UNIVERSITY OF SOUTHAMPTON

MODELLING COINTEGRATED I(2) SYSTEMS WITH AN APPLICATION TO MONEY AND EXCHANGE RATES

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This thesis is submitted in accordance with the requirements for the degree of Doctor of Philosophy

Department of Economics

October 2002

UNIVERSITY OF SOUTHAMPTON
ABSTRACT
FACULTY OF SOCIAL SCIENCE
ECONOMICS
Doctor of Philosophy

MODELLING COINTEGRATED I(2) SYSTEMS WITH AN APPLICATION TO MONEY AND EXCHANGE RATES

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The first chapter sets the scene for the empirical analyses that come later. In particular, we collect together some recent results on the representation, estimation and testing of cointegrated I(2) systems. In particular, this survey chapter extends Haldrup [1998] by including some recent developments such as tests on the cointegration parameters of Kongsted [1998, 2000] and by examining in more detail the influence of deterministic components in the I(2) model as well as providing an overview of the maximum likelihood procedure detailed in Johansen [1997].

In the second chapter we provide a cointegration analysis of UK money demand within a framework that allows for I(2) variables. The presence of I(2) variables is supported by a test for the integration indices of the model, which suggests two cointegrating relationships and one I(2) trend. We also find evidence for a nominal-to-real transformation to real money demand and a polynomial term involving nominal money and prices. This is in contrast to the common transformation used in the extant literature where the polynomial term contains the price variable alone. The I(1) analysis of the transformed information set provides two polynomial cointegrating vectors which are consistent with a real money demand and an excess demand relation that closely match those found from previous studies. Finally, we show that a parsimonious VAR that incorporates the two polynomial relations provides a good characterisation of the in-sample movements in real money demand and, moreover, provides superior out-of-sample forecasting power compared to a differenced VAR that excludes the long run relations.

In the third chapter we provide the first empirical examination of the monetary exchange rate model that allows for the presence of I(2) variables in the data. For the dollar-sterling exchange rate over the modern float we find support for the existence of two cointegrating relationships among the variables: the monetary exchange rate model; and a simple Taylor rule. Moreover, by formally testing for the number of I(1) and I(2) stochastic trends and finding evidence for two such I(2) trends in the data, the stationary cointegrating relations found correspond to polynomial relations where the linear combination of the differences of the variables are required in order to provide a stationary relationship. Finally, we show that by specifying an equilibrium correction model that incorporates the disequilibrium errors from the estimated cointegrating relations we obtain a model that is well-specified in-sample and, moreover, provides superior 1-step forecasting power compared to multivariate versions of simple autoregression models with drift. Thus, by careful attention to the time series properties of the variables in the monetary model we provide further evidence that Meese and Rogoff's [1983] criticism of the monetary model as a poor forecasting model should be laid to rest.

The final chapter provides a re-examination of the Cagan model of money demand over the German hyperinflation period. By allowing for the presence of I(2) variables we show that the restriction of long run price homogeneity in the Cagan model is not supported over the German hyperinflation period. Furthermore, we find support for the need to augment the model with both wages and foreign exchange rate depreciation as suggested by Michael et al. [1994] and Frenkel [1977] respectively. A key result is that we show that the Cagan model requires reinterpretation as a polynomial long run cointegrating relation in order to produce a stable equilibrium relation over the German hyperinflation. Finally, we find support for both prices and wages being weakly exogenous for the cointegration parameters over the period.

Acknowledgements

I would first of all like to thank the Department of Economics at Southampton for their support and friendship. Most notably of course my supervisor - Jan Podivinsky - who has provided unstinting support and numerous good times over a "quick orange juice". But there are many others in the Department who have made this thesis an enjoyable pursuit: John Aldrich; Ray O'Brien; Maozu Lu; Sally Srinivasan; Rob Hillman ("Dr. H") to name but a few.

Since leaving the cosy confines of Southampton in 1998 many people have generously allowed me to bend the rules of time and attendance. Derek Hodgson at HM Customs and Excise and Roger Clews at the Bank of England have been my chief partners in crime.

And finally, I owe more than I can say to the support and love of my family, especially my wife Rachel - *alter ipse amicus*.

To the future.

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Chapter 1

Cointegrated I(2) Systems: Representation, Estimation and Testing

In this introductory chapter we set the scene for the empirical analyses that come later. In particular, we collect together some recent results on the representation, estimation and testing of cointegrated I(2) systems. A comprehensive survey of tests for I(2) and an overview of cointegration in the I(2) model has already been provided by Haldrup [1998]. However, this survey chapter extends Haldrup [1998] by including some recent developments such as tests on the cointegration parameters of Kongsted [1998, 2000] and by examining in more detail the influence of deterministic components in the I(2) model as well as providing an overview of the maximum likelihood procedure detailed in Johansen [1997].

1.1 Representation

In this section we discuss the basic statistical vector autoregressive (VAR) model and show how a representation theorem (Johansen [1992]) defines the conditions under which the model allows the observed process to be integrated of order two (which we denote hereafter as I(2)). We continue by applying the representation theorem to yield a moving average (MA) representation of the I(2) model and discuss the individual processes contained therein. We also consider some useful reparameterisations of the VAR model including various equilibrium correction models (Johansen [2000]), a transformation to the I(1) model (Kongsted [2000]), and partial systems. These various representations are used later in the chapter when we consider estimation of the I(2) model and tests on the components of the model. We conclude this section by discussing the inclusion of deterministic components in the I(2) model through their impact on the moving average representation and on the individual processes in I(0), I(1) and I(2) spaces (Paruolo [1994, 1996] and Rahbek *et al.* [1999]).

Throughout we make use of the following notation: Δ denotes the difference operator such that $\Delta x_t = x_t - x_{t-1}$; for any $(p \times r), r < p$ matrix a of rank r let $\overline{a} \equiv a(a'a)^{-1}$such that $a'\overline{a} = I_r$; and let $a_\perp$ indicate a $p \times (p-r)$ matrix whose columns form a basis of the orthogonal complement of span(a) such that $a'_\perp a = 0$, $a_\perp = 0$ if $r = p$, and $a_\perp = I_p$ if $a = 0$.

1.1.1 Basic Statistical Model

Consider the p-dimensional kth order unrestricted VAR model given by

$$A(L)X_t = \varepsilon_t \tag{1.1}$$

where the errors, ε_t, are assumed to be independently Gaussian distributed with mean zero and variance Ω positive definite. To motivate the following note that the lag polynomial $A(z) = \sum_{i=0}^{k} A_i z^i$ can be expanded around the point $z = 1$ to yield

$$A(z) = A - (1-z)\dot{A} + \frac{1}{2}(1-z)^2 \ddot{A} + (1-z)^3 A_3(z) \tag{1.2}$$

where $\dot{A}(z) \equiv dA(z)/dz$ and $\ddot{A}(z) \equiv d^2A(z)/dz^2$ and where $A \equiv A(1)$, $\dot{A} \equiv \dot{A}(1)$ and $\ddot{A} \equiv \ddot{A}(1)$.

The conditions under which the process, X_t, is I(2) are set out in Johansen [1992]. First, we require that the roots of the characteristic equation $|A(z)| = 0$ lie outside the unit circle in the complex plane or at the point $z = 1$. (Thus, we exclude seasonal roots defined where $|z| = 1, z \neq 1$, and explosive roots defined where $|z| < 1$.) Second, we require the following reduced rank conditions on the impact matrix A and first-order derivative $\dot{A}$ at $z = 1$:

$$\Pi = \alpha\beta' \tag{1.3}$$

has reduced rank $r < p$, where $\Pi = -A$ and where α and β are $(p \times r)$ matrices of full rank r; and

$$\alpha_{\perp}\Gamma\beta_{\perp} = \xi\eta' \tag{1.4}$$

has reduced rank $s_1 < (p - r)$, where $\Gamma = -\dot{A} - \Pi$ and where ξ and η are $(p \times s_1)$ matrices of full rank s_1. This assumption allows for orders of integration greater than one. Last, we require the following full rank restriction to ensure that at most I(2) variables are generated:

$$\xi'_{\perp}\alpha'_{\perp}\Theta\beta_{\perp}\eta_{\perp} \tag{1.5}$$

is of full column rank $s_2 = (p - r - s)$. If the above assumptions all hold then the process X_t is I(2).

1.1.2 Moving Average Representation

Let $\beta_1 = \overline{\beta}_\perp \eta$, $\beta_2 = \beta_\perp \eta_\perp$, $\alpha_1 = \overline{\alpha}_\perp \xi$ and $\alpha_2 = \alpha_\perp \xi_\perp$. Note that $(\alpha, \alpha_1, \alpha_2)$ and $(\beta, \beta_1, \beta_2)$ are mutually orthogonal and that the parameter η depends on the choice of $\beta_\perp$ whereas $\beta_1$and β_2 are independent of this choice. Under the conditions set out in Johansen's representation theorem discussed above, the process X_t has the following moving average (MA) representation

$$X_t = C_2 \sum_{s=1}^{t} \sum_{i=1}^{s} \varepsilon_i + C_1 \sum_{i=1}^{t} \varepsilon_i + C^*(L)\varepsilon_t \tag{1.6}$$

where $C_2 = \beta_2(\alpha_2'\Theta\beta_2)^{-1}\alpha_2'$, $\beta' C_1 = \overline{\alpha}'\Gamma C_2$, $\beta_1' C_1 = \overline{\alpha}_1'(\Theta C_2 - I_p)$ and the lag polynomial $C^*(z) = \sum_{i=0}^{\infty} C_i^* z^i$ has all roots strictly outside the unit circle. From Eq.(1.6) we can make some comments on the different components of the process and their orders of integration. First, X_t will, in general, have s_2 common stochastic trends given by $\alpha_2' \sum_{s=1}^{t} \sum_{i=1}^{s} \varepsilon_i$. (In particular, note that $\beta_2{}' C_2 \neq 0$. Thus $\beta_2' X_t$ is simply a linear combination of the $\alpha_2' \sum_{s=1}^{t} \sum_{i=1}^{s} \varepsilon_i$ and thereby is itself a s_2-dimensional I(2) trend.) Second, since $(\beta, \beta_1)'\beta_2 = 0$ it follows that $(\beta, \beta_1)' C_2 = 0$ and hence $(\beta, \beta_1)'$reduces the integration order of X_t from two to one *i.e.* $(\beta, \beta_1)' X_t$ is I(1). Third, Johansen [1995] shows that the I(1) linear combination of levels $\beta' X_t$ forms a polynomial cointegrating relationship with the I(1) linear combination of differences $\beta_2' \Delta X_t$ such that $\beta' X_t - \delta \beta_2' \Delta X_t \sim$ I(0) where $\delta = \overline{\alpha}'\Gamma\overline{\beta}_2$. (Furthermore, as a special case, consider the othogonal complement of δ given by $\delta_\perp$ of dimension $r \times (r - s_2)$ such that $\delta'\delta_\perp = 0$. It then follows that $\delta_\perp' \beta' X_t \sim$ I(0) and thus we have direct, as opposed to multi-, cointegration from the I(2) level down to stationarity.) Last, note that the number of unit roots in the process is $s_1 + 2s_2 = \text{rank}(\beta_1) + \text{rank}(\beta_2)$.

1.1.3 Equilibrium Correction Models

Consider the following isomorphic representation of Eq.(1.1) where the impact matrices Π and Γ defined in section 1.1.1 are explicitly introduced into the model specification

$$\Delta^2 X_t = \Pi X_{t-1} - \Gamma \Delta X_{t-1} + \sum_{i=1}^{k-2} \Psi_i \Delta^2 X_{t-i} + \varepsilon_t \tag{1.7}$$

where $\Psi_i = \sum_{j=i+1}^{k} (j - i - 1)\Pi_j = -\sum_{j=i+1}^{k-1} \Gamma_j$. The specification given by Eq.(1.7) is termed an equilibrium correction model (EqCM) and provides a clearer representation of the I(2) VAR model given by Eq.(1.1) by specifying the model in terms of levels, differences and double differences.

By introducing the reduced rank condition given by Eq.(1.3) the following reparameterisation of the EqCM model in Eq.(1.7) is obtained

$$\Delta^2 X_t = \alpha\beta' X_{t-1} - \Gamma \Delta X_{t-1} + \sum_{i=1}^{k-2} \Psi_i \Delta^2 X_{t-i} + \varepsilon_t \tag{1.8}$$

where the cointegrating relations $\beta' X_{t-1}$ are clearly presented and where the parameter α is shown to be the adjustment to these relations.

As shown in section 1.1.2 the relations $\beta' X_{t-1}$ are, in general, I(1). To ensure that all the terms in the EqCM representation are stationary we make use of the following reparameterisation

$$\begin{aligned}\Delta^2 X_t &= \alpha(\rho'\tau' X_{t-1} + \psi'\Delta X_{t-1}) \\ &\quad +\Omega\alpha_\perp(\alpha_\perp'\Omega\alpha_\perp)^{-1}\kappa'\tau'\Delta X_{t-1} + \sum_{i=1}^{k-2}\Psi_i\Delta^2 X_{t-i} + \varepsilon_t\end{aligned} \tag{1.9}$$

where Johansen [1997] shows that Eq.(1.9) is a valid reparameterisation of Eq.(1.8) for $\beta = \rho\tau$ of dimension $(p \times r)$, $\xi = -\kappa'\overline{\rho}_\perp$ of dimension $(p - r) \times s_1$, $\eta = \beta_\perp'\tau\rho_\perp$ of dimension $(p - r) \times s_1$ and $\Gamma = -\Omega\alpha_\perp(\alpha_\perp'\Omega\alpha_\perp)^{-1}\kappa'\tau' - \alpha\psi'$ of dimension $(p \times p)$. The EqCM representation given in Eq.(1.9) has a number of advantages over the representation given by Eq.(1.8). First, all the terms entering the equation are stationary, namely the polynomial cointegrating term $\rho'\tau' X_{t-1} + \psi'\Delta X_{t-1}$, the direct cointegrating term $\tau'\Delta X_{t-1}$, and the double differenced terms $\sum_{i=1}^{k-2}\Psi_i\Delta^2 X_{t-i}$. Second, the parameters α and κ' are now clearly shown as the adjustment coefficients to the disequilibrium errors $\rho'\tau' X_{t-1} + \psi'\Delta X_{t-1}$ and $\tau'\Delta X_{t-1}$ respectively. Last, all the parameters $\alpha, \rho, \tau, \psi, \Omega, \Psi_1, \ldots, \Psi_{k-2}$ are now unrestricted. This feature is discussed in more detail later with reference to full information maximum likelihood estimation, see section 1.2.2 below.

In section 1.1.2 it was shown that the combination $\delta_\perp'(\beta' X_t - \delta\beta_2'\Delta X_t)$ cointegrates directly to form a stationary relation. Under the new parameter definitions given in Eq.(1.9) we have the analogous result that $\delta_\perp'(\rho'\tau' X_t - \delta\beta_2'\Delta X_t) = \delta_\perp'\rho'\tau' X_t \sim \mathsf{I}(0)$. This result is used in the following EqCM reparameterisation (see Engsted and Haldrup [1999]) such that

$$\begin{aligned}\Delta^2 X_t &= \alpha\overline{\delta}_\perp\delta_\perp'\rho'\tau' X_{t-1} + \alpha\overline{\delta}\delta'(\rho'\tau' X_{t-1} + \delta\tau_\perp'\Delta X_{t-1}) \\ &\quad +(\alpha\psi'\overline{\tau} + \Omega\alpha_\perp(\alpha_\perp'\Omega\alpha_\perp)^{-1}\kappa')\tau'\Delta X_{t-1} + \sum_{i=1}^{k-2}\Psi_i\Delta^2 X_{t-i} + \varepsilon_t\end{aligned} \tag{1.10}$$

now shows three disequilibrium errors corresponding to the directly cointegrating relations

$\delta'_{\perp}\rho'\tau' X_{t-1}$, the polynomial cointegrating relations $\delta'\rho'\tau' X_{t-1} + \delta'\delta\tau'_{\perp}\Delta X$, and the stationary relations $\tau'\Delta X_{t-1}$.

1.1.4 I(1) Transformations

Kongsted [2000] discusses the conditions required for a valid transformation of the I(2) model to I(1) while preserving the full set of cointegrating relations. This transformation is obviously of interest given that, if accepted, the analysis of the system can then proceed with the well known and comparatively simpler I(1) model.

Consider the EqCM model given in Eq.(1.9). The transformation that we consider is given by the following

$$Y_t = (\tau' X_{t,} \tau'_{\perp}\Delta X) \tag{1.11}$$

where the $p \times (r + s_1)$ matrix τ defines the linear combinations which, in general, reduce the order of integration from two to one; and the $p \times s_2$ matrix $\tau'_{\perp}$ corresponds to the condition that ΔX_t can be obtained from Y_t and ΔY_t. Kongsted [1999] shows that the conditions required for the transformation to be valid, while ensuring that the I(2) trends are eliminated and the cointegrating relations retained, are given by

$$b'(\beta, \beta_1) = 0 \tag{1.12}$$

Alternatively, if the loadings matrix to the I(2) trends, β_2, is equal to the known matrix τ then the transformation is also valid. The testing procedure for the transformation is discussed in full in section 1.3.3.

With the transformation in place we obtain the following I(1) model

$$\Delta Y_t = (\tau, \tau_{\perp})'\alpha(\rho', \delta)Y_{t-1} + \sum_{i=1}^{k-2} \Phi_i \Delta Y_{t-i} + (\tau, \tau_{\perp})'\varepsilon_t \tag{1.13}$$

where the coefficients $\sum_{i=1}^{k-2}\Phi_i$ are complicated functions of the parameters in the model. Note that the polynomial cointegrating coefficient $\delta = \psi'\overline{\tau}_\perp$ now forms part of the cointegrating vector in the I(1) model such that the number of cointegrating relations is greater than or equal to the number of polynomial cointegrating relations in the original I(2) model.

1.1.5 Partial Systems

The final set of reparameterisations we consider is those of partial systems *i.e.* those that decompose the I(2) VAR model into marginal and conditional models. Partial systems are used in both the two step reduced rank regression and full information maximum likelihood estimation procedures, see sections 1.2.1 and 1.2.2 respectively, and in deriving the conditions for weak exogeneity, see section 1.3.4.

Consider the EqCM model given by Eq.(1.8). Now, by pre-multiplying the EqCM model by $\overline{\alpha}' = (\alpha'\alpha)^{-1}\alpha$ and $\alpha'_\perp$ respectively we obtain the following set of equations

$$\overline{\alpha}'\Delta^2 X_t = \beta' X_{t-1} - \overline{\alpha}'\Gamma\Delta X_{t-1} + \sum_{i=1}^{k-2}\overline{\alpha}'\Psi_i\Delta^2 X_{t-i} + \overline{\alpha}'\varepsilon_t \tag{1.14}$$

and

$$\alpha'_\perp\Delta^2 X_t = -\alpha'_\perp\Gamma\Delta X_{t-1} + \sum_{i=1}^{k-2}\alpha'_\perp\Psi_i\Delta^2 X_{t-i} + \alpha'_\perp\varepsilon_t \tag{1.15}$$

The marginal model is simply that given by Eq.(1.15). The conditional model for $\overline{\alpha}'\Delta^2 X_t$ given $\alpha'_\perp\Delta^2 X_t$ and the set of past values can be easily derived from Eqs.(1.14)-(1.15) and is given by

$$\begin{aligned}\overline{\alpha}'\Delta^2 X_t &= \omega\alpha'_\perp\Delta^2 X_t + \beta' X_{t-1} \\ &\quad -(\overline{\alpha}' - \omega\alpha'_\perp)\Gamma\Delta X_{t-1} + \sum_{i=1}^{k-2}(\overline{\alpha}' - \omega\alpha'_\perp)\Psi_i\Delta^2 X_{t-i} + (\overline{\alpha}' - \omega\alpha'_\perp)\varepsilon_t\end{aligned} \tag{1.16}$$

for $\omega = \overline{\alpha}'\Omega\alpha_\perp(\alpha'_\perp\Omega\alpha_\perp)^{-1}$. Note that the error $(\overline{\alpha}' - \omega\alpha'_\perp)\varepsilon_t = (\alpha'\Omega^{-1}\alpha)^{-1}\alpha'\Omega^{-1}\varepsilon_t$ with variance $\Omega_1 = (\alpha'\Omega^{-1}\alpha)^{-1}$ and is independent of the error process in Eq.(1.15) which has a variance of $\Omega_2 = \alpha'_\perp\Omega\alpha_\perp$.

1.1.6 Deterministic Components

For ease of exposition we have so far ignored the possibility of deterministic components entering into the I(2) model. However, deterministic components may be required in order to adequately characterize movements in the process, both in the short run, through inclusion in the model specification, and in the long run, through correctly specified cointegrating relations. In light of this, this section provides a review of the literature on deterministic components in the I(2) model. In particular we focus on the effect of including deterministic processes on the stochastic representation of the I(2) observed process.

Model with constant

Paruolo [1994, 1996] extends the analysis of Johansen [1992] by allowing for the inclusion of a constant in the unrestricted VAR model such that

$$A(L)X_t = \mu + \varepsilon_t \tag{1.17}$$

The author considers the following factorization of the constant

$$\mu = \alpha\mu_0 + \alpha_1\mu_1 + \alpha_2\mu_2, \text{ with } \mu_d = \overline{\alpha}'_d\mu \tag{1.18}$$

whereby μ is projected onto the spaces spanned by α, α_1 and α_2 with dimensions r, s_1 and s_2 respectively and where μ_0 is a vector of intercepts in the stationary cointegrating relations with dimension $(r \times 1)$, μ_1 is a vector of linear trend slopes of dimension $(s_1 \times 1)$, and μ_2 is a vector of quadratic trend slopes of dimension $(s_2 \times 1)$. Paruolo [1994] considers the case where $\mu_2 \neq 0$, while Paruolo [1996] considers the case of a restricted constant where $\mu_2 = 0$. The impact of the restriction on the time series properties of the components of the I(2) model are discussed below.

Under Johansen's representation theorem, the MA representation of Eq.(1.17) is given by

$$X_t = C_2 \sum_{s=1}^{t} \sum_{i=1}^{s} (\varepsilon_i + \mu) + C_1 \sum_{i=1}^{t} (\varepsilon_i + \mu) + C^*(L)\varepsilon_i + a + \beta_2 bt \quad (1.19)$$

$$= \tau_2 t^2 + (\tau_2 + C_1\mu + \beta_2 b)t + C_2 \sum_{s=1}^{t} \sum_{i=1}^{s} \varepsilon_i + C_1 \sum_{i=1}^{t} \varepsilon_i + C^*(L)\varepsilon_i + a \quad (1.20)$$

where $\tau_2 = \frac{1}{2} C_2 \mu$ (*c.f.* Eq.(1.6) where the definitions of C_1, C_2 and C^* are the same). Here the parameter a depends on $C^*(1)\mu$ and the initial values while parameter b depends on the initial values only.

The MA representation given in Eq.(1.20) decomposes the process into quadratic and linear deterministic trends ($\tau_2 t^2$ and $(\tau_2 + C_1\mu + \beta_2 b)t$ respectively), integrated and cumulative stochastic trends ($C_2 \sum_{s=1}^{t} \sum_{i=1}^{s} \varepsilon_i$ and $C_1 \sum_{i=1}^{t} \varepsilon_i$ respectively) and a stationary process given by $C^*(L)\varepsilon_i$. It is clear that, as in Paruolo [1994], an unrestricted constant μ gives rise to a quadratic trend with coefficient τ_2. However, if $\mu_2 = 0$ as in Paruolo [1996], the quadratic trend disappears. This follows from noting that $C_2 = \beta_2(\alpha_2' \Theta \beta_2)^{-1} \alpha_2'$ so $\tau_2 = \frac{1}{2} C_2 \mu = \frac{1}{2} \beta_2 (\alpha_2' \Theta \beta_2)^{-1} \alpha_2' \alpha_2 \mu_2 = 0$, $\forall \mu_2 = 0$.

Considering the linear combinations β we have that $\beta' C_1 = \overline{\alpha}' \Gamma C_2$. If $\mu_2 = 0$ as in Paruolo [1996] then $\beta' X_t$ does not contain a linear trend and is dominated by the random walk $C_2 \sum \varepsilon_i$. However, if $\mu_2 \neq 0$ as in Paruolo [1994] $\beta' X_t$ now contains a linear trend in addition to the I(1) component and thus behaves as a random walk with drift. In both cases $\beta' X_t$ and $\beta_2' \Delta X_t$ are both dominated by the same random walk and are cointegrated with coefficient matrix δ such that polynomial cointegration occurs.

Regarding the linear combinations β_1 we have from above that $\beta_1' C_1 = \overline{\alpha}_1'(\Theta C_2 - I_p)$ that is the quadratic trend and integrated random walk components cancel so that $\beta_1' X_t$ is dominated by the linear trend. Note that under the additional restriction $\mu_1 = 0$ this linear trend disappears as $\beta_1' C_1 \mu = -\mu_1$.

Finally, in the direction β_2 the process is dominated by the I(2) stochastic components $\sum\sum \varepsilon_i$.

Model with restricted trend

Rahbek *et al.* [1999] extend the analysis of Paruolo [1994, 1996] to allow for a linear regressor in the unrestricted VAR model such that

$$A(L)X_t = \mu_0 + \mu_1 t + \varepsilon_t \tag{1.21}$$

In contrast to Paruolo [1994, 1996] the constant τ_0 is factorized into the spaces spanned by α and its othogonal complement $\alpha_\perp$ to give

$$\begin{aligned} \tau_0 &= \alpha\overline{\alpha}'\mu_0 + \overline{\alpha}_\perp \alpha_\perp' \mu_0 \\ &\equiv \alpha\kappa_0' + \overline{\alpha}_\perp \alpha_\perp' \mu_0 \end{aligned} \tag{1.22}$$

If the observed process X_t is integrated of order two then, in general, the unrestricted linear regressor $\mu_1 t$ cumulates to a cubic trend while, as above, the constant μ_0 allows for a quadratic trend. However, the authors show that by placing certain restrictions on μ_0

and μ_1 the observed process can be forced to display, at most, linear deterministic trends. These restrictions correspond to

$$\mu_1 = \alpha\beta_0' \tag{1.23}$$

where $\beta_0' = -\beta'\tau_1$ of dimension $r \times 1$; and

$$\alpha_\perp'\mu_0 = -\xi\eta_0' - (\alpha_\perp'\Gamma\overline{\beta})\beta_0' \tag{1.24}$$

where $\eta_0' = -(\overline{\beta}_\perp\eta)'\tau_1 = -\beta_1'\tau_1$ of dimension $s_1 \times 1$.

With these two restrictions in place the MA representation of the model is given by

$$X_t = C_2\sum_{s=1}^{t}\sum_{i=1}^{s}\varepsilon_i + C_1\sum_{i=1}^{t}\varepsilon_i + C^*(L)\varepsilon_i + \tau_0 + \tau_1 t \tag{1.25}$$

where the vectors τ_0 and τ_1 are functions of both the initial values and the parameters of the VAR model such that

$$(\beta, \beta_1)'\tau_1 = -(\beta_0, \eta_0)' \tag{1.26}$$

$$\beta'\tau_0 - \delta\beta_2'\tau_1 = -(\kappa_0' + \overline{\alpha}'\Gamma\overline{\beta}_1\eta_0' + (I_r + \overline{\alpha}'\Gamma\overline{\beta})\beta_0' \equiv \gamma \tag{1.27}$$

Rahbek *et al. op. cit.* provides the proof and full expressions for τ_0 and τ_1.

From Eqs.(1.25)-(1.27) we can see that X_t has at most linear trends in all directions. Specifically, we have: $-(\beta_0, \eta_0)'t$ for the I(1) linear combinations $(\beta, \beta_1)'X_t$; $\beta_2'\tau_1 t$ for the I(2) linear combinations $\beta_2'X_t$; and $(\beta_0't + \gamma)$ for the polynomial cointegrating relations $\beta'X_t - \delta\beta_2'\Delta X_t$. Finally, note that only $\beta_2'\tau_1 t$ is a function of the initial values.

1.2 Estimation

In this section we begin by providing a detailed discussion of the two step estimation procedure of Johansen [1995]. Our focus on this procedure stems from the availability of code in the CATS for RATS computer package (Hansen and Juselius [1992]) that implement the tests discussed later. As a result, the two step estimation procedure is employed in the empirical studies in the rest of this thesis. Next, in addition to the procedure outlined in Johansen [1995] we also consider in detail the inclusion of deterministic components. We conclude this section with an overview of the unrestricted maximum likelihood procedure of Johansen [1997]. The maximum likelihood procedure is not employed in this thesis as the implementation of the tests is still under development. However, it would clearly be of interest to employ these procedures when documented code becomes available.

1.2.1 Two Step Reduced Rank Regression

A two step estimation procedure model that requires only regression and reduced rank regression is provided by Johansen [1995]. In the first step, estimators of α, β and Ω together with the rank of Π are obtained by reduced rank regression. This is equivalent to the standard $\mathsf{I}(1)$ reduced rank analysis with a restricted linear term, see Johansen [1996, Section 6.2]. However, in contrast to the $\mathsf{I}(1)$ case, the estimators are not maximum likelihood but are asymptotically efficient in the sense that they have the same asymptotic limiting distribution as the maximum likelihood estimator. In the second step, estimators for the remaining parameters η, ξ and $\alpha'_{\perp}\Omega\alpha_{\perp}$ are provided from the reduced rank regression of $\alpha'_{\perp}\Gamma\beta_{\perp}$ conditional on the estimates of α, β and Ω obtained from the first step. A detailed discussion of these two steps is provided below.

Consider the EqCM representation given by Eq.(1.8) such that

$$\Delta^2 X_t = \alpha\beta' X_{t-1} - \Gamma\Delta X_{t-1} + \Phi Z_t + \varepsilon_t \tag{1.28}$$

where we have defined $\Phi Z_t = \sum_{i=1}^{k-2} \Psi_i \Delta^2 X_{t-i}$. The analysis begins by concentrating the parameters in ΦZ_t out of the likelihood function by regressing $\Delta^2 X_t, \Delta X_{t-1}$ and X_{t-1} on Z_t yielding the residuals R_{0t}, R_{1t} and R_{2t} and the equation

$$R_{0t} = \Gamma R_{1t} + \alpha\beta' R_{2t} + \widehat{\varepsilon}_t \tag{1.29}$$

Note that R_{0t} and $\beta' R_{2t}$ are stationary even though from section 1.1.2 we have that $R_{2t} = X_{t-1}$is $\mathsf{I}(2)$ and that $\beta' X_t$ and ΔX_t are $\mathsf{I}(1)$. Thus, the estimate of β is super-superconsistent as the linear combination of $\beta' R_{2t}$ transforms the process from $\mathsf{I}(2)$ to $\mathsf{I}(0)$.

In the first step, the procedure is performed under the restriction $\Pi = \alpha\beta'$ while ignoring the reduced rank restriction on Γ. The first step then corresponds to the reduced rank regression of R_{0t} on R_{2t} corrected for R_{1t} which is solved by the $(p+1)$-dimensional eigenvalue problem

$$\left|\lambda M_{22\cdot 1} - M_{20\cdot 1} M_{00\cdot 1}^{-1} M_{02\cdot 1}\right| = 0 \tag{1.30}$$

where $M_{ij\cdot h} = M_{ij} - M_{ih} M_{hh}^{-1} M_{hj}$ and $M_{ij} = T^{-1} \sum_{i=1}^{T} R_{it} R'_{jt}$ for $i, j, h = 0, 1, 2$ are conditional and unconditional sample product moment matrices respectively. The solution to the eigenvalue problem given in Eq.(1.30) provides ordered eigenvalues $1 > \widehat{\lambda}_1 > \ldots > \widehat{\lambda}_p > 0, \widehat{\lambda}_{p+1} = 0$ with corresponding eigenvectors $\widehat{V} = (\widehat{v}_1, \ldots, \widehat{v}_{p+1})$. The resulting estimators of α, β and Ω are given by

$$\begin{aligned}\widehat{\beta} &= (\widehat{v}_1, \ldots, \widehat{v}_r); \\ \widehat{\alpha} &= M_{02\cdot 1}\widehat{\beta}(\widehat{\beta}' M_{22\cdot 1}\widehat{\beta})^{-1}; \\ \widehat{\Omega} &= M_{00\cdot 1} - \widehat{\alpha}\widehat{\alpha}'\end{aligned} \tag{1.31}$$

In the second step α, β and Ω are assumed known and equal to the estimated values from the first step. The analysis proceeds by considering the partial system given by Eqs.(1.15)-(1.16) which splits the EqCM model into a marginal model for $\alpha'_\perp \Delta^2 X_t$ and a conditional model for $\overline{\alpha}'\Delta^2 X_t$ given $\alpha'_\perp \Delta^2 X_t$. However, we augment the marginal model by explicitly introducing the reduced rank condition on Γ, given by Eq.(1.4), to yield the following partial system

$$\alpha'_\perp \Delta^2 X_t = -\alpha'_\perp \Gamma \beta \overline{\beta}' \Delta X_{t-1} + \xi \eta' \overline{\beta}'_\perp \Delta X_{t-1} + \alpha'_\perp \Phi Z_t + \alpha'_\perp \varepsilon_t \tag{1.32}$$

$$\overline{\alpha}'\Delta^2 X_t = \beta' X_t + \overline{\alpha}'\Gamma \Delta X_{t-1} + \overline{\alpha}'\Phi Z_t + \overline{\alpha}'\varepsilon_t \tag{1.33}$$

where we have made use of the identity $I_p = \overline{\beta}\beta' + \overline{\beta}_\perp \beta'_\perp$. As in the first step the parameters in ΦZ_t are concentrated out of the likelihood function by regressing $\Delta^2 X_t, \Delta X_{t-1}$ and X_{t-1} on Z_t yielding the residuals S_{0t} and S_{1t} the equations

$$\alpha'_\perp S_{0t} = -\alpha'_\perp \Gamma \overline{\beta}(\beta' S_{1t}) - \xi \eta' (\overline{\beta}'_\perp S_{1t}) + \alpha'_\perp \widehat{u}_t \tag{1.34}$$

$$\overline{\alpha}' S_{0t} = \beta' S_{1t} - \overline{\alpha}'\Gamma S_{1t} + \overline{\alpha}'\widehat{u}_t \tag{1.35}$$

The likelihood analysis then proceeds as a reduced rank regression of $\alpha'_\perp S_{0t}$ on $\overline{\beta}'_\perp S_{1t}$ corrected for $\beta' S_{1t}$ which is solved by the $(p - r + 1)$-dimensional eigenvalue problem

$$\left| \varphi M_{\beta_\perp \beta_\perp \cdot \beta} - M_{\beta_\perp \alpha_\perp \cdot \beta} M^{-1}_{\alpha_\perp \alpha_\perp \cdot \beta} M_{\alpha_\perp \beta_\perp \cdot \beta} \right| = 0 \tag{1.36}$$

for ordered eigenvalues $1 > \widehat{\varphi}_1 > \ldots > \widehat{\varphi}_p > 0, \widehat{\varphi}_{p+1} = 0$ with corresponding eigenvectors $\widehat{W} = (\widehat{w}_1, \ldots, \widehat{w}_{p-r+1})$ and where the subscripts $\alpha_\perp, \beta_\perp$ and β refer to the variables $\alpha'_\perp S_{0t}, \beta'_\perp S_{1t}$ and $\beta' S_{1t}$ respectively. The estimators of η, ξ and $\alpha'_\perp \Omega \alpha_\perp$ are then given by

$$\begin{aligned} \widehat{\eta} &= (\widehat{w}_1, \ldots, \widehat{w}_{s_1}); \\ \widehat{\xi} &= M_{\alpha_\perp \beta_\perp \cdot \beta} \widehat{\eta}; \\ \widehat{\alpha'_\perp \Omega \alpha_\perp} &= M_{\alpha_\perp \alpha_\perp \cdot \beta} - \widehat{\xi\xi}' \end{aligned} \tag{1.37}$$

Treatment Of Deterministic Terms

The model of Paruolo [1994, 1996] is defined by the equations

$$\begin{aligned} \Delta^2 X_t &= \alpha\beta' X_{t-1} - \Gamma \Delta X_{t-1} + \Phi Z_t + \mu + \varepsilon_t \\ \alpha_\perp \Gamma \beta_\perp &= \xi\eta' \\ \mu &= \alpha\mu_0 + \alpha_1\mu_1 + \alpha_2\mu_2, \text{ with } \mu_d = \overline{\alpha}'_d \mu \end{aligned} \tag{1.38}$$

The case analyzed by Paruolo [1994] where the constant is completely unrestricted such that $\mu_2 \neq 0$ can be trivially covered by concentrating the constant out of the likelihood function in addition to the parameters in ΦZ_t. Thus in both steps we simply need to regress $\Delta^2 X_t, \Delta X_{t-1}$ and X_{t-1} on both Z_t and a constant and the rest of the analysis remains unchanged.

The case analyzed by Paruolo [1996] covers two cases where the parameter on the constant is restricted to yield a linear trend ($\mu_1 \neq 0$) and no linear trend ($\mu_1 = 0$). As in the case for no deterministics we begin by concentrating the parameters in ΦZ_t out of the likelihood function by regressing $\Delta^2 X_t, \Delta X_{t-1}, X_{t-1}$ and the constant μ on Z_t yielding the residuals T_{0t}, T_{1t}, T_{2t} and C_t and the equation

$$\begin{aligned} T_{0t} &= \Gamma T_{1t} + \alpha\beta' T_{2t} + \mu C_t + \widehat{v}_t \\ &= \Gamma T_{1t} + \alpha\beta' T_{2t} + (\alpha\mu_0 + \alpha_1\mu_1)C_t + \widehat{v}_t \end{aligned} \tag{1.39}$$

where we have imposed the restriction $\mu_2 = 0$ (*c.f.* Paruolo [1994]).

Consider the first case where $\mu_1 \neq 0$. In the first step of the procedure the constant is fitted unrestrictedly by regressing T_{0t}, T_{1t}, T_{2t} on C_t obtaining the residuals U_{0t}, U_{1t} and U_{2t} and the equation

$$U_{0t} = \Gamma U_{1t} + \alpha\beta' U_{2t} + \widehat{w}_t \tag{1.40}$$

Comparing Eq.(1.40) with Eq.(1.29) it is clear that one can proceed with the analysis discussed for the model with no deterministics save for substituting the residuals U_{0t}, U_{1t} and U_{2t} for the residuals R_{0t}, R_{1t} and R_{2t}.

Fixing α and β from the first step, the second step considers the equation

$$\begin{aligned} \alpha'_\perp T_{0t} &= -\alpha'_\perp \Gamma \overline{\beta}(\beta' T_{1t}) - \xi\eta'(\overline{\beta}'_\perp T_{1t}) + \xi\mu_1 C_t + \alpha'_\perp \widehat{v}_t \\ &= -\alpha'_\perp \Gamma \overline{\beta}(\beta' T_{1t}) - \xi(\eta', \mu_1)[(\overline{\beta}'_\perp T_{1t})', C_t] + \alpha'_\perp \widehat{v}_t \\ &= -\alpha'_\perp \Gamma \overline{\beta}(\beta' T_{1t}) - \xi\eta^{*\prime} d_t + \alpha'_\perp \widehat{v}_t \end{aligned} \tag{1.41}$$

where $d_t \equiv [(\overline{\beta}'_\perp T_{1t})', C_t]$ and $\eta^* \equiv (\eta', \mu_1)'$. The likelihood analysis of Eq.(1.41) for fixed α and β corresponds to a reduced rank regression of $\alpha'_\perp T_{0t}$ on d_t for fixed $\beta' T_{1t}$ which is solved by the eigenvalue problem

$$\left| \rho M^*_{dd\cdot\beta} - M^*_{d\alpha_\perp\cdot\beta} M^{*-1}_{\alpha_\perp\alpha_\perp\cdot\beta} M^*_{\alpha_\perp d\cdot\beta} \right| = 0 \tag{1.42}$$

for ordered eigenvalues $1 > \widehat{\rho}_1 > \ldots > \widehat{\rho}_p > 0, \widehat{\rho}_{p+1} = 0$ with corresponding eigenvectors $\widehat{S} = (\widehat{s}_1, \ldots, \widehat{s}_{p-r+1})$ where the superscript $*$ indicates that the moment matrix involves the residuals T_{it} and the subscript β and $\alpha_\perp$ refer to the variables $\beta' T_{1t}$ and $\alpha'_\perp T_{0t}$ respectively. Comparing Eq.(1.42) with Eq.(1.36) it is clear that the estimators for η^*, ξ and $\alpha'_\perp \Omega \alpha_\perp$ are given by

$$\begin{aligned} \widehat{\eta}^* &= (\widehat{s}_1, \ldots, \widehat{s}_{s_1}); \\ \widehat{\xi} &= M^*_{\alpha_\perp d \cdot \beta} \widehat{\eta}^*; \\ \widehat{\alpha'_\perp \Omega \alpha_\perp} &= M^*_{\alpha_\perp \alpha_\perp \cdot \beta} - \widehat{\xi\xi}' \end{aligned} \tag{1.43}$$

Consider the second case where $\mu_1 = 0$. In the first step we start by substituting $\mu_1 = 0$ into Eq.(1.39) to obtain

$$\begin{aligned} T_{0t} &= \Gamma T_{1t} + \alpha\beta' T_{2t} + \alpha\mu_0 C_t + \widehat{v}_t \\ &= \Gamma T_{1t} + \alpha(\beta', \mu_0)(T'_{2t}, C'_t)' + \widehat{v}_t \\ &= \Gamma T_{1t} + \alpha\beta^{*\prime} f_t + \widehat{v}_t \end{aligned} \tag{1.44}$$

where $\beta^* \equiv (\beta', \mu_0)'$ and $f_t \equiv (T'_{2t}, C'_t)'$. The statistical analysis of Eq.(1.44) for unrestricted Γ can be performed by a reduced rank regression of T_{0t} on f_t for fixed T_{1t} which is solved by the eigenvalue problem

$$\left|\gamma M^*_{ff\cdot 1} - M^*_{f0\cdot 1} M^{*-1}_{00\cdot 1} M^*_{0f\cdot 1}\right| = 0 \tag{1.45}$$

for ordered eigenvalues $1 > \widehat{\gamma}_1 > \ldots > \widehat{\gamma}_p > 0, \widehat{\gamma}_{p+1} = 0$ with corresponding eigenvectors $\widehat{G} = (\widehat{g}_1, \ldots, \widehat{g}_{p+1})$. From comparing Eq.(1.45) with Eq.(1.30) it is clear that the estimators of α, β^* and Ω are given by

$$\begin{aligned}\widehat{\beta}^* &= (\widehat{g}_1, \ldots, \widehat{g}_r); \\ \widehat{\alpha} &= M^*_{02\cdot 1}\widehat{\beta}^*(\widehat{\beta}^{*\prime} M^*_{22\cdot 1}\widehat{\beta}^*)^{-1}; \\ \widehat{\Omega} &= M^*_{00\cdot 1} - \widehat{\alpha}\widehat{\alpha}'\end{aligned} \tag{1.46}$$

The model of Rahbek *et al.* [1999] is defined by the equations

$$\begin{aligned}\Delta^2 X_t &= \alpha\beta' X_{t-1} - \Gamma\Delta X_{t-1} + \Phi Z_t + \mu_0 + \alpha\beta_0' t + \varepsilon_t \\ \alpha_\perp \Gamma \beta_\perp &= \xi\eta' \\ \alpha_\perp' \mu_0 &= -\xi\eta_0' - \alpha_\perp' \Gamma \overline{\beta}\beta_0'\end{aligned} \tag{1.47}$$

By concentrating the parameters in ΦZ_t out of the likelihood function and defining $X^*_{t-1} = (X'_{t-1}, t)'$ and $\beta^* = (\beta', \beta_0')'$ we obtain the residuals V_{0t}, V_{1t} and V_{2t} and the equation

$$V_{0t} = \Gamma V_{1t} + \alpha\beta^{*\prime} V_{2t} + \widehat{z}_t \tag{1.48}$$

The first step then corresponds to the reduced rank regression of V_{0t} on V_{2t} corrected for V_{1t} which is solved by the eigenvalue problem

$$\left|\theta M_{22\cdot 1} - M_{20\cdot 1} M^{-1}_{00\cdot 1} M_{02\cdot 1}\right| = 0 \tag{1.49}$$

The solution to the eigenvalue problem given in Eq.(1.49) provides ordered eigenvalues $1 > \widehat{\theta}_1 > \ldots > \widehat{\theta}_p > 0, \widehat{\theta}_{p+1} = 0$ with corresponding eigenvectors $\widehat{D} = (\widehat{d}_1, \ldots, \widehat{d}_{p+1})$. The resulting estimators of α, β^* and Ω are given by

$$\begin{aligned}\widehat{\beta^*} &= (\widehat{d}_1, \ldots, \widehat{d}_r); \\ \widehat{\alpha} &= M_{02\cdot 1}\widehat{\beta}(\widehat{\beta}' M_{22\cdot 1}\widehat{\beta})^{-1}; \\ \widehat{\Omega} &= M_{00\cdot 1} - \widehat{\alpha}\widehat{\alpha}'\end{aligned} \tag{1.50}$$

Fixing α, β and β_0 from the first step, the second step considers the equation

$$\alpha_\perp' V_{0t} = -\alpha_\perp' \Gamma \overline{\beta}(\beta^{*\prime} V_{1t}) - \xi \eta^{*\prime}(\overline{\beta}_\perp' V_{1t}) + \xi \eta^{*\prime} + \alpha_\perp' \widehat{z}_t \tag{1.51}$$

where the restriction on μ_0 is imposed. The statistical analysis of Eq.(1.51) can be performed by a reduced rank regression of $\alpha_\perp' V_{0t}$ on $(\overline{\beta}_\perp' V_{1t})$ for fixed $\beta^{*\prime} V_{1t}$ which is solved by the eigenvalue problem

$$\left| \kappa M^*_{\beta_\perp \beta_\perp \cdot \beta} - M^*_{\beta_\perp \alpha_\perp \cdot \beta} M^{*-1}_{\alpha_\perp \alpha_\perp \cdot \beta} M^*_{\alpha_\perp \beta_\perp \cdot \beta} \right| = 0 \tag{1.52}$$

for ordered eigenvalues $1 > \widehat{\kappa}_1 > \ldots > \widehat{\kappa}_p > 0, \widehat{\kappa}_{p-r+1} = 0$ with corresponding eigenvectors $\widehat{H} = (\widehat{h}_1, \ldots, \widehat{h}_{p-r+1})$. It follows that the estimators for η^*, ξ and $\alpha_\perp' \Omega \alpha_\perp$ are given by

$$\begin{aligned} \widehat{\eta}^* &= (\widehat{h}_1, \ldots, \widehat{h}_{s_1}); \\ \widehat{\xi} &= M^*_{\alpha_\perp \beta_\perp \cdot \beta} \widehat{\eta}^*; \\ \widehat{\alpha_\perp' \Omega \alpha_\perp} &= M^*_{\alpha_\perp \alpha_\perp \cdot \beta} - \widehat{\xi} \widehat{\xi}' \end{aligned} \tag{1.53}$$

1.2.2 Full Information Maximum Likelihood (FIML)

An analysis of the likelihood function of the I(2) model together with an algorithm for calculating the maximum likelihood estimator is provided by Johansen [1997]. In the following we provide an overview of the unrestricted maximum likelihood estimation procedure. Further details together with a discussion of tests within the maximum likelihood procedure can be found in Johansen [2000].

To begin, consider the reparameterisation of the I(2) model given in Eq.(1.9):

$$\begin{aligned} \Delta^2 X_t &= \alpha(\rho'\tau' X_{t-1} + \psi'\Delta X_{t-1}) \\ &\quad +\Omega\alpha_\perp(\alpha_\perp'\Omega\alpha_\perp)^{-1}\kappa'\tau'\Delta X_{t-1} + \sum_{i=1}^{k-2}\Psi_i\Delta^2 X_{t-i} + \varepsilon_t \end{aligned} \tag{1.54}$$

where the parameters in the model $\theta = (\alpha, \rho, \tau, \psi, \Omega, \Psi_1, \ldots, \Psi_{k-2})$ vary unrestrictedly given the full rank restriction on $\xi_\perp'\alpha_\perp'\Theta\beta_\perp\eta_\perp$ holds, see Eq.(1.5).

The algorithm for calculating the maximum likelihood estimator consists of two steps. In the first step α, and thus $\alpha_\perp$, is determined by fixing τ and solving an eigenvalue problem while the other parameters are found by regression. In the second step, the parameters estimated in step one are fixed and τ is determined by generalized least squares. By switching between these two steps until convergence the maximum likelihood estimator can be calculated. Note that by using a general result about consistency of the maximum likelihood estimator in a non-linear regression with independent Gaussian errors (Johansen [1997, Theorem A1]) it can be shown that the maximum likelihood estimator in the I(2) model defined by Eq.(1.9) is consistent. Furthermore, while the parameters $\widehat{\alpha}, \widehat{\kappa}, \widehat{\psi}, \widehat{\Psi}_1, \ldots, \widehat{\Psi}_{k-2}$ are estimated consistently, the parameters $\widehat{\beta}$ and $\widehat{\tau}$ are superconsistent.

The asymptotic distribution of the superconsistent estimators are proved to be asymptotically mixed Gaussian while those of the consistent estimators are asymptotically Gaussian. Moreover, as a corollary of the proof it is shown that the limiting distributions of α and β in the maximum likelihood analysis of the I(2) model are the same as those for the reduced rank estimators of α and β in the I(1) model. Thus, the two step estimation procedure discussed in section 1.2.1 is efficient for the estimation of α and β and one can ignore the second reduced rank condition, see Eq.(1.4), and simply fit the I(1) model as given by the reduced rank condition on Π, see Eq.(1.3). The efficiency of the two step procedure for the estimation of the remaining superconsistent parameters is given in Paruolo [1999]. A discussion of the asymptotic distributions of the maximum likelihood estimators of β, ψ and τ is provided in Johansen [1997].

1.3 Testing

In this section we discuss tests of various hypotheses of interest in the I(2) model. We begin by defining the hypothesis of cointegration and present the test for the determination of the integration indices in the I(2) model by Rahbek *et al.* [1999]. Next we consider various restrictions on the parameters of the I(2) model. After noting the similarity of tests on α and β with the standard I(1) model we cover tests on all the cointegrating relations as provided by Kongsted [1998]. Finally, we discuss the tests for weak exogeneity in cointegrated I(2) systems derived by Paruolo and Rahbek [1999].

1.3.1 Determination Of Integration Indices

Following Johansen [1995] let ξ denote the vector of parameters in $A_1, \ldots, A_k$ on and below the main diagonal of Ω and in turn let $\Xi \equiv \{\xi : \Omega \text{ p.d.}\}$denote the unconditional parameter space. With these definitions in place we can consider the following submodels of the I(2) model. First consider the submodel with the reduced rank restriction $\Pi = \alpha\beta'$ in place which we denote as

$$\Xi_r \equiv \{\xi : \xi \in \Xi, \Pi = \alpha\beta'; \alpha, \beta \in \Re^{p\times r}\} \tag{1.55}$$

As a special case consider the submodel within Ξ_r where α and β are of full rank r such that

$$\Xi_r^0 \equiv \{\xi : \xi \in \Xi_r,\ \text{rank}(\alpha) = \text{rank}(\beta) = r\} \tag{1.56}$$

Thus we have that $\Xi_r = \bigcup_{i=0}^{r} \Xi_i^0$, $\Xi_0^0 = \Xi_0$, $\Xi_i^0 = \Xi_i^0 \backslash \Xi_{i-1}^0$ and $\Xi_0 \subset \Xi_1 \subset \ldots \subset \Xi_r$. Next consider the submodel with the reduced rank restriction $\alpha_\perp \Gamma \beta_\perp = \xi\eta'$ in place which we denote

$$\Xi_{r,s_1} \equiv \{\xi : \xi \in \Xi_r^0, \alpha_\perp \Gamma \beta_\perp = \xi\eta'; \xi, \eta \in \Re^{p \times s_1}\} \tag{1.57}$$

Again we consider the special case of the submodel within Ξ_{r,s_1} where ξ and η are of full rank s such that

$$\Xi_r^0 \equiv \{\xi : \xi \in \Xi_{r,s}^0,\ \text{rank}(\xi) = \text{rank}(\eta) = s\} \tag{1.58}$$

Thus we have that $\Xi_{r,s} = \bigcup_{i=0}^{s} \Xi_{r,i}^0$, $\Xi_{r,0}^0 = \Xi_{r,0}$, $\Xi_{r,s}^0 = \Xi_{r,s_1}^0 \backslash \Xi_{r,s-1}^0$ and $\Xi_{r,0} \subset \Xi_{r,1} \subset \ldots \subset \Xi_{r,s}$. We are now in a position to construct the relationships between the various submodels shown below

$$
\begin{array}{ccccccccc}
\Xi_{0,0} & \subset & \Xi_{0,1} & \subset & \cdots & \subset & \Xi_{0,r} & = & \Xi_0^0 & = & \Xi_0 \\
 & & & & & & & & & & \cap \\
 & & \Xi_{1,0} & \subset & \cdots & \subset & \Xi_{1,r-1} & = & \Xi_1^0 & \subset & \Xi_1 \\
 & & & & \vdots & & & & & & \vdots \\
 & & & & & & & & & & \cap \\
 & & & & \Xi_{r-1,0} & \subset & \Xi_{r-1,1} & = & \Xi_{r-1,1}^0 & \subset & \Xi_{r-1} \\
 & & & & & & & & & & \cap \\
 & & & & & & & & & & \Xi_r = \Xi
\end{array}
$$

The hypotheses above of the form $H_{r,s_1} : \xi \in \Xi_{r,s_1}$ correspond to varying integration indices in the $\mathsf{I}(2)$ model. Specifically, we have r stationary cointegrating relations, s_1 common $\mathsf{I}(1)$ trends and thus the number of $(p - r - s_1) = s_2$ common $\mathsf{I}(2)$ trends. In particular, the hypothesis of cointegration is given by H_{r,s_1} against $H_r : \xi \in \Xi$ *i.e.* the unrestricted case.

With the nesting of models given above Rahbek et al. [1999] show that the likelihood ratio test of rank(Π)$\leq r$ against rank(Π)$\leq p$ is given by

$$Q_r = -T \sum_{i=r+1}^{p} \ln(1 - \theta_i) \tag{1.59}$$

from Eq.(1.49) above. Similarly the likelihood ratio test of $\text{rank}(\alpha_\perp \Gamma \beta_\perp) \leq s_1$ against $\text{rank}(\alpha_\perp \Gamma \beta_\perp) \leq p - r$ is given by

$$Q_{s_1} = -T \sum_{i=r+1}^{p} \ln(1 - \kappa_i) \tag{1.60}$$

from Eq.(1.52). Finally, Rahbek et al. [1999] show that the joint test for the hypothesis H_{r,s_1} against H_p is given by the union of the individual tests such that

$$S_{r,s_1} = Q_r + Q_{s_1} \tag{1.61}$$

1.3.2 Hypotheses On α and β

Following Johansen [1995] and Paruolo [1999] it can be shown that the asymptotic distribution of tests on α and β in the I(2) model are still χ^2. Thus one can ignore the second reduced rank condition given by Eq.(1.4) and calculate likelihood ratio tests of parameter restrictions on α and β exactly as in the standard I(1) model (see Johansen [1996, Section 7.2.1).

1.3.3 Hypotheses On τ

The hypothesis we consider on τ is that given by Kongsted [1998] of which a special case is the transformation to the I(1) model (Kongsted [2000]) discussed in section 1.1.4. Following Kongsted [1998], consider the hypothesis that all cointegrating relations in the I(2) model are subject to $p-q$ linear restrictions contained in the $p \times (p-q)$ matrix b such that $b'\tau = 0$ which is equivalent to $b'(\beta, \beta_1) = 0$. Now, consider the following parameterisation of $b'\beta$

$$\beta = B\varphi, \quad B = b_\perp \tag{1.62}$$

where φ is a $(r \times q)$ matrix of freely varying parameters. The orthogonal complement of β can then be constructed as

$$\begin{aligned} \beta_{\perp} &= (\overline{B}\varphi_{\perp}, b) \\ &= (B_1, b) \end{aligned} \tag{1.63}$$

With the parameterisation in place $b'\beta_1$ can then be formulated as a restriction on η in Eq.(1.4) such that

$$\begin{aligned} \eta &= \beta'_{\perp}\beta_1 \\ &= (B_1, b)'\beta_1 \\ &= \begin{pmatrix} B_1{}'\beta_1 \\ b'\beta_1 \end{pmatrix} \end{aligned} \tag{1.64}$$

where for the special case of Kongsted [2000] we have that $b'\beta_1 = 0$.

A test of $b'(\beta, \beta_1) = 0$ that uses the two step estimation procedure discussed in section 1.2.1 is provided in Kongsted [1998]. The first step estimates the model unrestrictedly as well as subject to the restriction on β. As discussed in section 1.3.2 this step is equivalent to testing restrictions on β in the standard I(1) model. Thus the likelihood ratio test, denoted Q_{b1}, is asymptotically distributed as χ^2 with $(p - q) \times r$ degrees of freedom subject to $b'\beta = 0$.

In the second step, conditional on the estimates of α and β in the first step, the model is estimated unrestrictedly as well as subject to the restriction on η such that $b'\beta_1 = 0$. The restriction is imposed in the second step by considering the parameterisation of η such that $\eta = H\theta$ with $H = (I_{q-r}, 0)'$ and θ being a $(q-r) \times s_1$ matrix of freely varying parameters. By defining

$$H^* = \begin{pmatrix} H & 0 \\ 0 & 1 \end{pmatrix} \tag{1.65}$$

the second step corresponds to solving the eigenvalue problem

$$\left| \kappa_b H^{*\prime} M^*_{\beta_\perp \beta_\perp \cdot \beta} H^* - H^{*\prime} M^*_{\beta_\perp \alpha_\perp \cdot \beta} M^{*-1}_{\alpha_\perp \alpha_\perp \cdot \beta} M^*_{\alpha_\perp \beta_\perp \cdot \beta} H^* \right| = 0 \tag{1.66}$$

for ordered eigenvalues $1 > \widehat{\kappa}_{b1} > \ldots > \widehat{\kappa}_{bp} > 0, \widehat{\kappa}_{bp-r+1} = 0$ with corresponding eigenvectors $\widehat{J} = (\widehat{j}_1, \ldots, \widehat{j}_{p-r+1})$. The restricted estimates are then $\eta = H\theta$ and $\beta_1 = \overline{\beta}_\perp \eta$ where $\theta = (\widehat{j}_1, \ldots, \widehat{j}_{s_1})$. Given the first step estimates of α and β the likelihood ratio test of $b'\beta_1 = 0$ is given by

$$Q_{b2} = T \sum_{i=1}^{s_1} \ln \left(\frac{1 - \kappa_{bi}}{1 - \kappa_i} \right) \tag{1.67}$$

which is asymptotically distributed as χ^2 with $(p - q) \times s_1$ degrees of freedom subject to $b'(\beta, \beta_1) = 0$ given that η is mixed Gaussian. Given the sequential nature of the test Kongsted [1998] suggests that one considers a rejection region that is the union of the rejection regions for Q_{b1} and Q_{b2}. Thus, given a rejection region of size v the size of each individual test can be chosen as $v/2$. As the separate tests are consistent, the sequential procedure is consistent against the alternative $b'(\beta, \beta_1) \neq 0$.

1.3.4 Testing Weak Exogeneity

Paruolo and Rahbek [1999] investigate weak exogeneity with respect to the cointegration parameters in the I(2) model. Consider the EqCM representation given in Eq.(1.7) such that

$$\Delta^2 X_t = \Pi X_{t-1} - \Gamma \Delta X_{t-1} + \Theta Z_t + \varepsilon_t \tag{1.68}$$

where we have defined $\Theta = (\Psi_1, \Psi_2, \ldots, \Psi_{k-2})$ and $Z_t = (\Delta^2 X_{t-1}, \Delta^2 X_{t-2}, \ldots, \Delta^2 X_{t-k+2})$ as before. To motivate the conditions required for weak exogeneity we make use of the following decomposition of $\Gamma \Delta X_{t-1}$ in Eq.(1.68)

$$\Gamma \Delta X_{t-1} = \alpha \delta \beta_2' \Delta X_{t-1} + (\varsigma_1, \varsigma_2)(\beta, \beta_1)' \Delta X_{t-1} \tag{1.69}$$

where $\varsigma_1 = \Gamma \overline{\beta}$, $\varsigma_2 = \Gamma \overline{\beta}_1$ and where we have made use of the property $\alpha_\perp \overline{\alpha}_\perp' \Gamma \beta_2 \overline{\beta}_2' = 0$. Inserting Eq.(1.69) into Eq.(1.68) yields

$$\Delta^2 X_t = \alpha[\beta' X_t + \delta \beta_2' \Delta X_{t-1}] + (\varsigma_1, \varsigma_2)[(\beta, \beta_1)' \Delta X_{t-1}] + \Theta Z_t + \varepsilon_t \tag{1.70}$$

which shows that we have the adjustment coefficient α to the stationary polynomial cointegrating relation given in the first square bracket and the adjustment coefficients $(\varsigma_1, \varsigma_1)$ to the stationary linear combinations in the second square bracket. Note that we may also define the second set of adjustment coefficients as (ς_1, α_1) since $\alpha_\perp \overline{\alpha}_\perp' \varsigma_2 = \alpha_\perp \overline{\alpha}_\perp' \Gamma \overline{\beta}_1 = \alpha_\perp \overline{\alpha}_\perp' \Gamma \beta_\perp \overline{\beta}_\perp' \overline{\beta}_1 = \alpha_1 \beta_1' \overline{\beta}_1 = \alpha_1$.

Paruolo and Rahbek [1999] show that under Johansen's representation theorem with $r > 0, s > 0$ a subset of $X_t, b'X_t$ is weakly exogenous for the cointegration parameters $\theta = (\beta, \beta_1, \delta)$ if and only if

$$b'(\alpha, \alpha_1, \varsigma_1) = 0 \tag{1.71}$$

Note two special cases: if $r = 0, s > 0$ we require only that $b'(\alpha_1, \varsigma_1) = 0$; while for $r > 0, s = 0$ we require only that $b'\alpha = 0$, that is the condition collapses to that in the standard $\mathsf{I}(1)$ case.

The first condition, $b'\alpha = 0$, implies that the polynomial cointegrating combinations do not appear in the equations for $b'\Delta^2 X_t$ and ensures asymptotic efficiency of the estimator for β in the first step of the two step reduced rank regression procedure.

The second condition $b'\alpha_1 = 0$ implies that the stationary combinations $\beta_1' X_t$ will be absent from the equations for $b'\Delta^2 X_t$. Taking the conditions $b'(\alpha, \alpha_1) = 0$ together results in the efficient estimation of the polynomial cointegrating parameters and guarantees that the cumulated innovations $\alpha_2' \sum_{s=1}^{t} \sum_{i=1}^{s} \varepsilon_i$ from Eq.(1.6) form the common stochastic I(2) trends in the system.

The last condition $b'\varsigma_1 = 0$ ensures that the stationary combinations $\beta' X_t$ remain absent from the equations for $b'\Delta^2 X_t$.

Paruolo and Rahbek [1999] suggest a sequential testing strategy for weak exogeneity based upon the two step reduced rank regression procedure discussed in section 1.2.1 so that departures from the maintained hypotheses can be identified.

The first hypothesis $b'\alpha = 0$ is tested within the first step of the two step reduced rank regression procedure. Specifically, we consider the model of $A'X_t$ conditional on $b'X_t$ where A is a $(p \times m)$-dimensional matrix. In the first step the estimates of α and β are subject to $\alpha = A\psi$ and are obtained by reduced rank regression of $A'\Delta^2 X_t$ on X_{t-1} corrected for $\Delta^2 X_t$ and Z_t as defined in Eq.(1.28). Denoting the eigenvalues that solve the restricted eigenvalue problem under $\alpha = A\psi$ as λ_i^*, under the hypothesis that $b'\alpha = 0$ the likelihood ratio test is given by

$$Q_{a1} = T \sum_{i=1}^{r} \ln \left(\frac{1 - \lambda_i^*}{1 - \lambda_i} \right) \tag{1.72}$$

The restricted estimate of α and the estimate of β are retained for use in the second step.

The second hypothesis $b'\alpha_1 = 0$ makes use of the partial systems representation presented in section 1.1.5. First, we construct the orthogonal complement of α such that

$$\begin{aligned} \alpha_{\perp} &= (\overline{A}\psi_{\perp}, b) \\ &= (A_1, b) \end{aligned} \tag{1.73}$$

Thus, since $\alpha_1 = \overline{\alpha}_{\perp}\xi$ we have that

$$\alpha_1 = (\overline{A}_1, \overline{b}) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \tag{1.74}$$

where under the restriction $b'\alpha_1 = 0$, $\xi_2 = 0$. With this parameterisation in place, the restricted estimate of ξ is obtained from the conditional model given by Eq.(1.16) by reduced rank regression of $A_1'\Delta^2 X_t$ on $\overline{\beta}_{\perp}\Delta X_{t-1}$ corrected for $\beta\Delta X_{t-1}, Z_t$ and $b'\Delta^2 X_t$. The likelihood ratio test under the hypothesis $b'\alpha_1 = 0$, denoted Q_{a2}, is then distributed as χ^2 with $(p-m) \times s_1$ degrees of freedom under the condition $b'(\alpha, \alpha_1) = 0$.

The final hypothesis $b'\varsigma_1 = 0$ is also tested in the second step but makes use of the marginal model given by Eq.(1.15). Paruolo and Rahbek [1999] show that the likelihood ratio test of $b'\varsigma_1 = 0$ with β fixed from the first step is simply a test of exclusion of some of the regressors in the marginal model. Specifically, defining S_{bb} as the residual product moment matrix corresponding to OLS of $b'\Delta^2 X_t$ on Z_t and similarly $S_{bb\cdot\beta}$ as the residual product moment matrix corresponding to OLS of $b'\Delta^2 X_t$ on $\beta'\Delta X_{t-1}$ and Z_t, the likelihood ratio test under the hypothesis $b'\varsigma_1 = 0$ is given by

$$Q_{a3} = T(\ln|S_{bb}| - \ln|S_{bb\cdot\beta}|) \tag{1.75}$$

and is distributed as χ^2 with $(p-m)r$ degrees of freedom under the condition $b'(\alpha, \alpha_1, \varsigma_1) = 0$.

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Chapter 2

An I(2) Cointegration Analysis Of UK Money Demand

2.1 Introduction

The current commitment to price stability provides a suitable backdrop for revisiting the literature on the specification of a money demand function for the UK. The importance of providing a money demand equation that displays both parameter constancy and is robust to changes of regime is that, though targeting money growth is no longer used as an explicit strategy for guiding the adjustment of interest rates in pursuit of price stability, a well-specified money demand equation stills provides an important role in the conduct of monetary policy. In fact, the current period of inflation targeting is not unique in requiring a robust money demand relationship. The estimation of econometric models of the demand for money have been a major focus of empirical monetary economics since the 1970s and it has received considerable focus through its use as the model of choice for the proponents of the general-to-specific methodology of Hendry [see Hendry [1995]). Turning to the econometric literature, early studies on this methodology estimated single equation models of UK money demand (Hendry [1979, 1985, 1988] and Hendry and Ericsson [1991]). However, more recently attention has been paid to the long run properties of money demand functions examined through cointegration (Ericsson et al. [1990, 1994], Hendry and Mizon [1993] and Hendry and Doornik [1994]). These studies have provided evidence that a cointegrating vector can be found that is interpretable as a money demand function and furthermore which satisfies certain restrictions suggested by economic theory. However, a weakness in this literature remains. To avoid the consequences of modelling nominal money and prices as I(2) variables, a transformation (long run price homogeneity) is imposed in order to proceed with the standard techniques of the I(1) model. In light of this, a number of recent studies have relaxed this restriction and tested for the number of cointegrating relations and I(2) trends within a cointegrated I(2) model (Johansen [1992], Paruolo [1996] and Rahbek *et al.* [1999]). In common with the I(1) studies support is found for a cointegrating relation that matches the theoretical propositions underpinning money demand functions. However, the evidence also suggests the presence of two I(2) trends. The importance of this result can be seen by noting that there are only two candidates for I(2) variables in the dataset, namely nominal money and prices. Thus, with two I(2) trends the price homogeneity restriction employed in the I(1) studies will not be sufficient to reduce the model down to the I(1) level.

The contribution of this chapter is to provide a formal test of the restriction employed by Hendry and Mizon [1993] and Hendry and Doornik [1994] in specifying an information set with long run price homogeneity imposed. In addition, we extend the analyses of Johansen [1992], Paruolo [1996] and Rahbek et al. [1999] by examining the dataset of Hendry and Mizon [1993] which includes certain dummy variables in the dataset (see below). In particular, we provide a long run stationary relation for money demand and show how the incorporation of this relation in a vector autoregression yields a parsimonious and congruent representation of the data over the full sample period. Finally, we show that this equilibrium correction model provides superior out-of-sample forecasting power compared to a multivariate autoregression model.

The structure of the Chapter is as follows. The rest of this section discusses the economic model and the choice of variables therein. Section 2.2 then presents the empirical analysis including testing for the appropriate integration indices of the model together with tests of parameter restrictions on the stationary relation and on the common trends in the system. We continue the empirical analysis by estimating an equilibrium model over the full sample period and show that it is congruent with respect to the dataset. The final empirical section provides evidence on the parameter constancy of the cointegrating relations and on the forecasting power of the equilibrium correction model compared to a simple multivariate time series model. Section 3 concludes.

2.1.1 The Economic Model

A long run money demand specification that is consistent with both inventory-theoretic and portfolio choice models of money demand can be represented in semi-log functional form by

$$m = f(p, y, \mathbf{R}) \tag{2.1}$$

where m is nominal money demand, p is the price level, y is a scale variable and $\mathbf{R}$ is a vector of returns on various assets. The function $f(\cdot)$ is increasing in p, y and $\mathbf{R}$ for the set of assets included in m and decreasing for the set of assets excluded from m.

As discussed above a common restriction applied to the function form in Eq.(2.1) is long run price homogeneity. In addition, studies that employ this restriction include inflation among the parameter set in $f(\cdot)$. It is clear that with price homogeneity imposed one can either include or exclude inflation from the model. However, excluding inflation imposes the further restriction that short-run and long-run elasticities of money demand with respect to prices are equal, a restriction that is rejected in Hendry and Ericsson [1991].

The scale variable used in the studies discussed above is real total final expenditure and its use derives from the fact that narrow money is mainly held for transactions purposes. Increases in real income will lead to a rise in the demand for nominal money to satisfy increases in aggregate consumption and also because of individuals wishing to hold a certain proportion of their wealth in liquid form (though this will be mitigated through progress in transactions technology which will force cash and other financial assets to be closer substitutes). The actual choice of scale variable is not clear and a number of alternatives have been used in the empirical literature including GDP and retail sales[1].

Our opportunity cost variable is the differential between the rate of return on 3 month local authority deposits and a learning adjusted rate of return on checkable interest-bearing accounts at commercial banks. The use of an interest rate differential is a natural measure of the excess return on an asset outside of M1 (3 month local authority deposits)

[1]Indeed the choice of scale variable may impact on the conclusion of the existence or otherwise of cointegration among the variables (see Juselius and Hargreaves [1992]).

compared to the return on the interest bearing component of M1 (checkable interest-bearing accounts at commercial banks). Following Hendry and Ericsson [1991], the introduction of interest-bearing sight deposits in 1984Q3 is modelled through an ogive weighting function L given by

$$L_t = (1 + \exp[a - b(t - T_1 + 2)])^{-1}, \quad t > T_1 \tag{2.2}$$

and zero otherwise and where $T_1 = 1984Q3$. With the weighting function our opportunity cost measure is then

$$R_t^* = R_t^{LA} - L_t R^{SD} \tag{2.3}$$

where R_t^{LA} and R^{SD} represent the rate of return on 3 month local authority deposits and the rate of return on checkable interest-bearing accounts at commercial banks respectively. As in Hendry and Ericsson [1991] the coefficients of the weighting function are assumed known and set at $a = 5$ and $b = 1.2$[2].

In addition to the variables discussed above, cointegration analyses since Hendry and Mizon [1993] have included certain deterministic components in the money demand model. First, a linear trend is included to approximate the growth in real income from the impact of growth in human and physical capital and proxy for changes in money velocity caused by cash-economising innovations in transactions technology. Thus, by including a linear trend one can avoid direct measures of financial innovation which are difficult to measure and may well be endogenous through their relationship with interest rates and income. The inclusion of a linear trend affects the limiting distribution of estimators and tests through its role as a nuisance parameter. However, Doornik *et al.* (1998) find that

[2]Ericsson [1998] estimates the coefficients by recursive nonlinear least squares and finds that $a = 3.2$ and $b = 0.75$. However, he shows that the parameters of the equilibrium correction model estimated in Hendry and Ericsson [1991] are not sensitive to the choice of coefficients in the weighting function.

restricting the trend to the cointegration space provides improved power and size of standard tests for the cointegration rank compared to the unrestricted case. Second, the studies cited above have included a number of impulse dummies representing exogenous shocks to the economy over the sample period (these dummies are defined and further discussed in Section 2 below). Though motivated by specific economic events such as the Heath-Barber boom and the two oil price shocks their purpose is to improve the fit of the system by accounting for the largest residuals over the sample. The inclusion of the impulse dummies is not trivial given their relative size compared to the regression residuals (one of the dummies is approximately $\sqrt{100}$ times the standard deviation of the model errors). However Doornik *et al.* [1998] show that if one enters the dummies unrestrictedly one can avoid severe size distortions.

2.2 Empirical Analysis

2.2.1 The Data

The variables in the dataset are quarterly and seasonally adjusted and cover the period 1963Q1 through 1989Q2. The variables correspond to nominal money (denoted m) as measured by the narrow monetary aggregate M1; real total final expenditure (y); its corresponding deflator (p) and a measure of the net opportunity cost of holding money (R^*) which includes an adjustment for the learning process following the introduction of interest-bearing checking accounts in 1984Q3 (see Hendry and Ericsson [1991] and Section 2.1 above for details). The variables in lower-case letters are in logs, while R^* is in fractions.

In addition to the stochastic variables in the information set, we also include a constant, time trend and two dummy variables. The dummy variables are the same as those constructed by Hendry and Mizon [1993]. Specifically, we denote the first dummy as D_y which takes the value unity in 1972Q4, 1973Q1 and 1979Q2 to account for the so-called "Barber Boom" and the impact of the first Thatcher government on output. The other dummy is denoted D_o and takes the value unity in 1973Q3, 1973Q4 and 1979Q3 to account for the two oil price crises and the VAT increase for inflation. Thus, our full information set is given by $X_t = (m, y, p, R^*, 1, t, D_y, D_o)$.

Figure 2.1 provides plots of the data in levels, first differences and second differences. We also present certain transformations of the data used in the literature corresponding to real money $(m-p)$ and the inverse velocity of money $(m-p-y)$. We note here that the plots of nominal money and prices in differences display evidence of non-stationarity with little evidence of mean reversion over long periods across the sample. In addition, imposing common restrictions corresponding to long run price homogeneity (to yield $(m-p)$) together with long run income homogeneity (to yield $(m-p-y)$) does not seem to render the series stationary. Finally, the plot of real income in levels suggests the need for a linear trend in the information set.

2.2.2 The Unrestricted Vector Autoregression

We begin by modelling the joint distribution of the variables within an unrestricted vector autoregression (VAR). By modelling the joint distribution it allows us test hypotheses both on the long run relationships between the variables in the form of restrictions on the cointegrating relations and on the loadings to these relationships to investigate the exogeneity status of the variables using the techniques discussed in Chapter 1.3.4.

The unrestricted VAR takes the form:

$$X_t = \sum_{j=1}^{k} A_j X_{t-j} + \Phi D_t + \varepsilon_t \tag{2.4}$$

where $X_t' = (m, y, p, R^*)$ is the four-dimensional vector containing our variables of interest; D_t is a vector of deterministic variables containing a constant, a trend and the two event dummies defined above; and ε_t is an innovation process which is independently distributed with mean zero and variance-covariance matrix Σ.

We begin the analysis by estimating unrestricted VAR for m, y, p and R^* given by Eq.(2.4) with $k = 5$ lags over the full sample from 1963Q1 to 1989Q2 (less observations used for lags)[3]. Given the frequency of the data an unrestricted VAR with five lags was assumed to be a sufficiently general starting point for the empirical analysis. This initial general system had therefore 106 unrestricted coefficients including the error variance-covariance matrix. (As in Hendry and Doornik [1994] the two regime shift dummies D_o and D_y were entered unrestricted.) A likelihood ratio test for the appropriate lag length of the unrestricted VAR suggested that $k = 3$ lags was sufficient (the test statistic for $k = 3$ lags against the alternative $k = 5$ being $\chi^2(32) = 40.5$ with a p-value > 0.14) .

Table 2.1 provides descriptive and diagnostic statistics for the estimated unrestricted VAR. The diagnostics correspond to p-values of the Lagrange multiplier test of residual serial correlation against 4th-order autoregression (Godfrey, [1978]) denoted $F_{ar}(\cdot,\cdot)$; the RESET test of functional form (Ramsey, [1969]) denoted $F_{reset}(\cdot,\cdot)$; the Jarque-Bera chi-square test of normality of regression residuals (Jarque and Bera [1980]) denoted $\chi^2_{norm}(\cdot,\cdot)$;

[3]The calculations and numerical results in the text were obtained using the computer packages CATS in RATS (Hansen and Juselius [1995]) and Microfit 4.0 (Peseran and Peseran [1997]).

and an equality of error-variances test denoted $F_{het}(\cdot,\cdot)$. The table shows that the initial system is generally well-specified with the only evidence of non-congruency coming from the interest rate equation in the form of heteroscedastic errors. Further evidence on the congruency of the initial system is provided by Figure 2.2 which plots the actual and fitted values and residuals for each equation. In particular, the residuals show no evidence of time dependency, consistent with stationary behaviour, with only isolated examples outside the error bands.

The dynamic properties of the unrestricted VAR are illustrated by the moduli of the eigenvalues of the companion matrix. The first eight of these are

$$(0.978, 0.953, 0.953, 0.791, 0.791, 0.515, 0.515, 0.451) \tag{2.5}$$

where the roots of the characteristic polynomial are the inverses of these eigenvalues. The eigenvalues suggest the presence of three, or perhaps five, unit roots in the dynamic system. Evidence on the roots of the characteristic equation is of direct use since conditioning on the dummy variables is likely to change the asymptotic distribution of the test for integration indices (see Chapter 1.3.1) to some (unknown) extent.

2.2.3 I(2) Cointegration Analysis

Testing for the integration indices

We now proceed to formally test for the integration indices in the system *i.e.* the number of r cointegrating vectors together with the number of s_1 I(1) trends and thus the number of $(p - r - s_1) = s_2$ I(2) trends using the test presented in Rahbek *et al.* [1999] and discussed in detail in Chapter 1.3.1. For our particular dataset we need to be careful when applying the formal test. Our model has relatively few observations and moreover contains two event dummies D_y and D_o. Thus using the asymptotic critical values from Rahbek *et al.* [1999] on their own may lead us to an accept an inappropriate set of integration indices. To help us in choosing the correct specification we first note that univariate analysis of the integration order of the single series suggests that we have up to two I(1) trends corresponding to real income and our interest rate and one or two I(2) trends corresponding to nominal money and prices (Hendry and Doornik [1994]). Moreover, the existing literature suggests that up to two cointegrating relations can be supported by the dataset: a money demand relation and an excess demand relation (Hendry and Mizon [1993], Hendry and Doornik [1994] and Hendry [1995]). Together, with the evidence from the eigenvalues of the companion matrix our prior is that two possibilities present themselves. First, that we have one cointegrating relation, a single I(1) trend, and two I(2) trends (*i.e.* five unit roots in the I(2) model) as suggested by Paruolo [1996] and Rahbek *et al.* [1999]. Second, that two cointegrating relationships exist with one I(1) trend and one I(2) trend (*i.e.*three unit roots) as suggested by Johansen [1992].

With this prior information in mind we now turn to the formal test. The test statistics for all combinations of r and s_1 together with the 95% quantiles of the asymptotic test distributions taken from Rahbek *et al.* [1999, Appendix C, Table 4] are presented in Table 2.2. (As indicated above, the reported quantiles do not take into account the presence of the two event dummies D_o and D_y.) Following, Paruolo [1996] and Rahbek *et al.* [1999] the test statistics are calculated under the assumption that the data may contain linear but not quadratic trends. This is consistent with the plots of the data in Figure 2.1 which shows that the variables display a zero mean in their second differences.

To determine the appropriate integration indices we start by testing the most restricted hypothesis, given by $(r, s_1, s_2) = (0, 0, 4)$, then, if this hypothesis is not rejected, we test successively less and less restricted hypotheses by continuing to the end of the first row and then by proceeding row-wise from left to right until the first rejection is found. The first submodel hypothesis corresponding to $(r, s_1, s_2) = (0, 0, 4)$ is easily rejected. In fact, all submodels for $r = 0$ are rejected. The first non-rejection is for the case $(r, s_1, s_2) = (1, 1, 2)$ with a test statistic of 62.2 which is below the 95% critical value of 68.2 (p-value of $> 10\%$). Though a non-rejection one has to be careful with the low p-value of this particular submodel due to the inclusion of the two event dummies and the small sample. That said, we find some support for the case for one cointegrating relation and two I(2) trends as suggested by Paruolo [1996] and Rahbek *et al.* [1999] for the model without dummies. As alluded to above this particular submodel corresponds to $s_1 + 2s_2 = 5$ unit roots and thus matches the results presented above on the roots of the characteristic polynomial. The next two non-rejections corresponding to $(r, s_1, s_2) = (1, 2, 1)$ and $(r, s_1, s_2) = (2, 0, 2)$, both contain only $s_1 + 2s_2 = 4$ unit roots, a result that is difficult to square with the roots of the characteristic polynomial reported in Eq.(2.5). However, the submodel representing our second prior of $(r, s_1, s_2) = (2, 1, 1)$ is also a non-rejection with a test statistic of 21.9 which is well below the 95% critical value of 34.4 (p-value of $> 50\%$).

Imposing the rank restrictions consistent with $(r, s_1, s_2) = (1, 1, 2)$ the first eight eigenvalues of the companion matrix are

$$(1, 1, 1, 1, 1, 0.362, 0.038, 0.037) \tag{2.6}$$

where it is clear that there are no remaining unit roots unaccounted for. Evidence that the fifth unit root is related to an I(2) trend and not to the rank deficiency of Π (defined in Chapter 1.1.1) can be seen by imposing $p - r = 3$ in the I(1) model. In this case the first seven eigenvalues become $(1, 1, 1, 0.860, 0.834, 0.527, 0.492)$ and we see two further near-unit roots emerge.

Similarly, imposing the model $(r, s_1, s_2) = (2, 1, 1)$ the first eight eigenvalues of the companion matrix are

$$(1, 1, 1, 0.591, 0.353, 0.245, 0.070)$$

where, as for the submodel above, there seems to be no unit roots present. Again, we find evidence that the third unit root is related to an I(2) trend and not to the rank deficiency of Π (defined in Chapter 1.1.1) by imposing $p - r = 2$ in the I(1) model. In this case the first seven eigenvalues become $(1, 1, 0.944, 0.744, 0.744, 0.506, 0.506)$ with one, or perhaps three, further near-unit roots appearing.

The above formal test for the integration indices of the I(2) model and the resulting eigenvalues of the companion matrix provide little discriminatory evidence on the appropriate integration indices for our dataset. Thus, we continue by presenting the unrestricted estimates from the two-step estimation procedure of Rahbek *et al.* [1999] based on Johansen [1995] for both submodels $(r, s_1, s_2) = (1, 1, 2)$ and $(r, s_1, s_2) = (2, 1, 1)$.

Imposing the integration indices $(r, s_1, s_2) = (1, 1, 2)$ yields the unrestricted parameter estimates in Table 2.3. The estimates have been decomposed into their respective I(0), I(1) and I(2) spaces and, for ease of exposition, we have normalised the β vector on narrow money m. The I(0) space corresponds to the estimate of the polynomial relation

$\beta' X_t - \delta' \beta_2' \Delta X_t$ provided in the first two columns in the top panel. Turning to the levels variables $\beta' X_t$ first, we see some evidence of long-run price homogeneity with strong interest rate effects while the small value of the parameter estimate on t indicates the possible absence of a deterministic trend in the polynomially cointegrating relation as in Paruolo [1996] and Rahbek *et al.* [1999]. With respect to the differenced variables $\delta' \beta_2' \Delta X_t$, the estimates suggest that maybe only the differences of narrow money and prices are needed to achieve stationarity. and furthermore that they might enter restricted as real money balances *i.e.* $\Delta(m - p)$. The adjustments to the disequilibrium error defined by the polynomial relation, denoted α, are presented in the first column of the bottom panel. They show that the polynomial relation has the strongest weight in the money demand equation which matches *a priori* expectations. In addition to the relations $\beta' X_t$ and $\delta' \beta_2' \Delta X_t$ the I(1) space contains the vectors α_1 and β_1. As discussed in Chapter 1.1.2 the linear combination $\beta_1' X_t$ corresponds to a cointegrating relation that reduces the order of integration from two to one *i.e.* a CI(2, 1) relation, while the α_1 vector provides the stochastic I(1) trend component. Both sets of estimates pick out the real income variable as the dominant force in this direction. The estimates of the two common I(2) trends are given by α_{2_1} and α_{2_2} (see Chapter 1.1.2 for a discussion of this component and β_2, for which see below). However note that the largest weights in the I(2) trends come from the twice cumulated residuals from the interest rate (α_{2_1}) and prices (α_{2_2}) . It seems that by allowing two I(2) trends in the model the interest rate is identified as an I(2) variable, a result which is extremely counterintuitive and serves to weaken the case for this particular submodel. For completeness we finally note the estimate of the weight with which the I(2) trends influence the variables of the system is given by β_{2_1} and β_{2_2}. A condition for a variable i in X_t to be $\mathsf{I}(2)$ is that $\beta_{i2\_j} \neq 0$ for $j = 1, \ldots, s_2$ and as expected *a priori* the estimates of β_2 seem to indicate that the $\mathsf{I}(2)$ trends are primarily driving nominal money and prices.

We next turn to the unrestricted parameter estimates derived from imposing the integration indices $(r, s_1, s_2) = (2, 1, 1)$, which are reported in Table 2.4. Note that since $r > s_2$ we now have one direct, as opposed to polycointegrating, relation from the $\mathsf{I}(2)$ level down to stationarity given by $\delta'_{\perp}\beta' X_t$ (see Chapter 1.1.2) and one polynomial relation given by $\delta'\beta' X_t - \delta'\delta\beta'_2\Delta X_t$. Following Hendry [1995] we have normalised the first cointegrating relation on nominal money and the second on real income. Comparing these unrestricted estimates from those in Table 2.3 under the submodel $(r, s_1, s_2) = (1, 1, 2)$ we see that the levels terms in the first cointegrating relation are broadly unchanged *i.e.* there is evidence of long-run price homogeneity with strong interest rate effects. The second relation matches the spirit of the excess demand relation posited in Hendry [1995] though the difference terms suggest a role for money as well as prices here. Moving on to $\mathsf{I}(1)$ space we see evidence that the CI$(2, 1)$ relation $\beta'_1 X_t$ is formed by a real money demand term with smaller effects from real income and the interest rate. The notable difference here with the first case is the large fall in the coefficient on real income. More importantly, we find that the estimate of the common $\mathsf{I}(2)$ trend given by α_2 points to nominal prices as the dominant variable. This finding sits better with our priors on the integration order of the individual series and, as a result, is evidence for $(r, s_1, s_2) = (2, 1, 1)$ as the correct integration indices for our dataset. Finally, as in the first case, the estimate of the weight of the $\mathsf{I}(2)$ trend given by β_2 indicates that the $\mathsf{I}(2)$ trends drive nominal money and prices alone.

Though there is little concrete evidence for or against the two alternative submodels, the identification of the interest rate as an I(2) variable in the submodel given by $(r, s_1, s_2) = (1, 1, 2)$ is, in our minds, hard to justify from economic considerations and from the existing evidence on the time series properties of this variable. As a result, the submodel corresponding to $(r, s_1, s_2) = (2, 1, 1)$ is maintained for the rest of the analysis.

Testing The Nominal To Real Transformation

The unrestricted estimates presented in Table 2.4 suggest a number of tests of restrictions on the parameters of the cointegrating relations. Though the order of testing is not clear, we first test for whether the I(2) system can be reduced to I(1) through a so-called nominal to real transformation proposed by Kongsted [2000] and summarised in Chapter 1.1.4. This is obviously of immediate interest since, if accepted, one can proceed with the standard I(1) framework which lends itself more readily to tests of parameter restrictions on the cointegrating relations and loadings.

The nominal-to-real transformation we examine is the hypothesis of long-run price homogeneity as imposed by a number of studies on this dataset (Hendry and Mizon [1993], Hendry and Doornik [1994] and Hendry [1995]) . The hypothesis corresponds to the set of parameter restrictions

$$\begin{aligned} \underline{\beta}_i &= (1, -1, *, *), \quad i = 1, \ldots, r \\ \beta_{1j} &= (1, -1, *, *), \quad j = 1, \ldots, s_1 \\ \beta_{2l} &= (1, 1, 0, 0)\,, \quad l = 1, \ldots, s_2 \end{aligned} \tag{2.7}$$

where, as discussed above, β and β_1 define the CI(2, 1) relations and β_2 is the loading matrix of the common I(2) trends in the system. Thus, the hypothesis implies that all cointegrating relations defined either by β or β_1 can be expressed as

$$\tau = H\varphi = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} \tag{2.8}$$

where $\tau = (\beta, \beta_1)$.

Before we test formally for the real-to-nominal transformation note that when normalised on m, we obtain the following point estimates for β, β_1 and β_2 from the two-step estimation

$$\begin{aligned} \underline{\beta}_1 &= (1, -0.838, 2.10, 10.23) \\ \underline{\beta}_2 &= (1, -1.47, -7.03, 2.54) \\ \beta_1 &= (1, -0.906, 0.252, -0.223) \\ \beta_2 &= (1, 1.08, -0.081, 0.000) \end{aligned}$$

Thus the point estimates in each of the cointegrating relations $\underline{\beta}_i, i = 1, 2$ and β_1 together with those from the loadings to the I(2) trend β_2 appear close to their theoretical values under the nominal-to-real hypothesis. With this in mind we now move on to the formal test of long-run price homogeneity proposed by Kongsted [1998,2000] and discussed in Chapter 1.3.3. We find that the first part of the test corresponding to the hypothesis $b'\beta = 0$ is easily accepted with a test statistic given by $Q_{b1} = 2.94$ with a corresponding p-value of 0.23. With the first step restriction in place the overall hypothesis is also accepted with

$Q_{b2} = 0.07$ with a p-value of 0.79. Thus, as suggested by the point estimates in the various vectors we find that a nominal-to-real transformation to the I(1) model is valid.

2.2.4 I(1) Cointegration Analysis

Following Kongsted [1998], with the nominal-to-real transformation in place the polynomial relations can be examined in a transformed I(1) VAR for $\tilde{X}_t$ where $\tilde{X}_t = (B'X_t, b'\Delta X_t)'$. Thus, we have that

$$\tilde{X}_t = (m_t - p_t, y_t, R_t^*, \Delta(m+p)_t)'$$

where the polynomial component of the cointegrating relation is given by $\Delta(m+p)_t$. Note the difference with the transformed information set used by Hendry and Doornik [1994] *i.e.* $X_t = (m_t - p_t, y_t, R_t^*, \Delta p_t)$. Our results suggest that the inclusion of Δm is required to provide a valid transformation to the I(1) model. Indeed, we find that the test for excluding $\Delta(m+p)_t$ from the cointegration space of the transformed I(1) VAR is firmly rejected with a test statistic of $\chi^2(2) = 24.1$ (p-value < 0.00).

The exactly identifying representation of the two cointegrating relations are provided in Table 2.5, Panel A. Following the previous literature and the point estimates from Panel A we test for a number of overidentifying restrictions given in Table 2.6. The first set of restrictions correspond to identifying a money demand relation with no trend (H_1), long run real income homogeneity to form the inverse money velocity relationship $m-p-y$ (H_2), and a semi-elasticity of real money demand with respect to the interest rate of 7 as in Hendry and Doornik [1994] (H_3). The second set of restrictions define an excess demand relationship which excludes real money (H_4), and assumes proportionality with respect to the the interest rate (H_5) and homogeneity with respect to the polynomial term (H_6). Table 2.6 shows that individually each restriction is accepted; taken together the set of restrictions is easily accepted with a test statistic of $\chi^2(4) = 2.03$ with corresponding p-value of 0.73.

With the restrictions in place, Table 2.5, Panel B provides the overidentifying structure of the polynomial relations with the standard errors on the unrestricted parameters in brackets. The first relation defines a real money demand equation of the form

$$p_1 = m - p - y + 7R^* + 3.1\Delta m + 3.1\Delta p$$

Hendry and Mizon [1993], Hendry and Doornik [1994] and Hendry [1995] find a similar cointegration relation given by

$$c_1 = m - p - y + 7R^* + 7\Delta p$$

Indeed, a plot of these two relations show that they track very closely, reflecting the strong correlation between money and prices over the period analysed. The second relation corresponds to what Hendry and Mizon [1993] term an excess demand relationship of the form

$$p_2 = y + R^* - \Delta m - \Delta p - 0.0067t$$

where the parallel relation in the above I(1) cointegration studies is given by

$$c_2 = y + 1.8R^* - 3.4\Delta p - 0.0063t$$

The difference in the excess demand relation between our study and that from the I(1) literature is more marked. Notably, we find smaller effects from the interest rate and inflation, but now the change in nominal money appears in the long run relationship. Though note that the weight on the linear trend is broadly unchanged between specifications. Figure 2.3 provides plots of the cointegrating relations p_1 and p_2 together with the I(2) trend given by $b'X_t = m_t + p_t$. The stationarity of the polynomial relations is apparent while the common I(2) trend displays the characteristic smooth time dependency of I(2) variables.

2.2.5 Modelling The I(0) PVAR

Kongsted [1998] shows that under the hypothesis $b'(\beta, \beta_1) = 0$ an EqCM for the transformed process $\tilde{X}_t = (B'X_t, b'\Delta X_t)'$ can be specified as

$$\Delta\tilde{X}_t = \widetilde{\alpha}\widetilde{\beta}'\widetilde{X}_{t-1} + \sum_{i=1}^{k-1}\widetilde{\Gamma}_i\Delta\widetilde{X}_{t-i} + \widetilde{\Phi}D_t + \widetilde{\varepsilon}_t \tag{2.9}$$

This specification is termed a parsimonious vector autoregression (PVAR) given the imposition of rank two on the cointegrating space and the reduction in lag length from $k = 3$ to $(k-1) = 2$. Here we have $\tilde{X}_t = (m_t - p_t, y_t, R_t^*, \Delta(m_t + p_t))$, $D_t = (1, t, D_y, D_o)$ and $\widetilde{\beta} = (p_1, p_2)'$.

Estimation of Eq.(2.9) over the full sample (less initial observations for constructing the first differences and the lag) yielded the diagnostics reported in Table 2.7. The table shows that the congruence of the original unrestricted VAR is maintained with only the real money demand equation showing evidence of misspecification.

One of the advantages in specifying a PVAR is that it provides a suitable framework for testing alternative models through parsimonious encompassing, see Hendry and Mizon [1993]. Testing whether simpler models can parsimoniously encompass the PVAR avoids the use of models that overly sample dependent and are not invariant to regime changes.

Specifically, we test whether a VAR in differences (DVAR) parsimoniously encompasses the PVAR by testing whether $\widetilde{\alpha} = 0$ in Eq.(2.9). This would result in a model that excludes the long run relationships estimated for the system given by the two cointegrating relations. The likelihood ratio test statistic for set of eight restrictions is given by $\chi^2(8) = 71.5$ (p-value < 0.00) and thus we reject that the DVAR parsimoniously encompasses the PVAR. This result implies that the zero frequency information contained in the cointegrating relations is required to model our variables of interest.

2.2.6 Parameter Constancy And Forecast Performance

In this final section we provide evidence on the constancy of the PVAR through an examination of the comparative forecast performance of our PVAR against a DVAR that does not contain the cointegrating relations. Following Hendry and Doornik [1994] we estimate multivariate dynamic 1-step ahead forecasts for the period from 1984Q3 to 1989Q2. The chosen forecast sample is a natural one to use given the regime shift brought about by the advent of interest-bearing sight deposit accounts. Note that in order to abstract from improvement in the PVAR's forecast performance as a result of the inclusion of the full sample estimates of the cointegrating relations (which incorporate information on the regime shift), we re-estimated the relations over the sample period 1963Q1 to 1984Q2 and retained these in the PVAR. The subsample estimates of the cointegrating relations are[4]

$$
\begin{aligned}
p_1^* &= m - p - y + 7R^* + 3.18\Delta m + 3.18\Delta p \\
p_2^* &= y + R^* - \Delta m - \Delta p - 0.0066t
\end{aligned}
$$

[4]The restrictions corresponding to H_1 to H_6 in Table 2.7 were accepted with a test statistic of 1.27 with a corresponding p-value of 0.87

A comparison of the above sub-sample estimates against those over the full sample discussed above shows that there is little difference in the estimates across the alternative sample periods. This is obviously of interest in itself given that it shows that the estimated cointegrating relations display parameter constancy over time.

Returning to the forecast comparison, Tables 2.8-2.9 report the in- and out-of-sample predictive performance for the DVAR and PVAR respectively. To compare the forecasting performance of our various competing models we employ the commonly used Root Mean Square Error (RMSE) loss function. Clements and Hendry [1993] show that one must be careful about using such a measure when comparing models that are simple non-singular scale-preserving linear transformations of a common linear system. In particular, they show that MSFE-type measures (of which RMSE is one) are not invariant to common transformations of linear systems such as VARs, EqCMs and cointegrating relations. In other words, MSFE-based measures can yield different rankings of the forecast models depending whether the level, difference or cointegrating relations are used as the basis of forecasting. Such problems carry over to the I(2) case as the model and restrictions therein are again non-singular scale-preserving linear transformations as discussed in Chapter 1.1. Bearing these difficulties in mind Tables 2.8-2.9 show that on the basis of the RMSEs of the forecast errors the best model for forecasting real money demand is the PVAR which retains the long run relations estimated above. The improvement is considerable with a 50 per cent reduction in the RMSE. Thus we have satisfied an important criterion of econometric modelling, namely that in specifying models for policy use we should aim to show that such models provide improved forecast performance compared to simple sample generated time-series models such as the DVAR.

2.3 Conclusion

In this chapter we provide a cointegration analysis of UK money demand within a framework that allows for I(2) variables. The presence of I(2) variables is supported by a test for the integration indices of the model, which suggests two cointegrating relationships and one I(2) trend. We also find evidence for a nominal-to-real transformation to real money demand and a polynomial term involving nominal money and prices. This is in contrast to the common transformation used in the extant literature where the polynomial term contains the price variable alone. The I(1) analysis of the transformed information set provides two polynomial cointegrating vectors which are consistent with a real money demand and an excess demand relation that closely match those found from previous studies. Finally, we show that a parsimonious VAR that incorporates the two polynomial relations provides a good characterisation of the in-sample movements in real money demand and, moreover, provides superior out-of-sample forecasting power compared to a differenced VAR that excludes the long run relations.

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	m	p	y	R
$\overline{R}^2$	0.999	0.999	0.997	0.882
$\widehat{\sigma}$	0.014	0.007	0.010	0.013
$F_{ar}(4, 83)$	[0.178]	[0.496]	[0.506]	[0.616]
$F_{reset}(1, 86)$	[0.710]	[0.220]	[0.503]	[0.304]
$\chi^2_{norm}(2)$	[0.462]	[0.225]	[0.982]	[0.100]
$F_{het}(1, 101)$	[0.062]	[0.650]	[0.086]	[0.002]

Table 2.1. UVAR. Diagnostics

$p-r$	r	S_{r,s_1}				Q_r
4	0	244.3	175.3	122.9	104.5	98.7
		137.0	113.0	92.2	75.3	62.8
3	1		122.4	62.2	50.9	44.7
			86.7	68.2	53.2	42.7
2	2			33.1	21.9	17.0
				47.6	34.4	25.4
1	3				11.6	6.7
					19.9	12.5
	s_2	4	3	2	1	0

Table 2.2. Formal test for integration indices

	β'	$\delta'\beta_2'$	β_1	β_{2_1}	β_{2_2}
m	1	−15.23	22.37	0.77	0.61
p	−0.84	14.27	−31.59	−0.56	0.78
y	2.10	−5.60	−126.86	0.25	−0.08
R	10.23	3.81	21.26	−0.17	0.02
t	−0.03				
	α		α_1	α_{2_1}	α_{2_2}
m	−0.06		0.00	0.14	0.37
p	0.02		0.00	−0.16	0.91
y	−0.01		−0.01	0.24	0.14
R	0.01		0.00	0.95	0.07

Table 2.3. Estimates of I(0), I(1) and I(2) spaces

$$(r, s_1, s_2) = (1, 1, 2)$$

	$\delta'_{\perp}\beta'$	$\delta'\beta'$	$\delta'\delta\beta'_2$	β_1	β_2
m	1	0.087	0.859	−45.7	0.679
p	−0.863	−0.016	0.927	41.4	0.733
y	1.75	1	−0.070	−11.5	−0.055
R	9.94	1.58	0.006	10.2	0.000
t	−0.022	−0.009			
	α_1	α_2		α_1	α_2
m	−0.014	0.246		0.000	−0.199
p	−0.002	0.203		−0.000	−0.814
y	0.007	−0.342		0.000	−0.467
R	−0.003	0.191		0.000	0.283

Table 2.4. Estimates of I(0), I(1) and I(2) spaces

$(r, s_1, s_2) = (2, 1, 1)$

Panel A	$m-p$	y	R^*	$\Delta(m+p)$	t
Money demand	1	0.514	8.83	1.76	−0.011
Excess demand	−0.160	1	−0.225	−1.54	0.005
Panel B	$m-p$	y	R^*	$\Delta(m+p)$	t
Money demand	1	−1	7	3.15	0
				(0.49)	
Excess demand	0	1	1	−1	−0.007
					(0.000)

Table 2.5. Cointegrating Relations of Transformed I(1) Model

Hypothesis		Test Stat.	dof	p-value
$H_1:$	$\widetilde{\beta}_{01} = 0$	0.00	1	0.99
$H_2:$	$\gamma_{11} + \gamma_{21} = 0$	1.21	1	0.27
$H_3:$	$\gamma_{31} - 7 = 0$	0.12	1	0.73
$H_4:$	$\gamma_{12} = 0$	0.00	1	0.99
$H_5:$	$\gamma_{22} - \gamma_{32} = 0$	0.21	1	0.65
$H_6:$	$\gamma_{22} + \gamma_{42} = 0$	0.49	1	0.48
$\cap_{i=1}^{6} H_i$		2.03	4	0.73

Table 2.6. Hypotheses on Cointegrating Relations

	$\Delta(m-p)$	Δy	ΔR	$\Delta^2(m+p)$
$\overline{R}^2$	0.63	0.42	0.06	0.58
$\widehat{\sigma}$	0.02	0.11	0.01	0.02
$F_{ar}(4,85)$	[0.56]	[0.78]	[0.69]	[0.57]
$F_{reset}(1,88)$	[0.20]	[0.28]	[0.10]	[0.82]
$\chi^2_{norm}(2)$	[0.44]	[0.51]	[0.21]	[0.80]
$F_{het}(1,100)$	[0.06]	[0.69]	[0.85]	[0.85]

Table 2.7. PVAR. Diagnostics

	DVAR			
	$\Delta(m-p)$	Δy	ΔR^*	$\Delta^2(m+p)$
1963Q4-1984Q2				
Mean	0.000	0.000	0.000	0.000
RMSE	0.017	0.011	0.012	0.017
1984Q3-1989Q2				
Mean	0.029	0.005	−0.003	0.001
RMSE	0.035	0.009	0.014	0.019

Table 2.8. DVAR. Forecast performance

	PVAR			
	$\Delta(m-p)$	Δy	ΔR^*	$\Delta^2(m+p)$
1963Q4-1984Q2				
Mean	0.000	0.000	0.000	0.000
RMSE	0.015	0.010	0.012	0.015
1984Q3-1989Q2				
Mean	0.010	−0.005	−0.011	0.005
RMSE	0.017	0.009	0.017	0.013

Table 2.9. PVAR. Forecast performance

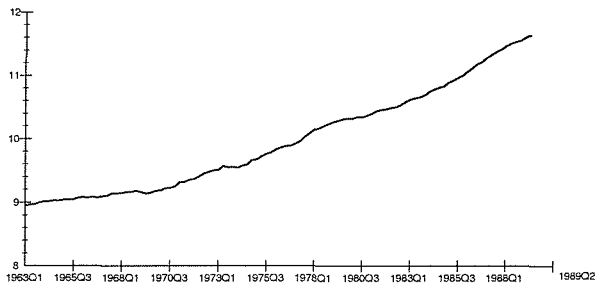

Figure 2.1(a) Plot of m 1963Q1-1989Q2

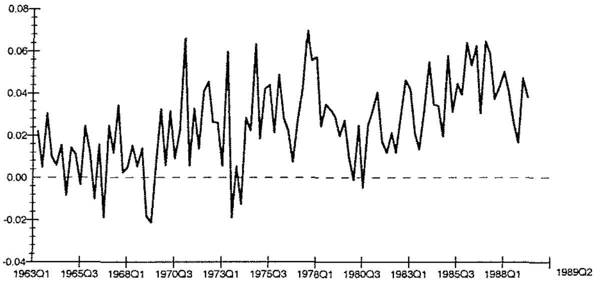

Figure 2.1(b) Plot of Δm 1963Q2-1989Q2

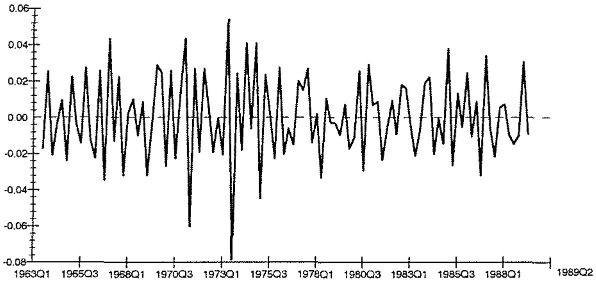

Figure 2.1(c) Plot of $\Delta^2 m$ 1963Q3-1989Q2

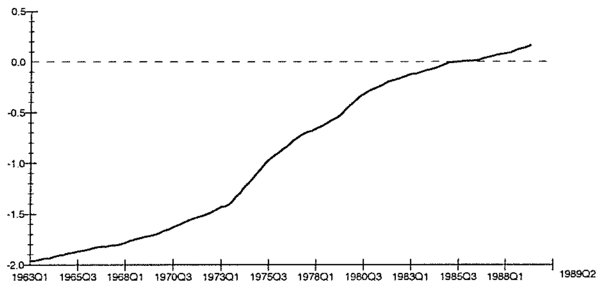

Figure 2.1(d) Plot of p 1963Q1-1989Q2

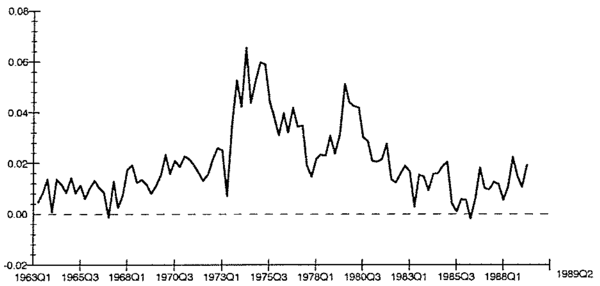

Figure 2.1(e) Plot of Δp 1963Q2-1989Q2

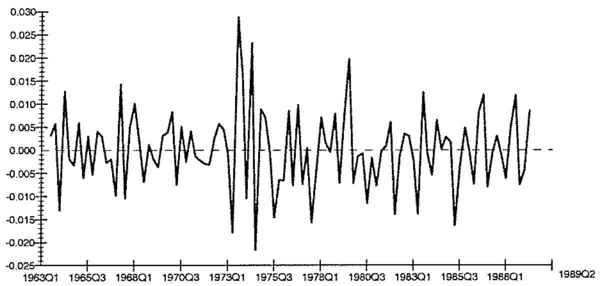

Figure 2.1(f) Plot of $\Delta^2 p$ 1963Q3-1989Q2

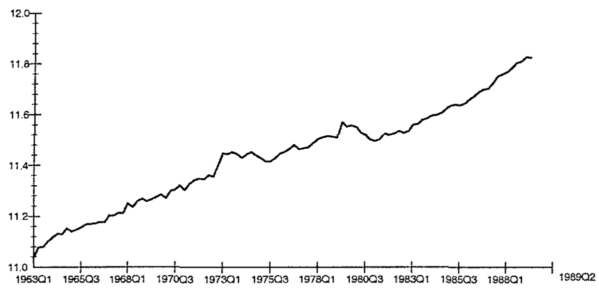

Figure 2.1(g) Plot of y 1963Q1-1989Q2

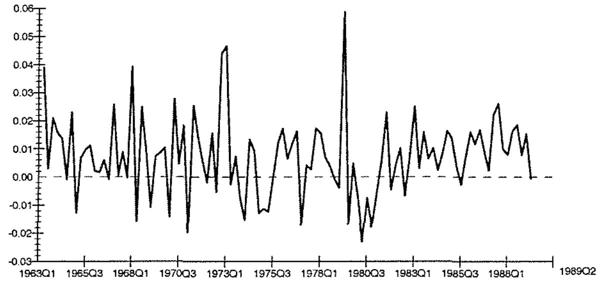

Figure 2.1(h) Plot of Δy 1963Q2-1989Q2

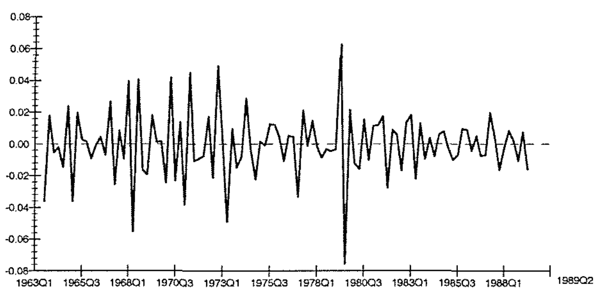

Figure 2.1(i) Plot of $\Delta^2 y$ 1963Q3-1989Q2

Figure 2.1(j) Plot of R^* 1963Q1-1989Q2

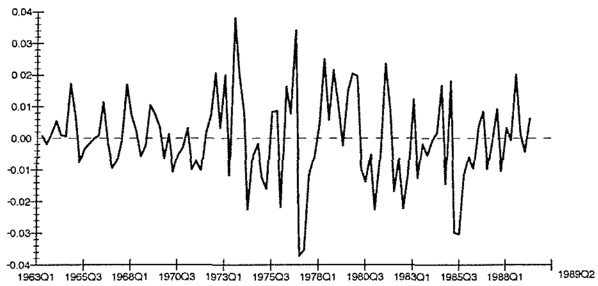

Figure 2.1(k) Plot of ΔR^* 1963Q2-1989Q2

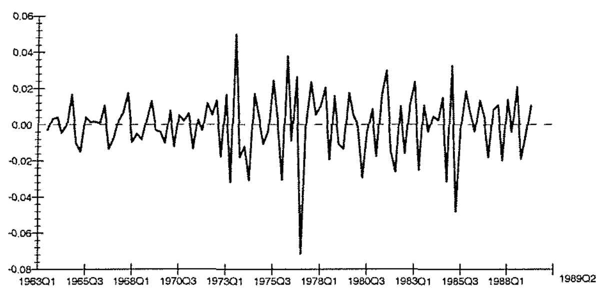

Figure 2.1(l) Plot of $\Delta^2 R^*$ 1963Q3-1989Q2

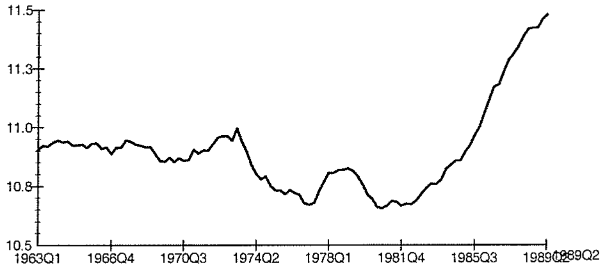

Figure 2.1(m) Plot of $(m-p)$ 1963Q1-1989Q2

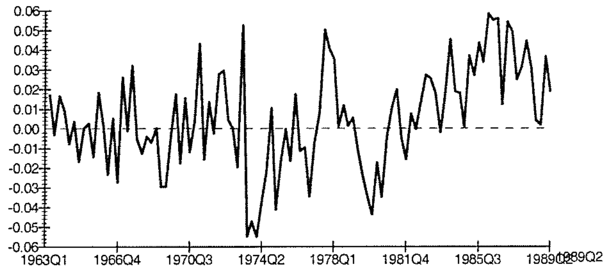

Figure 2.1(n) Plot of $\Delta(m-p)$ 1963Q2-1989Q2

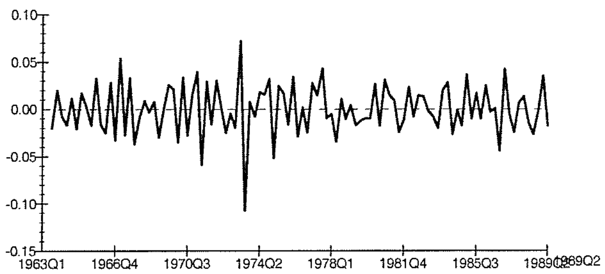

Figure 2.1(o) Plot of $\Delta^2(m-p)$ 1963Q3-1989Q2

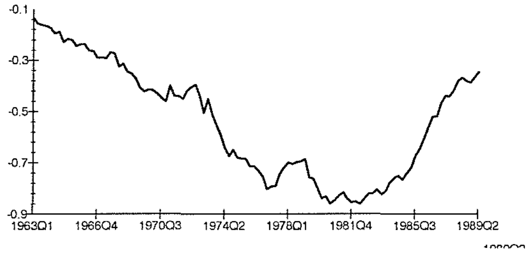

Figure 2.1(p) Plot of $(m-p-y)$ 1963Q1-1989Q2

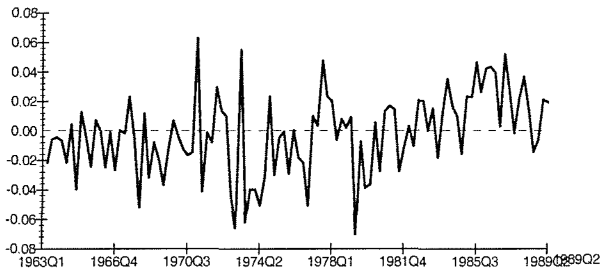

Figure 2.1(q) Plot of $\Delta(m-p-y)$ 1963Q2-1989Q2

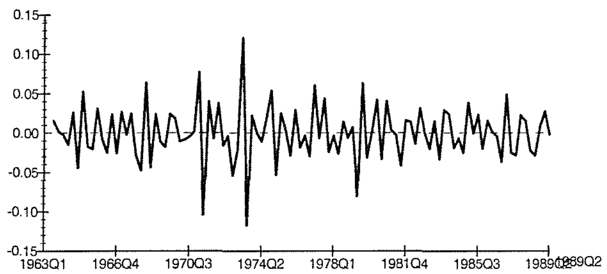

Figure 2.1(r) Plot of $\Delta^2(m-p-y)$ 1963Q3-1989Q2

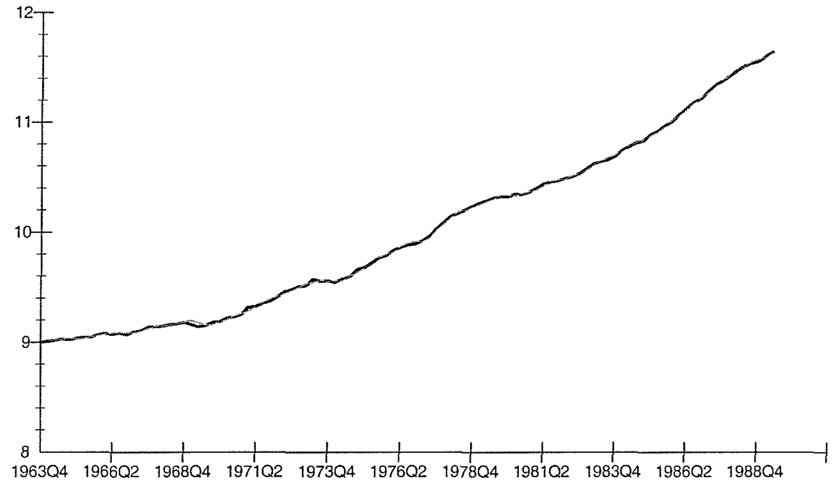

Figure 2.2(a) UVAR. m_t Actual and fitted 1963Q4-1989Q2

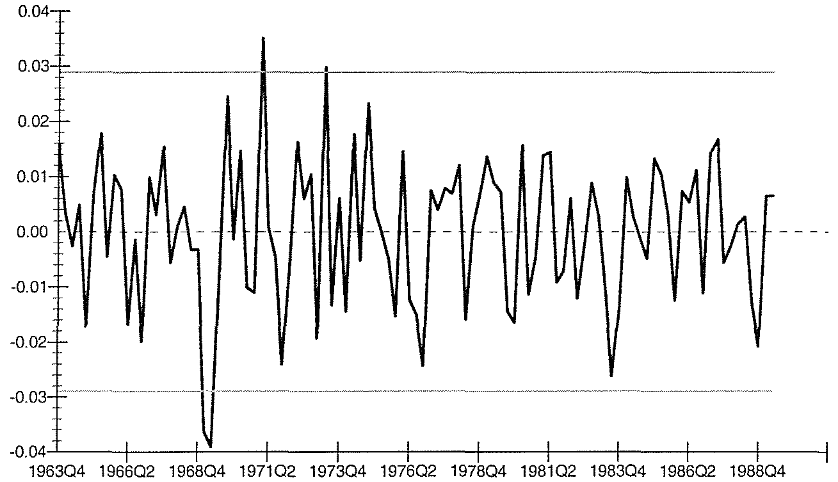

Figure 2.2(b) UVAR. m_t Residuals $\pm 2\widehat{\sigma}$ 1963Q4-1989Q2

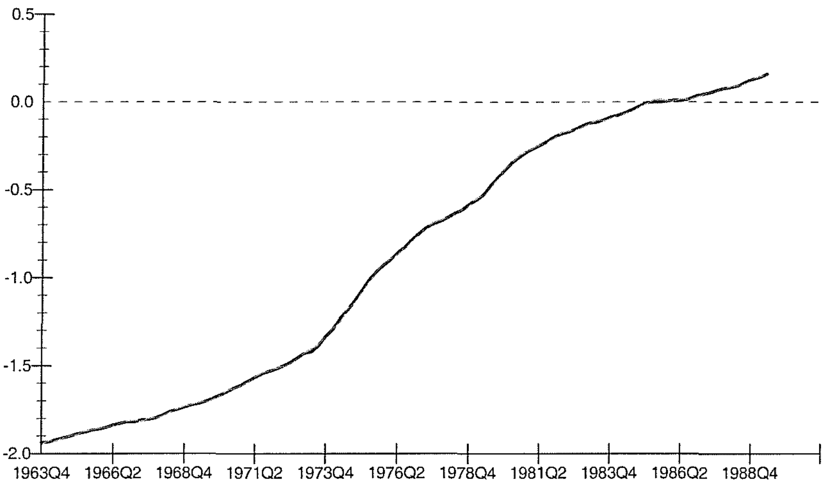

Figure 2.2(c) UVAR. p_t Actual and fitted 1963Q4-1989Q2

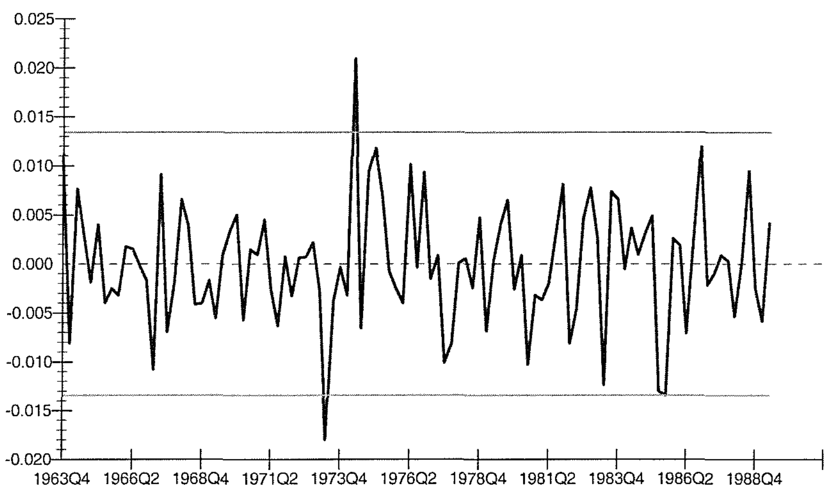

Figure 2.2(d) UVAR. p_t Residuals $\pm 2\widehat{\sigma}$ 1963Q4-1989Q2

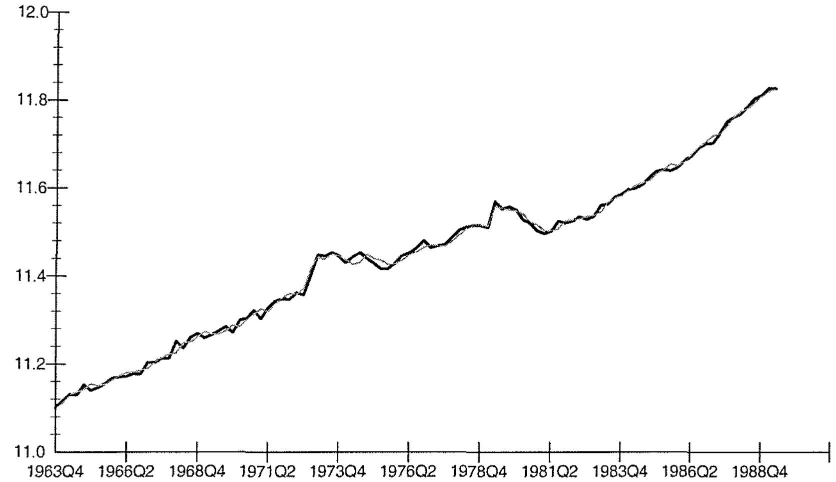

Figure 2.2(e) UVAR. y_t Actual and fitted 1963Q4-1989Q2

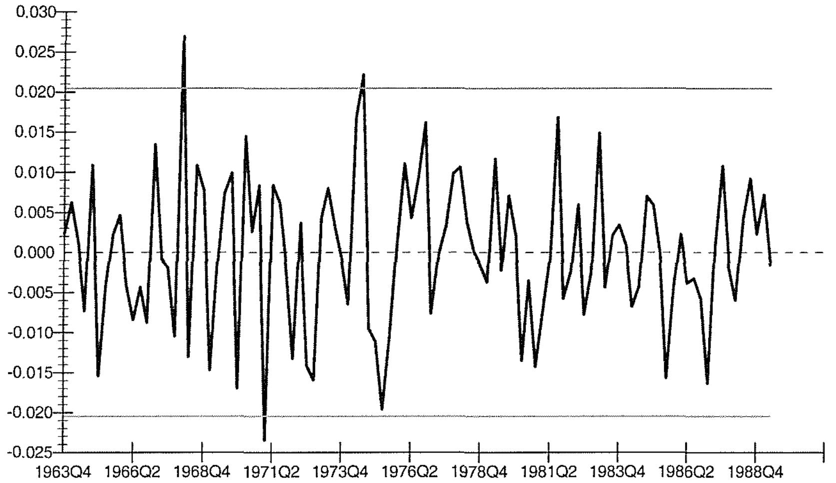

Figure 2.2(f) UVAR. y_t Residuals $\pm 2\widehat{\sigma}$ 1963Q4-1989Q2

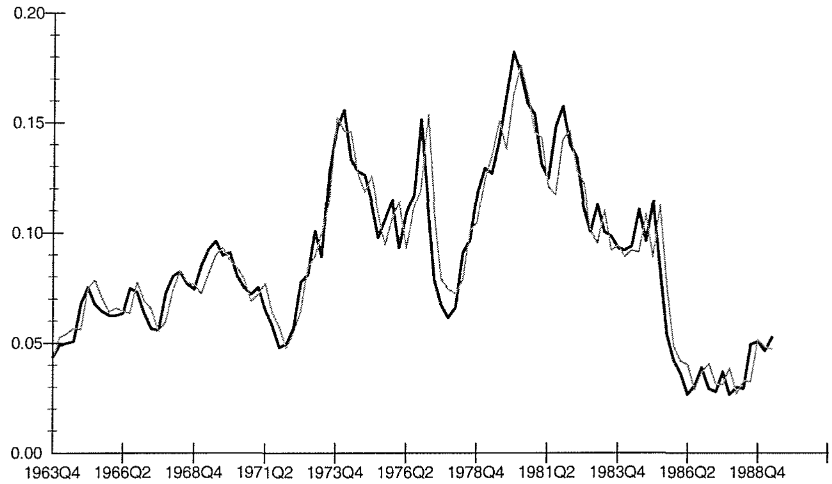

Figure 2.2(g) UVAR. R_t^* Actual and fitted 1963Q4-1989Q2

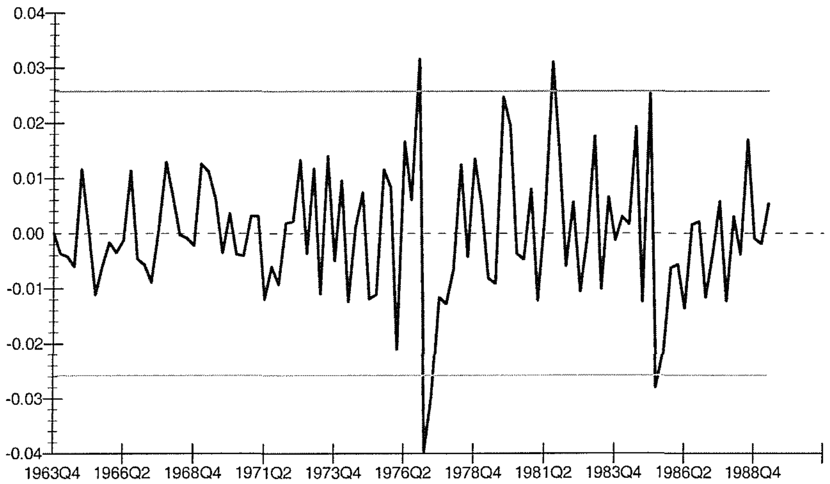

Figure 2.2(h) UVAR.R_t^* Residuals $\pm 2\hat{\sigma}$ 1963Q4-1989Q2

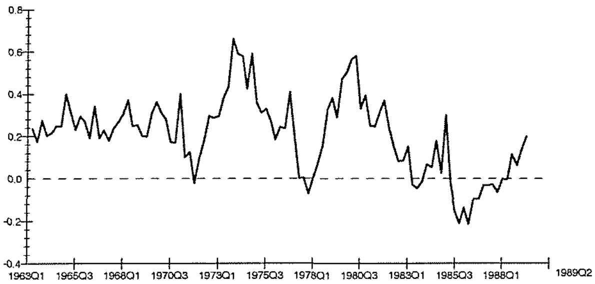

Figure 2.3(a) Polynomial cointegrating relation p_1

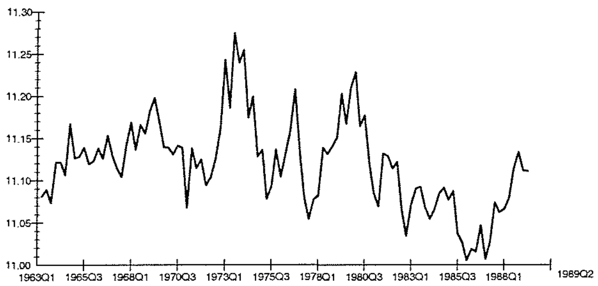

Figure 2.3(b) Polynomial cointegrating relation p_2

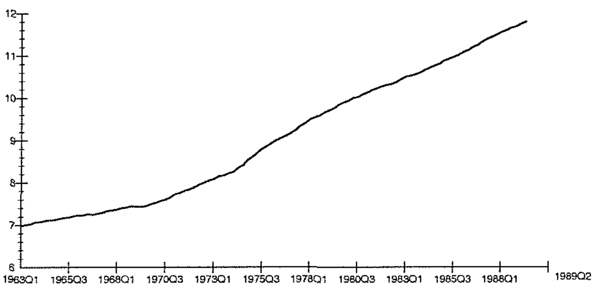

Figure 2.3(c) Common I(2) trend $b'X_t$

Chapter 3

Testing The Monetary Exchange Rate Model Within A Cointegrated I(2) System

3.1 Introduction

Since its conception in the 1970s the monetary exchange rate model has become the dominant theoretical model of exchange rate determination. However, empirical testing of the model has produced disappointing results. The common finding in empirical work is that the monetary model is plagued by unstable regression coefficients in terms of sign (Hayes and Stone [1981]), magnitude and significance. Furthermore, in a widely cited paper by Meese and Rogoff [1983] the authors show that a random walk with drift outperforms the monetary model in out-of-sample forecasting ability.

Recently, attention has shifted towards the ability of the monetary exchange rate model to adequately characterise long run movements in the exchange rate. In particular, following the work of Engle and Granger [1987], studies have been conducted to test the long run properties of the model using cointegration analysis. Unless such long run relationships exist, it is inappropriate to use the monetary model for forecasting and policy purposes. These studies can be separated into two broad strands: those that use relative variables[1] in the cointegrating vector; and those that include the variables without restriction. Studies that have used relative variables tend to find no evidence of a cointegrating vector [see Messe [1986], Boothe and Glassman [1987], Ballie and Selover [1987], Meese and Rogoff [1988], McNown and Wallace [1989] and Sarantis [1995]). In contrast, studies that use unrestricted variables find evidence of numerous cointegrating vectors [see MacDonald and Taylor [1991, 1994], Chrystal and MacDonald [1995] and Kanas [1997]) but these relations do not display the anticipated signs as suggested by the monetary model and often have extremely large magnitudes that are difficult to reconcile with economic theory. In spite of these difficulties, attention has continued to be paid to the monetary model. In particular, there is emerging evidence that equilibrium correction models based upon the monetary exchange rate model provide superior forecasting power compared to a random walk, with the degree of outperformance increasing as the forecast horizon is extended (see MacDonald and Taylor [1994] and Tawadros [2001]).

Our motivation for re-examining the monetary exchange rate model builds upon increasing evidence that variables used in the monetary model may well be integrated of order two. In particular, recent empirical work on modelling money demand functions suggest that UK nominal money over this period is $\mathsf{I}(2)$[2]. If this is the case then existing

[1] A relative variable corresponds to the restriction that domestic and foreign variables are equal and of opposite sign.

[2] See Johansen [1992], Haldrup [1994], Paruolo [1996] and Rahbek *et al.* [1999].

cointegration analyses of the monetary model will provide misleading results. This can be made clear by noting in the I(1) model the cointegrating vectors take the form $\beta' X_t \sim \mathsf{I}(0)$ while from Chapter 1.1.2 we know that in the I(2) model we have polynomial cointegration such that $\beta' X_t - \delta' \beta_2' \Delta X_t \sim \mathsf{I}(0)$. Specifically, the relation $\beta' X_t$ is now I(1) and requires a linear combination of the differenced variables ΔX_t to achieve stationarity.

To the authors knowledge only two studies have entertained the possibility of I(2) variables in the monetary model. Of note, Diamandis *et al.* [1998] provides an analysis of three US dollar bilateral exchange rates, including the dollar-sterling rate, using the I(2) test of Paruolo [1996] discussed in Chapter 1.2.1. However, the study finds no evidence in support for the presence of I(2) trends and so continues with a standard I(1) cointegration analysis. In addition, in an earlier paper McNown and Wallace [1994] find that the dollar exchange rate and nominal monies for Argentina, Chile and Israel over the modern float are integrated of order two. In order to avoid the difficulties of analysing the model using I(2) techniques the authors impose the untested restriction of long run domestic price homogeneity with respect to the exchange rate and again continue with the standard I(1) model.

In light of this it is clear that a thorough examination of the monetary model that uses recent techniques for estimating and testing restrictions on cointegrating relations that allow for the presence of I(2) variables is warranted. This Chapter provides such an analysis. The structure of the Chapter is as follows. In the remainder of this introduction we discuss the theoretical basis for the monetary exchange rate model and present the model to be used in the empirical section of the Chapter. In Section 3.2 we present an empirical analysis of the monetary model within a cointegrated I(2) system for the sterling-dollar exchange rate over the period 1976Q1 to 1999Q3. The section begins by specifying a congruent unrestricted vector autoregression for the dataset. Then, we test for the number of cointegrating relations and common I(1) and I(2) trends in the data using the test of Rahbek *et al.* [1999]. We continue by testing for some overidentifying restrictions on the cointegrating relations and give some intuition for the specifications we find support for. Finally, we provide a reparameterisation of the unrestricted vector autoregression that incorporates the cointegrating relations and analyse its forecasting ability compared to simple multivariate time series. Section 3.3 concludes.

3.1.1 The Monetary Exchange Rate Model

The asset approach continues to represent the dominant theory for explaining movements in the nominal exchange rate. Within the asset approach there are two conceptually distinct approaches to the importance of non-money assets in determining the exchange rate. The monetary approach is based on the assumption of perfect substitutability of non-money assets so that the exchange rate is determined only by relative excess money supplies. In contrast, the portfolio balance approach assumes that non-money assets are imperfect substitutes and drive movements in the exchange rate through economic agents maximising the expected return from their portfolio of assets. In this Chapter we concentrate only on the monetary class of exchange rate models and in particular on the flexible price monetary model which we outline below.

The standard monetary model used in the empirical literature was formulated by Frenkel [1976] and combines domestic and foreign money demand functions which a purchasing power parity identity to obtain a reduced form bilateral exchange rate function.

In addition, the model assumes that the income and interest rate elasticities for money demand are identical in the domestic and foreign countries and that the money supply is determined exogenously. With all these assumptions in place, the three behavioural equations take the form

$$\begin{aligned} m_t &= p_t + ay_t - br_t \\ m_t^* &= p_t^* + ay_t^* - br_t^* \\ s_t &= p_t - p_t^* \end{aligned} \tag{3.1}$$

where s is the spot exchange rate defined as the price of the domestic currency per unit of foreign currency; m is nominal money; p is the domestic price level; y is domestic real income and r is a suitably defined domestic short-term interest rate. In addition, lower case letters denote variables in logarithms and a superscript asterisk denotes the variables in the foreign country. Solving the money demand equations with respect to p_t and p_t^* and substituting into the purchasing power parity (PPP) identity yields the following reduced form exchange rate equation

$$s_t = (m_t - m_t^*) - a(y_t - y_t^*) + b(r_t - r_t^*) \tag{3.2}$$

Note that as a result of the PPP identity we have implicitly imposed one other restriction, namely that the coefficient on $(m_t - m_t^*)$ is unity *i.e.* there is a proportionate relationship between the nominal exchange rate and the relative money supply.

The assumptions embedded in Eq.(3.1) are extremely strong and have not been supported in the empirical literature. First, studies on modelling the demand for money in the UK and US (see Hendry and Ericsson [1991]) show that the response of money demand to changes in real income and interest rates are significantly different in the UK compared to the US. Second, PPP is a simple economic hypothesis that defines the long run equilibrium in the goods market. However, when used for empirical modelling, the effect of such issues as temporal aggregation and weak correspondence between theoretical and observed variables necessitates a more flexible specification than that given above. In particular, Dornbusch [1989] suggests that due to differing productivity trends in the tradeable and non-tradeable goods sectors and inter-country differences in consumption patterns, a secular decline in domestic prices relative to foreign prices could appear as a linear trend in the PPP relationship. In light of the above, we relax the restrictions in Eq.(3.1) to yield

$$\begin{aligned} m_t &= d_1 p_t + a_1 y_t - b_1 r_t \\ m_t^* &= d_2 p_t^* + a_2 y_t^* - b_2 r_t^* \\ s_t &= \theta_1 p_t - \theta_2 p_t^* + \tau t \end{aligned} \tag{3.3}$$

As before, we solve the money demand equations in Eq.(3.3) with respect to p_t and p_t^* and substitute these into the modified PPP relationship in Eq.(3.3) to obtain our reduced form for the empirical analysis

$$s_t = \left(\frac{\theta_1}{d_1}\right) m_t - \left(\frac{\theta_2}{d_2}\right) m_t^* - \left(\frac{\theta_1 a_1}{d_1}\right) y_t + \left(\frac{\theta_2 a_2}{d_2}\right) y_t^* + \left(\frac{\theta_1 b_1}{d_1}\right) r_t - \left(\frac{\theta_2 b_2}{d_2}\right) r_t^* + \tau t \tag{3.4}$$

which we simplify to

$$s_t = \alpha m_t - \alpha^* m_t^* - \beta y_t + \beta^* y_t^* + \gamma r_t - \gamma^* r_t^* + \tau t \tag{3.5}$$

for suitable parameter definitions. Note that this unrestricted reduced form allows us to test whether the restrictions implied by the standard monetary model (corresponding to $\alpha = \alpha^*$, $\beta = \beta^*$, $\gamma = \gamma^*$ and $t = 0$) are supported for our dataset.

3.2 Empirical Analysis

3.2.1 The Data

The variables in the dataset are quarterly observations covering the post Bretton-Woods period from 1976Q1 to 1999Q3. The variables correspond to the spot sterling-dollar exchange rate[3] (denoted s); the UK monetary aggregate M0 and the US monetary aggregate M1[4] (m and m^* respectively); UK and US real GDP at market prices (y and y^* respectively); and 3-month UK and US interest rates (r and r^* respectively). All the variables are in natural logarithms. In addition to the stochastic variables in the information set, we also include a constant, time trend and four dummy variables. The dummy variables take the value unity in 1978Q2, 1979Q2, 1980Q3 and 1982Q4 to account for the largest outliers in the unrestricted VAR (see below).

3.2.2 The Unrestricted Vector Autoregression

Our analysis begins by modelling the joint distribution of the variables within an unrestricted vector autoregression (VAR)[5]. The unrestricted VAR takes the form:

[3]Sources for the data can be found in Appendix A.

[4]Note that the UK M1 monetary aggregate was not used due to discontinuity in the measure caused by the steady conversion of UK building societies to public limited companies from 1989 (*e.g.* in July 1989 the conversion of the Abbey National building society increased M1 by 16% overnight).

[5]The calculations and numerical results in the text were obtained using the computer packages CATS in RATS (Hansen and Juselius [1995]) and Microfit 4.0 (Peseran and Peseran [1997]).

$$X_t = \sum_{j=1}^{k} A_j X_{t-j} + \Phi D_t + \varepsilon_t \tag{3.6}$$

where $X_t' = (s, m, m^*, y, y^*, r, r^*)$ is the seven-dimensional vector containing our variables of interest; D_t is a vector of deterministic variables containing a constant, a trend and the four dummies given above; and ε_t is an innovation process which is independently distributed with mean zero and variance-covariance matrix Σ.

The unrestricted VAR for X_t' given by Eq.(3.6) was estimated with $k = 3$ lags over the full sample from 1976Q1 to 1999Q3 (less observations used for lags). Given the frequency of the data it would have been natural to begin the analysis with five lags. This would ensure that the dynamics were rich enough to adequately characterise the persistence in the variables of interest. However, given the size of the system and the number of available observations one has to be careful to avoid over-fitting and mitigate the risk of small sample dependence. In light of these competing constraints we set a compromise choice of $k = 3$ lags so that the initial general system has 31 unrestricted coefficients including the error variance-covariance matrix in each equation which corresponds to approximately three observations per parameter. A likelihood ratio test for the lag length of the unrestricted VAR suggested that $k = 3$ is required (the test statistic for $k = 2$ lags against the alternative $k = 3$ being $\chi^2(49) = 101.8$ with a p-value < 0.001).

Table 3.1 provides descriptive and diagnostic statistics for the estimated unrestricted VAR. The diagnostics correspond to p-values of the Lagrange multiplier test of residual serial correlation against 4th-order autoregression (Godfrey, [1978]) denoted $F_{ar}(\cdot,\cdot)$; the RESET test of functional form (Ramsey, [1969]) denoted $F_{reset}(\cdot,\cdot)$; the Jarque-Bera chi-square test of normality of regression residuals (Jarque and Bera [1980]) denoted $\chi^2_{norm}(\cdot,\cdot)$; and an equality of error-variances test denoted $F_{het}(\cdot,\cdot)$. The table shows that the initial system is generally well-specified, with only marginal evidence of non-congruency in the equation for domestic income. In light of this, we continue with the unrestricted VAR as a valid general framework for our empirical analysis.

The dynamic properties of the unrestricted VAR are illustrated by the moduli of the eigenvalues of the companion matrix. The first eleven of the $(k \times n) = 21$ are

$$(0.990, 0.936, 0.927, 0.927, 0.918, 0.918, 0.754, 0.754, 0.703, 0.611, 0.611) \qquad (3.7)$$

where the roots of the characteristic polynomial are the inverses of these eigenvalues. The eigenvalues suggest the presence of at least six unit roots in the dynamic system and possibly eight. Evidence on the roots of the characteristic equation is used in parallel with the formal test of the integration indices (see below) since conditioning on the dummy variables may well change the asymptotic distribution of the test.

3.2.3 Cointegration Analysis

We continue by formally testing for the integration indices in the system *i.e.* the number of r cointegrating vectors together with the number of s_1 $\mathsf{I}(1)$ trends and thus the number of $(p - r - s_1) = s_2$ $\mathsf{I}(2)$ trends using the test presented in Rahbek *et al.* [1999] and discussed in detail in Chapter 1.3.1. Rahbek *et al.* [1999] show that the integration indices (r, s_1) should be determined jointly since the sequential approach of testing for the cointegration rank and then proceeding to test for the number of s_1 $\mathsf{I}(1)$ trends does not, in general, yield the correct asymptotic size for the test.

The test statistics for all combinations of r and s_1 together with the 95% quantiles of the asymptotic test distributions taken from Rahbek *et al.* [1999, Appendix C, Table 4] are presented in Table 3.2. Note that the 95% quantiles are not adjusted to account for the presence of the dummies. However, Doornik et al. [1998, Figure 4 and discussion] show that entering dummies unrestrictedly to the model when they are not present in the DGP has little impact on the tests for integration indices in the I(1) model and thus in the first step of the formal test for the integration indices of the I(2) system. In addition, following Paruolo [1996] and Rahbek *et al.* [1999] the test statistics are calculated under the assumption that the data contain at most linear trends.

To determine the appropriate integration indices we start by testing the most restricted hypothesis, given by $(r, s_1, s_2) = (0, 0, 7)$, then, if this hypothesis is not rejected, we test successively less and less restricted hypotheses by continuing to the end of the first row and then by proceeding row-wise from left to right until the first rejection is found. The first submodel hypothesis corresponding to $(r, s_1, s_2) = (0, 0, 7)$ is easily rejected. In fact, all submodels for $r = 0$ are rejected. The first non-rejection is for the submodel $(r, s_1, s_2) = (2, 2, 3)$ with a test statistic of 128.0 compared against the 95% critical value of 142.2. This non-rejection is a fairly strong one with a p-value of over 25%. Note also that this particular submodel corresponds to $s_1 + 2s_2 = 8$ unit roots which matches the results on the roots of the characteristic polynomial presented above. However, since the asymptotic tables may not provide a close approximation of the actual test distribution we also consider some of the other hypotheses to the right and below. The next two non-rejections correspond to $(r, s_1, s_2) = (2, 3, 2)$ and $(r, s_1, s_2) = (3, 1, 3)$ and thus imply

$s_1 + 2s_2 = 7$ unit roots which does not sit with roots of the characteristic polynomial in Eq.(3.7). In addition, there is no economic or empirical reason why these hypotheses should be preferred to the first non-rejection submodel.

Imposing the rank restrictions consistent with $(r, s_1, s_2) = (2, 2, 3)$ the first eleven eigenvalues of the companion matrix are

$$(1, 1, 1, 1, 1, 1, 1, 1, 0.59, 0.56, 0.56) \tag{3.8}$$

where it seems that we have accounted for all the unit roots in the system. Evidence in support of the need for allowing I(2) trends in the model can be seen by imposing $p - r = 5$ in the I(1) model. In this case the first nine eigenvalues become $(1, 1, 1, 1, 1, 0.95, 0.95, 0.86, 0.59)$ and thus we see three further near-unit roots appear. Thus, we proceed with $(r, s_1, s_2) = (2, 2, 3)$ as our maintained model.

Imposing the submodel corresponding to $(r, s_1, s_2) = (2, 2, 3)$ yields the unrestricted parameter estimates in Table 3.3. In the first four columns of the top panel of the table we have the estimates of the two polynomial cointegrating relations $\beta' X_t - \delta' \beta_2' \Delta X_t$ (where we have normalised the β vector on s). It is clear from the estimates of the levels variables $\beta' X_t$ that a number of the parameter values exhibit large magnitudes and signs that are in variance with those predicted by the monetary model. Furthermore, the estimates do not seem to lend any support for the basic monetary model where the domestic and foreign variables are entered in relative form. In order to provide more formal evidence we continue by testing for the validity of the parameter restrictions suggested by the basic monetary model *i.e.* $\alpha = \alpha^*$, $\beta = \beta^*$, $\gamma = \gamma^*$ and $t = 0$ in Eq.(3.5). The results of the tests, reported in Table 3.4, show that each of the restrictions is strongly rejected.

In order to impose some overidentifying restrictions on our polynomial cointegrating relations we also consider tests for excluding variables from each of the two cointegrating relations and tests for weak exogeneity. Table 3.5 presents the results. We can see that the first cointegrating relation seems to be a combination of only domestic money, income and the interest rate, perhaps representing a simplified money demand function or monetary authority reaction function. We return to this later when we examine the specific form of the first cointegrating relation. More importantly, the monetary model of the exchange rate is identified by the second cointegrating relation.

The results from testing for weak exogeneity of the variables of interest with respect to the cointegrating parameters show that both domestic and foreign income are weakly exogenous with test statistics of 3.83 (p-value of 0.15) and 2.02 (p-value of 0.36) respectively. A joint test for weak exogeneity of both variables (not shown in the table) results in a test statistic of 6.88 with a corresponding p-value of 0.14.

In summary, the above tests suggest that we can employ a series of overidentifying restrictions on our cointegrated system given by

$$H_\alpha = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

and

$$H_{\beta^1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, H_{\beta^2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}$$

The eight overidentifying restrictions are excepted with a test statistic of 10.02 (p-value > 0.25).

The restricted parameter estimates are presented in Table 3.6. For ease of exposition we have normalised the first cointegrating relation on the domestic interest rate and the second relation on the exchange rate. The first relation seems to correspond to a simple Taylor rule (a form of monetary authority reaction function, see Taylor [1993]) whereby the domestic interest rate changes positively in response to both inflation (domestic money is our proxy here) and output. Taylor [1993] suggests a response coefficient of 1.5 on inflation and 0.5 on output. As we can see from Table 3.6 the coefficient on domestic money is fairly close to 1.5, though the coefficient on income is a factor of ten larger.

The second cointegrating relation can be identified as the monetary exchange rate model, though the equilibrium relationship suggested by the monetary exchange rate model holds in a theoretically weaker sense given that the linear combination of the level variables is integrated of order one. Unfortunately, inkeeping with the extant empirical literature the magnitude and signs on the cointegrating parameters do not conform to their theoretical values.

In the rest of the top panel in Table 3.6 we have the estimates of the two CI(2,1) relations given by $\beta_1' X_t$ together with the estimates of the weight with which the I(2) trends influence the variables of the system is corresponding to $\beta_{2\_j}' X_t$,$j = 1, 2, 3$. It is interesting that the I(2) trends seem to be driving a number of the variables in the system as represented by the condition that $\beta_{2\_j}' X_t \neq 0$ for $j = 1, 2, 3$.

Moving on to the bottom panel of Table 3.6 we have the adjustments to the disequilibrium error defined by the polynomial relation together with the estimates of the stochastic I(1) and I(2) trend components given by α_i, α_{1_i} and α_{2_j}, respectively for $i = 1, 2$ and $j = 1, 2, 3$. We note here that the largest weight in the two I(1) trends come from the cumulated residuals in the exchange rate, while we have that the twice cumulated residuals from both domestic and foreign money supply and real income are the main components for the I(2) trends.

3.2.4 Modelling The I(0) PVAR

In the absence of any further possible tests of restrictions on the polynomial relations we continue by specifying a parsimonious reparameterisation that maps the system from I(2) to I(0) space. The specification follows Chapter 1.1.3 Eq.(1.9) in that we define an equilibrium correction model of the form

$$\begin{aligned}\Delta^2 X_t &= \alpha(\beta' X_{t-1} + \delta' \beta_2' \Delta X_{t-1}) + \Omega \alpha_\perp (\alpha_\perp' \Omega \alpha_\perp)^{-1} \kappa' \tau' \Delta X_{t-1} \\ &\quad + \sum_{i=1}^{k-2} \Psi_i \Delta^2 X_{t-i} + \Phi D_t + \varepsilon_t \qquad (3.9)\end{aligned}$$

where from Chapter 1.1.3 Eq.(1.9) we have that $\beta = \rho\tau$, $\delta\beta_2 = -\psi$ and $\tau = (\beta, \beta_1)$. This specification, denoted a parsimonious vector autoregression (PVAR), imposes the reduced rank condition on the cointegrating space and the reduction in lag length from $k = 3$ to $(k-2) = 1$. Note that the cointegrating relations in Eq.(3.9) are given by $S_t = \beta' X_{t-1} - \delta' \beta_2' \Delta X_{t-1}$ and $\tau' X_t = (\beta, \beta_1)' X_t$.

The PVAR was estimated over the full sample (less initial observations for constructing the second differences and the lag) and yielded the summary statistics and diagnostics reported in Table 3.7. The table shows that the initial system is generally well-specified, with only marginal evidence of non-congruency in the equation for domestic income and the foreign interest rate. A further test of the PVAR is provided by examining whether a model which excludes the long run relations *i.e.* a VAR in double differences (denoted a DDVAR) parsimoniously encompasses the PVAR following Hendry and Mizon [1993]. Testing whether simpler models can parsimoniously encompass the PVAR avoids the use of models that overly sample dependent and are not invariant to regime changes. The test corresponds to a likelihood ratio test for deleting the cointegrating relations such that $\alpha = \Omega\alpha_{\perp}(\alpha_{\perp}'\Omega\alpha_{\perp})^{-1}\kappa' = 0$ in Eq.(3.9). The test statistic for the set of restrictions is given by $\chi^2(42) = 377.1$ [0.00][6] and hence we reject that the DDVAR parsimoniously encompasses the PVAR. This result gives further support for the importance of the polynomial cointegrating relationships together with the other CI(2,1) relations when modelling our variables of interest.

[6]The individual test statistics, distributed as $\chi^2(7)$, for excluding the two polynomial cointegrating relation and each of the four directly cointegrating vectors are 75.2 [0.00], 55.8 [0.00], 43.4 [0.00], 35.3 [0.00], 64.4 [0.00] and 70.1 [0.00] respectively.

3.2.5 Forecasting Performance

As discussed in the introduction to this chapter an important criticism levelled at the monetary exchange rate model is its poor forecasting ability when compared to simple time series models such a random walk with drift (Messe and Rogoff [1983]). However, recent studies have suggested that by incorporating the monetary model within an equilibrium correction model one can obtain increased forecasting power compared to a simple random walk. In light of this, we now provide a comparison of the forecasting performance of our monetary exchange rate model driven PVAR with two multivariate examples of autoregression models with drift, namely our unrestricted VAR and the DDVAR. Specifically, we obtain multivariate dynamic 1-step ahead forecasts over the period 1995Q1 to 1999Q3. (The chosen forecast sample is arbitrarily set to correspond to approximately 20% of the available sample which is considered sufficient to compare the predictive power of the various models.) Tables 3.8-3.10 provide estimates of the first two moments of the forecast errors as given by the mean and the root mean squared error (RMSE) for each of the three models. As discussed in Chapter 2 we reiterate here that one needs to careful when employing MSFE-type measures (of which RMSE is one) when comparing the forecast performance of models that are simple transformations of linear systems such as VARs, EqCMs and cointegrating relations. This is due to the fact that MSFE-based measures can yield different rankings of the forecast models depending whether the level, difference or cointegrating relations are used as the basis of forecasting. With that in mind, Tables 3.8-3.10 show that the forecasting performance of the unrestricted VAR is poor. Not only is there evidence of under- and over-forecasting as measured the means but in addition the RMSE are seen to be in some cases fifty times that of the in-sample errors. Moving on to the DDVAR we see that the forecasting performance is much improved with little evidence of under- or over-forecasting and RMSEs in most cases similar or even lower than those of the in-sample errors. The DDVAR clearly poses a strong test for the PVAR in terms of forecasting power. However, as shown by Table 3.10 the forecasting performance of the PVAR matches that of the DDVAR. In particular, for our variable of interest, the dollar-sterling exchange rate, the PVAR provides a RMSE some 30% lower than the DDVAR and nearly half that of the unrestricted VAR.

The increased forecasting power of the PVAR compared to the unrestricted VAR is shown in Figure 3.1. As the figure shows, the unrestricted VAR not only consistently overforecasts the exchange rate but also mischaracterises the movement of the exchange rate over the forecast period. However, the PVAR provides a much closer fit over the period and generally tracks well the not insignificant volatile of quarterly movements in the series. In short, we have shown by specifying a congruent model that allows for the presence of I(2) variables we can generate forecasts of the exchange rate that compare well with multivariate versions of autoregression models with drift. Moreover, as the model incorporates the long run information in the data it is consistent with the theoretical basis for exchange rate movements as described by the monetary exchange rate model.

3.3 Conclusion

In this chapter we have provided the first empirical examination of the monetary exchange rate model that allows for the presence of I(2) variables in the data. We summarise our findings as follows. For the dollar-sterling exchange rate over the modern float we find support for the existence of two cointegrating relationships among the variables: the monetary exchange rate model; and a simple Taylor rule. Moreover, by formally testing for the number of $\mathsf{I}(1)$ and $\mathsf{I}(2)$ stochastic trends and finding evidence for two such $\mathsf{I}(2)$ trends in the data, the stationary cointegrating relations found correspond to polynomial relations where the linear combination of the differences of the variables are required in order to provide a stationary relationship. Finally, we show that by specifying an equilibrium correction model that incorporates the disequilibrium errors from the estimated cointegrating relations we obtain a model that is well-specified in-sample and, moreover, provides superior 1-step forecasting power compared to multivariate versions of simple autoregression models with drift. Thus, by careful attention to the time series properties of the variables in the monetary model we provide further evidence that Meese and Rogoff's [1983] criticism of the monetary model as a poor forecasting model should be laid to rest.

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	s	m	m^*	y	y^*	r	r^*
$\overline{R}^2$	0.865	0.999	0.999	0.998	0.999	0.912	0.956
$\widehat{\sigma}$	0.052	0.006	0.008	0.007	0.006	0.103	0.084
$F_{ar}(4,83)$	[0.555]	[0.244]	[0.146]	[0.009]	[0.532]	[0.932]	[0.358]
$F_{reset}(1,86)$	[0.938]	[0.772]	[0.006]	[0.102]	[0.408]	[1.000]	[0.384]
$\chi^2_{norm}(2)$	[0.088]	[0.466]	[0.636]	[0.914]	[0.266]	[0.468]	[0.807]
$F_{het}(1,101)$	[0.952]	[0.047]	[0.123]	[0.045]	[0.281]	[0.960]	[0.511]

Table 3.1. UVAR. Diagnostics

$p-r$	r	S_{r,s_1}							Q_r
7	0	522.1	424.1	358.8	311.7	272.7	245.6	237.4	231.7
		351.6	311.2	274.0	241.2	211.6	186.1	164.6	146.8
6	1		399.4	313.0	248.0	202.1	172.5	162.4	156.8
			269.2	233.8	202.8	174.9	151.3	130.9	115.4
5	2			239.9	174.3	128.0	108.6	103.4	102.6
				198.2	167.9	142.2	119.8	101.5	87.2
4	3				137.9	77.0	66.6	62.3	62.2
					137.0	133.0	92.2	75.3	62.8
3	3					122.6	72.0	44.3	38.9
						86.7	68.2	53.2	42.7
2	5						55.7	22.7	17.5
							47.6	34.4	25.4
1	6							7.7	2.7
								19.9	12.5
	s_2	7	6	5	4	3	2	1	0

Table 3.2. Formal test for integration indices

	β_1	β_2	$\delta'\beta_{2}\_1$	$\delta'\beta_{2}\_2$	β_{1}_1	β_{1}_2	β_{2}_1	β_{2}_2	β_{2}_3
s	1	1	−21.31	−16.86	−11.80	−17.91	0.686	−0.061	0.023
m	23.20	16.94	6.920	0.054	34.01	−59.64	−0.106	0.135	0.000
m^*	−0.136	−1.606	3.642	27.28	−8.282	−16.36	−0.643	−0.468	−0.151
y	44.56	1.121	3.322	−2.174	−7.425	11.97	−0.000	0.040	0.150
y^*	−13.57	−13.58	5.394	−4.171	41.71	−70.50	0.010	0.078	0.248
r	−5.905	1.317	30.21	−8.428	−17.92	16.14	−0.270	0.409	0.775
r^*	−1.888	−3.271	18.44	−5.086	−6.276	−3.064	−0.177	0.764	−0.540
t	−0.537	−0.131							
	α_1	α_2			α_{1}_1	α_{1}_2	α_{2}_1	α_{2}_2	α_{2}_3
s	−0.004	−0.011			−0.039	−0.019	0.053	0.022	−0.000
m	0.003	−0.001			0.000	−0.000	−0.520	−0.383	−0.590
m^*	−0.003	0.003			0.000	−0.000	0.076	−0.817	0.558
y	0.001	−0.005			−0.000	0.000	−0.575	−0.183	−0.021
y^*	0.000	0.004			0.000	−0.000	−0.624	0.390	0.581
r	0.059	−0.128			0.000	0.000	0.023	−0.000	0.047
r^*	0.064	0.126			−0.000	−0.000	0.021	−0.000	0.013

Table 3.3. Estimates of I(0), I(1) and I(2) spaces

Hypothesis	β	ν	Test statistic	p-value
H_α	$(*, 1, -1, *, *, *, *, *)$	2	18.5	0.00
H_β	$(*, *, *, 1, -1, *, *, *)$	2	21.2	0.00
H_γ	$(*, *, *, *, *, 1, -1, *)$	2	28.3	0.00
$t = 0$	$(*, *, *, *, *, *, *, 0)$	2	28.5	0.00

Table 3.4. Tests for overidentifying restrictions

	s	m	m^*	y	y^*	r	r^*	t
Exclusion								
$\chi^2(1), r = 1$	0.81	4.78	0.01	19.21	1.44	15.94	1.04	14.76
$\chi^2(2), r = 2$	4.34	18.19	4.87	19.79	7.50	29.16	13.98	28.46
Weak exogeneity								
$\chi^2(2), r = 2$		7.14	5.57	3.83	2.02	14.62	19.82	–

Table 3.5 Tests for exclusion and weak exogeneity

	β_1	β_2	$\delta'\beta_2\_1$	$\delta'\beta_2\_2$	β_1_1	β_1_2	β_2_1	β_2_2	β_2_3
s	0	1	1.508	−14.24	−16.84	13.33	−0.175	−0.177	0.553
m	−1.880	9.593	−0.945	−0.277	19.47	44.54	0.066	0.151	−0.072
m^*	0	−1.183	1.781	33.75	−6.90	16.70	−0.190	−0.161	−0.821
y	−6.163	−12.49	−0.436	−2.073	−8.61	−16.90	0.129	−0.050	0.000
y^*	0	−6.203	−0.931	−6.814	40.70	90.40	0.261	−0.094	0.089
r	1	2.730	−4.465	−13.29	−16.46	−20.42	0.919	−0.027	−0.086
r^*	0	−2.730	−3.893	−9.139	−4.299	5.682	−0.017	0.953	−0.018
t	0.070	0							
	α_1	α_2			α_1_1	α_1_2	α_2_1	α_2_2	α_2_3
s	0.062	−0.020			−0.043	0.012	−0.025	0.000	0.036
m	−0.029	0.004			0.000	0.000	0.808	−0.352	0.016
m^*	0.010	0.005			0.000	0.000	0.230	0.917	0.200
y	0	0			−0.000	−0.000	0.487	0.071	−0.399
y^*	0	0			−0.000	0.000	−0.138	0.168	−0.894
r	−0.187	−0.123			0.000	−0.000	−0.000	0.043	0.000
r^*	−0.654	0.160			−0.000	0.000	−0.034	0.017	0.000

Table 3.6 Restricted estimates of I(0), I(1) and I(2) spaces

	$\Delta^2 s$	$\Delta^2 m$	$\Delta^2 m^*$	$\Delta^2 y$	$\Delta^2 y^*$	$\Delta^2 r$	$\Delta^2 r^*$
$\overline{R}^2$	0.462	0.271	0.402	0.638	0.613	0.449	0.773
$\widehat{\sigma}$	0.054	0.007	0.008	0.007	0.006	0.124	0.104
$\text{F}_{ar}(4, 69)$	[0.746]	[0.440]	[0.722]	[0.095]	[0.333]	[0.562]	[0.050]
$\text{F}_{reset}(1, 72)$	[0.690]	[0.815]	[0.677]	[0.005]	[0.462]	[0.858]	[0.001]
$\chi^2_{norm}(2)$	[0.442]	[0.020]	[0.986]	[0.097]	[0.798]	[0.000]	[0.994]
$\text{F}_{het}(1, 90)$	[0.953]	[0.572]	[0.273]	[0.021]	[0.976]	[0.800]	[0.268]

Table 3.7. PVAR. Diagnostics

	Unrestricted VAR						
	s	m	m^*	y	y^*	r	r^*
1976Q4-1994Q4							
Mean	0.000	0.000	0.000	0.000	0.000	0.000	0.000
RMSE	0.044	0.005	0.006	0.005	0.004	0.083	0.072
1995Q1-1999Q3							
Mean	−0.052	0.037	−0.218	0.024	0.014	−0.070	0.525
RMSE	0.058	0.046	0.263	0.033	0.026	0.275	0.580

Table 3.8. UVAR. Forecast Performance

	DDVAR						
	$\Delta^2 s$	$\Delta^2 m$	$\Delta^2 m^*$	$\Delta^2 y$	$\Delta^2 y^*$	$\Delta^2 r$	$\Delta^2 r^*$
1976Q4-1994Q4							
Mean	0.000	0.000	0.000	0.000	0.000	0.000	0.000
RMSE	0.070	0.008	0.008	0.009	0.007	0.139	0.143
1995Q1-1999Q3							
Mean	−0.000	−0.001	−0.000	−0.001	−0.001	−0.021	−0.005
RMSE	0.047	0.007	0.009	0.004	0.006	0.077	0.089

Table 3.9. DDVAR. Forecast Performance

	PVAR						
	$\Delta^2 s$	$\Delta^2 m$	$\Delta^2 m^*$	$\Delta^2 y$	$\Delta^2 y^*$	$\Delta^2 r$	$\Delta^2 r^*$
1976Q4-1994Q4							
Mean	0.000	0.000	0.000	0.000	0.000	0.000	0.000
RMSE	0.051	0.007	0.007	0.007	0.005	0.121	0.098
1995Q1-1999Q3							
Mean	0.007	0.001	−0.003	0.003	−0.003	−0.000	−0.059
RMSE	0.034	0.006	0.009	0.005	0.007	0.060	0.098

Table 3.10 PVAR. Forecast Performance

Figure 3.1(a). UVAR. s_t actual (—) and forecast (- - -)

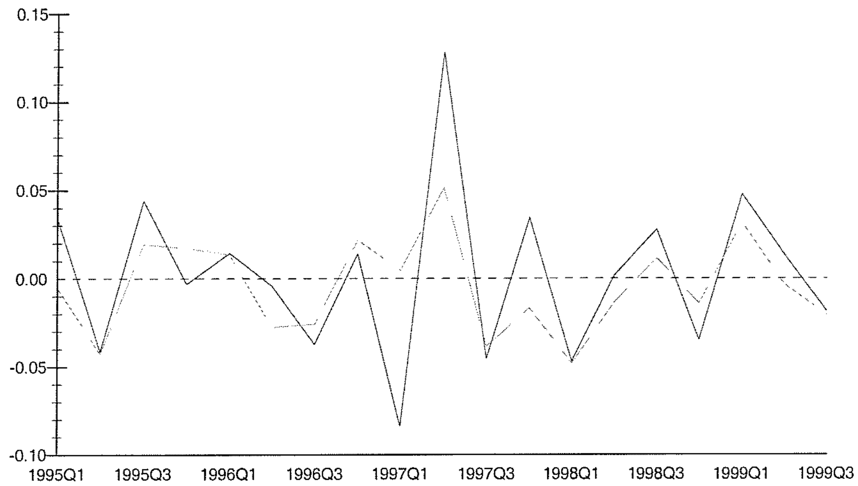

Figure 3.1(b). PVAR. s_t actual (—) and forecast (- - -)

Chapter 4

Testing Cagan's Money Demand Model Over The German Hyperinflation

4.1 Introduction

The 1920s are of particular interest to the applied economist since they mark the first hyperinflationary period where reliable data exist to test a range of economic hypotheses. In particular, following Cagan (1956) a great deal of attention has been paid to modelling the demand for money during the German hyperinflation period. The early literature (see Sargent and Wallace [1973], Sargent [1977], Frenkel [1977, 1979], Abel *et al.* [1979], Salemi [1979] and Salemi and Sargent [1979]) investigated the conditions required to identify the semi-elasticity parameter on expected inflation. These studies culminated in Salemi and Sargent [1979] who showed that three assumptions were required for identification, namely the exogeneity of the money supply, rational expectations on behalf of economic agents and random walk velocity shocks to money demand. Following Salemi and Sargent [1979] studies on the German hyperinflation period concentrated on testing for the presence of price bubbles[1] in the data. While early studies were inconclusive (*c.f.* Flood and Garber [1980] and Burmeister and Wall [1982]) later studies showed that when the assumptions on the exogeneity of money and the random walk for the velocity shock are relaxed the presence of bubbles is generally rejected and thus it is summised that market fundamentals were responsible for driving the German hyperinflation (see Casella [1987], Imrohoroglu [1993])[2]. The more recent empirical literature has focused on testing the long run properties of the Cagan model through cointegration (see Taylor [1991], Engsted [1993, 1994] and Michael *et al.* [1994]). Taylor [1991] shows that by simply assuming that agents expectational errors are stationary one can formulate the Cagan model as a cointegrating regression between real money balances and inflation[3]. Furthermore, if the hypothesis of cointegration holds then a superconsistent estimate of the elasticity parameter is obtained that is robust to simultaneity or omitted variables bias. The studies cited tend to find support for a cointegrating relation amongst the variables in the Cagan model though Taylor [1991] and Michael *et al.* [1994] find that the model needs to be augmented by other economic variables (for a discussion see below).

This Chapter extends these studies by re-examining Cagan's money demand model in light of recent work on modelling $\mathsf{I}(2)$ variables i.e. those variables which require differencing twice to achieve stationarity. In particular, we provide the first test of long run price homogeneity that is imposed in Cagan's model and also show that the model requires

[1]Flood and Garber [1980] define a price bubble where "self-fulfilling expectations of price changes drive actual price changes independently from market fundamentals" (*op. cit.* p.??).

[2]There is considerable evidence in support for the endogeneity of the money supply over the German hyperinflation period (see Sargent and Wallace [1973], Sargent [1977], Salemi and Sargent [1979] and a more recent treatment by Vazquez [1994]).

[3]Engsted [1993, 1994] shows that by imposing the further assumptions of rational expectations, no bubbles and stationary velocity shocks a further cointegrating relationship between real money balances and money growth can be derived.

reinterpretation as a polynomial long run cointegrating relation in order to produce a stable equilibrium relation over the German hyperinflation

The structure of this Chapter is as follows. Below we outline Cagan's [1956] model of money demand during hyperinflation and discuss the extensions to the model that we employ in the empirical analysis. In section 4.2 we present our analysis of the Cagan model within a cointegrated I(2) model. After testing for the integration indices we present tests on certain parameter restrictions suggested by the model and finally discuss the final form of the cointegrating relations and common trends within the system. Section 4.3 concludes.

4.1.1 The Demand For Money During Hyperinflation

Our model of money demand during the German hyperinflation follows that of Cagan [1956] who posits that real money balances in period t is related to the expected rate of inflation in period $(t+1)$ and a stochastic variable that represents shocks to money demand or velocity such that

$$(m-p)_t = \alpha - \beta \mathsf{E}_t(p_{t+1} - p_t) + \nu_t \tag{4.1}$$

where $(m-p)$ denotes real money demand, p corresponds to the price level and ν is the shock variable. To motivate the following discussion of alternate time series interpretations note that we can rewrite Eq.(4.1) as

$$(m-p)_t = \alpha - \beta^{-1}\Delta p_{t+1} + \beta^{-1}\eta_{t+1} \tag{4.2}$$

where $\eta_{t+1} = (p_{t+1} - \mathsf{E}_t p_{t+1}) - \beta^{-1}\nu_t$ represents the serially uncorrelated rational expectations error in period $(t+1)$. The extant empirical literature suggests that both money balances and prices are well described as I(2) variables. If this is true then it is clear that Eq.(4.2) defines a cointegrating regression such that testing the stationarity of η_{t+1} becomes a test of cointegration between real money balances and inflation. If this test is not rejected then the parameter of interest β, corresponding to the semi-elasticity of real money demand with respect to expected inflation, is estimated super-consistently (Stock [1987]) and is robust to simultaneity and omitted variable bias (Engle and Granger [1987]).

In early studies of Cagan's model of money demand during hyperinflation it was assumed that velocity shocks followed a random walk such that $\nu_t = \nu_{t-1} + \varepsilon_t$ as a result of the strong serial correlation observed in ν_t and also to facilitate tests for adaptive and rational expectations. It is clear from the preceding discussion that this assumption explicitly precludes the possibility of cointegration between real money balances and expected inflation or money growth, perhaps indicating the omission of an I(1) variable in the cointegrating regression.

One such variable to be proposed in the literature (Frenkel [1977]) is the expected rate of currency depreciation which is usually proxied by the forward foreign exchange premium. The subsequent literature has proposed three rationales for the inclusion of the foreign exchange premium. First, the premium is a natural proxy for the expected rate of inflation in that, following the literature on purchasing power parity, prices and exchange rates should move closely together during hyperinflations. Second, it has been argues that the true form of substitutability lies not between goods and money but rather between domestic and foreign monies. Last, it may well be that both goods and external assets would be considered as alternatives to holding domestic money such that both expected inflation and the forward premium should be included in the information set that determines real money balances.

An addition variable proposed in the literature is real income. In early studies, following Cagan [1956] and Frenkel [1977], it was felt that real income could be abstracted from the set of variables that influence real money balances as the relatively small movement in incomes over the hyperinflationary period were completely swamped by the large fluctuations in real money holdings and thus would provide little explanatory power. However, Michael *et al.* [1994] argues that this skein of logic is based upon the unreliable income data that was available to the authors of these early studies. In contrast, recently available data from the International Bureau of Labour shows a halving of real income between 1921 and the end of 1923. In summary then, a general form of the Cagan model can be given as

$$(m-p)_t = \alpha - \beta \Delta p_{t+1} + \gamma y_t - \lambda (f-s)_{t-1} + u_t \tag{4.3}$$

where y_t denoted real income and $(f-s)_{t-1}$ corresponds to the forward foreign exchange premium. Below we provide an analysis of Eq.(4.3) though we relax the restriction of long run price homogeneity. This now becomes a testable hypothesis on the parameter space in the $\mathsf{I}(2)$ model following Kongsted [2000].

4.2 Empirical Analysis

4.2.1 The Data

In this Chapter we use the standard database for analysing the German hyperinflation period beginning in February 1921 (the start of the float after the gold standard) and ending in June 1923. The data on nominal money balances (denoted m), wholesale prices (p), the index of skilled public workers real wages (w) and the expected depreciation in the exchange rate (defined as the ratio of the forward over the spot exchange rate and thus denoted $(f - s)$) are taken from Michael *et al.* [1994, p. 20].

4.2.2 Double Unit Root Tests

When analysing the Cagan model of hyperinflation it is a distinct possibility that some of the variables are explosive. This is a particular problem when analysing I(2) systems since in small samples the properties of explosive processes mimic those of I(2) series (see Haldrup [1998]). In light of this we employ three tests for double unit roots that have power against the explosive alternative. These are the parametric tests of Hasza and Fuller [1979] and Sen and Dickey [1987] together with the semi-parametric test of Shin and Kim [1999].

The Hasza-Fuller test uses the generalised ADF auxiliary regression

$$\Delta^2 x_t = (\hat{\alpha}_1 - 1)x_{t-1} + (\hat{\alpha}_2 - 1)\Delta x_{t-1} + \sum_{j=1}^{p-2} \hat{\theta}_j \Delta^2 x_{t-j} + u_t \tag{4.4}$$

where the joint hypothesis $H_0 : \alpha_1 = \alpha_2 = 1$ corresponds to x_t being I(2). The joint hypothesis is tested by a standard two-sided $F-$test which provides power against x_t being I(0), I(1) or explosive.

Sen and Dickey [1987] consider a symmetric version of the Hasza-Fuller $F-$test given by the SURE regression model

$$\Delta^2 x_t = (\hat{\alpha}_1 - 1)x_{t-1} + (\hat{\alpha}_2 - 1)\Delta x_{t-1} + \sum_{j=1}^{p-2} \hat{\theta}_j \Delta^2 x_{t-j} + u_t, \quad t = p+1, \ldots, n$$

$$\Delta^2 x_t = (\hat{\alpha}_1 - 1)x_{t-1} - (\hat{\alpha}_2 - 1)\Delta x_{t-1} + \sum_{j=1}^{p-2} \hat{\theta}_j \Delta^2 x_{t+j} + v_t, \quad t = 3, \ldots, n-p+2 \tag{4.5}$$

with the constraint that the parameters α_1, α_2 and θ_j are the same in each equation. The joint hypothesis $H_0 : \alpha_1 = \alpha_2 = 1$ for x_t being I(2) is again tested by a standard two-sided $F-$test. However, Sen and Dickey show that due to the test's symmetric nature power gains are achieved against stationary and explosive alternatives compared to the Hasza-Fuller test.

The final test we use is the semi-parametric test of Shin and Kim [1999]. Shin and Kim derive a semi-parametric version of the Sen-Dickey test discussed above where the jointly-estimated regressions take the form

$$\Delta^2 x_t = (\hat{\alpha}_1 - 1)x_{t-1} + (\hat{\alpha}_2 - 1)\Delta x_{t-1} + w_t, \qquad t = 3, \ldots, n$$

$$\Delta^2 x_t = (\hat{\alpha}_1 - 1)x_{t-1} - (\hat{\alpha}_2 - 1)\Delta x_{t-1} + z_t, \qquad t = 3, \ldots, n \tag{4.6}$$

i.e. Eq.(4.5) but with no lagged second differences. The specific form of the dependent and independent variables as well as the definition of the semiparametric $F-$testor is given in Shin and Kim [1999] and Haldrup [1998]. Shin and Kim show in a Monte Carlo study that their test improves on the size distortions found in other semiparametric tests such as that of Haldrup [1994] and Phillips and Perron [1988].

Table 4.1 reports the results from the double unit root tests on our variables of interest m, p, w and $(f - s)$. In order to obtain power against a range of deterministic alternatives the tests are employed with a constant, trend and quadratic trend in the regressions. The column headings in the Table determine which deterministic components were included. We find evidence in favour of the I(2) null for both nominal money and prices. This matches the results found by Haldrup [1998] using these and other tests. We also find some support for $(f - s)$ being I(2), though here the parametric and semiparametric tests disagree. Finally, the results for w are somewhat clearer suggesting that real wages are at most I(1). The results from the double unit root tests are of use for when we formally test for the integration indices of the I(2) system. In short, we find evidence for two or three I(2) and one or perhaps two I(1) components.

4.2.3 The Unrestricted VAR

We begin our investigation of the augmented Cagan model of money demand given by Eq.(4.3) by modelling the joint distribution of the variables within an unrestricted vector autoregression (VAR). The unrestricted VAR takes the form:

$$X_t = \sum_{j=1}^{k} A_j X_{t-j} + \Phi D_t + \varepsilon_t \tag{4.7}$$

where $X_t' = (m_t, p_t, w_t, (f - s)_{t-1})$ is the four-dimensional vector containing our variables of interest; D_t is a vector of deterministic variables containing a constant and a trend; and ε_t is an innovation process which is independently distributed with mean zero and variance-covariance matrix Σ.

Given the small sample size and the comparatively large information set for an analysis of this period the dimension of the VAR was restricted by assuming at most $k = 2$ lags. Estimating the unrestricted VAR over the period from February 1921 to May 1923[4]

[4]Estimation that included the last data point of June 1923 resulted in explosive roots which invalidate

resulted in the descriptive and diagnostic statistics presented in Table 4.2[5]. The diagnostics correspond to p-values of the Lagrange multiplier test of residual serial correlation against 4th-order autoregression (Godfrey [1978] denoted $F_{ar}(\cdot,\cdot)$; the RESET test of functional form (Ramsey [1969]) denoted $F_{reset}(\cdot,\cdot)$; the Jarque-Bera chi-square test of normality of regression residuals (Jarque and Bera [1980]) denoted $\chi^2_{norm}(\cdot,\cdot)$; and an equality of error-variances test denoted $F_{het}(\cdot,\cdot)$. The table shows that the restricted system is generally well-specified, though there is some evidence of non-congruency in the equation for the expected exchange rate depreciation.

The dynamic properties of the unrestricted VAR are illustrated by the moduli of the eigenvalues of the companion matrix which are given by

$$(1.000, 0.917, 0.917, 0.827, 0.827, 0.514, 0.514, 0.228) \tag{4.8}$$

where the roots of the characteristic polynomial are the inverses of these eigenvalues. The eigenvalues suggest the presence of at least three, and probably five, unit roots in the dynamic system. Evidence on the roots of the characteristic equation is used in parallel with the formal test of the integration indices below given the small sample size and that the test statistics in the formal test are derived asymptotically.

4.2.4 Cointegration Analysis

We continue by formally testing for the integration indices in the system *i.e.* the number of r cointegrating vectors together with the number of s_1 I(1) trends and thus the number of $(p - r - s_1) = s_2$ I(2) trends using the test presented in Rahbek *et al.* [1999]

Johansen's representation theorem for I(2) models (see Chapter 1.1.1).

[5] The calculations and numerical results in the text were obtained using the computer packages CATS in RATS (Hansen and Juselius [1995]) and Microfit 4.0 (Peseran and Peseran [1997]).

and discussed in detail in Chapter 1.3.1. Rahbek *et al.* [1999] show that the integration indices (r, s_1) should be determined jointly since the sequential approach of testing for the cointegration rank and then proceeding to test for the number of s_1 $\mathsf{I}(1)$ trends does not, in general, yield the correct asymptotic size for the test.

The test statistics for all combinations of r and s_1 together with the 95% quantiles of the asymptotic test distributions taken from Rahbek *et al.* [1999, Appendix C, Table 4] are presented in Table 4.3. Following Paruolo [1996] and Rahbek *et al.* [1999] the test statistics are calculated under the assumption that the data contain at most linear trends.

To determine the appropriate integration indices we start by testing the most restricted hypothesis, given by $(r, s_1, s_2) = (0, 0, 4)$, then, if this hypothesis is not rejected, we test successively less and less restricted hypotheses by continuing to the end of the first row and then by proceeding row-wise from left to right until the first rejection is found. The first near non-rejection is found for the submodel $(r, s_1, s_2) = (1, 1, 2)$ with a test statistic of 69.3 compared against the 95% critical value of 68.2. Though this particular submodel is only significant at the 10% level it does correspond to $s_1 + 2s_2 = 5$ unit roots which matches the results on the roots of the characteristic polynomial presented above. The next non-rejection corresponds to the submodel $(r, s_1, s_2) = (1, 2, 1)$ and therefore $s_1 + 2s_2 = 4$ unit roots. Though this submodel has a test statistic of 46.5 compared to the 95% critical value of 53.2 the conclusion of four unit roots does not tally with that suggested by the roots of the characteristic polynomial. Given the small sample size we conclude in favour of the submodel corresponding to $(r, s_1, s_2) = (1, 1, 2)$. Evidence in support of the need for allowing $\mathsf{I}(2)$ trends in the model can be seen by imposing $p - r = 3$ in the $\mathsf{I}(1)$ model.

In this case the first eight eigenvalues become $(1, 1, 1, 0.861, 0.779, 0.779, 0.347, 0.057)$ and three further near-unit roots appear.

4.2.5 Tests for Parameter Restrictions

We next consider tests of parameter restrictions on the various components of the model. First, we test the hypothesis of a nominal-to-real transformation following Kongsted [2000] and discussed in Chapter 1.1.3. The hypothesis corresponds to long run price homogeneity which is commonly used in empirical studies of Cagan's money demand model. The test starts by considering the validity of the restriction on β given by $\beta = (1, -1, *, *)$. We find that the restriction is strongly rejected with a test statistic given by $Q_{b1} = 7.98$ with a corresponding p-value of < 0.001. Thus, we have provided the first evidence that the imposition of long run price homogeneity in the Cagan money demand model may not be a valid restriction. In light of the rejection of long run price homogeneity we next test whether long run wage homogeneity is an acceptable transformation for our dataset. The first subhypothesis such that $\beta = (1, *, -1, *)$ is easily accepted with $Q_{b1} = 0.04$ with a corresponding p-value of 0.83. Following Kongsted [2000] we now test for the second subhypothesis such that $\beta_1 = (1, *, -1, *)$ and $\beta_2 = (1, *, 1, *)$. However, the test statistic for the second subhypothesis is given by $Q_{b2} = 11.89$ with a corresponding p-value of < 0.001. Thus, we reject the overall hypothesis of long run wage homogeneity. Our final test is for the absence of the linear trend in the cointegrating relation. The hypothesis is strongly accepted with a test statistic of $\chi^2(1) = 0.22$ with a corresponding p-value of 0.64. Thus, we exclude the linear trend from the β vector.

Our second set of tests for parameter restrictions corresponds to testing for the validity of weak exogeneity of the variables of interest for the cointegration parameters $\theta = (\beta, \beta_1, \delta)$. The form of the test is discussed fully in Chapter 1.3.4. To summarise, given the VAR model

$$\Delta^2 X_t = \alpha[\beta' X_t - \delta\beta_2' \Delta X_{t-1}] + (\varsigma_1, \varsigma_2)[(\beta, \beta_1)' \Delta X_{t-1}] + \Theta Z_t + \varepsilon_t \tag{4.9}$$

where $\Theta = (\Psi_1, \Psi_2, \ldots, \Psi_{k-2})$ and $Z_t = (\Delta^2 X_{t-1}, \Delta^2 X_{t-2}, \ldots, \Delta^2 X_{t-k+2})$ Paruolo and Rahbek [1999] show that under Johansen's representation theorem with $r > 0, s_1 > 0$ a subset of $X_t, b' X_t$ is weakly exogenous for the cointegration parameters $\theta = (\beta, \beta_1, \delta)$ if and only if

$$b'(\alpha, \alpha_1, \varsigma_1) = 0 \tag{4.10}$$

where $\varsigma_1 = \Gamma\overline{\beta}$. Choosing valid exogeneity restrictions is important since it results in a reduction in the number of short run variables in the EqCM and thus may avoid the need to model any complex stochastic properties exhibited in the marginal model due to interactions among the variables of interest. In addition, weak exogeneity tests enable the applied econometrician to learn more about the short run adjustment mechanisms within the general system. This is especially useful if the model is to be used as a tool to shed light on the short run implications of particular economic policies. The results of the tests from the sequential testing procedure are given in Table 4.4 where the individual test size is set at 1.67% to limit the overall test size at the conventional 5% level. The results provide strong evidence for m_t and marginal evidence for the other variables that the first sub-hypothesis $H_\alpha : b'\alpha = 0$ is satisfied. This would suggest that there is no level feedback from the cointegrating relation to any of our variable of interest. Moving on to the second sub-hypothesis $H_{\alpha_1} : b'\alpha_1 = 0$ we see that the restriction is rejected for m_t and $(f-s)_{t-1}$

but accepted for p_t and w_t. This suggests that the common I(2) trends are driven by the innovations to prices and wages. Finally, we turn to the sub-hypothesis $H_{\varsigma_1} : b'\varsigma_1 = 0$ where again the test accepts the hypothesis for the w_t and provides marginal support for p_t. Thus, we conclude that wages, and perhaps prices, are weakly exogenous for the cointegration parameters of the money demand model.

4.2.6 Cointegrating relations and common trends

We conclude our analysis of the Cagan money demand model within a cointegrated I(2) system by presenting the restricted parameter estimates of the cointegrating relations and common I(1) and I(2) trends following the tests of parameter restrictions above. Specifically, we impose weak exogeneity of prices and wages in α and long run wage homogeneity and the absence of a linear trend in β. Finally, imposing the submodel corresponding to $(r, s_1, s_2) = (1, 1, 2)$ yields the restricted parameter estimates in Table 4.5. The first two columns give the estimate of the polynomial cointegrating relation where we have normalised the β vector on narrow money m. Turning to the levels variables $\beta' X_t$ first, we see evidence of a strong effect from the foreign exchange rate depreciation variable while with respect to the differenced variables $\delta' \beta_2' \Delta X_t$, we see that the impact derives mainly from money, prices and wages. The adjustments to the disequilibrium error defined by the polynomial relation are presented in the first column of the bottom panel. Given the restrictions on α we have that the polynomial relation enters only the money demand and foreign exchange rate depreciation equations. Moving on to the linear combination $\beta_1' X_t$ which corresponds to a cointegrating relation that reduces the order of integration from two to one, *i.e.* a $\text{CI}(2, 1)$ relation, we have that money and prices virtually cancel to leave wages as the dominant component of the relation. This cancelling effect is also present in the α_1 vector which provides the stochastic I(1) trend component. Though the exchange rate depreciation variable has more of an effect here. The estimates of the two common I(2) trends are given by α_{2_1} and α_{2_2} and seem to be generated by a broadly equal weighting from the twice cumulated residuals from each variable . Finally, the estimate of the weight with which the I(2) trends influence the variables of the system is given by β_{2_1} and β_{2_2}. Here we find that the I(2) trends seem to drive only money, prices and wages which would appear to identify the foreign exchange rate depreciation as an I(1) variable.

A plot of the polynomial cointegrating relation together with the CI(2,1) relation $\beta' X_t$ is given in Figure 4.1. The plot clearly shows that the polynomial relation accounts for the extreme conditions at the end of the sample in contrast to the I(1) cointegrating relation $\beta' X_t$ which drops away markedly.

4.3 Conclusion

This chapter has provided a re-examination of the Cagan model of money demand over the German hyperinflation period. By allowing for the presence of I(2) variables we show that the restriction of long run price homogeneity in the Cagan model is not supported over the German hyperinflation period. Furthermore, we find support for the need to augment the model with both wages and foreign exchange rate depreciation as suggested by Michael et al. [1994] and Frenkel [1977] respectively. A key result is that we show that the Cagan model requires reinterpretation as a polynomial long run cointegrating relation in order to produce a stable equilibrium relation over the German hyperinflation. Finally, we find support for both prices and wages being weakly exogenous for the cointegration parameters over the period.

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		lags	c	c, t	c, t, t^2
m_t	HF-$F_{\alpha_1\alpha_2}$	4	5.56	5.64	9.70
	SD-$F_{\alpha_1\alpha_2}^{sym}$	4	3.04	4.05	4.77
	SK-$Z(F_{\alpha_1\alpha_2}^{sym})$	8	2.19	7.44	16.51
p_t	HF-$F_{\alpha_1\alpha_2}$	3	3.08	3.91	12.50
	SD-$F_{\alpha_1\alpha_2}^{sym}$	3	1.31	1.41	1.94
	SK-$Z(F_{\alpha_1\alpha_2}^{sym})$	8	0.66	4.53	69.24
w_t	HF-$F_{\alpha_1\alpha_2}$	1	13.81	17.81	17.66
	SD-$F_{\alpha_1\alpha_2}^{sym}$	1	30.20	29.96	34.41
	SK-$Z(F_{\alpha_1\alpha_2}^{sym})$	8	63.58	168.61	197.48
$(f-s)$	HF-$F_{\alpha_1\alpha_2}$	2	2.82	4.04	3.82
	SD-$F_{\alpha_1\alpha_2}^{sym}$	2	9.59	9.61	10.78
	SK-$Z(F_{\alpha_1\alpha_2}^{sym})$	8	24.26	88.13	307.25

Table 4.1. Double unit root tests

	m_t	p_t	w_t	$(f-s)_{t-1}$
$\overline{R}^2$	0.999	0.997	0.456	0.923
$\widehat{\sigma}$	0.032	0.127	0.142	0.015
$F_{ar}(12, 4)$	[0.081]	[0.376]	[0.586]	[0.003]
$F_{reset}(1, 15)$	[0.318]	[0.049]	[0.651]	[0.000]
$\chi^2_{norm}(2)$	[0.776]	[0.373]	[0.722]	[0.864]
$F_{het}(1, 24)$	[0.747]	[0.862]	[0.736]	[0.710]

Table 4.2. UVAR. Diagnostics

$p-r$	r	S_{r,s_1}				Q_r
4	0	206.3	147.7	107.4	81.4	78.9
		137.0	113.0	92.2	75.3	62.8
3	1		110.3	69.3	46.5	43.6
			86.7	68.2	53.2	42.7
2	2			46.5	23.6	19.4
				47.6	34.4	25.4
1	3				8.6	5.0
					19.9	12.5
	s_2	4	3	2	1	0

Table 4.3. Formal test for integration indices

Statistic	v	m_t	p_t	w_t	$(f-s)_t$
Q_{a1}	1	0.09 [0.77]	4.96 [0.03]	2.48 [0.11]	7.02 [0.01]
Q_{a2}	1	17.1 [0.00]	1.72 [0.19]	1.55 [0.21]	19.3 [0.00]
Q_{a3}	1	-	6.35 [0.01]	0.20 [0.65]	-

Table 4.4. Tests for weak exogeneity

	βX_t	$\delta\beta_2$	β_1	β_{2_1}	β_{2_2}
m_t	1	−2.40	−5.05	0.508	−0.500
p_t	−0.54	−2.17	5.01	0.644	−0.321
w_t	−1	1.08	−1.16	0.572	0.803
$(f-s)_{t-1}$	−23.5	−0.10	−0.28	−0.018	−0.048
t	–				
	α		α_1	α_{2_1}	α_{2_2}
m_t	−0.053		−0.052	0.498	−0.295
p_t	0		0.052	−0.132	−0.888
w_t	0		−0.077	−0.719	−0.221
$(f-s)_{t-1}$	0.056		−0.049	0.466	−0.276

Table 4.5. Restricted estimates of I(0), I(1) and I(2) spaces

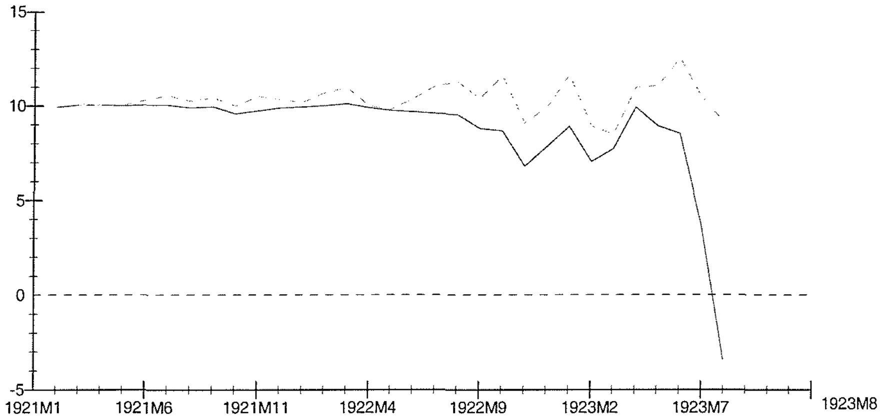

Figure 4.1. Polynomial (- - -) and CI(2,1) (—) cointegrating relations

Appendix A

Data definitions for Chapter 2

All data were downloaded from Datastream© (codes are in brackets)

s

Definition: reciprocal of the dollar-sterling spot exchange rate (USDOLLR)

m

Definition: UK M0 monetary aggregate, seasonally adjusted (UKM0....B)

m^*

Definition: US M1 monetary aggregate, seasonally adjusted (USM1....B)

y

Definition: UK real GDP at market prices (1990=100), seasonally adjusted (UKGDP...D)

y^*

Definition: US real GDP at market prices (1992=100), seasonally adjusted (USGDP...D)

r

Definition: UK 3-month London Interbank Offer Rate (LDNIB3M)

r

Definition: US 3-month Treasury Bill (FRTBS3M)