UNIVERSITY OF SOUTHAMPTON

Extensions of the AdS/CFT Correspondence

Recent developments

by

James Robert Babington

A thesis submitted for the degree of

Doctor of Philosophy

Department of Physics and Astronomy

October 2002

Dedicated to my parents

UNIVERSITY OF SOUTHAMPTON

ABSTRACT

FACULTY OF SCIENCE

PHYSICS

Doctor of Philosophy

Extensions of the AdS/CFT Correspondence
Recent developments

James Robert Babington

This thesis discusses several examples of deformations of the AdS/CFT correspondence and a direct proposal. $\mathcal{N}=4$ SYM is first studied on the Coulomb branch, where we find a simple parametrization of gauge theory operators in the corresponding gravity solution. Next we consider $\mathcal{N}=4$ SYM where a supersymmetric mass term is added to give a low energy $\mathcal{N}=2$ SYM. We try to make similar interpretations here, but find there is some difficulty in the interpretation of ultraviolet renormalization. Then a setup of wrapped branes is found to give a pure $\mathcal{N}=2$ SYM theory in the infrared. The operator parameterization works well in this case. In these $\mathcal{N}=2$ theories, we use a gauge theory result for the coupling to deduce the distribution of D branes in the transverse space. Finally a non-supersymmetric mass term is added to produce a more QCD like theory. The scalars are all given a mass which leaves pure glue in the infrared, and we successfully find an explicit gravity dual that is consistent.

Contents

List of Figures

List of Tables

Preface

The work described in this thesis was carried out in collaboration with Dr Nick Evans, James Hockings, and David Crooks. The following list details our original work and gives the references for the material.

- Chapter 2 and 3: J. Babington, N. Evans and J. Hockings, "Secrets of the metric in N = 4 and N = 2* geometries," JHEP **0107** (2001) 034 [arXiv:hep-th/0105235].
- Chapter 4: J. Babington and N. Evans, "Field theory operator encoding in N = 2 geometries," JHEP **0201** (2002) 016 [arXiv:hep-th/0111082].
- Chapter 5 J. Babington, D. E. Crooks and N. Evans, "A non-supersymmetric deformation of the AdS/CFT correspondence," arXiv:hep-th/0207076.

No claims to originality is made for the content of Chapter 1 which was compiled using a variety of other sources.

Acknowledgements

My deepest thanks must firstly go to my supervisor Nick Evans. He has given me a great deal of freedom to look at many things, including material not in this thesis and has always had the tenacity to keep on a problem. I am no longer frightened of the literature and feel confident at looking at new areas which at first sight appear daunting. Together with his encouragement and enthusiasm and the very close way which we have worked together, I feel very lucky.

Next I would like to thank James '007' Hockings and David, 'the Earl of' Crooks, my fellow student collaborators. We have learnt a great deal together and I must thank James for always listening to my rantings and David for his interest in what to work on. In addition, we have had many good beers together (mostly big ones!) and a lot of fun - Friday evenings were quite a life experience for me.

I would like to thank David 'daddy' Rayner, John Parry, Stefano 'party on' Arnone, Federico Mescia, Martin Kurth, David Lin and Oli Rosten for having a coffee time interest in what I was doing, and Dave Rayner in particular for giving much moral support.

I thank also the SHEP group for providing a very relaxed working environment and for supporting my trips to Istanbul, Berlin, and Cambridge. The group has been very generous and has facilitated my progress tremendously. I thank PPARC for providing my financial support over the last three years.

Finally I would like to thank my family; Annette and Robert my parents, Edward my brother and my grandparents for being so supportive over the years. Finally, finally I would like to thank my girlfriend Francesca for giving me all the encouragement and support I could want and the incentive to write my thesis in good time.

I wish you all very well.

> *"In the dark see past our eyes. Pursuit of truth, no matter where it lies. Gazing up to the breeze of the heavens. On a quest, meaning, reason. Come to be, how it begun. All alone in the family of the sun. Curiosity teasing everyone. On our home, third stone from the sun."*

- taken from 'Through the never' on the self titled 'Metallica' album.

Chapter 1

Introduction

1.1 Preliminaries

The AdS/CFT correspondence [4] has provided the first example of a fascinating duality between a particular strongly coupled gauge theory and a weakly coupled gravity background (to be described). It has immediately been of interest to extend the class of such dualities to other gauge theories and gravity backgrounds to understand how the duality manifests itself. A number of techniques have been used to push forward these explorations; finite temperature may be included by compactification of the time direction [5, 6], relevant deformations can be included by switching on appropriate supergravity fields that act as sources in the supersymmetric gauge theories e.g. [57], and new D brane structures with different world volume theories and their near horizon geometries may be constructed [70, 71]. It is natural to want to make a deeper investigation of some of these dualities. In principle two theories which are dual should simply be reparametrizations of the same "solution". Thus if we know the complete solution to some field theory the corresponding gravity dual should be uniquely determined. Understanding how this encoding occurs in some simple theories will hopefully lead to new tools for constructing a wider class of dualities.

The overview of this thesis is as follows. In this introductory chapter we give some motivation for studying the AdS/CFT conjecture based on 't Hoofts work. Supergravity is then discussed together with Branes that will be used throughout this work. One is required to make field theory connections so next the details of various SYM theories are given. Then, a description of how the conjecture is realised together with a discussion of using $D = 5$ supergravity. We conclude with a section on how and why brane probing is a necessary procedure.

Chapter 2 gives the first practical example of looking at a non-trivial vacuum structure of the gauge theory i.e. the $\mathcal{N}$=4 Super Yang Mills (SYM) on the Coulomb branch. We verify the D3- solution solves the field equations and check the amount of supersymmetry. Then the $D = 5$ supergravity solutions are discussed and their lift to $D = 10$. A brane probe is used to show the no-force property. We then show how the scalar vacuum expectation values (VEVS) are written in the harmonic functions and discuss the distribution of D3-branes. The specific solution found is then generalized to the whole of the moduli space.

Chapter 3 is in a similar spirit to the previous chapter. It should be thought of as complementary. In addition to that case, we are considering a supersymmetric mass term that has been added to the $\mathcal{N}$=4 theory so that in the infra red, we are left with an $\mathcal{N}$=2 theory. We also brane probe this $D = 10$ solution and deduce the gauge coupling from this. Additionally to the previous chapter, a gauge theory result that gives the gauge coupling in terms of the scalar VEVS is used to aid computing the distribution of D3-branes. In the ultraviolet one expects this to flow back to the $\mathcal{N}$=4 case, but we find additional logarithmic renormalization.

Chapter 4 is devoted to another $\mathcal{N}$=2 case. We discuss the wrapping of branes from both the field theory and gravity point of view to get the desired theory. We then solve the $D = 7$ fermion variations of supergravity variations and lift the solution to $D = 10$. Again this is brane probed and follows the same procedure as the previous chapter. We identify the gauge theory operators and rewrite the metric in terms of these and deduce the distribution of D5-branes.

Chapter 5 gives an example of a non-supersymmetric deformation. An equal mass is given to four of the scalars, whilst the two other scalars have the same tachyonic mass, twice that of the four scalars. In the infrared these decouple and we are left with a pure gauge theory. Numerical solutions are found in $D = 5$ supergravity. We construct an ansatz for the gravity fields that lifts the corresponding supersymmetry conditions in the minimal way. The field equations are then checked for consistency which is a nontrivial procedure. A brane probe is then performed and the probe potential deduced.

Chapter 6 gives a summary of what we have found and outlooks on what it would be interesting to look at next. Finally there are three appendices that would have interrupted the flow in the main text, but are interesting in themselves for some of the deeper issues only touched upon in the main body.

1.1.1 Motivation

As a way of motivating the AdS/CFT correspondence, we shall outline two fundamental ideas due to 't Hooft [11, 12], that are answered somewhat by the correspondence; namely *large-N gauge theory* [13] and *holography*. This discussion is based on [39].

As a motivation, we would like to be able to study Quantum Chromodynamics (QCD see [9]) and its implications such as how do quarks confine. Instead of working with $SU(3)$ consider working with $SU(N)$ (we now have an extra parameter that might be useful for control). One can then try to express physical quantities as an expansion in $(1/N)$. Consider Fig. 1.1 It would be nice if the residual interactions between the

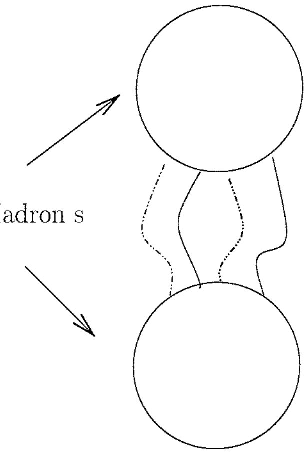

Figure 1.1: Expected behaviour of hadronic residual interactions.

hadrons behaved like $\sim (1/N)$. Then, one could separate the confinement problem from the residual hadronic interactions in the $N \to \infty$ limit. To this end let us consider a general field theory with degrees of freedom A_M that transforms in the adjoint representation of $U(N)$ (M is any type of symmetry index). In the large N limit $SU(N)$ differs from $U(N)$ by a $(1/N)$ factor and so are essentially the same (at the classical level). A general action will look like

$$S[A] = \int d^D x Tr \left((\partial A_M)^2 + g f_{MNP} A_M A_N (\partial A_P) + g^2 h_{MNPQ} A_M A_N A_P A_Q \right) \quad (1.1)$$

where g is a coupling constant and f_{MNP}, h_{MNPQ} are structure constants determining the interactions. We could also consider terms that involve derivatives. Now we consider doing a perturbation theory analysis of this theory using Feynman diagrams [9]. What is relevant for evaluating the Feynman Diagram N-dependence is

1. how g appears in $S[A]$,

2. A_M are in the adjoint representation.

Let us introduce the 't Hooft parameter $\lambda \equiv g^2 N$, such that

$$N \to \infty, \quad g^2 \to 0, \quad \lambda = fixed. \tag{1.2}$$

If we rescale the fields by $A_M \to A_M/g$ then the action Eqn. (1.1) becomes

$$S[A] = \frac{N}{\lambda} \int d^D x Tr \left((\partial A_M)^2 + f_{MNP} A_M A_N (\partial A_P) + h_{MNPQ} A_M A_N A_P A_Q \right) \tag{1.3}$$

This leads to the following properties when we evaluate a Feynman diagram;

1. a factor of (λ/N) for each propagator.
2. a factor of (N/λ) for each vertex.
3. a factor of $\sum_a \delta^a{}_a = N$ for each loop of group indices.

To evaluate the N-dependence of the diagrams we make the following definitions;

- V = number of vertices
- E = number of propagators
- F = number of group index loops

A Feynman diagram then contains the factor N^{V-E+F}, which tells us which diagrams are dominant. Since the adjoint representation can be viewed as a direct product of fundamental and anti-fundamental, a Feynman diagram becomes a *simplex* with the identifications

- F = number of faces
- E = number of edges

- V = number of vertices

Recalling Euler's Theorem [10] that

$$V - E + F = 2 - 2H \tag{1.4}$$

where H is the number of handles of the surface, one can see that a perturbative expansion is organized into a sum over *topologies*! Consider a vacuum amplitude (a diagram with no external legs); it will look like

$$A(N, \lambda) = \sum_{H=0}^{\infty} N^{2-2H} A_H(\lambda). \tag{1.5}$$

We see for $N \to \infty$, the amplitude is given by a *planar* diagram.

This is suggestive of perturbative string theory, in which amplitudes are given by a sum over two dimensional topologies [16, 18, 53]. The proposal of 't Hooft was that a large N gauge theory should admit a string description [11]. The AdS/CFT goes some way to realizing this.

We should now mention the second idea of 't Hooft [12] that is instrumental in the correspondence. From the Bekenstein formula for black hole entropy $S_{entropy} = A/4G_N$, where A is the area of the boundary of the region, and G_N is Newton's gravitational constant. One finds that it is possible to violate [43] the second law of thermodynamics. This relation implies that the degrees of freedom in the region enclosed by the area grows as the *area* and not the *volume*. One is led to a "*holographic*" principle [14, 15] which demands that for a quantum theory of gravity all the physics in the volume is to be described by degrees of freedom on the boundary. To satisfy the Bekenstein bound, there must be less than one degree of freedom per Planck area [12, 39, 43]. This then is the second feature that is incorporated into the AdS/CFT correspondence.

1.1.2 String Theory: Facts and Folklore

Having given a clear set of motivations for studying the AdS/CFT correspondence, let us now give a lightning discourse of string theory (see [16, 18, 53]). The starting point is to write down an action. The Nambu-Goto action is the natural one since it is simply the worldvolume of the string worldsheet. However this is difficult to quantize, and the classically equivalent Polyakov action

$$S[X,\psi] = \frac{1}{4\pi\alpha'}\int d^2\sigma(\eta^{ab}\partial_a X^M \partial_b X_M) + \frac{1}{4\pi}\int d^2\sigma(\bar{\psi}^M\gamma^a\partial_a\psi_M) \tag{1.6}$$

is preferred. A commentary on this action is as follows. The X^M are coordinates in the spacetime (a D-dimensional target space M^D) which are mappings from from the string worldsheet with coordinates $\sigma^a = (\sigma^1, \sigma^2)$. The metric on the worldsheet has been fixed by diffeomorphism and Weyl invariance to the conformal gauge $g_{ab} = \eta_{ab}$. This then is made supersymmetric [40] by adding in the second term to have an equal number of bosons and fermions. The ψ^M are two dimensional Weyl-Majoranna spinors, and the γ^a are 2d gamma matrices satisfying the Clifford algebra. Depending on what boundary conditions we put on these fields will determine whether the strings are open or closed. In particular closed ones must satisfy a periodicity condition $\sigma^1 \cong \sigma^1 + 2\pi$. The full action has other terms in it (such as the antisymmetric tensor B_{MN} and the 2d Ricci scalar), but these are not important for the present discussion [16, 18, 53]. The tension of the string is given by $T = 1/2\pi\alpha'$ and sets the energy scale for the problem. If this is going to be a quantum theory of gravity, then is must be the Planck scale.

Upon quantization one finds a different Hilbert space for the open and closed strings. The closed string sector contains the graviton and leads to a low energy supergravity theory, whilst the open strings contain a gauge field that is described at low energies by Super Yang Mills (SYM) theory. Depending on the boundary conditions that

	States	Fields
Bosons (NS-NS)	$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}$	Φ, B_{MN}, G_{MN}
Bosons (R-R)	$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}$	A_0, A_2, A_4
Fermions (NS-R)	$\mathbf{8}_s \oplus \mathbf{56}_s$	$\psi^{(1)}_M$, $\lambda^{(1)}$
Fermions (R-NS)	$\mathbf{8}_s \oplus \mathbf{56}_s$	$\psi^{(2)}_M$, $\lambda^{(1)}$

Table 1.1: This shows the massless bosonic and fermionic content of the IIB string theory.

are imposed on the Fermi fields (periodic or anti-periodic), one obtains the Ramond (R) sector or the Neveu-Schwarz (NS) sectors; the Ramond boundary condition gives fermionic states in spacetime. For closed string theory the Hilbert space of states consists of pairings of the left and right moving modes that gives four distinct sectors

- NS-NS and R-R are *bosonic* states.
- NS-R and R-NS are *fermionic* states.

For consistent quantization, we require the spacetime dimension $D = 10$. The field theory space consists of a finite number of massless states and an infinite number of heavy states with a scale set by α'. At this stage there is a still a problem due to a tachyon in the spectrum which the GSO projection solves and also ensures that we now have *spacetime* supersymmetry as well. By choosing the R-groundstates of left and right movers to have the same spacetime chirality, one obtains the IIB closed string theory which we shall be using throughout. It has $\mathcal{N}$=2 supersymmetry. The massless modes are then given by the covering group of the $SO(1,9)$ little group namely $Spin(8)$ which are $(\mathbf{8}_v \oplus \mathbf{8}_c) \times (\mathbf{8}_v \oplus \mathbf{8}_c)$. These states and their corresponding fields are given in Table 1.1. For the open string, the left and right moving modes are reflected into one another to produce standing waves. The massless states then are $(\mathbf{8}_v \oplus \mathbf{8}_c)$, and has the corresponding SYM fields (A_M, λ).

This concludes what we want to know from string theory, the essential point being

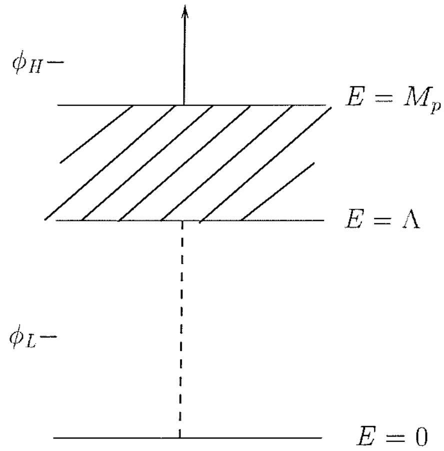

Figure 1.2: The heavy modes are integrated out, above the cutoff Λ.

that the closed IIB string theory has a low energy gravity description, whilst the open strings admit a SYM theory.

1.2 Supergravity and Branes

1.2.1 Low Energy Effective Theory

In this thesis we will be using string theory at low energies, which has its incarnation as supergravity. By this we mean the following; we know from the consistency of the superstring theory [16, 19] that we have a finite number of massless states (which includes the graviton, labeled by ϕ_L), and and infinite number of massive states starting at the Planck mass (labeled by ϕ_H). From our knowledge of effective field theory [9], we can say the following; suppose there exists a classical action $S[\phi_L, \phi_H]$. Then we can obtain a low energy effective action by integrating out the massive modes [17, 20] shown in Fig. 1.2 This is schematically given by

$$e^{-S_{eff}[\phi_L]} \sim \int [d\phi_H] e^{-S[\phi_L, \phi_H]} \tag{1.7}$$

and should be much easier to work with than the full theory. Note that if we had introduced a cutoff Λ slightly below the Planck mass, one would see there to be infinitely many terms [1] in the action. However, the immediate stumbling block which occurs is that $S[\phi_L, \phi_H]$ is not known! How then can a low energy effective action be obtained? Perturbative string amplitudes can be calculated for various processes [16, 18, 53]; if we restrict our attention to the purely massless sector, then the amplitudes

$$Amp = \langle final, \phi_L^i | initial, \phi_L^j \rangle, \tag{1.8}$$

describe the interactions between different massless particles (the i and j index the massless fields). One can then try to write down a classical action of fields which reproduce the amplitudes at the tree level. In particular, in the extreme low energy limit, the principal terms can be deduced by the symmetry of the system; gauge invariance and local supersymmetry [17, 19]; the high amount of supersymmetry completely determines the action. Following this prescription, the IIA and IIB supergravities were deduced (see bibliography in [17]). The $D = 11$ supergravity [22] obviously couldn't be obtained from these amplitudes, but by a reverse argument, suggests a yet more fundamental theory (M-theory) [21] exists of which $D = 11$ supergravity is its low energy limit. From there we could pass to IIA supergravity by compactifying on a circle [17]. At this stage, we would like to be able to write down supergravity actions which describe this low energy physics.

1.2.2 Supergravity Actions and Equations of Motion

Let us start to outline how to construct these actions from first principles. This is obviously a highly detailed process (for example, the precise form of the spin connection and four-Fermi terms etc, see [17, 23, 24]) but we can get quite far by 'building'

[1] I thank Stefano Arnone for clarifying a number of points on this.

upwards. Having got the idea of a supergravity theory from the string interactions in $D = 10$, lets first address the $D = 11$ action. Obviously we have the graviton field G_{MN}. This is the start of the local gauge invariance. Next, for supersymmetry we need to include fermions and have to be able to count spinors. In D-dimensions, the number of Dirac components is $2^{(D-1)/2}$ in odd dimensions, and $2^{D/2}$ in even dimensions (for an excellent complete treatment of spinors see [25]). These are not minimal, but are subject to Weyl and Majorana constraints in particular dimensions. These minimal spinors are used in the construction of the fermionic part of the action. So we know a minimal spinor in $D = 11$ has 32 real components [2]. In the case of the string supergravities, we know the degrees of freedom. We have to determine this for the $D = 11$ case. The spin-3/2 gravitino has 128 physical degrees of freedom, whilst the graviton has 44; 84 bosonic degrees of freedom are required, and this is incorporated in a three form A_3. We can now proceed to write down the bosonic part of the action

$$S_{11} = \frac{1}{2\kappa_{11}^2} \int d^{11}x\sqrt{G}[R - \frac{1}{2}|F_4|^2] + S_{CS}, \tag{1.9}$$

where $\kappa_{11}^2 = 8\pi G_{N(11)}$ and $G_{N(11)}$ is the $D = 11$ Newton constant. The last term (known as the Chern-Simons term) is a consequence of making the action supersymmetric; the other two terms correspond obviously to choosing a canonical form for the action. There is also a corresponding fermion action involving the Rarita-Schwinger term and a spin connection that has torsion [17].

Similarly, one can write down the supergravity actions in $D = 10$; In fact if the above

[2] One has to discuss consistency with $D = 4$. Since a supersymmetry transformation changes the spin of the particle by 1/2, we want all the helicity states to be *spin* ≤ 2 for consistency of interactions. This means that the maximum amount of *extended* supersymmetry (the number of supersymmetry generators) is $\mathcal{N} = 8$. Taking the minimal spinor in $D = 4$, having four real components, gives us 32 real spinors. If we were viewing the $D = 4$ theory as descended from higher dimensions, the constraint on the number of spinor components means that $D = 11$ is inevitably found to be the highest dimension possible.

supergravity theory is compactified on S^1, we get precisely the supergravity theory obtained from the tree level IIA string theory. This is the IIA supergravity; it has $\mathcal{N}$=2 supersymmetry and is non-chiral. In contrast, the IIB supergravity [27, 28]

$$\begin{aligned} S_{IIB} &= \frac{1}{4\kappa_{10}^2}\int \sqrt{G}e^{-2\Phi}[R+4(\partial\Phi)^2-\frac{1}{2}|H_3|^2] \\ &\quad -\frac{1}{2\kappa_{10}^2}\int \sqrt{G}[|F_1|^2+|F_3|^2+\frac{1}{2}|F_5|^2] \\ &\quad +\text{Cross Terms}+\text{Fermions}, \end{aligned} \tag{1.10}$$

(where $\kappa_{10}^2 = 8\pi G_{N(10)}$ and $G_{N(10)}$ is the $D = 10$ Newton constant) is chiral i.e. both spinors are of the same chirality. This supergravity cannot be derived from the above compactification, *but* it is T-dual to the IIA theory (see [18, 19, 53] for details on T-duality). This action contains the field strengths of the fields in Table 1.1. The R-R sectors are different in the two theories and they both have the same NS-NS sectors (which is the first line in the above). It is worth commenting on a technical issue that occurs for the 5-form, which is perhaps the most interesting case. In this case the above action is not strictly correct because the 5-form is self dual in Minkowski signature (see Eqn. (1.23) for a definition of the Hodge star)

$$F_5 = \star F_5, \tag{1.11}$$

which halves the degrees of freedom to agree with that of the string theory. This is inconsistent with the above covariant action, so the proviso is to write the action as usual and impose self-duality at the level of the field equations. See [26] for a formulation where writing a covariant action is pursued.

In the following chapters we will consider situations where only one of the forms is non-zero, so we will generally be considering actions of the form (where a_n =

$-(n-5)/2$) [35]

$$S_{10} = \frac{1}{2\kappa_{10}^2} \int \sqrt{G} \left[R - \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2n!} e^{a_n \Phi} F_n^2 \right], \tag{1.12}$$

which can be seen to be in the Einstein frame (in Eqn. 1.10, the action is in the string frame, where a factor of the dilaton multiplies the Ricci scalar. The Einstein frame is where this term is just the standard Einstein-Hilbert piece i.e. just the Ricci scalar. To put the action into this frame one re-scales the string frame metric by $G_{MN}(Einstein) = e^{-\Phi/2} G_{MN}(string)$). The important point is that it is a relatively simple system. This will give us a set of field equations

$$\begin{aligned} R_{MN} &= \frac{1}{2}\partial_M \Phi \partial_N \Phi + \\ &\quad \frac{1}{2n!} e^{a\Phi} (n F_M{}^{A_2 \cdots A_n} F_{N A_2 \cdots A_n} - \frac{n-1}{8} G_{MN} F_n^2), && (1.13) \\ \nabla_M \partial^M \Phi &= \frac{a}{2n!} F_n^2, && (1.14) \\ \nabla_M (e^{a\Phi} F^{M A_2 \cdots A_n}) &= 0, && (1.15) \end{aligned}$$

together with the Bianchi identity

$$dF_n = 0. \tag{1.16}$$

We will be interested in finding brane solutions of this system which are simple generalizations of the familiar $D = 4$ extremal Reissner-Nördstrom black holes.

1.2.3 Branes in Supergravity

A Dp-brane is a BPS object that preserves one-half of the spacetime supersymmetries and has an open string ending on it. It carries charge with respect to the $(p+1)$-form gauge potential from the R-R sector of type II superstring theory; their existence is required by non-perturbative string dualities [45]. When we consider N coincident branes the gauge group is enhanced from a $U(1) \to U(N)$ gauge theory on the world volume, reflecting the N^2 possibilities of how open strings can begin and end on the stack of branes. There are two aspects to looking at branes in string theory that we want to focus on here. Firstly how are these accounted for in a supergravity solution, their dynamics and coupling. Parallel to this, we want to know how branes enter into string theory. Let us first draw on gravitational aspects [30, 31, 32].

Unless otherwise stated, the brane solutions that are considered are flat translation-invariant static metric ansätze of the form

$$ds^2 = A^2(y^a)\eta_{\mu\nu}dx^\mu dx^\nu + B^2(y^a)\delta_{ab}dy^a dy^b. \tag{1.17}$$

Next we use the fact that the p-brane (where p refers to the number of spatial dimensions of the brane) is charged with respect to the R-R gauge potential A_{p+1}; this means the charge density Q_p of the p-brane is given by following covariant integral

$$Q_p = \int_{S^{d-1}} \star F_{p+2} \tag{1.18}$$

where d is the dimension of the transverse space and $d + p + 1 = 10$. In a brane solution which is extremal (this is when the horizon is at the origin), this charge is what will enter into the metric ansatz once it has been solved and so will give a scale to the problem. The above is the electric charge of the brane. A brane can also have magnetic charge (this is based on [29]'s account of electric and magnetic brane charges). For N p-branes we would simply have N times this basic charge.

In general, a p-brane couples to a $p+1$-form [29] with (in the language of differential forms)

$$A_{p+1} \to A_{p+1} + d\Lambda_p, \tag{1.19}$$

as the gauge transformation and

$$F_{p+1} = dA_p, \tag{1.20}$$

from which the Bianchi identity

$$dF_{p+1} \equiv 0 \tag{1.21}$$

follows immediately. In the absence of other interactions, the equation of motion for the $p+1$-form potential is

$$d^\star F_{10-(p+2)} =^\star J_{10-(p+1)}, \tag{1.22}$$

where the source J is a $p+1$-form. Here we have introduced the Hodge dual operation * which converts a $p+1$-form into a $10-(p+1)$-form, e.g.

$$(^\star J)^{M_1 M_2 \cdots M_{10-d}} \equiv \frac{1}{d!}\varepsilon^{M_1 M_2 \cdots M_{10}} J_{M_{10-d+1} \cdots M_{10}}, \tag{1.23}$$

where $\varepsilon^{M_1 \cdots M_{10}}$ is the 10-dimensional alternating symbol with $\varepsilon^{01 \cdots 9} = 1$.

Just as the usual Maxwell's equations imply the presence of an "electric" charge, i.e. a p-brane but no "magnetic" charge, we can choose to restore the duality symmetry by introducing a $(D-d-3)$-brane. We must then modify Eqn.(1.20) to

$$F_{p+2} = dA_{p+1} + \omega_{p+2}, \tag{1.24}$$

so that the Bianchi identity becomes

$$dF_{p+2} = X_{p+3}, \tag{1.25}$$

with

$$X_{p+3} = d\omega_{p+2}. \tag{1.26}$$

X may be singular

$$X_{123\cdots p+3} = \mathcal{P}_{10-(p+3)}\delta^{p+3}(y), \tag{1.27}$$

or may be smeared out so as to be regular at the origin. We then have

$$Q_p = \int\limits_{S^{10-(p+2)}} {}^*F_{10-(p+2)} = \int\limits_{M^{10-(p+1)}} {}^*J_{10-(p+1)}, \tag{1.28}$$

$$\mathcal{P}_{10-(p+2)} = \int\limits_{S^{p+2}} F_{p+2} = \int\limits_{M^{p+3}} X_{p+3}. \tag{1.29}$$

Lets now give some discussion of the fundamental dynamics of branes, with the goal of writing down the Dirac-Born-Infeld (DBI) action [18, 19, 53, 54]. It is natural to use the worldvolume of the brane in a similar fashion to the string action. Generalizing the Nambu-Goto action to p spatial dimensions gives an action proportional to the world volume

$$\begin{aligned} S_p &= -T_p \int_W d^{p+1}\xi \sqrt{-\det[\mathbf{P}(G_{MN})]} \\ &= -T_p \int_W d^{p+1}\xi \sqrt{-\det\left(G_{MN}\frac{\partial X^M}{\partial \xi^a}\frac{\partial X^N}{\partial \xi^b}\right)} \end{aligned} \tag{1.30}$$

Here, ξ^a are the coordinates on the brane with world volume W, X^M are the ten dimensional coordinates of the spacetime and $\mathbf{P}(G_{MN}) \equiv G_{ab}$ is the pullback of the metric. They in fact define an embedding into the spacetime, $X^M : W \to M^{10}$. As already mentioned, it is an electrically charged object so the Wess-Zumino (WZ) term needs to be added to describe the coupling

$$S_p = -T_p \int_W d^{p+1}\xi \sqrt{-\det[(G_{ab})]} + Q_p \int_W A_{p+1}. \tag{1.31}$$

This is quite close to the form we want, but we must recall a couple of string theory facts. First, we know that open strings end on a D-brane and therefore the low energy description is a $U(1)$ gauge field A_a. We also know from perturbative string theory that this low energy description is a $U(1)$ gauge theory so a term must be added to reproduce the gauge theory kinetic term. To fix it uniquely, one can use an argument based on T-duality invariance [18, 53] of the action to determine how it should be included. We should change the pullback to $G_{ab} \to G_{ab} + 2\pi\alpha' F_{ab}$ to achieve this. There is a further factor of the dilaton to be included since $e^{-\Phi} \sim g_s^{-1}$ occurs in the open string tree level action [18, 53] and this must coincide with the classical action. Finally we have for the bosonic part of the action [37]

$$S_p = -T_p \int_W d^{p+1}\xi e^{-\Phi} \sqrt{-\det[(G_{ab} + 2\pi\alpha' F_{ab})]} + Q_p \int_W A_{p+1} \qquad (1.32)$$

which has all the features required (in addition there can also be a pullback term B_{ab} coming from the antisymmetric tensor in string theory, but we shall omit this since it is zero for the examples we look at). It possesses various symmetries; gauge invariance and diffeomorphism invariance on the world sheet. Later we will be required to do some gauge fixing to render a Super Yang Mills (SYM) interpretation possible.

1.2.4 D branes and Open Strings

To answer the second aspect raised earlier about branes in string theory, we make the following definition. A Dp-brane is a $(p+1)$-dimensional hypersurface where open string ends are confined, that carries R-R charge Q_p. The open string ends satisfy a *Dirichlet* boundary condition [18, 19, 53]. Its low energy description is a SYM theory Fig. 1.3. In the last section we saw charges for the Dp-branes entering into the discussion. A string theory input has to be made to relate this to quantities in the string theory, so that the previous gravity solutions mean something in string theory (we have to fix the energy scale of the gravity solution in terms of the string scale α'). We now collect a number of facts.

Polchinksi's [36] calculation gives a relation between the Dp-brane charge and its

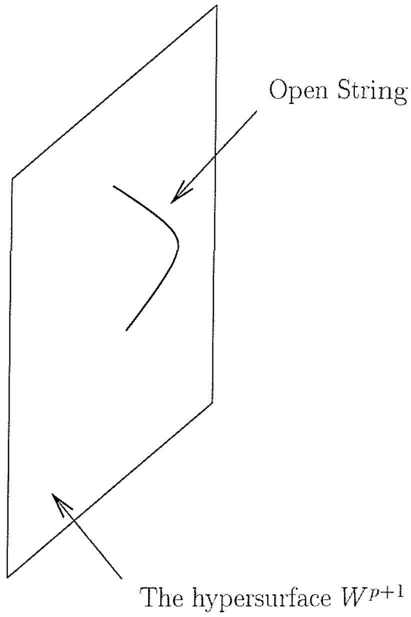

Figure 1.3: An open string ends on a Dp-brane.

tension,

$$Q_p = T_p\sqrt{16\pi G_{N(10)}} \tag{1.33}$$

and these are respectively given by a string theory calculation as

$$T_p = \frac{2\pi}{(2\pi l_s)^{p+1}g_s} \tag{1.34}$$

$$16\pi G_{N(10)} = (2\pi l_s)^8 g_s^2. \tag{1.35}$$

where g_s is the string coupling constant. Also note the following definitions which are used throughout

$$l_s^2 = \alpha' \tag{1.36}$$

$$16\pi G_{10} = 2\kappa_{10}^2. \tag{1.37}$$

The final specification in the scale is to just take account of the number of branes which implies

$$\frac{\sum Q_p}{T_p\sqrt{16\pi G_{10}}} = N. \tag{1.38}$$

1.3 Gauge Theory

In this section, we would like to make clear a set of gauge theory basics that will show up, not only in the construction of the AdS/CFT correspondence, but also in the subsequent investigations of other proposed dual solutions. Principally in this section, we would like to describe some of the properties of both $\mathcal{N}=4$ and $\mathcal{N}=2$ (SYM) gauge theory. We shall discuss field content, symmetries, operators, and ways the action can be written. This is based on [38, 39, 40, 43].

1.3.1 $\mathcal{N}=4$ super Yang-Mills theory

Non-supersymmetric 1+3 dimensional pure Yang-Mills theory is scale invariant, but it has $\beta(g) \neq 0$ (where $\beta = dg(\Lambda)/d\Lambda$ [9], with $g(\Lambda)$ the gauge coupling and Λ the renormalization group scale) at the quantum level [38, 41]. An interesting question is whether there are versions of this gauge theory that retain scale invariance even at the quantum level. The $\mathcal{N}=4$ SYM is known to have this property, so we shall now give some of the details that go into giving this remarkable result.

A natural starting point is to ask what fields go into making this SYM theory. The degrees of freedom of the theory are as follows:

i) A vector field A_μ in the adjoint representation of a gauged $SU(N)$ which is a singlet under the global symmetry $SO(6)$.

ii) Six real scalars X^i in the **6** vector representation of $SO(6)$, which transform in the adjoint representation of $SU(N)$.

iii) Four Weyl fermions λ_α^A transforming in the adjoint of $SU(N)$ and the **4** spinor representation of $SO(6)$ (corresponding to the fundamental representation of $SU(4)$, the covering group of $SO(6)$).

Notice that the bosonic and fermionic states balance on-shell. Next we should detail the symmetries of this gauge theory. It contains 16 supercharges, which under the Lorentz group transform as four spinors $(Q_\alpha^A, \bar{Q}_{\dot\beta}^A)$, $A = 1,2,3,4$, where Q_α, $\bar{Q}_{\dot\beta}$ are Weyl spinors. This gives us $\mathcal{N} = 4$ supersymmetry in $D = 4$. An $SU(4)$ rotation of the four spinors is an automorphism of the supersymmetry algebra. As a result, the Lagrangian is invariant under $SU(4)$ global transformations (R-symmetry) on the fermions, whilst the scalars transform under $SO(6)$.

The Lagrangian of the theory can be derived by dimensional reduction of $D = 10$ $\mathcal{N} = 1$ super Yang-Mills theory:

$$\mathcal{L} = -\frac{1}{2g_{\rm YM}^2} Tr[F_{MN}F^{MN}] - \frac{i}{g_{\rm YM}^2} Tr[\bar\lambda \Gamma^M D_M \lambda] \tag{1.39}$$

Here λ is a Majorana-Weyl **16** spinor of $SO(1,9)$. Upon reduction, we have the decomposition

$$SO(1,9) \to SO(1,3) \times SO(6)$$

under which

$$\mathbf{16} = (\mathbf{2},\mathbf{4}) \oplus (\bar{\mathbf{2}},\bar{\mathbf{4}})$$

The ten-dimensional gauge field gives rise to a 4d gauge field plus six scalar fields:

$$A_M = (A_\mu, X_i)\ , \qquad M = (\mu, i)\ ,$$

$$\mu = 0,1,2,3\ , \qquad i = 4,...,9\ .$$

The dimensionally reduced Lagrangian is then obtained as usual by assuming that fields depend only on x^μ. Performing this splitting of indices the Lagrangian reduces to

$$\mathcal{L} = -\frac{1}{2g^2_{\text{YM}}} Tr\Big(F_{\mu\nu}F^{\mu\nu} + 2D_\mu X_i D^\mu X_i - [X_i, X_j]^2\Big) - \frac{i}{g^2_{\text{YM}}} Tr\Big(\bar{\lambda}\Gamma^\mu D_\mu \lambda + i\bar{\lambda}\Gamma_i[X_i, \lambda]\Big) \tag{1.40}$$

It also has superconformal invariance [38]. This means that the Poincaré group is enlarged to the conformal group (we include dilatations and special conformal transformations) which in turn implies that the supersymmetry algebra is enlarged to account for these new generators. Thus we have the super-conformal group.

This action can also be written in an $\mathcal{N}$=1 formalism [42], with three chiral superfields Φ^i and a vector superfield V as

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{gauge} + \mathcal{L}_{scalar} \\ &= \text{Im} \int d^2\theta\, (\tau W^\alpha W_\alpha + H.C.) + \sum_{i=1}^{3} \int d^2\theta d^2\bar{\theta} (\text{Im}\,(\tau) \Phi_i^\dagger e^V \Phi_i) \\ &\quad - \int d^2\theta \sqrt{2}\Phi_1\Phi_2\Phi_3 + H.C. \end{aligned} \tag{1.41}$$

The τ function occurring in here is a combination of the gauge coupling and the theta angle

$$\tau = i\frac{4\pi}{g^2_{YM}} + \frac{\theta_I}{2\pi}. \tag{1.42}$$

To analyze the dynamical behavior of $\mathcal{N}$=4 theory, we look at the potential energy term,

$$V(X) = \frac{1}{2g^2_{YM}} \sum_{i,j} Tr[X^i, X^j]^2 \tag{1.43}$$

In view of the positive definite behaviour of the Cartan - Killing form on the compact gauge algebra $SU(N)$, each term in the sum is positive or zero. When the full potential is zero, a minimum is thus automatically attained corresponding to a $\mathcal{N}$=4 supersymmetric ground state. In turn, any $\mathcal{N}=4$ supersymmetric ground state is of this form,

$$[X^i, X^j] = 0, \qquad i,j = 1, \cdots, 6 \tag{1.44}$$

There are two classes of solutions to this equation,

- The *superconformal phase*, for which $\langle X^i \rangle = 0$ for all $i = 1, \cdots, 6$. The gauge algebra is unbroken. The superconformal symmetry $SU(2,2|4)$ is also unbroken. The physical states and operators are gauge invariant (i.e. $SU(N)$-singlets) and transform under unitary representations of $SU(2,2|4)$.

- The *spontaneously broken or Coulomb phase* (the Coulomb branch), where $\langle X^i \rangle \neq 0$ for at least one i. The detailed dynamics will depend upon the degree of residual symmetry. Generically, $SU(N) \to U(1)^{N-1}$, in which case the low energy behavior is that of $N-1$ copies of $\mathcal{N}=4$ $U(1)$ theory. Superconformal symmetry is spontaneously broken since the non-zero vacuum expectation value $\langle X^i \rangle$ sets a scale.

One usually speaks of branches in field theories; by having non-zero vacuum expectation values (VEVS), one can have a Higgs branch (where all the gauge bosons become massive) whilst as in this case, if some $U(1)$ subgroups remain massless we are on the Coulomb branch. The *moduli* space $\mathcal{M}$, where $V = 0$ for this theory is simply $\mathcal{M} = \mathbf{R}^6$

Conformal invariance persists even to the quantum level ($\beta(g) = 0$ to all orders). An outline of the 1-loop calculation is as follows; we know how to compute the QCD β function using dimensional regularization [9]. We also know that to obtain the $d = 4$ Lagrangian we just perform a dimensional reduction of the $D = 10$ Lagrangian. This fact lets us do the calculation simply, because we can "embed" the calculation in $D = 10$. Pretend we are doing QCD in $D = 10$ to evaluate the divergent pieces and then make the split to obtain the scalar contribution. For example, suppose we evaluate the self energy of the gauge field. We can obtain the scalar loop contribution by taking the Feynman diagram of a gluon loop, and then replacing the vector indices with $SO(6)$ indices. This amounts to replacing the spacetime metric with the $SO(6)$ metric, $\eta_{\mu\nu} \to \delta_{ab}$, and dropping any factors of momentum with vector indices. By this method we can get all the additional diagrams contributing with relative ease. In addition, the fermion contribution has to be adjusted to ensure we are working with the right type of spinors i.e. a degree of freedom count, but once this is done the β function at one loop is found to vanish [3].

1.3.2 Conformal group

The conformal group is the set of transformations that preserve the metric up to a position dependent scale factor, $g_{\mu\nu}(x) \to g_{\mu\nu}(x)\Omega^2(x)$. This group incorporates Poincaré transformations and scale transformations. The generators are the usual Lorentz generators $M_{\mu\nu}$, the Poincaré translation operators P_μ, and in addition generators D and K_μ. The conformal group is isomorphic to $SO(d,2)$, with the identification

$$\mathcal{M}_{\mu\nu} = M_{\mu\nu} \ , \qquad \mathcal{M}_{d\mu} = \tfrac{1}{2}(P_\mu - K_\mu) \ ,$$

$$\mathcal{M}_{\mu(d+1)} = \tfrac{1}{2}(P_\mu + K_\mu) \ , \qquad \mathcal{M}_{d(d+1)} = D \ .$$

The conformal (scaling) dimension Δ of an operator $\mathcal{O}(x)$ is dictated by the transformation rule under scaling of coordinates:

$$D: \qquad x^\mu \to \lambda x^\mu \ , \quad \mathcal{O}(x) \to \mathcal{O}'(x) = \lambda^\Delta \mathcal{O}(\lambda x)$$

[3] I thank Nick Evans for suggesting calculating this β-function some time ago, and being reminded of it as one of the problems in [38]

To combine the conformal algebra with the supersymmetry algebra, additional fermionic generators $\tilde{Q}$ must be included, which arise from $[K_\mu, Q] \sim \tilde{Q}$. As a result, the number of fermionic generators in the superconformal algebra is doubled with respect to the non-conformal case. For example, for a field theory with particles of spin ≤ 1, the maximal number of supercharges of the supersymmetry algebra is 16, and the maximal number of fermionic generators in a superconformal field theory is 32. All structure relations are rather straightforward, except the relations between the supercharges, which we now spell out. To organize the structure relations, it is helpful to make use of a natural grading of the algebra given by the scaling dimension of the generators,

$$
\begin{aligned}
[D] = [L_{\mu\nu}] = [T^A] = 0 \qquad & [P^\mu] = +1 \qquad [K_\mu] = -1 \\
& [Q] = +1/2 \qquad [S] = -1/2
\end{aligned}
\tag{1.45}
$$

Thus, we have

$$
\begin{aligned}
\{Q^a_\alpha, Q^b_\beta\} &= \{S_{\alpha a}, S_{\beta b}\} = \{Q^a_\alpha, \bar{S}^b_{\dot{\beta}}\} = 0 \\
\{Q^a_\alpha, \bar{Q}_{\dot{\beta} b}\} &= 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta_b{}^a \\
\{S_{\alpha a}, \bar{S}^b_{\dot{\beta}}\} &= 2\sigma^\mu_{\alpha\dot{\beta}} K_\mu \delta_a{}^b \\
\{Q^a_\alpha, S_{\beta b}\} &= \epsilon_{\alpha\beta}(\delta^a_b D + T^a_b) + \frac{1}{2}\delta^a_b L_{\mu\nu}\sigma^{\mu\nu}_{\alpha\beta}
\end{aligned}
\tag{1.46}
$$

It is now appropriate to make some definitions of the types of operators that one encounters in classifying representations of the superconformal algebra

Definition 1 *A Conformal Primary Operator is the lowest dimension operator that satisfies* $[K_\mu, \mathcal{O}] = 0,\ \mathcal{O} \neq 0$.

Definition 2 *A Super-Conformal Primary Operator (a Chiral Primary Operator) is* *the lowest dimension operator that satisfies* $[S, \mathcal{O}] = 0, \ \mathcal{O} \neq 0$.

Note that the conformal primary operators are defined by a weaker condition.

It is instructive to have explicit forms for the superconformal primary operators in $\mathcal{N} = 4$ SYM. The construction is most easily carried out by using the fact that a superconformal primary operator is not the Q-commutator of another operator. Thus, a key ingredient in the construction is the Q transforms of the canonical fields. We shall need these here only schematically,

$$\begin{aligned} \{Q, \lambda\} &= F^+ + [X, X] \qquad & [Q, X] &= \lambda \\ \{Q, \bar{\lambda}\} &= DX \qquad & [Q, F] &= D\lambda \end{aligned} \tag{1.47}$$

A local polynomial operator containing any of the elements on the right hand side of the above structure relations cannot be primary. As a result, chiral primary operators can involve neither the gauginos λ nor the gauge field strengths $F^{\pm}$. Being thus only functions of the scalars X, they can involve neither derivatives nor commutators of X. As a result, superconformal primary operators are gauge invariant scalars involving only X in a symmetrized way.

The simplest are the *single trace operators*, which are of the form

$$Tr\Big(X^{i_1} X^{i_2} \cdots X^{i_n}\Big) \tag{1.48}$$

where i_j, $j = 1, \cdots, n$ stand for the $SO(6)_R$ fundamental representation indices. Here, "Tr" denotes the symmetrized trace over the gauge algebra and as a result of this operation, the above operator is totally symmetric in the $SO(6)_R$-indices i_j. In general, the above operators transform under a reducible representation (namely the symmetrized product of n fundamentals) and irreducible operators may be obtained by isolating the traces over $SO(6)_R$ indices. Since $TrX^i = 0$, the simplest operators are

$$\begin{aligned} \sum_i TrX^iX^i &\sim \text{ Konishi multiplet} \\ sTrX^iX^j &\sim \text{ supergravity multiplet} \end{aligned} \tag{1.49}$$

where sTr stands for the traceless part only.

More complicated are the *multiple trace operators*, which are obtained as products of single trace operators. Upon taking the tensor product of the individual totally symmetric representations, we may now also encounter (partially) anti-symmetrized representations of $SO(6)_R$. There is a one-to-one correspondence between chiral primary operators and unitary superconformal multiplets, and so all state and operator multiplets may be labeled in terms of the superconformal chiral primary operators.

The unitary representations of the superconformal algebra $SU(2,2|4)$ may be labeled by the quantum numbers of the bosonic subgroup, listed below,

$$\begin{array}{ccccc} SO(1,3) & \times & SO(1,1) & \times & SU(4)_R \\ (s_+, s_-) & & \Delta & & [r_1, r_2, r_3] \end{array} \tag{1.50}$$

here $s_\pm$ are positive or zero half integers, Δ is the positive or zero dimension and $[r_1, r_2, r_3]$ are the Dynkin labels of the representations of $SU(4)_R$.

Two and three-point correlation functions of primary fields are entirely determined by conformal symmetry. For example, for an operator $\mathcal{O}$ with dimension Δ one finds

$$\langle \mathcal{O}(x)\mathcal{O}(x')\rangle = const. \frac{1}{|x-x'|^{2\Delta}}.$$

1.3.3 $\mathcal{N}$=2 Super Yang-Mills Theory

We will be interested in looking at the duals of $\mathcal{N}$=2 SYM. Just as for the $\mathcal{N}$=4 theory we can enumerate and classify all the fields in a similar manner. This system has 8 supercharges. The global R-symmetry group is now $SU(2)_R \times U(1)_R$ so the fields are classified as follows:

i) A vector field A_μ in the adjoint representation of a gauged $SU(N)$ which is a singlet under $SU(2)_R \times U(1)_R$.

ii) A complex scalar X in the fundamental representation of $U(1)_R$, which transform in the adjoint representation of $SU(N)$.

iii) 2 Weyl fermions λ^A_α transforming in the adjoint of $SU(N)$ and the **2** spinor representation of $SU(2)_R$.

Their degrees of freedom (bosonic and fermionic) match on shell. We can assemble these into a vector superfield V and a single chiral superfield Φ. The Lagrangian looks like in the $\mathcal{N}$=1 superspace formalism

$$= \frac{1}{4\pi}\text{Im}\left(\tau\int d^2\theta W^\alpha W_\alpha + H.C. + \int d^2\theta d^2\bar{\theta}\Phi^\dagger e^{2V}\Phi\right). \tag{1.51}$$

There is a similar moduli space here as for the $\mathcal{N}$=4 case given by the potential energy

$$V(X) = \frac{1}{2g^2_{YM}} Tr[X^\dagger, X]^2 \tag{1.52}$$

vanishing. Since we have two scalars the moduli space is simply $\mathcal{M} = \mathbf{C}$. Note that there are two different versions of this theory. There is the standard $\mathcal{N}$=2 SYM which is as above, and the $\mathcal{N} = 2^*$ theory were a term $m^2(\Phi_1^2 + \Phi_2^2)$ is added to the $\mathcal{N}$=4 theory. This then flows to an infrared $\mathcal{N}$=2 theory at low energy, and we also recover the standard $\mathcal{N}$=2 theory as $m \to \infty$. One should be careful to make this distinction. The size of the 't Hooft coupling has important consequences for the low energy theory. When it is large, the strong interaction scale at which we get the low energy theory is comparable to m, whereas when the coupling is small there is logarithmic running and the strong scale is much smaller than m. This means that the two theories are not equivalent.

1.4 Statement of the AdS/CFT Correspondence

Having collected some background thoughts, we can now discuss the AdS/CFT correspondence. In this section it will be essential to discuss some features that go into the correspondence and what makes it useful. This is based on [43, 44, 38].

As we have seen in Section(1.2.3), Dp-branes are $(p+1)$-dimensional hypersurfaces where open strings can end. They have a tension that behaves like $T_p \sim 1/g_s$. There are two descriptions of this system that show different facets of it.

- The massless spectrum of the open strings living on the world-volume is a $U(1)$ gauge theory in $(p+1)$-dimensions.
- Dp-branes can be considered as embeddings of p-brane solutions into supergravities, (i.e. *into* string theory) as they carry the same RR-charges.

Lets consider type IIB string theory in flat $D = 10$ Minkowski space with a stack of N parallel D3-branes that are very close to each other. As remarked earlier, if we consider the system at low energies ($\ll 1/l_s$) only massless string states can be excited and an effective Lagrangian description can be given in terms of two types of excitations:

- Massless closed string states living in the bulk $\rightarrow$ type IIB Supergravity.

- Massless open string states → $\mathcal{N}$=4, $U(N)$ SYM.

With this we can write down a schematic action [4]

$$S[effective] = S[bulk] + S[brane] + S[interaction]. \tag{1.53}$$

The bulk action is the type IIB supergravity + higher order derivatives, whilst the brane action is the SYM + higher order corrections. The massive modes can be figuratively thought of as having been integrated out. Consider a graviton fluctuation $G_{MN} = \eta_{MN} + \kappa h_{MN}$ that we put into the bulk action to get schematically

$$S[bulk] = \frac{1}{2\kappa^2}\int \sqrt{G}R \sim \int [(\partial h)^2 + \kappa h(\partial h)^2 + \cdots]. \tag{1.54}$$

For the brane action one has in similar fashion

$$S[brane] = S[\mathcal{N} = 4\ SYM] + \text{higher derivative corrections} \times \alpha'^n. \tag{1.55}$$

Now take the low energy limit $l_s \to 0$ (so that $\alpha' \to 0$, $\kappa \to 0$). This is done whilst keeping all the dimensionless parameters fixed, so that all higher order terms and interactions vanish. This decouples the system into two; free gravity in the bulk and 1 + 3 SYM living on the brane. Now let us consider this same system in the supergravity description. Firstly, write down the extremal D3 solution (where the horizon is at $r = 0$)

$$\begin{aligned} ds^2 &= H^{-1/2}(\eta_{\mu\nu}dx^\mu dx^\nu) + H^{1/2}(dr^2 + r^2 d\Omega_5^2), && (1.56)\\ H &= 1 + \frac{L^4}{r^4}, && (1.57)\\ A_4 &= H^{-1}dx^0 \wedge \cdots dx^3, && (1.58) \end{aligned}$$

[4] In reality the metricenters the brane action as couplings, so the separation of the action is heuristic.

where

$$L^4 = 4\pi g_s N \alpha'^2 \tag{1.59}$$

$$NQ_p = \int_{S^5} \star F_5. \tag{1.60}$$

The number of branes is $N \in \mathbf{Z}$ and is subject to a Dirac quantization condition. This expression then gives the length scale L in terms of the RR charge; going further we can use the the relation Eqn. (1.33) to put in the string content giving Eqn. (1.59). Since the SYM gauge coupling is identified with the string coupling $g_s = 2\pi g_{YM}^2$, we see that the length scale is given in terms of the 't Hooft parameter $\lambda = g_{YM}^2 N$. Note the choice of asymptotics in the solution - for $r \gg L$ the solution returns to flat $D = 10$ spacetime. We also see there is a large red-shift factor for an observer at infinity, due to the G_{00} component having a non-trivial r-dependence. Let E_0 be the energy of an object measured by an observer at $r = r_0$ and E_∞ be the energy of an object measured by an observer at infinity. Then the red-shift factor between the two energies is

$$E_\infty = \sqrt{G_{tt}} E_0 = H^{-1/4} E_0 \sim \frac{r}{l_s} E_0 \sim \frac{r}{l_s^2}(E_0 l_s). \tag{1.61}$$

This implies that an excitation with an arbitrary energy at $r = 0$ will look massless for an observer at infinity. Therefore, at low energies there are *two* regions of excitations; massless excitations in the asymptotically flat space which are described by supergravity, and excitations *near* the horizon. Any excitation close to the horizon will look like a low energy fluctuation to an outside observer because of this red-shift factor. The low energy fluctuations on the D3-branes are determined by the SYM theory. We now want to take the same low energy limit ($l_s \to 0$) in the metric in such a way as to retain some interesting structure. One can keep

$$\frac{r}{\alpha'} \equiv u \tag{1.62}$$

fixed so that $r \to 0$ and $H \to L^4/r^4$. The coordinate u is kept fixed because we

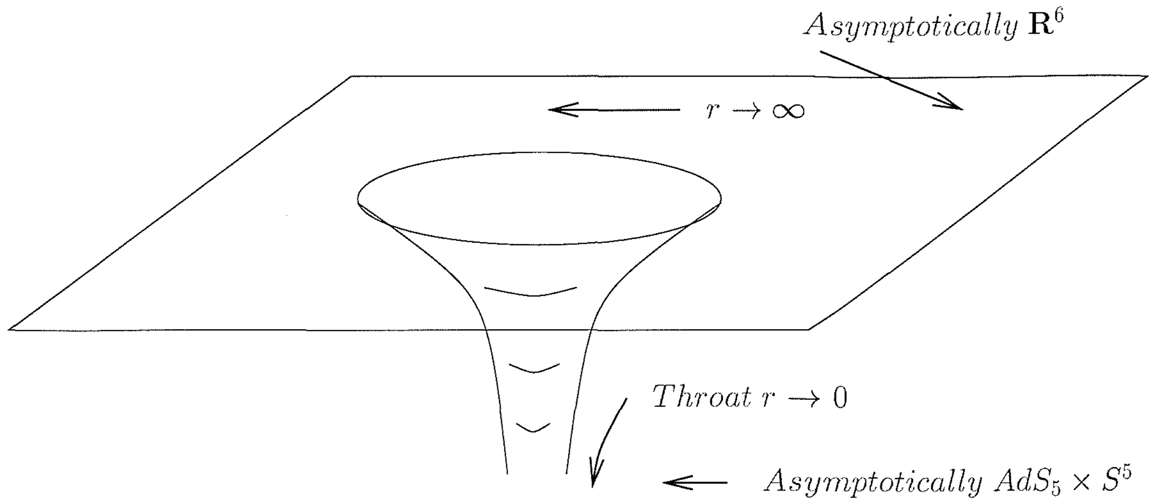

Figure 1.4: The different asymptotic's of the D3 brane solution.

want to keep fixed the energies of the objects in the throat region $E_0 l_s$ =fixed which implies that E_∞ is finite. Putting this limit into the metric gives

$$ds^2 = \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_5^2 \tag{1.63}$$

which is precisely $AdS_5 \times S^5$ (both with a length scale L)! See Appendix A.1 for a discussion of various aspects of AdS spaces. If we put this in the u-coordinates we have

$$ds^2 = \frac{u^2 l_s^2}{\sqrt{4\pi\lambda}}\eta_{\mu\nu}dx^\mu dx^\nu + \sqrt{4\pi\lambda}\, l_s^2 (\frac{du^2}{u^2} + d\Omega_5^2). \tag{1.64}$$

It is also worthwhile to mention the absorption cross-section [44] for a scalar particle. When it comes from infinity, it is found that $\sigma \sim E^3$, and hence the cross-section vanishes as we go to lower and lower energies. If one examines the potential energy of a particle in this "near horizon" limit, the potential barrier becomes very high so that the modes inside the horizon can't get out. This implies that at low energies the supergravity in the bulk and the fluctuations near the horizon decouple from one another.

The above discussion makes it clear what decouples from what in the the two different descriptions in Fig 1.4. We are led to conclude that in the string description

$$(\text{SUPERGRAVITY in the BULK}) + (\mathcal{N}{=}4,\ SU(N)\ \text{SYM})$$

is exactly equivalent to the supergravity description

$$(\text{SUPERGRAVITY in the BULK}) + (\text{Excitations near } AdS_5 \times S^5).$$

Therefore subtracting out the wordy expressions above we are led to Maldacena's conjecture [4] (the AdS/CFT correspondence) that

$$(\mathcal{N}{=}4,\ SU(N)\ \text{SYM}) \equiv (\text{Type IIB String Theory on } AdS_5 \times S^5).$$

This statement immediately implies a number of features. The gravity theory here is 5-dimensional and is being described in terms of a boundary 4-dimensional field theory - it is a holographic description (however a cutoff should be introduced in the fifth direction [46] in order for the action to remain finite). The gauge theory is defined non-perturbatively, so this can be thought of as a non-perturbative formulation of M-theory. We could also have chosen to look at $M2$ and $M5$ branes in $D = 11$ supergravity in which case we would have found spaces of the form $AdS_7 \times S^4$ and $AdS_4 \times S^7$. We could also replace the S^5 with a compact positive curvature Einstein space X^5; the effect of doing this would be to reduce the amount of supersymmetry preserved. The perturbative description of the SYM will be reliable when $g^2_{YM}N \ll 1$, whilst the supergravity description is valid when $g_s N \gg 1$. This is because for supergravity to be valid, the length scale must satisfy $L \gg l_s$ so that we can't resolve the string length. This means necessarily we are working at large 't Hooft coupling. These two different regions of couplings do not intersect, so there is no apparent contradiction. It is a strong/weak coupling duality and so a direct comparison of correlation functions is not possible, since one can't use perturbation theory at strong coupling. In addition since the string theory is perturbatively defined, it would not be possible to prove this conjecture in the manner of a standard mathematical theorem.

What is important, as evidence in support of the conjecture is that the symmetries match on both sides

1. Both the type IIB and the SYM have an $SL(2, \mathbf{Z})$ duality group.
2. The number of supersymmetries (32 real spinor parameters), The D3-solution has 16, as the brane halves the supersymmetry. This is enhanced to 32 as $r \to 0$ as a property of the AdS space, and the gauge theory acquires 16 more which are the superconformal charges,
3. The $SO(6)$ isometry of the S^5 corresponds to the $SU(4)$ R-symmetry of the SYM.
4. The $SO(2,4)$ isometry of the AdS_5 corresponds to the conformal symmetry group of the SYM, $SO(2,4) \cong SU(2,2)/\mathbf{Z_2}$

The AdS/CFT conjecture can be interpreted at different levels of strength. At the weakest level one would say that the gravity theory is equivalent to SYM for large $g_s N$. A slightly stronger version of this would be to say that it is valid for $g_s N = finite$, but $N \to \infty$ limit holding as well. The strongest form says that the two theories are exactly equivalent for all values of g_s and N (see Table 1.2).

At this point in setting up the correspondence we have a proposal for a duality symmetry, but lack a specification of how to compute quantities in this picture. This was made precise in [5, 8], where an identification between the partition function of the string theory with appropriate boundary conditions was made with the generating functional of connected Green's functions of the corresponding CFT,

$$Z_{string}[\phi(\partial AdS) = \phi_0] = \langle \exp \left(\int d^4x \phi_0(x) \mathcal{O}(x) \right) \rangle_{CFT} \tag{1.65}$$

• $\mathcal{N}=4$ conformal SYM all N, g_{YM} • $g_s = g_{YM}^2$	$\Leftrightarrow$	• Full Quantum Type IIB string theory on $AdS_5 \times S^5$ • $L^4 = 4\pi g_s N \alpha'^2$
• 't Hooft limit of $\mathcal{N}=4$ SYM $\lambda = g_{YM}^2 N$ fixed, $N \to \infty$ • $1/N$ expansion	$\Leftrightarrow$	• Classical Type IIB string theory on $AdS_5 \times S^5$ • g_s string loop expansion
• Large λ limit of $\mathcal{N}=4$ SYM (for $N \to \infty$) • $\lambda^{-1/2}$ expansion	$\Leftrightarrow$	• Classical Type IIB supergravity on $AdS_5 \times S^5$ • α' expansion

Table 1.2: The three forms of the AdS/CFT conjecture in order of decreasing strength

where ∂AdS_5 is the AdS boundary. A CFT does not have asymptotic states nor an S-matrix so the natural objects to consider are its operators. When the supergravity limit holds we have

$$Z_{string}(\phi_0) = \exp(-S_{supergravity}(\phi_0)) \tag{1.66}$$

This prescription immediately allows one to calculate CFT correlation functions; simply take functional derivatives on the supergravity partition function with respect to ϕ_0 and then set them to zero. One might wonder when using the IIB theory, that there might be some problem with a lack of a covariant action. This is not troublesome, since to obtain n-point Green's functions we regard these as coming from $(n-1)$th variation of the covariant field equations. Also note that only relative scales are meaningful; α' is not really a parameter, it simply sets the scale in string theory.

Let us also remark on some tests of the conjecture. We have already talked about the symmetries matching. In addition there are some correlation functions which are protected from quantum corrections and do not depend on λ. There is also the spectrum of chiral operators and the qualitative behaviour of the theory upon deformations (see [43] for general references on this).

There are five consistent string theories which are related to one another by various duality symmetries [45]. These are thought of as a moduli space of a yet more fundamental *M-theory* which lives in $D = 11$. The AdS/CFT hints at a possible way to define theses theories in a non-perturbative fashion.

1.4.1 Normalizable or Non-Normalizable?

On the AdS side, we shall decompose all 10-dimensional fields onto Kaluza-Klein towers on S^5, so that effectively all fields $\phi(r, x)$ are on AdS_5, and labeled by their dimension m_Δ (the different quantum numbers are implicit). Away from the bulk interaction region, it is assumed that the bulk fields are free asymptotically (just as this is assumed in the derivation of the LSZ formalism in flat space-time quantum field theory). The free field then satisfies $(\Box_{AdS} + m_\Delta^2)\phi_{r\to\infty} = 0$ for scalars. For the AdS metric (here we are using a different r-coordinate to that used in the previous section. They are related by a logarithmic mapping; see Appendix A.1)

$$ds^2 = e^{2r/L} dx^\mu dx_\mu + dr^2 \tag{1.67}$$

the scalar field equation is

$$\phi'' + \frac{4}{L}\phi' = m_\Delta^2 \phi \tag{1.68}$$

that has the solution

$$\phi = \mathcal{A}e^{-\Delta r/L} + \mathcal{B}e^{-(4-\Delta)r/L} \tag{1.69}$$

with

$$m_\Delta^2 = \Delta(\Delta - 4). \tag{1.70}$$

This last relation is a very important test of AdS/CFT because it relates masses of supergravity scalars, to the conformal dimension of the SYM operators that the scalars are dual to. The two independent solutions are characterized by the following asymptotics as $r \to \infty$,

$$\phi_0(r,x) = \begin{cases} e^{-\Delta r/L} & \text{normalizable} \\ \\ e^{-(4-\Delta)r/L} & \text{non-normalizable} \end{cases} \tag{1.71}$$

For $\Delta \geq 4$ this is appropriate terminology. We further restrict by considering only $\Delta \geq 2$ (note the system has the symmetry of $\Delta \to 4-\Delta$).Returning to the interacting fields in the fully interacting theory, solutions will have the same asymptotic behaviors as in the free case. The normalizable modes determine the vacuum expectation values of operators of associated dimensions and quantum numbers. The non-normalizable solutions on the other hand do not correspond to bulk excitations because they are not properly square normalizable. Instead, they represent the coupling of external sources to the supergravity or string theory. The precise correspondence is as follows [5]. The non-normalizable solutions $\phi_{N.N}$ define *associated boundary fields* $\bar{\phi}$ by the following relation

$$\bar{\phi}(x) \equiv \lim_{r\to\infty} \phi_{N.N}(r,x)e^{-(4-\Delta)r/L}. \tag{1.72}$$

Given a set of boundary fields $\bar{\phi}(x)$, it is assumed that a complete and unique bulk solution to string theory exists. We denote the fields of the associated solution ϕ.

To capture the features of operator insertions or VEVS, we generalize the above. The simplest possibility is to consider non-trivial dynamics for a scalar field in the 5d supergravity theory. We only allow the scalar to vary in the radial direction in AdS with the usual interpretation that this corresponds to renormalization group (RG) running of the source [5]. We look for solutions where the metric is described

[5]The radial direction in the AdS space should have a field theory interpretation. It has the scaling dimension of energy in the field theory; together with other arguments, this should be dual to the energy scale [15, 46, 50] of the field theory thereby giving a prescription to study RG flow in the field theory.

by [47, 84, 48, 49, 50, 51, 52]

$$ds^2 = e^{2A(r)/L} dx^\mu dx_\mu + dr^2 \tag{1.73}$$

and the scalar field has a Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - V(\phi) \tag{1.74}$$

There are two independent, non-zero, elements of the Einstein tensor (G_{00} and G_{rr}) giving two equations of motion plus there is the usual equation of motion for the scalar field

$$\phi'' + 4A'\phi' = \frac{\partial V}{\partial \phi} \tag{1.75}$$

$$6A'^2 = \phi'^2 - 2V(\phi) \tag{1.76}$$

$$-3A'' - 6A'^2 = \phi'^2 + 2V(\phi) \tag{1.77}$$

In fact only two of these equations are independent but it will be useful to keep track of all of them.

In the large r limit, where the solution will return to AdS_5 at first order and $\phi \to 0$ and $V \to m^2\phi^2$, only the first equation survives with solution Eqn.(1.69) and the mass relation Eqn. (1.70). If the solution retains some supersymmetry then the potential can be written in terms of a superpotential [57]

$$V = \frac{1}{8}\left|\frac{\partial W}{\partial \phi}\right|^2 - \frac{1}{3}|W|^2 \tag{1.78}$$

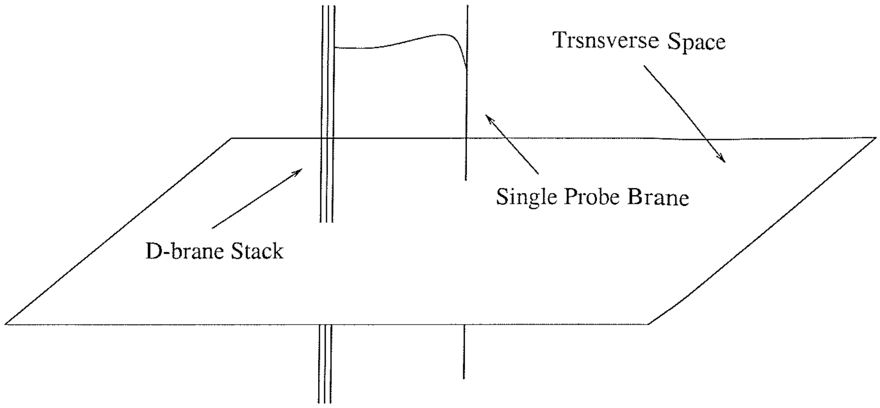

Figure 1.5: Branes distributed throughout the transverse space, with a single brane feeling the geometry.

and the second order equations reduce to the first order system

$$\phi' = \frac{1}{2}\frac{\partial W}{\partial \phi}, \qquad A' = -\frac{1}{3}W. \tag{1.79}$$

These first order equations have an important dual meaning, as they describe RG flow. The scalar equation describes the RG flow of the field operators, whilst the $A(r)$ function can be used to establish a c-theorem [38].

1.5 Brane Probing

A very important tool in the study of any gauge-gravity dual is the use of a brane probe [53]. Simply put, when some configuration of branes are present in a spacetime thereby producing a particular geometry, a single brane can be separated off the main configuration. This single brane will "feel" the background geometry without disrupting it when it is moved in a gentle way (see Fig. 1.5). The Dp-brane is described by the DBI action which takes a background spacetime metric, and produces a low energy $U(1)$ gauge theory on its worldvolume. A low energy $U(1)$ gauge theory lives on the surface and what this method provides is a way of determining gauge theory quantities, principally the three

- The gauge coupling
- The moduli space
- The physical coordinates.

It therefore provides a transparent link between a given supergravity background and the dual gauge theory. This is very much in the spirit of an experimental situation. If one thinks of measuring charges in electrostatics, or internal structure in say an atom, one has to use a smaller part of that system (electrons or photons), a probe, to determine the interactions. The idea then is to determine a low energy effective action that comes from the branes action.

Firstly then we must write down an action for the brane (with world volume W)

$$\begin{aligned} S_{brane} &= S_{DBI} + S_{WZ} \qquad (1.80) \\ &= -T_p \int_W d^{p+1}\xi e^{-\Phi} \sqrt{\det(G_{ab} + F_{ab})} + Q_p \int_W A_{p+1}. \qquad (1.81) \end{aligned}$$

As already remarked, this action captures the necessary features of a D-brane. Since we have diffeomorphism invariance one should gauge fix the action to get the physical degrees of freedom. The gauge chosen will always be the static gauge where

$$\begin{aligned} \xi^i &= x^\mu \\ \xi^m &= X^m(t) \qquad (1.82) \end{aligned}$$

If we feed a metric into the pullback and put it into the static gauge we see the scalar part of the action is

$$S[X] = -T_p \int_W d^{p+1}x\sqrt{-\det(g_{ab})}e^{-\Phi}(1 - g_{ij}\dot{X}^i\dot{X}^j/g_{00})^{1/2} + Q_p \int_W A_{p+1} \qquad (1.83)$$

where g_{ab} are the $D = 10$ metric components in the brane directions. At this point, an approximation is required to simplify matters to usual field theory; one considers a slow moving limit (the branes are moved slowly around the main brane configuration) where quadratic is the highest order retained in the expansion of the square root. Collecting the non-dynamical piece here (the "1" in the expansion) with the WZ gauge potential coupled to the brane, gives a potential for the scalar fields that schematically looks like $V(X) = A_{01\cdots p+1}/\sqrt{-\det(g_{ab})} - 1$. This then determines the moduli space of the theory, where

$$V(X) = 0. \qquad (1.84)$$

That this is possible is due to the relation of the p-brane tension with its electrical charge, as stated earlier. We can answer the other two points, that of the gauge coupling and the "physical" coordinates by making the expansion of the F_{ab} term; this just produces the usual gauge kinetic piece

$$S[A] = \int_W d^{p+1}x\sqrt{-\det(g_{ab})}e^{-\Phi}F_{\mu\nu}F^{\mu\nu}. \qquad (1.85)$$

One can read off the gauge coupling from the above. This should then be contrasted with the scalar piece, which will generally show that a change of coordinates is necessary. It is required that the scalar and gauge kinetic normalizations should coincide for it to be physical; the given scalars can then be reparametrized to make this so. We should qualify this here by saying that we can use this only for the case of $\mathcal{N}=2$ and 4, because its only here that there exists a relation between the normalizations of the scalar and gauge kinetic terms. The $U(1)$ gauge coupling is the same as the

Yang-Mills coupling, because of the two way process that there are no self interactions to change it, and the SYM fields are singlets with respect to the $U(1)$. Clearly a brane probe is a useful object for studying any gauge-gravity system.

Let us also comment on this when the background and gauge theory are supersymmetric. In this case generally W will have a flat world volume (see Appendix B.1.1) and so will support covariantly constant spinors. When we find a moduli space in the brane probe, and hence a no force property, this can be translated to the fact that the branes are BPS states [53]. A BPS condition sets a bound on the central charges with respect to the masses. For example, one can embed black holes into different supergravity theories [55], and these are found to satisfy a BPS condition. The brane has both a tension and a RR charge, and the BPS condition is the equality of these thereby saturating the bound.

Chapter 2

$\mathcal{N}$=4 on the Moduli Space

Having now described the general structure of the correspondence, it is instructive to consider a specific example. As mentioned in Section 1.4, there are two parts to moving away from the canonical case. Either an operator deformation or a vacuum expectation value of that operator can be considered, and the latter shall be looked at in this chapter. In particular we will begin by revisiting the gravity duals of $\mathcal{N}$=4 SYM on moduli space [57].

Let us overview this chapter. The first section gives an explicit demonstration of the D3-solution. This is intended both to demonstrate the multi-centre nature of the solutions in gravity language, and to give a precise meaning for it in the gauge theory picture. Having gained experience here, one is ready to look at systems where a particular $\langle \mathcal{O} \rangle$ is switched on. This is done firstly at the $d = 5$ level where the dual supergravity field is identified and supersymmetric solutions can be found. The solution can then be lifted to $D = 10$ by consistent truncation. Then, gauge theory questions can be addressed by brane probing (see Section 1.5). At this point the physical set of coordinates are identified, and we can now make a comparison with the multi-center solutions. The 5d supergravity solutions only describe a subset of the possible moduli space but the full set of 10d supergravity solutions needed to describe the full moduli space may be deduced. These metrics are indeed solutions of the 10d supergravity equations of motion.

2.1 The D3-brane Solution

Since a distribution of parallel D3-branes has a low energy description in terms of Yang-Mills theory (their separation placing the theory on its moduli space), it is interesting to see what the corresponding supergravity solution is. In fact this will turn out to be essential for understanding the latter findings of this chapter. Inspired by the work [38], let us verify the D3 solution. The fields we consider are $(G_{MN}, \Phi, A_{(0)}, A_{(4)})$. We now make an ansatz of the form:

$$\begin{aligned} ds^2_{10} &= H^{-2\alpha}\eta_{\mu\nu}dx^\mu dx^\nu + H^{2\beta}\delta_{ab}dy^a dy^b, \\ A_{(4)} &= H^{-\gamma}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \\ \Phi &= 0, \\ A_{(0)} &= 0. \end{aligned} \tag{2.1}$$

This is a solution of the IIB field equations provided H is a *harmonic* function on the transverse space $\mathbf{R}^6$

$$\Box_y H(y^a) = \frac{\partial^2}{\partial y^a \partial y^a} H(y^a) = 0, \tag{2.2}$$

and that we can fix the three constants (α, β, γ) by the consistency of the field equations with respect to the ansatz. Firstly, taking the dilaton and axion to be zero throughout all the field equations is consistent, as may be seen from Section 1.2.

Next we should write down the remaining field equations:

$$R^M{}_N = \frac{1}{2.4!} F^{MABCD} F_{NABCD}, \tag{2.3}$$

$$\partial_M(\sqrt{G} F^{MABCD}) = 0. \tag{2.4}$$

In addition the 5-form field strength must be self-dual, and satisfy the Bianchi identity

$$F_5 = dA_4 + *dA_4, \tag{2.5}$$

$$dF_5 \equiv 0 \tag{2.6}$$

Substituting the ansatz into Eqn. (2.4), and taking the (0123) component we get

$$\partial_a \left[H^{-4\alpha+6\beta}(H^{-2\beta}) H^{-\gamma-1} \partial_a H \right] = 0 \tag{2.7}$$

$\Rightarrow$

$$\Box_y H(y^a) = 0, \; for$$

$$4\alpha + 4\beta - \gamma - 1 = 0. \tag{2.8}$$

The Bianchi identity is also satisfied by virtue of it "secretly" containing the field equation for $A_{(4)}$. What now needs to be verified is that the Einstein field equations also contains this operator $\Box_y$, together with additional constraints on the constants. When calculating the Ricci tensor, two types of terms will be generated; one involving $(\partial^2 H)$, and one involving $(\partial H)^2$. This latter piece will have to cancel the piece coming from the energy momentum tensor.

We should now calculate the Ricci tensor. This will be done by using the *Cartan structure equations*, as familiarity with the vielbein formalism is essential [38]. The vielbein basis used is (hatted letters denote flat indices whilst unhatted indices are spacetime indices)

$$\begin{aligned} e^{\hat{\mu}} &= H^{-\alpha} dx^{\mu}, \\ e^{\hat{a}} &= H^{\beta} dx^{a}. \end{aligned} \tag{2.9}$$

Using Cartan's first structure equation (together with the torsion free condition $\omega^{\hat{a}\hat{b}} = -\omega^{\hat{b}\hat{a}}$)

$$de^{\hat{a}} + \omega^{\hat{a}}_{\ \hat{b}} \wedge e^{b} = 0, \tag{2.10}$$

we obtain the spin connection

$$\omega^{\hat{\mu}\hat{a}} = -\alpha(H^{-1-\alpha-\beta})\partial^{a} H dx^{\mu} \tag{2.11}$$

$$\omega^{\hat{a}\hat{b}} = \beta H^{-1}[\partial^{b} H dy^{a} - \partial^{a} H dy^{b}] \tag{2.12}$$

Using Cartan's second structure equation

$$R^{\hat{a}\hat{b}} = d\omega^{\hat{a}\hat{b}} + \omega^{\hat{a}}_{\ \hat{c}} \wedge \omega^{\hat{c}\hat{b}} = \frac{1}{2} R^{\hat{a}\hat{b}}_{\ \ \hat{c}\hat{d}} e^{\hat{c}} \wedge e^{\hat{d}}, \tag{2.13}$$

the Riemann tensor and by contraction the Ricci tensor can be found. The non-zero pieces contributing are

$$d\omega^{\hat{\mu}\hat{a}} = \alpha(1+\alpha+\beta)(H^{-2-2\beta})\partial_{b} H \partial^{a} H e^{\hat{b}} \wedge e^{\hat{\mu}} + (\partial\partial H) term \tag{2.14}$$

$$d\omega^{\hat{a}\hat{b}} = -\beta(H^{-2-2\beta})[\partial_{c} H \partial^{b} H \delta^{a}_{\ d} - \partial_{c} H \partial^{a} H \delta^{b}_{\ d}] e^{\hat{c}} \wedge e^{\hat{d}} + (\partial\partial H) term \tag{2.15}$$

$$\omega^{\hat{\mu}}_{\ \hat{c}} \wedge \omega^{\hat{c}\hat{a}} = -\alpha\beta(H^{-2-2\beta})[\partial_{b} H \partial^{a} H \eta^{\mu}_{\ \nu} - (\partial H)^{2} \delta^{a}_{\ b} \eta^{\mu}_{\ \nu}] e^{\hat{\nu}} \wedge e^{\hat{b}}. \tag{2.16}$$

We can now perform a simple piece of dimensional analysis, to further constrain the constants. Calculating the energy-momentum tensor for the spatial indices $M = \mu$, observe the behaviour

$$\frac{1}{2.4!}F^{MABCD}F_{NABCD} \sim [H]^{8\alpha-2\beta-2\gamma-2}(\partial H)^2. \tag{2.17}$$

From the pieces contributing to the Ricci tensor, they all behave as

$$R^M{}_N \sim (\partial\partial H)term + H^{-2-2\beta}(\partial H)^2. \tag{2.18}$$

If the corresponding pieces are to cancel, this clearly forces the constraint

$$4\alpha = \gamma \;\Rightarrow \beta = 1/4. \tag{2.19}$$

Since the ansatz is a 'warped' product manifold, the Ricci tensor will split into parallel and perpendicular pieces to the brane. For the parallel part one obtains

$$\begin{aligned} R^{\hat{\mu}\hat{a}}((\partial H)^2) &= (H^{-2-2\beta})[\alpha(1+\alpha+2\beta)\partial_b H\partial^a H - \alpha\beta\delta^a_b(\partial H)^2]\eta^{\hat{\mu}}_{\hat{\nu}}e^{\hat{b}} \wedge e^{\hat{\nu}} \\ &\Rightarrow \\ -\frac{1}{2}R^{\mu}{}_{\nu}((\partial H)^2) &= \eta^{\mu}_{\nu}(H^{-2-2\beta})[\alpha^2(\partial H)^2], \end{aligned} \tag{2.20}$$

Now calculating the corresponding piece of the energy-momentum tensor

$$\begin{aligned} T^{\mu}_{\nu} &= \frac{1}{2.4!}(H^{-2-2\beta})[-1][3!][4\alpha]^2(\partial H)^2\eta^{\mu}_{\nu} \\ &= -2\alpha^2(\partial H)^2\eta^{\mu}_{\nu} \end{aligned} \tag{2.21}$$

which is precisely $R^{\mu}{}_{\nu}$. So we see that the field equations in the D3-brane directions are consistent, and that these do not constrain the constants at all. Now considering directions transverse to the brane

$$\begin{aligned} R^{ab}((\partial H)^2) &= -\beta(H^{-2-2\beta})[\partial_c H\partial^b H\delta^a_d - \partial_c H\partial^a H\delta^b{}_d]e^{\hat{c}} \wedge e^{\hat{d}} \\ &\Rightarrow \end{aligned}$$

$$\frac{1}{2}R^a{}_b((\partial H)^2) = (H^{-2-2\beta})[-\partial^a H\partial_b H(-\beta) - (\partial H)^2\delta^a_b(\beta)] \qquad (2.22)$$

Doing the same as above for the energy momentum tensor

$$\begin{aligned} T^a{}_b &= \frac{1}{2.4!}(H^{-2-2\beta})[-1][3!][4\alpha]^2(\partial^a H\partial_b H - \delta^a_b(\partial H)^2) \\ &= 8\alpha^2[\partial^a H\partial_b H - (\partial H)^2\delta^a_b]. \end{aligned} \qquad (2.23)$$

The normalizations must match on both sides which implies $\beta = 4\alpha^2$. This gives $\alpha = \pm 1/4$. The sign ambiguity is fixed by considering the normalization of the 5-form and demanding that it is positive (this is what counts the number of branes involved) which gives $\gamma > 0$. Hence we find that a consistent solution exists for

$$\begin{aligned} \beta &= 1/4 \\ \gamma &= 1 \\ \alpha &= 1/4 \end{aligned} \qquad (2.24)$$

This completes the demonstration that the Einstein field equations reduce to the harmonic wave equation Eqn. (2.2). The non-singular solutions of this are just the familiar *multi-centre* solutions

$$\begin{aligned} H(y) &= 1 + \sum_{I=1}^{N} \frac{L^4 N_I}{|y - y_I|^4}, \\ N &= \sum_{I=1}^{N} N_I. \end{aligned} \qquad (2.25)$$

The "1" in the above expression is a boundary condition, so that asymptotically the solution returns to flat space. This has a very simple brane picture interpretation. The solution represents a collection of N D3-branes that are positioned throughout the transverse space with positions $y_I \in \mathbf{R}^6$; it is just a generalization of the Reissner-Nördstrom solution when it is extremal, to branes in higher dimensions. In addition

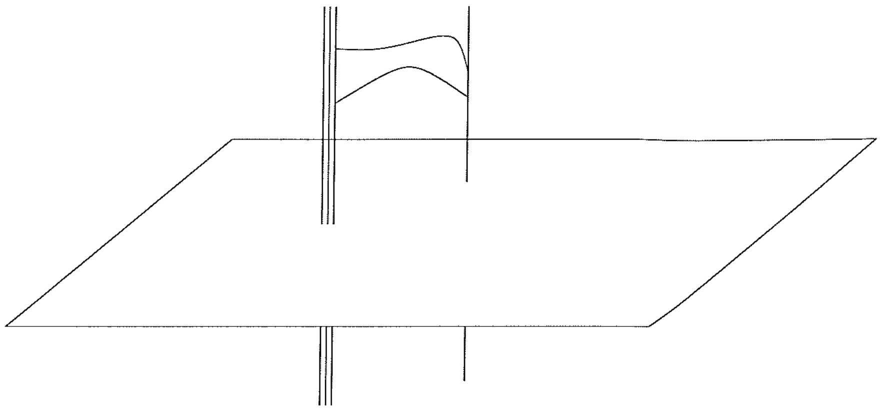

Figure 2.1: Branes distributed throughout the transverse space $\mathbf{R}^6$.

the L factor occurring here is the size of the space we are dealing with (in our case it would be the size of AdS_5). This had to be included in the above expression on dimensional grounds.

The multi-centre solution above can easily be generalized to describe a continuous distribution of D3-branes by taking a continuum limit where one finds

$$H(y) = 1 + \int_{\mathcal{M}} d^6y' \frac{L^4}{|y - y'|^4} \sigma(y') \tag{2.26}$$

$$N = \int_{\mathcal{M}} d^6y' \sigma(y'), \tag{2.27}$$

where $\sigma(y')$ is the density distribution of D3-branes over a space $\mathcal{M} \subset \mathbf{R}^6$. This shows something interesting for the correspondence of the six scalars in $\mathcal{N}$=4 SYM and the positions y_I of the D3-branes; one can write

$$y_I^i \equiv \langle X^i \rangle \tag{2.28}$$

where the $X \in$ Cartan-subalgebra. There appears to be a mismatch in this map, since there are only $N - 1$ generators in the $SU(N)$ Cartan-subalgebra. Had we been more careful one would remember that initially one has a $U(N)$ gauge group coming from the Chan-Paton factors of the open strings, and that in addition to the above identification, one should identify the $U(1) \subset U(N)$ with the centre of mass coordinate

$$y_{C.M.} = \frac{1}{N}\sum_{I=1}^{N} N_I y_I, \tag{2.29}$$

which isn't dynamical [38]. In the infrared, the $U(1)$ theory is free.

2.1.1 Supersymmetry preservation

Given we have found this simple solution to the field equations, it is necessary to ask how much supersymmetry is preserved [58]. Generally a solution will have to be very special if it is to admit a Killing spinor. To coincide with the notation in [29] we write the ansatz as

$$ds^2 = e^{2A}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2B}\delta_{mn}dy^m dy^n, \tag{2.30}$$

$$A_{0123} = e^C, \tag{2.31}$$

and we shall put in the specific form of the solution that was found in the last section. If this is a supersymmetric solution then we must have that $\delta\psi_M = 0$ and $\delta\lambda = 0$ for the IIB gravitino and complex spinor. From Appendix C.1 we see that the complex spinor equation is trivial, whilst the gravitino is not,

$$\delta\psi_M = \nabla_M\epsilon + \frac{i}{4.4!}\Gamma^A\Gamma^B\Gamma^C\Gamma^D\Gamma^E(\Gamma_M\epsilon)F_{ABCDE} = 0. \tag{2.32}$$

The spinor parameter is complex and subject to the IIB chirality condition

$$\Gamma_{11}\epsilon = -\epsilon. \tag{2.33}$$

We can divide this evaluation into two steps, one is to evaluate the spin connection, and the other is to put in the field strength. From Eqn. (2.1) and using the conversion

$\omega_M^{\hat{A}\hat{B}} dx^M = \omega^{\hat{A}\hat{B}}$ the non vanishing spin connection is

$$\omega_\mu^{\hat{\nu}\hat{m}} = e^{A-B}\partial^m A\delta_\mu^\nu \tag{2.34}$$

$$\omega_a^{\hat{m}\hat{n}} = \partial^n B\delta_a^m - \partial^m B\delta_a^n. \tag{2.35}$$

This allows us then to evaluate the covariant derivative given in Section C.1 as

$$\nabla_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{2}(\partial_m A)(\Gamma_\mu \gamma^m \epsilon), \tag{2.36}$$

$$\nabla_m \epsilon = \partial_m \epsilon + \frac{1}{2}(\partial_m B)\epsilon + \frac{1}{2}(\partial_n B)(\Gamma^n \gamma_m \epsilon). \tag{2.37}$$

Next we should compute the second term in Eqn. (2.32). In this we have to use $F_5 = dA_4 + \star(dA_4)$ for it to satisfy the self dual relation. The first piece is simple,

$$\Gamma^a\Gamma^0\Gamma^1\Gamma^2\Gamma^3(\Gamma_M\epsilon)(\partial_a e^C). \tag{2.38}$$

Because it is self dual, we might expect the extra piece would just simply be a "doubling" of this, but we should work through to see if this so;

$$(\star dA)_{ABCDE} = \frac{1}{5!}\sqrt{G}\epsilon_{MNPQRABCDE}F^{MNPQR}, \tag{2.39}$$

$$(\star dA_{a0123}) = e^{-4A+4B}\epsilon_{mnpqr}(\partial_a e^C) \tag{2.40}$$

where $m, n \cdots \neq a$. We can put these two terms together and we find

$$[\Gamma^a\Gamma^0\Gamma^1\Gamma^2\Gamma^3 + e^{-4A+4B}\Gamma^m\Gamma^n\Gamma^p\Gamma^q\Gamma^r](\Gamma_M\epsilon)(\partial_a e^C) \tag{2.41}$$

we now want to make use of an identity used in [56] to convert the transverse space quantity to the brane directions;

$$\sqrt{G}\Gamma^{M_1\cdots M_N} = -\frac{1}{(10-N)!}(-1)^{N(N-1)/2}\epsilon^{M_1\cdots M_{10}}\Gamma_{M_{N+1}\cdots M_{10}}\Gamma_{11} \tag{2.42}$$

For our case this reduces to

$$
\begin{aligned}
e^{-4A+4B}\Gamma^m\Gamma^n\Gamma^p\Gamma^q\Gamma^r &= -e^{-8A-2B}\Gamma_a\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_{11} \\
&= -\Gamma^a\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma_{11}. \qquad (2.43)
\end{aligned}
$$

Then simply anticommute the Γ_{11} and Γ_M, impose the chirality condition Eqn. (2.33), and we are left with the simple doubling that we guessed. The explicit variations then read

$$
\delta\psi_\mu = \partial_\mu\epsilon + \frac{1}{2}(\partial_m A)(\Gamma_\mu\gamma^m\epsilon) + \frac{i}{4.4}(\Gamma^a\Gamma^0\Gamma^1\Gamma^2\Gamma^3)(\Gamma_\mu\epsilon)(2\partial_a e^C), \qquad (2.44)
$$

$$
\begin{aligned}
\delta\psi_m &= \partial_m\epsilon + \frac{1}{2}(\partial_m B)\epsilon + \frac{1}{2}(\partial_a B)(\Gamma^a\gamma_m\epsilon) \\
&\quad + \frac{i}{4.4}(\Gamma^a\Gamma^0\Gamma^1\Gamma^2\Gamma^3)(\Gamma_m\epsilon)(2\partial_a e^C). \qquad (2.45)
\end{aligned}
$$

So these are the equations that need to be solved for the Killing spinors. Let us now turn to the symmetry content of the solution. Firstly to have $SO(1,3)$ invariance we must have $\partial_\mu\epsilon = 0$. Next, a projection has to be made to make the system algebraic; this is

$$
i\Gamma^{\hat{0}}\Gamma^{\hat{1}}\Gamma^{\hat{2}}\Gamma^{\hat{3}}\epsilon = \epsilon. \qquad (2.46)
$$

It is this condition that *halves* the total supercharges preserved. Thus the variation equations become

$$
\delta\psi_\mu = (\Gamma_\mu\Gamma^a\epsilon)[\frac{1}{2}(\partial_a A) - \frac{1}{2.4}(e^{-4A}\partial_a e^C)] = 0, \qquad (2.47)
$$

$$
\delta\psi_m = \partial_m\epsilon + \frac{1}{2}(\partial_m B)\epsilon - (\Gamma_m\Gamma^a\epsilon)[\frac{1}{2}(\partial_a B) + \frac{1}{2.4}(e^{-4A}\partial_a e^C)] = 0. \qquad (2.48)
$$

These can now be solved by inspection; $e^C = e^{4A}$, $B = -A$; in addition the 4-6 split can be made $\epsilon = \epsilon^{(4)} \otimes \epsilon^{(6)}$ and the Killing spinor is found to be

$$
\epsilon = e^{A/2}\epsilon_0^{(4)} \otimes \epsilon_0^{(6)} \qquad (2.49)
$$

This is all consistent with what was found from solving the field equations previously and identifying $e^A \equiv H^{-1}$. Thus the multi-centre solutions preserve 16 supercharges, which is the same number as for the $\mathcal{N}$=4 theory.

2.2 D=5 Gauged Supergravity and its lift to IIB

In this section, we will study gravity solutions describing $\mathcal{N}$=4 super Yang-Mills (SYM) theory on moduli space resulting from 5d supergravity [57]. We wish to study the gauge theory in the presence of a non-zero VEV for the scalar operator $sTrX^iX^j$ (see Section 1.3). This operator is symmetric and traceless, and transforms as the **20** of the global $SU(4)_R$ symmetry of the theory. In the 5d truncation of IIB supergravity on $AdS_5 \times S^5$ [59, 60] the lightest state is a scalar, α, in the **20** that acts as the source for $\langle sTrX^iX^j \rangle$ in the AdS/CFT correspondence. One may look for solutions of the 5d supergravity equations of motion with non-zero α and interpret them as gravity duals of the $\mathcal{N}$=4 theory with a scalar VEV switched on. In fact, considerable work is needed to arrive at the equations of motion since the scalars live in the coset $E_6/USp(8)$, the subtleties of which are discussed in [57]. We shall present the final results only here.

As an example let us consider the case of switching on $\langle sTrX^iX^j \rangle = \mathrm{diag}(1,1,1,1-2,-2)$. The appropriate supergravity scalar has been identified in [57]. In the supergravity theory the metric is dynamical and the scalar VEV cannot be considered in isolation. We parametrize the metric as

$$ds^2 = e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + dr^2 \tag{2.50}$$

where x describe four dimensional Minkowski space slices through the deformed AdS space, r is the radial direction, and in the AdS limit $A(r) = r/L$ with L the radius of the AdS space. The resulting supersymmetric equations of motion (for which the fermionic shifts vanish, see section C.1) are first order(where $\rho = e^{\alpha}$),

$$\frac{\partial \rho}{\partial r} = \frac{1}{3L}\left(\frac{1}{\rho} - \rho^5\right), \qquad \frac{\partial A}{\partial r} = \frac{2}{3L}\left(\frac{1}{\rho^2} + \frac{\rho^4}{2}\right). \tag{2.51}$$

These equations may be solved in the $\rho - A$ plane since

$$\frac{\partial \rho}{\partial A} = \frac{1}{2}\left(\frac{\rho - \rho^7}{1 + \frac{\rho^6}{2}}\right) \tag{2.52}$$

with solution

$$e^{2A} = \frac{l^2}{L^2}\frac{\rho^4}{\rho^6 - 1} \tag{2.53}$$

with l^2/L^2 a constant of integration. At this level the connection to the dual gauge theory is somewhat opaque.

Remarkably the solution has been lifted back to a $D = 10$ solution [57, 85] which takes the form

$$ds^2 = \frac{X^{1/2}}{\rho}e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{X^{1/2}}{\rho}\left(dr^2 + \frac{L^2}{\rho^2}\left[d\theta^2 + \frac{\sin^2\theta}{X}d\phi^2 + \frac{\rho^6\cos^2\theta}{X}d\Omega_3^2\right]\right), \tag{2.54}$$

where $d\Omega_3^2$ is the metric on a 3-sphere and

$$X \equiv \cos^2\theta + \rho^6\sin^2\theta \tag{2.55}$$

For consistency there must also be a non-zero A_4 potential of the form

$$A_4 = \frac{e^{4A}X}{g_s\rho^2}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \tag{2.56}$$

Again any relation to a dual theory is well hidden. In fact in [57] this metric was determined to be equivalent to the near horizon limit of a multi-centre solution around a D3 brane distribution. We wish to make the need for the transformation to these coordinates clear within the context of the duality. There are many coordinate redefinitions one could make and only a single set of coordinates can manifestly display the field theory duality. By brane probing this background one can find these coordinates and thereby show the physical parameterization.

2.3 Brane Probing The Solution

Brane probing [53] is most transparent in the original D3 brane construction for the AdS/CFT correspondence. Here there is a stack of N D3 branes at the origin with the $\mathcal{N}$=4 SYM as their world volume theory and $AdS_5 \times S^5$ as their near horizon geometry. If we imagine moving a single D3 brane from the stack and moving it in the space then, to first order, it will not effect the background metric. From the world volume field theory point of view, by separating a D3 brane we have introduced an adjoint scalar VEV breaking $SU(N) \to U(1) \times SU(N-1)$. The magic of D-branes is that the scalar fields' VEVS in the field theory are precisely identified with the position of the D3 brane in the surrounding spacetime. This is expressed by the Dirac Born Infeld (DBI) action for a D3 brane,

$$S_{probe} = -T_3 \int_{\mathcal{M}_4} d^4x \det[G_{ab} + 2\pi\alpha' e^{-\Phi/2} F_{ab}]^{1/2} + Q_3 \int_{\mathcal{M}_4} A_4, \qquad (2.57)$$

where G_{ab} is the pull back of the spacetime metric, F_{ab} the gauge field on the probes surface, Φ the dilaton (which is a constant in this solution) and $T_3 = Q_3/g_s$. Thus the DBI action allows us to translate the background metric to a potential for the scalar fields in the field theory. It is easy to identify the dimension of the field theory moduli space implied by the metric from where the DBI potential vanishes. In addition since the $U(1)$ theory lives on the probe's surface and is a non-interacting theory (photons do not self interact and there is only adjoint matter which for a $U(1)$ is chargeless), its coupling is that of the $SU(N)$ theory at the scale of the breaking VEV. The probe also therefore lets us determine the functional form of the coupling on moduli space.

We proceed to brane probe the 10d metric above by substituting (5.14)-(5.16) in (2.57). Allowing the brane to move slowly and concentrating on the scalar sector, we find the DBI action corresponds to the field theory,

$$S = -\frac{Q_3}{2g_s}\int_{\mathcal{M}_4} d^4x \left[\frac{Xe^{2A}}{\rho^2}\dot{r}^2 + \frac{L^2e^{2A}}{\rho^4}(X\dot{\theta}^2 + \sin^2\theta\ \dot{\phi}^2 + \rho^6\cos^2\theta\ \dot{\Omega}_3^2)\right]. \tag{2.58}$$

The immediate result is that we see there is no potential against motion of the probe in the full 6 dimensional transverse space corresponding in the field theory to the scalars having a 6d moduli space. This matches with our expectations for the $\mathcal{N}$=4 SYM theory where the six scalars have a potential of the form $tr[X^i, X^j]^2$ and so taking the VEVS to be in the Cartan sub-algebra, the six scalars may take arbitrary values.

The kinetic terms should be interpreted as the kinetic terms of the field theory scalars which in the $\mathcal{N}$=4 theory are given by $(1/8\pi)Im(\tau\Phi^\dagger\Phi)|_D$ (in $\mathcal{N}$=1 notation). The coefficient of the kinetic terms are therefore the gauge coupling which is known to be conformal in the $\mathcal{N}$=4 theory. We should expect the metric that the probe sees on moduli space to be flat which it manifestly isn't in (2.58). This is our hint as to the coordinate change we should make in order to pass to those coordinates where the duality is manifest. Forcing this relation we find a change of coordinates that makes the probe metric flat

$$(r,\theta) \rightarrow (u,\alpha) \tag{2.59}$$

such that

$$u^2\cos^2\alpha = L^2e^{2A}\rho^2\cos^2\theta, \qquad u^2\sin^2\alpha = L^2\frac{e^{2A}}{\rho^4}\sin^2\theta. \tag{2.60}$$

A small calculation shows that the metric in these coordinates takes the form

$$S = -\frac{Q_3}{2g_s}\int_{\mathcal{M}_4} d^4x \left[\dot{u}^2 + u^2(\dot{\alpha}^2 + \sin^2\alpha\;\dot{\phi}^2 + \cos^2\alpha\;\dot{\Omega}_3^2)\right]. \qquad (2.61)$$

This is a unique choice of coordinates and if the duality is to be manifest it must be in these coordinates where the coupling is seen to have the correct conformal property. It is therefore interesting to write the full metric in these coordinates

$$ds^2 = \left(\frac{\rho^2}{Xe^{4A}}\right)^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + \left(\frac{\rho^2}{Xe^{4A}}\right)^{1/2}\sum_{i=1}^{6}(du_i)^2 \qquad (2.62)$$

This is of the familiar form,

$$ds^2 = H^{-1/2}\eta_{ij}dx^i dx^i + H^{1/2}\sum_{i=1}^{6}du_i^2, \qquad A_4 = \frac{1}{Hg_s}dx^0\wedge dx^1\wedge dx^2\wedge dx^3 \quad (2.63)$$

From the coordinate transformations (2.60) and using (5.9), we can obtain an explicit expression for ρ in terms of (u,α)

$$\frac{u^2}{l^2}\sin^2\alpha\;\rho^{12} + \left(\frac{u^2}{l^2}\cos^2\alpha - \frac{u^2}{l^2}\sin^2\alpha - 1\right)\rho^6 - \frac{u^2}{l^2}\cos^2\alpha = 0 \qquad (2.64)$$

2.4 The Uses of Harmonic Functions

In [57] it was shown that in these coordinates $H(u)$ can be written as a multi-centre solution with a D3 density, σ,

$$H(u) = \int d^6x\;\sigma(x)\;\frac{L^4}{|\vec{u}-\vec{x}|^4} \qquad (2.65)$$

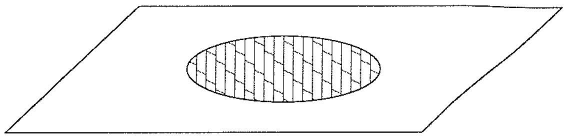

Figure 2.2: D3-branes distributed as a disk in the $\theta = \pi/2$ plane (an $\mathbf{R}^2 \subset \mathbf{R}^6$).

In this case the density is a 2 dimensional disk (see Fig. 2.2) of uniform density in the $\theta = \pi/2$ plane

$$\sigma(x) = \frac{1}{\pi l^2}\theta(l^2 - x^2) \tag{2.66}$$

We wish to make the connection to the field theory and instead consider the large u limit of (2.64) from which we obtain

$$\rho^6 = 1 + \frac{l^2}{u^2} + \left(\frac{l^2}{u^2}\right)^2 (1 - \sin^2\alpha) + \left(\frac{l^2}{u^2}\right)^3 (1 - 3\sin^2\alpha + 2\sin^4\alpha) + \mathcal{O}(\frac{l^8}{u^8}). \tag{2.67}$$

and hence from (2.62)

$$H(u) = \frac{L^4}{u^4}\left(1 + \frac{l^2}{u^2}(3\sin^2\alpha - 1) + \frac{l^4}{u^4}(1 - 8\sin^2\alpha + 10\sin^4\alpha)\right) + \mathcal{O}(\frac{L^4 l^6}{u^{10}}). \tag{2.68}$$

In this form it is possible for us to identify field theory operators [62]. The radial coordinate u has the scaling dimension of mass [5] so in each term in the expansion we can assign a scaling dimension to the coefficient. Further each term in the expansion is associated with a unique spherical harmonic[1]; the angular function in the $1/u^6$ term is the spherical harmonic in the **20** of $SU(4)_R$, that in the $1/u^8$ term the harmonic in the **50** and so forth. Note that by using the orthonormality of the spherical harmonics it is easy to show that each harmonic occurs only in a single term in the expansion.

[1]The spherical harmonics may be found by writing the 6 dimensional representation as a unit vector in the transverse space and then finding the symmetric traceless products $6 \times 6 = 20 + ..$, $6 \times 6 \times 6 \times 6 = 50 + ...$, etc

We can therefore identify the nth coefficient as having the dimension and symmetry properties of the field theory operator $\langle sTrX^n \rangle$ and further that the operator is not renormalized since there is no further function of u associated with the operator. Thus these solutions suggest the general form

$$H(u) = \frac{L^4}{u^4}(1 + \sum_n \frac{\langle sTrX^n \rangle}{u^n} Y_n). \tag{2.69}$$

where Y_n is the spherical harmonic obtained from the product of n 6 dimensional reps.

It is worth noting that at the level of the 5d supergravity theory we introduced only a VEV for the dimension 2 operator $\langle sTrX^2 \rangle$ yet after the lift to 10d the solution was forced to possess VEVS for higher dimension operators. If we returned to 5d the truncation would again remove these operators. The 5d supergravity metric gives specific relations between the operators as is explicit in (2.68) whilst in the field theory they are expected to be arbitrary reflecting the 6 dimensional moduli space. One may therefore try substituting the expansion with arbitrary coefficients into the supergravity field equations and they indeed turn out to be solutions [62]. Of course in this context this is no surprise because it is already known that the multi-centre solutions are solutions of the field equations for arbitrary D3 brane distributions. However, it is encouraging in this simplest case that one can deduce a full gravity description of the field theory from the 5d supergravity solutions. Further it is appealing that the metric is indeed seen to be a rewriting of the field theory solutions and it is of interest to see how this generalizes in theories with more complicated RG flow. In the next chapter we will study aspects of this generalization for the $\mathcal{N}=2^*$ theory.

Before moving on though we wish to note the power of the brane probing technique since it in fact is capable of deriving the above solutions on its own. In the $\mathcal{N}=4$ case if we wished to write down a metric dual to a point on moduli space we might begin by writing down an arbitrary 10d metric. If we then require the 6 dimensional moduli space and conformal coupling after a brane probe the metric is forced to take the form in (2.63). The supergravity field equations with this ansatz reduce to the transverse flat space Laplacian in 6 dimensions [38] ,

$$\Box_u H(u) = 0 \tag{2.70}$$

Which produces the multi-centre solutions. We see again that when we know sufficient information about the field theory the supergravity dual is uniquely determined.

2.5 Précis

This chapter has shown a very clean test of the AdS/CFT and has allowed us to build up a collection of tools that we can now use in more interesting situations. Having used the AdS/CFT map we have looked at $D = 5$ supergravity solutions that have been lifted by consistent truncation to $D = 10$. We are able then to brane probe this background and we recover the simple multi centre solutions. To identify a dual field of a gauge theory quantity and then to determine its (RG) properties from the supergravity point of view is quite non trivial.

Chapter 3

$\mathcal{N}$=2* SYM: an Operator Deformation

To test whether the encoding prescription is generic we move the techniques from the previous chapter across to the gravity dual of the $\mathcal{N}$=2* theory (the $\mathcal{N}$=4 theory with a mass term that breaks supersymmetry to $\mathcal{N}$=2 in the IR) which has more interesting RG flow properties. This is an example of an *operator deformation* $\mathcal{L} \to \mathcal{L} + \mathcal{O}$ and complements the previous chapter. The solutions are produced by including relevant deformations in the 5d supergravity theory [63, 65, 66]. These can again be lifted to 10d by consistent truncation [85]. The connection to the gauge theory of this set of solutions is far from apparent after the lift. The use of a brane probe to uncover the links was made in [66, 67]. The metric indeed describes the expected 2d moduli space of the field theory. The gauge coupling function on the moduli space is also revealed and, when the solution is placed in appropriately $\mathcal{N}$=2 coordinates, matches to field theory expectations. The set of solutions describe different points on moduli space with one corresponding to a singular point on moduli space where in the IR the gauge coupling diverges. This solution is of interest because it provides an example of the enhançon mechanism [68, 69] (there are points in the space where the tension of the probe falls to zero).

The content of this chapter is similar to that of the previous one with simple additions which will allow for it to describe this more difficult example. In the first section we look at the inclusion of the operator into the field theory, and the subsequent dual solution in $D = 10$. Then we go on to look at brane probing this background and write the metric on moduli space in the coordinates applicable to the field theory where it takes the form of a single function as in the $\mathcal{N}=4$ metrics multiplied by the gauge coupling function. It is natural to interpret the outstanding function according to the same prescription as in the $\mathcal{N}=4$ solution and read off field theory operators. In the field theory the gauge coupling encodes the only RG flow whilst the supergravity solution appears to describe additional renormalization of the scalar operators. In addition in the far UV the solution does not return to the $\mathcal{N}=4$ form but contains logarithmic renormalization. In the next section we highlight the discrepancy by following the prescription in [67] for deducing the D3 brane distribution from the expected field theory gauge coupling, as a function of position on moduli space, and the supergravity form for the coupling. We thus deduce the distribution for all the 5d supergravity lifts and can then calculate the expected scalar operators which again do not match with the function in the metric. Presumably there is some discrepancy in the prescription in this more complicated theory that has not yet been discovered.

3.1 The $D = 10$ Gravity Solution

We have seen that in the $\mathcal{N}=4$ duality there is a simple mapping between the field theory operators and the form of the metric. It would be interesting to understand how this mapping occurs in a more complicated theory with non-trivial renormalization group flow. The theory we choose to investigate in this light is the $\mathcal{N}=2^*$ theory where a mass term is introduced into the $\mathcal{N}=4$ theory that leaves an $\mathcal{N}=2$

supersymmetric theory in the IR. The operator deformation considered is

$$\mathcal{O}_m = m \sum_{i=1,2} \Phi_i^2, \tag{3.1}$$

together with the vacuum structure

$$\langle \mathcal{O}_1 \rangle = \langle \sum_{j=1}^{4} Tr(X^j X^j) - 2\sum_{j=5}^{6} Tr(X^j X^j) \rangle \tag{3.2}$$

as before. The 5d supergravity theory with the appropriate supergravity field deformations switched on was studied in [63, 65, 66]. Two supergravity scalars are needed, one describing the mass term and the other the possible VEV for the remaining two real scalar fields. The fields m, A and $\rho = e^{\alpha}$ are the supergravity fields given by the 5d supergravity equations of motion

$$\frac{\partial \alpha}{\partial r} = \frac{1}{3L}\left(\frac{1}{\rho^2} - \rho^4 \cosh(2m)\right) \tag{3.3}$$

$$\frac{\partial A}{\partial r} = \frac{2}{3L}\left(\frac{1}{\rho^2} + \frac{1}{2}\rho^4 \cosh(2m)\right) \tag{3.4}$$

$$\frac{\partial m}{\partial r} = -\frac{1}{2L}\rho^4 \sinh(2m) \tag{3.5}$$

which have solutions

$$e^A = k\frac{\rho^2}{\sinh(2m)} \tag{3.6}$$

$$\rho^6 = \cosh(2m) + \sinh^2(2m)\left(\gamma + \log\left[\frac{\sinh m}{\cosh m}\right]\right) \tag{3.7}$$

Although some connections were made between the field theory and these solutions the duality remained fairly opaque at the 5d level. A lift of this solution to 10d supergravity has again been provided [85, 65] and the summary of the solution is

$$ds^2 = \Omega^2(e^{2A}\eta_{\mu\nu}dx^\mu dx^\nu + dr^2) + \frac{L^2\Omega^2}{\rho^2}(\frac{d\theta^2}{c} + \rho^6\cos^2\theta(\frac{\sigma_1^2}{cX_2} + \frac{\sigma_1^2+\sigma_2^2}{X_1}) + \frac{\sin^2\theta}{X_2}d\phi^2) \tag{3.8}$$

where

$$\Omega^2 = (cX_1X_2)^{1/4}/\rho, \qquad c = \cosh 2m \tag{3.9}$$

$$X_1 = \cos^2\theta + \rho^6 c\sin^2\theta, \qquad X_2 = c\cos^2\theta + \rho^6\sin^2\theta \tag{3.10}$$

$$A_4 = \frac{e^{4A}X_1}{g_s\rho^2}dx^0\wedge dx^1\wedge dx^2\wedge dx^3 \tag{3.11}$$

The dilaton is non-trivial too. We write a complex scalar $\lambda = C_0 + ie^{-\Phi}$ and

$$\lambda = i\left(\frac{1-B}{1+B}\right), \qquad B = \left(\frac{b^{1/4}-b^{-1/4}}{b^{1/4}+b^{-1/4}}\right), \qquad b = \cosh(2m)\frac{X_1}{X_2} \tag{3.12}$$

The solution also has non-zero 2-forms [85] but they are zero in the $\theta = \pi/2$ plane which we will analyze below.

3.2 Brane Probing the Solution

Again brane probing is a necessity to make the duality with the field theory clear. In [67, 64] it was observed that after substituting the above 10d solution into the DBI action the potential vanishes in the $\theta = \pi/2$ plane. The moduli space for brane motion therefore matches the expected 2d moduli space of the $\mathcal{N}$=2* field theory which has two massless real scalars. From now on we will restrict our attention to this plane. Placing a brane probe off the moduli space corresponds in the field theory to giving a VEV to a massive scalar which is neither a vacuum of the theory nor supersymmetric. We know of no field theory results in the presence of such VEVS so there are no checks of the duality we can make.

On the moduli space a brane probe reveals the $U(1)$ field theory

$$\mathcal{L} = \frac{1}{2}\left(\rho^4 \cosh(2m)e^{2A}\dot{r}^2 + \frac{L^2\rho^4\cosh(2m)e^{2A}}{\rho^8}\dot{\phi}^2\right) + \frac{1}{4}T_3(2\pi\alpha')^2 e^{-\Phi}F^{\mu\nu}F_{\mu\nu} \quad (3.13)$$

In these coordinates the connection to the $\mathcal{N}$=2* theory is hidden but we can now find coordinates where the duality is manifest. The two scalar fields should have the same kinetic term with a common coefficient given by the gauge theory's running coupling, $1/g_{YM}^2(r)$. The first of these can be achieved by the change of coordinates

$$v = L\sqrt{\frac{\cosh 2m + 1}{\cosh 2m - 1}} \quad (3.14)$$

such that

$$\frac{\partial v}{\partial r} = \frac{\rho^4}{L}v \quad (3.15)$$

and we have

$$\mathcal{L} = \frac{1}{2}\frac{k^2L^2\cosh 2m}{\sinh^2 2m \;\; v^2}(\dot{v}^2 + v^2\dot{\phi}^2) \quad (3.16)$$

The solutions depend on two constants k and γ which correspond to the mass term and the scalar VEV [67] respectively. It is interesting to discuss the anatomy of these solutions at fixed k as a function of γ in the v coordinates. As in previous work [63, 64, 66, 67] we only consider $\gamma \leq 0$ since we can offer no physical interpretation of positive γ. Although, as we will see, $v - \phi$ are not the physical coordinates for the duality they have the benefit of an SO(2) symmetry in ϕ as can be seen from (3.16). The solutions with different choice of the parameter γ differ in the radial position at which the metric has divergences as a result of $\rho \to 0$. From (3.7) and (3.14) one may express γ in terms of this radius l as

$$\gamma = -\frac{l^2}{4L^2} + \frac{L^2}{4l^2} + \ln l/L \tag{3.17}$$

We expect the divergence in the metric to be associated with the presence of a disc D3 brane source and hence solutions with larger negative γ correspond to larger VEVS in the field theory. When $\gamma = 0$ the spacetime is good down to a radius $v = l = L$ where $\cosh 2m \to \infty$ and hence the coefficient of the scalar kinetic term falls to zero. This is the enhançon locus where the probes tension falls to zero (or in the field theory the coupling diverges) and according to lore [68, 69] we must excise the solution within. Only for this metric can the enhançon point be reached since the other, $\gamma < 0$, solutions have $\rho \to 0$ at a larger radius where the scalar kinetic terms coefficient is still regular.

As pointed out in [67] we can not yet formally make the connection to the gauge coupling because the U(1) theory is not in an $\mathcal{N}$=2 form since the coefficient of the $F_{\mu\nu}^2$ term is given by

$$e^{-\Phi} = \frac{c}{g_s|\cos\phi + ic\sin\phi|^2} \tag{3.18}$$

To obtain an $\mathcal{N}$=2 form we must make a holomorphic change of variables in the $v - \phi$ plane to equate the coefficients of the scalar and gauge field kinetic terms. The transformation is [67]

$$Y = \frac{kL}{2}(\frac{V}{L} + \frac{L}{V}) \tag{3.19}$$

where $V = ve^{i\phi}, Y = ye^{i\eta}$ are complex parameters on the 2d space. The low energy theory is then of the desired form with

$$\mathcal{L} = \frac{1}{g_{YM}^2(Y)}|\dot{Y}|^2 + \mathrm{Im}\left(\tau(F^{\mu\nu}F_{\mu\nu} + iF^{\mu\nu}\tilde{F}_{\mu\nu})\right) \tag{3.20}$$

with $4\pi/g_{YM}^2(Y) = Im\tau$ where

$$\tau = \frac{i}{g_s}\sqrt{\frac{Y^2}{Y^2 - k^2L^2}} \tag{3.21}$$

3.3 What the New Coordinates Say

In these coordinates the background takes the form

$$\begin{aligned} ds^2 &= \frac{1}{g_{YM}}\left(H^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H^{1/2}dYd\bar{Y}\right), && (3.22)\\ A_4 &= \frac{g_{YM}^2}{Hg_s}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, && (3.23)\\ T_3(2\pi\alpha')^2 e^{-\Phi} &= \frac{1}{g_{YM}^2} && (3.24) \end{aligned}$$

with

$$g_{YM}^2 H = \frac{\sinh^4 2m}{k^4\rho^{12}\cosh 2m} \tag{3.25}$$

All other fields are zero in the $\theta = \pi/2$ plane. In fact the brane probe does not uniquely fix the form of H since it can be rescaled by an arbitrary power of the Yang Mills coupling and still return the same probe theory. Since the coupling in (3.21) does not contain logarithms such a rescaling will not resolve the discrepancies discussed below.

We claim to have identified the unique coordinates in which in the $\theta = \pi/2$ plane a brane probe correctly matches the expected form for an $\mathcal{N}$=2 supersymmetric theory. In these physical coordinates we would expect the remainder of the metric to be a parametrization of field theory operators. To see the predictions for these operators we can expand the H function at large radius in these coordinates.

We note that the final transformation in (3.19) is rather strange since the circle $v = L$

is mapped to the real line of length $2kL$ and everything interior is mapped to exterior points to the line in Y space. Thus the V coordinates are a double cover of the Y space. In the v coordinates one can not take a probe through the enhançon so one should exclude the region $v < 1$.

At large y the v coordinate, from (3.19), is given by

$$v = \frac{2y}{k} - \frac{k\cos 2\eta}{2y} + \frac{k^3}{32y^3}(1 - 5\cos 4\eta) + \ldots. \tag{3.26}$$

Thus at large y we find, using (3.7) (3.19) and (3.25)

$$\begin{aligned} H \;=\; & \frac{L^4k^4}{16y^4} + \frac{L^6k^6}{64y^6}(-2 + 2\frac{l^2}{L^2} - \frac{2L^2}{l^2} + 8\ln(y/l) + 6\cos(2\eta)) \\ & + \frac{L^8k^8}{2^8y^8}[3(1 - \frac{l^2}{L^2} + \frac{L^2}{l^2} + 4\ln(y/L) - 2\cos 2\eta)^2 \\ & + 2\cos 2\eta(-2 + 2\frac{l^2}{L^2} - \frac{2L^2}{l^2} + 8\ln(y/l)) \\ & + (3 + 2\frac{l^2}{L^2} - 2\frac{L^2}{l^2} - 8\ln(y/L) - 8\cos 2\eta + 14\cos 4\eta)] + \ldots \end{aligned} \tag{3.27}$$

Finally we have arrived at the form for the metric we're interested in. The metric on moduli space, when written in the physical coordinates that explicitly display $\mathcal{N}$=2 supersymmetry in the brane probe, has two functions in it. One is the gauge coupling of the theory and the other, H, remains to be interpreted. We can read off the symmetry properties of operators from H using the same prescription as for the $\mathcal{N}$=4 solution; every factor of y carries mass dimension 1 and the η dependence can be interpreted as $SO(2)$ harmonics $\cos n\eta$ with charge n. Thus one would naturally like to interpret the coefficient of $\cos n\eta$, which has $U(1)$ charge n, as the operator $\langle TrX^n \rangle$ (with X the massless, two component, complex scalar field) and would expect it to be associated with a factor of $y^{(n+4)}$. The charge zero coefficients would correspond to $\langle Tr|X|^n\rangle$ again associated with a factor of y^{n+4}. There are also mixed operators of the form of a product of these two operator types as can be seen from the presence of a $\cos 2\eta$ term at order $1/y^8$. The presence of logarithms, though, undermines this interpretation. In the $l \to \infty$ limit one would expect the $\mathcal{N}$=2* theory to be on the edge of its moduli space and return to looking like the $\mathcal{N}$=4 metric. In fact at large l the leading terms in l do indeed take the form in Eqn. (2.68) but we can not neglect the $\log y$ terms in this limit which are absent from the $\mathcal{N}$=4 theory. There appears therefore to be UV logarithmic renormalization. Given that there is logarithmic renormalization we can not rule out power like renormalization either which would further confuse the interpretation.

We will make this discrepancy more manifest in the next section where we deduce the D3 brane distributions from the form of the gauge coupling and show that it does not predict the above form for the field theory operators. In the discussion we will suggest a few possible resolutions of the discrepancy.

3.3.1 D3 Distributions

To highlight the discrepancy between field theory expectations and the H function found in the $\mathcal{N}$=2* metric we will determine the D3 brane distribution function for spacetimes with different γ assuming the standard one loop renormalized expression for the prepotential governing the IR of the theory. The field theory is reviewed in [67] (see also Section 4.7) and the authors followed this logic for the special case $\gamma = 0$, where in Y space the D3 branes are distributed on a line. We extend the analysis to all γ. The prepotential for the $\mathcal{N}$=2* theory is expected to be

$$\mathcal{F} = \frac{i}{8\pi}\left[\sum_{i\neq j}(a_i - a_j)^2 \ln\left(\frac{(a_i - a_j)^2}{\mu^2}\right) - \sum_{i\neq j}(a_i - a_j + m)^2 \ln\left(\frac{(a_i - a_j + m)^2}{\mu^2}\right)\right] \tag{3.28}$$

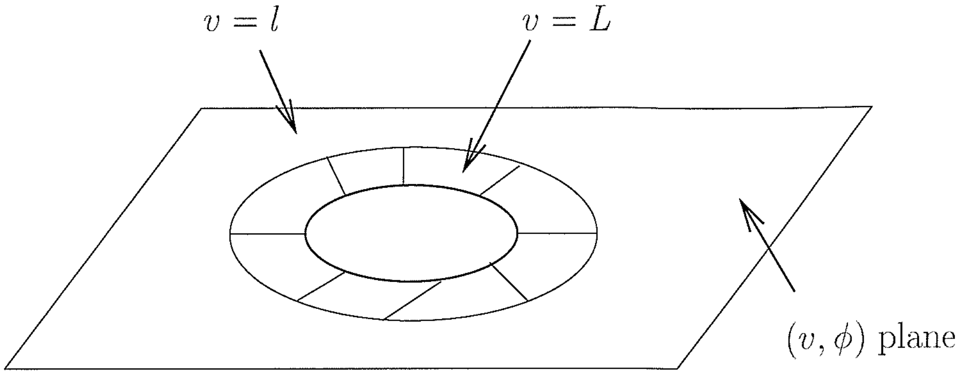

Figure 3.1: This shows the enhançon locus at $v = L$, and the branes distributed out to $v = l$.

where a_i are the scalar VEV eigenvalues and μ an RG scale. In the supergravity description of the $\mathcal{N}$=2* theory we expect the difference of the scalar VEVS to be large with respect to the mass term, and that the non-perturbative instanton corrections do not survive the large N limit [67]. With these assumptions the continuum limit found is

$$\tau(Y) = \frac{i}{g_s} + \frac{i}{2\pi}\int_{\mathcal{M}} d^2a\sigma(a)\frac{m^2}{(Y-a)^2} \tag{3.29}$$

where a is a complex 2d integral in Y space and σ the density of VEVS/D3 branes. To match with the supergravity we make the identification $m^2 = k^2\pi/L^2$ [67]. Using this ansatz we can determine the distributions that reproduce the supergravity solution expression for τ. In fact this is all but impossible in Y space since there is no spherical symmetry but we know that in V the distributions are circular out to l and cut off inside at $v = L$ (see Fig. 3.1).

Remarkably, a simple form for the density, σ, for each of the solutions, labelled by γ or equivalently l, can then be found by rewriting Eqn. (3.29) in V space using (3.19) and using

$$\sigma_v(V)vdvd\phi = \sigma_y(Y)ydyd\eta \tag{3.30}$$

Expanding the resulting expression as a power series at large y and inserting an expansion in powers of $1/v$ for $\sigma_v(V)$ one can show to all orders in the expansion that

$$\sigma_v(V) = \frac{1}{\pi(l^2 - L^4/l^2)}(1 + L^4/v^4 - 2L^2\cos(2\phi)/v^2) \tag{3.31}$$

reproduces the supergravity expression (3.21). Note that this result agrees with that of [67] for $\gamma = 0$, $l = L$; integrating with a measure vdv from $v = L$ to l and then taking the $l \to L$ limit we obtain an expression for the number of D3 branes of the form

$$N = \frac{N}{\pi}\int_0^\pi (1 - \cos 2\theta)d\theta \tag{3.32}$$

Changing variables to $y = kL\cos\theta$ this reproduces the line density in [67]

$$\sigma_y(Y) = \frac{2}{m^2}\sqrt{k^2L^2 - y^2} \tag{3.33}$$

Having identified the density we can then predict the expected scalar operators. Since the only renormalization in the $\mathcal{N}$=2* theory is that of τ we would expect the $\mathcal{N}$=4 expression for the metric quantity H when evaluated in the $\theta = \pi/2$ plane to display the full set of operators. Thus using (2.65) (with y rescaled to $2y/k$), performing the integration after a change of variables to V space using

$$y\cos\eta = \frac{kL}{2}(v + 1/v)\cos\phi, \qquad y\sin\eta = \frac{kL}{2}(1/v - v)\sin\phi \tag{3.34}$$

$$y^2 = \frac{k^2L^2}{4}(v^2 + 1/v^2 + 2\cos\phi - 2\sin\phi) \tag{3.35}$$

and further expanding at large y and evaluating the expression in the $\theta = \pi/2$ plane we obtain a prediction for H

$$
\begin{aligned}
H &= \frac{L^4k^4}{16y^4} + \frac{L^6k^6}{64y^6}(\frac{2l^2}{L^2} + \frac{2L^2}{l^2} + 6\cos(2\eta)) \\
&+\frac{k^8L^8}{2^8y^8}\left(3(\frac{l^4}{L^4} + 4 + \frac{L^4}{l^4}) + 16\frac{l^2}{L^4}(1 + \frac{L^4}{l^4})\cos 2\eta + 20\cos 4\eta\right) \qquad (3.36)
\end{aligned}
$$

This expression does not match that in (3.27) highlighting the apparent discrepancy in the interpretation of the coefficients as the scalar operators. There appears to be extra logarithmic and power renormalization in the supergravity theory that this simple field theory analysis has not explained.

3.4 Précis

Having learnt some useful techniques in the previous chapter to analyse dual solutions, we have applied it to the $\mathcal{N}$=2* gravity dual. Brane probing the solution reveals the 2d moduli space and, identifying the unique coordinates in which the U(1) theory on the probe takes an $\mathcal{N}$=2 form, the gauge coupling on that moduli space. These should be the physical coordinates in which the duality to the field theory is manifest in the rest of the metric. The metric indeed takes a form on the moduli space analogous to the metric on moduli space in the $\mathcal{N}$=4 theory except that the running of the gauge coupling is also encoded. There is one other function in the metric from which we can read off operators by their scaling dimension and their symmetry properties. In the field theory we expect the gauge coupling to be the only renormalized quantity and the operators $\langle TrX^n\rangle$ and $\langle Tr|X|^n\rangle$ to emerge as in the $\mathcal{N}$=4 case. In fact we find further renormalization including UV logarithmic renormalization.

The appearance of this extra renormalization is frustrating because it stops us from completely understanding the prescription for creating a gravity dual to a field theory even in the next simplest case to the $\mathcal{N}$=4 theory. The form of the metric on moduli space in (3.22) is highly suggestive that the prescription is to encode the running coupling as shown and then parametrizes the scalar VEVS in the field theory through H. It may be that the discrepancies we have seen are complications brought in by the 5d supergravity approach to constructing the dualities. One possibility is that we have not only introduced a mass term into the field theory. In the $\mathcal{N}$=4 theory when one attempts to introduce a dimension 2 operator at the level of 5d supergravity, after the lift to 10d, a whole host of higher dimension operators are found to be present to make the solution consistent (as can be seen in Eqn. (2.68)). Something similar may be happening here and the $\mathcal{N}=2^*$ solution is encoding both the field theory scalar vevs and this unknown tower of deformations.

An alternative possibility is that the 5d supergravity solution was created in the coordinates V which are a double cover of the physical coordinates Y. We have excised the solution interior to $v = L$ but possibly there is additional interior structure which in the Y coordinates is projected to large y. Possibly in the physical coordinates there are D3 branes through out the whole space!

As a final remark, note the holomorphic change of coordinates Eqn (3.19). We will see in the next chapter a pure $\mathcal{N}$=2 SYM theory that arises from a totally different construction, but that has this same change of coordinates.

Chapter 4

$\mathcal{N}$=2 SYM from Wrapped 5-Branes

Having considered a larger class of field theories by deforming the original $\mathcal{N}$=4 SYM with relevant operators, it is also interesting to look at direct proposals of other field theories with dual gravity descriptions. In this respect, we will discuss a proposal that is based on the *little string theory* [70, 71, 72, 73] which captures the features of a gauge theory in its IR limit (see also [74, 75]). Consequently, in the the UV limit it returns to being a string theory (contrast this with the original conjecture where the gauge theory remains intact at all energies).

Let us remark briefly here the features which will be discussed in detail subsequently. Firstly, facts about the little string theory will be reviewed so the proposal should have context. Then we discuss how branes are wrapped and appropriate Calabi-Yau theory [72] together with why the theory should be twisted so as to preserve supersymmetry. At this point we discuss gauged $D = 7$ supergravity [73], a consistent truncation which allows an unambiguous $D = 10$ solution. We talk about obtaining specific solutions by looking at the supersymmetry variations, and then analyzing the solution by brane probing and comparison to $\mathcal{N}$=2 gauge theory results. The end result is seeing that the supergravity fields are shown to parametrize the gauge theory

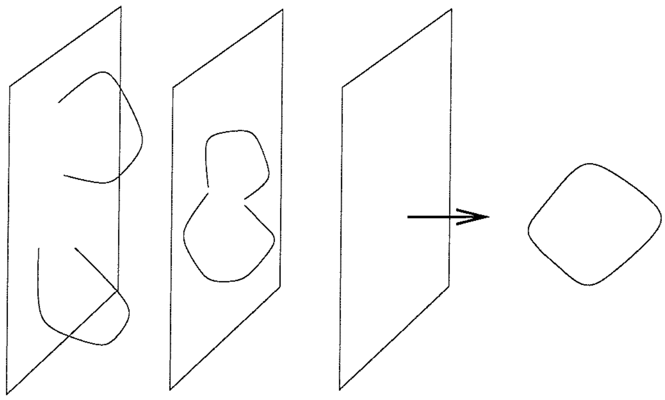

Figure 4.1: Two open strings on the brane come together to form a closed string, which is subsequently emitted into the bulk.

operators that have switched on.

4.1 Little String Theory (LST)

The starting point is to consider the decoupling limit of N parallel stacked NS5-branes [76, 77] that takes place in a vacuum of type IIB string theory. The five-branes halve the amount of supersymmetry from 32 to 16. To decouple the dynamics on the brane from that in the bulk a limit is chosen where modes on the brane that are emitted into the bulk as closed strings are suppressed (see fig. 4.1) i.e.

$$\begin{aligned} g_s &\rightarrow 0; \\ El_s &= fixed. \end{aligned} \tag{4.1}$$

Here, the emission process is at an energy E. Since the amplitudes for emission are proportional to g_s they will vanish. By considering the low energy limit on the branes we find an interacting QFT. This is $D = 6$ SYM with 16 supercharges [1]. So the full theory on the branes must be interacting. Compare this to the limit which is taken for D-brane physics; to decouple from the bulk a low energy limit is taken

$$\begin{aligned} g_s &= fixed; \\ El_s &\to 0. \end{aligned} \tag{4.2}$$

This is the decoupling limit of a QFT from gravity. If instead of a single D-brane, there were N, a similar decoupling limit to the LST is

$$\begin{aligned} g_s &\to 0; \\ El_s &= fixed \\ g_s N &= fixed. \end{aligned} \tag{4.3}$$

Here, $g_s N$ is the open string coupling constant and is fixed, so the theory on the open strings is interacting.

What can we write down as regards the little string theory? To study this theory a holographic dual description is used - this is a generalization of the AdS/CFT correspondence. It asserts that the LST is equivalent to 10d string theory in the background of five-branes. Therefore we must write down the metric, dilaton and NS B-field for the stack of NS5-branes [76]

$$\begin{aligned} ds^2 &= dx_6^2 + (1 + \frac{N\alpha'}{r^2})(dr^2 + r^2 d\Omega_3^2) \\ e^{2\Phi} &= g_s^2(1 + \frac{N\alpha'}{r^2}) \qquad (4.4) \\ H_{ijk} &= -\epsilon_{ijkl}\partial^l \Phi. \qquad (4.5) \end{aligned}$$

[1]One can see that this decoupling limit motivates the field theory. However, standard field theory in $D = 6$ does not make sense, since it's not renormalizable. The UV completion of this to an interacting string theory is able to fill out the sickness of the $D = 6$ SYM

This we see gives a breaking of the global symmetry group

$$SO(1,9) \to SO(1,5) \times SO(4), \tag{4.6}$$

which matches the $D = 6$ field theory requirements; the $SO(4)$ is the R-symmetry group for the 4-scalars (these parametrize brane motion in the transverse space) and the 8-fermions.

We now want to take the decoupling limit but to do so, the near horizon geometry $r \to 0$ needs to be taken at the same rate as the decoupling. With the change of coordinates $r \equiv g_s e^\rho$ this limit yields,

$$\begin{aligned} ds^2 &= dx_6^2 + N\alpha'(d\rho^2 + d\Omega_3^2) \\ \Phi &= -\rho. \end{aligned} \tag{4.7}$$

Here the B-field has been omitted. This concludes the discussion of the LST, the essential fact being the form of the metric and dilaton above in the decoupling limit.

4.2 NS5-Branes on S^2

So far, we have considered a brane whose world-volume is $\mathbf{R}^6$, and this is where the interacting SYM with 16 supercharges lives at low energy (in the UV, it is the non-local LST). To obtain a $D = 4$ theory, we dimensionally reduce this on a 2-cycle (a 2d compact submanifold Σ)

$$\mathbf{R}^6 \to \mathbf{R}^4 \times \Sigma, \tag{4.8}$$

In fact the 2-cycle is chosen to be S^2 for reasons to be discussed [72]. The metric to consider is then

$$dx_6^2 = dx_4^2 + N\alpha' e^{2h(\rho)} d\Sigma_2^2. \tag{4.9}$$

This embodies the above topology and that the N 5-branes have been wrapped on the Σ. The $e^{2h(\rho)}$ is a factor which will be given by the specific theory we want to study. For the moment though let us stick with Σ, and discuss when it's S^2 later. There are now two relevant points to ensure that we are really studying $\mathcal{N}$=2 SYM:

- How to ensure the field theory has $\mathcal{N}$=2 supersymmetry in $d = 4$, given that we are now on a partially compact manifold [70]
- what this means in the embedding of this in the supergravity derived from string theory [72].

The first is answered by making the field theory twisted, whilst the second is given by five-branes wrapping a 2-cycle in a Calabi-Yau 2-fold. These we now discuss.

4.2.1 Field Theory Considerations

Suppose we have a supersymmetric field theory, and we then allow it to be on a curved manifold. Then generically supersymmetry is no longer preserved as covariantly constant spinors are not necessarily admitted [70]

$$\nabla_M \epsilon = (\partial_M + \omega_M)\epsilon \neq 0. \tag{4.10}$$

Something has to be added to kill off the spin connection. Since the field theory we want to consider has a global R-symmetry group $SU(2)_R$ we can gauge this symmetry and thereby introduce an external gauge field which couples to the R-symmetry current. A new covariant derivative is formed

$$\nabla_M \epsilon = (\partial_M + \omega_M - A_M)\epsilon, \tag{4.11}$$

and for $\omega_M = A_M$, we now *can* find a covariantly constant spinor i.e. just a constant spinor. This theory is said to be "twisted" (our case is partially twisted because we are only twisting on the compact part of the product manifold) in that it changes all the spins of the fields, so the supersymmetry parameter becomes a scalar (for a fuller discussion of how a field theory is "twisted", see appendix B.1.1). This can be seen by noticing the $\mathcal{N}$=2 R-symmetry index becomes spinorial under the twist (in anticipation, this mechanism of branes wrapping cycles in string theory, is what allows some supersymmetry to be preserved [78]). Next we should ask about the details of this and how to retain the right amount of supersymmetry.

Firstly, the flat 5-branes preserve 16 supercharges. Next we perform the wrapping on the S^2, so the global symmetry groups are reduced to

$$SO(1,5) \times SO(4) \rightarrow SO(1,3) \times SO(2) \times SO(4). \tag{4.12}$$

Now the details of the twist are required. Following the approach of [70], we want to pick a $SO(2)$ in $SO(4)$ such that the breaking is $SO(4) \rightarrow SO(2)_1 \times SO(2)_2$. These will rotate the 4 coordinates in the 12-plane and 34-plane respectively. Now, write down a covariant derivative for a field Ψ that has spin s under the $SO(2)_\Sigma$ spin connection, and charge q under the $U(1) = SO(2)_1$ of the external gauge field. Then the covariant derivative in the Σ directions for the field is

$$\nabla_\mu \Psi = (\partial_\mu + is\omega_\mu + iqA_\mu)\Psi. \tag{4.13}$$

For spinors with $s = -q$ and $A_\mu = \omega_\mu$, it can be covariantly constant. So the twisting procedure turned them into scalars as the covariant derivative became a partial derivative. The tangent and normal bundles symmetry group $SO(1,5) \times SO(4)$ is decomposed to $SO(1,3) \times SO(2)_\Sigma \times U(1) \times SO(2)_2$. The preserved spinors transform as $(\mathbf{4}, \pm, \mp, \mathbf{2})$, so there are 8-spinors left as required to have $\mathcal{N}$=2 in $d = 4$. We now

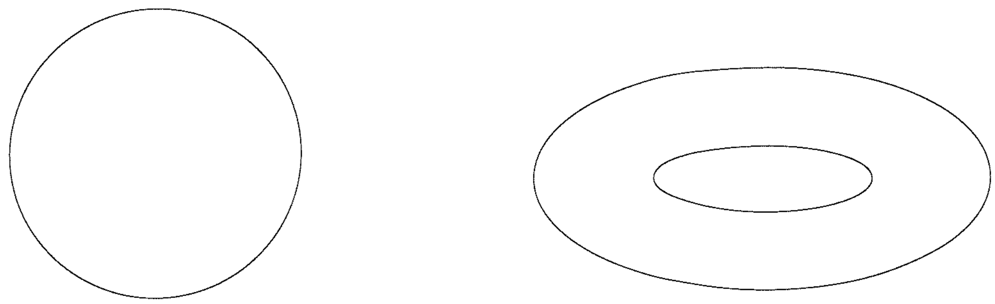

Figure 4.2: The genus $g = 0$ surface is rigid, and 2 of the 4 scalars have massive fluctuations; whilst for the torus, the interior circle can fluctuate giving 4 massless scalars.

need to check the scalars. These transform as **4** of $SO(4)$ that after twisting become $(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})$ of $SO(2)_1 \times SO(2)_2$. The S^2 is rigid so the first set of scalars have no zero modes; that is they can't fluctuate. The latter 2 scalars can so we get two massless $d = 4$ scalars. In the field theory this is clear, because the $U(1)$ "gauge field" we introduced will act as a mass term, whilst there is no such factor occurring for the $SO(2)_2$ symmetry. The gauge field on the S^2 also has no zero modes, so we are left with pure $\mathcal{N}$=2, SYM. If wrapping had been done on some genus-g 2-cycle, Σ_g, there would now be zero modes both from fluctuations of it and the gauge field. This would lead to g-additional hypermultiplets in the adjoint representation.

4.3 Gravity Considerations for Twisting

Having given a field theory description of how to obtain the gauge theory, this now needs to be translated into a gravity and brane picture; that is we want to move off the branes and into the bulk! This is going to be based on the discussion in [72, 78]. Firstly note for the case of a 5-brane whose world volume is W in the $D = 10$

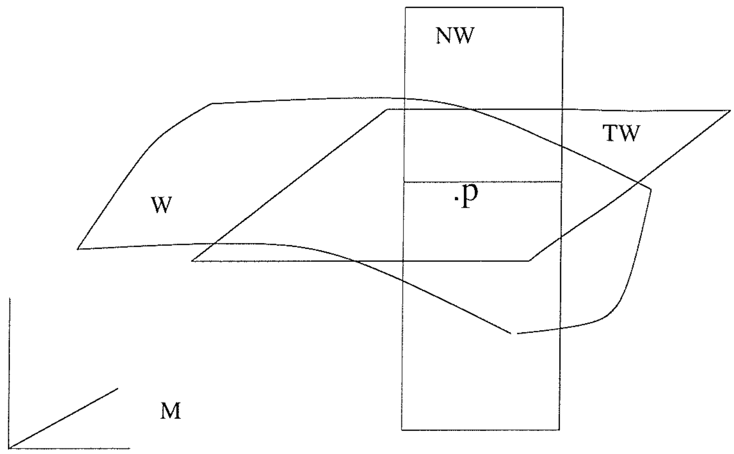

Figure 4.3: This shows how the tangent bundle is decomposed into normal and parallel pieces.

spacetime manifold M, we can decompose the tangent bundle as

$$TM = TW \oplus N_W \tag{4.14}$$

that is a normal bundle and "parallel" bundle. See Fig. 4.3. In fact if we recall that a section of a fiber bundle is a map from the fiber to the base space, we see that the 4 scalars are precisely the sections on the N_W. The normal bundle is dimension 4 and has an $SO(4)$ symmetry group (if we considered the spin connection here it would be a $SO(4)$ gauge field). This symmetry group corresponds to the $SO(4)_R$ symmetry group of the previous stated field theory. We wish to perform the same split here as was done for the field theory twisting. So one wishes to perform a split

$$SO(4) \to SO(2)_K \times SO(2)_T. \tag{4.15}$$

The last factor is going to describe the remaining flat directions in the transverse space, which is the remaining R-symmetry, whilst the first factor is going to allow us to perform the twist in a gravity context. This then is going to be a question of supersymmetry preserving, and immediately brings to the front the issue of Calabi-

Yau compactification [78].

Suppose we consider a spacetime of the form

$$M = \mathbf{R}^4 \times K^6. \tag{4.16}$$

It is clear since we are considering 5-branes, that the cycle Σ_g on which the branes are wrapped must be in the manifold K^6

$$\Sigma_g \subset K^6. \tag{4.17}$$

At this point, the supersymmetry mapping between the two systems need to be laid down. The Σ_g needs to be a supersymmetric cycle, so that supersymmetry is preserved on it. From the split made in Eqn. (4.15) and that we want 8 supercharges to be preserved, it is necessary that the cycle be in a Calabi-Yau 2-fold, so the geometry looks locally like

$$M = \mathbf{R}^4 \times (C.Y.)_{2-fold} \times \mathbf{R}^2. \tag{4.18}$$

The Calabi-Yau condition, that the *holonomy* group of the 2-fold is $SU(2)$ and not $U(2)$ (the $SO(4)$ symmetry of the normal bundle has the proper subgroup $U(2)$, which would be the holonomy group). The extra $U(1)$ piece, which can be seen to be the $SO(2)_K$ of the split, should therefore have its tangent bundle identified with the $SO(2)_\Sigma$ tangent bundle on the Σ_g. This is what corresponds to the field theory twisting, and we can easily see that they match up; the $SO(2)$ spin connection on the sphere is obvious, whilst the R-symmetry gauge field is the connection for the two normal directions to the brane in the Calabi Yau 2-fold.

4.4 The $\mathcal{N}$=2 Supergravity Solutions

4.4.1 The 7d Background

At this point in the construction the field theory and holographic dual features are clear. It is necessary now to follow the approach in [70, 71] in order to capture the details of the dual solution; namely, 7d gauged supergravity is used (we want to work with the gauged version because it will be these gauge fields that will allow us to perform the twisting in the supergravity theory) to find the specific solution, and then this can be lifted to 10d [79] by consistent truncation, to exhibit the full dual nature of the solution.

Let us collect here the relevant facts about the supergravity solutions obtained in [72, 73], using the conventions of [73]. The seven dimensional metric ansatz is (in the string frame)

$$ds_7^2 = dx_4^2 + N\alpha' e^{2h} d\Omega_2^2 + N\alpha' d\rho^2, \tag{4.19}$$

and to work with the supergravity, this should be put into the Einstein frame

$$ds_7^2 = e^{2f} dx_4^2 + N\alpha' e^{2g} d\Omega_2^2 + N\alpha' e^{2f} d\rho^2, \tag{4.20}$$

with $f = -2\Phi_7/5$ and $g = h - 2\Phi_7/5$.

By looking at the supersymmetry variations of the fermionic fields, and setting these to zero, first order equations can be obtained [73] for the bosonic fields describing $\mathcal{N}$=2 preserving deformations (see Appendix C.1 for the general ideas). Differently in [72], a first order Hamiltonian approach is used to obtain these first order equations, and the amount of supersymmetry preserved is checked at the 10-d level. The fields considered in [72, 73] are the scalars $(\lambda_1, \lambda_2, \tilde{\lambda}_2)$, the $U(1)$ gauge field $A_\mu^{(1)}$, a second $U(1)$ gauge field $A_\mu^{(2)}$ that is generally broken, and the metric ansatz Eqn. (4.20) (which is used in an obvious vielbein form to derive the spin connection). There are now two ways to obtain the $D = 7$ Lagrangian and supersymmetry variations. We could work directly with the $D = 7$, $SO(4)$ gauged supergravity in [82], just as would be done with the usual AdS/CFT picture. Or slightly non-obviously, we could perform a singular limit [80] in M-theory with M5-branes to reduce it to type IIB with NS5-branes [73]. The M-theory sector to consider initially is the truncation to $D = 7$ $SO(5)$ gauged supergravity [81, 70]. This arises from compactifying on S^4, and the singular limit to perform is [80]

$$S^4 \to S^3 \times \mathbf{R}. \tag{4.21}$$

This limit then has to be applied to the supersymmetry variations, which are found to be

$$\begin{aligned}
\delta\psi_\mu &= \nabla_\mu\epsilon + \frac{k}{2}(A_\mu^{(1)}\Gamma^{12} + A_\mu^{(2)}\Gamma^{34})\epsilon \\
&+\frac{1}{2}\gamma_\mu\gamma^\nu\partial_\nu(\lambda_1 + \frac{1}{2}(\lambda_2 + \tilde{\lambda_2}))\epsilon \\
&+\frac{1}{2}\gamma^\nu e^{-2\lambda_1}F_{\mu\nu}^{(1)}\Gamma^{12}\epsilon + \frac{1}{2}\gamma^\nu e^{-\lambda_2-\tilde{\lambda_2}}F_{\mu\nu}^{(2)}\Gamma^{34}\epsilon \\
\delta\lambda^{(1)} &= -\frac{1}{4}\gamma^\mu\partial_\mu(3\lambda_1 + \lambda_2 + \tilde{\lambda_2})\epsilon - \frac{1}{8}\gamma^{\mu\nu}e^{-2\lambda_1}F_{\mu\nu}^{(1)}\Gamma^{12}\epsilon + \frac{m}{4}e^{2\lambda_1}\epsilon \\
\delta\lambda^{(2)} &= -\frac{1}{4}\gamma^\mu\partial_\mu(2\lambda_1 + 2\lambda_2 + \tilde{\lambda_2}))\epsilon + \frac{m}{4}e^{2\lambda_2}\epsilon \\
\delta\lambda^{(3)} &= +similar(\lambda_2 \leftrightarrow \tilde{\lambda_2}).
\end{aligned} \tag{4.22}$$

Inserting into these variations the form of the metric and the twisting conditions (all gauge fields taken to be zero except the $A_\mu^{(1)}$ field), as well was imposing a specific projection ansatz on the spinors [73] reduces this system to a simple system of first order coupled equations

$$f' = -({\lambda_1}' + \frac{{\lambda_2}' + \tilde{\lambda}_2'}{2}), \tag{4.23}$$

$$g' = -({\lambda_1}' + \frac{{\lambda_2}' + \tilde{\lambda}_2'}{2}) + \frac{1}{2\Lambda}e^{f-2g-2\lambda_1}, \tag{4.24}$$

$$\lambda_2' + 2\tilde{\lambda}_2' + 2\lambda_1' = -\frac{1}{\Lambda} e^{f+2\tilde{\lambda}_2}, \tag{4.25}$$

$$2\lambda_2' + \tilde{\lambda}_2' + 2\lambda_1' = -\frac{1}{\Lambda} e^{f+2\lambda_2}, \tag{4.26}$$

$$3\lambda_1' + \lambda_2' + \tilde{\lambda}_2' = -\frac{1}{\Lambda} e^{f+2\lambda_1} + \frac{1}{2\Lambda} e^{f-2g-2\lambda_1}. \tag{4.27}$$

A prime indicates differentiation with respect to the radial coordinate ρ, and Λ is a dimensionful constant (in fact it can be seen from the LST that $\Lambda^2 = N\alpha'$, which will play the role of the strong coupling scale in the dual field theory).

Note that λ_2 and $\tilde{\lambda}_2$ enter the equations in a symmetrical way. This full set of equations was studied in [73] whilst in [72] only the case $\lambda_2 = \tilde{\lambda}_2$ was considered. Defining the dimensionless radial coordinate u, and making the change from $\rho \to u$;

$$u \equiv e^{2h}, \tag{4.28}$$

$$\frac{\Lambda}{H(u)} \equiv \frac{d\rho}{du} \equiv \Lambda e^{\lambda_1 - 1/2(\lambda_2 + \tilde{\lambda}_2)} \tag{4.29}$$

We can extract solutions for a number of quantities we shall use later. It is straightforward to show, from the field equations above, that

$$e^{-\lambda_2 + \tilde{\lambda}_2} = \frac{e^{2u} + b^2}{e^{2u} - b^2}. \tag{4.30}$$

where b is an integration constant, and

$$e^{-4\lambda_1 - 2\lambda_2 - 4\tilde{\lambda}_2} = c e^{-2u} \left(e^{2u} - b^2\right)^2 \tag{4.31}$$

where c is an integration constant, which only appears as an overall factor in the analysis below so we set it to one.

The final solution we will need is an expression for the function H which satisfies the equation

$$\frac{dH^2}{du} + H^2\left(\frac{1}{u} + 2\left(\frac{e^{4u}+b^4}{e^{4u}-b^4}\right)\right) = 2, \tag{4.32}$$

We can explicitly solve this finding

$$H^2(u) = \left[\frac{e^{4u}+b^4}{e^{4u}-b^4} - \frac{1}{2u} + \frac{2K}{u\Lambda^2}\left(\frac{e^{2u}}{e^{4u}-b^4}\right)\right]. \tag{4.33}$$

where K is again an integration constant (we have scaled K by Λ so it has the correct dimensions below). Note that when $\lambda_2 = \tilde{\lambda}_2$ as investigated in [72] $b = 0$ and in this limit we recover their solution

$$\lim_{b\to 0} H^2(u) = 1 - \frac{1}{2u} + \frac{2Ke^{-2u}}{u\Lambda^2}. \tag{4.34}$$

4.4.2 The 10d Background

To brane probe these solutions they must be lifted to $D = 10$ [79]. The lift was performed in [73] and we have the string frame solution

$$\begin{aligned} ds^2 &= ds_7^2 + e^{2\lambda_1+\lambda_2+\tilde{\lambda}_2}\Delta^{-1}[e^{-2\lambda_1}[d\mu_1^2 + d\mu_2^2 + \cos^2\theta(\mu_1^2+\mu_2^2)d\phi^2 \\ &\quad -2\cos\theta(\mu_1 d\mu_2 + \mu_2 d\mu_1)d\phi] + e^{-2\lambda_2}d\mu_3^2 + e^{-2\tilde{\lambda}_2}d\mu_4^2], \end{aligned} \tag{4.35}$$

with

$$e^{2\Phi} = e^{6\lambda_1+3\lambda_2+3\tilde{\lambda}_2}\Delta^{-1}, \tag{4.36}$$

$$\Delta = e^{2\lambda_1}(\mu_1^2+\mu_2^2) + e^{2\lambda_2}\mu_3^2 + e^{2\tilde{\lambda}_2}\mu_4^2. \tag{4.37}$$

The additional S^3 parameterization is given by the coordinates μ_i, such that $\sum_{i=1}^4 \mu_i^2 = 1$. These can be written in terms of the usual three angles,

$$(\mu_1, \mu_2) = \sin\psi(\cos\phi_1, \sin\phi_1) \tag{4.38}$$

$$(\mu_3, \mu_4) = \cos\psi(\cos\phi_2, \sin\phi_2) \tag{4.39}$$

These solutions describe the near horizon geometry of NS5 branes wrapped on S^2. To convert from a NS5 solution to a D5 solution (which is more appropriate in the IR) one performs the S-dual transformations [73]

$$\begin{aligned} \Phi_D &= -\Phi, \\ ds_D^2 &= e^{\Phi_D} ds_{NS}^2. \end{aligned} \tag{4.40}$$

There is also a 6 form potential for which the D5 branes are sources. The full expression is not given in [73] but the components in the D5 world volume, when $\mu_1 = \mu_2 = 0$, relevant to the brane probe analysis below is given by

$$A_6 = R^2 e^{2\Phi_D} u \; dx_4 \wedge d\Omega_2 \tag{4.41}$$

At this point then, we have the string frame $D5$ solution, which we can now proceed to brane probe. The full details of these solutions and lifts can be found in [73].

4.5 The $D = 2$ Moduli Space and Distributions

Each geometry, corresponding to a solution of (4.27), is expected to be dual to the $\mathcal{N} = 2$ SYM theory at a point on its 2d moduli space. Since the $\mathcal{N} = 2$ SYM theory has a moduli space, each of these solutions should then display a 2d space in which a probe D5 brane sees a flat potential. This corresponds to the theory knowing that any individual scalar vev may be changed at will on the moduli space whilst keeping a vacuum. At large N the changing of such a single VEV, or position of a D5, will leave the geometry unchanged. In [73] it was shown that such a $D = 2$ moduli space does indeed exist for all of these solutions.

For a single wrapped D5-brane, we have the low energy effective Born-Infeld action

$$S_{probe} = -T_5 \int d^6\xi e^{-\Phi_D} \sqrt{-\det(G_{ab} + F_{ab})} + Q_5 \int A_6. \tag{4.42}$$

where ξ are coordinates on the brane, G_{ab} is the pullback of the $10d$ spacetime metric, F_{ab} is the surface gauge field strength and $T_5 = Q_5 g_s^{-1}$.

As in [73], setting $\mu_1 = \mu_2 = 0$, and substituting in the background we find the gauge potential cancelling against the leading term from the expansion of the square root. Thus the (u, ϕ_2) plane is the moduli space. From henceforth we restrict ourselves to this moduli space since only on this space can we use field theory intuition in the probe world volume theory to find the correct coordinates in which to interpret the duality.

If we allow the probe brane to move slowly on the moduli space and also allow small gauge fields on its surface we can find the leading kinetic terms in the probe world volume theory. Passing to a new radial coordinate $u = \ln(z/\Lambda)$, we may write the kinetic piece in the form

$$S_{probe} = -T_5 R^2 \int d\Omega_2 d^4x \mathcal{L} \tag{4.43}$$

where

$$\begin{aligned}\mathcal{L} &= \frac{\ln(z/\Lambda)}{(z/\Lambda)^2} e^{-4\lambda_1 - 2\lambda_2 - 4\tilde{\lambda_2}}[\cos^2\phi_2 + e^{-2\lambda_2 + 2\tilde{\lambda_2}}\sin^2\phi_2](\dot{z}^2 + z^2\dot{\phi_2}^2) \\ &\quad + \frac{1}{4}\ln(z/\Lambda) F^{\mu\nu}F_{\mu\nu}.\end{aligned} \tag{4.44}$$

Note that it was the choice of coordinate transformation in (4.29), which allowed us to factor out the scalar terms between the coordinate u and the angular pieces μ_1, μ_2.

Using the field equations in (4.30)(4.31) we can evaluate this to be

$$\begin{aligned}\mathcal{L} &= \frac{\ln(z/\Lambda)}{(z/\Lambda)^4}[(z^2/\Lambda^2 - b^2)^2\cos^2\phi_2 + (z^2/\Lambda^2 + b^2)^2\sin^2\phi_2](\dot{z}^2 + z^2\dot{\phi}_2^2) \\ &\quad + \frac{1}{4}\ln(z/\Lambda) F^{\mu\nu}F_{\mu\nu},\end{aligned} \tag{4.45}$$

which can be written in terms of the complex coordinate $Z = ze^{i\phi_2}$ as

$$\mathcal{L} = \ln\left(\frac{|Z|}{\Lambda}\right)\left[\left(1 - \frac{b^2\Lambda^2}{Z^2}\right)\left(1 - \frac{b^2\Lambda^2}{\bar{Z}^2}\right)\right]|\dot{Z}|^2 + \frac{1}{4}\ln\left(\frac{|Z|}{\Lambda}\right)F^{\mu\nu}F_{\mu\nu}. \tag{4.46}$$

This form for the solution does not display the explicit $\mathcal{N} = 2$ form of the field theory. To find such a form we need to pass to a new set of coordinates, W, such that the scalar and gauge kinetic pieces appear with the right normalization

$$\mathcal{L} = \frac{1}{g_{YM}^2(W)}\left(|\dot{W}|^2 + \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right) \tag{4.47}$$

The appropriate Jacobian and hence the appropriate holomorphic change of variables [73] may be seen from (4.46)

$$W = Z + b^2\Lambda^2/Z. \tag{4.48}$$

The gauge coupling now reads

$$\frac{1}{g_{YM}^2(W)} = \ln\left(\frac{|Z(W)|}{\Lambda}\right) = \cosh^{-1}(\frac{W}{2b\Lambda}) + \ln b. \tag{4.49}$$

4.6 The Geometry of the Moduli Space and D5 Distributions

We have identified the unique set of coordinates on the moduli space, W, where the field theory duality is manifest. For the brane probe to leave the world volume theory (4.47) the geometry in the sub-space corresponding to the moduli space must take the form

$$ds_D^2 = e^{\Phi_D}[dx_4^2 + \frac{1}{g_{YM}^2(W)}R^2 d\Omega_2^2 + e^{-2\Phi_D} dW d\overline{W}] \tag{4.50}$$

$$A_6 = R^2 e^{2\Phi_D} \frac{1}{g_{YM}^2(W)} dx_4 \wedge d\Omega_2 \tag{4.51}$$

The background is described by two functions. One we have identified as the gauge coupling whilst the other, e^{Φ_D}, remains to be interpreted. We may find an explicit expression for the dilaton from the metric element G_{ww} which is G_{zz} times the Jacobian for the transformation. We find in the z coordinates

$$e^{2\Phi_D} = \frac{H^2\Lambda^2}{z^2}[(z^2/\Lambda^2 - b^2)^2 \cos^2\phi_2 + (z^2/\Lambda^2 + b^2)^2 \sin^2\phi_2]. \tag{4.52}$$

and our solution for H from (4.33) in these coordinates is

$$H^2(z) = \left[\frac{z^4/\Lambda^4 + b^4}{z^4/\Lambda^4 - b^4} - \frac{1}{\ln(z/\Lambda)} + \frac{2K}{\ln(z/\Lambda)} \left(\frac{z^2}{z^4 - b^4\Lambda^4} \right) \right]. \tag{4.53}$$

To attempt to interpret this function in terms of the field theory we must translate it to the coordinates appropriate to the duality, W. In fact, simply following through the holomorphic change of coordinates, we find

$$\begin{aligned} e^{2\Phi_D} &= \frac{W\bar{W}}{\Lambda^2} G^{1/2}\bar{G}^{1/2} \left[\frac{1 + G^{1/2}\bar{G}^{1/2}}{G^{1/2} + \bar{G}^{1/2}} - \frac{1}{g_{YM}^2(W)} \right. \\ &\quad \left. + \frac{2K}{g_{YM}^2(W)} \left(\frac{2}{W\bar{W}(G^{1/2} + \bar{G}^{1/2})} \right) \right], \end{aligned} \tag{4.54}$$

where

$$G = 1 - \frac{4b^2\Lambda^2}{W^2}. \tag{4.55}$$

Let us now consider the anatomy of the solution in both the Z and W coordinates. Firstly looking in the Z coordinates the solution has no ϕ_2 dependence so the D5 brane distribution must be symmetric in the Z plane. As can be seen from (4.53) there is always a singularity in the metric at $z = b$. For large K though there can be a singularity at larger z. In fact we can trade the parameter K for the radius of the singularity z_0

$$K = \frac{z_0^4 - b^4}{4z_0^2} - \ln(z_0/\Lambda) \left(\frac{z_0^4 + b^4\Lambda^4}{2z_0^2} \right). \tag{4.56}$$

The function $\ln(z_0/\Lambda)$ when translated to the physical W coordinates has the simple interpretation of $1/g_{YM}^2$ evaluated at the position of the singularity and we will write it henceforth as $1/g_{YM}^2(\text{sing})$.

We shall interpret the singularity as indicating the position of the D5 branes. Note that any given solution only describes the space $z_0 < z < \infty$. A probe is therefore restricted to this space and thus only for $z_0 = b = 1$ can it reach the enhançon locus (a circle here) where the coupling diverges. That distribution must correspond to a singular point on the field theory moduli space.

For large z_0 we may neglect b and the distribution is essentially a circle in the physical W coordinates (since $Z \simeq W$). As z_0 reduces, the coordinate transformation to W in (4.48) distorts the circle by squashing it in the imaginary W direction. When $z_0 = b$ the singularity lies on the real line between $w = \pm 2b$.

4.7 Gauge Theory Coupling

We would like to find the explicit distribution function for the D5 branes $\sigma_w(W)$ in the physical coordinates. We can attempt to do this using the supergravity expression for g_{YM}^2 and the form of the $\mathcal{N} = 2$ field theory prediction for the coupling as a function of scalar VEVS. To this end we now give some discussion following [72, 67].

Suppose we consider an $SU(N)$ Gauge Theory on the Coulomb Branch. The moduli space is then parameterised by the $(N-1)$ scalar expectations in the Cartan Subalgebra,

$$\begin{aligned} \langle \Phi \rangle &= diag(a_1, \cdots, a_N), \quad a_i \in \mathbf{C} \\ \sum_i a_i &= 0. \end{aligned} \tag{4.57}$$

This shows that generally the gauge group undergoes breaking to $SU(N) \to U(1)^{N-1}$ and gives a low energy effective action

$$S = \frac{1}{8\pi} \int d^4x \left(-\mathrm{Im} \left[\tau_{ij} \partial a^i \partial \bar{a}^j \right] + \frac{1}{2} \mathrm{Re} \left[\tau_{ij} (i F^i F^j + F^i \tilde{F}^j) \right] \right), \tag{4.58}$$

where the coupling is determined by the holomorphic prepotential

$$\tau_{ij} \equiv \frac{\partial^2 \mathcal{F}}{\partial a^i \partial a^j}. \tag{4.59}$$

There is both a perturbative part coming from an exact 1-loop result, and a non-perturbative part that comes from instanton corrections. This was argued in [67] to vanish in the large N limit (that is provided $|a_i - a_j| > \mathcal{O}(1/N)$). The perturbative expression for the prepotential is then

$$\mathcal{F} = \frac{i}{8\pi} \sum_{i \neq j} (a_i - a_j)^2 \ln\left[(a_i - a_j)^2/\Lambda^2\right]. \tag{4.60}$$

Going back to the brane probe in the previous section, there the gauge breaking was $SU(N+1) \to U(1)^{N-1} \times U(1)$ where the last factor is the brane probe. In terms of its position u the VEV can be written as

$$\langle \Phi \rangle = diag(u, a_1 - u/N, \cdots, a_N - u/N) \tag{4.61}$$

If we now compute the gauge coupling and take the large N limit, we get the simple expression

$$\tau(u) = \frac{\partial^2 \mathcal{F}}{\partial u^2} = \frac{i}{2\pi} \sum_i \ln\left[(u - a_i)^2/\Lambda^2\right]. \tag{4.62}$$

This can now be taken to a continuum limit since N is large giving

$$\begin{aligned} \tau(u) &= \frac{i}{2\pi} \int_{\mathcal{M}} da d\bar{a}\, \sigma(a) \ln\left[(u - a)^2/\Lambda^2\right] \\ N &= \int_{\mathcal{M}} da d\bar{a}\, \sigma(a). \end{aligned} \tag{4.63}$$

We are now in position to make comparisons between the two systems. From the above we can read off

$$\frac{4\pi}{g_{YM}^2(W)} = \frac{1}{\pi} \int dA d\bar{A}\, \sigma_w(A) \ln \left(\frac{A - W}{\Lambda} \right) \tag{4.64}$$

However, given the complicated shape of the distribution in W it is easier to transform this equation to the Z coordinates where we know we have spherical symmetry

$$\frac{4\pi}{g_{YM}^2(W)} = \frac{1}{\pi} \int dZ d\bar{Z}\, \sigma_z(Z) \ln \left(\frac{Z + \frac{b^2\Lambda^2}{Z} - W}{\Lambda} \right) \tag{4.65}$$

A degree of guess work is required to find the appropriate σ_z that reproduces the coupling in (4.49). In fact the simple guess that the distribution is just a ring at $z = z_0$ reproduces the supergravity result. Thus

$$\sigma_z(Z) = 2\pi\delta(z - z_0) \tag{4.66}$$

At this stage one must take on faith that the field theory expression is relevant to the supergravity solution. In other words we have assumed the duality to obtain this result. We will now explore the scalar operators encoded in the supergravity solution and show that they are consistent with this distribution function providing a non-trivial cross check of the duality.

4.8 Gauge Theory Operators

We have written the background on the moduli space of the theory in coordinates where the gauge coupling takes the explicit form expected in the dual field theory. The background involves one other function given by (4.54) in these coordinates. If the theories are truly dual we would expect them to be different parametrizations of the same information. We should therefore be able to interpret (4.54) in terms of field theory quantities.

The coordinate W transforms under two symmetries. The first is the scaling symmetry of the 4d gauge theory, familiar from the usual AdS/CFT. It is also present in this case, as can be seen from the way that W enters the gauge coupling as an energy scale or from the requirement of a consistent scaling of the metric, but it is broken by a number of parameters. Thus $|W|$ has mass dimension one. W also transforms under the U(1) symmetry of the 2d plane which corresponds in the field theory to the U(1) symmetry on the complex scalar. So looking in (4.54) we can identify the symmetry properties of the constants and hence match them to field theory quantities. Λ has mass dimension one and is a U(1) symmetry invariant - it plays the role of the strong coupling scale in the field theory as is apparent from its appearance in the coupling (4.49). K has dimension two and is uncharged under the U(1) symmetry - it contains two components which we will shortly show can be written as chargeless scalar operators or equivalently as moments of the D5 distribution. Finally the function G contains a dimension 2 operator of charge two which we shall again match to a scalar operator.

The $\mathcal{N} = 2$ field theory on moduli space should be described by the running coupling and the scalar operators. We have deduced the distribution function for the vevs σ_z above (4.66) from the form of the running coupling and hence can calculate these functions to see if they match those in (4.54). There are two dimension two operators we can calculate corresponding in the field theory to the chargeless $sTr|X|^2$ and the charge two $sTrX^2$. We must calculate these operators in the physical coordinates, W.

$$\mathcal{O}_2 = \frac{1}{4\pi}\int \sigma_w(W)W^2 dW d\bar{W} = \frac{1}{4\pi}\int \sigma_z(Z)W(Z)^2 dZ d\bar{Z} = 2b^2\Lambda^2 \tag{4.67}$$

$$\mathcal{O}_0 = \frac{1}{4\pi}\int \sigma_w(W) W W^\dagger dW d\bar{W} = \frac{1}{4\pi}\int \sigma_z(z)|W(Z)|^2 dZ d\bar{Z} = z_0^2 + \frac{b^4\Lambda^4}{z_0^2} \quad (4.68)$$

Pleasingly these functional forms precisely match the coefficient of the gauge coupling in K (4.56) and the operator in G (4.55). We are left to explain the form of the first term in K which is not one of these moments. However, it is clear from (4.54) that the solution contains the quantity $\sqrt{sTrX^2 sTrX^{\dagger 2}}$ which is chargeless and dimension two. This first term can be written as a combination of the two chargeless operators. Thus we can write

$$K = \frac{1}{4i}(\sqrt{\mathcal{O}_0^2 - \mathcal{O}_2\mathcal{O}_2^\dagger}) - \frac{1}{2g_{YM}^2(\text{sing})}\mathcal{O}_0 \quad (4.69)$$

and

$$G = 1 - \frac{2\mathcal{O}_2}{W^2}. \quad (4.70)$$

The encoding of the operators in (4.54) is quite complicated but it is encouraging that the correspondence can be made between the two duals. It is also nice that the distribution function determined above from the gauge coupling does indeed match to the functional form of the operators parametrized by the rest of the background. Note that this constitutes the first cross check of the assumption in (4.64) that the coupling of the probe world volume theory in the supergravity background is indeed governed by the field theory expression for the running coupling.

4.9 Précis

We have studied the supergravity solutions found in [72, 73] which were obtained by studying 7d gauged supergravity and then lifting the solutions to 10d. The solutions are expected to be the near horizon geometries of D branes wrapped on S^2 and to be dual to $\mathcal{N} = 2$ SYM theory in 4d. We have identified the unique coordinates in which the theory on the world volume of a probe D5 brane takes $\mathcal{N} = 2$ form. Restricting to the subspace of the background that describes the field theory's moduli space where these coordinates are known, we have shown that the background is described by two functions. One of these is the running gauge coupling of the field theory whilst we have shown the other parametrizes the field theory operators. Using the field theory expectation for the form of the running gauge coupling as a function of the D5 distribution, that distribution can be determined. We have shown that the scalar operators corresponding to the moments of this distribution function match the form of the parameters in the second function determining the background. The end result is remarkably clean showing that the two dual descriptions do indeed encode the same physical content, as has been previously observed in $\mathcal{N} = 4$ SYM on moduli space and its gravity dual. The result also confirms that the supergravity background is controlled by the gauge theory dynamics and that the only renormalization is through the gauge coupling.

Understanding how the gravity background encodes the dual field theory operators is hopefully a major step towards enlarging the class of known solutions. In particular the function G in the background (4.54) looks ripe to be interpreted in general as an harmonic function of the two dimensional Laplacian. To confirm whether such an extension of the solution is possible requires more work than that presented here since to test a solution of the supergravity equations one needs more than a restricted subspace of the solution as we have. Understanding these backgrounds off the field theory moduli space, where the field theory is less well understood, is an important challenge for the future.

Chapter 5

A Non-Supersymmetric Deformation of $\mathcal{N}$=4

In this chapter we return to deforming the original AdS/CFT correspondence. We will in fact introduce a mass term of the form $(X^1X^1 + X^2X^2 + X^3X^3 + X^4X^4 - 2X^5X^5 - 2X^6X^6)$ which is naively unbounded. This is precisely the operator $\mathcal{O} = sTrX^iX^j$ studied in Chapter 2 as a VEV, but considered here now as an operator insertion $\mathcal{L} \to \mathcal{L} + \mathcal{O}$. Our interest is in developing the technology to find and lift these solutions to 10d so we will not be so concerned by the runaway behaviour (although the 10d solution we provide correctly reproduces the expected behaviour). One might hope that there would be such backgrounds that are really stable since an $SO(6)_R$ singlet scalar mass term is not visible in the supergravity solution as it is not in a short multiplet. Its presence could stabilize the solution. Note that the supersymmetric deformations [84, 85, 86] already mentioned require this operator to be present. In fact our brane probe potential reveals the operator not to be present in our 10d lifts. Our solution is also of interest since it is probably the simplest example of a non-supersymmetric deformation; only the metric and four potential fields are non-zero. See also [87, 88, 89, 90].

In the next section we will discuss the introduction of our deformation at the 5d supergravity level. We then lift the full solution to 10 dimensions. Brane probing the background with a D3 brane shows that asymptotically the background indeed includes the operator we hoped to introduce showing the consistency of the techniques. Finally we plot the potential seen by the probe for the full solution.

5.1 Deformations in 5d Supergravity

5.1.1 A Scalar Operator

Let us now make a particular choice for the scalar field we will consider. We take a scalar from the multiplet in the **20** of $SO(6)_R$ as in chapter 2. In particular we will choose the scalar corresponding to the operator

$$\mathcal{O} = \sum_{i=1}^{4} X^i X^i - 2 \sum_{i=3}^{4} X^i X^i \tag{5.1}$$

This scalar has been studied in chapter 2 already in its role of describing an $\mathcal{N} = 4$ preserving scalar VEV and as a mixture of a mass term and a VEV in the $\mathcal{N} = 2^*$ gauge theory of chapter 3. The potential for the scalar, which we will write as $\rho = e^{\lambda/\sqrt{6}}$ is given by

$$V = -\frac{1}{\rho^4} - 2\rho^2 \tag{5.2}$$

and the three equations of motion become

$$\frac{\rho''}{\rho} - \left(\frac{\rho'}{\rho}\right)^2 + 4\frac{\rho'}{\rho}A' = \frac{\rho}{6}\frac{\partial V}{\partial \rho} \tag{5.3}$$

$$6A'^2 - 6\left(\frac{\rho'}{\rho}\right)^2 = -2V \tag{5.4}$$

$$A'' = -4\left(\frac{\rho'}{\rho}\right)^2 \tag{5.5}$$

The last of these is the sum of (1.76) and (1.77). The asymptotic ($r \to \infty$) solutions take the form

$$\lambda = \mathcal{A}e^{-2r} + \mathcal{B}re^{-2r} \tag{5.6}$$

with $\mathcal{A}$ the scalar VEV and $\mathcal{B}$ a mass term for the operator $\mathcal{O}$.

In the special case where only the first part of the solution is present the deformation preserves $\mathcal{N} = 4$ supersymmetry. The superpotential is

$$W = -\frac{1}{\rho^2} - \frac{1}{2}\rho^4 \tag{5.7}$$

and the second order equations reduce to the first order equations

$$\frac{\partial\rho}{\partial r} = \frac{1}{3}\left(\frac{1}{\rho} - \rho^5\right), \qquad \frac{\partial A}{\partial r} = \frac{2}{3}\left(\frac{1}{\rho^2} + \frac{1}{2}\rho^4\right) \tag{5.8}$$

with solution [57]

$$e^{2A} = l^2\frac{\rho^4}{\rho^6 - 1} \tag{5.9}$$

with l^2 a constant of integration.

5.1.2 Non-supersymmetric First Order Equations

In [91] it was pointed out that using Hamilton Jacobi theory the second order equations could be replaced by a system of first order equations. They further stated that a "superpotential", W, could be found which resulted in the equations (1.79) even for the non supersymmetric solution with only $\mathcal{B}$ switched on. A similar result was obtained in [92, 93] but as a requirement for the RG flow solution to be stable. Reducing the equations to first order would be very helpful, but the system we discuss here can not be.

Consider the UV of the theory where, expanding (5.2)

$$V = -3 - 2\lambda^2 + \sqrt{\frac{8}{27}}\lambda^3 + ... \tag{5.10}$$

we can attempt to find a superpotential W that reproduces this potential via the trial form

$$W = a + b\lambda^2 + c\lambda^3 + ... \tag{5.11}$$

Working to quadratic order one finds

$$a = -3, \qquad b = -2 \tag{5.12}$$

The solution for b comes from a quadratic equation with degenerate roots hinting at the two forms of the solution. However, it is then easy to show that at higher orders there is a unique series (eg $c = \sqrt{2/27}$) and it is simply the supersymmetric solution. We have therefore not been able to find a superpotential that describes the non-supersymmetric solution and are forced to numerically solve the second order equations. Of course our geometry is intrinsically unstable since we have introduced an unbounded operator in the field theory. Apparently the stability of the flow is essential for the system to reduce to first order.

5.1.3 Numerical Solutions

The second order equations of motion are easily solved. In figure 1 we show the numerical behaviour of ρ. For this plot we fix $\rho(r = \Lambda_{UV})$ and vary the derivative. The purely VEV supersymmetric solution ($\mathcal{B} = 0$) and purely masslike case ($\mathcal{A} = 0$) are labelled. The three regions (bounded by the $\mathcal{A} = 0$ and $\mathcal{B} = 0$ curves) correspond to

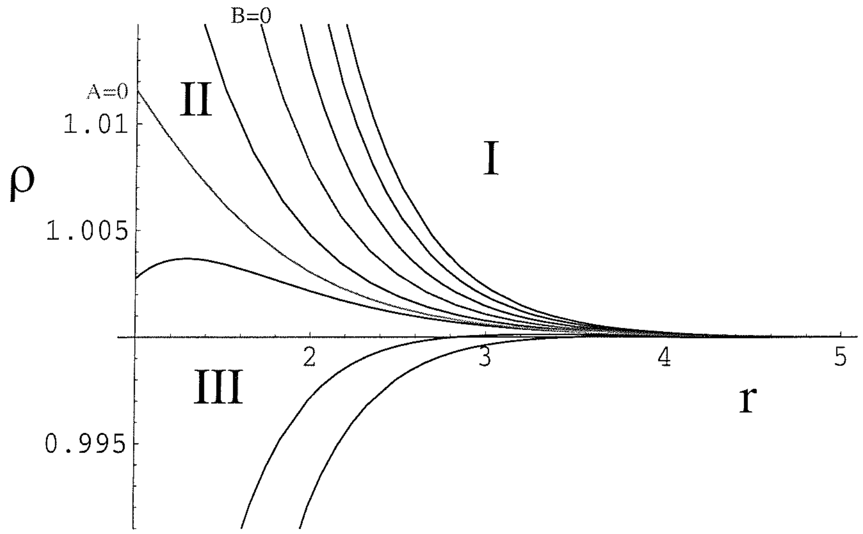

Figure 5.1: Plots of ρ vs r for a variety of initial conditions on ρ'. The VEV only initial condition solution is marked with $\mathcal{B} = 0$ and mass only initial condition with $\mathcal{A} = 0$. The marked regions are explained in (5.13).

$$\begin{array}{ccc} & \mathcal{A} & \mathcal{B} \\ & & \\ I & +ve & -ve \\ II & +ve & +ve \\ III & -ve & +ve \end{array} \tag{5.13}$$

In each case the function $A(r)$ deviates from $A(r) \sim r$ by a small amount so a plot is unrevealing. Note that most of these solutions become singular before $r = 0$. When lifted to 10d this singular point is expected to correspond to the position of the D3 brane sources in the transverse space. For most of these solutions there is a scalar VEV and so the D3 branes are expected to have moved away from the origin. The mass only solution ($\mathcal{A} = 0$) on the other hand can be extended to $r = 0$ which is consistent with the D3 branes being pinned at the origin.

It has proven difficult to extract aspects of the field theory from the 5d supergravity backgrounds. More success has been had at the 10d level where techniques such as brane probing can be used to connect to the field theory. We shall therefore move to discussing the lift of these solutions to 10d in the next section.

5.2 The 10d Background

To lift the 5d solution to 10d requires the procedure outlined in [94]. Finding the metric is complicated but we will be able to short cut the process since the lift of the 5d solution where the $\mathcal{N}=4$ theory is on moduli space has already been written down. In particular the solution where our scalar corresponds to a VEV has been studied in [57, 1] (it is also the limit of the metrics in [85, 86] with some of the fields switched off). That solution is given by

$$ds^2 = \frac{X^{1/2}}{\rho} e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{X^{1/2}}{\rho}\left(dr^2 + \frac{L^2}{\rho^2}\left[d\theta^2 + \frac{\sin^2\theta}{X} d\phi^2 + \frac{\rho^6 \cos^2\theta}{X} d\Omega_3^2 \right]\right), \tag{5.14}$$

where $d\Omega_3^2$ is the metric on a 3-sphere and

$$X \equiv \cos^2\theta + \rho^6 \sin^2\theta \tag{5.15}$$

For consistency there must also be a non-zero A_4 potential of the form

$$A_4 = \frac{e^{4A} X}{g_s \rho^2} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \tag{5.16}$$

Note that the solution has the same $SO(2) \times SO(4)$ symmetry as our operator Eqn. (5.1).

Clearly the lift of the full solution of the second order equations has this as a limit.

In fact the procedure for finding the form of the metric does not depend on the supersymmetric solution and we may take it over directly to our case. The A_4 potential though will change since the supersymmetric first order equations of motion were used in its derivation [57, 85].

In fact the 10d supergravity equations of motion we must concern ourselves with are relatively few since only the metric and A_4 are non-zero. There are the Einstein equations

$$R_{MN} = T_{MN} = \frac{1}{6} F_N{}^{PQRS} F_{PQRSM} \tag{5.17}$$

and

$$F_{(5)} =^* F_{(5)}, \qquad dF_{(5)} = 0 \tag{5.18}$$

The self duality condition can be imposed by using the ansatz

$$F_{(5)} = \mathcal{F} +^* \mathcal{F}, \qquad \mathcal{F} = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dw \tag{5.19}$$

where $w(r, \theta)$ is an arbitrary function.

There are three independent non-zero elements of R_{MN} which factorize into the useful equations

$$R^0{}_0 + R^r{}_r = \frac{1}{2} g^{00} g^{11} g^{22} g^{33} g^{rr} \left(\frac{\partial w}{\partial r} \right)^2 \tag{5.20}$$

$$R^0{}_0 - R^r{}_r = \frac{1}{2} g^{00} g^{11} g^{22} g^{33} g^{\theta\theta} \left(\frac{\partial w}{\partial \theta} \right)^2 \tag{5.21}$$

$$R^r{}_\theta = \frac{1}{2} g^{00} g^{11} g^{22} g^{33} g^{rr} \left(\frac{\partial w}{\partial \theta} \frac{\partial w}{\partial r} \right) \tag{5.22}$$

These are straightforward but laborious to explicitly calculate. We then use the second order equations of motion to eliminate ρ'', A'' and A'^2. The resulting background will therefore reproduce the full second order equations of motion. The middle of the above equations can be used to find the angular dependence of w giving

$$w(r,\theta) = \frac{e^{4A}}{\rho^2} - \frac{3\sin^2\theta\rho' e^{4A}}{\rho} - e^{4A}F(r) \tag{5.23}$$

Note that the supersymmetric limit corresponds to $F(r) = 0$ and ρ' replaced using the supersymmetric first order equation of motion (5.8). We should not be surprised that derivatives of ρ enter directly into the solution since introducing a mass term corresponds explicitly to introducing an extra degree of freedom via precisely this derivative.

F can then be found using either of the other two equations (the third equation providing a check on the consistency of the solution). It is the solution of

$$-2 - 2\rho^6 = -4\rho^2 A' + 4\rho^4 F A' + \rho^4 F' + 2\rho\rho' \tag{5.24}$$

We have not been able to solve this equation explicitly but in the UV limit the solution takes the form

$$F = \frac{1}{3}\left(\frac{1}{\rho} - \rho^5\right) - \rho' + ... \tag{5.25}$$

which clearly vanishes in the supersymmetric limit given (5.8). For a general numerical solution of the second order equations of motion we can set the boundary conditions on F using this asymptotic form and hence find F numerically.

The solution then faces its strongest test since $F_{(5)}$ must also satisfy its bianchi identity (5.18). At first sight this appears to be a challenge; since w contains a derivative of ρ the bianchi identity is a third order equation. In fact explicit computation shows that the second order equations of motion are a solution of this third order equation

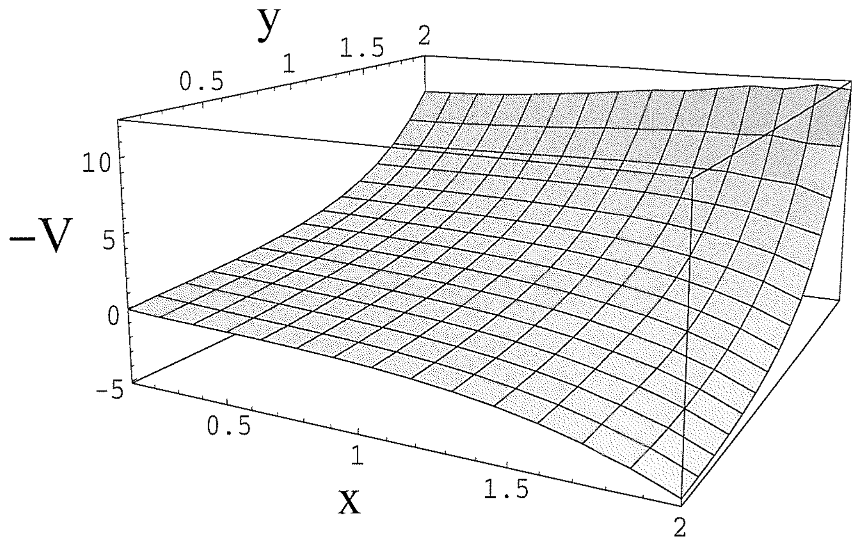

Figure 5.2: The probe potential plotted over the $r - \theta$ plane for the mass only case ($\mathcal{A} = 0$).

and the solution survives.

Given the complete 10d lift of our non-supersymmetric solutions we can study the background for signals that it correctly encodes the field theory dynamics.

5.3 Brane Probe Potential

It is now an automatic procedure to substitute this into the DBI action. The resulting scalar potential is given by

$$V_{probe} = -e^{4A}\left[\frac{X}{\rho^2} + \frac{3\sin^2\theta\rho'}{\rho} - \frac{1}{\rho^2} + F\right] \tag{5.26}$$

It is illuminating to evaluate this potential at leading order in the UV with

$$\rho = 1 + ve^{-2r} + m^2re^{-2r} + \ldots \tag{5.27}$$

We find

$$V = m^2 e^{2r}(2 - 6\sin^2\theta) + ... \tag{5.28}$$

The scalar VEV vanishes from the potential at this order consistent with the existence of the $\mathcal{N} = 4$ moduli space. The mass term reproduces precisely the mass operator we expected in Eqn. (5.1), and we conclude that the 10d background shows all the correct behaviour to be dual to the non-supersymmetric gauge theory with scalar masses.

Finally we numerically plot the probe potential in the $r - \theta$ plane for the mass only solution ($\mathcal{A} = 0$) in Figure 5.2. The plot fits well with the claim that the mass operator Eqn. (5.1) is present. The supersymmetric solutions ($\mathcal{B} = 0$) give a flat probe potential. Other non-supersymmetric solutions reproduce the form of Figure 5.2 up to a sign change dependent on the sign of $\mathcal{B}$.

5.4 Précis

We conclude that we have successfully found the 10d gravity dual of this simple non-supersymmetric deformation of the AdS/CFT Correspondence. Whilst it may appear an unphysical example, it has been an important step in constructing a non-supersymmetric dual example and a step closer to a QCD duality. It has also shown a novel feature, in the form of the ansatz for the 4-form gauge potential, and in moving away from a known supersymmetric case in a minimal way.

Chapter 6

Overview and Conclusions

We have now reached the end of our road. What the previous chapters have shown is that the AdS/CFT conjecture is a very concrete proposal, but not without its difficulties. One can see that when it is in the conformal phase, it is a very perfect system, but when we move away from here, complications set in. This is clearly illustrated in chapters 2 and 3. Everything is very clear when we are looking at the $\mathcal{N}$=4 Coulomb branch, but there are difficulties when we try and study non-conformal theories such as by an operator insertion. The direct proposal of a duality that we looked at for another non-conformal SYM didn't have the problems of chapter 3, and seemed to be a good example of a low energy SYM, albeit that very particular VEVS are switched on; it would be good to make a similar generalization as is Chapter 2, as an expansion of the harmonic function.

It also raises interesting questions for our outlook. In all that was considered, one noticed that certain quantities were renormalized, whilst others weren't. The general idea is that the radial coordinate in AdS should correspond to the energy scale in the field theory. This is somewhat of a vague notion, and I believe it is of high importance to make this identification precise. In [95], an approach is followed that is based on the local coupling approach in field theory. Local couplings have a very natural interpretation in AdS/CFT, as they are simply supergravity fields. It is understood in the conformal case, so the next step would be to make it clear in a non-conformal setting. One then may be able to make precise statements for field theories by using their gravity dual.

One might also wonder about QCD, and how it could be obtained from some dual picture. To this end, we have been looking at a non-supersymmetric deformation where all the gauginos are given a mass. The scalars then also get masses from loop corrections, so that in the IR we are left with pure glue. It should be stressed the level of complexity at hand as there are many fields switched on, and trying to find a solution without some special ansätze looks a hopeless task. If it does work however, it will provide a very good first step to looking at QCD. Some progress has been made at introducing quarks in AdS/CFT [96], which is very interesting phenomenologically because one may hope to compute a meson spectrum in a similar manner to computations of the glueball spectrum [43].

Appendix A

A.1 Maximally Symmetric Spaces

Consider a spacetime of dimension D with a cosmological term. Then the vacuum Einstein field equations are [97, 35, 43, 34, 44, 98]

$$R_{MN} - \frac{1}{2} G_{MN} R = -\frac{1}{2} \Lambda G_{MN}, \tag{A.1}$$

which implies that

$$R_{MN} = -\frac{\Lambda}{2-D} G_{MN}. \tag{A.2}$$

Spaces of this type, where the Ricci tensor is proportional to the metric tensor are called Einstein spaces. Clearly, there are a large class of solutions, so we make the further restriction to look at maximally symmetric solutions where

$$R_{MNPQ} = \frac{R}{D(D-1)} (G_{NQ} G_{MP} - G_{NP} G_{MQ}). \tag{A.3}$$

By considering this minimal construction of the Riemann tensor out of the metric (in terms of its symmetry properties), it can be seen the solutions admitted will be highly symmetric. In fact they are spheres, S^D, de Sitter spaces, dS_D ($\Lambda > 0$), and anti de

Sitter spaces, AdS_D ($\Lambda < 0$). They are maximally symmetric, homogeneous, isotropic solutions.The sphere is obviously a highly symmetric space, so in what follows it is useful to consider spheres at the same time. At each point on the way then there is a more familiar space to help us understand these new (AdS_D) hyperbolic spaces and their properties.

If we were to consider spherical spaces, we usually start by defining them via embedding in a space of one higher dimension. For a sphere we would embed the surface

$$X_0^2 + X_D^2 + \sum_{i=1}^{D-1} X_i^2 = L^2$$

in the flat $D+1$ dimensional space with metric,

$$ds^2 = dX_0^2 + dX_D^2 + \sum_{i=1}^{D-1} dX_i^2.$$

Similarly, this is done for the hyperbolic space AdS_D by embedding the surface,

$$X_0^2 + X_D^2 - \sum_{i=1}^{D-1} X_i^2 = L^2 \tag{A.4}$$

in the flat $D+1$-dimensional space in complete analogy. This space has the flat metric

$$ds^2 = -dX_0^2 - dX_D^2 + \sum_{i=1}^{D-1} dX_i^2. \tag{A.5}$$

Just as the sphere inherits its metric from from the Euclidean embedding space, this also happens for AdS_D. To obtain this metric we need to 'solve' (A.4), and put this constraint into the embedding space metric. If this were done for the case of the sphere, we would solve by using trigonometric functions at a constant radius R. Here, since there are two minus signs, it may be seen the need to use hyperbolic functions

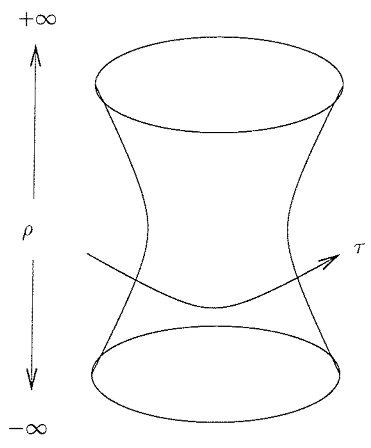

Figure A.1: The AdS space via embedding.

as well [43], justifying why we call it a hyperbolic space. Doing this we find

$$ds^2 = L^2(-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega_{D-2}^2). \tag{A.6}$$

These coordinates (called the global coordinates), cover the whole AdS sub-manifold once (see Fig A.1, when $\rho \geq 0$ and $0 \leq \tau \leq 2\pi$. Although it looked like there was two time-like directions, the above shows there is only one. Looking in the region $\rho = 0$ (where we see the topology of the space is $S^1 \times \mathbf{R}^{D-1}$), in the τ direction there are closed time-like curves (note that this space is not simply connected), which aren't usually allowed. If we unwrap this, the spacetime becomes causal (the universal covering space).

Let us discuss the symmetries of these spaces. Clearly the isometry group of AdS_D is $SO(2, D-1)$, since both the embedding space metric and (A.4) are invariant. Note that this is the conformal group in $(D-1)$ dimensions. There will therefore be $D(D+1)/2$ Killing vectors. Any theory defined on this space, will have an invariance group just as large as more familiar theories on flat spaces of the same dimension (the Poincare group). The maximal compact subgroup is $SO(2) \times SO(D-1)$, and these can be used to give representations to a particle spectrum (e.g. singletons [44]).

There are yet more interesting properties; let us discuss the *boundary* of this space (assuming we have conformally compactified it), which is a projective boundary. Suppose we consider a point $X \in AdS_D$ which is very large with respect to the length scale of the space. Define new coordinates

$$X^M = \gamma x^M \tag{A.7}$$

s.t. $\gamma \to \infty$. Therefore we see that the boundary must be

$$x_0^2 + x_D^2 - \sum_{i=1}^{D-1} x_i^2 = 0, \tag{A.8}$$

subject to the projective equivalence class

$$x \sim \lambda x. \tag{A.9}$$

This means that the boundary is $(D-1)$dimensional, as required [35]. It has the topology of $(S^1 \times S^{D-2})/\mathbf{Z}_2$ [5, 99]; we found that the AdS space having the topology $S^1 \times \mathbf{R}^{D-1}$ which for $\rho \to \infty$ explains the boundary topology. If we consider the universal cover then the boundary becomes $\mathbf{R}^1 \times S^{D-2}$. We should consider points at infinity, as we need to discuss conformal compactification. This is where we conformally rescale the metric such that. points at infinity, which are not points in the original metric are brought into finite distance in the new metric. With these points added, we have conformally compactified the spacetime. If we take $\mathbf{R}^D$ and add a point at infinity, this yields S^D. Similarly, $\mathbf{R}^1 \times S^{D-2}$ is the conformal compactification of $D-1$ Minkowski spacetime (a point at spacelike infinity has been added). This is relevant because, when looking at conformal field theories in Minkowski spacetime, its conformal compactification must be used; conformal transformations can map an ordinary point to infinity [5]. What we see is that the boundary of AdS is precisely a

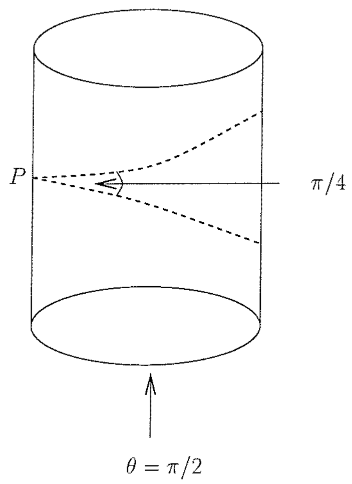

Figure A.2: The AdS Poincaré patch.

Minkowski spacetime of one lower dimension, appropriate for a conformal field theory.

Let us discuss some different coordinate systems, used to describe AdS. This is an essential requirement, as one frequently encounters different coordinate systems in the literature. It will also help clarify the causal structure of the spacetime to follow. Define coordinates,

$$X^0 + X^D \equiv Le^{r/L}, \tag{A.10}$$

$$X^0 - X^D \equiv Le^{-r/L} + e^{r/L}\eta_{\mu\nu}x^\mu x^\nu \tag{A.11}$$

$$X^\mu \equiv e^{r/L}x^\mu,\ \mu = 0, \cdots, D-1. \tag{A.12}$$

The idea here is to switch to a radial coordinate, and scale it out from a Minkowski looking line element. This explains the last definition, from which the first two can be deduced from consistency of the embedding equation. The metric then becomes

$$ds^2 = dr^2 + e^{2r/L}\eta_{\mu\nu}dx^\mu dx^\nu \tag{A.13}$$

known as the Poincare metric with (see Fig. A.2) as its Poincare patch [44]. Because of the type of exponential mapping, only half of the space gets covered, due to the positive definite coordinate changes (A.10, A.11). We can pass to another form of the metric by the radial redefinition $U \equiv e^{r/L}$, which gives

$$ds^2 = L^2 \frac{dU^2}{U^2} + U^2 \eta_{\mu\nu} dx^\mu dx^\nu. \tag{A.14}$$

This has the boundary structure of the single point $U = 0$, plus the plane at $U = \infty$, which is the Minkowski space $\mathbf{R}^{D-1}$. So the boundary of AdS space is really identical to the conformal compactification of Minkowski spacetime. Yet another is obtained by setting $z \equiv 1/U$, with the metric

$$ds^2 = \frac{1}{z^2}(L^2 dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu). \tag{A.15}$$

The boundary consists of a plane at $z = 0$, and a point at $z = \infty$; it is located at an infinite distance from any point of the space. A final set of coordinates, which can be obtained from (A.6), by passing to a new angular variable, as we do when looking at Minkowski space. Define

$$\tan\theta \equiv \sinh\rho, \; 0 \leq \theta \leq \pi/2, \tag{A.16}$$

from which (A.6) becomes

$$ds^2 = \frac{R^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_{D-2}^2). \tag{A.17}$$

So there are different sets of coordinates in which to describe the space, not all of which are a complete cover; let us conformally compactify (A.17) to look at the causal structure. This is because it looks like the Einstein static universe, which we can draw a Penrose diagram for. Rescaling by the conformal factor yields

$$ds^2 = (-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_{D-2}^2). \tag{A.18}$$

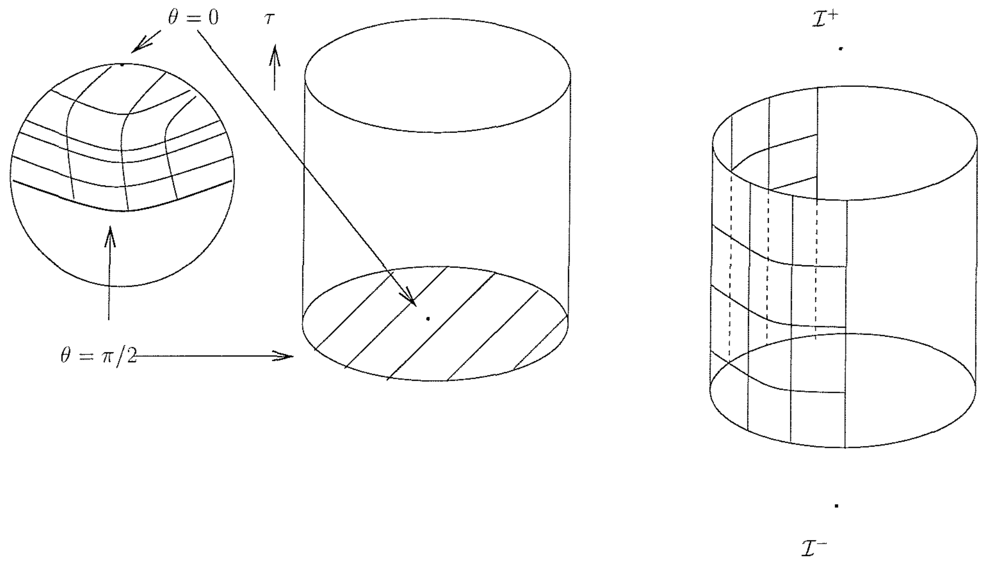

Figure A.3: The *AdS* space is put into the Einstein static Universe.

Since $0 \leq \theta \leq \pi/2$, the conformal mapping is only into half of the conformal compactification $\mathbf{R}^{1,D-1}$ (the Einstein static universe, see Fig. A.3).

This serves as a definition for the asymptotic regions of *AdS*; a space-time is *asymptotically AdS* if its conformal compactification has a boundary structure the same as one half of the Einstein static universe. This is a useful definition, for when deformed metrics are considered, they should have this asymptotic form. Considering a spacelike hypersurface, the boundary is at $\theta = \pi/2$, with a topology of S^{D-2}; however the full boundary extends in the time like direction, so the Cauchy problem requires specification by giving a boundary condition on the $\mathbf{R} \times S^{D-2}$ submanifold. From [97], note the following; we cant make a conformal transformation that brings time-like infinity to a finite point, so we represent them by the disjoint points $\mathcal{I}^+, \mathcal{I}^-$ (see Fig. A.4).

There is no Cauchy surface, so whilst we can put initial data on a spacelike hypersurface Σ, prediction past the Cauchy development $D^+(\Sigma)$ is hampered by the arrival of fresh information from $\mathcal{I}^-$ [100].

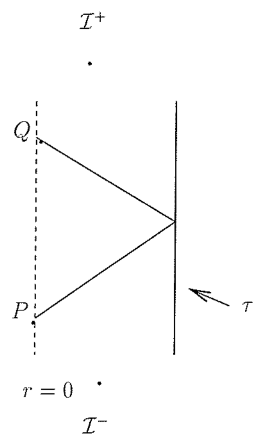

Figure A.4: Timelike infinity is represented by separate points.

As a closing remark, it is also apparent that massive particles moving along geodesics, can never reach the boundary, whilst light rays can reach the boundary and come back in finite time (as observed by an observer moving on a time-like geodesic). From Eq.(A.6), we see the coordinate time to reach the boundary is

$$\tau = \int_0^\infty d\rho \frac{1}{\cosh(\rho)} = \pi/2. \tag{A.19}$$

So light rays can travel from the centre of AdS space to the boundary and back again in a time π (with suitable boundary conditions). Had we looked at time-like geodesics, we would have found an infinite result.

Appendix B

B.1 Twisted Field Theories

B.1.1 The Basics

In this appendix, we give more details of the twisting which occurs in the supersymmetric field theories in 4. This is based on [101], where its connection to Donaldson-Witten theory is discussed.

Firstly consider the global symmetry group of $\mathcal{N}$=2 SYM in $\mathbf{R}^4$. The Lorentz covering (Euclideanised) group is

$$Spin(4) = SU(2)_+ \times SU(2)_-. \tag{B.1}$$

In addition, there is the R-symmetry group

$$\mathcal{R} = SU(2)_R \times U(1)_R. \tag{B.2}$$

With these, the supercharges can be classified i.e. they transform as

$$Q^I_\alpha \;\in\; \left(\frac{1}{2}, 0, \frac{1}{2}\right)^1 \tag{B.3}$$

$$\bar{Q}_{I\dot{\alpha}} \in \left(0, \frac{1}{2}, \frac{1}{2}\right)^{-1} \tag{B.4}$$

In addition, they satisfy the algebra

$$\{Q^I_\alpha, \bar{Q}_{J\dot{\beta}}\} = \delta^I_J P_{\alpha\dot{\beta}}. \tag{B.5}$$

We are now in a position to give a clear definition of twisting.

Definition 3 *The twisted theory is where the rotation group is taken to be $SU(2)^T \times SU(2)_-$, where $SU(2)^T$ is the diagonal subgroup of $SU(2)_+ \times SU(2)_R$.*

So the global symmetry index, I, becomes a spinorial index α, that is $Q^I_\alpha \to Q^\beta_\alpha$ and $\bar{Q}_{J\dot{\beta}} \to G_{\alpha\dot{\beta}}$. Clearly, there is a trace which can be defined as well $Q \equiv Q^\alpha_\alpha$. These new charges transform under the *new* global symmetry group $SU(2)^T \times SU(2)_- \times U(1)_R$ as

$$Q_{(\alpha\beta)} \in (1, 0)^1 \tag{B.6}$$

$$Q \in (0, 0)^1 \tag{B.7}$$

$$G_{\alpha\dot{\beta}} \in \left(\frac{1}{2}, \frac{1}{2}\right)^{-1} \tag{B.8}$$

At this point, all that has been done to the theory, is a fancy rearrangement of the symmetry groups with the appearance of a scalar symmetry generator Q. But this appearance is important because we can now pass to a curved manifold (e.g. covariant derivatives do not require a connection). If the energy momentum tensor can be written as some quantity under a Q transformation [101] then the theory is topological (twisted theories are regarded as Euclidean theories. The twisted algebra is now

$$\{Q^I_\alpha, \bar{Q}_{J\dot{\beta}}\} = \delta^I_J P_{\alpha\dot{\beta}} \to \{Q, G_{\alpha\dot{\beta}}\} = P_{\alpha\dot{\beta}}, \tag{B.9}$$

$$\{Q, Q\} \;=\; 0, \tag{B.10}$$

and shows this to be a *necessary* condition for it to be a topological theory. In models studied this is in fact true for the whole energy momentum tensor. In fact the algebra is precisely the basic equations that occur in a topological quantum field theory.

Now let us address the system on a general manifold. Whilst on $\mathbf{R}^4$ the original and twisted theories are equivalent, this is not true on a curved space because the energy momentum tensors are different [101, 102, 103]. The spin of the fields change since the R-symmetry index becomes spinor valued due to the twist. This implies that the couplings to the background metric M are altered. It may be viewed as arising from firstly *gauging* the $SU(2)_R$; this has the effect of coupling the new gauge field to the R-symmetry current and thus the Lagrangian is changed

$$\mathcal{L} \rightarrow \mathcal{L} + J^i A_i. \tag{B.11}$$

Clearly then this will change the energy momentum tensor. At this point quantities like correlators depend both on the spin connection and the gauge field. Identifying the gauge connection with the *spin* connection on M, will produce diagonal (topological) correlators [103]. That this is equivalent follows from noting the difference in energy momentum differs by a term involving the current J^i. This different coupling to gravity implies field's charges with respect to the current then must change their transformation law! Sequentially it produces a change in the spin connection and therefore the energy momentum tensor, and then in turn changes the couplings of the fields to gravity. See also [102]. When the gauge fields are set equal to the spin connection, covariant derivatives are modified so as to be compatible with the original twisted energy momentum tensor.

Let us state how the $\mathcal{N}$=2 fields are changed under the twisting operation;

$$A_{\alpha\dot{\beta}}\left(\frac{1}{2},\frac{1}{2},0\right)^0 \rightarrow A_{\alpha\dot{\beta}}\left(\frac{1}{2},\frac{1}{2}\right)^0, \tag{B.12}$$

$$\lambda_{i\alpha}\left(\frac{1}{2},0,\frac{1}{2}\right)^0 \rightarrow \chi_{\alpha\beta}(1,0)^{-1},\ \eta(0,0)^{-1}, \tag{B.13}$$

$$\bar{\lambda}_{i\dot{\alpha}}\left(0,\frac{1}{2},\frac{1}{2}\right)^1 \rightarrow \psi_{\alpha\dot{\beta}}\left(\frac{1}{2},\frac{1}{2}\right)^1, \tag{B.14}$$

$$B(0,0,0)^{-2} \rightarrow \lambda(0,0)^{-2}, \tag{B.15}$$

$$B^*(0,0,0)^{+2} \rightarrow \phi(0,0)^{+2}, \tag{B.16}$$

$$D_{ij}(0,0,1)^0 \rightarrow G_{\alpha\beta}(1,0)^0. \tag{B.17}$$

In Section 4, we are considering a *partially* twisted theory; our world volume M is a product manifold $M = \mathbf{R}^4 \times \Sigma$, and the twisting is performed on Σ. This then enables us to preserve some supersymmetry (there are killing spinors) and also have the usual field theory in the $\mathbf{R}^4$.

Appendix C

C.1 Supersymmetric Backgrounds

In this section, we would like to make to some general remarks on how some supersymmetry is preserved in a given background. This is very much in the spirit of chapter 15 of [17].

Suppose we consider local supersymmetry so the infinitesimal spinor supersymmetry parameter is some function of the coordinates $\epsilon_\alpha(x)$. To this there corresponds a conserved supercharge Q_α. An unbroken supersymmetry, Q_α is given by the condition

$$Q_\alpha|0\rangle, \tag{C.1}$$

where $|0\rangle$ is the vacuum state. This can be rephrased in terms of its behaviour with some other operator U. The above condition implies

$$\langle 0|[Q_\alpha U]_\pm|0\rangle = \delta U = 0. \tag{C.2}$$

When U is bosonic, this condition is simply satisfied by setting the fermions to zero. For a fermionic operator this gives a non-trivial equation. At tree level when δU and

$\langle|\delta U|0\rangle$ coincide for string theory, this amounts to solving two types of equation; one for the gravitino and one for the spin-1/2 fermions. As an example of these, we shall write down the IIB variations (which are used in chapter 2). These are [29]

$$\begin{aligned}\delta\psi_M &= \nabla_M\epsilon + \frac{i}{4.4.5!}\Gamma^{ABCDE}\Gamma_M\epsilon F_{ABCDE} + \\ &\quad \frac{1}{3.4!}(\Gamma_M{}^{NPQ}G_{NPQ} - 9\Gamma^{NP}G_{MNP})\epsilon^* + (Fermi)^2 \qquad (C.3)\\ \delta\lambda &= i\Gamma^M\epsilon^* P_M - \frac{i}{4!}\Gamma^{MNP}\epsilon G_{MNP} + (Fermi)^2 \qquad (C.4)\end{aligned}$$

These equations have a variety of uses, in finding particular solutions to the supergravity field equations. Into these one can put an arbitrary metric and dilaton ansatz in trying to find a supersymmetric solution ($\delta\psi_M = 0$ and $\delta\lambda = 0$), whilst the field strengths must satisfy certain Bianchi identities.

Bibliography

[1] J. Babington, N. Evans and J. Hockings, "Secrets of the metric in N = 4 and N = 2* geometries," JHEP **0107** (2001) 034 [arXiv:hep-th/0105235].

[2] J. Babington and N. Evans, "Field theory operator encoding in N = 2 geometries," JHEP **0201** (2002) 016 [arXiv:hep-th/0111082].

[3] J. Babington, D. E. Crooks and N. Evans, "A non-supersymmetric deformation of the AdS/CFT correspondence," arXiv:hep-th/0207076.

[4] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2** (1998) 231 [Int. J. Theor. Phys. **38** (1999) 1113] [arXiv:hep-th/9711200].

[5] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2** (1998) 253 [arXiv:hep-th/9802150].

[6] E. Witten, "Anti-de Sitter space, thermal phase transition, and confinement in gauge theories," Adv. Theor. Math. Phys. **2** (1998) 505 [arXiv:hep-th/9803131].

[7] E. Witten, "Baryons and branes in anti de Sitter space," JHEP **9807** (1998) 006 [arXiv:hep-th/9805112].

[8] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," Phys. Lett. B **428** (1998) 105 [arXiv:hep-th/9802109].

[9] M. E. Peskin and D. V. Schroeder, "An Introduction To Quantum Field Theory," *Reading, USA: Addison-Wesley (1995) 842 p.*

[10] M. Nakahara, "Geometry, Topology And Physics," *Bristol, UK: Hilger (1990) 505 p. (Graduate student series in physics).*

[11] G. 't Hooft, "A Planar Diagram Theory For Strong Interactions," Nucl. Phys. B **72** (1974) 461.

[12] G. 't Hooft, "Dimensional Reduction In Quantum Gravity," arXiv:gr-qc/9310026.

[13] Y. Makeenko, "Large-N gauge theories," arXiv:hep-th/0001047.

[14] L. Susskind, "The World as a hologram," J. Math. Phys. **36** (1995) 6377 [arXiv:hep-th/9409089].

[15] D. Bigatti and L. Susskind, "TASI lectures on the holographic principle," arXiv:hep-th/0002044.

[16] M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory. Vol. 1: Introduction," *Cambridge, Uk: Univ. Pr. (1987) 469 P. (Cambridge Monographs On Mathematical Physics).*

[17] M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies And Phenomenology," *Cambridge, Uk: Univ. Pr. (1987) 596 P. (Cambridge Monographs On Mathematical Physics).*

[18] J. Polchinski, "String Theory. Vol. 1: An Introduction To The Bosonic String," *Cambridge, UK: Univ. Pr. (1998) 402 p.*

[19] J. Polchinski, "String Theory. Vol. 2: Superstring Theory And Beyond," *Cambridge, UK: Univ. Pr. (1998) 531 p.*

[20] J. Polchinski, "Effective field theory and the Fermi surface," arXiv:hep-th/9210046.

[21] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, "M theory as a matrix model: A conjecture," Phys. Rev. D **55** (1997) 5112 [arXiv:hep-th/9610043].

[22] E. Cremmer, B. Julia and J. Scherk, "Supergravity Theory In 11 Dimensions," Phys. Lett. B **76** (1978) 409.

[23] P. Van Nieuwenhuizen, "An Introduction To Supergravity," Gen. Rel. Grav. **10** (1979) 211.

[24] Y. Tanii, "Introduction to supergravities in diverse dimensions," arXiv:hep-th/9802138.

[25] P. C. West, "Supergravity, brane dynamics and string duality," arXiv:hep-th/9811101.

[26] P. Pasti, D. P. Sorokin and M. Tonin, "On Lorentz invariant actions for chiral p-forms," Phys. Rev. D **55** (1997) 6292 [arXiv:hep-th/9611100].

[27] J. H. Schwarz and P. C. West, "Symmetries And Transformations Of Chiral N=2 D = 10 Supergravity," Phys. Lett. B **126** (1983) 301.

[28] J. H. Schwarz, "Covariant Field Equations Of Chiral N=2 D = 10 Supergravity," Nucl. Phys. B **226** (1983) 269.

[29] M. J. Duff, R. R. Khuri and J. X. Lu, "String solitons," Phys. Rept. **259** (1995) 213 [arXiv:hep-th/9412184].

[30] G. T. Horowitz and A. Strominger, "Black Strings And P-Branes," Nucl. Phys. B **360** (1991) 197.

[31] C. G. Callan, J. A. Harvey and A. Strominger, "Supersymmetric string solitons," arXiv:hep-th/9112030.

[32] C. G. Callan, J. A. Harvey and A. Strominger, "World Sheet Approach To Heterotic Instantons And Solitons," Nucl. Phys. B **359** (1991) 611.

[33] M. J. Duff and J. X. Lu, "The Selfdual type IIB superthreebrane," Phys. Lett. B **273** (1991) 409.

[34] M. J. Duff, "Lectures on branes, black holes and anti-de Sitter space," arXiv:hep-th/9912164.

[35] J. L. Petersen, "Introduction to the Maldacena conjecture on AdS/CFT," Int. J. Mod. Phys. A **14** (1999) 3597 [arXiv:hep-th/9902131].

[36] J. Polchinski, "Dirichlet-Branes and Ramond-Ramond Charges," Phys. Rev. Lett. **75** (1995) 4724 [arXiv:hep-th/9510017].

[37] R. G. Leigh, "Dirac-Born-Infeld Action From Dirichlet Sigma Model," Mod. Phys. Lett. A **4** (1989) 2767.

[38] E. D'Hoker and D. Z. Freedman, "Supersymmetric gauge theories and the AdS/CFT correspondence," arXiv:hep-th/0201253.

[39] J. G. Russo, "Large N field theories from superstrings," arXiv:hep-th/0008212.

[40] A. Bilal, "Introduction to supersymmetry," arXiv:hep-th/0101055.

[41] M. F. Sohnius and P. C. West, "Conformal Invariance In N=4 Supersymmetric Yang-Mills Theory," Phys. Lett. B **100** (1981) 245.

[42] P. Di Vecchia, "Large N gauge theories and ADS/CFT correspondence," arXiv:hep-th/9908148.

[43] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. **323** (2000) 183 [arXiv:hep-th/9905111].

[44] N. S. Deger, Lectures on AdS/CFT Duality, Istanbul Summer School on String Theory, September 2001.

[45] C. Vafa, "Lectures on strings and dualities," arXiv:hep-th/9702201.

[46] L. Susskind and E. Witten, "The holographic bound in anti-de Sitter space," arXiv:hep-th/9805114.

[47] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, "Novel local CFT and exact results on perturbations of N = 4 super Yang-Mills from AdS dynamics," JHEP **9812** (1998) 022 [arXiv:hep-th/9810126].

[48] M. Petrini and A. Zaffaroni, "The holographic RG flow to conformal and non-conformal theory," arXiv:hep-th/0002172.

[49] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, "The supergravity dual of N = 1 super Yang-Mills theory," Nucl. Phys. B **569** (2000) 451 [arXiv:hep-th/9909047].

[50] D. Z. Freedman, S. S. Gubser, K. Pilch and N. P. Warner, "Renormalization group flows from holography supersymmetry and a c-theorem," Adv. Theor. Math. Phys. **3** (1999) 363 [arXiv:hep-th/9904017].

[51] C. V. Johnson, K. J. Lovis and D. C. Page, "Probing some N = 1 AdS/CFT RG flows," JHEP **0105** (2001) 036 [arXiv:hep-th/0011166].

[52] C. V. Johnson, K. J. Lovis and D. C. Page, "The Kaehler structure of supersymmetric holographic RG flows," JHEP **0110** (2001) 014 [arXiv:hep-th/0107261].

[53] C. V. Johnson, "D-brane primer," arXiv:hep-th/0007170.

[54] G. W. Gibbons, "Aspects of Born-Infeld Theory and String/M-Theory," arXiv:hep-th/0106059.

[55] T. Mohaupt, "Black holes in supergravity and string theory," Class. Quant. Grav. **17** (2000) 3429 [arXiv:hep-th/0004098].

[56] M. J. Duff and J. X. Lu, "The Selfdual type IIB superthreebrane," Phys. Lett. B **273** (1991) 409.

[57] D. Z. Freedman, S. S. Gubser, K. Pilch and N. P. Warner, "Continuous distributions of D3-branes and gauged supergravity," JHEP **0007** (2000) 038 [arXiv:hep-th/9906194].

[58] K. S. Stelle, "BPS branes in supergravity," arXiv:hep-th/9803116.

[59] M. Günaydin, L.J. Romans and N.P. Warner, Phys. Lett. B154 (1982) 68; M. Pernici, K. Pilch and P. van Nieuwenhuizen, Nucl. Phys. B259 (1985) 460.

[60] M. Günaydin, L.J. Romans and N.P. Warner, Nucl. Phys. B272 (1986) 598.

[61] K. Pilch and N. P. Warner, "N = 2 supersymmetric RG flows and the IIB dilaton," Nucl. Phys. B **594** (2001) 209 [arXiv:hep-th/0004063].

[62] P. Kraus, F. Larsen and S. P. Trivedi, "The Coulomb branch of gauge theory from rotating branes," JHEP **9903** (1999) 003 [arXiv:hep-th/9811120].

[63] S. S. Gubser, "Curvature singularities: The good, the bad, and the naked," Adv. Theor. Math. Phys. **4** (2002) 679 [arXiv:hep-th/0002160].

[64] N. Evans, C. V. Johnson and M. Petrini, "The enhancon and N = 2 gauge theory/gravity RG flows," JHEP **0010** (2000) 022 [arXiv:hep-th/0008081].

[65] A. Brandhuber and K. Sfetsos, "An N = 2 gauge theory and its supergravity dual," Phys. Lett. B **488** (2000) 373 [arXiv:hep-th/0004148].

[66] N. Evans and M. Petrini, "AdS RG-flow and the super-Yang-Mills cascade," Nucl. Phys. B **592** (2001) 129 [arXiv:hep-th/0006048].

[67] A. Buchel, A. W. Peet and J. Polchinski, "Gauge dual and noncommutative extension of an N = 2 supergravity solution," Phys. Rev. D **63** (2001) 044009 [arXiv:hep-th/0008076].

[68] C. V. Johnson, A. W. Peet and J. Polchinski, "Gauge theory and the excision of repulson singularities," Phys. Rev. D **61** (2000) 086001 [arXiv:hep-th/9911161].

[69] C. V. Johnson, R. C. Myers, A. W. Peet and S. F. Ross, "The enhancon and the consistency of excision," Phys. Rev. D **64** (2001) 106001 [arXiv:hep-th/0105077].

[70] J. M. Maldacena and C. Nunez, "Supergravity description of field theories on curved manifolds and a no go theorem',' Int. J. Mod. Phys. A **16**, 822 (2001) [arXiv:hep-th/0007018].

[71] J. M. Maldacena and C. Nunez, "Towards the large N limit of pure N = 1 super Yang Mills',' Phys. Rev. Lett. **86**, 588 (2001) [arXiv:hep-th/0008001].

[72] J. P. Gauntlett, N. Kim, D. Martelli and D. Waldram, "Wrapped fivebranes and N = 2 super Yang-Mills theory',' Phys. Rev. D **64** (2001) 106008 [arXiv:hep-th/0106117].

[73] F. Bigazzi, A. L. Cotrone and A. Zaffaroni, "N = 2 gauge theories from wrapped five-branes", Phys. Lett. B **519** (2001) 269 [arXiv:hep-th/0106160].

[74] P. Di Vecchia, A. Lerda and P. Merlatti, "N = 1 and N = 2 super Yang-Mills theories from wrapped branes," arXiv:hep-th/0205204.

[75] N. Evans, M. Petrini and A. Zaffaroni, "The gravity dual of softly broken N = 1 super Yang-Mills," JHEP **0206** (2002) 004 [arXiv:hep-th/0203203].

[76] David Kutasov, Introduction to Little String Theory, Trieste Spring School on Superstrings and related matters, 2001 and references therein;

[77] O. Aharony, "A brief review of 'little string theories'," Class. Quant. Grav. **17** (2000) 929 [arXiv:hep-th/9911147].

[78] M. Bershadsky, C. Vafa and V. Sadov, "D-Branes and Topological Field Theories", Nucl. Phys. B **463** (1996) 420 [arXiv:hep-th/9511222].

[79] M. Cvetic, H. Lu and C. N. Pope, "Consistent Kaluza-Klein sphere reductions," Phys. Rev. D **62** (2000) 064028 [arXiv:hep-th/0003286].

[80] M. Cvetic, J. T. Liu, H. Lu and C. N. Pope, "Domain-wall supergravities from sphere reduction," Nucl. Phys. B **560** (1999) 230 [arXiv:hep-th/9905096].

[81] J. T. Liu and R. Minasian, "Black holes and membranes in AdS(7)," Phys. Lett. B **457** (1999) 39 [arXiv:hep-th/9903269].

[82] A. Salam and E. Sezgin, Phys. Lett. B **126** (1983) 295.

[83] M. Bershadsky, A. Johansen, V. Sadov and C. Vafa, "Topological reduction of 4-d SYM to 2-d sigma models", Nucl. Phys. B **448** (1995) 166 [arXiv:hep-th/9501096].

[84] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, "Confinement and condensates without fine tuning in supergravity duals of gauge theories," JHEP **9905** (1999) 026 [arXiv:hep-th/9903026].

[85] K. Pilch and N. P. Warner, "N = 2 supersymmetric RG flows and the IIB dilaton," Nucl. Phys. B **594** (2001) 209 [arXiv:hep-th/0004063].

[86] K. Pilch and N. P. Warner, "N = 1 supersymmetric renormalization group flows from IIB supergravity," Adv. Theor. Math. Phys. **4** (2002) 627 [arXiv:hep-th/0006066].

[87] J. Distler and F. Zamora, "Non-supersymmetric conformal field theories from stable anti-de Sitter spaces," Adv. Theor. Math. Phys. **2** (1999) 1405 [arXiv:hep-th/9810206].

[88] S. S. Gubser, A. A. Tseytlin and M. S. Volkov, "Non-Abelian 4-d black holes, wrapped 5-branes, and their dual descriptions," JHEP **0109** (2001) 017 [arXiv:hep-th/0108205].

[89] O. Aharony, E. Schreiber and J. Sonnenschein, "Stable non-supersymmetric supergravity solutions from deformations of the Maldacena-Nunez background," JHEP **0204** (2002) 011 [arXiv:hep-th/0201224].

[90] V. Borokhov and S. S. Gubser, "Non-supersymmetric deformations of the dual of a confining gauge theory," arXiv:hep-th/0206098.

[91] J. de Boer, E. Verlinde and H. Verlinde, "On the holographic renormalization group," JHEP **0008** (2000) 003 [arXiv:hep-th/9912012].

[92] K. Skenderis and P. K. Townsend, "Gravitational stability and renormalization-group flow," Phys. Lett. B **468** (1999) 46 [arXiv:hep-th/9909070].

[93] K. Behrndt and M. Cvetic, "Supersymmetric domain wall world from D = 5 simple gauged supergravity," Phys. Lett. B **475** (2000) 253 [arXiv:hep-th/9909058].

[94] A. Khavaev, K. Pilch and N. P. Warner, "New vacua of gauged N = 8 supergravity in five dimensions," Phys. Lett. B **487** (2000) 14 [arXiv:hep-th/9812035].

[95] J. Erdmenger, "A field-theoretical interpretation of the holographic renormalization group," Phys. Rev. D **64** (2001) 085012 [arXiv:hep-th/0103219].

[96] A. Karch and E. Katz, "Adding flavor to AdS/CFT," JHEP **0206** (2002) 043 [arXiv:hep-th/0205236].

[97] The Large Scale Structure of Space-time, S. W. Hawking and G. F. R. Ellis, Cambridge University Press (1973).

[98] TASI Lectures: Introduction to the AdS/CFT Correspondence, I. R. Klebanov, hep-th/0009139.

[99] Introduction to the AdS/CFT correspondence, E. T. Akhmedov, hep-th/9911095.

[100] Aspects of Quantum Field Theory in Curved Space-time, S. A. Fulling, Cambridge University Press (1989).

[101] J. M. Labastida and C. Lozano, "Lectures on topological quantum field theory", arXiv:hep-th/9709192.

[102] J. M. Labastida and M. Marino, "Twisted baryon number in N = 2 supersymmetric QCD", Phys. Lett. B **400** (1997) 323 [arXiv:hep-th/9702054].

[103] S. Cordes, G. W. Moore and S. Ramgoolam, "Lectures on 2-d Yang-Mills theory, equivariant cohomology and topological field theories", Nucl. Phys. Proc. Suppl. **41** (1995) 184 [arXiv:hep-th/9411210].