisons using this generalised inverse are given in Table 3.2. There are three forms for these variances depending on whether the two treatments being compared were in the affected block.

Table 3.3 shows the effect on variances of the pairwise treatment differences after the loss of a block from a selection of BIB designs. The first two designs in the Table have four treatments, each with three replicates. One has six blocks of two plots, and the other four blocks of three plots. The average variances for the complete designs are σ^2 and $0.75\sigma^2$

respectively. When a single block is lost from these two starting designs, the efficiencies of the resulting designs are 75% and 71% respectively, although the first design loses two units and the other three. However, the maximum of the variances for the first design increases to $2\sigma^2$, whereas the maximum is only $1.2\sigma^2$ for the second starting design. In the second design, all treatments occur together in two blocks of the starting design.

Comparing the two symmetric BIB designs with seven treatments and three or four replicates shows that the larger design loses less efficiency. This is because the second design has $\lambda = 2$, so that all pairs of treatments are always compared directly regardless of which block is lost. The average variance of a complete BIB design is $2k\sigma^2/\lambda v$, so the larger design has a substantially lower average variance.

Bhaumik and Whittinghill (1991) also considered the loss of two blocks from BIB designs, and the same approach could be used to derive expressions for the variances of treatment differences. The loss of efficiency will depend on the number of treatments common to the pair of missing blocks. They concluded that blocks should have as few treatments in common as possible to minimise the impact of losing two blocks.

3.3 Randomised Block Designs and Missing Values

The effect of drop-out on a randomised block design (RBD) is covered in this Section. Properties of the designs, $d(t)$, resulting from the removal of t observations depend on the configuration of the missing values. All distinct configurations of one, two, or three missing observations are considered, and the different resulting designs are denoted by $d(t; c)$, where t is the number of missing values and c is the sub-case corresponding to a particular configuration.

The average and maximum variances of pairwise treatment differences are derived theoretically in terms of some or all the design parameters, and numerically for all ways of losing up to three observations.

3.3.1 Loss of one plot from a RBD

The variances that are increased after the loss of a single observation depend on the actual treatment that has been lost. However, the average and maximum variances are the same for any configuration produced by removing one of the vr plots in the starting design.

Table 3.3: Average and pairwise variances (and efficiencies) for BIB designs where a complete block of observations become unavailable. The variances need to be multiplied by σ^2.

Parameters					Average	Variances of pairwise comparisons					
v	b	r	k	λ	Variance	(i)	num	(ii)	num	(iii)	num
4	6	3	2	1	1.3333	2.0000	1	1.2500	4	1.0000	1
					(0.7500)	(0.5000)		(0.8000)		(1.000)	
4	4	3	3	2	1.0500	1.2000	3	0.9000	3		
					(0.7143)	(0.6250)		(0.8333)			
5	10	4	2	1	0.9333	1.3333	1	0.9333	6	0.8000	3
					(0.8571)	(0.6000)		(0.8571)		(1.000)	
5	5	4	4	3	0.6788	0.7273	6	0.6061	4		
					(0.7857)	(0.7333)		(0.8799)			
5	10	6	3	3	0.4500	0.5000	3	0.4333	6	0.4000	1
					(0.8889)	(0.8000)		(0.9231)		(1.000)	
6	15	5	2	1	0.7333	1.0000	1	0.7500	8	0.6667	6
					(0.9091)	(0.6667)		(0.8889)		(1.000)	
6	10	5	3	2	0.5667	0.6667	3	0.5556	9	0.5000	3
					(0.8824)	(0.7500)		(0.8889)		(1.000)	
6	6	5	5	4	0.5044	0.5263	10	0.4605	5		
					(0.8261)	(0.7918)		(0.9049)			
6	15	10	4	6	0.2389	0.2500	6	0.2326	8	0.2222	1
					(0.9302)	(0.8888)		(0.9553)		(1.000)	
7	7	3	3	1	1.0714	1.5000	3	1.0714	12	0.8571	6
					(0.8000)	(0.5714)		(0.8000)		(1.000)	
7	7	4	4	2	0.6857	0.8000	6	0.6571	12	0.5714	3
					(0.8333)	(0.7143)		(0.8696)		(1.000)	
7	21	6	2	1	0.6095	0.8000	1	0.6286	10	0.5714	10
					(0.9375)	(0.7143)		(0.9090)		(1.000)	
7	7	6	6	5	0.4020	0.4138	15	0.3724	6		
					(0.8529)	(0.8287)		(0.9208)			
7	21	15	5	10	0.1502	0.1538	10	0.1473	10	0.1429	1
					(0.9512)	(0.9291)		(0.9701)		(1.000)	
8	28	7	2	1	0.5238	0.6667	1	0.5417	12	0.5000	15
					(0.9545)	(0.7500)		(0.9230)		(1.000)	
8	14	7	4	3	0.3619	0.4000	6	0.3583	16	0.3333	6
					(0.9211)	(0.8333)		(0.9302)		(1.000)	
8	8	7	7	6	0.3343	0.3415	21	0.3130	7		
					(0.8723)	(0.8542)		(0.9319)			
9	12	4	3	1	0.7500	1.0000	3	0.7778	18	0.6667	15
					(0.8889)	(0.6667)		(0.8572)		(1.000)	
9	36	8	2	1	0.4603	0.5714	1	0.4762	14	0.4444	21
					(0.9655)	(0.7777)		(0.9332)		(1.000)	
9	18	8	4	3	0.3156	0.3478	6	0.3156	20	0.2963	10
					(0.9388)	(0.8519)		(0.9388)		(1.000)	

Table 3.4: Variances of pairwise treatment comparisons in a RBD design where a single observation, assumed to correspond to the first treatment, is lost from the design.

Individual Pairwise Variances			
Treatment i	Treatment j	Variance	Number
1	$2, \cdots, v$	$\frac{2(bv-b-v+2)}{b(b-1)(v-1)}\sigma^2$	$(v-1)$
$2, \cdots, v$	$2, \cdots, v$	$\frac{2}{b}\sigma^2$	$(v-1)(v-2)/2$
Average Variance			
$\frac{2(vb-b-v+2)}{b(b-1)(v-1)}\sigma^2$			
Relative Efficiency			
$1-\frac{1}{(vb-b-v+2)}$			

Given a complete randomised block design, the $v-1$ non-zero eigenvalues, denoted by μ_i $(i = 1, \cdots, v-1)$, of the information matrix for treatment effects are all equal to b. When one observation is removed from the starting design, one of the eigenvalues decreases to $b-1$ for all resulting designs $d(1)$. To derive expressions for the variances of pairwise treatment differences, a generalised inverse of the information matrix is required. Setting $a = (b-1)/v$ in Theorem (3.1), and computing the inverse of the resulting matrix gives a generalised inverse

$$\Omega = \begin{bmatrix} \frac{1}{(b-1)} & \mathbf{0}'_{v-1} \\ \mathbf{0}_{v-1} & \frac{1}{b}\mathbf{I}_{v-1} + \frac{1}{b(b-1)(v-1)}\mathbf{J}_{v-1,v-1} \end{bmatrix} \tag{3.17}$$

This generalised inverse is used to generate the variances given in Table 3.4. To assess the effect of losing a single observation, the average and maximum variances are given in Table 3.5 for a range of design sizes, with efficiencies for both the values compared with those in the complete design. A starting design with three treatments and two blocks is seriously affected by the loss of one of its six plots. Assuming that one replicate of the first treatment is removed, then the average variance increases from σ^2 in the initial design, to $1.5\sigma^2$, and the largest of the pairwise variances is $1.75\sigma^2$. This design incurs a 33% reduction in overall efficiency. If an extra block were included in the design, then the loss of efficiency would be 20%. Additional blocks would reduce this loss further, but the benefits become smaller as the total number of blocks increases. The difference in robustness between designs with two and three blocks are noticeable for any number of

Table 3.5: Average and maximum variances of pairwise differences, and their relative efficiencies in parentheses for one missing value in RBDs with various combinations of treatments and blocks. All variances to be multiplied by σ^2.

	Number of Blocks						
Treatments	2	3	4	5	6	7	8
3	1.5000	0.8333	0.5833	0.4500	0.3667	0.3095	0.2679
	(0.6667)	(0.8000)	(0.8571)	(0.8889)	(0.9091)	(0.9231)	(0.9333)
	1.7500	0.9167	0.6250	0.4750	0.3833	0.3214	0.2768
	(0.5714)	(0.7273)	(0.8000)	(0.8421)	(0.8696)	(0.8889)	(0.9032)
4	1.3333	0.7778	0.5556	0.4333	0.3556	0.3016	0.2619
	(0.7500)	(0.8571)	(0.9000)	(0.9231)	(0.9375)	(0.9474)	(0.9545)
	1.6667	0.8889	0.6111	0.4667	0.3778	0.3175	0.2738
	(0.6000)	(0.7500)	(0.8182)	(0.8571)	(0.8824)	(0.9000)	(0.9130)
5	1.2500	0.7500	0.5417	0.4250	0.3500	0.2976	0.2589
	(0.8000)	(0.8889)	(0.9231)	(0.9412)	(0.9524)	(0.9600)	(0.9655)
	1.6250	0.8750	0.6042	0.4625	0.3750	0.3155	0.2723
	(0.6154)	(0.7619)	(0.8276)	(0.8649)	(0.8889)	(0.9057)	(0.9180)
6	1.2000	0.7333	0.5333	0.4200	0.3467	0.2952	0.2571
	(0.8333)	(0.9091)	(0.9375)	(0.9524)	(0.9615)	(0.9677)	(0.9722)
	1.6000	0.8667	0.6000	0.4600	0.3733	0.3143	0.2714
	(0.6250)	(0.7692)	(0.8333)	(0.8696)	(0.8929)	(0.9091)	(0.9211)
7	1.1667	0.7222	0.5278	0.4167	0.3444	0.2937	0.2560
	(0.8571)	(0.9231)	(0.9474)	(0.9600)	(0.9677)	(0.9730)	(0.9767)
	1.5833	0.8611	0.5972	0.4583	0.3722	0.3135	0.2708
	(0.6316)	(0.7742)	(0.8372)	(0.8727)	(0.8955)	(0.9114)	(0.9231)
8	1.1429	0.7143	0.5238	0.4143	0.3429	0.2925	0.2551
	(0.8750)	(0.9333)	(0.9545)	(0.9655)	(0.9722)	(0.9767)	(0.9800)
	1.5714	0.8571	0.5952	0.4571	0.3714	0.3129	0.2704
	(0.6364)	(0.7778)	(0.8400)	(0.8750)	(0.8974)	(0.9130)	(0.9245)
9	1.1250	0.7083	0.5208	0.4125	0.3417	0.2917	0.2545
	(0.8889)	(0.9412)	(0.9600)	(0.9697)	(0.9756)	(0.9796)	(0.9825)
	1.5625	0.8542	0.5938	0.4562	0.3708	0.3125	0.2701
	(0.6400)	(0.7805)	(0.8421)	(0.8767)	(0.8989)	(0.9143)	(0.9256)
10	1.1111	0.7037	0.5185	0.4111	0.3407	0.2910	0.2540
	(0.9000)	(0.9474)	(0.9643)	(0.9730)	(0.9783)	(0.9818)	(0.9844)
	1.5556	0.8519	0.5926	0.4556	0.3704	0.3122	0.2698
	(0.6429)	(0.7826)	(0.8438)	(0.8780)	(0.9000)	(0.9153)	(0.9265)

treatments given in Table 3.5. When there are a large number of treatments and blocks, the benefits of including another block in the starting design are small.

3.3.2 Two missing values in a RBD

There are three distinct classes of resulting design, denoted by $d(2;1)$, $d(2;2)$, and $d(2;3)$, when two observations are lost from a RBD. The structure of a RBD ensures that these three configurations of missing values can be investigated using only the first two blocks and treatments of the design, because other configurations can be adjusted by relabelling the treatments and/or the blocks to be the same as these three classes. The three cases occur when

1. the missing values are for different treatments and occur in the same block of the starting design, $d(2;1)$,
2. two observations are lost from different blocks and they correspond to different treatments, $d(2;2)$, and
3. two replicates of one treatment are removed from different blocks, $d(2;3)$.

In all of these three cases, two of the non-zero eigenvalues of the information matrix for treatment effects are altered, and can be written in the form

$$\mu_{\nu-2} = b - 1 + x \qquad \text{and} \qquad \mu_{\nu-1} = b - 1 - x \tag{3.18}$$

where x is dependant on the configuration of missing observations. Majorisation can be used on the vectors of eigenvalues corresponding to these three situations to show that the best class of resulting designs is $d(2;1)$, and that the worst configurations correspond to $d(2;3)$, when average variance or relative efficiency is used to compare the resulting designs.

The values of x for these configurations of missing values are given in Table 3.6. For $d(2;2)$, x is greater than zero for any number of treatments, and is strictly less than one if there are three or more treatments in the starting design. Formulae for the variances of the pairwise treatment comparisons are also given in Table 3.6, and these or the eigenvalues of the information matrix can be used to derive the average variance formulae shown in

Table 3.6: Formulae for all distinct variances of pairwise treatment comparisons when two observations are lost from a RBD.

Case	Treatments τ_i	τ_j	Pairwise Variance	Number of Comparisons
$d(2;1)$	1	2	$\frac{2}{b-1}\sigma^2$	1
	$1,2$	$3,\cdots,v$	$\frac{(2bv-4b-v+3)}{b(b-1)(v-2)}\sigma^2$	$2(v-2)$
	$3,\cdots,v$	$3,\cdots,v$	$\frac{2}{b}\sigma^2$	$(v-2)(v-3)/2$
Value of x			0	
Average Variance			$\frac{(vb-v-b+3)}{b(b-1)}\frac{2}{v-1}\sigma^2$	
Relative Efficiency			$\frac{(v-1)(b-1)}{(vb-v-b+3)}$	
No of Configurations			$\frac{bv(v-1)}{2}$	
$d(2;2)$	1	2	$\frac{2(v-1)}{bv-b-v}\sigma^2$	1
	$1,2$	$3,\cdots,v$	$\frac{(2b^2v^2-4b^2v+2b^2-3bv^2+7bv-4b+v^2-3v)}{b(bv-b-v+2)(bv-b-v)}\sigma^2$	$2(v-2)$
	$3,\cdots,v$	$3,\cdots,v$	$\frac{2}{b}\sigma^2$	$(v-2)(v-3)/2$
Value of x			$\frac{1}{v-1}$	
Average Variance			$\frac{(v^3b^2-2v^3b-3v^2b^2+8v^2b+3vb^2-10vb+v^3-5v^2-b^2+4b+6v)}{b(vb-b-v)(vb-v-b+2)}\frac{2}{v-1}\sigma^2$	
Relative Efficiency			$\frac{(v-1)(vb-b-v)(vb-v-b+2)}{(v^3b^2-2v^3b-3v^2b^2+8v^2b+3vb^2-10vb+v^3-5v^2-b^2+4b+6v)}$	
No of Configurations			$\frac{bv(v-1)(b-1)}{2}$	
$d(2;3)$	1	$2,\cdots,v$	$\frac{2(bv-b-v+2)}{b(b-2)(v-1)}\sigma^2$	$(v-1)$
	$2,\cdots,v$	$2,\cdots,v$	$\frac{2}{b}\sigma^2$	$(v-1)(v-2)/2$
Value of x			1	
Average Variance			$\frac{(vb-2v-b+4)}{b(b-2)}\frac{2}{v-1}\sigma^2$	
Relative Efficiency			$\frac{(v-1)(b-2)}{(vb-2v-b+4)}$	
Configurations			$\frac{bv(b-1)}{2}$	

the Table. All formulae are shown in terms of the number of blocks and treatments in the initial design only, because the number of replicates of each treatment and the block sizes are related to these two parameters. The number of configurations corresponding to each case is also given to show the distribution of average variances under the assumption of drop-out completely at random.

Tables 3.7 - 3.12 contain numerical values for the measures of interest for RBDs based on a range of different design parameters. Designs where there are only a small number of treatments and blocks are seriously affected by the removal of two observations. The best configuration of two missing values in a design with three treatments and three blocks suffers a reduction of 33% in efficiency. This loss of efficiency is reduced if there are more blocks in the starting design. The loss is 12.5% when the design has three treatments and eight blocks, but there are now 24 rather than 9 plots in the starting design. In the worst situation, where two replicates of one of the treatments are missing, the smallest tabulated design with three treatments and three blocks, loses 50% of its efficiency. The maximum variance of pairwise differences increases from $0.667\sigma^2$ in the complete design to $1.667\sigma^2$.

The reduction in efficiency due to the worst configuration of two missing observations is less severe when there are more treatments and/or blocks in the initial design. An eight treatment RBD in three blocks loses approximately 23% of its efficiency in the worst case, but the maximum variance of pairwise comparisons is $1.4286\sigma^2$ compared to $0.667\sigma^2$ for all comparisons in the complete design.

3.3.3 Three missing values

In the situation where three values are removed from a RBD, the many resulting designs correspond to one of six distinct configurations of missing values, labelled $d(3;i)$ $(i = 1,\cdots,6)$. The properties of these six cases of resulting design are studied separately to derive the average variance of the pairwise treatment differences. The six sub-cases can be investigated using the first three treatments and blocks of the starting design. Any configuration of three missing observations can be rearranged by switching the treatment labels and blocks to correspond to one of these six sub-cases. The configurations of missing values for these six cases are

Table 3.7: Average variance, individual variance of pairwise comparisons, and relative efficiencies for RBDs with three treatments and a variety of blocks. All variances to be multiplied by σ^2.

	Case d(2;1)			Case d(2;2)			Case d(2;3)		
Blocks	Var(Eff)		No.	Var(Eff)		No.	Var(Eff)		No.
	1.0000	(0.6667)	1	1.3333	(0.5000)	1	1.6667	(0.4000)	2
3	1.0000	(0.6667)	2	0.9333	(0.7143)	2	0.6667	(1.0000)	1
Average	1.0000	(0.6667)		1.0667	(0.6250)		1.3333	(0.5000)	
Number	9			18			9		
	0.6667	(0.7500)	1	0.8000	(0.6250)	1	0.8750	(0.5714)	2
4	0.6667	(0.7500)	2	0.6286	(0.7955)	2	0.5000	(1.0000)	1
Average	0.6667	(0.7500)		0.6857	(0.7292)		0.7500	(0.6667)	
Number	12			36			18		
	0.5000	(0.8000)	1	0.5714	(0.7000)	1	0.6000	(0.6667)	2
5	0.5000	(0.8000)	2	0.4762	(0.8400)	2	0.4000	(1.0000)	1
Average	0.5000	(0.8000)		0.5079	(0.7875)		0.5333	(0.7500)	
Number	15			60			30		
	0.4000	(0.8333)	1	0.4444	(0.7500)	1	0.4583	(0.7273)	2
6	0.4000	(0.8333)	2	0.3838	(0.8684)	2	0.3333	(1.0000)	1
Average	0.4000	(0.8333)		0.4040	(0.8250)		0.4167	(0.8000)	
Number	18			90			45		
	0.3333	(0.8571)	1	0.3636	(0.7857)	1	0.3714	(0.7692)	2
7	0.3333	(0.8571)	2	0.3217	(0.8882)	2	0.2857	(1.0000)	1
Average	0.3333	(0.8571)		0.3357	(0.8512)		0.3429	(0.8333)	
Number	21			126			63		
	0.2857	(0.8750)	1	0.3077	(0.8125)	1	0.3125	(0.8000)	2
8	0.2857	(0.8750)	2	0.2769	(0.9028)	2	0.2500	(1.0000)	1
Average	0.2857	(0.8750)		0.2872	(0.8705)		0.2917	(0.8571)	
Number	24			168			84		

Table 3.8: Average variance, individual variance of pairwise comparisons, and relative efficiencies for RBDs with four treatments and a variety of blocks. All variances to be multiplied by σ^2.

	Case d(2;1)			Case d(2;2)			Case d(2;3)		
Blocks	Var(Eff)		No.	Var(Eff)		No.	Var(Eff)		No.
	1.0000	(0.6667)	1	1.2000	(0.5556)	1	1.5556	(0.4286)	3
3	0.9167	(0.7273)	4	0.8952	(0.7447)	4	0.6667	(1.0000)	3
	0.6667	(1.0000)	1	0.6667	(1.0000)	1			
Average	0.8889	(0.7500)		0.9079	(0.7343)		1.1111	(0.6000)	
Number	18			36			12		
	0.6667	(0.7500)	1	0.7500	(0.6667)	1	0.8333	(0.6000)	3
4	0.6250	(0.8000)	4	0.6125	(0.8163)	4	0.5000	(1.0000)	3
	0.5000	(1.0000)	1	0.5000	(1.0000)	1			
Average	0.6111	(0.8182)		0.6167	(0.8108)		0.6667	(0.7500)	
Number	24			72			24		
	0.5000	(0.8000)	1	0.5455	(0.7333)	1	0.5778	(0.6923)	3
5	0.4750	(0.8421)	4	0.4671	(0.8563)	4	0.4000	(1.0000)	3
	0.4000	(1.0000)	1	0.4000	(1.0000)	1			
Average	0.4667	(0.8571)		0.4690	(0.8529)		0.4889	(0.8182)	
Number	30			120			40		
	0.4000	(0.8333)	1	0.4286	(0.7778)	1	0.4444	(0.7500)	3
6	0.3833	(0.8696)	4	0.3780	(0.8819)	4	0.3333	(1.0000)	3
	0.3333	(1.0000)	1	0.3333	(1.0000)	1			
Average	0.3778	(0.8824)		0.3790	(0.8796)		0.3889	(0.8571)	
Number	36			180			60		
	0.3333	(0.8571)	1	0.3529	(0.8095)	1	0.3619	(0.7895)	3
7	0.3214	(0.8889)	4	0.3176	(0.8997)	4	0.2857	(1.0000)	3
	0.2857	(1.0000)	1	0.2857	(1.0000)	1			
Average	0.3175	(0.9000)		0.3181	(0.8981)		0.3238	(0.8824)	
Number	42			252			84		
	0.2857	(0.8750)	1	0.3000	(0.8333)	1	0.3056	(0.8182)	3
8	0.2768	(0.9032)	4	0.2739	(0.9129)	4	0.2500	(1.0000)	3
	0.2500	(1.0000)	1	0.2500	(1.0000)	1			
Average	0.2738	(0.9130)		0.2742	(0.9116)		0.2778	(0.9000)	
Number	48			336			112		

Table 3.9: Average variance, individual variance of pairwise comparisons, and relative efficiencies for RBDs with five treatments and a variety of blocks. All variances to be multiplied by σ^2.

Blocks	Case d(2;1) Var(Eff)		No.	Case d(2;2) Var(Eff)		No.	Case d(2;3) Var(Eff)		No.
	1.0000	(0.6667)	1	1.1429	(0.5833)	1	1.5000	(0.4444)	4
3	0.8889	(0.7500)	6	0.8783	(0.7590)	6	0.6667	(1.0000)	6
	0.6667	(1.0000)	3	0.6667	(1.0000)	3			
Average	0.8333	(0.8000)		0.8413	(0.7925)		1.0000	(0.6667)	
Number	30			60			15		
	0.6667	(0.7500)	1	0.7273	(0.6875)	1	0.8125	(0.6154)	4
4	0.6111	(0.8182)	6	0.6049	(0.8266)	6	0.5000	(1.0000)	6
	0.5000	(1.0000)	3	0.5000	(1.0000)	3			
Average	0.5833	(0.8571)		0.5857	(0.8537)		0.6250	(0.8000)	
Number	40			120			30		
	0.5000	(0.8000)	1	0.5333	(0.7500)	1	0.5667	(0.7059)	4
5	0.4667	(0.8571)	6	0.4627	(0.8644)	6	0.4000	(1.0000)	6
	0.4000	(1.0000)	3	0.4000	(1.0000)	3			
Average	0.4500	(0.8889)		0.4510	(0.8870)		0.4667	(0.8571)	
Number	50			200			50		
	0.4000	(0.8333)	1	0.4211	(0.7917)	1	0.4375	(0.7619)	4
6	0.3778	(0.8824)	6	0.3751	(0.8886)	6	0.3333	(1.0000)	6
	0.3333	(1.0000)	3	0.3333	(1.0000)	3			
Average	0.3667	(0.9091)		0.3672	(0.9078)		0.3750	(0.8889)	
Number	60			300			75		
	0.3333	(0.8571)	1	0.3478	(0.8214)	1	0.3571	(0.8000)	4
7	0.3175	(0.9000)	6	0.3155	(0.9055)	6	0.2857	(1.0000)	6
	0.2857	(1.0000)	3	0.2857	(1.0000)	3			
Average	0.3095	(0.9231)		0.3098	(0.9222)		0.3143	(0.9091)	
Number	70			420			105		
	0.2857	(0.8750)	1	0.2963	(0.8438)	1	0.3021	(0.8276)	4
8	0.2738	(0.9130)	6	0.2723	(0.9179)	6	0.2500	(1.0000)	6
	0.2500	(1.0000)	3	0.2500	(1.0000)	3			
Average	0.2679	(0.9333)		0.2680	(0.9327)		0.2708	(0.9231)	
Number	80			560			140		

Table 3.10: Average variance, individual variance of pairwise comparisons, and relative efficiencies for RBDs with six treatments and a variety of blocks. All variances to be multiplied by σ^2.

	Case d(2;1)			Case d(2;2)			Case d(2;3)		
Blocks	Var(Eff)		No.	Var(Eff)		No.	Var(Eff)		No.
	1.0000	(0.6667)	1	1.1111	(0.6000)	1	1.4667	(0.4545)	5
3	0.8750	(0.7619)	8	0.8687	(0.7674)	8	0.6667	(1.0000)	10
	0.6667	(1.0000)	6	0.6667	(1.0000)	6			
Average	0.8000	(0.8333)		0.8040	(0.8291)		0.9333	(0.7143)	
Number	45			90			18		
	0.6667	(0.7500)	1	0.7143	(0.7000)	1	0.8000	(0.6250)	5
4	0.6042	(0.8276)	8	0.6004	(0.8327)	8	0.5000	(1.0000)	10
	0.5000	(1.0000)	6	0.5000	(1.0000)	6			
Average	0.5667	(0.8824)		0.5679	(0.8805)		0.6000	(0.8333)	
Number	60			180			36		
	0.5000	(0.8000)	1	0.5263	(0.7600)	1	0.5600	(0.7143)	5
5	0.4625	(0.8649)	8	0.4602	(0.8693)	8	0.4000	(1.0000)	10
	0.4000	(1.0000)	6	0.4000	(1.0000)	6			
Average	0.4400	(0.9091)		0.4405	(0.9081)		0.4533	(0.8824)	
Number	75			300			60		
	0.4000	(0.8333)	1	0.4167	(0.8000)	1	0.4333	(0.7692)	5
6	0.3750	(0.8889)	8	0.3734	(0.8927)	8	0.3333	(1.0000)	10
	0.3333	(1.0000)	6	0.3333	(1.0000)	6			
Average	0.3600	(0.9259)		0.3603	(0.9253)		0.3667	(0.9091)	
Number	90			450			90		
	0.3333	(0.8571)	1	0.3448	(0.8286)	1	0.3543	(0.8065)	5
7	0.3155	(0.9057)	8	0.3143	(0.9090)	8	0.2857	(1.0000)	10
	0.2857	(1.0000)	6	0.2857	(1.0000)	6			
Average	0.3048	(0.9375)		0.3049	(0.9370)		0.3086	(0.9259)	
Number	105			630			126		
	0.2857	(0.8750)	1	0.2941	(0.8500)	1	0.3000	(0.8333)	5
8	0.2723	(0.9180)	8	0.2714	(0.9210)	8	0.2500	(1.0000)	10
	0.2500	(1.0000)	6	0.2500	(1.0000)	6			
Average	0.2643	(0.9459)		0.2644	(0.9456)		0.2667	(0.9375)	
Number	120			840			168		

Table 3.11: Average variance, individual variance of pairwise comparisons, and relative efficiencies for RBDs with seven treatments and a variety of blocks. All variances to be multiplied by σ^2.

	Case d(2;1)			Case d(2;2)			Case d(2;3)		
Blocks	Var(Eff)		No.	Var(Eff)		No.	Var(Eff)		No.
	1.0000	(0.6667)	1	1.0909	(0.6111)	1	1.4444	(0.4615)	6
3	0.8667	(0.7692)	10	0.8625	(0.7730)	10	0.6667	(1.0000)	15
	0.6667	(1.0000)	10	0.6667	(1.0000)	10			
Average	0.7778	(0.8571)		0.7801	(0.8546)		0.8889	(0.7500)	
Number	63			126			21		
	0.6667	(0.7500)	1	0.7059	(0.7083)	1	0.7917	(0.6316)	6
4	0.6000	(0.8333)	10	0.5975	(0.8368)	10	0.5000	(1.0000)	15
	0.5000	(1.0000)	10	0.5000	(1.0000)	10			
Average	0.5556	(0.9000)		0.5562	(0.8989)		0.5833	(0.8571)	
Number	84			252			42		
	0.5000	(0.8000)	1	0.5217	(0.7667)	1	0.5556	(0.7200)	6
5	0.4600	(0.8696)	10	0.4584	(0.8725)	10	0.4000	(1.0000)	15
	0.4000	(1.0000)	10	0.4000	(1.0000)	10			
Average	0.4333	(0.9231)		0.4336	(0.9225)		0.4444	(0.9000)	
Number	105			420			70		
	0.4000	(0.8333)	1	0.4138	(0.8056)	1	0.4306	(0.7742)	6
6	0.3733	(0.8929)	10	0.3723	(0.8954)	10	0.3333	(1.0000)	15
	0.3333	(1.0000)	10	0.3333	(1.0000)	10			
Average	0.3556	(0.9375)		0.3557	(0.9371)		0.3611	(0.9231)	
Number	126			630			105		
	0.3333	(0.8571)	1	0.3429	(0.8333)	1	0.3524	(0.8108)	6
7	0.3143	(0.9091)	10	0.3135	(0.9113)	10	0.2857	(1.0000)	15
	0.2857	(1.0000)	10	0.2857	(1.0000)	10			
Average	0.3016	(0.9474)		0.3017	(0.9471)		0.3048	(0.9375)	
Number	147			882			147		
	0.2857	(0.8750)	1	0.2927	(0.8542)	1	0.2986	(0.8372)	6
8	0.2714	(0.9211)	10	0.2708	(0.9230)	10	0.2500	(1.0000)	15
	0.2500	(1.0000)	10	0.2500	(1.0000)	10			
Average	0.2619	(0.9545)		0.2620	(0.9543)		0.2639	(0.9474)	
Number	168			1,176			196		

Table 3.12: Average variance, individual variance of pairwise comparisons, and relative efficiencies for RBDs with eight treatments and a variety of blocks. All variances to be multiplied by σ^2.

	Case d(2;1)			Case d(2;2)			Case d(2;3)		
Blocks	Var	(Eff)	No.	Var	(Eff)	No.	Var	(Eff)	No.
	1.0000	(0.6667)	1	1.0769	(0.6190)	1	1.4286	(0.4667)	7
3	0.8611	(0.7742)	12	0.8581	(0.7769)	12	0.6667	(1.0000)	21
	0.6667	(1.0000)	15	0.6667	(1.0000)	15			
Average	0.7619	(0.8750)		0.7634	(0.8733)		0.8571	(0.7778)	
Number	84			168			24		
	0.6667	(0.7500)	1	0.7000	(0.7143)	1	0.7857	(0.6364)	7
4	0.5972	(0.8372)	12	0.5955	(0.8397)	12	0.5000	(1.0000)	21
	0.5000	(1.0000)	15	0.5000	(1.0000)	15			
Average	0.5476	(0.9130)		0.5481	(0.9123)		0.5714	(0.8750)	
Number	112			336			48		
	0.5000	(0.8000)	1	0.5185	(0.7714)	1	0.5524	(0.7241)	7
5	0.4583	(0.8727)	12	0.4572	(0.8749)	12	0.4000	(1.0000)	21
	0.4000	(1.0000)	15	0.4000	(1.0000)	15			
Average	0.4286	(0.9333)		0.4288	(0.9329)		0.4381	(0.9130)	
Number	140			560			80		
	0.4000	(0.8333)	1	0.4118	(0.8095)	1	0.4286	(0.7778)	7
6	0.3722	(0.8955)	12	0.3715	(0.8974)	12	0.3333	(1.0000)	21
	0.3333	(1.0000)	15	0.3333	(1.0000)	15			
Average	0.3524	(0.9459)		0.3525	(0.9457)		0.3571	(0.9333)	
Number	168			840			120		
	0.3333	(0.8571)	1	0.3415	(0.8367)	1	0.3510	(0.8140)	7
7	0.3135	(0.9114)	12	0.3129	(0.9130)	12	0.2857	(1.0000)	21
	0.2857	(1.0000)	15	0.2857	(1.0000)	15			
Average	0.2993	(0.9545)		0.2994	(0.9544)		0.3020	(0.9459)	
Number	196			1,176			168		
	0.2857	(0.8750)	1	0.2917	(0.8571)	1	0.2976	(0.8400)	7
8	0.2708	(0.9231)	12	0.2704	(0.9245)	12	0.2500	(1.0000)	21
	0.2500	(1.0000)	15	0.2500	(1.0000)	15			
Average	0.2602	(0.9608)		0.2602	(0.9607)		0.2619	(0.9545)	
Number	224			1,568			224		

1. three necessarily different treatments lost from one block of the starting design, $d(3;1)$,

2. missing observations removed from two blocks which correspond to three distinct treatments, $d(3;2)$,

3. two plots removed from one block, and the third from another block corresponding to one of the two treatments in the first block, $d(3;3)$,

4. three different treatments lose one replicate from three separate blocks, $d(3;4)$,

5. missing values spread over three blocks, and two replicates of one treatment are removed, $d(3;5)$, and

6. three replicates of the same treatment lost from different blocks of the initial design.

Table 3.13 shows the three affected eigenvalues of the information matrix for treatment effects for these six cases. The other non-zero eigenvalues are all equal to b, and for the cases where only one (two) treatments are involved, one (two) of the affected eigenvalues shown are also equal to b. Table 3.14 shows the average variance formulae and the relative efficiencies compared to the complete starting design for the six cases calculated using these eigenvalues. The Table also shows the number of configurations corresponding to each case.

A range of starting designs was considered for the loss of three observations. Tables 3.15 - 3.20 show the average variances and relative efficiencies for the six cases of realisable resulting designs and the corresponding number of configurations for the given starting designs. The values in these Tables illustrate the large reduction in efficiency suffered by small starting designs after the loss of three of their plots. Designs with more blocks, and consequently more replicates of each treatment, suffer a smaller loss of efficiency. However, a three treatment design with eight blocks has a relative efficiency of 77% in the worst situation when three replicates of one treatment are missing.

Best and worst configurations confirm the results of Whittinghill (1995) and Prescott and Mansson (2001b). Resulting designs with three different treatments removed from one block have the smallest average variance, and designs where three replicates of the same treatment are removed from separate blocks have the largest average variance.

Table 3.13: The three adjusted eigenvalues of the information matrix for the different configurations of three missing values, $d(3;1)$ - $d(3;6)$, in a RBD with v treatments and b blocks.

Case	Eigenvalues		
$d(3;1)$	$b-1$	$b-1$	$b-1$
$d(3;2)$	$(b-1)+\frac{\sqrt{2(v-1)(v-2)}}{(v-1)(v-2)}$	$b-1$	$(b-1)-\frac{\sqrt{2(v-1)(v-2)}}{(v-1)(v-2)}$
$d(3;3)$	b	$b-1$	$b-2$
$d(3;4)$	$(b-1)+\frac{2}{v-1}$	$(b-1)-\frac{1}{(v-1)}$	$(b-1)-\frac{1}{(v-1)}$
$d(3;5)$	b	$(b-\frac{3}{2})+\frac{\sqrt{v^2-2v+9}}{2(v-1)}$	$(b-\frac{3}{2})-\frac{\sqrt{v^2-2v+9}}{2(v-1)}$
$d(3;6)$	b	b	$b-3$

3.4 BIB designs and the loss of t observations

Whittinghill (1995) and Prescott and Mansson (2001b) showed that the properties of the designs resulting from the removal of t observations from balanced block designs depend on the specific configuration of the missing values. It is possible, see Whittinghill (1995), to identify the best and worst case scenarios for a fixed number of missing plots, but these do not completely summarise the effect of the loss of t observations on a BIB design. An experimenter would probably be more concerned about how many of the potential realisable designs were close, in a statistical sense, to the best and worst configurations of missing values. The influence of missing data on BIB designs is studied in this Section using the same approach as for RBDs, both through majorisation theory and derivation of the variances of pairwise treatment differences.

Some or all of the variances of the pairwise treatment comparisons are increased, while the others remain unchanged when t observations are lost from the starting design. The magnitude of these increases depends on the size and structure of the initial design, and the number and configuration of the t missing values. The robustness of a given BIB design could be investigated by studying the distribution of the variances of pairwise treatment differences for all possible configurations of t missing observations. This approach becomes computationally complex rapidly, because many thousands of configurations are realisable,

Table 3.14: Average variance, relative efficiency and number of resulting designs in the six cases of three missing values from a RBD.

Case	Average Variance Relative Efficiency Number of Configurations
$d(3;1)$	$\frac{(vb-v-b+4)}{b(b-1)}\frac{2}{v-1}\sigma^2$ $\frac{(v-1)(b-1)}{(vb-v-b+4)}$ $bv(v-1)(v-2)/6$
$d(3;2)$	$\frac{-v^3+3v^3b+v^3b^3-3v^3b^2+7v^2-4v^2b^3+15v^2b^2-18v^2b-12v+31vb+5vb^3-24vb^2-12b+12b^2-2b^3}{b(v^2+v^2b^2-2v^2b-3v+6vb-3vb^2-4b+2b^2)(b-1)}\frac{2}{v-1}\sigma^2$ $\frac{(v-1)(v^2+v^2b^2-2v^2b-3v+6vb-3vb^2-4b+2b^2)(b-1)}{-v^3+3v^3b+v^3b^3-3v^3b^2+7v^2-4v^2b^3+15v^2b^2-18v^2b-12v+31vb+5vb^3-24vb^2-12b+12b^2-2b^3}$ $bv(b-1)(v-1)(v-2)/2$
$d(3;3)$	$\frac{b^2(v-1)+3b(2-v)+2v-6}{b(b-1)(b-2)}\frac{2}{v-1}\sigma^2$ $\frac{(v-1)(b-1)(b-2)}{b^2(v-1)+3b(2-v)+2v-6}$ $bv(b-1)(v-1)$
$d(3;4)$	$\frac{(v^3b^2-2v^3b-3v^2b^2+10v^2b+3vb^2-14vb+v^3-7v^2-b^2+6b+12v)}{b(vb-b-v)(vb-v-b+3)}\frac{2}{v-1}\sigma^2$ $\frac{(v-1)(vb-b-v)(vb-v-b+3)}{(v^3b^2-2v^3b-3v^2b^2+10v^2b+3vb^2-14vb+v^3-7v^2-b^2+6b+12v)}$ $bv(b-1)(b-2)(v-1)(v-2)/6$
$d(3;5)$	$\frac{(v^3b^2-3v^3b-3v^2b^2+12v^2b+2v^3-10v^2+3vb^2-15vb+12v-b^2+6b)}{b(v^2b^2-3v^2b-2vb^2+6vb+2v^2-4v+b^2-3b)}\frac{2}{v-1}\sigma^2$ $\frac{(v-1)(v^2b^2-3v^2b-2vb^2+6vb+2v^2-4v+b^2-3b)}{(v^3b^2-3v^3b-3v^2b^2+12v^2b+2v^3-10v^2+3vb^2-15vb+12v-b^2+6b)}$ $vb(b-1)(b-2)(v-1)/2$
$d(3;6)$	$\frac{(vb-3v-b+6)}{b(b-3)}\frac{2}{v-1}\sigma^2$ $\frac{(v-1)(b-3)}{(vb-3v-b+6)}$ $bv(b-1)(b-2)/6$

Table 3.15: Average variance, and relative efficiencies for RBDs with three treatments and a variety of blocks when three observations become unavailable. Variances to be multiplied by σ^2.

	Cases					
Blocks	$d(3;1)$	$d(3;2)$	$d(3;3)$	$d(3;4)$	$d(3;5)$	$d(3;6)$
	0.7500	0.8333	0.8333	0.8000	0.9091	1.2500
4	0.6667	0.6000	0.6000	0.6250	0.5500	0.4000
	4	36	72	24	72	12
	0.5500	0.5833	0.5833	0.5714	0.6087	0.7000
5	0.7273	0.6857	0.6857	0.7000	0.6571	0.5714
	5	60	120	60	180	30
	0.4333	0.4500	0.4500	0.4444	0.4615	0.5000
6	0.7692	0.7407	0.7407	0.7500	0.7222	0.6667
	6	90	180	120	360	60
	0.3571	0.3667	0.3667	0.3636	0.3729	0.3929
7	0.8000	0.7792	0.7792	0.7857	0.7662	0.7273
	7	126	252	210	630	105
	0.3036	0.3095	0.3095	0.3077	0.3133	0.3250
8	0.8235	0.8077	0.8077	0.8125	0.7981	0.7692
	8	168	336	336	1,008	168

Table 3.16: Average variance, and relative efficiencies for RBDs with four treatments and a variety of blocks when three observations become unavailable. Variances to be multiplied by σ^2.

	Cases					
Blocks	$d(3;1)$	$d(3;2)$	$d(3;3)$	$d(3;4)$	$d(3;5)$	$d(3;6)$
	0.6667	0.6838	0.7222	0.6818	0.7436	1.0000
4	0.7500	0.7312	0.6923	0.7333	0.6724	0.5000
	16	144	144	96	144	16
	0.5000	0.5071	0.5222	0.5065	0.5296	0.6000
5	0.8000	0.7888	0.7660	0.7897	0.7553	0.6667
	20	240	240	240	360	40
	0.4000	0.4036	0.4111	0.4034	0.4145	0.4444
6	0.8333	0.8259	0.8108	0.8264	0.8042	0.7500
	24	360	360	480	720	80
	0.3333	0.3354	0.3397	0.3353	0.3415	0.3571
7	0.8571	0.8518	0.8411	0.8521	0.8366	0.8000
	28	504	504	840	1,260	140
	0.2857	0.2870	0.2897	0.2870	0.2908	0.3000
8	0.8750	0.8710	0.8630	0.8712	0.8598	0.8333
	32	672	672	1,344	2,016	224

Table 3.17: Average variance, and relative efficiencies for RBDs with five treatments and a variety of blocks when three observations become unavailable. Variances to be multiplied by σ^2.

	Cases					
Blocks	$d(3;1)$	$d(3;2)$	$d(3;3)$	$d(3;4)$	$d(3;5)$	$d(3;6)$
	0.6250	0.6313	0.6667	0.6315	0.6755	0.8750
4	0.8000	0.7920	0.7500	0.7918	0.7402	0.5714
	40	360	240	240	240	20
	0.4750	0.4776	0.4917	0.4778	0.4947	0.5500
5	0.8421	0.8375	0.8136	0.8372	0.8085	0.7273
	50	600	400	600	600	50
	0.3833	0.3847	0.3917	0.3848	0.3931	0.4167
6	0.8696	0.8665	0.8511	0.8663	0.8480	0.8000
	60	900	600	1,200	1,200	100
	0.3214	0.3222	0.3262	0.3223	0.3270	0.3393
7	0.8889	0.8868	0.8759	0.8866	0.8739	0.8421
	70	1,260	840	2,100	2,100	175
	0.2768	0.2773	0.2798	0.2773	0.2802	0.2875
8	0.9032	0.9016	0.8936	0.9015	0.8921	0.8696
	80	1,680	1,120	3,360	3,360	280

Table 3.18: Average variance, and relative efficiencies for RBDs with six treatments and a variety of blocks when three observations become unavailable. Variances to be multiplied by σ^2.

	Cases					
Blocks	$d(3;1)$	$d(3;2)$	$d(3;3)$	$d(3;4)$	$d(3;5)$	$d(3;6)$
	0.6000	0.6030	0.6333	0.6034	0.6378	0.8000
4	0.8333	0.8292	0.7895	0.8287	0.7839	0.6250
	80	720	360	480	360	24
	0.4600	0.4613	0.4733	0.4614	0.4749	0.5200
5	0.8696	0.8672	0.8451	0.8669	0.8423	0.7692
	100	1,200	600	1,200	900	60
	0.3733	0.3740	0.3800	0.3741	0.3807	0.4000
6	0.8929	0.8913	0.8772	0.8911	0.8755	0.8333
	120	1,800	900	2,400	1,800	120
	0.3143	0.3147	0.3181	0.3147	0.3185	0.3286
7	0.9091	0.9080	0.8982	0.9078	0.8971	0.8696
	140	2,520	1,260	4,200	3,150	210
	0.2714	0.2717	0.2738	0.2717	0.2740	0.2800
8	0.9211	0.9203	0.9130	0.9201	0.9123	0.8929
	160	3,360	1,680	6,720	5,040	336

Table 3.19: Average variance, and relative efficiencies for RBDs with seven treatments and a variety of blocks when three observations become unavailable. Variances to be multiplied by σ^2.

	Cases					
Blocks	$d(3;1)$	$d(3;2)$	$d(3;3)$	$d(3;4)$	$d(3;5)$	$d(3;6)$
	0.5833	0.5850	0.6111	0.5853	0.6137	0.7500
4	0.8571	0.8547	0.8182	0.8543	0.8147	0.6667
	140	1,260	504	840	504	28
	0.4500	0.4507	0.4611	0.4508	0.4620	0.5000
5	0.8889	0.8875	0.8675	0.8872	0.8658	0.8000
	175	2,100	840	2,100	1,260	70
	0.3667	0.3670	0.3722	0.3671	0.3726	0.3889
6	0.9091	0.9082	0.8955	0.9080	0.8945	0.8571
	210	3,150	1,260	4,200	2,520	140
	0.3095	0.3097	0.3127	0.3098	0.3129	0.3214
7	0.9231	0.9225	0.9137	0.9223	0.9130	0.8889
	245	4,410	1,764	7,350	4,410	245
	0.2679	0.2680	0.2698	0.2680	0.2700	0.2750
8	0.9333	0.9329	0.9265	0.9328	0.9260	0.9091
	280	5,880	2,352	11,760	7,056	392

Table 3.20: Average variance, and relative efficiencies for RBDs with eight treatments and a variety of blocks when three observations become unavailable. Variances to be multiplied by σ^2.

	Cases					
Blocks	$d(3;1)$	$d(3;2)$	$d(3;3)$	$d(3;4)$	$d(3;5)$	$d(3;6)$
	0.5714	0.5724	0.5952	0.5727	0.5969	0.7143
4	0.8750	0.8735	0.8400	0.8731	0.8377	0.7000
	224	2,016	672	1,344	672	32
	0.4429	0.4433	0.4524	0.4434	0.4529	0.4857
5	0.9032	0.9024	0.8842	0.9021	0.8831	0.8235
	280	3,360	1,120	3,360	1,680	80
	0.3619	0.3621	0.3667	0.3622	0.3669	0.3810
6	0.9211	0.9205	0.9091	0.9204	0.9084	0.8750
	336	5,040	1,680	6,720	3,360	160
	0.3061	0.3062	0.3088	0.3063	0.3090	0.3163
7	0.9333	0.9329	0.9251	0.9329	0.9247	0.9032
	392	7,056	2,352	11,760	5,880	280
	0.2653	0.2654	0.2670	0.2654	0.2671	0.2714
8	0.9423	0.9420	0.9363	0.9420	0.9360	0.9211
	448	9,408	3,136	18,816	9,408	448

even for relatively small initial designs and as few as three missing values. It is necessary in these circumstances to consider summary measures, e.g. the average variance and efficiencies, when comparing the robustness of different designs. It is also sensible to compute the maximum of the variances of pairwise treatment differences, because this varies more between different configurations of t missing observations. Also, if there are only a small number of comparisons that have the largest variance, consideration of the average only will not provide all relevant information.

Whittinghill (1995) used a majorisation relationship based on two ordered vectors, whose elements were the eigenvalues of the information matrix for treatment effects, to identify the best and worst configurations of two missing values and also to order all these cases. This method is valid because the sum of the eigenvalues of the information matrix, when any t observations are removed, is $v(v-1)\lambda/k - t$, and only the t smallest eigenvalues, excluding μ_v which is always zero, are affected. The particular configuration of missing values determines the alterations to these eigenvalues and the difference between the two affected eigenvalues will be shown to be an indicator of the magnitude of the increase in the variances of pairwise treatment comparisons when two observations are unavailable. In the numerical example which follows the theoretical results, the maximum of the individual variances is also computed, and this highlights that some cases have smaller average variances, but that the maximum variance of the pairwise treatment differences is larger. When more than two plots are lost, it is possible for more than two eigenvalues to decrease. Therefore the majorisation principle does not permit a complete ordering of all configurations of missing values for $t \geq 3$ in general, which will be demonstrated for the numerical example in Section 3.5. It will also be shown that the average variance can be expressed as a function of the three altered (smallest) non-zero eigenvalues of the information matrix for treatment effects, and the behaviour of this function can be studied.

3.4.1 One or two missing observations

In the case of one missing plot, the important summary measures under consideration for the resulting design $d(1)$ are the same irrespective of the position of the missing value in the starting design and the affected treatment. Assuming, without loss of generality, that the first treatment loses one replicate from a block containing the first k treatments, then

Table 3.21: Eigenvalues that are altered when one or two observations are removed from a BIB design.

(a) One missing value (t=1)		
Changed Eigenvalue	Average Variance	Relative Efficiency
$\frac{v\lambda}{k} - 1$	$\frac{2k}{v\lambda}\sigma^2 + \frac{2k^2}{v(v-1)\lambda(v\lambda-k)}\sigma^2$	$\left\{1 + \frac{k}{(v-1)(v\lambda-k)}\right\}^{-1}$

(b) Two missing values (t=2)		
Changed Eigenvalues	Average Variance	Relative Efficiency
$\frac{v\lambda}{k} - 1 \pm x$	$\frac{2k}{v\lambda}\sigma^2 + \frac{4k^2(v\lambda-k+kx^2)}{v(v-1)\lambda\{(v\lambda-k)^2-k^2x^2\}}\sigma^2$	$\left\{1 + \frac{2k(v\lambda-k+kx^2)}{(v-1)\{(v\lambda-k)^2-k^2x^2\}}\right\}^{-1}$

Case	Value of x	Configurations
1	0	$bk(k-1)/2$
2	$\frac{g}{k(k-1)},\quad g = 0, \cdots, k-1$	$(k-g)^2$
3	$\frac{k-g}{k(k-1)},\quad g = 1, \cdots, k-1$	$2g(k-g)$
4	$\frac{2k-g}{k(k-1)},\quad g = 2, \cdots, k$	$g(g-1)$
5	$\frac{k^2-2k+g}{k(k-1)},\quad g = 1, \cdots, k$	g

the information matrix for treatment effects of the resulting design $d(1)$ is given by

$$\mathbf{C}_{d(1)} = \begin{bmatrix} (\frac{\lambda v}{k} - 1) - \frac{(\lambda-1)}{k} & -\frac{(\lambda-1)}{k}\mathbf{1}'_{k-1} & -\frac{\lambda}{k}\mathbf{1}'_{v-k} \\ -\frac{(\lambda-1)}{k}\mathbf{1}_{k-1} & \frac{\lambda v}{k}\mathbf{I}_{k-1} - \{\frac{(\lambda-1)}{k} + \frac{1}{(k-1)}\}\mathbf{J}_{k-1,k-1} & -\frac{\lambda}{k}\mathbf{J}_{k-1,v-k} \\ -\frac{\lambda}{k}\mathbf{1}_{v-k} & -\frac{\lambda}{k}\mathbf{J}_{v-k,k-1} & \frac{\lambda v}{k}\mathbf{I}_{v-k} - \frac{\lambda}{k}\mathbf{J}_{v-k,v-k} \end{bmatrix} \tag{3.19}$$

The eigenvalue that is changed for this information matrix is given in Table 3.21. To solve these reduced normal equations and derive formulae for variances of the individual pairwise treatment comparisons, a generalised inverse needs to be chosen. A sensible choice will simplify the information matrix and lead to a simplified generalised inverse. Add $\frac{\lambda}{k}\mathbf{J}_{v,v}$ to $\mathbf{C}_{d(1)}$ and invert the resulting non-singular matrix to generate a particular

Table 3.22: Variances of the individual treatment comparisons when one observation is removed from a BIB design. It is assumed that the first k treatments occurred in the affected block.

One missing values (t=1)			
Treatment i	Treatment j	Variance	Number of Comparisons
1	$2,\cdots,k$	$\frac{k(2\lambda vk-2\lambda v+2k-k^2)}{\lambda v(\lambda vk-\lambda v-k^2+k)}\sigma^2$	$k-1$
1	$k+1,\cdots,v$	$\frac{k(2\lambda v-k-1)}{\lambda v(\lambda v-k)}\sigma^2$	$v-k$
$2,\cdots,k$	$2,\cdots,k$	$\frac{2k}{\lambda v}\sigma^2$	$(k-1)(k-2)/2$
$2,\cdots,k$	$k+1,\cdots,v$	$\frac{k(2\lambda vk-2\lambda v-2k^2+2k+1)}{\lambda v(\lambda vk-\lambda v-k^2+k)}\sigma^2$	$(k-1)(v-k)$
$k+1,\cdots,v$	$k+1,\cdots,v$	$\frac{2k}{\lambda v}\sigma^2$	$(v-k)(v-k-1)/2$

generalised inverse, denoted by $\mathbf{\Omega}$. The form of this matrix is

$$\mathbf{\Omega} = \begin{bmatrix} \frac{k(\lambda v-1)}{\lambda v(\lambda v-k)} & -\frac{k}{\lambda v(\lambda v-k)}\mathbf{1}'_{k-1} & \mathbf{0}'_{v-k} \\ -\frac{k}{\lambda v(\lambda v-k)}\mathbf{1}_{k-1} & \frac{k}{\lambda v}\mathbf{I}_{k-1} + \frac{k}{\lambda v(\lambda vk-\lambda v-k^2+k)}\mathbf{J}_{k-1,k-1} & \mathbf{0}_{k-1,v-k} \\ \mathbf{0}_{v-k} & \mathbf{0}_{v-k,k-1} & \frac{k}{\lambda v}\mathbf{I}_{v-k} \end{bmatrix} \tag{3.20}$$

Table 3.21 shows the formulae for the average variance and relative efficiency of these resulting designs. The average variance for one missing value is expressed as the average variance for the complete design and the increase due to losing data. Variances of the individual pairwise differences are shown in Table 3.22. There are five comparisons created by the loss of one plot from the starting design. The treatment that loses a replicate can be compared with another treatment from the affected block or one that does not occur in this block. The other $k-1$ treatments in the block can be compared against each other or against a treatment not in the block. The last comparison is between two treatments that do not occur in the block that loses a plot.

When two observations are unavailable, the situation is more complex because there are five types of configuration, all with different eigenvalues and average variances, that need to be considered individually. Four of these configurations have sub-cases based on g, the number of treatments that are common to the pair of blocks with the missing values. The cases, which are also given in Whittinghill (1995, page 28), are

1. two observations corresponding to different treatments in the same block of the starting design,

2. different treatments in two separate blocks, where neither of the affected treatments are common to the pair of blocks,

3. different treatments in different blocks, with one of the two treatments common to the blocks,

4. both treatments are different and are both common to the two different blocks, and

5. two replicates of one treatment are lost from different blocks.

The form of the information matrices for each of the five cases is complicated, but the two eigenvalues affected by the removal of two plots in any configuration can be expressed as

$$\mu_{v-2} = \frac{v\lambda}{k} - 1 + x \qquad \text{and} \qquad \mu_{v-1} = \frac{v\lambda}{k} - 1 - x \tag{3.21}$$

The value of this adjustment x, and the average variances and relative efficiencies of the different resulting designs are given in Table 3.21. Derivation of algebraic generalised inverses of the information matrices for treatment effects in the different cases is a non-trivial task, and the maximum of the pairwise variances will therefore only be calculated numerically for an example of a BIB design with 8 treatments in 14 blocks of 4 plots in section 3.5. Configurations corresponding to Case 1 suffer the smallest loss of efficiency if the average variance is compared to the complete design, and the sub-case of Case 2 where $g = 0$ also has the same reduction in efficiency, because $x = 0$ in this case and the two affected eigenvalues are the same as for Case 1. The resulting design with the lowest efficiency and the largest average variance occurs when two replicates of one treatment are removed from different blocks, which is denoted Case 5.

The results in this Section confirm the Theorem of Whittinghill (1995), where the different cases of two missing values in a BIB design were ordered based on the vector of eigenvalues of their information matrices. However, the real consequences of losing two observations are illustrated by considering the distributions of summary measures, e.g. the average or maximum variance of pairwise comparisons, instead of concentrating on the best and worst cases. In addition, the range of variances of the pairwise treatment

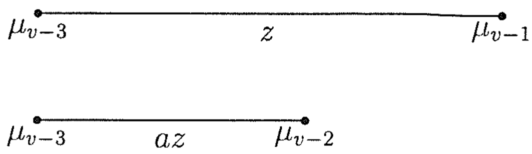

Figure 3.1: Graphical representation of a and z in terms of the three altered eigenvalues of the information matrix for treatment effects when three observations are lost from a BIB design.

differences should also be studied with a view to selecting designs where variances do not increase beyond a specified threshold for any configuration of one or two missing values. It will be shown that the average variance of pairwise treatment comparisons does not vary as substantially as the maximum of these variances for the different configurations of t missing values.

3.4.2 Three observations missing from the starting design

The situation when three observations are lost from the starting design is complicated because there are 25 potential cases that need to be considered separately. These cover the many different ways that three missing values may be configured for different treatments in either one, two, or three blocks. Sub-cases correspond to the number of treatments that are common to the pairs and triples of blocks containing the missing values. The properties of the resulting designs are also different, so only the average variance will be considered in general. For any of the resulting designs, denoted by $d(3)$, there are one, two, or three eigenvalues of the information matrix for treatment effects altered by the loss of three observations.

Let $\mu = (\mu_1, \mu_2, \cdots, \mu_{v-1}, \mu_v = 0)$ be an ordered vector of the eigenvalues of the information matrix for one of the resulting designs, where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_v$. Define z as the difference between two of the eigenvalues, μ_{v-3} and μ_{v-1}, and let $az (0 \leq a \leq 1)$ be the difference between μ_{v-3} and μ_{v-2}. This is shown graphically in Figure 3.1. Let $V = v\lambda/k - 1$. The eigenvalues of the information matrix for any design $d(3)$ can now be expressed as

- $\mu_1 = \cdots = \mu_{v-4} = \frac{v\lambda}{k}$

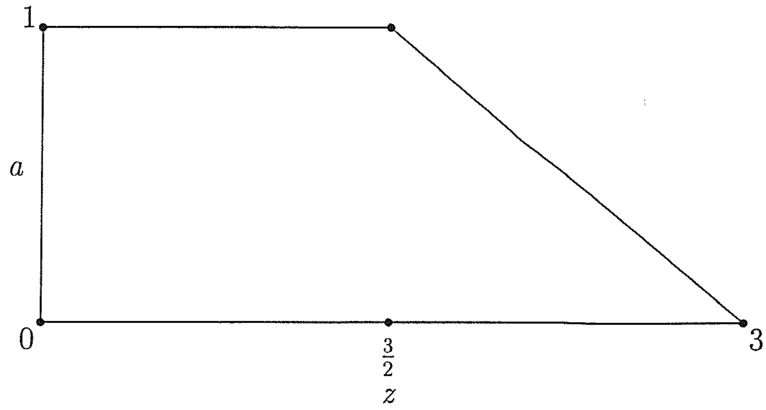

Figure 3.2: Range of values for variables a and z.

- $\mu_{v-3} = (\frac{v\lambda}{k} - 1) + \frac{(1+a)z}{3} = V + \frac{(1+a)z}{3}$
- $\mu_{v-2} = (\frac{v\lambda}{k} - 1) + \frac{(1-2a)z}{3} = V + \frac{(1-2a)z}{3}$
- $\mu_{v-1} = (\frac{v\lambda}{k} - 1) + \frac{(a-2)z}{3} = V + \frac{(a-2)z}{3}$
- $\mu_v = 0$

Note that $\mu_{v-3} \leq \lambda v/k$ and $\mu_{v-1} \geq 3$ for any configuration of three missing observations, and the three affected eigenvalues sum to $3(v\lambda/k - 1)$ for any resulting design. The variables a and z are non-negative and depend on the configuration of the missing values. The range of permissible values for a and z is shown by the region defined in Figure 3.2.

The best and worst cases can be identified by examining alterations to these three eigenvalues. A strict ordering of all possible resulting designs using the average variance of pairwise treatment comparisons, or their relative efficiencies, does not necessarily follow. This is because the majorisation principle of Whittinghill (1995) does not apply in every situation. This will be illustrated in the numerical example in Section 3.5.

When three observations become unavailable, it is possible to express the average variance of pairwise treatment differences using the alternative general expressions for the eigenvalues, with the variables a and z, as

$$\frac{2}{v-1}\left\{\frac{k(v-4)}{v\lambda} + \frac{1}{V + \frac{(1+a)z}{3}} + \frac{1}{V + \frac{(1-2a)z}{3}} + \frac{1}{V + \frac{(a-2)z}{3}}\right\}\sigma^2 \tag{3.22}$$

for any of the resulting designs, $d(3)$. The behaviour of this average variance function can be simplified by ignoring the additive and multiplicative constants, and the average variance is considered as a function of the two variables a and z given by

$$AV = \left\{ \frac{1}{V + \frac{(1+a)z}{3}} + \frac{1}{V + \frac{(1-2a)z}{3}} + \frac{1}{V + \frac{(a-2)z}{3}} \right\} \tag{3.23}$$

where $V = \lambda v/k - 1$ as defined earlier. This function can be studied to locate the values of a and z which correspond to the best and worst cases, in terms of average variance of pairwise treatment comparisons, of three missing observations. The partial derivative of this new function AV with respect to z is

$$\begin{aligned} \frac{\partial AV}{\partial z} &= \left\{ \frac{-\frac{(1+a)}{3}}{\{V + \frac{(1+a)z}{3}\}^2} + \frac{-\frac{(1-2a)}{3}}{\{V + \frac{(1-2a)z}{3}\}^2} + \frac{\frac{(2-a)}{3}}{\{V + \frac{(a-2)z}{3}\}^2} \right\} \\ &\geq \left\{ \frac{-\frac{(1+a)}{3}}{\{V + \frac{(1-2a)z}{3}\}^2} + \frac{-\frac{(1-2a)}{3}}{\{V + \frac{(1-2a)z}{3}\}^2} + \frac{\frac{(2-a)}{3}}{\{V + \frac{(1-2a)z}{3}\}^2} \right\} = 0 \end{aligned} \tag{3.24}$$

Thus for any fixed a, AV is an increasing function of z. The maximum value of AV, and also the maximum average variance, occurs when z reaches its largest potential value, i.e. $z = 3$ and $a = 0$. At this point of the region in Figure 3.2, the eigenvalues are $\mu_{v-3} = \mu_{v-2} = \lambda v/k$ and $\mu_{v-1} = \lambda v/k - 3$. This situation corresponds to the loss of three replicates of the same treatment from three identical blocks, which is not possible for a single replicate BIB design. Consider the right hand boundary in Figure 3.2, where $z = 3/(1+a)$. On this line, the three affected eigenvalues are given by

$$\mu_{v-3} = \frac{v\lambda}{k} \qquad \mu_{v-2} = \frac{v\lambda}{k} + z - 3 \qquad \mu_{v-1} = \frac{v\lambda}{k} - z$$

and for these configuration of missing observations, the function AV becomes

$$\frac{1}{\frac{v\lambda}{k}} + \frac{1}{\frac{v\lambda}{k} + z - 3} + \frac{1}{\frac{v\lambda}{k} - z} \tag{3.25}$$

The partial derivative of the function AV along the boundary line with respect to z is

$$\frac{\partial AV}{\partial z} = -\frac{1}{(\frac{v\lambda}{k} + z - 3)^2} + \frac{1}{(\frac{v\lambda}{k} - z)^2} \geq 0 \tag{3.26}$$

AV is an increasing function of z along the right hand boundary line $z = 3/(1+a)$. The function AV, and hence the average variance function, takes its maximum value when $a = 0$ and $z = 3$, with adjusted eigenvalues μ_{v-3}, μ_{v-2}, and μ_{v-1} stated earlier. If the function AV is partially differentiated with respect to a, then

$$\begin{aligned}\frac{\partial AV}{\partial a} &= \left\{\frac{-\frac{z}{3}}{\{V+\frac{(1+a)z}{3}\}^2} + \frac{\frac{2z}{3}}{\{V+\frac{(1-2a)z}{3}\}^2} + \frac{-\frac{z}{3}}{\{V+\frac{(a-2)z}{3}\}^2}\right\} \\ &= \frac{z}{3}\left\{\frac{-1}{\mu_{v-3}^2} + \frac{2}{\mu_{v-2}^2} + \frac{-1}{\mu_{v-1}^2}\right\} \\ &= \frac{z}{3}\left\{\left(\frac{1}{\mu_{v-2}^2} - \frac{1}{\mu_{v-3}^2}\right) - \left(\frac{1}{\mu_{v-1}^2} - \frac{1}{\mu_{v-2}^2}\right)\right\} \end{aligned} \tag{3.27}$$

This partial derivative is equal to zero when $z = 0$, or for some value of a in $[0, 1]$ for $z \neq 0$. When $z \neq 0$ and a is increasing, the function AV decreases to a minimum before increasing as a approaches its maximum value for the given value of z. The best situation, which corresponds to the smallest values of the average variance function, occurs when $z = 0$, where the three eigenvalues are all equal to $v\lambda/k - 1$. This is the case if three observations are lost from the same block of the initial design.

Certain combinations of a and z are possible for a particular BIB design. In general, the larger the distance between the extreme affected eigenvalues, the greater the average variance and consequently lower efficiency of the resulting design when compared to the starting design. In the following Section, an example is used to illustrate the reduction in efficiencies caused by losing up to three observations from a BIB design with eight treatments in blocks of four plots. Loss of efficiency is examined, and the efficiencies and number of configurations corresponding to each case for $t = 2$ and 3 are given.

3.5 Detailed BIB design Example

Consider a BIB design with the layout shown in Table 3.23. There are 8 treatments in the starting design, which are arranged in 14 blocks of 4 treatments per block. Each treatment has 7 replicates, and every pair of treatments occurs together in three blocks of the starting design. When the design is complete, the average variance of pairwise treatment comparisons is $\sigma^2/3$, and its efficiency relative to a completely randomised design with 8 treatments and 7 replicates is 85.71%. The seven non-zero eigenvalues of

Table 3.23: Layout of the BIB design used for illustrative purposes in §3.5.

Block	Treatments				Block	Treatments			
1	1	2	3	5	8	8	2	3	5
2	2	3	4	6	9	8	3	4	6
3	3	4	5	7	10	8	4	5	7
4	4	5	6	1	11	8	5	6	1
5	5	6	7	2	12	8	6	7	2
6	6	7	1	3	13	8	7	1	3
7	7	1	2	4	14	8	1	2	4

the information matrix for treatment effects are all equal to 6.

3.5.1 One missing value

When one observation is missing, one eigenvalue of the information matrix for treatment effects is reduced from 6 to 5, irrespective of which particular observation is removed from the starting design. The average variance of pairwise treatment differences for the resulting designs is computed using the formula in Table 3.21 to be

$$\frac{2\times 4}{8\times 3}\sigma^2 + \frac{2\times 4\times 4}{8\times 7\times 3\times(8\times 3-4)}\sigma^2 = \left(\frac{1}{3}+\frac{1}{105}\right)\sigma^2 = \frac{12}{35}\sigma^2$$

and the relative efficiency is 97.22%. The reduction in efficiency is less than 3% after the loss of one of the 56 units in the starting design, which is not a serious reduction. Variances of the individual pairwise treatment comparisons can be calculated from the formulae in Table 3.22 to determine the maximum of these variances, which is equal to $0.378\sigma^2$ for any of the resulting designs, and there are 3 comparisons out of the total 28 with this value.

3.5.2 Two missing values

In this BIB design it is not possible to realise all of the sub-cases of Whittinghill (1995) for two missing values. The possible situations for this design are listed in Table 3.24 with the average variance for each case and the corresponding efficiencies relative to the initial design. The number of configurations for each case are also shown.

The Table also shows the two eigenvalues that are affected by the missing data, and

Table 3.24: Eigenvalues of the information matrix for all possible configurations of two missing observations. Variances are multiplied by σ^2.

Case		Eigenvalues		x	Average variance	Relative efficiency	Number of designs	Maximum variance
1	g=0	5	5	0	0.352381	0.94595	84	0.4000
2(b)	g=1	5.0833	4.9167	1/12	0.352413	0.94586	189	0.3825
2(b)	g=2	5.1667	4.8333	1/6	0.352508	0.94560	252	0.3817
2(c)	g=3	5.25	4.75	1/4	0.352667	0.94518	7	0.3818
3(a)	g=3	5.0833	4.9167	1/12	0.352413	0.94586	42	0.4039
3(b)	g=2	5.1667	4.8333	1/6	0.352508	0.94560	504	0.4051
3(c)	g=1	5.25	4.75	1/4	0.352667	0.94518	126	0.4063
4(b)	g=3	5.4167	4.5833	5/12	0.353178	0.94381	42	0.4303
4(c)	g=2	5.5	4.5	1/2	0.353535	0.94286	126	0.4321
5(b)	g=1	5.75	4.25	3/4	0.355012	0.93894	21	0.4146
5(b)	g=2	5.8333	4.1667	5/6	0.355646	0.93726	126	0.4400
5(c)	g=3	5.9167	4.0833	11/12	0.356356	0.93539	21	0.4422

the values of x for the different configurations of two missing observations. The variable x is used in the majorisation approach to rank the different cases based on their vectors of eigenvalues. The results in Table 3.24 confirm the ordering given by Whittinghill (1995, page 29). The *best* configuration, where two observations are lost from the same block, has a relative efficiency of 94.59%, so the minimum reduction in efficiency is approximately 5%. The worst case is 5(c), where the efficiency is reduced by over 6%, when two replicates of one treatment become unavailable, e.g. treatment 2 from blocks 1 and 8 where there are 3 treatments common to the pair of affected blocks. There are, however, only 21 configurations corresponding to this situation.

There are 1,540 possible designs resulting from the loss of two plots, and frequencies of all the cases and their sub-cases are given in Table 3.24. The maximum variance of the pairwise treatment comparisons is also shown in the Table for every situation. It may be seen that the maximum of these occurs for Case 5(c), the configuration with the largest average variance. Whittinghill (1995)'s majorisation approach provides an exact ordering of the cases by their relative efficiencies for this number of missing observations. The ordering is illustrated in Figure 3.3, where the efficiency of the resulting designs is plotted against the distance between the two eigenvalues. The reduction in efficiency increases as the distance between these eigenvalues increases. The table shows that although the differences between average variances occurs in the third decimal place, the maxima of

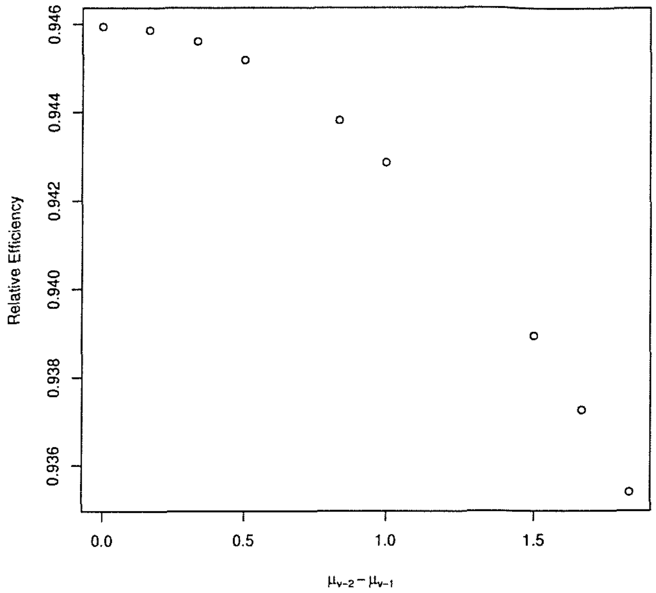

Figure 3.3: Efficiencies relative to the complete BIB design for all ways of losing two observations plotted against the difference between the eigenvalues that change.

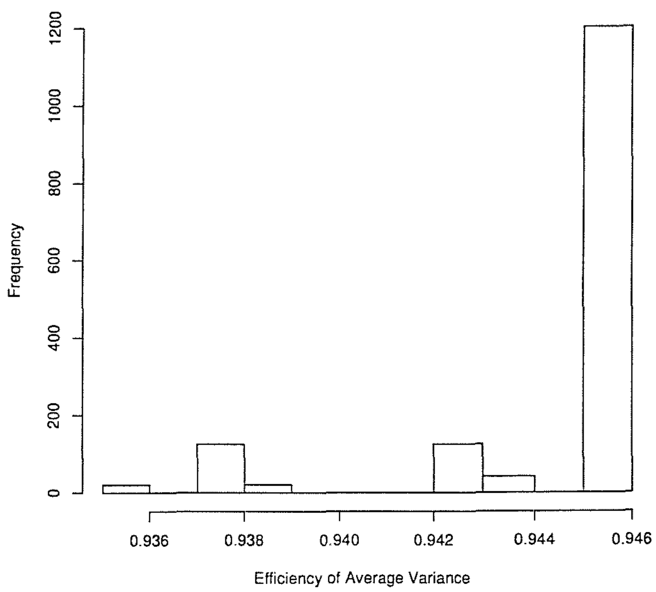

Figure 3.4: Histogram of efficiencies relative to the complete BIB design for all ways of losing two observations.

the variances differ substantially between the five cases and their sub-cases. It can also be seen that the ranking based on the vector of eigenvalues does not concur with ranking the cases based on the maximum variance.

A histogram of the relative efficiencies is shown in Figure 3.4. The majority of the configurations give rise to resulting designs with efficiencies that are close to the smallest reduction in efficiency for two missing values. There is only a small probability of obtaining the most severely affected resulting designs.

3.5.3 Three missing values

There are 27,720 ways that three observations can be lost from the BIB design. There are in general 25 specific cases, although these are not all available for this design. Best and worst cases can be identified for this example, but a complete ordering of all the cases using majorisation is not possible.

Table 3.25 illustrates the different configurations involving missing observations in one or two blocks of the initial design. The number of treatments common to the two blocks, g, is used to identify the different sub-cases. A particular example of each configuration is given with the average variance, relative efficiency, number of similar configurations, and the maximum of the variances of pairwise treatment comparisons.

The best configuration occurs when three observations, which necessarily correspond to different treatments, are lost from the same block of the starting design. Three of the non-zero eigenvalues of the information matrix for treatment effects are equal to 5, and the relative efficiency of this configuration of missing values is 92.11%. The worst of these situations covered by Table 3.25 corresponds to the removal of two replicates of one treatment and one of a different treatment, where three treatments are common to the pair of blocks. In this case, both of the affected treatments occur once in both blocks. The relative efficiency is reduced to 91.09% for this resulting design, and the maximum variance of pairwise comparisons increases to $0.4663\sigma^2$.

When the three missing observations occur in different blocks, various sub-cases are possible, based on three values of g, the number of treatments that are common to the three pairs of blocks. Examples of the three blocks in which the missing observations occur, the maximum average variance, mean average variance and efficiencies relative to a completely randomised design and the complete BIB design, are also shown in Table

Table 3.25: Average and maximum variances of pairwise treatment comparisons, and relative efficiencies when three observations are lost from one or two blocks of the starting design. Variances are to be multiplied by σ^2.

Cases and Sub-cases		Missing Values (treatment,block)			Average variance	Relative efficiency	Number of configurations	Maximum Variance
One Block		(1,1)	(2,1)	(3,1)	0.361905	0.92105	56	0.4000
Two Blocks								
g=1	(a)	(1,1)	(2,1)	(8,9)	0.362000	0.92081	378	0.4000
	(b)	(1,1)	(2,1)	(3,9)	0.362768	0.91886	126	0.4138
	(c)	(1,1)	(3,1)	(8,9)	0.362191	0.92032	378	0.4063
	(d)	(1,1)	(3,1)	(3,9)	0.364535	0.91441	126	0.4368
g=2	(a)	(1,1)	(5,1)	(4,2)	0.362287	0.92008	252	0.4000
	(b)	(1,1)	(2,1)	(4,2)	0.362096	0.92057	1,008	0.4051
	(c)	(1,1)	(5,1)	(2,2)	0.362287	0.92008	252	0.4119
	(d)	(2,1)	(3,1)	(4,2)	0.362287	0.92008	252	0.4052
	(e)	(1,1)	(2,1)	(3,2)	0.363645	0.91664	504	0.4404
	(f)	(1,1)	(2,1)	(2,2)	0.365237	0.91265	504	0.4465
	(g)	(2,1)	(3,1)	(2,2)	0.365439	0.91214	252	0.4660
g=3	(a)	(1,1)	(2,1)	(8,8)	0.362191	0.92032	42	0.4040
	(b)	(1,1)	(2,1)	(3,8)	0.362962	0.91837	84	0.4376
	(c)	(1,1)	(2,8)	(3,8)	0.362000	0.92081	42	0.4039
	(d)	(2,1)	(3,1)	(5,8)	0.364336	0.91490	42	0.4408
	(e)	(1,1)	(2,1)	(2,8)	0.366152	0.91037	42	0.4497
	(f)	(2,1)	(3,1)	(2,8)	0.365948	0.91088	84	0.4663

Table 3.26: Average variances of pairwise treatment comparisons when three observations are lost from different blocks. All variances are multiplied by σ^2.

Three blocks	Blocks	Max average variance	Mean average variance	Relative efficiency	Number of configurations
g=1,1,2	1,9,10	0.36546	0.36313	0.9179	2,688
g=1,2,2	1,9,11	0.36551	0.36349	0.9170	8,064
g=1,2,3	1,8,9	0.37421	0.36400	0.9158	2,688
g=2,2,2	1,2,3	0.37415	0.36393	0.9159	7,168
g=2,2,3	1,8,11	0.37516	0.36436	0.9148	1,344
g=2,2,3	1,8,2	0.37516	0.36446	0.9146	1,344
Total number of ways of losing three observations					27,720
Overall average variance					0.3637
Overall average efficiency relative to CRD					0.7857
Overall average efficiency relative to BIB design					0.9166
Maximum variance of pairwise treatment difference					0.5367

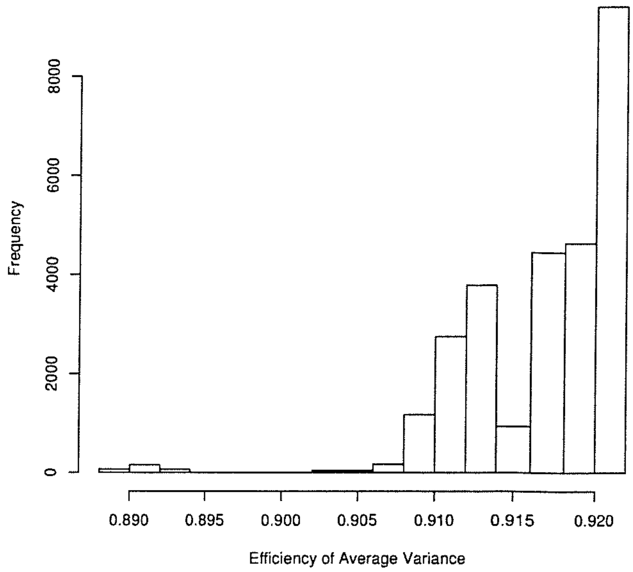

Figure 3.5: Efficiencies relative to the complete BIB design for all ways of losing three observations.

3.26.

One of the worst configurations is the loss of three replicates of one treatment, where the missing values occur in the three blocks in which the affected treatment is paired with one of the other seven treatments. For example, if treatment 1 is removed from blocks 1, 7, and 14, then there is no direct comparison of treatments 1 and 2. The affected eigenvalues in this configuration are reduced from 6 to 5.9167, 5.0862, and 3.2772 respectively, and the relative efficiency compared to the starting design is 88.85%.

The majorisation approach of Whittinghill (1995) cannot be extended to provide a complete ordering of the cases of three missing observations. This may be demonstrated by comparing the loss of treatments 5, 2, and 7 from blocks 1, 2, and 3, with the loss of treatments 2, 6, and 1 from blocks 5, 6, and 14. In the first situation, the affected eigenvalues are 5.1667, 5.1667, and 4.6667, compared with 5.4466, 4.8333, and 4.7201 for the second. Neither of the two vectors of eigenvalues majorise each other, and the average variances for the two cases need to be calculated for a comparison. The efficiencies are 92.01% and 91.93% respectively, although the first situation has the smallest affected eigenvalue.

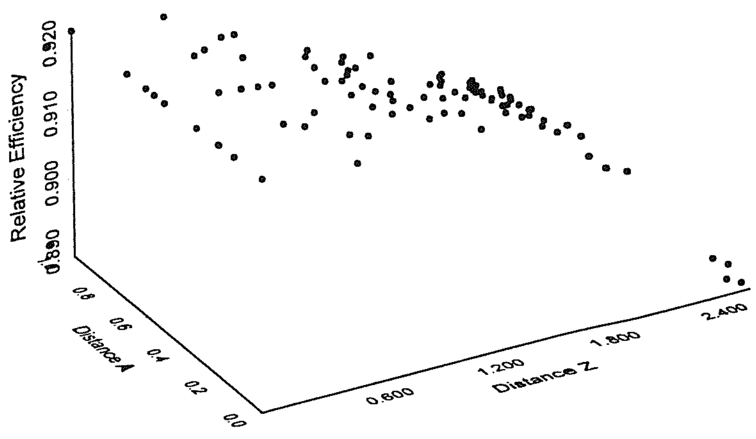

Figure 3.6: Efficiencies relative to the complete BIB design for all ways of losing three observations, plotted against a and z.

Figure 3.5 shows the distribution of relative efficiencies for the 27,720 configurations of three missing observations for the BIB design. The histogram shows that only a small number of configurations give rise to efficiencies as low as 89%, while the majority of the realisable designs have efficiencies between 90% and 92%.

Figure 3.6 shows the efficiencies plotted against the variables a and z, which also shows the small cluster of resulting designs with the largest loss of efficiency. These badly affected configurations correspond to large values of z, that is, situations where the distance between μ_{v-3} and μ_{v-1} is as large as possible.

3.6 Discussion

The effect of missing observations on a block design has been examined extensively in this Chapter. Randomised block and BIB designs have been considered to derive theoretical results for two patterns of missing data. The loss of complete blocks of observations was investigated initially and the effect of drop-out was measure by the average and maximum variance of the pairwise treatment differences. This extends the work of other authors who concentrated on the efficiency of the average variance as a measure of robustness to missing data.

The form of the information matrix for treatment effects has been derived and is given in this Chapter when a single block of observations becomes unavailable from a RBD or a BIB design. In the case of a RBD, the variances of pairwise treatment differences all increase from $2\sigma^2/r$ to $2\sigma^2/(r-1)$ which is equivalent to an efficiency of $(r-1)/r$, which is a serious loss of efficiency when the starting design has a small number of replicates of each treatment. When a single block is lost from a BIB design there are three types of treatment comparisons.

The other situation that has been studied is the loss of observations scattered throughout RBDs and BIB designs. Robustness of these designs is investigated by considering the adjustments to the information matrix for treatment effects caused by a variety of configurations of missing values. Previous work concentrated on the overall efficiency of the resulting design, which was defined using the non-zero eigenvalues of the information matrix. In this Chapter it has been shown that the variances of the pairwise comparisons can be derived algebraically with a suitable choice of generalised inverse. The maximum of the variances can also be found for various configurations of missing observations which provides more information on the effect of the loss of data, because the average reduces the impact of a small number of seriously affected treatment comparisons.

In the last Section of the Chapter, the aim was to extend the work of Whittinghill (1995) to consider the loss of three observations from a BIB design. When $t = 1$ or 2, it is possible to use majorisation to rank the different cases based on the average variance and to produce a complete ordering of the efficiencies based on the vectors of eigenvalues of the information matrices for treatment effects. The approach cannot be extended to the distinct cases of three or more missing observations. This was illustrated using two cases in the detailed example. In these cases it is still possible to identify the best and worst theoretical cases to provide the range of potential efficiencies, but a complete ordering of all cases cannot be made using the vector of eigenvalues. Efficiencies have to be computed for all of the configurations and this is a computationally expensive task even for only three missing values.

The example identified the best cases as those where observations are removed from the same block. Whittinghill (1995) showed that the worst configuration corresponds to the loss of the same treatment from identical blocks, which is not available for a single replicate of a BIB design, but it is possible for RBDs because of their structure. The most severely affected configurations in the BIB design example had more treatments common to the affected blocks than the other cases. Based on these results, it would appear prudent to select BIB designs where there are as few treatments common to blocks as possible to reduce the chance of substantial loss of efficiency due to only a small number of missing observations. This conclusion agrees with the results of Bhaumik and Whittinghill (1991). It is also sensible for pairs of treatments to occur together as often as possible, or there is the possibility that no direct comparisons between particular treatments can be made. This situation is one of the most severe for a BIB design.

Chapter 4

The effect of missing data on Latin Square designs

Although the effects of the loss of complete rows or columns, or of complete treatments from Latin square designs were investigated by Yates and Hale (1939), the majority of recent research has been concerned primarily with block designs, see previous Chapters for a detailed discussion. Low et al. (1999) investigated cross-over designs constructed using Latin squares to determine the effect of dropout on the variances of treatment comparisons. These designs had extra complications introduced by a carry-over term in the model.

The analysis of row-column designs is more complicated than block designs because there are two orthogonal blocking factors that need to be incorporated into the model, and adjusted for during the estimation of treatment effects. The full normal equations given by $(\mathbf{X}'\mathbf{X})\hat{\boldsymbol{\alpha}} = \mathbf{X}'\mathbf{Y}$ can be simplified by eliminating the overall mean μ, and the row and column parameters which are denoted by $\boldsymbol{\rho}$ and $\boldsymbol{\gamma}$ respectively. The reduced normal equations are general expressions that can also be used when there are missing values in the row-column design. The information matrix for treatment effects, after the elimination of the row and column parameters, for a general row-column design is given by

$$\mathbf{C} = \mathbf{t}^{\delta} - \mathrm{N}_1\mathbf{r}^{-\delta}\mathrm{N}_1' - (\mathrm{N}_2 - \mathrm{N}_1\mathbf{r}^{-\delta}\mathrm{N}_3')(\mathbf{c}^{\delta} - \mathrm{N}_3\mathbf{r}^{-\delta}\mathrm{N}_3')^{-}(\mathrm{N}_2' - \mathrm{N}_3\mathbf{r}^{-\delta}\mathrm{N}_1') \qquad (4.1)$$

where $\mathbf{t}^{\delta}$ is a diagonal matrix with elements equal to the number of replicates of each treatment in the design, $\mathbf{r}^{-\delta}$ is a diagonal matrix whose elements are the reciprocals of the number of plots in each row of the design, and $\mathbf{c}^{\delta}$ is also a diagonal matrix with elements equal to the number of plots in the columns of the design. The matrices $\mathbf{N}_1$ and $\mathbf{N}_2$ are incidence matrices for the row and column block designs, and $\mathbf{A}^-$ is a generalised inverse of the matrix $\mathbf{A}$, with the property $\mathbf{AA}^-\mathbf{A} = \mathbf{A}$. The matrices $\mathbf{N}_1$ and $\mathbf{N}_2$ are binary for the designs in this Chapter, because they are based on single Latin squares, where all treatments occur exactly once in every row and column of the starting design. $\mathbf{N}_3$ is a $(c \times r)$ matrix, where the *(i,j)th* element is equal to one if an observation is available for the plot in the *ith* column and *jth* row of the design, and zero otherwise.

When the row-column design is complete, the information matrix for treatment effects simplifies to

$$\mathbf{C} = \mathbf{t}^{\delta} - \mathbf{N}_1\mathbf{r}^{-\delta}\mathbf{N}_1' - \mathbf{N}_2\mathbf{c}^{-\delta}\mathbf{N}_2' + \frac{1}{n}\mathbf{tt}' \tag{4.2}$$

There are two conditions that need to be satisfied if Equation (4.2) is correct for the reduced normal equations. The necessary relationships are

$$\mathbf{tr}'/n = \mathbf{N}_2\mathbf{c}^{-\delta}\mathbf{N}_3 \qquad \text{and} \qquad \mathbf{tc}'/n = \mathbf{N}_1\mathbf{r}^{-\delta}\mathbf{N}_3' \tag{4.3}$$

These two identities are satisfied for a complete row-column design, and also for some of the restrictive patterns of missing observations. It will be shown that when complete treatments, or complete rows or columns become unavailable, these two conditions are satisfied and Equation (4.2) is used to derive the information matrix for treatment effects.

The orthogonality of a row-column design is lost when there are missing data, and Equation (4.1) is appropriate for deriving the reduced normal equations. To compare the different designs resulting from the loss of data, the variances of pairwise treatment differences need to be calculated. These can be found theoretically for a Latin square design of side r, by the generalised inverse method used in Chapter 3 with block designs. Estimates of the treatment differences are invariant to the choice of generalised inverse, and a convenient generalised inverse, which was also used in the block design Chapter, is given by

$$\mathit{\Omega} = (\mathbf{C} + a\mathbf{J}_{r,r})^{-1} \tag{4.4}$$

where a is a suitably chosen non-zero constant. Variances of the estimates of treatment differences are the diagonal elements of $\boldsymbol{\Gamma}\boldsymbol{\Omega}\boldsymbol{\Gamma}'$, where $\boldsymbol{\Gamma}$ is a matrix of all elementary contrasts between the r treatments. There are $r(r-1)/2$ pairwise treatment comparisons that have variances to be computed for a given resulting design. The increased variances depend on the actual configuration of missing observations.

The next two sections cover the loss of all plots related to one or more treatments, and the removal of every plot in one row or column of a Latin square design. In these two situations, the configuration of missing values ensures that Equation (4.2) can be used to derive the information matrix for treatment effects. Consideration of these restrictive situations of missing data is followed by an investigation into the effect of missing data scattered throughout the Latin square. Theoretical results are enumerated for a range of different sized squares in the various sections to quantify the impact of the loss of data on the starting design, which is measured using relative efficiencies of the variances of pairwise treatment comparisons.

4.1 Loss of a complete treatment from the starting design

Assume that all of the observations on one of the treatments in a design based on a Latin square of side r, become unavailable. The regular structure of Latin squares, where every symbol (treatment) occurs exactly once in every row and column, ensures that the form of the information matrix for treatment effects is the same, regardless of which particular treatment becomes unavailable in the starting design. Note that the remaining $r-1$ treatments have r replicates, and that there are exactly $r-1$ plots in every row and column of the resulting design. Denoting all the resulting designs d_1 and, using Equation (4.2), it can be shown that the information matrix for treatment effects is now given by

$$\begin{aligned}\mathbf{C}_{d_1} &= r\mathbf{I}_{r-1} - \frac{r}{r-1}\mathbf{J}_{r-1,r-1} - \frac{r}{r-1}\mathbf{J}_{r-1,r-1} + \frac{r^2}{r(r-1)}\mathbf{J}_{r-1,r-1} \\ &= r\mathbf{I}_{r-1} - \frac{r}{r-1}\mathbf{J}_{r-1,r-1} \end{aligned} \tag{4.5}$$

This formula is correct for this configuration of missing values because there are $r^2 - r$

observations in the resulting design and

$$\frac{\mathbf{tr}'}{n} = \frac{\mathbf{tc}'}{n} = \frac{1}{r(r-1)} \begin{bmatrix} 0 \\ r\mathbf{1}_{r-1} \end{bmatrix} \begin{bmatrix} (r-1) & (r-1)\mathbf{1}'_{r-1} \end{bmatrix} = \begin{bmatrix} \mathbf{0}'_r \\ \mathbf{J}_{r-1,r} \end{bmatrix} \tag{4.6}$$

and

$$\mathbf{N}_1\mathbf{r}^{-\delta}\mathbf{N}'_3 = \mathbf{N}_2\mathbf{c}^{-\delta}\mathbf{N}_3 = \begin{bmatrix} \mathbf{0}'_r \\ \mathbf{J}_{r-1,r} \end{bmatrix} \frac{1}{r-1}\mathbf{I}_r(\mathbf{J}_{r,r} - \mathbf{I}_r) = \begin{bmatrix} \mathbf{0}'_r \\ \mathbf{J}_{r-1,r} \end{bmatrix} \tag{4.7}$$

All complete row-column designs based on Latin squares are variance balanced. The resulting design remains variance balanced after the loss of any treatment, because the information matrix can be expressed in the form $c_1\mathbf{I} + c_2\mathbf{J}$. Following the terminology of Hedayat and John (1974), all Latin square designs of side r are classified as globally resistant of degree one. More generally, if $s \leq (r-2)$ treatments are lost from the starting design, the information matrix of treatment effects for any subset of s treatments, is

$$\mathbf{C}_{d_s} = r\mathbf{I}_{r-s} - \frac{r}{r-s}\mathbf{J}_{r-s,r-s} \tag{4.8}$$

Examination of the variances of the pairwise treatment differences shows that, although the loss of one or more complete treatments affects the information matrix, the variance of any individual comparison is unchanged as $2\sigma^2/r$. Any Latin square design with r treatments is globally resistant up to degree $r-2$ using the criterion of Hedayat and John (1974). This contrasts with the results for BIB designs, where the structure of the starting design, rather than only the design parameters, determined whether it was globally resistant, locally resistant or susceptible. Hedayat and John (1974) gave an example of a BIB design with these properties.

4.2 Unavailability of observations in a row or column

The results in this section are related to those for Youden square designs given by Das and Kageyama (1992). The layout of a Latin square ensures that the loss of a complete row of observations has the same effect on the initial design as the removal of a complete column. When a single row (column) is lost then one replicate of every treatment is lost, and every column (row) has $r-1$ plots remaining. Assuming that the observations are removed from one row (column) of the starting design, every pair of treatments occurs together in the remaining $r-1$ rows (columns) and in $r-2$ columns (rows) of the resulting design, denoted by d^r. Equation (4.2) is used to derive the normal equations because

$$\frac{\mathbf{tr}'}{n} = \frac{1}{r(r-1)} \begin{bmatrix} (r-1) \\ (r-1)\mathbf{1}_{r-1} \end{bmatrix} \begin{bmatrix} 0 & r\mathbf{1}'_{r-1} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_r & \mathbf{J}_{r,r-1} \end{bmatrix} \tag{4.9}$$

$$\frac{\mathbf{tc}'}{n} = \frac{1}{r(r-1)}(r-1)\mathbf{1}_r(r-1)\mathbf{1}'_r = \frac{r-1}{r}\mathbf{J}_{r,r} \tag{4.10}$$

and

$$\mathbf{N}_1\mathbf{r}^{-\delta}\mathbf{N}'_3 = (\mathbf{J}_{r,r} - \mathbf{I}_r)\frac{1}{r-1}\mathbf{I}_r \begin{bmatrix} \mathbf{0}_r & \mathbf{J}_{r,r-1} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_r & \mathbf{J}_{r,r-1} \end{bmatrix} \tag{4.11}$$

$$\mathbf{N}_2\mathbf{c}^{-\delta}\mathbf{N}_3 = \begin{bmatrix} \mathbf{0}_r & \mathbf{J}_{r,r-1} \end{bmatrix} \begin{bmatrix} 0 & \mathbf{0}'_{r-1} \\ \mathbf{0}_{r-1} & \frac{1}{r}\mathbf{I}_{r-1} \end{bmatrix} \begin{bmatrix} \mathbf{0}_r \mathbf{J}_{r,r-1} \end{bmatrix} = \frac{r-1}{r}\mathbf{J}_{r,r} \tag{4.12}$$

The two conditions in Equation (4.3) are satisfied for this resulting design. The important matrices in Equation (4.2) are given by

$$\begin{aligned} \mathbf{t}^{\delta} &= (r-1)\mathbf{I}_r \\ \mathbf{N}_1\mathbf{r}^{-\delta}\mathbf{N}'_1 &= \frac{(r-1)}{r}\mathbf{J}_{r,r} \\ \mathbf{N}_2\mathbf{c}^{-\delta}\mathbf{N}'_2 &= \frac{1}{(r-1)}\{\mathbf{I}_r + (r-2)\mathbf{J}_{r,r}\} \\ \frac{1}{n}\mathbf{tt}' &= \frac{(r-1)}{r}\mathbf{J}_{r,r} \end{aligned}$$

Substituting these in Equation (4.2) gives the information matrix for treatment effects

$$\begin{aligned} \mathbf{C}_{d^r} &= (r-1)\mathbf{I}_r - \frac{(r-1)}{r}\mathbf{J}_{r,r} - \frac{1}{(r-1)}\{\mathbf{I}_r + (r-2)\mathbf{J}_{r,r}\} + \frac{(r-1)}{r}\mathbf{J}_{r,r} \\ &= \frac{r(r-2)}{(r-1)}\mathbf{I}_r - \frac{(r-2)}{(r-1)}\mathbf{J}_{r,r} \end{aligned} \tag{4.13}$$

after the loss of a row (column) of observations. The resulting design is variance balanced after the loss of any row or column of observations from the starting design. The non-zero eigenvalues of this information matrix are $r(r-2)/(r-1)$ with multiplicity $r-1$. The

Table 4.1: Average variances and efficiencies for Latin squares where $r = 3, 4, \cdots, 10$ after a complete row (or column) is lost. Average variances are multiplied by σ^2.

r	3	4	5	6	7	8	9	10
A.V. Complete	0.667	0.500	0.400	0.333	0.286	0.250	0.222	0.200
A.V. Block Missing	1.333	0.750	0.533	0.417	0.343	0.292	0.254	0.225
R.E.	0.500	0.667	0.750	0.800	0.833	0.857	0.875	0.889

average variance of all pairwise treatment comparisons for this design is

$$\frac{2\sigma^2}{(r-1)} \times (r-1) \times \frac{(r-1)}{r(r-2)} = \frac{2(r-1)}{r(r-2)}\sigma^2 \tag{4.14}$$

and when compared to the average variance for a complete Latin square design, the relative efficiency of any resulting design d^r is $(r-2)/(r-1)$. Table 4.1 shows the relative efficiencies of Latin square based designs of side $r = 3, \cdots, 10$, when a single row or column of observations becomes unavailable. A design with ten treatments suffers a reduction in efficiency of just over 11%, but small designs are very seriously affected by the removal of a complete row or column, e.g. a five treatment Latin square design loses 25% efficiency when a row of data is lost.

4.3 Missing observations scattered throughout the initial design

In most situations, the actual pattern of missing observations will be unknown to the experimenter at the outset. Suppose that the t missing plots are scattered throughout the starting design. In this section, the case of $t \leq 3$ missing observations will be examined in detail and results will also be provided for $t > 3$. The potential resulting designs for these configurations of missing data will be analysed by evaluating the average variance of the pairwise treatment differences, and the corresponding efficiencies relative to the complete design. For $t > 1$, a variety of configurations of missing values, which result in different information matrices, need to be considered, and the average variances and efficiencies are calculated to compare the distinct resulting designs for t missing observations. This will give an overview of the range of efficiencies due to the unavailability of t observations. It will also be shown that the distribution of variances is independent of the particular Latin square of side r that is selected. Individual variances of pairwise treatment differences are derived for all configurations of one and two missing values, using generalised inverses of the information matrices for treatment effects and the matrix of all elementary pairwise contrasts, $\mathbf{\Gamma}$. The maximum of these variances will also be used as an extra measure of robustness of the designs to missing observations.

Denote the planned design d, and suppose that the loss of t observations, scattered randomly throughout the starting design, produces a resulting design denoted by $d(t) \in D$, where D is the set of all possible resulting designs. The size of D increases for increasing r and t, and there are $\binom{r^2}{t}$ available designs when t observations become unavailable, because there are r^2 plots in a single replicate of a Latin square. The properties of all $d(t) \in D$ for $t = 1, 2$, and 3 will be examined in the following subsections, theoretically and numerically for given Latin squares. The cases of t missing values will be distinguished in subsequent Sections by an extra number, e.g. $d(2;1)$ corresponds to Case 1 of two missing observations.

4.3.1 One missing value

When one plot becomes unavailable from the starting design, there are r^2 possible resulting designs in D, and one of these resulting designs is denoted by $d(1)$. The effect on the form of the information matrix for treatment effects and the properties of the resulting design is the same whichever observation is lost. Assume, without loss of generality, that the missing observation is the plot in the first row and first column of the starting design and that it corresponds to the first treatment. The components of the information matrix for treatment effects in Equation (4.1) are now

$$\mathbf{t}^{\delta} = \mathbf{r}^{\delta} = \mathbf{c}^{\delta} = \begin{bmatrix} (r-1) & \mathbf{0}'_{r-1} \\ \mathbf{0}_{r-1} & r\mathbf{I}_{r-1} \end{bmatrix} \tag{4.15}$$

and

$$\mathbf{N}_1 = \mathbf{N}_2 = \mathbf{N}_3 = \begin{bmatrix} 0 & \mathbf{1}'_{r-1} \\ \mathbf{1}_{r-1} & \mathbf{J}_{r-1,r-1} \end{bmatrix} \tag{4.16}$$

Substitution of these matrices into Equation (4.1), and simplification of the resulting algebra gives the form of the information matrix for treatment effects of $d(1)$ as

$$\mathbf{C}_{d(1)} = \begin{bmatrix} (r-2) & -\frac{(r-2)}{(r-1)}\mathbf{1}'_{r-1} \\ -\frac{(r-2)}{(r-1)}\mathbf{1}_{r-1} & r\mathbf{I}_{r-1} - \frac{(r^2-2r+2)}{(r-1)^2}\mathbf{J}_{r-1,r-1} \end{bmatrix} \tag{4.17}$$

where $\mathbf{1}_p$ is a p dimensional vector of 1s. The elements of $\mathbf{C}_{d(1)}$ can be rearranged to derive the information matrix when a different plot is lost from the design. Variances of pairwise treatment comparisons that are increased by the loss of an observation will depend on the actual treatment missing from the initial design, but the average and maximum of the variances are the same for all r^2 resulting designs, $d(1)$. A generalised inverse of the information matrix $\mathbf{C}_{d(1)}$ is necessary to express these variances in terms of the number of treatments in the starting design. Letting $a = (r^2 - 2r + 2)/(r-1)^2$ in Equation (4.4) and inverting the non-singular matrix gives

$$\Omega = \begin{bmatrix} \frac{(r-1)^3}{r(r^3-4r^2+6r-4)} & -\frac{(r-1)}{r(r^3-4r^2+6r-4)}\mathbf{1}'_{r-1} \\ -\frac{(r-1)}{r(r^3-4r^2+6r-4)}\mathbf{1}_{r-1} & \frac{1}{r}\mathbf{I}_{r-1} + \frac{1}{r(r-1)(r^3-4r^2+6r-4)}\mathbf{J}_{r-1,r-1} \end{bmatrix} \tag{4.18}$$

The average variance of pairwise treatment comparisons is the same for all resulting designs $d(1) \in D$, and when the Latin square is of side r, is given by

$$\frac{2(r^2-3r+3)}{r(r-1)(r-2)}\sigma^2 \tag{4.19}$$

The $(r-1)(r-2)/2$ pairwise treatment comparisons that do not involve the affected treatment, which is treatment one for illustrative purposes, are unaltered by the loss of data, and their variances remain at $2\sigma^2/r$, but the variances of all differences between treatment one and the other $r-1$ treatments increase to

$$\frac{(2r^2-5r+4)}{r(r-1)(r-2)}\sigma^2 \tag{4.20}$$

Average and maximum pairwise variances, and the efficiencies of designs of side $r = 3, \cdots, 10$ after the loss of a single observation, are shown in Table 4.2. Reduction in efficiency for a design with three treatments is approximately 33%, which is a serious loss of accuracy resulting from the unavailability of one of the nine observations in the initial design. This loss of efficiency decreases rapidly as the number of treatments in the

Table 4.2: Average variances, maximum variances of pairwise treatment comparisons, and relative efficiencies for Latin squares of side $r = 3, \cdots, 10$ when either one or two observations are missing. Also shown are the frequencies for the three different cases of two missing values.

	One missing	Two missing					
r	$d(1)$	$d(2;1)$	count	$d(2;2)$	count	$d(2;3)$	count
	$1.0000\sigma^2$	$1.3333\sigma^2$				$2.0000\sigma^2$	
	(0.6667)	(0.5000)				(0.3334)	
3	$1.1667\sigma^2$	$1.3333\sigma^2$	18	Disconnected	9	$2.6667\sigma^2$	9
	(0.5714)	(0.5000)				(0.2500)	
	$0.5833\sigma^2$	$0.6667\sigma^2$		$0.7083\sigma^2$		$0.7500\sigma^2$	
	(0.8571)	(0.7500)		(0.7059)		(0.6667)	
4	$0.6667\sigma^2$	$0.7500\sigma^2$	48	$1.0000\sigma^2$	48	$1.0000\sigma^2$	24
	(0.7500)	(0.6667)		(0.5000)		(0.5000)	
	$0.4333\sigma^2$	$0.4667\sigma^2$		$0.4714\sigma^2$		$0.4889\sigma^2$	
	(0.9231)	(0.8571)		(0.8485)		(0.8182)	
5	$0.4833\sigma^2$	$0.5333\sigma^2$	100	$0.6000\sigma^2$	150	$0.6222\sigma^2$	50
	(0.8276)	(0.7500)		(0.6667)		(0.6429)	
	$0.3500\sigma^2$	$0.3667\sigma^2$		$0.3677\sigma^2$		$0.3750\sigma^2$	
	(0.9523)	(0.9089)		(0.9064)		(0.8888)	
6	$0.3833\sigma^2$	$0.4167\sigma^2$	180	$0.4444\sigma^2$	360	$0.4583\sigma^2$	90
	(0.8696)	(0.8000)		(0.7500)		(0.7273)	
	$0.2952\sigma^2$	$0.3048\sigma^2$		$0.3051\sigma^2$		$0.3086\sigma^2$	
	(0.9678)	(0.9373)		(0.9364)		(0.9258)	
7	$0.3190\sigma^2$	$0.3429\sigma^2$	294	$0.3571\sigma^2$	735	$0.3657\sigma^2$	147
	(0.8957)	(0.8332)		(0.8001)		(0.7813)	
	$0.2560\sigma^2$	$0.2619\sigma^2$		$0.2620\sigma^2$		$0.2639\sigma^2$	
	(0.9766)	(0.9546)		(0.9542)		(0.9473)	
8	$0.2738\sigma^2$	$0.2917\sigma^2$	488	$0.3000\sigma^2$	1,344	$0.3056\sigma^2$	224
	(0.9131)	(0.8570)		(0.8333)		(0.8181)	
	$0.2262\sigma^2$	$0.2302\sigma^2$		$0.2302\sigma^2$		$0.2313\sigma^2$	
	(0.9823)	(0.9654)		(0.9652)		(0.9607)	
9	$0.2401\sigma^2$	$0.2540\sigma^2$	648	$0.2593\sigma^2$	2,268	$0.2630\sigma^2$	324
	(0.9255)	(0.8749)		(0.8570)		(0.8450)	
	$0.2028\sigma^2$	$0.2056\sigma^2$		$0.2056\sigma^2$		$0.2063\sigma^2$	
	(0.9862)	(0.9730)		(0.9729)		(0.9697)	
10	$0.2139\sigma^2$	$0.2250\sigma^2$	900	$0.2286\sigma^2$	3,600	$0.2313\sigma^2$	450
	(0.9350)	(0.8889)		(0.8749)		(0.8647)	

Case 1

1	2	X	⋯	X

Case 2

1	X	X	⋯	X
X	2	X	⋯	X

Case 3

1	X	X	⋯	X
X	1	X	⋯	X

Figure 4.1: Potential configurations of two missing values in a Latin square design. The letter X refers to any other possible treatment in the design.

design increases, because the total number of observations in the experiment increases substantially as the number of treatments increases. The difference between the maximum and average variance is large when the initial design has a small number treatments, but becomes smaller for large Latin squares.

4.3.2 Two values missing from a Latin square design

When two observations become unavailable, the many resulting designs, denoted by $d(2)$, fall into three distinct sub-cases, each of which needs to be considered separately to identify the properties of these designs. Sub-cases of D correspond to the particular configuration of two missing observations. Rows, columns, and treatment labels of the design may be interchanged, so the configurations corresponding to these three sub-cases will be identified in terms of treatments 1 and 2, and the rows and columns in which the missing observations occur. An example of a configuration of missing values for each of these three cases, listed in order of increasing average variance of pairwise treatment comparisons, is given in Figure 4.1. The regulated structure of a Latin square ensures that Case 1 includes the situation where two different treatments are lost from the same column.

Table 4.3 shows the number of configurations corresponding to each of the three sub-cases, and also important measures of robustness - variances of the individual pairwise treatment comparisons, the numbers of each of the comparisons, and the overall average of these variances.

Derivation of the results for the sub-case with the largest average variance, $d(2;3)$, is illustrated below, and the same approach can be used to produce the many formulae shown in Table 4.3 for the other two sub-cases. There are six matrices in Equation (4.1),

Table 4.3: Average and pairwise variances for Latin square designs of side r for the three configurations of two missing observations. The number of configurations corresponding to each case is also shown.

Case	Treatment i	Treatment j	Pairwise Variance	Number
$d(2;1)$	1	2	$\frac{2(r-1)}{r(r-2)}\sigma^2$	1
	$1,2$	$3,\cdots,r$	$\frac{(2r^2-7r+7)}{r(r-2)^2}\sigma^2$	$2(r-2)$
	$3,\cdots,r$	$3,\cdots,r$	$\frac{2}{r}\sigma^2$	$(r-2)(r-3)/2$
Average Variance	$\frac{2(r^2-3r+4)}{r(r-1)(r-2)}\sigma^2$			
Configurations	$r^2(r-1)$			
$d(2;2)$	1	2	$\frac{2(r-2)}{r(r-3)}\sigma^2$	1
	$1,2$	$3,\cdots,r$	$\frac{(2r^3-11r^2+23r-22)}{r(r-3)(r^2-3r+4)}\sigma^2$	$2(r-2)$
	$3,\cdots,r$	$3,\cdots,r$	$\frac{2}{r}\sigma^2$	$(r-2)(r-3)/2$
Average Variance	$\frac{2(r-2)(r^3-5r^2+11r-11)}{r(r-1)(r-3)(r^2-3r+4)}\sigma^2$			
Configurations	$r^2(r-1)(r-2)/2$			
$d(2;3)$	1	$2,\cdots,r$	$\frac{2(r^2-3r+4)}{r(r^2-4r+4)}\sigma^2$	$(r-1)$
	$2,\cdots,r$	$2,\cdots,r$	$\frac{2}{r}\sigma^2$	$(r-1)(r-2)/2$
Average Variance	$\frac{2(r^2-4r+6)}{r(r^2-4r+4)}\sigma^2$			
Configurations	$r^2(r-1)/2$			

and for configurations of missing values in sub-case $d(2;3)$, under the assumption that a replicate of treatment one is lost from row 1, column 1 and also from row 2, column 2, these can be shown to be

$$\mathbf{t}^\delta = \begin{bmatrix} (r-2) & \mathbf{0}'_{r-1} \\ \mathbf{0}_{r-1} & r\mathbf{I}_{r-1} \end{bmatrix} \qquad \mathbf{r}^\delta = \mathbf{c}^\delta = \begin{bmatrix} (r-1)\mathbf{I}_2 & \mathbf{0}_{2,r-2} \\ \mathbf{0}_{r-2,2} & r\mathbf{I}_{r-2} \end{bmatrix} \qquad (4.21)$$

and

$$\mathbf{N}_1 = \mathbf{N}_2 = \begin{bmatrix} \mathbf{0}_2' & \mathbf{1}_{r-2}' \\ \mathbf{J}_{r-1,2} & \mathbf{J}_{r-1,r-2} \end{bmatrix} \qquad \mathbf{N}_3 = \begin{bmatrix} \mathbf{J}_{2,2} - \mathbf{I}_2 & \mathbf{J}_{2,r-2} \\ \mathbf{J}_{r-2,2} & \mathbf{J}_{r-2,r-2} \end{bmatrix} \tag{4.22}$$

The information matrix for treatment effects is found by substituting the expressions in Equations (4.21) and (4.22) into Equation (4.1) to give

$$\mathbf{C}_{d(2;3)} = \begin{bmatrix} r - 2 - \frac{r-2}{r} - \frac{2(r-1)(r-2)}{r(r^2-2r+2)} & \{\frac{2(r-2)}{r(r^2-2r+2)} - \frac{r-2}{r}\}\mathbf{1}_{r-1}' \\ \{\frac{2(r-2)}{r(r^2-2r+2)} - \frac{r-2}{r}\}\mathbf{1}_{r-1} & r\mathbf{I}_{r-1} - \{\frac{2}{r-1} + \frac{r-2}{r} + \frac{2(r-2)}{r(r-1)(r^2-2r+2)}\}\mathbf{J}_{r-1,r-1} \end{bmatrix} \tag{4.23}$$

for a resulting design corresponding to sub-case $d(2;3)$. One particular generalised inverse that can be used to find the variances of pairwise treatment comparisons is given by

$$\boldsymbol{\Omega} = \begin{bmatrix} \frac{(r^2-2r+2)^2}{r(r^4-6r^3+16r^2-24r+16)} & -\frac{2(r^2-2r+2)}{r(r^4-6r^3+16r^2-24r+16)}\mathbf{1}_{r-1}' \\ -\frac{2(r^2-2r+2)}{r(r^4-6r^3+16r^2-24r+16)}\mathbf{1}_{r-1} & \frac{1}{r}\mathbf{I}_{r-1} + \frac{4}{r(r^4-6r^3+16r^2-24r+16)}\mathbf{J}_{r-1,r-1} \end{bmatrix} \tag{4.24}$$

Variances of all the individual pairwise treatment differences can be found by selecting the appropriate elements from this generalised inverse. These are listed in Table 4.3, which also shows the average of these individual variances. The relative efficiency is calculated simply by comparing the average variance formula for a given case to the average for the complete design, which is $2\sigma^2/r$ for a Latin square design with r treatments.

Table 4.2 shows that a small square with three treatments incurs a 66% reduction in efficiency and a corresponding large increase in average and maximum variance for the nine ways of achieving the worst configuration of two missing values, $d(2;3)$. Loss of efficiency for larger starting designs is smaller, e.g. it is less than 5% when there are 9 or more treatments in the initial design, which is partly due to the large number of observations in these situations, 81 and 100 respectively for nine and ten treatments respectively. The difference in efficiency between the best case, when two observations are removed from the same row or column, and configurations corresponding to the worst case decreases as the size of the square increases. The increased size of the starting design reduces the impact of losing any two observations.

Case 1

1	2	3	$\cdots$	X

Case 2

1	2	X	$\cdots$	X
X	3	X	$\cdots$	X

Case 3

1	X	X	$\cdots$	X
X	2	X	$\cdots$	X
X	3	X	$\cdots$	X

Case 4

1	X	X	$\cdots$	X
X	2	X	$\cdots$	X
X	X	3	$\cdots$	X

Case 5

1	2	X	$\cdots$	X
X	1	X	$\cdots$	X

Case 6

1	X	X	$\cdots$	X
X	1	X	$\cdots$	X
X	2	X	$\cdots$	X

Case 7

1	X	X	$\cdots$	X
X	1	X	$\cdots$	X
X	X	2	$\cdots$	X

Case 8

1	X	X	$\cdots$	X
X	1	X	$\cdots$	X
X	X	1	$\cdots$	X

Figure 4.2: Potential configurations of three missing values in a single replicate Latin square design. The letter X refers to any other possible treatment in the design.

4.3.3 Three missing values

The loss of three observations from a Latin square is a more complicated situation. There are eight distinct configurations of three missing observations, each producing a sub-case of resulting designs with different properties, in particular, the average and maximum variance of pairwise treatment differences. Best and worst cases, in terms of the relative efficiency of the sub-case of designs, can be identified by considering these eight sub-cases separately. The situation is more straightforward than for a BIB design, where there are twenty-five cases, many with sub-cases, to be considered individually. Once again, because rows and columns may be interchanged in a Latin square and treatment labels may be permuted without altering the properties of the resulting design, the eight configurations of missing observations can be studied using the first three treatments and first three rows and columns of the Latin square. The 8 cases are illustrated in Figure 4.2 and the number of configurations of each of these cases is shown in Table 4.4 in terms of the size of the square, r.

For each of the eight sub-cases of resulting designs, denoted by $d(3)$, the effect on the information matrix for treatment effects, $\mathbf{C}$, is investigated by considering the changes to the elements of Equation (4.1). After the form of $\mathbf{C}$ has been determined, the variances of individual pairwise treatment differences, their average, and the relative efficiency for the particular resulting design in the sub-case may be found using a suitable generalised inverse $\boldsymbol{\Omega}$ of $\mathbf{C}$. The derivation of the theoretical formulae for the variances of pairwise

Table 4.4: The number of configurations corresponding to each case of three missing observations.

Case	Configurations
$d(3;1)$	$r^2(r-1)(r-2)/3$
$d(3;2)$	$r^2(r-1)(r-2)$
$d(3;3)$	$r^2(r-1)(r-2)(r-3)$
$d(3;4)$	$r^2(r-1)(r-2)(r^2-6r+10)/6$
$d(3;5)$	$r^2(r-1)$
$d(3;6)$	$2r^2(r-1)(r-2)$
$d(3;7)$	$r^2(r-1)(r-2)(r-3)/2$
$d(3;8)$	$r^2(r-1)(r-2)/6$

treatment comparisons and their efficiencies, is illustrated for a resulting design in the first sub-case only, and corresponding results for the other seven sub-cases may be found using the same approach. Numerical results are calculated for different square sizes at the end of the subsection to illustrate the possible effect of three observations becoming unavailable.

Assume, without loss of generality, that three observations corresponding to the first three treatments are lost from the same row (column) of the design, and that they occurred in the first three columns (rows) of the starting design. To compute the form of the information matrix for treatment effects, the following matrices are required

$$\mathbf{t}^{\delta} = \mathbf{c}^{\delta} = \begin{bmatrix} (r-1)\mathbf{I}_3 & \mathbf{0}_{3,r-3} \\ \mathbf{0}_{r-3,3} & r\mathbf{I}_{r-3} \end{bmatrix} \qquad \mathbf{r}^{\delta} = \begin{bmatrix} (r-3) & \mathbf{0}'_{r-1} \\ \mathbf{0}_{r-1} & r\mathbf{I}_{r-1} \end{bmatrix} \tag{4.25}$$

and

$$\mathbf{N}_1 = \mathbf{N}_3 = \begin{bmatrix} \mathbf{0}_3 & \mathbf{J}_{3,r-1} \\ \mathbf{1}_{r-3} & \mathbf{J}_{r-3,r-1} \end{bmatrix} \qquad \mathbf{N}_2 = \begin{bmatrix} \mathbf{J}_{3,3} - \mathbf{I}_3 & \mathbf{J}_{3,r-3} \\ \mathbf{J}_{r-3,3} & \mathbf{J}_{r-3,r-3} \end{bmatrix} \tag{4.26}$$

On substitution of the matrices in Equations (4.25) and (4.26) into Equation (4.1), and after simplification, the information matrix for treatment effects is given by

$$\mathbf{C}_{d(3;1)} = \begin{bmatrix} \frac{r(r-2)}{(r-1)}\mathbf{I}_3 - \{\frac{(r-1)}{r} - \frac{1}{r(r-1)}\}\mathbf{J}_{3,3} & -\{\frac{(r-1)}{r} - \frac{1}{r(r-1)}\}\mathbf{J}_{3,r-3} \\ -\{\frac{(r-1)}{r} - \frac{1}{r(r-1)}\}\mathbf{J}_{r-3,3} & r\mathbf{I}_{r-3} - \{\frac{1}{(r-3)} + \frac{(r-1)}{r} + \frac{3}{r(r-1)(r-3)}\}\mathbf{J}_{r-3,r-3} \end{bmatrix} \tag{4.27}$$

A sensible choice of generalised inverse for this information matrix is given by

$$\Omega = \begin{bmatrix} \frac{(r-1)}{r(r-2)}\mathbf{I}_3 & \mathbf{0}_{3,r-3} \\ \mathbf{0}_{r-3,3} & \frac{1}{r}\mathbf{I}_{r-3} + \frac{1}{r(r-2)(r-3)}\mathbf{J}_{r-3,r-3} \end{bmatrix} \tag{4.28}$$

Variances of differences between treatments that do not lose a replicate remain unchanged at $2\sigma^2/r$. For comparisons that involve treatments 1, 2, and 3, the variances are increased to $2(r-1)\sigma^2/r(r-2)$. The comparisons between one of the first three treatments and any of the other $r-3$ treatments have variances equal to $(2r-5)\sigma^2/r(r-3)$. The average variance of pairwise treatment differences for designs in this sub-case is given by

$$\frac{2(r^2-3r+5)}{r(r-1)(r-2)}\sigma^2 \tag{4.29}$$

for a Latin square of side r. Table 4.5 shows average and maximum pairwise variances, and relative efficiencies for Latin square designs with between four and ten treatments for all 8 cases given in Figure 4.2. These results suggest that, for smaller designs with four or five treatments, there is a large range in relative efficiencies over the different cases, and that, even for the best case with three missing observations, there is a reduction of over 30% for a Latin square of side $r = 4$. The maximum pairwise variances are increased substantially for the smallest design of four treatments, which could lead to ruination of the experiment. In the worst case for this design, the efficiency drops to 40%. As the size of the design increases, the effects of losing any three observations decrease as expected due to the larger number of plots in the starting design. The range of variances of pairwise treatment comparisons also decreases as the size of the square increases. A design constructed from a Latin square with ten treatments is not severely affected when the average variance is considered, but the maximum variance of pairwise treatment comparisons increases from $0.2\sigma^2$ to $0.2536\sigma^2$, corresponding to a reduction in efficiency of 21%.

Table 4.5: Average and maximum variances of pairwise treatment comparisons, and relative efficiencies (in parenthesis) for Latin squares where $r = 4, \cdots, 10$ for the eight cases of three missing values. Variances are multiplied by σ^2.

	CASE							
r	$d(3;1)$	$d(3;2)$	$d(3;3)$	$d(3;4)$	$d(3;5)$	$d(3;6)$	$d(3;7)$	$d(3;8)$
	0.7500	0.7833	0.9167	0.8500	0.9167	0.8500	1.0833	1.2500
	(0.6667)	(0.6383)	(0.5455)	(0.5882)	(0.5455)	(0.5882)	(0.4616)	(0.4000)
4	0.7500	1.0000	1.2500	1.0000	1.2500	1.2000	2.0000	2.0000
	(0.6667)	(0.5000)	(0.4000)	(0.5000)	(0.4000)	(0.4167)	(0.2500)	(0.2500)
	0.5000	0.5033	0.5133	0.5125	0.5333	0.5233	0.5400	0.6000
	(0.8000)	(0.7947)	(0.7792)	(0.7805)	(0.7500)	(0.7644)	(0.7407)	(0.6667)
5	0.5333	0.6000	0.6233	0.6000	0.6667	0.6933	0.8100	0.9000
	(0.7500)	(0.6667)	(0.6417)	(0.6667)	(0.6000)	(0.5770)	(0.4938)	(0.4444)
	0.3833	0.3840	0.3859	0.3861	0.3944	0.3918	0.3949	0.4167
	(0.8695)	(0.8680)	(0.8637)	(0.8632)	(0.8450)	(0.8506)	(0.8441)	(0.7999)
6	0.4167	0.4444	0.4503	0.4444	0.4722	0.4984	0.5385	0.5833
	(0.7999)	(0.7501)	(0.7402)	(0.7501)	(0.7059)	(0.6688)	(0.6190)	(0.5715)
	0.3143	0.3145	0.3150	0.3151	0.3190	0.3181	0.3190	0.3286
	(0.9090)	(0.9085)	(0.9070)	(0.9066)	(0.8955)	(0.8981)	(0.8957)	(0.8695)
7	0.3429	0.3571	0.3593	0.3571	0.3786	0.3923	0.4111	0.4357
	(0.8332)	(0.8001)	(0.7952)	(0.8001)	(0.7547)	(0.7283)	(0.6950)	(0.6558)
	0.2679	0.2679	0.2681	0.2682	0.2702	0.2699	0.2701	0.2750
	(0.9333)	(0.9331)	(0.9325)	(0.9323)	(0.9251)	(0.9264)	(0.9255)	(0.9091)
8	0.2917	0.3000	0.3010	0.3000	0.3167	0.3248	0.3351	0.3500
	(0.8570)	(0.8333)	(0.8306)	(0.8333)	(0.7894)	(0.7697)	(0.7460)	(0.7143)
	0.2341	0.2342	0.2342	0.2343	0.2354	0.2353	0.2354	0.2381
	(0.9491)	(0.9490)	(0.9486)	(0.9485)	(0.9437)	(0.9445)	(0.9440)	(0.9332)
9	0.2540	0.2593	0.2598	0.2593	0.2725	0.2777	0.2840	0.2937
	(0.8749)	(0.8570)	(0.8554)	(0.8570)	(0.8155)	(0.8002)	(0.7825)	(0.7566)
	0.2083	0.2083	0.2084	0.2084	0.2091	0.2090	0.2091	0.2107
	(0.9600)	(0.9599)	(0.9598)	(0.9597)	(0.9564)	(0.9568)	(0.9566)	(0.9492)
10	0.2250	0.2286	0.2289	0.2286	0.2393	0.2428	0.2470	0.2536
	(0.8889)	(0.8749)	(0.8737)	(0.8749)	(0.8358)	(0.8237)	(0.8097)	(0.7886)

Table 4.6: Average variance, maximum pairwise variance, and relative efficiencies (in parenthesis) of the best and worst configurations of t missing values for Latin squares with $r = 5, \cdots, 10$ treatments. All variances are multiplied by σ^2.

	Number of Missing Values (t)							
	4		5		6		7	
r	Best	Worst	Best	Worst	Best	Worst	Best	Worst
	0.5333	0.9333	-	-	-	-	-	-
	(0.7500)	(0.4286)	-	-	-	-	-	-
5	0.5333	1.7333	-	-	-	-	-	-
	(0.7500)	(0.2308)						
	0.4000	0.5000	0.4167	0.7500	-	-	-	-
	(0.8333)	(0.6667)	(0.8000)	(0.4444)	-	-	-	-
6	0.4167	0.8333	0.4167	1.5833	-	-	-	-
	(0.7999)	(0.4000)	(0.7999)	(0.2105)				
	0.3238	0.3619	0.3333	0.4286	0.3429	0.6286	-	-
	(0.8823)	(0.7894)	(0.8571)	(0.6667)	(0.8332)	(0.4545)	-	-
7	0.3429	0.5524	0.3429	0.7857	0.3429	1.4857	-	-
	(0.8332)	(0.5172)	(0.8332)	(0.3636)	(0.8332)	(0.1923)		
	0.2738	0.2917	0.2798	0.3194	0.2857	0.3750	0.2917	0.5417
	(0.9131)	(0.8570)	(0.8935)	(0.7826)	(0.8750)	(0.6667)	(0.8570)	(0.4615)
8	0.2917	0.4167	0.2917	0.5278	0.2917	0.7500	0.2917	1.4167
	(0.8570)	(0.6000)	(0.8570)	(0.4737)	(0.8570)	(0.3333)	(0.8570)	(0.1765)
	0.2381	0.2476	0.2421	0.2619	0.2460	0.2857	0.2500	0.3333
	(0.9332)	(0.8974)	(0.9178)	(0.8484)	(0.9033)	(0.7777)	(0.8889)	(0.6667)
9	0.2540	0.3365	0.2540	0.4008	0.2540	0.5079	0.2540	0.7222
	(0.8749)	(0.6604)	(0.8749)	(0.5544)	(0.8749)	(0.4375)	(0.8749)	(0.3077)
	0.2111	0.2167	0.2139	0.2250	0.2167	0.2375	0.2194	0.2583
	(0.9474)	(0.9231)	(0.9351)	(0.8889)	(0.9231)	(0.8421)	(0.9116)	(0.7743)
10	0.2250	0.2833	0.2250	0.3250	0.2250	0.3875	0.2250	0.4917
	(0.8889)	(0.7060)	(0.8889)	(0.6154)	(0.8889)	(0.5161)	(0.8889)	(0.4068)

4.3.4 Extension to more than three missing values

When more than three observations are lost from a Latin square design, there is a large number of distinct configurations leading to different sub-cases, that need to be considered. Full enumeration of these would be a computationally expensive task for many designs. However, based on the results of this section, it would appear reasonable to assume that (1) the best of these cases is the removal of $t(< r)$ plots from the same row or column of the starting design, and that (2) the worst situation would occur if t replicates of one of the treatments become unavailable.

The form of the information matrix could be derived theoretically for these cases, but in this subsection, the results will be calculated numerically for different Latin square

Table 4.7: Distribution of eight cases after the loss of three values for Latin squares where $r = 4, \cdots, 10$ treatments.

	CASE							
r	$d(3;1)$	$d(3;2)$	$d(3;3)$	$d(3;4)$	$d(3;5)$	$d(3;6)$	$d(3;7)$	$d(3;8)$
4	0.0571	0.1714	0.1714	0.0571	0.0857	0.3429	0.0857	0.0286
5	0.0435	0.1304	0.2609	0.1087	0.0435	0.2609	0.1304	0.0217
6	0.0336	0.1008	0.3025	0.1681	0.0252	0.2017	0.1513	0.0168
7	0.0266	0.0798	0.3191	0.2261	0.0160	0.1596	0.1596	0.0133
8	0.0215	0.0645	0.3226	0.2796	0.0108	0.1290	0.1613	0.0108
9	0.0177	0.0532	0.3190	0.3278	0.0076	0.1063	0.1595	0.0089
10	0.0148	0.0445	0.3117	0.3711	0.0056	0.0891	0.1558	0.0074

designs and different numbers of missing values. Table 4.6 shows the relative efficiencies for these two situations when $t = 4, 5, 6$ or 7 observations are lost from Latin square designs with $r = 5, \cdots, 10$. There is large variation between the efficiencies in the eight cases for the smaller designs, which again highlights the severity of losing data from a Latin square design. Relative efficiencies are very low for the worst case for many of the designs. The efficiencies for the maxima of the variances of pairwise treatment comparison are reduced more than those of the averages. For example, the loss of seven replicates of one treatment for a design of side $r = 10$ is very serious, representing a reduction in efficiency to 77%. The variances of all comparisons involving the affected treatment are much larger than those of the other comparisons, $0.4917\sigma^2$ compared with the average $0.2583\sigma^2$.

4.4 Observations missing completely at random

Suppose that drop-out from a Latin square design can be assumed to occur completely at random, see Diggle and Kenward (1994). Under these circumstances it is possible to produce a distribution of relative efficiencies for the loss of t observations from a Latin square of side r. Formulae for the number of configurations in each sub-case and hence the probabilities for these cases, when $t = 2$ and 3 observations are lost, have been given in Sections 4.3.2 and 4.3.3. The expected average variance given two missing values can also be expressed theoretically using the derived formulae as

$$E[\text{A.V.}|t = 2] = \frac{2(r^7 - 10r^6 + 42r^5 - 92r^4 + 95r^3 + 6r^2 - 118r + 88)}{r(r^2-1)(r^2-4r+4)(r-3)(r^2-3r+4)}\sigma^2 \qquad (4.30)$$

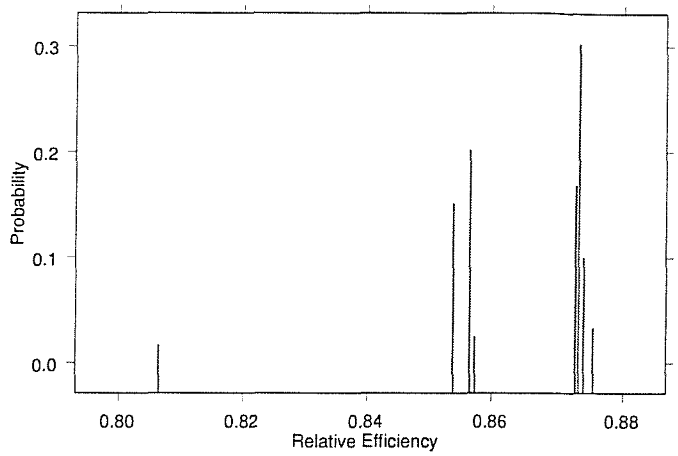

Figure 4.3: Distribution of relative efficiencies for the eight subclasses of three missing observations for a Latin square of size 6.

The distribution of efficiencies relative to the starting design have been calculated for Latin squares of side $4, \cdots, 10$ for three missing observations and are tabulated in Table 4.7. Probabilities of obtaining the best (Case 1) and worst (Case 8) sub-cases decrease as the number of treatments in the initial design increases. The most common configurations of three missing values become Cases 3, 4, and 7 as the size of the square increases.

Figure 4.3 shows the distribution of efficiencies for $t = 3$ missing observations for a Latin square design with six treatments. The figure shows that while a small proportion of the configurations give rise to efficiency losses of nearly 20%, the majority (over 96%) of resulting designs have a reduction in efficiency of under 16%. It is possible to produce similar graphs for other sizes of square and/or numbers of missing values to investigate the robustness of the initial design to any configuration of t missing observations.

4.5 Discussion

The results in this Chapter illustrate the effect of data becoming unavailable in a Latin square experimental design. Loss of complete treatments, rows or columns, and missing values scattered throughout the starting design have been considered. A complete design with three or four treatments suffers a large reduction in efficiency when one or two observations are missing. Large designs are affected less seriously by the loss of a small number of observations, but as the number of missing values increases, the reduction in efficiency increases correspondingly. The structure of Latin square designs ensures that the overall properties of the resulting designs are the same regardless of the square that is selected. The results suggest that the use of replicate squares to construct a design may be beneficial if the experiment has a small number of treatments, to minimise the effect of missing data.

Chapter 5

Missing values in Replicate Latin Squares

The results in the previous Chapter indicated the benefits of constructing designs based on more than one Latin square. Small designs where there are only three or four treatments were found to suffer a large reduction in efficiency, when one or two observations became unavailable from the starting design. In this Chapter, designs based on any number of Latin squares, where k is used to denote the number of squares, are examined for their robustness against the loss of up to three observations scattered throughout the starting design. The results presented in this Chapter are reported in Mansson and Prescott (2001b). It will be seen that the overall distribution of resulting designs when t observations are lost is independent of the choice of squares used to construct the starting design. For example, when $k = 2$, there is no difference in the effect of missing data if two identical squares or two different squares were chosen to construct the design.

The information matrix for the treatment effects is used to evaluate the variances of pairwise treatment differences for each resulting design, and is expressed in terms of the number of missing values and the size of the initial design. The resulting averages of these variances are used to assess the overall robustness of the designs to the loss of up to three observations. There are many similarities in potential configurations of missing values between single and multiple square designs, with the main differences being extra configurations introduced by the increased number of plots in the columns of the chosen designs, and the multiple replicates of each treatment in the column block design. In general, there are 5 and 16 different situations for the cases of two and three missing values respectively when there are at least three Latin square replicates in the starting design. It is possible to determine algebraic expressions for the variances for all possible configurations, which has been done for all cases of two missing observations, but only best and worst cases are given in detail for three missing values. A numerical illustration is provided with the average variances, relative efficiencies, minimum and maximum variances, and the frequencies of the distinct configurations of $t = 2$ and 3 missing observations, showing the effects of the missing observations for Latin squares of side $r = 4$ and up to four squares.

The results in this Chapter can be compared with those in the previous Chapter as reported by Mansson and Prescott (2001c) by setting $k = 1$, which is equivalent to a design based on a single Latin square. In the next Section, the theory used to identify the properties of the resulting designs and to compare the different realisable designs is covered briefly.

5.1 Designs constructed with multiple Latin Squares

In practice, the small designs based on single squares studied in Chapter 4 can be extended either by using several squares or by allocating several subjects to each treatment sequence (row) of the Latin square. The aim of this approach is to reduce the increases to the variances of pairwise treatment comparisons caused by missing data. Following this approach, suppose that a row-column design is constructed using up to k replicates of a $(r \times r)$ Latin square, or alternatively of k different Latin squares of the same size, so that the complete design with r periods (columns) applied to a group of kr patients (rows) is such that each patient receives a sequence of the r treatments. An additive model with row, column, and treatment effects is assumed, and there are $n = kr^2$ observations in the starting design. When the design is complete, the information matrix for treatment effects is given by $\mathbf{C}_d = kr\mathbf{I}_r - k\mathbf{J}_{r,r}$.

Assume that t observations are removed from the starting design, and that they are scattered throughout the design. The general form of the information matrix for treatment effects when t values are missing, shown in Chapter 2, is given by

$$\mathbf{C}_{d(t)} = \mathbf{t}^{\delta} - \mathbf{N}_1\mathbf{r}^{-\delta}\mathbf{N}_1' - (\mathbf{N}_2 - \mathbf{N}_1\mathbf{r}^{-\delta}\mathbf{N}_3')(\mathbf{c}^{\delta} - \mathbf{N}_3\mathbf{r}^{-\delta}\mathbf{N}_3')^{-}(\mathbf{N}_2' - \mathbf{N}_3\mathbf{r}^{-\delta}\mathbf{N}_1') \qquad (5.1)$$

where the main differences between the incidence matrices for the replicated square and single square designs are the dimensions of the matrices $\mathbf{N}_1$ and $\mathbf{N}_3$, and the elements of the column incidence matrix, $\mathbf{N}_2$. For a complete design constructed from k Latin squares, there are k plots corresponding to every treatment in each of the columns of the starting design, so that $\mathbf{N}_2$ is a $(r \times r)$ matrix of elements all equal to k. The elements of $\mathbf{N}_1$ are all equal to 1 when the design is complete as all treatments occur exactly once in every row of the starting design.

In the next section, the properties of designs resulting from the loss of up to three observations are developed. Formulae for the average of the variances are determined under all configurations for either one or two missing values, and the best and worst cases are given in detail for three missing values. The theoretical section is followed by a numerical example to illustrate the potential benefits of replicate squares for a four treatment experiment. The minimum and maximum of the variances of pairwise treatment differences are also computed to assess the most severely affected treatment comparison, which is not provided when the average variance is used as a measure of robustness.

5.2 Changes to the variances of pairwise treatment differences when observations are lost

The effect of missing observations on the information matrix for treatment effects may be investigated by evaluating the alterations to the six components of the right-hand side of Equation (5.1). Different configurations of the missing values may arise, which are dependent on the rows, columns, and treatments that are affected by the loss of data. As discussed in earlier Chapters, configurations are not regarded as essentially different, when the variances of pairwise treatment comparisons are analysed, if one can be obtained from another by simply rearranging the rows, columns, and treatment labels appropriately. In each of the cases, the properties of a basic configuration of t missing value are determined using specific row, column, and treatment labels, and subsequently deriving the number of similar situations for k replicates of a $(r \times r)$ Latin square.

The increase in variances of the individual pairwise treatment comparisons will depend on which specific treatments are affected by the given configuration of missing values.

Except for the case of a single missing value, which is relatively trivial, the changes to the individual components of Equation (5.1) will be identified, and then these will be combined appropriately to produce all possible configurations of two missing values for row-column design constructed from k Latin squares of side r. Only the best and worst cases, measured by their average variance, are investigated for the situation where three plots become unavailable.

5.2.1 One value missing from the starting design

The removal of a single observation from the initial design gives rise to one of a possible kr^2 resulting designs, all of which are identified by $d(1)$ say, where the information matrix has a basic structure which is common to all the resulting designs. Assuming that the plot in the first row and first column of the starting design, which corresponds to the first treatment, is lost, the six components in the formula for the information matrix for treatment effects in this case are given by

$$\mathbf{t}^{\delta} = \begin{bmatrix} (kr-1) & \mathbf{0}'_{r-1} \\ \mathbf{0}_{r-1} & kr\mathbf{I}_{r-1} \end{bmatrix} \qquad \mathbf{r}^{\delta} = \begin{bmatrix} (r-1) & \mathbf{0}'_{kr-1} \\ \mathbf{0}_{kr-1} & r\mathbf{I}_{kr-1} \end{bmatrix} \qquad \mathbf{c}^{\delta} = \begin{bmatrix} (kr-1) & \mathbf{0}'_{r-1} \\ \mathbf{0}_{r-1} & kr\mathbf{I}_{r-1} \end{bmatrix} \tag{5.2}$$

and

$$\mathbf{N}_1 = \begin{bmatrix} 0 & \mathbf{1}'_{kr-1} \\ \mathbf{1}_{r-1} & \mathbf{J}_{r-1,kr-1} \end{bmatrix} \qquad \mathbf{N}_2 = \begin{bmatrix} (k-1) & k\mathbf{1}'_{r-1} \\ k\mathbf{1}_{r-1} & k\mathbf{J}_{r-1,r-1} \end{bmatrix} \qquad \mathbf{N}_3 = \begin{bmatrix} 0 & \mathbf{1}'_{kr-1} \\ \mathbf{1}_{r-1} & \mathbf{J}_{r-1,kr-1} \end{bmatrix} \tag{5.3}$$

The form of the information matrix is found by substituting these six expressions into Equation (5.1), and after simplification, is

$$\mathbf{C}_{d(1)} = \begin{bmatrix} (kr-1) - \frac{(kr-1)}{r} - \frac{(r-1)}{r(kr-1)} & -\{\frac{(kr-1)}{r} - \frac{1}{r(kr-1)}\}\mathbf{1}'_{r-1} \\ -\{\frac{(kr-1)}{r} - \frac{1}{r(kr-1)}\}\mathbf{1}_{r-1} & kr\mathbf{I}_{r-1} - \{\frac{1}{(r-1)} + \frac{(kr-1)}{r} + \frac{1}{r(r-1)(kr-1)}\}\mathbf{J}_{r-1,r-1} \end{bmatrix} \tag{5.4}$$

These reduced normal equations can be solved by selecting a generalised inverse. One complicated generalised inverse of this information matrix is given by

$$\boldsymbol{\Omega} = \begin{bmatrix} \frac{(r-1)(kr-1)^2}{kr(k^2r^3-kr^2(k+3)+2r(2k+1)-4)} & -\frac{kr-1}{kr(k^2r^3-kr^2(k+3)+2r(2k+1)-4)}\mathbf{1}'_{r-1} \\ -\frac{kr-1}{kr(k^2r^3-kr^2(k+3)+2r(2k+1)-4)}\mathbf{1}_{r-1} & \frac{1}{kr}\mathbf{I}_{r-1} + \frac{1}{kr(r-1)(k^2r^3-kr^2(k+3)+2r(2k+1)-4)}\mathbf{J}_{r-1,r-1} \end{bmatrix} \tag{5.5}$$

The average variance of pairwise treatment comparisons for a general design based on k squares can be found using either the non-zero eigenvalues of this information matrix or the generalised inverse and matrix of all pairwise treatment contrasts. The formula for the average variance of a design based on k Latin squares of side r is

$$\text{A.V.} = \frac{2(kr^2 - kr - 2r + 3)}{kr(r-1)(kr-2)}\sigma^2 \tag{5.6}$$

Variances of individual treatment comparisons will depend on the treatment associated with the missing plot, but the average and maximum are the same for all kr^2 resulting designs. Equation (5.6) can be simplified for a design based on a single square, where $k = 1$, to

$$\text{A.V.} = \frac{2(r^2 - 3r + 3)}{r(r-1)(r-2)}\sigma^2 \tag{5.7}$$

which corresponds to the result derived in Chapter 4. Variances of the $r - 1$ pairwise treatment comparisons which involve the affected treatment are increased to

$$\frac{(2kr^2 - 2kr - 3r + 4)}{kr(r-1)(kr-2)}\sigma^2 \tag{5.8}$$

and the other variances all remain unchanged as $2\sigma^2/kr$.

5.2.2 Two missing values

In a design based on a single Latin square there are three distinct configurations of two missing observations that may occur, as discussed in greater detail in Chapter 4. This increases to five cases when the starting design is constructed from more than one square. These missing values can occur in either the same or different squares of the starting design, and in the same or different rows and columns. It is also possible to lose two replicates of the same treatment, or one replicate of each of two different treatments. The design can be considered to have kr rows rather than differentiating between observations

Case 1

1	2	X	···	X

Case 2

1	X	X	···	X
X	2	X	···	X

Case 3

1	X	X	···	X
2	X	X	···	X

Case 4

1	X	X	···	X
X	1	X	···	X

Case 5

1	X	X	···	X
1	X	X	···	X

Figure 5.1: Potential configurations of two missing values in a k replicated Latin square design. The letter X refers to any other possible treatment in the design.

in the different squares to identify distinct configurations of missing observations. The five cases are shown in Figure 5.1 using only the first two treatments, and the first two rows and columns of the starting design. When there is only one square, the properties of Case 3 are similar to those of Case 1, and Case 5 is not possible because there is only one replicate of each treatment in all the columns of the starting design.

The average variance of pairwise treatment differences for these five cases can be found by considering the changes to the components of the information matrix for treatment effects, as given in Equation (5.1). Although there are five different situations, the forms of some of the matrices are similar for the different cases of missing values. These are illustrated using the first two treatments, the first two rows and columns of the starting design. For example, the matrix $\mathbf{t}^\delta$ takes only two forms, identified by (a) and (b) below.

$$(a) \qquad \mathbf{t}^\delta = \begin{bmatrix} (kr-1)\mathbf{I}_2 & \mathbf{0}_{2,r-2} \\ \mathbf{0}_{r-2,2} & kr\mathbf{I}_{r-2} \end{bmatrix} \qquad (b) \qquad \mathbf{t}^\delta = \begin{bmatrix} (kr-2) & \mathbf{0}'_{r-1} \\ \mathbf{0}_{r-1} & kr\mathbf{I}_{r-1} \end{bmatrix} \tag{5.9}$$

The expression in (a) corresponds to one replicate of two different treatments becoming unavailable, while (b) occurs when two replicates of the same treatment are lost from the starting design. There are also two different forms for $\mathbf{r}^\delta$ and $\mathbf{c}^\delta$ that are based on the number of plots removed from one or two rows or columns of the initial design. These matrices are given by

$$(a) \qquad \mathbf{r}^{\delta} = \begin{bmatrix} (r-2) & \mathbf{0}'_{kr-1} \\ \mathbf{0}_{kr-1} & r\mathbf{I}_{kr-1} \end{bmatrix} \qquad (b) \qquad \mathbf{r}^{\delta} = \begin{bmatrix} (r-1)\mathbf{I}_2 & \mathbf{0}_{2,kr-2} \\ \mathbf{0}_{kr-2,2} & r\mathbf{I}_{kr-2} \end{bmatrix} \tag{5.10}$$

where (a) occurs when the 2 plots are in the same row, and (b) is the loss of plots from different rows, and

$$(a) \qquad \mathbf{c}^{\delta} = \begin{bmatrix} (kr-1)\mathbf{I}_2 & \mathbf{0}_{2,r-2} \\ \mathbf{0}_{r-2,2} & kr\mathbf{I}_{r-2} \end{bmatrix} \qquad (b) \qquad \mathbf{c}^{\delta} = \begin{bmatrix} (kr-2) & \mathbf{1}'_{r-1} \\ \mathbf{1}_{r-1} & kr\mathbf{I}_{r-1} \end{bmatrix} \tag{5.11}$$

where (a) corresponds to observations in different columns, and (b) is two observations from one column of the starting design. There are three situations to be considered for the form of $\mathbf{N}_1$, the incidence matrix for treatments and rows. The three situations correspond to different treatments in the same row of the starting design, denoted I(r), different treatments in different rows, II(r), and the same treatment in separate rows, III(r). The forms of the matrix $\mathbf{N}_1$ in the three situations are given in Equations (5.12), (5.13), and (5.14), assuming without loss of generality that the two missing observations occurred in the first two rows and columns of the starting design and correspond to the first two treatments.

$$\text{I(r)}: \qquad \mathbf{N}_1 = \begin{bmatrix} \mathbf{0}_2 & \mathbf{J}_{2,kr-1} \\ \mathbf{1}_{r-2} & \mathbf{J}_{r-2,kr-1} \end{bmatrix} \tag{5.12}$$

$$\text{II(r)}: \qquad \mathbf{N}_1 = \begin{bmatrix} \mathbf{J}_{2,2} - \mathbf{I}_2 & \mathbf{J}_{2,kr-2} \\ \mathbf{J}_{r-2,2} & \mathbf{J}_{r-2,kr-2} \end{bmatrix} \tag{5.13}$$

$$\text{III(r)}: \qquad \mathbf{N}_1 = \begin{bmatrix} \mathbf{0}'_2 & \mathbf{1}'_{kr-2} \\ \mathbf{J}_{r-1,2} & \mathbf{J}_{r-1,kr-2} \end{bmatrix} \tag{5.14}$$

The extra replicates of each treatment in the columns of the designs imply that there are four situations to be considered for the columns. The four cases for the incidence matrix $\mathbf{N}_2$ of the column component correspond to different treatments in different columns, I(c), different treatments in a single column, II(c), the same treatment in separate columns, III(c), and the same treatment in one column of the starting design, IV(c). Equations (5.15) - (5.18) show the forms of the incidence matrix $\mathbf{N}_2$ for the four situations. It is

Table 5.1: The five cases of two missing observations from designs constructed using multiple Latin squares and the components of their respective information matrices

Case	$\mathbf{t}^\delta$	$\mathbf{r}^\delta$	$\mathbf{c}^\delta$	N_1	N_2	N_3
1	a	a	a	I(r)	I(c)	I(rc)
2	a	b	a	II(r)	I(c)	II(rc)
3	a	b	b	II(r)	II(c)	III(rc)
4	b	b	a	III(r)	III(c)	II(rc)
5	b	b	b	III(r)	IV(c)	III(rc)

again assumed that the missing values correspond to the first two treatments and the first two columns of the starting design.

$$\mathrm{I(c)}: \qquad \mathbf{N}_2 = \begin{bmatrix} k\mathbf{J}_{2,2} - \mathbf{I}_2 & k\mathbf{J}_{2,r-2} \\ k\mathbf{J}_{r-2,2} & k\mathbf{J}_{r-2,r-2} \end{bmatrix} \tag{5.15}$$

$$\mathrm{II(c)}: \qquad \mathbf{N}_2 = \begin{bmatrix} (k-1)\mathbf{1}_2 & k\mathbf{J}_{2,r-1} \\ k\mathbf{1}_{r-2} & k\mathbf{J}_{r-2,r-1} \end{bmatrix} \tag{5.16}$$

$$\mathrm{III(c)}: \qquad \mathbf{N}_2 = \begin{bmatrix} (k-1)\mathbf{1}_2' & k\mathbf{1}_{r-2}' \\ k\mathbf{J}_{r-1,2} & k\mathbf{J}_{r-1,r-2} \end{bmatrix} \tag{5.17}$$

$$\mathrm{IV(c)}: \qquad \mathbf{N}_2 = \begin{bmatrix} (k-2) & k\mathbf{1}_{r-1}' \\ k\mathbf{1}_{r-1} & k\mathbf{J}_{r-1,r-1} \end{bmatrix} \tag{5.18}$$

The final component of the information matrix for treatment effects is $\mathbf{N}_3$, and there are three different situations. Figure 5.1 illustrates possible locations of missing observations in the rows and columns of the initial design. The first is where observations are in the same row and different columns, I(rc), the second is different rows and columns, II(rc), and the last is different rows but the same column, III(rc). In these three circumstances the forms of the matrix $\mathbf{N}_3$ are given by

$$\mathrm{I(rc)}: \qquad \mathbf{N}_3 = \begin{bmatrix} \mathbf{0}_2 & \mathbf{J}_{2,kr-1} \\ \mathbf{1}_{r-2} & \mathbf{J}_{r-2,kr-1} \end{bmatrix} \tag{5.19}$$

$$\mathrm{II(rc)}: \qquad \mathbf{N}_3 = \begin{bmatrix} \mathbf{J}_{2,2} - \mathbf{I}_2 & \mathbf{J}_{2,kr-2} \\ \mathbf{J}_{r-2,2} & \mathbf{J}_{r-2,kr-2} \end{bmatrix} \tag{5.20}$$

$$\text{III(rc)}: \qquad \mathbf{N}_3 = \begin{bmatrix} \mathbf{0}_2' & \mathbf{1}_{kr-2}' \\ \mathbf{J}_{r-1,2} & \mathbf{J}_{r-1,kr-2} \end{bmatrix} \tag{5.21}$$

These components can be combined in an appropriate manner to obtain the information matrix, $\mathbf{C}_{d(2)}$, for the various configurations of two missing observations. Table 5.1 shows the combinations of matrices needed for the five distinct configurations of missing values given in Figure 5.1. Corresponding expressions for the average variances of the pairwise treatment differences are given in Table 5.2. Variances of the individual pairwise treatment comparisons are shown in this Table, together with the number of ways of achieving each of the five cases for a design constructed with k replicates of a Latin square of side r.

5.2.3 Three missing values

There are only eight distinct configurations involving three missing values in a single Latin square design, see Chapter 4 and Mansson and Prescott (2001c) for details. This increases to fifteen when the design is composed of two squares, and a further case arises if the design is based on three or more squares. This additional case occurs when three replicates of the same treatment become unavailable in one column of the starting design, which requires three or more squares, i.e. $k \geq 3$. Configurations of missing values in these sixteen cases are displayed in Figure 5.2 using the labels 1, 2, and 3 to identify the treatments corresponding to the missing observations. The missing observations are also in the first three rows and columns of the design for illustrative purposes only. Similar configurations can be compared by a rearrangement of the rows and columns of the starting design and by switching the treatment labels. Comparing Cases 6 and 10, for example, shows that the properties of the resulting designs are the same when the starting design is constructed from only one square.

The form of the information matrix for treatment effects, and the alterations to the variances of pairwise treatment differences, can be determined for all cases by considering the various ways that the three observations can be lost over the rows, columns, and treatments of the design. The algebra involved in finding some of the general expressions is heavy, and many of the final equations are quite complicated. The procedure for deriving these formulae will be illustrated for the best and worst cases of three missing values, but all cases will be considered numerically for the example in Section 5.3.

Table 5.2: Variances of pairwise treatment comparisons for Latin squares of side r for the five configurations of two missing observations.

Case	Treatment i	Treatment j	Pairwise Variance	Number
$d(2;1)$	1	2	$\frac{2(kr-1)}{kr(kr-2)}\sigma^2$	1
	$1,2$	$3,\cdots,r$	$\frac{2r^2k-3r-4kr+7}{kr(kr-2)(r-2)}\sigma^2$	$2(r-2)$
	$3,\cdots,r$	$3,\cdots,r$	$\frac{2}{kr}\sigma^2$	$\frac{(r-2)(r-3)}{2}$
A.V.	$\frac{2(r^2k-kr-2r+4)}{kr(kr-2)(r-1)}\sigma^2$			
Frequency	$kr^2(r-1)/2$			
$d(2;2)$	1	2	$\frac{2(kr-k-1)}{kr(kr-k-2)}\sigma^2$	1
	$1,2$	$3,\cdots,r$	$\frac{2k^2r^3-4k^2r^2-7kr^2+2k^2r+15kr+6r-8k-14}{kr(kr-k-2)(kr^2-kr-2r+4)}\sigma^2$	$2(r-2)$
	$3,\cdots,r$	$3,\cdots,r$	$\frac{2}{kr}\sigma^2$	$\frac{(r-2)(r-3)}{2}$
A.V.	$\frac{2(16+6k-16kr-16r-3k^2r^3+3k^2r^2-k^2r+14kr^2+4r^2-4kr^3+k^2r^4)}{kr(r-1)(kr-k-2)(kr^2-kr-2r+4)}\sigma^2$			
Frequency	$kr^2(r-1)(kr-k-1)/2$			
$d(2;3)$	1	2	$\frac{2(r-1)}{r(kr-k-1)}\sigma^2$	1
	$1,2$	$3,\cdots,r$	$\frac{2k^2r^3-7kr^2-4k^2r^2+15kr+2k^2r-8k+4r-6}{kr(kr-k-1)(kr^2-3r-kr+4)}\sigma^2$	$2(r-2)$
	$3,\cdots,r$	$3,\cdots,r$	$\frac{2}{kr}\sigma^2$	$\frac{(r-2)(r-3)}{2}$
A.V.	$\frac{2(6k+10-16kr-11r-k^2r+3k^2r^2-3k^2r^3+14kr^2+3r^2+k^2r^4-4kr^3)}{kr(r-1)(kr-k-1)(kr^2-3r-kr+4)}\sigma^2$			
Frequency	$k^2r^2(r-1)/2$			
$d(2;4)$	1	$2,\cdots,r$	$\frac{2(kr^2-kr-2r+4)}{kr(kr^2-kr-3r+4)}\sigma^2$	$(r-1)$
	$2,\cdots,r$	$2,\cdots,r$	$\frac{2}{kr}\sigma^2$	$\frac{(r-1)(r-2)}{2}$
A.V.	$\frac{2(kr^2-kr-3r+6)}{kr(kr^2-kr-3r+4)}\sigma^2$			
Frequency	$k^2r^2(r-1)/2$			
$d(2;5)$	1	$2,\cdots,r$	$\frac{2(kr^2-3r-kr+4)}{kr(kr-4)(r-1)}\sigma^2$	$(r-1)$
	$2,\cdots,r$	$2,\cdots,r$	$\frac{2}{kr}\sigma^2$	$\frac{(r-1)(r-2)}{2}$
A.V.	$\frac{2(kr^2-kr-4r+6)}{kr(kr-4)(r-1)}\sigma^2$			
Frequency	$kr^2(k-1)/2$			

Case 1

1	X	X	$\cdots$	X
1	X	X	$\cdots$	X
1	X	X	$\cdots$	X

Case 2

1	X	X	$\cdots$	X
1	X	X	$\cdots$	X
X	1	X	$\cdots$	X

Case 3

1	X	X	$\cdots$	X
X	1	X	$\cdots$	X
X	X	1	$\cdots$	X

Case 4

1	X	X	$\cdots$	X
1	X	X	$\cdots$	X
2	X	X	$\cdots$	X

Case 5

1	X	X	$\cdots$	X
1	X	X	$\cdots$	X
X	2	X	$\cdots$	X

Case 6

1	X	X	$\cdots$	X
2	X	X	$\cdots$	X
X	1	X	$\cdots$	X

Case 7

1	X	X	$\cdots$	X
X	1	X	$\cdots$	X
X	X	2	$\cdots$	X

Case 8

1	X	X	$\cdots$	X
1	2	X	$\cdots$	X

Case 9

1	X	X	$\cdots$	X
2	1	X	$\cdots$	X

Case 10

1	X	X	$\cdots$	X
X	1	2	$\cdots$	X

Case 11

1	X	X	$\cdots$	X
2	X	X	$\cdots$	X
3	X	X	$\cdots$	X

Case 12

1	X	X	$\cdots$	X
2	X	X	$\cdots$	X
X	3	X	$\cdots$	X

Case 13

1	X	X	$\cdots$	X
X	2	X	$\cdots$	X
X	X	3	$\cdots$	X

Case 14

1	2	X	$\cdots$	X
X	3	X	$\cdots$	X

Case 15

1	2	X	$\cdots$	X
X	X	3	$\cdots$	X

Case 16

1	2	3	$\cdots$	X

Figure 5.2: Potential configurations of three missing values in a k replicated Latin square design. The letter X refers to any other possible treatment in the design.

The loss of one replicate of any three different treatments in a single column is the least badly affected configuration in terms of the average variance of pairwise treatment comparisons. For $k = 1$, this is not essentially different from the loss of one replicate of three treatments from one row of the starting design. Assuming, without loss of generality, that the first three treatments lose an observation in the first column of the design (Case 11), and the plots are also lost from the first three rows, the six components of the information matrix for treatment effects become

$$\mathbf{t}^\delta = \begin{bmatrix} (kr-1)\mathbf{I}_3 & \mathbf{0}_{3,r-3} \\ \mathbf{0}_{r-3,3} & kr\mathbf{I}_{r-3} \end{bmatrix} \qquad \mathbf{r}^\delta = \begin{bmatrix} (r-1)\mathbf{I}_3 & \mathbf{0}_{3,kr-3} \\ \mathbf{0}_{kr-3,3} & r\mathbf{I}_{kr-3} \end{bmatrix} \qquad \mathbf{c}^\delta = \begin{bmatrix} (kr-3) & \mathbf{0}'_{r-1} \\ \mathbf{0}_{r-1} & kr\mathbf{I}_{r-1} \end{bmatrix} \tag{5.22}$$

and

$$\mathbf{N}_1 = \begin{bmatrix} \mathbf{J}_{3,3} - \mathbf{I}_3 & \mathbf{J}_{3,kr-3} \\ \mathbf{J}_{r-3,3} & \mathbf{J}_{r-3,kr-3} \end{bmatrix} \qquad \mathbf{N}_2 = \begin{bmatrix} (k-1)\mathbf{1}_3 & k\mathbf{J}_{3,r-1} \\ k\mathbf{1}_{r-3} & k\mathbf{J}_{r-3,r-1} \end{bmatrix}$$

$$\mathbf{N}_3 = \begin{bmatrix} \mathbf{0}'_3 & \mathbf{1}'_{kr-3} \\ \mathbf{J}_{r-1,3} & \mathbf{J}_{r-1,kr-3} \end{bmatrix} \tag{5.23}$$

The information matrix for treatment effects, denoted $\mathbf{C}_{d(3;11)}$, is given by substituting these matrices in Equation (5.1) to give

$$\mathbf{C}_{d(3;11)} = \begin{bmatrix} \frac{r(kr-k-1)}{(r-1)}\mathbf{I}_3 - \frac{k^2r^2-k^2r-5kr+6k+r}{(r-1)(kr-3)}\mathbf{J}_{3,3} & -\frac{k(kr^2-kr-4r+6)}{(r-1)(kr-3)}\mathbf{J}_{3,r-3} \\ -\frac{k(kr^2-kr-4r+6)}{(r-1)(kr-3)}\mathbf{J}_{r-3,3} & kr\mathbf{I}_{r-3} - \frac{k^2r^3-k^2r^2-3kr^2+6kr-18}{r(r-1)(kr-3)}\mathbf{J}_{r-3,r-3} \end{bmatrix} \tag{5.24}$$

One sensible choice of generalised inverse of this information matrix that can be used to compute the variances of the treatment differences is

$$\boldsymbol{\Omega} = \begin{bmatrix} \frac{(r-1)}{r(kr-k-1)}\mathbf{I}_3 - \frac{(k-1)(r-1)}{kr(kr-k-1)(kr^2-kr-4r+6)}\mathbf{J}_{3,3} & \mathbf{0}_{3,r-3} \\ \mathbf{0}_{r-3,3} & \frac{1}{kr}\mathbf{I}_{r-3} + \frac{1}{kr(kr^2-kr-4r+6)}\mathbf{J}_{r-3,r-3} \end{bmatrix} \tag{5.25}$$

Variances of comparisons between two treatments that do not lose a replicate remain unchanged as $2\sigma^2/kr$, while the variances for the pairwise differences between any two treatments that lose one of their replicates increase to

$$\frac{2(r-1)}{r(kr-k-1)}\sigma^2 \tag{5.26}$$

and the differences between treatments 1, 2, and 3 and any of the other $r-3$ treatments have variances given by

$$\frac{(2k^2r^3 - 4k^2r^2 - 9kr^2 + 21kr + 2k^2r - 12k + 5r - 8)}{kr(kr-k-1)(kr^2-kr-4r+6)}\sigma^2 \tag{5.27}$$

The average variance (A.V.) over all these comparisons for a design based on k Latin squares of side r is given by

$$\text{A.V.} = \frac{2(9k + 3k^2r^2 - k^2r - 23kr + 21 - 19r + 19kr^2 - 3k^2r^3 - 5kr^3 + k^2r^4 + 4r^2)}{kr(kr-k-1)(r-1)(kr^2-kr-4r+6)}\sigma^2 \tag{5.28}$$

The worst situation is where three replicates of one treatment are lost from the same column of the starting design. This configuration is only possible when there are three or more squares used to construct the design. To compute the information matrix for treatment effects, the following matrices are needed

$$\mathbf{t}^\delta = \mathbf{c}^\delta = \begin{bmatrix} (kr-3) & \mathbf{0}'_{r-1} \\ \mathbf{0}_{r-1} & kr\mathbf{I}_{r-1} \end{bmatrix} \qquad \mathbf{r}^\delta = \begin{bmatrix} (r-1)\mathbf{I}_3 & \mathbf{0}_{3,kr-3} \\ \mathbf{0}_{kr-3,3} & r\mathbf{I}_{kr-3} \end{bmatrix} \tag{5.29}$$

and

$$\mathbf{N}_1 = \mathbf{N}_3 = \begin{bmatrix} \mathbf{0}'_3 & \mathbf{1}'_{kr-3} \\ \mathbf{J}_{r-1,3} & \mathbf{J}_{r-1,kr-3} \end{bmatrix} \qquad \mathbf{N}_2 = \begin{bmatrix} k-3 & k\mathbf{1}'_{r-1} \\ k\mathbf{1}_{r-1} & k\mathbf{J}_{r-1,r-1} \end{bmatrix} \tag{5.30}$$

These six expressions are substituted into Equation (5.1), and after simplification, the information matrix for treatment effects of configurations of three missing values for the resulting designs $d(3;1)$ is given by

$$\mathbf{C}_{d(3;1)} = \begin{bmatrix} (kr-3) - \frac{(kr-3)}{r} - \frac{9(r-1)}{r(kr-3)} & -\{\frac{(kr-3)}{r} - \frac{9}{r(kr-3)}\}\mathbf{1}'_{r-1} \\ -\{\frac{(kr-3)}{r} - \frac{9}{r(kr-3)}\}\mathbf{1}_{r-1} & kr\mathbf{I}_{r-1} - \{\frac{3}{(r-1)} + \frac{(kr-3)}{r} + \frac{9}{r(r-1)(kr-3)}\}\mathbf{J}_{r-1,r-1} \end{bmatrix} \tag{5.31}$$

To find a generalised inverse of this information matrix, let the constant $a = (kr-3)/r - 9/r(kr-3)$ in Theorem 3.1. The resulting non-singular matrix is then inverted yielding the generalised inverse

$$\boldsymbol{\Omega} = \begin{bmatrix} \frac{kr-3}{kr(kr-6)} & \mathbf{0}'_{r-1} \\ \mathbf{0}_{r-1} & \frac{1}{kr}\mathbf{I}_{r-1} + \frac{3}{kr(kr-6)(r-1)}\mathbf{J}_{r-1,r-1} \end{bmatrix} \tag{5.32}$$

Assuming that three replicates of the first treatment are lost, the comparisons between treatment one and the other $r-1$ treatments all have an increased variance equal to

$$\frac{(2kr^2 - 2kr - 9r + 12)}{kr(kr-6)(r-1)}\sigma^2 \tag{5.33}$$

and the average variance of all pairwise treatment differences is given by

$$\frac{2(kr^2 - kr - 6r + 9)}{kr(kr-6)(r-1)}\sigma^2 \tag{5.34}$$

because the variances of all other pairwise treatment comparisons are unchanged at $2\sigma^2/kr$. These theoretical results are illustrated for a design based on Latin squares of side $r = 4$, for up to four replicates, in the next Section to demonstrate the advantages of constructing designs with multiple squares.

5.3 Numerical Illustration

To illustrate the effect of missing observations on the variances of pairwise treatment comparisons in a Latin square based design, consider a design with four treatments and four periods (columns). To investigate any benefits introduced by replication, designs constructed from a single replicate and up to four Latin squares will be examined in detail. Replicated designs can be formed from the same square used k times or from k different squares, because all treatments will occur exactly once in every row of the complete design, and exactly k times in each of the columns (periods). For the complete designs, the average variances of pairwise treatment differences, with $k = 1, 2, 3$, and 4, are $0.5\sigma^2$, $0.25\sigma^2$, $0.167\sigma^2$, and $0.125\sigma^2$ respectively.

Results for a single missing value, obtained using Equation (5.6), are listed in Table 5.3, which also shows the relative efficiency, minimum and maximum variances for a treatment difference and the frequency of the particular configuration. For a single replicate the

Table 5.3: The average variance, relative efficiency, and the minimum and maximum pairwise variances for up to four replicates of a Latin square of side 4, when there is one missing observation.

k	Average Variance	Relative Efficiency	Minimum Variance	Maximum Variance	Relative Efficiency	Count
1	$0.5833\sigma^2$	0.8571	$0.5000\sigma^2$	$0.6667\sigma^2$	0.7500	16
2	$0.2639\sigma^2$	0.9474	$0.2500\sigma^2$	$0.2778\sigma^2$	0.8999	32
3	$0.1722\sigma^2$	0.9677	$0.1667\sigma^2$	$0.1778\sigma^2$	0.9376	48
4	$0.1280\sigma^2$	0.9767	$0.1250\sigma^2$	$0.1310\sigma^2$	0.9542	64

loss of a single observation incurs a reduction in efficiency of over 14%. The benefit of replicating the squares for such a small design is quite evident - the efficiency increases to over 94% as soon as two or more squares are used. The difference between the maximum variances is also obvious, for a single square there is an increase from $0.5\sigma^2$ to $0.667\sigma^2$ and when $k = 2$, the increase is from $0.25\sigma^2$ to $0.278\sigma^2$. There are, of course, more plots in the starting design, but the inclusion of these extra observations appears to offer more protection against drop-out. Consideration of the efficiency of the maximum of the variances of pairwise treatment differences provides further support for the use of multiple squares. When the design is constructed from a single square of side $r = 4$, the maximum variance increases from $0.5\sigma^2$ to $0.667\sigma^2$, which represents a reduction of 25% efficiency. The loss of efficiency is reduced to 10% when there are two squares.

The computations for two missing observations obtained from the expressions in Table 5.2 are shown in Table 5.4. Recall that Case 5 is not possible with a single square design, and that Case 3 is equivalent to Case 1 in terms of the analysis for a single Latin square.

For a single square there is a minimum reduction of 25% in efficiency, and more than 30% in Case 4, which is the worst case for a design where $k = 1$. Even when two squares are used, the efficiency loss is more than 10%, and is almost 15% in Case 5, corresponding to a configuration where the same treatment loses two replicates from one column. In the worst case, the maximum variance increases to $0.33\sigma^2$ from $0.25\sigma^2$ for the complete design, which corresponds to a loss of 25% in efficiency. The maximum variance is doubled from $0.5\sigma^2$ to σ^2 for over half of the configurations of two missing values for the single square design. The loss of efficiency is severe for designs constructed from one or two Latin squares of side $r = 4$. The probabilities associated with the occurrence of these situations, under the assumption that the observations are missing completely at random, can be

Table 5.4: Summary of the variances and efficiencies for the five cases of two missing values for Latin squares of side $r = 4$. Data are given for one, two, three and four replicates of the squares. Efficiencies are in parenthesis.

k	Case	Average Variance	Minimum Variance	Maximum Variance	Count
1	1	$0.6667\sigma^2$	$0.5000\sigma^2$	$0.7500\sigma^2$	48
		(0.7500)		(0.6667)	
	2	$0.7083\sigma^2$	$0.5000\sigma^2$	$1.0000\sigma^2$	48
		(0.7059)		(0.5000)	
	4	$0.7500\sigma^2$	$0.5000\sigma^2$	$1.0000\sigma^2$	24
		(0.6667)		(0.5000)	
2	1	$0.2778\sigma^2$	$0.2500\sigma^2$	$0.2917\sigma^2$	48
		(0.9000)		(0.8570)	
	2	$0.2792\sigma^2$	$0.2500\sigma^2$	$0.3125\sigma^2$	240
		(0.8955)		(0.8000)	
	3	$0.2771\sigma^2$	$0.2500\sigma^2$	$0.3000\sigma^2$	96
		(0.9023)		(0.8333)	
	4	$0.2813\sigma^2$	$0.2500\sigma^2$	$0.3125\sigma^2$	96
		(0.8889)		(0.8000)	
	5	$0.2917\sigma^2$	$0.2500\sigma^2$	$0.3333\sigma^2$	16
		(0.8571)		(0.7501)	
3	1	$0.1778\sigma^2$	$0.1667\sigma^2$	$0.1833\sigma^2$	72
		(0.9375)		(0.9094)	
	2	$0.1781\sigma^2$	$0.1667\sigma^2$	$0.1905\sigma^2$	576
		(0.9359)		(0.8751)	
	3	$0.1776\sigma^2$	$0.1667\sigma^2$	$0.1875\sigma^2$	216
		(0.9385)		(0.8891)	
	4	$0.1786\sigma^2$	$0.1667\sigma^2$	$0.1905\sigma^2$	216
		(0.9333)		(0.8751)	
	5	$0.1806\sigma^2$	$0.1667\sigma^2$	$0.1944\sigma^2$	48
		(0.9231)		(0.8575)	
4	1	$0.1310\sigma^2$	$0.1250\sigma^2$	$0.1339\sigma^2$	96
		(0.9545)		(0.9335)	
	2	$0.1311\sigma^2$	$0.1250\sigma^2$	$0.1375\sigma^2$	1,056
		(0.9538)		(0.9091)	
	3	$0.1309\sigma^2$	$0.1250\sigma^2$	$0.1364\sigma^2$	384
		(0.9551)		(0.9164)	
	4	$0.1313\sigma^2$	$0.1250\sigma^2$	$0.1375\sigma^2$	384
		(0.9524)		(0.9091)	
	5	$0.1319\sigma^2$	$0.1250\sigma^2$	$0.1389\sigma^2$	96
		(0.9474)		(0.8999)	

Table 5.5: Average variances of pairwise treatment comparisons, and relative efficiencies for the sixteen distinct configurations of three missing values, illustrated for one, two, and three replicates of a Latin square of side $r = 4$.

	$k = 1$		$k = 2$		$k = 3$	
Case	A.V. (R.E.)	Count	A.V. (R.E.)	Count	A.V. (R.E.)	Count
(1)	-	-	-	-	$0.1944\sigma^2$ (0.8575)	16
(2)	-	-	$0.3173\sigma^2$ (0.7879)	96	$0.1882\sigma^2$ (0.8858)	432
(3)	$1.2500\sigma^2$ (0.4000)	16	$0.3036\sigma^2$ (0.8235)	128	$0.1859\sigma^2$ (0.8967)	432
(4)	-	-	$0.3045\sigma^2$ (0.8210)	96	$0.1857\sigma^2$ (0.8977)	432
(5)	-	-	$0.3109\sigma^2$ (0.8041)	192	$0.1870\sigma^2$ (0.8914)	1,008
(6)	$0.8500\sigma^2$ (0.5882)	192	$0.2955\sigma^2$ (0.8460)	960	$0.1842\sigma^2$ (0.9050)	3,456
(7)	$1.0833\sigma^2$ (0.4616)	48	$0.2988\sigma^2$ (0.8367)	768	$0.1848\sigma^2$ (0.9021)	3,024
(8)	-	-	$0.3056\sigma^2$ (0.8181)	96	$0.1861\sigma^2$ (0.8958)	288
(9)	$0.9167\sigma^2$ (0.5454)	48	$0.2959\sigma^2$ (0.8449)	192	$0.1842\sigma^2$ (0.9050)	432
(10)	Part of Case (6)		$0.2953\sigma^2$ (0.8466)	384	$0.1842\sigma^2$ (0.9050)	864
(11)	$0.7500\sigma^2$ (0.6667)	32	$0.2893\sigma^2$ (0.8642)	128	$0.1827\sigma^2$ (0.9124)	432
(12)	$0.9167\sigma^2$ (0.5454)	96	$0.2940\sigma^2$ (0.8503)	768	$0.1838\sigma^2$ (0.9070)	3,024
(13)	$0.8500\sigma^2$ (0.5882)	32	$0.2955\sigma^2$ (0.8460)	448	$0.1842\sigma^2$ (0.9050)	1,824
(14)	$0.7833\sigma^2$ (0.6383)	96	$0.2922\sigma^2$ (0.8556)	384	$0.1834\sigma^2$ (0.9089)	864
(15)	Part of Case (12)		$0.2959\sigma^2$ (0.8449)	288	$0.1842\sigma^2$ (0.9050)	720
(16)	Part of Case (11)		$0.2917\sigma^2$ (0.8570)	32	$0.1833\sigma^2$ (0.9094)	48

assessed from the frequency count values given in Table 5.4. The advantages of replicating the design four times are evident from the results given in Table 5.4 for $k = 4$. The loss of efficiency is less than 6% for all five configurations of two missing observations.

Table 5.5 covers the many cases of three missing values and their average variances. For a single square, there are only 8 possible cases and Case 1 is not possible for a design based on two squares. The average variances of pairwise treatment differences for $k = 1, 2$, and 3 are $0.5\sigma^2$, $0.25\sigma^2$, and $0.167\sigma^2$ respectively when the design is complete. The results in Table 5.5 for a single square represent a reduction in efficiency of between 30% and 60%. When the design consists of two replicates, the efficiencies are reduced by between 14% and 22%, and for $k = 3$ the loss in efficiency due to three missing observations varies between 8% and 15%. Evidently, if there is the possibility that some observations might be lost in the experiment, replication of the Latin squares, particularly small squares, is advisable to minimise the impact of the missing data. The relative efficiencies for the resulting designs in the sixteen cases shows the severity of the loss of three observations from the starting design.

5.4 Discussion

The effect of losing observations from designs formed from k Latin squares has been considered by evaluating the alterations to the information matrix for the treatment effects. Designs resulting from all possible configurations of up to three missing values have been investigated and formulae for the pairwise and overall average variances of the treatment differences for some cases have been determined. These results extend work from Chapter 4 in which a single replicate of a Latin square was considered. The advantages of using replication to overcome the considerable loss of efficiency encountered when observations are lost from small designs, has been supported by the numerical example used to illustrate the results for up to four replicated squares.

Chapter 6

Youden Square designs and the loss of data

If a balanced incomplete block (BIB) design for v treatments in b blocks (rows) with $b = v$ and $k < v$ plots per row, can be arranged such that each column contains every treatment once, then it is called a Youden square. In this case k, the number of plots in each row, is equal to r, the number of replicates of each treatment. A Youden square can be used as a row-column design with the row component a BIB design and the column component a RBD. The design has three parameters $v(= b)$, $k(= r)$, and λ, the number of times each pair of treatments occurs together in the same row (or block) of the starting design. These three design parameters satisfy the relationship $\lambda = k(k-1)/(v-1)$.

A Youden square is a row-column design, and consequently the general form of the information matrix for treatment effects is similar to the Latin square based designs covered in Chapters 4 and 5, and is given by

$$\mathbf{C} = \mathbf{t}^{\delta} - \mathbf{N}_1\mathbf{r}^{-\delta}\mathbf{N}_1' - (\mathbf{N}_2 - \mathbf{N}_1\mathbf{r}^{-\delta}\mathbf{N}_3')(\mathbf{c}^{\delta} - \mathbf{N}_3\mathbf{r}^{-\delta}\mathbf{N}_3')^{-}(\mathbf{N}_2' - \mathbf{N}_3\mathbf{r}^{-\delta}\mathbf{N}_1') \qquad (6.1)$$

where the individual matrices have been defined in previous Chapters. The robustness of a Youden square design to the loss of data is investigated using the same approach as for block designs and Latin square designs. Das and Kageyama (1992) considered the removal of a complete row, i.e. one block of the BIB component, from a Youden square design, and derived the form of the information matrix for treatment effects. They obtained the average variance of the pairwise treatment differences and the relative efficiency using the non-zero eigenvalues of this information matrix. The results of Das and Kageyama (1992) are complemented in this Chapter with formulae for the individual variances of pairwise treatment comparisons. These expressions are used to calculate the effect of losing a single block from a wide range of Youden square designs. It will be shown that the loss of efficiency is substantial for small starting designs, especially when the number of treatments is less than eight.

The case of one missing value is studied theoretically for designs based on a single Youden square. Formulae for the variances of individual treatment comparisons and the number of each *type* of comparison are derived using the generalised inverse approach and are expressed in terms of the three design parameters. Numerical results are given when two observations are missing from a variety of different Youden squares. It is shown that frequencies of the eight cases of two missing values depend on the form of the initial design as well as the design parameters. The results for the loss of a complete block of observations and missing data scattered throughout the starting design are reported by Prescott and Mansson (2001a) and Mansson and Prescott (2001a) respectively.

6.1 Loss of complete treatments

Hedayat and John (1974) and Most (1975) considered the construction of BIB designs that were robust to the loss of all observations corresponding to one or more treatments. The necessary conditions for robust designs to exist were derived, where robustness was defined as variance balance of the resulting design. The ideas in these papers and others can be applied to designs based on Youden squares.

Let the design d be a Youden square design, where the row component is a symmetric BIB (v, k, λ) design, and the column component is a randomised block design with v treatments in k blocks of v plots. Consider the loss of one treatment, and denote the missing treatment by x. The column component now has $v-1$ treatments in k blocks, but it remains a RBD with $v-1$ treatments independent of the treatment that becomes unavailable. The row component can be partitioned into two sets of blocks using the missing treatment. Let the set of blocks that contain the missing treatment x be denoted by $d(x)$, and the set of the remaining blocks be $d(\bar{x})$. Denote the design formed by removing the occurrences of the treatment x by $d'(x)$. Hedayat and John (1974) gave two necessary conditions based on the form of the information matrix for treatment effects for a BIB design to be variance balanced. It must be possible to express the information matrix in the form $c_1\mathbf{I}+c_2\mathbf{J}$, and the non-zero eigenvalues of this matrix are consequently all equal. A design d is classified as locally resistant of degree n if the resulting design is variance balanced with respect to some of the subsets of n treatments only. The design is globally resistant of degree n if all resulting designs formed by the loss of any subset of n treatments are variance balanced. A susceptible design is one that is neither locally or globally resistant of any degree.

To investigate the effect of losing a complete treatment, consider the two subdesigns of the row component of the resulting design, $d'(x)$ and $d(\bar{x})$, separately. The part of Equation (6.1) that relates to the information matrix for treatment effects of the row component is $\mathbf{t}^{\delta} - \mathbf{N}_1\mathbf{r}^{-\delta}\mathbf{N}_1'$, and this remains variance balanced after the removal of one treatment when all elementary contrasts are estimated with the same precision. The information matrices for $d'(x)$ and $d(\bar{x})$ can be derived individually and then added to find the form of $\mathbf{t}^{\delta} - \mathbf{N}_1\mathbf{r}^{-\delta}\mathbf{N}_1'$ for a particular resulting design.

Consider a Youden square design d with seven treatments replicated four times, where all pairs of treatments occur together in two of rows of the initial design. One particular layout of this design is shown below, and the two subdesigns formed by the removal of treatment 1 are also given.

d	1	2	3	5	$d'(x)$	2	3	5	$d(\bar{x})$	2	3	4	6
	2	3	4	6		4	5	6		3	4	5	7
	3	4	5	7		6	7	3		5	6	7	2
	4	5	6	1		7	2	4					
	5	6	7	2									
	6	7	1	3									
	7	1	2	4									

All pairs of treatments do not occur together the same number of times in either of the subdesigns. If the information matrix for treatment effects is computed, its forms is

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2.8137 & -0.5833 & -0.5833 & -0.5833 & -0.4804 & -0.5833 \\ 0 & -0.5833 & 2.8137 & -0.4804 & -0.5833 & -0.5833 & -0.5833 \\ 0 & -0.5833 & -0.4804 & 2.8137 & -0.5833 & -0.5833 & -0.5833 \\ 0 & -0.5833 & -0.5833 & -0.5833 & 2.8137 & -0.5833 & -0.4804 \\ 0 & -0.4804 & -0.5833 & -0.5833 & -0.5833 & 2.8137 & -0.5833 \\ 0 & -0.5833 & -0.5833 & -0.5833 & -0.4804 & -0.5833 & 2.8137 \end{bmatrix} \tag{6.2}$$

and the partition of the matrix relating to treatments two to seven cannot be expressed as $c_1\mathbf{I}+c_2\mathbf{J}$, because the off-diagonal elements are not all equal. This design is not locally resistant to the loss of treatment 1. An example of a design that is variance balanced is the Youden square formed by removing the last column from the standard (6×6) Latin square. The structure of the row component of this design and the two subdesigns created by the loss of the first treatment are

$$\begin{array}{cccccc} d & 1 & 2 & 3 & 4 & 5 \\ & 6 & 1 & 2 & 3 & 4 \\ & 5 & 6 & 1 & 2 & 3 \\ & 4 & 5 & 6 & 1 & 2 \\ & 3 & 4 & 5 & 6 & 1 \\ & 2 & 3 & 4 & 5 & 6 \end{array} \qquad \begin{array}{ccccc} d'(x) & 2 & 3 & 4 & 5 \\ & 6 & 2 & 3 & 4 \\ & 5 & 6 & 2 & 3 \\ & 4 & 5 & 6 & 2 \\ & 3 & 4 & 5 & 6 \end{array} \qquad \begin{array}{cccccc} d(\bar{x}) & 2 & 3 & 4 & 5 & 6 \end{array}$$

When the design is complete, the information matrix for treatment effects can be written in the form $4.8\mathbf{I}_6 - 0.8\mathbf{J}_{6,6}$. The information matrix for the reduced design $d'(x) \cup d(\bar{x})$ is $4.7368\mathbf{I}_5 - 0.9474\mathbf{J}_{5,5}$, and the design is locally resistant of degree one to the loss of treatment 1. The two subdesigns $d'(x)$ and $d(\bar{x})$ are both balanced block designs when considered individually. Their information matrices can therefore be expressed as a combination of an identity matrix and a matrix of ones. It can also be shown that this design is locally resistant to the loss of any of its six treatment, so the design is also globally resistant of degree one.

Conjecture 6.1 *The row component of a Youden square design, which is a symmetric BIB design, is variance balanced after the loss of a treatment if both the subdesigns $d'(x)$*

and $d(\bar{x})$ are balanced designs.

Proof: The starting design is a symmetric BIB design, so a pair of treatments occurs together in λ rows of the initial design d. The subdesign $d'(x)$ has $v-1$ treatments allocated to the $k-1$ plots of k blocks, and all these treatments have λ replicates. The other subdesign $d(\bar{x})$ also has $v-1$ treatments arranged in $v-k$ blocks of k plots, and every treatment in $d(\bar{x})$ is replicated $k-\lambda$ times.

Consider the two subdesigns separately. A pair of treatments, say h and i, occurs together in $\lambda_{hi}(d'(x))$ of the set of blocks that used to contain the missing treatment, and in $\lambda_{hi}(d(\bar{x}))$ of the remaining blocks of the starting design. If these pairing parameters are both independent of h and i then the two subdesigns are pairwise balanced, and consequently the off-diagonal elements of the concurrence matrices are the same for both subdesigns. The information matrix for treatment effects of the resulting design $d'(x) \cup d(\bar{x})$ can also be expressed in the form $c_1\mathbf{I} + c_2\mathbf{J}$. ∎

The subdesign $d'(x)$ is a BIB $(v-1, k, \lambda, k-1, \lambda(k-2)/(v-2))$ design if the original design d is locally resistant to the loss of a given treatment x. A consequence of this condition is that $k \geq v-1$ and $\lambda > 1$, which substantially reduces the number of designs that may be locally/globally resistant. When $\lambda = 1$, it is impossible for both of the subdesigns to have all treatment comparisons within their blocks. All Youden square designs where $k = v-1$ can be constructed by removing one column from a Latin square of side v.

There are k blocks in $d(x)$, and, after a complete treatment is lost there are $k-1$ plots and $(k-1)(k-2)/2$ treatment comparisons within every block. Hence there are $k(k-1)(k-2)/2$ comparisons over the k blocks of the subdesign. The resulting design has $v-1$ treatments, so that there are now $(v-1)(v-2)/2$ elementary pairwise contrasts involving the treatments. The number of available comparisons in $d'(x)$ must be a positive multiple of the total number of pairwise treatment comparisons for the design to be variance balanced.

The column component also needs to be studied to decide whether a given design is resistant to the loss of a treatment. This component corresponds to the remainder of Equation (6.1), which is given by

$$(\mathbf{N}_2 - \mathbf{N}_1\mathbf{r}^{-\delta}\mathbf{N}_3')(\mathbf{c}^{\delta} - \mathbf{N}_3\mathbf{r}^{-\delta}\mathbf{N}_3')^{-}(\mathbf{N}_2' - \mathbf{N}_3\mathbf{r}^{-\delta}\mathbf{N}_1') \tag{6.3}$$

Conjecture 6.2 *The Youden square design d is variance balanced after the loss of a treatment if both subdesign, $d'(x)$ and $d(\bar{x})$, are balanced and $d'(x)$ is a BIB $(v-1, k, \lambda, k-1, \lambda(k-2)/(v-2))$ design.*

Proof: Consider the middle matrix $(\mathbf{c}^\delta - \mathrm{N}_3\mathbf{r}^{-\delta}\mathrm{N}_3')$ separately. For any Youden square design after the removal of the first treatment, assumed without loss of generality to occur once in each of the first k rows of the initial design, the three matrices in this expression are given by

$$\mathbf{c}^\delta = (v-1)\mathbf{I}_k$$
$$\mathbf{r}^{-\delta} = \begin{bmatrix} \frac{1}{(k-1)}\mathbf{I}_k & \mathbf{0}_{k,v-k} \\ \mathbf{0}_{v-k,k} & \frac{1}{k}\mathbf{I}_{v-k} \end{bmatrix}$$
$$\mathrm{N}_3 = \begin{bmatrix} (\mathbf{J}_{k,k} - \mathbf{I}_k) & \mathbf{J}_{k,v-k} \end{bmatrix}$$

The matrix $(\mathbf{c}^\delta - \mathrm{N}_3\mathbf{r}^{-\delta}\mathrm{N}_3')$ can therefore be expressed as

$$\left\{(v-1) - \frac{1}{k-1}\right\}\mathbf{I}_k + \left\{\frac{(k-2)}{(k-1)} + \frac{(v-k)}{k}\right\}\mathbf{J}_{k,k} \tag{6.4}$$

A generalised inverse is necessary, and the choice is arbitrary once the generalised inverse is pre- and post-multiplied by $(\mathrm{N}_2 - \mathrm{N}_1\mathbf{r}^{-\delta}\mathrm{N}_3')$ and $(\mathrm{N}_2' - \mathrm{N}_3\mathbf{r}^{-\delta}\mathrm{N}_1')$ respectively. A sensible choice of generalised inverse is $(v-1-(k-1)^{-1})^{-1}\mathbf{I}_k$. With this generalised inverse, Equation (6.3) reduces (apart from a multiplier $(v-1-(k-1)^{-1})^{-1}$) to

$$(\mathrm{N}_2 - \mathrm{N}_1\mathbf{r}^{-\delta}\mathrm{N}_3')(\mathrm{N}_2' - \mathrm{N}_3\mathbf{r}^{-\delta}\mathrm{N}_1') = \mathrm{N}_2\mathrm{N}_2' - 2\mathrm{N}_2\mathrm{N}_3\mathbf{r}^{-\delta}\mathrm{N}_1' + \mathrm{N}_1\mathbf{r}^{-\delta}\mathrm{N}_3'\mathrm{N}_3\mathbf{r}^{-\delta}\mathrm{N}_1' \tag{6.5}$$

The column component of the resulting design is a randomised block design with $v-1$ treatments and k blocks of $v-1$ plots. Therefore, the matrix $\mathrm{N}_2\mathrm{N}_2'$ can be written in the form

$$\mathrm{N}_2\mathrm{N}_2' = \begin{bmatrix} \mathbf{0}_k' \\ \mathbf{J}_{v-1,k} \end{bmatrix} \begin{bmatrix} \mathbf{0}_k & \mathbf{J}_{k,v-1} \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{0}_{v-1}' \\ \mathbf{0}_{v-1} & k\mathbf{J}_{v-1,v-1} \end{bmatrix} \tag{6.6}$$

and for any Youden square design, the second term of Equation (6.5) is given by

$$\mathrm{N}_2\mathrm{N}_3\mathbf{r}^{-\delta}\mathrm{N}_1' = \begin{bmatrix} 0 & \mathbf{0}_{v-1}' \\ \mathbf{0}_{v-1} & k\mathbf{J}_{v-1,v-1} \end{bmatrix} \tag{6.7}$$

The last part of the expression depends on the structure of the two subdesigns, $d'(x)$ and $d(\bar{x})$. In general, it can be shown that

$$\mathbf{r}^{-\delta}\mathrm{N}_3'\mathrm{N}_3\mathbf{r}^{-\delta} = \begin{bmatrix} \frac{1}{(k-2)^2}\mathbf{I}_k + \frac{(k-2)}{(k-1)^2}\mathbf{J}_{k,k} & \frac{1}{k}\mathbf{J}_{k,v-k} \\ \frac{1}{k}\mathbf{J}_{v-k,k} & \frac{1}{k}\mathbf{J}_{v-k,v-k} \end{bmatrix} \tag{6.8}$$

and when this is pre- and post-multiplied by the matrices N_1 and N_1' respectively, the resulting matrix can be written as

$$\begin{bmatrix} 0 & \mathbf{0}_{v-1}' \\ \mathbf{0}_{v-1} & \frac{1}{(k-1)^2}\mathbf{I}_{v-1} + \frac{k^4-2k^3+k^2-k+k\lambda-\lambda^2}{k(k-1)^2}\mathbf{J}_{v-1,v-1} \end{bmatrix} \tag{6.9}$$

if the pairing parameters $\lambda_{hi}(d'(x))$ and $\lambda_{hi}(d(\bar{x}))$ of the two subdesigns are independent of the treatments h and i. If all the preceding matrices are combined and the resulting matrix simplified, the expression $(\mathrm{N}_2 - \mathrm{N}_1\mathbf{r}^{-\delta}\mathrm{N}_3')(\mathbf{c}^{\delta} - \mathrm{N}_3\mathbf{r}^{-\delta}\mathrm{N}_3')^{-}(\mathrm{N}_2' - \mathrm{N}_3\mathbf{r}^{-\delta}\mathrm{N}_1')$ is given by

$$\begin{bmatrix} 0 & \mathbf{0}_{v-1}' \\ \mathbf{0}_{v-1} & \frac{1}{(vk-v-k)(k-1)}\mathbf{I}_{v-1} + \frac{\lambda k-\lambda^2-k}{k(k-1)(vk-v-k)}\mathbf{J}_{v-1,v-1} \end{bmatrix} \tag{6.10}$$

When this is combined with the information matrix for treatment effects for the row component, it can be seen that it has the required form for the design to be locally resistant to the loss of treatment 1. The same argument can be used for any of the other treatments. ■

The conditions in this Section indicate that for a Youden square design to be locally or globally resistant of degree one, the blocks have no fewer than $v-1$ plots, i.e. $k \geq v-1$. Therefore a globally resistant design can be constructed by removing a column from any Latin square design.

6.2 Loss of a row

Das and Kageyama (1992) derived the information matrix for treatment effects when a row of k observations becomes unavailable in a Youden square design. Equation (4.2) can be used in this situation, because the two identities $\mathbf{tr}'/n = \mathrm{N}_2\mathbf{c}^{-\delta}\mathrm{N}_3$ and $\mathbf{tc}'/n = \mathrm{N}_1\mathbf{r}^{-\delta}\mathrm{N}_3'$ are both satisfied for this configuration of missing data. Assuming, without loss

Table 6.1: Variances of pairwise treatment comparisons in a Youden square design when a single block containing treatments $1, \cdots, k$ is removed from the initial design. These results are the same for the loss of any block in the starting design, but the individual treatments in the comparisons will vary based on whether they were in the affected block.

Individual Pairwise Variances				
	Treatment i	Treatment j	Variance	Number of Comparisons
(a)	$1, \cdots, k$	$1, \cdots, k$	$\frac{2(v-1)}{v(k-2)}\sigma^2$	$k(k-1)/2$
(b)	$1, \cdots, k$	$k+1, \cdots, v$	$\frac{(2k^2-3k-1)(v-1)}{vk(k-1)(k-2)}\sigma^2$	$k(v-k)$
(c)	$k+1, \cdots, v$	$k+1, \cdots, v$	$\frac{2(v-1)}{v(k-1)}\sigma^2$	$(v-k)(v-k-1)/2$
Average Variance			$\frac{2(vk-2v+1)}{v(k-1)(k-2)}\sigma^2$	
Relative Efficiency			$\frac{(v-1)(k-2)}{(vk-2v+1)}$	

of generality, that the first row contains one replicate of each of the first k treatments, the information matrix for treatment effects, as given by Das and Kageyama (1992), can be expressed in the form

$$\mathbf{C} = \begin{bmatrix} \{\frac{\lambda v}{k} - \frac{v}{(v-1)}\}\mathbf{I}_k - \{\frac{\lambda}{k} - \frac{v}{k(v-1)}\}\mathbf{J}_{k,k} & -\frac{\lambda}{k}\mathbf{J}_{k,v-k} \\ -\frac{\lambda}{k}\mathbf{J}_{v-k,k} & \frac{\lambda v}{k}\mathbf{I}_{v-k} - \frac{\lambda}{k}\mathbf{J}_{v-k,v-k} \end{bmatrix} \tag{6.11}$$

The eigenvalues of this information matrix can be found using Lemma (3.3), and they are given in Das and Kageyama (1992). A generalised inverse can be identified to solve the reduced normal equations and to derive the variances of pairwise treatment comparisons. The choice of generalised inverse is arbitrary, and after adding $\frac{\lambda}{k}\mathbf{J}_{v,v}$ to the information matrix and inverting the resulting non-singular matrix, we obtain a particular generalised inverse of $\mathbf{C}$ as

$$\mathbf{\Omega} = \begin{bmatrix} \frac{k(v-1)}{(\lambda v^2-\lambda v-vk)}\mathbf{I}_k - \frac{kv}{\lambda v(\lambda v^2-\lambda v-vk)}\mathbf{J}_{k,k} & \mathbf{0}_{k,v-k} \\ \mathbf{0}_{v-k,k} & \frac{k}{\lambda v}\mathbf{I}_{v-k} \end{bmatrix} \tag{6.12}$$

Table 6.1 shows particular variances of the pairwise treatment comparisons in terms of v, the number of treatments, and k, the number of plots in each row, for the three separate types of comparisons depending on whether the two treatments occurred in the missing row (or block) of the starting design. The comparisons are based on losing one replicate

Table 6.2: Average variances (A.V.) and relative efficiencies (R.E.) of pairwise treatment differences for a variety of Youden square designs after the loss of one row (block) of observations containing treatments $1, \cdots, k$.

Parameters				Pairwise treatment differences		
$v = b$	$k = r$	λ	A.V.(R.E.)	(a)	(b)	(c)
4	3	2	$1.2500\sigma^2$ (0.6000)	$1.5000\sigma^2$ (0.5000)	$1.0000\sigma^2$ (0.7500)	
5	4	3	$0.7333\sigma^2$ (0.7273)	$0.8000\sigma^2$ (0.6666)	$0.6333\sigma^2$ (0.8421)	
6	5	4	$0.5278\sigma^2$ (0.7895)	$0.5556\sigma^2$ (0.7500)	$0.4722\sigma^2$ (0.8825)	
7	3	1	$1.1429\sigma^2$ (0.7500)	$1.7143\sigma^2$ (0.5000)	$1.1429\sigma^2$ (0.7500)	$0.8571\sigma^2$ (1.0000)
7	4	2	$0.7143\sigma^2$ (0.8000)	$0.8571\sigma^2$ (0.6667)	$0.6786\sigma^2$ (0.8420)	$0.5714\sigma^2$ (1.0000)
8	7	6	$0.3417\sigma^2$ (0.8537)	$0.3500\sigma^2$ (0.8334)	$0.3167\sigma^2$ (0.9211)	
9	8	7	$0.2910\sigma^2$ (0.8727)	$0.2963\sigma^2$ (0.8572)	$0.2725\sigma^2$ (0.9321)	
10	9	8	$0.2536\sigma^2$ (0.8873)	$0.2571\sigma^2$ (0.8751)	$0.2393\sigma^2$ (0.9402)	
11	6	3	$0.4091\sigma^2$ (0.8889)	$0.4545\sigma^2$ (0.8000)	$0.4015\sigma^2$ (0.9056)	$0.3636\sigma^2$ (1.0000)
11	10	9	$0.2247\sigma^2$ (0.8989)	$0.2273\sigma^2$ (0.8887)	$0.2134\sigma^2$ (0.9466)	
13	4	1	$0.6923\sigma^2$ (0.8889)	$0.9231\sigma^2$ (0.6667)	$0.7308\sigma^2$ (0.8421)	$0.6154\sigma^2$ (1.0000)
13	9	6	$0.2527\sigma^2$ (0.9130)	$0.2637\sigma^2$ (0.8752)	$0.2454\sigma^2$ (0.9405)	$0.2308\sigma^2$ (1.0000)
15	7	3	$0.3378\sigma^2$ (0.9211)	$0.3733\sigma^2$ (0.8334)	$0.3378\sigma^2$ (0.9211)	$0.3111\sigma^2$ (1.0000)
15	8	4	$0.2889\sigma^2$ (0.9231)	$0.3111\sigma^2$ (0.8573)	$0.2861\sigma^2$ (0.9322)	$0.2667\sigma^2$ (1.0000)
16	6	2	$0.4062\sigma^2$ (0.9231)	$0.4688\sigma^2$ (0.7999)	$0.4141\sigma^2$ (0.9056)	$0.3750\sigma^2$ (1.0000)
16	10	6	$0.2240\sigma^2$ (0.9302)	$0.2344\sigma^2$ (0.8887)	$0.2201\sigma^2$ (0.9464)	$0.2083\sigma^2$ (1.0000)
19	9	4	$0.2519\sigma^2$ (0.9403)	$0.2707\sigma^2$ (0.8748)	$0.2519\sigma^2$ (0.9401)	$0.2368\sigma^2$ (1.0000)
19	10	5	$0.2237\sigma^2$ (0.9412)	$0.2368\sigma^2$ (0.8889)	$0.2224\sigma^2$ (0.9465)	$0.2105\sigma^2$ (1.0000)
21	5	1	$0.5079\sigma^2$ (0.9375)	$0.6349\sigma^2$ (0.7500)	$0.5397\sigma^2$ (0.8823)	$0.4762\sigma^2$ (1.0000)
23	11	5	$0.2010\sigma^2$ (0.9519)	$0.2126\sigma^2$ (0.8998)	$0.2010\sigma^2$ (0.9517)	$0.1913\sigma^2$ (1.0000)
25	9	3	$0.2514\sigma^2$ (0.9545)	$0.2743\sigma^2$ (0.8750)	$0.2552\sigma^2$ (0.9404)	$0.2400\sigma^2$ (1.0000)
27	13	6	$0.1672\sigma^2$ (0.9597)	$0.1751\sigma^2$ (0.9166)	$0.1672\sigma^2$ (0.9599)	$0.1605\sigma^2$ (1.0000)
31	6	1	$0.4032\sigma^2$ (0.9600)	$0.4839\sigma^2$ (0.8000)	$0.4274\sigma^2$ (0.9057)	$0.3871\sigma^2$ (1.0000)
31	10	3	$0.2231\sigma^2$ (0.9639)	$0.2419\sigma^2$ (0.8892)	$0.2272\sigma^2$ (0.9467)	$0.2151\sigma^2$ (1.0000)
31	15	7	$0.1432\sigma^2$ (0.9653)	$0.1489\sigma^2$ (0.9281)	$0.1432\sigma^2$ (0.9653)	$0.1382\sigma^2$ (1.0000)

of the first k treatments. Table 6.2 shows these variances computed for a range of Youden square designs based on some of the symmetric BIB designs listed by Raghavarao (1971, Table 5.10.1). The pairwise treatment differences (a), (b), and (c) in Table 6.2 correspond to the particular treatment comparisons listed in Table 6.1. When $v = k+1$, corresponding to Youden square designs constructed by removing a single column from a Latin square of side v, only comparisons (a) and (b) are possible, because there is only one treatment that does not lose a replicate.

6.3 Loss of one observation

The effect of the loss of one plot from a Youden square design can also be studied by considering the alterations to the components of the information matrix for treatment effects. When the design is based on a single Youden square, there are vk units and consequently vk potential realisable designs each with one missing value. The overall effect of the missing data is the same for each of the realisable resulting designs, but the variances of particular treatment comparisons vary. The derivation of the normal equations is similar for all of these possible resulting designs, so we may assume that the missing observation corresponds to treatment 1 in the first column and first row of the starting design. The following four matrices are required to derive the form of the information matrix for treatment effects corresponding to Equation (6.1).

$$\mathbf{t}^{\delta} = \begin{bmatrix} (k-1) & \mathbf{0}'_{v-1} \\ \mathbf{0}_{v-1} & k\mathbf{I}_{v-1} \end{bmatrix} \tag{6.13}$$

$$\mathbf{N}_1\mathbf{r}^{-\delta}\mathbf{N}_1' = \begin{bmatrix} \frac{(k-1)}{k} & \frac{(\lambda-1)}{k}\mathbf{1}'_{k-1} & \frac{\lambda}{k}\mathbf{1}'_{v-k} \\ \frac{(\lambda-1)}{k}\mathbf{1}_{k-1} & \frac{(k-\lambda)}{k}\mathbf{I}_{k-1} + \{\frac{(\lambda-1)}{k} + \frac{1}{(k-1)}\}\mathbf{J}_{k-1,k-1} & \frac{\lambda}{k}\mathbf{J}_{k-1,v-k} \\ \frac{\lambda}{k}\mathbf{1}_{v-k} & \frac{\lambda}{k}\mathbf{J}_{v-k,k-1} & \frac{(k-\lambda)}{k}\mathbf{I}_{v-k} + \frac{\lambda}{k}\mathbf{J}_{v-k,v-k} \end{bmatrix} \tag{6.14}$$

$$\mathbf{N}_2 - \mathbf{N}_1\mathbf{r}^{-\delta}\mathbf{N}_3' = \begin{bmatrix} -\frac{(k-1)}{k} & \{1 - \frac{(k-1)}{k}\}\mathbf{1}'_{k-1} \\ \{1 - \frac{(k-1)}{k}\}\mathbf{1}_{k-1} & \{1 - \frac{1}{(k-1)} - \frac{(k-1)}{k}\}\mathbf{J}_{k-1,k-1} \\ \mathbf{0}_{v-k} & \mathbf{0}_{v-k,k-1} \end{bmatrix} \tag{6.15}$$

$$\mathbf{c}^{\delta} - \mathbf{N}_3\mathbf{r}^{-\delta}\mathbf{N}_3' = \begin{bmatrix} (v-1) - \frac{(v-1)}{k} & -\frac{(v-1)}{k}\mathbf{1}_{k-1}' \\ -\frac{(v-1)}{k}\mathbf{1}_{k-1} & v\mathbf{I}_{k-1} - \{\frac{1}{(k-1)} + \frac{(v-1)}{k}\}\mathbf{J}_{k-1,k-1} \end{bmatrix} \tag{6.16}$$

These expressions, when substituted into Equation (6.1), give the adjusted information matrix for treatment effects $\mathbf{C}_{d(1)}$ as

$$\begin{bmatrix} (k-1) - \frac{(k-1)}{k} - \frac{(k-1)}{k(v-1)} & -\frac{(k^2-k-v)}{k(v-1)}\mathbf{1}_{k-1}' & -\frac{\lambda}{k}\mathbf{1}_{v-k}' \\ -\frac{(k^2-k-v)}{k(v-1)}\mathbf{1}_{k-1} & \frac{v(k-1)}{(v-1)}\mathbf{I}_{k-1} - \frac{(k^3-2k^2+k+v)}{k(k-1)(v-1)}\mathbf{J}_{k-1,k-1} & -\frac{\lambda}{k}\mathbf{J}_{k-1,v-k} \\ -\frac{\lambda}{k}\mathbf{1}_{v-k} & -\frac{\lambda}{k}\mathbf{J}_{v-k,k-1} & \frac{v(k-1)}{(v-1)}\mathbf{I}_{v-k} - \frac{\lambda}{k}\mathbf{J}_{v-k,v-k} \end{bmatrix} \tag{6.17}$$

To solve these reduced normal equations, a suitable generalised inverse is required. One practical choice is obtained by adding λ/k to every element of $\mathbf{C}_{d(1)}$ and inverting the resulting non-singular matrix. This particular generalised inverse $\mathbf{\Omega}$ of $\mathbf{C}_{d(1)}$ is given by

$$\mathbf{\Omega} = \begin{bmatrix} \frac{(k^2-k-1)(v-1)}{vk(k-1)(k-2)} & -\frac{(v-1)}{vk(k-1)(k-2)}\mathbf{1}_{k-1}' & \mathbf{0}_{v-k}' \\ -\frac{(v-1)}{vk(k-1)(k-2)}\mathbf{1}_{k-1} & \frac{(v-1)}{v(k-1)}\mathbf{I}_{k-1} + \frac{(v-1)}{vk(k-1)^2(k-2)}\mathbf{J}_{k-1,k-1} & \mathbf{0}_{k-1,v-k} \\ \mathbf{0}_{v-k} & \mathbf{0}_{v-k,k-1} & \frac{(v-1)}{v(k-1)}\mathbf{I}_{v-k} \end{bmatrix} \tag{6.18}$$

The variances of individual pairwise treatment differences can be identified using this generalised inverse. These are shown in Table 6.3 for a Youden square with one replicate of the first treatment missing, where the row from which the observation was removed contained the first k treatments. It is possible to show that Youden square designs created by removing a column from a Latin square have no pairwise comparisons corresponding to Case (e) when a single observation becomes unavailable, because $v = k + 1$ in this situation.

The average variance of pairwise treatment comparisons for a Youden square with v treatments with a single value missing is given by

$$\frac{2(vk - 2v - k + 3)}{v(k-1)(k-2)}\sigma^2 \tag{6.19}$$

Table 6.4 shows individual variances of pairwise treatment comparisons, the number of each type of comparison, and the average of all these variances, when one observation on treatment 1 is missing for the range of Youden squares tabulated by Raghavarao (1971) and used in Table 6.2. The pairwise comparisons (a) to (e) correspond to the treatment

Table 6.3: Variances of the individual treatment comparisons when the observation on treatment 1 is removed from a block containing the first k treatments in a Youden square design.

One missing value (t=1)				
	Treatment i	Treatment j	Variance	Number of Comparisons
(a)	1	$2, \cdots, k$	$\frac{(v-1)(2k^2-5k+4)}{v(k-1)^2(k-2)}\sigma^2$	$k-1$
(b)	1	$k+1, \cdots, v$	$\frac{(v-1)(2k^2-3k-1)}{vk(k-1)(k-2)}\sigma^2$	$v-k$
(c)	$2, \cdots, k$	$2, \cdots, k$	$\frac{2(v-1)}{v(k-1)}\sigma^2$	$(k-1)(k-2)/2$
(d)	$2, \cdots, k$	$k+1, \cdots, v$	$\frac{(v-1)(2k^3-6k^2+4k+1)}{vk(k-1)^2(k-2)}\sigma^2$	$(k-1)(v-k)$
(e)	$k+1, \cdots, v$	$k+1, \cdots, v$	$\frac{2(v-1)}{v(k-1)}\sigma^2$	$(v-k)(v-k-1)/2$

differences listed in Table 6.3. The comparisons (c) and (e) can be combined into a single column in this Table because the numerical values are the same for these two types of comparisons. The loss of efficiency is more than 5% for only the first five Youden square designs in Table 6.4. The majority of the designs are relatively robust to the loss of a single observation.

6.4 The unavailability of two or more values

In Chapter 4 it was shown that when two observations are removed from Latin square based designs with v treatments, the distribution of the average variances of pairwise treatment differences is not related to the structure of the initial design. There are three distinct configurations of missing values for a single Latin square, and five for designs based on two or more squares. When the design is based on a Youden square different cases of resulting designs, all with different properties, also occur. The number of each type of design resulting from the loss of two observations depends on the structure of the starting design, even for designs with the same design parameters. These configurations are based on whether the missing values occur in the same or different columns, which treatments are affected, and when the two observations occur in different rows, the number of treatments common to the affected rows of the initial design.

Table 6.4: Variances of pairwise treatment differences and their efficiencies, frequencies, and the overall average variance (A.V.) for a variety of Youden square designs after the loss of one observation on treatment 1.

Parameters				Variances of pairwise treatment differences (R.E.)							
v	k	λ	A.V.(R.E.)	(a)		(c,e)		(b)		(d)	
4	3	2	$1.0000\sigma^2$ (0.7500)	$1.3125\sigma^2$ (0.5714)	2	$1.0000\sigma^2$ (0.7500)	1	$0.7500\sigma^2$ (1.0000)	1	$0.8125\sigma^2$ (0.9231)	2
5	4	3	$0.6000\sigma^2$ (0.8889)	$0.7111\sigma^2$ (0.7500)	3	$0.6333\sigma^2$ (0.8421)	1	$0.5333\sigma^2$ (1.0000)	3	$0.5444\sigma^2$ (0.9796)	3
6	5	4	$0.4444\sigma^2$ (0.9375)	$0.5035\sigma^2$ (0.8276)	4	$0.4722\sigma^2$ (0.8824)	1	$0.4167\sigma^2$ (1.0000)	6	$0.4201\sigma^2$ (0.9917)	4
7	3	1	$1.0000\sigma^2$ (0.8571)	$1.5000\sigma^2$ (0.5714)	2	$1.1429\sigma^2$ (0.7500)	4	$0.8571\sigma^2$ (1.0000)	7	$0.9286\sigma^2$ (0.9231)	8
7	4	2	$0.6190\sigma^2$ (0.9231)	$0.7619\sigma^2$ (0.7500)	3	$0.6786\sigma^2$ (0.8421)	3	$0.5714\sigma^2$ (1.0000)	6	$0.5833\sigma^2$ (0.9796)	9
8	7	6	$0.3000\sigma^2$ (0.9722)	$0.3257\sigma^2$ (0.8955)	6	$0.3167\sigma^2$ (0.9211)	1	$0.2917\sigma^2$ (1.0000)	15	$0.2924\sigma^2$ (0.9976)	6
9	8	7	$0.2593\sigma^2$ (0.9796)	$0.2782\sigma^2$ (0.9130)	7	$0.2725\sigma^2$ (0.9320)	1	$0.2540\sigma^2$ (1.0000)	21	$0.2543\sigma^2$ (0.9985)	7
10	9	8	$0.2286\sigma^2$ (0.9844)	$0.2431\sigma^2$ (0.9256)	8	$0.2393\sigma^2$ (0.9403)	1	$0.2250\sigma^2$ (1.0000)	28	$0.2252\sigma^2$ (0.9990)	8
11	6	3	$0.3727\sigma^2$ (0.9756)	$0.4182\sigma^2$ (0.8696)	5	$0.4015\sigma^2$ (0.9057)	5	$0.3636\sigma^2$ (1.0000)	20	$0.3652\sigma^2$ (0.9959)	25
11	10	9	$0.2045\sigma^2$ (0.9877)	$0.2160\sigma^2$ (0.9351)	9	$0.2134\sigma^2$ (0.9467)	1	$0.2020\sigma^2$ (1.0000)	36	$0.2022\sigma^2$ (0.9993)	9
13	4	1	$0.6410\sigma^2$ (0.9600)	$0.8205\sigma^2$ (0.7500)	3	$0.7308\sigma^2$ (0.8421)	9	$0.6154\sigma^2$ (1.0000)	39	$0.6282\sigma^2$ (0.9796)	27
13	9	6	$0.2335\sigma^2$ (0.9882)	$0.2493\sigma^2$ (0.9256)	8	$0.2454\sigma^2$ (0.9403)	4	$0.2308\sigma^2$ (1.0000)	34	$0.2310\sigma^2$ (0.9990)	32
15	7	3	$0.3156\sigma^2$ (0.9859)	$0.3474\sigma^2$ (0.8955)	6	$0.3378\sigma^2$ (0.9211)	8	$0.3111\sigma^2$ (1.0000)	43	$0.3119\sigma^2$ (0.9976)	48
15	8	4	$0.2698\sigma^2$ (0.9882)	$0.2921\sigma^2$ (0.9130)	7	$0.2861\sigma^2$ (0.9320)	7	$0.2667\sigma^2$ (1.0000)	42	$0.2671\sigma^2$ (0.9985)	49
16	6	2	$0.3812\sigma^2$ (0.9836)	$0.4313\sigma^2$ (0.8696)	5	$0.4141\sigma^2$ (0.9057)	10	$0.3750\sigma^2$ (1.0000)	55	$0.3766\sigma^2$ (0.9959)	50
16	10	6	$0.2101\sigma^2$ (0.9917)	$0.2228\sigma^2$ (0.9351)	9	$0.2201\sigma^2$ (0.9467)	6	$0.2083\sigma^2$ (1.0000)	51	$0.2085\sigma^2$ (0.9993)	54
19	9	4	$0.2387\sigma^2$ (0.9921)	$0.2559\sigma^2$ (0.9256)	8	$0.2519\sigma^2$ (0.9403)	10	$0.2368\sigma^2$ (1.0000)	73	$0.2371\sigma^2$ (0.9990)	80
19	10	5	$0.2120\sigma^2$ (0.9931)	$0.2251\sigma^2$ (0.9351)	9	$0.2224\sigma^2$ (0.9467)	9	$0.2105\sigma^2$ (1.0000)	72	$0.2107\sigma^2$ (0.9993)	81
21	5	1	$0.4841\sigma^2$ (0.9836)	$0.5754\sigma^2$ (0.8276)	4	$0.5397\sigma^2$ (0.8824)	16	$0.4762\sigma^2$ (1.0000)	126	$0.4802\sigma^2$ (0.9917)	64
23	11	5	$0.1923\sigma^2$ (0.9950)	$0.2030\sigma^2$ (0.9424)	10	$0.2010\sigma^2$ (0.9519)	12	$0.1913\sigma^2$ (1.0000)	111	$0.1914\sigma^2$ (0.9995)	120
25	9	3	$0.2414\sigma^2$ (0.9941)	$0.2593\sigma^2$ (0.9256)	8	$0.2552\sigma^2$ (0.9403)	16	$0.2400\sigma^2$ (1.0000)	148	$0.2402\sigma^2$ (0.9990)	128
27	13	6	$0.1611\sigma^2$ (0.9965)	$0.1684\sigma^2$ (0.9531)	12	$0.1672\sigma^2$ (0.9597)	14	$0.1605\sigma^2$ (1.0000)	157	$0.1605\sigma^2$ (0.9997)	168
31	6	1	$0.3903\sigma^2$ (0.9917)	$0.4452\sigma^2$ (0.8696)	5	$0.4274\sigma^2$ (0.9057)	25	$0.3871\sigma^2$ (1.0000)	310	$0.3887\sigma^2$ (0.9959)	125
31	10	3	$0.2159\sigma^2$ (0.9959)	$0.2300\sigma^2$ (0.9351)	9	$0.2272\sigma^2$ (0.9467)	21	$0.2151\sigma^2$ (1.0000)	246	$0.2152\sigma^2$ (0.9993)	189
31	15	7	$0.1386\sigma^2$ (0.9974)	$0.1439\sigma^2$ (0.9604)	14	$0.1432\sigma^2$ (0.9653)	16	$0.1382\sigma^2$ (1.0000)	211	$0.1383\sigma^2$ (0.9998)	224

Table 6.5: Two examples of constructing a Youden square design with five treatments in blocks of four plots, where all pairs of treatments occur together in three blocks of the starting design.

Design 6.5(a)

1	2	3	4
5	1	2	3
4	5	1	2
3	4	5	1
2	3	4	5

Design 6.5(b)

1	2	3	4
2	1	4	5
3	4	5	2
4	5	1	3
5	3	2	1

Table 6.5 shows two Youden square designs with $v = b = 5$, $k = r = 4$, and $\lambda = 3$, identified as Designs 6.5(a) and 6.5(b). For both complete designs, the average variance of pairwise treatment differences is $0.5333\sigma^2$. Each design consists of 20 plots, so there are 20 possible resulting designs when only one observations is unavailable. For both designs and for each of the 20 resulting designs with one missing observation, the average variance of pairwise treatment comparisons increases to $0.6\sigma^2$, corresponding to a reduction in efficiency of 11%. The maximum variance is $0.7111\sigma^2$, which corresponds to a 25% loss of efficiency.

The situation is more complicated when two values are missing. For each design there are 190 ways in which these two observations might be lost, but there are two different distributions for the average and maximum variances for the designs. For Design 6.5(a), there are seven distinct configurations of resulting design with two missing values, but there is an extra case to be considered for Design 6.5(b). Table 6.6 shows average variances of pairwise comparisons, relative efficiencies, maximum variances, and the frequencies of the seven/eight configurations for all of the realisable situations when two observations are unavailable for these two designs. Although the distributions of configurations are different for the two designs, the variances of pairwise treatment comparisons are the same for the seven cases/sub-cases that are common to the resulting designs when two values are unavailable.

The single configuration that occurs only in Design 6.5(b) corresponds to the loss of treatment 2 from row one, column two, and the loss of treatment 5 from row four, column two. In this configuration, the two missing values correspond to different treatments that occur in the same column of the initial design, and neither of the two treatments is common to the pair of rows. Consideration of every other pair of rows shows that there is

Table 6.6: Distinct configurations of two missing observations for the Designs 6.5(a) and 6.5(b), and the average and maximum variances for the cases. The number of configurations of each case for the two different designs are given in the last two columns.

Two missing values (t=2)				
Case	A.V.(R.E.)	M.V.(R.E.)	6.5(a)	6.5(b)
1	$0.666667\sigma^2$ (0.800000)	$0.800000\sigma^2$ (0.666667)	30	30
2(a)i	$0.733333\sigma^2$ (0.727273)	$0.844444\sigma^2$ (0.631579)	0	1
2(a)ii	$0.679167\sigma^2$ (0.785276)	$0.779167\sigma^2$ (0.684492)	20	18
2(a)iii	$0.660952\sigma^2$ (0.806916)	$0.838095\sigma^2$ (0.636364)	20	21
2(b)i	$0.666667\sigma^2$ (0.800000)	$0.733333\sigma^2$ (0.727273)	10	9
2(b)ii	$0.672381\sigma^2$ (0.793201)	$0.864762\sigma^2$ (0.616740)	40	42
2(b)iii	$0.704167\sigma^2$ (0.757396)	$1.066667\sigma^2$ (0.500000)	40	39
2(c)	$0.729167\sigma^2$ (0.731429)	$1.066667\sigma^2$ (0.500000)	30	30

no pair of treatment labels that has a similar configuration of treatments within Design 6.5(b), and also that there are no such configurations in Design 6.5(a). This particular resulting design has the largest average variance of pairwise treatment comparisons, but not the largest individual pairwise variance.

In general, there are a maximum of eight distinct configurations or classes of resulting design with two missing values for any Youden square based design. Within these eight classes, different members can be transformed into any other member of the class by rearranging the rows, columns, or treatment labels of the starting design. These eight classes for two missing observations are described below, firstly by identifying two cases with the missing values occurring in the same row (Case 1), or in different rows (Case 2), and then subdividing Case 2 into a series of different Sub-cases 2(a), 2(b), and 2(c), depending on which treatments are missing and where they are positioned within the starting design. There are three further sub-cases corresponding to 2(a) and 2(b) that depend on whether the missing treatments are common to both the affected rows.

Cases and sub-cases of two missing observations from a general Youden square design

Case 1 Same row of the initial design, necessarily different columns, and also different treatments. There are $vk(k-1)/2$ configurations of this kind for a single replicate Youden square design.

Case 2 Different rows of the starting design.

(a) Same column necessarily different treatments. The number of configurations of two missing observations in this case is $vk(v-1)/2$. Sub-cases correspond to the situations where

i. neither affected treatment occurs in both rows,

ii. one affected treatment occurs in both rows,

iii. the two affected treatments occur in both rows.

(b) Different columns with different treatments. There are $vk(k-1)(v-2)/2$ resulting designs of this type. The sub-cases correspond to the situations where

i. neither affected treatment occurs in both rows,

ii. one affected treatment occurs in both rows,

iii. the two affected treatments occur in both rows.

(c) Different columns, but the same treatment. There are a total of $vk(k-1)/2$ configurations for this sub-case.

Different Youden squares may lead to different frequencies of occurrence of resulting design within these eight classes, as shown by Designs 6.5(a) and 6.5(b). In particular, for a Youden square with $\lambda = 1$ (only one treatment common to all pairs of rows), two of these eight cases, 2(a)iii and 2(b)iii, are not attainable. This is because it is not possible for both of the affected treatments to appear in both of the two rows when $\lambda = 1$.

6.5 Influence of the structure of a Youden square on its robustness to missing values

In this section, we consider a family of Youden squares with similar parameters to illustrate the effect of the structure of the design on the distributions of the designs resulting from the loss of two observations. Consider Youden square designs with seven treatments allocated to the units of seven blocks of four plots, where all treatments are replicated four times. There are six essentially different Youden square design with these parameters

Table 6.7: Representative members of the six isotropy classes of (7×4) Youden squares given by Preece (1995).

Design 1

1	2	3	5
2	3	4	6
3	4	5	7
4	5	6	1
5	6	7	2
6	7	1	3
7	1	2	4

Design 2

1	2	3	5
2	4	6	3
5	3	4	7
6	1	5	4
7	5	2	6
3	6	7	1
4	7	1	2

Design 3

1	2	3	5
6	4	2	3
5	3	7	4
4	5	6	1
2	7	5	6
3	6	1	7
7	1	4	2

Design 4

1	2	3	5
6	3	2	4
5	7	4	3
4	6	5	1
2	5	7	6
3	1	6	7
7	4	1	2

Design 5

1	2	3	5
6	3	2	4
5	4	7	3
4	5	6	1
2	6	5	7
3	7	1	6
7	1	4	2

Design 6

1	2	3	5
6	3	2	4
5	4	7	3
4	5	6	1
2	7	5	6
3	6	1	7
7	1	4	2

which may be constructed from (7×7) Latin squares. Table 6.7 shows representative members of the six main classes of Youden squares as described by Preece (1966, 1995). To study the effect that two missing observations have on these six designs, the distributions of average and maximum variances of pairwise treatment comparisons were computed for all possible configurations of two missing observations for all the six designs.

All the eight cases and sub-cases described earlier are possible for each of these six designs, but the frequencies of occurrence of these cases and sub-cases are different except for Designs 1 and 4 which have identical results. Table 6.8 shows the average variances, relative efficiencies, and maximum variances of pairwise treatment differences for each case/sub-case and also the frequencies for each design.

Although the distributions of the average variances are different for these designs, the numbers of configurations in total corresponding to Cases 2(a) and 2(b) are the same for all six designs. The distributions within these two cases depend on the structure of the Youden square. Design 2 has the largest number of configurations with the smallest average variance, $0.6667\sigma^2$, but it also has the largest number of resulting designs where the average variance is $0.6964\sigma^2$. Design 5 has fewest configurations corresponding to the smallest average variance, 2(a)ii, 2(a)iii, and 2(b)i, but also fewer resulting designs in Sub-case 2(b)iii with an average variance of $0.6964\sigma^2$. The mean values of the average and

Table 6.8: Average pairwise variances, relative efficiencies, maximum pairwise variances and frequencies of occurrence for the eight cases/subcases produced when there are two observations missing from the six (7 × 4) Youden squares. Variances are to be multiplied by σ^2.

Two missing values (t=2)							
Case	A.V.(R.E.)	M.V.(R.E.)	1/4	2	3	5	6
1	0.6667 (0.8571)	0.8571 (0.6667)	42	42	42	42	42
2(a)i	0.6845 (0.8348)	0.8348 (0.6845)	21	9	19	27	25
2(a)ii	0.6667 (0.8571)	0.8286 (0.6897)	42	66	46	30	34
2(a)iii	0.6667 (0.8571)	0.9524 (0.6001)	21	9	19	27	25
2(b)i	0.6667 (0.8571)	0.7857 (0.7273)	63	75	65	57	59
2(b)ii	0.6721 (0.8502)	0.9265 (0.6167)	126	102	122	138	134
2(b)iii	0.6964 (0.8205)	1.1429 (0.5000)	21	33	23	15	17
2(c)	0.7083 (0.8067)	1.1429 (0.5000)	42	42	42	42	42
	Mean average variance		0.6758	0.6758	0.6758	0.6757	0.6758
	Mean maximum variance		0.9169	0.9151	0.9166	0.9177	0.9174

maximum variances of pairwise treatment comparisons for the distributions of two missing values are given in Table 6.8 for the six designs. Although the frequencies are different, the means of the distributions are similar. Ordering the designs from the smallest mean average variance to the largest gives 5, 6, 1/4, 3, and 2, but the ordering is reversed when the mean maximum variance is used. When an experimenter is concerned about the magnitude of the maximum variance of pairwise treatment differences, it is prudent to avoid using Design 2, because it has the largest proportion of resulting designs attaining the maximum variance of $1.1429\sigma^2$, which has an efficiency of only 50%.

All the six designs have the same set of four treatments in each of their seven rows (blocks). The differences in distributions of average variances between the designs are due to the position of the treatments within the columns of the starting design. The sub-cases of Cases 2(a) and 2(b) are defined by whether the affected treatments are common to the pair of blocks in which the two observations occur. To investigate the differences between the distributions of these cases, consider Designs 1 and 2 and the number of configurations in each of the Sub-cases 2(a) and 2(b) for every pair of blocks in the starting designs. Table 6.9 shows these distributions, and it can be seen that for Design 1, the numbers of configurations corresponding to each sub-case is the same for every pair of rows. For each pair of rows (blocks) in Design 1 there are 4 resulting designs in Sub-case 2(a) and

Table 6.9: Distribution of configurations within Cases 2(a) and 2(b) for Designs 1 and 2 of the (7×4) Youden square designs. This identifies where the differences in the numbers of each case occur based on all pairs of rows.

Row		Design 1						Design 2					
		2(a)			2(b)			2(a)			2(b)		
i	j	(i)	(ii)	(iii)	(i)	(ii)	(iii)	(i)	(ii)	(iii)	(i)	(ii)	(iii)
1	2	1	2	1	3	6	1	0	4	0	4	4	2
1	3	1	2	1	3	6	1	0	4	0	4	4	2
1	4	1	2	1	3	6	1	0	4	0	4	4	2
1	5	1	2	1	3	6	1	1	2	1	3	6	1
1	6	1	2	1	3	6	1	1	2	1	3	6	1
1	7	1	2	1	3	6	1	0	4	0	4	4	2
2	3	1	2	1	3	6	1	1	2	1	3	6	1
2	4	1	2	1	3	6	1	0	4	0	4	4	2
2	5	1	2	1	3	6	1	1	2	1	3	6	1
2	6	1	2	1	3	6	1	0	4	0	4	4	2
2	7	1	2	1	3	6	1	1	2	1	3	6	1
3	4	1	2	1	3	6	1	1	2	1	3	6	1
3	5	1	2	1	3	6	1	1	2	1	3	6	1
3	6	1	2	1	3	6	1	0	4	0	4	4	2
3	7	1	2	1	3	6	1	0	4	0	4	4	2
4	5	1	2	1	3	6	1	0	4	0	4	4	2
4	6	1	2	1	3	6	1	1	2	1	3	6	1
4	7	1	2	1	3	6	1	0	4	0	4	4	2
5	6	1	2	1	3	6	1	0	4	0	4	4	2
5	7	1	2	1	3	6	1	0	4	0	4	4	2
6	7	1	2	1	3	6	1	1	2	1	3	6	1
Totals		21	42	21	63	126	21	9	66	9	75	102	33

Table 6.10: Two examples of (13 × 4) Youden square designs that are given by Preece (1966).

Design 1

1	3	4	8
2	4	5	9
3	5	6	10
4	6	7	11
5	7	8	12
6	8	9	13
7	9	10	1
8	10	11	2
9	11	12	3
10	12	13	4
11	13	1	5
12	1	2	6
13	2	3	7

Design 2

1	3	4	8
4	2	5	9
5	6	3	10
6	4	7	11
8	7	12	5
9	8	13	6
7	10	9	1
10	11	8	2
12	9	11	3
13	12	10	4
11	5	1	13
2	1	6	12
3	13	2	7

10 in Sub-case 2(b). These divide into 1, 2, and 1 occurrences of Sub-cases 2(a)i, ii, and iii respectively, and into 3, 6, and 1 occurrence of Sub-cases 2(b)i, ii, and iii respectively. Although this also happens for some of the pairs of rows of Design 2, for some pairs (rows 1 and 2 for example) the partition of the 4 and 10 resulting designs in Sub-cases 2(a) and 2(b) produces 0, 4, and 0 for Sub-cases 2(a)i, ii, and iii and 4, 4, and 2 resulting designs for Sub-cases 2(b)i, ii, and iii respectively. Thus Design 2 has a greater number of occurrences of 2(a)ii and fewer of 2(a)i and 2(a)iii than Design 1, and also a greater number of resulting designs in Sub-cases 2(b)i and 2(b)iii.

Preece (1966) gave two (13 × 4) Youden square designs, for which $\lambda = 1$, as shown in Table 6.10. From the earlier considerations of such designs, only six of the eight cases/sub-cases of resulting designs with two missing values are possible. The distributions of the average and maximum variances of pairwise treatment differences for the six cases/sub-cases for these two 13 treatment designs are as shown in Table 6.11. The number of configurations corresponding to each case/sub-case is the same for these two designs. The loss of efficiency varies between 8% and 11% when considering the average variance, but the loss is greater for the maximum variance in all situations.

Preece (1966) also gave three examples of Youden squares with 11 treatments in blocks of 5, all with $\lambda = 2$, see Table 6.12 for details of the structure of these designs. The distributions of average and maximum variances of pairwise treatment comparisons for

Table 6.11: Average variances of pairwise treatment differences, relative efficiencies, maximum pairwise variances, and frequencies of the cases/sub-cases of two missing observations for the designs in Table 6.10.

Two missing values (t=2)				
Case	A.V.(R.E.)	M.V.(R.E.)	1	2
1	$0.66667\sigma^2$ (0.92308)	$0.92308\sigma^2$ (0.66666)	78	78
2(a)i	$0.66886\sigma^2$ (0.92004)	$0.85714\sigma^2$ (0.71795)	156	156
2(a)ii	$0.66667\sigma^2$ (0.92308)	$0.93590\sigma^2$ (0.65753)	156	156
2(a)iii	N/A (N/A)	N/A (N/A)	0	0
2(b)i	$0.66667\sigma^2$ (0.92308)	$0.84615\sigma^2$ (0.72728)	546	546
2(b)ii	$0.67033\sigma^2$ (0.91803)	$0.99780\sigma^2$ (0.61674)	312	312
2(b)iii	N/A (N/A)	N/A (N/A)	0	0
2(c)	$0.68750\sigma^2$ (0.89510)	$1.09135\sigma^2$ (0.56387)	78	78

Table 6.12: Three examples of (11×5) Youden square designs given by Preece (1966).

Design 1

1	2	3	5	8
2	3	4	6	9
3	4	5	7	10
4	5	6	8	11
5	6	7	9	1
6	7	8	10	2
7	8	9	11	3
8	9	10	1	4
9	10	11	2	5
10	11	1	3	6
11	1	2	4	7

Design 2

1	2	3	5	8
4	6	2	3	9
3	5	10	7	4
5	8	4	6	11
9	7	6	1	5
8	10	7	2	6
11	3	9	8	7
10	9	8	4	1
2	11	5	9	10
6	1	11	10	3
7	4	1	11	2

Design 3

1	2	3	5	8
4	3	2	9	6
3	5	4	10	7
6	4	11	8	5
5	6	7	1	9
8	7	6	2	10
7	8	9	3	11
10	9	8	4	1
9	10	5	11	2
11	1	10	6	3
2	11	1	7	4

Table 6.13: Average variances of pairwise treatment differences, relative efficiencies, maximum pairwise variances, and frequencies of the cases/sub-cases of two missing observations for the designs in Table 6.12.

	Two missing values (t=2)				
Case	A.V.(R.E.)	M.V.(R.E.)	1	2	3
1	$0.48485\sigma^2$ (0.93750)	$0.60606\sigma^2$ (0.75000)	110	110	110
2(a)i	$0.48623\sigma^2$ (0.93483)	$0.56234\sigma^2$ (0.80831)	110	68	105
2(a)ii	$0.48468\sigma^2$ (0.93783)	$0.59822\sigma^2$ (0.75983)	110	194	120
2(a)iii	$0.48485\sigma^2$ (0.93750)	$0.64394\sigma^2$ (0.70588)	55	13	50
2(b)i	$0.48485\sigma^2$ (0.93750)	$0.57576\sigma^2$ (0.78947)	385	427	390
2(b)ii	$0.48544\sigma^2$ (0.93635)	$0.62365\sigma^2$ (0.72885)	550	466	540
2(b)iii	$0.48779\sigma^2$ (0.93184)	$0.68182\sigma^2$ (0.66666)	55	97	60
2(c)	$0.49374\sigma^2$ (0.92062)	$0.70707\sigma^2$ (0.64286)	110	110	110

these three (11×5) designs when two observation become unavailable are different, and are shown in Table 6.13. It may be seen that the total number of configurations for the sub-cases corresponding to Case 2(a) and Case 2(b) for the three designs are the same, but the distributions within the cases are different. Design 2 has the largest number of configurations that give rise to resulting designs with maximum variance of $0.68182\sigma^2$, but it also has the smallest number of configurations corresponding to Sub-case 2(a)iii which has the largest maximum variance within Case 2(a).

6.6 Discussion

Das and Kageyama (1992) derived the information matrix for treatment effects after the loss of a complete block (row) from a Youden square. The eigenvalues of this matrix were used to derive an expression for the average variance of pairwise treatment differences in the incomplete design. Their results for the case of a missing block have been extended by identifying a simple generalised inverse of the information matrix, and using it to obtain expressions for the variance of any individual pairwise treatment comparison, and consequently the average of these variances. These formulae have been used to produce numerical results for the loss of a block from a range of Youden square designs with 4 to 31 treatments. It can be seen that small designs are those most seriously affected by the loss of data when the efficiencies are computed.

The effects of losing t observations from a Youden square based design have also been considered. For $t = 1$, the reduced normal equations were derived in general terms and variances of pairwise treatment differences were determined for a wide range of Youden square using the design parameters v, b, and k. The loss of two plots is shown to lead to up to eight possible resulting incomplete designs whose properties need to be considered separately. Not all of these situations can be achieved for all designs. In particular, designs in which pairs of treatments occur together in the same block only once cannot have resulting designs belonging to Sub-cases 2(a)iii and 2(b)iii. It was also shown that for some Youden squares, the structure of the starting design dictates the number of configurations corresponding to certain sub-cases of resulting designs.

Youden squares representing those available for certain numbers of treatments and block sizes, have been investigated in detail in this Chapter. The first set of designs had five treatments in blocks of four units, four replicates of each treatment, and every pair of treatments occurred in three of the blocks of the starting design. It was shown that there were seven cases of two missing values for one design, but there was an extra case for the other (5×4) design and the frequencies of occurrences of the sub-cases were slightly different. The class of six representative (7×4) Youden squares, derived from (7×7) Latin squares, has also been considered and shown to give generally different distributions of resulting designs when two observations are missing.

Larger designs have also been compared, and large average variances correspond to cases where the missing observations are for two different treatments in different rows and in different columns. Therefore it is desirable to choose an initial design in which pairs of treatments occur in the same two rows and two columns as often as possible. For example, in Design 5 of Table 6.7 there are nine pairs of treatments with this property so that there are only 15 resulting designs in Sub-case 2(b)iii whereas for Design 2 there are none, so that this sub-case contains 33 resulting designs with average variance $0.696\sigma^2$.

Chapter 7

Diallel Cross Designs and the unavailability of data

In a genetic or mating experiment, an investigator studies the general combining ability of p lines (parents) to identify the best combination of parentage. In general, there are p^2 potential crosses between the p lines, but in this Chapter we do not consider the effect of combining an individual line with itself (e.g. 1×1), and reciprocal crosses such as 1×2 and 2×1 are assumed to be the same in the analysis. Consequently, there are $p(p-1)/2$ pairs of lines (or crosses) that are investigated in the diallel cross experiment. An overview of designs used for genetic experiments is provided by Hinkelmann (1975).

A complete diallel cross (CDC) design is an experiment in which all potential crosses occur at least once, although they do not have to be replicated the same number of times. Ghosh and Desai (1998) constructed CDC designs using BIB designs, and Ghosh and Desai (1999) considered constructing CDC designs with unequal replication of the crosses. Gupta and Kageyama (1994) also considered constructing optimal complete diallel cross designs, and Agarwal and Das (1987) gave a method for constructing diallel cross designs from two incomplete block designs. When the number of lines is large, there may be only enough experimental units to accommodate a subset of the $p(p-1)/2$ crosses in the starting design. In these situations, a partial diallel cross (PDC) design involving fewer crosses may be chosen to estimate the general combining ability of the p lines. The methods for constructing diallel cross designs, both complete and partial, use partially balanced incomplete block designs - both triangular and group divisible designs, and are considered in detail in the next Section.

The loss of a block of crosses is studied in Section 7.3 for particular diallel cross designs. Variances of pairwise line differences are calculated to assess the impact of the missing observations. This work complements Ghosh and Desai (1998, 1999) and Ghosh and Biswas (2000), where the loss of one block from complete diallel cross designs was considered. The effect of missing crosses scattered throughout the starting design is studied for the same designs in Section 7.4.

7.1 Construction of Diallel Cross Designs

Agarwal and Das (1987) considered the construction of balanced n-ary designs, which can subsequently be used as CDC designs by combining a BIB design and a two associate partially balanced incomplete block (PBIB) triangular design. The first design is a BIB design with p treatments where there are $p(p-1)/2$ blocks of two units such that every pair of treatments occurs in one block of the BIB design. The blocks of the BIB design are then considered to contain crosses of the two treatments in them, e.g. if block one has treatments 1 and 2 then these now correspond to the cross 1×2. The other design is a PBIB design with $p(p-1)/2$ treatment labels, and these labels are replaced by their corresponding block from the BIB design to produce a CDC design. Each observation in the PBIB design is replaced by a cross, e.g. all plots corresponding to treatment 1 are replaced by the cross 1×2, treatment 2 is replaced by 1×3, etc. The designs generated by this approach are not in general binary, because although no cross occurs more than once in any block of the CDC design, the lines can occur more than once in a given block and consequently the elements of the incidence matrix are not all zero or one. For example, line 1 could occur in two crosses, say 1×2 and 1×3, of one block of the starting design.

Gupta and Kageyama (1994) also constructed diallel cross designs using symmetric BIB designs. Their series of designs are formed by cyclically developing an initial block of crosses. When there is an odd number of lines in the experiment, the following steps will generate a CDC design.

1. Consider a BIB $(v = 2t + 1, b = 2t + 1, k = 2t, r = 2t, \lambda = 2t - 1)$ design,

2. take an initial block $(1, 2t), (2, 2t - 1), \cdots, (t, t + 1)$, and

3. develop this set of lines cyclically $mod(2t + 1)$.

Table 7.1: Example of a cyclically generated design with an odd number of lines using the method of Gupta and Kageyama (1994). The rows of the array correspond to blocks.

1×6	2×5	3×4
0×2	3×6	4×5
1×3	0×4	5×6
2×4	1×5	0×6
3×5	2×6	0×1
4×6	0×3	1×2
0×5	1×4	2×3

Table 7.2: Example of a cyclically generated design with an even number of lines using the method of Gupta and Kageyama (1994). The rows of the array correspond to blocks.

1×4	2×3	0×6
0×2	3×4	1×6
1×3	0×4	2×6
2×4	0×1	3×6
0×3	1×2	4×6

To demonstrate this method, consider $t = 3$, which gives a BIB $(7, 7, 6, 6, 5)$ design. The initial generating block of the design has the crosses 1×6, 2×5, and 3×4. The second block of the design has three crosses 0×2, 3×6, and 4×5. The other five blocks are generated similarly and the resulting diallel cross design is shown in Table 7.1, where the rows correspond to blocks.

Gupta and Kageyama (1994) gave a procedure for generating an optimal diallel cross with $p-1$ blocks and $p/2$ crosses in each block, when there is an even number of lines in the proposed design.

1. The lines are coded $0, 1, \cdots, p-2, p$,
2. the elements of the $p-1$ blocks are $(j+l, p-1-j+l), (l, p)$; $j = 1, \cdots, p/2-1$; $l = 0, \cdots, p-2$, and
3. the first $p/2-1$ crosses of each block are developed $mod(p-1)$.

An example of a six line design in five blocks of three crosses, generated using this method, is given in Table 7.2.

PBIB designs were used by Ghosh and Divecha (1997) to construct PDC designs. There are k treatments in every block of the PBIB design, and to construct the PDC, all of the distinct pairs of treatments within each block are generated. This yields $k(k-1)/2$ crosses per block of the resulting PDC design. For example, if the first block of the PBIB design had treatments 3, 5, and 8 then the corresponding block in the PDC design has three crosses 3×5, 3×8, and 5×8. The partial balance of the incomplete block design creates the partial completeness of the diallel cross. Table 7.3 shows two PDC designs generated from PBIB designs with two association classes, which are investigated in later Sections to assess the impact of missing data.

7.2 Analysis of Diallel Cross Designs

To describe the data collected from a diallel cross design, an additive model is assumed, where the effect of the *ith* line under investigation in the planned experiment is represented by $g_i; i = 1, \cdots, p$. The model can be expressed as

$$y_{ijl} = \mu + g_i + g_j + \beta_l + \epsilon_{ijl} \tag{7.1}$$

$$(i = 1, 2, \cdots, p; j = 1, 2, \cdots, p; i \neq j; l = 1, 2, \cdots, b)$$

where y_{ijl} is the response due to the *ith* and *jth* lines being crossed in the *lth* block, μ is an overall mean effect, g_i and g_j are the effects of the *ith* and *jth* lines respectively, β_l is the effect of the *lth* block, and ϵ_{ijl} is random error. The usual least squares analysis is performed to derive the full normal equations to estimate the model parameters. The design matrix $\mathbf{X} = (\mathbf{1}|\mathbf{X_g}|\mathbf{X}_\beta)$ for all model parameters can be partitioned into a column of ones, and design matrices for the line and block effects respectively. The design matrix for lines is different from the usual design matrix for treatments in the general block experiment. Each *observation* in a diallel cross design is composed of two lines so there are two non-zero elements in each row of the design matrix for lines, which is denoted by $\mathbf{X}_g$. Consider Design (b) given in Table 7.3. The first row of the design matrix $\mathbf{X_g}$, which corresponds to the allocation of the cross 1×9 to the first unit of the first block of the design, is given by $(1, 0, 0, 0, 0, 0, 0, 0, 1)$.

Table 7.3: (a) 12 line PDC design of Ghosh and Divecha (1997) and (b) PDC generated using the same method from a group divisible design with 9 lines in 9 blocks.

Design (a)

1×2	1×3	1×4	2×3	2×4	3×4
7×10	5×7	4×7	5×10	4×10	4×5
6×11	6×9	4×6	9×11	4×11	4×9
1×7	1×6	1×8	6×7	7×8	6×8
5×11	2×11	8×11	2×5	5×8	2×8
9×10	3×10	8×10	3×9	8×9	3×8
1×11	1×10	1×12	10×11	11×12	10×12
2×9	7×9	9×12	2×7	2×12	7×12
3×5	5×6	5×12	3×6	3×12	6×12

Design (b)

1×9	2×9	1×2
3×4	3×5	4×5
6×7	6×8	7×8
3×9	6×9	3×6
1×4	1×7	4×7
2×5	2×8	5×8
4×9	8×9	4×8
1×5	1×6	5×6
2×3	2×7	3×7

The reduced normal equations for line effects after the removal of block effects is

$$\mathbf{C} = \mathbf{R} - \mathbf{N}\mathbf{K}^{-\delta}\mathbf{N}' \tag{7.2}$$

where $\mathbf{R}$ is not a diagonal matrix. The off-diagonal elements of $\mathbf{R}$ record the number of times that any two lines occur as a cross in the starting design. $\mathbf{K}^{\delta}$ is defined as a diagonal matrix whose elements are the number of crosses, and not the number of occurrences of the lines, in each of the b blocks of the starting design, and $\mathbf{K}^{-\delta}$ is the inverse of $\mathbf{K}^{\delta}$. $\mathbf{N}$ is the incidence matrix for the lines in the blocks of the diallel cross design. The elements of $\mathbf{N}$, $\{n_{il}\}$, are not necessarily 0 or 1, because any line can occur in more than one of the crosses in a given block.

7.2.1 Complete diallel cross designs

Consider the balanced n-ary designs of Agarwal and Das (1987), which are formed by combining a BIB ($v_0 = p, b_0 = p(p-1)/2, r_0 = p-1, k_0 = 2, \lambda_0 = 1$) design and a two associate PBIB ($v' = p(p-1)/2, b', r', k', \lambda_1', \lambda_2'$) design. The resulting CDC design has p lines in b' blocks of $2k'$ lines or k' crosses, and each line is replicated $r'r_0$ times as crosses with other lines in the starting design. The treatments in the PBIB have r' replicates, so that all $p(p-1)/2$ crosses are replicated r' times in the resulting CDC design. The **R** matrix component of the information matrix for line effects can be expressed in the form

$$\mathbf{R} = \begin{bmatrix} r'r_0 & \dots & r' \\ \vdots & \ddots & \vdots \\ r' & \dots & r'r_0 \end{bmatrix} = r'(r_0 - 1)\mathbf{I}_p + r'\mathbf{J}_{p,p} \qquad (7.3)$$

and the diagonal matrix of block sizes simplifies to $\mathbf{K}^\delta = k'\mathbf{I}_{b'}$, because there are k' crosses in all blocks of the starting design. The *ith* diagonal element of the concurrence matrix, $\mathbf{NN}'$, is $S = \sum_{l=1}^{b'} n_{il}^2$ and the *(i,j)th* off-diagonal element of this matrix is $\Delta = \sum_{l=1}^{b'} n_{il}n_{jl}$. The structure of the CDC designs of Agarwal and Das (1987) ensures that these two values, S and Δ, are independent of i and j, so that the concurrence matrix for the complete design with p lines can be simplified to

$$\mathbf{NN}' = \begin{bmatrix} S & \dots & \Delta \\ \vdots & \ddots & \vdots \\ \Delta & \dots & S \end{bmatrix} = (S - \Delta)\mathbf{I}_p + \Delta\mathbf{J}_{p,p} \qquad (7.4)$$

Combining Equations (7.3), (7.4), and $\mathbf{K}^{-\delta}$ gives the information matrix for line effects in the complete design as

$$\mathbf{C} = \{r'(r_0 - 1) - (S - \Delta)/k'\}\mathbf{I}_p + \{r' - \Delta/k'\}\mathbf{J}_{p,p} \qquad (7.5)$$

A generalised inverse can be used to solve these reduced normal equations to estimate (non-uniquely) the line effects. A suitable choice of generalised inverse to simplify the expressions for the variances of treatment differences, using Theorem (3.1), is derived by subtracting $(r' - \Delta/k')$ from all the elements of **C** and inverting the resulting non-singular matrix to give

$$\Omega = \{r'(r_0 - 1) - (S - \Delta)/k'\}^{-1}\mathbf{I}_p \qquad (7.6)$$

For the CDC design when there are no missing values, all variances of pairwise line differences are equal to

$$2\sigma^2\{r'(r_0-1)-(S-\Delta)/k'\}^{-1} \tag{7.7}$$

Gupta and Kageyama (1994) gave procedures for constructing CDC designs with an even or odd number of lines. These are generated from symmetric BIB designs with $2t+1$ treatments with $2t$ plots in every block. The lines of the CDC design are replicated $2t$ times and occur once as a cross with every other line. The information matrix for line effects is given by

$$\mathbf{C} = (2t-1-1/t)\mathbf{I}_{2t+1} + (1/t-1)\mathbf{J}_{2t+1,2t+1} \tag{7.8}$$

and all pairwise line comparisons are estimated with a variance of $2t\sigma^2/(2t^2-t-1)$. In the case where there are an even number of lines, there are in general p lines allocated to $p-1$ blocks of $p/2$ crosses. The information matrix for line effects in this situation can be expressed in the form

$$\mathbf{C} = (p-2)\mathbf{I}_p + (1-2(p-1)/p)\mathbf{J}_{p,p} \tag{7.9}$$

where the variances of pairwise line differences are $2\sigma^2/(p-2)$.

7.2.2 Partial diallel cross designs

Ghosh and Divecha (1997) considered constructing PDC designs using PBIB designs. The initial design had p treatments replicated r times, allocated to the k plots of b blocks. The association scheme for the PBIB design was $(\lambda_1, \lambda_2, n_1, n_2, p^i_{jk}; i=j=k=1,2)$, where one of the two pairing parameters, λ_1 or λ_2, was assumed to be zero. In each block of the PBIB design all distinct pairs of treatments are produced, and there are $k(k-1)/2$ crosses within each block of the diallel cross design. The PDC design has p lines replicated in $r(k-1)$ of the crosses, and $k(k-1)/2$ crosses within each of the b blocks of the starting design. The association parameters of this new PDC design are $\Lambda_1 = \lambda_1(k-1)^2$ and $\Lambda_2 = \lambda_2(k-1)^2$ for the members of the two associate classes.

To derive the elements of the information matrix for line effects, consider the matrices $\mathbf{R}$, $\mathbf{K}^\delta$, and $\mathbf{NN}'$ of Equation (7.2) individually. Note that $\mathbf{K}^\delta = k(k-1)/2\mathbf{I}_b$, because all blocks have the same number of crosses when the design is complete. The elements of $\mathbf{R}$ depend on whether pairs of lines are first or second associates, and are consequently given by

$$r_{ij} = \begin{cases} r(k-1) & i = j \\ \lambda_1 & i \neq j \text{ and lines i and j are first associates} \\ \lambda_2 & i \neq j \text{ and lines i and j are second associates} \end{cases} \tag{7.10}$$

and the elements of the concurrence matrix, $\mathbf{NN}'$, can be similarly separated into three groups as

$$(\mathbf{NN}')_{ij} = \begin{cases} r(k-1)^2 & i = j \\ \lambda_1(k-1)^2 & i \neq j \text{ and lines i and j are first associates} \\ \lambda_2(k-1)^2 & i \neq j \text{ and lines i and j are second associates} \end{cases} \tag{7.11}$$

Combining these expressions and $\mathbf{K}^{-\delta}$ gives the form of the elements of the information matrix for line effects. They are

$$\mathbf{C}_{ij} = \begin{cases} r(k-1)(k-2)/k & i = j \\ -\lambda_1(k-2)/k & i \neq j \text{ where lines i and j are first associates} \\ -\lambda_2(k-2)/k & i \neq j \text{ where lines i and j are second associates} \end{cases} \tag{7.12}$$

which depend on the number of times the crosses occur in the design. Some of the entries are zero because Ghosh and Divecha (1997) made the assumption that one of λ_1 and λ_2 is zero.

Consider a particular class of PBIB designs, group divisible designs, where the p treatments are divided into m_2 groups of m_1 treatments. Two treatments in the same group occur together in the same block λ_1 times and two treatments in different groups occur together λ_2 times. The information matrix for line effects can be partitioned into matrices relating to m_2 different groups of m_1 lines for the designs under consideration, and, after relabelling and rearrangement the matrix can be expressed using Equation

(7.12) as

$$\mathbf{C} = \frac{(k-2)}{k}\begin{bmatrix} a_1\mathbf{I}_{m_1} - \lambda_2\mathbf{J}_{m_1,m_1} & a_2\mathbf{I}_{m_1} - \lambda_2\mathbf{J}_{m_1,m_1} & \cdots & a_2\mathbf{I}_{m_1} - \lambda_2\mathbf{J}_{m_1,m_1} \\ a_2\mathbf{I}_{m_1} - \lambda_2\mathbf{J}_{m_1,m_1} & a_1\mathbf{I}_{m_1} - \lambda_2\mathbf{J}_{m_1,m_1} & \cdots & a_2\mathbf{I}_{m_1} - \lambda_2\mathbf{J}_{m_1,m_1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2\mathbf{I}_{m_1} - \lambda_2\mathbf{J}_{m_1,m_1} & a_2\mathbf{I}_{m_1} - \lambda_2\mathbf{J}_{m_1,m_1} & \cdots & a_1\mathbf{I}_{m_1} - \lambda_2\mathbf{J}_{m_1,m_1} \end{bmatrix} \tag{7.13}$$

where

$$a_1 = r(k-1) + \lambda_2 \tag{7.14}$$

and

$$a_2 = \lambda_2 - \lambda_1 \tag{7.15}$$

To identify a generalised inverse of this information matrix, add $\lambda_2\mathbf{J}_{v,v}$ and invert the resulting non-singular matrix to give

$$\mathbf{\Omega} = \frac{k}{(k-2)}\begin{bmatrix} \alpha_1\mathbf{I}_{m_1} & \alpha_2\mathbf{I}_{m_1} & \cdots & \alpha_2\mathbf{I}_{m_1} \\ \alpha_2\mathbf{I}_{m_1} & \alpha_1\mathbf{I}_{m_1} & \cdots & \alpha_2\mathbf{J}_{\cdot m_1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_2\mathbf{I}_{m_1} & \alpha_2\mathbf{I}_{m_1} & \cdots & \alpha_1\mathbf{I}_{m_1} \end{bmatrix} \tag{7.16}$$

where

$$\alpha_1 = \frac{a_1 + a_2(m_2 - 2)}{[a_1 + a_2(m_2 - 1)][a_1 - a_2]} \tag{7.17}$$

and

$$\alpha_2 = \frac{-a_2}{[a_1 + a_2(m_2 - 1)][a_1 - a_2]} \tag{7.18}$$

There are two types of pairwise line comparison based on whether the lines are first or second associates. If two lines are first associates, i.e. they occur together in λ_1 blocks of the original Group Divisible design and as λ_1 crosses, the variance of the pairwise line difference is given by

$$\frac{2k(\alpha_1 - \alpha_2)}{(k-2)}\sigma^2 = \frac{2k}{(k-2)(rk - r + \lambda_1)}\sigma^2 \tag{7.19}$$

Table 7.4: Five treatment CDC design given by Agarwal and Das (1987).

1×3	4×5	2×4	2×5
4×5	1×2	1×3	2×3
2×5	1×4	3×4	1×3
2×4	1×3	3×5	1×5
1×2	3×5	4×5	3×4
2×3	1×5	1×4	4×5
3×4	2×5	1×5	1×2
1×4	2×3	2×5	3×5
3×5	2×4	1×2	1×4
1×5	3×4	2×3	2×4

and when two lines are second associates, the variance of their pairwise line difference is

$$\frac{2k\alpha_1}{(k-2)}\sigma^2 = \frac{2k[r(k-1)+\lambda_2+(\lambda_2-\lambda_1)(m_2-2)]}{(k-2)[r(k-1)+\lambda_2+(\lambda_2-\lambda_1)(m_2-1)][r(k-1)+\lambda_1]}\sigma^2 \tag{7.20}$$

7.3 Removal of blocks from a Diallel Cross Design

Consider the five line CDC design given by Agarwal and Das (1987), which is shown in Table 7.4. There are ten crosses in the starting design, and these are allocated to the four units of the ten blocks. All lines are replicated 16 times and appear in four crosses with each of the other four lines in the starting design. Variances of pairwise line comparisons for the complete design are given by Equation (7.7), and, for this CDC design with five lines, are all equal to $0.1778\sigma^2$.

The loss of all four crosses in any of the ten blocks in the starting design leads to the same overall average variance of pairwise line differences. The variances of individual pairwise line comparisons depend on the crosses that are lost, and the replicates of each line that become unavailable. One of these variances remains unchanged at $0.1778\sigma^2$, three increase to $0.1951\sigma^2$, and the other six increase to $0.2047\sigma^2$. As an illustration, consider the loss of the first block of four crosses from the starting design. In this situation, lines 1 and 3 occur once in the missing block and the variance of the comparison between lines 1 and 3 remains unchanged as $0.1778\sigma^2$. A direct comparison of these two lines does not occur in this block because their effects are indistinguishable within the cross. The other three lines lose two replicates in the remaining three crosses of the block. The variances of the three pairwise line differences involving these three crosses, 4×5, 2×4, and 2×5 respectively, are increased to $0.1951\sigma^2$, and the other six pairwise line comparisons that do not relate to crosses in the missing block are all increased to $0.2047\sigma^2$. The losses in efficiency for these two types of comparison are 9% and 13% respectively. The average variance of pairwise line differences is $0.1991\sigma^2$, which corresponds to a reduction in efficiency of 11%.

Ghosh and Desai (1999) considered constructing diallel cross designs from singular group divisible (SGD) designs. The crosses in the blocks of the new CDC design are formed by considering every pair of treatments within the blocks of the original SGD design. Ghosh and Desai (1999) derived the normal equations for these diallel cross designs generally, and expressions for the variances of pairwise line comparisons. These are CDC designs but the number of replicates of each cross is unequal in the starting design. An example of a design with eight lines in twelve blocks of six crosses is given in Table 7.5. The majority of the crosses have two replicates, but four of them are replicated six times in the starting design. When this design is complete, the average variance of pairwise line differences is $0.2024\sigma^2$. The crosses 1×5, 2×6, 3×7, and 4×8 are all replicated six times and consequently the variances of comparisons between these four pairs of lines are $0.1667\sigma^2$. The variances of all other pairwise line comparisons are $0.2083\sigma^2$.

Consider the loss of a single block of crosses, which in this situation corresponds to six crosses. The average variance of pairwise line comparisons for this CDC design increases to $0.2238\sigma^2$, and the maximum variance of pairwise line differences is $0.2667\sigma^2$, representing a loss in efficiency of 22%. Now consider removing block one from the starting design, which involves lines 1, 3, 5, and 7 and the six crosses formed from these four lines. There are five distinct variances of pairwise line comparisons. The variance of any comparison involving one treatment from the missing block and one of the other four, e.g. 1×2, increases from $0.2083\sigma^2$ to $0.2271\sigma^2$, representing a reduction in efficiency of 8%. The comparisons involving crosses 1×5 and 3×7, which are replicated six times in the starting design, have an increased variance of $0.2\sigma^2$, corresponding to a loss of 17% in efficiency. Pairwise line differences corresponding to the other four crosses in the affected block have an increased variance of $0.2667\sigma^2$, representing a reduction of 22% in efficiency. Variances of all the pairwise comparisons involving lines 2, 4, 6, and 8 remain unaltered.

Table 7.5: Complete diallel cross design with an unequal number of crosses taken from Ghosh and Desai (1999). Four of the crosses, 1×5, 2×6, 3×7, and 4×8 are replicated six times and the other 24 crosses are all replicated twice. The design has 12 blocks, which correspond to the rows of the diagram, each with six crosses.

1×5	1×3	1×7	3×5	5×7	3×7
2×6	2×4	2×8	4×6	6×8	4×8
3×7	1×3	3×5	1×7	5×7	1×5
4×8	2×4	4×6	2×8	6×8	2×6
1×5	1×2	1×6	2×5	5×6	2×6
3×7	3×4	3×8	4×7	7×8	4×8
2×6	1×2	2×5	1×6	5×6	1×5
4×8	3×4	4×7	3×8	7×8	3×7
1×5	1×4	1×8	4×5	5×8	4×8
2×6	2×3	2×7	3×6	6×7	3×7
4×8	1×4	4×5	1×8	5×8	1×5
3×7	2×3	3×6	2×7	6×7	2×6

Table 7.3 gives two examples of PDC designs constructed from group divisible designs using the approach of Ghosh and Divecha (1997). The first, labelled Design (a) in the table, has twelve lines allocated to the crosses of nine blocks of six observations, and each line has nine replicates in the initial design. All the lines appear in crosses with nine of the other lines, but do not occur as a cross with the remaining two lines. For example, line 1 does not appear in a cross with lines 5 and 9. When this PDC design has no missing values, the variances of the pairwise line comparisons are either $0.4074\sigma^2$ or $0.4444\sigma^2$. These depend on whether the cross corresponding to the particular comparison occurs in the starting design, e.g. lines 1 and 5 or 9 are compared with a variance of $0.4444\sigma^2$. The average variance of all 66 pairwise line differences is $0.4141\sigma^2$. When a block of six crosses becomes unavailable, the average variance increases to $0.4918\sigma^2$ and the maximum variance of the individual pairwise line differences is $0.6875\sigma^2$. The overall effect, in terms of average and maximum variance, is the same irrespective of the block that is missing from the starting design. This corresponds to a reduction of approximately 16% in efficiency for the average and approximately 36% for the maximum of the variances, which is a severe loss of accuracy.

Design (b) from Table 7.3 is also constructed using a group divisible design. In this case there are nine lines in nine blocks of three crosses, and every line occurs in a cross with six of the other eight lines. The variances of pairwise line comparisons are either $0.8889\sigma^2$ or σ^2, with an average of $0.9167\sigma^2$ for this design when there are no observations missing. If a block of crosses is lost, the average variance of pairwise line differences increases to $1.1\sigma^2$, with relative efficiency of 83%, and the maximum of the variances is $1.6\sigma^2$ (62% efficiency). Although only three crosses are lost from the affected block, the loss of efficiency is substantial, e.g. 38% for the worst case. The overall effect is the same regardless of the block that becomes unavailable.

Consider the situation where a block of crosses becomes unavailable from a PDC design of Ghosh and Divecha (1997), similar to those shown in Table 7.3. Assuming that the first k lines occurred in the missing block, all crosses between the k lines are lost. The information matrix for line effects can now be expressed as

$$\begin{bmatrix} d_1\mathbf{I}_k & d_2\mathbf{I}_k - d_2\mathbf{J}_{k,k} & d_2\mathbf{I}_k - d_2\mathbf{J}_{k,k} \\ d_2\mathbf{I}_k - d_2\mathbf{J}_{k,k} & d_3\mathbf{I}_k - d_2\mathbf{J}_{k,k} & d_2\mathbf{I}_k - d_2\mathbf{J}_{k,k} \\ d_2\mathbf{I}_k - d_2\mathbf{J}_{k,k} & d_2\mathbf{I}_k - d_2\mathbf{J}_{k,k} & d_3\mathbf{I}_k - d_2\mathbf{J}_{k,k} \end{bmatrix} \tag{7.21}$$

where

$$d_1 = (r-1)(k-1)(k-2)/k \tag{7.22}$$

$$d_2 = \lambda_2(k-2)/k \tag{7.23}$$

$$d_3 = (r-1+\lambda_2)(k-2)/k \tag{7.24}$$

A particular generalised inverse is found using the same approach as for the starting design. In this situation, $d_2\mathbf{J}_{v,v}$ is added to the singular information matrix and the resulting non-singular matrix is inverted to give

$$\mathbf{\Omega} = \begin{bmatrix} \delta_1\mathbf{I}_k + \delta_2\mathbf{J}_{k,k} & \delta_3\mathbf{I}_k + \delta_4\mathbf{J}_{k,k} & \delta_3\mathbf{I}_k + \delta_4\mathbf{J}_{k,k} \\ \delta_3\mathbf{I}_k + \delta_4\mathbf{J}_{k,k} & \delta_5\mathbf{I}_k + \delta_6\mathbf{J}_{k,k} & \delta_7\mathbf{I}_k + \delta_8\mathbf{J}_{k,k} \\ \delta_3\mathbf{I}_k + \delta_4\mathbf{J}_{k,k} & \delta_7\mathbf{I}_k + \delta_8\mathbf{J}_{k,k} & \delta_5\mathbf{I}_k + \delta_6\mathbf{J}_{k,k} \end{bmatrix} \tag{7.25}$$

where the expressions for $\delta_1, \cdots, \delta_8$ are complicated, and are given in Table 7.6 in terms of d_1, d_2, and d_3. There are four cases for the variance of the pairwise line comparisons, which depend on whether the affected lines appeared in the missing crosses and whether

Table 7.6: General expressions for $\delta_1, \cdots, \delta_8$ when a block of crosses becomes unavailable in a PDC design.

Element	Formula
δ_1	$\frac{d_2+d_3}{d_1d_2+d_1d_3-2d_2^2}$
δ_2	$\frac{d_2(d_2+d_3)^2}{(2d_2^2-d_1d_2-d_1d_3)(d_1d_2+d_1d_3+kd_2^2+kd_2d_3-2d_2^2)}$
δ_3	$-\frac{d_2}{d_1d_2+d_1d_3-2d_2^2}$
δ_4	$-\frac{d_2^2(d_2+d_3)}{(2d_2^2-d_1d_2-d_1d_3)(d_1d_2+d_1d_3+kd_2^2+kd_2d_3-2d_2^2)}$
δ_5	$-\frac{d_2^2-d_1d_3}{(d_2-d_3)(2d_2^2-d_1d_2-d_1d_3)}$
δ_6	$\frac{d_2^3}{(2d_2^2-d_1d_2-d_1d_3)(d_1d_2+d_1d_3+kd_2^2+kd_2d_3-2d_2^2)}$
δ_7	$\frac{(d_2-d_1)d_2}{(d_2-d_3)(2d_2^2-d_1d_2-d_1d_3)}$
δ_8	$\frac{d_2^3}{(2d_2^2-d_1d_2-d_1d_3)(d_1d_2+d_1d_3+kd_2^2+kd_2d_3-2d_2^2)}$

the two lines are first or second associates. The first case corresponds to two lines in the missing block, with a variance of $2\delta_1\sigma^2$. The second case occurs if only one line occurred in the affected block, and if the two lines are first associates, the variance is $(\delta_1 + \delta_2 - 2\delta_3 - 2\delta_4 + \delta_5 + \delta_6)\sigma^2$, whereas it is $(\delta_1 + \delta_2 - 2\delta_4 + \delta_5 + \delta_6)\sigma^2$ when they are second associates. The third case corresponds to two unaffected lines that are second associates, and the variances of any pairwise line comparison of this type are $2\delta_5\sigma^2$. The last is where the two lines are first associates and do not lose any replicates. Here the variance is $2(\delta_5 - \delta_7)\sigma^2 = 2k\sigma^2/(r-1)$.

7.4 Missing observations scattered throughout the starting design

Consider the situation when t crosses, scattered throughout the starting design, become unavailable for the analysis. The number of distinct configurations of t missing values and the properties of these cases depend on the particular CDC design, i.e. which crosses

Table 7.7: CDC design with five treatments used by Ghosh and Biswas (2000).

1×2	3×4		1×4	3×5
1×2	3×5		1×5	2×3
1×2	4×5		1×5	2×4
1×3	2×4		1×5	3×4
1×3	2×5		2×3	4×5
1×3	4×5		2×4	3×5
1×4	2×3		2×5	3×4
1×4	2×5			

are lost and the number of replicates of the lines in the starting design, the number of crosses common to the pairs/triples of blocks, etc. Numerical results, i.e. the average and maximum variances of pairwise line differences, for the loss of one, two, and three crosses are presented for the designs discussed in the previous Section. The efficiencies of these measures have also been computed to quantify the impact of the missing data.

7.4.1 One observation missing from the starting design

The design in Table 7.4 constructed by Agarwal and Das (1987) has two distinct ways of losing one observation that lead to resulting designs with different properties. In all ten blocks of the starting design there are two replicates of three different lines which compose three of the crosses in the block, and the fourth cross is a combination of the other two lines. In the first block of the design there is the cross 1×3, and the other three observations correspond to all distinct pairs of lines 2, 4, and 5, i.e. the crosses 2×4, 2×5, and 4×5. For example, if the cross 1×3 is lost from the first block, the average variance of pairwise line comparisons increases to $0.1905\sigma^2$, and the maximum variance is equal to $0.1989\sigma^2$. The efficiencies for these variances are 93.3% and 89.4% respectively. There are two distinct types of line comparison in the resulting design. The four comparisons between lines that occur as crosses in the block are unchanged, while the other six are increased. When one of the other three crosses becomes unavailable, e.g. 4×5, the average variance is $0.1829\sigma^2$, and the maximum of the variances of pairwise line differences increases to $0.1895\sigma^2$. The efficiencies are 97.2% and 93.8% respectively. Comparisons between lines 1 and 2 and between 4 and 5 remain unchanged. Comparisons between the lines replicated once in the block, 1 or 3, and line 2 suffer a small loss of

Table 7.8: Summary of the effect of one, two, and three missing observations on the CDC design given by Ghosh and Biswas (2000). Average variances, maximum variances of pairwise line differences and relative efficiencies are given for the distinct cases of missing values, with an example of each configuration.

Missing Values				Number
Block(s)	Cross(es)	A.V.(R.E.)	M.V.(R.E.)	of Cases
(a) One missing value				
1	1×2	$0.29091\sigma^2$ (0.91668)	$0.31515\sigma^2$ (0.84617)	30
(b) Two missing values				
1 1	1×2 3×4	$0.29091\sigma^2$ (0.91668)	$0.31515\sigma^2$ (0.84617)	15
1 4	1×2 1×3	$0.31515\sigma^2$ (0.84617)	$0.36364\sigma^2$ (0.73334)	60
1 5	1×2 1×3	$0.31667\sigma^2$ (0.84211)	$0.37333\sigma^2$ (0.71430)	240
1 2	1×2 1×2	$0.32976\sigma^2$ (0.80868)	$0.40000\sigma^2$ (0.66668)	120
(c) Three missing values				
1 1 4	1×2 3×4 1×3	$0.31515\sigma^2$ (0.84617)	$0.36364\sigma^2$ (0.73334)	60
1 1 5	1×2 3×4 1×3	$0.31667\sigma^2$ (0.84211)	$0.37333\sigma^2$ (0.71430)	240
1 1 2	1×2 3×4 1×2	$0.32976\sigma^2$ (0.80868)	$0.40000\sigma^2$ (0.66668)	120
1 4 7	1×2 1×3 1×4	$0.33939\sigma^2$ (0.78573)	$0.36364\sigma^2$ (0.73334)	40
1 4 8	1×2 1×3 1×4	$0.34245\sigma^2$ (0.77871)	$0.38655\sigma^2$ (0.68987)	480
1 5 9	1×2 1×3 1×4	$0.34359\sigma^2$ (0.77613)	$0.38564\sigma^2$ (0.69150)	480
1 5 14	1×2 1×3 2×4	$0.34444\sigma^2$ (0.77421)	$0.44444\sigma^2$ (0.60001)	160
1 2 4	1×2 1×2 1×3	$0.35577\sigma^2$ (0.74956)	$0.46192\sigma^2$ (0.57731)	960
1 2 8	1×2 1×2 1×4	$0.35614\sigma^2$ (0.74878)	$0.41228\sigma^2$ (0.64682)	480
1 2 6	1×2 1×2 1×3	$0.35903\sigma^2$ (0.74275)	$0.41639\sigma^2$ (0.64043)	480
1 2 9	1×2 1×2 1×4	$0.37826\sigma^2$ (0.70499)	$0.52174\sigma^2$ (0.51112)	480
1 2 3	1×2 1×2 1×2	$0.40952\sigma^2$ (0.65118)	$0.49524\sigma^2$ (0.53847)	80

efficiency. The four pairwise line differences 1-4, 1-5, 3-4, and 3-5 have the same variance, but the comparisons 2-4 and 2-5 are different because lines 4 and 5 have lost one of their replicates from the affected block. There are 10 configurations corresponding to the more severely affected first case and 30 to the second case.

Ghosh and Biswas (2000) investigated the effect of losing a block of crosses on CDC designs, and gave an example of a five line design where the ten crosses are replicated three times, which is shown in Table 7.7. The effect of missing observations scattered throughout the starting design is illustrated in Table 7.8. When a single observation becomes unavailable, the average variance of pairwise line differences increases from $0.2667\sigma^2$ to $0.2909\sigma^2$, representing an 8% loss of efficiency. The maximum of the individual variances of pairwise line comparisons is substantially increased to $0.3152\sigma^2$, which is approximately a reduction of 15% in efficiency. For this design, the loss of one observation is equivalent to the loss of a complete block because there are only two crosses in each block of the starting design. This is similar to the situation for the block and row-column designs discussed in earlier Chapters.

The CDC design with an unequal replication of crosses constructed by Ghosh and Desai (1999) has different properties from other CDC designs considered. For example, for the design in Table 7.5 there are two types of resulting design with different variances, depending on the particular cross that is lost. When the missing value corresponds to one of the four crosses (e.g. 2×5) that is replicated six times in the starting design, the average variance of pairwise line comparisons increases to $0.2087\sigma^2$, representing a loss of 3% in efficiency, and the maximum variance increases to $0.2304\sigma^2$, which corresponds to approximately 10% reduction in efficiency. In the other case, where one of the 24 crosses (e.g. 1×3) replicated twice becomes unavailable, the average variance increases to $0.205\sigma^2$, and the maximum variance is $0.2176\sigma^2$, representing efficiencies of 98.7% and 95.7% respectively.

The two PDC designs given in Table 7.3 have also been investigated to assess the impact of one or more missing observations on the variances of pairwise line comparisons. There are 54 potential resulting designs when a single observation is lost from Design (a), and 27 for Design (b). In all of the 54 configurations for the first design, the average variance is increased to $0.4263\sigma^2$, corresponding to a reduction of less than 3% in efficiency, and the maximum variance increases to $0.4733\sigma^2$. When the pairwise line comparisons are considered individually for one missing value, there are eight different variances which are summarised using the average and maximum variances. When there is a single value missing from Design (b), the average variance increases to $1.008\sigma^2$ (a 9% loss of efficiency), and the maximum variance is $1.422\sigma^2$. The latter is very serious as it corresponds to a reduction of 30% in efficiency compared to the initial design.

7.4.2 Two or more observations missing

The design of Agarwal and Das (1987) (see Table 7.4) has also been investigated to assess the effect of the loss of two crosses on the starting design. There are fifteen different configurations of two missing crosses that can occur, and these are listed in Table 7.9 with a particular example of each configuration. Different configurations of two missing values occur when the crosses are in the same or different blocks and depend on the

Table 7.9: An example of each of the fifteen configurations of two missing values for the CDC design of Agarwal and Das (1987), and the average and maximum variance of pairwise line comparisons.

Missing Values						Number
Block	Cross	Block	Cross	A.V.(R.E.)	M.V.(R.E.)	of Cases
9	1×2	10	2×4	$0.18807\sigma^2$ (0.94529)	$0.19629\sigma^2$ (0.90570)	120
1	4×5	2	1×3	$0.18817\sigma^2$ (0.94478)	$0.19145\sigma^2$ (0.92860)	15
1	4×5	3	1×4	$0.18817\sigma^2$ (0.94478)	$0.19675\sigma^2$ (0.90358)	60
9	1×2	10	2×3	$0.18823\sigma^2$ (0.94448)	$0.20230\sigma^2$ (0.87879)	60
1	4×5	2	1×2	$0.18850\sigma^2$ (0.94313)	$0.19739\sigma^2$ (0.90065)	60
9	1×2	10	3×4	$0.18889\sigma^2$ (0.94118)	$0.20364\sigma^2$ (0.87301)	60
1	4×5	1	2×4	$0.18895\sigma^2$ (0.94088)	$0.19829\sigma^2$ (0.89657)	30
9	2×4	10	2×4	$0.18922\sigma^2$ (0.93954)	$0.19827\sigma^2$ (0.89666)	30
1	1×3	1	4×5	$0.19481\sigma^2$ (0.91258)	$0.20472\sigma^2$ (0.86841)	30
1	1×3	2	1×2	$0.19562\sigma^2$ (0.90880)	$0.21070\sigma^2$ (0.84376)	60
1	1×3	5	3×5	$0.19583\sigma^2$ (0.90783)	$0.21216\sigma^2$ (0.83795)	120
1	1×3	5	4×5	$0.19754\sigma^2$ (0.89997)	$0.21556\sigma^2$ (0.82474)	60
1	1×3	2	1×3	$0.19909\sigma^2$ (0.89296)	$0.21754\sigma^2$ (0.81723)	30
1	1×3	5	1×2	$0.20343\sigma^2$ (0.87391)	$0.22222\sigma^2$ (0.80002)	30
1	1×3	2	4×5	$0.20748\sigma^2$ (0.85685)	$0.23007\sigma^2$ (0.77272)	15

lines that are lost. Table 7.9 shows that there are differences in average and maximum variances of pairwise line comparisons between these fifteen cases. In the *best* case, the average variance increases to $0.1881\sigma^2$, a reduction in efficiency of 5.5%, and in the *worst* case the average variance is $0.2075\sigma^2$, which corresponds to approximately a 15% loss of efficiency. In the worst situation, the two affected crosses are common to both of the affected blocks in the starting design.

There are four distinct configurations of two missing observations for the design of Ghosh and Biswas (2000), whose properties are shown in Table 7.8. The case with the smallest reduction of efficiency corresponds to the loss of both observations in one of the blocks of the initial design. The other three cases are equivalent to losing two blocks from the starting design, because when one observation is lost from a block in this CDC design, the increase in average variance is equivalent to losing both observations in the two blocks. Reduction in efficiency ranges from 8% to 19% for these four cases. The maximum variance is also substantially increased, especially when two replicates of the same cross become unavailable from two different blocks of the starting design. The maximum variance increases to $0.4\sigma^2$ for this configuration of missing values corresponding to a reduction in

Table 7.10: Summary of the effect of one and two missing observations on the PDC designs given in Table 7.3. Average and maximum variances of pairwise line comparisons are given with their efficiencies relative to the starting design for all distinct configurations of one and two missing values. The number of configurations corresponding to each case is also given, as well as an example of how each configuration occurs.

Missing Values				Number
Block(s)	Cross(es)	A.V.(R.E.)	M.V.(R.E.)	of Cases
Design (a)				
(i) One Missing Value				
1	1×2	$0.42632\sigma^2$ (0.97143)	$0.47331\sigma^2$ (0.86076)	54
(ii) Two Missing Values				
1 2	1×2 4×7	$0.43876\sigma^2$ (0.94389)	$0.49074\sigma^2$ (0.83019)	432
1 2	1×2 7×10	$0.43933\sigma^2$ (0.94266)	$0.49998\sigma^2$ (0.81484)	864
1 1	1×2 3×4	$0.44003\sigma^2$ (0.94116)	$0.54745\sigma^2$ (0.74419)	27
1 1	1×2 1×3	$0.44034\sigma^2$ (0.94050)	$0.58667\sigma^2$ (0.69444)	108
Design (b)				
(i) One Missing Value				
1	1×2	$1.00833\sigma^2$ (0.90909)	$1.42222\sigma^2$ (0.62500)	27
(ii) Two Missing Values				
1 1	1×9 2×9	$1.10000\sigma^2$ (0.83334)	$1.60000\sigma^2$ (0.55555)	27
1 4	1×9 3×9	$1.10025\sigma^2$ (0.83315)	$1.57143\sigma^2$ (0.63636)	54
1 2	1×9 3×4	$1.10150\sigma^2$ (0.83220)	$1.43860\sigma^2$ (0.61789)	54
1 2	1×9 3×5	$1.10606\sigma^2$ (0.82877)	$1.66667\sigma^2$ (0.60000)	27
1 4	1×9 6×9	$1.11979\sigma^2$ (0.81861)	$1.60590\sigma^2$ (0.55351)	54
1 4	1×9 3×6	$1.12667\sigma^2$ (0.81361)	$1.85778\sigma^2$ (0.47847)	108
1 6	1×9 5×8	$1.23333\sigma^2$ (0.74325)	$2.32222\sigma^2$ (0.38278)	27

efficiency of 33%.

When three observations are lost from this design, there are twelve potential realisable resulting designs, all listed with an example of the configuration of three missing crosses in Table 7.8. The loss of efficiency in the *best* situation is 15%, where the average variance has increased to $0.3152\sigma^2$, which is less serious than two of the configurations of two missing observations. The most severely affected configurations occur when the three missing values are in different blocks of the starting design, which in this example is equivalent to the loss of three blocks. In these cases, the loss of efficiency is over 20%, and the maximum of the average variances increase to over $0.4\sigma^2$ representing a loss of 35% in efficiency.

The PDC designs shown in Table 7.3 have also been studied to assess the influence of two missing observations. For both of these designs, all configurations corresponding to

Table 7.11: Example of a CDC design with unequal replication of crosses constructed by Ghosh and Desai (1999). The effect of one or two observations becoming unavailable is summarised using the average and maximum variances of pairwise line differences, and the number of configurations corresponding to the distinct cases.

Missing Values Block(s)	Cross(es)	A.V.(R.E.)	M.V.(R.E.)	Number of Cases
(i) One Missing Value				
1	1×3	$0.2050\sigma^2$ (0.9871)	$0.2176\sigma^2$ (0.9574)	48
2	2×6	$0.2087\sigma^2$ (0.9698)	$0.2304\sigma^2$ (0.9043)	24
(ii) Two Missing Values				
1 1	1×3 5×7	$0.2071\sigma^2$ (0.9770)	$0.2250\sigma^2$ (0.9259)	24
1 2	1×3 2×4	$0.2077\sigma^2$ (0.9745)	$0.2176\sigma^2$ (0.9574)	240
1 5	1×3 1×2	$0.2078\sigma^2$ (0.9737)	$0.2205\sigma^2$ (0.9450)	768
1 1	1×3 1×7	$0.2079\sigma^2$ (0.9732)	$0.2181\sigma^2$ (0.9553)	48
1 3	1×3 1×3	$0.2083\sigma^2$ (0.9714)	$0.2292\sigma^2$ (0.9091)	48
1 2	1×5 2×4	$0.2113\sigma^2$ (0.9577)	$0.2304\sigma^2$ (0.9043)	192
1 5	1×3 1×5	$0.2113\sigma^2$ (0.9577)	$0.2327\sigma^2$ (0.8953)	768
1 3	1×5 1×3	$0.2113\sigma^2$ (0.9577)	$0.2397\sigma^2$ (0.8693)	96
1 1	1×5 1×3	$0.2118\sigma^2$ (0.9554)	$0.2483\sigma^2$ (0.8391)	96
1 1	1×5 3×7	$0.2143\sigma^2$ (0.9444)	$0.2500\sigma^2$ (0.8333)	12
1 2	1×5 2×6	$0.2150\sigma^2$ (0.9414)	$0.2304\sigma^2$ (0.9043)	48
1 5	1×5 1×5	$0.2156\sigma^2$ (0.9385)	$0.2381\sigma^2$ (0.8750)	192
1 3	1×5 3×7	$0.2177\sigma^2$ (0.9297)	$0.2619\sigma^2$ (0.7955)	24

the loss of one observation have the same overall effect on the variances of pairwise line differences. There are four distinct configurations of two missing values for Design (a), and these are listed in order of increasing average variance in Table 7.10. The average variances of pairwise line comparisons are very similar for these cases, corresponding to a loss of approximately 6% in efficiency, but there are substantial differences in the maximum of the individual pairwise line variances. In the *best* situation, the maximum variance is $0.4907\sigma^2$, a reduction of 17% in efficiency, while the *worst* situation has a maximum variance of $0.5867\sigma^2$, which is an efficiency loss of approximately 31%.

The second PDC plan, Design (b), with nine lines has seven cases of two missing values to be computed separately. These are also shown in Table 7.10, ordered by average variance, and it can be seen that there is a substantial loss of efficiency caused by the unavailability of two observations. The *worst* case corresponds to an average variance of $1.2333\sigma^2$ and a maximum variance of $2.3222\sigma^2$, which is disastrous when compared to the starting design. The relative efficiencies for these two variances are 74.3% and 38.3%

respectively.

There are 13 distinct configurations of two missing observations when the CDC design of Ghosh and Desai (1999), see Table 7.5, is considered. These cases correspond to missing crosses in the same or different blocks of the starting design, and the number of crosses common to the pair of affected blocks. The various cases and an example of how each of the cases occurs are given in Table 7.11. It can be seen that the loss of efficiency varies between 2% and 7% for the average variance, but there is greater variation between the maximum of the variances of pairwise line comparisons.

7.5 Discussion

In genetic experiments, the use of diallel cross designs for crossing lines or parents is increasing. As in other practical situations, these designs are subject to missing values. The loss of blocks from diallel cross designs has been considered in terms of the average variance of pairwise line comparisons by other authors, and in this Chapter it has been shown that many designs suffer a small reduction in efficiency, but that some of the individual comparisons may incur a substantial loss of efficiency. For PDC designs, the variances of pairwise line differences are derived algebraically for a subset of these designs.

The loss of observations scattered throughout the initial design has been considered in this Chapter. The distribution of average and maximum variances have been enumerated numerically for different types of diallel cross designs. It is shown that the loss of efficiency is frequently substantial after the loss of one, two, or three crosses.

Chapter 8

Summary and Future Work

In this final Chapter, there are two questions to be considered. Firstly, what conclusions may be drawn from the research in the thesis. Secondly, how can the research be extended and/or be applied to different designs.

8.1 Conclusions

The aim of the thesis has been to consider the impact of missing data on different classes of designed experiments. To assess the influence of drop-out on the starting designs, the increases to the variances of pairwise treatment comparisons are calculated numerically or developed theoretically. Three patterns of missing data have been investigated extensively to study the effect on block and row-column designs.

Treatment effects are estimated using ordinary least squares with general formulae for the reduced normal equations. These equations can be used when the design is complete or when there are missing observations. The variances of pairwise treatment comparisons can be derived and expressed algebraically for most patterns of missing data, and also evaluated numerically for many sets of design parameters. The average of these variances is considered as an overall measure of robustness to missing data, but the maximum should also be calculated because a small number of comparisons may be severely affected by the loss of data.

When blocks of observations become unavailable in block designs, it has been shown that small starting designs suffer the largest reduction in efficiency. Loss of identical blocks is the worst situation, which occurs for RBDs. Less severe is the loss of blocks with a disjoint set of treatments, which may occur for incomplete block designs. The analysis is more involved for BIB designs because all treatments do not occur in every block of the starting design. Loss of a row or column of observations from a Latin square design has also been studied, and it has been shown that the efficiency of the resulting designs are in general substantially reduced. For a Latin square of side $r = 10$, the efficiency is 88.9% when compared to the starting design. Youden square designs were also investigated for the loss of a row of observations, and it was shown that small designs suffer a large reduction in efficiency in this situation.

More general patterns of missing observations have also been considered, where the missing values are scattered throughout the starting design. When considering a BIB design, it appears prudent to choose a design where blocks have as few treatments common to each other as possible. It is also sensible to have pairs of treatments occurring together in blocks as frequently as is practical.

The average variance of pairwise treatment comparisons is substantially increased when a Latin square with five or fewer treatments loses one or two observations. This loss of efficiency is reduced when there are larger starting designs, although the efficiency of the maximum variance of all pairwise comparisons is always substantially reduced. For a ten treatment Latin square design there are 45 pairwise comparisons, and only nine of these are affected in the worst case which is concealed if the average and not the maximum is computed.

To reduce the impact of drop-out on row-column designs constructed from Latin squares, it may possible to use multiple squares where the choice of squares does not affect the overall robustness properties of the design. Consider a four treatment design with up to four squares. When one value becomes unavailable, the loss of efficiency is 25% for a single square which decreases to 10% if there are two squares used to construct the design. If two observations become unavailable then the loss of efficiency is only small if three or more squares are used, and the situation is even more serious when three values are lost.

The effect of three patterns of missing data on single replicate Youden square designs has also been studied. Conditions for these designs to remain variance balanced after the loss of all observations corresponding to one of the treatments have been derived. It was shown that Youden squares generated by removing one column from a Latin square are globally resistant of degree one.

The last class of designs considered are diallel cross designs used in mating experiments, where a set of lines are crossed to discover the best combination. When a block of crosses is lost, the efficiency of the average variance is not substantially reduced. The maximum variance does suffer a large reduction in efficiency.

8.2 Future Work

Theoretical results were provided for the loss of a complete block of crosses from PDC designs. Ghosh and Desai (1998) used BIB designs to construct CDC designs, and it may be possible to derive formulae for the variances of pairwise line comparisons when a single cross is lost.

The work in Chapter 7 concentrated on assessing the influence of missing data on diallel crosses in block designs. Gupta and Choi (1998) provided four series of optimal row-column designs that could be used for diallel cross experiments. It would be beneficial to consider the impact of drop-out on these designs using the same approach.

Bibliography

Agarwal, S. C. and Das, M. N. (1987) A note on Construction and Application of Balanced n-ary Designs. *Sankhyā - The Indian Journal of Statistics Series B*, **49**, 192–196.

Bhaumik, D. K. and Whittinghill, D. C. (1991) Optimality and Robustness to the Unavailability of Blocks in Block Designs. *Journal of the Royal Statistical Society Series B*, **53**, 399–407.

Das, A. and Kageyama, S. (1992) Robustness of BIB and extended BIB designs against the unavailability of any number of observations in a block. *Computational Statistics and Data Analysis*, **14**, 343–358.

Dey, A. and Dhall, S. P. (1988) Robustness of Augmented BIB Designs. *Sankhyā - The Indian Journal of Statistics Series B*, **50**, 376–381.

Diggle, P. and Kenward, M. G. (1994) Informative Drop-out in Longitudinal Data Analysis. *Journal of the Royal Statistical Society Series B*, **43**, 49–93.

Duan, X. and Kageyama, S. (1995) Robustness of Augmented BIB Designs against unavailability of some observations. *Sankhyā - The Indian Journal of Statistics Series B*, **57**, 405–419.

Ghosh, D. K. and Biswas, P. C. (2000) Robust designs for diallel crosses against the missing of one block. *Journal of Applied Statistics*, **27**, 715–723.

Ghosh, D. K. and Desai, N. R. (1998) Robustness of complete diallel crosses plans to the unavailability of one block. *Journal of Applied Statistics*, **25**, 827–837.

— (1999) Robustness of a complete diallel crosses plan with an unequal number of crosses to the unavailability of one block. *Journal of Applied Statistics*, **26**, 563–577.

Ghosh, D. K. and Divecha, J. (1997) Two associate class partially balanced incomplete block designs and partial diallel crosses. *Biometrika*, **84**, 245–248.

Graybill, F. A. (1983) *Matrices with Applications in Statistics.* Belmont, California 94002: Wadsworth, Inc.

Gupta, S. and Choi, K. C. (1998) Optimal Row-column designs for complete diallel crosses. *Communications in Statistics - Theory and Methods*, **27**, 2827–2835.

Gupta, S. and Kageyama, S. (1994) Optimal complete diallel crosses. *Biometrika*, **81**, 420–424.

Gupta, V. K. and Srivastava, R. (1992) Investigations on Robustness of Block Designs against Missing Observations. *Sankhyā - The Indian Journal of Statistics Series B*, **54**, 100–105.

Hedayat, A. and John, P. W. M. (1974) Resistant and Susceptible BIB Designs. *The Annals of Statistics*, **2**, 148–158.

Hinkelmann, K. (1975) Design of Genetical Experiments. In *A Survery of Statistical Design and Linear Models* (ed. J. N. Srivastava), 243–269. North-Holland Publishing Company.

John, P. W. M. (1976) Robustness of Balanced Incomplete Block Designs. *The Annals of Statistics*, **4**, 960–962.

Lal, K., Gupta, V. K. and Bhar, L. (2001) Robustness of designed experiments against missing data. *Journal of Applied Statistics*, **28**, 63–79.

Little, R. J. A. and Rubin, D. B. (1987) *Statistical analysis with missing data.* New York: Wiley.

Low, J. L., Lewis, S. M. and Prescott, P. (1999) Assessing robustness of crossover designs to subjects dropping out. *Statistics and Computing*, **9**, 219–227.

Mansson, R. A. and Prescott, P. (2001a) Missing Observations in Youden Square designs. *Submitted to Computational Statistics and Data Analysis.*

— (2001b) Missing Values in Replicated Latin Squares. *Journal of Applied Statistics*, **28**, 743–757.

— (2001c) The effect of missing data on Latin square designs. *Utilitas Mathematica (to appear).*

Most, B. M. (1975) Resistance of Balanced Incomplete Block Designs. *The Annals of Statistics*, **3**, 1149–1162.

Preece, D. A. (1966) Classifying Youden Rectangles. *Journal of the Royal Statistical Society Series B*, **28**, 118–130.

— (1995) How many 7x7 Latin squares can be partitioned into Youden squares? *Discrete Mathematics*, **138**, 343–352.

Prescott, P. and Mansson, R. A. (2001a) Efficiency of pairwise treatment comparisons in incomplete block experiments subject to the loss of a block of observations. *Communications in Statistics - Theory and Methods (to appear).*

— (2001b) Robustness of BIB Designs to randomly missing observations. *Journal of Statistical Planning and Inference*, **92**, 283–296.

Raghavarao, D. (1971) *Constructions and Combinatorial Problems in Design of Experiments.* New York, USA: Wiley.

Shah, S. M. and Gujarathi, C. C. (1977) On a Locally Resistant BIB design of degree 1. *Sankhyā - The Indian Journal of Statistics Series B*, **39**, 406–408.

— (1983) Resistant BIB design. *Sankhyā - The Indian Journal of Statistics Series B*, **45**, 225–232.

Srivastava, R., Parsad, R. and Gupta, V. K. (1996) Robustness of block designs for making test treatment - control comparisons against a missing observation. *Sankhyā - The Indian Journal of Statistics Series B*, **58**, 407–413.

Whittinghill, D. C. (1989) Balanced Block Designs Robust Against the Loss of a Single Observation. *Journal of Statistical Planning and Inference*, **22**, 71–80.

— (1995) A Note on the Optimality of Block Designs Resulting from the Unavailability of Scattered Observations. *Utilitas Mathematica*, **47**, 21–31.

Yates, F. and Hale, R. W. (1939) The Analysis of Latin Squares when Two or More Rows, Columns, or Treatments are Missing. *Journal of the Royal Statistical Society, Supplement*, **6**, 67–79.