DETERMINANTS OF THE FAIR VALUE OF DEBT SUBJECT TO DEFAULT RISK

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ABSTRACT

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by Marco Realdon

ABSTRACT

This thesis contributes to the theory of the fair valuation of the firm's debt subject to default through structural models of credit risk. The focus has been on four determinants of the firm's debt value:

◊ the presence of growth opportunities embedded in the firm's assets;
◊ the lack of perfect information about the firm's assets risk;
◊ the existence of the implicit option to renegotiate and extend debt maturity;
◊ the interactions between interest rate risk and default risk.

The main results follow.

When the growth option is exercised so as to maximise equity value, debt value is often higher than when the growth option is exercised so as to maximise the growth option value. Financing the cost of new investment by new subordinated debt rather than by new equity can increase both equity value and senior debt value.

When debt holders are uncertain about the debtor's assets risk (volatility), the cost of borrowing increases and such increase can be very sensitive to the assumed default condition, to the local convexity of the debt value function and to the process the default free short rate is assumed to follow. Assuming high constant assets volatility may not be a prudent assumption when valuing subordinated and subordinated convertible debt in the presence of uncertainty about assets risk. The sensitivity of debt value to (uncertainty about) assets volatility can heavily depend on the stochastic process followed by the default free short interest rate and on the instantaneous correlation between assets value and the short rate.

When, as the firm approaches financial distress, debt maturity can be renegotiated and extended, a valuable "implicit option" to extend debt maturity is present and can significantly alter debt value (and equity value). It is shown that the value of the "extension option" is very sensitive to default conditions and to possible exercise policies. The presence of the "extension option" can increase short-term credit spreads thus improving the predictions of structural models of credit risk.

When debt value is sensitive to changes in default free interest rates as well as to changes in default risk, "interactions" between default risk and interest rate risk cause different processes for the default free short rate to imply different credit spreads on corporate bonds and different values for credit derivatives. Such "interactions" are a source of "interest rate model risk" and eliminating them seems desirable. "Interactions" can be eliminated by "separating" the modelling of default risk from the modelling of interest rate risk, which seems a convenient simplification for practical pricing purposes and allows simple closed form solutions for pricing corporate bonds.

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INTRODUCTION

This dissertation contributes to the theory of the valuation of the firm's debt subject to default risk. Generally, default risk is the risk that the debtor may breech a debt covenant. Usually default consists in the debtor missing or delaying payment of debt interest and/or principal. This thesis focuses on the debt of firms, which encompasses primarily bank loans to firms as well as corporate bonds.

In the European Union and in the United States firms' debt obligations, which are subject to default risk, are less valuable than Government bonds that promise the same cash flows, because default risk is minimum for Government bonds. Thus lending to a firm commands a "credit spread". A "credit spread" signifies the extra yield (over the default free yield of Government bonds) required by investors/lenders as compensation for bearing the risk that the debtor may default on its debt obligation.

Then the aim of this dissertation is to deal with the question: **"what is the fair value (or fair credit spread) of a firm's debt that is subject to default risk?"**

This question, as well as this dissertation, abstracts from liquidity risk and taxation regimes, which can affect debt value, too. The fundamental research question hinges on the concept of fairness. But, before defining fairness, is it necessary to motivate the fundamental research question of this dissertation.

1. THE MOTIVATION OF THE FUNDAMENTAL RESEARCH QUESTION

Many academics, regulators and the financial community face the same fundamental question considered in this thesis.

The theory of the fair value of the firm's debt, to which this thesis tries to contribute, is still developing and is the object of much ongoing academic research (see Sundaresan (2000)).

The fair value of the firm's debt is an ever more pressing problem for the financial community due to the rising volumes of corporate bonds in the US and European markets. In the US alone the Bond Market Association have estimated that outstanding corporate debt was worth 3,915.5 Billion Dollars at the beginning of the year 2002. The interest of the financial community in the fair valuation of firms' debt is stimulated also by the global growth of the market for syndicated loans, which are loans sponsored by a pool of banks. For syndicated loans as well as for other types of bank loans, active secondary markets are developing, whereby banks can exchange loans nearly in the same way as they exchange bonds. In the US and in Europe the interest in assessing the credit risk and fair value of syndicated loans is witnessed by the fact that such loans often have an official credit rating.

The fair valuation of the firm's debt also concerns regulators, as credit risk is at the heart of the new Basel Accord among the major central banks. The accord redefines the capital requirements for commercial banks and makes them dependent on the credit risk of the loans and bonds held by each bank. Thus, the new Basel accord is a potent stimulus for regulators and for banks to understand and measure the credit risk and fair value of bonds and loans.

Beyond previous motivations, there is another concern of the financial community that explains why the fair valuation of firms' debt is more of an issue than in the past: there has been a dramatic increase in the number of worldwide defaults on corporate bonds in the years 2000 and 2001. Globally the default rates on public corporate bonds have grown to record levels in the year 2000 and even worse will be the record in the year 2001. For example, in the first six months of 2001, 136 corporate issuers of public bonds defaulted worldwide, which means the payment obligations promised by bonds worth 54 billion dollars were not honoured. Moreover credit rating agencies forecast default rates to grow even more. For example, Moody's forecast global default rates for rated corporate bonds to rise to about 3.5% in early 2002, which is exceptionally high compared to the historical average default rate of 1.4% for the last twenty years (see Moody's Default Commentary for the second quarter of 2001). These figures mean that the number of defaulting corporations in a year is rising to 3.5 out of 100. Clearly such events boost the interest of the financial community in the fair valuation of debt subject to default risk.

Though, it must not be forgotten that default risk is a matter of concern not only because some rare time default actually occurs, but especially because changes in the "perceived default risk" of an issuer drive continual changes in the prices of corporate bonds. In the last few years the overall deterioration in credit quality of public corporate bonds has increased the "perceived default risk" of bonds, which is confirmed by lower credit ratings.

Finally in the European context the valuation of firms' debt is an important issue for two particular reasons. First, even if more than half of the defaults on publicly held debt are due to US issuers, European investors are exposed to the risk of such defaults since they often hold bonds issued in the US. Second, a proper market for high yield bonds has developed in Europe in the past few years, lead by huge amounts of newly issued "telecom bonds". Such bonds yield some 1,5% to 3% more than similar Government bonds and there is great interest in assessing whether such yields are fair compensation for the high default risk borne by investors.

After motivating the interest in valuing firms' debt, "fair" debt value is next defined.

2. THE FAIR VALUE OF DEBT: STRUCTURAL MODELS OF CREDIT RISK

In the context of the markets for corporate bonds and bank loans, fairness recalls some sort of equilibrium between the value or credit spread of debt and the creditworthiness of the borrower. This equilibrium has mainly been understood in two different ways, although either way presupposes that arbitrage opportunities should be ruled out of the market. Hence two different concepts of fair debt value and hence of fair credit spread prevail:

- the fair credit spread is equal to the market credit spread of other debts with the same credit rating; this is the notion of fairness underpinning some "reduced form models of credit risk";
- the fair credit spread properly reflects the debtor's (risk neutral) default probability and loss given default, which depend on the debtor's assets and leverage; this is the notion of fairness implied by the so called "structural models of credit risk".

It is the second definition of fair credit spread and debt value that guides this thesis. This choice is below motivated in many ways, but the main reason is that structural models are causal models, in other words they relate the fair value of debt to the determinants of the risk of default of the debtor: the debtor firm's assets risk and capital structure. This makes structural models more theoretically rigorous than reduced form models, since reduced form models simply believe in the fairness of ratings and market prices without "explaining" such ratings and market prices. Reduced models simply infer the default probability from observed market prices, let alone that reliable market prices are available only for a subset of debt instruments. Instead, this thesis is a theoretical enquiry about the causes of the default probability and of debt value. Hence it is the notion of fairness implied by structural models the one that underlies this whole thesis.

Other reasons motivating the structural approach

The structural approach to credit risk (also known as Contingent Claims Analysis) is the methodology common to all theoretical analyses of debt valuation in this thesis. Some secondary reasons that support the choice of the structural approach are the following.

A) Structural models do not waste information: they make use of information from the financial statements as well as from the equity and bond markets.

B) Unlike in reduced models, in structural models default is not a complete "surprise", but the consequence of the debtor progressively "approaching" default.

C) Structural models can consistently provide the entire term structure of credit spreads for any debtor firm.

D) Structural models can price the whole capital structure, even equity, in a consistent fashion. This is important since default often depends on the value of equity.

E) Since structural models relate debt and equity valuation to the economic fundamentals of the firm, equity holders' investment and financing decisions affecting such economic fundamentals affect the valuation of debt and equity too. Then structural models can measure the impact of equity holders' decisions on debt value.

F) In structural models debt valuation is a direct function of the design of debt indenture provisions (such as protective covenants), so that debt design decisions can be explicitly driven by debt valuation.

G) Structural models can value privately placed debt and debt with special terms for which there are no observed market prices.

H) Structural models have some support from empirical tests. The most recent models are capable of predicting credit spreads similar to the yield spreads on corporate debt observed in the market.

After motivating the structural approach to debt valuation, the specific research questions are reviewed and the conclusions anticipated.

3. THE FOUR RESEARCH QUESTIONS AND CONCLUSIONS

The principal four research questions aim to advance the theory of the fair valuation of debt subject to default risk and focus on four determinants of debt value:

◊ the presence of growth opportunities embedded in the firm's assets;

◊ the absence of perfect information about the firm's assets risk;

◊ the presence of an implicit option to extend debt maturity;

◊ the interactions between interest rate risk and default risk.

The four questions and conclusions of the thesis are now anticipated as follows.

1) What is the value of debt when the debtor firm has a valuable growth option?

Sub-question:

What is the value of debt when the exercise of the growth option is financed by new subordinated debt rather than by new equity?

Main results

Especially when free cash flows are low, the debt induced tax shield can lead equity holders of a levered firm to exercise the growth option earlier rather than later than they would optimally do in the absence of leverage. In such case, the leverage induced change in the growth option exercise policy increases debt value. Financing the cost of new investment by new subordinated debt rather than by new equity can benefit both equity holders and senior debt holders.

In short: ***Financing the exercise of growth opportunities through subordinated debt, rather than through equity, can be in the interest of senior debt holder.***

2) How is the value and cost of the firm's debt affected by the fact that debt holders do not know the assets volatility parameter with certainty but expect volatility to remain within a bounded range?

Main results

When debt holders are uncertain as to the debtor's assets volatility, the cost of borrowing increases and such increase is sensitive to the assumed default condition. The value of subordinated debt and subordinated convertible debt under worst-case volatility can well be lower than in the case of highest constant volatility. The sensitivity of debt value to assets volatility can heavily depend on the stochastic process followed by the default free short interest rate and on the instantaneous correlation between assets value and the short rate.

In short: ***the extra cost of debt due to uncertainty about assets volatility is significantly affected by the default barrier, by the local convexity of the debt value function and by the short rate process.***

3) What is the value of debt in the presence of an implicit option to renegotiate and extend debt maturity as the firm approaches distress?

Main results

When debt maturity can be renegotiated and extended, as observed in practice and allowed by some bankruptcy codes, a valuable "implicit option" to extend debt maturity is present. It is shown that the value of the "extension option" is very sensitive to the possible exercise policies and default conditions and that its presence can increase the short-term credit spreads of outstanding debt. This seems a partial remedy to the underestimation of short-term credit spreads that is typical of structural models.

In short: ***debt value should reflect the presence of the implicit option to extend debt maturity.***

4) What is the effect of "interactions" between interest rate risk and default risk in the valuation of bonds and credit derivatives and how to eliminate such "interactions"?

Main results

When debt value depends on stochastic default free interest rates as well as on the debtor's assets, "interactions" between default risk and interest rate risk cause different processes for the default free short rate to imply different credit spreads on corporate bonds and different values for credit derivatives. Such "interactions" are a source of "interest rate model risk" and eliminating them seems desirable. "Interactions" are eliminated by "separating" the modelling of default risk from the modelling of interest rate risk, which seems a convenient simplification for practical pricing purposes, and one which allows simple closed form solutions for pricing corporate bonds.

In short: ***"interactions" introduce excessive "interest rate model risk" in the credit risk model; default risk can be separated from interest rate risk thus eliminating "interactions", making bond valution simpler and making credit spreads independent of the interest rate model.***

After reviewing the questions and anticipating the conclusions, now the four research questions are motivated.

Justification for the research questions

The research questions are now repeated and justified. To each of the four research questions corresponds a distinct chapter in the dissertation.

Research question for chapter 2: What is the value of debt when the debtor firm has a valuable growth option?

Sub-question: What is the value of debt when the exercise of the growth option is financed by new subordinated debt rather than by new equity?

The research questions of chapter 2 are motivated by a basic observation. Often debt holders, be they banks or bond investors, know that the firm has valuable growth opportunities, which are viewed as "growth options" or "real options" to undertake profitable investments in the future. These growth options represent the capability of the firm to expand the scale of current operations and may constitute a major portion of the firm's total value.

The presence of growth options should be reflected in the fair valuation of the firm's currently outstanding debt. In general, the fact that the firm will or will not undertake profitable investments in the future and the way the firm is going to finance such investments bears on the probability of default of the debtor.

So chapter 2 compares the cases in which future investment is financed by new equity or by new subordinated debt and measures the effects of either financing policy on the value of currently outstanding debt. This is of interest since financing new investments by issuing subordinated high yield debt is quite frequent and often causes the downgrading of previously outstanding debt (see "Default and recovery rates of corporate bond issuers: 2000" by Moody's Investors Service at page 13).

Then debt valuation in the presence of growth options needs to account for the incentive for equity holders to under-invest. In other words equity holders and debt holders may have conflicting interests. Equity holders may forego profitable investments if they are to bear the full investment cost and to share investment value with debt holders. Thus in chapter 2 not only the investment decision by equity holders is endogenous in the model and affects debt value, but the presence of debt affects the investment decision.

Research question for chapter 3: How is the value of the firm's debt affected by the fact that debt holders do not know assets risk with certainty but expect assets volatility to remain within a bounded range?

Chapter 3 studies the effects on the value and cost of debt due to creditors' uncertainty about the debtor's assets risk (volatility). Structural models for debt valuation require an estimate of the volatility of the market value of the firm's total assets. But volatility is not easy to estimate and future volatility may differ from past volatility. This problem characterises also the pricing of financial options, but it seems even more acute in the pricing of medium to long-term debt through structural models.

KMV, Ericsson and Reneby (2001), Pan (2001) and others provided convincing methods to estimate assets volatility for debtors whose stock is traded in the market, but for debtors whose stock is not traded equally convincing solutions are not available due to lack of stock market data. Moreover management may substitute present assets with new and riskier assets, thus increasing assets volatility and making previously issued debt more risky.

The above reasons explain why creditors are often uncertain about the debtor's assets volatility. Then chapter 3 studies how creditors are compensated for their uncertainty about volatility as well as for the risk of default they bear. Chapter 3 envisages that creditors require a specific compensation for their uncertainty through a prudent valuation of debt in which uncertain creditors assume a prudent volatility scenario.

The fundamental assumption is that uncertain debt holders can expect the volatility parameter to remain within a bounded range, so worst-case volatility is the upper bound of the volatility range, since higher debtor's assets risk decreases debt value. *Per contra* chapter 3 shows that this is not always the case when debt is subordinated or convertible.

Chapter 3 first shows how creditors' uncertainty about assets volatility increases the cost of debt and how such increase depends on the assumed default condition. Then it is shown how the sensitivity of debt value to uncertainty in assets volatility depends on the fact that default free rates be stochastic rather than deterministic. Different models for the default free short rate are shown to imply significantly different sensitivities of debt value to (uncertainty in) assets volatility.

Research question for chapter 4: What is the value of debt in the presence of an implicit option to renegotiate and extend debt maturity?

Some recent literature has highlighted how the value of the firm' s debt depends on the possibility for the borrower and the creditors to renegotiate the debt contract (e.g. Anderson and Sundaresan (1996), Mella-Barral and Perraudin (1997), Mella-Barral (1999), Fan and Sundaresan (2001), etc…). But the important and frequent case in which debt re-negotiation (in the proximity of financial distress) entails extending debt maturity has so far been neglected by the debt valuation literature that adopts a continuous time finance approach.

In fact firms in distress do renegotiate their contractual debt obligations and often have the original contractual maturity of debt extended in order to allow the firm to overcome temporary problems (e.g. lack of liquidity). This often happens through so called distressed exchanges, which allow postponing default and the ensuing costly assets liquidation. Then chapter 4 values the firm's debt when debt holders and equity holders have an "implicit re-negotiation option" to extend debt maturity.

Research question for chapter 5: What is the effect of "interactions" between interest rate risk and default risk and how to eliminate them from the valuation of bonds and credit derivatives?

This chapter recognises that the value of the firm's debt is subject to interest rate risk as well as to default risk. Past literature has proposed structural models of credit risk in which the default free short rate is stochastic as well as the value of the firm's assets (e.g. Kim-Ramaswamy-Sundaresan (1993), Longstaff-Schwartz (1995)), but these structural models exhibit "interactions" between the default risk and interest rate risk. "Interactions" mean that different models and different parameters for the process followed by the default free short rate imply different credit spreads. Thus chapter 5 ascertains whether these "interactions" introduce excessive "interest rate model risk" in the valuation of corporate bonds and of claims contingent on bonds (credit derivatives). Then chapter 5 looks for ways to conveniently eliminate such interactions.

The specific research questions of this thesis have been motivated. Next the usefulness of this research is discussed.

4. THE USEFULNESS OF THIS RESEARCH

The theoretical analysis of this dissertation is useful for valuing publicly or privately held corporate bonds and bank loans to firms. Thus such analysis is of interest to borrowing firms, institutional investors, commercial banks and credit rating agencies.

The models can be more easily applied to the valuation of debts of public firms, for which stock market data are available. Some of the models provide realistic term structures of credit spreads (e.g. in chapter 5). In fact structural models can consistently explain the whole term structure of credit spreads for every debtor, which is a major advantage over reduced form models.

The structural models in chapters 2 and 4 not only support debt valuation, but also decision making. In chapters 2 and 4 debt value depends on endogenous decisions by equity holders of the debtor firm. The endogenous decisions by equity holders are assumed to maximise equity value. Such decisions are:

- when to exercise a growth option (chapter 2);
- when to renegotiate debt maturity (chapter 4).

Finally the models and analysis in this thesis can enhance the credit rating process.
By predicting credit spreads, the structural models in this thesis imply a credit rating for the firm's debt. Thus the predictions of fair credit spreads in this thesis can be a complement to traditional credit rating methodologies, at least in so far as credit rating is about assessing credit spreads rather than actual default probabilities. But, unlike traditional credit rating, structural models offer the benefit of predicting credit spreads even without estimating the actual default probabilities of the debtor, which means even without a statistical forecasts of the debtor's future cash flows.

Here the usefulness of this thesis has been outlined, but theoretical insights, rather than immediate practical applications, that are likely to be the main outcome of this thesis.

CHAPTER 1: LITERATURE REVIEW

INTRODUCTION

This chapter reviews the literature on contingent claims analysis and structural models of credit risk. The term "contingent claims analysis" usually denotes the earlier literature, whereas the term "structural models" usually denotes the later literature, but the two terms will be used interchangeably hereafter. Structural models value claims contingent on the firm's assets value, in particular they value debt subject to default risk. This thesis is an attempt to contribute to the literature on structural models and this chapter reviews such literature.

Unlike alternative models for valuing the firm's debt, structural models are grounded on a sound theory of the firm's default. The firm defaults when a state variable (usually is the firm's assets value) following a diffusion process reaches a lower barrier from above. This approach has at least two important consequences.

The first consequence is that the probability of default is a function of the "distance" of the state variable from the default barrier. This means that structural models are causal models of default rather than statistical models of default: structural models do not treat default as an unexplained surprise.

The second consequence is that, if some traded portfolio tracks the dynamics of the state variable, then the risk of default can be hedged by trading in the tracking portfolio. This means that the valuation of claims contingent on the firm's assets can employ the arbitrage pricing theory of financial derivatives. Then investors' risk preferences would not affect the valuations of the firm's equity and debt.

The two mentioned theoretical consequences make structural model enticing.

This chapter is organised as follows. Section 1 reviews the chronological development of structural models and tries to shed light on how such models seem to evolve. The remaining part of this chapter focuses on specific aspects of structural models, which are relevant for the analyses of the following chapters in this thesis. Such aspects are:

- the default condition, which is a stylised description of the event of default (section 2);
- the loss in debt value when default occurs (section 3);
- the role of default free interest rates (section 4);
- the valuation of the firm's debt in complete markets (section 5);
- the data required by and the empirical testing of structural models (section 6);
- the relation between debt value and investment decisions by equity holders (section 7).

Section 2 concentrates on the ways default has been modelled in the past. This is relevant since the following chapters will employ the different default conditions proposed by past literature. Different default conditions heavily affect the results in chapters 2 and 4.

Section 3 concentrates on the loss given default and the capital structure of the debtor. The following chapters will assume that the loss give default is either endogenous or exogenous to the model according to the complexity of the debtor's capital structure. These assumptions are explained in section 3.

Section 4 analyses the role of default free interest rates in structural models. The results reviewed in section 4 are the basis for the analysis of chapters 5, in which the short rate is assumed stochastic.

Section 5 justifies the arbitrage pricing and risk neutral valuation employed in contingent claims analysis and throughout this thesis in order to value the firm's debt.

Section 6 discusses the problems of estimating the parameters required by structural models and of testing the debt valuations of structural models against bond market prices. Debt holders are often uncertain about the debtor's assets volatility parameter and this problem is studied in chapter 3. The problem of having the structural model predict realistic credit spreads is faced in chapters 4 and 5.

Section 7 reviews structural models in which endogenous investment decisions by equity holders affect debt value. This problem is relevant since it characterises chapter 2.

1. CHRONOLOGICAL DEVELOPMENT OF THEORETICAL STRUCTURAL MODELS

This section examines the evolution over time of the theory of debt valuation through structural models. In my opinion the following contributions are the "backbone" of the literature:

- Black and Scholes (1973), Merton (1974, 1977), Black and Cox (1976), Geske (1977), Brennan and Schwartz (1984), Kim-Ramaswamy-Sundaresan (1993), Nielsen and Saa-Requejo and Santa Clara (1993), Leland (1994, 1996, 1998), Longstaff-Schwartz (1995), Anderson and Sundaresan (1996), Ericsson and Reneby (1998), Saa-Requeho and Santa Clara (1999), Tauren (1999), Dufresne and Goldstein (2001).

These contributions are reviewed below.

Black and Scholes (1973),

In the same article in which Black and Scholes solved the problem of pricing European options, they propose to employ their option pricing theory to value also the firm's equity and debt.

Merton (1974-1977)

Merton builds on the insight of Black and Scholes. He makes many simplifying assumptions, which allow him to prove that Modigliani and Miller's proposition I (M&M1) holds even in the presence of possible bankruptcy. M&M1, which states that total firm value does not depend on the firm's capital structure, holds even if debt is subject to the risk of default, provided default is costless (no bankruptcy costs). So Merton concludes that, if contingent claims analysis is used to value the firm's liabilities, if no arbitrage opportunities exist and if taxes and bankruptcy costs are null, the capital structure choice cannot affect total firm value. Merton also provides a closed form solution for the value of zero-coupon debt and studies the term structure of credit spreads. In 1977, he offers an alternative proof of M&M1.

Merton's analysis is theoretically important, but is suffers from a number of limitations:

- the default free short interest rate is assumed constant;
- default cannot occur before debt maturity;
- the firm's capital structure is extremely simple (the only debt is a zero coupon bond);
- the credit spreads predicted by the model are too low.

The subsequent contributions have addressed all these limitations.

Black and Cox (1976)

Black and Cox address some of the limitations of Merton's analysis. They allow default to occur as soon as the value of the firm's assets hits a default barrier from above. In this way default may take place even before debt maturity. Then they value both senior and subordinated debt.

Black and Cox conclude that issuing subordinated zero coupon debt, with maturity not shorter than the maturity of senior debt, does not harm senior debt holders. In 1993 Kim, Ramaswamy and Sundaresan prove this conclusion to be not true if subordinated debt is coupon debt rather than zero coupon debt.

Black and Cox are the first to show that, in the case of perpetual debt, the default barrier may be equal to the value of the firm's assets at which equity holders stop contributing new funds to keep the firm solvent. The full consequences of this default condition are analysed by Leland in the 1990's.

The paper by Black and Cox is relevant especially to chapter 3. In fact chapter 3 builds on the insight by Black and Cox according to which the value of subordinated debt may be a locally increasing function of the firm's assets volatility.

Geske (1977)

Geske proposes a model in which equity is viewed as compound call option and periodic debt service is financed by issuing new equity rather than by selling the firm's assets. Like in Black and Cox, default is the result of the decision of equity holders to stop contributing new funds to keep the firm solvent. Unlike in Black and Cox, Geske's analysis assumes a time dependent setting in which debt pays periodic coupons and has finite maturity. The model by Geske requires the solution of a multiple integral.

Brennan and Schwartz (1984)

Brennan and Schwartz look for optimal investment and financing policies that maximise equity value. The unique feature of this model is that the continuous changes in leverage are endogenous in the model, since equity holders control the rates of change in the book values of assets and of debt. Brennan and Schwartz then assume that:

- the stochastic factor is the return on the book value of assets;
- equity holders continuously change the book value of assets and debt so as to maximise the value of equity;
- sales of the firm's assets are forbidden by debt indentures;
- no taxes and bankruptcy costs are considered;
- all debt is due back on one future date, which implies a finite time horizon and time dependency.

Brennan and Schwartz conclude that optimal dynamic financial policy should consider not only the initial capital structure, but also the optimal debt indenture and the optimal continuous changes to the initial capital structure. This paper has been reviewed because it assumes that the face value of total debt grows exponentially and section 3 of chapter 5 makes the same assumption.

Kim, Ramaswamy and Sundaresan (KRS) (1993)

KRS value coupon bonds of finite maturity and assume default is triggered as soon as the instantaneous cash flow generated by the firm's assets is not enough to service continuous coupons payments. This means that default is triggered by a lack of liquidity (cash flow shortage) while assuming a fixed investment policy and capital structure. But in reality equity holders may have the interest and the capability to alter investment policy and capital structure in order to raise the funds necessary to remedy a temporary cash flow shortage. Other points concerning this default condition are the following:

- the model by KRS is able to produce credit spreads between corporate bonds and sovereign bonds that are closer to those empirically observed in the market than the credit spreads predicted by previous models;

- the free cash flow generated by the firm's assets need not be a constant proportion of the value of the firm's assets as assumed by KRS;
- the free cash flow generated by the firm's assets in excess of continuous coupon payments need not be all instantaneously paid out as dividends as assumed by KRS;
- when subordinated coupon bonds co-exist with senior debt in the capital structure, periodic coupon payments to subordinated bonds make senior bonds more exposed to the risk of default due to a cash flow shortage.

KRS assume that the short-term default free interest rate follows the CIR[1] (1985) process. KRS show that stochastic interest rates affect the decision of the debtor to "call" outstanding debt and that "calling" debt reduces bond holders' exposure to default risk. In fact, falling default free interest rates trigger the debt call decision by equity holders, but the call event eliminates debt holders' exposure to the firm's default risk.

The paper by KRS is relevant to chapter 5, in which the default free short interest rate is assumed to be stochastic, but also to chapters 2, 3, and 4 in which default is triggered by cash flow shortage.

Nielsen, Saa-Requejo and Santa Clara (1993)

This paper assumes that the default free interest rate follows the Vasicek process and that the default barrier follows its own stochastic process correlated with both the value of assets and the short rate. This paper tries to capture the interactions between default risk and interest rate risk. For this reason this paper is very relevant to chapter 5, which is again concerned with such interactions.

Longstaff and Schwartz (1995)

Longstaff and Schwartz propose an important model whose main features are:

- as in KRS the default free short interest rate is stochastic, although it follows the Vasicek process rather than the CIR process;
- deviations from the absolute priority rule (APR) in bankruptcy are allowed, which is often the case in reality;
- closed form solutions for bonds paying fixed or floating coupons are provided;
- the value of a coupon bond is equal to the value of a portfolio of zero-coupon bonds;
- the model allows to value single debt issues belonging to complex capital structures;

- the model predicts a negative relation between the level of the default free short interest rate and the level of credit spreads on corporate bonds.

Longstaff and Schwartz assume an exogenous and constant default barrier for the value of the firm's assets. This simplifying assumption is convenient since it allows closed form solutions to the debt valuation problem.

The loss given default is an exogenous input to the model. The model focuses on default risk rather than on both default and recovery risk, hence it does not require that all senior debt issues in the firm's capital structure be simultaneously valued. This model feature overcomes previous criticism according to which it is problematic to apply structural models when the debtor's capital structure is complex.

Other recent papers follow Longstaff and Schwartz in assuming an exogenous loss given default (e.g. Ericsson and Reneby (1998), Saa-Requeho and Santa Clara (1999), Tauren (1999), Dufresne and Goldstein (2001)).

The paper by Longstaff and Schwartz is relevant especially to chapter 5 of this thesis, where the default free short interest rate is stochastic. Unlike in Longstaff and Schwartz, the analysis of chapter 5 is not limited to the case in which the default free short interest rate follows the tractable process assumed by Vasicek.

Leland (1994), Leland and Toft (1996)

Leland writes two articles in which the valuation of the firm's debt and the choice of the optimal capital structure are simultaneously analysed. Leland provides insightful closed form solutions. In the first article the analysis is in a time independent setting whereby debt has indefinite maturity. In the second article, debt is assumed to have finite maturity and to be continuously "rolled-over" so that the nominal capital structure remains constant over time: empirical evidence about bond credit spreads, capital structures and default rates supports the predictions of this model.

Leland shows that the choice of the amount and maturity of debt involves tradeoffs between tax savings, bankruptcy costs and agency costs of debt due to the incentive for equity holders to increase assets risk. Leland manages to measure these tradeoffs. Leland views total firm value as made up of the value of assets plus the value of tax savings minus the value of bankruptcy costs. Equity is simply the difference between total firm value and debt value.

Leland (1994) highlights the impact on debt value and optimal capital structure of different default conditions. He compares the endogenous decision to default by equity holders with the

[1] Cox-Ingersoll-Ross (CIR).

exogenous default event triggered by a "positive net worth" protective covenant. In both the cases of endogenous and exogenous default, the default barrier is constant since the firm's nominal capital structure is assumed constant. This seems a restrictive assumption, which is removed in the recent papers by Tauren (1999), Dufresne and Goldstein (2001), Ericsson (2001). Moreover, Leland simplifies the analysis by assuming that the default free interest rate and assets volatility are constant.

The contribution by Leland (1994) is especially relevant to chapter 2 in this thesis. The model in chapter 2 is an extension of the model by Leland (1994) whereby the firm holds and investment real option (growth option) embedded in the firm's assets.

Leland (1998) and Ericsson (2000)

Leland (1998) and Ericsson (2000) relax the assumption of constant assets volatility. They both assume that equity holders can switch assets volatility so as to maximise equity value.

Leland (1998) studies optimal capital structure in a setting that accounts for corporate taxes, bankruptcy costs and agency costs of debt due to assets substitution. The model is time independent, in that the time variable does not appear even though debt has finite average maturity. Equity holders decide a permanent increase in the firm's assets volatility so as to maximise the value of equity. Leland attempts to capture the interaction between the choice of capital structure and the decision to increase assets volatility. This model accounts also for dynamic debt restructuring when debt is assumed to be callable.

Ericsson (2000) develops an analysis similar to that by Leland (1998) in a similar setting, but the work by Ericsson has important differences:

- debt is not callable, so no debt restructuring is considered;
- default is triggered by a cash flow shortage condition rather than by worthless equity as in Leland (1998).

This different default condition leads Ericsson to predict higher agency costs of debt due to assets risk switching than in Leland (1998). Unlike Leland, Ericsson determines not only the optimal amount of debt, but also the optimal average maturity of debt.

The analyses by Leland (1998) and Ericsson (2000) are relevant especially to chapter 3, which is concerned with the valuation of the firm's debt when debt holders are uncertain about assets volatility. Whereas Leland and Ericsson focus on the incentive for equity holders to maximised equity value by increasing the firm's assets volatility, chapter 3 focuses on the prudent valuation of debt by uncertain debt holders. Whereas Leland and Ericsson confine their analyses to the case of a permanent increase in the firm's assets volatility, chapter 3 values subordinated and convertible debt under "worst case" scenarios in which assets volatility can change an unlimited number of times.

Anderson and Sundaresan (1996)

Anderson and Sundaresan (1996) start a new family of structural models in which equity holders and debt holders can strategically renegotiate the debt contract. Anderson and Sundaresan value the firm's debt in discrete time using a binomial lattice. The important feature of their model is the endogenous determination of the default barrier, which is the result of a game played by self interested perfectly informed equity holders and debt holders. This game hinges on the fact that liquidation the firm's assets after default is costly and debt holders may want to avoid liquidation costs. Then equity holders can obtain partial debt forgiveness by threatening default and costly liquidation of the firm's assets. This new perspective allows Anderson and Sundaresan to show how the design of the debt contract affects the endogenous default decision and hence debt value. More recent contributions along this same line of strategic debt service are Mella-Barral and Perraudin (1997), Mella Barral (1999), Fan and Sundaresan (2001).

The analysis by Anderson and Sundaresan is relevant especially to chapter 4 of this thesis, in which the debt contract is renegotiated. Though, unlike in Anderson and Sundaresan, chapter 4 assumes that equity holders and debt holders renegotiate debt maturity rather than the amount of periodic coupons or of debt principal. In fact the renegotiation of debt of firms in financial distress often leads equity holders and debt holders to agree to extend debt maturity. This aspect has been so far been neglected by the structural models that allow debt renegotiation.

Ericsson and Reneby (1998)

Ericsson and Reneby (1998) propose a barrier option approach to the pricing of the firm's debt. They show how the payoffs to a portfolio of barrier options are the same as the payoffs to the firm's debt, so that the value of the portfolio of barrier options must be the same as the value of the firm's debt. This barrier option framework encompasses some previous models with constant default barrier as special cases. This framework is useful to derive closed form solutions for the firm's debt, but the assumption of constant default free interest rates seems restrictive. The assumption of constant capital structure to justify a constant default barrier may also be restrictive.

Saa-Requeho and Santa Clara (1999)

Saa-Requeho and Santa Clara (1999) try to model the dynamic evolution of the firm's nominal capital structure and propose a structural model whereby both the value of the firm's assets and the value of the firm's total debt follow diffusion processes. As in Longstaff and Schwartz (1995), the state variable capturing default risk is the ratio of assets value to the default barrier. In Saa-Requeho and Santa Clara the default barrier equals the value of total liabilities. By assuming that assets and liabilities are tradable securities, Saa-Requeho and Santa Clara can value debt while assuming the risk neutral drifts of both assets and liabilities are driven by the default free short interest rate[2]. As in other recent structural models, the loss given default is an exogenous input to the model.

Saa-Requeho and Santa Clara separate default risk from interest rate risk in the case in which the value of assets is not correlated with the default free short rate. This is relevant especially to chapter 5 in which default risk is again separated from interest rate risk.

Tauren (1999), Dufresne and Goldstein (2001)

The papers by Tauren and by Dufresne and Goldstein are other recent attempts to account for the dynamic evolution of the firm's nominal capital structure. The two papers propose structural models whereby the firm's assets value follows a diffusion process and the book value of the firm's debt follows some sort of exogenous mean reverting process, which is in accord with some empirical evidence. In these models default takes place when leverage soars too high, which again seems to correspond to empirical observations.

The credit spreads predicted by these models seem more realistic than those predicted by previously proposed models. Credit spreads are reasonably high even for relatively safe firms. The volatility of spreads is lower when leverage reverts to a long-term mean, since debt value becomes less sensitive to assets value. The spreads of short-term debt are more sensitive to the current leverage ratio, whereas the spreads of long-term debt are more sensitive to the target leverage ratio. Spreads increase with lower speed of mean reversion. Overall, allowing for mean reverting leverage seems to improve the empirical performance of structural models and accounts for the fact that firms can and do adjust their capital structures.

The work of Tauren does not allow the risk neutral probability of default to depend on the default free short rate. This seems a limitation, which Dufresne and Goldstein overcome. Dufresne and Goldstein show how credit spreads in their model are negatively correlated with the default free short rate. This negative correlation has some empirical support.

In order to retain tractability, the models by both Tauren and by Dufresne and Goldstein have the common limitation of assuming that the default free short rate follows the Vasicek process. But it

[2] The default risk-free interest rate is assumed stochastic.

may be more appropriate to assume another process for the default free short rate, such as CIR (1985) or Ahn-Gao (1999).

Over the last thirty years significant progress in the evolution of structural models has been made, but much can still been done. Here the chronological sequence of main contributions to the theory of structural models has been outlined, whereas in what follows the various contributions are analysed in more detail according to various aspects. In the next section the modelling of the default event is the aspect of interest.

2. THE DEFAULT EVENT

The condition that triggers the debtor's default is at the core of any structural model. Default may be triggered by a debt covenant being breached, by equity becoming worthless, by a cash flow shortage, by the outcome of a strategic game played by equity holders and debt holders. In any of these cases, default is triggered by (at least) one state variable that follows a stochastic process and finally reaches a threshold level (default barrier). The state variable often is the market value of the firm's assets, whereas the default barrier is a function of the amount of outstanding debt. Before default occurs, structural models can measure a sort of "distance from default". In this way default is not a surprise event as in reduced form models[3] of credit risk. In structural models the default probability is not an input as in reduced form models, but an output.

What follows reviews various model elements that bear on the event of default: state variables, stochastic processes and default barriers. Finally strategic structural models and models that assume dynamic nominal capital structures are reviewed.

2.1) Alternative state variables

Default risk may be captured by state variables that are different from the market value of the firm's assets. Such alternative state variables are discussed.

In some structural models default is triggered by the price of the firm's output (e.g. Mella-Barral and Perraudin (1997), Fries-Miller-Perraudin (1997), Mella-Barral (1999)). The price follows geometric Brownian motion, it triggers default when it hits a lower barrier, it may be observable in the markets and it may be tracked by some traded security.

In other models the state variable is given by the firm's instantaneous earnings (e.g. Wilmott

[3] A consequence of this is that the risk of default could in principle be dynamically hedged by taking appropriate short positions for example in the firm's traded stock.

(1998)) or instantaneous free cash flows, which again follow a Brownian motion. The problem with these choices is that, although earnings and cash flows are observable, they are subject to the accounting and financial policies of the firm, so that estimating adequate parameters for their stochastic process can be an arbitrary effort. In these models default would occur when equity value approaches zero. Finally, it is worth mentioning the model proposed by Cathcart and El-Jahel (1998), in which the state variable refers to a general signalling process rather than to the firm's assets value.

2.2) The stochastic process of the state variable

Usually structural models assume that the state variable follows a diffusion process, which most often is a geometric Brownian motion with constant volatility[4]. This choice seems reasonably realistic and lends tractability to the debt valuation problem.

Sometimes, the value of the firm's assets has been assumed to follow a jump process or a jump-diffusion process. Examples are Mason and Battacharya (1981), who assume a jump process for the firm's assets value, and Zhu (1997, 2001), who assumes a jump-diffusion process. Jumps are particularly suitable to account for sudden unexpected losses precipitating financial distress. Downward jumps can account for the fact that default may come as a surprise to debt holders. Jumps increase short-term credit spreads. This alleviates the notorious shortcoming of structural models predicting too low short-term credit spreads. On the other hand, the probability and magnitude of jumps is difficult to estimate.

2.3) The default barrier

The seminal structural model by Merton (1974) assumes no default barrier before debt maturity. This model implies too low credit spreads for the firm's debt. Then Black and Cox (1976) introduce a default barrier that allows default to occur also before debt maturity. Most successive structural models assume a default barrier. The default barrier may be exogenously determined by a cash flow shortage condition or by a positive net worth covenant. Alternatively the default barrier may be endogenously determined by equity holders (Leland (1994, 1996, 1998)) or by debt holders (Kim-Ramaswamy-Sundaresan (1993)). The cases in which the default barrier is endogenous and exogenous are now separately discussed.

The exogenously determined barrier may be reached when:

- the firm's assets are exhausted; this may happen when assets are sold to finance payouts to

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[4] A noteworthy exception to the constant volatility assumption is Mason (1994), in which volatility is itself assumed to follow a Brownian motion.

security holders and honour periodic debt service obligations, although this is not very realistic since the firm's assets are not fungible, not liquid, not conveniently disposable without disrupting the firm's operations; moreover protective debt covenants often restrict sales of assets (Smith and Warner 1979);

- the firm's assets value is equal to the market value of debt or a fraction of the book value of debt (Nielsen, Saa-Requeho-Santa Clara 1993);
- the firm's free cash flows are not enough to honour debt service obligations and neither new securities can be issued nor assets can be sold (Kim-Ramaswamy-Sundaresan (1993)); in this way default occurs as soon as the firm becomes "illiquid", but it seems too restrictive to assume that the firm cannot procure the needed liquidity either by issuing new securities or by altering its investment policy so as to increase free cash flows.

Setting the level of the default barrier exogenously circumvents the problem of determining the default barrier endogenously, i.e. the problem of determining when the debtor's equity becomes worthless. The endogenously determined default barrier is such that equity-holders stop contributing any more funds to help the firm meet debt service obligations. In continuous time this means that equity holders will stop contributing more funds when equity becomes worthless (Leland 1994, 1996, 1998). Alternatively, endogenous default may be triggered by periodic rather than continuous debt service obligations and equity may be viewed as a compound option (Geske (1977)).

The difference between the endogenous default events in Leland and in Geske can be expressed as follows. Whereas Leland views default as the exercise by equity holders of the option to stop contributing funds to firm, Geske views default as the equity holders' omission to exercise one of the options to keep the firm solvent. When the capital structure is complex or stationary over time, it is more convenient to view endogenous default as per Leland.

2.4) Strategic structural models

In a recent family of structural models default and assets liquidation are preceded by renegotiations of the debt contract. These models recognise that, even if the firm does not honour debt service obligations, debt holders may not find it convenient to force costly liquidation of the firm's assets. Moreover, equity holders may threaten liquidation in order to extort concessions from debt holders.

So in these "strategic structural models" the firm and its creditors can renegotiate the debt contract. Principal repayment can be partially forgiven, coupons payments can be curtailed, collateral can be pledged to "nervous" creditors, and attempts can be made to "settle out of court".

The rules of the bankruptcy code will clearly affect these re-negotiations of the debt contract. The models by Anderson and Sundaresan (1996), Mella-Barral and Perraudin (1997), Mella-Barral (1999), Fan and Sundaresan (2001) are examples of such strategic structural models.

Anderson and Sundaresan (1996) study the design of debt in a time dependent setting employing a binomial lattice. They model an optimal sequence of renegotiations under symmetric information and assume that all bargaining power lies with the equity holders.

Mella-Barral and Perraudin (1997) analyse strategic debt service along the same lines as Anderson and Sundaresan in 1996. Their model is in a time independent setting in that time is not an independent variable. This feature allows them to derive closed form solutions for debt and equity. In particular, they model the game between equity holders and debt holders. The decision to liquidate the firm's assets is endogenously determined in the model. Mella-Barral (1999) extends such analysis by exploring how different types of concessions by debt holders affect the decision by equity holders to liquidate the firm's assets.

Fan and Sundaresan (2001) analyse a time dependent setting as well as a time independent one. They consider the case in which the bargaining power is shared by equity holders and debt holders. They explore the issue of debt capacity and optimal endogenous dividend policy.

The merit of these strategic structural models is that they account for debt renegotiations, which in reality take place, and are able to explain the high credit spreads that even relatively safe corporate bonds yield in the markets. Furthermore, strategic structural models rationalise the absolute priority rule (APR) violations[5] that most often occur in bankruptcy. While strategic structural models have so far assumed that debt coupon and principal are reduced when the debt contract is renegotiated, it would be interesting to consider the quite common case in which renegotiation leads to postponing repayment of debt principal. This is the theme of chapter 4 of this thesis.

2.5) Dynamic nominal capital structure

The default event crucially depends on the total amount of the firm's debt, which may change over time. But most structural models assume that the nominal capital structure does not change. For example, some models assume that "no new securities are issued while there are outstanding claims other than stock"[6]. Other models assume that "only equity or subordinated claims are issued

[5] The absolute priority rule (APR) dictates that, during a voluntary or enforced liquidation procedure, equity holders receive just the liquidation proceeds which are left over after selling all of the firm's assets and paying all of the firm's debts. Violations to APR entail that equity holders receive some of the liquidation proceeds, even if debts have not been repaid in full.

[6] Cox and Rubinstein (1985).

to finance the payouts to security holders"[7]. In general the assumption of constant nominal capital structure is compatible with a constant default barrier, which allows closed form solutions for the firm's debt value. But assuming constant nominal capital structure seems too restrictive.

Thus there have been attempts to account for dynamic nominal capital structures and to value debt accordingly. These attempts are:

- Brennan and Schwartz (1984), who model continuous changes in leverage and account for protective debt covenants that constrain the evolution of the capital structure;
- Fisher, Heinkel and Zechner (1989), who model endogenous lumpy changes in the capital structure that are driven by the magnitude of transaction costs;
- Leland (1998), who allows restructuring by calling outstanding debt when the value of the firm's assets grows sufficiently;
- Fries, Miller and Perraudin (1997), who allow leverage to increase (but not to decrease) when the firm's output price grows to a new maximum;
- Schobel (1999), who assumes leverage to be constant in the risk neutral measure rather than in the real measure;
- Saa-Requejo and Santa Clara (1999), who allow the values of both assets and liabilities to follow correlated diffusion processes;
- Tauren (1999), who assumes that the leverage ratio follows a mean reverting process in accordance with some empirical evidence;
- Dufresne-Goldstein (2001), who assume that leverage follows a mean reverting process that depends also on the level of the default free short rate;
- Goldstein, Ju and Leland (2001), who provide closed form solutions for debt and equity when the firm has the option to increase leverage through discrete increments in the nominal amount of debt.

The papers by Brennan and Schwartz, by Tauren and by Dufresne and Goldstein have in common the fact that they allow smooth and continuous changes to the firm's nominal capital structure. These three important papers have already been commented in the previous section.

After reviewing the modelling of the default event, the attention is next turned to the losses incurred when default takes place.

3. THE LOSS GIVEN DEFAULT AND THE DEBTOR'S CAPITAL STRUCTURE

To value a firm's debt it is necessary not only to determine when default occurs, but also to

[7] E.g. Black and Cox (1976).

determine the so called "loss given default" (LGD) or the recovery value of debt upon default. Analysis of LGD in structural models depends on the complexity of the capital structure of the debtor:

- 1) some models value the aggregate debt issued by firms with relatively simple capital structures; in these models the LGD for the aggregate debt of the firm is endogenous;
- 2) some other models value single debt issues belonging to complex capital structures; in these models the LGD for the single debt issue is exogenous.

Either case is now discussed.

3.1) Simple capital structures and endogenous LGD

Assuming simple capital structures, often just equity and one class of risky debt, has allowed many models to make LGD an endogenous prediction of the model. When the LGD is endogenously determined, it depends on the following main determinants:

- the assets value at which default occurs (default barrier);
- debt seniority;
- absolute priority rule (APR) violations;
- bankruptcy and/or assets liquidation costs.

The last three determinants of LGD are now commented in turn.

Debt seniority

To determine the LGD for each single debt claim we need to consider all senior debt claims on the firm's assets. Often the capital structure is complex, as it is made up of many claims with different maturities, seniorities and covenants, so that it is hard to endogenously determine the LGD. In practice reasonable estimates of LGD can be derived from historical recovery rates for similar bonds or bank debts with the same seniority.

APR violation and liquidation costs to justify high observed yield spreads

The LGD depends on the violations to the absolute priority rule (APR) and on liquidation costs. The first generation of structural models of the Merton's type did not consider APR violations or liquidation costs, so that such structural models were not capable to reproduce the high yield spreads observed in the debt markets. Instead the more recent models have attempted to produce higher yield spreads in two ways:

- 1) by incorporating APR violations that increase the LGD suffered by debt holders,
- 2) by incorporating strategic debt service motivated by the presence of liquidation costs.

In the first way the priority structure of claims is not respected in case of default. In the second way, the threat of assets liquidation costs or of having to operate a distressed business leads debt holders to make concessions to allow the debtor to avoid default.

Although these models with simple capital structures allow to endogenously determine the LGD, the real capital structures of most firms are far more complex, with many debt classes, many debt payment obligations, many covenants. These complications cast doubts about the applicability of this family of structural models.

After this brief review of LGD in structural models with simple capital structures, now attention is turned to the case of complex capital structures.

3.2) Complex capital structures and exogenous LGD

In the attempt to overcome the limitations of structural models that assume too simple capital structures, Longstaff and Schwartz (1995) started a series of important papers in which LGD is exogenous. After Longstaff and Schwartz, several other papers took the same approach: Ericsson and Reneby (1998), Cathcart and El-Jahel, Schobel (1999), Saa-Requejo (1999), Tauren (2000), Dufresne and Goldstein (2001). When LGD is an exogenous input, debt can be easily valued even if it belongs to a complex capital structure. Actually to value single debt issues it is not necessary to value the whole capital structure of the firm, which is a major and necessary simplification for the practical applicability of structural models.

Exogenous LGD simplifies debt valuation also because it makes the value of coupon debt equal to the value of a portfolio of zero coupon bonds.

After reviewing LGD, the focus of analysis now shifts on how the literature of structural models deals with the problem of stochastic default free interest rates.

4. THE DEFAULT FREE INTEREST RATES

Default free interest rates are key to the valuation of any debt. Many proposed structural models assume that the default free short rate is constant, even if such assumption is too restrictive to value bonds, especially to value investment grade bonds as Jones-Mason-Rosenfeld (1984) first noted.

Then chapter 5 of this thesis more realistically assumes that the default free short interest rate is stochastic. This section is related to chapter 5 since various structural models are here reviewed in which the default free short interest rate follows a diffusion process, usually either the process proposed by Vasicek (1977) or by Cox-Ingersoll-Ross (1985). Examples of such structural models are the following:

- Brennan and Schwartz (1980);
- Ramaswamy and Sundaresan (1986);
- Shimko, Tejima and Van Deventer (1991);
- Kim-Ramaswamy and Sundaresan (1993);
- Longstaff and Schwartz (1995);
- Briys and De Varenne (1997);
- Cathcart and El-Jahel (1998);
- Schobel in (1999)[8].

In 1980 Brennan and Schwartz value convertible bonds allowing for stochastic default free interest rates and default risk. They provide an early attempt to account for a stochastic short rate in a structural model and use finite differences to solve the resulting differential equation for bond value.

In 1986, Ramaswamy and Sundaresan attempt to value corporate floating rate bonds.

In 1991 Shimko, Tejima and Van Deventer provide a closed form solution for the value of a zero coupon debt when the default free short rate is stochastic (as per Vasicek), but they make restrictive assumptions since there is no default barrier prior to maturity and the zero coupon debt is the only debt in the capital structure as in Merton (1974).

In 1993 KRS study callable debt when the short rate follows a diffusion process. They show the important interaction between the default free rate and the risk of default: low default free rates increase the value of debt until the firm finds it convenient to call the debt, thus eliminating debt holders' exposure to default risk.

In 1995 Longstaff and Schwartz (LS) propose an important model to value bonds with either fixed or floating coupons. Their model fits the empirically observed phenomenon of negative correlation between the yield spreads on an index of corporate bonds and the level of default free short interest rates. Longtaff and Schwartz come to a conclusion similar to Chance (1990) and Leland (1996): the duration of default risky debt is sensitively shorter than the duration of default free debt.

In 1997 Briys and De Varenne provide closed form solutions for a firm' s zero-coupon bonds when interest rates are stochastic, but they need to assume too simple a capital structure.

In 1998 Cathcart and El-Jahel provide solutions for debt values when the default free short interest rate is as per Cox-Ingersoll and Ross (CIR 1985) and the state variable triggering default is not the firm's assets value, but some other generic state variable. Since the state variable is not the firm's assets, Cathcart and El-Jahel need not assume that the risk neutral drift of the state variable depends on the short rate. This allows them to use the CIR short rate process and separate the variables (the variables being the short rate and the firm's assets value). Their article shows how the debt valuation problem becomes much simpler when the drift of the state variable triggering default does not depend on the short rate. Though, an important limitation of the article by Cathcart and El-Jahel is that they assume no correlation between the state variable and the short rate.

In 1999, Schobel provides an adjustment to the model by Longstaff and Schwartz and is able to derive simple closed form solutions for zero coupon bonds and coupon bonds, while still allowing for correlation to exist between the firm's assets value and the short interest rate. Whereas Longstaff and Schwartz assume a constant default barrier, Schobel assumes constant expected leverage in the risk neutral measure. This adjustment of Schobel shows how structural models can provide the same benefit as reduced form models, namely the value of a default risky bond would simply be equal to the value of a default risk free bond times a function of the value of the firm's assets. This beneficial property could be termed "separation of variables". Separation of variables holds even if the two variables, short rate and assets value, are correlated. Separation of variables implies that a model to value default risky bonds is just and extension of a one factor model to value default free bonds. In other words, all expertise and resources deployed to deal with market risk (interest rate risk) could be easily deployed also to deal with corporate bonds.

A limitation of all the above structural models is that they all assume either the Vasicek or the CIR short rate process for the default frees short rate. Other processes, such as the one proposed by Ahn and Gao (1999), may be more appropriate and realistic.

Finally, apart from the model of Cathcart and El-Jahel (1998), all the structural models in this

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[8] The work of Cakici Chaterjee (1996) may be added to this list.

section exhibit interactions between interest rate risk and default risk. In other words, the credit spreads predicted by the models depend on the stochastic process that the short rate is assumed to follow. This "interest-rate-model-dependence" of credit spreads needs further research. Such is the research in chapter 5 of this thesis.

An alternative approach to uncertain interest rates

Wilmott (1998) proposes an alternative approach to account for uncertain interest rates in structural models. Instead of assuming the short rate follows a given stochastic process, Wilmott assumes that the short rate is an uncertain parameter lying within a bounded range and varying with bounded speed. When the default free short interest rate is an uncertain parameter, worst-case pricing can be applied (see Wilmott (1998) chapter 40). Wilmott shows in his book that this approach may reduce the "interest-rate-model-dependence" of the structural model.

After dealing with the problem of stochastic or uncertain default free interest rates, the focus turns to the problem of justifying the valuation methodology of structural models.

5. MARKET COMPLETENESS AND RISK NEUTRALITY

Structural models usually assume either market completeness or risk-neutrality of investors. Either assumption leads to the same formulation of the debt valuation problem[9]. The complete markets assumption, which is employed also in all the following chapters of this thesis, is now discussed.

Structural models usually assume that the value of debt and equity is contingent on the value of the firm's assets. Then the firm's assets are assumed to be replicable by an observable, traded and continuously rebalanced portfolio of securities. Such assumption is equivalent to market completeness.

Ericsson and Reneby (1998) point out that, if the firm's equity is publicly traded, market completeness would not be an assumption, but an outcome of structural models. The argument of Ericsson and Reneby can be summarised as follows:

- if the firm's stock is traded, the firm's assets value can be replicated by trading in the stock and borrowing or lending at the risk free rate of interest;

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[9] The debt valuation problem is usually formulated in two equivalent ways. Debt value either is the solution to a partial differential equation or the solution to a conditional expectation with respect to risk neutral probabilities. The latter formulation goes under the name of risk neutral valuation and consists in valuing debt as the expected discounted value of the future cash flows to the debt claim, where the discount rate is the default free short rate and the expectation is with respect to risk neutral probabilities.

- if the firm's asset value can be replicated by trading in the stock, also the value of any other claim on the firm's assets value can be replicated by trading in the firm's stock.

Ericsson and Reneby conclude that the firm's debt value can be replicated by trading in the firm's stock. This allows to value debt while avoiding the problem of estimating any market price of risk associated with the firm's assets value.

This result by Ericsson and Reneby supports structural models that assume complete markets. Their result highlights that, if only the firm's equity is traded, market completeness can be assumed even if the firm's assets are not a traded security.

Instead, when the firm's equity is not traded, the market price of risk associated with the state variable (the firm's assets value) is needed. When the firm's equity is not traded, also a result by Merton (1997) can be invoked to value the firm's liabilities. Merton extends the framework for valuing contingent claims to cases in which the underlying asset is neither continuously tradable nor continuously observable.

After reviewing the assumption of market completeness and the argument by Ericsson and Reneby, some problems associated with the estimation of parameters and the empirical performance of structural models are now explored.

6. IMPLEMENTATION OF STRUCTURAL MODELS

Structural models have provided important theoretical insights, but their early implementations did not prove as useful for valuing the firm's debt. So this section discusses the main problems with the implementation of structural models, namely:

- the estimation of model parameters;
- the testing of the bond values and credit spreads predicted by the model against the market.

Both problems are discussed below.

6.1) Parameters estimation

A strength but at the same time a weakness of structural models is the data they require. Such models require few parameters and variables[10] (a strength), but such variables and parameters can be hard to observe or estimate (a weakness).

In fact structural models usually assume knowledge of the firm's assets market value (which cannot be observed) and its volatility, of the default barrier, of payouts to security holders and of correlation between assets value and the default free short rate.

The debtor's assets value can be estimated from market data such as the market value of equity (equity is a claim on the firm's assets) and from accounting data such as published financial statements. Some literature (e.g. Pan (2001)) approximates assets value as the sum of debt book value plus equity market value.

Sometimes the state variable is more simply the ratio between the firm's assets value the default barrier, a ratio that can be implied by bond market prices so that assets and default barrier need not be separately estimated. This important simplification figures in the models by Longstaff and Schwartz (1995), Cathcart and El-Jahel (1998), Tauren (1999) and Dufresne and Goldstein (2001).

The debtor's assets volatility can be estimated from time series data about the firm's traded equity. KMV Corporation, Moody's, Pan (2001) have been successful in doing so and have reported that assets volatilities are lower and more stable than the respective equity volatilities. But, there are problems with assuming constant volatility of the firm's assets.

In practice present assets volatility is often uncertain, due to difficulty in estimating assets volatility, especially when the firm's stock is not floated. Moreover, future volatility may differ from present or past volatility in an unpredictable way.

These two problems cast doubts about the assumption of known and constant assets volatility. So chapter 3 of this thesis studies the valuation of debt under uncertain assets volatility.

Having reviewed some of the problems with estimating model parameters, next the empirical performance of structural models is considered.

6.2) Empirical tests

Empirical tests have provided mixed evidence as to the capability of structural models to predict the spreads observed in the bonds and loans markets. The chronological development of the main empirical tests of structural models is now outlined.

Jones-Mason-Rosenfeld (1984)[11]

In 1984 Jones-Mason-Rosenfeld show that their structural model can explain just the values of corporate bonds that are rated below investment grade. They find that the yield spreads implied by their model are lower than the observed yield spreads, especially for high-grade bonds. Jones-Mason-Rosenfeld conclude that:

- structural models are likely to be more appropriate to value low grade debts than high grade

[10] "Reduced" models may require even fewer parameters.

[11] Noteworthy is also the method proposed by Ronn and Verma (1986) to estimate assets value and volatility from equity prices.

debts;

- for high grade debts the main source of risk are interest rates rather than default.

Sarig and Varga (1989)

In 1989 Sarig and Varga find that the observed term structures of yield spreads have shapes similar to the term structures predicted by Merton's 1974 model. Sarig and Varga find that:

- high grade bonds (low leverage firms) display positive sloping term structures of yield spreads;
- medium to low grade bonds (medium to high leverage firms) display hump shaped or downward sloping term structures of yield spreads.

Though this empirical analysis is limited to zero coupon bonds and a small sample of observations.

Titman and Torous (1989)

Timan and Torous apply a structural model to the valuation of commercial mortgages. They allow the default free short rate to follow the risk neutral process of Cox-Ingersoll and Ross (1985), and find that the mortgage valuations of the model fit empirical observations. Their model explains the time series variations in yield spreads commanded by commercial mortgages over Treasury yields. This is the most successful of the early empirical tests of structural models. Such success is probably due to the fact that valuing commercial mortgages does not require modelling the complex capital structures of the corporations that issue bonds.

Bohn (1999)

Bohn, using a very extensive data set, shows that empirical data support the qualitative predictions of structural models as to credit spreads:

- high grade bonds display upward sloping term structure of credit spreads;
- low grade bonds display hump shaped or downward sloping term structure of credit spreads;
- changes in credit quality tracked by the models account for much of the changes in observed credit spreads; this is consistent with the predictions of the theory underpinning structural models.

Anderson and Sundaresan (1996 and 2000)

In 1996 Anderson and Sundaresan show that structural models of strategic debt service in the presence of liquidation costs can produce theoretical yield spreads much closer to those observed in the market. In 2000, Anderson and Sundaresan report encouraging empirical results, according to which structural models with endogenously determined default barriers manage to explain time series variations in corporate bond yield spreads. They also highlight the importance of modelling the liquidity premium on corporate bonds and of allowing for stochastic default free interest rates when the default barrier is endogenously determined. Their encouraging conclusion is that changes in assets risk and leverage explain changes in credit spreads.

Ho, Eom, Helwege and Huang (2002)

Ho, Eom, Helwege and Huang compare the empirical performance of four models: Merton's 1974 one, Geske's 1977 one, Longstaff and Schwartz's 1995 one and Leland and Toft's 1996 one. The first two models are shown to understate yield spreads, whereas the two latter ones (in particular Leland and Toft's) are shown to overstate yield spreads. These researchers suggest how allowing for payouts to security holders and correlation between assets value and the short rate can improve the empirical performance of structural models (yield spreads predictions improve).

Ericsson and Reneby (2001)

In 2001 Ericsson and Reneby propose a structural model that allows for increasing nominal debt over time. They use equity data to estimate model parameters by maximum likelihood methods and report that the credit spreads predicted by their model are unbiased predictors of observed credit spreads. Their model predicts changes in credit spreads highly correlated with the changes in observed credit spreads. They also suggest that the poor empirical performance of other structural models may be due to the way in which model parameters have been estimated in past studies. They argue that, if maximum likelihood methods are employed to estimate model parameters, structural models should better predict observed credit spreads. This study provides the best empirical results so far and the maximum likely-hood approach seems the most promising for future estimation of model parameters.

In 2000, Sundaresan summarises the main empirical phenomena that structural models cannot explain. The observed yield spreads between firms' debt and similar default-risk-free debt are higher than the spreads predicted by structural models, especially for short debt maturities (under 1 or 2 years). Structural models fail, at least as yet, to model the re-negotiation of the debt contract between the firm and its creditors, which is aimed at avoiding costly bankruptcy as the firm approaches financial distress. Finally, structural models fail to model the observed negative relation[12] between default-free interest rates and credit spreads. Overcoming these problems is a challenge for future research.

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[12] Empirical evidence about the presence and significance of this negative relation is not conclusive.

In this section we have discussed the practical problems that characterise the application of structural models by reviewing the parameter estimation and empirical test of the models. The empirical tests inspire some modelling choices in the following chapters, especially the choice to assume a default barrier before debt maturity and, in chapter 5, the choice to assume a stochastic default free short rate. In agreement with the reviewed empirical evidence, in chapters 3 and 5 hump shaped term structures of credit spreads are generated by the models.

Next the attention turns to a particular family of structural models in which investment decisions are endogenous.

7 ENDOGENOUS INVESTMENT DECISIONS AND DEBT VALUE

This section reviews the literature of structural models in which endogenous investment decisions by equity holders, or by management acting in the best interests of equity holders, affect the values of debt and equity. This literature is relevant to chapters 2 and 3, which explore how the exercise of growth opportunities and changes in assets risk by equity holders can affect debt value.

Particularly in the past ten years, structural models have been proposed in which investment decisions by equity holders are endogenously determined as a function of the firm's capital structure (amount of debt, maturity of debt, covenants of debt). Thus in such models the investment policy is not taken as given and Modigliani and Miller's proposition I cannot hold: capital structure affects investment decisions, which in turn affect total firm value, debt value and equity value.

These models can quantify the agency costs of debt, often measured as the decrease in total firm value due to the fact that equity holders decide to invest so as to maximise equity value rather than total assets value. Endogenous investment decisions mainly concern:

- when to invest, i.e. when to exercise a growth option (Mauer and Ott, 1996);
- when to suspend or restart productive activity (Mello and Parsons, 1992);
- when to liquidate the firm's assets (Mella-Barral and Perraudin 1997, Mella-Barral 1999);
- when to increase the firm's assets risk (Leland 1998, Ericsson 2000).

These models are now reviewed in more detail, but first their relation with Modigliani and Miller's proposition I is highlighted.

Structural models and Modigliani and Miller's proposition I (M&M 1)

Merton (1977) showed that M&M 1 can hold even when debt and equity are valued as claims contingent on the firm's assets and even if the firm's debt is exposed to the risk of bankruptcy.

Then, assuming no taxes and bankruptcy costs, total firm value does not depend upon of the firm's capital structure (amount and maturity of debt).

Thereafter structural models could be classified into those that assume Modigliani and Miller's proposition 1 holds and those that do not. Assuming that proposition I holds is a simplification. In practice proposition I does not hold for a number of reasons, in particular because:

- a) bankruptcy costs as well as debt induced tax savings do exist;
- b) the investment policy depends on the firm's capital structure.

These are the reasons why M&M 1 does not hold in the structural models reviewed below.

Agency costs of debt due to sub-optimal operating decisions

In Mello and Parsons (1992) the value of the firm depends on operating decisions (to resume production, to suspend production, to abandon the business), which are in turn affected by the capital structure. The model by Mello and Parsons can quantify the agency costs of debt, i.e. the loss in firm value due to sub-optimal operating decisions induced by the presence of debt. Actually, when debt is present equity holders tend to choose operating policies so as to maximise equity value rather than total firm value.

Agency costs of debt[13] due to under-investment

Mauer and Ott (1996-2000) analyse debt-induced under-investment of the type first studied by Myers in 1977. The agency cost of debt in this setting is a consequence of the sub-optimal investment policy equity holders would select when the firm is levered. Sub-optimal investment policy is again due to the fact that the investment decision is not taken so as to maximise the value of the firm, but the value of equity.

Mauer and Ott's model is in a time independent setting in that time is not an independent variable in their model. This simplification allows them to find closed form expressions for debt and equity and to quantify the effect of debt induced under-investment. They confirm under-investment occurs when equity holders bear the full cost of the investment, but share the benefits of investing with pre-existing debt holders. Mauer and Ott suggest that debt financing of the new investment mitigates under-investment. Their study is similar to the analysis in chapter 2 of this thesis.

[13] Trigeorgis (1993) carries out a somewhat similar analysis but in a setting in which time is an independent variable.

Agency costs of debt due to assets substitution

Recently two important papers, one by Leland (1998) and one by Ericsson (2000), analyse the agency costs of debt due to assets substitution and how such costs affect the optimal capital structure. The new feature of the two papers by Leland and Ericsson is that assets risk is endogenously increased so as to maximise equity value. The capital structure in place is shown to affect the decision to increase assets risk. These papers have already been reviewed in section 1.

The structural models reviewed in this section show how debt value depends on endogenous investment decisions by equity holders. These models enhance investment decisions, financing decisions, debt valuation and allow important insights for creditors and debtors to better design debt contracts. Next general conclusions regarding the state of the art of structural models are drawn.

CONCLUSION

This chapter has reviewed a selection of the most significant contributions to the theory of structural models of credit risk. Such review has lead to the following conclusions, which inspire the subsequent chapters of this thesis.

The first conclusion is that the stream of papers started by Longstaff-Schwartz (1995) has overcome the problem of valuing bonds that belong to complex capital structures. The key point is to separate the problem of determining the debtor's risk-neutral probability of default from the problem of determining the loss given default to be incurred on each bond. This approach is used in chapter 5.

The second conclusion is that the most recent research is trying to model default risk while relaxing the restrictive assumption of constant nominal capital structure. In chapters 2 and 5 such restrictive assumption is relaxed.

The third conclusion is that the structural models that allow for debt contracts to be renegotiated have not allowed debt maturity to be extended. This seems an important limitation and is addressed in chapter 4.

The fourth conclusion is that valuing the firm's debt under the market completeness or risk neutrality assumptions seems adequate especially when the debtor's equity is floated. In some cases in chapters 3 and 5 the market completeness assumption is dropped.

The fifth conclusion is that structural models exhibit interactions between default risk and interest rate risk. These interactions need further research and chapter 5 moves along this direction.

The sixth conclusion is that recently important progress has been made in the implementation (estimation and testing) of structural models.

Unlike reduced form models, which require bond market data, structural models mainly require equity market data. Estimation of structural parameters from equity market data is progressing significantly, mainly through the work of KMV and Ericsson and Reneby, but it remains problematic when the debtor's equity is not traded. The recent empirical tests of structural models have provided encouraging results.

The seventh conclusion is that some structural models allow assessing how investment decisions by equity holders impact debt value. Chapter 2 proposes one such structural model. Moreover structural models can measure the agency costs of debt and determine the optimal capital structure by trading off debt induced tax shields, bankruptcy costs and agency costs of debt.

After reviewing the main literature about the theory of structural models in this chapter, the following chapters contribute to develop selected aspects of such theory.

CHAPTER 2:

DEBT VALUE AND THE DEBTOR'S GROWTH OPTIONS

INTRODUCTION

This chapter is a theoretical analysis of how the firm's growth options affect the value of the firm's debt. Closed form formulas are derived for debt and other claims. The analysis shows the following main points.

- Corporation taxes can lead equity holders of a levered firm to exercise a growth option (i.e. invest) earlier (rather than later) than they would do in the absence of leverage. When the cost of the exercise of the growth option is financed by new equity, investing earlier makes debt more valuable.
- Financing the cost of the exercise of the growth option by new subordinated debt, rather than by new equity, can increase the value of senior debt already issued.
- Reducing debt average maturity reduces the credit spreads on senior debt both when investment is financed by new equity and when investment is financed by new subordinated debt.

The setting is a levered firm that holds a growth option[14] and has some assets in place, too. The growth option is the discretionary right of the firm to expand the scale of current operations. When the firm exercises the growth option, the value of the firm's assets in place increases substantially. The firm can finance the cost of exercising the growth option by issuing either new equity or new debt that is subordinated to previously issued debt.

This chapter is organised as follows. First the more directly relevant literature is compared to the analysis presented here. Then, in section 1 the basic model is developed: debt has indefinite maturity and the focus is on the classic case in which investment is financed by a new equity issue. Section 2 extends the analysis of section 1 to the case in which debt has finite average maturity. Section 3 concerns the case in which investment is financed by issuing new subordinated coupon debt instead of new equity. Then the main conclusions follow.

THE LITERATURE

The literature directly relevant to this analysis is that on structural models of credit risk and that on real options. In this literature two theoretical issues appear to which the present analysis attempts to contribute:

- a) the issue of valuing risky debt when the debtor's nominal capital structure changes over time; in the present analysis financing the cost of new investment either through new equity or new debt changes the capital structure;

- b) the issue of growth options held by a levered firm rather than by an un-levered firm; there exist interaction effects between investment policy and capital structure.

The real options literature has devoted attention to the interactions between growth options and capital structure: see mainly Myers (1977), Trigeorgis (1993), Mauer and Ott (1996 and 2000). Myers (1977) showed how debt can lead to under-investment and how under-investment decreases the total value of the firm and the value of equity. Trigeorgis (1993) developed a somewhat similar analysis to stress the interactions between the firm's growth options and the "equity default option".

Both Myers' and Trigeorgis' analyses are time dependent, in that the growth option is exercised at a pre-set time before debt maturity. On the other hand, the model in this chapter is time independent.

Myers' (1977) analysis and many related papers assume new investment is to be financed by issuance of new equity. On the other hand, the analysis below encompasses also the case of new investment financed by issuance of new subordinated debt.

The literature on structural models of credit risk often simply assumes that the firm's investment policy is given and is independent of the firm's capital structure. These assumptions may be too restrictive and some attempts to relax them are found in Mello and Parson (1992), in Leland (1998), in which the firm can increase assets volatility, and in Mauer and Ott (1996-2000), in which the agency costs of debt due to Myers' under-investment are measured.

Thus the analysis below is a further attempt to make the investment policy endogenous in the structural model of credit risk. This chapter is similar to Mauer and Ott's work in that it considers a levered firm with an option to expand the scale of operations. The main differences from Mauer and Ott's work are the following ones:

- here attention is devoted to the optimal investment policies of the levered and of the un-levered firm, rather than on the difference between the equity value maximising and the firm value maximising investment policies of a levered firm;
- the growth option is here assumed to be lost upon default and bankruptcy;
- default is triggered by a lack of liquidity rather than by worthless equity;
- the cases are considered in which investment is financed by issuing subordinated debt;
- the issue of optimal capital structure is not addressed.

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[14] The terms growth "option" and growth "opportunity" have the same meaning in this work.

The model in this chapter is an extension of Leland's 1994 model. Leland derived closed form solutions for claims contingent on the firm's assets and derived important results on risky debt, capital structure, endogenous bankruptcy and protective debt covenants. The analysis in this chapter expands Leland's model by including an investment real option (i.e. a growth option) similar to the one modelled by McDonald and Siegel (1986). The growth option is embedded in the firm's assets, but is lost if the firm defaults before investing. As in Leland, the present model is time independent and allows closed form solutions for the values of all claims contingent on the value of the firm's assets in place.

1. WHEN INVESTMENT IS FINANCED BY ISSUING NEW EQUITY

In this section the basic model is developed. Debt value is shown to depend on the value and exercise policy of the growth option, while investment cost is financed by new equity. Debt D(V), equity E(V), debt induced future tax savings T(V), bankruptcy costs BC(V), the growth option Ro(V) are all functions of the value of the firm's "assets in place" (V). In particular T(V) is the discounted expected value of the flow of possible future tax savings due to the fact that debt coupons are a cost deductible from the firm's taxable income. BC(V) is the value of possible future bankruptcy costs to be incurred when default takes place.

The model hinges on the exercise of the growth option (investment), which takes place when V rises to the upper value V*. V* will be endogenously determined later on and is supposed to be set by equity holders so as to maximise equity value.

The functions $T^a(V), BC^a(V), D^a(V), E^a(V)$ depend on V ante investment, which means before V rises to V*. These functions are derived below. $T^p(V), BC^p(V), D^p(V), E^p(V)$ are functions of V post investment, which means after V has risen to V* and the growth option has been exercised. Such functions are known from Leland (1994).

Before investment the market value balance sheet identity is

1) $E^a = T^a - BC^a + V + Ro - D^a$.

V follows a diffusion stochastic process with constant volatility. Equation 1) includes Ro(V) as distinct from the value of the firm's assets in place (V). Ro(V) can be thought of as an opportunity to expand the present scale of the business by a given percentage (q). Ro(V) is like a perpetual American call option (e.g. see McDonald and Siegel 1986) and the value of its underlying asset is a constant fraction (q) of V. Thus the value of the growth option changes over time as V changes over time. The investment cost (I), which is the call option exercise price, is a constant and is financed by issuing fairly priced new debt or new equity.

After investment the market value balance sheet identity is

2) $E^p = T^p - BC^p + V^p \cdot (1+q) - D^p$.

So, whereas before investment the firm has assets worth V and plus a growth option worth Ro(V), after investment the firm has assets worth (1+q) V.

When V = V* investment takes place and the following conditions must hold:

3.1) $q \cdot V^* - I = Ro(V^*)$, which is the payoff to the firm from the exercise of the growth option;

3.2) $Eq^p(V^*) - I = Eq^a(V^*)$, which is the payoff to equity holders from the exercise of the growth option;

3.3) $T^p(V^*) = T^a(V^*)$, which states that, when the growth option is exercised, the value of the tax shield before investment "converges" to the value of the tax shield after investment;

3.4) $BC^p(V^*) = BC^a(V^*)$, which states that, when the growth option is exercised, the value of expected bankruptcy costs before investment "converges" to the value of expected bankruptcy costs after investment;

3.5) $D^p(V^*) = D^a(V^*)$, which is the payoff to debt holders upon exercise of the growth option by equity holders.

The above conditions imply that:

$$Ro(V^*) + I + V^* + T^a(V^*) - BC^a(V^*) = D^a(V^*) + Eq^a(V^*) + I =$$

$$V^* \cdot (1+q) + T^p(V^*) - BC^p(V^*) = D^p(V^*) + Eq^p(V^*).$$

Before investment default is triggered when V drops to the default level V_B^a. The boundary conditions for default before investment are:

4.1) $Ro(V_B^a) = RoK = 0$

(RoK is the recovery value of the growth option upon default and is set equal to 0 in this chapter);

4.2) $E^a(V_B^a) = 0$;

4.3) $T^a(V_B^a) = 0$;

4.4) $BC^a(V_B^a) = a \cdot V_B^a$, where "a" denotes bankruptcy costs expressed as a percentage of assets value upon default;

4.5) $D^a(V_B^a) = (1-a) \cdot (V_B^a + RoK)$;

4.6) $Ro(V_B^a)+V_B^a+T^a(V_B^a)-BC^a(V_B^a)=Debt^a(V_B^a)+Eq^a(V_B^a)$.

K^a and K^p denote the levels of the default barrier respectively ante and post exercise of the growth opportunity:

5.1) $K^a = RoK + V_B^a$ ($V_B^a = K^a$ since RoK = 0);

5.2) $K^p = V_B^p \cdot (1+q)$,

with V_B^p being the level of V triggering default post investment.

The value of the firm's "assets in place" (V) follows the "risk neutral process" $\partial V = V \cdot (r-d) \cdot \partial t + s \cdot V \cdot \partial z$, in which ∂z is the increment of a Wiener process, "s" is the volatility parameter, "r" is the default risk free short interest rate (which is assumed constant) and "d" is the instantaneous income generated by the firm's assets. Since all and only current income is distributed to security holders, "d" coincides with the assets pay-out rate.

When the firm exercises its growth option, it increases the scale of its current operations by a fraction "q". Thus "qV" is the increase in the value of assets in place due to exercising Ro(V) and the payoff to the growth option is Ro(V*) = q V* – I. Assuming that "q" is constant is the usual assumption of the real options literature when valuing options to expand the scale of a project or business. After exercise of the growth option, the value of the firm's total assets follows the risk neutral process $\partial(V(1+q)) = V(1+q) \cdot (r-d) \cdot \partial t + s \cdot V(1+q) \cdot \partial z$.

Default conditions

This chapter considers mainly the case in which default is triggered by a liquidity constraint, thus:

5.3) $V_B^a = \dfrac{C \cdot (1-t) + m \cdot P}{d + m \cdot (1-a)}$ before investment and $V_B^p = \dfrac{C \cdot (1-t) + m \cdot P}{[d + m \cdot (1-a)] \cdot (1+q)}$ after investment,

where "t" denotes the corporate income tax rate.

Conditions 5.3) are similar to the cash flow shortage condition of Ericsson (1993) and imply that default occurs as soon as the instantaneous assets payout cannot cover interest charges net of tax savings. As in Ericsson (2000), every year a fraction (m) of outstanding debt is continuously retired at par value (m P) and substituted by an equal nominal amount of new debt issued at market value [m $D^a(V)$]. If the difference [m $D^a(V)$-m P] is negative, it is a cash net outflow to be financed by the assets generated cash flow (Vd). If [m $D^a(V)$-m P] is positive, it is a cash net inflow to be distributed to equity holders. C and P are constant over time, so that the value of total debt [$D^a(V)$] is time independent.

Before proceeding with the model, a list of assumptions follows.

1.1) List of main assumptions

By delaying investment, the firm foregoes an instantaneous assets pay-out of " $d \cdot V \cdot q \cdot \partial t$ ", which is the opportunity cost of delaying investment for a short period (dt) as opposed to investing immediately. The assumption that all and only current income is distributed is consistent with assuming a constant cost (I) for an investment whose gross value is a constant proportion of assets value (qV).

The other main assumptions follow:

- markets are perfect, frictionless and complete, continuous trading is possible at all times;
- the payment of coupons on debt is continuous at an yearly rate of C and is financed by the assets generated cash flow "d"; dividends are distributed to equity holders at a rate equal to (Vd – C) dt;
- the face value of debt (P) and debt coupons are constant;
- in this section investment cost (I) is financed by issuing new equity.

1.2) The model

After investment the model becomes the same as per Leland (1994), so the functions $T^p(V), BC^p(V), D^p(V), E^p(V)$ are already known from Leland. Then the problem is finding V* and the functions $T^a(V), BC^a(V), D^a(V), E^a(V)$. These are derived through solving the following ordinary differential equations, whose solutions are reported in Appendix II.

Employing standard arguments for the valuation of contingent claims (e.g. see Appendix I), the function D(V) before investment can be shown to satisfy the following ODE:

$$6)\ \frac{1}{2} D^a_{VV} \cdot V^2 \cdot s^2 + (r-d) \cdot D^a_V \cdot V - r \cdot D^a + C + m \cdot (P - D^a) = 0$$

subject to

$$6.1)\ D^a\left(V^a_B\right) = \min\left[(1-a) \cdot V^a_B, P\right]$$

$$6.2)\ D^a\left(V^*\right) = \frac{C+mP}{r+m} + \left[-\frac{C+mP}{r+m} + \min\left[(1-a) \cdot K^p, P\right]\right]\left[\frac{V^*}{V^p_B}\right]^{bb^-_p}.$$

with

$$bb_p^- = \frac{-\left(r-d-\frac{s^2}{2}\right)-\sqrt{\left(r-d-\frac{s^2}{2}\right)^2+2\cdot(r+m)\cdot s^2}}{s^2}$$

and with $V_B^a = \frac{P\cdot m + C\cdot(1-t)}{d+(1-a)m} = K^a$ [15], with $V_B^p = V_B^a \cdot \frac{1}{(1+q)} = \frac{K^p}{(1+q)}$,

where V* denotes the optimal investment policy.

Condition 6.1) follows from condition 4.5), whereas condition 6.2) is a consequence of 3.5) and of assuming that the function $D^p(V)$ is as per Leland.

The function T(V) before investment satisfies the following ODE:

$$7)\ \frac{1}{2}T_{VV}^a \cdot V^2 s^2 + (r-d)\cdot T_V^a \cdot V - r\cdot T^a + t\cdot C = 0,$$

subject to

$$7.1)\ T^a(V_B^a) = 0$$

$$7.2)\ T^a(V^*) = \frac{t\cdot C}{r} - \left(\frac{t\cdot C}{r}\right)\cdot\left(\frac{V^*}{V_B^p}\right)^{b_p^-}, \text{ where}$$

$$b_p^- = \frac{-\left(r-d-\frac{s^2}{2}\right)-\sqrt{\left(r-d-\frac{s^2}{2}\right)^2+2\cdot r\cdot s^2}}{s^2}.$$

Condition 7.1) follows from condition 4.3), whereas condition 7.2) is a consequence of 3.3) and of assuming the function $T^p(V)$ is as per Leland.

The function BC(V) before investment satisfies the following ODE:

$$8)\ \frac{1}{2}BC_{VV}^a \cdot V^2 s^2 + (r-d)\cdot BC_V^a \cdot V - r\cdot BC^a = 0,$$

subject to:

$$8.1)\ BC^a\left(V_B^a\right) = a\cdot V_B^a$$

$$8.2)\ BC^a(V^*) = a\cdot K^p \cdot\left(\frac{V^*}{V_B^p}\right)^{b_p^-}.$$

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[15] This condition is the same as in Ericsson (2000) and implies that default is again triggered by lack of liquidity.

Condition 8.1) follows from condition 4.4), whereas condition 8.2) is a consequence of 3.4) and of assuming the function $BC^p(V)$ is as per Leland.

Before investment the function Ro(V), which is like a perpetual American call option, satisfies

9) $\frac{1}{2}Ro_{VV} \cdot V^2 s^2 + (r-d)Ro_V \cdot V - r \cdot Ro = 0$,

with boundary conditions

9.1) $Ro(V_B^a) = 0$

9.2) $p \cdot V^* - I = Ro(V^*)$.

Condition 9.1) follows from condition 4.1), whereas condition 9.2) is the payoff to the growth option. The important feature is that V* maximises $E^a(V)$ rather than Ro(V), because equity holders, or management acting in the equity holders' best interest, will want to choose V* so as to maximise the value of equity rather than the value of the growth option. If and only if the firm has no debt, the V* maximising equity value maximises also the value of the growth option.

Once $T^a(V), BC^a(V), D^a(V)$ and Ro(V) have been found, $E^a(V)$ is also found if only we invoke equation 1). $E^a(V)$ can be shown to satisfy the following ODE:

10) $\frac{1}{2} E^a_{VV} \cdot V^2 \cdot s^2 + (r-d) \cdot E^a_V \cdot V - r \cdot E^a + d \cdot V - C \cdot (1-t) + m \cdot (D^a - P) = 0$,

with boundary conditions

10.1) $E^a\left(V_B^a\right) = 0$

10.2)

$$E^a(V^*) = V^* \cdot (1+p) + t \cdot \frac{C}{r}\left[1 - \left(\frac{V^*}{V_B^p}\right)^{\bar{b}_p}\right] - a \cdot K^p \cdot \left(\frac{V^*}{V_B^p}\right)^{\bar{b}_p} +$$

$$-\left\{\frac{C+mP}{r+m} + \left[-\frac{C+mP}{r+m} + \min\left[P, K^p \cdot (1-a)\right]\right]\left[\frac{V^*}{V_B^p}\right]^{\bar{bb}_p}\right\} - I$$

10.3) $\left[E^a(V, V^*)\right]_{V^*} = 0$.

Condition 10.1) follows from condition 5.4), whereas condition 10.2) is a consequence of condition 3.2) and of assuming that equity after investment, $E^p(V)$, is as per Leland (1994). Condition 10.3) is the "smooth pasting" condition and ensures that V* maximises equity value. V* is found numerically.

Predictions of the model

In the following part of this section and in the next sections, various conclusions are discussed by analysing a selection of scenarios. Such conclusions have been confirmed also in a number of unreported scenarios. For simplicity, in this section debt maturity is assumed to be indefinite (m=0), so that debt can be thought of as either a perpetual claim or as being continuously "rolled over" at a constant interest rate.

Table 2.1 displays numerical results obtained through the formulas of Appendix II, by substituting realistic parameter values used in previous studies (e.g. Fan and Sundaresan (2001), Leland (1998), Ericsson (2000)), and by assuming liquidity default: V_B^a, V_B^p are determined by conditions 5.3). The base case scenario is taken to be (P=50, q=100%, d=6%, s=20%, V=100, m =0%, r=5%, a=20%, C/P=6%).

About taxes and bankruptcy costs

It is important to notice the behaviour of bankruptcy costs (BC(V)) and the debt induced tax shield (T(V)) to understand some of the following points. Table 2.1 shows that the tax shield is lower before investment than after investment (respectively 14.7 and 17.6 in the base case scenario), whereas bankruptcy costs are greater before investment than after investment (respectively 1.9 and 1.1 in the base case scenario). This is due to the fact that the risk neutral probability of default is greater before investment than after investment, because the default barrier drops after investment ($K^a = 32.5$ and $K^p = 16.25$ in the base case scenario). Thus before investment bankruptcy costs are more likely to be incurred, whereas tax savings are more likely to be lost.

1.3) About the optimal investment policy: over-investment and under-investment

The optimal investment policy consists in investing as soon as V rises to the level V*, where V* is numerically determined by satisfying condition 10.3). V* is unique and is such that it maximises the value of equity before investment $E^a(V)$. Table 2.1 shows that V*=168 in the base case scenario, while for a twin but un-levered firm V*=166.7. But when, ceteris paribus, the assets pay-out decreases from d=6% to d=3%, then V*=248 while for a twin but un-levered firm V*=272.1.

HYPOTHESES TABLE 2.1: WHEN INVESTMENT COST IS FINANCED BY NEW EQUITY

a	20%	I = q 100 (Ro(V) is at the money at V=100)	
r	5%	m	0%
s	20%	t	35%
P	50	C/P	6%

Default is triggered by lack of liquidity

	Default is triggered by lack of liquidity				Default is triggered by lack of liquidity				Default is triggered by lack of liquidity				Default is triggered by lack of liquidity			
q	50%				50%				100%				100%			
d	6%				3%				6%				3%			
V	50	100	150	200	50	100	150	200	50	100	150	200	50	100	150	200
Ante investment																
BC	4.2	2.0	1.1			6.4	3.1	1.6	4.2	1.9	0.9			6.4	3.0	1.5
T	7.4	14.5	17.5			10.6	16.0	18.4	7.4	14.7	18.0			10.7	16.1	18.6
Ro	1.3	9.1	25.6			13.3	31.4	51.8	2.6	18.3	51.1			26.6	63.0	103.8
E	16.5	72.1	137.7			61.4	136.3	209.7	17.8	81.2	163.1			74.8	168.0	261.9
D	38.0	49.5	54.3			56.0	58.0	58.9	38.0	49.8	55.1			56.1	58.1	59.0
Total assets	54.5	121.7	191.9			117.4	194.3	268.6	55.8	131.1	218.2			130.9	226.1	321.0
V_B ante investment	32.5	32.5	32.5			65.0	65.0	65.0	32.5	32.5	32.5			65.0	65.0	65.0
Credit spread of D	2.90%	1.06%	0.53%			0.36%	0.17%	0.09%	2.89%	1.02%	0.44%			0.35%	0.16%	0.08%
V* levered	168.1	168.1	168.1			241.0	241.0	241.0	168.0	168.0	168.0			248.0	248.0	248.0
V* unlevered	166.7	166.7	166.7			272.1	272.1	272.1	166.7	166.7	166.7			272.1	272.1	272.1
Post investment																
BC	2.8	1.4	0.9			3.5	1.8	1.2	2.1	1.1	0.7			2.2	1.2	0.7
T	11.9	16.5	18.0			15.4	18.1	19.1	14.2	17.6	18.7			17.4	19.1	19.8
E	38.8	112.4	186.9			104.6	182.6	258.9	63.1	162.1	261.7			156.9	258.9	359.6
D	45.3	52.6	55.1			57.3	58.6	59.1	49.0	54.5	56.3			58.3	59.1	59.4
Total assets	84.1	165.0	242.0			161.9	241.2	318.0	112.1	216.5	318.0			215.2	318.0	419.1
V_B post investment	16.3	16.3	16.3			32.5	32.5	32.5	16.3	16.3	16.3			32.5	32.5	32.5
Credit spread of D	1.63%	0.70%	0.45%			0.23%	0.12%	0.08%	1.13%	0.51%	0.33%			0.15%	0.08%	0.05%

	Positive net worth triggers default				Positive net worth triggers default				Positive net worth triggers default				Positive net worth triggers default			
q	50%				50%				100%				100%			
d	6%				3%				6%				3%			
V	50	100	150	200	50	100	150	200	50	100	150	200	50	100	150	200
Ante investment																
BC		4.7	2.4			6.4	3.1	1.6		4.6	2.0			6.4	3.0	1.5
T		11.1	15.9			10.6	16.0	18.4		11.4	16.8			10.7	16.1	18.6
Ro		8.6	25.4			13.3	31.4	51.8		17.2	50.8			26.6	63.0	103.8
E		64.4	133.7			61.4	136.3	209.7		73.2	159.6			74.8	168.0	261.9
D		50.6	55.1			56.0	58.0	58.9		50.9	56.0			56.1	58.1	59.0
Total assets		115.0	188.9			117.4	194.3	268.6		124.1	215.6			130.9	226.1	321.0
V_B ante investment		50.0	50.0			65.0	65.0	65.0		50.0	50.0			65.0	65.0	65.0
Credit spread of D		0.93%	0.44%			0.36%	0.17%	0.09%		0.89%	0.36%			0.35%	0.16%	0.08%
V* levered		158.7	158.7			241.0	241.0	241.0		160.2	160.2			248.0	248.0	248.0
V* unlevered		166.7	166.7			272.1	272.1	272.1		166.7	166.7			272.1	272.1	272.1
Post investment																
BC		3.3	2.2			3.5	1.8	1.2		2.5	1.7			2.2	1.2	0.7
T		14.0	16.3			15.4	18.1	19.1		15.8	17.5			17.4	19.1	19.8
E		107.3	183.6			104.6	182.6	258.9		158.3	259.2			156.9	258.9	359.6
D		53.3	55.6			57.3	58.6	59.1		55.0	56.7			58.3	59.1	59.4
Total assets		160.7	239.1			161.9	241.2	318.0		213.3	315.8			215.2	318.0	419.1
V_B post investment		33.3	33.3			32.5	32.5	32.5		25.0	25.0			32.5	32.5	32.5
Credit spread of D		0.62%	0.40%			0.23%	0.12%	0.08%		0.45%	0.29%			0.15%	0.08%	0.05%

So when the assets pay-out ratio is low, the levered firm would invest earlier than a twin but un-levered firm would do (over-investment), i.e. the presence of debt induces equity holders to anticipate investment. Then increasing leverage and coupon obligations decreases V*, which confirms the rule "the more the debt, the earlier the firm invests". Thus the levered firm would exercise growth options that would not (yet) be exercised by the un-levered firm.

This conclusion may appear to contradict the debt induced under-investment studied by Myers (1977), but in fact it does not. Myers, in the absence of corporate taxes, showed that the levered firm foregoes the exercise of growth options that would instead be exercised by the un-levered firm. Myers showed that the levered firm under-invests because equity holders decide to invest so as to maximise equity value rather than total firm value. Also in the model of this chapter equity holders exercise the growth option so as to maximise equity value rather than total firm value, which confirms Myers's most fundamental conclusion: debt induces equity holders to chose a sub-optimal (i.e. non firm value maximising) investment policy. But the present analysis shows that, unlike in Myers, such sub-optimal investment policy may mean that the levered firm invests earlier (more) rather than later (less) than the un-levered firm. The reason is the very inclusion of corporate taxes in this analysis.

After equity financed investment takes place, new assets enter the firm and default becomes less likely, making debt induced future tax savings more likely and more valuable, while bankruptcy costs become more unlikely and less valuable. Hence, the investment policy of the levered firm (V*) produces two effects on $E^a(V)$ when compared with the investment policy of the un-levered firm:

- on one hand it decreases $E^a(V)$ by reducing Ro(V)[16] because V* does not maximise Ro(V)[17];
- on the other hand it increases $E^a(V)$ by increasing $T^a(V)$, since earlier investment decreases the probability of default.

It is the latter effect that can induce equity holders of the levered firm to invest earlier than they would do in the absence of leverage. And the sooner the growth option is exercised, the sooner the default probability decreases, the sooner debt becomes safer and the lower the credit spread is. The credit spread is equal to $[\frac{C}{D^a(V)} - r]$. We can conclude that the leverage induced change in the optimal exercise policy of the growth option may increase as well as decrease debt value. Over-investment increases debt value.

1.4) About debt value and the investment policy

Equity holders are assumed to decide the investment policy so as to maximise equity value, but equity value is quite insensitive to deviations from the optimal investment policy. Thus equity holders may well be expected to exercise the growth option either when V>V* or when V<V*. But debt value is quite insensitive to such deviations from the optimal investment policy too. In the base case V*=168, $E^a(V)$=81.22 and the credit spread is 1.02%. But if the investment option is exercised at V=150, then $E^a(V)$=80.73 and the spread is 0.97%, while if the option is exercised at V=190, then $E^a(V)$=80.77 and the spread is 1.06%. Again, the sooner equity financed investment takes place, the sooner the default probability decreases, the sooner debt becomes safer and the lower the credit spread is.

If the firm finances the investment cost by issuing new equity, then the investment cost (I) affects debt credit spread only through a change in the optimal investment policy (V*). Given that

[16] Anticipating investment also reduces the probability of default and thus of losing the growth option upon default.

[17] Only for the un-levered firm the investment policy (V*) maximising E(V) would maximise also

the growth option [Ro(V)] is "at the money" when V=I/q, the lower I is, the lower V* is and the lower debt credit spread is, too.

Debt credit spread before investment depends also on the magnitude (q) of the growth option. But Table 2.1 shows that V* as well as debt credit spread are quite insensitive to the magnitude of the growth option. When q = 0.5, V* is virtually the same as when q = 1 and the credit spread just rises from 1.02% to 1.06%. The credit spread rises when the magnitude and hence value of the growth option diminishes[18]. In fact the "smaller" the growth option is, the smaller is the decrease in the default barrier from K^a to K^p and the decrease in the default probability after investment.

1.5) About debt value and the assets pay-out rate

Table 2.1 shows that the lower the assets generated cash flow (d) is, the lower the credit spread on debt is: with d=6% the spread is 1.02%, with d=3% the spread is 0.35%. This may be unexpected. Higher "d" decreases the default barrier (V_B^a=32.5 with d=6% and V_B^a=65 with d=3%) and the lower default barrier tends to decrease the probability of default (as well as to decrease the recovery value of assets upon default). Moreover, higher "d" increases the foregone income due to delaying investment, so that investment is anticipated (V*=168 with d=6% and V*=248 with d=3%). Anticipating investment reduces the probability of default and makes debt more valuable. But higher "d" also increases the dividends paid out to equity holders, which tends to increase the probability of default. It is this effect that usually does increase the credit spreads as it does in this case.

1.6) About a positive net worth covenant to protect debt holders

Often a positive net worth or similar protective debt covenant sets the default barrier (K) equal to the face value of debt (P) or to a multiple of that. Given base case parameters, the covenant sets $K^a = K^p =$ P=50, while $K^a = K^p$=32.5 when default is triggered by lack of liquidity. So, assuming base case parameters, the positive net worth covenant increases the default barrier and the risk neutral probability of default, with the following consequences:

- the values of the growth option [Ro(V)] and of the debt induced tax savings [$T^a(V)$], which are lost upon default, decrease;
- V* decreases from 168 to 160 as Table 2.1 shows; protected debt leads equity holders to anticipate investment, which highlights how the optimal investment policy depends on the

Ro(V).

[18] Note that in Table 2.1 "I" changes as "q" changes so that the growth option remains "at the

default condition.

Protected debt increases the probability of default, but it also increases the recovery value of assets upon default. In fact it is often maintained that a positive net worth covenant protects debt holders by allowing them to trigger default when the firm's assets, with which they can satisfy their claim, are still valuable. The higher the debt covenant sets K, the more creditors are supposed to be protected. The protective covenant is felt to be important especially when the assets payout ratio (d) is high and the firm can sell its assets to meet debt service obligations. Thus it is no surprise that, in the base case of Table 2.1, the credit spread for protected debt is lower than for unprotected debt (0.89% rather than 1.02%). In general inserting the net worth covenant in the debt contract does protect debt holders, since the value of un-protected debt is lower when default is triggered by a lack of liquidity[19]. We have considered just the case in which the positive net worth covenant increases the default barrier, i.e. $V_B^a \leq P$. The case in which $V_B^a \geq P$ is not of interest because default would again be triggered by lack of liquidity and the positive net worth covenant would be ineffective.

Finally, unlike Leland (1994) who concludes that, with protected debt, equity is a concave function of V, the growth option in this model often makes equity before investment a convex function of V even if debt is protected by a positive net worth covenant and even if default is triggered by a lack of liquidity. If equity is convex in V, equity is not incentive compatible in that equity holders will have an incentive to increase assets volatility, thus increasing equity value to the detriment of debt holders. When a growth option is present and debt is protected, equity is often convex in V before investment and then concave in V after investment.

In this section we have focused on debt value and investment policy when debt has indefinite maturity. Next the analysis focuses on the case in which debt has finite average maturity.

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money" at V=100, i.e. qV - I = 0 for V=100. Thus I = q 100 in table 2.1.

[19] This conclusion would not be valid if default were triggered endogenously but the worthless equity condition as per Leland (1994).

2. WHEN INVESTMENT IS FINANCED BY NEW EQUITY AND DEBT HAS FINITE AVERAGE MATURITY

In this section, the model is the same as that in section 1, but the focus is on the case in which debt has finite average maturity, rather than indefinite maturity. Table 2.2 displays numerical results with base case parameters (d=6%, P=50, p=100%, r=5%, a=20%, t=35%, C/P=6%) and with m=20% and V=100.

HYPOTHESES TABLE 2.2: WHEN DEBT HAS FINITE AVERAGE MATURITY

a	20%	I = q 100 (Ro(V) is at the money at V=100)				
r	5%	d	6%			
s	20%	t	35%	Default is triggered by lack of liquidity		
P	50	C/P	6%			

q	100%			50%			100%			50%		
m	20%			20%			40%			40%		
V	50	100	150	60	100	140	60	100	140	60	100	140
Ante investment												
BC		5.3	2.0	9.8	5.4	3.0	11.1	5.9	2.8	11.1	6.1	3.2
T		10.8	17.2	2.1	10.5	15.2	0.8	10.3	16.0	0.8	10.0	15.1
Ro		16.5	50.1	0.8	8.1	20.6	0.6	15.9	41.3	0.3	7.8	20.1
E		71.5	163.3	7.5	62.6	121.1	3.5	69.7	143.4	3.1	61.0	121.0
D		50.6	51.9	45.6	50.6	51.7	46.9	50.6	51.1	46.9	50.6	51.1
Total assets		122.1	215.2	53.0	113.2	172.8	50.4	120.3	194.5	50.0	111.6	172.0
VB ante investment		54.3	54.3	54.3	54.3	54.3	57.8	57.8	57.8	57.8	57.8	57.8
Credit spread of D		0.70%	0.03%	3.52%	0.70%	0.13%	4.03%	0.46%	0.05%	4.03%	0.46%	0.03%
V* levered		150.4	150.4	145.9	145.9	145.9	147.1	147.1	147.1	140.8	141.1	141.1
V* unlevered		166.7	166.7	166.7	166.7	166.7	166.7	166.7	166.7	166.7	166.7	166.7
Post investment												
BC		3.0	2.0	6.6	3.9	2.8	5.6	3.3	2.4	7.4	4.4	3.2
T		15.3	17.2	8.3	13.4	15.6	10.9	14.9	16.7	7.5	12.9	15.2
E		160.6	263.3	41.8	107.9	170.9	74.5	160.5	243.2	39.8	107.5	171.0
D		51.8	51.9	50.0	51.5	51.8	50.9	51.1	51.1	50.3	51.0	51.1
Total assets		212.3	315.2	91.8	159.5	222.8	125.3	211.6	294.3	90.1	158.5	222.0
VB post investment		27.2	27.2	36.2	36.2	36.2	28.9	28.9	28.9	38.5	38.5	38.5
Credit spread of D		0.10%	0.03%	1.01%	0.23%	0.09%	0.22%	0.03%	0.01%	0.73%	0.09%	0.02%

Changing debt maturity changes total firm value $[D^a(V)+E^a(V)]$ only by changing K^a, K^p and V*. Table 2.2 shows that if debt has average maturity of 5 years (1/m=1/20%=5) rather than infinite maturity as in section 1, then debt value before investment increases (from 49.8 to 50.6), debt credit spread decreases (from 1.02% to 0.70%), equity value decreases (from 81.2 to 71.5) and the optimal investment policy drops from V*=168 to V*=150.4.

The decrease in V* confirms that, when investment cost is equity financed, shortening debt maturity induces equity holders to anticipate investment (the higher "m" is, the lower V* is). In fact anticipating the exercise of the growth option anticipates the increase in debt value $[D^a(V)]$ and thus the increase in the cash flow $[m D^a(V)-mP]$ due to "rolling debt over", which is beneficial to equity holders. Anticipating exercise of the growth option clearly increases debt value. In general, shorter debt maturity reduces the distance $[V^*-K^a]$ also because it increases K^a.

The decrease in credit spreads[20] (from 1.02% to 0.70%) is significant and corresponds to the

[20] The credit spread on a debt with value D(V) is defined as the yield on debt subject to default minus the yield on default free debt that promises the same cash flows. Hence in our case, when

intuition that shorter maturity makes debt safer. Shorter debt maturity decreases credit spreads by reducing both the probability of default on outstanding debt and the loss given default. In fact, shorter maturity entails that the firm is more likely to default *after* repaying outstanding debt. Moreover, increasing "m" increases the default barriers [K^a and K^p] and decreases the default free value of debt [$\frac{C + mP}{r + m}$]: both such effects decrease the loss in debt value when default occurs and hence the credit spreads on debt.

Debt credit spread is again quite insensitive to the change in the optimal investment policy. Given base case parameters, investing at V*=168 rather than V*=150.4 raises the credit spread just by 0.02%. Thus, shorter debt maturity induces equity holders to anticipate investment, but anticipating investment has a negligible effect on debt value. Moreover, if equity holders invest so as to maximise total firm value rather than equity value, they invest at V*=149.3 rather than V=150.4. Such slight difference hardly changes debt value and total firm value. In general, the shorter debt maturity is, the smaller the increase in debt credit spread due to the sub-optimal investment policy of the firm that invests so as to maximise equity value rather than total firm value.

So far we have analysed debt value and investment policy when investment is financed by new equity, but now the analysis proceeds by considering the case in which investment cost is financed by subordinated debt rather than equity.

3. WHEN INVESTMENT COST IS FINANCED BY NEW SUBORDINATED DEBT

In this section the same model as in section 1 is studied, but investment cost is partly or wholly financed by issuing new subordinated debt, SD(V), instead of new equity. Subordinated debt is of interest because senior debt holders often protect themselves by requiring that, if more debt is to be issued in the future, it be subordinated to their own claim in case of default and assets liquidation.

Again equity holders invest so as to maximise equity value, which now entails that investment takes place when $V = V' \neq V^*$. Denoting with "u" the fraction of investment cost (I) financed by

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m > 0, the credit spread (cs) is equal to issuing fairly priced subordinated debt, and since subordinated debt is issued at par value, we have

11) $SD(V') = u\,I = P_{SD}$.

SD(V) has average maturity equal to $\frac{1}{m_{SD}}$[21], has face value of P_{SD}, continuously pays coupons at an yearly rate of C_{SD}, receives a payoff of $\min\left[\max\left[K^{P} - P, 0\right], P_{SD}\right]$ upon bankruptcy, and causes the default barrier after investment to be $K^{P} = V^{P}_{B'}(1+q)$. Default after investment is triggered at $V = V^{P}_{B'}$ when instantaneous cash inflows equal instantaneous cash outflows (liquidity default), thus:

$$d\cdot(1+q)\cdot V^{P}_{B'} + m_{SD}\cdot SD\left(V^{P}_{B'}\right) + m\cdot D^{P}\left(V^{P}_{B'}\right) = \left(C + C_{SD}\right)\cdot(1-t) + m_{SD}\cdot P_{SD} + m\cdot P$$ [22].

In particular:

- case 1) if $K^{P}(1-a) \geq \left(P + P_{SD}\right)$, then $V^{P}_{B'} = \frac{\left(C + C_{SD}\right)\cdot(1-t)}{d\cdot(1+q)}$;
- case 2) if $P \leq K^{P}(1-a) \leq \left(P + P_{SD}\right)$, then $V^{P}_{B'} = \frac{\left(C + C_{SD}\right)\cdot(1-t) + m_{SD}\cdot\left(P + P_{SD}\right)}{\left[d + m_{SD}\cdot(1-a)\right]\cdot(1+q)}$;
- case 3) if $(1-a)\cdot K^{P} \leq P$, then $V^{P}_{B'} = \frac{\left(C + C_{SD}\right)\cdot(1-t) + m\cdot P + m_{SD}\cdot P_{SD}}{\left[d + m\cdot(1-a)\right]\cdot(1+q)}$.

In this section we focus on the common and usual case in which $K^{P}(1-a) \leq \left(P + P_{SD}\right)$ and assume that $m = m_{SD}$. From cases 2) and 3) above, it follows that, in this section, default is triggered at $V = V^{P}_{B'} = \frac{\left(C + C_{SD}\right)\cdot(1-t) + m\cdot\left(P + P_{SD}\right)}{\left[d + m\cdot(1-a)\right]\cdot(1+q)}$.

To account for the change in financing policy, boundary conditions 6.2), 7.2), 8.2), 9.2) change in turn as follows. Condition 6.2) becomes

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$$cs = \frac{C + m(P-D)}{D} - \frac{C + m\left(P - \frac{C + mP}{r+m}\right)}{\frac{C + mP}{r+m}} = \frac{C + m(P-D)}{D} - r.$$

[21] m_{SD} is the yearly debt retirement for subordinated debt.

[22] This liquidity default condition is again similar to the one in Ericsson (2000).

$$12)\ D^{a}(V') = \frac{C + m \cdot P}{r + m} + \left[-\frac{C + m \cdot P}{r + m} + \min\left[(1 - a) \cdot K^{p}, P\right]\right]\left[\frac{V'}{V_{B'}^{p}}\right]^{bb_{p}^{-}}.$$

Condition 7.2) becomes $T^{a}(V') = \frac{t \cdot (C + C_{SD})}{r} - \left(\frac{t \cdot (C + C_{SD})}{r}\right) \cdot \left(\frac{V'}{V_{B'}^{p}}\right)^{b_{p}^{-}}$.

Condition 8.2) becomes $BC^{a}(V') = a \cdot K^{p} \cdot \left(\frac{V'}{V_{B'}^{p}}\right)^{b_{p}^{-}}$.

It can be shown that equation 11) can be re-written as

13)

$$SD(V') = \frac{C_{SD} + m_{SD} \cdot P_{SD}}{r + m_{SD}} +$$

$$+\left[-\frac{C_{SD} + m_{SD} \cdot P_{SD}}{r + m_{SD}} + \min\left[\max\left[(1 - a) \cdot K^{p} - P, 0\right], P_{SD}\right]\right]\left[\frac{V'}{V_{B'}^{p}}\right]^{sb_{p}^{-}} = u \cdot I = P_{SD}$$

with $sb_{p}^{-} = \frac{-\left(r - d - \frac{s^{2}}{2}\right) - \sqrt{\left(r - d - \frac{s^{2}}{2}\right)^{2} + 2 \cdot (r + m_{SD}) \cdot s^{2}}}{s^{2}}$.

Then, in order for equation 13) to hold, subordinated debt coupon must be set equal to

14)

$$C_{SD} = -m_{SD} \cdot P_{SD} + \frac{(r + m_{SD}) \cdot \left\{u \cdot I - \min\left[\max\left[(1 - a) \cdot K^{p} - P, 0\right], P_{SD}\right]\left[\frac{V'}{V_{B'}^{p}}\right]^{sb_{p}^{-}}\right\}}{1 - \left[\frac{V'}{V_{B'}^{p}}\right]^{sb_{p}^{-}}}.$$

Table 2.3 displays numerical results for the model variant of this section, in which the issuance of subordinated debt finances a fraction (u) of the cost (I) of exercising the growth option. What follows comments the scenario with base case parameters (d=6%, P=50, q=100%, m=0%, s=20%, r=5%, a=20%, C/P=6%) and with u=100%. In this scenario P=50 is the nearly optimal firm leverage, since it maximises the sum of the value of debt and equity[23]. $D^a(V)$ stands again for the value of senior debt issued before investment.

3.1) About subordinated debt financing and the value of senior debt

Table 2.3 shows that, assuming base case parameters, senior debt value before investment can be higher when investment cost (I) is to be financed by issuing new subordinated debt rather than by issuing new equity. Under base parameters, if new equity is issued, Table 2.1 shows that $D^a(V)$=49.8 (senior debt credit spread is 1.02%) and $E^a(V)$=81.2, while if new subordinated debt is issued, Table 2.2 shows that $D^a(V)$=50.2 (senior debt credit spread is 0.97%) and $E^a(V)$=87.4. The increase in equity value is primarily due to the higher debt induced tax savings after the exercise of the growth option.

HYPOTHESES TABLE 2.3: WHEN INVESTMENT COST IS FINANCED BY NEW SUBORDINATED DEBT

a 20% I = q 100 (Ro(V) is at the money at V=100)
r 5% m 0%
s 20% t 35% Default is triggered by lack of liquidity
P 50 C/P 6%

u	100%															
q	50%				50%				100%				100%			
d	6%				3%				6%				3%			
V'	50	100	150	200	50	100	150	200	50	100	150	200	50	100	150	200
Ante investment																
BC	4.4	3.0	3.9			7.1	4.8	4.4		4.1	7.0			7.7	6.1	6.6
T	8.0	19.0	30.0			13.1	21.8	28.0		23.7	43.2			15.6	27.7	37.8
Ro	1.3	9.1	25.4			13.2	31.3	51.6		18.0	50.4			26.5	62.7	103.4
E	16.9	74.8	145.1			63.4	140.8	217.2		87.4	180.4			78.6	177.0	276.8
D	38.1	50.3	56.3			55.8	57.5	58.0		50.2	56.2			55.7	57.4	57.8
Total assets	55.0	125.1	201.4			119.1	198.3	275.2		137.6	236.6			134.3	234.4	334.6
VB ante investment	32.5	32.5	32.5			65.0	65.0	65.0		32.5	32.5			65.0	65.0	65.0
Credit spread of D	2.88%	0.97%	0.33%			0.38%	0.22%	0.17%		0.97%	0.34%			0.38%	0.23%	0.19
V' levered	156.5	156.5	156.5			229.9	229.9	229.9		153.0	153.0			235.2	235.2	235.2
V' unlevered	166.7	166.7	166.7			272.1	272.1	272.1		166.7	166.7			272.1	272.1	272.1
Post investment																
BC	13.0	6.5	4.3			16.9	8.9	5.7		11.1	7.4			28.8	15.1	9.6
T	3.2	24.1	31.1			11.5	24.4	29.7		32.2	44.2			10.4	32.4	41.3
E	4.3	72.8	145.7			44.1	135.1	216.6		84.7	181.0			36.9	164.1	275.2
SD	10.2	39.4	49.2			47.5	49.1	49.8		81.7	99.3			92.9	97.4	99.3
C of SD	3.4	3.4	3.4			2.5	2.5	2.5		6.7	6.7			5.1	5.1	5.1
D	50.7	55.4	56.9			53.0	56.3	57.6		54.7	56.5			51.8	55.7	57.3
Total assets	65.2	167.6	251.8			144.6	240.5	324.0		221.1	336.8			181.6	317.2	431.7
VB post investment	46.5	46.5	46.5			80.1	80.1	80.1		52.7	52.7			88.0	88.0	88.0
Credit spread of D	0.92%	0.42%	0.27%			0.66%	0.33%	0.20%		0.48%	0.31%			0.79%	0.39%	0.24
Credit spread of SD	28.65%	3.71%	1.98%			0.35%	0.18%	0.11%		3.23%	1.77%			0.51%	0.26%	0.16

The first reason why subordinated debt financing decreases senior debt credit spread (from 1.02% with equity financing to 0.97%) is that it gives equity holders a stronger incentive to anticipate investment: the optimal investment policy becomes V'=153 (< V*=168) so that debt becomes safer sooner than in section 1, which makes senior debt more valuable than in section 1. If instead investment took place at V* rather than V', senior debt credit spread would decrease just from 1.02% to 1.01% rather than from 1.02% to 0.97%. The reason why V'<V* is that issuing subordinated debt rather than equity greatly increases the value of the tax shield after investment to 32.2 (from 17.6): higher coupon obligations would increase the value of the tax shield and of

[23] The results in the previous sections assumed P=50, too. So comparisons with previous results

equity after exercise of the growth option, which motivates equity holders to anticipate investment.

The second reason why subordinated debt financing decreases senior debt credit spread (when compared to equity financing) is that it increases the interest burden and the default barrier from 32.5 (before investment) to 52.7 (after investment). The higher default barrier after investment entails a higher recovery value of assets after investment ((1-a) $V^{p}_{B'}(1+q)$), with which senior debt holders can satisfy their claim.

Thus subordinated debt financing of new investment, as opposed to equity financing, can decrease senior debt credit spreads because $V' < V^*$ and because $V^{p}_{B'} > V^{p}_{B}$.

But subordinated debt financing can also increase senior debt credit spreads. This is apparent in Tables 2.1 and 2.3 when d = 3%: when the assets payout rate (d) decreases, V^{p}_{B} and $V^{p}_{B'}$ increase. In these cases $V^{p}_{B'} > V^{p}_{B}$ implies that subordinated debt financing increases the default probability after investment, but may not necessarily decrease the loss given default for senior debt holders. In particular if, after investment, $(1\text{-}a)\,V^{p}_{B}(1+q) > P$, the loss given default for senior debt holders would be equal to $\frac{C+mP}{r+m} - P$, and, if $(1\text{-}a)\,V^{p}_{B'}(1+q) > P$, the loss would still be equal to $\frac{C+mP}{r+m} - P$. Thus, when "d" is low, subordinated debt financing of the investment cost may increase credit spreads on senior debt, because subordinated debt financing would increase the default probability, but would not decrease the loss given default on senior debt.

The above results suggest that covenants prohibiting issuance of new subordinated debt to finance new investments must be used with care. Senior debt holders may or may not be better off when new investment is financed by new subordinated debt rather than by new equity.

3.2) When senior and subordinated debt have finite maturity

Finally we consider the case in which senior and subordinated debt have finite average maturity, i.e. $m = m_{SD} = 20\% > 0$. Table 2.4 shows the results for this case. Comparison between Tables 2.3 and 2.4 reveals that V' increases as "m" rises from 0% to 20%.

Thus, although when investment is equity financed shorter debt maturity induces equity holders to anticipate investment, the opposite is true when investment is financed by subordinated debt. The reason is that shorter debt maturity increases $V^{p}_{B'}$, which increases the probability of default

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assume constant leverage, but only in this section P=50 represents optimal leverage.

HYPOTHESES TABLE 2.4: WHEN INVESTMENT COST IS FINANCED BY NEW SUBORDINATED DEBT AND DEBT HAS FINITE MATURITY

a	20%				I = q 100 (Ro(V) is at the money at V=100)											
r	5%															
s	20%				t	35%			Default is triggered by lack of liquidity							
P	50				C/P	6%										
m	20%															
u	100%															
q	50%				50%				100%				100%			
d	6%				3%				6%				3%			
V	50	100	150	200	50	100	150	200	50	100	150	200	50	100	150	200
Ante investment																
BC		7.2	8.0			6.6	4.4	4.1		8.6	12.2			7.2	5.9	6.4
T		11.8	20.0			13.3	21.3	26.8		13.7	25.8			15.6	26.6	35.5
Ro		8.3	25.3			13.7	31.7	52.0		16.7	50.6			27.4	63.4	104.0
E		62.3	135.7			68.8	146.6	222.8		71.2	162.6			84.2	182.2	281.2
D		50.5	51.6			51.7	51.9	52.0		50.5	51.6			51.7	51.9	52.0
Total assets		112.9	187.3			120.5	198.6	274.8		121.7	214.2			135.9	234.2	333.1
VB ante investment		54.3	54.3			62.9	62.9	62.9		54.3	54.3			62.9	62.9	62.9
Credit spread of D		0.73%	0.18%			0.16%	0.04%	0.02%		0.73%	0.20%			0.16%	0.04%	0.02%
V* levered		170.4	170.4			248.8	248.8	248.8		172.4	172.4			249.7	249.7	249.7
V* unlevered		166.7	166.7			272.1	272.1	272.1		166.7	166.7			272.1	272.1	272.1
Post investment																
BC		15.6	10.4			18.4	9.7	6.1		26.2	17.5			32.9	17.3	11.0
T		11.4	21.1			10.0	23.5	29.0		11.5	27.9			6.4	29.9	39.5
E		49.1	134.6			41.0	137.1	220.9		44.4	160.1			24.4	161.2	276.7
SD		45.5	49.4			49.6	49.9	50.0		90.0	98.7			98.7	99.7	99.9
C of SD		2.8	2.8			2.5	2.5	2.5		5.7	5.7			5.0	5.0	5.0
D		51.2	51.8			51.0	51.8	51.9		50.9	51.7			50.5	51.6	51.9
Total assets		145.8	235.8			141.6	238.8	322.8		185.3	310.4			173.5	312.6	428.5
VB post investment		72.1	72.1			82.7	82.7	82.7		81.0	81.0			92.7	92.7	92.7
Credit spread of D		0.38%	0.12%			0.50%	0.12%	0.04%		0.54%	0.17%			0.76%	0.18%	0.06%
Credit spread of SD		3.18%	0.92%			0.19%	0.05%	0.02%		3.50%	1.00%			0.35%	0.08%	0.03%

after investment, decreases $T^p(V)$, increases $BC^p(V)$ and decreases $E^p(V)$. Since equity value after investment decreases, equity holders have less of an incentive to anticipate investment when "$m_{SD} = m$" rises.

Finally, when investment is financed by subordinated debt, reducing debt maturity reduces the credit spreads on senior debt, which was the case also when investment was equity financed.

CONCLUSIONS

This chapter has analysed debt valuation in relation to the debtor's growth options. The main results for the case in which investment cost is financed by new equity are the following.

By including taxes and bankruptcy costs in the analysis and by assuming either liquidity default or default triggered by a net worth covenant, it has been shown that optimal investment by the levered firm may anticipate optimal investment by the same but un-levered firm.

Debt credit spread and the optimal investment policy are quite insensitive to the magnitude of the growth option. Debt credit spread is quite insensitive also to deviations from the optimal investment policy.

If the levered firm finances the cost of new investment by issuing subordinated debt, rather than by issuing equity, before investment the value of senior debt previously issued can increase.

Shorter debt average maturity induces equity holders to anticipate investing, when the investment cost is equity financed, and to postpone investing, when the investment cost is financed by subordinated debt.

If investment cost is financed by equity or by subordinated debt, shorter debt average maturity decreases credit spreads on senior debt before investment.

Finally, future research may study the issue of optimal capital structure before investment. This is a difficult issue because optimal leverage maximises the firm's total value and has to be determined jointly with the optimal investment policy, with the optimal debt average maturity and possibly with the equity value maximising endogenously determined default barrier.

APPENDIX I: DERIVATION OF THE VALUATION EQUATIONS

The standard contingent claims pricing arguments employed to derive equations 6, 7, 8, 9 are now shown. Consider equation 6) for $D^a(V)$. Since $\partial V = V\left[(m - d)\cdot\partial t + s\,\partial z\right]$ and since $D^a(V)$ is a function of V, by applying Ito's lemma

A.1) $\partial D^a(V) = \frac{1}{2} D^a_{VV} \cdot (\partial V)^2 + D^a_V \cdot \partial V = \frac{1}{2} D^a_{VV} \cdot s^2 \cdot V^2 \cdot \partial t + D^a_V \cdot V \cdot d$

(here ∂t stands for a very small time increment). Then a risk free portfolio (Π) can be set up, such that

A.2) $\Pi = D^a(V) - D^a_V \cdot V$.

This portfolio requires selling short D^a_V units of V, where V (or a "twin" of V) is assumed to be traded in the market. That portfolio Π is risk-less can be easily seen from A.1) and A.2), which lead to

A.3) $\partial\Pi = \partial D - D^a_V \cdot \partial V = \frac{1}{2} D^a_{VV} \cdot s^2 \cdot V^2 \cdot \partial t$.

Since $\partial\Pi$ is deterministic and arbitrage opportunities are ruled out, the portfolio Π needs to earn but the default risk free interest rate (r), so

A.4) $\partial\Pi = r \cdot \partial t \cdot \Pi$.

But the cost of holding the short position for a short time period must be accounted for [i.e. $D^a_V \cdot V \cdot d \cdot \partial t$] as well as the coupon ($C \cdot \partial t$) and the instantaneous cash flows due to "rolling debt over", i.e. $\left(P - D^a(V)\right) \cdot m \cdot \partial t$, perceived by debt holders. Thus

A.5) $\partial\Pi = r \cdot \partial t \cdot \Pi + D^a_V \cdot V \cdot d \cdot \partial t - C \cdot \partial t - \left(P - D^a(V)\right) \cdot m \cdot \partial t$.

Substituting, simplifying and rearranging from A.2), A.3) and A.5) yields

A.6) $\frac{1}{2} D_{VV} \cdot V^2 \cdot s^2 + (r - d) \cdot D_V \cdot V - r \cdot D + C + m \cdot (P - D^a) = 0$.

Equation A.6) is equal to equation 6) of section 1 for the value of debt ante-investment. Then the solution to equation 6) is

A.7) $D^a(V) = \dfrac{C + mP}{r + m} + c_{1D} \cdot V^{b_a^+} + c_{2D} \cdot V^{b_a^-}$

with

$$b_a^{\pm} = \frac{-\left(r - d - \frac{s^2}{2}\right) \pm \sqrt{\left(r - d - \frac{s^2}{2}\right)^2 + 2 \cdot (r + m) \cdot s^2}}{s^2}$$

and with

c_{1D}, c_{2D} constants to be determined by boundary conditions 6.1) and 6.2) in section 1.

By similar arguments, also equations 7, 8, 9 and 10 can be derived subject to the respective boundary conditions: 7.1, 7.2, 8.1, 8.2, 9.1, 9.2, 10.1 and 10.2.

APPENDIX II: CLOSED FORM SOLUTIONS

Closed form solutions for sections 1 and 2

The solution to equation 6 subject to 6.1 and 6.2 is:

$$D^a(V) = \frac{C + mP}{r + m} + c_{1D} \cdot V^{bb_a^+} + c_{2D} \cdot V^{bb_a^-}$$

with

$$bb_a^{\pm} = \frac{-\left(r - d - \frac{s^2}{2}\right) \pm \sqrt{\left(r - d - \frac{s^2}{2}\right)^2 + 2 \cdot (r + m) \cdot s^2}}{s^2}$$

with

$$c_{1D} = -c_{2D} \cdot (V_B^a)^{(bb_a^- - bb_a^+)} + \frac{\left[-\frac{C + Pm}{r + m} + \min\left[P, (1 - a) \cdot V_B^a\right]\right]}{(V_B^a)^{bb_a^+}}$$

with

$$c_{2D} = \frac{-\left[-\frac{C + Pm}{r + m} + \min\left[P, (1 - a) \cdot V_B^a\right]\right] \cdot \left(V_B^a\right)^{-bb_a^+} + \frac{\left[-\frac{C + Pm}{r + m} + \min\left[P, (1 - a) \cdot K^p\right]\right] \cdot \left(\frac{V^*}{V_B^p}\right)^{bb_p^-}}{(V^*)^{bb_a^+}}}{\left[(V^*)^{(bb_a^- - bb_a^+)} - (V_B^a)^{(bb_a^- - bb_a^+)}\right]}$$

The solution to equation 7 subject to 7.1 and 7.2 is:

$$T^a = \frac{t \cdot C}{r} + c_{1T} \cdot V^{b_a^+} + c_{2T} \cdot V^{b_a^-},$$

with

$$b_a^{\pm} = \frac{-\left(r - d - \frac{s^2}{2}\right) \pm \sqrt{\left(r - d - \frac{s^2}{2}\right)^2 + 2 \cdot r \cdot s^2}}{s^2}$$

with

$$c_{1T} = -\left[\frac{tC}{r} \cdot \left(V_B^a\right)^{-b_a^+} + c_{2T} \cdot \left(V_B^a\right)^{b_a^- - b_a^+}\right]$$

with

$$c_{2T} = \frac{\frac{t \cdot C}{r}\left[1 - \left(\frac{V^*}{V_B^a}\right)^{b_a^-}\right] + \frac{t \cdot C}{r}\left[\left(\frac{V^*}{V_B^p}\right)^{b_p^-} - 1\right]}{\left[(V^*)^{b_a^-}(V_B^a)^{(b_a^- - b_a^+)} - (V^*)^{b_a^+}\right]}.$$

The solution to equation 8 subject to 8.1 and 8.2 is:

$BC^a = c_{1BC} \cdot V^{b_a^+} + c_{2BC} \cdot V^{b_a^-}$, with

$$c_{1BC} = -c_{2BC} \cdot \left(V_B^a\right)^{b_a^- - b_a^+} + a \cdot \left(V_B^a\right)^{1 - b_a^+}$$

and with

$$c_{2BC} = \frac{-a \cdot \left(V_B^a\right)^{1 - b_a^+} \cdot (V^*)^{b_a^+} + a \cdot K^p \cdot \left(\frac{V^*}{V_B^p}\right)^{b_p^-}}{\left[(V^*)^{b_a^-} - (V^*)^{b_a^+} \cdot \left(V_B^a\right)^{b_a^- - b_a^+}\right]}.$$

The solution to equation 9 subject to 9.1 and 9.2 is:

$$Ro = c_{1Ro} \cdot (qV)^{b_a^+} + c_{2Ro} \cdot (qV)^{b_a^-},$$

$$c_{1Ro} = -c_{2Ro} \cdot (qV_B^a)^{(b_a^- - b_a^+)} + RoK \cdot (qV_B^a)^{(-b_a^+)},$$

$$c_{2Ro} = \frac{(qV^*)^{(1-b_a^+)} - I \cdot (V^*)^{(-b_a^+)} - RoK \cdot (qV_B^a)^{-b_a^+}}{\left[(V^*)^{(b_a^- - b_a^+)} - (V_B^a)^{(b_a^- - b_a^+)}\right]}.$$

The value of equity before exercise of the growth option can be found through equation 1).

Closed form solutions for section 3

When the exercise of the growth option is financed by issuing subordinated debt as in section 3, then formula for the value of senior before investment $[D^{a}(V)]$ is the same as the formula for section 1, but for the fact that $V^{p}_{B'}$ now substitutes V^{p}_{B}, V' substitutes V*.

When the exercise of the growth option is financed by subordinated debt, the formula for the value of bankruptcy costs is the same as the formula for section 1, but for the fact that $V^{p}_{B'}$ substitutes V^{p}_{B} and V' substitutes V*. Then the formula for the value of the debt induced tax shield now becomes

$$T^{a} = \frac{t \cdot C}{r} + c_{1T} \cdot V^{b^{+}_{a}} + c_{2T} \cdot V^{b^{-}_{a}},$$

with

$$c_{1T} = -\left[\frac{t \cdot C}{r} \cdot \left(V^{a}_{B}\right)^{-b^{+}_{a}} + c_{2T} \cdot \left(V^{a}_{B}\right)^{b^{-}_{a} - b^{+}_{a}}\right] \text{ and}$$

$$c_{2T} = \frac{\dfrac{t \cdot C}{r}\left[-1 + \left(\dfrac{V'}{V^{a}_{B}}\right)^{b^{+}_{a}}\right] + \dfrac{t \cdot \left(C + C_{SD}\right)}{r}\left[-\left(\dfrac{V'}{V^{p}_{B'}}\right)^{b^{-}_{p}} + 1\right]}{\left[-\left(V'\right)^{b^{+}_{a}}\left(V^{a}_{B}\right)^{\left(b^{-}_{a} - b^{+}_{a}\right)} + \left(V'\right)^{b^{-}_{a}}\right]}.$$

CHAPTER 3:
DEBT VALUATION WHEN ASSETS VOLATILITY IS UNCERTAIN FOR DEBT HOLDERS.

INTRODUCTION

This chapter studies the change in the value and cost of debt due to debt holders' uncertainty about the firm's assets risk (assets volatility). The analysis in this chapter shows that:

- the increase in the cost of debt due to debt holders' uncertainty about assets volatility is very sensitive to the assumed default condition. When default is endogenous, such cost can be much higher than when default is triggered by lack of liquidity;
- the values of subordinated debt and of convertible debt are not minimised by assuming highest constant assets volatility; the reason is that the values of these types of debt are locally convex functions of assets value; the constant volatility assumption seems particularly inadequate when valuing subordinated convertibles;
- the sensitivity of debt value to (uncertainty about) assets volatility may heavily depend on the default free short rate model when the short rate is stochastic and instantaneously correlated with the value of the firm's assets; if the short rate follows the Ho-Lee process, the decrease in debt value due to uncertainty can be much lower than if the short rate follows the Vasicek process.

The literature on structural models of credit risk (e.g. Leland in 1994) has shown that debt value can be very sensitive to the firm's assets volatility, especially when debt has long maturity and when a default barrier for the value of the firm's assets is assumed. But debt holders are often uncertain about the firm's assets volatility:

- they often do not know or cannot estimate present volatility; this is the case especially when the debtor firm is not traded in the stock market, since assets volatility cannot be estimated from equity prices[24];
- even if they do know present assets volatility, debt holders never know future assets volatility; assets volatility in a few years is never known in advance; actions by management or exogenous factors may cause assets volatility to change, while debt holders have no direct control over volatility.

In this chapter the fundamental assumption is that creditors do not know the assets volatility parameter value, but that creditors do know that the volatility parameter lies and remains within a bounded range. This assumption reflects the fact that, although debt holders may not have perfect information about assets volatility, they may still have enough data to be able to specify a region of

[24] Companies such as KMV may provide reliable estimates of present and past assets volatility for traded firms.

values for the volatility parameter. Creditors require higher credit spreads to be compensated for their uncertainty about volatility.

The organisation of the chapter is as follows. Section 1 briefly reviews the most relevant past literature. Section 2 considers a time independent model and discusses the effects of uncertainty about assets volatility on the cost of debt. In this setting alternative default barriers are considered. Sections 3, 4 and 5 study the effect of uncertain assets volatility on the valuation of single debt claims, such as subordinated debt, convertible debt and zero coupon debt. Section 3 studies worst-case value and worst case yield of subordinated debt, given uncertain assets volatility. Section 4 studies the prudent valuation of subordinated convertible debt and its sensitivity to assets volatility. Section 5 highlights the importance of the instantaneous correlation between the default free stochastic short interest rate and the firm's assets value: such correlation is shown to affect debt sensitivity to volatility in important ways. The conclusions follow.

1. THE PROBLEM AND THE MOST RELEVANT LITERATURE

In this chapter the problem is one of studying the value and cost of debt when debt holders are uncertain about the risk (volatility) of the debtor firm's assets. It is important to stress that uncertainty is assumed to persist in that it is not assumed to be resolved over time.

Debt holders are uncertain but can expect assets volatility to remain within a bounded range. When the debtor's equity is floated, the bounds of the assets volatility range may be given by estimates from the prices of the debtor's equity and from the prices of credit derivatives (see Pan (2001)). When the debtor' s equity is not floated, the bounds of the assets volatility range may be given by volatility estimates for firms in the same industry[25].

Uncertain debt holders are assumed to prudently value debt by envisaging the highest volatility in the bounded range, since higher volatility decreases debt value. But this is not always the case as we shall see later on.

The literature

In 1995 Avellaneda-Levy-Paras (ALP) propose a model for pricing options in which the volatility of the underlying is uncertain, remains within a bounded range and evolves in the worst possible

[25] For example KMV provide estimates of assets volatility for different industries and for different firm's assets sizes.

way within the given bounded range. This entails that, when the gamma[26] of the contingent claim becomes negative (positive), the volatility of the underlying switches from lowest (highest) to highest (lowest). So for ALP volatility switches are not random, but occur deterministically so as to minimise the value of an option or of a portfolio. Also in this chapter, precisely in sections 3 and 4, volatility is assumed to switch in a worst-case fashion as per ALP, so as to obtain the most prudent debt valuation with respect to uncertain volatility.

Leland (1998) and Ericsson (2000) propose two structural models that recognise the incentives for equity holders to increase assets volatility and value debt claims by recognising such incentive. Leland and Ericsson both assume symmetric information and model the increase in assets volatility in a similar manner: a single irreversible upward switch in assets volatility, which is endogenously decided by equity holders so as to maximise equity value. This chapter differs from Leland (1998) and Ericsson (2000) in that creditors are assumed to be imperfectly informed. Uncertain creditors assume a prudent volatility scenario that tends to minimise debt value rather then to maximise equity value.

This chapter draws also from the most recent literature about structural models of credit risk, such as Longstaff and Schwartz (1995) and Ericsson and Reneby (1998), where a constant default barrier is assumed. But, unlike in these articles, here assets volatility is assumed to be uncertain rather than known and constant.

This chapter analyses the prudent valuation of subordinated debt and subordinated convertible debt by uncertain debt holders. The highest volatility scenario does not necessarily minimise the values of subordinated debt and convertible debt. This point follows from a previous insight by Black and Cox (1976) who showed that subordinated debt value may increase with higher assets volatility when the firm's assets have low value. As for convertible debt, the conclusion by Brennan and Schwartz (1988), according to whom convertibles are relatively insensitive to the volatility of the firm's assets, is challenged.

2. UNCERTAINTY ABOUT ASSETS VOLATILITY, DEFAULT CONDITIONS AND COST OF DEBT

This section shows how the cost of debt increases when creditors are uncertain about the firm's assets volatility and how such increase in the cost of debt heavily depends on the assumed default condition.

[26] The gamma is the second derivative of the value of the claim with respect to the value of the underlying.

The increase in the cost of debt due to creditors' uncertainty is equal to the cost of debt under uncertainty minus the cost of debt under certainty. Under certainty conditions, debt holders know assets volatility and volatility remains constant over time. Under uncertainty conditions, debt holders do not know assets volatility and just expect volatility to remain within a bounded range. Debt holders require compensation for their uncertainty by valuing debt prudently. It is assumed that uncertain prudent creditors value debt by envisaging that volatility is equal to the upper bound of the volatility range.

A model in a time independent setting

In the rest of this section, the following model is employed to study the increase in the cost of debt due to creditors' uncertainty. First the following notation and assumptions are introduced:

- V is the value of the firm's assets, whose "risk neutral process" is $\partial V = V \cdot (r-d) \cdot \partial t + V \cdot s \cdot \partial z_V$, where ∂z_V is the differential of a Wiener process, "r" is the default free short interest rate, "d" is the instantaneous assets payout rate, "s" is assets volatility;
- "s^+" is the upper bound of the range in which creditors expect volatility to lie;
- "r" is assumed constant over time;
- "a" denotes bankruptcy costs expressed as a percentage of V;
- D(V) denotes the value of debt as a function of V; creditors receive a continuously paid annual coupon of C; the face value of debt is P;
- "t" denotes the corporate tax rate;
- V_B denotes the values of V at which default is triggered (default barrier);
- "m" is the inverse of debt average maturity, i.e. it is the fraction of debt that is retired and newly issued every year; debt is continuously retired and newly issued ("rolled over") so the total nominal amount of debt is constant as per Leland (1998) and Ericsson (2000).

Under certainty conditions, we know from Leland (1998) that debt value (D(V)) satisfies the following ordinary differential equation:

2.1) $\frac{1}{2} \cdot (s \cdot V)^2 \cdot D_{VV} + (r-d) \cdot V \cdot D_V - r \cdot D + C + m \cdot (P-D) = 0$ [27]

subject to

2.2) $D(V_B) = (1-a) \cdot V_B$,

[27] The subscript of D(V) and other value functions denotes the derivatives of the functions.

$$2.3)\ D(V \to \infty) \to \frac{C + m \cdot P}{r + m}.$$

The solution to 2.1), 2.2) and 2.3) is

$$2.4)\ D(V) = \frac{C + mP}{r + m} + \left[-\frac{C + mP}{r + m} + (1 - a) \cdot V_B \right] \cdot \left[\frac{V}{V_B} \right]^{b^-},$$

where b^- is a parameter that depends on the true volatility (s):

$$b^- = \frac{-\left(r - d - \frac{s^2}{2} \right) - \sqrt{\left(r - d - \frac{s^2}{2} \right)^2 + 2 \cdot s^2 \cdot (r + m)}}{s^2}$$

Debt is assumed to be issued at par value, which means that C is such that D(V) = P when debt is issued, thus

$$2.5)\ C = \frac{(r + m) \cdot \left[P - (1 - a) \cdot V_B \cdot \left(\frac{V}{V_B} \right)^{b^-} \right]}{\left[1 - \left(\frac{V}{V_B} \right)^{b^-} \right]} - mP.$$

So far nothing is new.

Under uncertainty conditions, debt holders do not know present volatility and resort to prudently assuming the level of present and future volatility. Since debt holders expect volatility to remain within a bounded range (e.g. $20\% \le s \le 40\%$), they cautiously value their debt claim by assuming highest constant volatility (s = 40%). Then, using the subscript or superscript "u" to denote uncertainty conditions, debt value for uncertain debt holders is

$$2.6)\ D^u(V) = \frac{C^u + mP}{r + m} + \left[-\frac{C^u + mP}{r + m} + (1 - a) \cdot V_B^u \right] \cdot \left[\frac{V}{V_B^u} \right]^{b_u^-}$$

where b_u^- is the same as b^- but for the fact that it assumes "s^+" instead of "s", where $D^u(V)$ denotes debt value when debt holders are uncertain about assets volatility, where $D^u(V, C^u) = P =$ D(V,C) since debt is assumed to be issued at par value (P), where $C^u > C$ and

$$2.7)\ C^{u} = \frac{(r+m)\cdot\left[P-(1-a)\cdot V_B^u\cdot\left(\frac{V}{V_B^u}\right)^{b_u^-}\right]}{\left[1-\left(\frac{V}{V_B^u}\right)^{b_u^-}\right]} - mP.$$

It can then be seen that, under uncertainty conditions, the cost of debt is altered because b^- and V_B change to b_u^- and V_B^u.

Different default conditions

Either under certainty or under uncertainty conditions, default can be triggered in at least three different ways, which will be shown to imply significantly different increases in the cost of debt due to creditors' uncertainty.

When default is triggered by a positive net worth covenant, then $V_B = V_B^u = P$, i.e. the default barrier is the same both under certainty and under uncertainty about assets volatility.

When, following Ericsson (2000), default is triggered by lack of liquidity, the default barrier under certainty and under uncertainty respectively are

$$2.8)\ V_B = \frac{P\cdot m + C\cdot(1-t)}{d+(1-a)\cdot m} \le V_B^u = \frac{P\cdot m + C^u\cdot(1-t)}{d+(1-a)\cdot m}$$

When, following Leland (1994b), default is endogenous, the default barrier under uncertainty and under certainty conditions respectively are

$$2.9)\ V_B^u = \frac{-\frac{C^u + m\cdot P}{r+m}\cdot b_u^- + \frac{C^u}{r}\cdot t\cdot bb_u^-}{1-a\cdot bb_u^- - (1-a)\cdot b_u^-} \le V_B = \frac{-\frac{C+m\cdot P}{r+m}\cdot b^- + \frac{C}{r}\cdot t\cdot bb^-}{1-a\cdot bb^- - (1-a)\cdot b^-},$$

where b_u^- is the same as b^- but for the fact that it assumes "s^+" instead of "s", where

$$bb^- = \frac{-\left(r-d-\frac{s^2}{2}\right)-\sqrt{\left(r-d-\frac{s^2}{2}\right)^2+2\cdot s^2\cdot r}}{s^2}$$

and where bb_u^- is the same as bb^- but for the fact that it assumes "s^+" instead of "s".

Next a numerical example shows the impact creditors' uncertainty on the cost of debt under the above three different default conditions.

A numerical example under different default conditions

Table 3.1 refers to a base case scenario with average realistic parameters[28]. With base case parameters and debt protected by a positive net worth covenant, debtors' uncertainty about volatility increases the credit spread from 0.07% to 0.79%. With liquidity default, uncertainty increases the credit spread from 0.07% to 0.68%. With endogenous default, uncertainty increases the credit spread from 0.07% to 0.99%.

Endogenous default entails the highest increase in the cost of debt due to creditors' uncertainty about assets volatility. The reason is that higher volatility entails a lower endogenous default barrier and hence a lower recovery value of assets upon default. Instead, when default is triggered by a positive net worth covenant, the default barrier and the recovery vale of assets do not depend on assets volatility. In such cases higher volatility increases just the risk neutral default probability, but not the loss given default. The same can be said when default is triggered by lack of liquidity, since in this case the default barriers under uncertainty and under certainty are very close.

Finally, whatever the default condition, the shorter the debt average maturity is, the lower is the increase in the cost of debt due creditors' uncertainty about assets volatility. For example, assuming liquidity default and base case parameters, uncertainty increases the credit spread from 0.07% to 0.68% when "m" equals 20%, whereas the spread increases from 0.02% to 0.32% when m equals 40%.

This section has shown that the increase in the cost of debt due to creditor's uncertainty about assets volatility is very sensitive to the assumed default condition. When default is endogenous, the increase in the cost of debt can be much higher than when default is triggered by lack of liquidity. So, when default is endogenous, the debtor can reduce the cost of creditors' uncertainty by reducing debt maturity, by inserting a positive net worth covenant in the debt indenture or by inserting a cash flow based covenant that triggers default when the assets generated cash flows fall below the level of interest charges.

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[28] In this work the chosen parameter values are similar to the values used in previous studies, e.g. Fan and Sundaresan (2001), Ericsson (2000), Leland (1998), etc.. .

TABLE 3.1: THE INCREASE IN THE COST OF DEBT AT ISSUANCE DUE TO UNCERTAINTY ABOUT ASSETS VOLATILITY

Uncertain creditors assume highest volatility (20% =< s <= 40%)	Uncertainty conditions	Certainty conditions	Uncertainty conditions	Certainty conditions
a (bankruptcy costs as fraction of K)	20%	20%	20%	20%
r (default risk-free spot interest rate)	4%	4%	4%	4%
s (volatility of V)	**40%**	**20%**	**40%**	**20%**
d (total payout to security holders)	6%	6.0%	6%	6.0%
t (tax rate)	35%	35%	35%	35%
m (annual constant proportional debt retirement rate)	**20%**	**20%**	**40%**	**40%**
P (face value of debt)	20.0	20.0	20.0	20.0
V (assets value)	100.0	100.0	100.0	100.0
WHEN DEFAULT IS TRIGGERED ENDOGENOUSLY				
C (annual coupon, which is paid coutinuously)	1.00	0.81	0.91	0.80
Vb (value of assets triggering default)	11.1	14.6	13.6	16.9
bb+	1.6	2.7	1.6	2.7
bb-	- 0.3	- 0.7	- 0.3	- 0.7
D (debt)	20.0	20.0	20.0	20.0
b+ (D)	2.5	4.6	3.1	5.8
b- (D)	- 1.2	- 2.6	- 1.8	- 3.8
Yield spread [C+m(P-D)]/D-r	**0.99%**	**0.07%**	**0.57%**	**0.02%**
WHEN DEFAULT IS TRIGGERED BY LACK OF LIQUIDITY				
C (annual coupon, which is paid coutinuously)	0.94	0.81	0.86	0.80
Vb (value of assets triggering default)	20.9	20.6	22.5	22.4
D (debt)	20.0	20.0	20.0	20.0
b+ (D)	2.5	4.6	3.1	5.8
b- (D)	- 1.2	- 2.6	- 1.8	- 3.8
Yield spread [C+m(P-D)]/D-r	**0.68%**	**0.07%**	**0.32%**	**0.02%**
WHEN DEFAULT IS TRIGGERED BY A POSITIVE NET WORTH COVENANT				
C (annual coupon, which is paid coutinuously)	0.96	0.81	0.90	0.80
Vb (value of assets triggering default)	20.0	20.0	20.0	20.0
D (debt)	20.0	20.0	20.0	20.0
b+ (D)	2.5	4.6	3.1	5.8
b- (D)	- 1.2	- 2.6	- 1.8	- 3.8
Yield spread [C+m(P-D)]/D-r	**0.79%**	**0.07%**	**0.51%**	**0.02%**

3. UNCERTAIN ASSETS VOLATILITY AND SUBORDINATED DEBT VALUE

This section studies the value and credit spread of subordinated debt of finite maturity when creditors are again uncertain about assets volatility. Since subordinated debt can be a locally convex function of the firm's assets value, high constant volatility generally does not minimise the value of subordinated debt. Then this section analyses the value and credit spreads of subordinated debt when volatility is highest and constant and when volatility evolves in such a way so as to minimise subordinated debt value (worst-case volatility scenario). The worst-case volatility scenario is such that, when debt gamma[29] is positive, volatility is lowest, and when debt gamma is negative, volatility is highest. For subordinated debt, debt gamma is positive when assets value is low and time to maturity is short, whereas gamma is negative when assets value is high and time to maturity is long.

Subordinated debt credit spreads under worst-case volatility provide an upper bound to credit spreads under uncertainty and can be well higher than credit spreads under highest constant volatility.

The valuation of subordinated debt

Before proceeding with the analysis, some more notation and assumptions are introduced. We assume[30]:

- a single senior debt issue with maturity "T", principal equal to "P" and continuously paying a coupon at an yearly rate "C";
- "SD" is the value of a single subordinated debt issue with maturity T, principal "SP" and continuously paid coupon "SC";
- bankruptcy costs are now null, for simplicity;
- assets volatility "s" still remains within a bounded range;
- V* is the default barrier before maturity T; when V=V* default is triggered.

The assets generated cash flow rate, "Vd", is already net of taxes. The cost of total debt net of the tax shield is (1-t) (SC+C). If default before maturity is triggered by a cash flow shortage, then V* = (1-t) (SC+C)/d. For "d" sufficiently high, V* may well be below the face value of senior and subordinated debt (P+SP). Similarly, if the firm's assets were perfectly fungible and could be easily sold to finance coupon payments, the default barrier before debt maturity would be V*= 0, because default would take place just when all assets had been sold. On the other hand, at debt maturity (T) the firm's assets need be more valuable than the face value of the debt that falls due (SP+P). Then it can be shown that

3) $SD_t + \frac{1}{2} \cdot s^2 \cdot V^2 \cdot SD_{VV} + (r-d) \cdot V \cdot SD_V - r \cdot SD + SC = 0$

subject to the terminal condition

3.1) SD(V,T) = min [SP, max[V(T) - P, 0]],

and subject to the boundary conditions

3.2) $SD(V \to \infty, t) \to \frac{SC}{r} \cdot \left(1 - e^{-r(T-t)}\right) + SP \cdot e^{-r(T-t)}$,

3.3) SD(V*,t) = min [SP, max[V* - P, 0]].

When V*<P and the absolute priority rule is enforced, all assets would be assigned to senior debt holders upon default before T, i.e. SD(V*, t) = 0. Whenever V*<P, debt gamma [SD_{VV}] is not single signed and the worst case valuation of subordinated debt requires solving the following system of equations

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[29] Gamma is the second derivative of debt value with respect to the firm's assets value.

[30] As before, "r" is the (constant) default free interest rate, "V" is the value of the firm's assets, "d" is the assets payout rate, "s" is the volatility if the firm's assets. In this section T denotes debt

$$3.4)\ SD_t^w + \frac{1}{2}\cdot\left(s^+\cdot V\right)^2\cdot SD_{VV}^w + (r-d)\cdot V\cdot SD_V^w - r\cdot SD^w + SC = 0,\ \text{if } SD_{VV} \le 0,$$

$$3.5)\ SD_t^w + \frac{1}{2}\cdot\left(s^-\cdot V\right)^2\cdot SD_{VV}^w + (r-d)\cdot V\cdot SD_V^w - r\cdot SD^w + SC = 0,\ \text{if } SD_{VV} \ge 0.$$

The terminal and boundary conditions remain the same as in 3.1) to 3.3) above. The superscript "w" denotes the value of "SD" under worst-case volatility. The superscript "+" or "-" above "s" indicate respectively highest or lowest volatility in the assumed bounded volatility range. Explicit finite differences allow to numerically solve 3.4) and 3.5) subject to 3.1), 3.2), 3.3). Figure 3.1 displays the results in a base case scenario.

The credit spreads

Figure 3.1 shows that credit spreads on subordinated debt with worst-case volatility are higher than with highest constant volatility. This is due to the fact that the gamma of subordinated debt changes sign. Figure 3.1 shows the typical hump shape term structure of credit spreads that is found also in other studies. Such hump shape is found both when highest constant volatility is assumed and when worst-case volatility is assumed. The credit spreads commanded by subordinated debt can often be substantially greater with worst-case volatility than with highest constant volatility: the difference in the spreads widens for lower assets values and narrows for higher assets values. The same applies also to the so called "relative spread", i.e. the ratio given by the spread obtained with worst volatility divided by the spread obtained with highest volatility. Such ratio rises at lower assets values and drops to one for higher assets values. Thus, the nearer the assets value is to the default barrier, the less prudent it is to value subordinated debt with highest constant volatility. When instead the assets value is far from the default barrier, valuing subordinated debt with highest constant assets volatility can be sufficiently prudent.

The effect of coupons

The difference between subordinated debt value with highest volatility and subordinated debt value with worst volatility is lower for coupon debt than for zero coupon debt. Actually, coupons have a mitigating effect on the worst-case value of debt. Coupons make the subordinated debt value function more "concave" with respect to the firm's assets value. The more (and the higher) the coupons yet to be paid out before maturity, the smaller the probability that subordinated debt gamma may turn positive. So, the longer the debt maturity, the more coupons are yet to be paid

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maturity.

and the smaller is the difference between subordinated coupon debt value under worst case volatility and under highest constant volatility.

Numerical experiments, some of which are unreported, allow to conclude that worst case volatility can lead to sensitively higher credit spreads for subordinated debt than highest constant volatility would imply, especially for zero coupon subordinated debt. Hence prudent uncertain subordinated debt holders may indeed not want to simply assume highest constant assets volatility.

The default barrier and the results about uncertain assets volatility

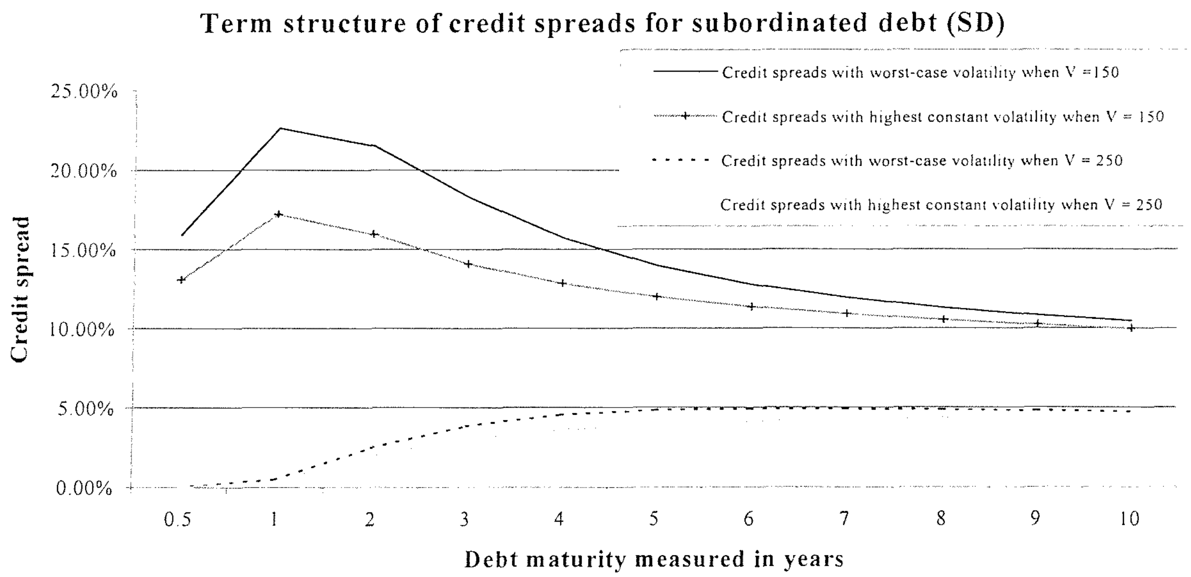
Finally, it must be stressed that the results in this section depend on the assumption that $V^*<P$. When $V^* \geq P$ subordinated debt value with highest constant volatility and with worst-case volatility coincide. Then raising the default barrier (V^*) has in general two effects on subordinated debt. Firstly it increases the "vega"[31] of subordinated debt, where the "vega" is the sensitivity of debt value to a change in assets volatility. Secondly it decreases the difference in subordinated debt value with highest constant volatility and with worst-case volatility. After this analysis of subordinated debt, next subordinated convertible debt is analysed.

[31] The vega is meant to be the first derivative of debt value with respect to the volatility of the firm's assets.

FIGURE 3.1: CREDIT SPREADS ON SUBORDINATED DEBT UNDER UNCERTAIN VOLATILITY

INPUT PARAMETERS		
Assets value		250 or 150
Default barrier		65.0
Assets payout rate		6%
Lower volatility bound		10%
Upper volatility bound		30%
Default risk free interest rate		4%
Debt maturity (mesured in years)		10
Subordinated debt face value		10
Senior debt face value		100
Coupon on subordinated debt		10%
Coupon on senior debt		5%
No. asset steps	(Multiple of 8)	160

	Bond values				Credit spreads			
	V=150		V=250		V=150		V=250	
Years to maturity	Worst volatility	Maximum volatility	Worst volatility	Maximum volatility	Worst volatility	Maximum volatility	Worst volatility	Maximum volatility
0.5	9.53	9.66	10.30	10.30	15.87%	13.09%	0.00%	0.00%
1	8.54	8.99	10.54	10.55	22.63%	17.21%	0.48%	0.38%
2	7.57	8.36	10.66	10.77	21.51%	15.94%	2.48%	1.92%
3	7.31	8.13	10.57	10.81	18.30%	14.08%	3.87%	3.01%
4	7.31	8.01	10.49	10.83	15.72%	12.84%	4.55%	3.59%
5	7.37	7.94	10.47	10.85	13.97%	11.98%	4.84%	3.94%
6	7.44	7.90	10.51	10.89	12.77%	11.36%	4.90%	4.13%
7	7.49	7.87	10.57	10.93	11.95%	10.90%	4.91%	4.25%
8	7.55	7.86	10.65	10.98	11.31%	10.52%	4.87%	4.32%
9	7.60	7.85	10.74	11.03	10.83%	10.24%	4.81%	4.37%
10	7.64	7.85	10.84	11.15	10.46%	9.99%	4.74%	4.31%



4. UNCERTAIN VOLATILITY AND SUBORDINATED CONVERTIBLE DEBT VALUE

This section studies how uncertain assets volatility affects the value of subordinated convertible debt. Convertible debt is often subordinated. This type of debt is of interest because the conversion option, when "deep in the money", causes the gamma of subordinated debt to turn positive. This fact can contribute to make the constant assets volatility assumption an inadequate one. Creditors are again assumed to be uncertain about assets volatility, and to know that assets volatility remains within a bounded range. The bounds of the assets volatility range are all we need, irrespective of the actual dynamics of volatility within the range.

As Brennan and Schwartz (1988) first pointed out, debt cum warrant and convertible debt may be issued because such debts are less sensitive to the firm's assets volatility than other types of debt are. Actually, the vega[32] of convertible debt may be lower than the vega of similar non convertible (straight) debt. Brennan and Schwartz concluded that convertibles would then be less sensitive to assets substitution and/or to asymmetric information about the true volatility of the firm's asset, which explains why convertibles are often issued by fast growing and risky firms that are quite "opaque" to outside investors. Convertibles would allow such firms to limit the cost of borrowing[33] when assets volatility is uncertain for debt holders.

Brennan and Schwartz did not consider that convertible debt may be a locally concave function of asset value when default risk is considered, i.e. they did not consider that higher assets volatility may sometimes decrease rather than increase the value of the convertible.

More precisely, when assets value is next to the default barrier, subordinated debt can be a locally convex function of assets value as seen above. Then, when assets value rises, the value of the subordinated convertible becomes a locally concave function of assets value. But, when assets value keeps rising, the conversion option becomes "in the money" and the subordinated convertible becomes again a locally convex function of assets value. As a consequence, an increase in volatility may increase the value of a near default subordinated convertible, decrease the value of a subordinated convertible that is not "in the money", and increase the value of an "in the money" subordinated convertible. For simplicity just convertibles to be converted at maturity are here considered, so this analysis applies to convertibles with "European" conversion options and possibly to convertibles with "American" conversion options when no dividends are distributed to equity holders.

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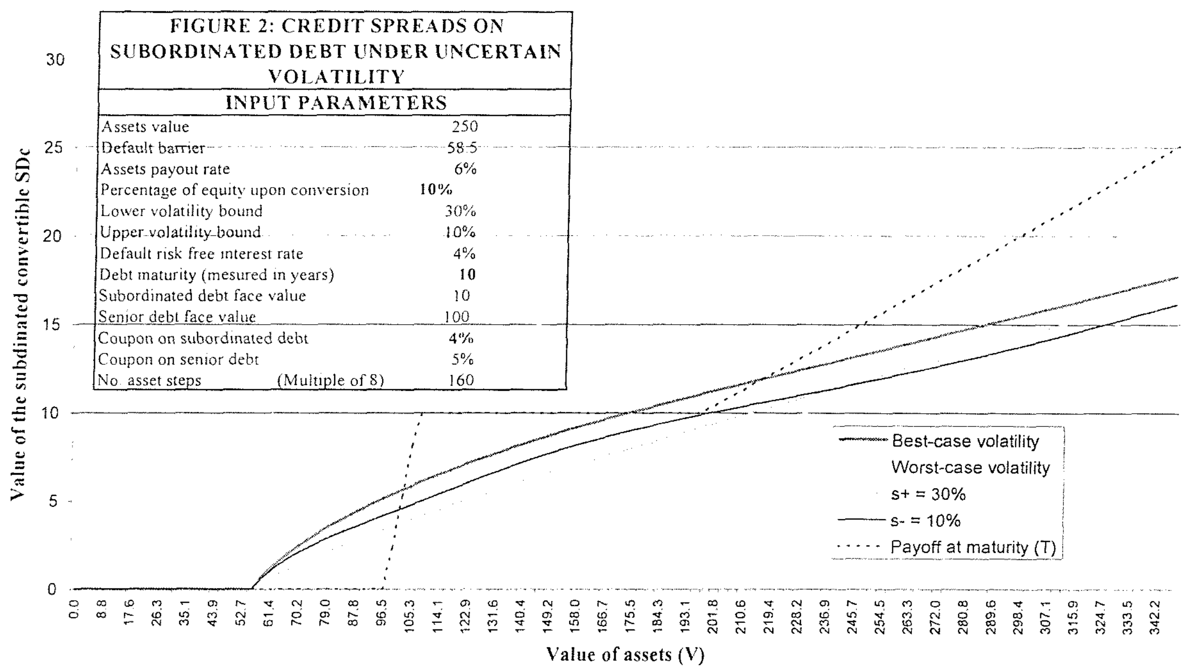
[32] The vega is the first derivative of convertible debt value with respect to assets volatility.
[33] More precisely, when assets volatility increases, the required coupons on convertible debt would increase less than the required coupons on "straight" debt.

The valuation model and results

We now assume the same model as in the previous section and value a subordinated convertible debt rather than a plain subordinate debt. SDc(V, t) denotes the value of the subordinated convertible, which again satisfies equations 3), 3.4), 3.5) and conditions 3.2) and 3.3), but condition 3.1) is now substituted by:

4.1) $SDc(V,T) = \max\{z\,(V(T) - P),\ \min\,[SP,\ \max[V(T) - P,\ 0]]\}$,

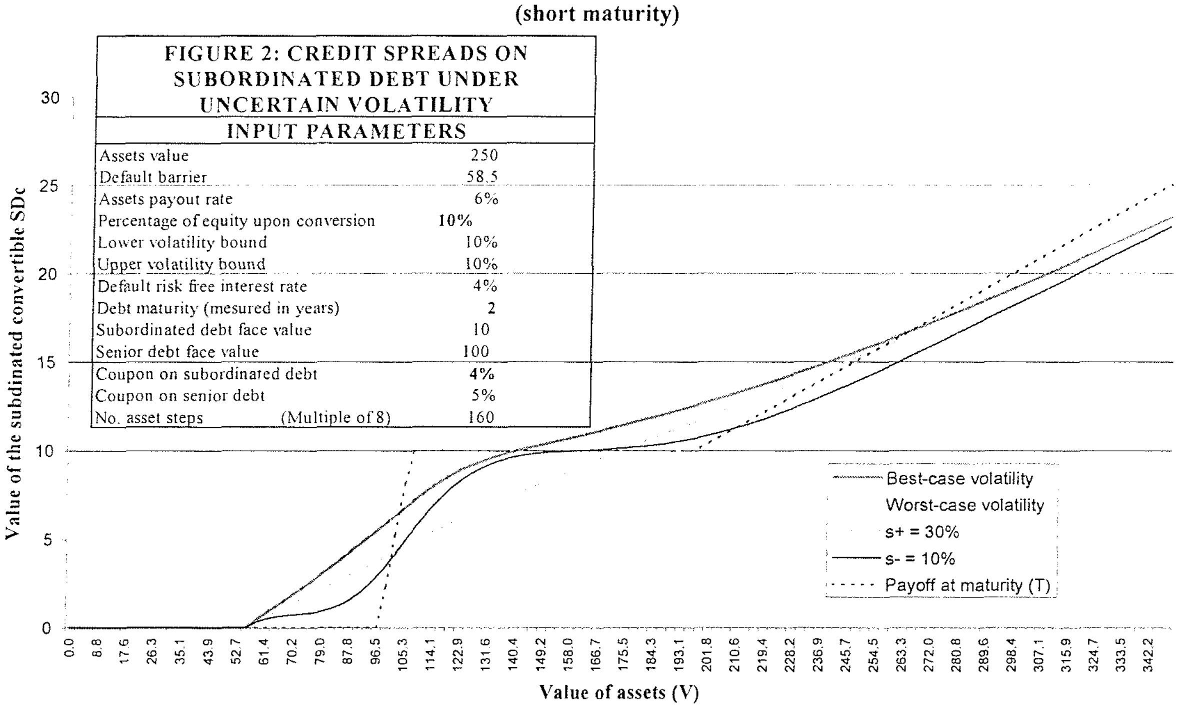
Figure 3.2.1: Value of subordinated convertible debt with best, worst and constant volatility (long maturity)



where "z" is the percentage of equity into which the subordinated convertible can be converted at maturity (T). Figures 3.2 display some of the results of the explicit finite difference numerical solutions to the convertible valuation problem. Figures 3.2 show the following points. The value of the convertible with highest constant volatility crosses over the value of the convertible with lowest constant volatility. When assets volatility is constrained within a bounded range but need not be constant over time, the value of the convertible under the worst-case volatility scenario can be significantly lower than when volatility is constrained to remain constant over time. Hence assuming volatility is constant may cause us to significantly underestimate the sensitivity of the convertible to uncertainty in assets volatility. Moreover, assuming constant volatility can imply a materially higher value for the convertible than the most prudent (worst) volatility scenario would imply. Then Figure 3.2.2 highlights how the value of subordinated convertible debt becomes more sensitive to the different volatility assumptions as maturity gets nearer.

The conclusion for this section is that prudent investors may want to value subordinated convertible debt by assuming worst-case assets volatility. Worst-case volatility implies that the range of possible values of the convertible is wider than the range encompassed between highest constant volatility and lowest constant volatility. Hence, assuming worst-case volatility implies that the value of the convertible can be significantly more sensitive to uncertainty about assets volatility than was suggested in the analysis by Brennan and Schwartz (1988). In particular this is true in the "out of the money" region of the convertible, i.e. the region in which convertibles are usually priced at issuance.

Figure 3.2.2: Value of subordinated convertible debt with best, worst and constant volatility (short maturity)



As a consequence, if the convertible is valued assuming worst-case volatility, the argument by Brennan and Schwartz about the insensitivity of convertibles to assets volatility may be challenged, and the anecdotally reported under-pricing of convertibles at issuance may be perhaps explained.

5. DEBT VALUE AND ASSETS VOLATILITY WHEN DEFAULT FREE INTEREST RATES ARE STOCHASTIC

In this section it is shown how the sensitivity of the firm's debt value to uncertainty in assets volatility depends on the fact that default free interest rates are stochastic and instantaneously correlated with the value of the firm's assets[34]. Given the firm's asset value is correlated with the short rate, it is shown how different short rate models imply different sensitivities of debt value to (uncertainty in) assets volatility. Two popular Gaussian short rate models are assumed: the Merton-Ho-Lee and Vasicek models. The Merton-Ho-Lee short rate model implies that medium to long-term debt is much more sensitive to assets volatility than the Vasicek short rate model would imply. This section focuses on zero-coupon bonds, but the results can be readily extended also to coupon bonds, which can be viewed and valued as portfolios of zero coupon bonds.

The fact that debt sensitivity to assets volatility depends on the short rate process characterises other recent contributions such as in Cakici and Chatterjee (1993)[35], Longstaff and Schwartz (1995), Tauren (1999), Dufresne and Golstein (2001). These papers assume that the short rate follows either the Vasicek or the CIR process. Instead, this section assumes that the short rate follows either the Vasicek or the Merton-Ho-Lee process.

Then, this section, unlike the above literature and the previous sections, assumes markets to be incomplete, i.e. the firm's assets neither are a traded security nor can be replicated by a traded portfolio. So the "risk neutral" process for the firm's assets value is:

5.1) $\partial V = V\,[(y - l\,s - d)\,dt + s\,\partial z_V]$,

where "y" is the real drift of V, "l" is the market price of "V risk", "d" is the constant assets payout rate, "s" is assets volatility and ∂z_V is the Wiener process for V. The assumption of incomplete markets is here convenient because it allows to "separate" the variables (assets value and short rate) as per equation 5.3) below, thus simplifying the analysis.

Then, assume the short rate [r] follows Vasicek's model

5.2) $\partial r = (n - r\,g)\,dt + w\,\partial z_r$,

where n, g, w are constants, dt is the time increment and ∂z_r is the increment of the Wiener process for the short rate.

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[34] Longstaff and Schwartz (1995) already noted the importance of recognising such correlation for valuing debt.

[35] Cakici and Chatterjee (1996) highlight this interaction through numerical solutions to the debt pricing equation.

If V and r are instantaneously correlated such that $\partial z_V \cdot \partial z_r = \rho \cdot dt$, then it can be shown[36] that the value of a zero coupon bond (D*(r,V,t)) issued by a firm is:

5.3) D*(r,V,t) = Z(r,t) H(V,t),

where Z(r,t) is the value of a default free zero coupon bond as per Vasicek, and where H(V,t) must satisfy the following

$$5.4)\ H_t + \frac{1}{2}(V \cdot s)^2 H_{VV} + \left(-\rho \cdot w \cdot s \cdot \frac{1}{g}[1 - e^{-g(T-t)}] + y - ls - d\right) \cdot V \cdot H_V = 0$$

subject to

5.4.1) H(V, t=T)=1 (the face value of the bond is normalised at 1)

5.4.2) H(V=L, t)=1-a (L is the default barrier and “a” is the loss given default expressed as a percentage of Z)

5.4.3) $H(V \to \infty, t) \to 1$.

Instead, if the short rate follows Merton-Ho-Lee’s process, 5.2) is replaced by

5.5) $\partial r = q(t)\,dt + w\,\partial z_r$, with q(t) being a function of time; q(t) is usually calibrated to the observed term structure and need not be known for our purposes.

Ceteris paribus, the Ho-Lee model implies that 5.3) becomes[37]

5.6) D*(r,V,t) = Z'(r,t) H(V,t)

where Z'(r,t) is the value of a default free zero coupon bond as per Ho-Lee, and where H(V,t) must now satisfy the following

$$5.7)\ H_t + \frac{1}{2}(V \cdot s)^2 H_{VV} + (-\rho \cdot w \cdot s \cdot (T-t)] + y - ls - d) \cdot V \cdot H_V = 0$$

subject to the boundary conditions 5.4.1), 5.4.2) and 5.4.3).

In all that follows, we assume that l=0 for the sake of simplicity. In equations 5.4) and 5.7) assets volatility (s) not only appears in the coefficients of the second derivatives, but also in the coefficients of the first derivatives. The coefficients of the first derivatives cause debt value and debt value sensitivity to (uncertainty about) assets volatility to depend on the correlation between the firm’s assets and the short rate. In particular:

- 1) if the firm’s assets have no correlation with the short rate, an increase in assets volatility decreases H(V,t) and D*(V,t);

[36] The results are proved in Appendix IV of chapter 5 if only we substitute "y" with "m", "L" with "V_B" and "d" with "b".

[37] Again the proof of these results is in Appendix IV of Chapter 5 (see the previous note).

- 2) if the firm's assets are positively correlated with interest rates, an increase in assets volatility decreases H(V,t) and D*(V,t) more than in the case of no correlation (higher "s" decreases the coefficients of the first derivatives in equations 5.4) and 5.7));
- 3) if the firm's assets are negatively correlated with interest rates, an increase in assets volatility decreases H(V,t) and D*(V,t) less than in the case of no correlation (higher "s" increases the coefficients of the first derivatives in equations 5.4) and 5.7)).

Let us now concentrate on the coefficients of the first derivatives in equations 5.4) and 5.7). The effect of assets volatility on such coefficients, and hence on debt value (D*(V, r, t)), depends on the short rate process. If $\rho \neq 0$, such coefficients are more sensitive to "s" when (T-t) is higher, i.e. when debt is of longer maturity. As a result, the longer debt maturity is, the greater the impact of different short rate processes on H(V,t), on debt value and on debt sensitivity to assets volatility.

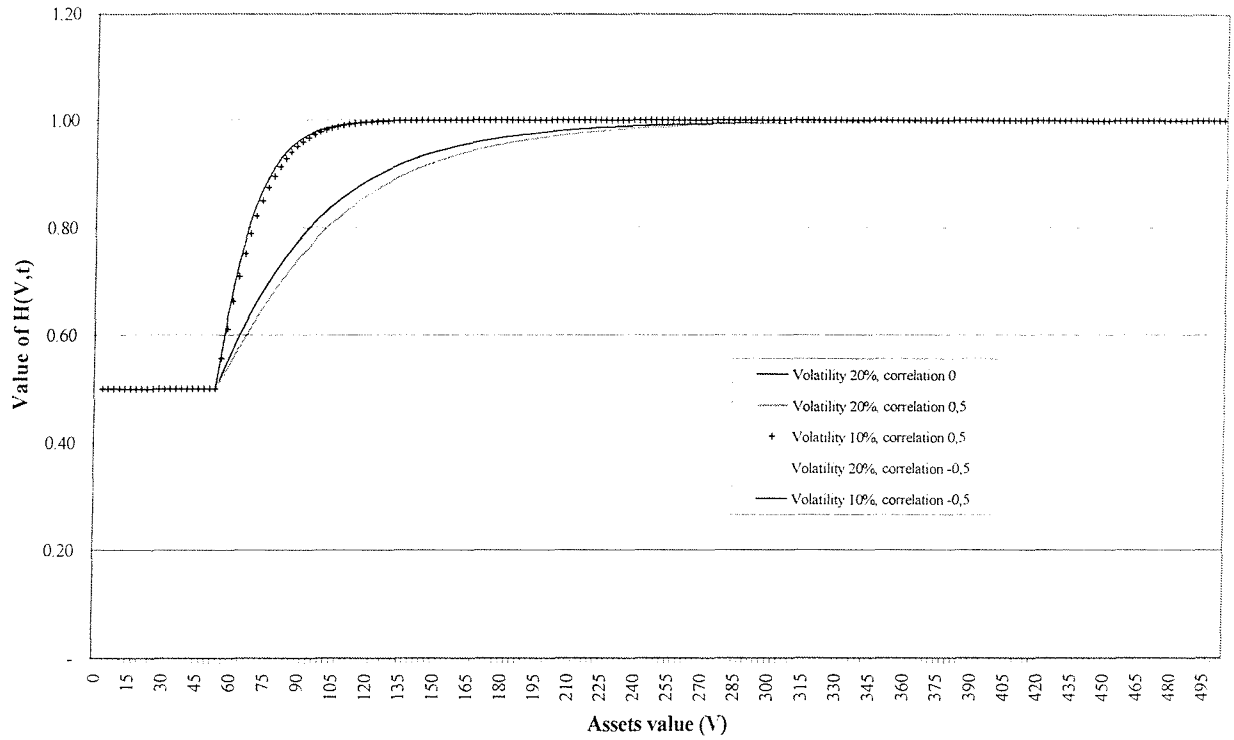
If the assets value and the short rate are correlated and the short rate behaves according to the Ho-Lee model, debt is much more sensitive to assets volatility than in the case of no correlation. Instead, if the assets value and the short rate are correlated, but the short rate behaves according to the Vasicek model, debt sensitivity to assets volatility is quite similar to debt sensitivity in case of no correlation. In other words the Vasicek model predicts that the correlation between the short rate and the firm's assets has a smaller effect on debt value and debt sensitivity to firm's assets volatility than the Ho-lee model would predict.

A numerical example

Figure 3.3 shows the results of finite differences numerical solutions to equations 5.4) and 5.7), under the following parameters: T = 10 years, L = 50, y = 6%, l = 0, s = 10% or 20%, ρ = 0.5 or -0.5, d = 5%, r = 5%, g = 0.5 and w = 2%. Figure 3.3 shows that, assuming the Ho-Lee model and V = 100, an increase in assets volatility from 10% to 20% causes H(V,t) to drop from 0,9885 to 0,8684 if the correlation coefficient (ρ) is –0,5. But, if the correlation coefficient is 0,5, the same increase in volatility causes H(V,t) to drop from 0,985 to 0,8063. Thus, if correlation is -0,5, H(V,t) and D*(V,t) drop by 12,15%. If correlation is 0,5, H(V,t) and D*(V,t) drop by 18,14%. This example shows that, if the short rate follows the Merton-Ho-Lee process, the correlation (ρ) can have a material impact on debt value and debt sensitivity to (uncertainty in) assets volatility.

Assuming the above base case parameters and the Vasicek process for the short rate, an increase in assets volatility from 10% to 20% means H(V,t) drops from 0,9911 to 0,8476 if the correlation coefficient is –0,5. Instead H(V,t) drops from 0,9885 to 0,8294 if the correlation coefficient is 0,5. Thus, if correlation is 0,5, H(V,t) and D*(V,t) drop by 16,1%, whereas if correlation is -0,5, H(V,t) and D*(V,t) drop by 14, 47%. The Vasicek model implies that correlation has less of an impact on debt sensitivity to assets volatility.

FIGURE 3.3: H(V,t) WHEN THE SHORT RATE FOLLOWS THE MERTON-HO-LEE PROCESS.
(T=10 years, V=100, L=50, y=6%, d=5%,r=5%, w=2%)



Importantly, the above examples show also that the higher the assets volatility, the more debt value (D*(V,t)) is sensitive to the short rate model. Actually, at 20% volatility and 0,5 correlation, when assuming the Ho-Lee model H(V,t) equals 0,8063 (0,985 with 10% volatility), whereas when assuming the Vasicek model, H(V,t) equals 0,8294 (0,9885 with 10% volatility).

The results of this section may provide guidance as to the choice of the short rate model when valuing a firm's debt. The Merton-Ho-Lee model is simple and easy to calibrate to the observed term structure, but it may artificially increase medium to long-term debt value sensitivity to the firm's assets volatility when assets value and short rate are instantaneously correlated. As a result, when assuming the Merton-Ho-Lee model, there is a risk, especially for long-term debt, to overstate the increase in the cost of debt due to creditors' uncertainty about the debtor's assets volatility.

CONCLUSIONS

This chapter has analysed debt value and debt yield when debt holders are uncertain about the debtor's assets volatility. Assuming assets volatility at any time remains within a bounded range, the following conclusions have been derived.

The increase in debt credit spreads due to debt holders' uncertainty about assets volatility is very sensitive to the assumed default condition. When default is endogenous, such cost can be much higher than when default is triggered by lack of liquidity. The debtor can reduce the cost of uncertainty by reducing debt maturity, by inserting a positive net worth covenant in the debt indenture or by inserting a cash flow based covenant.

When debt gamma changes sign, highest constant volatility does not minimise debt value. This sensitively affects the prudent (i.e. worst case) valuation of subordinated debt and of subordinated convertible debt by uncertain debt holders. Such debt holders may want to require a higher yield than they would require under the assumption of highest constant assets volatility. As for subordinated debt, higher coupons, longer debt maturity, higher default barrier before maturity and higher assets value all imply subordinated debt value with highest constant volatility is closer to worst case subordinated debt value. As for subordinated convertible debt, assuming constant volatility may lead us to underestimate the sensitivity of the convertible to uncertainty in assets volatility, especially when the convertible is "out of the money".

When the default free interest rate is assumed stochastic and correlated with the firm's assets value, the choice of the short rate model may materially alter the sensitively of debt to changes in (and to uncertainty about) assets volatility. For long-term debt such sensitivity is materially higher if the Merton-Ho-Lee short rate model is assumed rather than the Vasicek short rate model. Moreover, the presence of correlation between assets and short rate entails that the higher the assets volatility is, the greater is the difference in debt value under the different short rate models.

APPENDIX I: THE CODE

The Visual Basic code that implements the numerical solutions to the valuation problems in this chapter is now provided. The numerical schemes employ explicit finite differences.

```
'The following is the code to value subordinated debt under uncertain assets volatility

Private VOld(0 To 200) As Double
Private Gamma(0 To 200) As Double

'The parameters of the algorithm have the following meaning:
'Asset denotes the value of the firm's assets
'SeniorFace denotes the face value of senior debt
'JuniorFace denotes the face value of junior debt
'Maturity denote debt maturity
'Volatility1 denotes the lower assets volatility bound
'Volatility2 denotes the upper assets volatility bound
'Intrate denotes the default free interest rate
'SeniorCoupon denotes the yearly coupon rate on senior debt
'JuniorCoupon denotes the yearly coupon rate on subordinated debt
'Payout denotes the assets payout rate
'NoAssetSteps denotes the number of intervals of equal length in which the solution domain in the V 'direction is
divided

Function SubDebtValue(Asset As Double, SeniorFace As Double, JuniorFace As Double, Maturity As Double,
Volatility1 As Double, Volatility2 As Double, Intrate As Double, _
SeniorCoupon As Double, JuniorCoupon As Double, Payout As Double, _
NoAssetSteps As Integer)
Dim VNew(0 To 200) As Double
Dim Delta(0 To 200) As Double
Dim Theta(0 To 200) As Double
Dim S(0 To 200) As Double
Dim Ssqd(0 To 200) As Double
halfvol1sqd = 0.5 * Volatility1 * Volatility1
halfvol2sqd = 0.5 * Volatility2 * Volatility2

Barrier = (1 - 0.35) * (SeniorCoupon * SeniorFace + JuniorCoupon * JuniorFace) / Payout
'The tax rate is assume to be 35% so that the cost of total debt net of the tax shield is:
'(1 - 0.35) * (SeniorCoupon * SeniorFace + JuniorCoupon * JuniorFace).

assetstep = 8 * Barrier / NoAssetSteps
mini = Barrier / assetstep

NearestGridPt = Int(Asset / assetstep)
dummy = (Asset - NearestGridPt * assetstep) / assetstep
volfortimestep = Application.Max(Volatility1, Volatility2)
timestep = assetstep * assetstep / volfortimestep / volfortimestep / (64 * Barrier * Barrier)
NoTimesteps = Int(Maturity / timestep) + 1
timestep = Maturity / NoTimesteps

For i = mini To NoAssetSteps
  S(i) = i * assetstep
  Ssqd(i) = S(i) * S(i)
  VOld(i) = Application.Max(S(i) - SeniorFace, 0) - Application.Max(S(i) - (SeniorFace + JuniorFace), 0)
Next i

For j = 1 To NoTimesteps
  For i = (mini + 1) To NoAssetSteps - 1
    Delta(i) = (VOld(i + 1) - VOld(i - 1)) / (2 * assetstep)
```

```
        Gamma(i) = (VOld(i + 1) - 2 * VOld(i) + VOld(i - 1)) / (assetstep * assetstep)
  If Gamma(i) >= 0 Then
        VNew(i) = VOld(i) + timestep * (halfvol1sqd * Ssqd(i) * Gamma(i) + _
        (Intrate - Payout) * S(i) * Delta(i) - Intrate * VOld(i))
        Else
        VNew(i) = VOld(i) + timestep * (halfvol2sqd * Ssqd(i) * Gamma(i) + _
        (Intrate - Payout) * S(i) * Delta(i) - Intrate * VOld(i))
  End If
     Next i
     VNew(mini) = Application.Min(JuniorFace, Application.Max(S(mini) - SeniorFace, 0))
     VNew(NoAssetSteps) = 2 * VNew(NoAssetSteps - 1) - VNew(NoAssetSteps - 2)

  For i = 0 To NoAssetSteps
     If S(i) > Barrier Then
     VOld(i) = VNew(i) + JuniorCoupon * JuniorFace * timestep
     Else
     VOld(i) = VNew(i)
     End If
  Next i

  For i = 1 To NoAssetSteps - 1
  Gamma(i) = (VOld(i + 1) - 2 * VOld(i) + VOld(i - 1)) / (assetstep * assetstep)
  Next i

  Next j

  SubDebtValue = (1 - dummy) * VOld(NearestGridPt) + dummy * VOld(NearestGridPt + 1)

  End Function
```

The following code values subordinated convertible debt under uncertain assets volatility.

```
Private VOld(0 To 200) As Double
Private Gamma(0 To 200) As Double

'The parameters of the algorithm have the following meaning:
'Asset denotes the value of the firm's assets
'SeniorFace denotes the face value of senior debt
'JuniorFace denotes the face value of junior debt
'Maturity denote debt maturity
'Volatility1 denotes the lower assets volatility bound
'Volatility2 denotes the upper assets volatility bound
'Intrate denotes the default free interest rate
'SeniorCoupon denotes the yearly coupon rate on senior debt
'JuniorCoupon denotes the yearly coupon rate on subordinated debt
'Payout denotes the assets payout rate
'Conversion denotes the percebatge of equity into which the convertible can be converted at maturity.
'NoAssetSteps denotes the number of intervals of equal length in which the solution domain in the V 'direction is
divided

'Option Explicit

Private VOld(0 To 200) As Double
Private Gamma(0 To 200) As Double

Function ConvSubDebtValue(Asset As Double, SeniorFace As Double, JuniorFace As Double, Maturity As Double, _
Volatility1 As Double, Volatility2 As Double, Intrate As Double, _
SeniorCoupon As Double, JuniorCoupon As Double, Payout As Double, _
Conversion As Double, NoAssetSteps As Integer)
```

```
    Dim VNew(0 To 200) As Double
    Dim Delta(0 To 200) As Double
    Dim Theta(0 To 200) As Double
    Dim S(0 To 200) As Double
    Dim Ssqd(0 To 200) As Double
    halfvol1sqd = 0.5 * Volatility1 * Volatility1
    halfvol2sqd = 0.5 * Volatility2 * Volatility2

    Barrier = (1 - 0.35) * (SeniorCoupon * SeniorFace + JuniorCoupon * JuniorFace) / Payout
    assetstep = 8 * Barrier / NoAssetSteps
    mini = Barrier / assetstep

    NearestGridPt = Int(Asset / assetstep)
    dummy = (Asset - NearestGridPt * assetstep) / assetstep
    volfortimestep = Application.Max(Volatility1, Volatility2)
    timestep = assetstep * assetstep / volfortimestep / volfortimestep / (64 * Barrier * Barrier)
    NoTimesteps = Int(Maturity / timestep) + 1
    timestep = Maturity / NoTimesteps

    For i = mini To NoAssetSteps
       S(i) = i * assetstep
       Ssqd(i) = S(i) * S(i)
       'VOld(i) = Application.Max(S(i) - SeniorFace, 0) - Application.Max(S(i) - (SeniorFace + JuniorFace), 0)
    VOld(i) = Application.Min(JuniorFace, Application.Max(S(i) - SeniorFace, 0))
    VOld(i) = Application.Max(VOld(i), Conversion * (S(i) - SeniorFace))
    Next i

    For j = 1 To NoTimesteps
       For i = (mini + 1) To NoAssetSteps - 1
          Delta(i) = (VOld(i + 1) - VOld(i - 1)) / (2 * assetstep)
          Gamma(i) = (VOld(i + 1) - 2 * VOld(i) + VOld(i - 1)) / (assetstep * assetstep)
    If Gamma(i) >= 0 Then
          VNew(i) = VOld(i) + timestep * (halfvol1sqd * Ssqd(i) * Gamma(i) + _
          (Intrate - Payout) * S(i) * Delta(i) - Intrate * VOld(i))
          Else
          VNew(i) = VOld(i) + timestep * (halfvol2sqd * Ssqd(i) * Gamma(i) + _
          (Intrate - Payout) * S(i) * Delta(i) - Intrate * VOld(i))
    End If
       Next i
       VNew(mini) = Application.Min(JuniorFace, Application.Max(S(mini) - SeniorFace, 0))
       VNew(NoAssetSteps) = 2 * VNew(NoAssetSteps - 1) - VNew(NoAssetSteps - 2)

    For i = 0 To NoAssetSteps
       If S(i) > Barrier Then
       VOld(i) = VNew(i) + JuniorCoupon * JuniorFace * timestep
       Else
       VOld(i) = VNew(i)
       End If
    Next i

    For i = 1 To NoAssetSteps - 1
    Gamma(i) = (VOld(i + 1) - 2 * VOld(i) + VOld(i - 1)) / (assetstep * assetstep)
    Next i

    Next j

    ConvSubDebtValue = (1 - dummy) * VOld(NearestGridPt) + dummy * VOld(NearestGridPt + 1)

    End Function
```

The following code values debt when the debtor's assets value is instantaneously correlated with the default free short rate and the short rate follows either the Ho-Lee or the Vasicek process.

```
Option Explicit

Private VOld(0 To 300) As Double
Private Gamma(0 To 300) As Double

The meaning of the parameters of the algorithm "DebtValueCorrelationHoLee" is the following:
'AssetsValue denotes the value of the firm's assets (V)
'Barrier denotes the level of the default barrier (L)
'DebtFaceValue denotes the face value of debt (which is assumed to be equal to 1)
'DebtMaturity denotes debt maturity (T)
'AssetsVolatility denotes assets volatility (s)
'AssetsDrift denotes the risk neutral assets drift of the value of the firm's assets (y - l s)
'AssetsPayout denotes the assets payout rate (d)
'IntRateVolat denotes the present level of the default free interest rate (w)
'Correlation denotes the degree of correlation between the short rate and assets value (ρ)
'CouponRate denotes the interest rate of debt coupons (which is assumed to be 0)
'DebtRecoveryValue denotes the value of debt that is recovered upon default (1-a)
'NoAssetsValueSteps denotes the number of intervals of equal length into which the solution domain in 'the V
direction is divided

Function DebtValueCorrelationHoLee(AssetsValue As Double, Barrier As Double, DebtFaceValue As Double, _
DebtMaturity As Double, AssetsVolatility As Double, AssetsDrift As Double, AssetsPayout As Double, _
IntRateVolat As Double, Correlation As Double, CouponRate As Double, DebtRecoveryValue As Double,
NoAssetsValueSteps As Integer)

Dim VNew(0 To 300) As Double
Dim Delta(0 To 300) As Double
Dim Theta(0 To 300) As Double
Dim S(0 To 300) As Double
Dim Ssqd(0 To 300) As Double

Dim halfvolsqd As Double
Dim AssetsValuestep As Double
Dim NearestGridPt As Integer
Dim dummy As Double
Dim timestep As Double
Dim NoTimesteps As Integer
Dim mini As Integer

Dim i As Integer
Dim j As Integer

halfvolsqd = 0.5 * AssetsVolatility * AssetsVolatility

AssetsValuestep = 10 * Barrier / NoAssetsValueSteps
NearestGridPt = Int(AssetsValue / AssetsValuestep)
dummy = (AssetsValue - NearestGridPt * AssetsValuestep) / AssetsValuestep
timestep = AssetsValuestep * AssetsValuestep / AssetsVolatility / AssetsVolatility / (100 * Barrier * Barrier)
NoTimesteps = Int(DebtMaturity / timestep) + 1
timestep = DebtMaturity / NoTimesteps
mini = Barrier / AssetsValuestep

For i = mini To NoAssetsValueSteps
  S(i) = i * AssetsValuestep
  Ssqd(i) = S(i) * S(i)
  VOld(i) = DebtFaceValue
```

```
Next i

For j = 1 To NoTimesteps
For i = (mini + 1) To NoAssetsValueSteps - 1
    Delta(i) = (VOld(i + 1) - VOld(i - 1)) / (2 * AssetsValuestep)
    Gamma(i) = (VOld(i + 1) - 2 * VOld(i) + VOld(i - 1)) / (AssetsValuestep * AssetsValuestep)
    VNew(i) = VOld(i) + timestep * (halfvolsqd * Ssqd(i) * Gamma(i) + _
    (AssetsDrift - AssetsPayout - Correlation * IntRateVolat * j * timestep * AssetsVolatility) * S(i) * Delta(i))
Next i

VNew(mini) = DebtRecoveryValue
VNew(NoAssetsValueSteps) = 2 * VNew(NoAssetsValueSteps - 1) - VNew(NoAssetsValueSteps - 2)

For i = 0 To NoAssetsValueSteps
  If Not (S(i) < Barrier) Then
  VOld(i) = VNew(i) + CouponRate * timestep
  Else
  VOld(i) = VNew(i)
  End If

Next i
Next j

DebtValueCorrelationHoLee = (1 - dummy) * VOld(NearestGridPt) + dummy * VOld(NearestGridPt + 1)

End Function

'IntSpeed denotes the mean reversion speed parameter in the Vasicek model (g)

Function DebtValueCorrelationVasicek(AssetsValue As Double, DebtFaceValue As Double, Barrier As Double, _
DebtMaturity As Double, AssetsVolatility As Double, AssetsDrift As Double, IntRateVolat As Double, IntSpeed As
Double, _
Correlation As Double, Payout As Double, CouponRate As Double, DebtRecoveryValue As Double, _
NoAssetsValueSteps As Integer)

Dim VNew(0 To 200) As Double
Dim Delta(0 To 200) As Double
Dim Theta(0 To 200) As Double
Dim S(0 To 200) As Double
Dim Ssqd(0 To 200) As Double

Dim halfvolsqd As Double
Dim AssetsValuestep As Double
Dim NearestGridPt As Integer
Dim dummy As Double
Dim timestep As Double
Dim NoTimesteps As Integer
Dim mini As Integer

Dim i As Integer
Dim j As Integer


halfvolsqd = 0.5 * AssetsVolatility * AssetsVolatility

AssetsValuestep = 10 * Barrier / NoAssetsValueSteps
NearestGridPt = Int(AssetsValue / AssetsValuestep)
dummy = (AssetsValue - NearestGridPt * AssetsValuestep) / AssetsValuestep
timestep = AssetsValuestep * AssetsValuestep / AssetsVolatility / AssetsVolatility / (100 * Barrier * Barrier)
NoTimesteps = Int(DebtMaturity / timestep) + 1
timestep = DebtMaturity / NoTimesteps
```

```
mini = Barrier / AssetsValuestep

For i = mini To NoAssetsValueSteps
   S(i) = i * AssetsValuestep
   Ssqd(i) = S(i) * S(i)
   VOld(i) = DebtFaceValue
Next i

For j = 1 To NoTimesteps
For i = (mini + 1) To NoAssetsValueSteps - 1
      Delta(i) = (VOld(i + 1) - VOld(i - 1)) / (2 * AssetsValuestep)
      Gamma(i) = (VOld(i + 1) - 2 * VOld(i) + VOld(i - 1)) / (AssetsValuestep * AssetsValuestep)

      VNew(i) = VOld(i) + timestep * (halfvolsqd * Ssqd(i) * Gamma(i) + _
      (AssetsDrift - Payout - Correlation * IntRateVolat * (1 / IntSpeed * (1 - Exp(-IntSpeed * j * timestep))) *
AssetsVolatility) * S(i) * Delta(i))

Next i

      VNew(mini) = DebtRecoveryValue
      VNew(NoAssetsValueSteps) = 2 * VNew(NoAssetsValueSteps - 1) - VNew(NoAssetsValueSteps - 2)

For i = 0 To NoAssetsValueSteps
If Not (S(i) < Barrier) Then
VOld(i) = VNew(i) + CouponRate * timestep
Else
VOld(i) = VNew(i)
End If

Next i
Next j

DebtValueCorrelationVasicek = (1 - dummy) * VOld(NearestGridPt) + dummy * VOld(NearestGridPt + 1)

End Function
```

CHAPTER 4:
ABOUT DEBT AND THE OPTION TO EXTEND DEBT MATURITY

INTRODUCTION

Firms that approach financial distress may renegotiate their debt obligations. Such debt renegotiations can entail extending the original contractual maturity of debt in order to allow the firm to overcome temporary problems such as a temporary lack of liquidity. The re-negotiation of debt maturity as the debtor approaches financial distress has been neglected by the debt valuation literature that adopts a continuous time finance approach. Such literature has instead concentrated on the re-negotiation of contractual coupon payments or of debt principal.

In this chapter the problem is the valuation of the firm's debt (and equity) when debt holders and equity holders may have the ability to extend debt maturity in order to avoid default and costly assets liquidation. In such case debt holders and equity holders have an "implicit option to renegotiate and extend debt average maturity".

The results of the analysis are:

- the option available to equity holders to extend the average maturity of debt increases equity value more than it decreases debt value; such option can materially increase equity value, while often causing a non negligible increase or decrease in the yield required by debt holders;
- provided debt maturity is extended before assets liquidation, the different rational "extension option exercise policies" do not alter total firm value, but they significantly affect the extension option values;
- as in Longstaff (1990), sometimes it is possible for both equity holders and debt holders to benefit from the extension of debt maturity as the firm approaches distress;
- different default conditions, either worthless equity or cash flow shortage, can materially affect the values of the extension option and imply different incentives for debt holders and equity holders to re-negotiate and extend debt maturity;
- the presence of an implicit extension option can improve the prediction of the structural model by inflating short-term credit spreads.

The analysis of this chapter is split into time dependent and time independent settings. In a time independent setting, single debt issues are continuously refunded as they continuously fall due at maturity. Thus the nominal amount of debt outstanding at any time is constant, which makes the valuation of debt and equity a problem independent of time. Later, instead, the "extension option" is analysed in a time dependent setting in which debt is *not* refunded at maturity, in which the nominal amount of debt outstanding is *not* constant, in which the default probability is lower and in which the extension option is less valuable.

Past literature

Extendible debt was valued by Brennan and Schwartz in 1977 and by Ananthanarayanan and Schwartz in 1980, but these two papers assume that debt is default free. Two other papers deal with debt that is subject to default risk and whose maturity can be extended by equity holders or by debt holders. The first paper is by Franks and Torous (1989) and considers the implicit option for equity holders to file for US Code Chapter 11 reorganisation, which entails suspending all payments of coupons and principal to debt holders. Franks and Torous show that recognising this implicit option to file for chapter 11 makes contingent claims models capable of predicting credit spreads on corporate debt that more closely approximate those observed in the bond market.

The second paper is by Longtaff (1990) and provides closed form solutions for a similar option for equity holders to extend debt maturity. Longstaff also considers the option for debt holders to spontaneously extend debt maturity in order to avoid costly assets liquidation as the debtor defaults.

Both Franks/Torous and Longstaff restrict their attention to a Mertonian setting in which default and extension of debt maturity can take place just on the contractually agreed debt maturity date. Instead, this chapter considers the case in which default and extension of debt maturity can take place at any time. Central to this chapter is the case in which debt maturity can be renegotiated by equity holders and debt holders. In fact the re-negotiation of debt maturity has been neglected by the debt valuation literature concerned with strategic debt service (e.g. Anderson and Sundaresan (1996), Anderson and Sundaresan and Tychon (1996), Mella-Barral and Perraudin (1997)).

In sections 1 to 3 the analysis of the option to extend debt maturity is carried out in a time independent setting in which default is triggered by cash flow shortage or by worthless equity, whereas in section 4 the same analysis is carried out in a time dependent setting in which default is triggered by worthless equity.

1. THE GENERAL MODEL IN A TIME INDEPENDENT SETTING

1.1) Results from past literature

Now we introduce the notation and some results of past literature. These results are the basis for the subsequent analysis of the option to extend debt maturity.

Let us assume that equity holders and debt holders are risk neutral and have perfect information. V is the value of the firm's assets, whose risk neutral process is a geometric Brownian motion, i.e.:

1) $dV = (r-d)\, V\, dt + s\, V\, dz$,

where:

- s is the volatility of the firm's assets;
- d is the firm's assets pay-out rate;
- r is the default free interest rate, which is assumed constant
- dz differential of a Wiener process.

In addition, let us assume that:

- a denotes bankruptcy/liquidation costs proportional to assets value;
- K is the fixed cost of assets liquidation/bankruptcy,
- "tax" is the corporate income tax rate,
- C is the annual coupon,
- P is the face value of debt,
- $c = C / P$,
- m is the fraction of outstanding debt that is retired and substituted with newly issued debt every year (in short m is the debt retirement rate),
- ff(V) is the value of the firm's debt when extension of debt maturity is not possible,
- BC(V) is the value of the firm's bankruptcy costs,
- e(V) is the value of the firm's equity,
- TT(V) is the value of the tax shield,
- V_B is the default the barrier (i.e. the value of the firm's assets at which default occurs).

Following Leland (1998) and Ericsson (2000), but adding fixed bankruptcy costs (K), it is possible to show that

$$2)\ ff(V) = \frac{C + m \cdot P}{r + m} + \left[-\frac{C + m \cdot P}{r + m} + (1 - a) \cdot V_B - K \right] \cdot \left[\frac{V}{V_B} \right]^b$$

3)

$$e(V) = V + tax \cdot \frac{C}{r} \cdot \left\{ 1 - \left[\frac{V}{V_B} \right]^j \right\} - (a \cdot V_B + K) \cdot \left[\frac{V}{V_B} \right]^j - \frac{C + mP}{r + m} \left\{ 1 - \left[\frac{V}{V_B} \right]^b \right\} - \left[(1 - a) \cdot V_B - K \right] \cdot \left[\frac{V}{V_B} \right]^b$$

with

$$4)\ b = \frac{-\left(r - d - \frac{s^2}{2}\right) - \sqrt{\left(r - d - \frac{s^2}{2}\right)^2 + 2 \cdot (r + m) \cdot s^2}}{s^2}$$

$$5)\ j = \frac{-\left(r - d - \frac{s^2}{2}\right) - \sqrt{\left(r - d - \frac{s^2}{2}\right)^2 + 2 \cdot r \cdot s^2}}{s^2}.$$

Total firm value is then equal to

$$6)\ ff(V) + e(V) = V + tax \cdot \frac{C}{r} \cdot \left\{1 - \left[\frac{V}{V_B}\right]^j\right\} - \left(a \cdot V_B + K\right) \cdot \left[\frac{V}{V_B}\right]^j.$$

The important aspect is that every year a fraction "m" of debt is continuously refunded as it falls due. Then average debt maturity is equal to 1/m years.

1.2) When average debt maturity can be extended

The above results are next modified to account for the possibility that equity holders and debt holders renegotiate the debt contract and extend debt average maturity by rescheduling the payments of debt principal. Re-negotiation may take place in an informal workout or in a formal bankruptcy proceeding. It is important to remark that in what follows it is assumed that all single debt issues comprising the firm's total outstanding debt have their respective time to maturity extended at the same time and by the same proportion. For example, if there are two outstanding debt issues, one with a residual life of 1 year and the other one of 2 years, their respective residual lives are *simultaneously* extended to 2 years and 4 years.

Equity holders and debt holders may want to renegotiate the debt contract and extend debt maturity before default or as soon as default takes place, where default here means missing a payment that is due to creditors. By agreeing with creditors to postpone repayment of debt principal, equity holders may avoid default, insolvency or difficult and costly refunding through issuance of new debt.

On the other hand, also debt holders may be enticed to renegotiate the debt contract as explained later in section 2. An important assumption underlies all the analysis: equity holders always keep paying the contractually agreed coupons to debt holders until debt principal is eventually paid back.

Let us now assume V_R denotes the value of the firm's assets at which debt average maturity is extended. For now we take V_R as given, but in section 2 it will be shown how V_R can be determined. Anyway V_R cannot be lower than V_B, otherwise debt maturity would not be extended and there would be no difference from the analyses of past literature, thus

7) $V_R \geq V_B$.

On the other hand, it is here assumed that

8) $V_R < V_0$,

where V_0 denotes the firm's assets value today. Condition 8) does not imply a great loss in generality as it will become apparent later.

Then $1/m_R$ is debt average maturity after "extension" and $1/m$ is debt average maturity before "extension". The change from m to m_R is irreversible, and we can write:

9) $\infty \geq m \geq m_R \geq 0$.

For simplicity, in all this chapter it is assumed that debt average maturity can be extended just once.

When $V \geq V_R$, debt value before "extension", F(V), must satisfy

10) $\frac{1}{2} \cdot s^2 \cdot V^2 F_{VV} + (r - d) \cdot V \cdot F_V - r \cdot F + C + m \cdot (P - F) = 0$,

subject to $F(V \to \infty) \to \frac{C + m \cdot P}{r + m}$ and to $F(V_R) = f(V_R)$,

where debt value after "extension", f(V), must satisfy

11) $\frac{1}{2} \cdot s^2 \cdot V^2 f_{VV} + (r - d) \cdot V \cdot f_V - r \cdot f + C + m_R \cdot (P - f) = 0$,

subject to $f(V \to \infty) \to \frac{C + m_R \cdot P}{r + m_R}$ and to $f(V_{BR}) = (1 - a) \cdot V_{BR} - K$,

where V_{BR} denotes the default barrier after debt average maturity has been extended. In general V_{BR} is lower than V_B, since by extending debt maturity default is postponed.

The solutions to equations 10) and 11) are respectively

12)

$$F(V) = \frac{C+mP}{r+m} + \left[-\frac{C+m\cdot P}{r+m} + \frac{C+m_R\cdot P}{r+m_R} + \left[-\frac{C+m_R\cdot P}{r+m_R} + (1-a)\cdot V_{BR} - K \right]\cdot\left[\frac{V_R}{V_{BR}}\right]^h \right]\cdot\left[\frac{V}{V_R}\right]^b$$

$$13)\ f(V) = \frac{C+m_R\cdot P}{r+m_R} + \left[-\frac{C+m_R\cdot P}{r+m_R} + (1-a)\cdot V_{BR} - K \right]\cdot\left[\frac{V}{V_{BR}}\right]^h,$$

with b as per equation 4) and with

$$14)\ h = \frac{-\left(r-d-\frac{s^2}{2}\right) - \sqrt{\left(r-d-\frac{s^2}{2}\right)^2 + 2\cdot(r+m_R)\cdot s^2}}{s^2}.$$

Then, the value of equity in the presence of the "extension option" [E(V)] is equal to total firm value in the presence of the "extension option", which is given by equation 17) below, minus the value of debt before "extension", thus:

$$15)\ E(V) = V + tax\cdot\frac{C}{r}\cdot\left\{1-\left[\frac{V}{V_{BR}}\right]^j\right\} - (a\cdot V_B + K)\cdot\left[\frac{V}{V_{BR}}\right]^j - F(V).$$

Instead, the value of equity after the "extension option" has been exercised [ER(V)] is equal to total firm value as per equation 17) below, minus the value of debt after "extension", thus:

$$16)\ ER(V) = V + tax\cdot\frac{C}{r}\cdot\left\{1-\left[\frac{V}{V_{BR}}\right]^j\right\} - (a\cdot V_B + K)\cdot\left[\frac{V}{V_{BR}}\right]^j - f(V).$$

When debt maturity can be extended the above formulas give the values of debt and equity.

1.3) Modigliani and Miller' s proposition 1 and the option to extend debt maturity

Now, since corporate taxes and bankruptcy costs have been assumed, Modigliani - Miller' s proposition 1 cannot hold. This entails that total firm value changes due to the presence of the "extension option": total firm value would no longer be given by equation 6) but by

$$17)\ F(V) + E(V) = V + tax\cdot\frac{C}{r}\cdot\left\{1-\left[\frac{V}{V_{BR}}\right]^j\right\} - (a\cdot V_B + K)\cdot\left[\frac{V}{V_{BR}}\right]^j.$$

This equation is the same as equation 6), but for the fact that the default barrier is now lower since debt maturity is extended before or at default. This means that total firm value is now higher than total firm value as per 6), because a lower default barrier entails higher expected value of the debt induced tax shield [$TT(V) = tax \cdot \frac{C}{r} \cdot \left\{1 - \left[\frac{V}{V_{BR}}\right]^j\right\}$] and lower expected value of bankruptcy costs [$BC(V) = (a \cdot V_B + K) \cdot \left[\frac{V}{V_{BR}}\right]^j$].

Now, since longer debt maturity implies higher total firm value, it is not clear why firms should ever be interested in an option to extend debt maturity if they could simply choose to issue debt of longer maturity in the first place. As Leland (1996) puts it, longer-term debt may not be incentive compatible, in other words it is too sensitive to assets substitution or to other agency costs. Anyhow, corporate debt usually does have finite maturity.

Equation 17) also reveals that total firm value does not depend on V_R as long as condition 7 holds. In other words, *given the presence of corporate taxes and/or bankruptcy costs, total firm value only depends on "whether or not" debt maturity is extended not later than default, but not on "when" debt maturity is extended.* Later numerical examples will confirm this statement.

1.4) The payoff and the values of the option to extend debt maturity

The value of debt whose maturity can be extended (F(V)) can be viewed as the value of debt whose maturity cannot be extended (ff(V)) plus a position in the option to extend debt maturity: hereafter the value of this (often short) position is denoted by O(V).

In the same way the value of equity when debt maturity can be extended (E(V)) can be thought of as the value of equity when debt maturity cannot be extended (e(V)) plus a position in the option to extend debt maturity: hereafter the value of this (often long) position is denoted by OE(V).

At this point we can specify the payoffs for OE(V) and O(V) when it is equity holders who decide as to the exercise of the "extension option" and determine V_R :

18) $OE(V_R) = \max\{E(V_R) - e(V_R), 0\}$,

19) $O(V_R) = F(V_R) - ff(V_R)$.

Instead when it is debt holders who exercise the "extension option", which is a possibility as is noticed later on, then[38]:

20) $OE(V_R) = E(V_R) - e(V_R)$,

21) $O(V_R) = \max\{F(V_R) - ff(V_R), 0\}$.

The above allows to derive the expression for O(V) as the difference between F(V) and ff(V) and the expression for OE(V) as the difference between E(V) and e(V):

$$22)\ O(V) = \left[-\frac{C + m \cdot P}{r + m} + \frac{C + m_R \cdot P}{r + m_R} + \left[-\frac{C + m_R \cdot P}{r + m_R} + (1-a) \cdot V_{BR} - K \right] \left[\frac{V_R}{V_{BR}} \right]^h \right] \cdot \left[\frac{V}{V_R} \right]^b -$$

$$\left[-\frac{C + m \cdot P}{r + m} + (1-a) \cdot V_B - K \right] \left[\frac{V}{V_B} \right]^b .$$

23)

$$OE(V) = -O(V) + tax \cdot \frac{C}{r} \cdot \left\{ -\left[\frac{V}{V_{BR}} \right]^j + \left[\frac{V}{V_B} \right]^j \right\} + \left\{ -\left(a \cdot V_{BR} + K\right) \cdot \left[\frac{V}{V_{BR}} \right]^j + \left(a \cdot V_B + K\right) \cdot \left[\frac{V}{V_B} \right]^j \right\} .$$

Equation 23) highlights how OE(V) is different from O(V). In the jargon of options: the value of the (often long) position in the "extension option" (OE) is different from the value of the (often short) position in that same "extension option" (O). This unusual asymmetry is again due to the fact that Modigliani and Miller's proposition 1 does not hold, because taxes and bankruptcy costs are assumed to exist.

2. CONDITIONS FOR EXTENSION OF DEBT MATURITY AND FOR DEFAULT

As stated above, given that $V_{BR} \le V_B \le V_R$, when the value of the firm's assets (V) declines down to V_R, debt average maturity is extended. Possible ways to determine V_R are now discussed and then possible ways to determine V_B and V_{BR} are discussed too.

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[38]Clearly these payoffs imply that $E[V=V_{R1}] = ER[V=V_{R1}]$ with $F[V=V_{R1}] = f[V=V_{R1}]$.

2.1) The conditions for debt maturity to be extended

As for V_R , there are at least four possible ways to determine when debt maturity can be extended.

2.1.1) Take-it-or-leave-it offers

Debt maturity can be extended when equity holders make the following "take-it-or-leave-it" hostile offer to debt holders: "If you, debt holders, want us, equity holders, to keep servicing outstanding debt, you must concede that debt average maturity be extended!". This hostile offer is similar in spirit to the "take-it-or-leave-it" offer assumed by that Anderson and Sundaresan (1996). If equity holders stopped servicing debt, debt holders would need to satisfy their claim through costly liquidation of the firm's assets.

We now assume that equity holders make their hostile offer to debt holders when $V = V_{R1}$. Then debt holders will concede an "extension" only if assets recovery value upon immediate liquidation is lower than debt value with extended maturity, i.e.

24) $f(V_{R1}) \geq X(V_{R1})$

where $X(V_{R1})$ is the assets recovery value if default is forced when $V = V_{R1}$ and $f(V_{R1})$ is the value of debt with extended maturity. The assumption about assets recovery is

25) $X(V) = (1-a)\cdot V - K$,

where K denotes the fixed costs of assets liquidation and "a" denotes the proportional costs of liquidation. From 24) and 25) the condition for debt holders to accept the "take-it-or-leave-it" offer by equity holders can be restated as

26) $f(V_{R1}) \geq (1-a)\cdot V_{R1} - K$.

Condition 26) implies that all bargaining power during re-negotiation of the debt contract lies with equity holders.

Then, equity holders will want to have debt average maturity extended just if the extension option (OE(V)) is "in the money", i.e. if

27) $E(V_R) = ER(V_R) > e(V_R)$.

Equity holders may want to optimally choose V_{R1}, while making sure that condition 26) is met. This means that equity holders would determine V_{R1} as

28) $\max_{V_{R1}} \{E[V, V_{R1}]\}$

subject to conditions 26) and 27). As it will be apparent later, this often implies that equity holders choose V_{R1} as the highest value of the firm's assets (V) at which condition 26) is met. Thus 26) is often a binding constraint.

Condition 26) is more easily met when fixed liquidation costs (K) are high. The same is not always true if proportional liquidation costs are high (i.e. if "a" is high). Condition 26) is more easily met also when V_B is low. If V_B is low, also V_{R1} can be low even without violating constraint 7) (i.e. $V_B \leq V_{R1}$). Then the lower V_{R1} implies the lower $X(V_{R1})$ and condition 26) is more likely to hold.

2.1.2) When also debt holders gain from extension of debt maturity

Equity holders and debt holders can agree to renegotiate the debt contract and to extend debt maturity even if equity holders do not make the hostile offer implied by condition 26). This is the case when debt holders (as well as equity holders) are better off by extending maturity, i.e. when the value of debt with longer average maturity surpasses the value of debt with shorter average maturity. Then debt maturity would be extended at V_{R2}, where V_{R2} is determined as

29) $\max\limits_{V_{R2}} \left\{ E\left[V, V_{R2}\right] \right\}$

subject to 27) and to

30) $f(V_{R2}) \geq F(V_{R2})$.

2.1.3) Extension of debt maturity upon default

Default can take place without being preceded by the extension of debt maturity. This may be the case whenever equity holders and debt holders cannot renegotiate the debt contract, due for example to the high number of creditors involved or to asymmetric information between debtor and creditors. But, when default takes place, debt holders may spontaneously concede an extension of debt maturity to avoid immediate and costly assets liquidation. Then debt maturity would be extended at V_{R3}, where V_{R3} would be determined by the two simultaneous conditions:

31) $V_{R3} = V_B$,

and again

32) $f(V_{R3}) \geq (1-a) \cdot V_{R3} - K$.

2.1.4) Explicit option to extend debt maturity

Debt holders may be unconditionally subjected to the decision of equity holders as to the extension of debt maturity. This theoretical limit case applies when the debt contract or the bankruptcy code concede an "explicit" option to equity holders to extend debt maturity at any time. The debt indenture may concede one such option in some issues of "extendible debt" giving equity holders the unilateral right to extend debt maturity. A hypothetical bankruptcy code may concede to equity holders the right to voluntary file for an official reorganisation proceeding that, *without the approval of creditors*, would grant a moratorium to the debtor. The moratorium would allow the debtor to temporarily suspend debt payments and thus to stretch the effective maturity of debt.

Such extension options are theoretical limit cases and are explicit in that they are provided by the debt contract or by the code. Instead, the previous extension options are implicit in that debt maturity is extended through re-negotiation or through a unilateral concession by debt holders upon default. Anyway, an explicit extension option would allow equity holders to unilaterally decide to extend debt maturity so as to maximise equity value[39]. Equity holders could then extend debt maturity at V_{R4}, where V_{R4} is such that

33) $$\max_{V_{R4}}\left\{E\left[V,V_{R4}\right]\right\}.$$

Usually $V_{R4} \geq V_{R1}$ since condition 26) is not required in this case. For realistic parameters V_{R4} is usually an internal value internal value, i.e.: $V_B \leq V_{R4} \leq V_0$. Equity holders choose $V_{R4}=V_0$ when they want to immediately extend debt maturity. This may be the case especially when assets volatility is high. Instead, when the rate of debt coupons is very high equity holders would never want to extend debt maturity (i.e. $V_B \geq V_{R4}$) as they would want to minimise the number of high coupons to be paid and refinance at cheaper interest rates. Finally, when liquidation costs are exceptionally high, constraint 26) is not binding so that $V_{R4}= V_{R1}$.

Of course $V_{R1}, V_{R2}, V_{R3}. V_{R4}$ all imply rationality and symmetric information for both equity holders and debt holders. V_{R1}, V_{R2} and V_{R4} can be found by numerical algorithms. For V_{R3} also closed form solutions are available as becomes apparent next.

[39] Equity holders are assumed to extend the maturity of all outstanding debt at the same time.

2.2) The default barriers

Now the ways to determine the default barriers V_B and V_{BR} are discussed.

As for V_B, default before debt maturity is extended can take place at different possible values of the firm's assets (V), in particular:

- at V_{BI} when default is triggered by a cash flow shortage that makes the firm insolvent;
- at V_{BE} when default is triggered by equity becoming worthless .

As for V_{BR}, default after debt maturity has been extended can again be determined either by a cash flow shortage or by equity becoming worthless. In the first case default takes place at V_{BIR}, whereas in the second case default takes place at V_{BER}.

Then, V_{BI} would be determined by the following cash flow shortage condition:

34) $d \cdot V_{BI} + m \cdot f(V_{BI}) = d \cdot V_{BI} + m \cdot [(1 - a) \cdot V_{BI} - K] = C \cdot (1 - tax) + m \cdot P$,

which implies

35) $V_{BI} = \dfrac{m \cdot (K + P) + C \cdot (1 - tax)}{d + m \cdot (1 - a)}$.

Conditions 34) and 35) are the same as in Ericsson (2000) and state that default occurs when the firm becomes insolvent, i.e. when the instantaneous inflows to the firm are equal to in the instantaneous outflows from the firm. Condition 34) presupposes that debt average maturity cannot be extended. But, if debt average maturity is extended not later than default, i.e. $V_{BI} \leq V_R$, then the default barrier becomes

36) $V_{BIR} = \dfrac{m_R \cdot (K + P) + C \cdot (1 - tax)}{d + m_R \cdot (1 - a)}$.

Finally, when default is triggered by worthless equity, equity holders are assumed to endogenously determine the default barrier as per Leland (1998). In this case, and if debt maturity cannot be extended, the default barrier is determined by the following conditions

37) $[E]_{V=V_{BE}} = 0$,

38) $[E_V]_{V=V_{BE}} = 0$

that imply

39) $V_{BE} = \dfrac{\left(tax \cdot \dfrac{C}{r} + K\right) \cdot j - \dfrac{C + m \cdot P}{r + m} \cdot b - K \cdot b}{(1 - a \cdot j - (1 - a) \cdot b)}$.

If instead debt maturity is extended before default or at default, i.e. $V_{BE} \leq V_R$, then

$$40)\ V_{BER} = \frac{\left(tax \cdot \frac{C}{r} + K\right) \cdot 1 - \frac{C + m_R \cdot P}{r + m_R} \cdot b - K \cdot b}{\left(1 - a \cdot 1 - (1 - a) \cdot b\right)}.$$

In this section the conditions for debt maturity to be extended and the default barriers have been determined. In the next section such conditions are discussed with reference to a base case scenario in which realistic average parameter values are assumed. Different conditions for extension of debt maturity and different default barriers are shown to heavily affect the values of debt, equity and the extension options.

3. NUMERICAL ANALYSIS IN A TIME INDEPENDENT SETTING WHEN DEFAULT IS TRIGGERED BY CASH FLOW SHORTAGE OR BY WORTHLESS EQUITY

The following analysis builds on a base case scenario, which is of interest since it assumes realistic average parameters. Such parameters are similar *inter alia* to those in Fan and Sundaresan (2001), Ericsson (2000), Leland (1998) and are displayed in italics in Table 4.1. The significant effect of different default conditions and different policies to extend debt maturity is highlighted.

3.1) Base case scenario when default is triggered by a cash flow shortage (liquidity default)

The base case scenario with liquidity default reveals that $V_{BIR} < V_{BI} < V_{R4} < V_0 < V_{R2}$ ($V_{BI} =$ 49.8, $V_{R4} = 82.8$, $V_0 = 100$, $V_{R2} = 123.5$) where V_0 denotes the value of the firm's assets today. V_{R1} and V_{R3} are non existent since condition 26) is never met when $V \geq V_{BI}$. The fact that V_{R1} and V_{R3} are non-existent means that debt holders will always choose immediate liquidation rather than extension of debt maturity, even if extending debt maturity would in fact postpone default by lowering the default barrier from $V_{BI} = 49.8$ to $V_{BIR} = 44.8$. The fact that $V_{BI} < V_{R4}$ means that, if equity holders can unilaterally decide when to extend debt maturity in an unconstrained fashion, they will do so at V_{R4} before default. On the other hand, the fact that 49.8 $= V_{BI} < V_{R2} = 123.5$ reveals that debt holders may accept an offer to extend debt maturity before default when $V \geq V_{R2}$, i.e. when the firm is very far from default. If debt maturity was extended at V_{R2}, then OE(V) = 1.72 and O(V) = 0.

In this scenario, equity holders may want to have an "explicit option" to unilaterally impose an extension of debt maturity to debt holders. Such explicit option would be optimally exercised at $V = V_{R4} = 82.8$, since V_{R4} maximises OE(V) (and E(V)) and minimises O(V) and (F(V)).

Results for $V_R = V_{R4}$ are displayed in Table 4.1 Panel A[40]. Total firm value increases and equity value (E(V)) rises by some 4.1% (from 55 to 57.2), while debt value (F(V)) decreases just slightly (from 50.5 to 50). A slight increase in the annual coupon rate (0.24%) would be enough to compensate debt holders for conceding the explicit "extension option" (i.e. c = 6.24% implies ff(V) = F(V)). This case is an example of the result that generally, *given taxes and bankruptcy costs, the "extension option" increases the value of equity well more than it decreases the value debt.*

3.2) When assets volatility is low

Now assets volatility is assumed to be equal to 10% rather than 20% and all other things are equal to the base case scenario. This new scenario implies that now:

$V_{BIR} \leq V_{BI} = V_{R3} < V_{R1} < V_{R2} < V_{R4} < V_0$ ($V_{R3} = V_{BI} = 49.8$, $V_{R1} = 52.4$, $V_{R2} = 52.7$, $V_{R4} = 63.4$, $V_0 = 100$). So, unlike in the base case scenario, V_{R1} and V_{R3} exist since condition 25) can be met even when $V \geq V_{BI}$. The reason why condition 26) can now be met is that lower assets volatility makes debt less risky and more valuable and hence the value of debt with extended maturity (f(V)) is now more likely to be higher than the assets recovery value (X(V)).

When condition 26) is met, debt holders will prefer to have debt maturity extended rather than outright assets liquidation. Anyway, since in this case $V_{R1} < V_{R2}$, debt holders may accept to have debt maturity extended even if equity holders do not make the "take-it-or-leave-it" offer mentioned above. In fact, when $V \leq V_{R2} = 52.7$ debt of longer maturity (f(V)) is not less valuable than debt of shorter maturity F(V).

Then, in this scenario debt holders are interested in spontaneously extending debt maturity at default, i.e. at $V = V_{BI} = V_{R3}$, in order to avoid assets liquidation (this case is illustrated in Table

[40] The Panels of Table 1 are separately reproduced here below and the entire Table 1 is displayed

4.1 Panel B). This important point is similar to that of Longtaff (1990), who assumes that, as the firm defaults, debt holders may prefer to extend debt maturity rather than costly liquidation of the firm's assets. Though the analysis of Longstaff is limited to the classic Mertonian setting: a zero coupon bond is the only debt and default cannot occur before debt maturity. So, when $V_{BI} = V_{R3}$ *the analysis by Longstaff is being extended to a time independent setting in which the firm has multiple debt issues that are continuously refunded at maturity and in which default can take place at any time. As in Longstaff, even in this setting debt holders can prefer extension of debt maturity to liquidation.*

COPY OF TABLE 4.1: PANELS A AND B (THE EFFECTS OF THE EXTENSION OPTION)			
Input data in italics	No extension option	Ante extension	Post extension
a (bankruptcy costs as fraction of V)	*15%*	*15%*	*15%*
r (default risk-free interest rate)	*5%*	*5%*	*5%*
s (volatility of V)	*20%*	*20%*	*20%*
d (assets total payout to security holders)	*7.0%*	*7.0%*	*7.0%*
tax (tax rate)	*35%*	*35%*	*35%*
K (fixed liquidation costs)	*0*	*0*	*0*
C (annual coupon, which is paid coutinuously)	*3.00*	*3.00*	*3.00*
Coupon rate (c = C/P)	*6.00%*	*6.00%*	*6.00%*
m (percentage of P that is refinanced every year)	***20%***	***20%***	***10%***
P (face value of debt)	*50.0*	*50.0*	*50.0*
V_0 (today's assets value)	*100.0*	*100.0*	*100.0*
PANEL A: Base case with irreversible extension of debt maturity and cash flow shortage default			
$V_R = V_{R4}$ (value of asset at which debt maturity is extended)		82.8	
V_{BI} and V_{BIR} (value of assets triggering default)	49.8	44.8	44.8
BC (present value of future possible bankruptcy costs)	4.1	3.3	3.3
T (present value of future tax savings on debt coupons)	9.6	10.6	10.6
OE (value of the "extension option" for equity holders)		**2.26**	
E (value of equity)	55.0	57.2	57.2
O (value of the "extension option" for debt holders)		**-0.53**	
F (value of debt ante extension)	50.5	50.0	
f (value of debt post extension)		48.6	50.1
$X(V_{R4})$ (assets recovery value at V_{R4})		70.4	
Total assets = Total liabilities	105.49	107.21	107.21
Credit spread: [C+m(P-F)]/F-r or [C+m(P-f)]/f-r	0.74%	1.02%	0.98%
PANEL B: All as in panel A exept for assets volatility and $V_r = V_{r3}$ rather than $V_r = V_{r4}$			
s (volatility of V)	*10%*	*10%*	*10%*
$V_R = V_{R3}$ (value of asset at which debt maturity is extended)		49.8	
V_{BI} and V_{BIR} (value of assets triggering default)	49.8	44.8	44.8
BC (present value of future possible bankruptcy costs)	2.6	2.0	2.0
T (present value of future tax savings on debt coupons)	13.8	14.9	14.9
OE (value of the "extension option" for equity holders)		**1.65**	
E (value of equity)	59.5	61.2	60.5
O (value of the "extension option" for debt holders)		**0.01**	
F (value of debt ante extension)	51.7	51.7	
f (value of debt post extension)		42.8	52.4
$X(V_{R3})$ (assets recovery value at V_{R3})		42.3	
Total assets = Total liabilities	111.21	112.88	112.88
Credit spread: [C+m(P-F)]/F-r or [C+m(P-f)]/f-r	0.14%	0.14%	0.26%

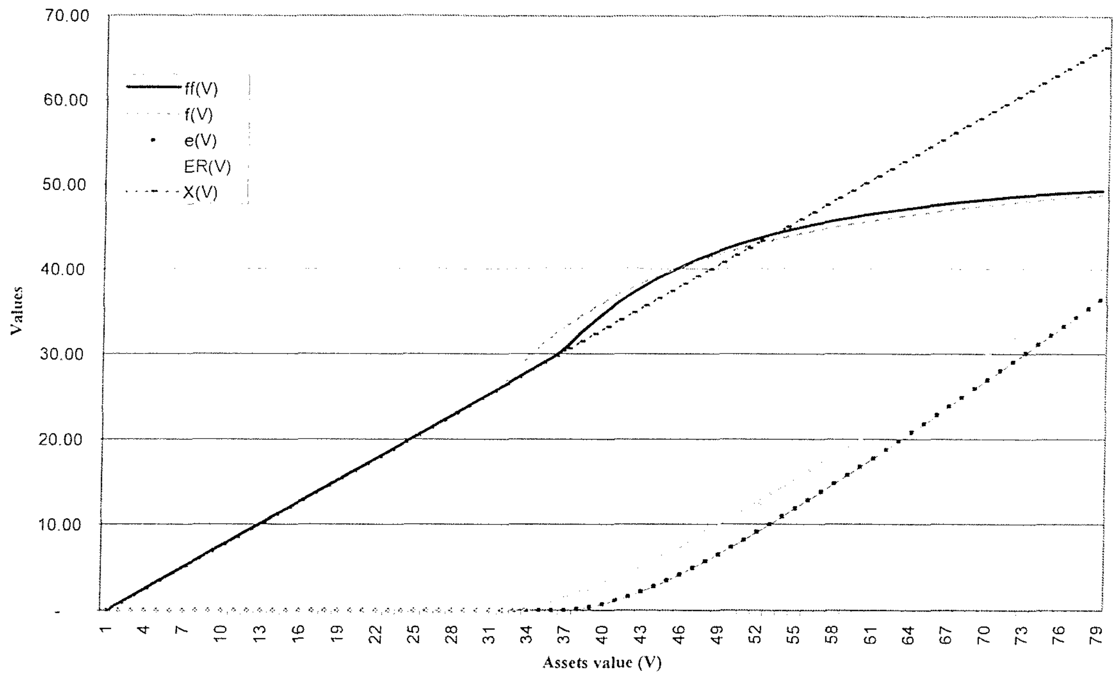
also at the end of the chapter for direct comparisons across different cases.

3.3) Base case scenario when default is triggered by worthless equity

Base case scenario parameters now imply that: $V_{BER} \leq V_{BE} = V_{R3} < V_{R2} < V_{R1} < V_{R4} < V_0$ ($V_{BER} = 30.4$, $V_{BE} = V_{R3} = 35.5$, $V_{R2} = 44.5$, $V_{R1} = 50.5$, $V_{R4} = 76.5$, $V_0 = 100$). Table 4.1 Panels C, D and E illustrate this scenario when debt maturity is extended at V_{R1} or V_{R2} or V_{R3}. Thus, unlike when default is triggered by a cash flow shortage, when default is triggered by worthless equity V_{R3} and V_{R1} exist even with base case scenario parameters. In fact, when default is triggered by worthless equity, default takes place at lower values of the firm's assets ($V_{BER} \leq V_{BIR}$ and $V_{BE} \leq V_{BI}$) so that condition 26) is more likely to obtain before or at default. In other words, *as the firm approaches default, debt holders are more likely to prefer to concede an extension of debt maturity (rather than assets liquidation) when default is triggered by worthless equity than when default is triggered by a cash flow shortage*. Moreover, optimal leverage is higher when the explicit option to extend maturity is present as opposed to when such option is absent, and the higher the firm leverage is, the stronger is the incentive for debt holders to renegotiate and concede a maturity extension.

Figure 4.1 shows the values of debt and equity, assuming base case scenario parameters, when debt maturity is extended at $V_{R1} = 50.5$. V_{R1} is such that $f(V_{R1}) = X(V_{R1})$ and is the highest value of V at which condition 26) is met before the firm defaults, i.e. before the value of equity in the absence of the extension option drops to 0: in this case $V_B \leq V_{R1} = V_R$. Debt maturity can be extended only if the recovery value of assets (X(V)) is not greater than debt value after exercise of the extension option (f(V)) and only if equity (e(V)) has not yet become worthless.

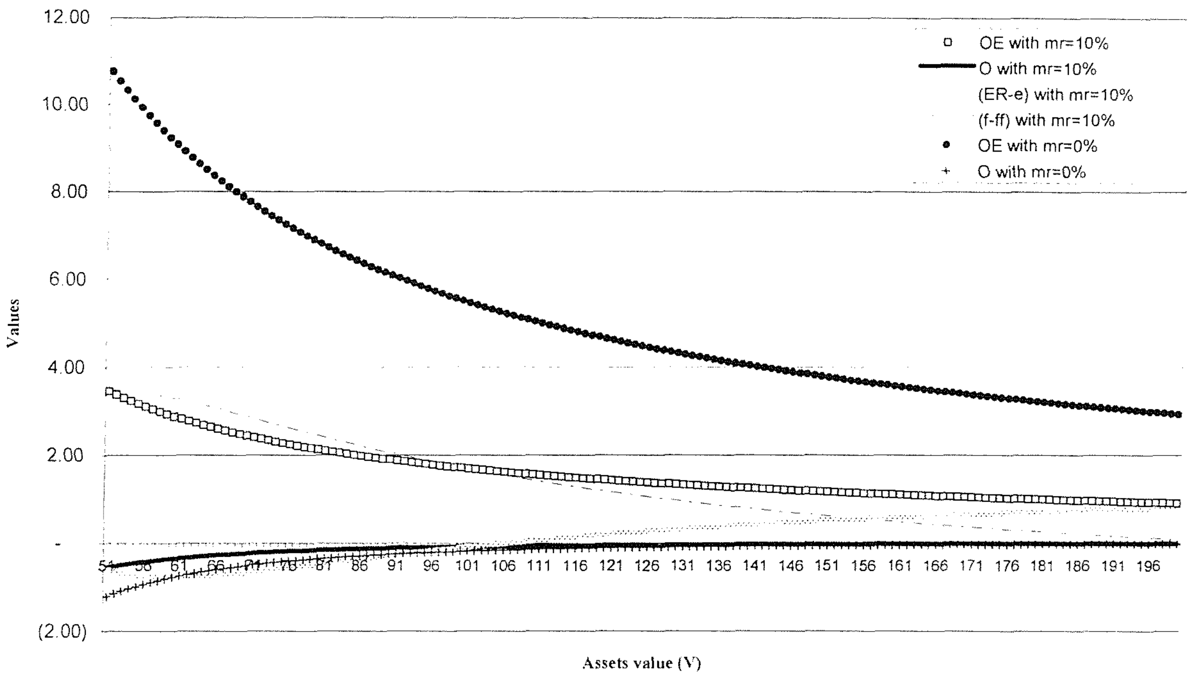
Figure 4.1: Base case scenario with default triggered by worthless equity



For this same case, Figure 4.2 displays the values of $OE(V, V_{R1})$ and $O(V, V_{R1})$[41] and their respective payoffs (ER(V)-e(V), f(V)-ff(V)). Before "extension" we can see that $\lfloor ER(V, V_{R1}) - e(V, V_{R1}) \rfloor > OE(V, V_{R1})$, so it is clear that $OE(V, V_{R1})$ is not optimally exercised. In fact, equity holders can extend debt maturity only when condition 26) is met. Figure 4.2 also shows the case in which, ceteris paribus, debt of extended maturity is a perpetuity so that $m_R = 0$: the longer the "extension" is, the more valuable $OE(V, V_{R1})$ is and the less valuable $O(V, V_{R1})$ is.

[41] $O(V, V_{R1})$ and $OE(V, V_{R1})$ denote the option values when debt maturity is extended at V_{R1}.

Figure 4.2: Values of the extension option in the base case scenario with default triggered by worthless equity



Panels C, D and E of Table 4.1 show the effect of different exercise policies of the extension option with base case parameters and when default is triggered by worthless equity. In particular:

- $OE(V = 100, V_{R1}) = 171$ and $O(V = 100, V_{R1}) = -0.09$;
- $OE(V = 100, V_{R2}) = 161$ and $O(V = 100, V_{R2}) = 0$;
- $OE(V = 100, V_{R3}) = 144$ and $O(V = 100, V_{R3}) = 0.17$.

$OE(V = 100, V_{R1})$ denotes the value of the extension option when debt maturity is extended at V_{R1}. It is then clear that equity holders have an incentive to exercise their bargaining power by renegotiating debt maturity as soon as condition 26) is met, i.e. at V_{R1}, since this increases the extension option value. On the other hand $O(V = 100, V_{R1})$ is negative, which means that the detriment of debt holders if debt maturity were extended at V_{R1}.

If debt maturity was extended at V_{R2}, debt holders would neither lose nor gain, so the extension option would be worthless for debt holders in such case: $O(V = 100, V_{R2}) = 0$.

On the other hand, when maturity is extended at $V < V_{R2} = 44.5$, debt holders too would gain from an extension of debt maturity and indeed they would gain the most if maturity were extended just at default, i.e. at $V_{BE} = V_{R3}$.

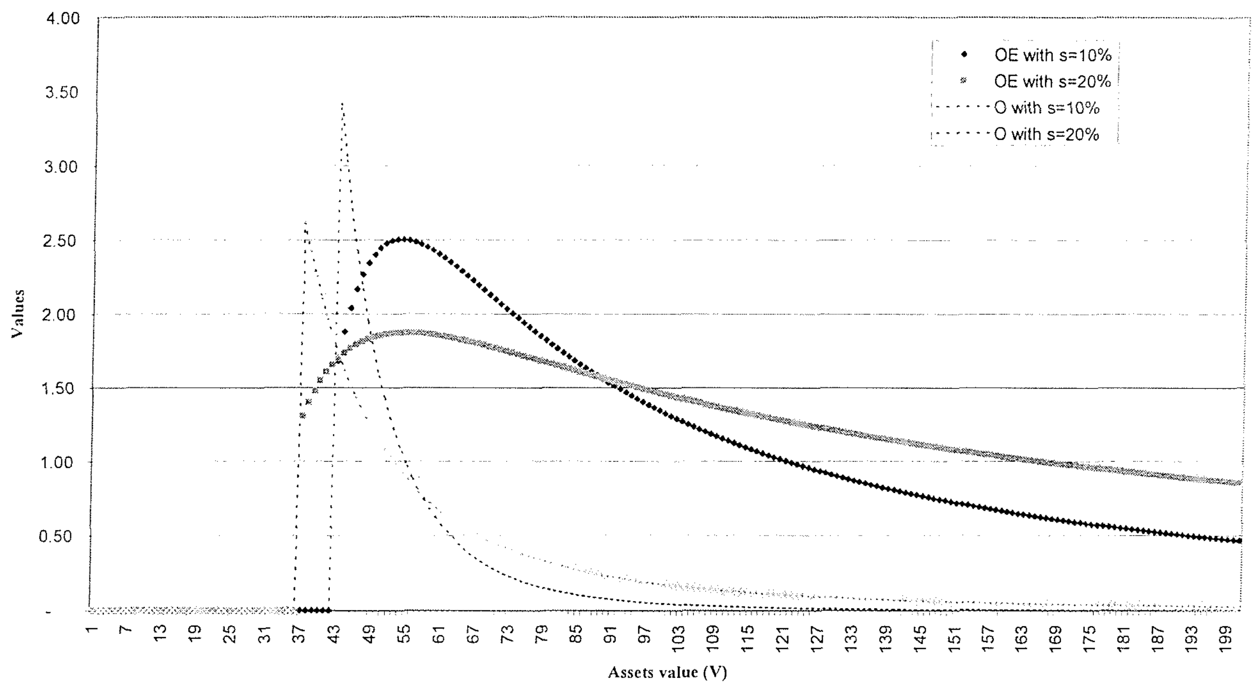
In these cases equity holders would have to share with debt holders the benefit of having debt maturity extended (i.e. the increase in total firm value). Panels C, D and E of Table 4.1 show that:

COPY OF TABLE 4.1: PANELS C, D AND E (THE EFFECTS OF THE EXTENSION OPTION)			
Input data in italics	No extension option	Ante extension	Post extension
a (bankruptcy costs as fraction of V)	*15%*	*15%*	*15%*
r (default risk-free interest rate)	*5%*	*5%*	*5%*
s (volatility of V)	*20%*	*20%*	*20%*
d (assets total payout to security holders)	*7.0%*	*7.0%*	*7.0%*
tax (tax rate)	*35%*	*35%*	*35%*
K (fixed liquidation costs)	*0*	*0*	*0*
C (annual coupon, which is paid coutinuously)	*3.00*	*3.00*	*3.00*
Coupon rate (c = C P)	*6.00%*	*6.00%*	*6.00%*
m (percentage of P that is refinanced every year)	***20%***	***20%***	***10%***
P (face value of debt)	*50.0*	*50.0*	*50.0*
V_0 (today's assets value)	*100.0*	*100.0*	*100.0*
PANEL C: Base case, but extension of debt maturity at Vr = Vr1 and default when equity is worthless			
$V_R = V_{R1}$ (value of asset at which debt maturity is extended)		50.5	
V_{BE} and V_{BER} (value of assets triggering default)	35.5	**30.4**	30.4
BC (present value of future possible bankruptcy costs)	2.2	1.6	1.6
T (present value of future tax savings on debt coupons)	12.5	13.6	13.6
OE (value of the "extension option" for equity holders)		**1.71**	
E (value of equity)	59.7	61.4	61.4
O (value of the "extension option" for debt holders)		**-0.09**	
F (value of debt ante extension)	50.6	50.5	
f (value of debt post extension)		42.9	50.5
$X(V_{R1})$ (assets recovery value at V_{R1})		42.9	
Total assets = Total liabilities	110.32	111.94	111.94
Credit spread: [C+m(P-F)]/F-r or [C+m(P-f)]/f-r	0.68%	0.72%	0.83%
PANEL D: Base case, but extension of debt maturity at Vr = Vr2 and default when equity is worthless			
$V_R = V_{R2}$ (value of asset at which debt maturity is extended)		44.5	
V_{BE} and V_{BER} (value of assets triggering default)	35.5	**30.4**	30.4
BC (present value of future possible bankruptcy costs)	2.2	1.6	1.6
T (present value of future tax savings on debt coupons)	12.5	13.6	13.6
OE (value of the "extension option" for equity holders)		**1.61**	
E (value of equity)	59.7	61.3	61.4
O (value of the "extension option" for debt holders)		**0.00**	
F (value of debt ante extension)	50.6	50.6	
f (value of debt post extension)		40.1	50.5
$X(V_{R2})$ (assets recovery value at V_{R2})		37.8	
Total assets = Total liabilities	110.32	111.94	111.94
Credit spread: [C+m(P-F)]/F-r or [C+m(P-f)]/f-r	0.68%	0.68%	0.83%
PANEL E: Base case, but extension of debt maturity at Vr = Vr3 and default when equity is worthless			
$V_R = V_{R3}$ (value of asset at which debt maturity is extended)		35.5	
V_{BE} and V_{BER} (value of assets triggering default)	35.5	**30.4**	30.4
BC (present value of future possible bankruptcy costs)	2.2	1.6	1.6
T (present value of future tax savings on debt coupons)	12.5	13.6	13.6
OE (value of the "extension option" for equity holders)		**1.44**	
E (value of equity)	59.7	61.1	61.4
O (value of the "extension option" for debt holders)		**0.17**	
F (value of debt ante extension)	50.6	50.8	
f (value of debt post extension)		32.9	50.5
$X(V_{R3})$ (assets recovery value at V_{R3})		30.2	
Total assets = Total liabilities	110.32	111.94	111.94
Credit spread: [C+m(P-F)]/F-r or [C+m(P-f)]/f-r	0.68%	0.59%	0.83%

$OE(V,V_{R1})+O(V,V_{R1})=OE(V,V_{R2})+O(V,V_{R2})=OE(V,V_{R3})+O(V,V_{R3})$.

Finally Figure 4.3 displays how higher assets volatility does not necessarily increase the option values $OE(V,V_{R3})$ and $O(V,V_{R3})$. Higher volatility increases $O(V,V_{R3})$ when default is far, moreover it can decrease $OE(V,V_{R3})$ since $OE(V,V_{R3})$ becomes a locally concave function of V as default nears. Higher volatility implies a lower default barrier. Figure 4.3 shows the values of the extension option, $OE(V, V_{R3})$ and $O(V, V_{R3})$, in the base case scenario with default triggered by worthless equity and maturity extended just upon default: $V_{R3} = V_{BE}$, volatility is equal either to 10% or to 20%.

Figure 4.3: Values of the extension option in the base case scenario with default triggered by worthless equity and maturity extended just upon default



3.4 When debt holders gain from having debt maturity extended

It is here reminded that V_{R2} is the value of assets at which debt holders would be indifferent as whether to have debt maturity extended or not, because V_{R2} is such that $F(V_{R2})=f(V_{R2})$. It may appear surprising that V_{R2} equals 44.5 in the base case scenario when default is triggered by worthless equity, given V_{R2} equals 123.5 when default is triggered by a cash flow shortage condition (see above). The reason for this difference is that, when default is triggered by worthless equity, there are in reality two values of the firm's assets that make debt of shorter maturity of equal value to debt of longer maturity in this base case scenario. So there are two values for V_{R2}: one is 44.5 and the other one is 106.6. More precisely, when V > 106.6 debt of longer maturity is more valuable than debt of shorter maturity ($f(V) > F(V)$): this is because when V grows, debt becomes safer and the contractual coupon over-remunerates the risk of default of debt (debt value rises above par). When this is the case, debt holders will want to extend debt maturity in order to get such over-remuneration for a longer period. In the base case when default is due to a cash flow shortage this happened when $V > 123.5$ rather than when $V > 106.6$.

Then, when $44.5 < V < 106.6$ debt of longer maturity is less valuable than debt of shorter maturity ($f(V) < F(V)$): this is because when V falls below 106.6, the risk of default increases in such a way that the contractual coupon under-remunerates the risk of default of debt. In this case debt holders will not want to extend debt maturity in order to limit the period in which they are under-remunerated.

Then again, when $V < 44.5$, debt of longer maturity becomes again more valuable than debt of shorter maturity: this is because longer maturity postpones default by implying a lower default barrier and hence a lower probability of default. In the base case when default is due to a cash flow shortage this never happened since, when $V < 123.5$, debt of shorter maturity was always more valuable than debt of longer maturity, even if debt of shorter maturity implied a higher default barrier.

The above analysis has covered a time independent setting. The following analysis covers a time dependent setting.

4. A TIME DEPENDENT SETTING

The main new assumption in this setting is that debt is not continuously refunded so as to keep the nominal capital structure constant and independent of time as in the previous section. Rather, the (continuous) payment of principal is funded by assets generated cash flows and/or by issuance of new equity. Now time is an explicit independent variable. In a time dependent setting closed form solutions for the values of debt, equity and the option to extend debt maturity are no longer possible, so explicit finite differences are employed to provide numerical solutions to the relevant valuation equations.

4.1) The model in a time dependent setting

Some more notation before proceeding:

- $P(t)$ is the face value of debt outstanding at time t;
- c is the annual coupon rate on debt;
- $C(t)$ is the instantaneous coupon payment at time t, $C(t) = c\,P(t)$;

- t denotes time; to highlight time dependence the notation changes to $V_R(t)$, $V_B(t)$, $V_{BR}(t), e(V,t), E(V,t), ER(V,t), f(V,t), ff(V,t), F(V,t)$;
- without loss of generality today's date is set equal to t = 0, e.g. P(0) denotes today's outstanding debt;
- T is the contractually agreed time at which debt amortisation is completed;
- T*(>T) is the time at which debt amortisation is completed after debt maturity has been extended;
- M is the rate at which debt principal is continuously amortised, so that P(T) = P(0) - M T =0; unlike in Appendix 2, it is here assumed that P(T) = 0, so that debt principal is completely paid back by time T.

Now the problem of valuing equity and debt whose maturity can be extended is reformulated in a time dependent setting. Before debt maturity is extended, when $V \geq V_R(t)$, debt value before "extension" (F(V, t)) must satisfy

$$41)\, F_t + \frac{1}{2} \cdot s^2 \cdot V^2 \cdot F_{VV} + (r-d) \cdot V \cdot F_V - r \cdot F + c \cdot [P(0) - M \cdot t] + M = 0,$$

with $F(V,T) = 0$, with P(T) = 0, with $F(V_R(t), t) = f(V_R(t), t)$ and with

$$42)\, F(V \to \infty) \to \int_0^T e^{-r \cdot t} \{M + c \cdot [P - M \cdot t]\} dt =$$

$$= \frac{-e^{-rT} \cdot r \cdot (M + c \cdot P) + e^{-rT} \cdot c \cdot M \cdot (r \cdot T + 1) + (M + c \cdot P) \cdot r - c \cdot M}{r^2}.$$

Condition 42) states that, as V grows infinitely, debt value approaches the value of a default free debt that promises the same cash flows, i.e. {M + c [P - M t}dt in every small period "dt". Then t* is the first time at which V reaches $V_R(t)$ from above. t* is a random variable that depends on the future path of V. For every $0 \leq t^* \leq T$, debt value after the re-negotiation, f(V, t), must satisfy

$$43)\, f_t + \frac{1}{2} \cdot s^2 \cdot V^2 \cdot f_{VV} + (r-d) \cdot V \cdot f_V - r \cdot f + c \cdot [P(t^*) - M_R \cdot (t - t^*)] + M_R = 0,$$

with $f(V, T^*) = 0$, with P(T*) = 0, with $T^* = t^* + \frac{P(t^*)}{M_R}$, with

$$44)\, f(V_{BR}(t), t) = (1-a) \cdot V_{BR}(t) - K,$$

and with

45)

$$f(V \to \infty, t) \to \int_{t^*}^{T^*} e^{-r\cdot(t-t^*)} \cdot \left[M_R + c\cdot P(t^*) - c\cdot M_R \cdot (t - t^*)\right] dt =$$

$$= \frac{-e^{-r\cdot(T^*-t^*)}\cdot r\cdot\left(M_R + c\cdot P\right) + e^{-r\cdot(T^*-t^*)}\cdot c\cdot M_R\left(r\cdot(T^*-t^*)+1\right) + \left(M_R + c\cdot P\right)\cdot r - c\cdot M_R}{r^2}$$

.

M_R is such that $M_R < M$ and is the rate at which debt principal is repaid after debt maturity has been extended. In appendix 1 the problem is reformulated for the case in which $M_R = 0$. Condition 45) states that, as V grows infinitely, debt value approaches the value of a default free debt that promises the cash flows equal to $\left[M_R + c\cdot P(t^*) - c\cdot M_R(t - t^*)\right] dt$ in every small period "dt" after t*.

Then, as in section 2, we are left with the problem of determining $V_R(t), V_B(t)$ and $V_{BR}(t)$, where it is again assumed that $V_0 \geq V_R(t) \geq V_B(t)$[42]. Such problem is solved by valuing the firm's equity, which is done next. Hereafter E(V,t) denotes equity value before debt maturity has been extended and ER(V,t) denotes equity value after debt maturity has been extended. Then:

46) $E_t + \frac{1}{2}\cdot s^2\cdot V^2\cdot E_{VV} + (r-d)\cdot V\cdot E_V - r\cdot E + V\cdot d - (1-tax)\cdot c\cdot[P - M\cdot t] - M = 0$,

with $E(V \to \infty, t) \to V$, with $E(V,T) = V - P(T) = V$ (since P(T) = P(0) – M T = 0), with

47) $E(V_R(t), t) = ER(V_R(t), t)$,

for $\forall t, 0 \leq t \leq T$ max $V_R(t)$, subject to

48) $ER(V_R(t), t) \geq E(V_R(t), t)$,

48.1) $V_0 \geq V_R(t) \geq V_B(t)$,

49) $f(V_R(t), t) \geq (1-a)\cdot V_R(t) - K$.

50)

$$ER_t + \frac{1}{2}\cdot s^2\cdot V^2\cdot ER_{VV} + (r-d)\cdot V\cdot ER_V - r\cdot ER + V\cdot d - (1-tax)\cdot c\cdot\left[P(t^*) - M_R\cdot(t-t^*)\right] - M_R = 0$$

with $ER(V \to \infty, t) \to V$, with $ER(V, T^*) = V$, with

51) $ER\left(V_{BR}(t), t\right) = 0$,

52) $\left[ER(V, t)\right]_{V=V_{BR}(t)} = 0$.

Explicit finite differences allow to solve equations 41) to 52) simultaneously. $V_R(t)$ is determined for every time "t" as the highest assets value at which conditions 48) 48.1 and 49) are all satisfied: these conditions ensure that debt maturity is extended in such a way that equity value, E(V), is maximised subject to condition 49) and provided default has not yet taken place. Condition 49) is similar to condition 26) and must hold if the option to extend debt maturity is implicit in the possibility of debt re-negotiation. This is the case we focus on below and the maturity extension policy [$V_R(t)$] is comparable to V_{R1} in the time dependent setting of the previous sections. For some parameter values, there is no $V_R(t)$ satisfying conditions 48), 48.1) and 49); in such case debt maturity cannot be extended and E(V,t) is equal to e[V,t] as defined below.

Conditions 51) and 52) grant that equity value is always non-negative and that it be maximised by the choice of $V_B(t)$. Conditions 51) and 52) are similar to conditions 37), 38) and to condition 17) in Mello and Parsons (1992) at page 1891.

Assuming it is equity holders who decide as to the exercise of the extension option, the payoff to equity holders is:

53) $OE(V, t^*) = \max\left\{ER\left[V_R(t^*), t^*\right] - e\left[V_R(t^*), t^*\right], 0\right\}$, with $ER\left[V_R(t^*), t^*\right] = E\left[V_R(t^*), t^*\right]$,

with $e\left[V_R(t^*), t^*\right]$ denoting the value of equity deprived of the extension option.

Then e[V,t] must satisfy the same equation as E(V,t), but the lower boundary condition is the default condition (since debt maturity cannot be extended before default):

54) $e_t + \frac{1}{2} \cdot s^2 \cdot V^2 \cdot e_{VV} + (r - d) \cdot V \cdot e_V - r \cdot e + V \cdot d - (1 - tax) \cdot c \cdot [P(0) - M \cdot t] - M = 0$,

with $e(V \to \infty, t) \to V$, $e(V, T) = Max\left[V - (P(0) - M \cdot T), 0\right]$ and with

55) $\left[e(V, t)_V\right]_{V=V_B(t)} = 0$,

56) $\left[e(V, t)\right]_{V=V_B(t)} = 0$.

Moreover, upon extension the payoff to debt holders is:

57) $O(V, t^*) = \left\{f\left[V_R(t^*), t^*\right] - ff\left[V_R(t^*), t^*\right]\right\}$.

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[42] V_0 denotes the value of the firm's assets today.

with $f[V_R(t^*),t^*] = F[V_R(t^*),t^*]$ and with $ff[V_R(t^*),t^*]$ denoting the value of debt in the absence of the "extension option". Then ff[V,t] must satisfy the same equation as F(V,t), but the lower boundary condition is the payoff upon default (since debt maturity cannot be extended before default):

$$58)\ ff_t + \frac{1}{2}\cdot s^2\cdot V^2\cdot ff_{VV} + (r-d)\cdot V\cdot ff_V - r\cdot ff + c\cdot[P(0) - M\cdot t] + M = 0,$$

with $ff(V,T) = 0$ since P(T) = 0, with $ff(V_B(t),t) = (1-a)\cdot V_B(t) - K$

and with

$$59)\ ff(V\to\infty,t) \to \int_t^T e^{-rt}\cdot[M + c\cdot P(0) - c\cdot M\cdot t]\,dt =$$

$$= \frac{-e^{-rT}\cdot r\cdot(M + c\cdot P) + e^{-rT}\cdot c\cdot M\cdot(r\cdot T+1) + (M + c\cdot P)\cdot r - c\cdot M}{r^2}.$$

We have formulated the time dependent model. Next numerical results with base case parameters are examined.

4.2) Base case scenario in a time dependent setting when default is triggered by worthless equity

The base case scenario parameters assumed in section 3 are here employed again[43]. Here again ER(V,t=0) is greater than or equal to E(V,t=0) for every value of the firm's assets (V). In fact extending debt maturity increases equity value by increasing the value of the tax shield, since more coupons must be paid if debt maturity is extended. Equity value increases also because equity is here similar to some sort of compound call option that is continuously exercised as debt is continuously serviced: thus extending debt maturity increases equity value also by increasing the time value of the equity compound call option. The endogenous default barrier drops from $V_B(t)$, for $t < t^*$, to $V_{BR}(t)$ for $t > t^*$.

Since ER(V,t=0) is greater than or equal to E(V,t=0), equity holders will have an incentive to renegotiate debt maturity as soon as condition 49) is met. If condition 49) is satisfied, debt holders have incentives to voluntarily concede extensions of debt maturity before default. The values of the model parameters determine whether or not condition 49) is satisfied.

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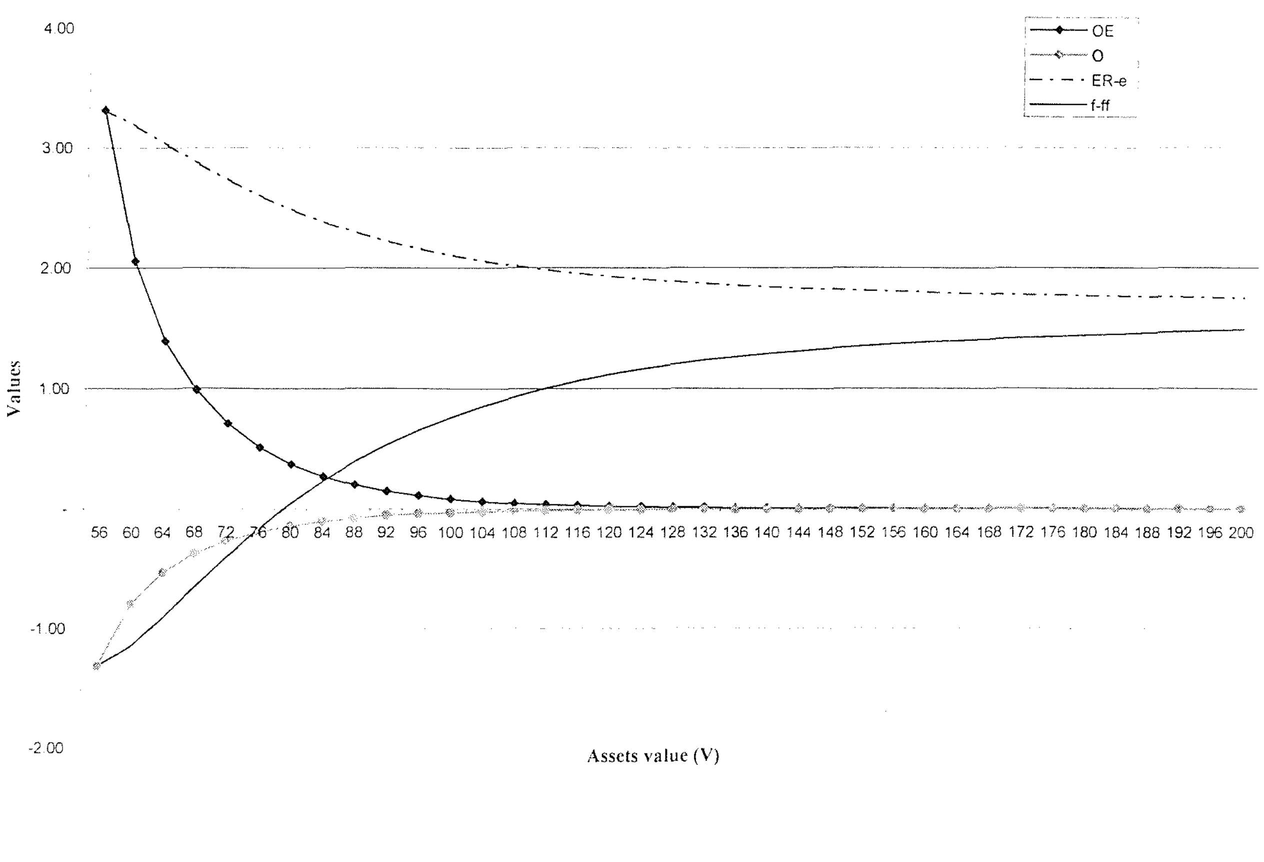
[43]In the previous time independent settings debt average maturity "1/m" was doubled upon exercise of the "extension option": similarly in the base case scenario of this time dependent setting the average maturity of debt is doubled at t*, so that $M_R = \frac{M}{2}$. Moreover, T is now chosen so that the initial debt average maturity (T/2) is such that T/ 2 = (1/m) = 5.

Then, the base case scenario in this setting reveals that debt both before and after default is more valuable (F(V=100, t=0) = 52.02 and f(V=100, t=0) = 52.82) than debt before and after default as per the base case scenario of section 3 (respectively F(V=100) = 50.54 and f(V=100) = 50.52). This is mainly due to the fact that the probability of default is now lower since assets pay-outs are mainly used to pay back debt principal, whereas in section 3 debt was refunded and a greater share of assets pay-outs could be destined to be distributed as dividends rather than to repaying debt principal.

Since debt is now more valuable, the extension option is much less valuable for equity holders than in the base case of section 3. In fact, *the riskier debt is, the more valuable the extension option for equity holders is*. The base case scenario now gives OE(V=100, t=0) = 0.08 instead of OE(V=100) = 1.71, and O(V=100, t=0) = -0.03 instead of O(V=100) = -0.09. Figure 4.4 displays the values of the extension option OE(V, t=0) and O(V, t=0) assuming base case scenario parameters in the present time dependent setting: due to constraint 49), [ER(V, t) - e(V, t)] > OE(V, t=0). Unlike in the time independent setting, now nominal outstanding debt is not constant and the time at which debt maturity is extended affects total firm value. An explicit finite differences scheme is employed with asset-step = 4 and time-step < (1 year /100). See appendix 3 for the case in which debt maturity is extended at V_{R2} (t).

Figure 4.4: Values of the extension option with base case scenario parameters in the time dependent setting

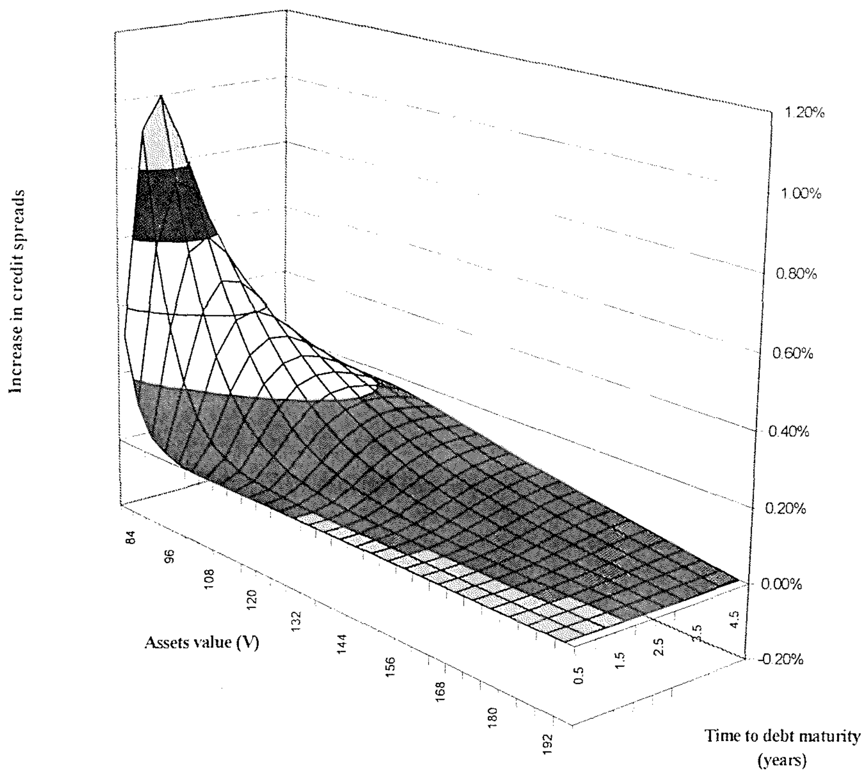


4.3) The term structure of credit spreads

In a time dependent setting the term structure of credit spreads can be analysed. Figure 4.5 displays the differential credit spreads due to an implicit option to re-negotiate and extend debt maturity when bankruptcy costs are high (K=10, a=15%) and debt is not amortised (M=0). It is interesting that the implicit extension option causes a significant increase in short-term credit spreads (lower assets values entail a more accentuated increase). In fact it is precisely such short-term credit spreads that traditional structural models, which do not account for debt re-negotiation, systematically understate. So, *these results suggest that structural models may understate short-term credit spreads because they neglect the presence of the implicit option to extend debt maturity.*

But it may not be apparent why short-term credit spreads should increase more than long-term credit spreads when an implicit extension option is recognised. The reason is that, for high leverage, debt market value (f(V, t)) is below debt face value (P), but as debt maturity approaches, debt market value is "pulled to par" if the debtor is solvent. This means that, when V is low, the payoff of the extension option (O(V, t*) = {f[V_R (t*), t*] - ff[V_R (t*), t*]} = {F[V_R (t*), t*] - ff[V_R (t*), t*]}) increases as t* approaches original debt maturity (T): exercising late implies a higher option payoff. Thus, as maturity draws near, O(V, t) becomes more valuable and its presence implies a higher increase in short-term credit spreads. On the other hand, if it is a few months before maturity and V is high enough, the implicit extension option is going to expire out of the money as the probability of the recovery value of assets dropping below ff[V_R (t), t] gradually vanishes. So immediately before maturity O(V, t) is too low to imply any significant increase in credit spreads. These arguments explain the shape in Figure 4.5 of the increase in the short-term credit spreads due to the presence of an implicit extension option.

Figure 4.5: Increase in short term credit spreads due to the "implicit" extension option



CONCLUSIONS

This chapter has focused on the value of debt given an option to renegotiate and/or extend debt maturity before default or just at default. The analysis has covered a time independent setting in which the firm keeps a constant nominal capital structure and a time dependent setting in which the firm's nominal capital structure is not constant.

The main result in a time independent setting is that an implicit or explicit extension option increases equity value more than it decreases debt value. Such option can cause a material increase in the value of equity and may also cause a non-negligible increase or decrease in the yield required by debt holders when the firm is far from default.

Under some conditions, equity holders and debt holders can both be better off by re-negotiating and extending debt maturity, which extends a previous result by Longstaff in a simple Mertonian setting. This may often be the case when debt maturity is extended soon before or just at default in order to avoid costly liquidation of the firm's assets.

In a time independent setting it has also been shown that different default conditions heavily affect the value of the implicit option to re-negotiate debt maturity and the incentive for debt holders to accept re-negotiation: when default is triggered by cash flow insolvency the implicit option to extend debt maturity may easily be worthless if debt holders are not enticed to accept re-negotiation by the threat of high bankruptcy costs.

Finally, in a time dependent setting it has been shown that when the firm does not refund debt with new debt, the probability of default decreases making debt more valuable and the extension option less valuable. Moreover, *in a time dependent setting the presence of the implicit "extension option" boosts the short-term credit spreads on the firm's debt thus partially overcoming the typical problem of structural models predicting too low short-term credit spreads.*

Future research could extend the above analysis and valuation of "extension options" to the case in which default free interest rates are stochastic. Future research may also consider:

1. the impact of the option to extend debt maturity on the choice of optimal capital structure;
2. the case in which the extended maturity of debt is endogenously determined so as to maximise equity value rather than being exogenous as it has been assumed in this chapter.

APPENDIX I: THE OPTION TO EXTEND MATURITY AND CREDIT SPREADS

The presence of the option to extend debt maturity (O(V)) implies a change in debt credit spread (dY), where

$$60)\ dY = \frac{C + m \cdot [P - F(V)]}{F(V)} - \frac{C + m \cdot [P - ff(V)]}{ff(V)}$$

and where F(V) is debt value (as per equation 12) when the extension option is present and ff(V) is debt value (as per equation 2) when the extension option is absent. The expressions $m \cdot [P - F(V)]$ and $m \cdot [P - ff(V)]$ denote the cash flows to and from debt holders due to continuously rolling debt over.

Equity holders may compensate debt holders for the option to renegotiate and extend debt maturity by promising a higher coupon (C^R) that would make F(V) = ff(V). Substituting for F(V) and ff(V) from equations 2) and 12), this gives:

61)

$$\frac{C^R + mP}{r+m} + \left[-\frac{C^R + mP}{r+m} + \frac{C^R + m_R P}{r+m_R} + \left[-\frac{C^R + m_R P}{r+m_R} + (1-a)\cdot V_{BR} - K \right] \cdot \left[\frac{V_R}{V_{BR}} \right]^h \right] \cdot \left[\frac{V}{V_R} \right]^b$$

$$= \frac{C + mP}{r+m} + \left[-\frac{C + mP}{r+m} + (1-a)\cdot V_B - K \right] \cdot \left[\frac{V}{V_B} \right]^b$$

Then root finding numerical algorithms can easily find C^R by solving 61).

APPENDIX II: ANOTHER CONDITION TO EXTEND MATURITY

Debt in the time dependent setting of section 4 is safer and more valuable than in the previous time dependent settings, so a coupon rate of 6% (i.e. 1% credit spread) over-remunerates debt holders for the risk of default they bear in the base case. Then debt holders will want this over-remuneration to last as long as possible. Then debt holders will want, at some point, to extend the maturity of debt that pays such generous coupons. In particular, they will desire to have maturity extended whenever

$$62)\ f[V(t), t] \geq F[V(t), t].$$

This condition can be satisfied at two points for every time t: $V_{R2.1}(t) \leq V_0 \leq V_{R2.2}(t)$.

Then, if conditions 48) 48.1) and 49) are substituted, by the following

for $\forall t, 0 \leq t \leq T$ max $V_{R2.1}(t)$, subject to

48.a) $ER(V_{R2.1}(t), t) \geq E(V_{R2.1}(t), t)$,

48.1.a) $V_0 \geq V_{R2.1}(t) \geq V_B(t)$,

49.a) $f(V_{R2.1}(t), t) \geq (1-a) \cdot V_{R2.1}(t) - K$

for $\forall t, 0 \leq t \leq T$ $\min V_{R2.2}(t)$, subject to

48.b) $ER(V_{R2.2}(t), t) \geq E(V_{R2.2}(t), t)$,

48.1.b) $V_0 \leq V_{R2.2}(t)$,

49.b) $f(V_{R2.2}(t), t) \geq (1-a) \cdot V_{R2.2}(t) - K$,

and if all other equations are the same as in the system of equations 41) to 52), we can find the values of equity and debt given that debt maturity is extended as soon as it is advantageous for both the debtor and the creditors to do so. The values $V_{R2.1}(t) \leq V_{R2.2}(t)$ make debt holders indifferent between holding debt of shorter or longer average maturity. Then, as in section 3, equity holders could convincingly propose to debt holders to have debt maturity extended as soon as $V(t) \leq V_{R2.1}(t)$ or $V(t) \geq V_{R2.1}(t)$. Though, in section 4 $V_{R2.1}(t=0)$ is about 48 and $V_{R2.2}(t=0)$ is about 80 as opposed to respectively 44.5 and 106.6 in the time independent setting of section 3 with base case parameters.

APPENDIX III: WHEN DEBT AMORTISATION STOPS

Given the time dependent setting of section 4, if $M_R = 0$ then the continuous amortisation of debt principal stops at t = t* and all debt principal still outstanding mat be repaid at T through a single "balloon" payment. Both when $M_R = \frac{M}{2}$ and when $M_R = 0$ with repayment at T, the average maturity of debt still outstanding at time t* is effectively double as long as when $M = M_R$. When $M_R = \frac{M}{2}$ the rate at which debt principal is amortised is halved, when $M_R = 0$ the repayment of debt principal is suspended until T.

When $M_R = 0$, the conditions for equation 43) change, since debt holders receive P(t*) at T and coupons at a rate $c \cdot P(t^*) \cdot dt$ between t* and T. Hence, condition 45) is substituted by

63)

$$f(V\to\infty,t)\to\int_t^T e^{-r\cdot(t-t^*)}c\cdot P(t^*)\cdot dt+e^{-r\cdot(T-t)}\cdot P(t^*)=\frac{1-e^{-r(T-t^*)}}{r}c\cdot P(t^*)+e^{-r(T-t)}\cdot P(t^*)$$

and the final condition is no longer $f(V,T^*)=0$, but

64) $f(V,T)=P(t^*)$ if V(T)> P(t*), or

65) $f(V,T)=\min[P(t^*),V\cdot(1-a)]$ if V(T)< P(t*).

Then, if $M_R = 0$ and P(t*) is due at T, the final condition for equation 50) is no longer

$ER(V,T^*)=V$, but

66) $ER(V,T)=\text{Max}\{V-P(t^*),0\}$.

Table 4.1: Debt value when debt maturity can be extended (a time independent setting). The firm's assets value is normalised at 100 and the face value of debt is assumed to be equal to 50. Panel A (and in the same way the other panels) is to be interpreted as follows: if debt average maturity is extended at V_{R4} (from a 5 years to a 10 years), equity (E(V)) rises from 55 to 57.2 and debt (F(V)) drops from 50.5 to 50. Extending debt maturity decreases default barrier from $V_{BI} = 49.8$ to $V_{BIR} = 44.8$, increases total firm value and the expected value of the tax shield (from TT(V)=13 to TT(V)=13.7) and decrease the expected value of bankruptcy costs (from BC(V)=2.9 to BC(V)=2.3).

TABLE 4.1: SUMMARY OF THE EFFECTS OF THE PRESENCE OF THE EXTENSION OPTION			
PANEL A: Base case with irreversible extension of debt maturity and cash flow shortage default			
Input data in italics	No extension option	Ante extension	Post extension
a (bankruptcy costs as fraction of V)	*15%*	*15%*	*15%*
r (default risk-free interest rate)	*5%*	*5%*	*5%*
s (volatility of V)	*20%*	*20%*	*20%*
d (assets total payout to security holders)	*7.0%*	*7.0%*	*7.0%*
tax (tax rate)	*35%*	*35%*	*35%*
K (fixed liquidation costs)	*0*	*0*	*0*
C (annual coupon, which is paid coutinuously)	*3.00*	*3.00*	*3.00*
Coupon rate (c = C/P)	*6.00%*	*6.00%*	*6.00%*
m (percentage of P that is refinanced every year)	***20%***	***20%***	***10%***
P (face value of debt)	*50.0*	*50.0*	*50.0*
V_0 (today's assets value)	*100.0*	*100.0*	*100.0*
$V_R = V_{R4}$ (value of asset at which debt maturity is extended)		82.8	
V_{BI} and V_{BIR} (value of assets triggering default)	49.8	44.8	44.8
OE (value of the "extension option" for equity holders)		**2.26**	
E (value of equity)	55.0	57.2	57.2
O (value of the "extension option" for debt holders)		**-0.53**	
F (value of debt ante extension)	50.5	50.0	
f (value of debt post extension)		48.6	50.1
$X(V_{R4})$ (assets recovery value at V_{R4})		70.4	
Credit spread: [C+m(P-F)]/F-r or [C+m(P-f)]/f-r	0.74%	1.02%	0.98%
PANEL B: All as in panel A exept for assets volatility and $V_r = V_{r3}$ rather than $V_r = V_{r4}$			
s (volatility of V)	*10%*	*10%*	*10%*
$V_R = V_{R3}$ (value of asset at which debt maturity is extended)		49.8	
V_{BI} and V_{BIR} (value of assets triggering default)	49.8	44.8	44.8
OE (value of the "extension option" for equity holders)		**1.65**	
E (value of equity)	59.5	61.2	60.5
O (value of the "extension option" for debt holders)		**0.01**	
F (value of debt ante extension)	51.7	51.7	
f (value of debt post extension)		42.8	52.4
$X(V_{R3})$ (assets recovery value at V_{R3})		42.3	
Credit spread: [C+m(P-F)]/F-r or [C+m(P-f)]/f-r	0.14%	0.14%	0.26%
PANEL C: All as in panel A exept for default when equity is worthless and $V_r = V_{r1}$			
$V_R = V_{R1}$ (value of asset at which debt maturity is extended)		50.5	
V_{BE} and V_{BER} (value of assets triggering default)	35.5	**30.4**	30.4
OE (value of the "extension option" for equity holders)		**1.71**	
E (value of equity)	59.7	61.4	61.4
O (value of the "extension option" for debt holders)		**-0.09**	
F (value of debt ante extension)	50.6	50.5	
f (value of debt post extension)		42.9	50.5
$X(V_{R1})$ (assets recovery value at V_{R1})		42.9	
Credit spread: [C+m(P-F)]/F-r or [C+m(P-f)]/f-r	0.68%	0.72%	0.83%
PANEL D: All as in panel A exept for default when equity is worthless and $V_r = V_{r2}$			
$V_R = V_{R2}$ (value of asset at which debt maturity is extended)		44.5	
V_{BE} and V_{BER} (value of assets triggering default)	35.5	**30.4**	30.4
OE (value of the "extension option" for equity holders)		**1.61**	
E (value of equity)	59.7	61.3	61.4
O (value of the "extension option" for debt holders)		**0.00**	
F (value of debt ante extension)	50.6	50.6	
f (value of debt post extension)		40.1	50.5
$X(V_{R2})$ (assets recovery value at V_{R2})		37.8	
Credit spread: [C+m(P-F)]/F-r or [C+m(P-f)]/f-r	0.68%	0.68%	0.83%
PANEL E: All as in panel A exept for default when equity is worthless and $V_r = V_{r3}$			
$V_R = V_{R3}$ (value of asset at which debt maturity is extended)		35.5	
V_{BE} and V_{BER} (value of assets triggering default)	35.5	**30.4**	30.4
OE (value of the "extension option" for equity holders)		**1.44**	
E (value of equity)	59.7	61.1	61.4
O (value of the "extension option" for debt holders)		**0.17**	
F (value of debt ante extension)	50.6	50.8	
f (value of debt post extension)		32.9	50.5
$X(V_{R3})$ (assets recovery value at V_{R3})		30.2	
Credit spread: [C+m(P-F)]/F-r or [C+m(P-f)]/f-r	0.68%	0.59%	0.83%

APPENDIX IV: THE CODE

The following is the Visual Basic code that implements the numerical solutions to the valution problems in this chapter. The numerical schemes employ explicit finite differences.

The following algorithm "Renegotiation" solves the equations 41 to 59. The values of VoldE, VoldD, VoldER, VOldDR found by the algorithm correspond respectively to E, F, ER, f.

```
Option Explicit

Private VOldE(0 To 200) As Double 'This is the value of equity before debt maturity is extended
Private VOldD(0 To 200) As Double 'This is the value of debt before debt maturity is extended
Private VOldER(0 To 200) As Double 'This is the value of equity after debt maturity is extended
Private VOldDR(0 To 200) As Double 'This is the value of debt after debt maturity is extended
Private Recovery(0 To 200) As Double ' This is the firm's assets recovery value after costly liquidation

'The meaning of the parameters of the algorithm is the following:
'Asset indicates the value of the firm's assets (V),
'Principal indicates the face value of debt (P),
'Alpha indicates the loss given default (a),
'k indicated the fixed bankruptcy costs (K),
'Expiry indicates debt maturity (T),
'Volatility indicates assets volatility (s),
'IntRate indicates the default free short interest rate (r),
'Payout indicates the assets pay-out rate (d),
'tax indicates the corporate tax rate (tax),
'CouponRate indicates the debt coupon rate [C],
'PaymentAfter indicates the yearly rate at which debt principal is continuously repaid [ M_R ] after debt 'maturity is extended,
'Collateral indicates the minimum liquidation value of the firm's assets,
'NoAssetSteps indicates the number of intervals of equal length into which the solution domain in the 'in the V direction is divided.

Function Renegotiation(Asset As Double, Principal As Double, Alpha As Double, _
k As Double, Expiry As Double, Volatility As Double, IntRate As Double, _
Payout As Double, tax As Double, CouponRate As Double, PaymentAfter As Double, Collateral As Double,
NoAssetSteps As Integer)

Dim GammaE(0 To 200) As Double
Dim VNewE(0 To 200) As Double
Dim DeltaE(0 To 200) As Double
Dim ThetaE(0 To 200) As Double

Dim GammaER(0 To 200) As Double
Dim VNewER(0 To 200) As Double
Dim DeltaER(0 To 200) As Double
Dim ThetaER(0 To 200) As Double

Dim S(0 To 200) As Double
Dim Ssqd(0 To 200) As Double

Dim GammaD(0 To 200) As Double
Dim VNewD(0 To 200) As Double
Dim DeltaD(0 To 200) As Double
Dim ThetaD(0 To 200) As Double

Dim GammaDR(0 To 200) As Double
Dim VNewDR(0 To 200) As Double
```

```
Dim DeltaDR(0 To 200) As Double
Dim ThetaDR(0 To 200) As Double

Dim i As Integer
Dim j As Integer
Dim u As Integer
Dim v As Integer
Dim mini As Integer
Dim Assetstep As Double
Dim NearestGridPt As Integer
Dim dummy As Double
Dim Timestep As Double
Dim NoTimeSteps As Integer
Dim halfvolsqd As Double

Dim Repayment As Double
Dim CumulatedRepayment As Double
Dim RepaymentR As Double
Dim CumulatedRepaymentR As Double

halfvolsqd = 0.5 * Volatility * Volatility

Assetstep = 4 * Principal / NoAssetSteps
NearestGridPt = Int(Asset / Assetstep)
dummy = (Asset - NearestGridPt * Assetstep) / Assetstep
Timestep = Assetstep * Assetstep / (Volatility) / (Volatility) / (16 * Principal * Principal)
NoTimeSteps = Int(Expiry / Timestep) + 1
Timestep = Expiry / NoTimeSteps
mini = 0

    Repayment = (Principal / Expiry) 'Annual repayment rate of principal
    CumulatedRepayment = Principal 'By the debt maturity date principal is assumed to have been completely repaid

For i = mini To NoAssetSteps
    S(i) = i * Assetstep
    Ssqd(i) = S(i) * S(i)
    Recovery(i) = Application.Max((1 - Alpha) * S(i) - k, Collateral)
    VOldE(i) = Application.Max(S(i) - (Principal - CumulatedRepayment), 0)
    If VOldE(i) > 0 Then
    VOldD(i) = Principal - CumulatedRepayment
    Else
    VOldD(i) = Application.Min(Principal - CumulatedRepayment, Recovery(i))
    End If
Next i

For j = 1 To NoTimeSteps
For i = (mini + 1) To NoAssetSteps - 1
      DeltaE(i) = (VOldE(i + 1) - VOldE(i - 1)) / (2 * Assetstep)
      GammaE(i) = (VOldE(i + 1) - 2 * VOldE(i) + VOldE(i - 1)) / (Assetstep * Assetstep)

      DeltaD(i) = (VOldD(i + 1) - VOldD(i - 1)) / (2 * Assetstep)
      GammaD(i) = (VOldD(i + 1) - 2 * VOldD(i) + VOldD(i - 1)) / (Assetstep * Assetstep)

      VNewE(i) = VOldE(i) + Timestep * (halfvolsqd * Ssqd(i) * GammaE(i) + _
      (IntRate - Payout) * S(i) * DeltaE(i) - IntRate * VOldE(i))

      VNewD(i) = VOldD(i) + Timestep * (halfvolsqd * Ssqd(i) * GammaD(i) + _
      (IntRate - Payout) * S(i) * DeltaD(i) - IntRate * VOldD(i))

Next i
```

```
        VNewE(mini) = 0
        VNewE(NoAssetSteps) = 2 * VNewE(NoAssetSteps - 1) - VNewE(NoAssetSteps - 2)

        VNewD(mini) = Application.Min((Principal - CumulatedRepayment), Recovery(mini))
        VNewD(NoAssetSteps) = 2 * VNewD(NoAssetSteps - 1) - VNewD(NoAssetSteps - 2)


'****************************************************************************

    RepaymentR = Repayment * PaymentAfter 'Renegotiated annual repayment rate of principal repayment
    CumulatedRepaymentR = CumulatedRepayment '+ RepaymentR * Int(1 / PaymentAfter) * j * timestep
 'Renegotiated debt principal repaid by maturity

 For u = mini To NoAssetSteps
    VOldER(u) = Application.Max(S(u) - (Principal - CumulatedRepaymentR), 0)
    If VOldER(u) > 0 Then ' Corrected mistake: If VOldE(u) > 0 Then
    VOldDR(u) = Principal - CumulatedRepaymentR
    Else
    VOldDR(u) = Application.Min(Principal - CumulatedRepaymentR, Recovery(u))
    End If
 Next u

 For v = 1 To NoTimeSteps '(Int(1 / PaymentAfter) * j)

 For u = (mini + 1) To NoAssetSteps - 1
       DeltaER(u) = (VOldER(u + 1) - VOldER(u - 1)) / (2 * Assetstep)
       GammaER(u) = (VOldER(u + 1) - 2 * VOldER(u) + VOldER(u - 1)) / (Assetstep * Assetstep)

       DeltaDR(u) = (VOldDR(u + 1) - VOldDR(u - 1)) / (2 * Assetstep)
       GammaDR(u) = (VOldDR(u + 1) - 2 * VOldDR(u) + VOldDR(u - 1)) / (Assetstep * Assetstep)

       VNewER(u) = VOldER(u) + Timestep * (halfvolsqd * Ssqd(u) * GammaER(u) + _
       (IntRate - Payout) * S(u) * DeltaER(u) - IntRate * VOldER(u))

       VNewDR(u) = VOldDR(u) + Timestep * (halfvolsqd * Ssqd(u) * GammaDR(u) + _
       (IntRate - Payout) * S(u) * DeltaDR(u) - IntRate * VOldDR(u))

Next u

      VNewER(mini) = 0
      VNewER(NoAssetSteps) = 2 * VNewER(NoAssetSteps - 1) - VNewER(NoAssetSteps - 2)

      VNewDR(mini) = Application.Min((Principal - CumulatedRepaymentR), Recovery(mini))
      VNewDR(NoAssetSteps) = 2 * VNewDR(NoAssetSteps - 1) - VNewDR(NoAssetSteps - 2)

For u = 0 To NoAssetSteps

'When default is triggered by equity becoming worthless:
VOldER(u) = Application.Max(VNewER(u) + (S(u) * Payout - ((Principal - CumulatedRepaymentR) * CouponRate *
(1 - tax) + RepaymentR)) * Timestep, 0)
If VOldER(u) > 0 Then
VOldDR(u) = VNewDR(u) + (Principal - CumulatedRepaymentR) * CouponRate * Timestep + RepaymentR *
Timestep
Else
VOldDR(u) = Application.Min(Principal - CumulatedRepaymentR, Recovery(u))
End If

'When default is triggered by insolvency:
'If (S(u) * Payout - (Principal - CumulatedRepaymentR) * CouponRate * (1 - tax) - RepaymentR) > 0 Then
'VOldER(u) = Application.Max(VNewER(u) + (S(u) * Payout - (Principal - CumulatedRepaymentR) * CouponRate *
(1 - tax) - RepaymentR) * timestep, 0)
```

```
'VOldDR(u) = VNewDR(u) + (Principal - CumulatedRepaymentR) * CouponRate * timestep + RepaymentR *
timestep
'Else
'VOldE(u) = Application.Max(Recovery(u) - (Principal - CumulatedRepaymentR), 0)
'VOldDR(u) = Application.Min(Principal - CumulatedRepaymentR, Recovery(u))
'End If

Next u

  CumulatedRepaymentR = CumulatedRepaymentR - RepaymentR * Timestep

Next v

'******************************************************************************

For i = 0 To NoAssetSteps

'When default is triggered by equity becoming worthless:
VOldE(i) = Application.Max(VNewE(i) + (S(i) * Payout - (Principal - CumulatedRepayment) * CouponRate * (1 -
tax) - Repayment) * Timestep, 0)
If VOldE(i) > 0 Then
  VOldD(i) = VNewD(i) + (Principal - CumulatedRepayment) * CouponRate * Timestep + Repayment * Timestep
Else
  VOldD(i) = Application.Min(Principal - CumulatedRepayment, Recovery(i))
End If

'When default is triggered by insolvency:
'If (S(i) * Payout - (Principal - CumulatedRepayment) * CouponRate * (1 - tax) - Repayment) > 0 Then
'VOldE(i) = Application.Max(VNewE(i) + (S(i) * Payout - (Principal - CumulatedRepayment) * CouponRate * (1 -
tax) - Repayment) * timestep, 0)
'VOldD(i) = VNewD(i) + (Principal - CumulatedRepayment) * CouponRate * timestep + Repayment * timestep
'Else
'VOldE(i) = Application.Max(Recovery(i) - (Principal - CumulatedRepayment), 0)
'VOldD(i) = Application.Min(Principal - CumulatedRepayment, Recovery(i))
'End If

'If Renegotiation takes place the first time the recovery value of debt is lower than the value of 'renegotiated debt:
'If (Recovery(i) =< VOldDR(i)) Then
'  If (VOldER(i) > VOldE(i)) Then
'  If (VOldE(i) > 0) Then
'  VOldE(i) = Application.Max(VOldER(i), VOldE(i))
'  VOldD(i) = VOldDR(i)
'  End If
'  End If
'End If

'If Renegotiation takes place the first time equity becomes worthless:
'If (Recovery(i) < VOldDR(i)) Then
'  If (VOldER(i) > VOldE(i)) Then
'  If VOldE(i) = 0 Then
'  VOldE(i) = Application.Max(VOldER(i), VOldE(i))
'  VOldD(i) = VOldDR(i)
'  End If
'  End If
'End If

'If Renegotiation takes place the first time before default in which both debt holders and equity holders 'both gain from
renegotiationg
If (VOldD(i) < VOldDR(i)) Then
  If (VOldER(i) > VOldE(i)) Then
  If VOldE(i) > 0 Then
```

```
    VOldE(i) = Application.Max(VOldER(i), VOldE(i))
    VOldD(i) = VOldDR(i)
    End If
    End If
  End If

  Next i

   CumulatedRepayment = CumulatedRepayment - Repayment * Timestep    'This updating condition is here not to
  distort the amount of coupons above

  Next j

  Renegotiation = (1 - dummy) * VOldE(NearestGridPt) + dummy * VOldE(NearestGridPt + 1)

  End Function
```

The following algorithm "Renegotiation1" solves the equations 41 to 59 when debt is not amortised before maturity. The values of VoldE, VoldD, VoldER, VOldDR found by the algorithm correspond respectively to E, F, ER, f. The meaning of the parameters of the algorithm is the same as before. This is the code used to generate the data in Figure 4.5.

```
Option Explicit

Private VOldE(0 To 200) As Double 'This is the value of equity before debt maturity is extended
Private VOldD(0 To 200) As Double 'This is the value of debt before debt maturity is extended
Private VOldER(0 To 200) As Double 'This is the value of equity after debt maturity is extended
Private VOldDR(0 To 200) As Double 'This is the value of debt after debt maturity is extended
Private Recovery(0 To 200) As Double ' This is the firm's assets recovery value after costly liquidation

Function Renegotiation1(Asset As Double, Principal As Double, Alpha As Double, _
k As Double, Expiry As Double, Extension As Double, Volatility As Double, IntRate As Double, _
Payout As Double, tax As Double, CouponRateBefore As Double, CouponRateAfter As Double, _
Collateral As Double, NoAssetSteps As Integer)

'Constant capital structure
'In this algorithm debt maturity is extended upon renegotiation There is no amortisation of debt 'principal, at least
before renegotiation and extension of debt maturity.

Dim GammaE(0 To 200) As Double
Dim VNewE(0 To 200) As Double
Dim DeltaE(0 To 200) As Double
Dim ThetaE(0 To 200) As Double

Dim GammaER(0 To 200) As Double
Dim VNewER(0 To 200) As Double
Dim DeltaER(0 To 200) As Double
Dim ThetaER(0 To 200) As Double

Dim S(0 To 200) As Double
Dim Ssqd(0 To 200) As Double

Dim GammaD(0 To 200) As Double
Dim VNewD(0 To 200) As Double
Dim DeltaD(0 To 200) As Double
Dim ThetaD(0 To 200) As Double
```

```
Dim GammaDR(0 To 200) As Double
Dim VNewDR(0 To 200) As Double
Dim DeltaDR(0 To 200) As Double
Dim ThetaDR(0 To 200) As Double

Dim i As Integer
Dim j As Integer
Dim u As Integer
Dim v As Integer
Dim mini As Integer
Dim Assetstep As Double
Dim NearestGridPt As Integer
Dim dummy As Double
Dim Timestep As Double
Dim NoTimeSteps As Integer
Dim NoExtTimesteps As Integer
Dim halfvolsqd As Double

Dim RepaymentR As Double
Dim CumulatedRepaymentR As Double

halfvolsqd = 0.5 * Volatility * Volatility

Assetstep = 4 * Principal / NoAssetSteps
NearestGridPt = Int(Asset / Assetstep)
dummy = (Asset - NearestGridPt * Assetstep) / Assetstep
Timestep = Assetstep * Assetstep / (Volatility) / (Volatility) / (16 * Principal * Principal)
NoTimeSteps = Int(Expiry / Timestep) + 1
NoExtTimesteps = Int(Extension / Timestep) + 1
Timestep = Expiry / NoTimeSteps
mini = 0

For i = mini To NoAssetSteps
   S(i) = i * Assetstep
   Ssqd(i) = S(i) * S(i)
   Recovery(i) = Application.Max((1 - Alpha) * S(i) - k, Collateral)
Next i

'*****************************************************************************
'The following code is to value ER and DR (i.e. equity and debt post renegotiation of the debt contract)

   RepaymentR = 0 'The renegotiated annual repayment rate of principal is zero
   CumulatedRepaymentR = 0 '(RepaymentR * NoExtTimesteps * Timestep)  'No debt principal is 'repaid before debt
maturity

For u = mini To NoAssetSteps
   VOldER(u) = Application.Max(S(u) - (Principal - CumulatedRepaymentR), 0)
'The following are the payoff conditions at the "extended" debt maturity
   If VOldER(u) > 0 Then
   VOldDR(u) = Principal - CumulatedRepaymentR
   Else
   VOldDR(u) = Application.Min(Principal - CumulatedRepaymentR, Recovery(u))
   End If
Next u

For v = 1 To NoExtTimesteps

For u = (mini + 1) To NoAssetSteps - 1
     DeltaER(u) = (VOldER(u + 1) - VOldER(u - 1)) / (2 * Assetstep)
     GammaER(u) = (VOldER(u + 1) - 2 * VOldER(u) + VOldER(u - 1)) / (Assetstep * Assetstep)
```

```
        DeltaDR(u) = (VOldDR(u + 1) - VOldDR(u - 1)) / (2 * Assetstep)
        GammaDR(u) = (VOldDR(u + 1) - 2 * VOldDR(u) + VOldDR(u - 1)) / (Assetstep * Assetstep)

        VNewER(u) = VOldER(u) + Timestep * (halfvolsqd * Ssqd(u) * GammaER(u) + _
        (IntRate - Payout) * S(u) * DeltaER(u) - IntRate * VOldER(u))

        VNewDR(u) = VOldDR(u) + Timestep * (halfvolsqd * Ssqd(u) * GammaDR(u) + _
        (IntRate - Payout) * S(u) * DeltaDR(u) - IntRate * VOldDR(u))

  Next u

        VNewER(mini) = 0
        VNewER(NoAssetSteps) = 2 * VNewER(NoAssetSteps - 1) - VNewER(NoAssetSteps - 2)

        VNewDR(mini) = Application.Min((Principal - CumulatedRepaymentR), Recovery(mini))
        VNewDR(NoAssetSteps) = 2 * VNewDR(NoAssetSteps - 1) - VNewDR(NoAssetSteps - 2)

  For u = 0 To NoAssetSteps

  'When default is triggered by equity becoming worthless:
  VOldER(u) = Application.Max(VNewER(u) + (S(u) * Payout - ((Principal - CumulatedRepaymentR) *
  CouponRateAfter * (1 - tax) + RepaymentR)) * Timestep, 0)
  If VOldER(u) > 0 Then
  VOldDR(u) = VNewDR(u) + (Principal - CumulatedRepaymentR) * CouponRateAfter * Timestep + RepaymentR *
  Timestep
  Else
  VOldDR(u) = Application.Min(Principal - CumulatedRepaymentR, Recovery(u))
  End If

  'When default is triggered by insolvency (cash flow shortage):
  'If (S(u) * Payout - (Principal - CumulatedRepaymentR) * CouponRateAfter * (1 - tax) - RepaymentR) '> 0 Then
  'VOldER(u) = Application.Max(VNewER(u) + (S(u) * Payout - (Principal - CumulatedRepaymentR) *
  CouponRateAfter * (1 - tax) - RepaymentR) * Timestep, 0)
  'VOldDR(u) = VNewDR(u) + (Principal - CumulatedRepaymentR) * CouponRateAfter * Timestep + RepaymentR *
  Timestep
  'Else
  'VOldER(u) = 0
  'VOldDR(u) = Application.Min(Principal - CumulatedRepaymentR, Recovery(u))
  'End If

  Next u

   CumulatedRepaymentR = CumulatedRepaymentR - RepaymentR * Timestep

  Next v

'***************************************************************************

'This terminal condition implies that debt can be renegotiated at maturity just if the firm is still solvent 'at maturity
For i = mini To NoAssetSteps
If S(i) > Principal Then
VOldE(i) = Application.Max(S(i) - Principal, 0)
VOldD(i) = Principal
Else
VOldE(i) = Application.Max(S(i) - Principal, 0)
VOldD(i) = Application.Min(Principal, Recovery(i))
End If

'If S(i) > Principal Then
If VOldE(i) < VOldER(i) Then
If VOldDR(i) > Recovery(i) Then
```

```
VOldE(i) = VOldER(i)
VOldD(i) = VOldDR(i)
End If
End If
'End If
Next i

For j = 1 To NoTimeSteps
For i = (mini + 1) To NoAssetSteps - 1
    DeltaE(i) = (VOldE(i + 1) - VOldE(i - 1)) / (2 * Assetstep)
    GammaE(i) = (VOldE(i + 1) - 2 * VOldE(i) + VOldE(i - 1)) / (Assetstep * Assetstep)

    DeltaD(i) = (VOldD(i + 1) - VOldD(i - 1)) / (2 * Assetstep)
    GammaD(i) = (VOldD(i + 1) - 2 * VOldD(i) + VOldD(i - 1)) / (Assetstep * Assetstep)

    VNewE(i) = VOldE(i) + Timestep * (halfvolsqd * Ssqd(i) * GammaE(i) + _
    (IntRate - Payout) * S(i) * DeltaE(i) - IntRate * VOldE(i))

    VNewD(i) = VOldD(i) + Timestep * (halfvolsqd * Ssqd(i) * GammaD(i) + _
    (IntRate - Payout) * S(i) * DeltaD(i) - IntRate * VOldD(i))

Next i

    VNewE(mini) = 0
    VNewE(NoAssetSteps) = 2 * VNewE(NoAssetSteps - 1) - VNewE(NoAssetSteps - 2)

    VNewD(mini) = Application.Min(Principal, Recovery(mini))
    VNewD(NoAssetSteps) = 2 * VNewD(NoAssetSteps - 1) - VNewD(NoAssetSteps - 2)

For i = 0 To NoAssetSteps

'When default is triggered by equity becoming worthless:
VOldE(i) = Application.Max(VNewE(i) + (S(i) * Payout - Principal * CouponRateBefore * (1 - tax)) * Timestep, 0)
If VOldE(i) > 0 Then
VOldD(i) = VNewD(i) + (Principal * CouponRateBefore * Timestep)
Else
VOldD(i) = Application.Min(Principal, Recovery(i))
End If

'When default is triggered by insolvency (cash flow shortage):
'If (S(i) * Payout - (Principal * CouponRateBefore * (1 - tax))) > 0 Then
'VOldE(i) = Application.Max(VNewE(i) + (S(i) * Payout - (Principal * CouponRateBefore * (1 - tax))) '* Timestep, 0)
'VOldD(i) = VNewD(i) + Principal * CouponRateBefore * Timestep
'Else
'VOldE(i) = 0
'VOldD(i) = Application.Min(Principal, Recovery(i))
'End If

'When debt holders extend maturity at default, not earlier and not later, and default is triggerd by 'insolvency:
'If Not (S(i) * Payout - (Principal * CouponRateBefore * (1 - tax))) > 0 Then
'When debt holders extend maturity at default, no earlier and not later, and default is triggered by worthless 'equity:
If VOldE(i) = 0 Then
If (Recovery(i) < VOldDR(i)) Then
If (VOldER(i) > VOldE(i)) Then
If VOldE(i + 1) > 0 Then
VOldE(i) = VOldER(i)
VOldD(i) = VOldDR(i)
End If
End If
End If
'End If
```

```
'End If
Next i

Next j

Renegotiation1 = (1 - dummy) * VOldD(NearestGridPt) + dummy * VOldD(NearestGridPt + 1)

End Function
```

CHAPTER 5:

"INTERACTIONS" BETWEEN DEFAULT RISK AND INTEREST RATE RISK IN STRUCTURAL MODELS OF CREDIT RISK

INTRODUCTION

Structural models of credit risk usually exhibit "interactions" between default risk and interest rate risk, in that different processes for the default free short rate imply different credit spreads on corporate bonds and different values for credit derivatives. Such "interactions" are a source of "interest rate model risk". This chapter shows that "interactions" can be significant and proposes ways to eliminate them when markets are either complete or incomplete. This allows to obtain simple closed form solutions for the value of bonds subject to both interest rate risk and default risk.

Over the past decade, a number of structural models of credit risk have been proposed to value corporate bonds. Usually the building block of these models is the "risk neutral" value process of the firm's assets, which is assumed to follow a Geometric Brownian motion with "risk neutral drift". The "risk neutral drift" implies that the expected growth rate of the firm's assets value is equal to the default free short rate of interest (hereafter "short rate"). The "risk neutral drift" is required by the absence of arbitrage opportunities in a complete market setting and implies that the lower the short rate is, the higher the bond credit spreads are. This implication does not have strong empirical support, since the empirically observed negative relation[44] between changes in short rate and changes in credit spreads seems weak for non-callable bonds, and especially for high grade ones (see G.Duffie (1998) and Ericsson-Reneby (2001)).

Some structural models also assume instantaneous correlation between the Wiener process driving the debtor's assets value and the one driving the short rate (e.g. Kim-Ramaswamy-Sundaresan (1993) and Longstaff-Schwartz (1995)).

Both "risk neutral drift" and instantaneous correlation between assets and short rate entail "interactions" between interest rate risk and default risk. "Interactions" mean that different processes for the short rate imply different credit spreads on corporate bonds and different values for claims contingent on corporate bonds, such as credit derivatives.

This chapter focuses on the "interactions" that are due to the "risk neutral drift" rather than to instantaneous correlation. The main results of the chapter are the following.

- Assuming that the short rate may follow either the Vasicek (1977) or the CIR (1985) process, it is shown that a typical structural model predicts credit spreads and values of "credit puts" that are exposed to "interest-rate-model-risk". In fact "interactions" between interest rate risk and default risk cause disturbing differences in the values of bonds and credit derivatives predicted by a structural model under different short rate processes.

- "Interactions" can be eliminated from a structural model in at least three simple ways. One way assumes incomplete markets, the other two assume complete markets. All three ways entail separating interest rate risk from default risk. This gives simple closed form solutions for the value of corporate bonds.

Comparison with the literature

The problem of "interactions" between interest rate risk and default risk in bond valuation is highlighted by Nielsen, Saa'-Requejo and Santa Clara (1993), but these authors only provide numerical solutions to the bond valuation problem. By eliminating the mentioned "interactions", this chapter proposes simple closed form solutions to a similar bond valuation problem. Closed forms are of interest for they allow easy calibration of model parameters to observed bond market prices, calibration being essential for practical pricing purposed.

Other structural models of credit risk have provided closed form solutions for bonds subject to both interest rate risk and default risk, in particular: Shimko D. and Naohiko T. and Van Deventer (1993), Longstaff and Schwartz (1995), Bris and deVarenne (1997), Cathcart and El-Jahel (1998), Schlogel (1999), Saa'-Requejo and Santa Clara (1999), Tauren (1999), Dufresne and Goldstein (2001)[45]. But the closed form solutions proposed is this chapter are the only ones thus proposed with *all* the following merits:

- they allow the presence of both a default barrier and a stochastic short rate, unlike in Shimko D., Naohiko T. and Van Deventer (1993);
- they do not require the default barrier to be constant in the risk neutral measure, unlike in Schlogel (1999);
- they do not require the firm to have issued just a single zero coupon bond, unlike in Bris and deVarenne (1997);
- they do not require that the short rate necessarily follow the Vasicek process, unlike in Shimko D. and Naohiko T. and Van Deventer (1993), Longstaff and Schwartz (1995), Bris and deVarenne (1997), Schlogel (1999), Tauren (1999), Dufresne and Goldstein (2001);
- they do not involve the complicated type of closed form solutions found in Longstaff and Schwartz (1995), Tauren (1999), Dufresne and Goldstein (2001);
- they do not require to specify how the Wiener process driving the short rate affects the value of the firm's liabilities, unlike in Saa'-Requejo and Santa Clara (1999);

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[44] See Longstaff-Schwartz (1995) and Duffie (1998).

[45] The closed form solutions of Leland (1996) and Ericsson (1998) assume that the short rate is constant rather than stochastic.

- they generate realistic term structures of credit spreads;
- they allow us to "separately" model default risk and interest rate risk.

A special case of the closed form solution provided in this chapter under incomplete markets is also the solution to the model by Cathcart and El-Jahel (1998).

The chapter is organised as follows. In section 1 a typical structural model with "interactions" is discussed. Section 2 experiments with the model of section 1 to show significant "interactions" in the valuation of bonds and credit derivatives under different processes for the short rate. In section 3 structural models without "interactions" are proposed. Then the conclusions follow.

1. ASSUMPTIONS AND VALUATION MODEL WITH "INTERACTIONS"

This section discusses a typical structural model with "interactions" and its assumptions. This model synthesizes past contributions and is used to value not only bonds but also credit derivatives such as a "credit put".

1.1) Assumptions and notation

Before proceeding, the notation and main assumptions are introduced. Some notation first:

V = market value of the debtor's assets;

V_B = default barrier; when $V = V_B$ default is triggered;

r = default free short-term interest rate (the short rate);

t = time;

T = debt maturity;

D(V, r, t) = value of debt subject to both default risk and interest rate risk; "F" is the face value of debt and "c" is the debt coupon rate; coupons are paid continuously;

Z(r, t) = value of debt that promises the same cash flows on the same dates as D(V, r, t), but that is default free;

a = debt loss given default expressed as a percentage of Z(r, t); it holds that $0 \le a \le 1$; debt recovery value after default is equal to (1-a)Z(r, t).

The main assumptions follow in the next page.

A) The default barrier is known and constant over time at V_B (at least in sections 1 and 2). This assumption is common (e.g. Kim-Ramaswamy-Sundaresan (1993), Longstaff-Schwartz (1995), Ericsson (1998), etc..) and often implies a constant nominal capital structure, which may be a restrictive assumption to value long-term bonds. But bonds can be periodically re-valued as and if the debtor's nominal capital structure changes[46]. In section 3 the barrier is allowed to be an exponential function of time in such a way that "interactions" are eliminated.

B) Following Longstaff-Schwartz (1995), the debt recovery value upon default is:

1) $D(V_B, r, t) = (1 - a) \cdot Z(r, t)$.

Equation 1) entails that the valuation of a bond issue does not require the valuation of the whole capital structure of the debtor, because "a" is an exogenous parameter.

C) Securities markets are perfect and dynamically complete. Equity holders and debt holders have symmetric information. These assumptions are common and it follows that the risk neutral process of the debtor's assets value is:

2) $\partial V = (r - b) \cdot V \cdot dt + s \cdot V\, \partial z_V$, where:

∂z_V is the increment of a Wiener process;

"s" is the volatility of the debtor's assets value;

"b" is the firm's assets pay-out rate expressed as a percentage of V;

"r" is the short rate.

The presence of "r" in equation 2) causes "interactions" between default risk and interest rate risk. In section 3 such interactions will be eliminated.

D) The short rate follows the following process:

3) $\partial r = u \cdot (n - r) \cdot dt + r^g \cdot w \cdot \partial z_r$, where:

∂z_r is the increment of a Wiener process;

"n" is the mean reversion level of "r";

"u" is the speed of mean reversion;

"w" is the volatility of "r";

"g" is the elasticity of variance for "r"; when the short rate is "Vasicek" then g = 0; when the short rate is "CIR then g = 0.5.

E) For simplicity this chapter assumes that ∂z_V and ∂z_r have no instantaneous correlation. Such correlation is difficult to estimate and is another source of "interactions" effects when it is assumed to be present.

Given these assumptions, the following valuation models can be derived.

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[46] Alternatively, Tauren (1999) and Goldstein-Dufresne (2001) assume the dynamics of the

1.2) The valuation model for corporate bonds

Given dynamically complete markets, the value of a default-free bond Z(r, t) that continuously pays coupons at an yearly rate $c \cdot F$ satisfies[47]:

$$4)\ Z_t + \frac{1}{2} \cdot \left(w \cdot r^g\right)^2 \cdot Z_{rr} + u \cdot (n - r) \cdot Z_r - r \cdot Z + c \cdot F = 0,$$

subject to:

4.1) $Z(r, t = T) = F$

4.2) $Z(r \to \infty, t) \to 0$

4.3) if g=0.5 and if $w < \sqrt{2 \cdot u \cdot n}$, then $Z(r \to 0, t) \to \text{finite}$; if g=0, then $Z(r \to -\infty, t) \to \text{finite}$[48].

By adjusting equation 53) in Ingersoll (1987) at page 445, it can be shown that the value of debt that promises the same cash-flows as Z(r, t), but that is subject to default (here denoted by D(V, r, t)), satisfies:

$$5)\ D_t + \frac{1}{2} \cdot s^2 \cdot V^2 \cdot D_{VV} + \frac{1}{2} \cdot \left(w \cdot r^g\right)^2 \cdot D_{rr} + u \cdot (n - r) \cdot D_r + (r - b) \cdot V \cdot D_V - r \cdot D + c \cdot F = 0$$

subject to:

5.1) $D(V \to \infty, r, t) \to Z(r, t)$,

5.2) $D(V_B, r, t) = (1 - a) \cdot Z(r, t)$,

5.3) $D(V, r, t = T) = F$,

5.4) $D(V, r \to \infty, t) \to 0$,

5.5) if g=0.5 and if $w < \sqrt{2 \cdot u \cdot n}$, then $D(V, r \to 0, t) \to \text{finite}$; if g=0, then $D(V, r \to -\infty, t) \to \text{finite}$.[49]

This model is similar to that of Longstaff-Schwartz (1995), with the difference that the debtor's assets and the short rate are not correlated, coupons are paid continuously rather than discretely and the short rate may follow processes different from Vasicek's.

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debtor's leverage is mean reverting.

[47] This equation is similar to the one in Wilmott (1998) at page 452 for "range notes".

[48] These lower boundary conditions are explained by Wilmott (1998) at page 431. For the implementation of these boundary conditions in the numerical solution to the differential equation, see Tavella (2000) at pages 121-132. If g=0.5, the condition $w < \sqrt{2 \cdot u \cdot n}$ always obtains in this chapter.

1.3) The valuation model for credit puts

A structural model can be used to value also credit derivatives. This is of interest since section 2 will show that important "interactions" can affect the valuation of credit derivatives even more than the valuation of bonds.

As an example, we value a "full protection credit put". Its value is denoted by O(V, r, t). The credit put expiry date is T*, with T > T*, and K is the strike price of the credit put expressed as a percentage of Z(r, t), with 0 < K < 1. A credit put is similar to an insurance that pays off $O(V_B, r, t) = \max\left[K \cdot Z(r,t) - D(V_B, r, t), 0\right]$ in case a reference bond defaults before the option expiry date, i.e. in case $V = V_B$ at some t < T*. A credit put offers full protection from credit risk when it insures bond holders not only from the event of default but also from the risk of a widening credit spread on the reference bond. If the credit put offers full protection, it also has a terminal payoff on the option expiry date: $O(V, r, T^*) = \max\left[K \cdot Z(r, T^*) - D(V, r, T^*), 0\right]$. At expiry the put can be exercised if the value of the reference bond has decreased sufficiently. Then O(V, r, t) can be shown to satisfy an equation that is similar to 5) but for the inhomogeneous term ($c \cdot F$):

$$6)\ O_t + \frac{1}{2} \cdot s^2 \cdot V^2 \cdot O_{VV} + \frac{1}{2} \cdot \left(w \cdot r^g\right)^2 \cdot O_{rr} + u \cdot (n - r) \cdot O_r + (r - b) \cdot V \cdot O_V - r \cdot O = 0$$

subject to:

6.1) $O(V \to \infty, r, t) \to 0$,

6.2) $O(V_B, r, t) = \max\left[K \cdot Z(r,t) - D(V_B, r, t), 0\right]$,

6.3) if g=0.5 and if $w < \sqrt{2 \cdot u \cdot n}$, then $O(V, r \to 0, t) \to$ finite ; if g=0, then $O(V, r \to -\infty, t) \to$ finite [50],

6.4) $O(V, r \to \infty, t) \to 0$,

6.5) $O(V, r, t = T^*) = \max\left[K \cdot Z(r, T^*) - D(V, r, T^*), 0\right]$.

Condition 6.4) says that when the short rate is very high, the debtor's assets risk neutral drift is very high, the default probability vanishes as well as the credit spread, and the credit put will never be exercised.

The interest in valuing a credit put is due also to the fact that a credit default swap, which is the most heavily traded of credit derivatives, can be seen as a special case of a credit put. In fact, a

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[49] See note 48.

[50] See note 48.

credit default swap with up-front payment[51] of the premium (as opposed to periodic payments) is like a credit put with $K \leq (1-a)$, so that the put is never exercised at maturity, and with condition 6.2) replaced by:

6.6) $O(V_B, r, t) = \max\left[F - D(V_B, r, t), 0\right]$,

where F denotes debt principal. Condition 6.6) is the payoff to the default swap upon default.

Next the above valuation models for corporate bonds and for credit puts are solved through explicit finite differences under different parameterisations for the short rate in order to measure the "interactions" between interest rate risk and default risk. The relevant code that implements the finite difference solutions is reported in Appendix 5 of this chapter.

2. "INTERACTIONS": CREDIT SPREADS ON BONDS AND THE VALUE OF CREDIT PUTS

This section analyses credit spreads on corporate bonds and the value of credit puts under alternative popular processes for the short rate, namely the Vasicek (1977) and the CIR (1985) processes. The purpose here is to ascertain how the choice between these different short rate models is not a matter of indifference since it can alter the credit spreads on bonds and the value of credit derivatives. Thus the purpose is not to ascertain which short rate process is more appropriate to get the structural model to predict realistic credit spreads, for the prediction of realistic credit spreads mainly depends on the credit risk component of the model rather than on the interest rate risk component.

2.1) "Fitting" the Vasicek and CIR processes to the default free term structure

Before employing the Vasicek and CIR short rate processes in the structural models of equations 5) and 6), the parameters "n", "u" and "w", which are assumed constant over time, are "best fitted" to the UK default free yield curve observed on 20th February 2002[52]. The three-month rate is taken to be the observed short rate "r", which is equal to 3.83%. The fitted values of "n", "u" and "w" for the Vasicek and CIR processes are such that: 7) $\arg \underset{n,u,w}{\mathrm{MIN}}\left(\sum_{i=1}^{12} \left|O_i - ydeffCIR_i\right|\right)$,

8) $\arg \underset{n,u,w}{\mathrm{MIN}}\left(\sum_{i=1}^{12} \left|O_i - ydeffVas_i\right|\right)$

[51] Credit default swaps often require the payment of periodic instalments rather than a single up front payment of the option premium.

where: O_i is the observed yield for maturity "i" of the default free term structure;

$1 < i < 12$ is an ordinal index to denote different maturities;

ydefCIR = yield on a bond subject to default given the CIR short rate;

ydeffCIR = yield on a default free bond given the CIR short rate (for each maturity "i");

ydefVas = yield on a bond subject to default given the Vasicek short rate;

ydeffVas = yield on a default free bond given the Vasicek short rate (for each maturity "i").

TABLE 5.1: Calibration of Vasicek and CIR processes to the UK default free term structure observed on 20th February 2002

"i" = ordinal index of points on the default free yield curve		1	2	3	4	5	6	7	8	9	10
Debt maturity expressed in years	0.25	1	2	3	4	5	6	7	8	9	10
g	0										
w	**3.00%**										
u	**1.32**										
n	**5.13%**										
Vasicek zero coupon bond	0.9900	0.9569	0.9110	0.8661	0.8231	0.7822	0.7433	0.7063	0.6711	0.6377	0.6060
Debt maturity expressed in years	0.25	1	2	3	4	5	6	7	8	9	10
g	0.5										
w	**27.2%**										
u	**0.77**										
n	**5.49%**										
CIR zero coupon bond	0.9901	0.9579	0.9127	0.8679	0.8246	0.7831	0.7437	0.7061	0.6705	0.6366	0.6045
Observed yield curve	3.83%	3.97%	4.66%	4.75%	4.96%	4.99%	5.01%	5.01%	5.00%	5.00%	4.97%
Yield curve per Vasicek		4.40%	4.66%	4.79%	4.87%	4.913%	4.94%	4.97%	4.98%	5.00%	5.01%
Yield curve per CIR		4.30%	4.57%	4.72%	4.82%	4.889%	4.94%	4.97%	5.00%	5.02%	5.03%

Table 5.1 reports data about "fitting" and provides the "fitted" parameter values of "n", "u" and "w" for the Vasicek and CIR models. The "fitted" CIR and Vasicek models produce term structures of yields that are very close to the observed term structure of yields.

2.2) "Interactions" and credit spreads

Now we study "interactions" effects on the credit spreads of zero coupon bonds obtained through the model of equation 5). The focus on zero coupon bonds is in the interest of simplicity and does not entail much loss in generality. "Interactions" are measured as I = (ydefVas – ydeffVas) - (ydefCIR – ydeffCIR) and are now studied in a base case scenario[53] in which a hypothetical corporate bond is valued as per equation 5) and in which a = 50%, s = 20%, b = 6%, $x = \frac{V}{V_B} = 1.5$, c = 0, T = 10, F = 1. For the CIR and Vasicek short rate processes we assume the "fitted" parameter values of Table 5.1. Then, the structural model for corporate bonds (equation 5)) produces credit spreads of 3.91% with CIR short rate and 3.99% with Vasicek short rate. This implies that "interactions" are equal to I = 0.08% = [3.99% - 3.91%] in the base case scenario. Thus, *even if the "fitted" Vasicek and the CIR short rate processes produce just slightly different default free yield*

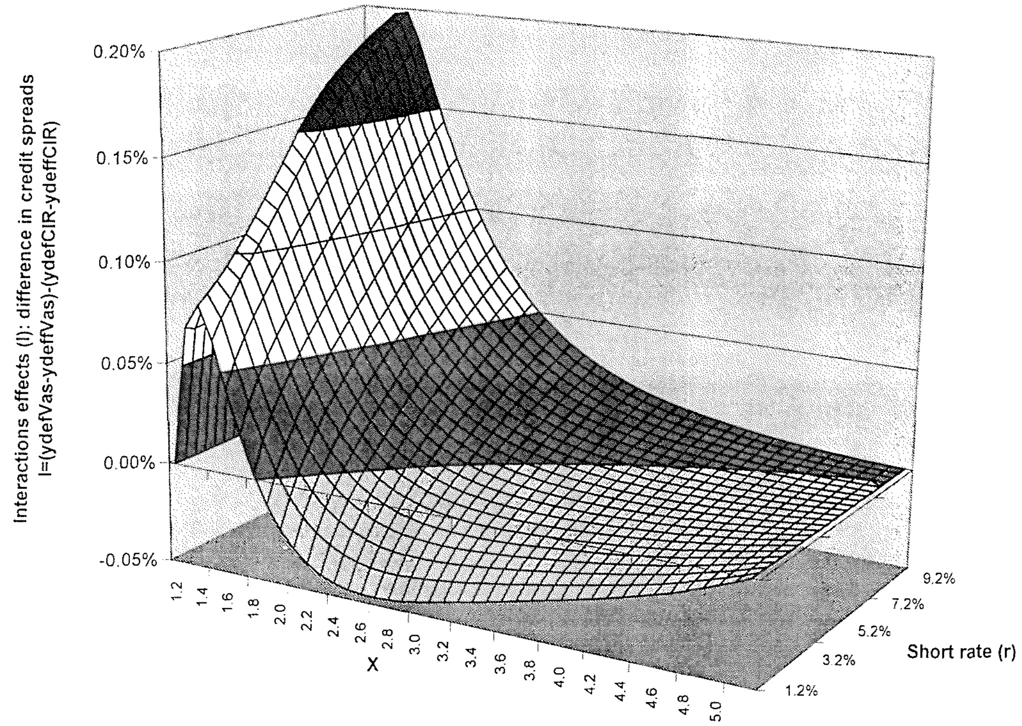
[52] The curve was provided at the Bloomberg website.

[53] In this work the chosen parameter values are similar to the values used in previous studies, e.g. Longstaff and Schwartz (1995), Fan and Sundaresan (2001), Ericsson (2000), Leland (1998), etc.. .

curves, they can produce quite different credit spreads on bonds subject to both interest rate risk and default risk.

Figure 5.1 shows that, given the base case parameters, "interactions" are stronger when the firm's assets value is low. When assets value is very high credit spreads vanish as well as "interactions" effects on credit spreads.

Figure 5.1: Interactions between credit risk and interest rate risk given base case parameters (Grid parameters: r = 0% to 20%, dr = 20%/50; X = 0 to 5, dX = 5/50; T = 10, dt = 1/2505).



2.3) "Interactions" and the value of a credit put

Now it is shown that "interactions" can be significant also when valuing a credit put as per equation 6). In the case of a credit put "interactions" are measured by the difference in the value of the put under alternative short rate processes (CIR and Vasicek in our case). Given the above base case scenario parameters, Figure 5.2 displays the value of a full protection credit put when the "fitted" CIR process is assumed. A higher short rate implies a higher risk-neutral drift, a lower risk-neutral probability of default and a slightly lower value for the credit put. When the value of the debtor's assets is high, the default probability is low, debt is more valuable, the credit put is less valuable, the *absolute* difference in credit put value under the alternative short rate processes decreases, whereas the *percentage* difference in credit put value increases. For example, when X = 1.5 (X = 4), the value of the credit put under the Vasicek short rate is 0.2017 (0.0151), which is 1.9% (2.6%) lower than the value of the credit put under the CIR process, which is 0.1979 (0.0155).

Figure 5.2: Value of a "full protection" credit put given base case parameters and K=1, T*=1, g=0.5 (Grid parameters: r = 0% to 20%, dr = 20%/50; X = 0 to 5, dX = 5/50; T = 10, dt = 1/2505).

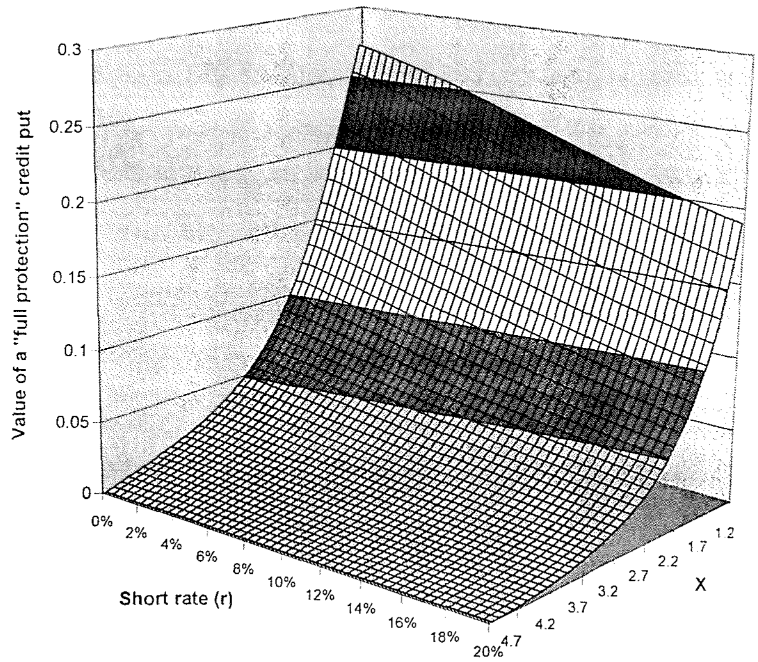
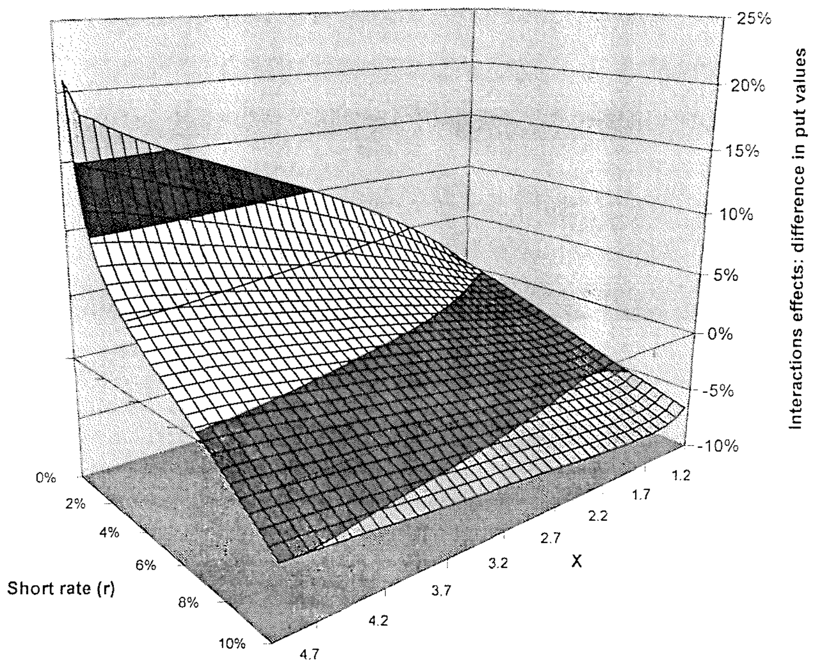


Figure 5.3, which again assumes base case scenario parameters, shows that at high assets values,

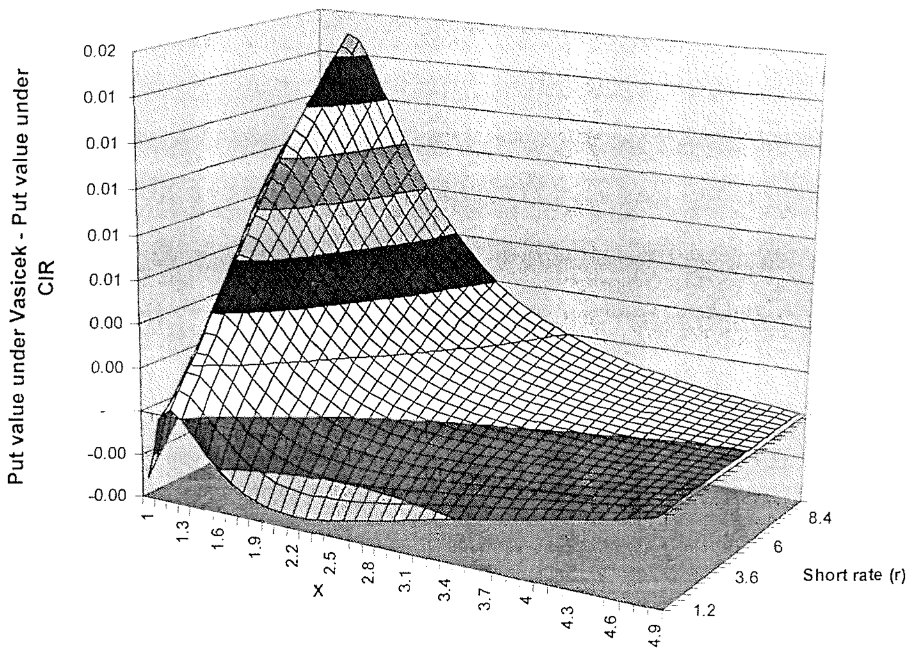
Figure 5.3: Percentage difference in credit put value (base case parameters, K=1, T*=1) (Put value under CIR - Put value under Vasicek)/Put value under CIR. (Grid parameters: r = 0% to 20%, dr = 20%/50; X = 0 to 5, dX = 5/50; T = 10, dt = 1/2505).



the percentage difference in credit put value due to "interactions" increases. Thus important "interactions" affect also the value of credit puts especially when the probability of default is relatively low, which is usually the case in practice.

Figure 5.4 shows that in the base case scenario the absolute difference in credit put value under the Vasicek and CIR processes increases at low assets values.

Figure 5.4: Difference in credit put value (base case parameters K=1, T*=1)
(Put value under Vasicek - Put value under CIR)
(Grid parameters: r = 0% to 20%; dr = 20%/50, X = 0 to 5, dX = 5/50; T = 10, dt = 2505)



These results highlight that the valuation of a credit derivative through a "typical" structural model exhibits disturbing "interest-rate-model-risk".

3. ELIMINATING "INTERACTIONS" BETWEEN DEFAULT RISK AND INTEREST RATE RISK

In this section "interactions" between default risk and interest rate risk are eliminated from the valuation of corporate bonds and of credit derivatives. Eliminating "interactions" simplifies the bond valuation problem and has two major merits:

- it entails that the choice between different short rate processes has no bearing on credit spreads and on the values of credit derivatives predicted by a structural model, i.e. the structural model becomes "robust" with respect to different assumptions about the short rate;
- it allows simple closed form solutions for the value of bonds subject to both default risk and interest rate risk, since the modelling of interest rate risk becomes independent of the modelling of default risk.

Next "interactions" are eliminated in three ways: one assumes incomplete markets, the other two assume complete markets. Then the bond valuation models are analysed through comparative statics, calibrated and finally criticised.

3.1) Eliminating "interactions" when markets are not complete

The first way to eliminate "interactions" is to recognise that the firm's assets neither are a traded security nor can be perfectly replicated by a portfolio of traded securities, i.e. markets are incomplete. This, as Ericsson and Reneby (1999) argue, entails that the risk neutral value process[54] for such assets is no longer given by equation 2), which assumes complete markets, but by

9) $\partial V = (m - b - l \cdot s) \cdot V \cdot dt + s \cdot V\, \partial z_V$,

where "m" denotes the real drift of V and "l" is a constant that denotes the "market price of V risk". The short rate is now absent from the drift term of the assets value process. It is this feature that will allow us to eliminate "interactions" in this incomplete markets setting.

We again focus on zero coupon bonds for simplicity, and we do so without loss of generality since the models in this section imply that the value of a coupon bond is equal to the value of a portfolio of zero coupon bonds. Given equation 9) and by accordingly modifying equation 53) in Ingersoll (1987) at page 445, a zero coupon bond will satisfy[55]

10) $D^*_t + \frac{1}{2} \cdot s^2 \cdot V^2 \cdot D^*_{VV} + \frac{1}{2} \cdot \left(w \cdot r^g\right)^2 \cdot D^*_{rr} + u \cdot (n - r) \cdot D^*_r + (m - b - l \cdot s) \cdot V \cdot D^*_V - r \cdot D^* = 0$

subject to:

10.A.1) $D^*(V \to \infty, r, t) \to Z(r,t)$, where Z(r,t) is the value of a similar default free zero coupon bond,

10.A.2) $D^*(V_B, r, t) = (1 - a) \cdot Z(r,t)$,

10.A.3) $D^*(V, r, T) = 1$,

10.A.4) $D^*(V, r \to \infty, t) \to 0$,

10.A.5) $D^*(V, r \to 0, t) \to \text{finite}$.

$O^*(V, r, t)$ will again satisfy equation 10), but subject to the different conditions:

---

[54] V may also be thought of as the value of net assets, i.e. total assets net of current liabilities. In such case the default barrier would be proportional just to the face value of the firm's long-term debt.

[55] The "star" superscript is used to denote the values of bonds and credit derivatives in the absence of interactions.

10.B.1) $O^*(V \to \infty, r, t) \to 0$,

10.B.2) $O^*(V_B, r, t) = \max\left[K \cdot Z(r,t) - D^*(V_B, r, t), 0\right]$,

10.B.3) $O^*(V, r \to 0, t) \to \text{finite}$,

10.B.4) $O^*(V, r \to \infty, t) \to 0$,

10.B.5) $O(V, r, t = T^*) = \max[K \cdot Z(r, T^*) - D(V, r, T^*), 0]$.

Then $O^*(V, r, t)$ = Z(r. t) Q(V, t), where Q(V, t) is a function with no simple closed form solution. Hereafter we focus instead on the solution for the bond value, which can be written as

11) $D^*(V, r, t) = Z(r, t)\, H(V, t) = Z(r, t) \cdot \{1 - a\,[1 - P(V, t)]\}$,

since H(V, t) = 1 - a [1 - P(V, t)]. P(V, t) is the risk neutral probability that default occur *after* the bond maturity date (T). Equations 11) and 12) imply that interaction effects are absent since the credit spread (cs) on D(V, r, t) does not depend on r:

12) $$cs = \frac{-\ln[H(V,t)]}{T-t} = \frac{-\ln\left[D^*(V,r,t)\right]}{T-t} - \frac{-\ln[Z(r,t)]}{T-t}.$$

Moreover, P(V,t) must satisfy the following PDE:

13) $P_t + \frac{1}{2} \cdot s^2 \cdot V^2 \cdot P_{VV} + (m - b - l \cdot s) \cdot V \cdot P_V = 0$,

subject to:

13.1) $P(V \to \infty, t) \to 1$,

13.2) $P(V_B, t) = 0$,

13.3) $P(V,T) = 1$ for $V(T) \geq V_B(T)$.

P(V, t) can be interpreted as the probability that the first passage time of a geometric Brownian motion to a constant absorbing lower barrier is greater then T. Such probability is already known in the literature (see Ingersoll (1987) at page 353 or Musiela and Rutkowski (1997) at page 470):

14) $$P(V,t) = N\left(\frac{-y + vt}{s\sqrt{t}}\right) - e^{2 \cdot v \cdot y \cdot s^{-2}} \cdot N\left(\frac{y + vt}{s\sqrt{t}}\right)$$

where N(x) is the cumulative standard normal density with x as the upper limit of integration, y = $\ln \frac{V_B}{V}$ and $v = m - b - l \cdot s - \frac{1}{2} \cdot s^2$. Thus equation 14) is the solution to equation 13).

Equation 13) is the same as the one proposed by Cathcart and El-Jahel (1998) (page 68) in that the drift of the debtor's assets does not depend on the short rate. But unlike in Cathcart and El-

Jahel, the solution to problem 13) does not require the evaluation of the Bromwich integral. The solution as per equation 14) is equivalent, but is expressed in a more tractable way since it involves just the integral of the standard normal density function. Equation 14) produces exactly the same results as in Cathcart and El-Jahel. Hence Cathcart and El-Jahel have already shown the empirical validity of the credit spreads implied by this model.

Then, we can easily accommodate a nominal amount of debt growing exponentially over time at rate "q" (as in Ericsson and Reneby 2001), so that [$F(t) = F(t = 0) \cdot e^{qt}$]. This just requires substituting "v" with $v_1 = v - q$ in equation 14), provided that $\frac{V_B}{F} = \frac{V_B(t)}{F(t)}$ is a constant.

3.1.1) Justification of the value process of the firm's assets

The above result hinges on equation 9), which is now justified. The drift term in equation 9) is "risk neutralised" by subtracting the risk premium "$l \cdot s$" (i.e. the market price of risk "l" times the assets volatility "s") from the real drift of V. In fact in order to value D(V, r, t), the assumed process for V must be the risk neutral process. But also the drift term in equation 2) is "risk neutralised". Then, it may not be clear which risk neutral processes for V to assume.

If we assume complete markets, then even if V is not a traded security itself, it can be replicated by trading in market securities. In this case V is a price process and its risk neutral drift must be as per equation 2). Instead if we assume incomplete markets, then V is not a traded security and neither can it be replicated by trading in market securities. In this case V is not a price process and its risk neutral drift is as per equation 9) (see Ericsson and Reneby (1999) for more on this).

Then, equation 9) assumes incomplete markets, which is the case in reality. In particular, market completeness is further from reality when no bond or stock written against the assets of the firm is a traded security. And even if the firm's stock were traded, it would be difficult to replicate the value process of the firm's assets by trading in the firm's stock, as suggested by Ericsson and Reneby (1999), since we do not know with certainty the precise function linking stock value to assets value.

Complete markets are usually assumed in order to obtain preference free pricing as per Black and Scholes and thus to simplify debt valuation. But the point is that *the complete markets assumption introduces "interactions"* between interest rate risk and default risk in the valuation of debt and credit derivatives through structural models. So the complete markets assumption may arguably make the valuation of bonds more complicated rather than simpler. *Per contra*, incomplete markets on one hand involve the new problem of estimating or calibrating the risk neutral drift of assets, but on the other hand eliminate the mentioned "interactions" effects.

Moreover, the new problem of estimating/calibrating the new "risk neutral drift" (m - l s) under incomplete markets can be less undesirable than it may seem. Indeed Elton and Gruber (2001) indicate that the main driver of the credit spreads seems to be the compensation for exposure to systematic risk. The compensation for systematic risk exposure should be reflected in the market price of the firm's assets risk, so that it seems important to explicitly bring the parameter "l" to bear. Then the "risk neutral drift" parameters "m" and "l" provide the flexibility to calibrate the credit spreads predicted by the model to the credit spreads observed in the market, in much the same way as the risk neutral drifts in the Vasicek or CIR default free short rate models allow calibration to the default free term structure of yields.

3.2) Eliminating "interactions" when markets are complete

So far "interactions" have been eliminated by assuming incomplete markets, which implies that the assets value process is given by equation 9) rather than equation 2). Though, there is an interesting special case, in a complete market setting, in which the variables can be "separated" as per equation 11) and "interactions" can be eliminated even if the value of the firm's assets is a price process. This is the case when $V_B(t)$ is a constant fraction of debt face value $F(t)$ and $F(t)$ is such that

15) $F(t) = F(t=0) \cdot e^{(r+h+q)t}$,

where "h" denotes the average credit spread on aggregate debt whose total face value is F. Assuming that the default barrier is a fraction of the face value of outstanding debt is consistent with Dufresne and Goldstein (2001) and with Ericsson and Reneby (2001). Equation 15) links the growth in the nominal amount of debt to the cost of debt itself. The instantaneous cost on aggregate debt is approximately [F(r + h) dt] and depends on the floating default free short rate and the average credit spread. If q = 0, the cost of debt tends to be financed entirely through issuance of new debt. If q < 0, only part or none of the cost of debt is financed through issuance of new debt. If we assume equation 15), the solution for P(V,t) in this case is the same as in equation 14), but for the fact that "v" is now substituted by $v_2 = -\left(q + b + h + \frac{1}{2} \cdot s^2\right)$. So we just need to know the value of the sum of the parameters "q", "b" and "h" instead of their individual values. Appendix II shows the solution for the case in which $F(t)$ evolves in a non-deterministic fashion over time. The next sub-section shows also another argument to eliminate "interactions" when assuming complete markets.

3.2.1) Another way to eliminate "interactions" when markets are complete

When assuming complete markets, the firm's assets expected return is $m = r + l \cdot s$, so that equation 9) becomes $\partial V = (r - b) \cdot V \cdot dt + s \cdot V\, \partial z_V$. But, we may view the assets payout rate as $b = m - j$, with $j \leq m$, where "j" is a constant that denotes the part of "m" ("m" being the expected return on assets, coinciding, when Modigliani and Miller's proposition 1 holds, with the weighted average cost of capital) that is not paid-out to security holders as dividends or interest charges. Then equation 9) would become $\partial V = (-l \cdot s + j) \cdot V \cdot dt + s \cdot V\, \partial z_V$. This result has the advantage that, if equation 15) reduces to $F(t) = F(t = 0) \cdot e^{q \cdot t}$, i.e. even if the growth in the nominal amount of debt no longer depends on instantaneous cost of debt as approximated by F[h+r]dt, then we can substitute "v" in equation 14) with $v_3 = -l \cdot s + j - \frac{1}{2} s^2 - q$. This again allows to conveniently eliminate "interactions" from the valuation of corporate bonds, while requiring less specific assumptions about the aggregate cost of debt and the growth in the nominal amount of debt.

3.3) Model extension to boost short-term credit spreads

We consider an extension of the model without "interactions" proposed in sub-section 3.2). Such model extension still allows closed form solutions for P(V, t) while predicting higher (i.e. more realistic) short-term credit spreads. In fact, it is well know that structural models tend to predict too low short-term credit spreads (e.g. see comment by Sundaresan 2000 page 1591).

First we can assume that the default barrier at bond maturity "jumps upward", i.e.:

16) $V_B(T) \geq V_B(T - dt)$, where "dt" denotes an infinitesimal time interval.

This condition causes the structural model to predict higher short-term credit spreads. K-R-S (1993) show that condition 16) may hold when default before debt maturity is triggered by a cash flow shortage, and default at maturity is triggered by a positive net worth condition because debt principal falls due. Condition 16) may hold also if we allow the debt contract to be renegotiated just at maturity due to the presence of bankruptcy costs[56]: the re-negotiation barrier, which would apply at maturity, is usually well higher than the default barrier, which would apply before maturity. If condition 16) applies, P(V, t) in equation 11) must change into P"(V,t) and P"(V,t) is again provided by Musiela and Rutkowski (1997) at page 470. Then substituting symbols into their formula at Corollary B.3.4, if $V_B(T) = k \cdot V_B(T - dt)$, with k > 1, P"(V, t) can be shown to be

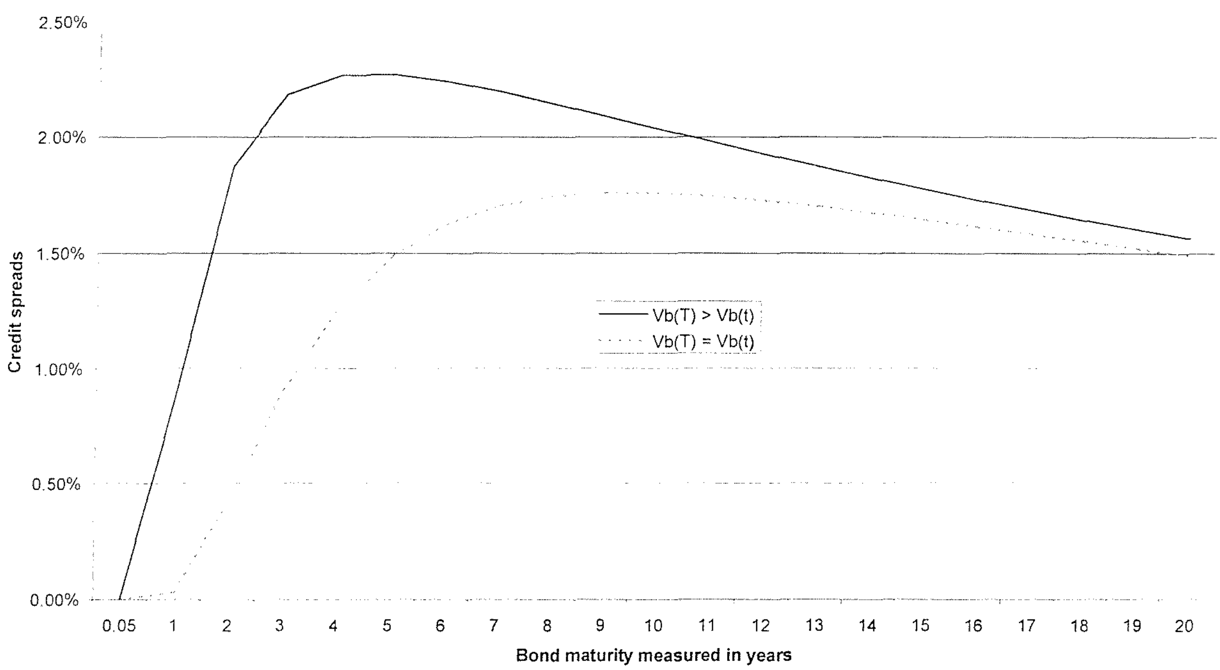
[56] See Sundaresan 1996, but in Sundaresan the debt contract can be renegotiated also before debt maturity.

$$17)\ P''(V,t)=N\left(\frac{\ln\left(\frac{V}{V_B(t=0)\cdot k}\right)+vt}{s\sqrt{t}}\right)-\left(\frac{V}{V_B(t=0)}\right)^{1-2\cdot v\cdot s^{-2}}\cdot N\left(\frac{\ln\left(\frac{V_B(t=0)}{V\cdot k}\right)+vt}{s\sqrt{t}}\right).$$

If equation 15) holds, v_2 substitutes "v" in equation 17).

Equation 17) causes the structural model to generate higher short-term credit spreads (see Figure 5.5) while retaining closed form solutions for bond values.

Figure 5.5: Term structure of credit spreads in the absence of interactions (a=50%, s=20%, v=1%, X=2)



3.4) Comparative statics with respect to the risk neutral drift

We have seen above that eliminating "interactions" from the valuation of corporate bonds entails that we need to know "drift parameters" such as "v" or "v_1" or "v_2" or "v_3" as defined above.

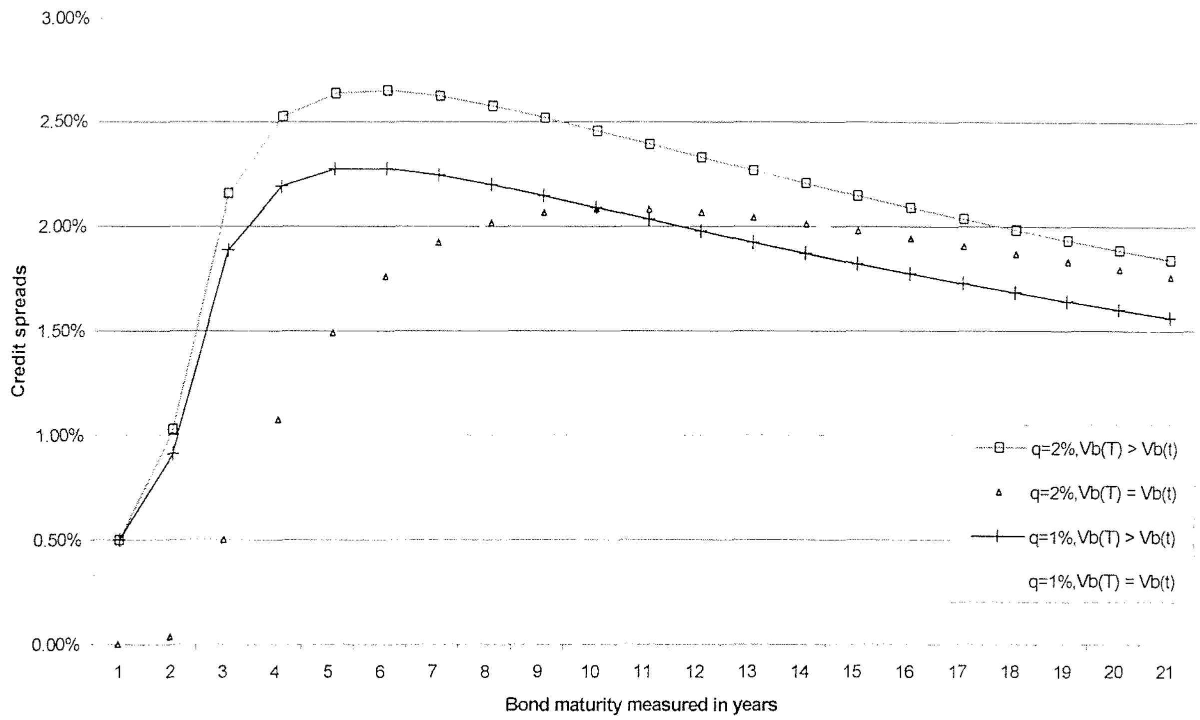
Then it is instructive to assess how such drift parameters affect the term structure of credit spreads. What follows displays results for different values of v, which are valid also for different values of v_1, v_2 and v_3.

When P"(V,t) substitutes P(V,t) in equation 12), credit spreads on zero coupon bonds are

$$19)\ cs=\frac{-\ln[1-a[1-P''(V,t)]]}{T-t}.$$

Figure 5.6 shows the effect of the drift parameter on the term structure of credit spreads as per equation 19). It is apparent that even a slight increase in q (and corresponding decrease in v,

Figure 5.6: Term structure of credit spreads in the absence of interactions (a=50%, s=20%, X=2).



v_1, v_2 and v_3) can significantly increase medium to long-term credit spreads, but not short term credit spreads. Then it seems important to relax the assumption of constant nominal capital structure in order to value medium to long-term bonds. More generally, the credit spreads predicted by the proposed models without "interactions" heavily depend on the estimates of the drift parameters (v, v_1, v_2 and v_3).

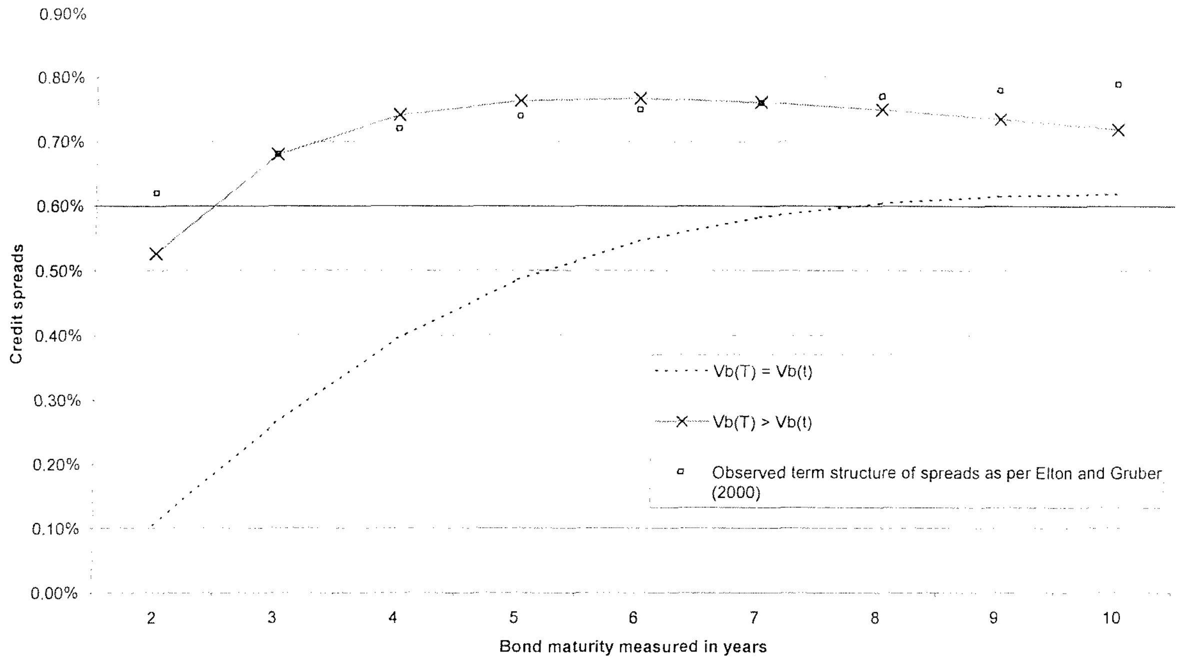
3.5) Calibration to an observed term structure of credit spreads

Now we calibrate model parameters to an observed term structure of credit spreads. Observed spreads are the average credit spreads estimated by Elton and Gruber (2000) for A rated industrial bonds for the period 1987-1996. Calibration consists in the choice of parameters values (v,s,a,k) that minimise the sum of the squares of the differences between observed credit spreads and the credit spreads predicted by the model. If the model is realistic, it should predict the observed credit spreads without requiring unrealistic values for the calibrated parameters.

Figure 5.7 shows observed credit spreads and calibrated credit spreads using the models in this section. The results are encouraging because the calibrated parameters and variables all assume realistic average values: X = 1.95, s = 17.5%, a = 22.5%, k = 1.26, v = 1%. In other words, if the model is fed with realistic average parameter values, it produces realistic average credit spreads.

Thus, the proposed structural models not only eliminate interactions and have simple solutions, but they can also predict realistic credit spreads.

Figure 5.7: Calibration of credit spreads generated by the model to the credit spreads reported in Elton and Gruber (2000) for A rated industrial bonds.



3.6) Limitations of the proposed models

Finally we review the main disadvantages of the structural models proposed in this section.

A) Having eliminated "interactions" between interest rate risk and default risk, the structural models in this section do not predict any negative relation between the level of credit spreads and the level of the short rate. But this may not be a problem if, as G.Duffie (1998) and Ericsson-Reneby (2001) maintain, the empirically observed negative correlation between changes in credit spreads and in the short rate is not strong. Moreover, if such empirical correlation is week, it may be over-stated by structural models that do feature "interactions". Empirical tests of the proposed models are clearly needed.

B) Introducing a positive correlation between assets and the short rate in the above models can lead to higher credit spreads being associated with lower levels of the short rate, but such correlation would scupper the tractability of the models proposed in this section (see Appendix IV).

C) If the short rate is instantaneously correlated with the firm's assets, "interactions" between interest rate risk and default risk still exist[57]. Though, if the short rate follows the Merton-Ho-Lee or the Vasicek processes, then equation 11) still holds even if "interactions" persist. H(V,t) must then be found numerically (see Appendix V).

D) The risk neutral drift implied by equation 9) (m – b – l s) increases the parameterisation of the model and must be either estimated from equity data or calibrated to the observed prices of bonds and credit derivatives.

This section has shown three simple ways to eliminate the problem of "interactions" between interests rate risk and default risk from structural models. Future empirical tests may shed light as to which one of the three ways, if any, is to be preferred. The proposed bond valuation models retain tractability, remain parsimonious and do not seem to predict less realistic credit spreads than past structural models. On the other hand such models cannot reproduce the empirically observed negative correlation between changes in default free interest rates and changes in credit spreads.

CONCLUSIONS

This chapter has focused on the issue of "interactions" between interest rate risk and default risk in the valuation of corporate bonds and credit derivatives. The two main conclusions are the following.

A) Typical structural models of credit risk, which assume complete markets, exhibit disturbing "interactions" between default risk and interest rate risk. The choice between the Vasicek or CIR short rate processes can significantly affect the credit spreads on corporate bonds predicted by a structural model. "Interactions" affect also the valuation of credit derivatives though a typical structural model. The value of credit put options on "high-grade" debt has been shown to change by some 2%-3% when assuming the Vasicek rather than the CIR short rate process. "Fitting" the Vasicek and CIR short rate processes to the same default free yield curve does not eliminate "interactions". Overall, "interactions" cause a typical structural model to exhibit "interest-rate-model-dependence", which seems

[57] But also in reduced form models this kind of interactions is present in so far as the short rate is instantaneously correlated with the diffusion process followed by the instantaneous default intensity.

an undesirable type of *model risk* in estimating credit spreads and in valuing credit derivatives.

B) Three simple ways to eliminate "interactions" and the associated "interest-rate-model-risk" from a structural model have been proposed: one way assumes incomplete markets, the other two assume complete markets. All these ways entail modelling default risk in isolation from interest rate risk. Also cutting-edge market practice models credit risk in isolation (see Pan (2001)). Simple closed form solutions for the valuation of single issues of corporate debt have been obtained. These closed forms simplify the valuation of credit derivatives and allow easy "calibration" of model parameters to observed market prices, which seems important for practical applications.

Future research may try and indicate *the one and as yet unknown* most appropriate short rate process to be assumed in a structural model. Yet more research needs to deal with the problem of "interactions" between default risk and interest rate risk that are due not to the risk neutral drift of assets, but to instantaneous correlation between the short rate and the debtor's assets.

APPENDIX I: ABOUT THE RISK NEUTRAL DRIFT OF ASSETS

For simplicity now we assume the short rate is constant, but the result in this appendix can be shown to apply also when the short rate follows a diffusion process.

Assuming that V is neither a traded security, nor can be replicated by a dynamic trading strategy, and assuming there are two traded securities that are claims contingent on the value the firm's assets [denoted by D1(V, t) and D2(V, t)], then past literature has shown that (e.g. see Hull(1997) in chapter 14 or Ingersoll(1989) pages 381-383) the following result must hold:

A.I.1) $$\frac{D1_t + \frac{1}{2}\cdot s^2 \cdot V^2 \cdot D1_{VV} - r \cdot D1}{D1_V} = \frac{D2_t + \frac{1}{2}\cdot s^2 \cdot V^2 \cdot D2_{VV} - r \cdot D2}{D2_V} = l,$$

where l is a constant that goes under the name of « market price of V risk ».

If V is a traded security or if it can be replicated by a dynamic trading strategy (complete markets assumption), we can regard D1(V,t) as equal to assets value, i.e. D1(V,t) = V. Then A.I.1) reduces to the familiar Black and Scholes equation

A.I.2) $$0 = D2_t + \frac{1}{2}\cdot s^2 \cdot V^2 \cdot D2_{VV} + r \cdot V \cdot D2_V - r \cdot D2.$$

If the short rate "r" is stochastic as per equation 3), A.I.2) becomes equation 5) (with c = 0) and the term $r \cdot V \cdot D2_V$ causes "interactions" between interest rate risk and default risk.

APPENDIX II : WHEN THE FACE VAUE OF DEBT EVOLVES IN A NON-DETERMINISTIC FASHION

In this appendix the result in sub-section 3.2) is extended to the case in which the face value of debt evolves in a non-determinstic fashion such as

A.II.1) $$\partial F = (r + h + q)\cdot F \cdot dt + s_F \cdot F \cdot \partial z_F,$$

with $\partial z_F \cdot \partial z_V = \rho \cdot dt$. The solution to this problem is the same as in equation 14), but for the fact that "v" is substituted by $v_4 = -\left[q + b + sp + \frac{1}{2}\left(s_V^2 - s_F^2\right)\right]$ and "s" is substituted by $\sqrt{s_V^2 - 2\cdot\rho\cdot s_V \cdot s_F + s_F^2}$. If $dz_F \cdot dz_r \neq 0$, then a closed form solution for $D^*(V,r,t)$ is no longer possible.

Equation A.II.1) resembles the stochastic process for the default barrier proposed by Nielsen, Saa'-Requeho and Santa Clara (1993) (N-SR-SC). The differences from N-SR-SC are that:

- here F is the face value of total debt rather than the markets value of total debt ;
- F does not depend on the stochastic process of the short rate;
- default takes place when assets value is a pre-determined fraction of F ;
- a net assets payout rate (b) is allowed in this model.

APPENDIX III : ACCOUNTING ALSO FOR LIQUIDITY RISK

This appendix shows how the model in section 3 can account for liquidity risk. Ericsson and Reneby (2001) argue that short-term spreads mainly reflect a liquidity premium.

If a bond is not liquid, bond holders run the risk of having to sell the bond at a discount should they need to liquidate their position. Let us assume that "f" is a constant and that in a very short time period (dt) there is a constant probability (f dt) that the bond holder needs to sell the bond at a fraction of the market value of the bond (1-y). Here "y" measures the percentage discount required to sell the bond immediately (0 < y < 1). Then y can also be an increasing function of time y(t), since as time (t) draws near to maturity (T) liquidity can be expected to decrease and the liquidity discount (y(t)) can be expected to increase. Then it can be shown that equation 10) becomes

A.III.1)

$$D_t^* + \frac{1}{2} \cdot s^2 \cdot V^2 \cdot D_{VV}^* + \frac{1}{2} \cdot \left(w \cdot r^g\right)^2 \cdot D_{rr}^* + u \cdot (n-r) \cdot D_r^* + (m - b - l \cdot s) \cdot V \cdot D_V^* - [r + f \cdot y(t)] \cdot D^* = 0 .$$

The solution to A.III.1) can be written as

A.III.2)

$$D^*(V, r, t) = Z(r,t) \cdot e^{-f\int_t^T y(j)dj} \cdot H(V, t)$$

so that accounting for liquidity risk can often allow to retain closed form solutions.

Equation A.III.1) is obtained by setting up a portfolio $\Pi = D - \Delta \cdot Z - \Delta_1 V$, with $\Delta = D_Z, \Delta_1 = D_V$ (Z now being a zero coupon bond) and by setting the expected return on Π equal to $E(\partial\Pi) = f(-y \cdot D) + \partial\Pi = r \cdot \Pi \cdot dt - D_V \cdot b \cdot V \cdot dt$, with $\partial\Pi = \partial D - \Delta \cdot \partial Z - \Delta_1 \partial V$. Then equation A.III.1) is obtained by applying Ito's lemma to find ∂D and ∂Z and remembering that, if Z is a zero coupon bond it follows from equation 4) that

$$Z_t + \frac{1}{2} \cdot \left(w \cdot r^g\right)^2 \cdot Z_{rr} = -u \cdot (n-r) \cdot Z_r + r \cdot Z .$$

APPENDIX IV: VALUE PROCESS OF THE FIRM'S ASSETS CORRELATED WITH THE SHORT RATE

This appendix shows that, even if V and r are instantaneously correlated and the market is incomplete, the result in equation 11), i.e. D(r,V,t) = Z(r,t) H(V,t), still applies when r follows the Ho-lee or Vasicek process, but closed form solutions for H(V, t) are not available.

Assume that

A.IV.1) $\partial z_v \cdot \partial z_r = \rho \cdot dt$. Then the value of a default risky zero coupon bond D(r,V,t) can be shown to satisfy

A.IV.2)

$$D_t + \frac{1}{2}(V \cdot s)^2 D_{VV} + D_{rV} \cdot w \cdot V \cdot s \cdot \rho + \frac{1}{2} w^2 \cdot D_{rr} + (m - l \cdot s - b) \cdot V \cdot D_V + u \cdot (n - r) \cdot D_r - r \cdot D = 0$$ subject to conditions 10.1) to 10.5). A.IV.2) becomes

A.IV.3)

$$\begin{aligned}&(ZH)_t + \frac{1}{2}(V \cdot s)^2 (ZH)_{VV} + (ZH)_{rV} \cdot w \cdot V \cdot s \cdot \rho + \frac{1}{2} w^2 \cdot (ZH)_{rr} + (m - l \cdot s - b) \cdot V \cdot (ZH)_V + \\ &+ (u - \lambda \cdot w) \cdot (ZH)_r - r \cdot (ZH) = 0\end{aligned},$$

and then

$$\begin{aligned}&H \cdot \{ Z_t + \frac{1}{2} w^2 \cdot Z_{rr} + u \cdot (n - r) \cdot Z_r - r \cdot Z \} + (ZH)_{rV} \cdot w \cdot V \cdot s \cdot \rho = \\ &= -Z \cdot \left\{ H_t + \frac{1}{2} s^2 \cdot V^2 \cdot H_{VV} + V \cdot (m - l \cdot s - b) \cdot H_V \right\}\end{aligned}.$$

But we know that a default risk free zero coupon bond satisfies

A.IV.4) $Z_t + \frac{1}{2} w^2 \cdot Z_{rr} + u \cdot (n - r) \cdot Z_r - r \cdot Z = 0$

subject to

A.IV.4.1) Z(r,T) = 1,

hence

A.IV.5) $-\frac{Z_r}{Z} \cdot w \cdot \cdot \rho = \frac{\left\{ H_t + \frac{1}{2} s^2 \cdot V^2 \cdot H_{VV} + V \cdot (m - l \cdot s - b) \cdot H_V \right\}}{V \cdot s \cdot H_V}$.

Equation A.IV.5) says that the right hand side term is a function of r and t and that it is equal to the left hand side term, which is a function of V and t. For this to be possible, both sides can be at most only a function of t, so we can write

A.IV.6) $-\frac{Z_r}{Z}\cdot w\cdot\cdot\rho=\frac{\left\{H_t+\frac{1}{2}s^2\cdot V^2\cdot H_{VV}+V\cdot(m-l\cdot s-b)\cdot H_V\right\}}{V\cdot s\cdot H_V}=f(t).$

This allows us to "separate" the variables (r and V) and to derive the following system of ODE's

A.IV.7) $-\frac{Z_r}{Z}\cdot w\cdot\cdot\rho-\frac{m\text{-}l\cdot s-b}{s}=f(t),$

A.IV.8) $\frac{\left\{H_t+\frac{1}{2}s^2\cdot V^2\cdot H_{VV}\right\}}{V\cdot s\cdot H_V}=f(t).$

Interest rate models such as Ho-Lee or Vasicek satisfy A.V.7). If we assume that the Ho-Lee model applies, then A.IV.7) becomes

A.IV.9) $(T-t)\cdot w\cdot\rho-\frac{m\text{-}l\cdot s-d}{s}=f(t),$

and equation A.IV.8) becomes

A.IV.10) $H_t+\frac{1}{2}(V\cdot s)^2H_{VV}+(-\rho\cdot w\cdot s\cdot(T-t)+m\text{-}l\cdot s-b)\cdot V\cdot H_V=0$

subject to the boundary condition for a zero coupon bond:

A.IV.10.1) H(V,T) = 1,

A.IV.10.2) $H(V\to\infty)\to 1$,

A.IV.10.3) H(V = V_B,t) = 1-a.

We have in fact separated the variables, so that we can again write D(r,V,t) = Z(r,t) H(V,t), even if V and r are correlated.

If the Vasicek model is assumed, then equation A.IV.7) becomes

A.IV.11) $\frac{1}{u}[1-e^{-u(T-t)}]\cdot w\cdot\cdot\rho-\frac{m\text{-}l\cdot s-b}{s}=f(t),$

and equation A.IV.8) becomes

A.IV.12) $H_t+\frac{1}{2}(V\cdot s)^2H_{VV}+\left(-\rho\cdot w\cdot s\cdot\frac{1}{u}[1-e^{-u(T-t)}]+m\text{-}l\cdot s-b\right)\cdot V\cdot H_V=0.$

Finally, since the drift term in equations A.IV.12) and A.IV.10) depends on time, no closed form solutions for A.IV.10) and A.IV.12) are known. "Interactions" here persist in that different short rate models imply different functions f(t) and hence different values and credit spreads of bonds subject to default.

APPENDIX V: THE CODE

The following Visual Basic code implements the numerical solutions to the valution problems in this chapter. The numerical scheme is an explicit finite difference scheme in three dimensions: time [t], short interets rate [r] and assets value [V].

The following algorithm "Valuation" solves the equations 4, 5 and 6 numerically. The values of VoldD, VoldZ, VoldO found by the algorithm correspond to D(V, r, t), Z(r, t) and O(V, r, t).

```
Option Explicit

Private VOldD(0 To 200, 0 To 200) As Double 'This is the value of debt subject to default
Private VOldZ(0 To 200) As Double 'This is the value of default free debt
Private VOldO(0 To 200, 0 To 200) As Double 'This is the value of a full protection credit put

'The meaning of the parameters of the algorithm is the following:
'Asset indicates the value of the firm's assets (V),
'Barrier indicates the default barrier ( V_B ),
'Volatility indicates assets volatility (s),
'Payout indicates the assets pay-out rate (b),
'Principal indicates the face value fo debt (F),
'Maturity indicates debt maturity (T),
'CouponRate indicates the debt coupon rate [c],
'OptionExpiry indicates the option expiry date (T*),
'Strike indicates indicates the strike price (K) of the credit put [O(V, r, t)],
'Alpha indicates the loss given default (a),
'IntRate indicates the default free short interest rate (r),
'MeanIntRate indicates the mean reversion level (n),
'IntRateVolat indicates the volatility of the default free short rate (w),
'IntSpeed indicates the mean reversion speed (u),
'NoAssetSteps indicates the number of intervals of equal length into which the solution domain in the 'V direction is divided,
'NoIntRateSteps indicates the number of intervals of equal length into which the solution domain in the
'r direction is divided.

Function CodeChapter5(Asset As Double, Barrier As Double, Volatility As Double, Payout As Double, _
Principal As Double, Maturity As Double, CouponRate As Double, OptionExpiry As Double, _
Strike As Double, Alpha As Double, IntRate As Double, MeanIntRate As Double, _
IntRateVolat As Double, IntSpeed As Double, NoAssetSteps As Integer, NoIntRateSteps As Integer)

Dim GammaD(0 To 200, 0 To 200) As Double
Dim VNewD(0 To 200, 0 To 200) As Double
Dim DeltaD(0 To 200, 0 To 200) As Double
Dim rDeltaD(0 To 200, 0 To 200) As Double
Dim rGammaD(0 To 200, 0 To 200) As Double

Dim VNewZ(0 To 200) As Double
Dim DeltaZ(0 To 200) As Double
Dim GammaZ(0 To 200) As Double

Dim GammaO(0 To 200, 0 To 200) As Double
Dim VNewO(0 To 200, 0 To 200) As Double
Dim DeltaO(0 To 200, 0 To 200) As Double
Dim rDeltaO(0 To 200, 0 To 200) As Double
Dim rGammaO(0 To 200, 0 To 200) As Double

Dim O(0 To 200, 0 To 200) As Double

Dim V(0 To 200) As Double
Dim Vsqd(0 To 200) As Double
Dim r(0 To 200) As Double
```

```
Dim u(0 To 200) As Double
Dim w(0 To 200) As Double

Dim i As Integer
Dim j As Integer
Dim k As Integer

Dim M As Double

Dim dV As Double 'AssetStep
Dim dr As Double 'IntRatestep
Dim NearestGridPt As Integer
Dim NearestGridPR As Integer
Dim Timestep As Double
Dim NoTimeSteps As Integer
Dim NoExtTimesteps As Integer
Dim halfvolsqd As Double
Dim mini As Integer
Dim Mi As Integer

halfvolsqd = 0.5 * Volatility * Volatility

dV = 5 * Barrier / NoAssetSteps
dr = 0.2 / NoIntRateSteps
NearestGridPt = Int(Asset / dV)
NearestGridPR = Int(IntRate / dr)
Timestep = 1 / 2505
NoTimeSteps = Int(Maturity / Timestep) + 1
Timestep = Maturity / NoTimeSteps
mini = Barrier / dV
Mi = NoIntRateSteps

For i = mini To NoAssetSteps
   V(i) = i * dV
   Vsqd(i) = V(i) * V(i)
Next i

For j = 0 To NoIntRateSteps
   r(j) = j * dr
   u(j) = (MeanIntRate - r(j)) * IntSpeed 'Non calibrated drift
   w(j) = IntRateVolat '* WorksheetFunction.Power(r(j), 0.5)
Next j

'Payoffs to D, O and Z
For j = 0 To NoIntRateSteps
VOldZ(j) = Principal
For i = mini + 1 To NoAssetSteps
VOldD(i, j) = Principal
VOldO(i, j) = 0
Next i
VOldD(mini, j) = (1 - Alpha)
VOldO(mini, j) = 0
Next j

For k = 1 To NoTimeSteps
For j = 0 To NoIntRateSteps
Next
'''
'Z
For j = 1 To NoIntRateSteps - 1
GammaZ(j) = (VOldZ(j + 1) - 2 * VOldZ(j) + VOldZ(j - 1)) / (dr * dr)
```

```
If u(j) > 0 Then
DeltaZ(j) = (-3 * VOldZ(j) + 4 * VOldZ(j + 1) - VOldZ(j + 2)) / (2 * dr)
Else
DeltaZ(j) = (3 * VOldZ(j) - 4 * VOldZ(j - 1) + VOldZ(j - 2)) / (2 * dr)
End If

VNewZ(j) = VOldZ(j) + Timestep * (-r(j) * VOldZ(j) + 1 / 2 * GammaZ(j) * w(j) * w(j) + _
DeltaZ(j) * u(j))
Next j

GammaZ(0) = (VOldZ(2) - 2 * VOldZ(1) + VOldZ(0)) / (dr * dr)
DeltaZ(0) = (-3 * VOldZ(0) + 4 * VOldZ(1) - VOldZ(2)) / (2 * dr)
VNewZ(0) = VOldZ(0) + Timestep * (-r(0) * VOldZ(0) + 1 / 2 * GammaZ(0) * w(0) * w(0) + _
DeltaZ(0) * u(0))

'Alternative approximation to the lower boundary condition: VNewZ(0) = 2 * VNewZ(1) - VNewZ(2)
VNewZ(NoIntRateSteps) = 2 * VNewZ(NoIntRateSteps - 1) - VNewZ(NoIntRateSteps - 2)

'D
For i = (mini + 1) To NoAssetSteps - 1
For j = 1 To NoIntRateSteps - 1
If (r(j) - Payout) > 0 Then
DeltaD(i, j) = (VOldD(i + 1, j) - VOldD(i - 1, j)) / (2 * dV) '(-3 * VOldD(i, j) + 4 * VOldD(i + 1, j) - VOldD(i + 2, j))
/ (2 * dV)
Else
DeltaD(i, j) = (3 * VOldD(i, j) - 4 * VOldD(i - 1, j) + VOldD(i - 2, j)) / (2 * dV)
End If
'DeltaD(i, j) = (VOldD(i + 1, j) - VOldD(i - 1, j)) / (2 * dV)
GammaD(i, j) = (VOldD(i + 1, j) - 2 * VOldD(i, j) + VOldD(i - 1, j)) / (dV * dV)
rGammaD(i, j) = (VOldD(i, j + 1) - 2 * VOldD(i, j) + VOldD(i, j - 1)) / (dr * dr)
If u(j) > 0 Then
rDeltaD(i, j) = (-3 * VOldD(i, j) + 4 * VOldD(i, j + 1) - VOldD(i, j + 2)) / (2 * dr)
Else
rDeltaD(i, j) = (3 * VOldD(i, j) - 4 * VOldD(i, j - 1) + VOldD(i, j - 2)) / (2 * dr)
End If

VNewD(i, j) = VOldD(i, j) + Timestep * (halfvolsqd * Vsqd(i) * GammaD(i, j) + _
(r(j) - Payout) * V(i) * DeltaD(i, j) - r(j) * VOldD(i, j) + _
1 / 2 * rGammaD(i, j) * w(j) * w(j) + rDeltaD(i, j) * u(j))
Next j
Next i

For j = 0 To NoIntRateSteps
VNewD(mini, j) = (1 - Alpha) * VNewZ(j)
VNewD(NoAssetSteps, j) = VNewZ(j) '2 * VNewD(NoAssetSteps - 1, j) - VNewD(NoAssetSteps - 2, j)
Next j

For i = mini + 1 To NoAssetSteps - 1
'rGammaD(i, 0) = rGammaD(i, 1) '=(VOldD(i, 2) - 2 * VOldD(i, 1) + VOldD(i, 0)) / (dr * dr)
'rDeltaD(i, 0) = (-3 * VOldD(i, 0) + 4 * VOldD(i, 1) - VOldD(i, 2)) / (2 * dr)
'VNewD(i, 0) = VOldD(i, 0) + Timestep * (halfvolsqd * Vsqd(i) * GammaD(i, 0) + _
'(r(0) - Payout) * V(i) * DeltaD(i, 0) - r(0) * VOldD(i, 0) + _
'1 / 2 * rGammaD(i, 0) * w(0) * w(0) + rDeltaD(i, 0) * u(0))

'Subsitute the linearity upper boundary condition for the interest rate with the following:
rGammaD(i, Mi) = 0 ' Alternatively we can posit rGammaD(i, Mi) = rGammaD(i, Mi - 1) when Vasicek
rDeltaD(i, Mi) = (3 * VOldD(i, Mi) - 4 * VOldD(i, Mi - 1) + VOldD(i, Mi - 2)) / (2 * dr)
VNewD(i, Mi) = VOldD(i, Mi) + Timestep * (halfvolsqd * Vsqd(i) * GammaD(i, Mi) + _
(r(Mi) - Payout) * V(i) * DeltaD(i, Mi) - r(Mi) * VOldD(i, Mi) + _
1 / 2 * rGammaD(i, Mi) * w(Mi) * w(Mi) + rDeltaD(i, Mi) * u(Mi))
```

```
'Alternatively the linearity boundary conditions are:
VNewD(i, 0) = 2 * VNewD(i, 1) - VNewD(i, 2)
'VNewD(i, NoIntRateSteps) = 2 * VNewD(i, NoIntRateSteps - 1) - VNewD(i, NoIntRateSteps - 2)
Next i

'O
For i = (mini + 1) To NoAssetSteps - 1
For j = 1 To NoIntRateSteps - 1
DeltaO(i, j) = (VOldO(i + 1, j) - VOldO(i - 1, j)) / (2 * dV)
GammaO(i, j) = (VOldO(i + 1, j) - 2 * VOldO(i, j) + VOldO(i - 1, j)) / (dV * dV)
rGammaO(i, j) = (VOldO(i, j + 1) - 2 * VOldO(i, j) + VOldO(i, j - 1)) / (dr * dr)
If u(j) > 0 Then
rDeltaO(i, j) = (-3 * VOldO(i, j) + 4 * VOldO(i, j + 1) - VOldO(i, j + 2)) / (2 * dr)
Else
rDeltaO(i, j) = (3 * VOldO(i, j) - 4 * VOldO(i, j - 1) + VOldO(i, j - 2)) / (2 * dr)
End If
VNewO(i, j) = VOldO(i, j) + Timestep * (halfvolsqd * Vsqd(i) * GammaO(i, j) + _
(r(j) - Payout) * V(i) * DeltaO(i, j) - r(j) * VOldO(i, j) + _
1 / 2 * rGammaO(i, j) * w(j) * w(j) + rDeltaO(i, j) * u(j))
Next j
Next i

For j = 0 To NoIntRateSteps
VNewO(mini, j) = WorksheetFunction.Max(Strike * VOldZ(j) - VOldD(mini, j), 0)
VNewO(NoAssetSteps, j) = 2 * VNewO(NoAssetSteps - 1, j) - VNewO(NoAssetSteps - 2, j)
Next j

For i = mini + 1 To NoAssetSteps
rGammaO(i, 0) = (VOldO(i, 2) - 2 * VOldO(i, 1) + VOldO(i, 0)) / (dr * dr)
rDeltaO(i, 0) = (-3 * VOldO(i, 0) + 4 * VOldO(i, 1) - VOldO(i, 2)) / (2 * dr)
VNewO(i, 0) = VOldO(i, 0) + Timestep * (halfvolsqd * Vsqd(i) * GammaO(i, 0) + _
(r(0) - Payout) * V(i) * DeltaO(i, 0) - r(0) * VOldO(i, 0) + _
1 / 2 * rGammaO(i, 0) * w(0) * w(0) + rDeltaO(i, 0) * u(0))

'Alternative approximation to the lower boundary condition: VNewO(i, 0) = 2 * VNewO(i, 1) - VNewO(i, 2)
VNewO(i, NoIntRateSteps) = 2 * VNewO(i, NoIntRateSteps - 1) - VNewO(i, NoIntRateSteps - 2)
Next i

'Time stepping
For j = 0 To NoIntRateSteps
VOldZ(j) = VNewZ(j) + Principal * CouponRate * Timestep
Next j
For i = mini To NoAssetSteps
For j = 0 To NoIntRateSteps
VOldD(i, j) = VNewD(i, j) + Principal * CouponRate * Timestep
If i = mini Then VOldD(mini, j) = VNewD(mini, j)
If k * Timestep <= (Maturity - OptionExpiry) Then
If (k + 1) * Timestep > (Maturity - OptionExpiry) Then
VOldO(i, j) = WorksheetFunction.Max(VOldZ(j) * Strike - VOldD(i, j), 0)
Else
VOldO(i, j) = VNewO(i, j)
If i = mini Then VOldO(mini, j) = WorksheetFunction.Max(VOldZ(j) * Strike - VOldD(mini, j), 0)
End If
End If

Next j
Next i
Next k
CodeChapter5 = VOldD(NearestGridPt, NearestGridPR)

End Function
```

CONCLUSIONS

This dissertation has attempted to advance the theory of the fair valuation of the firm's debt subject to default through structural models of credit risk and has focused on four determinants of debt value:

◊ the growth opportunities embedded in the firm's assets (chapter 2);

◊ the lack of perfect information about the firm's assets risk (chapter 3);

◊ the implicit option to renegotiate and extend debt maturity (chapter 4);

◊ the interactions between interest rate risk and default risk (chapter 5).

Hereafter the contributions of chapters 2 to 5 are commented in detail. Then unresolved problems and avenues for future research are discussed.

1 THE CONTRIBUTIONS OF THIS THESIS

The main contributions of this dissertation are the following.

Contribution about debt valuation in the presence of the debtor's growth opportunities

Chapter 2 has provided results about the valution of debt when the debtor has important growth opportunities or growth options.

The first result is that corporation taxes can induce equity holders of a levered firm to exercise growth options (i.e. to invest) earlier than they would optimally do in the absence of leverage. In such case, the leverage induced change in the optimal exercise policy of the growth option increases debt value.

This result is consistent with Myers' debt induced under-investment, since equity holders of a levered firm still have incentives to exercise growth options sub-optimally in order to maximise equity value rather than total firm value.

The second result is that, when the debtor has important growth options, protective debt covenants such as the prohibition to issue any further debt may not really protect debtholders' interests in that they may decrease rather than increase debt value. Financing the cost of new investment by new subordinated debt rather than by new equity has been shown to increase equity value through higher tax savings and to increase senior debt value.

The third result confirms that, when investment cost is financed by issuing new equity, shorter debt average maturity induces equity holders to anticipate the exercise of a growth option. Anticipated exercise increases debt value.

Contribution about the valuation of debt under imperfect information about assets risk

Chapter 3 has recognised that debt holders are often not perfectly informed about assets risk and hence are uncertain about assets volatility. By assuming that at any time assets volatility remains within a bounded range, the following main results have been derived.

The first result is that the increase in debt credit spreads due to creditors' uncertainty about assets volatility is very sensitive to the assumed default condition. When default is endogenous, such cost can be much higher than when default is triggered by lack of liquidity.

The second result is that, when debt holders are uncertain about assets volatility, the most prudent valuation scenario of long term subordinated debt or subordinated convertible debt does not coincide with the scenario assuming highest constant volatility. The reason is that subordinated debt and subordinated convertible debt are locally convex functions of debt value.

As for subordinated debt, higher coupons, longer debt maturity, higher default barrier before maturity and higher assets value all imply that subordinated debt value with highest constant volatility is closer to subordinated debt value under the worst case volatility scenario.

As for subordinated convertible debt, assuming that assets volatility is constant over time may underestimate the sensitivity of the convertible to uncertainty in assets volatility, especially when the convertible is "out of the money". As a consequence the value of convertible debt may be more sensitive to uncertainty about assets volatility than previously suggested by Brennan ans Schwartz (1982).

The third result is that, if the value of the debtor's assets is instantaneously correlated with the default free short rate, the stochastic process of the short rate affects the sensitivity of debt value to (uncertainty in) assets volatility. For long-term debt such sensitivity is materially higher if the Merton-Ho-Lee short rate model is assumed rather than the Vasicek short rate model.

Overall chapter 3 has shown that the increase in the cost of debt due to creditors' uncertainty about assets volatility heavily depends on the default barrier, the concavity/convexity of the debt value function and the process of the default free short rate.

Contribution about the valution of debt whose maturity can be extended

Chapter 4 has recognised that borrowers and creditors often have an implicit option to extend debt maturity. This implicit "extension option" is associated with the possibility for debtors and creditors to renegotiate the debt contract, either in distressed exchanges or in a formal bankruptcy proceeding. The analysis has confirmed that debt holders often have an interest in conceding "extensions" to the borrower.

Thus it is often important to recognise the existence of a valuable "debt maturity extension option" in order to value debt, especially low grade debt. In a time independent setting with constant nominal capital structure, it has been shown that:

- the "extension option" causes an increase in equity value that is usually higher than the decrease in debt value;
- the value of the "extension option" is very sensitive to different possible exercise policies;
- when default is triggered by worthless equity the "extension option" is more valuable than when default is triggered by a liquidity shortage.

In a time dependent setting the presence of the "extension option" has been shown to increase the short-term credit spreads of outstanding debt. This seems a partial remedy to the underestimation of short-term credit spreads that is typical of structural models.

Contribution about the valuation of debt and interactions between interest rate risk and default risk

Chapter 5 recognises that, when valuing single debt issues, interest rate risk should be taken into account as well as default risk. But, structural models of credit risk usually exhibit "interactions" between default risk and interest rate risk, in that different processes for the default free short rate imply different credit spreads on corporate bonds and different values for credit derivatives. Such "interactions" are a source of "interest rate model risk" and eliminating them seems desirable.

Chapter 5 has shown that "interactions" can be significant and has proposed ways to eliminate them when markets are either complete or incomplete. Eliminating interactions has meant "separating" the modelling of default risk from the modelling of interets rate risk, which seems a convenient simplification for practical pricing purposes and allows simple closed form solutions for pricing corporate bonds that are subject to both interest rate risk and default risk.

The contribution in chapter 5 is the most immediately relevant for credit market practitioners. The closed form solutions allow easy calibration of the bond values predicted by the model to the bond prices observed in the market.

Having reviewed the contributions of this thesis, unresolved problems and future research are discussed next.

2 UNRESOLVED ISSUES

This section discusses the main unresolved problems encountered in this thesis.

Too low short-term credit spreads and liquidity risk

In chapters 4 and 5 of this thesis the attempt has been made to boost the short-term (one or two years to maturity) credit spreads predicted by structural models. In fact typical structural models predict too low short-term credit spreads when compared with market credit spreads.

In chapter 4 recognising the presence of an option to extend debt maturity has increased short-term credit spreads. In chapter 5 the increase in the default barrier at debt maturity has again increased short-term credit spreads.

Anyway, more is to be done to increase the predicted short-term yield spreads to the levels observed in the bond markets. Following Ericsson and Renault (2001), a possible solution to this problem may be to recognize that the observed yield spreads incorporate not only a credit risk premium but also a liquidity risk premium that rises for shorter debt maturities.

Stochastic short rate in a time independent setting

In order to retain tractability, the valuation of debt and equity in *a time independent setting* usually assumes that the default free short rate is a constant. Such assumption, which is common to this thesis and to the literature, seems restrictive and has not yet been removed.

Dynamic capital structure

A major unresolved issue in this thesis and in the literature of structural models is how to account for the dynamic evolution of the debtor's nominal capital structure in the valuation of debt.

Recent contributions (Tauren (1999), Golstein and Dufresne (2001)) have suggested that the debtor's leverage may follow a mean reverting process. Instead, chapter 5 of this thesis has suggested that the face value of debt may either grow exponentially or follow its own diffusion process.

Despite these efforts, debt valution as a function of the dynamics of the nominal capital structure is still a open issue.

Valuing debt when parameters are time dependent

In chapter 5 the debt valuation model makes use of some closed form solutions for the valuation of barrier options or for Brownian motion passage times. But such closed form solutions apply only when the default barrier, assets volatility and drift parameters are constant. Unfortunately no closed form solutions are available when the default barrier, assets volatility and drift parameters are functions of time. If such solutions were available, present structural models could be extended in a number of ways without losing tractability.

After reviewing the main unresolved problems in this thesis, the next section indicates directions for worthy future research into the valution of firms' debt subject to default through structural models of credit risk.

3 FUTURE RESEARCH

Future research along the lines of this thesis can develop in two main directions:

- valuing debt by modelling financial distress more accurately;
- improving the valution of debt subject to both default risk and interst rate risk.

Both research directions are now discussed in turn.

3.1) Valuing debt by modelling financial distress more accurately

This line of reserch can deepen our understanding of how the value of the firm's debt is affected by different default conditions, by the option to liquidate the firm's assets, by the possibility that a distressed firm be reorganised, by the renegotiation of the debt contract.

Debt value and the option to liquidate the firm's assets

This problem is again about relating the value of debt to the characteristics of the firm's assets. The firm's assets embed a valuable real option, but a liquidation option rather than a growth option as in chapter 2. The liquidation (or collateral) value of assets can be estimated from the firm's accounts and lenders do consider such value when pricing debt.

Debt value and "once-off" re-negotiation of the debt contract

In chapter 4 debt value depended on the option to renegotiate the debt contract and to extend debt maturity. Future research may study how debt value depends on the possibility that the debtor attempt a distressed exchange in order to avoid default. A distressed exchange would then be similar to a *"once-off"* re-negotiation of the maturity *and* coupon of the debt contract, as opposed to the continual renegotiation that is often assumed in the so called "strategic" structural models of credit risk (Anderson-Sundaresan (1996), Mella-Barral and Perraudin (1997), etc...). Similarly future research may study how debt value depends on the *"once-off"* re-negotiation of the debt contract that takes place within an official bankruptcy proceeding rather than within an informal workout. This would relate debt value to the characteristics of the bankruptcy code.

The above are the ways in which future research can value debt as a more accurate function of the debtor's financial distress.

3.2) Improving the valuation of debt subject to both default risk and interest rate risk

The second research direction aims at improving the valuation of debt that is subject to both interest rate risk and default risk. The following are the main open issues.

About "interactions" between default risk and interest rate risk

Chapter 5 has studied interactions between interest rate risk and default risk in structural models. Then chapter 5 has proposed two ways to eliminate interactions and the associated "interest rate model risk". Alternatively future research may eliminate interactions and retain tractability by employing a different state variable to capture default risk (e.g. leverage) whose process should not depend on the default free short rate of interest.

Testing the proposed models and the problem of estimating structural parameters from equity prices

The models proposed in chapter 5 eliminate "interactions" between default risk and interest rate risk, but need empirical testing against bond market prices. The empirical tests should follow the maximum likelihood approach proposed by Ericsson and Reneby (2001): their approach has provided the most promising results. But such approach uses equity market prices to estimate structural parameters[58] and requires knowledge of a closed form relating equity value to structural parameters. Such closed form for equity is available only when the default free short rate is assumed constant over time *and* there is not default barrier before maturity. Per contra, the models in chapter 5 assume stochastic interest rates *and* a default barrier. Thus, the lack of a closed form solution for equity under assumptions that are consistent with the models in chapter 5 has been the obstacle in the way of empirical testing the proposed models. Future research could provide closed form solutions for equity assuming stochastic default free interest rates *and* a default barrier. Such closed forms could then be employed to estimate structural parameters from equity prices and test the models proposed in section 5.

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[58] Such parameters typically are: firm's assets volatility and payout rate. The value of the firm's assets, which is a variable rather than a parameter, need also be estimated. The default barrier need not always be estimated because may be it is endogenous in the structural model.

The above are the ways in which future research can improve the valuation of debt subject to both default risk and interest rate risk.

CONCLUSION OF THE CONCLUSIONS

The aim of this thesis is to advance the theory of the fair valuation of the firm's debt subject to default risk.

The thesis has developed selected aspects of the theory of structural models and has moved towards two ends. The first end is to develop practical models to price single issues of corporate debt and possibly credit derivatives. This need calls for simple tractable structural models requiring few parameters.

The second end is to shed more light on aspects of the theory of corporate finance: investment policy and debt value, debt renegotiation and debt value. This need calls for structural models that value the whole capital structure and accommodate endogenous decisions by equity holders and/or debt holders. Sundaresan (2001) calls these models "strategic structural models".

In the future the theory of structural models of credit risk is likely to witness a more marked differentiation between these two different types of structural models.

Finally the theory of the valuation of the firm's debt that is based on structural models of credit risk not only can explain what is (observed in the markets), but it can also attempt to predict what should be (observed in idealised markets). Structural models seem capable to explain how the determinants of credit risk can affect the fair value of the firm's debt.

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