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FACULTY OF SOCIAL SCIENCES

RESEARCH AND GRADUATE SCHOOL OF EDUCATION

LEARNING TO TEACH ALGEBRA: SECONDARY TRAINEE-TEACHERS' KNOWLEDGE OF STUDENTS' ERRORS AND DIFFICULTIES

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ABSTRACT

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This study investigates secondary trainee-teachers' knowledge of students' errors and difficulties in algebra. There is a great deal of research about students' errors and difficulties in algebra but only a few studies of teacher knowledge about students' errors. There are even fewer studies of secondary mathematics trainee-teachers' knowledge of students' errors and difficulties. This area of research therefore forms the focus of this study because it is less well-documented by research despite the fact that secondary trainee-teachers constitute the next generation of teachers.

To investigate what trainee-teachers know about students' errors and difficulties in algebra, a national survey in which the participants were asked to explain students' errors and suggest ways for helping students who make the errors was carried out. Participants were also asked to predict the most likely errors that students might make when working out given algebra problems. In addition, they were asked to rank-order the problems according to their relative difficulty. Finally, they were asked to give their expectations about students' success in working out the problems.

The study was carried out by administering a questionnaire to collect the majority of data for the study followed by taking a small number of interviews to extend the data from the questionnaire survey in order to reveal the reasons participants have for their answers and to justify their choices.

This study suggests that secondary mathematics trainee-teachers are able to suggest a teaching sequence that takes into account the relative difficulty of the algebraic ideas that are part of the high school mathematics curriculum. For example, most can correctly identify rank-order algebra questions from those that particular groups of students find the easiest to those they find most difficult. The trainee-teachers are also able to predict the most likely errors that secondary students are known to make when encountering algebra in secondary school.

Nevertheless, the study indicates that less than one-fifth of secondary mathematics trainee-teachers are able to identify major sources of students' errors (i.e. the reason why such errors occurred). Consequently, most secondary mathematics trainee-teachers are unable to suggest ways that challenge students' thinking and make the students realise their faults in advance of the teacher attempting to add additional knowledge. Instead, trainee-teachers are likely to devote time and energy to explaining the whole topic again. Similarly, less than one-fifth of trainee-teachers seem to understand the sort of characteristics that determine the complexity level of an algebra problem, such as the number of variables involved, the nature of the elements in the problem and students' possible interpretations of the letters used in algebraic expressions.

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Definitions

Data: the raw data of a study.

Information: research results, suggestions, ideas, etc.

Student: a school child, at any stage, unless otherwise specified.

Trainee-teacher: a university student wishing to qualify as a secondary school teacher through a successful completion of a one year Post Graduate Certificate in Education (PGCE).

Error: an incorrect answer to a mathematics problem.

Common error: an error committed by a large number of students.

Resistant error: an error committed by students in different year groups. It is called resistance because it lasts for a long time and cannot be easily corrected.

Systematic error: an error committed by the same student in different occasions.

Random error: an error occurred due to a mental slip. It may not occur again.

Misconception: sources of the errors. Any misconception consists of a family of errors rather than just one. For example, saying that 0.234 is larger than 0.41 means making an error. If in different pairs of decimal numbers a student said that, in each pair, the longer number (has more digits after the decimal point) is the larger then that student holds the misconception that "the longer decimals are the larger" (Stacey et al, 2001). Another example is that simplifying the fraction (x+1)/(x+2) to ½ is an error and simplifying x/(2x+y) to 1/(2+y) is another error. There are many more examples in which algebraic fractions are simplified incorrectly in the same way (see Matz, 1980). These errors are caused by the same difficulty, which is cancellation of similar letters in the denominator and numerator of a fraction. Consequently, they form a misconception. A third example is that evaluating 4x, when x=6, to 46 is an error and evaluating xy, when x=5 and y=3, to 53 is another error. Both errors are caused by the misconception of using the same idea of concatenation in arithmetic in algebra (Matz, 1982).

Difficulty: a general term that includes errors and misconceptions and other things. If a student made an error or has a misconception then s/he has a difficulty. Also, inability to answer a problem means having a difficulty.

Chapter 1

INTRODUCTION

1.1 Introduction

There is considerable research that indicates that teacher knowledge can be thought of as consisting of several types of knowledge which teachers utilise in their teaching in order to teach well (Grossman, 1990; Gudmundsdottir, 1991; Fennema & Franke, 1992). One element of teachers' knowledge is their knowledge about students. This knowledge is made up of several sub-components such as how students learn algebra, what strategies they use for solving algebra problems, and what errors and misconceptions they have about a particular topic in algebra. This latter sub-component forms the focus of this study.

Understanding students' strategies and errors is a crucial component of knowledge for teachers to acquire (Shulman, 1986). Research results support the suggestion that having more or less of this knowledge affects teaching (Carpenter et al, 1989). This knowledge is therefore worth investigation.

Research on teacher knowledge about students indicates that experienced teachers are generally more aware about students' errors and difficulties than novices or trainee-teachers (Leinhardth & Smith, 1985; Borko et al, 1989). This seems to suggest that teachers develop this knowledge mainly from their teaching experience. However, Tirosh (2000, p.6) argues that "experience acquired during the course of teaching is the primary but not the only possible source of teachers' knowledge of students' common conceptions and misconceptions." Consequently, she suggests that the pre-service period is a crucial time to examine and develop such knowledge. This study therefore focuses on trainee-teachers rather than teachers in schools.

This study also uses students' errors and difficulties in algebra as a springboard to study trainee-teachers' knowledge about student errors. The reasons why this study focuses on trainee-teachers and algebra are described in the next section. Following that, the importance of teacher knowledge about students' errors and difficulties in algebra is highlighted. Then in section 1.4, different indications (both positive and negative) for teacher knowledge about the errors are extracted and analysed. The research problem and the various rationales for this study are then discussed in the subsequent section. Finally, the chapter ends with discussion about the purpose and the research questions.

1.2 Why this study focuses on trainee-teachers and algebra

One feature of this study is that it focuses on trainee-teachers as the population of the study and on algebra as the field of the study. The reasons for studying trainee-teachers rather than teachers are discussed in the next paragraph. The reasons for concentrating on algebra are discussed afterwards.

To start with, in the field of understanding students' cognition, such as their strategies and errors, some researchers focused on teachers (Even & Markovits, 1993; Wanjala, 1996; Leu, 1999). Other researchers chose to focus on trainee-teachers (e.g. Tirosh, 2000) and others on the comparison between novice teachers and experienced teachers (Leinhardt & Smith, 1985; Borko et al, 1989). In the present study, however, the decision was made to focus on trainee-teachers. One reason for this decision is that trainee-teachers comprise the next generation of teachers and therefore, in studying how teachers gain knowledge of students' errors, it is important to begin with trainees. Another reason is that trainee-teachers' knowledge of students' errors and difficulties has not been extensively examined by research in the field of algebra. In fact, of the few studies that have been conducted in this area of research, such as Tirosh (2000), Nathan and Koedinger (2000), Even and Markovits (1993) and Wanjala (1996), most are of teachers rather than trainee-teachers. The case of trainee-teachers is therefore less researched than the case of teachers. Studying trainee-teachers fills a gap in knowledge and provides comparative information to the studies about teachers in order to see how knowledge about students' errors and difficulties develops in the domain of algebra.

Now we come to the discussion about the reasons that this study focuses on algebra. Algebra is an important part of the mathematics curriculum in school. The following quote gives a general answer to the question: why is learning algebra important?

"Learning algebra is an important milestone in a student's mathematical development. It opens the door to organised abstract thinking and supplies a tool for logical reasoning. It gives a student the satisfaction of finding simplicity in what appears to be complex and finding generality in a collection of particulars" (Stacey & MacGregor, 1997, p. 252).

There are more specific reasons why algebra is important than those specified in the quote above. First, algebra is widely used in other areas of mathematics such as geometry, trigonometry, and statistics as well as in some other subjects outside mathematics such as science and geography. Even some social science subjects such as economics and religion may unexpectedly contain some forms of algebra. In Islam, for instance, algebra sometimes makes the calculations related to the distribution of inheritances to the living relatives, such as sons, daughters, and parents, according to the Islamic system, much easier than arithmetic. It was for this purpose that Al-Khawarizmi, one of the ancient algebraists, developed some algebraic strategies for simplifying and solving word problems, which translated to equations of different complexities (Charbonneau & Lefebvre, 1996).

The second reason that makes algebra important is because some students need it in their higher education such as those wishing to be mathematicians, scientists or engineers. In fact, sometimes it is considered as the key element to higher education. For example, "In the United States, algebra is often spoken of as the gatekeeper to higher education" (Rhine & Samek, in press).

The third reason is that algebra is sometimes considered to be important because it facilitates different aspects of life, helps create more productive and beneficial citizens. In this case, it becomes important for most people and not just for a few of them (Edwards, 1990).

The above reasons about the usefulness of algebra are usually considered as aims for the teaching and learning of algebra. Orton (1994) classified aims of teaching and learning mathematics (including algebra) into two broad categories: "societal goals" and "goals for students". Examples of the "societal goals" are mathematics helps people to better understand their environment and it helps create a modern society through its support to science and technology. The goals for students include the facts that mathematics makes students better problem solvers and gives them opportunities to reason and think mathematically.

Because algebra is believed to be important, it is therefore compulsory, as mathematics is compulsory in most, if not all, schools of the world. Students start studying algebra at different ages across the world (see, for example, Thompson, 1988; Howson, 1991). However, it appears that there is no agreement on when students start thinking algebraically. Introducing students to letters might not be the actual start of their encounter with algebra (see, Janvier, 1996).

This study focuses on algebra and not on any other area of mathematics because of the special importance of algebra. Another reason is that researchers have explored students' difficulties in algebra and identified interesting findings in this area of research, which can be used by trainee-teachers (and teachers) to inform their teaching practices. However, as was noted above, there is a shortage of research about trainee-teachers' knowledge in the domain of students' difficulties and errors in algebra. Further research, such as the present study, is therefore needed.

1.3 Students' errors and teaching

The discussion in this section presents students' difficulties and errors as a crucial element for better teaching when appreciated by teachers. It aims to answer the question: why this study focuses on this type of knowledge?

There is a great deal of research evidence about students' errors and difficulties in algebra, which is extensively reviewed in the next chapter. The authors of these studies believe that this knowledge is important for teaching with understanding. For example, Booth (1984) says:

> "It is important to know about the nature of the errors that children [students] make and how common they are because this way provides information concerning the ways in which the child [student] views the problem, and the procedures that are used in attempting to solve the problem. This information is of interest not only because it might suggest ways of helping children [students] to avoid these errors, but also because it might explain children's [students'] apparent lack of progress in attaining higher levels of understanding in algebra" (Booth, 1984, p.3).

Following findings and recommendations from the above studies, the usefulness of students' errors for teaching has become increasingly appreciated (Resnick, 1985; Shulman, 1986; Nesher, 1987; Carpenter et al, 1988; Ridgway, 1988; Peterson et al, 1991; Askew & William, 1995; Borasi, 1996; Schunk, 2000;

Tirosh, 2000). In addition to this research evidence, some mathematics teachers have expressed their beliefs that students' errors should be addressed in teaching. For example, Michele Linnecor, a mathematics teacher, says:

"I feel that an important part of the planning process is to acquaint oneself with the common mistakes children [students] make and tackle them head on in the classroom. I believe that misconceptions should not be left unchallenged, as they will become deep rooted. I believe that they should be tackled repeatedly to reinforce pupils' [students'] understanding" (Linnecor, undated, p.1)

Knowledge about students' errors is important for many reasons, such as:

- Knowing the sort of errors that students are likely to make in a particular topic of mathematics helps teachers to address these errors in their teaching before they occur.
- Once these errors occur, this knowledge helps teachers to give the right diagnosis and remediation to the right student.
- Students' errors can be used as a starting point for reflection about mathematical topics. For example, why expanding $\sqrt{(ab)}$ to $\sqrt{a}\sqrt{b}$ is correct whereas expanding $\sqrt{(a+b)}$ to $\sqrt{a} + \sqrt{b}$ is not. Another example, $3(x+2) = 3x+6$ is correct whereas $\text{Sin}(a+b) = \text{Sin}(a) + \text{Sin}(b)$ is not. The two expressions, $\sqrt{(a+b)} = \sqrt{a} + \sqrt{b}$ and $\text{Sin}(a+b) = \text{Sin}(a) + \text{Sin}(b)$ are systematic errors for many students (Matz, 1982).

While the above points are extensively discussed with examples in chapters 2 and 3, it is sensible to address some remarks to the first two points above because these are less obvious than the third one. To start with, consider the error $6s=p$ which many students make when working out the following problem:

"Write an equation using s and p to describe the following statement. 'At this university there are 6 times as many students as there are professors.' Use s for the number of students and p for the number of professors" (Clement, 1982, p.17).

There are several explanations for the error $6s=p$ in the literature (see for example, Clement, 1982). For some students, the error could be a careless or a random mistake. In this case, there is no problem, the error can easily be corrected (Maurer, 1987).

For some other students, the error appears as a result of a syntactic translation of the problem, that is, changing key words to symbols from left to right in the problem statement (Clement, 1982; Wollman, 1983; Herscovics, 1989). For those students, the error is not deeply rooted (Ibid). In this case, telling them to think about the problem statement instead of just "blindly" translating it to algebra might be useful. Alternatively, substituting numbers for either variables in 6s=p and then checking if the answers to the other variable and the problem statement have the same meaning might be enough to show the students their mistake.

For a third group of students, the error 6s=p could be understood as "6 students with every one professor" (Clement, 1982; Herscovics, 1989). In this case, "6s" is understood as "6 students" rather than "6 times the number of students" and "1p" as "one professor" rather than "one times the number of professors". Those students have no problem in understanding that the problem statement and the "equation" both mean that there are more students than professors (e.g. Herscovics, 1989). Therefore, the error for them is deeply rooted and it cannot be easily corrected as in the first two cases. For this form of error it is not enough to show the students the correct procedure for obtaining the answer or to compare their incorrect answer with the problem statement, or even to say to them that their answer is wrong because the right answer is s=6p (Rosnick & Clement, 1980). Other ways of addressing this type of error are required. Teachers' knowledge of this form of error can be useful in that it enables them to have a better understanding of students' thinking and strategies for solving algebraic problems. Through this knowledge, they can understand individual cases of students and so give the right diagnosis and remedy to the right student.

Another example is that many students simplify, for example, (x+2)/(x+1) to 2; a common error that comes as a result of the incorrect cancellation of the two x's together (Matz, 1982). In the same area of algebraic fractions, other students are expected to simplify √(a+b)/√a to √b because they would simplify √(a+b) to √a+√b and then cancel √a from the numerator and denominator (Ibid). The knowledge of students' errors in fractions and the underlying difficulties may help teachers to prepare a better teaching-plan for teaching algebraic fractions. The new teaching plan should takes into account the students' expected misconceptions and errors when working with fractions, trying to avoid or at least minimise such misconceptions, capitalising on the correct responses and so on.

If teachers are not aware of students' strategies and errors then, according to Leu (1999), they would:

- Select teaching materials and sequences that do not match with students' learning.
- Teach and evaluate students in a way that does not match with their learning.
- Fail to identify and address students' misconceptions and would overestimate or underestimate the difficulties students encountered while working out problems.

Leu (1999) found that the majority of teachers in his study did not know that students have intuitive strategies to solve fraction problems and when they knew, they could not specify the strategies. For example, they thought that 4th grade students could not divide a circle into two equal halves before they had been introduced to the concepts of diameter and radius. However, according to Leu, students can do that intuitively either visually or by actually folding it into two equal halves. Thus, the teachers in this example ignored students' intuitive approaches because they thought that students would use the analytical approach that teachers use. Another example, from Wanjala (1996), is that teachers were given some algebra problems and asked to put them in order of students' problem solving difficulty from the easiest to the hardest. To do this reasonably well, they should have some knowledge about students' difficulties and ways of thinking in algebra. For instance, to decide which one of the two expressions: (a-b)+b and 3a-(b+a) is easier to simplify, they should appreciate the complexity of the minus sign before the bracket in the second expression. Many students incorrectly simplify this expression to 4a-b (24% in Wanjala's study, 1996). Similarly, to say that the equation 4x-3 = x-11 is harder to solve than the equation 7x = 5, they should know that when the unknown occurs on both sides of the equality sign this introduces more difficulty in the equation than when the unknown occurs on just one side (Filloy & Rojano, 1984, 1985a, 1985b; Herscovics & Linchevski, 1994). Without this knowledge it is very difficult for trainee-teachers to plan their teaching to take account of students' difficulties and errors.

Another thing is that teachers who have limited knowledge of students' strategies and errors might teach and evaluate students in a way that does not match with their learning. For example, they might prepare a test and have their students score up to 90% because of some misconceptions that they have and which teachers failed to diagnose rather than because students understand mathematics (Nesher, 1987). This also raises the point that such teachers may help develop students' misconceptions when they evaluate student answers as "correct" whilst obtaining them for the wrong reason.

1.4 Teacher knowledge about students' errors

Two important findings have been made in the previous section. First, the great deal of research evidence on students' errors and difficulties in algebra, and second, the importance of such knowledge for teaching with understanding. These two findings raise an important question, which is, are teachers aware of students' errors and difficulties in algebra?

At present, the above question cannot be answered easily because there is not much research information available about teacher knowledge about students (Brophy, 1991, Aubrey, 1994, Even & Tirosh, 1995). One reason is that "pedagogical content knowledge" (including knowledge about students) is a relatively new area of research (Shulman, 1986). Many of its components were recognised as important elements for teaching only very recently because of the way that this knowledge used to be approached by research (Ball, 1991). This lack of research about teacher knowledge of students' errors and difficulties in algebra explicitly suggests that more research should be undertaken in this area. This study therefore follows this suggestion as it aims to shed some light on teacher knowledge about students' difficulties and errors in algebra.

Despite the fact that teacher knowledge about students' difficulties and errors in algebra has not been extensively examined by research, there are some indications in the literature that this knowledge is sometimes not satisfactory.

One indication comes from the fact that students have enormous difficulties in algebra and some of them may arise from the way that teachers teach algebra. For example, many students understand algebraic letters as names for things such as apples and bananas rather than as variables or generalised numbers (Küchemann, 1981). This problem is believed to have its origin in school (MacGregor, 1991).

Evidence from other studies which investigated students' errors and difficulties in algebra (Matz, 1980, 1982; Sleeman, 1982, 1984; Booth, 1984; Herscovics & Linchevski, 1994; Wanjala, 1996; Trigueros & Ursini, 1999) suggest that some of the students' errors in algebra are resistant to change and may last for a long time, sometimes up to the university level. This could happen because these errors remain unchallenged by current teaching methods at school.

"These results suggest that instead of promoting a deep understanding of variable and the development of intuitive algebraic ideas, current teaching practices seem to obstruct them. Also it is important to stress that errors committed by algebra beginners are not remedied by instruction and they prevail up to university levels" (Trigueros and Ursini, 1999, p.280)

Another example is Herscovics and Linchevski (1994) who explored, in a case study, students' intuitive strategies when working with linear equations of one unknown and, at the end of their study, they came to the following conclusion:

"... it seems that many teachers and textbook authors are unaware of the serious cognitive difficulties involved in the learning of algebra. As a result, many students do not have the time to construct a good intuitive basis of ideas of algebra or to connect these with the pre-algebra ideas they have developed in primary school; they fail to construct meaning for the new symbolism and are reduced to performing meaningless operations on symbols they do not understand." (Herscovics & Linchevski, 1994, pp.59-69).

One fact about the above studies is that they were not specifically designed to investigate teacher knowledge about students' errors in algebra. Therefore, their suggestions mostly came from student observations. In this case, teacher awareness is measured in terms of teaching product, that is, teacher ability to transfer what s/he knows about students' errors to his/her teaching.

Studies of teacher knowledge of student errors (Even & Markovits, 1993; Wanjala, 1996; Sleeman et al, 1996; Tirosh, 2000) generally indicated that teacher knowledge about student errors is limited or unsatisfactory since most teachers, for example, did not know how to address the errors except by re-teaching the whole topic again. Another point that comes from these studies, as well as from others that attempted to compare novice teachers with experienced teachers (Leinhardt & Smith, 1985; Borko et al, 1989), is that the experienced teachers are more aware about students' errors than the novices. This suggests that teachers develop their knowledge about student errors from their teaching experience. However, some researchers may disagree that teaching is the only important source for this type of knowledge (Tirosh, 2000). Moreover, not all the experienced teachers possess a "good" teaching knowledge, for example, knowledge about students (Bishop & Whitfield, 1972). It follows that the pre-service period is worth considering in any investigation about teacher knowledge of student errors.

There is a crucial difference between the sets of studies described above. One set of studies tended to measure knowledge in terms of teaching product. In other words, students encounter a large number of difficulties in algebra and this has been taken as an indication of lack of knowledge on the side of mathematics teachers. In another set of studies, teachers' awareness was measured in terms of their knowledge about students' errors. For example, Wanjala (1996), measured teachers' knowledge in terms of their ability to arrange algebraic problems in order of difficulty, predict students' errors in a number of mathematical problems and explain some of the students' errors. This does not imply that teachers who possess this sort of knowledge will necessarily transfer this knowledge into their teaching. Teaching is far more complex than that and cannot adequately be achieved through just teacher knowledge about students' errors (Askew et al, 1997; Raymond, 1997). To understand the position of this type of knowledge in teaching practices, a theoretical framework is developed in chapter 3.

1.5 Research problem and rationales

One requirement of the initial teacher training curriculum for both primary and secondary mathematics in England is to educate trainee-teachers about students' errors in mathematics. In the government regulations (DfEE, 1998, p.57), it is stated clearly that trainee-teachers must be taught to "recognise common pupil [student] errors and misconceptions in mathematics, and to understand how these arise, how they can be prevented and how to remedy them". Similarly, the Key Stage 3 mathematics framework (DfEE, 2001, p.55) suggests that considerations of many questions, some of which are listed below, can help to identify approaches to teaching and student activities that can be used to ensure that the objectives of unit of work can be met:

"What difficulties or misconceptions might pupils [students] have? How can these be avoided and resolved? How can the plenary sessions be used to probe misconceptions?" (DfEE, 2001, p.55)

This belief, of course, comes from the fact that teacher knowledge about students' errors is important (section 1.2). Despite this belief, this knowledge has rarely been addressed by research. Furthermore, there are indications in the literature that this knowledge is sometimes unsatisfactory (section 1.3). The need for more studies in this area of research is therefore recommended (Brophy, 1991). The present study is among the first few studies, which attempted to address trainee-teacher knowledge about students' errors and difficulties in algebra.

One rationale for this study is to inform all those interested in mathematics education about this state of affairs. This research is also important because it opens the door for future similar and complementary research to be conducted in the same area of trainee-teacher knowledge about students' errors. For example, some researchers might become interested in conducting some sort of action research by, for example, educating trainee-teachers about students' errors and then measuring the effect of that in their teaching.

Another possible implication for this study is that, like other studies about teacher knowledge, it can be used as an indication for teachers' knowledge of the relevant literature. Trainee-teachers whose knowledge about the well-documented accounts of students' errors in the literature is limited could be interpreted as being unaware of the relevant research in their subject. This could be one reason why research, in its majority, is not translated into practice (Kennedy, 1997; Sowder, 2000).

1.6 Research purpose and questions

This study endeavours to investigate trainee-teachers' knowledge in relation to the following areas:

- Their expectations about students' success in solving algebraic problems.
- The way they rate algebraic content in order of student difficulty.
- Their explanations of students' errors.

- Their suggestions for addressing students' errors in algebra.
- Their ability to predict students' errors in algebra.

The first two points in the list above measure trainee-teachers' knowledge of the order of difficulty of algebraic content. For example, given the task 3a-(b+a), the trainee-teachers are expected to appreciate the complexity of the minus sign before the bracket and consequently to judge that the task is more complex than, for instance, (a-b)+b (Wanjala, 1996). Such knowledge helps teachers to determine what students already know or how they, as individuals, approach tasks. For example, knowing that 3a-(b+a) is often simplified to "3a-b+a" is useful in understanding students' ways of thinking. Without this knowledge, the students "may be expected to learn either something that they already know or something that is too complex for their current state knowledge" (Askew et al, 1997, p.20).

The last three points in the list above aim to extract trainee-teacher knowledge in terms of their ability to predict students' errors, analyse them to identify their sources and suggest suitable ways for addressing the errors. Such knowledge is crucial for any investigation on teacher knowledge about students' errors because, "discovering where misconceptions exist and analysing their sources are important skills for teachers to acquire" (Ridgway, 1988, p.48). It is not surprising to find therefore that sources of students' errors and the ways that these errors are addressed in teaching have been the focus of many studies in this area of research (e.g. Even & Markovits, 1993; Wanjala, 1996; Tirosh, 2000).

Following the above discussion, the research questions that this study attempt to answer are now listed:

- What are trainee-teachers' expectations about students' success in solving given algebraic problems?
- How do trainee-teachers use their knowledge of students' difficulties in algebra to rank-order algebra questions?
- What explanations do trainee-teachers give for students' errors in algebra?
- What strategies do trainee-teachers use, or conceive, for tackling students' errors in algebra?
- How successful are trainee-teachers in predicting students' errors in algebra?

1.7 Outline of the structure of the thesis

This chapter introduced the thesis. It has identified "trainee-teachers", "algebra", and "students' errors and difficulties" as closely related terms in this study. The reasons why this study focuses on these specific terms are discussed in section 1.2 and section 1.3. Then, in section 1.4, knowledge about students' errors and difficulties is discussed by giving indications from the literature to suggest that this knowledge is sometimes unsatisfactory. Section 1.5 raises the research problem and suggests rationales for the study. Finally, the research purpose and questions are discussed in section 1.6. The outline of the structure of the rest of the thesis is described in what remains of this section.

Due to the lack of research information about trainee-teachers' knowledge of students' errors, chapter 2 reviews the literature on students' errors and difficulties in common approaches to algebra as well as the domains of learning algebra. Generalisation, problem-solving, modelling and functional approaches are reviewed in this chapter for the purpose of identifying possible sources of students' errors and difficulties in these approaches. In the process of learning algebra, the review concentrates more on equations, expressions and word problems as they are most related to this study. The chapter ends with a discussion about possible ways for addressing students' errors and difficulties in algebra.

Chapter 3 develops a theoretical framework for this study. To do this, it starts by identifying and reviewing different components of teacher knowledge including subject matter knowledge, pedagogical content knowledge and knowledge about students. In addition, it gives a brief review about teachers' beliefs as they affect the teaching and learning of algebra. This taxonomy about components of knowledge is then used to outline the theoretical framework of this study. In this theoretical framework, knowledge about students' errors and difficulties is part of knowledge about students and this appears as a sub-component of teacher pedagogical content knowledge.

Chapter 4 presents the methodological issues as related to this study. It starts by setting the research within an overall approach to research. Aspects of survey, descriptive research, longitudinal research and qualitative research are found to be relevant characteristics of research and hence discussed in turn. The chapter also introduces and justifies the research methodology and instruments: questionnaire and interview. The questionnaire development, validity and reliability, population and sample, and the detailed procedure for administering the questionnaire are all discussed in turn in this chapter.

Chapters 5 and 6 discuss the analysis procedures and show the research findings for the five areas investigated: trainee-teachers' expectations, putting problems in order of difficulty, explaining students' errors, addressing students' errors and predicting students' errors. Finally, Chapter 7 summaries the findings and suggests possible implications for the research.

1.8 Summary

This first chapter provides a rationale for focussing on trainee-teachers, algebra and students' errors in this study. A range of research evidence, given above, suggests that trainee-teacher (and teacher) knowledge about students' errors is sometimes inadequate. Thus, further research in this area is needed.

Chapter 2

STUDENTS' ERRORS AND DIFFICULTIES IN THE PROCESS OF TEACHING AND LEARNING ALGEBRA

2.1 Introduction

Algebra is difficult for students to learn at all school levels. Cockcroft (1982, p.60), for example, said, "algebra is a source of considerable confusion and negative attitudes among pupils [students]", and Herscovics (1989, p.60) stated that "algebra is a major stumbling block for many students in secondary school." Orton and Frobisher (1996) suggested two reasons that might make algebra a source of confusion and a stumbling block for students, one is the "unhelpful teaching" and the other is "learning difficulties". Both reasons are therefore addressed in the literature review in this chapter.

This chapter is divided into four parts. In the first part, the overall approaches to algebra are reviewed. This includes four types of teaching approaches, namely, generalisation, modelling, problem solving and functional. The review of these teaching approaches is not meant to be a comprehensive review, but rather it centres on just one particular aspect, the difficulties that cause students' errors and misconceptions in algebra. Thus, although some examples might be given in each case, these are used to enhance the understanding of the discussion of the difficulties.

The second part of the literature review concentrates on what is known about students' errors and cognitive difficulties in algebra. Research on student learning of equations, expressions, and equation-word problems of two variables are examined as these form the focus of this study. However, this review must be accompanied by a review of students' understanding of letters and the difficulties they usually encounter on the transfer from arithmetic to algebra. Thus, a clear picture of the problem of students' difficulties in algebra is obtained first by reviewing common teaching approaches to algebra and second by reviewing the obstacles of students' understanding of algebra.

In the third part of this chapter, a discussion about possible ways for addressing students' errors and difficulties is given. Finally, part four, summarises the main points in the chapter.

It should be noted that this chapter does not incorporate a review of trainee-teachers' knowledge of students' errors and difficulties in algebra. This is reserved for Chapter 3.

2.2 Students' errors and difficulties in the teaching approaches to algebra

(a) Generalisation approaches to algebra

Radford (1996) defined the term "generalisation" from a purpose perspective, that is, as a way for producing "new results" from a set of particulars or known facts, he says:

"A goal in generating geometric-numeric patterns is to obtain a new result. Conceived in this form, generalization is not a concept. It is a procedure allowing for the generation, within a theory and beginning with certain results, of new results." (Radford, 1996, p.108).

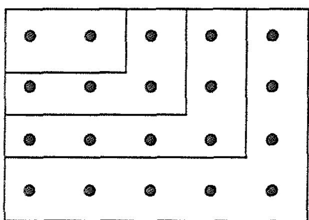
Generalisation therefore is a way for introducing school algebra through generalising activities. These activities usually take the form of numeric or geometric patterns.

From an epistemological point of view, "generalization is the life-blood, the heart of mathematics" (Mason, 1996a, p.74). Generalisation is so central to mathematics that Lee (1996, p.103) believes that "algebra and indeed all of mathematics is about generalizing patterns." Consequently, if this is the case then students and teachers should encounter generalisation very frequently. This, however, does not mean that they always recognise its presence. In fact, most of the time it goes unnoticed because "many professionals no longer notice its presence in what is, for them, elementary." (Mason, 1996a, p.66). For example, many students do not recognise the sentence "*The* angle-sum of *a* triangle equals 180 degrees" as an example of generalisation in mathematics because, in many classrooms, teachers may not know what they should be stressing or ignoring (Gattegno, 1990). Thus, they often work on many examples involving triangles before students receive the message that the angle-sum of any possible triangle is always 180 degrees. Mason (1996) observed that many teachers stress the "180 degrees" in their classes to facilitate its memorisation and leave out what is more important to students, that is, *a* and *the*: the generalisation tools in the sentence.

Another example of generalisation in mathematics is "a+b = b+a". Students are guided to this form by going through many examples of arithmetic equalities, such as 6 + 5 = 5 + 6, until they recognise the pattern that led them to the conclusion of "a+b = b+a". However, Mason (1996a, p.70) noted that students may arrive at this conclusion without understanding as they might know, for example, that 6+5 equals 5+6 because both equal 11 and not because "order does not matter". In this case, using bigger numbers such as 66+75 = 75+66 may be helpful to test students' understanding of the general form.

The following is an example of a geometrical pattern. It would help explain some of the difficulties encountered when students attempt to generalise geometric/numeric patterns.

Figure 2.1: The dot rectangle problem (Lee, 1996, p.94)



The rectangle at the left represents a set of overlapping rectangles.

The first contains	2 dots.
The second contains	6 dots.
The third contains	12 dots.
The fourth contains	20 dots.

How many dots in the fifth rectangle?
How many dots in the hundredth rectangle? How do you know?
How many dots in the nth rectangle? How do you know?

The above problem asks three questions, two of them are in arithmetic and the last one is in algebra. Asking students to answer some arithmetic questions before they go to the algebra questions can be very helpful for a number of reasons. First, students usually spend several years studying arithmetic before they are introduced to algebra and so these types of questions might facilitate their understanding of algebra because they make use of their familiarity with arithmetic. Second, answering an easy question can motivate students to answer the next difficult question and then the next one and so by the time they reach algebra question, they will face it with some motivation and encouragement.

Third, arithmetic questions help develop a sense that algebra is a very powerful subject because it summarises infinitely many cases in just one case (the general formula) which can then be used to answer any arithmetic question about the problem in systematic and easier way than arithmetic.

Some geometric activities do not start with arithmetic questions, that is, they only ask for the general expression (see Mason, 1996a, pp.84-85 for examples). In these cases, there might be a rush to symbolisation, which for many students can be, according to Mason, a "nightmare" even then writing the final expression may be achieved without understanding.

Another difficulty comes from the fact that geometric/numeric patterns may be perceived in different ways, which consequently lead to different approaches for establishing the general forms. The problem is that some of these approaches are not useful for writing the general expression. For example, Lee (1996) identified three approaches among the high school students that she interviewed about the dot rectangle problem (Figure 2.1). According to Lee, although the dot pattern can be used to answer the first two questions of the problem, only one of the patterns is useful for writing the general expression. Thus, Lee concluded that, in order to write a correct general expression, it is important that students perceive the algebraically useful pattern rather than any pattern.

English and Warren (1998) came to the same conclusion from a study that they conducted with 430 students aged 12-15 years. They tested the students in both shape and number patterns. They asked students various questions such as describing the rules that they used to generalise the pattern in their language, write the rules in an algebraic form, try to think of other rules for the same pattern and say why these equivalent rules are equal. They also interviewed some individual students from those who participated in the test. They found that, although the pattern approach seems encouraging in that it provides a meaningful introduction to algebra, students encountered many difficulties in expressing the general form. They also found that, students used different strategies to determine the generality and that some of these strategies were inappropriate to write algebraic rules.

Radford (1996) agreed that there are many ways or procedures that lead to the conclusion α (the general form), depending on the student's way of thinking or how the "observed facts" are interpreted. He also questioned the validity of the conclusion α that students arrive at through a generalisation procedure. He seems to say that the generalisation approach not only places difficulties on students who may come to a conclusion or a perception that is not useful to express the general form, but on teachers as well who may not have the time and ability to judge various conclusions. Furthermore, Radford appears to be worried about the role of the teacher and that of the students, sometimes being mixed together. According to him, there are certain beliefs on the side of many students that make the judgement of a conclusion sometimes incorrect or inappropriate.

Finally, there are two objections about the generalisation approach, suggesting that it should be integrated with other approaches such as problem solving. The first is that in all generalisation activities students often (may be always) work with whole numbers. If one tries to use numbers other than whole numbers then perhaps the student loses meaning. For example, in geometric patterns, it is difficult to ask students to find the number of dots in the 5.23th rectangle or the number of matchsticks needed to build 5.1 squares. Lee (1996, p.106) seems to suggest the same thing when she questioned: "How would these students conceptualise functions when the input was other than a whole number?"

The second objection to the generalisation approach comes from its limitations. According to Radford (1996), not all algebraic concepts can be reached through geometric-numeric generalisations. He argues that because generalisation seeks to find a general form or an expression, letters are always used as variables rather than as unknowns. Therefore, concepts like *variable*, *expression* and *formula* can "easily" be approached through generalisation. On the other hand, the concepts of *unknown* and *equation* can best be reached through problem solving because, according to him, the purpose of the latter is to find an unknown, not a variable. Consequently, he suggested integration of the two approaches.

(b) Problem solving in algebra

Problem solving as an approach to algebra is very important. It has made a major contribution to the development of algebra and many people would argue that "the evidence of history is definitely on the side of the problem-solving approach to algebra" (Wheeler, 1996, p.147). Teachers' understanding of what constitutes a mathematical problem can be very varied. It follows therefore that their approach to teaching problem solving will also vary. Some teachers might limit problem solving to the forming and solving of equations using word problems (e.g. Bednarz & Janvier, 1996). Others might teach problem solving in a much broader way by asking students to explore different aspects of the problem, such as generating other problems from the same problem (e.g. Bell, 1996b).

As stated above, the teaching of problem solving takes many different forms because people have different beliefs about what a mathematical problem means. One can identify four categories of mathematical problems according to the three components of a problem: the problem situation (the main body of the problem), the goal (what the problem asks for) and the path (any algorithm that can be used to solve the problem). These categories are now described in turn. In the first category, the problem is seen as a mathematical situation that clearly expresses a goal and has a known algorithm that leads the problem solver to the goal of the problem (Frobisher, 1994). In this case, the problem is usually posed by the teacher and students are taught and practise with similar problems. The type of problems to be solved is therefore not completely new or unfamiliar to them.

In the second category, the problem can be described as a new situation facing the problem solver for the first time, and s/he does not have a ready answer for it nor does s/he necessary know any solving algorithm. For example, Kantoski (1981, p.113) says, " a problem is a situation which differs from an exercise in that the problem solver does not have a procedure or algorithm which will certainly lead to a solution."

Some researchers go further than that and put a new restriction for a mathematical situation to be a problem, that is, it must be a "real" problem for the problem solver (see, for example, Mason, 1996a, p.83).

In the third category, a mathematical problem is one in which the situation is determined but the goal and the path are not really determined, arising naturally during the problem solving.

Finally, there is the ideal stage where the student does everything. S/he chooses a situation, a goal and explores the different possible paths for attacking his own problems. The following definition of a mathematics problem seems to be very close to the ideal stage:

"a problem is a situation that has interest and appeal to the child [student], who therefore wishes to explore the situation more fully in order to gain understanding of it. Goals arise naturally during the exploration and are determined not by the setter of the problem but by the child [student]. The child [student] in turn surveys the problem situation before exploring avenues of interest, following paths which may or may not lead to a satisfactory conclusion." (Frobisher, 1994, p.154)

These different views about mathematical problems influence the way mathematics is taught and consequently the sort of beliefs which students have about mathematics, its teaching, and learning. For example, many students learn mathematics by an approach which emphasises memorisation of rules and procedures followed by applying them "blindly" to mathematical "problems". Those students might resist, in the future, any creative and unusual activity that tries to present mathematics as a way of inquiry rather than as a set of rules to be memorised. Thus, when Borasi (1996) tried to approach mathematics in an unusual way using students' errors as starting points, they disliked the new way of teaching and expressed very negative reactions about it though the author believed in its usefulness and creativeness. In this case, the new way of teaching contradicted the sort of expectations that the students had about mathematics and its teaching and learning.

Another difficulty that students might encounter in this approach arises when teaching mathematics emphasises problems involving only one answer. Frobisher (1994) believes that this is still the case in many countries across the world. These sorts of problems (mainly those of the first category above) tend to develop a belief in students that every problem in mathematics has a unique solution and can be solved by only one procedure. Those students might therefore have a difficulty when they encounter a mathematics problem that has more than one solution.

Finally, Bednarz and Janvier (1996) differentiate between arithmetic and algebraic problems in that the former are "connected" whereas the latter are "disconnected". An example of each type is given in Figure 2.2. According to Bednarz and Janvier, the first problem in the figure is arithmetic because it is "connected" since using the known data in the problem (the relationships and the given state) students can easily find the unknowns in the problem. So, they can easily calculate b and c respectively as $12 \times 5 = 60$ and $12+16 = 28$ and hence the total number of fish as $12+60+28 = 100$. The authors also argue that the reasoning used to solve this type of problem is arithmetic since students start from the known (in this case 12: the number of fish of type a) to the unknowns (b, c, and the total number of fish).

Figure 2.2: Example of arithmetic and algebraic problems.

PROBLEM 1 (arithmetic problem):
A man bought three different types of fish: a, b, and c. He bought 5 times as many fish of b as he bought of a, 16 more fish of type c than a, and 12 fish of type a. How many fish altogether did he buy?
PROBLEM 2 (algebraic problem):
A man bought 100 fish from three different types: a, b, and c. He bought 5 times as many fish of b as he bought of a, and 16 more fish of type c than a. How many fish of each type did he buy?

They call the second problem algebraic because it is "disconnected" since the known data in the problems (relationships and the total number of fish) cannot be used as easily as in the first problem to calculate the unknowns. The problem can be solved using algebraic reasoning (going from the unknown to the known by first expressing all the unknowns in terms of just one unknown, say a, and then operating on the unknown as if it is known). Students, however, as Bednarz and Janvier noted, mainly use arithmetic approaches to solve algebraic problems at the introductory stage of algebra despite the fact that these approaches are more difficult to use for these problems than algebraic ones. When this happens, some errors occur because students select the harder way to solve the problems.

(c) Modelling approaches to algebra

There are two options for introducing algebra at school according to the different meanings of letters. In the first option, students start using letters as undetermined numbers in formulae (e.g. Area = πr^2) and unknowns in equations before they start using letters as generalised numbers in expressions and variables in functions. This move is supported by problem solving approaches (e.g. Bednarz & Janvier, 1996). In the second option, students start using letters as variables right from the beginning. This is supported by modelling and functional approaches, which are mainly concerned with developing the notion of variable. People who favour this approach argue that students overcome many of the difficulties that they usually experience when algebra is introduced the other way round, from letters as unknowns to letters as variables. This, however, does not mean that modelling and functional approaches are without their own difficulties.

According to Janvier (1996), modelling involves two phases: a formulation phase followed by a validation phase. The formulation phase is very important since it leads to the creation of the model. This phase usually starts by investigating a phenomenon (e.g. the growth of a plant over time) so that important relationships between variables (e.g. the height of the plant versus time) are identified. Such relationships are established on the basis of many hypotheses, which come from careful observations or measurements or by performing imaginary actions on the situation under investigation. After that, the formulation phase passes through a crucial stage, which leads to the creation of the model by expressing the relationships between the variables of the situation in terms of a symbolic expression, a graph or a table of numbers.

The validation phase, on the other hand, is used to ensure that the formulation phase is complete and correct. It involves therefore going back to the reality of the situation and doing some investigation which might lead to a minor or major correction in the formulation phase. Nemirovsky (1996a) calls the minor correction of a graph-line an "adjustment" and the major correction a "repair". Thus, on the plant growth example above, one might assume after the first few measurements of the plant height over a period of time that it grows at a constant rate and consequently creates a model according to this assumption. Later on, s/he might observe that the plant started to grow at a different rate and therefore *adjust* or *repair* the previously constructed model.

The two phases through which modelling is created and developed give modelling a double meaning. On the one hand, it has an abstract meaning inherent in the main body of the model, which is obtained through the construction process and expressed in the form of a symbolic expression, a graph, or a table of numbers. On the other hand, it has a concrete meaning because a model is meant to represent a physical phenomenon and so it should be interpreted and described within this framework. When the two meanings become independent of each other then the mathematical model is only seen from the abstract side. Thus, many of the students' difficulties in algebra, according to Janvier (1996), come as a result of applying rules in their abstract sense and without referring back to their concrete meaning.

As noted above, the modelling approach involving placing students in a situation which allows them to construct meaning for different algebraic representations, such as graphs. The situation, which the graph or any other representation is said to represent, gives the model its concrete meaning and hence stimulates students' concrete thinking rather than their abstract thinking. Thus, although they may learn more easily from these concrete situations than the abstract ones, it might be difficult for them to use this knowledge later in abstract settings. This worrying point has already been confirmed by studies that use geometrical models to teach algebra (Filloy & Rojano, 1985a, 1985b) or a computer environment (Thompson & Thompson, 1987).

Modelling activities depend on magnitudes and these create understanding problems for students. These problems are to be discussed in the remainder of this section. Nemirovsky (1996a) carried out many teaching experiments where students worked on activities involving graphs of the motion of a car, the growth of a plant, etc. It was assumed that these graphs were constructed from binary pairs, which are described by Janvier (1996) as "concrete numbers" or "magnitudes". Thus, magnitudes refer to measuring numbers such as 10km/h for the speed of a car, 10cm for the height of a plant, etc. Unlike pure numbers, which are not accompanied by a measuring unit, magnitudes make the model concrete.

Magnitudes are classified into categories (Rouche, 1992), which clearly show that they provide different levels of concreteness to mathematical models and that students are more familiar with some magnitudes than others. The problem is that even when students work on a model of a certain magnitude and become familiar with it, their experience might not transfer to another model, which requires the same thinking processes. For example, Janvier (1996, p.230) found that familiarity with the "price" linear model does not transfer to the "speed" linear model despite the fact that the two models offer problems, which require the same solution logic.

Finally, there is a major difficulty associated with models and magnitudes which arises from the continual shift between the abstract and the concrete meanings of a model. Any magnitude, such as 10m, can be used as a "concrete" number (with the measuring unit) and as a pure number (without the measuring unit). In algebra, however, those letters which represent the magnitude unit should be replaced with algebraic letters, or simply used with a different meaning. For example, algebraic notation indicates that 10m should not be used as 10-metres but rather as 10-times-the number of metres, 20s should not be used as 20-students but rather as 20-times-the number of students, etc. Thus, letters in the two domains, algebra and magnitudes, have different meanings even if they are the same. Students who do not realise this fact may suffer from using algebraic letters as names for things (measuring units) instead of using them as unknowns or variables (see for example, Küchemann, 1981; Clement, 1982).

(d) Functional approaches to algebra

Functional approaches, like modelling approaches, introduce algebra via the notion of variable. The concentration in any functional approach is therefore in using letters as variables, that is, letters representing infinitely many values rather than just one or few values (Küchemann, 1981). A functional approach may not exactly mean the study of functions as contemporary functional approaches are usually extended to include other topics such as equations and inequalities.

The discussion in this section concentrates on two different examples of functional approaches, namely, *Computer Intensive Algebra (CIA)* and *CARAPACE* (a computer environment that includes three representations: algorithmic, tabular and graphic). For a discussion on the two approaches respectively refer to Heid (1996) and Kieran et al (1996). They are chosen to be the focus of this discussion because they are among the most important functional approaches and they are used in many schools. Despite this fact, several students' difficulties can be identified when learning algebra through the two approaches. These are discussed below.

The *CIA* approach (Heid, 1996) is like the modelling approach in that both emphasise the use of real world situations for introducing algebra. Nemirovsky (1996b) questions the role played by real world problems in the learning of algebra. According to him, what may count as a real problem for someone may not be so for someone else. Some of the questions which he raised in his analysis regarding the role played by real world problems in the study of algebra, were the following: What makes a real problem real? For whom it is real? Why? Janvier (1996) alerts us that those real world problems tend to develop misunderstanding of the term variable as they involve using magnitudes (see under modelling).

Another difficulty associated with the *CIA* and *CARAPACE* is discussed after the two terms: procedural and structural are clarified. Kieran (1992) uses the two terms procedural-structural in correspondence with Sfard's (1991) terms process-structural. On the one hand, both authors agree that the term procedural (or process) means recognising, for example, the expression $3(x+4)$ as a set of instructions and approaching the expression procedurally means to them evaluating it by numbers to yield numbers. On the other hand, both authors, agree that recognising the structure of the expression means having a global perception of it, that is, recognising how the numbers and letters are arranged, for example, knowing that $12 + 3x$ and $3x + 12$ represent a single "thing".

In the two functional approaches, *CIA* and *CARAPACE*, students start approaching algebra procedurally by constructing a tabular form for a function and then converting that to its graphic representation. It has to be noted, however, that students can ask the computer to draw the graph of the function directly from its symbolic representation after they gain some experience with the different representations of a function. The structural notion of the function is emphasised later on by asking students to explore the relation between the symbolic and the graphic representations and noting that, for example, the sign of 2 in $y = 2x^2$ changes the direction of the function. Thus, the movement from the procedural to the structural representation of a function characterises both approaches to algebra. Tall et al (1999) suggested that this movement marks the normal way of teaching algebra at school. This movement, however, is not without its difficulties despite the fact that it is supported by the historical development of mathematical concepts (Sfard, 1991). Many researchers argue that approaching algebra procedurally creates many difficulties for students because the move from the procedural to the structural approaches is not a smooth one (Booth, 1988; Sfard, 1991; Kieran, 1992; Sfard & Linchevski, 1994).

Even (1993) identified two essential characteristics for the concept of function: *arbitrariness* and *univalence*. She refereed the *arbitrariness* to the relationship between the two sets of a function and also to the two sets themselves. According to her, the *arbitrariness* of the relationship means that "the function does not have to exhibit regularity, be expressed in a specific expression or presented in a particular shaped graph" (p.96). And the *arbitrariness* of the two sets means that they do not have to be sets of numbers or any other specific elements. The *univalence* of a function, on the other hand, means that every element in the domain has only one image in the opposite domain.

The two terms above are not emphasised equally in the two functional approaches under discussion. In fact, they are not explicitly defined to students in the two approaches simply because it is assumed that students will develop the correct meaning for a concept (e.g. function) from their exploration of that concept. Heid (1996), for example, said: "The *Computer-Intensive Algebra* curriculum is designed to develop concepts gradually and through examples rather than through definitions and applications of those definitions." (p.247). One can, however, suggest that both approaches pay more attention to the *univalence* of a function than to its *arbitrariness.* In the first case, the *univalence* property is indirectly addressed by the two approaches by engaging students on many examples involving use of tabular and graphical representations of functions and which implicitly suggest the *univalence* of a function.

The *arbitrariness*, on the other hand, seems not to be addressed in the same way that Even (1993) described it. In the two approaches to algebra, students work with examples of functions, which emphasise the regularity of a function rather than not being necessarily regular. The domain and the opposite domain contain number elements rather than something different. Perhaps working with computers involves that the two sets are sets of numbers, the function expressed as an expression, a regular graph or a regular table of numbers, in order to facilitate the construction of other representations of the function and to allow continual access from one representation to another.

2.3 Students' errors and difficulties in the process of learning algebra

The discussion in this section is arranged under six sub-headings: the transition to algebra, the nature of algebra, overgeneralisation of rules and procedures, students' understanding of letters in algebra, translating a word problem to an equation, equations and expressions. Although most of the errors and difficulties discussed under these sub-headings are not directly involved in this investigation, the literature reviewed in this chapter is necessary to fully inform this study. It helps in understanding the cognitive difficulties of students in algebra and this helps to build the research instruments, analyse data of this study and identify examples of errors to the suggested errors by the trainee-teachers.

(a) The transition to algebra

Students start learning arithmetic long before they take any formal courses in algebra. This is because arithmetic knowledge is considered to be "a necessary and sufficient prerequisite to algebra" (Herscovics & Linchevski, 1994, p.61). Nevertheless, this arithmetic knowledge causes them difficulties and errors when they fail to discriminate the continuities and discontinuities between arithmetic and algebra. The literature review in this section concentrates on the obstacles that students experience at the time of the transfer from arithmetic to algebra, a phenomenon that is usually described in the literature as a "jump" (e.g. Cortes et al, 1990).

School algebra is usually defined as "generalised arithmetic" (e.g. Booth, 1984; Kieran, 1984). This is because it enables us to make generalisations for numbers or for processes on numbers. For example, the algebraic expression a+b is a generalisation for the process (or product) of adding two numbers in arithmetic such as '4+3'. However, the two expressions, a+b and 4+3, require different understanding. The plus sign in the arithmetic expression means that the two numbers 4 and 3 are to be added. In algebra, the two numbers, a and b, cannot be added unless both of them are given (Kieran, 1988a). This conceptual change between processes in arithmetic and that of algebra is one difficulty that causes students' errors in algebra. Other conceptual changes between arithmetic and algebra are given in the next few paragraphs.

Matz (1980, 1982) identified three conceptual changes between arithmetic and algebra that cause students' errors in algebra, namely, the nature of symbolic values, the expanded interpretation of the equal sign, and the changing nature of problem solving. Introducing the symbolic values, according to Matz, forms the critical transition step because they require students to denote and manipulate abstractions. She added that "this involves first recognising that letters have referents (that they are abstractions of something and not just arbitrary entities) and next figuring out how to operate with abstractions and to denote and interpret results" (Matz, 1982, p.38). When students do not realise that a letter is representing a number then they may fail to operate on or with the letter. For example, when Davis (1975) asked Henry to multiply x by an expression, the student could not do it and said "How can I multiply by x when I don't know what x is". Furthermore, many students find it difficult to accept, for example, x+1 as an answer for a problem rather than as a process that needs to be carried out in order to get a "proper" answer (Booth, 1984).

Another characteristic of the symbolic value is concatenation. This is used in both arithmetic and algebra. Matz (1982), however, noted that concatenation in algebra means something different to that in arithmetic. In arithmetic, it is used in place-value notation and also in the notation for mixed fractions where it denotes implicit addition; for example, 43 and $4\frac{3}{4}$ mean 40+3 and 4+¾ respectively. In algebra, it denotes symbolic multiplication; for example, xy and 3x means $x \times y$ and $3 \times x$ respectively. According to Matz, students whose thinking is still arithmetic are therefore expected to err when they encounter concatenation in algebra; for example, concluding that 4x equals 46 when x = 6 because they are still using their arithmetic interpretation of concatenation. Research results strongly support this expectation. For example, Herscovics and Chalouh (1984) found that the students in their study did not perceive 3n as $3 \times n$ because when asked to substitute 2 and 5 for n they got 32 and 35 respectively.

Regarding the equal sign, Matz (1982) found that in arithmetic it is mainly used to connect a problem with its numerical result and, in a less frequent way, to connect two equivalent processes (e.g. $2 \times 7 = 7+7$) or to connect the solution steps together (e.g. $2(6-4) = 2 \times 2 = 4$). In algebra, the equal sign can also be used to connect a problem with its result (e.g. 2x = 6), two processes together (e.g. x+3 = 12–2x), or the solution steps of a problem (e.g. when expanding (x+1)(x-1)). However, Matz differentiates between the equal sign in arithmetic and algebra in that, in arithmetic, it is always connecting universally true statements (e.g. $2\times7 = 7+7$), in algebra, it is also connecting universally true statements (e.g. $(x+1)^2 = x^2+2x+1$) as well as denoting constraints (e.g. $x+3 = 7$).

In relation to the changing nature of problem solving, Matz argues that solving arithmetic problems involves executing algorithms. In algebra, however, it involves planning and then executing the plan. Kieran (1990) seems to make a similar observation when talking about the representation of word problems by equations. She found that in arithmetic, students are thinking procedurally, that is, executing the explicit operations, from left to right taking into account the priority of each operation over the others. In algebra, however, it involves first representing the problem situation symbolically and then solving the problem by executing implicit operations opposite to the ones that they have in the problem. For example, solving the equation $2x + 3 = 7$ requires subtracting three from both sides of the equation and then dividing both sides by 2, that is, using the opposite operations of addition and multiplication.

Stacey and MacGregor (1997) suggest a different point of view for the difference between problem solving in arithmetic and algebra. They say, students usually have many options to work out arithmetic problems but in algebra, they have only one option (see page 254 of their article for an example).

Finally, Bednarz and Janvier (1996) suggest that problem solving in arithmetic requires forward operations (working from the known to the unknown) whereas in algebra involves backward operations (working from the unknown to the known in the problem). Examples of these were given in section "(b): Problem solving in algebra".

The views above agree in that problem solving in arithmetic differs from problem solving in algebra. This difference between the two domains, arithmetic and algebra, creates problems for students when they encounter algebra.

The last difficulty that is raised in this section is the changing nature between the required answers in arithmetic and that of algebra. Booth (1984) identified three differences between algebraic and arithmetic answers. Firstly, algebraic answers are not always numerical values but they sometimes contain letters. Secondly, algebraic answers may come in different but equivalent forms. For example, the expressions 2(x+3), 2×(x+3), (x+3)×2, 6+2×x, 2x+6 and 6+2x are all equal and any one of them can be a correct answer for the same problem. Thirdly and more importantly, any algebraic expression may appear as an answer and as a question at the same time. The conflict between students' familiarity with arithmetic answers and the nature of the required answers in algebra is responsible for many of students' errors in algebra. When students are not aware of these conventions, they seek to provide numerical answers for algebraic problems, they do not know that two expressions are equal and they show confusion between expressions such as 2x+3 as an answer and as a set of instructions (Sfard & Linchevski, 1994). Booth (1984) and Küchemann (1981) reported many cases in which students tried to escape from the unclosed algebraic answers by substituting numbers for the letter in order to get numerical answers. In many of these cases, students chose the position of the letter in the alphabet as a value for that letter. For example, some of the students in Küchemann's study substituted n=14 when asked to find the perimeter of a shape with n sides and each side is of length 2 and so they avoided getting the answer 2n.

(b) The nature of algebra

Some of the students' difficulties in algebra are caused by the nature of algebra itself. Algebra allows different interpretations of concepts and this fact makes the understanding of algebra more difficult for students to achieve. Consequently, they make errors when confronted with algebraic problems. Teachers' knowledge of students' interpretations of concepts may be helpful in that it enables them to understand students' thinking and the strategies they use for tackling algebraic problems. They would then be able to think of better ways to help reinforce students' understanding of algebra. In this section, three examples are selected and discussed in turn to show that they have different interpretations. These examples include algebraic expressions, letters and the equality sign.

Expressions

Any algebraic expression, for example, 3(x+5)+1, can be interpreted in different ways and this makes the understanding of the expression more difficult for students because they can fail to discriminate between the different meanings (Sfard & Linchevski, 1994). According to Sfard and Linchevski, the above expression can be interpreted in four different ways. First, it might be seen as a process (first you add 5 to x then multiply the result by 3 and finally add 1 to it). Second, as a product, that is, as a fixed but unknown number that comes as a result of the process described above. This number becomes known once x is known. Thirdly, the expression can be seen as a function that provides for every number x in the domain a corresponding number 3(x+5)+1 in the opposite domain. Finally, the expression can be seen just as an object that stands for nothing but can be manipulated and combined with other expressions.

The above discussion about the different meanings of the expression 3(x+5)+1 shows that algebra allows different interpretations of expressions. Do teachers know about these so that they can understand which interpretation the students in their classes are using? Do students, in fact, interpret algebraic expressions always in that way? There is evidence that some students interpret algebraic expressions in different ways to those described above. Küchemann (1981), among others, found that students sometimes interpret, for example, 2a+5b as two apples and 5 bananas. This interpretation of letters is believed to originate in schools as teachers sometimes encourage students to think of letters as abbreviations for concrete objects (MacGregor, 1991). Many students will, of course, find it easier to manipulate, for example, 2a +5a when they think of a as an apple because this way provides them with a concrete sense than when they think of a as a generalised number or as a variable (Küchemann, 1978, 1981). On the other hand, using letters as names for things not only obscures the letters from their actual meanings but also encourages students to err when this method becomes no more helpful. When Küchemann (1981), for example, asked a group of students to add 2a+5b then many of them answered 7ab. He interpreted this answer as a result of adding 2-apples and 5-bananas and when mixed together in a bag, it gives 7-apples-and-bananas. Another example is that the students could not simplify an expression, which starts with a subtraction sign, such as –a+3b+5a, because, to them, it was not possible to start by subtracting an 'apple' when they have no apples, at the beginning, to start with (Küchemann, 1981).

Letters

Letters in algebra are used in many different ways depending on the pedagogical context in which these letters occur. Thus, the same letter (for example, x) could have different meanings according to the mathematical activities in which students are working. Janvier (1996) classified algebraic letters according to their meanings into four categories: Letters as undetermined numbers, letters as unknowns, letters as variables, and letters as objects.

Janvier argues that letters as undetermined numbers are encountered in formulae, such as $C=2\pi r$ for the circumference of a circle. He added that in this circumstance, both the r and C are not expected to vary and so they are not variables. Instead, r is expected to be given in the problem or the exercise rather than to be calculated as an unknown or as a variable. Similarly, C assumed to be known once r is given. So, both letters are undetermined numbers.

Letters as unknowns are encountered in equations such as $3x+4 = 10$. Here, x is expected to be calculated and its value to be identified. Sometimes, equations include letters that are best described as undetermined numbers or as givens. For example, the letters a, b and c in the equation $ax^2+bx+c = 0$ are expected to be given and, consequently, they are known as parameters (Sfard and Linchevski, 1994).

Letters as variables are clearly used in functions such as $y = 3x+2$. Here, y clearly varies as x is expected to vary and so both letters are variables.

Sometimes a letter is used without any reference to the three meanings above. For example, expanding brackets such as $(x-1)^2 = x^2-2x+1$ is usually done without thinking about what the letter x might actually mean. In this case, x is used more as an object or as a placeholder.

Janvier used his classification of letters to argue that algebraic reasoning does not start when students first encounter letters as undetermined numbers in formulae but rather when they first start using letters as unknowns in equations.

The equality sign

Students first encounter the equality sign in arithmetic in sentences such as $2+3 = 5$. In measurement, students also meet the equal sign in sentences such as 1m =

100cm. These two examples show two different meanings for the equal sign. In the first example, the equal sign is used, as in calculators, to execute the answer. In this case it means "it gives" or "the answer is" (Behr et al, 1980). In the second example, however, the equal sign is used differently. It is used to compare two measurement quantities of different units. Note that the letters in "1m = 100 cm" are not algebraic letters and the sentence, 1m = 100cm, is not an equation. The equal sign can be understood as "consists of".

By the time students are introduced to algebra, they meet the equal sign in simple equations with one unknown, such as x+2 = 5 and 2x+5 = 8-x, and later on in functions, such as *f*(x) = 2x+1. In these three examples, the equal sign has different meanings.

The first equation, x+2 = 5, is usually received by students as "how much you should add to 2 to give 5". Sometimes the equation requires some simplifications before it appears in the form ax+b = c. Yet, after simplification, it is still possible to solve it in the same way as x+2 = 5. This method of solving equations is usually known in the literature as "the cover-up method" (Bernard et al, 1988) or sometimes called "use of number facts" (Kieran, 1992) or even the "counting up approach" (Booth, 1981). From my experience, many teachers in Oman encourage students to solve linear equations such as x+2 = 5 in the way described above to help them find the value of the unknown more easily. Also, they usually present the equation ax+b = c to the students in that way rather than as c = ax+b. In cases like these, students may have less opportunity to see the equal sign as a balance between the two sides of the equation. As a result, many of them continue to see the equal sign as "it gives" even by the time they go into college (Mevarech & Yitschak, 1983).

When the unknown occurs in both sides of the equality, such as in the second equation 2x+5 = 8-x, students obviously cannot just read the equation from right to left to find the value of the unknown as they used to do in simple equations of the type ax+b = c. In other words, they could not use the "cover–up method" to solve this type of equation and so they need to know a new way of thinking about the equation and the equality sign as well as a new way of solving procedure. In this case, the equal sign behaves like a balance between the two sides. It reflects the symmetry of the two quantities on both sides of the equation (Herscovics & Kieran, 1980; Kieran, 1981).

In functions, however, the equal sign plays the part of a definition more than as symmetry or a balance of the two sides of the equality (Cortes et al, 1990). Hence various understandings of the equal sign are possible and sometimes within the same example. Take, for instance, the sentence $x = 2$ and think how you understand it? For some people, $x = 2$ is an equation but for others it is not. The second group of people might think of $x = 2$ in the same way as it is used in the computer programme MATHEMATICA where $x = 2$ means "x is now assigned the value 2". The second view of the equal sign is clearly preferable by school students as they usually reject sentences like $2 = x$ and instead they prefer it as $x = 2$ (Behr et al, 1980).

(c) Overgeneralization of rules and procedures

One source of students' errors is the overgeneralisation of rules and procedures they use to solve algebraic problems. For example, students are taught the distributive law at school and how it is used to simplify things by breaking them into parts such as:

$\sqrt{(ab)} = \sqrt{a}\times\sqrt{b}$

$2(x-y) = 2x-2y$

$(ab)^2 = a^2\, b^2$

Yet many students misuse the distributive law when they attempt to simplify things as in the following examples:

- Computing Sin(a+b) to be Sin(a)+Sin(b)
- Computing $\sqrt{(a+b)}$ to be $\sqrt{a}+\sqrt{b}$
- Computing $(a+b)^2$ to be a^2+b^2

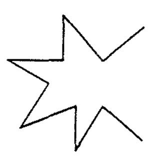
In this case, the distributive law is overgeneralised (see; Maurer, 1987). Another area where students overgeneralise rules is when they simplify algebraic fractions. One common error associated with fractions is when they simplify, for example, $\frac{x+1}{x+2}$ to ½ because they often told at school to replace $\frac{x}{x}$ by 1.

The above are only a few examples of students' errors in algebra and Matz (1980) lists and discusses 33 of such errors. Kieran (1990) has noted students' temptation to overgeneralise rules and procedures when she reported that:

"... the research on students' ability to cope with the continuities and discontinuities between arithmetic and algebra has shown that many of the errors that students make in algebra are the result of reasonable, though unsuccessful, attempts to use or adapt previously acquired knowledge to a new situation." (Kieran, 1990, p.112)

Booth (1981) has observed students using different strategies to answer algebraic problems than the ones they are taught at schools. These informal strategies, which students use, are described by Booth as being *improper*, *naïve* and *intuitive*. Despite the fact that these strategies may be used to answer simple algebraic problems, they sometimes lead to incorrect understanding and errors. This is best illustrated by an example. Students are usually asked to solve simple equations such as x+6 = 11 when they are first introduced to the concept of equation solving. For these simple equations, they do not usually use formal methods (such as subtracting 6 from both sides of the equality sign) to solve the equations but rather by using some intuitive strategies such as trial-and-error or number-fact methods (Kieran, 1992). Teachers usually introduce students to a more formal method by the time they reach more advanced problems so that the students appreciate the need for a procedure that can be generalised for all problems. However, many researchers argue that students continue using their informal methods even beyond this stage and so they encounter difficulties because their methods cannot usually be generalised (Petitto, 1979; Booth, 1984). For example, Booth (1981) observed students solving the above equation by a strategy that she named the "counting on approach". She argues that the same strategy was used by students to answer other problems for which this strategy does not work such as Figure 2.3.

Figure 2.3: The perimeter of a shape



Part of this figure is not drawn. There are n sides altogether all of length 2. Find the perimeter (Küchemann, 1981, p.102)

According to Küchemann, 18% of the students that he tested gave numerical answers such as 32 and 34 to the question in Figure 2.3. He added that they arrived at these answers by counting the number of sides in the figure, or by adding a few more sides in order to close the figure first and then counting the number of sides and multiplying that by 2 to get a numerical value and avoid getting 2n.

(d) Students' understanding of letters in algebra

In section (b) above, four different uses of letters were identified and discussed according to the pedagogical context in which they are used. The discussion in this section is about students' interpretations of letters and their role in generating some of the misconceptions and errors in algebra.

To begin with, Thorndike (cited in Kieran, 1989) recognised that the different interpretations of letters confuse students and therefore suggested in the 1920s that different letters should be reserved for different interpretations. Whether this suggestion is good or not, it has never been used anyway. As a result, further observations about students' difficulties with letters in algebra have been reported since the time of Thorndike. For example, Matz (1982, p.38) says, "Lumping together symbolic constants, parameters, unknowns, arbitrary symbolic values, and pattern variables as simply "variables" draws attention only to a single common feature-their abstractness."

Three studies reviewed in this section (Küchemann, 1981; Booth, 1984; Warren, 1999). These are discussed in more detail than others because they are referred to repeatedly in this chapter and in the rest of the thesis.

Küchemann's (1981) study is part of a large study known as the "Concepts in Secondary Mathematics and Science" (CSMS). The CSMS study investigated students' difficulties and errors as well as their hierarchies of understanding in secondary mathematics including many topics such as algebra, measurement, fractions, decimals and so on (Hart, 1981). In this study, paper-and-pencil tests were administered to large representative samples of students aged from 11 to 16 years. The tests were developed on the basis of many aspects such as textbook analysis, discussion with teachers, relevant research literature and individual interviews with students. The interviews were used to determine the language and format of the test items and, more importantly, understand the kind of conceptions (both positive and negative) of students and the strategies they used to work out mathematics problems. In the case of algebra, the interviews were conducted with 27 students aged 13-15 years and the paper-and-pencil test was administered to a total of 3550 students in different schools across England. The findings, as reported in Küchemann (1978, 1981), indicated that students used and interpreted algebraic letters in six different ways ranging from letters evaluated to letters used as variables. These will be discussed following reviews of the studies by Booth (1984) and Warren (1999).

Booth's study (1984) was a follow up to the CSMS study. It consisted of two parts. In the first part it investigated the reasons behind particular errors in algebra, namely, those errors which had been suggested by the previous CSMS study as common errors among 2nd, 3rd and 4th graders in English secondary schools. To achieve this aim, two phases of individual interviews with students from five comprehensive schools were conducted. Students for the interview were selected by administering the CSMS test to identify students who make the kind of errors suggested by the CSMS study and then make the selection according to the type of errors that students made. Having selected the interviewees, about 16 students from each year group were interviewed in phase one to determine causes of students' errors. In phase two, 17 of the students who were interviewed in phase one were interviewed again to further enhance understanding of the errors. In the second part of the study, the effectiveness of short teaching modules designed to overcome students' errors and difficulties was investigated. In total, six 35-minute lessons were developed and tried initially with three groups of five to six students from two schools and finally with larger groups consisting of half classes. These teaching lessons involved introducing algebra as generalised arithmetic right from the beginning rather than introducing it from letters representing unknowns to letters representing variables. The results of this study are not discussed separately here but are incorporated with findings from other studies in the discussion given later in this section and in other relevant places in this thesis.

Warren (1999) explored students' understanding of the concept of variable by first administering a written test to 379 students and then selecting 12 of them for semi-structured interviews. The students were aged between 12 years 2 months and 15 years 10 months and were selected from two schools in Australia. The test consisted of eight questions about generalising from visual patterns, generalising from tables of values and understanding the concept of variable.

Returning now to the discussion about students' interpretations of letters as suggested by Küchemann (1981), these are listed and discussed in some detail below. Relevant evidences from other studies are also incorporated.

Evaluating letters.

According to Küchemann, the students evaluated the letter in items such as "find a if a+3 = 7" without having to operate on the unknown. In fact, students need only to use number fact method (Kieran, 1992) or counting up approach (Booth, 1981) to find an answer to simple equations such as the above one. One common mistake connected to the evaluation of letters occurs when students are asked to describe expressions such as 3+2x, here they tended to evaluate x with a number such as 1 to get a numerical answer for the expression (Warren, 1999).

Ignoring letters.

Some of the students' answers, according to Küchemann, involved ignoring the letters and ignoring their actual meanings. For example, when asked to "find the value of a+b+2 given that a+b = 43", they only replaced 'a+b' in the first expression by 43 to get '43+2=45'. Warren (1999) has also observed students' tendency to ignore the letters or at best acknowledge their existence when they, for example, simplify the expression 3x+2y+7x to 12xy (adding up all the numbers and then writing down all the existing letters).

Using letters as objects.

Using letters as objects has already been discussed briefly in various parts of the literature review in this chapter (part (c) in section 2.1 and part (b) in section 2.2). More will be said about it in this section.

Students are said to use letters as objects when they, for example, think of adding 2a+5b as adding together two apples and five bananas or 2 a's and 5 b's (Küchemann, 1981). Using letters in this way is a common practice of students

(Booth, 1984; Warren, 1999; MacGregor, 1991). Booth found that 48% of the students that she interviewed used letters as objects in places where they should use them as numbers. MacGregor confirmed this tendency among the Australian students. Warren (1999), however, obtained different results from a written test compared to that of semi-structured interviews. She asked students two equivalent questions: In the test, she asked them to choose, from a list of alternatives, all the meanings of c if "c+d = 10". In the interview, she asked them to choose all the possible meanings of p if "p+m = 12". She found that 22.7% and 21.4% of the 379 students allowed c in "c+d = 10" to stand for "an object like a cabbage" and "an object like an orange" respectively. However, when 12 of those students were interviewed, not one of them allowed p in "p+m = 12" to be "shorthand for pear". Even when a student initially chose p to represent an "object like a pear", when prompted by the interviewer, it was found that "a pear" meant "one pear" to the student. From this study, Warren concluded that when students allow a letter to stand for an object in a written test they are usually aware that this letter is representing a hidden number. However, the other reported studies in this section also used the interview to elicit students' understanding of letters in algebra and confirmed that many of them use letters as objects rather than numbers. This apparent contradictory results between Warren's study and the other studies can be related to the framing of the questions being asked. In Warren's study, students were asked first whether p in "p+m = 12" could take any of the values 4, 12, 15, 0, 3.9, -2 before she asked them whether it could also mean "an object like a pear". The students then had the opportunity to think of the letter as a number first and then as an object. This was not the case in Küchemann and Booth's studies. Also, students' understanding of "a pear" as "one pear" does not necessarily mean "one times the number of pears" to them and so this does not contradict what is reported elsewhere in the literature.

Using letters as unknowns

Some of the students in Küchemann's study were able to use letters as unknowns. However, even when students start thinking of letters as numbers they tend to think in specific cases rather than general cases. For example, they may see the expression 3a-b+a as representing just one or few numbers rather than a family of numbers. This point was made clear by Booth (1984) when she reported that:

"Children [students] tend to think in terms of specific cases rather than generalities, e.g. 'any rectangle' is used by the teacher to mean the general case, i.e. what can be said of any rectangle in the world. Children [students], however, tend to interpret 'any rectangle' or 'any number' to mean the particular one you choose" (Booth, 1984, p.109).

Warren (1999) further investigated this issue by giving a group of students the following question:

- Which one of the expressions t+t or t+4 is larger and when?

The students' answers to the above question clearly showed that they were thinking in terms of specific cases. For example, Adams said "t+t is bigger when t=5, t+4 is bigger when t=2 and they are the same when t=4"(p.318). Another student said "t+t is bigger when t = 5, 6, 7, 8, 9, .."

Using letters as generalised numbers

Few of the students tested by Küchemann (1981) used letters as generalised numbers. For example, about 11% of them said "c<5" to the question "What can you say about c if c+d=10 and c is less than d?" One misconception associated with letters as generalised numbers is using letters as distinct sets of numbers, that is, no two letters can be used to represent the same number (Küchemann, 1981; Booth, 1984). This misconception is not only specific to English speakers. Fujii (1993) interviewed a group of Japanese students to see if they hold the same misconception as English students. In the interview, he asked them to judge whether or not the three pairs of numbers, (6, 10), (9, 7) and (8, 8) are appropriate for x and y so that the equation x+y = 16 is correct. The students who chose (6, 10) and (9, 7) to be correct answers but did not choose (8, 8) were said to hold the misconception: different letters must represent different numbers. Fujii found another misconception: the same letter can represent different numbers at the same expression or equation. For example, x in the right hand side of the equation 3x+5 = 6-x can takes different values than x in the left hand side of the same equation.

Using letters as variables.

Here the letter takes infinitely many values. Research results from Küchemann (1981) and Trigueros and Ursini (1999) suggest that students undergo very slow progress through understanding of the different uses of letters (unknown in equations, general number in expressions and variable in functional relationships).

(e) Translating a word problem into an equation

While the research literature is rich in information about students' errors and difficulties when working out different types of word problems, it was decided to restrict this review to word problems translated to simple equations of two variables because this is the most relevant literature to this research. The review in this section helped in analysing trainee-teachers' explanations and suggestions for addressing students' errors when working out this type of word problem. In particular, it helped identify appropriate categories for the trainee-teachers' explanations and suggestions, such as *semantic* and *syntactic*, which are discussed later in this section.

One of the word problems in which students were found to have major difficulties in formulating an equation from a word problem is the following:

"Blue pencils cost 5 pence each and red pencils cost 6 pence each. I buy some blue and some red pencils and altogether it costs me 90 pence. If b is the number of blue pencils bought and if r is the number of red pencils bought, what can you write down about b and r?" (Küchemann, 1981, p.107)

According to Küchemann, only 10% of the 14 years old students answered this problem correctly and that "b+r = 90", "6b+10r = 90" and "12b+5r = 90" are examples of the committed errors in this problem. The individual interviews revealed that "b+r = 90" was used as "blue pencils and red pencils cost 90 pence" and this statement although incorrect mathematically makes sense to students. The other two answers: 6b+10r = 90 and 12b+5r = 90 were understood as "six blue pencils and 10 red pencils cost 90 pence" and "6 blue pencils and 10 red pencils cost 90 pence" respectively. These examples clearly show that the letters b and r were used as labels for the blue and red pencils respectively. Using letters in this way is believed to have its origin in measurement relations when students learn to write statements such as 10mm = 1cm and in this context the use of letters is acceptable because they are not algebraic letters (Kieran, 1990).

The same thing was found to happen in the USA by Clement (1982) and his colleagues (Clement, Narode, & Rosnick, 1981; Clement, Lochhead, & Monk, 1981). Their famous problem, known in the literature as the "students and professors problem", reads as follows:

> Write an equation using the variables s and p to represent the following statement: "There are six times as many students as professors at this university". Use s for the number of students and p for the number of professors. (Clement, 1982, p.17)

Clement (1982) reported that 63% of 150 freshmen engineering students gave a correct answer to this problem. Using another group of students, Clement, Lochhead and Monk (1981) found that only 43% of 47 non-science majors taking college algebra solved the problem correctly. Using the same problem or adaptation of it, later studies across the world have confirmed students difficulties in writing a correct equation from a given word problem (Wollman, 1983; Fisher, 1988; MacGregor, 1991; Wanjala, 1996).

The correct answer to the "students and professors problem" is $s = 6p$ and so the numerical number associated with the bigger variable is one. Word problems that involve integer numbers bigger than one to be associated with the two variables appear to be more difficult for students. For example, consider the following problem:

> "Write an equation using the variables c and s to represent the following statement: "At Mindy's restaurant, for every four people who ordered Cheesecake, there are five people who ordered Strudel." Let c represent the number of Cheesecake and s represent the number of Strudels ordered." (Rosnick & Clement, 1980, p.4).

Clement (1982) and Clement, Lochhead and Monk (1981), using the same group of 150 freshmen engineering students as described before, found that the percentage of students who gave correct answers to this problem was only 27% compared to 63% for the "students and professors problem".

The above studies have also found that the most common errors in students' answers were $6s = p$ and $4c = 5s$ for the first and second problems respectively instead of the correct answers $s = 6p$ and $5c = 4s$. This error is called the "reversal error" (Rosnick & Clement, 1980, p.5) or the "reversal equation" (Sims-Knight & Kaput, 1983, p.561). Clement (1982) found that about 68% of the students' errors for both problems were of this type. The clinical interviews conducted by Clement and his co-workers with individual students revealed that this error is so strong and persistent. Even when students were given the hint "Be careful. Some students put a number in the wrong place in the equation", their answers had only improved a little (Clement, 1982, p.18).

A similar thing is reported in Sims-Knight and Kaput (1983, p.562) who found that "problems in which the literal were first-letter abbreviations appeared to encourage these errors to a greater extent than problems in which literals to be used were x and y". MacGregor (1991, p.27), referring to the work of Kaput (1987), identified three possible grammatical uses for number words which might have some responsibility for students' errors. These grammatical forms are adjectives (e.g. four people in the cheesecake problem), nouns (e.g. four is less than five) and as a "third function" which appears in statements like "six times five is thirty". According to MacGregor, numbers which come in their adjective forms, such as 6s, encourage the "automatic tendency to assume that 6s means six students" (p.27). Using the last suggestion as well as that of Sims-Knight and Kaput above, one can argue that there is something special with this type of problem, such as the "students and professors problem", which cause even the university students to make errors and produce a low level of correct answers. Therefore, if the problems were simplified and made easier to read and comprehend then high levels of correct responses might be obtained. However, there is doubt concerning the validity of this suggestion given that students were found to have difficulty with simple problems such as the following:

"s and t are numbers. s is eight more than t. Write an equation showing the relation between s and t". (MacGregor, 1991, p.222)

The letters here are clearly not first letter abbreviations and were used as nouns and not as adjective or "third function" forms. Nevertheless, MacGregor (1991) found that the percentage of students (14 years old, n = 281) who correctly answered this problem was only 27% and the reversal error was still common (40% of the total errors).

The discussion above concentrated on just one type of errors namely the reversal error, but there are other types of errors which are less common than the reversal one. Some errors produced by adding the two quantities together (e.g. 6s+p, 6p+s, 6p+p and s+p for the students and professors problem) despite the fact that the problem asks for an equation (Clement, Narode, & Rosnick, 1981; Kaput & Sims-Knight, 1983; Sims-Knight & Kaput, 1983). Clement, Narode and Rosnick named this error the "total error" (p.41) whereas Kaput and Sims-Knight named it the "additive error" or "lack of equation" in places where the numerical value was omitted (e.g. s+p). In Kaput and Sims-Knight's study, 19% of 181 first year secondary students who were in their last month of their first algebra course gave the additive error.

MacGregor (1991) and MacGregor and Stacey (1993) reported similar errors made by students (e.g. s8+t or s^8+t for the s and t problem above). Another type of error which they reported included those in which the equal sign was replaced by ">". These errors, however, differ to the reversal error in that they reflect a misunderstanding of the problem rather than a misconception since they do not form equations. Hence, they may not be deeply rooted, as can be the reversal error.

The reversal error is not only common when students simplify a word problem into an equation, but also when they convert an equation into a word sentence (Lochhead, 1980; Clement et al, 1981), a diagram into an equation (MacGregor, 1990) and a table of values to an equation (Rosnick & Clement, 1980; Clement, 1982). This fact encouraged many authors, as discussed below, to investigate possible cognitive obstacles behind these errors. Two important explanations have been given. Both aimed to understand the way that students translate a word problem to an equation. Consequently, they are known in the literature as syntactic and semantic translations.

Syntactic

Syntactic translation of a word problem occurs when students translate key words of the problem from left to right to their corresponding algebraic symbols without paying much attention to the meaning of the problem (Clement et al, 1981; Clement, 1982; Herscovics, 1989; MacGregor, 1991). Although Clement (1982) and Herscovics (1989) argue that this type of translation would always produce an incorrect answer for the "students and professors problem", clearly one can rewrite the problem in a way so that any syntactic translation would produce a correct answer. The following is an example:

Write an equation using the variables s and p to represent the following statement: "The number of students is equal to six times the number of professors at this university". Use s for the number of students and p for the number of professors.

Semantic

This is the other way of translating a word problem into an equation with two variables. There are several research reports describing this method of translation (Rosnick & Clement, 1980; Clement, 1982; Wollman, 1983; Herscovics, 1989; MacGregor, 1991). Students here show a considerable understanding of the problem in contrast to those who follow the first approach. For example, Clement (1982) noticed that student S4 comprehended the relative sizes of the quantities in the "students and professors problem" and that it was clear to him from the problem that there are more students than there are professors. Herscovics (1989) emphasised that some students who erred did in fact understand the relative sizes of the two groups, "students" and "professors", as they tended to draw a bigger circle for the number of students in the problem and a smaller one for the number of professors.

In this type of translation, students do not attempt to match the words of the problem with their corresponding symbols as they do when they syntactically translate the problem into an equation. Instead, they perceive the problem as if there are two sets, in this case, one for students and the other for professors and that the first set has only "six students" and the other set has only "one professor". So the s and p are used as names or labels for the two sets and not as variables that could take any positive integer number. Students also use the equal sign in order to link the two sets together and not in the way it should be used.

Rosnick and Clement (1980) used another version of the "students and professors problem" in order to test the resilience of the difficulty on the student side. They gave a group of students the question and the correct answer, $s = 6p$, and asked them to choose from multiple choices what the s and p in the equation stand for. More than 22% of 152 calculus students chose "s stands for professor" instead of the correct choice "s stands for the number of students". Thus, the authors concluded that "the tendency on the part of many students to write the reversal equation, 6s = p, is not only a common one but is one that is deeply entrenched" (p.420).

(f) Equations

Many students' errors in equations arise from their interpretations of algebraic letters and from the limitations of the strategies they use to solve one type of equations when attempting to use the same strategies to solve another type of equations for which they are no longer appropriate (Booth, 1984). Since students' difficulties with algebraic letters and their overgeneralisation of rules and procedures which have already been discussed in earlier sections of this chapter, they will not be discussed here further. Instead, this section will consider students' understanding of the equal sign and of the equivalence of equations. These are discussed in the next two sub-headings. Firstly, however, some of the students' errors which they make when working out linear equations of one variable are described.

Sleeman and his colleagues studied students' errors in equations of one variable and reported them in a series of studies (Sleeman, 1984; Moore & Sleeman, 1987; Sleeman et al, 1991). These are classified into four categories: manipulative errors, parse errors, clerical errors and unexplained errors (Sleeman, 1984). According to Sleeman, the first category includes errors that occur when a student overgeneralises a correct manipulative rule. For example, x = m/n instead of x = n/m for simplifying the equation m×x = n. Another example is when a student changes the side of a term in the equation but does not change the sign.

The second category, according to Sleeman, includes errors which occur due to students' misunderstanding of algebraic notations. For example, transforming the equation mx = nx +p to x+x = m+n+p. The third category includes errors that are due to mental slips (random errors) and the last category includes unexplained or unclassified errors.

Understanding the equal sign

Many students do not understand the equal sign in the way that they should (Behr et el, 1980; Cortes et al, 1990; Stacey & MacGregor, 1997). Behr and colleagues explored how students view the equal sign by conducting unstructured individual interviews with students aged six to twelve years and found that the students could not assign any meaning to sentences of the form 3 = 3 or □ = 4+5. Most of them changed 3 = 3 to 0+3 = 3 or 3+3 = 6 or 3-3 = 0 and □ = 4+5 to 5+4 = □ so that it had a meaning for them. Hence, the authors concluded that the students viewed the equal sign as an operator or as "a do something signal" and not as a sign that indicates the sameness or the equivalence of the two sides of an equation. This fact can easily be seen in arithmetic when students use the equal sign to connect the solutions stages of a problem as in the sentence 12+2 = 14×5 = 70.

By the time they encounter algebra, many of the students continue using the equal sign in the same way they used to do in arithmetic; that is, to execute answers or to connect the solution stages. For example, Kieran (1990) observed students solving the equation, 2x+3 = 5+x, where the equal sign is used to connect the left and right sides of the equation. She summarised their solution steps as follows:

Solve for x

2x+3 = 5+x

2x+3-3 = 5+x

2x = 5+x-x-3

x = 2 Kieran (1990, p.101)

Stacey and MacGregor (1997) argue that using the equal sign as an operator is not something unexpected or unusual to students because this is how it is sometimes used in and outside school. They say that students encounter many examples of the form "hard work = success" and "6×5 =..." where the equal sign works as a result executor and that this meaning is further emphasised by the use of a calculator where students press the equal sign to get the answer. In school, Cortes et al (1990) identified four meanings for the equal sign, namely, it gives the result (e.g. 6×5 =...), shows the equivalence of two sets (e.g. 3x+5 = 12-7x), shows the identity (e.g. a+b=b+a), and shows the specification or definition (e.g. $f(x)$ = 3x+2).

The equivalence of equations

This concept of the equivalence of equations involves the understanding that two equations are equal if they have the same surface or systemic structure (see

Kieran, 1988b, 1989, for a discussion on surface and systemic structure). To test students' ability to recognise that two equations are equal if they have the same structure, Wagner (1981) interviewed 30 students aged 10 to 16 1/2 years and asked them if the solution of the two equations $7\times w+22 = 109$ and $7\times n+22 = 109$ would be the same. She found that only about 40% of the students were conservative, saying that the two equations should have the same solution. The other students were either unsure (17%) or stated that the two equations are different and so the two letters w and n should have different values. Some typical answers to the question "which one is bigger n or w" that Wagner received from the students were "w, because n comes first in the alphabet", "neither, because it does not matter what letter you use" and "you can't tell without solving the equations". The first answer was classified as a non-conservative, the second as conservative and the last as transitional.

In another study, Wagner et al (1984) reported a similar finding to that above, students were unable to recognise that the equation $4(2r+1)+7=35$ has the same solution as $4x+7 = 35$ when solved for $(2r+1)$ except by resolving the equation. This was considered by the authors as a failure to recognise that the two equations have the same semantic structure.

Steinberg et al (1990) investigated students' understanding of the equivalence of equations. Ninety-six 8^{th} and 9^{th} graders who had studied before how to solve linear equations by performing the same operation to both sides of the equality participated in this study. Their role was to judge whether or not the equations in each pair of 21 pairs of equations given to them were equivalent and to reason their answers for 13 of the 21 pairs. The authors found that the average of the correct answers for grade 8^{th}, middle 9^{th} grade, and high 9^{th} grade students were 51%, 67%, and 88% respectively. According to them, about third of the students judged the equivalence of equations by solving both equations in each pair and then comparing their solutions to see if they were equal. Also, about half of the 8^{th} graders and few of the 9^{th} graders did not give good reasons of why two equations were equivalent.

(g) Expressions

Many errors performed by students in algebra appear to be a result of the general difficulty that they have with the various meanings of expressions (Sfard & Linchevski, 1994) and algebraic letters (Küchemann, 1981) as well as their inattention to the structure of algebraic expressions and equations (Thompson & Thompson, 1987). Two meanings that continuously cause problems for students in algebra are expressions as processes and as products. This is one of the discontinuity points between arithmetic and algebra. In arithmetic, expressions come either as processes or products and it is always possible to separate the process from the product (Kieran, 1989). For example, 2+3 is the process of the product 5. Students, after so many years of their study in arithmetic, are unable to accept in algebra that, for example, x +3 as an answer for an algebraic problem rather than as a process (Davis, 1975). Davis calls students' inability to hold an expression as a name for the answer and as a process at the same time the "name-process dilemma". Later on, Sfard and Linchevski (1994) chose to call it the "process-product dilemma".

A similar thing was reported by Collis (1975) who found that many students were unable to accept a+b as an answer for combining the two sets: a and b together. Instead they tried to get rid of the plus sign by evaluating the letters in order to obtain a numerical answer or other answers such as ab. Collis calls students inability to accept expressions such as a+b as an answer "lack of closure".

Booth (1984) further explored the process-product dilemma. She argued that this difficulty has a basis in arithmetic where many students do not accept, for example, 6+4 (rather than 10) as an answer for combining two sets containing 6 and 4 elements. Many of the students she interviewed had problems in accepting the "unclosed" algebraic tasks as answers for algebraic problems. For example, when she asked them to choose all the right answers for the question "add 2 to 5a", many of them chose 7a and/or 10a instead of the right ones: "5a+2" and "2+5a". More interestingly, many of those who erred described "5a+2" and "2+5a" as "not real answers since they contain a plus sign".

Working with expressions involving brackets is another area of difficulty that many students are unable to cope with. Booth (1984) conducted individual interviews with students of different age groups using the idea of area of a rectangle to test their ability to multiply a number with brackets. In one of the questions, she gave them a rectangle of width 3 units and length m+4 units and asked them to write down the area of the rectangle. Only 10% of them gave correct answers such as 3(m+4) and 3m+12. The most common error was $3\times m+4$. In a similar question, she gave them a rectangle of width 5 and length e+2 units and asked them to choose all expressions that represent correctly the area of the rectangle from a set of expressions given to them including some correct and some incorrect answers. In this case, 19% of them answered correctly, many of the rest chose $5\times e+2$ and/or $e+2\times 5$ to be correct answers for 5(e+2). From these two exercises (and others), Booth concluded that students find brackets extremely difficult to understand. Even when she tried to address their errors by teaching, many of them resisted the idea that brackets are useful and important in mathematics (see part a in section 2.4 for more discussion).

Another point from Booth's study was that many of the students who failed to write a correct answer were able to describe verbally how to obtain the area of the rectangle, indicating a difficulty in transferring knowledge from a verbal form to a written form. The last finding is consistent with that of English and Warran (1998) who also found that students find it easier to express a general expression verbally than symbolically. The added that any generalisation can usually be expressed in many ways, both verbally and symbolically, but some verbal expressions do not translate easily into an algebraic expression.

2.4 Addressing students' errors and misconceptions

While conducting the literature review in this chapter, special attention was given to the suggested ways of addressing students' errors since this is part of this thesis. Such review was useful in understanding and analysing the suggested ways for help as given by the trainee-teachers in this study. In this literature review, three different approaches to students' errors and misconceptions in mathematics were identified. The first approach addresses the errors once they occur. It treats the errors as something harmful to students and therefore should be "diagnosed" and "remedied". This approach to errors involves the study of what individuals are thinking, what type of errors they commit and why. This is sometimes accomplished through Artificial Intelligence computer programs which are specifically written to quickly diagnose the sort of errors a given student has. Having committed an error, the student is then given the opportunity to realise where and why his/her error had occurred.

The second approach to errors also starts with known students' errors and uses these as starting points to teach students mathematics rather than just helping them to overcome the errors.

The third approach to errors argues that most students' errors and misconceptions are natural results of their improper understanding of algebra. Hence, it is assumed that students' difficulties can be overcome by improving their understanding of algebra through better ways of teaching than the ones usually used at school. The following is a review of the above three approaches.

(a) Addressing students' errors using specifically designed tutorials

Sleeman et al (1991) conducted four field studies with experienced teachers in order to investigate how they diagnose and remediate students' errors in algebra. In the first study, they presented four teachers with a list of students' errors and asked them to suggest suitable ways of addressing the errors. All the teachers except one did not look for a common error (i.e. a misconception) in the set of tasks presented to them and gave suggestions on a task by task basis. Also, they only suggested remedies for 50% of the errors. Their suggestions were mainly procedural (based on re-teaching students the correct procedure).

In the second study, they observed a teacher addressing a set of errors made by eight individual students. One of these errors was flipped division (e.g. solving the equation $5x = 3$ to $x = 5/3$). The teacher first tried to determine the reason behind this error by probing students' understanding of fractions and of the general procedure for solving this type of equation and then giving procedurally based remediation.

In the third study, they interviewed three teachers about how they taught and remediate students' errors. As in the first two studies, these teachers also suggested re-teaching the correct procedure to the students.

The fourth study was a workshop held with a group of algebra teachers. From this workshop as well as the above studies, the authors concluded that it is common practice to teach algebra procedurally.

On the basis of the four studies above, Sleeman et al designed a computer program called PIXIE to address students' errors in algebra. For example, if a student simplified the equation 6x = 3 to x = 2 then PIXIE is able to tell what mal-rule (manipulative rule) the student used (in this case x = a/b instead of x = b/a in the equation ax = b). It also highlights the error by focussing attention on the incorrect step(s) the student made and then re-teaches him/her the correct procedure. According to Sleeman et al, PIXIE is able to guess 34% of students' mal-rules. In cases where it fails to recognise a mal-rule, it just re-teaches the student the correct procedure.

The problem with PIXIE is that it is based on mal-rules and a large number of mal-rules are unstable (Birenbaum et al, 1992). The term "unstable" refers to mal-rules which occur only once or few times but not all the time. For example, all the errors x = b, x = a-b, x = b-a, x = -(a+b), and x = a+b are mal-rules for the equation ax = b (Ibid.) but some of them may not occur the next time the equation is worked out by students.

An alternative approach to errors is the rule space (Tatsuoka, 1983). This approach is an attempt to group students' mal-rules into categories and then identify the non-mastery of the sub-skill that caused each of the categories. Thus, it seeks to identify a more global explanation to the errors than PIXIE by refocusing attention on the source of the difficulty that caused the error rather than the mal-rule that the student used. For example, all the errors listed above for the equation ax = b are caused by the students' non-mastery of the sub-skill of dividing both sides of the equation by the coefficient of x (Birenbaum et al, 1992). This non-mastery of the sub-skill is therefore the difficulty that causes all the identified mal-rules of the equation. According to Birenbaum et al, this explanation of errors is more stable than that of PIXIE.

Another study designed to address students' errors in algebra is reported in Zehavi (1997). The study started by administering three tests to students in grades 7, 8, and 9 in Israel and then identifying students' errors and misconceptions in these tests. On the basis of these, remedial activities were designed to address the errors and to enhance students' understanding of the structure in algebra. The students of each grade worked on two types of activities using a computer program called "Derive". The first type involved replicating the steps of fully worked tasks using forward calculations, and the second type involved partially worked tasks, with some missing steps where students applied backward and forward calculations in order to complete the solution steps. The computer does not allow a student to progress from one step to the next unless that step was completed correctly, thus providing instant feedback to the student. For example, if a student simplifies an algebraic expression such as 3(x+6) to 3x+6 then that student was treated as not mastering the distributive law and the instructor assigned him/her an activity that involved substituting numbers in expressions such as 4(3-2x). The student's role was to complete the missing steps in the activity.

Booth (1984) designed teaching experiments to overcome students' errors and difficulties in the following areas:

- Interpretation of letters, such as using letters as objects and as specific unknowns.
- Conventions and notation, such as calculating a+m as *am* and ignoring use of brackets at places where they should be used.
- Using inappropriate methods to make generalisations in algebra, such as using informal methods to work out complex problems for which the methods do not work.

In these teaching experiments a model of "mathematics machine" in which all the instructions are written in mathematics was used to focus students' attentions on the fact that the procedure for working out a problem should be made explicit. The terms input, output and "processing unit" were used to describe the solution steps of a problem. The students' role was to give the processing instructions to the machine (e.g. add 3 to any number) and then input that in mathematics format (e.g. 3+n) to get the output (n+3). In some other activities, students' were given the instructions and asked to guess the output before they press the input button (e.g. write the output for the instruction: multiply n+2 by 3). The instructions sometimes required the use of brackets as well as letters. As explained above, these teaching experiments were tried first with small groups of five to six students and then with larger groups of about 15 students before they were used in normal class settings (see page 38 for more detail).

Three types of tests were used to assess the effectiveness of the teaching experiments described above. They included a pre-test administered before the teaching experiments started, an immediate post-test administered on the day following the final session and a delayed post-test administered two and four months after the immediate post-test for the larger and the smaller groups respectively. The results of the study for the four year groups students (12 to 15 years old) indicated that they significantly improved their performance between the pre and immediate post tests and this performance was generally maintained between the immediate and delayed post tests. In relation to students' errors and difficulties, the teaching programme showed both successful and weak points. On the one hand, it was successful in developing the notion of "acceptance of lack of closure", such as accepting x+3 as an answer to a problem rather than closing the expression by substituting a specific number for x to obtain a numerical answer. On the other hand, the programme was less successful in addressing students' difficulties when working with expressions involving brackets. Even some of the higher achiever students continued to believe that using brackets is optional because "expressions are the same with and without brackets" (Booth, 1984, p.54). When an attempt was made to convince the students that calculating expressions with different operations in different orders would sometimes give different answers unless brackets were used, they still did not accept that brackets are important in mathematics.

(b) Using students' errors as an approach to teach mathematics

Unfortunately, studies that deal with this type of approach to mathematics errors are limited. However, Nesher (1987) and Borasi (1996) provided many examples to explain how students' errors might be used as "springboards for inquiry" about mathematics. Two of these examples from Nesher (1987) are discussed in section 3.3. One of the examples that Borasi (1996) provided is discussed below.

Borasi (1996) found that one of the most common errors in arithmetic is cancellation of similar digits in fractions. The following is an example:

$$\frac{16}{64} = \frac{1}{4}$$

The inquiry starts by asking students whether the above method of simplification of the fraction is correct or not and giving counterexamples in both cases. So, if a student said it is correct because 16/64 is indeed equal to ¼ then other examples with the participation of the students in the class were given to show that this method is, most of the time, incorrect. Alternatively, if a student said the method is incorrect then questions like the following were raised. Why it works for this example? Are there other fractions for which this method works? And if yes or no, how did they know?

Borasi tried the above inquiry with 20 students (mostly eleventh graders) and gave them opportunity to reflect about their knowledge of some mathematical concepts. The inquiry led them to connect arithmetic with algebra by constructing linear equations of one and two variables and to think of "novel" ways to solve equations at the time when they had only one equation but two variables. The equations that they constructed were used to investigate other fractions for which that "crazy" cancellation method works. The equations were established by representing the numerator and denominator of a fraction using a general form (algebraic letters). This method of inquiry, according to Borasi, is supported by the history of mathematics where some of the historical errors led to the development of mathematics, e.g. the realisation that Euclidean geometry is not the only one that represent spatial relationships. On the other hand, this is an unusual approach to mathematics and so it might contradict students' expectations about the teaching of mathematics as they are used to the traditional way of teaching. This can be a real problem if the errors to be studied do not belong to the students. For these two reasons, many students in Borasi's study expressed negative reactions about the new approach.

(c) Addressing students' errors using better ways of teaching

Word problems

There are some trials which aimed to change students' misconceptions and to deepen their understanding and ability to formulate an equation with two variables from a word problem (Rosnick & Clement, 1980; Wollman, 1983). Rosnick and Clement chose a sample of nine students to test the resilience of their misconceptions about the meaning of letters in equations. Most of these students had studied calculus and all had reversed the "students and professors problem" and similar ones. In their teaching tutorials, they taught the students how to create equality between two quantities when they are not equal, how to read correctly an equation and how to check it for correctness. In addition, they worked individually with the students and told them that their answers were reversals, that s and p are variables and represent the "number of students" and the "number of professors" respectively rather than just "students" and "professors". They also told them that because there are more students than professors the number of professors should be multiplied by six in order to make equality and the final answer should be tested by substituting numbers for s and p and by drawing graphs and tables for reflection. They found that the students' misconceptions were resilient and could not be easily corrected and that even when the students corrected themselves, they went back again to the reversal equation when prompted by the interviewer.

Wollman (1983) used, in a series of six studies, numerical computations and comparison questions to check students' understanding of a given word problem (such as the "students and professors problem"), an algebraic equation and to help them successfully translate a problem into an equation. He found that the students in the first two studies had generally no difficulty in understanding the problem or the equation but a large proportion of them (52%) missed the question when asked to translate the "students and professors problem", or similar ones, to an equation in study three. However, when another group of students were asked the same translation questions as in study three after having been asked a computation and/or a comparison question, Wollman observed a significantly greater proportion wrote correct equations than students who had only been asked the translation questions. In his last study, Wollman gave three equivalent word problems to 43 students. He found that 17 out of the 43 students wrote incorrect equations for the first problem. However, only one student out of the 43 wrote an incorrect equation for the third problem when the students were first given the chance to answer some questions about the second problem, including questions on computation of variables, comparison, translation and verifying answers. Also "after only 10 minutes and without benefit of a tutor prompting, 16 of the 17 students who initially erred were able to arrive at a correct equation"(p.179). This conclusion was criticised by Kaput, Sims-Knight and Clement (1985). Their point was that those students who succeeded may still have the same thinking that previously caused them to err and their high success can be related to Wollman's teaching strategy. In other words, Wollman was able to change their performance for the better but their thinking can only be elicited by conducting individual interviews in the same way as Rosnick and Clement (1980).

Equations

Herscovics and Kieran (1980) (see also Kieran, 1981) used an arithmetic model to introduce students (12-14 years old) to the concept of equation. They started by asking students to create arithmetic equalities with one, two or more operations on each side such as:

$2\times6 = 3\times4$ (same operation on both sides).

$2\times3 = 3+3$ (different operations on both sides).

$2\times8+5-3 = 6\times8-1+7$ (multiple operations on each side).

Herscovics and Kieran noticed that the students initially resisted accepting these equalities without writing the answer but eventually they were able to handle the equal sign as a relational sign rather than as "a do something signal". The students were then asked to hide one of the numbers in the arithmetic identities they had created, initially by a finger, then by a box and finally by a letter to arrive at an algebraic equation. According to them, this model enabled the students to create more complicated trivial and non-trivial equations and to have a better understanding of the equal sign and the surface structure of an equation. The term trivial equations refers to equations in which the unknown occurs only on one side of the equality and the term non-trivial equations refers to equations in which the unknown occurs on both sides of the equality.

Other researchers have used concrete models to introduce students to the concept of equation (Filloy & Rojano, 1985a, 1985b; Cortes et al, 1990). Cortes and colleagues designed their teaching experiment to overcome the six conceptual difficulties encountered by the students (Y7 & Y8) in their study, namely, the concept of the unknown, the meaning of the equal sign, the homogeneity of an equation, knowledge of arithmetic facts, and the conventions of algebraic calculations. The students were divided into small groups, presented with a problem and guided to work through the problem in stages. This strategy introduces students to the concept of equation step by step, stresses the homogeneity of an equation, teaches them to represent the unknown by any letter and makes them familiar with multiple operations on each side of the equation.

Filloy and Rojano (1985a, 1985b) used the area and the balance models to teach students (12-13 years old) equations and equation solving of the form $ax \pm b = cx$ and $ax \pm b = cx \pm d$. The students had already solved equations of the type $x \pm b = c$ and $ax \pm b = c$. The model has shown some success on clarifying the abstraction of operating on the unknown but, on the other hand, it kept them fixate on the model without being able to go back to the abstraction again.

Some researchers have used a computer environment to teach students equations and equation solving (Sleeman, 1982; Brown, 1985; Thompson & Thompson, 1987; Roberts et al, 1989). Roberts et al taught two classes of 6th graders the concepts of variable, equation, function and graphing using Logo. Their aim was to see how much success there would be if computers were used to introduce these concepts to the students at an earlier time than usual. Their results were not encouraging in the sense that the students could not transfer what they learned from Logo to the traditional algebra format. The students, according to Roberts et al, had a problem in learning two things at the same time: algebra and Logo. This result supports that of Filloy and Rojano (1985a, 1985b) in the sense that the students in these studies could not use what they learned from the geometrical model or the computer in abstract situations. However when a computer environment is used for older students than those of Roberts et al (1989) then evidently they can do better in understanding the surface structure of an equation (Thompson & Thompson, 1987).

Expressions

Kieran (1989) observed that the number of studies expressing themselves only to algebraic expressions is limited compared with the number of studies dealing with equations and equation solving. Most of the studies reported here used either geometrical models (Chalouh and Herscovics, 1984, 1988; English & Warren, 1998) or a computer environment (Thopmson & Thompson, 1978).

Chalouh and Herscovics (1984) carried out a teaching experiment which aimed at introducing algebraic expressions for students (6^{th} & 7^{th} graders) in a meaningful way, taking into account the cognitive obstacles encountered by them during their study of algebra such as "students lack of acceptance of closure". They used three types of geometrical models in their teaching including rectangular arrays of dots, line segments and the area of a rectangle. According to them, these models improved the students' understanding of expressions but most of them were unable to understand that 2a+5a and 7a equivalent. In reviewing this study, Kieran (1989, p.43) concluded that "constructing meaning for algebraic expressions does not necessarily lead to spontaneous development of meaning for the simplification of algebraic expressions."

English and Warren's (1998) study was already reviewed earlier at this chapter (see part (a) in section 2.1). MacGregor and Stacey (1992) compared the performance of students who learnt algebra by the "pattern-based" approach and other approaches. In their test, they only included table patterns but not shape or other number patterns. They found that, the students (Y7 to Y10, n=512) generally had problems in writing a rule that expressed the relationship between two variables when presented in table format and that the pattern-based approach students did not do significantly better than the other students in the test.

Finally, Thompson and Thompson (1987) used a computer environment to teach students the concepts of variable, equation and expression. After teaching, the eight 7^{th}-graders improved their understanding, which was seen in terms of students' ability to recognise the structure of expressions. Also, they did not overgeneralise rules.

2.5 Summary

This chapter reviews students' errors and difficulties in the two domains: "unhelpful teaching" and "learning difficulties". The former domain includes review of four common approaches to algebra for the purpose of identifying possible difficulties that cause students' errors. One difficulty lies in the nature of these approaches. For example, modelling approaches tend to make students understand letters as objects because they usually involve working with concrete (or real) situations in which concrete numbers are used. Similarly, generalisation activities may develop a belief that algebra involves only generalising whole numbers because they usually involve working with whole numbers rather than fractions. In problem solving, however, understanding of what constitutes a mathematical problem affects the way mathematics is taught and this affects the learning of mathematics. Thus, a certain problem solving approach might lead students to believe that every mathematical problem has one and only one answer or that it can only be solved once an algorithm is given. Functional approaches engage students in activities that often involve substituting numbers in a function to yield numbers and for this reason they implicitly stress the *univalence* rather than the *arbitrariness* of the domain and the opposite domain.

In relation to learning difficulties, many of the students' errors and difficulties occur at the time of the transfer from arithmetic to algebra because of the continuous and discontinuous points between the two domains of mathematics. Some other errors and difficulties occur because of the nature of algebra itself. Still some other errors and difficulties occur because of the tendency on the side of many students to overgeneralise algebraic rules and procedures.

In addition to the above, this chapter explains three different ways for addressing students' errors including the use of specifically designed tutorials, using students' errors as an approach to teach mathematics and using better ways of teaching mathematics.

Chapter 3

TEACHER KNOWLEDGE: TOWARD A THEORETICAL FRAMEWORK

3.1 Introduction

This chapter presents the second part of the literature review, which has been started in the previous chapter. It aims to build a theoretical framework for this study by arguing that teaching is a complex activity which calls upon several types of knowledge that teachers should incorporate into their teaching in order to teach well. These types of knowledge, including subject matter knowledge, pedagogical content knowledge and knowledge about students, are discussed in turn with the emphasis being on knowledge about students as this incorporates students' errors and difficulties. Teachers' beliefs and their influence on teaching are also discussed in relation to the three components of teacher knowledge. The chapter ends with a model summarising the important aspects which emerged through the discussion, and set the scene for the theoretical framework of this study.

It is important to note that this chapter incorporates a review about teachers and trainee-teachers rather than just trainee-teachers. This is because the literature review about teachers was considered useful for this study partly because it provides further directions and guidelines, but mainly because it covers the shortage of research about trainee-teachers in the areas under investigation.

3.2 What makes effective teaching "effective"?

The broader aim of education is to help others to learn. Teachers are central to the education process because it is through them that the education aim is achieved. Commonsense suggests that teachers vary in their abilities to teach well and consequently in their ability to achieve the public aim in education. Therefore, they are assigned characteristics such as "good" or "bad", "effective" or "ineffective", "veteran" or "novice", which aim to differentiate between different types of teachers. Although these descriptions clearly suggest variability in teachers' ability to teach well, they do not tell us exactly what makes an effective teacher "effective". Even when people do suggest certain competencies for good teachers, they do so according to their understanding and beliefs, which may result in different points of view. For example, they may disagree on what and how much information teachers should know about a particular topic in order to be able to teach it effectively. This point is emphasised by Fennema and Franke (1992, p.147) who say, "there is no consensus on what critical knowledge is necessary [for teachers] to ensure that students learn mathematics".

Despite the above disagreement, recent researchers agree that the quality of teaching is determined by several types of knowledge (e.g. Shulman, 1986; Ball, 1991; Jones, 2000). Yet they may still place different emphases on the role played by a particular type of knowledge or belief. Therefore, Fennema and Franke (1992) argued that, for some educators, in-depth subject knowledge is the most important component of teacher knowledge since they believe that teachers cannot teach what they do not know. They go on to argue that for some other educators, the knowledge about how to teach mathematics and present it in a comprehensible way to others is an important component for teachers to acquire. No less important than that is the knowledge about students, that is, how students learn mathematics, what conceptions and misconceptions they bring to the classroom and how students' misconceptions should be addressed in teaching. Still some other educators believe that the knowledge about the culture of ethnic groups and how they live is an essential component of teacher knowledge. Others may emphasise the role played by teachers' beliefs in their teaching. There will be more discussion about the above components of knowledge after the following section concerning phases of research on teacher knowledge.

3.3 Phases of research on teacher knowledge

In the search for what makes some teachers more effective than others, Ball (1991) identified three phases of research on teaching and cited a quote by Medley (1979) to describe the first phase:

"Driven by commonsense and conventional wisdom about teaching, the earliest research compiled characteristics of teachers whom others perceived as effective." (Medley, 1979 in Ball, 1991, p.2).

In the first phase, Ball noted that research in teaching has commonly focused on the characteristics of the so-called "good" teachers and the expected influence on their teaching. At that time, teachers were judged to be "good" or "less good" according to their students' assessments and without referring to what teachers actually do in classroom or what students actually learn from them. Therefore, the results of the earlier studies were soon recognised as being far less useful than originally assumed. Such recognisable weakness of the early studies led to a drastic modification in the methodology, which appears in the second phase of the research.

In the second phase of research, according to Ball (1991), researchers attempted to establish relationships between teachers' behaviour at classroom and student learning. Thus, they studied, for example, the effect of teachers' questioning, reinforcement, and explanation, as well as qualities such as clarity, directness, and enthusiasm. The results of such studies were again not satisfactory to many educators because they tended to draw a picture for effective teaching as summarised by Rosenshine (1979):

"Large groups, decision making by the teacher, limited choice of materials and activities by students, orderliness, factual questions, limited exploration of ideas, drill, and high percentage of correct answers."" (Rosenshine, 1979, p.47).

In the most recent research, the third phase, researchers have attempted to investigate teachers' thinking and decision making in relation to their subject matter areas. Teaching was therefore redefined in terms of thought and processes which take place before, during, and after the teaching instruction. Ball (1991) has observed that studies of this phase of research placed different emphasis on teachers' thinking and decision-making by, for example, investigating teachers' knowledge and beliefs about the subject matter, its teaching, and learning, or by concentrating on subject matter knowledge as a critical variable on teaching mathematics. Thus, as the time went on researchers have become increasingly aware of the complexities of classroom and teaching practice by moving the emphasis from what effective teachers like, to what effective teachers do and finally to what effective teachers do and think about their work.

3.4 Discussion on different types of teacher knowledge

Shulman (1986) calls upon several types of knowledge to be incorporated in any attempt to examine teacher knowledge. Gudmundsdottir (1991) identified seven types of knowledge including content knowledge, general pedagogical knowledge, curriculum knowledge, knowledge of learners, knowledge of educational context, knowledge of educational aims and pedagogical content knowledge. Another example is Fennema and Franke (1992) who classified teacher knowledge into four components: content knowledge, knowledge of learning, knowledge of mathematics representations, and pedagogical knowledge.

Sometimes teacher knowledge is classified into two components: knowledge about the content of a subject and knowledge about how to teach the content and present it in a comprehensible way to the learners (e.g. Jones, 2000). The first one is known in the literature as *subject matter knowledge* (SMK) and the second one is known as *pedagogical content knowledge* (PCK). In this case, the term "pedagogical content knowledge" is usually used to include many types of knowledge. For example, Grossman (1990) defines PCK as comprised of four components: (a) knowledge and beliefs about the purposes for teaching a subject at different grade levels, (b) knowledge of students' understanding, conceptions, and misconceptions of subject matter, (c) knowledge of curriculum materials available for teaching a subject and knowledge of horizontal and vertical curricula for the subject, and (d) knowledge of instructional strategies and representations for teaching particular topics.

Both SMK and PCK are influenced by teachers' orientations or beliefs about the subject, its teaching, and learning. In our case, teachers' beliefs include their beliefs about the nature of mathematics, how students learn mathematics and how mathematics can best be taught (Thompson, 1984). These types of knowledge interact together to determine the quality of teaching.

For the purpose of this research, teacher knowledge and beliefs are discussed in four sections as follows

- Subject matter knowledge.
- Pedagogical content knowledge.
- Knowledge about students.
- Teachers' beliefs.

Classifying teacher knowledge into different components does not mean that they are independent from each other but is done to simplify the discussion and to have a deeper understanding of each individual component. These components are equally important for teachers and the lack of understanding of one of them may result in less successful teaching. For example, without a firm understanding of students' ideas in mathematics, the teacher is less likely to incorporate the other elements in his/her teaching successfully (Aubrey, 1994).

(a) Subject matter knowledge

Subject matter knowledge plays a key role in teaching mathematics and hence it should receive a good attention from research into teacher knowledge (Shulman, 1986). Some relevant restudies on this area of research are reviewed later in this section.

But what does SMK include? Ball (1991) suggested that SMK includes knowledge *of* the subject and knowledge *about* the subject. The former, according to Ball, consists of the prepositional and procedural knowledge of mathematics such as the understanding of particular topics (e.g. decimals and functions), procedures (e.g. solving an equation and multiplying two expressions), and concepts (e.g. function and infinity). It also includes mathematics structures and connections plus knowledge of the relationships between these topics, procedures and conceptions. Ball (Ibid) named this dimension of SMK *substantive knowledge.* The other dimension, knowledge *about* mathematics, includes "understandings about the nature of mathematical knowledge and activity" (p.7)

Similarly, Aubrey (1994) reported that subject matter knowledge includes knowledge *of* the subject (substantive knowledge) and knowledge *about* the subject. She explained the latter as follows:

"... knowing about the fundamental activities and discourse of a particular discipline, showing an awareness of competing perspectives and central ideas within the field as well as understanding how seemingly incompatible views can be justified and validated" (Aubrey, 1994, p.3).

Grossman et al (1989) found that subject matter knowledge includes three types of knowledge: content knowledge, substantive knowledge and syntactic knowledge. The first consists of factual information, primary concepts, organising principles and ideas, which make up the disciple. The second consists of knowledge of explanatory models or paradigms, the conceptual tools used to guide enquiry conducted in the field or make sense of data. And the third consists of knowledge of relevant forms of methodology, the ways in which new knowledge is brought into the field, including the "canons of evidence and proof and rules governing how they are applied".

Although the above views about the subject matter knowledge may place different emphasis on different aspects of it, they all agree on the importance of knowledge *of* and *about* mathematics. This importance is recognised elsewhere in the literature. For example, Ball (1988, p.12) says: "Knowledge of mathematics is obviously fundamental to being able to help someone else learn it." Similarly, Post et al (1988, p.210) stated that "a firm grasp of the underlying concepts is an important and necessary framework for the elementary teacher to possess."

Although most researchers believe in the importance of SMK for teachers, they vary in their methodological approaches to the study of the SMK. Consequently, some researchers have measured teachers' subject knowledge quantitatively by the number of courses teachers take at university level or their scores in a standardised test (Ball, 1991; Even, 1993). Others may judge teacher knowledge in terms of students' success or failure by relating students' achievement in a test to their teacher subject knowledge so that if students did good in the test then that means their teacher subject knowledge is good too and vice versa (Fennema & Franke, 1992).

Ball (1991) criticised the way that earlier studies approached teacher subject knowledge and suggests that in time, researchers have moved away from the quantitative measurement of teacher subject knowledge to the qualitative sort of measurement by studying teachers' understanding of concepts and procedures.

Fennema and Franke (1992) found that despite the evidence that teachers' knowledge of mathematics is not very good, particularly those at elementary and middle schools, earlier studies about teacher subject knowledge did not report significant correlation between teacher SMK and student learning mainly because of the methodology they used. When the methodology of research studies about teacher knowledge was changed to the qualitative nature, Fennema and Franke identified stronger evidence of the importance of teacher subject knowledge in teaching and student learning. Some of these studies can now be reviewed. These studies all point to the fact that the subject matter knowledge of trainee-teachers is not satisfactory.

Even (1993) studied prospective teachers' understanding of the concept of function and how that influenced their pedagogical content knowledge. In particular she studied their understanding of the *arbitrariness* and *univalence* aspects of a function. These two terms have been discussed in the literature review concerning functional approaches to algebra (Chapter 2). Even found that many of the prospective teachers in her study did not have a "modern" conception of function, that is, they did not understand the arbitrariness and univalence of function as they should do, and that this understanding was reflected in their pedagogical thinking.

Ball (1990) analysed 19 prospective teachers' understanding of division. Findings of this study indicated that their knowledge of division was generally poor. That even when they were able to provide a correct answer to a division problem, they often failed to attach any meaning to the process or represent the problem in a way that school students could understand it.

Putt (1995) and Stacey et al (2001) investigated pre-service teachers' understanding of decimals. The former found that only 9 students from 29 university students enrolled in a mathematics course in Australia were able to place in order, from smallest to largest, the numbers 0.606, 0.0666, 0.6, 0.66 and 0.0660. The latter distributed a written test to 553 pre-service elementary school teachers in four universities in Australia and New Zealand. The test consisted of seven types of comparison items such as pairs of numbers in which the larger number is shorter (e.g. 0.75/0.8), the larger number is longer (e.g. 0.754/0.43), the two numbers are of equal length (e.g. 0.45/0.63) and so on. The pre-service teachers in this study did better than the pre-service in the above study in that 76 to 86% of them scored high in all the seven comparison types in the four universities respectively. In addition, the participants in the two studies were found to hold some misconceptions in decimals including "longer decimals are larger" and "shorter decimals are larger". These are the same misconceptions that Nesher (1987) named "Jermy's misconception" and Ruth's misconception" respectively (discussed later in this chapter).

Borko et al (1992) described SMK and PCK of a student teacher in relation to the division of fraction. The influence of her beliefs and the university courses that she took on her teaching was also examined. Although she was found to hold a "modern" view about mathematics and its teaching and learning, she was unable to teach mathematics accordingly because she had a limited knowledge about the division of fraction as well as the pedagogical aspects related to the same topic. The authors related this 'poor' knowledge of both content and pedagogy mainly to the nature of teacher education programs. According to them, these programs encourage no reflection on SMK or PCK nor do they cause student teachers to examine their beliefs in relation to both content and pedagogy.

Nissen (1994) investigated the mathematical knowledge of a group of 445 Jamaican primary teachers who were found to have a "number of over-simplistic concepts which lead them to make some serious errors" (p.699). Nissen used this finding to question the assumption that all students' errors are self-generated.

(b) Pedagogical content knowledge

"... teachers' subject-matter knowledge does not directly determine the nature or quality of their instruction. Instead, how teachers teach particular topics is determined by the pedagogical content knowledge that they develop through experience in teaching those topics to particular types of students." (Brophy, 1991, p.350).

The above quote suggests that PCK is an important and complementary component to teacher knowledge in that it determines how a particular topic should be taught so that students can understand it. It also suggests that teachers do not only develop this type of knowledge from university courses but also from their teaching experience, their experience as students at school before university and from their interaction with other teachers of mathematics, colleagues and other people. This suggestion is emphasised by Tirosh (2000). Similarly, Fennema and Franke (1992) point out that teachers' knowledge which they develop from their education at school cannot be ignored nor it is a trivial one, since teachers spend many years at school before they go to university.

Brophy (1991) made the same argument as Fennema and Franke when she reported that students' knowledge and beliefs about a subject are partially shaped by the knowledge and beliefs of their teachers. According to her, if a teacher understands mathematics as a set of rules and procedures to be memorised and applied blindly to mathematics problems then that teacher would try to develop the same understanding of mathematics in his or her students' minds. Alternatively, if a teacher understands mathematics as a piece of enquiry or as a set of skills and concepts that need to be understood and not memorised then that teacher would try to transfer the same meaning to students. School experience is an important factor in formulating PCK as well as other components of teacher knowledge.

Teaching experience is also important in developing PCK. Because of this, Fennema and Franke (1992) argued that at the beginning of their in-service teaching, teachers rely heavily on their pre-service pedagogical knowledge and as time goes on, they refine this knowledge and become more dependent on their teaching experience. This may happen, for example, when the new teachers attempt to follow the same pattern of the experienced teachers and as they gain more teaching experience, certain aspects of their teaching might change according to whether these work well or not for them.

The above argument is that teacher knowledge develops from many sources of knowledge including important ones such as university courses, school experience and teaching experience. Also that the understanding of what is good or bad in teaching plays a significant role in shaping teacher knowledge. Therefore, Brophy (1991) argued that teachers' pedagogical content knowledge is "subjectively constructed" in the same way as teachers' orientations toward a subject are constructed and hence she questioned the validly of such knowledge. But what does a pedagogical content knowledge include? This question is answered in what remains of this section.

Pedagogical content knowledge for mathematics teachers includes many elements. The most important ones are the teaching strategies and procedures that can be used to teach mathematics, the different ways for representing mathematical ideas, and teacher decision making. These are discussed below because of their special importance for teaching.

Teaching strategies.

One important aspect of PCK is the knowledge of particular teaching strategies by which teachers present the content in a comprehensible way to students (Leinhardt & Smith, 1985). In their literature review of PCK of science teachers, Smith and Neale (1991) identified several important strategies for facilitating students' understanding of science. These teaching strategies are not only useful for science teachers but for mathematics teachers as well. According to Smith and Neale, they include:

"(a) eliciting students' preconceptions and predictions about phenomena, (b) asking for clarification and explanation, (c) providing discrepant events, (d) encouraging debate and discussion about evidence, and (e) clearly presenting alternative scientific explanation" (Smith & Neale, 1991, p.191).

I suggest that mathematics teachers should also encourage discussion and exploration of ideas, be able to represent mathematics in various ways, probe students' understanding of a topic and use students' preconceptions to improve the quality of teaching.

Mathematics representation.

Mathematics representations are what teachers use to explain something (e.g. picture, analogy and concrete objects). They are very important since "mathematics is composed of a large set of highly related abstractions, and if teachers do not know how to translate those abstractions into a form that enables learners to relate the mathematics to what they already know, they will not learn with understanding" (Fennema & Franke, 1992, p.153).

In algebra, as is also the case in other mathematics domains, teachers use, for instance, real world situations to model algebraic concepts such as functions (e.g. Nemirovsky, 1996a; Heid, 1996) and equations (e.g. Bednarz & Janvier, 1996). Heid (1996) found that real world situations are particularly important for introducing families of functions (linear, quadratic, exponential and rational) and the study of their properties. She has also used them to motivate students to the study of other algebraic concepts such as equations and inequalities.

Another type of representation that teachers use to enhance students' understanding of mathematics is pictorial representation. For example, Mason

(1996a) used geometric patterns to help students construct algebraic expressions and to make them appreciate how powerful and useful algebra is in making generalisations for arithmetic numbers and procedures. Another example is Lee (1996), who not only used geometric patterns to help students construct general formulae, but also to encourage them to model different representations for semantically the same expression, for example 3(x+1) and 3x+3, and then to reason why they are equal.

Other mathematics teachers may use concrete objects such as Cuisenaire rods and Dienes blocks to represent mathematical concepts especially at the early stages of their introduction. For example, Hiebert and Weearne (1986) illustrated how Dienes blocks can be used to teach students decimal numbers.

Teachers general knowledge of teaching and decision making.

Leinhardt and her colleagues (Leinhardt et al, 1991) described a type of pedagogical content knowledge which they called *knowledge of lesson structure.* They described four important elements of lesson structure, namely, agendas (teacher mental plan of a lesson), curriculum scripts (goals and actions for teaching a particular topic), explanations (how teachers communicate subject matter content to students) and representations (what teachers use to represent a lesson). The authors compared the teaching of experts and novices and found that the teaching of experts was much richer than the teaching of novices in terms of the four elements described above. For example, in their agendas the expert teachers provided more details about their teaching plans, made more explicit references to students and gave mathematics of different nature and amount than the novices.

Fennema and Franke (1992) recognised the importance of the decisions teachers take before, during and after their teaching practices in classroom. According to them, teachers' pre-instruction decisions are included in their plans (mental or written), which usually include references to lesson goals, teaching materials, selective activities, teaching procedures and lesson evaluation. During teaching, teachers take many interactive decisions, for example, deciding when to move from one activity to another, what questions to ask and to which student and how to examine understanding of a particular student in the class. Finally, teachers take decisions after the interaction with students to reflect on their own teaching. Bishop and Whitfield (1972) gave a detailed discussion on the nature and types of *pre-lesson* and *within-lesson* decisions. According to them, "good" teachers should not be understood in terms of *what* they do but rather in terms of *how they decide* what to do in any particular situation. They provided extensive examples of teaching situations in which teachers should take important decisions.

(c) Knowledge about students

This component of knowledge consists of the following:

"the conceptual and procedural knowledge that students bring to the learning of a topic, the misconceptions about the topic that they may have developed, and the stages of understanding that they are likely to pass through in moving from a state of having little understanding of the topic to the mastery of it. It also includes knowledge of techniques for assessing students' understanding and diagnosing their misconceptions, knowledge of the instructional strategies that can be used to enable students to connect what they are learning to the knowledge they already possess, and knowledge of instructional strategies to eliminate the misconceptions they may have developed" (Carpenter et al, 1988, p.386).

Students bring knowledge to the classroom that is both formal and informal (Booth, 1981), right and wrong (e.g. Peterson et al, 1991). Researchers have focused on how this knowledge affects the students' learning and how it can be developed or changed (Resnick, 1985).

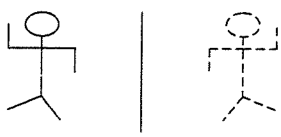
There is a belief that by knowing, for example, what preconceptions students have about a mathematical topic, teachers can plan their lessons so that they make use of what students already know about the topic, capitalise on their "good" preconceptions, and correct or minimise their errors (Wanjala, 1996). Given this importance, Shulman (1986) recognised students' misconceptions and difficulties to be at the heart of teacher pedagogical content knowledge:

"Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons. If these preconceptions are misconceptions, which they so often are, teachers need knowledge of the strategies most likely to be fruitful in reorganizing the understanding of learners, because those learners are unlikely to appear before them as blank slates". (Shulman 1986, p.9)

The researchers above emphasised the role played by students' errors and difficulties in the process of teaching-learning mathematics. Why is this so?

Several authors attempted to answer this question (Nesher, 1987; Ridgway, 1988; Askew & Wiliam, 1995; Borasi, 1996; Schunk, 2000). Askew and Wiliam (1995) presented a short, practical discussion of a wide range of topics, such as addressing common misconceptions in mathematics. In their review of relevant research, they argue that learning is more effective when common misconceptions are addressed. Similarly, Schunk (2000) emphasised the fact that challenging and helping students to correct their misconceptions is essential to effective learning. Ridgway (1988, p.46) argued that "the analysis of misconceptions offers a major stimulus to the growth of knowledge about maths for both pupils [students] and teachers". He suggested that misconceptions are very important in teaching as well as testing and consequently "discovering where misconceptions exist and analysing their sources are important skills for teachers to acquire" (p.48). Borasi (1996) provided practical examples on how students' errors and misconceptions can be used as an approach to teach mathematics. Nesher (1987) discussed the contribution of errors and misconceptions in the process of teaching-learning mathematics from psychological, epistemological and cognitive perspectives. She cited many examples to make her argument. It is worth going through two of them. In one of the examples, Nesher observed a geometrical lesson in which students learned from their mistakes. In that lesson, the students were given an exercise which consisted of a given shape and a given line of reflection (Figure 3.1). Their task was to draw the reflected figure in the place where they think it should fall, then fold the paper along the reflection line and puncture the original figure to see if their drawings were right or wrong. The author goes on to describe how knowledge about students' errors in reflection transformation helped to create a supportive environment in which the teacher contributed additional knowledge or led the students who erred to realise for themselves their mistakes.

Figure 3.1: Reflection in mathematics (Source: Nesher, 1987, p.34)



In another example, Nesher analysed two misconceptions in comparing or ordering decimals. She named the first one "Jeremy's misconception" which is the number with the longer number of digits (after the decimal point) is the larger number (in value). She named the other one "Ruth's misconceptions" which is the number with the shorter number of digits is the larger number. Consider the following tasks in which the author asked Y6 students to mark the larger number in each case:

Case I	0.4	vs.	0.234
Case II	0.4	vs.	0.675

In her analysis of the two misconceptions above, Nesher found that students who held "Jeremy's misconception" (35% of the sample population) were correct in all cases similar to Case II but were wrong in all cases similar to Case I. The opposite was true for students who held "Ruth's misconception" (34% of the sample population). Nesher concluded from the individual interviews that the first group of student was still using the idea of comparing whole numbers where the number with the longer number of digits is indeed the larger number. Also, those students with the second misconception were still using the idea of comparing ordinary fractions of equal numerators, such as 7/10 and 7/22, where the number with the smaller denominator is the larger one.

From the above analysis, Nesher argued that, unless something is done, the students' "success" or "failure" in a test will depend on the actual pairs of numbers given to them in that test. She added that, unless the teacher is aware of the two misconceptions, students may be rewarded for getting the right answer for the wrong reason. As a result, the two misconceptions may get deeply rooted in students' minds. Furthermore, they are hard to detect by unaware teachers. This is because not every pair of numbers is good to discriminate between the various types of misconceptions. For example, 0.4 and 0.234 cannot be used to discriminate between students who hold "Ruth's misconception" and the experts (those who know the order of decimals) as both groups will get the right answer. However, the same pair of numbers is good to discriminate between students who hold "Jeremy's misconceptions" and the experts. Another example is that the two numbers 0.234 and 0.672 cannot be used to detect either of the two misconceptions because in this case students tended to compare the two numbers as if they were whole numbers. Nesher calculated the probability of giving a test that can detect the two misconceptions if the teacher is not aware of the problem and found that:

"when pairs of numbers are randomly selected from all the possible pairs of numbers having at most three digits after the decimal point, the probability of getting items that will discriminate Jeremy's error was 0.10, and Ruth's error 0.02. Thus both Jeremy and Ruth can score up to 90% on a test composed by their teacher if the teacher is not aware of the problem" (Nesher, 1987, p.36).

Research on teacher knowledge about students

Despite the argument above about the contribution of teacher knowledge about student errors in the process of teaching/learning mathematics, Brophy (1991) noted that research in this type of knowledge is limited. Consequently, she recommended the undertaking of more research in the same general area and in mathematics in particular.

Most of the studies in this field of research were conducted in areas of mathematics other than algebra. So, on the one hand, there is a good deal of research in algebra that provides information about: (a) students' preconceptions (e.g. the intuitive strategies they use for solving simple equations), (b) the errors and misconceptions that they might have developed (e.g. misusing the equal sign) and (c) what can be done to assess and improve students' understanding of algebra. On the other hand, there is not much information available about what teachers know about students' preconceptions and errors in algebra, what strategies they use or conceive for correcting students' misconceptions and how they might alter their teaching if educated about students' difficulties and misconceptions in algebra. Thus, there are many questions that need answers and any research in this area might therefore be useful.

Research information about teacher knowledge of student knowledge came from studies which are reviewed in the remainder of this section. This review gives more detail about these studies because they are closely related to the present study and are returned to later in the methodological chapter as well as in the discussion of the results.

Carpenter and colleagues conducted a series of studies that provided some of the information we seek to know about teacher knowledge of student knowledge

(Carpenter et al, 1988, 1989; Peterson et al, 1991). The focus of their studies was addition and subtraction problems. In the first study (Carpenter et al, 1988), they asked 40 teachers to do different things. Firstly, they asked them to write six word problems that can best be represented by six given arithmetic sentences (e.g. 5+7 = ?) Secondly, they gave them 16 pairs of word problems and asked them to predict the harder problem in each pair. Thirdly, they asked them to predict abilities and strategies of individual students when working out different addition and subtraction problems. The results of this study indicated that, on the one hand, the teachers were able to distinguish between some types of addition and subtraction problems, write a word problem for a given arithmetic sentence and predict some of the most frequent strategies of students. On the other hand, they did not develop a coherent framework for classifying problems nor did they seem to categorise problems in terms of students' thinking.

In the next study (Carpenter et al, 1989), the authors investigated whether educating teachers about the research findings on addition and subtraction would make a difference in their teaching. They selected two groups of teachers, experimental and control, and started their study by asking both groups to predict how the students of their own classes would solve given addition and subtraction problems. The experimental teachers were then given access to the available knowledge about various aspects of students' thinking (e.g. strategies and errors) about addition and subtraction problems. At the end of the study, the two groups of teachers were tested once again in terms of the knowledge they had about the mental processes used by students when working out addition and subtraction problems and how teachers might use this knowledge to improve their teaching. The authors found that the experimental teachers did better than the control teachers in many aspects. For example, they increased their ability to predict students' thinking of a particular problem, they spent less time in drill teaching, and they spent more time in problem solving and in listening to students. Still more interesting than this is their suggestion that the students in the experimental classes also learned more than did the students in the control classes, both in computational skills and in problem solving.

Another study is by Leu (1999) which investigated primary school teachers' understanding of knowledge of students' cognition in fractions. The research instruments consisted of questionnaire and interview. A convenient sample of 136 elementary school teachers taking a summer course in a teacher college in Taiwan completed the questionnaire. A smaller sample of 33 teachers was selected from the questionnaire sample according to gender, teaching experiences and teaching grade for semi-structured, one to one, interview. Questions of both the questionnaire and the interview required teachers to identify students' problem solving strategies, possible difficulties and reasons for having these difficulties in a number of student tasks in fractions. For example, one task was about four regular and irregular shapes, which were designed to teach students equivalent fractions. In this task, the teachers were asked to put the 4 shapes in order of difficulty from the easiest to the hardest one if the task was to be given to the 2nd, 3rd and 4th grade students. They were also asked to reason why they put them in that order. The task requires understanding of students' thinking tendency, to decide that, for example, an irregular shape may be answered by more students than a regular one despite the fact that the former looks more complex than the latter. In another task, the problem $3/4 \div 1/8$ was given in this format and as a word problem. The response rate obtained by giving the task to students was also included with both formats. The teachers were asked to judge if the response rate for each format was reasonable and say why, and then identify all the possible strategies that students might use for obtaining the correct answer.

The results of Leu's study suggested that 27% of the teachers understood students' thinking tendency and order of difficulty, 63% of them did not understand students' thinking tendency but understood the order of difficulty and 10% of them did not understand either of them. Thus, these results suggest that 90% of the teachers understood order of difficulty despite the fact that only 27% of them understood students' thinking tendency. Another finding of the study was that 14% of the teachers were aware of (a) the diversity of students' problem solving strategies and (b) managed to identify these strategies, 34% of them either aware of (a) or (b) and 52% of them neither aware of (a) or (b).

Even and Markovits (1993) carried out a study about two teachers and two 11th grade students. One teacher had taught junior high school mathematics for eleven years and the other had taught mathematics for five years in junior and senior high school. Two interviews were taken: one with the students and the other with the teachers. Each interview consisted of two parts. In the first part, three mathematics situations were presented in written form. They were based on students' ways of thinking and misconceptions in functions and asked the subjects to explain students' misconceptions and then suggest how to help students construct their knowledge (overcome their difficulties). For example, situation C reads:

"A student is asked to give an example of a function that satisfies $f(2) = 3, f(3) = 4$ and $f(9) = 15$. The student says that there is no such function.

- Why do you think the student answered this way?
- How would you respond? (Even & Markovits, 1993, p.534)".

After responding to the three situations and in the second part of the interview the subjects were presented with responses from other teachers to the three situations and asked to react to them. Relevant to report here are the findings that teachers were often unaware or ignored students' sources of errors. That they reinforced only "right" or "wrong" and that even when they were aware of a student difficulty, they tended to either "emphasise mechanical thinking or explain the whole topic again" (p.532).

Nathan and Koedinger (2000) administered a questionnaire to 105 elementary, middle and high school mathematics teachers and asked them to rank a set of mathematics problems based on their expectations of relative problem-solving difficulty. They also asked them to rate their levels of agreement to a variety of statements concerning their beliefs about teaching and learning of mathematics. In relation to problem difficulty ranking, the authors found that the teachers correctly ranked algebra problems as more difficult than arithmetic problems. In this context, algebra problems refer to symbolic or verbal problems in which the unknown occurs at the beginning (e.g. solve for x: $x \times 6 + 66 = 81.90$). Also, arithmetic problems refer to symbolic or verbal problems in which the unknown occurs last (e.g. solve for x: $(81.90-66)/6 = x$). The authors also found that, on average, teachers incorrectly ranked verbal problems as more difficult than their corresponding symbolic format problems. The teachers based their suggestion on the fact that, unlike symbolic problems, students should translate verbal problems to their symbolic format before they can solve them. However, the authors found that verbal problems were easier than symbolic problems because students tended to use intuitive methods more with the verbal problems than with the symbolic problems. In relation to teachers' beliefs, the authors found that the participants generally agreed with the fact that *students invented solution methods are effective* and *teachers should encourage students to invent their own solution methods*. On the other hand, they disagreed with the views that challenged the reform-based views, such as *students' answers are more important than their problem solving processes*.

Wanjala (1996) carried out a study which aimed to investigate teacher knowledge of students' strategies and errors in algebra. The subjects were 67 mathematics teachers in Kenya (43 males and 24 females). Their ages ranged from 23 to 45 years and their teaching experience ranged from 1 to 25 years. As a sample, they possess all the teacher qualifications in Kenya. Data gathering involved administering a questionnaire in which teachers were asked to (a) arrange given algebraic problems in order of difficulty for students, (b) predict students' errors in a number of tasks, (c) identify students' sources of errors and (d) suggest ways of addressing the errors. These components are similar to the areas that are investigated in the present study. Wanjala found that, on the one hand, the majority of teachers empathised strongly with students' difficulties as they managed to put the students' tasks in the same order as the one that had been empirically tested with students. They also predicted the most likely errors in the given tasks. On the other hand, only 33% of them identified sources of students' errors and 7% of them gave suggestions that took into account students' cognition. Wanjala saw this as a natural result of the nature of mathematics teaching in Kenya, where teachers pay more attention to accuracy of answers rather than to student understanding.

There are four small studies carried out by Sleeman et al (1991) about teachers' ways of addressing students' errors in algebra. They have been reviewed in section 2.3. Their findings indicated that most teachers address students' errors by teaching the procedure to the students rather than considering their needs. This suggestion is consistent with that of Wanjala (1996) and Even and Markovits (1993) reported above.

Research about science teachers can be used as an indication of awareness on the side of mathematics teachers because of the similarities in many aspects between the two subjects, mathematics and science. As in the case of most studies about mathematics teachers, research about science teachers suggests that science teachers are unaware of students' errors and difficulties in science. For example, in their literature review about science teachers, Smith and Neal (1991) reported that:

"less knowledgeable high school science teachers failed to recognise student errors and actually corrected a (simulated) student who was especially insightful about underlying principles in the lesson. In primary science, teachers are rarely aware of students' preconceptions, or of their power to interfere with science learning. ... Even when informed of these student ideas, teachers often assume that they can be ignored or easily changed." (Smith & Neil, 1991, p.191).

While the above studies were conducted with teachers, Tirosh (2000) chose to focus on prospective teachers' knowledge of division of fractions as well as their awareness of the nature and the sources of related misconceptions held by students. Her sample consisted of a class of 30 prospective female teachers in their second year of a four-year teacher education program in a teacher college in Israel. At the beginning of the academic year, the participants completed a questionnaire and were then individually interviewed to assess their subject matter knowledge (SMK) and pedagogical content knowledge (PCK) of rational numbers. After that they enrolled in a course designed to develop their SMK and PCK of rational numbers. For example, the course attempted to develop their knowledge of conceptions and misconceptions in fractions, sources of the misconceptions and how they might be addressed. At the middle and the end of the year, the participants completed home assignments providing further data for the research. Tirosh's findings indicated that the prospective teachers were unaware of major sources of students' misconceptions in this domain. For example, some of them made similar errors to that of students such as arguing incorrectly that $320 \div 1/3 = 106.666$. However, when educated about students' misconceptions in the division of fractions, most of the prospective teachers became familiar with various sources of incorrect responses. This last finding is consistent with that of Carpenter et al (1989) discussed above in that both the teachers and the prospective teachers in the two studies developed their knowledge about students' errors and difficulties after they were educated about this type of knowledge. However, while Carpenter's et al study indicated improvement in teachers' teaching, Tirosh's study did not say how the development that occurred in the prospective teachers' knowledge about students' difficulties and errors in the division of fractions affected their teaching.

In some studies expert teachers were compared with novices or trainee-teachers. In these cases the expert teachers were more aware than the novices or trainee-teachers in many aspects. In one of these studies (Leinhardt & Smith, 1985), four expert teachers were compared with four novices in their understanding of fractions and in the ways they used to represent that to students. They found a considerable variation between the two groups of teachers and to a less extent within the expert teachers in their understanding of fractions (e.g. equivalence of fractions), in their ability to provide alternative ways of representing a fraction and in their knowledge of students' conceptions of fractions. In relation to the latter, they found that the expert teachers knew better than the novices about students' difficulties in fractions, for example, students' misconception that 1/8 is bigger than ¼ because eight is bigger than four.

Borko et al (1989) compared three trainee-teachers with three co-operative teachers in terms of their lesson planning, teaching and post-lesson reflections. Relevant to report here is their conclusion that expert teachers are more likely to anticipate problems and obstacles, for example student misconceptions. In their own words, they say that:

"Experts are also better able than novices to predict where in a course students are likely to have problems. Their better-developed prepositional structures for content knowledge, pedagogical content knowledge and knowledge of learners, and the more extensive interconnections among these schemata, enable them to predict misconceptions that students may have and areas of learning these misconceptions are likely to affect" (Borko et al, 1989, p.491).

(d) Teachers' beliefs

"to understand how teachers' subject-matter knowledge affects their teaching, one needs to ask not only what a teacher knows about a subject, but also what is the teacher's orientation toward the subject." (Brophy, 1991, p.351).

Many educators argue that teachers' beliefs play a significant role in teaching practices because they determine how and what to teach of the subject. For example, Thompson (1984, p.105) reported that "there is a strong reason to believe that in mathematics, teachers' conceptions (their beliefs, views, and preferences) about the subject matter and its teaching play an important role in affecting their effectiveness as the primary mediators between the subject and the learner". Cooney et al (1998, p.306) also found that "as a profession we have come a long way in realising the importance of the relationship between what teachers believe about mathematics and teaching mathematics and the way teachers teach mathematics". As a result of that, many studies were conducted about teachers' beliefs. These studies reported various degrees of consistencies between the beliefs that teachers hold and what they do in classroom. In fact, some of these studies reported high consistency whereas some others reported sharp inconsistency between teachers' beliefs and their teaching practice. A good example of the first type of study is Thompson (1984) who conducted a case study of three junior high school teachers. At the end of the study, she reported that "examination of the relationship between conceptions and practice showed that the teachers' beliefs, views, and preferences about mathematics and its teaching played a significant, albeit subtle, role in shaping their instructional behaviour" (p.105).

On the other side, Cooney (1985) is a good example of a study which reported large conflicts between beliefs and practice. Cooney observed a novice teacher while he was teaching mathematics. He found that although the teacher believes that mathematics is mainly problem solving, he did not teach students in class by the means of problem solving.

Askew et al (1997) found in a study on effective teachers of numeracy that mathematics teachers have different beliefs about numeracy, its teaching and learning. As a result, teachers were found to hold three orientations towards numeracy: *"connectionist orientation"*, *"transmission orientation"* and *"discovery orientation"*. Teachers were also classified into three categories: highly effective, effective and moderately effective. One important aspect of their findings was that effective teachers have different beliefs and practice than the others. Most of the highly effective teachers were described as having strong connectionist orientation whereas the others were described as having strong transmission, strong discovery or showed no strong orientation.

Ernest (1988) described three views that teachers have about mathematics. These are *problem solving, instrumentalist* and *platonist views*. However, these three views can be connected with the previous three orientations of Askew and his team. The instrumentalist view, as described by Ernest, corresponds to the transmission orientation of Askew et al. In both cases, teachers see mathematics as a set of facts, concepts, skills and routines. Hence they encourage students to perform standard procedures and routines for solving mathematics problems and by using pencil and paper most of the time. Teachers usually stand at the front of the class and teach students by means of a lecture. Teachers are the main source of information and questions in the class. Students are expected to work individually and apply the procedures and routines, which they have learned and memorised in order to get correct answers.

The problem solving view identified by Ernest corresponds to the connectionist orientation of Askew at al. Here teachers see mathematics as a process of inquiry. Students need not memorise the routines and procedures in order to solve mathematics problems. Instead, students are encouraged to find many ways for solving mathematics problems but at the same time to choose the most convenient and appropriate approach for them. During teaching, teachers connect mathematical ideas together whenever it is possible and provide links between the different aspects of mathematics so that students learn by understanding and not by memorising. This approach also has positive effect in students' attitudes and beliefs. For example, students learn not to give way when confronted with hard mathematics and when there is no obvious way or procedure for solving a mathematical problem.

The third view, the platonist view of Ernest, is illustrated by Thompson (1992) who says:

"Mathematics is a static but unified body of knowledge, a crystalline realm of interconnecting structures and truths, bound together by flaments of logic and meaning. Thus mathematics is a monolith, a static immutable product. Mathematics is discovered not created." (Thompson, 1992, p.132).

Thompson's view cannot be connected easily with the discovery orientation as described by Askew and his team. But the last point, mathematics is discovered not created, fits with what the teachers with discovery orientation do when they teach students mathematics. They use practical apparatus and individual activity based on actions on objects.

3.5 Theoretical framework for this study

There is a range of ways in which teacher knowledge can be classified and studied. The most useful way is to classify it as subject matter knowledge, pedagogical content knowledge and knowledge about students. These categories are used to inform the methodology and the analysis of this study and are returned to later in this thesis.

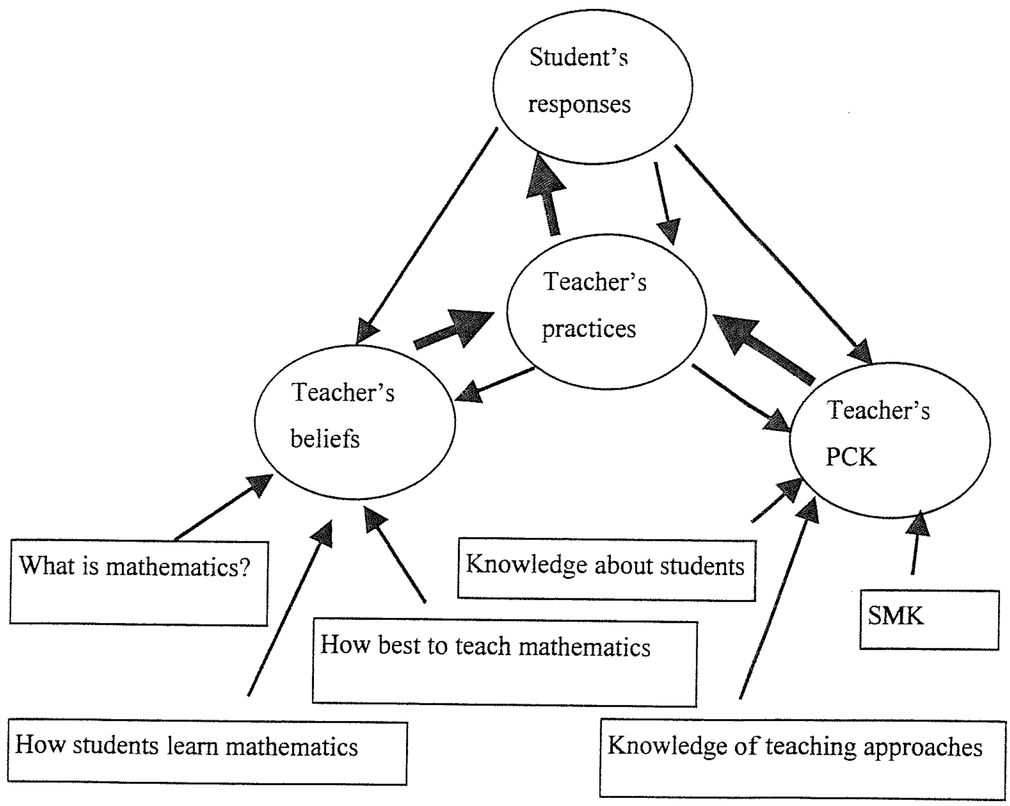
To give an idea about how the theoretical framework in this chapter was used to inform this study, a model is shown below in an attempt to fit teacher knowledge about students' errors with other forms of knowledge and beliefs. This model was adapted from a similar model about effective teachers of numeracy (Askew et al, 1997). Because of this, it has some limitations for this study in that, for example, it assumes that both trainee-teachers' beliefs and PCK have the same influence on their teaching practices. However, trainee-teachers' beliefs are likely to have a stronger influence on their teaching practices than their pedagogical content knowledge (PCK) simply because the latter may not have yet developed in a great deal in comparison with that of the experienced teachers.

While some researchers may consider teachers' knowledge about students' errors as a sub-component of their knowledge about students (e.g. Gudmundsdottir, 1991) or their PCK (e.g. Grossman, 1990), in this model it is not a trivial exercise to decide how this knowledge fits with the other components of knowledge and beliefs. For people influenced by the fact that knowledge about students' errors is a crucial component of teacher knowledge and that it should be used as an approach to teach mathematics, this knowledge is strongly linked to their beliefs. Examples of this group of people are Nesher (1987) and Borasi (1996). Alternatively, and for the same people, this knowledge may be seen as a new approach to teaching and consequently becomes part of their knowledge of teaching approaches (see the model). For another group of people, who may have a considerable knowledge about students' errors but are not yet encouraged to use it in their teaching, this knowledge may fit more appropriately with their PCK or knowledge about students. Trainee-teachers and novices could be examples of this group of people.

Another way to fit knowledge about students' errors with other types of knowledge and beliefs is to break it into sub-components and then place them in the most appropriate boxes in the model. For this purpose, I need to refer to the definition of Carpenter et al (1988, p.386) which is quoted in section (c) of this chapter. From this definition, "the misconceptions about the topic that they [students] may have developed" fits more appropriately with knowledge about students. Whereas, "knowledge of techniques for ... diagnosing their misconceptions" and "knowledge of instructional strategies to eliminate the misconceptions they may have developed." fit more appropriately with PCK. This discussion could only add to the suggestion above that this model has some limitations to this study. Another suggestion that this discussion adds is that the relationship between teacher knowledge, beliefs and teaching practice is complex and even more complex than the model shows.

Figure 3.2: Relationships between knowledge, beliefs and teaching practices.

PCK = pedagogical content knowledge and SMK = subject matter knowledge.



The bold arrows in the model (Figure 3.2), such as the one that goes from teacher's beliefs to teacher's practices, suggest stronger influences than the other arrows. It is assumed in this model that teacher's practice in the classroom, in lessons, is the major factor that influences student learning as it involves the direct interaction between the teacher and the students. On the other hand, teacher's beliefs and pedagogical content knowledge determine, to a large extent, what happens in the classroom.

According to the model, teachers' subject matter knowledge, knowledge about students and knowledge of teaching approaches are different sources for their pedagogical content knowledge. This knowledge (PCK) feeds back and influences teachers' beliefs and practices. However, the model assumes that this interaction goes in both directions and therefore PCK is also influenced by teachers' beliefs and practices in the classroom in lessons and outside the classroom through informal interaction with the students. This view confirms what was said earlier in this chapter: that PCK is "subjectively constructed" because of the influence of different aspects of teachers' beliefs (Brophy, 1991) and that teachers develop their pedagogical content knowledge as they get more teaching experience (Fennema & Franke, 1992).

The above model helps us to understand the role played by teacher knowledge about students (e.g. their misconceptions) in the process of teaching/learning mathematics. In this process, this knowledge is important as it helps build teachers' PCK and beliefs and consequently influences student learning. Without this knowledge, teachers are unlikely to incorporate the other elements successfully in their teaching. For example, "if a teacher does not have a good knowledge of pupils [students] in terms of what they already know or how they, as individuals, approach tasks, the pupils [students] may be expected to learn either something that they already know or something that is too complex for their current knowledge state" (Askew et al, 1997, p.20).

The above model also helps us to understand that while teacher knowledge about students is important in shaping lessons, it is only part of the story. Other aspects of teacher knowledge as well as teachers' beliefs also contribute in significant ways. Thus, a trainee-teacher might have a sound knowledge about students but s/he does not convey this knowledge in his/her teaching because, for example, his/her subject matter knowledge is not as good as it should be. This fact helps us to set the limitations of this study.

3.6 Summary

This chapter lists and discusses several components of teacher knowledge including subject matter knowledge, pedagogical content knowledge, knowledge about students and teacher beliefs. It then uses these components to build a model for the theoretical framework of this study. It attempts to fit knowledge about students' errors with other forms of knowledge and beliefs in the model.

Chapter 4

METHODOLOGY

4.1 Introduction

It is important to examine the assumptions upon which a research study is based as failure to do so could lead to research being no more than a set of methods and procedures applied to a defined research problem (Usher, 1996). In addition, awareness of one's theoretical base helps with collecting and analysing data purposefully, systematically and coherently (Bogdan & Biklen, 1982).

This chapter examines the theoretical underpinnings of this study. It starts by discussing the basis on which the research model of this study was selected. The advantages and disadvantages of the selected research model are highlighted. The discussion then goes on to describe different aspects of the methodology including selecting, developing and piloting the research instruments, changes made to the questionnaire, the population and sample, the return rate, preparing data for analysis, validity, reliability and ethical issues and other detailed procedures. This chapter, however, does not discuss data analysis procedures as they are presented together with the research findings in the next chapter.

4.2 A survey/descriptive/qualitative research model

This study is designed to investigate trainee-teachers' knowledge of students' errors and difficulties in algebra. It aims to find significant answers to the following research questions:

- What are trainee-teachers' expectations about students' success in solving given algebraic problems?
- How do trainee-teachers use their knowledge of students' difficulties in algebra to rank-order algebra questions?
- What explanations do trainee-teachers give for students' errors in algebra?
- What strategies do trainee-teachers use, or conceive, for tackling students' errors in algebra?

- How successful are trainee-teachers in predicting students' errors in algebra?

In order to select the most appropriate research model for this study, different research designs were considered. This was done by examining the research methodologies of the relevant studies, which have been reviewed in section 3.3 and by consulting literature on educational research methods.

Starting by considering types of research methods in education, Robson (1993) classified research into three broad categories: experimentation, surveys and case studies. He added that there could be an overlap between these three categories as some research designs are obtained using, for example, a combination of survey and case study designs. Moreover, the main categories of research are normally divided into smaller categories. For example, surveys are often divided into analytic and descriptive (e.g. Oppenheim, 1992) or cross-sectional and longitudinal (e.g. Wallen & Fraenkel, 2001), depending on the perspective being used.

On the other hand, the review of previous studies in the same area as this research revealed that they incorporated all the three research designs described by Robson above. The research design of using experimental and control groups then testing both groups before and after a formal teaching intervention about students' strategies and errors to measure the effect of this teaching on teachers' knowledge, was used by Carpenter et al (1989). A similar design was used by Tirosh (2000) but with no control group.

The case study design characterised studies of Sleeman et al (1991), Even and Markovits (1993) and Tirosh (2000). The number of subjects in these studies varied from just one teacher (Sleeman et al, 1991) to a class of 30 prospective teachers (Tirosh, 2000). Their aim was exploratory and hence concentrated on just a few subjects and only one topic such as fractions (Tirosh, 2000) or functions (Even & Markovits, 1993). Tirosh's study is classified as both experimental and a case study because, on the one hand, it involved measuring the change in one variable (knowledge about students' errors) by manipulating another variable (teaching intervention) and on the other hand, it involved only one class of prospective teachers in its sample.

The third research design used by previous studies is survey (Wanjala, 1996; Leu, 1999). As will be deduced from the discussion below, this is the most appropriate model for this study and was therefore adopted. The experimental design was totally rejected because the purpose of this study is descriptive rather than measuring causal relationships between two (or more) variables, such as the effect of educating trainee-teachers about students' errors in their teaching. Unlike Carpenter et al and Tirosh's studies, this study does not involve any teaching intervention. A case study could have been used. However, the fact that little is known about trainee-teacher (and teacher) knowledge of students' errors in algebra encouraged the researcher to use the survey approach so that a "general" view could be obtained by covering a considerable number of trainee-teachers from different institutions. Once more information becomes available in this domain case studies can then be conducted to explore this knowledge in greater detail.

The main characteristics of this study are that it is descriptive and qualitative. Hence the title "A survey/descriptive/qualitative model" was introduced. These terms in the title are discussed in what remains of this section. The purpose is to show their relevance to this study so that more characteristics of this research model become apparent. Their advantages and disadvantages are also highlighted. However, the latter is stressed more in the following discussion because one's awareness of the research limitations is extremely important.

4.3 Characteristics of the research model: advantages and disadvantages

(a) Survey research

Most surveys possess the following three characteristics. Firstly, data are collected from a sample rather than from every member of the population. Secondly, data are collected for descriptive purposes rather than exploratory and explanatory purposes. Thirdly, data are collected by administering questionnaires and/or interviews (see for example, Robson, 1993; Borg & Gall, 1996; Wallen & Fraenkel, 2001). These characteristics also apply to the present study. This fact carries many advantages and disadvantages for this study. Some of them are associated with the questionnaire and the interview which are used for collecting data in this study (see the research instruments below). For example, the questionnaire raises many issues about the validity of the data collected (see the validity issues in section 4.12 and the purpose of the interview in section 4.5).

The potential limitations that the adopted research model brings to this study do not only lie in the research instruments but also in its purpose. This study is not expected to provide extensive details about trainee-teachers' knowledge of students' errors because, unlike case studies, its main aim is not exploratory.

Some researchers may disagree with the fact that survey research can be qualitative. It seems that they recognise surveys to be more like quantitative than qualitative research and their purpose is to generalise findings. Consider the two definitions:

- Survey is a study that involves administering questionnaires and/or interviews to "collect data from participants in a sample about their characteristics, experiences, and opinions in order to generalize the findings to a population that the sample is intended to represent" (Borg & Gall, 1996, p.289).
- "Survey is a study that involves "collection of standarized information from a specific population, or some sample from one, usually but not necessary by means of questionnaire or interview" (Robson, 1993, p.49).

According to the above definitions, this study may not be called survey because it is not intended to *generalise* the findings nor it is intended to collect *standardised* data. However, this study is called *survey* because this is the nearest approximation term to describe its research model. It possesses many characteristics of survey as described above. In addition, it follows the "survey research cycle" described by Rosier (1997) with the exception that data preparation and analysis are integrated together where qualitative data are involved. This is because in qualitative analysis, establishing and editing categories goes in parallel with data analysis.

In regard to the generalisability issue raised above, Wallen and Fraenkel (2001, p.377) make it clear that in a survey research "the researcher is entitled to generalize the findings to a larger group if the sample surveyed was randomly selected from that group and not too many are "lost"". Similarly, Cohen et al (2000, p.172) said that "Non-probability samples tend to be avoided in surveys if generalizability is sought; probability sampling will tend to lead to generalisability of the data collected". The research sample in this study was not randomly selected. Hence, there was no attempt to generalise the findings. It follows that the findings of this study do not provide definite proof of the situation concerning trainee-teachers' knowledge about students' errors and difficulties in algebra.

(b) Descriptive research

The present study is descriptive in the sense that Oppenheim (1992) explained it:

"The purpose of the descriptive survey is to count. When it cannot count everyone, it counts a representative sample and then makes inferences about the population as a whole. ... The important point is to recognize that descriptive surveys chiefly tell us how many (what proportion of) members of a population have a certain opinion or characteristic or how often certain events occur together (that is, are associated with each other); they are not designed to 'explain' anything or to show causal relationships between one variable and another" (Oppenheim, 1992, p.12).

Although this study uses mainly qualitative analysis procedures (except for responses in section 2 of the questionnaire), the final findings are presented in the form of statistical tables. These tables should tell us, for example, what proportion of the trainee-teachers suggested re-teaching the procedure for addressing students' errors. This study is not designed to examine causal relationships between certain variables but responses may sometimes be compared together, for example, in terms of gender and qualifications in order to reveal further characterisations of the classified responses.

Another characteristic of a descriptive approach is that it looks at "individuals, groups, institutions, methods and materials in order to describe, compare, contrast, classify, analyse and interpret the entities and the events that constitute their various field of inquiry" (Cohen et al, 2000, p.169). This study clearly looks at individuals, classifies their responses into categories and, where appropriate, compares these categories. One type of comparison in this study occurs between different categories such as those involving different strategies for addressing students' errors.

Another form of comparison that is relevant to this study is the comparison between this study and that of Wanjala (1996). To remind the reader, Wanjala (1996) studied teachers' abilities to predict and identify students' errors as well as their ability to empathise with students' difficulties in algebra. This type of comparison may help us understand how knowledge about students' errors and difficulties in algebra develops as teachers gain more teaching experience.

Comparison between two or more studies is very useful (Keeves & Adams, 1997). On the other hand, official descriptive studies in comparative education are beset with many problems such as the matter of "uncertainties about the consistency or meaningfulness of data returned, because of the tendency for ministry officials to wish to present the conditions of education within their country in the best possible light" (Ibid, p.33). It is arguable, however, that this suggestion is not prominent in this study because it is not official and because the researcher is foreign to the country since he is an Omani and the study was carried out in England.

(c) Qualitative research

Qualitative and quantitative research are distinguished from each other by comparing their general characteristics. Wallen and Fraenkel (2001) listed these characteristics and contrasted them with one another. Similarly, Borg and Gall (1989) described the characteristics of qualitative research. I shall refer to the most relevant characteristics in order to suggest that this research is qualitative. These are summarised in the following points:

- In qualitative research, the hypothesis is not usually formulated beforehand but rather develops as the analysis proceeds toward the end of the study. This description applies to the present study because it does not involve hypothesis testing but rather aims at structuring a picture of what trainee-teachers know and think about different aspects of students' errors and difficulties.
- In qualitative research, data are collected in the form of words and pictures and then analysed in that form, but in quantitative research data are reduced to numbers before analysis. In this research, data are mainly qualitative because they take the form of words, although in one section the analysis is partly quantitative.
- The analysis in qualitative research is inductive rather than deductive, that is, it starts by examining answers to open questions for the purpose of identifying important categories, dimensions and interrelationships. In this study, the analysis also involves categorising answers and then applying descriptions on these categories.

- Qualitative researchers want to know "what the participants of the study are thinking and why. ... Assumptions, motives, reasons, goals, and values are all of interest and likely to be the focus of the researcher's questions" (Fraenkel & Wallen, 1996, p.443). Indeed, this study is concerned with "what" and "why". For example, trainee-teachers are given an erroneous answer by a student and asked, first, to identify what the student has done wrong and secondly, suggest why the student has produced that answer.

Since this study possesses many characteristics of qualitative research, it could be categorised as qualitative research. However, data from section 2 of the questionnaire might be seen as exceptional because it shares two characteristics with quantitative research: first responses are reduced to numbers and second, they are analysed statistically. However, section 2 also has some characteristics of qualitative research, for example, it is not designed to test any hypothesis about trainee-teachers' expectations.

4.4 Selecting the research instruments

The research instruments of this study were chosen according to the research theoretical framework in chapter 3 and the procedure: Firstly, review the methodology of previous studies on teacher knowledge about students' errors and difficulties. Secondly, search for as many different choices of methods as possible. Finally, make the selection according to the purpose and the nature of this study.

Regarding previous studies on teacher knowledge about students' errors, which have been reviewed in Chapter 3, a questionnaire was utilised in studies which adopted a survey research design similar to that of the present study. The type of questionnaire used in these studies is open-ended (Wanjala, 1996; Leu, 1999). Some other studies in the same area used open-ended questionnaire and interview together (Tirosh, 2000) or interview only (Sleeman, 1991; Even & Markovits, 1993) or observation only (Sleeman et al, 1991). Note that Sleeman et al was quoted more than once because they reported four of their studies rather than just one.

Having described the research instruments of the previous studies, the reader should now be reminded that the purpose of this study is to find answers for the research questions above. They are related to different aspects of trainee-teachers' knowledge of students' errors and difficulties in algebra (see also section 4.12). To serve this purpose, there are a number of methods that can be used. The following are examples:

- Observe trainee-teachers to determine how they address a list of students' errors and difficulties in algebra. This list of errors might be prepared by the researcher and then given to them prior to their teaching.
- Observe trainee-teachers to see how they interact with students who make errors in their classes.
- Use students' work (in and outside school) and trainee-teachers' corrections to such work as a springboard to study trainee-teachers' knowledge of students' errors and difficulties in algebra.
- Analyse teaching plans to study how trainees plan for addressing students' errors and difficulties prior to their teaching.
- Select individual students from each trainee-teacher's classes. Give a test to the students and analyse their answers in order to determine the type and nature of the errors they made in the test. Use the same test to interview trainee-teachers about the students who took the test. By comparing their responses to students' answers, the researcher might be able to measure their knowledge about students' errors and difficulties in algebra.
- Give a questionnaire to collect the majority of the data for the study, analyse the responses and then conduct interviews to identify the reasons that the participants give for their answers.

Each of the methods listed above has its advantages and disadvantages. The researcher therefore has to choose the method that best suits the research purpose and at the same time, consumes a reasonable amount of time, money and effort.

To start with, the first two points in the list above require carrying out productive observations, which are avoided in our case for the following reasons. Firstly, to give the observed teacher-trainees a list of errors means that they have to adjust their teaching to fit the purpose of the researcher. In fact, it may be the case that they have to prepare their teaching plans according to the needs of the researcher. Such an intervention is obviously not very welcome by the observed or by the school. An alternative way would be to attend some classes without interrupting the teaching plans. One problem with this is that observations have to be delayed until the observed-sample start teaching algebra. In fact, they should only be taken sometimes near the end of their courses to allow time for teaching experience to develop. Another problem is that the researcher has to go to school for a long time to conduct his observations in order to pick up good examples about trainee-teachers' behaviour toward students' errors. This is because students' errors may not occur very frequently during teaching and when they do occur they may go unnoticed by the observed and/or by the observer.

The second reason for which the first two points in the list above are avoided is because they go beyond the purpose of this study. The purpose of this study is not to know how trainee-teachers translate their knowledge about students' errors into practice. Furthermore, teaching is a very complex activity and calls upon different aspects to be incorporated and recalled together at the time of teaching (see the theoretical framework in chapter 3). Hence, it would have been very difficult to suggest that, for example, what happened in the classroom resulted from a specific type of knowledge.

The third and fourth points in the list above are also not useful because they require, as in the case of the first two points, teaching experience. This experience is important to study effectively students' work, analyse their errors and then come up with something good that can be used to address these errors in the next teaching lesson. In addition to that, teaching experience is important to prepare a good teaching plan that takes students' errors into account. Moreover, it is hypothesised that most trainee-teachers do not make detailed plans, which cover students' errors, nor do they check students' work for the purpose of improving their teaching. One reason for this is, of course, their time constraint as well as their limited experience. An alternative method would be to interview trainee-teachers about their plans and/or students' incorrect work. However, this was thought to be time consuming and unproductive and so avoided.

The fifth point, which is about comparing trainee-teachers' responses to the interview with students' performance in the test, was also rejected. This method requires preparing a test followed by analysing and classifying the errors, which is again beyond the scope of this thesis.

Finally, the last point in the list above represents the actual method that is used in this research. This method overcomes some of the difficulties of the other listed methods. The questionnaire has many advantages for this study, the most significant being the ability to collect a considerable amount of data in a relatively short time. Thus, although the researcher has to wait until the very end of trainee-teachers' courses, he can still gather the data required. This instrument was therefore used because it was believed to be the most appropriate tool for this study. This selection is consistent with what was suggested above that this study is survey-based, a research design that often associated with questionnaires and interviews as the main tools for collecting data (e.g. Borg & Gall, 1989; Robson, 1993; Cohen et al, 2000; Wallen & Fraenkel, 2001). It is also consistent with the fact that the questionnaire was used as a method for collecting data in the majority of the previous studies about teacher knowledge about students' strategies and errors (Wanjala, 1996; Leu, 1999; Tirosh, 2000).

While some of the studies reviewed above used either a questionnaire or interview, in the present study the two instruments are combined to obtain a more in-depth view about trainee-teachers' knowledge of students' errors.

Researchers usually come across the hard question of why use a questionnaire and not interviews. In this study, the questionnaire was used as the main instrument rather than the interview for several reasons. Firstly, by using a questionnaire a larger sample can be covered, respondents can complete it when and where they want (Bolton, 1988). Secondly, as discussed above, it can save time on data collection. Thirdly, the questionnaire is more economical than the interview in terms of money and effort (Oppenheim, 1992).

Despite the many advantages of the questionnaire, it has several defects or disadvantages. Firstly, understanding its questions depends very much on question wording. Cohen et al (2000) gave many examples of questions which are poorly worded. This difficulty can obviously be overcome in an interview because it is always possible to repeat the same question in another way. Secondly, respondents are required to record their answers to a questionnaire in some way whereas in an interview they give verbal answers which is easier than writing. This is even more difficult if open-ended questions are involved as in the present study because responses to these questions can be ambiguous or they may even be left out without answers. In short, the interview has many advantages over the questionnaire. Cohen et al (2000) identified three purposes for an interview in the following quote:

"First, it may be used as the principal means of gathering information having direct bearing on the research objective. ... Second, it may be used to test hypotheses or to suggest new ones or as an explanatory device to help identify variables and relationships. And third, the interview may be used in conjunction with other methods in a research undertaking" (Cohen et al, 2000, p.268).

The interview in this research was used as a supplementary method to the questionnaire. I believe that verbal data was very valuable when complemented by the questionnaire. I believe too that not all the data needed could be acquired by the questionnaire. The purpose of the interview was then to expand on the data collected by the questionnaire.

Seliger and Shohamy (1989) pointed out that interviews can probe for information and obtain data that have often not been foreseen. They added that much of the data obtained during an open or a semi-structured interview is incidental and comes out as the interview proceeds. In this study, the interview was found to be useful and there are many examples where it provided unforeseen data in almost all sections of the questionnaire. This is discussed in the next section.

4.5 Developing the research instruments

The discussion in this section focuses on the design of the questionnaire and its relevance to the research questions. It also gives a brief idea about the interview schedule. Before proceeding further with this discussion, it should be noted that this discussion reflects the initial thinking about the questionnaire. Changes made to the questionnaire are discussed in a separate section.

(a) The questionnaire

Section I: Background information

The questionnaire consists of six sections (see Appendix A). Three of them (3, 4 and 5) were adapted from Wanjala (1996) and the other three were constructed. The intention in the first section of the questionnaire is to obtain some biographical data about the respondents. Their gender, degree qualifications and teaching year groups are variables of the first section. These variables (and others, for example, institution) may affect responses to the questionnaire and consequently are used to study consistency in the responses.

This section was placed at the beginning of the questionnaire because it is the easiest one to complete. Starting the questionnaire with the easiest section is what Robson (1993) and Youngman (1982) recommend. The pilot study of the questionnaire did not show any difficulty or unwillingness on the part of the respondents to complete this section and so there was no evidence that it should be placed elsewhere.

Section II: Predicting percentages of Y9 students

The second section aims at measuring trainee-teachers' expectations about Y9 students. In other words, it aims to know the extent to which trainee-teachers can identify the proportion of students who can answer a particular algebra problem correctly.

The term "expectations" was introduced because of the suggestion that some trainee-teachers may not teach Y9 group despite the fact they are required to do so after they graduate. Teaching to this year group is usually given to the experienced teachers because of the important examinations that students have at the end of the year. The question of why chose Y9 and not any other year group is returned to below.

One way to find out about trainee-teachers' expectations is to ask them to give examples of algebra problems which can be solved by a few Y9 students, most Y9 students, about half of Y9 students and so on. An advantage of this way is that trainees can suggest problems from their experience. A disadvantage is that each trainee might suggest different problems and the result is likely to be a large number of problems, which introduces difficulties with analysis. Furthermore, these problems should be tested with Y9 students in order to establish a criterion by which the given suggestions can then be compared. It was decided to avoid such a process for these reasons.

An alternative way to the above is to ask trainee-teachers to present their expectations about the percentages of Y9 students who can answer correctly a set of selected problems. Although this method gives the researcher some control on the responses, it still has many difficulties. Firstly, a "correct answer" should be defined because it can come in different forms. Secondly, a criterion is still needed to help analyse the responses. Thirdly, how can the problems be selected or, in other words, what problems to choose? Finally, some sort of categorisation is needed to help classify the responses and to make the section easier to complete. The first difficulty can be overcome by providing specific answers to the problems. The second difficulty can be overcome by either trying the problems with students or getting the results from a reliable study. The first option was avoided because it means testing a large number of students for the purpose of either generalising the test's results about all Y9 students or obtaining results from a sample of students taken from each respondent's classes. It was thought that obtaining general results about students from a small-scale study with one researcher is more difficult than, for example, a national study with a team of researchers. The third difficulty above might be overcome by selecting a considerable number of problems so that, after piloting, some problems can be rejected. The fourth difficulty can be overcome by splitting the percentages, from 0 to 100, into categories.

To begin constructing section II of the questionnaire, 26 algebraic problems, along with their solutions, were chosen from different sources. Of the 26 problems, 21 were selected from Küchemann (1981), three from Fischbein and Barash (1993), one from the APU (1985) and one from MacGregor (1991). Most of the problems were selected from Küchemann for two reasons. The first reason is that it is part of a well-known study, the CSMS (see part d of section 2.2). The second reason is that it is an English study carried out with English students. The algebra problems which came from other sources were chosen in an attempt to cover different algebra topics. Also, respondents are asked to focus their thinking on Y9 students because the reported percentages in these studies are just for Y9 students.

Now I come to the question of how the problems were selected. Only seven of the reported problems in the Küchemann 's study were not included in the initial version of the questionnaire. Four of them are about calculating the area or the perimeter of a regular shape, and another two are word problems. These were not chosen because the questionnaire already contained one or more examples of such problems. So in that sense, they represented a clear repetition. The last problem is about comparing two equations, $(x+1)^3+x = 349$ and $(5x+1)^3+5x = 349$, and suggesting the value of x in one equation if x = 6 in the other equation. This problem was not chosen because it was decided to include only first-degree equations in this study, otherwise the questionnaire would be too long.

Another feature of section 2 of the questionnaire is categorisations of the responses. It was decided to close this section by using a 10-category scale to represent different categories of students' abilities. In other words, the percentages 0-100 were divided into 10 equal intervals so that respondents can choose the interval which they think best represents Y9 students who would answer each of the problems correctly as required. A 10-category scale was used because, on the one hand, I want a reasonably fine category and, on the other hand, I cannot expect reasonable judgements with too many categories. Therefore, 10 is a good compromise between the two conflict criteria.

Section III: Putting problems in order of difficulty

This section aims to extract trainee-teachers' knowledge about comparative difficulties of different algebra problems. In other words, given different algebra problems, can trainees identify which problems are the easiest, which problems are the hardest, and which problems occur between the two extremes? Küchemann (1981) told us that every algebra problem has some characteristics which can be used to determine, to a large extent, the complexity level of the problem. One of these characteristics is the number of variables in the problem. Another characteristic is the nature of the elements in the problem (large numbers, small numbers, letters, etc.).

To serve the purpose above, there were two suggestions:

- Ask trainees to give examples of algebra problems and then to indicate the order of the relative difficulties of these problems; and
- Ask trainees to indicate the order of the relative difficulty for a given set of problems.

The first suggestion is clearly too open because it gives the option for trainees to choose the types of problems that they want to compare and the result is likely to be a long list of such problems. Thus, although it allows trainee-teachers to choose problems from their experience, it introduces difficulties in the analysis because a criterion order is needed to analyse the responses (see section 5.3). For this reason, the second option in the list above was used.

Now assume that there are 5 problems for which the order of the comparative difficulty is theorised but not yet tested. Assume too that these problems are good as test items. Now comes the stage of testing these problems on students so that a good approximation of the order of these problems can be obtained. Now it should be realised at this stage that there are 120 (five factorial) possible responses if these problems are given to the respondents to indicate the order of their difficulty. Thus, before that stage these problems should be allocated into levels of difficulty so that, for example, two problems might occur in the same level. The importance of this is that it helps categorising the responses (see the analysis section in the next chapter). In this study, it was decided that, since such a criterion was already established by a recent study, there was no need to go through that lengthy and complicated process because of the limitation of time available to the researcher. For that reason, the three questions in this section of the questionnaire were adapted from Wanjala (1996). They are respectively about expressions, fractions and equations.

Section IV: Explaining student errors and suggesting remediation

The fourth section of the questionnaire is designed to answer the research questions:

- What explanations do trainee-teachers give for students' errors in algebra?
- What strategies do trainee-teachers use or conceive for tackling students' errors in algebra?

To answer the two questions above, different strategies can be used:

- Ask trainees to give examples of students' errors, then identify sources of these errors and finally, state what strategies they should use to address the errors.
- Examine the work of some students in the classes of the trainees to find examples of errors and then ask trainees about these errors.
- Attend some teaching lessons of the trainees to pick up examples of students' errors and then ask the trainees about these errors.
- Use the literature to find examples of students' errors in algebra and then ask trainees about these errors.

In this research, it was decided to use the last option rather than the first three in the list above. The first suggestion, although it looks attractive in the sense that it takes trainees' experience into consideration, leads to a major difficulty: there is no control on what the respondents might suggest. For example, a trainee might suggest trivial errors (i.e. not common) from just one topic of algebra, say, equations. Another one might suggest all the errors from topics that are not covered in this study. In order to overcome this difficulty, one can give detailed instructions about what exactly is required but such instructions are likely to be long and difficult to follow.

The next two suggestions in the list above have problematic issues because they explicitly suggest a different questionnaire for each individual respondent. They also mean that the researcher has to wait, possibly a long time, until trainees start their teaching practice at school before he can start designing the questionnaire. Thus, the last suggestion in the list above is used because it enables all the research questions to be suitable addressed. Once such a decision was taken, the researcher faced the difficulty of selecting appropriate algebra problems from the literature. This is addressed in the following discussion.

Because this section requires the most writing from the respondents, it was decided to start with one algebraic problem in order to check the appropriateness of asking open-ended questions in the questionnaire. If the pilot study supported the usefulness of this section for answering the intended research questions, more algebra problems could then be added and the questionnaire could be piloted again. The unique problem in this section came from Clement (1982). An incorrect answer to this problem is also included from the same source. The problem was chosen from Clement because the same problem (or adaptation of it) has been tried by many other authors across the world (e.g. Wollman, 1983; Fisher, 1988; MacGregor, 1991, Wanjala, 1996) and detailed analyses about students' errors have been given. This can help in analysing trainee-teachers' responses (e.g. semantic and syntactic are two categories used to describe students' errors and the same terms are used in the data analysis).

There were four questions under the unique algebra problem: the first two questions provide data for the first research question above and the last two questions provide data for the next research question above. The first question asks the respondents to identify what the student has done wrong when attempting to answer that problem. The second question is important because it concentrates the attention on the fact that what is required is not the direct description of the error but rather the difficulty behind that error. Thus, anyone that answered the first question as, for example, "6s = p is wrong because 6 is associated with s" should realise in the second question that a deeper answer than that is required. The third question asks the respondents to suggest ways for helping students who make such errors. And the last question attempts to provide more data about the strategies given in response to the third question.

Section V: Predicting student errors and suggesting remediation

The purpose of section five is to gather data to answer the following research questions:

- How successful are trainee-teachers in predicting students' errors in algebra?
- What strategies do trainee-teachers use or conceive for tackling students' errors in algebra?

The second research question is also covered in the previous section of the questionnaire. The wisdom behind covering the same question in section five of the questionnaire is because what is needed to do so (students' errors and some problems) is readily available.

In total, three questions are asked in this section. The first one is about suggesting the most likely errors when students work out given algebra problems. This is to answer the first research question above. The second question is to say how the errors, which they suggested as the most likely in the first question, should be addressed. The third question is to give reasons for their answers to the second question. The last two questions aim to answer the second research question above.

Now I come to the discussion about the process of choosing the algebra problems in this section. Because designing a questionnaire is not independent of the way it is analysed, this can also help in choosing the content of the questionnaire. In our case, a criterion is needed to analyse the responses in this section. This will help us, for example, to say that error-1 is suggested by about 70% of the respondents but had been committed by 20% of students. There are two ways to establish this criterion. The first one is to prepare a test that can be given to students in different year groups in order to find out types of errors and the average percentages of students who make these errors. The next thing is to choose from the test the most appropriate problems to include in the questionnaire. This process needs a careful study if the aim is to get reliable results. An alternative way is to refer to a previous study which has already established such a criterion. In the case of this questionnaire, three problems were chosen from Wanjala (1996) and another one from Küchemann (1981).

Section VI: Comments and suggestions

Finally, section six asks for comments about the questionnaire. It aims to highlight the weakest points of the questionnaire as seen by the respondents. These points might help to improve the questionnaire after the pilot study or suggest some questions for the interview.

(b) The interview

The purpose of the interview was discussed in section 4.4 above. Briefly speaking, it aims to provide data in places where the questionnaire failed to do so. As described in the subsequent chapters, the analysis of the questionnaire revealed that there were some issues that should be addressed in the interview. These issues soon became the questions of the interview. This means that all the interviews were conducted upon completion of the analysis of the questionnaire.

Any questionnaire suffers from a range of disadvantages, the most significant here being the validity of the responses. I asked the respondents in section 2 to give their expectations but the question is how do I know whether or not these expectations were given at random, i.e. without much thinking. Similarly, in section 3, the respondents rank-ordered the sub-questions but the question would be how did they place a sub-question before or after another. In summary, in sections 2, 3, 4 and 5, i.e. the main body of the questionnaire, there were some important questions for the interview.

4.6 The pilot study

The pilot study is defined as a preliminary study undertaken prior to some major project or investigation. Sometimes it is intended as a feasibility study and sometimes used to practice the proposed methods or to try out alternatives while there is still an opportunity to modify the research questions and/or the content of the instruments (Rowntree, 1981). This view is supported by many authors in the field of research methodology (e.g. Borg & Gall, 1989; Oppenheim, 1992). In this study, both the questionnaire and the interview were piloted.

(a) The questionnaire

Piloting the questionnaire has enormous advantages. A long list of such advantages can be found in Cohen et al (2000, pp.260-261). In this investigation, the following are the most important functions for which the questionnaire was piloted:

- To test the feasibility of the questions.
- To test whether the questions have the same meaning to the respondents and to the researcher.
- To check the order of the questions and the different sections.

Developing the pilot study involved three stages and ended with different versions of the questionnaire. It was decided, however, not to go through the detailed differences between these versions but rather discuss the main changes made to the questionnaire in a separate section.

In the first stage, the questionnaire was tried with 29 mathematics trainee-teachers and another 10 mathematics students on a Masters course, giving a total number of 39. This took place at the Research and Graduate School of Education, University of Southampton in October and November 1999 for the two groups respectively. Fourteen copies were completed and returned from both groups (10 and 4 respectively), giving valuable feedback on the questionnaire. For example, this was seen by checking the completed questionnaires to see if every question had an answer and that the given answers match the questions. This stage ended with some changes to both the layout and the content of the questionnaire and as a result a new version was established.

The new version of the questionnaire was given to five teachers in the second stage of the pilot study. Two copies were completed and returned to the researcher. They suggested no modifications to the questionnaire.

The first two stages of the pilot study revealed that only 16 copies out of the 44 copies distributed were completed. This gives a response rate of about 36%. At that time, it was decided to test the questionnaire again by distributing a considerable number of copies to teachers. Consequently, 65 copies were mailed to different schools in the second half of February 2000. Unfortunately, none of the questionnaires were returned. Part of the problem was thought to be the Easter holiday in March.

The above problem was investigated by interviewing two teachers (one had completed the questionnaire before the interview and the other had tried but had encountered difficulties). They both agreed that section 2 of the questionnaire (prediction of students' success) was too difficult to answer and gave several suggestions. The first was to reduce the number of algebra problems in section 2 so that it becomes easier to complete. Another suggestion was to write out the 26 problems on separate cards so that they become 26 pieces of paper altogether and then ask the respondents to order these cards instead of choosing the category which represents the likely number of percentage of students. The idea behind this suggestion is to provide a physical sense of the problems and to make it possible, if they wish, to spread the cards on the surface of a table, compare them together and then decide upon the order of the problems. A third suggestion was to ask teachers, in a covering letter, if they prefer, to talk about the questionnaire rather than to complete it in writing. These suggestions helped issuing a third version of the questionnaire. This encouraged two more teachers to complete it in the third stage of the pilot study.

Upon completion of the three stages of the pilot study, two workshops were held at the Mathematics Education Department, University of Southampton to validate and discuss some issues about the questionnaire (see section 4.12 for more details). They further informed the questionnaire.

(b) The interview

The interview pilot study aimed mainly to practice the instrument and to get a sense about the interviewees' knowledge about students' errors and difficulties in algebra. In total, six interviews were undertaken with individual teachers (two males and four females). Four of them had completed the questionnaire before the interview and the other two did not complete the questionnaire. The interviews were videotaped and later on transcribed to practice the full process needed in the main study. The interview questions came from the analysis of the pilot study of the questionnaire as well as the research questions.

4.7 Changes made to the questionnaire

Some modifications were made to the questionnaire, which are discussed in this section. It is necessary to point out here that the most important modifications were piloted in the second and the third stages of the pilot study.

The first modification is that the headings of sections 2 to 6 were worded in order to make them more comprehensible to the respondents. Secondly, there were some alterations within most sections. In section one, the question that asks for "teaching experience" was replaced by a question that asks for the year-groups for which the respondents taught algebra. The reason is that the focus of the main study is about trainee-teachers rather than the experienced teachers.

In section 2, the 26 problems were reduced to just eight problems. This is because the pilot study of the questionnaire and the interview with two teachers revealed that this section was the most difficult and time-consuming one and the number of problems should be cut down. The criterion, which was used for selecting the problems, is as follows. Firstly, all the problems which came from foreign studies were omitted because these problems might show different complexity levels in England than abroad. Secondly, the pilot study revealed that almost all the responses indicated that some problems were either more complex or less difficult than the criterion (Küchemann, 1981). Hence, three problems were chosen from each type. The other two problems were chosen because they represented the closest fit between the responses and the literature. Also, problems which have only one definite answer from the literature and which did not represent a repetition were considered first in this selection.

Another modification to the questionnaire was omission of unnecessary questions. In total, two questions were removed from sections 4 and 5. Both questions asked the respondents to say why they think their suggestions for addressing students' errors would work. They were deleted partly because they were not answered by most of the respondents in the pilot study but largely because they duplicate other questions.

A fourth change made to the questionnaire was the addition of three more algebra problems to section 4. It was described above that this section initially contained one algebra problem for the purpose of testing the feasibility of open-ended questions in the first stage of the pilot study. Thus, the new algebra problems were added because this strategy proved to be useful. The procedure of selecting the new problems went as follows. Firstly, a list of all possible options was prepared. All the problems in this list came from Küchemann (1981) for the same reason described above, namely, it is a national study. Also, the associated errors were made by a considerable number of students (more than 10%). Finally, they occur within the algebra topics, which are covered in this study (equation word problems of two variables, equations, fractions and expressions). Once this was done, the problems in the list were then grouped into word problems, equations, fractions and expressions. After that, the problems with the most common errors in each group were chosen. This procedure ended with only three problems (no fractions because no examples of fractions were given in the criterion study). Once the new problems were chosen, they were each printed on a separate page to provide enough space for the answers. They were printed before the questions rather than after as in the first draft. This last alteration was done according to some suggestions from the respondents.

In section 5, the problem "L+M+N = L+P+N Always?, Sometimes?, Never?" was omitted for reasons related to analysis. It was borrowed from Küchemann (1981) but because this study only gives few examples of students' errors, the problem was deleted. The analysis of the responses in this section requires comparing the frequency of a predicted error with the frequency of the students who had committed the same error.

Another change in section 5 is that two more problems were added, giving a total of five problems. All five problems were taken from Wanjala (1996). However, some of these problems are also found in Küchemann (1981) and Booth (1984) but these two studies do not list all the errors committed and the percentages of students associated with these errors as in the case of Wanjala.

The last alteration in section 5 is that question B, which is about suggesting ways for helping students who make the predicted errors, was made optional. This is because the same question was covered in section 4.

4.8 Population and sample

(a) The questionnaire

The target population of this study is secondary school trainee-teachers enrolled in pre-service teacher preparation courses in the academic year 2000/2001 (in 2000, this number was 1162). They were distributed over all teacher training institutions across England. The rationales behind selecting trainee-teachers rather than teachers are discussed in section 1.2 of Chapter 1.

Persons wishing to qualify as secondary school teachers of mathematics, in England and Wales can attain Qualified Teacher Status (QTS) through several routes, including successful completion of a one year Post Graduate Certificate in Education (PGCE). In this study, the term "trainee-teachers" refers to the PGCE students only. They are usually of different ages and degree qualifications because some of them attend the programme directly after obtaining the minimum requirement of mathematics subject knowledge whereas others attend only after spending many years working or obtaining higher degree qualifications from different university departments such as economics and engineering.

Once the research population is determined, the next step is to decide about the sampling size and the sampling strategy in order to draw the research sample (see for example, Oppenheim, 1992; Robson, 1993; Gall et al, 1996; Cohen et al, 2000). In relation to the sample size, Cohen et al (2000) discussed this matter in detail to conclude that there is no definite answer as to how many subjects should be sampled. They added that this depends on many factors such as the research design adopted and on the accuracy of results that the researcher seeks to achieve. If a questionnaire is used, such as in the present study, the authors advise researchers to overestimate rather than to underestimate the size of the sample required because some of the participants may fail to return the questionnaire. Furthermore, the "sampling error" decreases as the sample size gets larger, suggesting that a large sample is preferable in some studies. On the other hand, a very large sample would inevitably complicate the analysis without necessarily improving the accuracy of results. Moreover, factors such as time, money and number of researchers should also be considered when selecting the sample size. For these reasons, a sample size of about a quarter of the target population was thought to be reasonable in the present study to obtain the desired results and to keep the workload to manageable proportion for one researcher. It may be important to note that this sample size is larger than that of the previous studies reviewed in section 3.3.

In relation to the sampling strategy, a *convenience* sample was drawn. A convenience sample is defined as the one in which the researcher chooses "the nearest individuals to serve as respondents and continuing that process until the required sample size has been obtained" (Cohen et al, 2000, p.102). This strategy was used in order to sample only major institutions. Add to that their willingness to co-operate by encouraging their trainee-teachers to complete the questionnaire. This last point is useful for obtaining a good return rate and more valid data (see the next section). Although this study was not designed to generalise the findings, sampling only major institutions helps by suggesting that any gaps in knowledge revealed by the study also apply to the trainee-teachers in other institutions.

The procedure for selecting the sample started by contacting 15 institutions to ask if their trainee-teachers would be willing to complete the questionnaire. This selection was made according to what has been said above, that is, they are likely to co-operate more than any other institutions because of the co-operation relationship between these institutions and Southampton University, in which this research is based. Only institutions, which replied positively to the e-mail letter, were included in the sample and a covering letter was then sent to them together with the questionnaires. More detail about this procedure is given in the next section.

(b) The interview

There was an option to choose the interviewees either from the trainee-teachers this year (2001-2002) or from first year teachers who completed the questionnaire last year (2000-2001). It was hypothesised that since the interviews were to be conducted in January and February, the trainee-teachers would not have yet developed enough teaching experience to complete the questionnaire and to answer the interview questions. The sample was therefore selected from first year teachers.

Once the target population for the interview was determined, a sample population should then be selected. For this purpose, consistency in the responses to the questionnaire was investigated because this might suggest some criteria for selecting the interviewees (see section 5.7). However, such analysis did not end with any explicit criterion that could be used in this selection. It was decided therefore to invite volunteers to take part in the interview rather than inviting others who may not be willing to do so. Although it was not easy to make the arrangements with the teachers for several reasons such as the time constraint on teachers and the place of the interview, their co-operation and willingness helped conducting the interviews.

Although the aim was to select about 10% of the questionnaire's sample (i.e. 25 teachers), it was only possible to interview five teachers due to the difficulty in identifying volunteers and making the arrangements. In addition, this number (n=5) was found to be reasonable enough for two reasons. Firstly, the purpose of the interview was well defined. Taking more interviews does not necessarily mean better results. The second reason is that the interview questions were designed to yield detailed, qualitative data. Thus, only a small number of participants could realistically be interviewed given the time scale of this study.

4.9 The procedure for administering the research instruments

(a) The questionnaire

It was decided not to mail the questionnaire directly to the respondents but rather to the course tutors at the sample institutions. The purpose was to obtain a good return rate and more accurate data. Each course tutor received a covering letter with the instructions for administering the questionnaire as well as a stamped and addressed envelope for their return. The questionnaire was administered in June and July 2001, that is, just before the end of the teaching courses to enable the respondents to have some teaching experience.

The questionnaire was distributed by the course tutors during one of their sessions and collected upon its completion. This took the form of an activity, as in the following example, which represents the actual way in which the questionnaire was administered at the University of Southampton in the presence of the researcher. Firstly, trainees were reminded about the importance of teacher knowledge of students' errors and difficulties. Also, they were told that such knowledge is part of their course and were shown the evidence from DfEE (1998) and DfEE (2001). Secondly, the questionnaire was given to them, section by section, and a discussion took place upon their completion of each section so there was a chance to hear from them and to take notes on what they saw as difficulties in the questionnaire. Thirdly, by the end of this activity, the completed questionnaires were gathered, photocopied and then returned to them later for their retention. As described above, this procedure was used to obtain a good return rate and valid data.

(b) The interview

All the interviews were videotaped and transcribed within one week from the time of the interview. Each interview lasted for about one hour. The interviewees were given the option whether to complete the interview in one go or in two consecutive intervals. The interviews took place either in the school or in the university, depending on the interviewees' preference. In both cases, they were carried out in a suitable room, pre-booked for this purpose to ensure that conversation went smoothly and in a quiet place.

Each interview started with an introduction, thanking the participants for taking part in the interview. Then they were informed that the interview was about the same questionnaire that they had completed before. To make them feel that their job was easier in the interview than when they first completed the questionnaire, they were told that they would not be asked to write down their answers in section 4 since it requires the most writing. Also, in this introduction, they were reminded that the purpose of the interview was to gather data for research and not for a test. It was stressed that there were no definite, right or wrong answers to the questionnaire or the interview questions. Hence, they were encouraged to give any suggestions.

The strategy that was adopted throughout the interview was "think aloud and tell me why you make such a judgement". This strategy was useful because:

- It enabled the interviewer to copy the answers to his questionnaire to facilitate conversation in the interview.
- It enabled the interviewer to intervene (ask questions) at suitable points in time.
- It enabled the interviewer to hear their (interviewees') thinking and to understand the strategy followed for completing the questionnaire.
- It helped to check the consistency of their answers by asking questions that were already covered in the think aloud strategy.

In the interview, two copies of the questionnaire were made available: one for the interviewee and the other for the interviewer. The questionnaire was completed section by section and each section was followed by some interview questions during or after completion of that section. In the second and fifth sections, the relevant questions were asked after they completed the sections. In the third and the fourth sections, the interview questions started upon completion of each part in these sections. In both cases, the interviewees were told before they completed a section about the plan of the interview in that section to encourage them to think about the questionnaire for the purpose of obtaining more accurate data. When they know that they would be asked, for example, about their predicted errors in section 5 then this might encourage them not to suggest the errors at random. There are more details about the interview protocol in Appendix B.

4.10 The return rate

Nine out of the twelve co-operating institutions returned the questionnaire, giving a total of 159 questionnaires (about 63%). This return rate appears to be high when compared with the return rates of some educational studies despite the fact that they were carried out in a different environment and about a different research topic than the present study. For example, Al-Sumih (1999) obtained a return rate of 51.2% in a study about job satisfaction and dissatisfaction amongst a university staff members in Saudi Arabia. However, Al-Saadi (1996) and Al-Harbi (1994) in similar studies to that of Al-Sumih obtained return rates of 76% and 70.4% respectively. Although the return rates of these studies look larger than that of the present study, their questionnaires were structured in that they mainly asked the respondents to present their opinions using a Likert-scale of strongly agree to strongly disagree or to choose an answer from a set of alternatives. The present study's questionnaire is mostly open-ended and definitely requires more time and thought to complete than the above studies. Hence, a return rate of 63% is good for this study's questionnaire. This return rate was obtained because of the methodology employed for administering the questionnaire.

4.11 Preparing data for analysis

Cohen et al (2000) discussed three editing tasks which researchers should apply to the completed questionnaires (or to interview transcriptions) before and during the analysis. They involve checking the questionnaires for *completeness*, *accuracy*, and *uniformity* to decide which of them should or should not be analysed and to pick up anything that could reduce the validity of the data. Briefly speaking, the term *completeness* means checking whether every question has an answer. The term *accuracy* means trying to understand if the given answers match the questions. The term *uniformity* means ensuring that data has been assigned the appropriate codes or categories.

Starting from the last term above, data from this study have been checked for *uniformity* by running a double check during the analysis to see if the different bits of data have been allocated to the appropriate categories or given the appropriate codes. After all the questionnaires were checked for *completeness* and *accuracy*, 13 required discarding: eleven copies were not completed and the other two were partly completed (the first three sections) but represented a clear carelessness on the part of the respondents. In these two questionnaires, section 2 was completed by placing a tick in the boxes of the same column and section 3 was completed by ordering the sub-questions from the first to the last one rather than from the easiest to the hardest one.

There was a third questionnaire in which section 2 was answered in the same way as above. To decide upon it, Oppenheim (1992) described two types of deletion, which he termed as *listwise deletion* and *pairwise deletion.* According to him, the former refers to the deletion of the questionnaire altogether from the analysis and the latter refers to the deletion of part of the questionnaire from the analysis. He added that the listwise deletion "should never be done lightly, not only because of the inadvertent creation of the possible bias but also for some good, practical reasons" (p.280). Returning to our questionnaire, the responses to the other sections were examined to see which type of deletion should be applied. This process ended with no practical evidence other than that of section 2 to exclude the entire questionnaire from the analysis. Only section 2 was therefore not analysed. There were other partly completed questionnaires. In all of them, a pairwise deletion was *not* applied because there was no other reason to exclude them from the analysis. Thus, 146 questionnaires were subject to analysis.

The analysis of the questionnaire involved categorising the responses except in section 2 where the statistical package SPSS was used because of the special nature of data in this section. This computer programme is the most commonly used computer package in educational research (Borg & Gall, 1996). In this study, there was an option to choose between either SPSS or Spreadsheets and the former was used because the data contained some missing entries and this option deals with this issue scientifically whereas the other treats them as zeros. The analysis procedures adopted are discussed in sufficient details in the next chapter.

4.12 Validity, reliability and ethical issues

The term validity, as commonly used by researchers in education, means:

"the degree to which a test measures what it purports to measure. ... The prospective test user should ask not 'Is this test valid?' but 'Is this test valid for the purposes to which I wish to put it'" (Borg & Gall, 1989, pp.249-250).

Because the purpose of research is not merely collecting data but rather drawing a conclusion or a hypothesis about the people (or other things) on whom the data was collected, several kinds of validity have been established to ensure that the whole process of the research is valid. Cohen et al (2000), for example, listed 18 different kinds of validity. Three of them are face validity, content validity and construct validity. These are relevant to this study and therefore discussed in turn below.

Face validity "is concerned with the degree to which a test appears to measure what it purports to measure" (Borg & Gall, 1989, p.256). This worry about validity may be less apparent in qualitative research, such as this one, because as Drew et al (1996) described it:

"validity [is found] to be a strength of qualitative research, because it sets findings within natural settings, free from the contaminating effects of control or variable manipulation." (Drew et al, 1996, p.169).

One might argue that because the questions of the questionnaire and the interview of this study are mostly open-ended, they are likely to have more face validity than closed questions which dominate quantitative research. This is because closed questions force respondents to choose an answer from a limited number of choices. Therefore, they may choose something when they want to say something different. This, however, does not suggest that open questions have no threats to validity. A respondent might offer an inaccurate answer by giving the first thing that comes to his/her mind at that time. Cohen and colleagues (Cohen et al, 2000) have noted this problem and another one that they raised in their discussion about the validity of the questionnaire. The second point that they raised is about whether the distribution of answers for those who failed to return the questionnaire would be the same as the returnees. In order to overcome this accuracy kind of problem in research, Cohen et al suggested taking follow-up interviews. This suggestion is used in the present study.

The discussion about the problem of inaccurate data has led us to another type of validity, which is the construct validity. The construct validity is defined as:

"the extent to which a particular test can be shown to measure a hypothetical construct, that is, a theoretical construction about the nature of human behavior. Psychological concepts-such as an intelligence, anxiety, creativity- are considered hypothetical constructs because they are not directly observable but rather are inferred on the basis of their observable effects on behavior" (Borg & Gall, 1989, p.225).

The above definition appears to suggest that the construct validity is more relevant to quantitative research than qualitative research. This is because it stresses the need to test a hypothesis that is formulated beforehand and then tested using the research instrument (s). However, the main purpose of the construct validity, as well as the other types of validity, is to provide evidence of the degree to which a hypothesis or research represents the truth, whether suggested from the beginning or derived from the analysis of data (Wallen & Fraenkel, 2001). In this sense, the construct validity has some relevance to this study. In order to obtain good construct validity, Wallen and Fraenkel suggested three steps:

- The researcher formulates a clear definition of the variable being measured.
- Hypotheses, based on a theory underlying the variable, are formed about how people who posses a "lot" versus a "little" of the variable will behave in a particular situation.
- The hypotheses are tested both logically and empirically. (Wallen & Fraenkel, 2001, p. 96)

The three points listed above further clarify the definition of the construct validity by Borg and Gall above. In relation to the first point, the dependent variable in this study is trainee-teachers' knowledge of students' errors and difficulties in algebra. This is seen in terms of the five areas under investigation (listed below). In addition, independent variables such as degree qualification, area of study, gender and institution affect the dependent variable and are hence used as guidelines to study possible consistency in the responses to the questionnaire.

The second point is about the research hypothesis. As stated above, the theory in this study develops from data analysis and is not formulated from a previous observation. Because of this and due to the fact that the independent variables above are not the only ones that can affect responses to the questionnaire, linking points two and three above together is not a straightforward exercise. However, because this research employs two research instruments, the problem of construct validity might be overcome by validating the theory that is developed from the analysis of the responses to the questionnaire by conducting interviews. This is sometimes known as criterion-related evidence (Wallen & Fraenkel, 2001).

The third and last kind of validity in this discussion is the content validity. This is defined as:

"the degree to which the sample of test items represents the content that the test is designed to measure. Content validity should not be confused with face validity, which refers to the evaluator's appraisal of what the content of the test measures" (Borg & Gall, 1989, p.250)

Wallen and Fraenkel (2001) discussed the content validity and classified it as two key elements, question sampling, as in the definition above, and format of the instrument. These are discussed in turn and in relation to this study. In order to understand the first element, the reader should be reminded of the purpose of this study. The purpose is to investigate trainee-teachers' knowledge in relation to the following intended areas:

- Their expectations about students' success in solving algebraic problems.
- The way they rate algebraic content in order of student difficulty.
- Their explanations of students' errors.
- Their suggestions for addressing students' errors in algebra.
- Their ability to predict students' errors in algebra.

In each case, a number of questions are asked in the questionnaire in order to answer the research questions that belong to the five areas above. These questions represent a sample from other possible questions that could have been asked. Thus, the content validity concerns whether this sample is adequate to reveal the "truth" about trainee-teachers' knowledge in relation to the five areas above. In other words, does this sample measure what it is supposed to measure?

The other element of the content validity refers to the format of the instrument. In this case, things like the language of the questions, adequacy of the space provided for answers, the order of the questions and clarity of the printing become important. The point is that, a good format of the instrument can help provide more valid answers. To obtain a good format for the questionnaire one should consult different books in educational research. In the case of this study, recommendations from different authors on how to design a good questionnaire are considered (e.g. Oppenheim, 1992; Robson, 1993). Moreover, Wallen and Fraenkel (2001) pointed out that a common way to obtain a content-related evidence of validity is to have someone to judge its (questionnaire) suitability for the research purpose both in terms of the sample of questions selected and the instrument format. To follow this recommendation, two consecutive workshops were held at the Department of Mathematics Education, University of Southampton to validate the questionnaire. Prior to that, two different versions of the questionnaire (one containing all the questions and the other some of the questions) were given to the participants in the workshops. The two versions were accompanied by the research questions, the purpose of the study and some issues or concerns for discussion. The participants were academic staff and research students. Their role was to judge both the face and the content validity and to suggest anything that could improve both the content and the layout of the questionnaire. The purpose of submitting two versions of the questionnaire was to judge which one should be validated and used for the main study. The recommendations they gave in the first workshop were followed and a copy from the parts of the questionnaire that were covered during the first workshop was submitted to them before the second workshop was held. After the second workshop, a new copy of the questionnaire was shown to some of them to ensure that all recommended alterations had been made. This process helped to prepare the final copy of the questionnaire.

The reliability, on the other hand, is used to describe the possibility for one's results to be replicated if a similar study or the same one is to be carried out again under similar conditions. Drew et al (1996) put it more clearly when they said that:

"Reliability in qualitative designs has much the same general meaning as it does in quantitative designs: under similar circumstances, can the study be replicated with similar results? Are the results an accurate reflection or rendition of what actually occurred, of relationships, of observed interactions...?" (Drew et al, 1996, p.168).

To check reliability in quantitative research, a number of techniques are used such as the test-retest method where the same test is given twice to the same respondents within a period of time and the results from both times are compared together. In this case, the test is said to be reliable if the same people who scored high the first time also scored high the next time and so on. In qualitative research, however, this is less obvious. But because the present study uses two instruments, results from the questionnaire are compared with that of the interview. So, this is one technique that can be used to check the reliability and validity of the results. Other techniques that are used are concerned with the administration of the questionnaire/interview and some ethical issues. The most important ones are listed below:

- Give a small introduction before giving the questionnaire or conducting interviews. This is to stress things like the purpose of the questionnaire or the interview, anonymity and confidentiality of their responses and the importance of giving accurate answers.
- Explain any ambiguity in the questions of both the questionnaire and the interview. This is done by the interviewer (researcher) and by the course tutor in the case of the questionnaire.
- Videotape all interviews so that full transcriptions can be obtained and double-checked, and the actual questions that are being asked in each interview can be compared to the answers.

Relevant to the discussion here is the fact that the respondents were asked to write their names on the questionnaire. On the one hand, this presents an ethical issue because some of the respondents may not be willing to give their names. On the other hand, this was important for the study because some of the respondents to the questionnaire were invited to take part in the interview. Thus, it was made clear to them before completing the questionnaire that the purpose of giving their names was to invite some of them for the interview.

Another point to stress here is that the participants were not "forced" to complete the questionnaire and/or take part in the interview but rather encouraged to do so by the course tutors. In fact, they were convinced that knowing about students' errors and difficulties was part of their course (see section 4.9).

4.13 Summary

This chapter introduces the research methodology of this study, including the selection of the research model, its advantages and disadvantages, the selection, development and piloting of the research instruments, changes made to the questionnaire and research population and sample. The end of the chapter includes sections about preparing data for analysis, validity, reliability and ethical issues and other detailed procedures. The next chapter introduces the findings and their analysis procedures.

Chapter 5

QUESTIONNAIRE ANALYSIS AND DISCUSSION

5.1 Introduction

The discussion in this chapter spans five main sections related to the investigated areas including trainee-teachers' expectations, order of difficulty, explaining students' errors, addressing student' errors and predicting students' errors. They aim to answer the key research questions:

- What are trainee-teachers' expectations about students' success in solving given algebraic problems?
- How do trainee-teachers use their knowledge of students' difficulties in algebra to rank-order algebra questions?
- What explanations do trainee-teachers give for students' errors in algebra?
- What strategies do trainee-teachers use, or conceive, for tackling students' errors in algebra?
- How successful are trainee-teachers in predicting students' errors in algebra?

5.2 Expectations

The first purpose of the study is to investigate trainee-teachers' expectations about the proportion of Year 9 students who could successfully work out given algebra problems.

Table 5.1 shows part of the table that the respondents were asked to complete by indicating, in the appropriate box, the proportion of Year 9 students who they would expect to be able to give the specified answer.

(a) Analysis procedure

The analysis procedure involved comparing the responses obtained from every individual respondent with that of the criterion study (Küchemann, 1981). This was done in two different ways. The first way started by coding the five categories: few, some, about half, most and almost all in Table 5.1 using the numbers 1 to 5 respectively. Then, all the responses were coded using the same numbers: those in the first category were labelled 1, those in the second category were labelled 2 and so on. Then, sets of numbers representing the expectations of all the respondents were established. The set X = {2, 4, 2, 1, 5, 5, 4, 2} represented the criterion expectation (CE) by which all the other sets were compared. There are eight elements in this set referring respectively to the criterion expectations for the eight algebra problems in the relevant section of the questionnaire. The first element means some (20-40%) of the students were expected to answer the *first* algebra problem. The second element means most (60-80%) of the students were expected to answer the *second* algebra problem and so on. The other sets were given the names M1 to M146 so that, for example, M1 = {3, 2, 3, 2, 4, 4, 3, 5} refers to the respondent M1. These sets consisted of *nominal* data in which numbers are used to "identify or to 'name' the attribute or category being described" (Cramer, 1998, p.16).

Table 5.1: The structure of section 2 of the questionnaire.

			Few	Some	About half	Most	Almost all
N	Question	Required answer	0-20 %	20-40 %	40-60 %	60-80 %	>80 %
1	(a-b) + b = ...	A					

Once the above process was completed, the Spearman's rho correlation coefficient for the relation between the criterion set and every other set was calculated using SPSS for the reasons described in the previous chapter. As a result, 146 numbers were obtained, summarising the expectations of the 146 respondents. The Spearman's rho correlation coefficient was used rather than Pearson's R because data were not normally distributed and were *nominal* as explained above.

(b) Findings

The output from the SPSS revealed that: $-0.389 \leq R \leq 0.936$, $0 \leq R^2 \leq 0.876$, R was negative for 15 trainees. To give an idea about the distribution of these numbers, 19 trainees (13.01%) obtained $R^2 > 0.6$, 11 trainees (7.53%) obtained $0.5 < R^2 < 0.6$ and the remaining, 116 trainees (79.54%), obtained $R^2 < 0.5$. This means that there was little agreement between the trainees' suggestions and Küchemann's (1981) findings. In other words, the majority of the trainee-teachers failed to give close expectations to those of the criterion study.

The correlation coefficient does not allow us to know much about the nature of the given expectations in relation to the individual algebra problems. An alternative way is to show the number (and the percentage) of the trainee-teachers giving the various expectations (few, some, about half, most and almost all) in relation to each of the algebra problems. This is shown in Table 5.2. The shaded boxes represent the proportions of the trainees who gave the CE in each case. This table is useful in that it compares the facility levels (complexity) of the problems as indicated by the criterion study with the respondents' expectations.

Table 5.2: Number (and percentage) of the trainees giving the various expectation in each problem

	Number (percentage) of trainees selecting each category						
	Few	Some	About half	Most	Almost all	Missing	Total of
Problem	0-20%	20-40%	40-60%	60-80%	>80%	values	Trainees
N1	4 (2.7)	**33 (22.6)**	66 (45.2)	36(24.7)	7(4.8)	0	146 (100)
N2	6 (4.1)	35 (24.0)	59 (40.4)	**40 (27.4)**	5 (3.4)	1 (0.7)	146 (100)
N3	7 (4.8)	**39 (26.7)**	57 (39.0)	36 (24.7)	6 (4.1)	1 (0.7)	146 (100)
N4	**20 (13.7)**	52 (35.6)	65 (44.5)	7 (4.8)	2 (1.4)	0	146 (100)
N5	5 (3.4)	27 (18.5)	49 (33.6)	46 (31.5)	**19 (13.0)**	0	146 (100)
N6	12 (8.2)	25 (17.1)	49 (33.6)	45 (30.8)	**14 (9.6)**	1 (0.7)	146 (100)
N7	9 (6.2)	52 (35.6)	50 (34.2)	**29 (19.9)**	6 (4.1)	0	146 (100)
N8	0	8 (5.5)	36 (24.7)	65 (44.5)	**37 (25.3)**	0	146 (100)
Total	63 (43.2)	271 (185.6)	431 (295.2)	304 (65.8)	96 (65.8)	3 (2.1)	1168 (800)

As can be seen from Table 5.2, a great proportion of the trainees targeted the middle category, "About half", despite the fact that this category does not match the CE in all the questions. This might be a natural result of the way the questionnaire was presented, and might raise a question about its validity.

Table 5.2 also suggests that the majority of the respondents did not give the CE in all the eight algebra problems. In these cases, 27.4% is the largest proportion while 9.6% is the smallest proportion of the trainees who gave the CE in N2 and N6 respectively. In addition, all the possible expectations were given in all cases, meaning that some of them were at the other extreme (or not close) to the CE. For example, the CE for N6 is "almost all" but the given expectation for 12 trainees (8.2%) was "few" and for another 25 trainees (17.1%) was "some". This means that about a quarter of the trainees (25.3%) suggested that the problem is very complex whereas the criterion study suggested that it is very easy. In some other cases, they suggested the opposite, that is, they said the problem is too easy when the CE said it is too difficult. For instance, 28.8% of them gave the expectations "most" and "almost all" in N3 compared to the CE, which is "some". This seems to reject the assumption that most of the respondents obtained low scores because:

- They thought of a certain level of students, such as high ability, middle ability and low ability students, when completing the questionnaire despite the fact that they were told to think of the full range of classes.
- They assumed that the present mathematics teaching places more emphasis on understanding than on letter manipulation. Consequently, students currently do better in mathematics than about 20 years ago when the criterion study was conducted.

Instead, the following assumption seems to be more likely to explain the fact that most of the trainee-teachers obtained low scores:

- They did not understand the sort of characteristics that determine the complexity of an algebra problem such as the number of variables in the problem, the nature of the elements involved and importantly, students' interpretations of the letters (Küchemann, 1981).

The interviews are designed to examine the above assumptions by attempting to understand the strategies used by the respondents to give their expectations.

5.3 Order of difficulty

The second purpose of this study is to investigate trainee-teachers' ability to rank-order the relative difficulty of algebraic questions in a way that takes into account the relative difficulties of the algebraic ideas that are part of the secondary school mathematics. In order to achieve this, the following research question has been formulated:

- How do trainee-teachers use their knowledge of students' difficulties in algebra to rank-order algebra questions?

(a) Analysis procedure

Section three of the questionnaire, which is analysed here, consists of three questions in which there are 4, 5 and 6 algebra problems which the respondents were asked to put in order from the easiest to the hardest for a full range of students. It was found that arranging these problems in an increasing order of difficulty would lead to many possible answers. Theoretically, there are 24 (4!), 720 (6!) and 120 (5!) different ways in which the problems in questions 1, 2 and 3 respectively can be arranged. Because it was not practical to report on these cases individually, the idea of categorising the responses was therefore used.

Levels of difficulties

To provide a basis for the order of the problems and hence for the prospective categories, the problems should be allocated to general levels of difficulties rather than a specific difficulty index. Each level of difficulty might contain one or more of the problems so that the problems of the same level can be arranged in any order without affecting the main criterion for categorising the responses.

Wanjala (1996) established three levels of difficulties, which he referred to as *bands*. The first band is the easiest band and consists of the easiest problems that only present difficulty for the low ability students. The second band is the next difficult band and consists of problems that present difficulty for the average as well as the low ability students. The third band is the hardest band and consists of problems that present difficulty for students of all ability levels. These bands can be used to categorise trainee-teachers' responses.

Categories of trainee-teachers' responses

Wanjala categorised teachers' responses in his study to strong empathy, moderate empathy and weak empathy. He defined the three categories as follows:

- *Strong empathy* consisted of the responses in which the problems were arranged from the easiest to the hardest bands but the flexibility in this order is that the problems of the same band can be arranged in any order without affecting the main criterion for placing the responses in this category.
- *Moderate empathy* consisted of the responses which failed to meet the point above, but in which the problems of the easiest band were still placed before the problems of the hardest band.
- *Weak empathy* consisted of the responses in which any problem of the hardest band was placed before any problem of the easiest band.

In the present study, I shall use the same categories above but with different names to suit the purpose. As the purpose in this study is to know how trainee-teachers use their knowledge to specify order of difficulty, I shall refer to the three categories above as:

- Good Knowledge (GK) to say that trainee-teachers in this category appear to have a good knowledge of the sequence of algebraic ideas and the difficulties inherent in algebraic questions.
- Moderate Knowledge (MK) to indicate less success in determining the sequence of algebraic ideas than the first group but more success than the third group.
- Weak Knowledge (WK) to suggest that those trainee-teachers appear to be weak and they require assistance to be able to determine students' difficulties.

(b) Findings

The findings are presented according to the relevant questions in section 3 of the questionnaire.

Question 1

The 24 possible orders to this question were placed in the three categories of knowledge using the following three bands of levels of difficulties:

- Easiest band: problems iv and i.
- Middle band: problem ii.

- Hardest band: problem iii.

These bands were used to categorise the responses as described below.

Good Knowledge

This category consisted of the responses in which i and iv were placed first in either order, followed by ii in the third place and finally iii in the fourth place. There were only two possible orders in this category, i, iv, ii, iii and iv, i, ii, iii.

Moderate Knowledge

This category consisted of the responses in which either (1) ii was placed before the easiest problems, that is either i or iv or both but iii was placed last or (2) i and iv were placed first in either order but iii was placed before ii. Examples: i, ii, iv, iii and i, iv, iii, ii.

Weak Knowledge

This category consisted of the responses in which the hardest problem, iii, was placed before either i or iv or both. Examples: i, iii, iv, ii and ii, iii, i, iv.

Frequencies of the responses

Table 5.3 shows the number and percentage of the responses in each category. The percentages were calculated according to the total number of the responses (n = 148) rather than the respondents (n = 146). It was found that two of the trainees gave two different answers in this question and hence the number of responses is larger than the number of respondents.

Table 5.3: Frequencies of the categories.

Type of teaching sequence	GK	MK	WK	Vague	NR	Total
Number of responses	88	45	13	2	0	148
Percentage of responses	59.46	30.41	8.78	1.35	0	100.00

In Table 5.3, the "vague" category consisted of the ambiguous responses in which either one or more of the problems were missing from the suggested order (e.g. iv, i, ii) or that a problem was repeated twice (e.g. i, ii, iv, ii). Also, the NR (No Response) consisted of cases in which the question was left blank. As can be seen from the table, all the respondents answered this question.

From the above table, the GK category (Good Knowledge) consisted of the largest number of the responses, that is, 88 out of the total 148 responses. This suggests that more than half of the respondents were aware that, for example, the introduction of brackets in algebraic expressions increases the level of difficulty for students (Booth, 1984). Their placement of (a-b)+b before 3a-(b+a) suggested that they were able to differentiate between different expressions involving brackets and that they appreciate the complexity of the minus sign when it comes before a bracket (Ibid).

The MK category (Moderate Knowledge) consisted of 45 responses (30.41%), which was the next larger number of the responses in all the categories. They were successful in placing the easiest band problems (i and iv) before the hardest band problem (iii) but they placed the middle band problem (ii) either before i or iv or both or after iii. However, it was noted that in 42 out of the 45 responses in this category, the problem "(ii) (a-b)+b" was placed before either "(iv) 2a+5b" or "(i) 3x+8y+x" or both. Only in the remaining 3 responses, it was placed after "(iii) 3a-(b+a)". The reasons for giving such answers are discussed in the interview findings.

From the above table, 13 responses (8.78%) occurred in the WK category (Weak Knowledge). In these responses, the hardest band problem was placed before either or both of the easiest band problems.

Question 2

The 720 possible orders for this question were placed in the three categories of knowledge using the following three bands of levels of difficulties:

- Easiest band: problems i, ii and iv.
- Middle band: problems v and vi.
- Hardest band: problem iii.

These bands were then used to categorise the responses as described below.

Good Knowledge

This category consisted of the responses in which the first three spaces were filled by i, ii and iv in any order, the fourth and the fifth places were filled by v and vi in either order, and the last space was filled by iii. Examples include i, iv, ii, v, vi, iii and iv, i, ii, vi, v, iii.

Moderate Knowledge

This category consisted of the responses in which the easiest band problems (i, ii, and iv) were filled before the hardest band problem (iii), but either one or both of the middle band problems (v and vi) were filled before any of the easiest band problems or after the hardest band problem. Examples: i, ii, iv, v, iii, vi and v, i, iv, ii, vi, iii.

Weak Knowledge

This category consisted of the responses in which the hardest band problem (iii) was placed before any of the easiest band problems (i, ii, iv). Examples: i, iii, iv, ii, v, vi and ii, i, vi, iii, iv, v.

Frequencies of the responses

Table 5.4 shows the number and the percentage of the responses in each category.

Table 5.4: Frequencies of the categories.

Type of teaching sequence	GK	MK	WK	Vague	NR	Total
Number of responses	43	98	36	3	1	181
Percentage of responses	23.76	54.14	19.89	1.66	0.55	100.00

As can be seen from the table, about a quarter of the trainees were able to rank-order the algebraic problems in the question to reflect the learning difficulties for students. About half of them were able to rank-order the problems in a way that takes into account the easiest and the hardest problems but not the middle band problems. Approximately one fifth of them ranked-order the problems in a way that only reflected their weakness and inability to differentiate between the easiest and the hardest band problems. A further investigation revealed that in 82 out of the 98 responses in the MK category, the easiest band problems (i, ii and iv) were placed in the first three places but the hardest problem (iii) was placed before either or both of the middle band problems (vi and v). Only in the remaining 16 responses in this category, were either or both of the middle band problems placed before any of the easiest band problems but the hardest band problem was filled last. This is another area in which the interview can be used to understand why the majority of the respondents think that iii is easier than v and vi.

Question 3

The 120 possible orders to this question were placed in the three categories of knowledge using the following three bands of levels of difficulties:

- Easiest band: problems iv and ii.
- Middle band: problem iii.
- Hardest band: problems v and i.

These bands were then used to categorise the responses as described below.

Good Knowledge

This category consisted of the responses in which iv and ii were placed first in either order, followed by iii in the third place and finally by v and i in the fourth and the fifth places, in either order. There are four possible orders in this category, two of them are iv, ii, iii, v, i and ii, iv, iii, v, i.

Moderate Knowledge

This category consisted of all the responses in which iv and ii were filled before v and i but iii was filled either, (1) before either iv or ii or both or (2) after either v or i or both. Examples of the responses in this category are iv, ii, v, iii, i and iii, iv, ii, v, i.

Weak Knowledge

This category consisted of the responses in which at least one of the hardest band problems (v and i) was placed before at least one of the easiest band problems (iv and ii). Examples: iv, iii, i, ii, v and ii, i, v, iv, iii.

Frequencies of the responses

Table 5.5 shows the number and the percentage of the responses in each category.

Table 5.5: Frequencies of the categories.

Type of teaching sequence	GK	MK	WK	Vague	NR	Total
Number of responses	76	54	29	1	0	160
Percentage of responses	47.50	33.75	18.13	0.63	0	100.00

Table 5.5 suggests that:

- About half of the responses were in the GK category.
- About one third of the responses were in the MK category.
- About one fifth of the responses were in the WK category.

The respondents in the GK category were able to sequence the five problems from the easiest band problems to the hardest band problems.

It was found that in 42 of the responses in the MK category, the easiest band problems (iv and ii) were filled in before the hardest band problems (v and i) but the middle band problem (iii) was filled in before either or both of the easiest band problems. Only in the remaining 14 responses in this category, were the easiest band problems placed before the hardest band problems but the middle band problem was placed after either or both of the hardest band problems. This fact seemed to suggest that the majority of the respondents perceived the fractional equation as being the hardest and this is consistent with the criterion order.

In regard to the WK category, it was found that either or both of the hardest band problems were placed before either or both of the easiest band problems. By examining these problems, one can see that the equations $7x = 5$ and $x/2 = 10$ are "trivial" equations whereas the equation 4x-3 = x-11 is a "non-trivial" one. In the trivial equations, the unknown occurs on just one side of the equality whereas in the non-trivial equations the unknown occurs on both sides of the equality (Booth, 1981). These terms, trivial and non-trivial equations, correspond to Sfard and Lincheviski's (1994) terms: "arithmetic" and "non-arithmetic" equations. Those authors found that the former is accessible to most students but not the latter. In fact, this is where the "didactic cut" (Filloy & Rojano, 1984, 1985a, 1985b) and the "cognitive gap" (Herscovics & Linchevski, 1994) between arithmetic and algebra occur. Relative to this discussion is Moncur (1994) who examined students' understanding and abilities to solve different types of equations and found that these equations can be ordered from the easiest to the hardest as follows:

- Non-fractional equations with a single term in x.
- Non-fractional equations with more than one term in x.
- Equations solvable by cross-multiplication.
- Equations containing more than two fractional terms.

The first two findings in the list above are clearly consistent with that of Booth (1981), Filloy and Rojano (1984, 1985a, 1985b) and Herscovics and Linchevski (1994). The other two findings suggest that fractional equations are generally more complex than non-fractional equations. From these studies one can conclude that 7x = 5 and x/2 = 10 are non-fractional equations and the unknown occurs on just one side of the equality and for this reason they are generally easier than the other equations in the question. The equation 7-3x = 1 is also non-fractional and the unknown still occurs on just one side of the equality. However, the minus sign before the unknown might be the reason why this equation, according to the criterion order, is more difficult than x/2 = 10 and 7x = 5. The equation 4x-3 = x-11 is more difficult than the above three equations because the unknown occurs in both sides of the equality. Finally, the equation 5/(3x+7) = 7/3 is a fractional equation and hence it is the hardest (Moncur, 1994) but it can still be placed in the same level of difficulty as the problem 4x-3 = x-11 (Wanjala, 1996).

Returning now to the discussion about the trainee-teachers in the WK category, it seems that they were not aware of the facts discussed above. For example, their placement of 4x-3 = x-11 before 7x = 5 and x/2 = 10 suggested that they did not know that trivial equations are generally easier than the non-trivial ones. Also, their placement of 5/(3x+7) = 7/3 before the equations 7x = 5 and x/2 = 10 suggested that their knowledge of the fact that fractional equations are generally harder than non-fractional equations was limited.

(c) Overview

The trainee-teachers' responses revealed various types of knowledge including good, moderate and weak. The "Good Knowledge" indicates close ordering of algebraic content to match students' difficulties. The "Moderate Knowledge" reflects partial awareness of students' difficulties and this could lead to failure in the sequential of algebraic content. Finally, the "Weak Knowledge" suggests a complete lack of awareness of students' difficulties and this raises doubt as whether the trainee-teachers involved could rank-order algebraic questions according to students' ability to work out the questions.

In this study, less than a fifth of the trainee-teachers occurred in the WK category. The majority of them occurred in the GK and MK categories. This indicates that most of them were able to rank-order the algebra problems taking into account the relative difficulty of the algebraic ideas that are part of high school mathematics.

Although the questionnaire was useful in identifying the extent to which the trainee-teachers were able to identify the correct order of relative difficulty, it was not designed to understand the reasons that the participants have about their order of relative difficulty. The interviews, on the other hand, aim to investigate such reasons.

(d) Comparison with other studies

For the purpose of comparison, Table 5.6 presents a summary of the results of this study and compares them with results from Wanjala (1996). The table seems to suggest that:

- The teachers in Wanjala's study were more successful than the trainee-teachers in the present study in specifying the order of relative difficulty that takes into account not only the easiest and the hardest algebra problems but the middle band problems as well. This is because there were more teachers than there were trainee-teachers in the GK category in all the three questions that have been discussed.

- Even when the trainee-teachers were able to decide about the easiest and hardest algebra problems, they were still less successful than the teachers in deciding about the middle band problems. This is because there were more trainee-teachers than there were teachers in the MK category in all the three questions.
- The number of the trainee-teachers in the WK category is larger than that of the teachers in all but the first question in which the opposite is true.

Table 5.6: Comparison of results in this study and Wanjala (1996)

Question	This study			Wanjala (1996)		
	GK	MK	WK	GK	MK	WK
1	59.46%	30.41%	8.78%	70.1%	10.4%	17.9%
2	23.76%	54.14%	19.89%	37.3%	46.3%	7.5%
3	47.50%	33.75%	18.13%	56.7%	23.9%	11.9%
Average	43.57%	39.43%	15.60%	54.70%	26.87%	12.43%

Thus, the comparison between the two studies indicates that the teachers understand the order of difficulty better than the trainee-teachers. This is even more apparent in Leu's (1999) study in which only 10% of the teachers did not know order of difficulty in arithmetic fractions. Similarly, most teachers were able to identify the harder problem in each pair of the 16 pairs of addition and subtraction problems that were given to them by Carpenter et al (1988).

The fact that the trainee-teachers in the present study did less well than the teachers in the previous studies is something expected because teachers have more experience in planning their lessons than have trainee-teachers. In addition, their subject matter knowledge and pedagogical content knowledge have developed more than that of the trainees. Hence, unlike teachers, trainee-teachers are likely to depend more on their beliefs than their pedagogical content knowledge or subject matter knowledge to determine students' difficulties or sequential of algebraic ideas.

Although the present study and the above cited studies indicate that trainee-teachers and teachers accurately judged students' performance abilities in some mathematics problems, Nathan and Koedinger (2000) found that teachers misjudged them in some others. They asked 67 high school teachers to rank-order the relative difficulty of six types of mathematics problems that are varied along two dimensions, the position of the unknown in the problem (either at the beginning or last) and the problem format (symbolic or verbal). The authors found that 84% of the teachers correctly predicted that problems in which the unknown occurs at the beginning (e.g. $x \times 6+66 = 81.90$) are more difficult for students than problems in which the unknown occurs last (e.g. $(81.90-66)/6 = x$)). However, 70% of them incorrectly said that verbally presented problems are more difficult than their equivalent symbol problems.

5.4 Explaining students' errors

This section aims to answer the third research question:

- What explanations do trainee-teachers give for students' errors in algebra?

To answer the above question, the fourth section of the questionnaire is analysed.

(a) Analysis procedure

Because the data that are dealt with in this section (and in the next section) are qualitative, the analysis procedure is essentially the same as that recommended by many authors, namely, coding and establishing genuine categories (e.g. Robson, 1993; Strauss & Corbin, 1998). In the case of the present study, these categories were derived from three sources: the research questions, the literature review in chapter 2 and the raw data. Firstly, the research questions helped in establishing the main categories by focusing the analysis process in something specific rather than anything. For instance, the research question above helped to differentiate between the responses which attempted to explain students' errors and other responses which only stressed the correct answer without describing possible sources of the errors. Similarly, the research question "What strategies do trainee-teachers use, or conceive, for addressing students' errors in algebra?" helped focuse the attention on the suggested strategies. Thus, "error-explained" and "suggestion-given" are two main categories in the analysis process.

The literature review helped subcategorising the main categories. For example, *semantic* and *syntactic* are two terms used to explain students' errors in algebra. The same terms are also used in this research to describe trainee-teachers' explanations of students' errors. Another example is that the different ways used for addressing students' errors in the literature are found to be useful in describing trainee-teachers' suggestions. Finally, some of the sub-categories were established from the raw data without having any obvious link to the literature review.

In this sub-section, two analysis procedures are presented and discussed in turn. The first one is that of Wanjala (1996) and the second one is an improved version from the same procedure. Through this discussion, the following problem is referred to as the "students and professors problem":

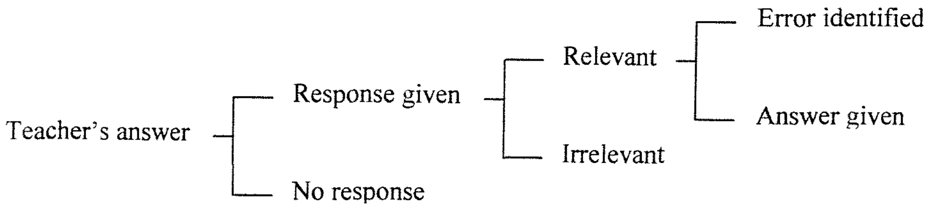
Write an equation using the variables s and p to represent the following statement: "There are six times as many students as professors at this university". Use s for the number of students and p for the number of professors.

The respondents were given the above problem as well as the incorrect answer 6s = p and requested to identify sources of the student error.

The first method of analysis

Wanjala (1996) analysed the responses to his questionnaire according to the following network system:

Figure 5.1: Network for analysing responses on error identification (Source: Wanjala, 1996, p.223)



The relevant responses were classified into "error identified" or "answer given". According to Wanjala, the former consisted of responses in which the misconception leading to the error was mentioned whereas the latter consisted of all responses in which the concentration was on giving or stressing the right answer to the question. Thus, all the responses were put under the four categories: error identified, answer given, irrelevant response and no response.

The above method of analysis has both advantages and disadvantages. On the one hand, it is easy and quick because it has only four destination categories. On the other hand, Wanjala assumed or counted just one response for every individual teacher. Thus, for five questions that he gave to 67 teachers, he counted 335 responses (5*67 = 335). This number included 57 missing answers, which he put under the category of the "No response". In the present study, however, some of the trainee-teachers were counted more than once because they gave different explanations for the same error. One reason for this is that they were asked two different questions:

- Identify what the student has done wrong?
- Suggest reason(s) which caused the student to produce this erroneous solution.

The first question above was borrowed from Wanjala (1996) whereas the second question was constructed for this study. To see how these two questions could produce different answers, let us take an example. When asked to explain the error 6s = p, S2 answered the two questions above as follows:

- "Answer should be s = 6p." (S2, first question-answer given)
- "Mainly the order of wording: six...students...professors. Also 'six times as many students' suggests 'six-times-students'." (S2, second question-error identified).
- "They think s = student, and p = professor, instead of s = number of students, p = number of professors. They may think they have to increase the number of students to get number of professors." (S2, second question-error identified)

The first response above is clearly of the type "answer given" because it stresses the right answer to the problem rather than focusing on the error itself. The other two responses were classified as "error identified" because they suggest different sources for the student error. This example shows that the number of the responses should not be restricted to the number of the respondents as some of them gave different explanations for the same error. It also shows that question (a) above was less successful than question (b) in eliciting S2's ability to explain the student error.

Another fact about Wanjala's study is that he did not attempt to subcategorise the "error identified" category. This was because he was only interested in knowing the extent to which the teachers were able to identify the difficulty leading to the students' errors. The present study aims to extend on this point by identifying not only trainee-teachers' abilities to explain the errors but also the different explanations they give for students' errors. Before I discuss how this might happen let us consider, once again, the following example:

- "They think s = student, and p = professor, instead of s = number of students, p = number of professors. They may think they have to increase the number of students to get number of professors." (S2)
- "Mainly the order of wording: six...students...professors. Also 'six times as many students' suggests 'six-times-students'." (S2).

Previously, both responses above were marked "error identified" but in fact they are different because they suggest two different sources for the student error. Hence, they could be placed in two different subcategories as will be seen in the following alternative way of analysis.

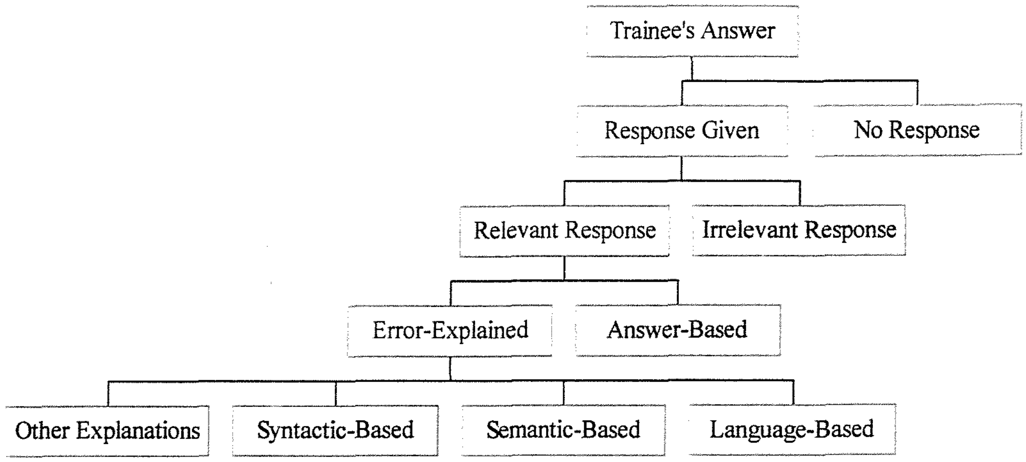
The second method of analysis

Figure 5.2 shows the alternative method. The shaded boxes represent the final destinations under which all the responses were placed. According to this model, the relevant responses are categorised into answer-based and error-explained. These two categories were used instead of "Answer given" and "error identified" in the previous method. Although the names "error-explained" and "error identified" mean exactly the same, the former was used here rather than the latter because the respondents were actually given the error and asked to identify sources of the error. Similarly, the name "answer-based" was used rather than "answer given" because the former included all the responses in the latter as well as other responses in which the focus was still on giving the right answer. This point can best be discussed with examples:

- "Answer should be s = 6p." (S2)

- "Put the s and the p in the wrong places-they should be the other way around." (P2).
- "6 needs to be in front of p not 6" M123.

Figure 5.2: Analysis of the responses on error explanation



Previously, S2's answer above was classified as "answer given". The same answer is now classified as "answer-based" since it is clearly stressing the right answer. In the next two responses, P2 and M123 did not give the right answer but, instead, they described how the given answer differs from the actual answer. In addition, they did not describe how the student has possibly arrived to the error. Such responses were also classified as answer-based.

The error-explained category was subcategorised into language-based, syntactic-based, semantic-based and "other" explanations. These terms are further clarified with examples in appropriate places in the next sub-section. Here, they are only described in general. The syntactic-based category consisted of responses which related the errors to the incorrect application of the procedure for working out the problems. In the case of word problems, the responses explained the errors in terms of the incorrect way of translating key words in the problem statement to their corresponding mathematical symbols. For example, C1 described how the errors $6s = p$ produced from the "students-professors" problem by suggesting that "Question says 'six times as many students', i.e. 6s" (C1). In the case of non-word problems, the responses related the errors to the incorrect application of the procedure or convention for working out the problem. For example, the errors

4n+5 and n+20 occurred because the student "Did not multiply whole expression by 4" (L2) in "multiply n+5 by 4". Another example is that "The student has only multiplied part of the formula by 4" (K2). This means that the student misused the mathematical convention for multiplying a number with brackets or with an expression.

The semantic-based category consisted of responses, which attempted to identify major sources of the error. This was seen in terms of incorrect use of algebraic letters as objects in word problems or attempted to provide other general explanations for the errors in the non-word problems. These responses appeared to explain students' errors from the point of view of students' understanding of algebra rather than the "blind" translation of algebra problems, which seemed to be the case in the syntactic-based explanations. To further illustrate this meaning, let us take examples. A2 said that the error 6s = p occurred because "At first glance, it looks acceptable. It looks logical. The text suggests that 6s is equal to 1p". Another example is that the two errors "n+20" and "4n+5" for the problem "multiply n+5 by 4" occurred because of:

"The lack of closure of the expression. In arithmetic, children are used to and can reasonably easily deal with multiplying known amounts. These clearly fall within their own experience. Unfortunately they find the concept of variable and unclosed expression very difficult to understand. I believe that they still use arithmetic schemata to deal with these types of quantities." (N1).

The name "lack of closure", in the quote above, is also used in the literature (see for example, Chalouh & Herscovics, 1984; Booth, 1984; Sfard & Linchevski, 1994). It generally means that some students are unable to hold the unclosed expressions. Thus, when faced with an expression they attempt to close that expression by, for example, substituting a number for the variable so that they obtain a numerical answer or by just omitting the operation signs in that expression. For example, they might simplify the expression x+3 to x3 or they might just put 5 for x to obtain 8. So in this case, those students still think arithmetic because in arithmetic, all the expressions can be closed, for example, 8+6 is simplified to 14. After this discussion, read the quote above again and you will notice that the quote and this discussion, to some extent, convey the same meaning. Therefore the above quote was classified as semantic because it tries to locate the error within a global category, which consists of class of errors rather than just one or two particular errors. In other words, the semantic-based explanations attempt to identify the misconception behind the errors. In this sense, they are more useful for teaching than any other explanations because they are more "stable" (Birenbaum et al, 1992).

The language-based explanations related the errors to the ill wording of the problems. In other words, they suggested that there is something special with some algebraic problems that cause students to make the errors. For example, referring to the error "6s = p", T1 said: "I think it is just because of language. ...".

The "other explanations" consisted of all other responses, which remained from the main category, error-explained. For example, A1 seemed to suggest that the error 6s = p occurred because of the students' difficulty in understanding the equal sign. He said "This should be a ratio not an equals sign".

(b) Findings

Table 5.7 shows the percentage of the responses in each category of error explanations in relation to the four algebra problems in the relevant section of the questionnaire. In addition, it shows the overall findings by, for example, counting all the syntactic-based explanations given in all the questions (n = 295) and then calculating the percentage according to the total number of responses in all the questions (n = 955). The following discussion is arranged according to the categories in the table.

Syntactic-based explanations

The syntactic-based category consisted of the highest proportion of the responses in all the cases, except in N2 for the reason explained later under this heading. The fact that the syntactic-based category consisted of the largest proportion of the responses indicates that the trainee-teachers relied more heavily on explaining students' errors in terms of the students' incorrect application of the relevant procedures for working out the problems. The nature of the given responses is discussed below briefly with few examples of the given responses.

In N1 and N2, the trainee-teachers related the errors to the incorrect procedure for translating the word problems to equations. According to them, students jump to the answer in just one or a few steps by changing key words in the problem statement to their mathematical symbols. They added that students should arrive at the answer in several steps by, for example, generating a table of values for the two variables and then using the pattern in the table to write the equation. The following are examples of their responses:

- "The student has jumped in (AS I DID !!!) and wrote the 6 next to 'students' as it appears in the worded problem statement" (G1, N1).
- The student "just follow literally as the last sentence = If b is the number of blue pencils ..." (M117, N2).

Table 5.7: Percentages of the responses in the categories of error explanations

Type of response	N1	N2	N3	N4	Overall
Syntactic-Based	36.86	12.50	36.33	38.94	30.89
Semantic-Based	10.59	17.34	15.92	19.03	15.71
Language-based	5.93	9.27	11.02	1.33	7.02
Other Explanations	8.47	17.34	9.80	8.85	11.20
Answer-Based	31.36	31.05	15.51	18.58	24.19
Irrelevant Response	4.66	9.27	6.94	5.31	6.60
No Response	2.12	3.23	4.49	7.96	4.40
Total	100	100	100	100	100

They added that the errors in N3 occur when students do not follow correctly the procedure for multiplying a number with an expression (or with brackets). Hence, this was seen as the major difficulty behind the errors. The following are examples of their responses:

- "Student has either done 4×n+5 or n+5×4 and calculated the answer from left to right, i.e. has not multiplied the whole thing by 4" (M118, N3).
- For 4n+5, the student "Simply multiplied the first part only by 4" and for n+20, the student "multiplied second part by 4" (M116, N3).

In N4, they described that the error occurs when students compare the two equations (n-246 = 762 and n-247 = ?) together and notice that 247 is one more than 246 and so the students automatically add one to the answer instead of subtracting one. Thus, the participants described how students arrive at the error from the problem. The following are examples of their responses:

- "Student has seen the number on the LHS of the equation increase by 1, and so has increased the number on the RHS by 1" (M116, N4).
- "Added 1 to the answer as 1 is added to 246" (M113, N4).

It has been pointed out above that the syntactic-based category is the largest one in Table 5.7 in all the cases except N2. The reason seemed to be that the participants found the error b+r = 90 harder to explain using their most common explanation (i.e. syntactic-based) compared to the other students' errors and hence attempted to identify other explanations or simply give irrelevant responses or answer-based responses. In fact the respondents who attempted to explain the error b+r = 90 syntactically were not very clear about how changing key words in the problem might lead to the error. For example, M117 said that students "just follow literally as the last sentence = If b is the number of blue pencils ..." This answer is still not very clear as to how translating the second sentence leads to the student error.

Semantic-based explanations

The semantic-based category consisted of less than one fifth of the responses in all the cases. In these responses, the participants were successful in identifying major sources of students' errors. In N1 and N2, they recognised the students' errors as representing a serious cognitive difficulty when students use letters as objects. According to them, the error 6s = p could be understood by students as "6 students with every one professor". The added that although this meaning matches the problem statement, mathematically it is not valid because, for example, 6s should be understood as "6 times the number of students" rather than "6 students". For example, M66 said:

- "I think the student understood the statement but hasn't understood that the expression does not reflect what should be presented. Six lots of students equals one professor-student has stated that for every professor there are six students" M66.

Similarly, the participants suggested that the letters b and r in b+r = 90 may have been used as names for the prices of the blue and the red pencils respectively rather than their numbers. Hence, the error may be understood as "blue pencils and red pencils cost 90 pence". In this case, it obviously has some meaning to students. This view is strongly supported by Küchemann (1981). The following are examples of the responses in the semantic-based category:

- "Could have understood the question as number of blue and red pencils bought cost 90 pence" (M109, N2).
- "It says some blue & some red pencils and these have a total cost of 90p" (M91, N2).

In N3, several semantic-based explanations were given for the student errors. Some of the trainee-teachers suggested that students might not be able to multiply 4 with an unknown number or with an expression. For example, M100 pointed that "For n+20, the pupil seems to think you can multiply numbers together but not numbers with letters". Similarly, M62 said "Some students may think that you only need to multiply one term or numbers are only multiplied by other numbers, not with letters". These suggestions confirm what was found in the literature that some students believe that it is not possible to multiply, for example, 3 with x unless you know what the value of x is (David, 1975). Another group of trainee-teachers appeared to say that algebraic expressions are stressed in school as processes rather than as objects and consequently a student may not perceive n+5 as a number but rather as a process. Furthermore, making the errors depends on whether the problem is perceived as 4(n+5) or as (n+5)×4 because, according to them, the former ends with the error 4n+5 and the latter ends with the error n+20. Thus, it is the student's perception of the problem and of the expression that accounts for the two errors. The following statements convey all or part of this meaning:

- "The idea of dealing with each term separately is stressed in algebra and in a same manner, the students has attempted to do this" (M110, N3).
- "n is a variable and 5 is a constant. The students may think only one of them should be multiplied and not the other. This may depend on whether the 4 is in the RHS or LHD" (M83, N3).

In N4, the trainee-teachers also gave several semantic-based explanations for the student error. Most of them argued that the error has its origin in the idea of balancing which is stressed in school because students are often told to keep a balance between the two sides of the equality by performing the same operation on both sides. However, this idea is overgeneralised in the present mathematical problem by subtracting one more from the left hand side of the second equation but adding one more to the right hand side of the same equation. Some other respondents attempted to convey the same meaning but in a different way. For them, the error could be generated from the idea of "change side-change sign" which is often used in school to solve algebraic equations. More specifically, the pattern in the present mathematical problem suggests that there is one more subtracted from the left-hand side of the second equation (n-247 =...) compared to that of the first equation (n-246 = 762). Hence, a student may overgeneralise the above rule by adding one to 762 instead of subtracting. The following are examples of the responses in the semantic-based category:

- "Probably the idea of balancing, i.e. that you need to add one to both sides" (M110, N4).
- "Using equations always told to do same thing to each side. This is an equation I must do the same" (M108, N4).
- "Rule: what you do to one side you do to the other. Add one to number on LHS then add one to number on RHS" (M63, N4).

Language-based explanations

The language-based category consisted of 67 responses (7.02%). They suggested that the language of the problems is confusing and consequently responsible for the students' errors. There were 14, 23, 27 and 3 responses in the language-based category in the four algebra problems respectively. This means that the second and the third were found to be the most ill worded problems whereas the first and the last were found to be the least ill worded problems. Some examples of the responses in this category are the following:

- "Language construction of the question has misled students' understanding of what their answers actually means" (M85, N1).
- "It's a wording question. Student has not necessary got a mathematical problem, but cannot extract the mathematical information from the way the question is presented" (M103, N2).
- "Not understanding that n+5 is one component which is being multiplied by 4. The lack of brackets may have caused confusion" (M94, N3).

Other explanations

This category consisted of all the explanations that could not be categorised as syntactic, semantic or language-based. They formed about tenth of the responses in all the cases except in N2 where they formed about one fifth of the responses given in the same question. This fact confirms the suggestion above that the error in the second algebra problem was found to be the most difficult to explain syntactically and so the respondents were forced to think of other explanations. Most of them explained the errors in terms of students' confusion, forgetfulness and misunderstanding of the question being asked. Some of the examples are the following:

- "Misunderstanding what the statement is saying. Not reading the statement properly-not thinking about it before writing down the answer" (M108, N1).
- "Student has confused number of pencils with cost ~ forgotten to combine number of pencils with unit prices" (M114, N2).
- "May not understand that (n+5) is a number" (M90, N3).
- "Cannot see the linkage between the two questions-inability to see help that has been given" (M104, N4).

Answer-based responses

This category consisted of the responses that failed to explain the errors and instead they stressed the right answer to the problems. This may happen because it is a common practice in school to evaluate student answers in terms of only "right" or "wrong" and without attempting to understand why the wrong answers are wrong or whether students who gave the right answers really understand mathematics (Nesher, 1987).

In about a quarter of the overall responses, the trainee-teachers stressed the correct answers. Some examples of their responses have been given at the beginning of this section. Some other examples are the following:

- "Needs integers in front of b and r, e.g. 5b+6r = 90" (M123, N2).
- The answer "Should be 5b+6r = 90" (M116, N2).

Missing and irrelevant responses

Finally, there were 4.4% missing responses and 6.6% irrelevant responses in all the four algebra problems. It is noticeable that the number of missing responses increases toward the end of section 4.

The irrelevant responses were either ambiguous or did focus on the question being asked. Some examples of such responses can be given:

- "Multiplied the wrong variables" (M88, N1).
- "It is true that b+r is the total number of pencils" (M79, N2).

(c) Overview

The discussion above suggests that in the overall:

- About one third of the trainee-teachers explained the errors in terms of the students' incorrect application of the relevant procedures for working out algebra problems.
- About one fifth of the trainee-teachers explained the errors in terms of the language of the problems or students' confusion and misunderstanding of the question being asked.
- Less than one fifth of the trainee-teachers were able to identify major sources of the students' errors (i.e. semantic-based explanation).
- About a quarter of the trainee-teachers stressed the correct answer but did not explain the errors.
- About one tenth of the trainee-teachers gave either irrelevant or missing responses.

The points above suggest that most of the respondents were unable to identify major sources of the students' errors. The most commonly used explanation for students' errors was the students' incorrect application of the relevant procedures for working out the problems. The interviews are designed to explore all the possible explanations of students' errors that the participants are able to give. This can then be used to know whether they know the major sources of the errors or whether they would also rely more heavily on explaining the errors in terms of the incorrect application of the relevant procedures.

(d) Comparison with other studies

Although the analysis procedures used in the present study and Wanjala's (1996) are not identical, some common points from the findings can still be found. First, both studies indicate that the majority of the respondents attempted to explain students' errors. However, only one fifth of the trainee-teachers in the present study were able to identify the main sources of the students' errors. The majority of them explained the errors in terms of the incorrect application of the relevant procedures for working out the questions. This finding is consistent with findings from Tirosh (2000), more than three quarters of the prospective teachers explained students' errors in fractions in terms of incorrect application of the standard division-of-fraction algorithm. In comparison, about one third of the teachers in Wanjala's study identified major sources of the students' errors. Thus, trainee-teachers seem to be less able to identify students' major difficulties and, instead rely more frequently on explaining students' errors in terms of the incorrect use of the relevant procedures for working out algebra problems. This discrepancy between teachers and trainee-teachers' abilities to explain students' errors and identify their sources have been observed elsewhere in the literature. Leinhardt and Smith (1985), for example, found that the expert teachers knew better than the novices about students' difficulties in fractions, such as students' misconception that 1/8 is bigger than ¼ because eight is bigger than four. Similarly, Borko et al (1989) concluded that expert teachers are more likely to anticipate problems and obstacles, for example, student misconceptions.

Another point for comparison between the present study and Wanjala's (1996) is that, in about a quarter of the responses in the present study as well as Wanjala's study, the correct answer was given without actually explaining the error. A third comparison point is that there were 27% irrelevant responses and 17% missing responses in Wanjala's study compared to only 6.6% irrelevant responses and 4.4% missing responses in the present study. The procedure for administering the questionnaire in the two studies seems to be responsible for obtaining these results. Also, the fact that the order of the questionnaires is not the same in the two studies might have had an effect.

5.5 Addressing students' errors

This section attempts to answer the research question:

- What strategies do trainee-teachers use, or conceive, for addressing students' errors in algebra?

As in the previous section, this section presents the analysis procedure followed by the findings. It also shows relationships between error explanations, as discussed in the previous section and suggestions for help, discussed in this section.

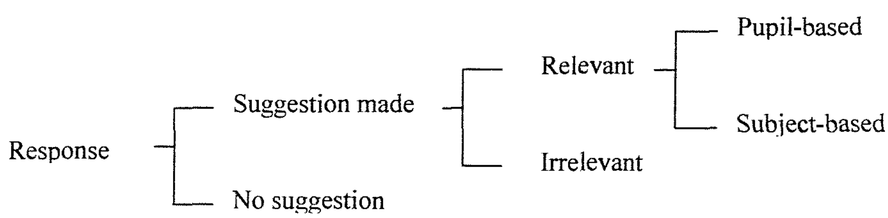
(a) Analysis procedure

Two analysis procedures are presented and discussed in turn in this sub-section. The first one is that of Wanjala (1996) and the second one is an improved version from the same method.

The first method

To analyse teachers' suggestions for helping, Wanjala developed a similar network system to the one that he used to analyse their responses for error identification. His new model is shown below (Figure 5.3).

Figure 5.3: Categories of the responses on suggestions for help (Source: Wanjala, 1996, p.225)



The categories are self-explanatory except, perhaps, pupil-based and subject-based. According to Wanjala, the former consisted of teachers' suggestions on which the concentration was on students' experiences such as using of analogue examples or suggesting ways for checking students' answers. The subject-based category consisted of responses in which the emphasis was on re-teaching students, who erred, the former steps for obtaining the correct answers. Thus, he used four destination categories to analyse all his data, namely, pupil-based, subject-based, irrelevant suggestion and no suggestion.

Examples of the responses in the above categories can be given from the present study and in relation to the students-professors problem (see the previous section). An example of a pupil-based suggestion is "Ask them to check their answer. For example, if there are five students, how many professors does that make" (G1). An example of a subject-based suggestion is "Explain that for the two sides to be equal you need to multiply the number of professors by 6 to have the same number of students" (E2). Finally, an example of an irrelevant suggestion is "say it more concisely, e.g. the number of students is 6 times the number of professors". The reason is that "it is a more logical way of saying it, and you write it down as it is said; i.e. s = 6p" (C1).

The last example above shows one disadvantage of this method. This suggestion is very useful from a teaching point of view but it was judged as irrelevant because it seemed to change the statement of the problem to another one where perhaps the syntactic translation of the new statement would produce a correct answer (MacGregor & Stacey, 1993). Thus, this is one area where this method seems to neglect important information.

Another area where the above method neglects important information can be seen from the fact that it categorises the relevant responses into pupil-based and subject-based. These two categories are the most important ones because they are related directly to the research question. Although they are useful because they tell us that some of the strategies are pupil-based and some others are subject-based, they do not tell us exactly what these strategies are. It would be more useful if the "relevant" responses are categorised in a way that can tell us, for example, how many trainee-teachers suggested re-teaching the algebra problems to the students who did not succeed in obtaining the answers required. The alternative method, which is presented underneath, deals exactly with this issue.

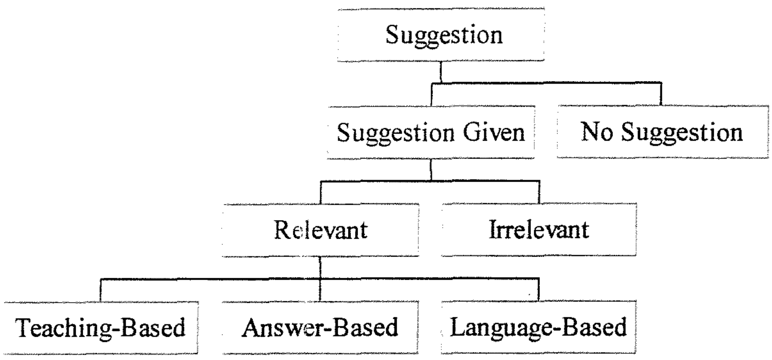
The second method

This method is basically the same as that of Wanjala but instead of classifying the relevant responses into pupil-based and subject-based, they are classified here into language-based, teaching-based and answer-based. Figure 5.5 clarifies this meaning.

The model is explained here briefly with a few examples. More discussion and more examples are given in appropriate places in the findings. The language-based consisted of responses, which recommended re-wording of the problem. The following is an example:

- C1 thinks that the problem should be rewritten in an easier way avoiding the complication of language. He believes that teachers should *"say it more concisely, e.g. the number of students is 6 times the number of professors"*. His reason is that *"it is a more logical way of saying it, and you write it down as it is said; i.e. s = 6p" (C1, students-professors problem).*

Figure 5.4: Analysis of the suggestions for help



The teaching-based category consisted of responses, which suggested re-teaching the problem to explain the apparent difficulties. Some, for example, stressed the idea of giving more examples and exercises to students to make them practice mathematical skills and rules. Others attempted to explain some form of algorithms to enable students obtaining the correct answers. The following is an example of the responses in this category:

- "Explain that for the two sides to be equal you need to multiply the number of professors by 6 to have the same number of students" (E2, students-professors problem).

The answer-based category consisted of responses, which suggested checking the answer by substituting numbers or using counter examples to make students recognise their faults. For example:

- "Suggest a numerical example like, if there are 50 students, s, how many professors, p, are there. So they would see that p = 300 ...and then hopefully realise their mistake" (E1, students-professors problem)

(b) Findings

Table 5.8 shows the percentages of the responses in each category of addressing students' errors in relation to the individual algebra problems. It also gives an overview of the findings by, for example, counting all the suggestions that belong to the teaching-based category in all the questions (n = 340) and then calculating the percentage according to the total number of suggestions in all the questions (n = 600).

Table 5.8: Percentages of the responses on the categories of addressing the error

Type of suggestion	N1	N2	N3	N4	Overall
Teaching-based	47.68	63.33	52.29	63.70	56.67
Answer-based	21.85	6.67	7.19	5.48	10.33
Language-based	5.96	2.67	20.26	2.74	8.00
Irrelevant suggestion	8.61	9.33	3.27	6.85	7.00
No Suggestion	15.89	18.00	16.99	21.23	18.00
Total	100	100	100	100	100

Teaching-based suggestions

It is evident from Table 5.8 that the teaching-based category consisted of the highest proportion of suggestions. This indicates that the trainee-teachers paid more attention to the idea of re-teaching the relevant procedures that already taught to the students. Their idea was that if students apply the procedure correctly then this might help students obtain the correct answers and consequently avoid making the errors.

The most commonly recommended procedure in N1 and N2 was the one that involves generating a table of values for the two variables and then using the pattern in the table to study the relationship between the two variables and write the general statement. The following are examples of such suggestions:

- "Put in values i.e. if there is 1 professor then there are 6 students, 2 professors then there are 12 students, 3 professors then there are 18 students and then apply algebra" (M118, N1).
- "Ask questions about the cost of 1 blue pencil then 2, 3, then ask how they worked it out-so for any number of blue pencils? Same for red pencils-then join together with altogether it equals 90 p." (M115, N2).

In N3, most of the trainee-teachers suggested explaining, in different ways, the procedure for multiplying a number with an expression. Finally, in N4, the majority of them recommended re-teaching the idea of balancing in an equation. The following are examples of the given responses in the last algebra problems:

- "Area of a garden extended by n meters.... area of each part? Area of total (add together each bit)" (M115, N3). (This is only part of his suggestion, which recommended using the area of a rectangle in which one side is 4 and the other side is 5+n. This suggestion is similar to the area model shown in Booth, 1984).
- "Build up from easier examples using number bonds the students already know. E.g. 11-3 = 8, 11-4 = 7, and then algebra involving smaller numbers. E.g. n-2 = 7, n-3 = 6, talk about the idea of taking an extra one away" (M111, N4).

There were other less frequently recommended procedures such as giving extensive exercises similar to the questions that students find difficult to enable them practice mathematical skills and routines. Others recommended re-reading the question several times to ensure that students understand it. Still some other trainees recommended telling students about the larger and the smaller variables in the question to make them translate this meaning in their answers.

Answer-based suggestions

The trainee-teachers in this category seemed to be able to suggest ways that challenge students' thinking and make the students aware of their faults. The number of such suggestions is considerably higher in the first algebra problem than in the others. Thus, whereas about one fifth of the suggestions were in the answer-based category in N1, less than a tenth of the suggestions were in the answer-based category in all the other cases. A further investigation found that within the answer-based category, the most frequently suggested procedure in N1 and N2 is the idea of substituting numbers for either variable in the students' answer and then comparing the outcome with the problem statement to convince the students that their answer does not have the same meaning as the problem statement. Similar suggestions were also given in N3 to show students that 4n+5 and n+20 are not the correct answers for the question "multiply n+5 by 4". There are other less frequently suggested procedures such as the one that focuses students' attention to the fact that the error b+r = 90 does not represent an equation because it is not a homogenous one since the left hand side of the equality is about the number of blue and red pencils whereas the right hand side of the equality is about their prices. Some of the responses in this category are given below:

- "Encourage the student to check answers by substituting values for s or p and see if the answer makes sense" (M116, N1).
- "Get pupils to write out equation in terms of the units, i.e. [pencils]+[pencils] = [pence]. They should then be able to see that both sides need to be pencils or pence to make sense" (M136, N2).
- "Get them to use numbers e.g. 4 times (5+4). They can then see that 4*9 = 36 and can understand that the answer cannot be 20+4 or 5+16" (M136, N3)
- "Get students to check their answers for themselves, work out value of n" (M110, N4).

The language-based suggestions

The language-based suggestions concerned with re-wording the algebra problems so that they become more understandable to students. The third algebra problem received considerably more of such suggestions than the other algebra problems. This seems to be a normal finding since the same algebra problem was suggested to be the most ill structured in the previous section. Therefore, there is some consistency here between the trainee-teachers' explanations of the errors and their suggestions for help. However, they suggested N2 as the second most ill structured problem in the previous section but, here, only few of them suggested that the language of N2 should be simplified and made clearer for students.

- "Rewriting the statement could be an option. If we expressed the statement as number of students equals six times as number of professors" (M66, N1).
- "Re-write question so that it is more of obvious that more than 1 was bought, use price as unknown variable" (M64, N2).
- "May say multiply (throughout both terms) n+5 by 4. Or change the representation of the question 4(n+5)" (M11, N3).
- "Put equation into words: n take away 246 = ...so n take away 247 is ..." (M79, N4).

Irrelevant suggestions and no suggestions

The pattern in Table 5.8 suggests that the number of no suggestions generally increases toward the end of section 4 of the questionnaire. It also shows that the first two algebra problems involved more irrelevant responses despite the fact that there were less than a tenth of such responses in all the questions. In the previous section, however, the second algebra problem also showed the most irrelevant responses on error explanation. As explained above, some of the students' errors were found easier to explain and to give suggestions for help.

Further discussion on suggestions for help

In this sub-section, findings from question B in section 5 of the questionnaire are presented in Table 5.9. The respondents were asked in that question to suggest ways for helping students who make the errors which they (the respondents) suggested the most likely in question A in the same section. Findings from this question are presented separately than those of section 4 of the questionnaire for the following reasons:

- For the purpose of making comparisons between error explanations and suggestions for help (the next sub-section). This was possible in section 4 because the errors were given to the respondents and they were asked both to explain and suggest ways to help. In section 5, however, they were only asked to suggest ways to help.
- To show that most of the missing responses on suggestions for help came from question B of section 5 because it was an optional question and occurred at the end of the questionnaire.

Table 5.9: Types of suggestions given on the optional question

Type of suggestion	Number of suggestions	Percentage
Teaching-based	329	45.07
Answer-based	26	3.56
Language-based	0	0.00
Irrelevant suggestion	38	5.21
No Suggestion	337	46.16
Total	730	100.00

As can be seen from Table 5.9 that most of the responses occurred in just two categories: teaching-based and no suggestion. The table also shows that there were 38 (5.21%) irrelevant suggestions and only 26 (3.56%) answer-based suggestions.

The finding that about half of the suggestions occurred in the teaching-based category confirmed what was said above, most of the respondents emphasised re-teaching algorithms rather than understanding. It was found that most of the responses were proceeded by words such as explain, discuss, review, tell them, remind them, emphasise, talk about, practice, write out and revisit. In addition, the trainee-teachers hardly ever referred to the students' errors when explaining the procedures. It may be useful to give some examples (the errors in the brackets are those for which the suggestions were made):

- "Should explain sign rules. Show them sign: 3a-(+b+a) = 3a-b-b" M78. (Error: 3a-b+a. Problem: "Simplify 3a-(b+a)).
- "Show/inform students that a and b are different letters and hence cannot be added" M78. (Error: 7ab. Problem: Simplify if possible: 2a+5b).
- "Go back to earlier stage. Discuss/practice bracket rules" M79. (Error: 3a-b+a = 4a-b. Problem: Simplify 3a-(b+a)).
- "Suggest that a and b are two unlike objects that cannot be summed" M82. (Error:7ab. Problem: Simplify if possible: 2a+5b).
- "Explain that (a+x) & (b+x) make one value and cannot be separated" M93. (Error: a/b+1. Problem: Simplify if you can: (a+x)/(b+x)).
- "Explain that the question is squaring everything within the brackets, squaring is multiplying everything within the brackets" M93. (Error: $3x+2^2$. Problem: Write $(3x+2)^2$ without brackets).
- "Give more examples without unknown variables introduced to ensure concept is clear" M95. (Errors: 10e and 10+e. Problem: Calculate the area of a rectangle of length e+2 and width 5).

Although the strategy of answer checking appears to be useful in the sense that it can make students recognise that their answers are incorrect, only few respondents gave answer-based suggestions. The majority of them gave teaching-based suggestions which mainly focused on re-teaching the relevant procedure that already introduced to students.

(c) Relationship between error explanations and suggestions

In this sub-section, an attempt was made to establish relationships between the different ways of "error explanations" and "suggestions for help" by calculating the proportions of responses for all possible combinations of the two variables. The seven categories of the variable "error explanations" and the five categories of the variable "suggestions for help" lead to 35 possible combinations (see Table 5.10).

Table 5.10: Percentages of the combinations

	Suggestions for help				
Explanations	Teaching-based	Answer-based	Language-based	Irrelevant	No suggestion
Syntactic-based	20.02	3.54	2.73	1.92	2.83
Semantic-based	10.11	1.72	1.52	0.51	1.42
Language-based	4.04	0.51	1.52	0.51	0.61
Other explanation	7.48	1.31	1.01	0.81	0.81
Answer-based	13.75	3.13	1.31	2.12	3.34
Irrelevant	2.63	0.61	0.81	1.31	1.52
No response	0.61	0.10	0.10	0.10	3.64

In relation to the percentages of the combinations in Table 5.10, some examples of these combinations are:

- A syntactic-based explanation was given and a teaching-based suggestion was made, which included 198 responses.
- A syntactic-based explanation was given and an answer-based suggestion was made, which included 35 responses.
- A syntactic-based explanation was given and a language-based suggestion was made, which included 27 responses.
- A syntactic-based explanation was given and an irrelevant suggestion was made, which included 19 responses.

- A syntactic-based explanation was given and no suggestion was made, which included 28 responses.

Percentage of a combination = $N/989\times100$ where N = Number of the responses in a combination and 989 = total number of the responses in all the combinations. The percentages for the examples 1 to 5 in the list above were thus calculated as 20.02%, 3.54%, 2.73%, 1.92% and 2.83% respectively. Table 5.10 shows the percentages of all the 35 possible combinations.

It was decided to use the "percentage" of a combination rather than the "probability" of a combination due to the imprecise nature of the data. Calculating the probability of a combination requires knowing the total number of *possible* responses rather than the total number of responses in the combinations. This was avoided in this study since many of the trainees gave more than one explanation and more than one suggestion.

The sum of the rows and columns in Table 5.10 was not calculated because it does not have any meaning in this case. This is because of the way the responses in any combination were counted. To illustrate this meaning, assume that a respondent gave syntactic-based and semantic-based explanations, and teaching-based and answer-based suggestions. To find the possible combinations in this example means counting each of the explanations twice and each of the suggestions twice, for example, counting the syntactic-based explanation once with the teaching-based suggestion and then with the answer-based suggestion. Hence, the total number of the syntactic-based explanations will not be the same as that in Table 5.7.

The numbers in the first column of the table above are the highest when compared to the numbers in the same row. This indicated that the trainee-teachers relied more frequently on the suggestion of re-teaching the procedure no matter what type of explanation they gave for an error. It is interesting to note that even when the language of the problem was suggested to be the main source for the students' errors, the majority of the trainee-teachers still recommended re-teaching of the procedure rather than re-wording the problem. Similarly, although in 15.71% of the responses the trainee-teachers recognised students' errors as representing a serious cognitive difficulty, such as understanding letters as names for things, only in 1.72% of the suggestion did they recommend ways that challenge students' thinking. In the majority of the responses (10.11%), they assumed that this difficulty might be overcome by correctly applying the procedures for working out the problem.

(d) Overview

Table 5.8 summarises the findings in this section as follows:

- More than half of the trainee-teachers were in the teaching-based category. They suggested re-teaching of the relevant procedure for the purpose of helping students to get the correct answer and avoid making errors when working out algebra problems.
- About a tenth of the trainee-teachers were able to suggest ways that challenge students' thinking and make the students recognise their faults.
- Less than a tenth of the trainee-teachers suggested that the aids for addressing students' errors can be found in the good wording of algebra problems.
- About a quarter of the trainee-teachers either gave irrelevant suggestions or no suggestions.

The interviews are designed to explore the kind and nature of all the possible strategies that might be used for addressing students' errors.

(e) Comparison with other studies

As in the previous section, I shall draw on some common points between this study and Wanjala's study (1996). Wanjala found that there were 31% irrelevant suggestions and 36% no suggestion compared to only 7% irrelevant suggestions and 18% no suggestions in the present study. These differences in findings may be related to the procedure for administering the questionnaire in the two studies. In the present study, it was possible to get the respondents together in one room, give them the questionnaire and then collect it soon after they had completed it. In Wanjala's study, the respondents were given the freedom to complete it in their spare time. In addition, the order of the questions in the present study is not the same as that of Wanjala.

The most important part of the findings in both studies is that about a quarter of the teachers in Wanjala's study saw the aids for addressing student errors in providing them with rules and procedures. In comparison, about half of the trainee-teachers in the present study emphasised re-teaching of rules and procedures for addressing the errors. On the other hand, less than 10% of the teachers and the trainee-teachers in both studies gave suggestions according to students' needs, that is, aimed directly for correcting the errors.

The above findings are further strengthened by findings from Even and Markovits (1993) who suggested that teachers often ignored students' misconceptions and that they were unable to address the errors other than by explaining the whole topic again. Similarly, Sleeman et al (1991) found that most teachers addressed students' errors by re-teaching the procedure to the students rather than considering their needs. Although these studies are not about trainee-teachers, the overall suggestion seems to be that trainee-teachers are more reliant on teaching the procedure and ignoring students' ways of thinking than are teachers.

5.6 Predicting students' errors

In section 5 of the questionnaire, five algebra problems were given and the respondents were asked to suggest the most likely error (and others, if relevant) that students might make when working out each of the algebra problems. The question therefore allowed the respondents to predict more than one error for the same problem. The responses to the question were analysed for the purpose of answering the research question:

- How successful are trainee-teachers in predicting students' errors in algebra?

As in the previous section, this section presents the analysis procedure followed by the research findings.

(a) Analysis procedure

The obvious way to start the analysis is to make a list of all the predicted errors. From the list, we should be able to know the number of respondents who predicted a certain error. But the question is "is this useful information?" The fact is that if these errors occur at school then such knowledge is obviously important because errors can then be addressed before they occur. It is assumed here that if students' errors are not addressed in time then they may become deeply rooted (Linnecor, undated) and, as a result, they may prevail up to university levels (Trigueros & Ursini, 1999). But how do we now if the predicted errors occur at school or not. The best thing to do is to try and group these errors and then have a criterion that can tell us which errors are likely to occur at school and which errors are not. The criterion that was used here is Wanjala (1996) because it is the only known study that lists the errors and the percentages of the students who committed the errors in a great detail. Thus, even errors that were made by 3% of the students were listed in this study.

Categorising the responses

The list of the predicted errors was divided into two categories. The first category consisted of errors which rarely occur or do not occur at school. The second category consisted of errors which occur at school a considerable number of times. There is also a third category of errors, which did not appear in the list of the predicted errors. This category consisted of errors, which occur in school but were not predicted by the respondents. Wanjala (1996) calls the three categories: hypothetical, coincident and unnoticed errors respectively. He defined them as:

- Coincident errors: Errors, which had been committed by the students and suggested by the respondents.
- Unnoticed errors: Errors, which had been committed by the students but not suggested by the respondents.
- Hypothetical errors: Errors, which had not been committed by the students but suggested by the respondents.

In the present study I used the same categories. The rest of the categories that were considered here consist of the vague responses and no responses. The former refers to the ambiguous responses or those which lacked the focus on the task. The latter refers to the missing responses.

(b) Findings

Algebra problem 1

Simplify where possible:

3a-(b+a)

Frequencies of the predicted errors

A total of fourteen errors were predicted for this question. Table 5.11 shows the number and percentage of the trainee-teachers predicting each error as well as the percentage of the students committing the same error. (Note: most the tables in this section are split into two parts to reduce the occupied space). The last column in the table was borrowed from Wanjala (1996) for comparison.

Table 5.11: Frequencies of the predicted errors.

PE*	NT	PT	PS	PE	NT	PT	PS
4a-b	77	52.74	24.0	4a	1	0.68	
3a-b+a	65	44.52		3a-ba	1	0.68	5.0
4a+b	4	2.74		3a-b+4a	1	0.68	
3a-b-a	3	2.05		3a-b+2a	1	0.68	
2a+b	3	2.05	4.0	-4ab	1	0.68	
2a-b	2	1.37		3ab+3aa	1	0.68	
-3b	1	0.68		Vague	9	6.16	
3a	1	0.68		NR*	12	8.22	

Table notes (also apply to most tables in this section):

PE = Predicted Error, NT = Number of trainees, PT = Percentage of trainees, PS = Percentage of Students, NR = No Response.

Some of the predicted errors in the Table 5.11, such as 4a-b and 3a-b+a, are just different forms of the same expression. Thus, there is a question whether it can be assumed that, for example, those who predicted the error 4a-b also predicted the error 3a-b+a. In other words, can we assume that those who suggested the error 3a-b+a did so because they were thinking that this is the error that they would observe in the classroom rather than 4a-b? Because it was difficult to answer such a question from the completed questionnaires, it was decided to report the findings using both possibilities. In addition, less attention was given on the discussion to the errors that were predicted by less than 3% of the trainees because they do not seem to make a pedagogical issue. The 3% limit was also used by Wanjala (1996). However, unlike Wanjala's study, these errors were still reported in the present study for future reference by other studies.

Coincident errors

Three errors were found in this category: 4a-b, 2a+b and 3a-ba. The error 4a-b was predicted by about half of the trainees and, at the same time, it had been committed by about a quarter of the students, the highest proportion in Table 5.11. However, a larger proportion of the trainees can be obtained by subsuming the prediction of the error 4a-b in the prediction of the error 3a-b+a since both errors are different forms of the same expression. This process should takes into account the number of the trainees who actually suggested both errors (n = 25). Once this is done, the new percentage is 80.14%. This suggests that more than three quarters of the trainees were aware of the error 4a-b which is commonly observed in students works (Booth, 1984).

The other two errors in this category (coincident), 2a+b and 3a-ba, were only predicted by 3 (2.05%) and one (0.68%) trainees respectively and, at the same time, they were less common than the first error as they had been made by only 4% and 5% of the students respectively.

Unnoticed errors

There were four errors that had been committed by the students but not predicted by the trainee-teachers, namely, $3ab-3a^2$, $-3ab-3a^2$, 3ab-ba and 3-b. According to Wanjala (1996), the first three errors had been committed by 9% of the students and the last one by 5% of them. From these percentages, it could be argued that they are not random errors since they had been committed by a considerable number of the students. The non-prediction of these errors by the trainee-teachers suggested that they were not aware of them. Hence, they may go unnoticed in their classroom. This raised the point that their assessment procedures are not diagnostic enough to reveal the difficulties and the misconceptions held by students.

Hypothetical errors

Eleven errors were found in this category: 3a-b+a, 4a+b, 3a-b-a, 2a-b, -3b, 3a, 4a, 3a-b+4a, 3a-b+2a, -4ab and 3ab+3aa. These errors predicted by the trainees but had not been committed by the students. The error 3a-b+a predicted by 65 trainees (44.52%) or, as suggested above, 117 trainees (80.14%).

The two expressions 3a-b-a and 2a-b were predicted as errors but in fact they are the correct answers. This might have happened because, for example, the respondents misread the question, did not know what constituted an error or did not know what constituted a correct answer. However, they were only predicted by 3 and 2 trainees respectively. In fact, those who predicted 2a-b also predicted 3a-b-a. Thus, at most, 3 trainees can be assumed to have suggested both expressions.

The error 4a+b was predicted by 4 of the trainees (2.74%). The rest of the errors listed above were predicted by one trainee each.

The above hypothetical errors can be understood in different ways. Firstly, they can be taken as an indication of the trainee-teachers' ability to be imaginative about students' errors and difficulties (Wanjala, 1996). Secondly, they can be taken as an indication of the trainee-teachers' inability to be precise in their predictions since they predicted errors that are unlikely to occur at school. Finally, they may raise the point that these errors had not been committed by the students because they had been addressed by the experienced teachers. This last point suggests that being aware of the errors is still useful though they might not occur in some schools.

The vague and the no responses

There were 12 missing responses (8.22%) indicating that the respondents were unable or unwilling to respond or run out of time to complete the questionnaire. It is worth knowing, however, that nine of them did not respond to any of the questions regarding error prediction. Furthermore, they were from different institutions rather than from the same institution.

There were 9 vague responses (6.16%) which did not give the predicted errors in their symbolic forms but rather described them in words. Hence, they were imprecise in their descriptions of the errors. The following are examples of the responses in this category:

- "Not converting the plus sign to a minus" M132.
- "Ignore presence of the brackets" M135.
- "Not multiplying the negative sign" M91.
- "Not multiplying everything in bracket by -" M58.

Algebra problem 2

Simplify where possible:

2a+5b.

Frequencies of the predicted errors

A total of fourteen errors were predicted on this question. Table 5.12 shows the number and percentage of the trainee-teachers predicting each error as well as the percentage of the students committing the same error.

Table 5.12: Frequencies of the predicted errors.

PE	NT	PT	PS
7ab	104	71.23	10.0
7a+b	12	8.22	
10ab	8	5.48	2.0
7a	7	4.79	
7b	6	4.11	
7	4	2.74	
7+a+b	3	2.05	
7 (a+b)	2	1.37	

PE	NT	PT	PS
10 (a+b)	2	1.37	
a+7b	1	0.68	
10a	1	0.68	
10b	1	0.68	
5ab	1	0.68	
25ab	1	0.68	
Vague	7	4.79	
NR	10	6.85	

Coincident errors

Two errors were found in this category: 7ab and 10ab. They occur because students are unable to hold unclosed expressions such as 2a+5b. Hence, they might attempt to simplify the expression by adding on all the numbers and then writing down all the existing letters. Alternatively, they might multiply the numbers together and then write down the letters. Such a strategy is commonly used by students (Warren, 1999). In the CSMS study, the errors 7ab (and 8ab) had been committed by 45% of the 13-years-old students (reported in Booth, 1984, p.3). The prediction of the error 7ab by about three quarters of the trainees is a good indication of their awareness of the error. The error 10ab was suggested by eight trainees (5.48%) and had been committed by 2% of the students.

Unnoticed errors

Only the error 7+ab was not predicted by the trainees but had been committed by 2% of the students. However, this small proportion of the students who had committed the error suggests that it is not a common error and, therefore, it can hardly be observed in the classroom.

Hypothetical errors

Twelve errors were found in this category: 7a+b, 7a, 7b, 7, 7+a+b, 7(a+b), a+7b, 10(a+b), 10a, 10b, 5ab and 25ab. These errors were predicted by the trainees but had not been committed by the students. The error 7a+b is the most frequently predicted error in this category as it was suggested by 12 of the trainees (8.22%). The idea behind it seemed to be that some students might add the numerical numbers first in the expression 2a+5b and then place what remained from that expression, i.e. a+b next to it so that the final answer would be 7a+b.

The errors: 7a and 7b were less frequently predicted than 7a+b but they were still predicted by considerable proportions of the trainees (4.79% and 4.11% respectively). The remaining errors in this category were predicted by less than 3% of the trainees.

The vague and the no responses

Ten of the trainees (6.87%) gave no suggestion. Nine of them were the same as those who did not respond to any of the questions regarding error predictions. Seven of the trainees gave vague responses (4.79%). One of them suggested that students would recognise that the expression 2a+5b cannot be simplified and hence, they would make no errors. The rest of the trainees described the errors in words rather than in their symbolic forms. They were ambiguous and consequently marked as vague. The following are examples:

- "Adding the coefficient despite the different variables" M132.
- "May add these two terms together" M91.
- "a & b cannot be added together" M92.
- "Adding a's and b's" M58.

Algebra problem 3

Simplify if you can:

$$\frac{a+x}{b+x}$$

Frequencies of the predicted errors

Eleven errors were predicted on this question. Table 5.13 summarises the findings.

Table 5.13: Frequencies of the predicted errors.

PE	NT	PT	PS
$\frac{a}{b}$	97	66.44	32.0
$\frac{a}{b}+1$	14	9.59	
$\frac{a}{b}+\frac{x}{x}$	8	5.48	
$\frac{a}{b}+x$	3	2.05	
$\frac{a+1}{b+1}$	2	1.37	
$\frac{a}{x}$	2	1.37	
1	2	1.37	

PE	NT	PT	PS
$\frac{ax}{bx}$	2	1.37	3.0
a+b+2x	1	0.68	
$\frac{a}{b}\times x$	1	0.68	
$\frac{ax+bx}{bx}$	1	0.68	
Vague	6	4.11	
NR	15	10.27	

Coincident errors

Two of the predicted errors had been committed by the students: a/b and (ax)/(bx). They were respectively predicted by 97 (66.44%) and 2 (1.37%) of the trainees. It was found from the questionnaires that the two trainees who predicted the latter did not predict the former. However, this might have occurred in their minds and therefore the trainees who predicted the error a/b could have been aware of the students' interpretation of a+x and b+x in the problem as ax and bx, only they proceeded in their minds and cancelled the two x's together without writing the error (ax)/(bx). Also, those who predicted the error (ax)/(bx) might have been aware of the error a/b, only that they did not cancel the two x's together because they assumed that this is too obvious. Therefore, a total of 99 trainees (67.81%) could be assumed to have predicted both errors.

From Table 5.13, the error a/b had been committed by a high proportion of the students (32%), indicating that it occurs so frequently at school. The prediction of the error by about two thirds of the trainees is therefore encouraging as it suggested that the majority of them were aware of it and of its sources. Matz (1982) also found that this error is a systematic error for many students. The second error, (ax)/(bx), had been committed by 3% of the students.

Unnoticed errors

There were no errors in this category because the trainees predicted all the errors that had been committed by the students.

Hypothetical errors

A total of nine errors were found in this category: a/b+1, a/b+x/x, a/b+x, (a+1)/(b+1), a/x, a/b×x, (ax+bx)/(bx), 1 and a+b+2x. The prediction of any of the errors a/b+1, a/b+x/x and (ax+bx)/(bx) can be subsumed in the prediction of the others. Using this assumption, they were predicted by 21 trainees (14.38%) instead of 14, 8 and 1 trainees for the three errors respectively. The rest of the errors in the list above were predicted by less than 3% of the trainees.

The vague and the no responses

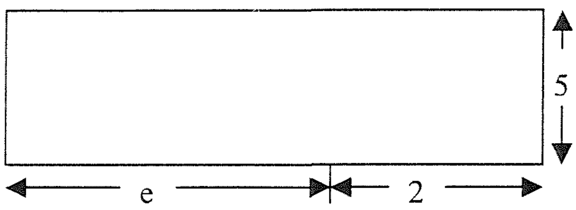
There were 15 missing responses and 6 vague responses on this problem given by 10.27% and 4.11 of the trainees respectively. Some of the vague responses were:

- "Split into two fractions" M129.
- "May cancel the x down" M91.
- "Cancel wrongly" M57.

The above responses were "vague" because they were given in words and were not clear enough to be classified in other categories.

Algebra problem 4

What is the area of this shape?



Frequencies of the predicted errors

Table 5.14 shows all the 28 predicted errors on this problem. As can be seen from the table, most of the predicted errors are different forms of the same expression. For example, the errors 10e, 2e×5, 2×e×5, and 5(2e) represent the same expression. Also, the majority of them were predicted by less than 3% of the trainees.

Coincident errors

Three coincident errors were predicted on this problem. The errors 10e, 10+e and 5×e+2 were predicted by 32.88%, 17.12% and 1.37% of the trainees and had been committed by 12%, 3% and 3% of the students respectively. However, the error 10e can be assumed to have been predicted by the trainee-teachers who predicted the errors 2e×5, 2×e×5 and 5(2e). In this case, 55.48% of the trainees predicted the error 10e as well as 2e×5, 2×e×5 and 5(2e). These errors appear to suggest that students might attempt to find the area of the rectangle by multiplying the numbers and the letters together because they think that identifying the area means multiplying everything together (Booth, 1984). Alternatively, they might attempt to multiply 5 and 2 together and then add the letter afterward to end with the error 10+e. In the CSMS study, the errors 5e2, e10, 10e and e+10 had been committed by 42% of the 13-years-old students (reported in Booth, 1984, p.3). In the present study, the error 10+e was predicted by 17.12% of the trainees. However, 23.97% of the trainees can be said to have predicted the error since it might also have been predicted by the trainees who predicted the errors 5×2+e and (5×2)+e.

Table 5.14: Frequencies of the predicted errors.

PE	NT	PT	PS
10e	48	32.88	12.0
10+e	25	17.12	3.0
5e+2	18	12.33	
2ex 5	17	11.64	
2×e×5	14	9.59	
5+2+e	12	8.22	
5×(e+2)	11	7.53	
7+e	9	6.16	
14+2e	7	4.79	
5×2+e	8	5.48	
2e+5	6	4.11	
10	3	2.05	
7e	3	2.05	
5e+10	3	2.05	
5×e+2	2	1.37	3.0

PE	NT	PT	PS
5(2e)	2	1.37	
(5×2)+e	2	1.37	
15e	2	1.37	
5+5+2+e+2+e	1	0.68	
5e+2e	1	0.68	
52e	1	0.68	
5e2	1	0.68	
2e+10+2e+10	1	0.68	
4e+20	1	0.68	
2e+5+2e+5	1	0.68	
2(5+2+e)	1	0.68	
2(7+e)	1	0.68	
14+e	1	0.68	
Vague	6	4.11	
NR	13	8.90	

Finally, the error 5×e+2 was predicted by 2 trainees. They appear to assume that students would either ignore the brackets, thinking that they are not important, but still understand 5×e+2 as 5×(e+2) or that they would attempt to multiply 5 with only one part of the expression. Both interpretations are used by students (Booth, 1984). The same thing can be said about the trainee-teachers who predicted the error 5e+2 (n = 17). In fact, one trainee suggested both 5×e+2 and 5e+2. It can therefore be assumed that 19 trainees (13.01%) predicted 5×e+2 as well as 5e+2.

Unnoticed errors

Several errors, all numerals, had been committed by the students but they were not predicted by the trainees including 5×5×2 = 50, 9.5×5 and 5(2+2). Although they reflected various interpretations of the area of the rectangle, they all involved substituting a number for the letter. Such interpretations are common in students' works (Küchemann, 1981, Booth, 1984). There is one numeral response in Table 5.14, 10, which was suggested by 3 of the trainees. However, it does not involve substituting a numeral number for the letter e. It seemed to be a result of multiplying 5 and 2 in 5×(2+e) and then omitting the letter. Wanjala (1996) found that the above errors together had been committed by 17% of the students. Thus, it was discouraging to find that the trainee-teachers were not aware of the above interpretations in algebra.

Hypothetical errors

All the errors in Table 5.14 were hypothetical except three, namely 10e, 10+e and 5×e+2 which had been committed by the students. In total, there were 25 hypothetical errors. They can be discussed in groups. The first group consists of 2ex5, 2×e×5 and 5(2e). The proportions of the trainees who predicted these errors were 11.64%, 9.59% and 1.37% respectively. However, it was suggested above that they could have predicted by 55.48% of the trainees because they point to the same interpretation of the area of the rectangle.

The second group of errors consists of 5×2+e and (5×2)+e which were respectively predicted by 5.48% and 1.37% of the trainees. As discussed above, the two errors might have been predicted by 23.97% of the trainees.

The third group of errors consists of 5+2+e and 7+e that were suggested by 12 (8.22%) and 9 (6.16%) trainees respectively. Both errors appears to suggest that students would totally miss out the area and instead, attempt to calculate half of the perimeter, i.e. add together the numbers and the letters actually given. For this reason, it can be assumed that they were predicted by 11.64% of the trainees (n = 17). In the completed questionnaires, four of the trainees actually suggested both errors.

The fourth group of errors consists of 14+2e, 5+5+2+e+2+e, 2(5+2+e) and 2(7+e). There was one trainee who actually suggested all the errors. There were another 6 trainees who predicted the first error but could also be assumed to have predicted the others because they all based on the assumption that students would calculate the whole perimeter instead of the area of the rectangle.

Other errors in this category that were suggested by a considerable number of the trainees consist of 5e+2 and 2e+5. They were suggested by 18 (12.33%) and 6 (4.11%) of them respectively. However, it was said before that the prediction of the error 5e+2 could be subsumed in the prediction of the error 5×e+2. In this case, a total of 19 (13.01%) trainees predicted it.

Another point to report is that 5e+10 and 5(e+2) were suggested respectively by 3 and 11 trainees as errors but in fact they represent the actual area of the rectangle. A further investigation revealed that those who suggested 5e+10 did not suggest 5(e+2) though this might happened in their minds. A similar thing happened in the first algebra problem. However, none of the trainees made the same mistake twice. Therefore, it is likely that this occurred due to a mental slip rather than anything else.

Finally, the rest of the hypothetical errors, which have not yet been discussed, were suggested by less than 3% of the trainees (see Table 5.14).

The vague and the no responses

There were 13 missing responses (8.9%) and six vague responses (4.11%) for this problem. Some of the vague responses were:

- "Calculate perimeter, miscalculate area" M129.
- "Not getting the length is 2+e" M91.
- "Finding perimeter" M58.

Algebra problem 5

Write $(3x+2)^2$ without brackets

Frequencies of the predicted errors

A total of 22 errors were predicted on this problem. As in the above algebra problems, some of the predicted errors can be subsumed in the prediction of others. Furthermore, 12 of them were suggested by less than 3% of the trainees. Table 5.15 shows all the predicted errors together with the proportions of the trainees and the students.

Table 5.15: Frequencies of the predicted errors.

PE	NT	PT	PS	PE	NT	PT	PS
$9x^2+4$	35	23.97	12.0	6x+2	2	1.37	
$3x+2^2$	28	19.18		$6x^2+4$	1	0.68	
$3x^2+4$	26	17.81	3.0	$3x^2+4+12x$	1	0.68	
6x+4	21	14.38	10.0	$9x^2+4x+4$	1	0.68	
3x+4	17	11.64		$6x^2+2x+4$	1	0.68	
$3x^2+2^2$	9	6.16		$3x^2+4+6x$	1	0.68	
9x+4	8	5.48	3.0	9x+6x+6x+4	1	0.68	
3x+2	5	3.42		$3x+4^2$	1	0.68	
9x+2	3	2.05		$9x^2+2x+4$	1	0.68	
$(3x+2)\times 2$	3	2.05		$9x^2+6x+2$	1	0.68	
$3x^2+2$	2	1.37		Vague	4	2.74	
$9x^2+2$	2	1.37		NR	14	9.59	

Coincident errors

There were four of the predicted errors that had been committed by the students, namely, $9x^2+4$, $3x^2+4$, 6x+4 and 9x+4. The first error was predicted by 23.97% of the trainees. This is the most common error because it had been committed by

12% of the students, the highest proportion in the Table 5.15. It seemed that about three-quarters of the trainees were not aware of it.

The error $3x^2+4$ was suggested by 17.81% of the trainees (n = 26) and had been committed by 3% of the students. Another predicted error to compare is $3x^2+2^2$ which was suggested by 6.16% of the trainees (n = 9). Although this error had not been committed by the students, its prediction can be subsumed in the prediction of $3x^2+4$ since both errors assume that students would square only the letter and the 2 in $(3x+2)^2$. The fact is that one trainee suggested both errors. Therefore, a total of 34 trainees (23.29%) can be assumed to have predicted the error $3x^2+4$ as well as $3x^2+2^2$.

The error 6x+4 was predicted by 14.38% of the trainees (n = 21) and had been committed by 10% of the students. A relevant error to compare is $(3x+2)\times2$ which was suggested by 3 trainees. Both errors assume that students would multiply the expression 3x+2 by 2 instead of squaring it. A further investigation revealed that one of the trainees who suggested the error $(3x+2)\times2$ also suggested the other. Hence, 23 trainees (15.75%) may have predicted both errors.

Finally, the error 9x+4 was predicted by 5.48% of the trainees and had been committed by 3% of the students.

Unnoticed errors

There were no errors in this category because trainees predicted all the errors that had been committed by the students.

Hypothetical errors

There were 18 errors in this category. Five of them predicted by more than 3% of the trainees including $3x^2+2^2$, $(3x+2)\times2$, $3x+2^2$, 3x+4 and 3x+2. The first two errors have been discussed above. The two errors $3x+2^2$, 3x+4 were respectively suggested by 28 (19.18%) and 17 (11.64%) of the trainees. It was found that eight of them suggested both errors. Thirty-seven trainees were therefore assumed to have predicted both errors (25.34%). The error 3x+2 was suggested by 3.42% of the trainees (n = 5). The rest of the errors in Table 5.15, which have not been discussed so far, were suggested by less than 3% of the trainees.

The vague and the no responses

There were 14 missing responses (9.59%) and four vague responses (2.74%) for this question. Two of the vague responses were:

- "Multiply by Φ" M129.
- "Multiplying by 2, just squaring each term in the brackets" M132.

(c) Overview

This section shows that 4a-b, 7ab, a/b, 10e and $9x^2+4$ are the most likely errors in the algebra problems that have been discussed above. The first four errors were predicted by about half to three quarters of the trainee-teachers. The last one was predicted by about a quarter of the trainees. In addition, the trainee-teachers predicted most of the errors that had been committed by the students in Wanjala's (1996) study including the less likely errors. The above analysis also shows that there were some of the predicted errors that had not been committed by the students. However, most of them seemed to be based on some expectations toward students' behaviour when working out algebra problems.

The interviews are designed to explore whether the participants are able to interpret their predicted errors, i.e. explaining how the errors might be generated from the question. I also need the interviews to know whether the participants have observed their predicted errors in the classroom and to know which of the equivalent expressions, that might be given, are the most likely to occur in the classroom.

(d) Comparison with other studies

Three types of predicted errors have been discussed above, namely, coincident, unnoticed and hypothetical errors. The coincident errors consisted of errors, which were suggested by the trainee-teachers and had been committed by the students. It was found that, in all five algebra problems that have been presented, both the trainee-teachers in this study and the teachers in Wanjala's (1996) study predicted the most likely error in each case. The most likely errors refer to the errors that had been committed by the largest proportion of the students. They consisted of 4a-b, 7ab, a/b, 10e and $9x^2+4$. In regard to these errors, it was found that:

- The teachers seemed to be more aware of the errors a/b and $9x^2+4$ than the trainees because 73.1% and 40.3% of the teachers respectively predicted the two errors compared to 66.44% (or 67.81%, see below) and 23.97% of the trainees.
- The teachers and the trainee-teachers seemed to be equally aware of the error 7ab which was suggested by 70.1% and 71.23% of the teachers and the trainees respectively.
- Depending on the perspective discussed below, the teachers might do better than the trainee-teachers in predicting the errors 4a-b and 10e which were predicted by 80.6% and 46.3% of the teachers compared to 52.74% (or 80.14%) and 32.88% (or 55.48%) of the trainees respectively.

In the three points above, the percentage in brackets represents the prediction of the error after it was subsumed in the prediction of another error (or errors). As explained before, although some of the predicted errors are just different forms of the same expression, it was not possible to know from the questionnaire whether, for example, 5×2+e and 10+e were based on the idea of multiplying 5 with only one term of the expression 2+e or whether the former based on a different expectation than the latter such as the idea of ignoring the brackets in 5×(2+e). It was decided therefore to report the findings using both possibilities. However, the same thing did not seem to be the case in Wanjala's (1996) study. This is because he rarely reported any of such examples despite the fact that there were plenty in the present study. It appeared that he assumed that, for example, those who predicted the error 5(2e) also predicted the error 10e and hence he only reported the latter.

The fact that most of the above errors predicted by more than half of the trainee-teachers and the teachers does not interfere with the suggestion that some of the student errors go unnoticed in the classroom (Nesher, 1987). For example, the error $9x^2+4$ was only predicted by about a quarter of the trainees and less than half of the teachers. This suggests that the other three-quarters of the trainees and more than half of the teachers were not aware of it. Another example is that the error 10e was predicted by about half of the trainees and the teachers. Thus, the other half of them did not seem to be aware of this error.

The most likely errors, discussed above, form only one part of the coincident errors. Table 5.16 shows all the coincident errors and compares the percentages of the trainees who predicted the errors with those of the teachers.

Table 5.16: The percentages of trainee-teachers and teachers who predicted a coincident error.

Problem	Error	Trainees	Teachers	Problem	Errors	Trainees	Teachers
N1	4a-b	52.74 (80.14)	80.6	N4	10e	32.88 (55.48)	46.3
	2a+b	2.05	4		10+e	17.12 (23.97)	9
	3a-ba	0.68	-		5×e+2	1.37 (13.01)	7.5
N2	7ab	71.23	70.1	N5	$9x^2+4$	23.97	40.3
	10ab	5.48	6		$3x^2+4$	17.81	14.9
N3	a/b	66.44 (67.81)	73.1		6x+4	14.38 (15.75)	29.9
	(ax)/(bx)	1.37	-		9x+4	5.48	9

As can be seen from Table 5.16, most of the errors predicted by the trainees were also predicted by the teachers. Only the errors 3a-ba and (ax)/(bx) were predicted by the trainees but not by the teachers. However, both of them were predicted by less than 3% of the trainees. This suggests that they might have been predicted by a few of the teachers but not reported for this reason. In fact, Wanjala (1996) told us that he did not report the errors that were predicted by less than 3% of the teachers. In the present study, however, all the predicted errors were reported.

Although trainees in the present study and teachers in Wanjala (1996) predicted most of the errors that had been committed by the students, some of these errors remained unpredicted by either or both of them. They were referred to as unnoticed errors. This finding supports the suggestion above that some of the student errors are not diagnosed in the classroom.

For the purpose of comparison, Table 5.17 shows all the unnoticed errors in both this study and Wanjala's study. The table suggests that most of the unnoticed errors occurred in N1. In addition, most of the unnoticed errors are the same for both the trainees and the teachers.

Table 5.17: The unnoticed errors in this study and Wanjala (1996)

Problem	Error	Trainees	Teachers
N1	$3ab-3a^2$	No	No
	$-3ab-3a^2$	No	No
	3ab-ba	No	No
	3-b	No	No
	3a-ba	Yes	No

Problem	Error	Trainees	Teachers
N2	7+ab	No	No
N3	(ax)/(bx)	Yes	No
N4	?	No	No
N5	Nil	-	-

- Table notes:
- No = The error was not predicted. Yes = The errors was predicted.
- ? = Different responses, all numerals, involving substituting a numerical number for the letter e in the rectangle area.

In addition to the coincident and unnoticed errors, there were many hypothetical errors that were predicted by the trainee-teachers and/or the teachers but had not been committed by the students. In all five algebra problems that have been discussed the trainee-teachers predicted more of these errors than the teachers. In numbers, they predicted 11, 12, 9, 25 and 18 errors in problems 1 to 5 respectively. In comparison, the teachers predicted only 1, 5, 1, 4 and 3 errors in the same problems respectively. However, it should be stressed that Wanjala (1996) did not report the errors which were suggested by less than 3% of the teachers. Thus, if I apply the same criterion in the present study then the trainee-teachers predicted 1, 3, 2, 9 and 4 hypothetical errors in problems 1 to 5 respectively. This seems to suggest that they predicted about the same number of hypothetical errors as the teachers in all the problems except in problem 4 (identify the area of the rectangle) in which 9 hypothetical errors predicted by the trainees but only 4 hypothetical errors predicted by the teachers. The errors predicted by the trainees include 2e×5, 2×e×5, 5×2+e, 5+2+e, 7+e, 14+2e, 5e+2, 2e+5 and 5(e+2). In comparison, the errors suggested by the teachers include 5e+2, 5(e+2), 5×2 = 10 and 2(2e+5). It can be noted that, in the first group, the prediction of the error 2×e×5 can be subsumed in the prediction of the error 2e×5 and similarly for the two errors 5+2+e and 7+e. In the second group, however, there are no errors that can be subsumed in the prediction of others. This might happen because Wanjala (1996) assumed that, for example, those who predicted 5×2+e also predicted the error 10+e and consequently he only reported the latter.

To have an idea about the hypothetical errors predicted by the trainees and/or the teachers in all cases, Table 5.18 presents and compares these errors after applying the 3% criterion.

Table 5.18: Hypothetical errors predicted by either or both of the trainees and the teachers

Problem	Error	Trainees	Teachers
N1	3a-b+a	Yes	No
	4ab	No	Yes
N2	7a+b	Yes	Yes
	7a	Yes	Yes
	7b	Yes	Yes
	7(a+b)	No	Yes
	2+5	No	Yes
N3	a/b+1	Yes	No
	a/b+x/x	Yes	No
	(a+b)/(b+1)	No	Yes
N4	2e×5	Yes	No
	2×e×5	Yes	No
	5×2+e	Yes	No
	5+2+e	Yes	No
N4	7+e	Yes	No
	14+2e	Yes	No
	5e+2	Yes	Yes
	2e+5	Yes	No
	5(e+2)	Yes	Yes
	5×2 = 10	No	Yes
	2(2e+5)	No	Yes
N5	$3x+2^2$	Yes	No
	3x+4	Yes	Yes
	$3x^2+2^2$	Yes	No
	3x+2	Yes	No
	$9x^2+12x+4$	No	Yes
	10x	No	Yes

Table 5.18 shows that there were 28 hypothetical errors that were predicted by either or both of the trainees and the teachers. Several points can be concluded from the table:

- Only six hypothetical errors were predicted by the trainees and the teachers. The remaining 22 errors were predicted by either of them: 14 by the trainees and 8 by the teachers. This can be understood in different ways as discussed in section 5.6.

- The first problem showed the least hypothetical errors and the fourth problem showed the most hypothetical errors. This may be related to the nature of the problems and the participants' experience with each one.

Relevant to the discussion in this section is the vague and the no responses. Table 5.19 shows the proportion of the vague and the no responses given by the trainees and the teachers in relation to the five algebra problems. From the table, the teachers gave more vague responses than the trainees in all the questions except in N2. This may have occurred due to the wording of the questions in both studies. In the present study, the trainees were asked to write the predicted errors in their symbolic forms rather than describing them in words. This was not the case in Wanjala's (1996) study.

Table 5.19: Percentages of vague and no responses in this study and Wanjala (1996)

	This study		Wanjala (1996)	
Problem	Vague	No response	Vague	No response
N1	6.16	8.22	10.4	1.5
N2	4.79	6.87	4.5	9.0
N3	4.11	10.27	10.4	9.0
N4	4.11	8.9	10.4	10.4
N5	2.74	9.59	9.0	14.9
Average	4.38	8.77	8.94	8.96

The average proportion of the no responses in both studies is about the same despite the fact that there were variations in this proportion across the questions. For example, only 1.5% of the teachers failed to respond in N1 compared to 8.22% of the trainees. Another example is that 14.9% of the teachers failed to respond in N5 compared to 9.59% of the trainees. One reason for this variation is that the order of the questions is not the same in Wanjala's study and in the present study.

5.7 Comparing trainee-teachers' responses and mathematical background

In this study, some information were collected about the respondents, including their gender, qualifications and institutions and the year groups which they taught, with a view to investigate possible relationships between these variables and the participants' mathematical background. To do this, the top 10% of the respondents and the bottom 10% of the respondents in each section of the questionnaire were selected. For example, those who obtained scores greater than R=0.8 (n=16) in section two and those who obtained negative correlation coefficients (n=15) in the same section were selected. Their responses to all the questions of the questionnaire were then examined to see how consistent they were across the questionnaire. Similarly in section three, those who were in the GK (good knowledge) category in all the questions (n=14) and those who were in the WK (weak knowledge) category in at least two of the questions (n=8) were also selected.

It was found that only respondents M13 and M82 were consistent in sections two and three of the questionnaire. The former obtained R = 0.84 in section two and occurred in the GK category in all the questions in section three. He was therefore in the top 10% in both sections. M82 obtained R = -0.327 in section two and she was in the WK category in two of the questions in section three. She was therefore in the bottom 10% in both sections.

Other than these two cases, no consistency was found across the respondents. Those in the top 10% in section two came from the same type of institution and possessed the same type of qualifications as those in the bottom 10% in section two. Despite being in the top 10% in section two, those respondents did not necessarily do well in section three and/or other sections of the questionnaire. In fact, many of them occurred in the bottom 10% in one or more sections of the questionnaire. The same thing can be said about the respondents in the bottom 10% in section two. They were sometimes in the top in other sections. Speaking in general, there was no consistency across respondents. This fits with other studies that have shown, for instance, that those with the best qualifications are not necessarily the ones with the greatest expertise (for example, see Askew et al, 1997).

One reason that this study did not show any relationship between the participants' institutions and their mathematical background could be that there are many variables that can affect this relationship and that these variables have not been under control in the present study. For example, trainee-teachers' subject matter knowledge affects the responses to the questionnaire. However, trainee-teachers do not acquire this type of knowledge only from their training institutions. School experience also adds to this knowledge. Thus, trainee-teachers are likely to have different backgrounds of subject matter knowledge even within the same institution. Another reason could be that because the questionnaire is about different research topics, it seems to be natural that trainee-teachers who do well in specifying order of relative difficulty may not do so well in explaining students' errors because different kinds of knowledge operate in different proportions in the two topics. This point can again be linked to the institutions in which trainee-teachers may learn about one kind of knowledge more than another kind of knowledge. The present study is not designed to investigate how and which variables affect the responses to the questionnaire. However, future studies might attempt to do this.

5.8 Summary

This chapter presents the research findings from the questionnaire survey. It suggests that the majority of the secondary mathematics trainee-teachers:

- Explain students' errors in terms of the students' incorrect application of the relevant procedures for working out algebra problems.
- Suggest a teaching strategy that involves re-teaching of the error-prone procedures to the students for the purpose of helping students overcome their difficulties.
- Identify correctly the rank-order of the algebra questions from those that particular groups of students find easiest to those that they find most difficult.
- Predict the most likely errors that students are known to make when encountering algebra in secondary school.

On the other hand, only a few trainee-teachers can:

- Identify major sources of students' errors such as using "6s" as "6 students" instead of "6 times the number of students".
- Suggest ways for helping students that challenge their thinking and make them understand their faults.
- Understand the sort of characteristics that determine the complexity level of an algebra problem such as the number of variables involved, the nature of the elements in the problem and students' interpretations of the letters.

Chapter 6

INTERVIEWS ANALYSIS AND DISCUSSION

6.1 Introduction

The previous chapter presents the research findings from the questionnaire survey. It also raises a number of issues concerning the findings. The present chapter aims to explore these issues by presenting data from a small number of interviews. The interviewees were five first-year teachers, three males and two females, from three secondary schools. Three of the teachers completed the questionnaire twice: during the interview and about eight months before the interview, i.e. when they were doing their PGCE course (see section 4.8 for more detail). As explained in section 4.9, each interview lasted about one hour and was video recorded. Appendix B shows the interview protocol. During the interviews participants did not answer all the questions that were posed in the same detail, but according to their abilities and willingness to give such detail for any one question.

6.2 Expectations of the respondents

Several assumptions have been raised in section 5.2, together with some indications from the analysis of the questionnaire to support or reject such assumptions. They aim to understand whether the reasons that the participants give for their expectations relate to Küchemann's (1981) findings concerning the characteristics that determine the complexity level of an algebra problem such as brackets, the number of variables in the problem, students' interpretations of the letters, etc. It is for this purpose that the interviews were designed.

To start with, respondents M115, M114, M107, E147 and J148 completed the questionnaire during the interview and were probed to explore their ways of thinking. Analysis of their responses suggests that the following categories of knowledge might have influenced their answers:

- Knowledge about students. This includes knowledge about students' strategies such as their interpretation of the letters in algebra or the way that they might work out a question. For example, a participant might suggest that the majority of Y9 students would work out "Multiply n+5 by 4" incorrectly as n+20 and then use this suggestion to say that this is a difficult question.

- Subject matter knowledge. The participants analyse the structure of the question and note the type of mathematical operations involved, brackets, number of terms and other inherent difficulties, to suggest the facility level of the question. Sometimes they might also compare the question with another one that they know better in order to help establish their judgement.
- Pedagogical content knowledge. The participants use their experience and beliefs about teaching and learning mathematics to make their judgements. For example, if they believe that Y9 students would have solved many examples similar to "(a-b)+b = " then they would suggest that this is an easy question. Another example is that there are many equation solving methods to solve 7-3x = 1 (Bernard et al, 1988) and depending on which method belongs to the teaching and learning experience of a participant, the question might be suggested as easy or difficult.

In the interview, respondent M115 appeared to be the most able to explain "why", then M114 and J148 and finally, M107 and E147. In fact, M107 and E147 hardly ever explained their answers without being asked questions despite the fact that they were reminded several times in the interview to think aloud.

I shall start the discussion about the strategies used by the participants by considering the following two algebra problems, followed by some other problems. The discussion is arranged according to students' interpretations of letters, as suggested by Küchemann (1981), rather than the order in which the algebra problems appear in the questionnaire.

N1 (a-b)+b =...

N2 What can you say about m if m = 3n+1 and n = 4?

Over the interview, M115 appeared consistent in his thinking about N1 and N2. Generally, in N1 he stressed the complexity of the brackets and the fact that the problem contains two different letters that cannot be evaluated. In N2, he emphasised that although the problem contains two different letters, one of them can be evaluated in terms of the other and hence the problem becomes only a matter of adding together two numerical values.

The above view is consistent, to some extent, with Küchemann's (1981) findings that *letter evaluated* applies to N2. Although the problem initially looks complex because the first equation (m = 3n+1) has two "unknowns", this complexity is resolved as soon as students reach the second equation (n = 4). Consequently, they do not need to operate on the unknown in this problem but, instead, they only need to substitute the value of n from the second equation into the first equation to identify the value of m.

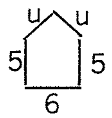
In relation to N1, Küchemann argued that this problem is quite difficult as it was answered only by 23% of the 14 year old students. He explained that the letters in the problem cannot be evaluated or used as objects. In addition, the brackets in (a-b) focus the attention in this expression, which cannot be simplified. Thus, both M115 and Küchemann identified the brackets and the inability to evaluate the letters in N1 as the inherent difficulties that make this problem very difficult.

Another participant, M107, appeared initially to be unable to explain his judgements except to suggest that *all algebra problems are difficult for Y9 students because letters are confusing.* However, later on he noticed that this strategy is not very useful because all the problems contain letters. Thus, he started to think about other ways to make better judgements. This made him change his mind toward his initial expectation in regard to N2. His new suggestion was that N2 is easier than N1 because the former involves direct substitution whereas the latter contains brackets.

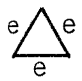
The other participants relied more heavily on their teaching experience about the extent to which students are familiar with the above questions. This is because they suggested that both N1 and N2 are easy questions because the former is about simplifying an expression and the latter is about substituting a value and even Y7 and Y8 students are familiar with such questions. Consequently, they did not see the brackets in N1 as representing a major difficulty for Y9 students. In fact, E147 and J148 argued that N1 is easier than N2 despite the fact that this contradicts Küchemann's finding.

Now consider the following problems. In regard to N6 (see below), M115 was worried that students might not see the help provided, that is, they may attempt to find an answer for the letters in a+b = 43 before they proceed to the next part of the question (a+b+2 = ?). Consequently, he suggested that this question would be answered by about half of the Y9 students. The other participants said that the letters in N6 do not need to be manipulated and can be ignored by adding on another 2. Thus, they anticipated the same idea given by Küchemann that although the problem has two unknowns, students do not need to use these letters in the sense that they can ignore them by taking the value of "a+b" from the first equation and substituting it in the second equation to get 43+2 = 45. This problem is therefore easy and was answered by 97% of the 14 year old students in Küchemann's study.

N6 If a+b = 43 then a+b+2 =...

N7 

What is the perimeter of this shape?

N5 

What is the perimeter of this shape?

N8 2a+5a =...

Although both questions N5 and N7 require the working out of a perimeter of a shape, the latter was suggested as a more difficult question than the former by all the interviewees. According to them, questions which require dealing with either numbers on their own or letters on their own are generally easier for students than other questions which require dealing with numbers and letters together within the same question. However, this idea does not seem to work for all the questions because, for example, N1 requires dealing with letters on their own but, according to Küchemann, it is a much more difficult question than N7 which requires dealing with numbers and letters. Despite this fact, respondent M114 showed no flexibility when he suggested that the above idea still apply to questions N1 and N7 in the sense that the former is easier than the latter for the same reason. However, the other participants recognised that their suggestion above has some limitations but were unable to suggest any alternative idea that could apply to a wide range of questions. Thus, all the participants seemed to be unable to suggest that the letters in the last three problems in the list above, according to Küchemann, can be regarded as names or labels rather than as unknown numbers. Thus, the letters in N7 and N5 can be used as labels for the sides and then collected in that sense rather than being regarded as the unknown lengths of the sides. In the eighth problem, the letter can be used as a shorthand for, say, an apple or simply collected as 2 a's and 3 a's are 5 a's.

Finally, consider the last two problems in this discussion:

N3 Add 4 to 3n.

N4 Multiply n+5 by 4.

According to Küchemann, the letter in both problems cannot be evaluated, ignored or used as an object. Instead, it should be used as an unknown number. In addition, the answer to the first problem 3n+4 appears to be very simple but, according to Küchemann, it is unsatisfactory to many students because of their attempt to close it by giving answers such as 7n or 7. He added that although the two problems require use of letters as unknowns, the structural complexity of N4 is harder than N3. This is because 4 needs only to be "attached" to the 3n to get 3n+4 in the former whereas in the latter it consciously needs to multiply both elements in n+5 to get the answer 4n+20. In the CSMS study, N3 and N4 were answered respectively by 36% and 17% of the 14 year old students.

In the present study, all the participants, except J148, agreed with the finding that N3 is easier than N4 for Y9 students. M115 explained his answer that N3 is difficult for students because of their inability to hold the expression 3n+4 in this form and, instead, they might attempt, for example, to evaluate the letter to obtain a numerical answer. However, he identified N4 to be even more difficult than N3 because "written in that way it is quite confusing to middle ability pupils and some of high ability". He was aware that students needs to consciously multiply 4 with both terms in N4 otherwise they would end up with incorrect answers such as 4n+5 and n+20 and that for this reason N4 is harder than N3.

In contrast with M115, M114 and E147 recognised N3 as an easy question despite the fact that they were aware of some students' errors in this problem. As suggested above, they relied quite often on their teaching experience to make their judgements and thus they also used this experience to suggest that N3 is an easy question. In N4, however, they recognised the need for multiplying a number by an expression to be the most difficult thing to deal with and that this makes it more difficult than N3. This is in contrast to their suggestion that N1 is very easy, although both questions require dealing with the idea of brackets. They seemed to see the brackets as representing a difficulty only when they are associated with multiplication as in N4. In N1, however, one can ignore the brackets without affecting the final answer.

In regard to the same questions, respondent M107 suggested that N4 is more difficult than N3 because "In N4 they don't realise that they need to multiply the whole lot by 4" but in N3 "they know that they need to add 4 to all the 3n".

Finally, respondent J148 suggested that both questions, N3 and N4, are very easy and almost all Y9 students can do them. Her reason was that both questions require only a few readings and occur within students' experience. She also thought that the fact that N4 is written in words helps students do better than if it is written with brackets.

The above discussion shows that the participants used several strategies to complete this section of the questionnaire including some of Küchemann's findings in relation to the inherent difficulties that make an algebra problem easy or difficult for students. In addition, they used their teaching experience, beliefs and the strategy of comparison between two problems. They varied in their reliance on the above strategies but generally, M115 seemed to be the most able to explain his answers scientifically by examining the structure of a problem, considering the full range of students and incorporating that with his knowledge about the National Curriculum (NC) and beliefs. M114 and J148 seemed to be more dependent on their beliefs and knowledge about the NC though sometimes they examined the structure of a problem for the purpose of identifying students' difficulties. Finally, M107 and E147 seemed to be the least able to explain why but, nevertheless, they were generally able to compare two problems together and identify the most difficult one. In this sense, they did not contradict the criterion study in most of the problems despite the fact that their expectations may be different.

Overview

Analysis from the questionnaire survey in the previous chapter revealed that the majority of the trainee-teachers obtained low scores when correlated with Küchemann's (1981) findings in relation to the facility level of the algebra problems. In this chapter a small number of interviews has been analysed to explore the reasoning used by the respondents in completing the questionnaire. This analysis supports the idea that the given expectations were based on some criteria such as beliefs, teaching experience, the structure of an algebra problem and comparison between two problems. Generally, the interviewees seemed to be unable to anticipate the deep analysis given by Küchemann in regard to the structural complexity of an algebra problem despite the fact that they understood, for example, that the introduction of brackets increases the level of difficulty for students. They were also less able to suggest that students interpret letters in algebra in different ways and use this to predict the effect on students' ability to answer algebra questions.

6.3 Order of difficulty

The main purpose of the interview in this section is to understand the reasons why the participants place algebra problems in a particular order of difficulty.

Question 1

M115, E147 and M107 agreed that i and iv should come in the first two places because they do not have brackets whereas the other two problems, ii and iii, have brackets which make them more difficult for students. However, they disagreed about the order of i and iv. Respondent M115 placed i first and then iv whereas respondents M107 and E147 placed them the other way round. Nevertheless, both orders are acceptable as they do not affect the main criterion order.

Once M115 and E147 placed i and iv in the first two places, they put ii next and iii last because, they argued, the minus sign before the bracket in iii makes it the hardest. This is in contrast with M107 who placed iii in third place and ii in the final place because, as he said, iii and ii "really look the same. I don't see any difference between the two". He suggested that iii would be more difficult than ii if it was written as (a+b)-3a instead of 3a-(b+a). He appeared not to consider that the minus sign might represent a difficulty when it comes before the bracket rather than after. Thus, M107 occurred in the MK (Moderate Knowledge) category whereas M115 and E147 occurred in the GK (Good Knowledge) category.

The other two participants, M114 and J148 placed the four problems in the order iv, ii, i, iii and ii, i, iv, iii respectively. The fact that they placed ii before i contradicts the criterion order and puts them in the MK category. Although they agreed with the criterion study that the minus sign before the bracket in iii makes this sub-question the hardest, they disagreed that the brackets in ii make this sub-question more difficult than iv and i. Their reason was that ii can be simplified to *a* without much work, i.e. by ignoring the brackets and then noting that –b and +b cancel each other out. This is in contrast with i which requires carrying out the process of simplifying and then knowing when to stop this process so that, for example, students do not simplify 4x+8y to 12xy. Respondent J148 further explained that ii is easier than iv because in ii "the letters automatically cancel each other out" whereas iv requires appreciation that, although the question is about simplifying, iv cannot be simplified.

Question 2

In the second question, M115 said that iii (a/(2(a+b)+b/(a+b)) is the easiest "because the denominators are the same, so the answer is going to be the same denominator and it's just simply a plus b". This is enough reason to place M115 in the WK (Weak Knowledge) category because, according to the criterion order, iii is the hardest.

After iii, M115 placed i, ii and iv in the second, third and fourth places respectively. His reason was that i involves direct cancellation of the two x's together whereas ii and iv involve one more step before students can cancel, that is, writing a^2 as $a \times a$. In addition, students should not ignore or forget putting 1 in the numerator after they cancel the letter in iv otherwise the answer will be wrong.

This makes it slightly harder than ii. In fact, all the three problems belong to the easiest band in the criterion study.

M115 placed vi and v in the last two places. According to him, vi is more difficult than all the above problems because students need to find a common denominator in order to simplify it. He added that v is the hardest because "you've got to add x on the top, add x on the bottom and the temptation would be simply to cross out the added x".

Another three participants, M114, J148 and E147, placed i, ii and iv in the first three spaces just like the main criterion order. Their reason was that students can obtain the correct answer in these questions simply by cancelling similar letters. Another reason was that students can easily replace the letters with numbers if they wish to experiment with their answers. The last three spaces in this order should be filled with the remaining sub-questions, that is vi, v and iii. However, to meet the criterion order, the sub-question iii should come last. Only J148 suggested this when she said that iii is the hardest because "it just looks horrible". Consequently, she occurred in the GK category. The other two participants occurred in the MK category because M114 placed iii before vi and E147 placed iii before vi and v. To M114, vi requires finding a common denominator but not iii and that makes vi the hardest. The same thing was suggested by E147 when she also placed iii before vi. In addition, she placed iii before v because she was worried that students might attempt to incorrectly cancel the two x's in v to obtain an incorrect answer.

Finally, M107 appeared to be the least able to think about the question. He changed his mind several times about the order in which the algebra problems should be placed. One reason for this is that he failed to differentiate between the cancellation of letters that leads to the correct answer and the cancellation that leads to the wrong answer. Another reason is that he appeared to think about some of the sub-questions in a way that led him to suggest that they are too hard to work out. For example, he suggested that in order to simplify a/a^2, students should think about the "negative indices", that is simplifying a/a^2 as $a^1 \times a^{-2} = a^{-1} = 1/a$ rather than writing a^2 as $a \times a$ and then do the cancellation. A third reason is that his subject matter knowledge appeared to be very weak when he simplified iii and vi incorrectly saying that he could not remember how to add together two algebraic fractions. Nevertheless, M107 occurred in the MK category when his final answer was considered. However, the interview had had an effect on his final answer otherwise he would have been placed in the WK category.

Question 3

The participants J148, M107, M114 and E147 placed iv and ii in the first two places. The first two participants said that iv is the easiest because students need only to multiply both sides of the equation by 2 to solve it but would need to divide 7 by 5 to solve ii and 7/5 is a "nasty" number because students might not accept it as a final answer and might attempt to write it as a decimal number. The other two participants argued that ii is the easiest because students tend to approach multiplication more confidently than division despite the fact that finding the answer in both sub-questions requires applying the opposite operation to the one in the question. Nevertheless, either of the sub-questions can be placed first without affecting the main criterion order.

Once J148, M107, M114 and E147 placed iv and ii in the first two places, they then put iii, v and i in the last three spaces respectively. They put iii in third place because the minus sign before the 3x and the fact that it has three terms make it more difficult than iv and ii which have only two terms. They were also convinced that v should come in the fourth place because it is the only equation that has the unknown on both sides of the equality. According to them, the sub-question i should come last because it is a fractional equation and this makes it the hardest. Those participants met the criterion order and consequently occurred in the GK category.

The last participant in this discussion is M115 who agreed with the other participants in the sense that he placed iv in the first place and v and i in the last two spaces respectively. However, M115 placed iii rather than ii in the second place. His reason was that in iii students "can look at the sum and say 7 take away 1 is 6. So, 3x is 6 and 6 divided by 3 gives you the answer." This is not the same in ii which "for the majority of pupils is difficult because 7 times something is 5 and you can't do that. ... The problem says 7 times something is 5 and it's not a whole number therefore it is hard". He was obviously thinking of a particular equation-solving method, namely, the "cover-up" (Bernard et al, 1988). In this order, M115 occurred in the MK category.

Overview

As explained in the previous chapter, the findings from the questionnaire survey indicated that most trainee-teachers were able to correctly rank-order algebra questions from those that particular groups of students find the easiest to those that they find most difficult. The reasons that participants give for placing a sub-question before or after another one were explored by conducting interviews.

Analysis of the interviews suggests that the same categories listed in section 6.2 influence the way that algebra questions are put in order of relative difficulty. These categories include subject matter knowledge, pedagogical content knowledge, knowledge about students and beliefs. There are several examples that can be taken from the above analysis to support this conclusion. One example is that respondent M107 said that students would simplify a/a^2 using the "negative indices", that is writing a/a^2 as $a^1 \times a^{-2} = a^{-1} = 1/a$. When he was asked why not apply the cancellation strategy, he said because this is how he teaches students how to simplify algebraic fractions because he thinks that this way is easier to do than writing a^2 as $a \times a$ and then applying the cancellation strategy.

6.4 Explaining students' errors

One focus of the interviews was eliciting all the possible explanations that the participants could give so that one could understand whether or not they were able to identify major sources of students' errors. Thus, they were asked, if possible, to give more than one explanation for the same error. The result of this process is discussed below in relation to the four algebra problems.

In the first algebra problem, three explanations were given for the error $6s = p$. First, M115 said that the student has multiplied the number of students by 6 instead of multiplying the number of professors by 6. This is an answer-based explanation since it does not really identify sources of the student error. Second, M115, M107, E147 and J148 suggested that the student has written the answer in the same order as the question reads, i.e. by changing the key words in the problem statement to their mathematical symbol. This is a syntactic-based explanation. Finally, M114 suggested that the problem is misleading because it "leads a lot of pupils to believe that 6 times the number of students gives the number of professors". This is a language-based explanation.

In the second algebra problem, four explanations were given for the error b+r = 90. The first one is an answer-based explanation since it only stresses that b and r are the number of pencils bought and 90 is their total price and the student has forgotten to take into account the price of each individual pencil (M114, M107, E147). The second one is a syntactic-based explanation because it suggests that the student has only translated the second sentence "I buy some blue and some red pencils and altogether it costs me 90 pence" and ignored the rest of the problem (M114, M107). The third one is a language-based explanation because it focuses on the fact that the question is a very long question and consequently by the time students reach the last sentence, they forgot about the first sentence (M114, E147, J148). The last one is a semantic-based explanation because it explains that the error means to the students that the number of blue and red pencils they bought cost them 90 pence which, to their minds, is correct (M115).

In the third algebra problem, syntactic-based and answer-based explanations were given for the errors 4n+5 and n+20. In the former, participants suggested that the errors occurred because students do not have a solid understanding of algebraic manipulation, that is, they do not master the standard algorithm for multiplying a number with an expression (or with brackets). In the latter, they only stressed that the correct answer should be 4n+20.

In the fourth algebra problem, all the participants said that students have seen that the number 246 on the left hand side of the equation n-246 = 762 has increased by 1 so they automatically added 1 to the number on the right hand side to obtain 763 instead of 761. They further suggested that the students have misused the "rule" that says if you take away a larger number from the left hand side of an equation then the outcome (the number on the right hand side) is getting smaller. This is a syntactic-based explanation.

Overview

The syntactic-based explanation is the most common one in all the interviews. In addition, respondents M115, M114 and M107, who completed the questionnaire twice, were consistent in their responses to the questions. However, the interview encouraged them to give more explanations than when they first completed the questionnaire. This is seen as a natural result because it was stressed in the interview that the participants should give all the possible explanations for every student error. However, most of the alternative explanations given in the interview either emphasised the correct answer without actually explaining the errors or duplicated other explanations that had already been given.

6.5 Addressing students' errors

In the first algebra problem, all the participants gave teaching-based suggestions for addressing the error 6s = p. These suggestions are based on the idea of generating a table of numerical values for the variables, s and p, from the problem statement and then using that to write the general statement (the answer). When they were asked to give another suggestion, two of the participants said that they did not know another way (M107 and J148). The other participants attempted to do so but visually only gave another teaching-based suggestion. For example, M115 said that he would ask the students who succeeded in getting the correct answer to explain to their classmates how they had worked out the question. Another example is that M114 suggested the strategy of making a table of values, as explained above, accompanied by a physical demonstration such as getting some students out of their seats and then asking them, for example, if the number of students is 18, how many professors there should be. In addition to the teaching-based suggestion, M115 recommended the strategy of number substitution to convince students that their answer is wrong. Another suggestion is re-wording of the problem statement (M114).

In the second algebra problem, all the participants recommended a teaching-based suggestion for addressing the error b+r = 90. This is similar to the one above but instead of generating numerical values for the number of students and professors, here they generate values for the number of blue and red pencils. As M114 put it, students should arrive at the answer in a couple of steps rather than jumping to the answer in only one step and then getting the wrong answer. As another way for addressing the error, they explained the same idea again using some sort of physical demonstrations such as two boxes of red and blue pencils in which students only know the price of each type of pencil but not how many pencils are inside each box.

An important point to report here is the fact that M115 made the same error that students do when he used b and r as objects rather than as letters representing unknown quantities of blue and red pencils. To my surprise, this teacher was the only one who explained the error b+r = 90 by suggesting that some students use letters incorrectly as objects and this makes the error sound correct to their minds. He added that the error would have meaning to students if they understand it as "I bought this number of blue pencils and this number of red pencils and it costs me 90 pence". Thus, on the one hand, he appeared to understand major sources of the error. On the other hand, he committed the same error (using letters as objects) when he came to describe how he would address the error b+r=90. He suggested asking students "if I bought one blue pencil and one red pencil, what would it cost me?" He went on to describe that he would make students write his sentence as an "equation": 1b+1r=11. Then he continued giving more examples of "equations": 2b+2r=22 and 3b+2r=27. Only when he was asked whether the letters in his examples would still mean the same as in the problem statement, he started to think and then noted that they are not. There were 6 trainee-teachers who committed the same sort of error in the questionnaire survey.

In the third algebra problem, all the participants suggested reinforcing the procedure of multiplying a number with brackets. They suggested that this should be done either in its abstract way (M115 and M114) or by using physical demonstrations (all the teachers). For example M107 said "I might say a bag with pens plus another 5 more [pens] and I would say I'm going to multiply all of this by 4, how many bags and how many pens should I have. And they should think that you should have 4 bags and 20 pens. So, you multiply the whole thing by 4". Another example is that M114 said "The other thing you can do, you reinforce the method of expanding brackets".

In the fourth algebra problem, all the teachers suggested giving numerical examples, such as 7-4 =? And 7-5 =? to make students realise that the larger the number they take away, the smaller the answer they get. If this idea did not work with all students, the participants suggested using physical demonstrations. For example, "I might use the same cubes again and use smaller numbers and actually use a number for n. So, say if I've got a row of 10 cubes and I take a way 4, how many of them left. If I've a row of 10 cubes and I take away 5, how many cubes left" (M114).

Overview

As in the case of the questionnaire survey, the most commonly given suggestion in the interviews is explaining the whole topic again to the students so that they might overcome their difficulties. Only a few of the interviewees were able to suggest other strategies that challenge students' thinking and make them aware of their mistakes. One such strategy is substituting numbers for either variable in 6s=p and then comparing the results with the problem statement to see that they do not match in meaning. Participants were sometimes able to suggest alternatives such as physical demonstrations, but only when pressed to do so.

6.6 Predicting students' errors

The participants were asked to:

- List the most likely errors and others, if relevant.
- Say whether they have observed their predicted errors.
- Give their interpretations of the errors.
- If equivalent expressions were given then they were asked to suggest which are the most likely to be committed by students.

The interviews revealed that a total of 22 errors were predicted, constituting 10 coincident errors and 12 hypothetical errors. This section concentrates more on justifying answers than listing and classifying the predicted errors.

In the first algebra problem, four errors were predicted including 3a-b+a, 4a-b, 2ab and 4ab. All the participants said that they have observed the error 4a-b.

They were aware that this error occurs when students simplify 3a-(b+a) to 3a-b+a. In fact, M115 and E147 listed the error 3a-b+a and then simplified it to 4a-b, saying that this is more likely to be committed by students. The last two predicted errors, 2ab and 4ab, were suggested by M107. He suggested that once students arrive at the error 4a-b, they would simplify it to 4ab. According to him, even if students arrived at the right answer 2a-b, they would write it as 2ab. He added that he has seen students simplifying expressions such as 2u+16 to 18u.

In the second algebra problem (2a+5b=), three errors were suggested including 7ab, 10ab and 7. The first one was predicted and observed by all the participants. They explained that the error occurs when students add all the numbers and then write down the letters. The last two errors were predicted by M114, E147 and J148. M114 put it nicely when he explained the source of the error 10ab by suggesting that when students are first introduced to addition in expressions, for example 2a+5b, they are usually able to tell that 2a cannot be added to 5b because they are unlike terms. However, when they start doing multiplication, they learn that 2a × 5b equals 10ab. Then going back to addition, they get confused and start thinking that 2a+5b equals 10ab as well. In relation to the error 7, the participants explained that this error is based on the idea of ignoring the letters when simplifying the expression.

In the third algebra problem, a/b and (ax)/(bx) were the only predicted errors. The former was suggested by all the respondents. M114 and M107 said that they have observed students making similar errors. M115, J148 and E147 said that they have not observed it but they strongly believe that students would make it. The other predicted error, (ax)/(bx), was suggested by E147. According to Matz (1980), both errors come from students' interpretation of a+x as ax and b+x as bx. Consequently, some of the students might then cancel the two x's together to obtain a/b. Alternatively, they might cancel the two x's right at the beginning without thinking of a+x and b+x as ax and bx. This last suggestion seemed to be the one that best captures the thinking of the participants as they actually cancelled the two x's, without first writing a+x and b+x as ax and bx, to show that this is what students usually do in the classroom.

In the fourth algebra problem (about the area of the rectangle in which one side is 5 and the other side is 2+e), eight errors were suggested, mostly by M115. They include 10e, 10+e, 5e+2, 15e, 5×2+e, 7+e, 7e, 14+2e and 16e. These predicted errors can be put into two groups. The first group consists of 10e, 10+e, 5×2+e, 5e+2 and 15e which occur when students correctly attempt to find the area of the rectangle rather than, for example, the perimeter. However, they were based on four different assumptions. Firstly, the error 10e occurs when students multiply 2 and 5 together and then put the letter afterward. This is because they understand that identifying the area means multiplying everything together just like the perimeter means adding everything together (Booth, 1984). This explanation was also offered by M114 who predicted the error 10e. Secondly, the error 10+e, which was predicted by M115, occurs when students simplify 5(2+e) to 10+e by multiplying 5 with only one part of the expression. Thirdly, the error 5e+2 was predicted by M114 and E147 who explained that students might ignore the brackets, thinking that they are not important. This means that they might still understand 5e+2 as 5(e+2). The same suggestion was given by M107 in regard to the error 5×2+e. An alternative suggestion is that students might multiply 5 with only one part of the expression e+2 (or 2+e) and still end up with the two errors 5×2+e and 5e+2. In this case, these two errors are produced in the same way as 10+e. In fact, both interpretations of ignoring use of brackets and multiplying 5 with one part of the expression are commonly used by students (Booth, 1984; Zehavi, 1997). Fourthly, the error 15e occurs when students correctly find the area of the rectangle as 10+5e but then simplify it to 15e.

The second group of errors consists of 7+e, 7e, 14+2e and 16e. All of them were suggested only by M115. According to him, they occur when students miss out the area altogether and, instead, attempt to find the perimeter. In this case, they either identify the whole perimeter and end up with 14+2e or part of the perimeter, by adding only the numbers and the letter actually given, and end up with 7+e. These two expressions may further by simplified incorrectly to 7e and 16e.

In the above algebra problem, most participants said that they have not observed the errors that they suggested. Only M115 said that he has observed some of the errors including 15e and 10+e. In addition, he has seen students simplifying expressions such as 14+2e to 16e and 7+e to 7e.

In the fifth and the last algebra problem (write $(3x+2)^2$ without brackets), five errors were predicted including 9x+4, $9x^2+4$, $3x+2^2$, 3x+4 and 15x+10. The first two errors were coincident and predicted only by M114, E147 and J148. They point to the fact that students would treat the expression as two separate terms in which either the numerical numbers are squared to get 9x+4 or both terms are successfully squared and then joined together to get $9x^2+4$. The next two errors, $3x+2^2$ and 3x+4, were suggested by both M115 and M107. They assumed that students would literally interpret the question. Since the question says "write without brackets", students would write the expression as the question reads to get $3x+2^2$ and then some may simplify this to 3x+4. The last error in this discussion is 15x+10, which was suggested by M115. He assumed that students would understand that $(3x+2)^2$ is 3x(3x+2) + 2(3x+2) but then simplify 3x(3x+2) to 9x+6 and 2(3x+2) to 6x+4 and then add the results together to get 15x+10. He added that "I definitely seen 3x multiplied by 3x is 9x, I definitely seen that in the classroom". The other errors, however, have not been seen by any of the interviewees because, as they suggested, this type of questions is no longer used at school.

Overview

The participants observed some of their predicted errors and were able to justify their answers. For example, they were able to explain that the error $3x+2^2$ occurs when students literally interpret the question "write $(3x+2)^2$ without brackets". Another example is that the error 15e occurs when students correctly identify the area of the rectangle of base e+2 and width 5 as 5e+10 but then simplify it incorrectly to 15e. However, such interpretations of the errors may not be realistic in that they may not reflect students' difficulties. For example, some of the participants suggested that the errors 5e+2 and 5×2+e occur when students ignore the brackets in the expression 5(e+2) because they may think that brackets are not important. However, the two errors could also be made when students multiply 5 with only one part of the expression e+2.

It appears that the majority of the explanations given by the participants for their predicted errors were syntactic in nature, but this need further investigation.

6.7 Summary

The analysis of the interview indicates that:

- The participants used several categories of knowledge and strategies to give their expectations and to rank-order algebra questions. These include pedagogical content knowledge, subject matter knowledge, knowledge of students' errors and difficulties and beliefs. These components of knowledge and beliefs may not be applied distinctively and instead, a mixture of them may be used to suggest that an algebra question is easy or difficult for particular groups of students.
- Although the participants were probed to give all the possible explanations that they know about students' errors, most of the participants explained the errors in terms of the incorrect use of the relevant procedures for working out algebra questions. Only a few of the participants correctly identified major sources of the students' errors or knew why such errors occur.
- Most of the participants suggested that students' errors can best be corrected by the re-teaching of the relevant procedures that have already been taught to the students. Only a few of them suggested ways for addressing the errors that challenge students' thinking and make them aware of their mistakes.
- Participants were normally able to give their reasons for their answers in relation to predicting students' errors. For example, they were able to explain that the predicted error 5e+2 occurs when students ignore the brackets in the expression $5 \times (e+2)$. However, this may not be the case because students can also make the error by multiplying 5 with only one part of the expression.

Chapter 7

SUMMARY, CONCLUSION AND RECOMMENDATIONS

7.1 Introduction

The aim of this chapter is to provide an overall picture of this research study into secondary mathematics trainee-teachers' knowledge of students' errors and difficulties in algebra. In addition, it attempts to indicate some of the conclusions which have emerged from this study by seeking to highlight some overall points, make an evaluation statement on the research, and provide some recommendations for future studies in the area of teacher knowledge about students' errors and difficulties.

7.2 Overview of the research area of this study

Algebra

Algebra is the most powerful tool for making generalisations in mathematics. Not only is algebra widely used across many areas of mathematics, but it also plays an important role in other subjects, from science to geography, which frequently rely on a good deal of algebra. Algebra is sometimes considered to be one of the gatekeepers to higher education. Without competence in algebra, learners cannot progress with a whole range of subjects.

Obstacles to learning algebra

While the importance of algebra is undeniable, it is widely recognised that "algebra is a source of considerable confusion and negative attitudes among pupils" (Cockcroft, 1982, p.60). As Herscovics explains, algebra is "a major stumbling block for many students in secondary school." (Herscovics, 1989, p.60). Orton and Frobisher (1996) suggest two reasons that make algebra a source of confusion and a stumbling block for students, one is the "learning difficulties" associated with learning algebra and the other is "unhelpful teaching". The former has been extensively examined by research to find that students have serious cognitive difficulties with algebra, such as the errors that they make when working out algebra problems. The latter, the development of teaching methods, is less well-documented and this study explores one aspect of that issue, namely trainee-teachers' knowledge of students' errors in algebra.

Secondary trainee-teacher knowledge about students' errors

While there are some research studies which have examined teaching methods in algebra, only a few studies have so far been conducted about teacher knowledge of students' errors and difficulties. The research that has been conducted into this element of teacher knowledge has focused on the teaching of a range of mathematical topics, including investigating teacher knowledge of students' errors and difficulties in fractions (Tirosh, 2000) and algebra (Nathan & Koedinger, 2000; Even & Markovits, 1993; Wanjala, 1996). These studies have indicated that the knowledge teachers have of student errors is sometimes inadequate and that this lack of knowledge is more apparent in the case of trainee-teachers and novices than with experienced teachers. The present study focuses on trainee-teachers because their case has been subject of less research, despite the fact that trainees form the next generation of teachers. The following research questions have been formulated for this study:

- What are trainee-teachers' expectations about students' success in solving given algebraic problems?
- How do trainee-teachers use their knowledge of students' difficulties in algebra to rank-order algebra questions?
- What explanations do trainee-teachers give for students' errors in algebra?
- What strategies do trainee-teachers use, or conceive, for tackling students' errors in algebra?
- How successful are trainee-teachers in predicting students' errors in algebra?

To answer the above questions, a national survey was undertaken in which participants were asked to explain students' errors and suggest ways for helping students who make the errors. In addition, participants were given some algebra problems and asked to predict the most likely errors that students might make when working out the problems. Participants were also asked to rank-order the problems in a way that takes into account inherent difficulties with the problems.

Finally, participants were asked to give their expectations about the proportion of Y9 students who would be able to answer each of the problems correctly.

The study was carried out by administering a questionnaire to a national sample of 251 trainee-teachers across 12 institutions. Nine of the co-operating institutions returned the questionnaire, giving a total of 159. From these, 146 questionnaires were analysed. Once this process was completed, five first-year teachers were interviewed about six months after they had started teaching in their first post. Three of the five teachers had completed the questionnaire before they graduated, i.e. during their pre-service course. The aim of the interviews is to explore the reasons that the participants have for their answers. The main findings of the study are summarised and discussed below.

Findings

This study indicates that secondary mathematics trainee-teachers are able to rank-order algebra questions in a way that takes into account the relative difficulty of the algebraic ideas that are part of the high school mathematics curriculum. For example, most can "correctly" (in relation to the criterion studies) rank-order algebra questions, from those that particular groups of students find the easiest to those they find most difficult. The trainee-teachers are also able to predict the most likely errors that secondary students are known to make when encountering algebra in secondary school.

The findings of the study also indicate that the majority of secondary mathematics trainee-teachers explain students' algebraic errors in terms of the students' incorrect application of the relevant procedures for working out algebra problems. Probably as a consequence, the teaching strategy most often recommended by trainees is the re-teaching of the error-prone procedures to the students so that the students might overcome their difficulties. Less than one-fifth of secondary mathematics trainee-teachers appear able to identify major sources of students' errors or the reason why such errors occurred. Consequently, most secondary mathematics trainee-teachers are unable to suggest ways that challenge students' thinking and make the students realise their faults in advance of the teacher attempting to add additional knowledge. Instead, trainee-teachers are likely to devote time and energy to explaining the whole topic again. Similarly, less than one fifth of trainees seem to understand the sort of characteristics that determine the complexity level of an algebra problem, such as the number of variables involved, the nature of the elements in the problem and students' possible interpretations of the letters used in algebraic expressions.

In addition to the main findings above, the teachers interviewed were able to give reasons for their answers in relation to their expectations, rank-order of algebra questions and prediction of students' errors. However, their reasons do not necessarily reflect the underlying nature of students' difficulties in algebra.

Discussion

Several factors are thought to affect trainee-teachers' actions and decision making including pedagogical content knowledge, subject matter knowledge, knowledge about students and beliefs. The relation between these factors is discussed in the model that has been developed for this study from previous research (see Chapter 3). The effect of these factors in teachers' expectations and order of difficulty has been noted and discussed with examples in the analysis of the interviews in the previous chapter. The objective now is to say why trainee-teachers were judged to be good at specifying the order of difficulty but not so good at specifying their expectations. The results of this study suggest that most trainee-teachers can correctly rank-order algebra questions but only few of them can give accurate expectations about the percentage of students who could answer given questions correctly. The question is why is this the case?

Part of the answer to the above question might be that predicting the percentage of students is cognitively more demanding than rank-ordering the problems. For example, knowing that simple equations, in which the unknown occurs on just one side of the equality, are easier for students than equations in which the unknown occurs on both sides of the equality (Sfard & Lincheviski, 1994), is enough to place $x/2 = 10$ before $4x-3 = x-11$ in their order of relative difficulty. However, this is not enough to suggest the percentage of students who would be able to solve each of the equations correctly. Instead, the participants might give a number (percentage) that summarises how easy or difficult they think each of the questions is for 13-14 year old students.

The above components of knowledge and belief may also play a significant role in determining the sequence of algebraic ideas. Nathan and Koedinger (2000) found that teachers accurately judge students' abilities when working out some problems but misjudge them on some others. For example, they found that teachers rank-order mathematics problems presented in verbal format as more difficult for students than the same problems presented in symbolic format. This is because verbal problems, according to the teachers, should be translated first to their symbolic format and then solved using *formal* (taught) methods. However, Nathan and Koedinger found that verbal problems are easier than symbolic problems because students tend to use informal methods, such as guess-and-test and unwinding (undo) methods, more with verbal problems than with symbolic problems. According to them, these informal methods show more success than formal methods. They added that these informal methods may not be known to the teachers. Instead, teachers tended to judge that symbolic problems are easier than verbal problems according to the structure of school textbooks which place symbolic problems before verbal problems.

Returning now to the present study, it is not hard to suggest why trainee-teachers did better in ordering algebra problems than in their expectations about the percentage of students who could work out a given question. The fact is that the former involved working with symbolic problems only, whereas the latter involved working with symbolic problems, word problems and other problems that contain geometrical shapes. Although the word problems are not just another format of the symbolic problems, the participants might still find it difficult to go from one form to another. Consequently, they might have done better if they were asked to give their expectations for, say, symbolic problems only. This point is worth further systematic study in the future.

The length of teaching experience that trainee-teachers had might have also has played a significant role in determining the nature of their expectations. The fact is that trainee-teachers might find it difficult to give their expectations because they had little experience of teaching 13-14 year old students.

To interpret the other results of the present study, I first need to consider the effect of the teaching grade level (elementary, middle and high school) as described by Nathan and Koedinger (2000). These researchers found that there were pronounced variations between different teachers' abilities to predict students' performance at solving mathematical problems. For example, they found that while students' informal methods were effective, many of the high school teachers did not agree with students using such methods. In addition, they found that 31% of the high school teachers disagreed or strongly disagreed that students should be encouraged to invent their own strategies. This finding is important when interpreting the results of the present study. Given that about one third of the experienced teachers may not support the use of student-invented strategies, then it is possible that trainee-teachers know little about students' informal problem-solving strategies. In fact, research results support the suggestion that experienced teachers know more about students' informal problem-solving methods than trainee teachers (Leinhardt & Smith, 1985; Borko et al, 1989; Nathan & Koedinger, 2000). This suggests that, when compared with experienced teachers, at least as many, and perhaps more, trainee teachers might stress the use of formal methods to solve algebra questions. Such a conclusion is supported by the finding of the present study that most trainee-teachers explained students' errors in terms of the incorrect application of the relevant procedures that are taught to the students in advance. The above view also corresponds to the finding that most of the trainee-teachers recommend re-teaching of the error-prone procedures to the students so that they correct their faults.

In addition to the above, this study indicates that trainee-teachers were able to predict the most likely errors that secondary students are known to make when encountering algebra. This, however, does not mean that all the trainee-teachers predicted all the most likely errors. For example, the error involving $9x^2+4$ was only predicted by about a quarter of the trainee-teachers despite the fact that it occurs so frequently in the classroom (Wanjala, 1996). In addition to the most likely errors, the trainee-teachers failed to predict some of the students' errors that occur less frequently in the classroom. This supports the suggestion that some of the students' errors remained unknown to teachers (Nesher, 1987).

7.3 Conclusion from the study

In conclusion, this study reveals that secondary mathematics trainee-teachers' knowledge in relation to explaining and addressing students' errors in algebra and their expectations about the complexity level of an algebra problem, needs further professional development and refinement. Trainee-teachers' knowledge in relation to predicting students' errors and identifying the order of difficulty seems to be more satisfactory. However, even when trainees correctly identify the order of difficulty, one that accords with what is known about student learning of algebraic ideas, trainee-teachers are less able to explain why a certain algebra problem is more complex than another one. It is therefore suggested that in order to develop a highly effective teacher, in-service training during the early stages of teaching could usefully build on secondary mathematics teachers' knowledge in relation to the investigated areas in the present study.

7.4 The implications of the present research

This section presents the contribution of this study in several categories; there are contributions to the teacher education; to the field of knowledge about students' errors and difficulties and to the field of research methodology.

The contribution to the teacher education

The results of this research suggest that the one year postgraduate teacher training course does not equip trainee-teachers with all the necessary knowledge about students' errors and difficulties. While such courses may equip trainees with knowing "pupils most common errors and misconceptions" (DfEE, 1998, 2001), further knowledge is required to be a really effective teacher. In the light of the present research results, the following points are important:

- Trainee-teachers need help to develop and refine their knowledge of major sources of students' errors. Currently, the majority of them appear either unable to explain students' errors or they interpret the errors in terms of the incorrect application of the relevant procedures for working out algebra problems.
- Trainee-teachers need help to develop their knowledge about the most effective ways of addressing students' errors instead of relying solely on addressing the errors by re-teaching the relevant procedures that have already been taught to students.

- Trainee-teachers need help to develop and refine their knowledge about inherent difficulties in the mathematics curriculum such as the sort of characteristics that make an algebra problem easier or more complex than another one. This would help them, for example, prepare better teaching plans that take into account the relative difficulty of algebraic ideas that are part of high school mathematics.
- Although this study indicated that the majority of trainee-teachers know the most likely errors that students might make when working out algebra problems, there are some trainee-teachers who are unable to predict the most likely errors. Their knowledge about the diagnostic procedures therefore needs further development and refinement so that they become more effective in predicting students' errors and other inherited difficulties that delay students' progress in learning algebra.

The contribution to the field of knowledge about students' errors

The contribution of this study to the substantive field of secondary trainee-teachers' knowledge of students' errors and difficulties in mathematics generally and algebra specifically can be highlighted through the following points:

- The present research provides valuable and relevant information about trainee-teachers' knowledge of students' errors and difficulties in algebra at a time when additional research in this area is badly needed (Brophy, 1991).
- The literature review, in Chapter 2, highlights the role of both teaching and learning difficulties in generating students' errors. In comparison, previous studies carried out literature reviews only about the learning difficulties and often ignored the teaching side on generating students' errors. The literature review of the present research therefore helps in understanding the previous research and in presenting an even clearer picture about students' errors and difficulties in algebra. This might also help future studies in the same area of research in participating more effectively and in attempting to provide a fuller and a more refined picture about the situation of teacher knowledge of students' errors.

- Chapter 3 builds a model suitable for describing trainee-teachers' knowledge of students' errors and its interaction with other components of knowledge and beliefs. In comparison, some of the previous studies did not attempt to build such a model. In fact, they did not carry out a literature review about teacher knowledge of students' errors and difficulties (e.g., Sleeman et al, 1991; Wanjala, 1996). Thus, the literature review in the present study should help future studies in participating more effectively in the field of knowledge about students' errors and difficulties.

The methodological contribution

The methodological contribution of the present research can be highlighted through the following points:

- The use of two separate methodological procedures, i.e. questionnaire and interview, whereas previous studies, except Tirosh (2000), settled for only one method which is either a questionnaire (Wanjala, 1996; Leu, 1999; Nathan & Koedinger, 2000) or interview (Sleeman, 1991; Even & Markovits, 1993) or observation (Sleeman et al, 1991).
- The development of a procedure for analysing trainee-teachers' explanations of students' errors. This procedure might be of value to future studies in the same area of research.
- The development of a procedure for analysing trainee-teachers' suggestions for addressing students' errors. This procedure can be used in future studies by other researchers that attempt to do the same investigation.
- The development of a questionnaire that can be used to measure trainee-teachers' knowledge of students' errors and difficulties in algebra such as trainee-teachers' understanding of the complexity level of an algebra problem. This questionnaire and the accompanied analysis procedures should be of help to future studies.
- The development of an interview protocol that can be used with the questionnaire to investigate trainee-teachers' knowledge of students' errors and difficulties.

The contribution to the development of theory

The final point in this section is that although the present study is not designed to test the theoretical model in Chapter 3, its results seem to support the view that trainee-teachers' beliefs about mathematics, its teaching and learning do not influence their teaching practices in the same way as for experienced teachers. As explained above, it may be the case that more trainee-teachers than experienced teachers believe that students should be encouraged to use formal methods to work out mathematics problems. This point is worth further investigation in the future.

7.5 Evaluation of the study

The questionnaire was designed to elicit trainee-teachers' knowledge in relation to trainees' expectations, order of difficulty, explaining, addressing and predicting students' errors. Thus, one strong point of the questionnaire lies in its ability to identify wide ranging information.

The questionnaire was successful in studying trainee-teachers' knowledge of students' errors and difficulties, in so far as it revealed information about:

- The extent to which trainee-teachers are able to determine the complexity level of algebra questions.
- The kind of knowledge that trainee-teachers have about order of difficulty.
- The kind of explanations that trainee-teachers use to interpret students' errors.
- The kind of strategies that trainee-teachers suggest for counteracting students' errors.
- The different errors of which trainee-teachers are aware.

Nevertheless, although this study has been carefully planned to gather a large volume of relevant data concerning trainee-teachers' knowledge of students' errors and difficulties, only a small number of interviews were undertaken because of the time limitation available for this research. The interviews proved useful in clarifying many issues related to the questionnaire survey and in gathering data that the questionnaire failed to accomplish. For example, it was able to reveal the reasons participants give for their order of relative difficulty and expectations in sections two and three of the questionnaire. I therefore recommend the development of interview procedures for eliciting more detailed information about this area of research. For example, a large sample could be interviewed, or interviews could concentrate on one or two aspects only, to obtain deeper insights into teachers' reasoning about students' errors.

The present study is dependent on some previous studies as criteria for analysing the responses on trainee-teachers' expectations, order of difficulty and predicting students' errors. I felt sometimes that these criteria need to be revised. For example, the responses in section three of the questionnaire were analysed according to the finding from Wanjala (1996) that the problem 1/(3x)+2/x is easier than the problem a/(2(a+b))+b/(2(a+b)) without knowing why this is so. However, some of the participants suggested the opposite for the reason that 1/(3x)+2/x does not have a common denominator whereas the other one already has a common denominator and it only requires adding the numerators together. Such a suggestion seems to be reasonable but needs to be tested by research. Another example is that the present study uses a hierarchy of difficulty that was established by the CSMS study to judge trainee-teachers' expectations. However, O'Reilly (1990) criticised the CSMS study for that:

- The hierarchy of difficulty is not independent of the way students are taught.
- The hierarchy of difficulty is not independent of the methodology used to determine the hierarchy.

Moreover, the CSMS study was undertaken at a time when the school mathematics curriculum was different from the present one in some significant ways.

Although the criterion studies may raise some questions about their current validly and reliability, it was not possible in the present study to establish an independent criterion within the time available to complete this research. Nevertheless, future studies might attempt to do this.

Some participants criticised the questionnaire as it being too long to complete. However, because the questionnaire covers five areas of investigation related to the research questions, it was not possible to make it any shorter without reducing its productivity. Future studies, however, should consider this by, for example, covering only some of the investigated areas, but in greater depth perhaps.

Finally, this study may be criticised for the fact that it uses the term "errors" which may not be the ideal term. For example, those who believe that students should be encouraged to invent their own problem-solving strategies may take students' errors as an indication of their attempt to establish such strategies. Consequently, they may see students' errors as unsuccessful trials rather than faults that need corrections and remedies.

7.6 Recommendations for further studies

This study opens the door for future studies that might be conducted in the field of teacher knowledge of students' errors and difficulties. For example, there is a need to conduct longitudinal studies where knowledge about students' errors and difficulties is examined at different points in time. Such studies are important to measure growth in knowledge or test the assumption that most of this knowledge is acquired during in-service rather than pre-service teaching. Currently, there are only few studies that attempted to do this by conducting cross-sectional rather than longitudinal studies (e.g., Tirosh, 2000). However, longitudinal studies are best suited for measuring constancy and change in education than any other studies (Cohen et al, 2000). The present study attempts to do this to a small extent by selecting two samples and testing each sample at a different point in time. However, the in-service sample was very small compared to the pre-service sample. In addition, the period between the two points in time may be too small for knowledge to develop to a significant level. In short, the present study is not designed to measure the change in knowledge over time. Nevertheless, it maybe useful for future studies to recall that the three participants who completed the questionnaire twice obtained higher scores for their expectations. The first time of answering, respondents M115, M114 and M107 obtained $R = 0.510$, $R = 0.284$ and $R = 0.660$ respectively. At the second point, they obtained $R = 0.762$, $R = 0.563$ and $R = 0.758$ respectively. Thus, their scores on the second occasion are considerably higher than their scores on the first occasion. Several reasons may have contributed to this result. One reason is the teaching experience that they gained in school between the two points in time. This teaching experience might have improved their expectations about students' success in working out algebra problems. The differences in their scores could also be related to the different techniques used to collect data in the first occasion (self completed questionnaire) and in the second occasion (the questionnaire completed in the interview).

Given the limitations of the present study, future longitudinal studies should consider taking a larger sample and allowing enough time for knowledge to develop before taking the second measure. They should also consider applying a successive measure at more than two points in time.

Alternatively, some researchers may choose to conduct case studies for the purpose of exploring teacher knowledge about students' errors and difficulties in more detail than the present study. Case studies of a small number of participants would also help the researchers keep the workload to a manageable level because they then do not have to analyse, for example, a large number of questionnaires.

Case studies can give different kinds of insight from those produced from surveys (Cohen et al, 2000; Wallen & Fraenkel, 2001)

This study revealed that trainee-teachers' knowledge of students' errors and difficulties needs further development and refinement. Future studies should consider the best possible way for doing this. Creating a web site can be an option for action research. Such a web site should allow the participants to communicate and share knowledge about students' errors and difficulties. In addition, it may give them access to some relevant research. This would help educate the participants about students' errors and difficulties given that one reason for research being not translated into practice is because teachers do not have access to relevant research (Kennedy, 1997; Sowder, 2000).

More specifically, the present study investigates trainee-teachers' ability to give accurate expectations and correct order of relative difficulty and to explain, address and predict students' errors. In regard to their expectations and order of difficulty, this study suggests that participants understand order of difficulty more than expectations about students' ability to answer algebra questions. This finding needs further investigation to identify the reasons behind it. Some possible reasons are given above but they should be checked by research. In addition, future studies might attempt to answer the questions:

- To what extent do subject matter knowledge, pedagogical content knowledge, beliefs and knowledge about students (e.g. knowledge of students' errors) each affect trainee-teachers' expectations and their suggested teaching sequences?
- How can teacher education best help trainee-teachers improve their expectations about students' success and order of relative difficulty?

In regard to explaining, addressing and predicting students' errors, future studies might attempt to answer the questions:

- How can trainee-teachers be trained to become better at diagnosing errors, that is, helping them to become more effective in assessing students' work and the likely difficulties that students might have in mathematics?
- How can trainee-teachers be educated to become more knowledgeable about major sources of students' errors and the best ways for dealing with students' errors?
- How do trainee-teachers use their knowledge about students' errors practically in the classroom?

The final suggestion that might be said here is that, in spite of the fact that the model of the theoretical framework in Chapter 3 has been developed for this study, this does not mean that this model has no relevance to future studies. It would be useful, however, if the model is further clarified by future research by, for example, attempting to understand where trainee-teachers' knowledge about students' errors and difficulties fits more appropriately into the model. Several suggestions have been given in the present research (see Chapter 3). However, they need to be tested by future research.

7.7 A final comment

Having reached the end of this chapter, the researcher may say that all the five research questions have been addressed and hopefully this research has been able to provide some insight about trainee-teachers' knowledge of students' errors and difficulties in algebra. It also hoped that this study has filled a gap in its area of research and that it will provide guidance for similar studies in the future.

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Appendix A

Student Teacher Questionnaire

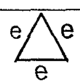
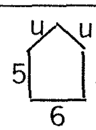
Student Teacher Questionnaire

SECTION I: BACKGROUND INFORMATION

a) Your name: ....
b) You are: 1. Male (.......) 2. Female (..........)
c) Please indicate the year groups for which you have taught algebra topics
1. Y7 ( ) 2. Y8 ( ) 3. Y9 ( ) 4. Y10 ( ) 5. Y11 ( ) 6. A-level ( )
d) Please indicate **all the qualifications** you have. For the **last four choices**, please write down the main **area of study**:
1. O-Level/GCSE ( ) 2. A-Level ( ) 3. Bachelors degree ( ) ....
4. Masters degree ( ) .... 5. Doctorate ( ) .... 6. Other (specify) ....

SECTION II: PREDICTING THE PERCENTAGE OF STUDENTS WHO COULD ANSWER CORRECTLY A GIVEN PROBLEM

Please indicate, in the appropriate box, the proportion of **Year 9 students** who you would **expect** to be able to give the answer specified.

N	Question	Required answer	Few 0-20 %	Some 20-40 %	About half 40-60 %	Most 60-80 %	Almost all >80 %
1	(a-b) + b = ...	a					
2	What can you say about m if m=3n+1 and n=4 ?	m=13					
3	Add 4 to 3n	3n+4					
4	Multiply n+5 by 4	4n+20 or 4(n+5)					
5	 What is the perimeter of this shape?	3e					
6	If a+b = 43 then a+b+2 =......	45					
7	 What is the perimeter of this shape?	2u+16					
8	2a+5a=.......	7a					

SECTION III: PUTTING PROBLEMS IN ORDER OF DIFFICULTY

In thinking about planning for a range for classes, for each of the algebra questions below, indicate, by placing the Roman numerals on the adjacent line, the order of relative difficulty of the various sub-questions for students.

Question | Order of difficulty

1. Simplify where possible

(i) $3x + 8y + x$ (ii) $(a - b) + b$
(iii) $3a - (b+a)$ (iv) $2a + 5b$

Easiest .... Hardest

2. Simplify where you can.

(i) $\frac{ax}{bx}$ (ii) $\frac{a^2}{a}$
(iii) $\frac{a}{2(a+b)} + \frac{b}{2(a+b)}$ (iv) $\frac{a}{a^2}$
(v) $\frac{a+x}{b+x}$ (vi) $\frac{1}{3x} + \frac{2}{x}$

Easiest .... Hardest

3. Solve the following.

(i) $\frac{5}{3x+7} = \frac{7}{3}$ (ii) $7x = 5$
(iii) $7-3x = 1$ (iv) $\frac{x}{2} = 10$
(v) $4x - 3 = x - 11$

Easiest .... Hardest

SECTION IV: EXPLAINING STUDENTS' ERRORS AND SUGGESTING REMEDIATION

Below are erroneous solutions to some algebraic problems. On the basis of your experience, please complete sections (a), (b) and (c) for each of the four algebra problems.

Algebra problem 1: **Write an equation using the variables s and p to represent the following statement: "There are six times as many students as professors at this university". Use s for the number of students and p for the number of professors.**

<u>Student Answer</u>: 6s = p.

(a) Identify what the student has done wrong. ....

....

(b) Suggest reason(s) which caused the student to produce this erroneous solution. ....

....

....

....

....

....

(c) Can you suggest way(s) for helping a student who makes such an error.

....

....

....

....

....

....

Algebra problem 2: **Blue pencils cost 5 pence each and red pencils cost 6 pence each. I buy some blue and some red pencils and altogether it costs me 90 pence. If b is the number of blue pencils bought and if r is the number of red pencils bought, what can you write down about b and r?**

<u>Student answer</u>: b+r = 90

(a) Identify what the student has done wrong. ....

....

(b) Suggest reason(s) which caused the student to produce this erroneous solution. ....

....

....

....

....

....

(c) Can you suggest way(s) for helping a student who makes such an error.

....

....

....

....

....

....

Algebra problem 3: **Multiply n+5 by 4**

Student answer: TWO answers are given, 4n+5 and n+20

(a) Identify what the student has done wrong. ....
....

(b) Suggest reason(s) which caused the student to produce these incorrect answers ....
....
....
....
....
....

(c) Can you suggest way(s) for helping a student who makes such errors.
....
....
....
....
....
....

Algebra problem 4: **If n – 246 = 762 then n – 247 = ...**

<u>Student answer</u>: 763

(a) Identify what the student has done wrong. ....

....

(b) Suggest reason(s) which caused the student to produce this erroneous solution ....

....

....

....

....

(c) Can you suggest way(s) for helping a student who makes such an error.

....

....

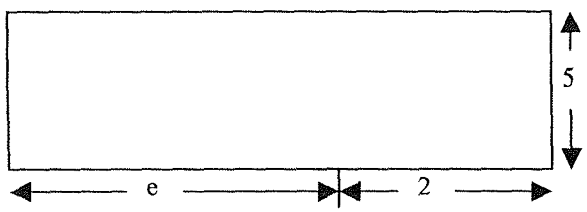
....

....

....

SECTION V: PREDICTING STUDENT ERRORS AND SUGGESTING REMEDIATION

A. Please suggest the most likely **error** (and others, if relevant) that students might make when working out **each** part of the following three questions. Please write the errors in their symbolic form rather than describing them in words:

Question	Student Error(s)
1. Simplify where possible:	
(a) 3a - (b + a)	....
(b) 2a + 5b	....
(c) $\dfrac{a+x}{b+x}$	....
2. What is the area of this shape?	
	....
3. Write $(3x + 2)^2$ without brackets	....

Optional Question

B. Please suggest **ways of helping students who make the error that you suggest is the most likely**.

Question

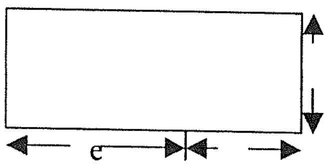
Suggestions for help

1(a) 3a - (b + a) ....

1(b) 2a + 5b ....

1(c) $\frac{a+x}{b+x}$ ....

2. What is the area of the shape? ....



3. Write $(3x + 2)^2$ without brackets ....

SECTION VI: YOUR COMMENTS

If you have any comment about any of the sections of this questionnaire, please use the space below:

....

....

....

THANK YOU VERY MUCH

Appendix B

Interview Schedule

The interview Schedule

This appendix shows the specific details about the interview protocol used when participants were considering the individual sections of the questionnaire.

Section II (expected student facility rate with algebra problems)

The main questions of the interview were the following:

- Why do you suggest that N3 would be answered by this percentage of students?
- You expect N1 to be answered by the same/smaller/larger/ number of students as N2. Why do you think N1 is harder/easier than N2?
- Why do you think that this proportion of students would answer N4 but this proportion of students would answer N5?
- Why do you think that the same/different proportion of students would answer N6 and N7?

The above questions served as common questions that were given to all the participants. They were selected because the letters in the algebraic problems of the questions involve different interpretations among students (Küchemann, 1981) and the aim was to know whether the participants take these interpretations into account when they explain their expectations.

Section III (order of difficulty)

In general, the participants were asked to give their reasons why they placed a sub-question before or after another. The following are examples of the interview questions in this section:

- Why did you place ii before/after iv and/or i in Question 1?
- Why do you think that iii is easier/more difficult than v and/or vi in Question 2?
- Why did you place iii before/after ii and/or iv in Question 3?

Section IV (explaining and addressing student errors)

The participants were only asked to talk about this section rather than completing it in writing because it requires the most writing in the entire questionnaire. The interview focused on eliciting all the possible explanations for students' errors and all the strategies that they know for addressing the errors. For example, if a participant said that the error 6s=p occurs when students literally translate the "students and professors problem", then s/he was asked to suggest other reasons, if possible.

Section 5 (predicting student errors)

The main questions are the following:

- How do you think students might interpret these errors?
- Have you observed these errors in the classroom?
- Which of these errors are the most likely?
- You suggest the error $6x + 2^2$ as the most likely in problem 5. Do you think students might also make the error 6x+4? If not, why not?