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A Gauge Invariant Flow Equation

by

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ABSTRACT

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A Gauge Invariant Flow Equation

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Given a Quantum Field Theory, with a particular content of fields and a symmetry associated with them, if one wants to study the evolution of the couplings via a Wilsonian renormalisation group, there is still a freedom on the construction of a flow equation, allowed by scheme independence.

In the present thesis, making use of this choice, we first build up a generalisation of the Polchinski flow equation for the massless scalar field, and, applying it to the calculation of the beta function at one loop for the $\lambda\phi^4$ interaction, we test its universality beyond the already known cutoff independence. Doing so we also develop a method to perform the calculation with this generalised flow equation for more complex cases.

In the second part of the thesis, the method is extended to $SU(N)$ Yang-Mills gauge theory, regulated by incorporating it in a spontaneously broken $SU(N|N)$ supergauge group. Making use of the freedom allowed by scheme independence, we develop a flow equation for a $SU(N|N)$ gauge theory, which preserves the invariance step by step throughout the flow and demonstrate the technique with a compact calculation of the one-loop beta function for the $SU(N)$ Yang-Mills physical sector of $SU(N|N)$, achieving a manifestly universal result, and without gauge fixing, for the first time at finite N.

Dedicated to my family

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Preface

Original work is contained in the last section of chapter 2 and in chapters 4, 5 and 6 (in collaboration with Tim Morris and Stefano Arnone) and it can also be found in:

(i) S.Arnone, A. Gatti and T.R. Morris, JHEP **0205** (2002) 059
(ii) S.Arnone, A. Gatti and T.R. Morris, hep-th/0205156
(iii) S.Arnone, A. Gatti and T.R. Morris, hep-th/0207153
(iv) S.Arnone, A. Gatti and T.R. Morris, hep-th/0207154
(v) S.Arnone, A. Gatti and T.R. Morris, hep-th/0209162

No claim to originality is made for the content of the rest of chapter 2 and chapter 3, which were compiled using a variety of other sources.

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Chapter 1

Introduction

The Exact Renormalisation Group (ERG) [1]-[4], is a powerful tool to control the infinities arising in quantum field theories. Once one has started introducing a Lagrangian which defines a certain theory, the calculation of possible predictions from it, such as scattering amplitudes or lifetime of particles, have in most of the cases to pass through the process of perturbative expansion, this being often, the only practical possibility. These quantities are evaluated at different approximations by truncating an expansion on the coupling constant, assumed to be small. Infinities come out already trying to calculate the second approximation, when one faces the task to integrate over all the possible momenta of the virtual particles taking part in the process. Rather than integrating out all the momenta at once, by introducing a cutoff at a certain scale of momenta Λ_0 (here imposed via a cutoff function) which regulates these integrals, making the theory finite, one introduces another scale Λ (much lower than the first one), and the integral of the partition function is made from this new scale up to the first one, we are left with an integral between zero momenta and this new scale Λ. This integral can still be expressed as a partition function, but the previous action (called the *bare action*) which is usually chosen as simple as possible, is replaced by a complicated *effective action*. This *transformation*

of the action in the partition function due to *integrating out* momenta is a transformation of the so called Wilsonian **Renormalisation Group** (RG) (see fig.1.1). Imposing the invariance of the partition function under such transformations, one can find an equation, whose solution (with set boundary condition at Λ_0 being the bare action of the theory), the so called *Wilsonian action*, describes the RG flow of the action. Since the equation is written non-perturbatively, the approach is called the *Exact RG*. The limit that the cutoff tends to infinity of the solution of this equation would be the action of the theory at any scale. This will be explained in detail in the next section for the case of a massless scalar field theory.

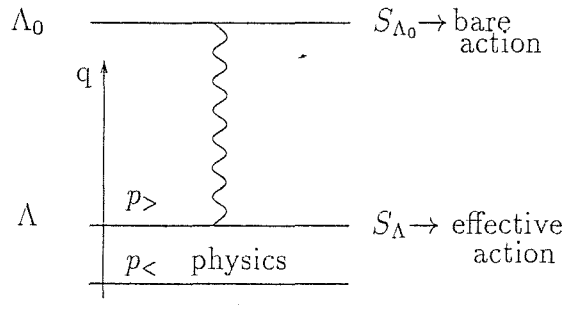

Figure 1.1: Flow of S_Λ as momenta are integrated out

When this process is applied to gauge theories, a further problem arises: the cutoff functions usually used, break the gauge invariance. The usual approach at this point was to recover this symmetry when the cutoff was removed. In this way each step is done *without* gauge invariance, which is reinserted at the end (this approach will be briefly reviewed later in section 3.1). The "temporary" non-gauge invariance limits the study of the theory in particular for what concerns non-perturbative studies. Since this is an interesting direction for a better understanding of gauge theories another way can be worth exploring, the one pioneered by Dr. Tim Morris in [4, 6, 7], which is based on studying an ERG for a gauge theory preserving this symmetry step by step.

This thesis is organised as follows. In the second chapter will be considered the Polchinski approach to the Exact RG applied to scalar field theory. A review of the concept of scheme independence will be done, and through the freedom derived from it, a flow equation more general than the one by Polchinski, will be introduced. In the last section of the chapter, we will calculate making use of this generalised flow equation, the one loop beta function for the massless scalar field theory, developing a method to perform the calculation in this more general framework, which will be possible to apply also to the more complicated gauge field case.

The third chapter will be dedicated to the description of the attempts done to regularise gauge theories. Starting from the attempts mentioned earlier in which gauge invariance was first broken and then restored, and then going on to the methods involving covariant derivatives with the addition of Pauli-Villars fields. In the last section of the chapter will be also reviewed one of the most recent methods pioneered by Dr. Tim Morris et. al, in which the regularisation is performed embedding the gauge group of the theory in a bigger graded group. The chapter will then start with some concepts related to this peculiar group and it will explain the mechanism through which its subgroup of physical interest comes out regulated in a gauge invariant way.

The fourth chapter will mainly be concerned with the build up of a flow equation for the theory constructed on this bigger gauge group. Making use of the freedom allowed by scheme independence, the equation has been constructed in order to preserve gauge invariance through the flow.

The fifth and sixth chapters involve a check on the flow equation introduced in the previous chapter. Adapting the method used for the scalar field to the gauge case, we calculate the one loop beta function for the $SU(N)$ Yang-Mills theory without fixing the gauge at any stage.

Chapter 2

The Scalar Field Case

Before we start with the attempt to write a flow equation for a $SU(N|N)$ gauge theory, it might be useful to have a look at the way a flow equation has been worked out in a much simpler case, which is the massless scalar field theory.

Of the many possible ERG formulations, we have chosen to describe the one by Polchinski, as in [6, 7], since it appeared more suitable for extracting a generalisation of its flow equation, using scheme independence. The concept of scheme independence, which is going to be described in the second section of this chapter, will be central for the development of this whole work.

As it will appear clear through this chapter, the simple massless scalar theory can give great insights on how to proceed for more sophisticated cases such as gauge theories. As an example, the calculation of the one loop β-function performed in the last section, with a generalised Polchinski-like flow equation for the scalar theory, will set up a method which will be adapted to the analogous calculation for $SU(N)$ Yang-Mills.

Let us start now with a brief introduction to the Polchinski ERG for the massless

scalar field.

2.1 The Polchinski ERG equation

The central object in the path integral formulation of quantum field theories is the partition function $\mathcal{Z}$ from which it is possible to extract information such as the vacuum expectation value of a product of fields (*a.k.a.* correlators). These are related to physical objects like cross sections and so on. The partition function of a theory describing a field ϕ defined by an action $S[\phi]$ is[1]:

$$\mathcal{Z}[j] = \int \mathcal{D}\phi \; e^{-S[\phi]+j\cdot\phi} \tag{2.1}$$

Taking derivatives with respect to the current j and setting it to zero, gives expectation values of fields. When these integrals have to be performed, depending on the form of S, there is sometimes the need to make use of perturbative methods, and, as it was mentioned in the previous chapter, this gives rise to divergent integrals.

When these infinities arise from large momenta in the integrals of the particles appearing in loops, the usual procedure is to regularise the bare action to have finite quantum corrections, namely by substituting $S \to S_{\Lambda_0}$ (see fig.1.1). The partition function regulated via Λ_0 is then:

$$\mathcal{Z} = \int \mathcal{D}\phi \; e^{-S_{\Lambda_0}} \tag{2.2}$$

where the action of the theory is taken at Λ_0 to be (bare action), in the momentum space:

$$S_{\Lambda_0} = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4} p^2 c^{-1}(p^2/\Lambda_0^2)\tilde{\phi}(p)\tilde{\phi}(-p) + S^{int}[\tilde{\phi}, \Lambda_0] \tag{2.3}$$

[1] If not otherwise specified we will always be working in Euclidean space

where $\tilde{\phi}(p)$ are the Fourier component of the field $\phi(x)$ and $c(p^2/\Lambda^2)$ is a *smooth*, *i.e.* infinitely differentiable, ultra violet cutoff profile. The cutoff which modifies propagators $1/p^2$ to c/p^2, satisfies $c(0) = 1$ in such a way that low energy is unaltered and $c(p^2/\Lambda^2) \to 0$ as $p^2/\Lambda^2 \to \infty$, fast enough in order to make all the Feynman diagrams ultra-violet regulated. $S^{int}[\tilde{\phi}, \Lambda_0]$ is the interaction part of the bare action, containing all the relevant (and marginal) interactions compatible with the symmetries of the theory, considered to be the only non zero interactions at the scale Λ_0. For the case considered in section 2.3, we would normally choose the following (see also [35]):

$$S^{int}[\phi, \Lambda_0] = \frac{\lambda_0}{4!}\int d^4x\ \phi^4 + \frac{m_0}{2}\int d^4x\ \phi^2 \tag{2.4}$$

To motivate the later strategy, setting an intermediate cutoff scale Λ, we can (at least heuristically) separate the fields into the ones with momentum greater than Λ ($\phi_>$) and smaller than it ($\phi_<$) and rewrite the partition function with the new measure. We can then perform the integral on $\phi_>$ for a certain Λ to get: :

$$\mathcal{Z} = \int \mathcal{D}\phi_> \mathcal{D}\phi_<\ e^{-S_{\Lambda_0}} = \int \mathcal{D}\phi_<\ e^{-S_\Lambda} \tag{2.5}$$

where:

$$e^{-S_\Lambda} = \int \mathcal{D}\phi_>\ e^{-S_{\Lambda_0}} \tag{2.6}$$

In principle now S_Λ could contain all possible interactions compatible with the symmetries of the theory. In our case, the RG transformation amounts to changing the cutoff from Λ_0 to $\Lambda << \Lambda_0$ in eq.(2.3):

$$S_\Lambda = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4} p^2 c^{-1}(p^2/\Lambda^2)\tilde{\phi}(p)\tilde{\phi}(-p) + S^{int}[\tilde{\phi}, \Lambda] \tag{2.7}$$

where now $S^{int}[\tilde{\phi}, \Lambda]$ is a more complicated functional of $\tilde{\phi}$. (The "tilde" for the Fourier components of $\phi(x)$ will be dropped from now on.) The choice of a flowing

kinetic term which keeps the same form as the corresponding one in the bare action, is the choice performed in [2] and it is just for simplicities sake. Now, changing the intermediate scale Λ, the interaction term S_Λ^{int}, transforms (flows) as we integrate out momenta (RG transformation). One way to get the equation describing its flow is the following. Demanding that physics be invariant under such a scale change, follows from asking the partition function $\mathcal{Z}$ to be independent on Λ. If we then require its variation under the RG transformation to vanish

$$\delta \mathcal{Z} = 0, \tag{2.8}$$

we are led to a flow equation for S_Λ^{int} (Polchinski's for scalar field, see [2]):

$$\Lambda \partial_\Lambda S_\Lambda^{int} = \underbrace{-\frac{1}{\Lambda^2}\frac{\delta S_\Lambda^{int}}{\delta \phi}\cdot c' \cdot \frac{\delta S_\Lambda^{int}}{\delta \phi}}_{\text{Classical Term}} + \underbrace{\frac{1}{\Lambda^2}\frac{\delta}{\delta \phi}\cdot c' \cdot \frac{\delta S_\Lambda^{int}}{\delta \phi}}_{\text{Quantum Correction}} \tag{2.9}$$

(c') is the derivative of the cutoff function with respect to its argument (p^2/Λ^2) and the following notation has been introduced: given two functions $f(x)$ and $g(y)$ and a momentum space kernel $W(p^2/\Lambda^2)$ (Λ is the effective cutoff),

$$f \cdot W \cdot g = \int\int d^4x d^4y f(x) W_{xy} g(y), \tag{2.10}$$

where $W_{xy} = \int \frac{d^4p}{(2\pi)^4} W(p^2/\Lambda^2) e^{ip\cdot(x-y)}$.

The solution of the (exact) equation (2.9), with boundary condition $S_{\Lambda=\Lambda_0}^{int} = S_{\Lambda_0}^{int}$, in the continuum limit ($\Lambda_0 \to \infty$) would be the action of the theory at any scale (as we were observing earlier in the thesis).

2.2 Scheme Independence

The equation derived for the interaction part of the effective action, in the previous section was indeed a consequence of the request $\delta\mathcal{Z} = 0$ under a RG transformation. Following the example of J.I.Latorre and T.R.Morris in [23], it is worth at this point making an observation. This one flow equation is not necessarily the most general consequence of such a requirement. There is a more general statement, which can be extracted from it and this introduces a particular viewpoint on the concept of Scheme Independence (SI).

Let us first consider the effective kinetic term of the scalar field theory, in the notation introduced in eq.(2.10):

$$\frac{1}{2}\partial_\mu\phi \cdot c^{-1} \cdot \partial_\mu\phi \tag{2.11}$$

This will be referred to as "seed action" and denoted with $\hat{S}$. The total effective action can be written then (dropping the Λ) as: $S = \hat{S} + S^{int}$. Defining the combination $\Sigma = S - 2\hat{S}$, the Polchinski flow equation (2.9), can be rewritten (up to a vacuum energy term, discarded in [2]) as:

$$\Lambda\partial_\Lambda S = -\frac{1}{\Lambda^2}\frac{\delta S}{\delta\phi} \cdot c' \cdot \frac{\delta\Sigma}{\delta\phi} + \frac{1}{\Lambda^2}\frac{\delta}{\delta\phi} \cdot c' \cdot \frac{\delta\Sigma}{\delta\phi} \tag{2.12}$$

The invariance of the partition function is manifest from the previous equation, since it is possible to recognise that eq.(2.12) can be recast as:

$$\Lambda\partial_\Lambda e^{-S} = -\frac{1}{\Lambda^2}\frac{\delta}{\delta\phi} \cdot c' \cdot \left(\frac{\delta\Sigma}{\delta\phi}e^{-S}\right) \tag{2.13}$$

i.e. the infinitesimal RG transformation on the partition function is a change in the integrand corresponding to a total derivative. From the previous equation we can

also notice that:

$$\delta\mathcal{Z} = \frac{\partial\mathcal{Z}}{\partial\Lambda}\delta\Lambda = \delta\Lambda\int\mathcal{D}\phi\left(\frac{\delta}{\delta\phi}\cdot\psi - \psi\cdot\frac{\delta S}{\delta\phi}\right)e^{-S} \tag{2.14}$$

where $\psi = -\frac{1}{\Lambda^2}c'\frac{\delta\Sigma}{\delta\phi}$. This establishes another result: integrating out degrees of freedom correspond to a redefinition of the fields in the theory [23]. In the case we have been considering here, the change in the partition function due to a transformation under the RG, corresponds to the variation due to the change of variables (field redefinition):

$$\phi \to \phi + \delta\Lambda\psi \tag{2.15}$$

Recognising the first term in eq.(2.14) as arising from the Jacobian and the second as arising from the variation of S. ψ is called the *kernel* of the RG transformation. Different kernels, lead to different flow equations. If these flow equations come from different choices of kernels connected via a field redefinition, they describe the same physical system. This gives a great freedom on the form of the flow equation. First of all there is a choice of the form of Σ, which could be chosen as a polynomial in S. A reason for choosing it at least linear in the effective action, as it is done here, is to ensure a quadratic term in S on the RHS of the flow equation, which can give fixed point solutions to the flow equation. After this first choice is made, the freedom is on the "seed action" $\hat{S}$ (which will be widely used in the gauge case) and on the choice of the cutoff function, which in principle can now contain interactions (as will be the case for the "wines" which will be introduced for the gauge case, following [6, 7]), higher functional derivatives, and/or other more complex dependences on S.

Physical quantities should be independent of these choices. One of the main purposes of this thesis is to check that the equation derived from this more general formulation of the ERG, can give the same results that were found with previous ones. The universal quantity examined here is the first coefficient of the beta function for both the massless scalar field and the $SU(N)$ Yang-Mills theory. In the former case this

check leads to a proof of universality for the beta function at one loop beyond the change of the cutoff function and allows to develop the right procedure to deal with calculations in this more general scheme. In the case of gauge theories, since SI allows to write a flow equation which preserves the symmetry for each step of the flow, the check represents also the first calculation of such a quantity in a gauge invariant way at finite N.

As shown above, in the Polchinski case, the "seed action" coincided with the kinetic term, as eq.(2.12) is equivalent to eq.(2.9) for this choice of $\hat{S}$. For our purposes, in the calculation of the beta function at one loop for the massless scalar field, the "seed action" will be chosen much more general. First of all we require it to preserve the symmetry $\phi \to -\phi$, so it must be an even functional of the fields. Second, if one wants the effective kinetic term to flow as in eq.(2.7), the bilinear term of $\hat{S}$ must be still equal to (2.11). For all the other interaction vertices with $n > 2$, we just ask them to be infinitely differentiable (Taylor expandable to any order) to ensure they do not introduce infrared singularities, and that they do not lead to ultraviolet divergent momentum integrals, so that the flow described by the equation can be interpreted as integrating out momenta. We will see from the next section that the first coefficient of the beta function is blind to the introduction of all these extra parameters, which can be always eliminated in favour of the physically meaningful vertices of the effective action.

2.3 One loop β-function with general $\hat{S}$: the scalar field case

Before we start the discussion for the super-gauge field we will consider the massless scalar field case, in the present formulation and show that starting with the new form of the flow equation (2.12), we can get the correct β-function at one loop, without

specifying $\hat{S}$ and without any strong constraint on it.

As we have mentioned already we expect the physical quantities to be universal, *i.e.* independent of the renormalization scheme. In particular, they should not be sensitive to the particular choice of the RG kernel, *e.g.* on the form of the cutoff function or the expression for the seed action. We aim to calculate one of those, the one-loop contribution to the β function, while keeping as general a seed action as possible. As we will see, an elegant, clear cut way of achieving such a result is to make use of the flow equations for the effective couplings in order to get rid of the seed action vertices.

As usual, universal results are obtained only after the imposition of appropriate renormalization conditions which allow us to define what we mean by the physical (more generally renormalised) coupling and field. (The renormalised mass must also be defined and is here set to zero implicitly by ensuring that the only scale that appears is Λ.)

We write the vertices of S as

$$S^{(2n)}(\vec{p};\Lambda) \equiv S^{(2n)}(p_1,p_2,\cdots,p_{2n};\Lambda) \doteq (2\pi)^{8n}\frac{\delta^{2n}S}{\delta\phi(p_1)\delta\phi(p_2)\cdots\delta\phi(p_{2n})}, \qquad (2.16)$$

(and similarly for the vertices of $\hat{S}$). In common with earlier works [2, 42], we define the renormalised four-point coupling λ by the effective action's four-point vertex evaluated at zero momenta: $\lambda(\Lambda) = S^{(4)}(\vec{0};\Lambda)$. This makes sense once we express quantities in terms of the renormalised field, defined (as usual) to bring the kinetic term into canonical form $S^{(2)}(p,-p;\Lambda) = S^{(2)}(0,0;\Lambda) + p^2 + O(p^4/\Lambda^2)$. The flow equation can then be taken to be of the form [24, 25]:

$$\Lambda\partial_\Lambda S - \frac{\gamma}{2}\,\phi\cdot\frac{\delta S}{\delta\phi} = -\frac{1}{\Lambda^2}\frac{\delta S}{\delta\phi}\cdot c'\cdot\frac{\delta\Sigma}{\delta\phi} + \frac{1}{\Lambda^2}\frac{\delta}{\delta\phi}\cdot c'\cdot\frac{\delta\Sigma}{\delta\phi}. \qquad (2.17)$$

We have used the short hand defined in eq.(2.10), and as usual the anomalous dimension $\gamma = \frac{1}{Z}\Lambda\partial_\Lambda Z$, where Z is the wavefunction renormalization. As emphasised

in refs. [4, 23], although eq. (2.17) is not the result of changing variables $\phi \mapsto \phi\sqrt{Z}$ in eq. (2.12), it is still a perfectly valid flow equation and a more appropriate starting point when wavefunction renormalization has to be taken into account. This is in fact a small example of the immense freedom we have in defining the flow equation. (The new term on the left hand side arises from replacing $\partial_\Lambda|_\phi$ with a partial derivative at constant renormalised field, but in order to produce the right hand side, and in order to reproduce the same $\hat{S}$, we need to start with the alternative cutoff function cZ in eqs. (2.3) – (2.12). Alternatively, for the purposes of computing the β function, we could have simply taken account of the wavefunction renormalization afterwards as in ref. [26].)

We now rescale the field ϕ to

$$\phi = \frac{1}{\sqrt{\lambda}}\,\tilde{\phi}, \tag{2.18}$$

so as to put the coupling constant in front of the action. This ensures the expansion in the coupling constant coincides with the one in $\hbar$, the actual expansion parameter being just $\lambda\hbar$. The resulting expansion is more elegant, being no longer tied at the same time to the order of expansion of the field ϕ. It is also analogous to the treatment pursued for gauge theory in refs. [4, 6, 7] (where gauge invariance introduces further simplifications in particular forcing $\gamma = 0$ for the new gauge field). The following analysis thus furnishes a demonstration that these ideas also work within scalar field theory.

The bare action (2.3) rescales as

$$S_{\Lambda_0}[\phi] = \frac{1}{\lambda}\left[\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\,p^2c^{-1}(\tfrac{p^2}{\Lambda_0^2})\,\tilde{\phi}^2 + \frac{1}{4!}\int d^4x\,\tilde{\phi}^4\right] \doteq \frac{1}{\lambda}\tilde{S}_{\Lambda_0}[\tilde{\phi}]. \tag{2.19}$$

Defining the "rescaled" effective and seed actions as $S[\phi] = \frac{1}{\lambda}\tilde{S}[\tilde{\phi}]$, $\hat{S}[\phi] = \frac{1}{\lambda}\tilde{\hat{S}}[\tilde{\phi}]$, and

absorbing the change to $\partial_\Lambda|_{\tilde{\phi}}$ in a change to $\tilde{\gamma}$, the flow equation (2.12) reads

$$\Lambda\partial_\Lambda\left(\frac{1}{\lambda}\tilde{S}\right)-\frac{\tilde{\gamma}}{2\lambda}\,\tilde{\phi}\cdot\frac{\delta\tilde{S}}{\delta\tilde{\phi}}=-\frac{1}{\lambda\Lambda^2}\frac{\delta(\tilde{S}-2\tilde{\hat{S}})}{\delta\tilde{\phi}}\cdot c'\cdot\frac{\delta\tilde{S}}{\delta\tilde{\phi}}+\frac{1}{\Lambda^2}\frac{\delta}{\delta\tilde{\phi}}\cdot c'\cdot\frac{\delta(\tilde{S}-2\tilde{\hat{S}})}{\delta\tilde{\phi}}. \quad (2.20)$$

Expanding the action, the beta function $\beta(\Lambda)=\Lambda\partial_\Lambda\lambda$ and anomalous dimension, in powers of the coupling constant:

$$\begin{aligned}\tilde{S}[\tilde{\phi}] &= \tilde{S}_0+\lambda\tilde{S}_1+\lambda^2\tilde{S}_2+\cdots,\\ \beta(\Lambda) &= \beta_1\lambda^2+\beta_2\lambda^3+\cdots,\\ \tilde{\gamma}(\Lambda) &= \tilde{\gamma}_1\lambda+\tilde{\gamma}_2\lambda^2+\cdots\end{aligned}$$

yields the loopwise expansion of the flow equation[2]

$$\Lambda\partial_\Lambda S_0=-\frac{1}{\Lambda^2}\frac{\delta S_0}{\delta\phi}\cdot c'\cdot\frac{\delta(S_0-2\hat{S})}{\delta\phi}, \quad (2.21)$$

$$\Lambda\partial_\Lambda S_1-\beta_1 S_0-\frac{\gamma_1}{2}\,\phi\cdot\frac{\delta S_0}{\delta\phi}= -\frac{2}{\Lambda^2}\frac{\delta S_1}{\delta\phi}\cdot c'\cdot\frac{\delta(S_0-\hat{S})}{\delta\phi}+\frac{1}{\Lambda^2}\frac{\delta}{\delta\phi}\cdot c'\cdot\frac{\delta(S_0-2\hat{S})}{\delta\phi}, \quad (2.22)$$

etc. γ_1 and β_1 may now be extracted directly from eq. (2.22), as specialised to the two-point and four-point effective couplings, $S^{(2)}(\vec{p};\Lambda)$ and $S^{(4)}(\vec{p};\Lambda)$ respectively, once the renormalization conditions have been taken into account.

We impose the wavefunction renormalization condition in the new variables:

$$S^{(2)}(p,-p;\Lambda)=S^{(2)}(0,0;\Lambda)+p^2+O(p^4/\Lambda^2). \quad (2.23)$$

Bearing in mind that the coupling constant has been scaled out, we impose the

[2]In order to simplify the notation, the tildes will be removed from now on.

condition

$$S^{(4)}(\vec{0};\Lambda) = 1. \tag{2.24}$$

Both conditions eq. (2.23) and eq. (2.24) are already saturated at tree level. (To see this it is sufficient to note that, since the theory is massless, the only scale involved is Λ. Since $S_0^{(4)}$ is dimensionless it must be a constant at null momenta, thus $S_0^{(4)}(\vec{0};\Lambda) = S_0^{(4)}(\vec{0};\Lambda_0) = 1$. Similar arguments apply to $S_0^{(2)}$.) Hence the renormalization condition implies that we must have no quantum corrections to the four-point vertex at $\vec{p} = \vec{0}$, or to the $O(p^2)$ part of the two-point vertex, *i.e.*

$$S_n^{(4)}(\vec{0};\Lambda) = 0 \quad \text{and} \quad S_n^{(2)}(p,-p;\Lambda)\Big|_{p^2} = 0 \qquad \forall n \geq 1, \tag{2.25}$$

where the notation $|_{p^2}$ means that one should take the coefficient of p^2 in the series expansion in p. The flow equations for these special parts of the quantum corrections thus greatly simplify, reducing to algebraic equations which then determine the β_i and γ_i. In particular, from the flow of $S_1^{(4)}$ at null momenta:[3]

$$\beta_1 + 2\gamma_1 = \frac{8c_0'}{\Lambda^2}\Big[1 - \hat{S}^{(4)}(\vec{0})\Big]S_1^{(2)}(0) - \frac{1}{\Lambda^2}\int_q c'(\tfrac{q^2}{\Lambda^2})\Big[S_0^{(6)} - 2\hat{S}^{(6)}\Big](\vec{0},q,-q), \tag{2.26}$$

where $c_0' = c'(0)$ and $\int_q \doteq \int \frac{d^4q}{(2\pi)^4}$, and from the flow of $S_1^{(2)}$ expanded to $O(p^2)$:

$$\beta_1 + \gamma_1 = -\frac{1}{\Lambda^2}\int_q c'(\tfrac{q^2}{\Lambda^2})\Big[\, S_0^{(4)} - 2\hat{S}^{(4)}\Big](p,-p,q,-q)\Big|_{p^2}. \tag{2.27}$$

Note that contrary to the standard text book derivation our one-loop anomalous dimension is not zero, picking up a contribution from the general field reparametrization [23] induced by higher point terms in $\hat{S}$ and a contribution $-\beta_1$ due to the field rescaling eq. (2.18).

In order to evaluate eq. (2.26), we need to calculate $S_1^{(2)}(0)$ and $S_0^{(6)}(\vec{0},q,-q)$. We

[3]Here and later we suppress the Λ dependence of the S and $\hat{S}$ vertices.

would also need $\hat{S}^{(4)}(\vec{0})$ and $\hat{S}^{(6)}(\vec{0},q,-q)$, but we will see that we can avoid using explicit expressions for them, and thus keep $\hat{S}$ general, by using the flow equations to express them in terms of the effective vertices $S_0^{(4)}$ and $S_0^{(6)}$.

However, as explained in the previous section, our $\hat{S}$ is not completely arbitrary. Apart from some very general requirements on the differentiability and integrability of its vertices, for convenience we restrict $\hat{S}$ to have only even-point vertices, as in fact already used in eqs. (2.26) and (2.27), and constrain its two-point vertex so that the two-point effective coupling keeps the same functional dependence upon Λ as the bare one (as in eq. (2.7)). This last condition reads

$$S_0^{(2)}(p) = p^2 c^{-1}(\tfrac{p^2}{\Lambda^2}) \tag{2.28}$$

and from the two-point part of eq. (2.21), we immediately find

$$\hat{S}^{(2)}(p) = p^2 c^{-1}(\tfrac{p^2}{\Lambda^2}). \tag{2.29}$$

Let us start with the calculation of $S_1^{(2)}(0)$. From eq. (2.22), its equation reads

$$\Lambda\partial_\Lambda S_1^{(2)}(0) = \frac{1}{\Lambda^2}\int_q c'(\tfrac{q^2}{\Lambda^2})\Big[S_0^{(4)} - 2\hat{S}^{(4)}\Big](0,0,q,-q), \tag{2.30}$$

where eqs. (2.29) and (2.28) have been already used to cancel out the classical terms. Pursuing our strategy, we get rid of $\hat{S}^{(4)}$ by making use of the flow equation for the four-point effective coupling at tree level

$$\Lambda\partial_\Lambda S_0^{(4)}(\vec{p}) = \frac{2}{\Lambda^2}\sum_i \frac{p_i^2 c'_{p_i}}{c_{p_i}}\hat{S}^{(4)}(\vec{p}), \tag{2.31}$$

where $c_{p_i} \doteq c(\frac{p_i^2}{\Lambda^2})$ and the invariance of $S_0^{(4)}(\vec{p})$ under permutation of the p_i's (which it has without loss of generality) has been utilised. Specialising the above equation

to $\vec{p} = (0, 0, q, -q)$, eq. (2.30) becomes

$$\begin{aligned}
\Lambda\partial_\Lambda S_1^{(2)}(0) &= \frac{1}{\Lambda^2}\int_q c'_q S_0^{(4)}(0,0,q,-q) - \int_q \frac{c_q}{2q^2}\Lambda\partial_\Lambda S_0^{(4)}(0,0,q,-q) \\
&= -\int_q \frac{1}{2q^2}\Lambda\partial_\Lambda\Big(c_q\, S_0^{(4)}(0,0,q,-q)\Big) \\
&= -\Lambda\partial_\Lambda \int_q \frac{c_q\, S_0^{(4)}(0,0,q,-q)}{2q^2}. \qquad (2.32)
\end{aligned}$$

In the above, the derivative with respect to the cutoff may be taken after integrating over the loop momentum since the integral is regulated both in the ultraviolet and in the infrared as a result of the properties of the effective couplings. Eq. (2.32) may be now integrated to give

$$S_1^{(2)}(0) = -\int_q \frac{c_q\, S_0^{(4)}(0,0,q,-q)}{2q^2}, \qquad (2.33)$$

with no integration constant since for a massless theory, there must be no other explicit scale in the theory apart from the effective cutoff.

Let us now move on to the tree-level six-point function. From (2.21) we get

$$\begin{aligned}
\Lambda\partial_\Lambda S_0^{(6)}(\vec{0},q,-q) &= \frac{4q^2}{\Lambda^2}\frac{c'_q}{c_q}\hat{S}^{(6)}(\vec{0},q,-q) \\
&\quad -\frac{8c'_0}{\Lambda^2}\Big[1-\hat{S}^{(4)}(\vec{0})\Big]S_0^{(4)}(0,0,q,-q) + \frac{8c'_0}{\Lambda^2}\hat{S}^{(4)}(0,0,q,-q) \\
&\quad -\frac{12}{\Lambda^2}c'_q\, S_0^{(4)}(0,0,q,-q)\Big[S_0^{(4)} - 2\hat{S}^{(4)}\Big](0,0,q,-q). \qquad (2.34)
\end{aligned}$$

Using eq. (2.31), and solving for $\hat{S}^{(6)}(\vec{0},q,-q)$,

$$\begin{aligned}
\hat{S}^{(6)}(\vec{0},q,-q) &= \frac{\Lambda^2}{4q^2}\frac{c_q}{c'_q}\Big\{\Lambda\partial_\Lambda S_0^{(6)}(\vec{0},q,-q)) + \frac{8c'_0}{\Lambda^2}\Big[1-\hat{S}^{(4)}(\vec{0})\Big]S_0^{(4)}(0,0,q,-q) \\
&\quad -2c'_0\frac{c_q}{q^2c'_q}\Lambda\partial_\Lambda S_0^{(4)}(0,0,q,-q) \\
&\quad -\frac{6}{q^2}S_0^{(4)}(0,0,q,-q)\Lambda\partial_\Lambda\Big[c_q\, S_0^{(4)}(0,0,q,-q)\Big]\Big\}. \qquad (2.35)
\end{aligned}$$

We will see that substituting eqs. (2.33) and (2.35) into eq. (2.26) will cause almost all the non universal terms to cancel out. The remaining ones will disappear once γ_1 is substituted using eq. (2.27), leaving just the precise form of the one-loop beta function.

Note that in eq. (2.35) and later, it appears at first sight that we need to be able to take the inverse $1/c'_q$. This would mean that in addition to the general restrictions on $\hat{S}$ outlined earlier we would also require that c' does not vanish at finite argument. In fact, we could arrange the calculation more carefully so that $1/c'$ never appears, thus *e.g.* here we can recognize that only $c'_q\hat{S}^{(6)}(\vec{0},q,-q)$ is needed for eq. (2.26) and that from eq. (2.31), $\Lambda\partial_\Lambda S_0^{(4)}(0,0,q,-q)$ has a factor of c'_q. For clarities sake, we will continue to write $1/c'$ in intermediate results but it is easy to check that all such inverses can be eliminated.

Returning to the calculation in detail, the first term in (2.35) and the $S_0^{(6)}$ term in (2.26) may be paired up into

$$\Lambda\partial_\Lambda \int_q \frac{c_q}{2q^2}\, S_0^{(6)}(\vec{0},q,-q), \tag{2.36}$$

where again, due to the properties of the effective action vertices, the order of the derivative and integral signs can be exchanged. Moreover, as the integrand in eq. (2.36) is dimensionless, there cannot be any dependence upon Λ after the momentum integral has been carried out, hence the result vanishes identically! Also, the second term in (2.35), when substituted into (2.26), exactly cancels the first term of the latter once (2.33) is used. One is then left with

$$\begin{aligned}\beta_1+2\gamma_1 &= -c'_0\int_q \frac{c_q^2}{q^4c'_q}\Lambda\partial_\Lambda S_0^{(4)}(0,0,q,-q) - 3\int_q \frac{c_q}{q^4}S_0^{(4)}(0,0,q,-q)\Lambda\partial_\Lambda\Big\{c_q\, S_0^{(4)}(0,0,q,-q)\Big\}\\ &= -c'_0\int_q \frac{c_q^2}{q^4c'_q}\Lambda\partial_\Lambda S_0^{(4)}(0,0,q,-q) - \frac{3}{2}\int_q \frac{1}{q^4}\Lambda\partial_\Lambda\Big\{c_q\, S_0^{(4)}(0,0,q,-q)\Big\}^2. \end{aligned}\tag{2.37}$$

In order to cancel out the first term in eq. (2.37), the one-loop contribution of the wave function renormalization coming from eq. (2.27) must be taken into account. Again making use of eq. (2.31) to rid us of the hatted four-point coupling,

$$\frac{1}{\Lambda^2}\,\hat{S}^{(4)}(p,-p,q,-q)\Big|_{p^2} = \frac{c_q}{4q^2c_q'}\Lambda\partial_\Lambda S_0^{(4)}(p,-p,q,-q)\Big|_{p^2} - c_0'\left(\frac{c_q}{2q^2c_q'}\right)^2\Lambda\partial_\Lambda S_0^{(4)}(0,0,q,-q), \tag{2.38}$$

and substituting back in eq. (2.27),

$$\beta_1+\gamma_1 = \frac{1}{2}\Lambda\partial_\Lambda\int_q c_q\, S_0^{(4)}(p,-p,q,-q)\Big|_{p^2} - \frac{c_0'}{2}\int_q c_q'\left(\frac{c_q}{q^2c_q'}\right)^2\Lambda\partial_\Lambda S_0^{(4)}(0,0,q,-q). \tag{2.39}$$

The first term on the right hand side of eq. (2.39) vanishes as it is a dimensionless UV and IR convergent integral, and therefore γ_1 takes the form

$$\gamma_1 = -\beta_1 - \frac{c_0'}{2}\int_q c_q'\left(\frac{c_q}{q^2c_q'}\right)^2\Lambda\partial_\Lambda S_0^{(4)}(0,0,q,-q). \tag{2.40}$$

Finally, substituting (2.40) in (2.37) yields

$$\begin{aligned}\beta_1 &= \frac{3}{2}\int_q\frac{1}{q^4}\,\Lambda\partial_\Lambda\left\{c_q S_0^{(4)}(0,0,q,-q)\right\}^2 &(2.41)\\ &= -\frac{3}{2}\frac{\Omega_4}{(2\pi)^4}\int_0^\infty dq\,\partial_q\left\{c_q\, S_0^{(4)}(0,0,q,-q)\right\}^2 \\ &= \frac{3}{16\pi^2}, &(2.42)\end{aligned}$$

which is the standard one-loop result [27].[4] Note that in the top equation the Λ derivative cannot be taken outside the integral, as this latter would not then be properly regulated in the infrared. Moreover, had that been possible, it would have resulted in a vanishing beta function, as the integral is actually dimensionless.

[4]The term in braces depends only on q^2/Λ^2. Ω_4 is the four dimensional solid angle. The last line follows from the convergence of the integral and normalisation conditions $c(0)=1$ and (2.24). As far as independence with respect to the choice of cutoff function is concerned, this is standard.

Chapter 3

Regularising Gauge Theories

The first step towards an ERG approach for a quantum field theory is to construct a regularised effective action, with a regulator suppressing the high modes and maintaining the symmetries of the theory. As far as a scalar field theory is concerned, the problem to solve is quite easy and it was developed in the previous chapter. It is well known that for gauge theories this task represents a more complicated issue.

The notion standing at the base of the ERG is in fact the division between small and large momenta (with respect to some effective cutoff Λ), being the high ones those that are integrated out. This separation operated in the momentum space is at odds with the concept of gauge invariance [38]. A way to notice it is to consider a homogeneous gauge transformation Ω acting on a field $\phi(x)$:

$$\phi \to \Omega(x)\phi(x). \tag{3.1}$$

In the momentum space, $\phi(p)$ is mapped through this transformation into a convolution with the gauge transformation, and any division between low and high momenta is not preserved by gauge transformations. In order to overcome the problem, there are two options left: either one breaks gauge invariance trying to recover it in the limit

$\Lambda \to \infty$ or tries to find a generalisation of the ERG. The former approach, which will be briefly reviewed in the next section, has been the one mainly followed so far, as can be also found in [38, 39, 43]. The second one, starts on writing gauge invariant cutoff functions with addition of Pauli-Villars fields, by A. Slavnov *et al.* , and is continued by T.R. Morris *et al.* with the introduction of the supergauge theory $SU(N|N)$ as a gauge invariant regulator for the Yang-Mills theory. This will be reviewed in detail in the last few sections of this chapter.

A gauge theory regulated in a gauge invariant way is then a solid basis to build a flow equation capable to preserve this feature while extracting information from it. This is going to be the content of the last two chapters.

3.1 Breaking the gauge symmetry and the Quantum Action Principle

If one chooses the first possibility, and introduces a scale Λ to regularise the effective action, the result is that whilst the classical action is invariant under the gauge transformation, the cutoff effective action is not. The consequence is a breaking of the effective Ward-Takahashi identities, or Slavnov-Taylor identities, for the non-Abelian case. This complicates the issue but it is not a problem as long as it is possible to recover gauge symmetry when the cutoffs are removed. Rephrasing it, it is not a problem if it is possible to identify a functional of the effective action, representing the explicit breaking term, which satisfies the equation $\Delta_{eff}[S_\Lambda, \Lambda] = 0$, in the "physical" limit $\Lambda = 0$ and $\Lambda_0 \to \infty$.

In order to derive this symmetry breaking term, it is possible to invoke the Quantum Action Principle. This method is used to study the response of the action of a Quantum Field Theory under a field transformation and it can be used to construct

theories with a given symmetry. In the case of gauge theories and for the present purpose, one has to consider the response of the regularised effective action under a gauge transformation and make sure that the term arising from such a change is zero in the physical limit described above. We will illustrate here just the procedure for constructing an action symmetric under a simple transformation which could then in principle be specified. Let us consider a theory described by an action $S[\phi]$, and the corresponding generating functional:

$$Z[J] = \int \mathcal{D}\phi \; e^{-S[\phi]+J_A\phi_A}, \tag{3.2}$$

where a source term has been added to the action. Consider now the following infinitesimal continuous transformation of the fields:

$$\delta\phi_A = \epsilon P_A[\phi], \tag{3.3}$$

where P_A are (anticommuting) polynomials in the fields , which in the case of gauge theories can correspond to a BRS transformation and ϵ is an anticommuting parameter. Adding to the action a source-type term for the variation of the field of the form $-\eta_A P_A$ and performing the field transformation on the generating functional, we get:

$$\int dx \; J_A \frac{\delta Z}{\delta \eta_A} = \int \mathcal{D}\phi \; \Delta[\phi,\eta] e^{-S[\phi]+\eta_A P_A[\phi]} \tag{3.4}$$

where we indicate with Δ the following:

$$\Delta[\phi,\eta] = \int dx \; \frac{\delta^2 S}{\delta\phi_A(x)\delta\eta_A(x)} - \int dx \; \frac{\delta S}{\delta\phi_A(x)} \frac{\delta S}{\delta\eta_A(x)} \tag{3.5}$$

As one can notice, the first term is due to the Jacobian of the transformation, while the second term takes into account the change in the action due to the variation of the fields. The response of the system is then given by the insertion of the local operator Δ. Eq.(3.4) is known as the Quantum Action Principle.

For our purposes, as anticipated, in order to get Δ_{eff} one would have to follow a similar procedure, with a regularised effective generating functional, performing a cutoff field transformation. The full calculation for the present case will not be shown here, and can be found in the literature (see *e.g.* [44]).

It is possible to prove that the breaking term obeys the following equation:

$$\Lambda\partial_\Lambda\Delta_{eff} = \mathcal{M}[\Delta_{eff}] \tag{3.6}$$

where $\mathcal{M}$ is a linear operator. This implies that, if it is possible to impose at some Λ_R zero boundary conditions for eq.(3.6), the breaking term vanishes at any Λ. The main point is then to set to zero at Λ_R those for the relevant part of Δ_{eff}. This procedure usually overdetermines the vertices of S_{Λ_R} thus the number of independent constraints has to be reduced making use of consistency conditions (algebraic identities coming from anticommutativity of the operator $\frac{\delta}{\delta\eta_A}\frac{\delta}{\delta\phi_A}$).

In this picture it is crucial the way the relevant parts are defined. If the boundary conditions are set at $\Lambda_R \neq 0$, the relevant parts of $\Delta_{eff}(\Lambda_R)$ can be extracted by expanding the vertices around zero momenta even in presence of massless particles. What one gets at this point is that the consistency conditions constrain some of the couplings in the relevant part of $\Delta_{eff}(\Lambda_R)$, which via a tuning of the relevant couplings of the effective action must match with their set of relations. This procedure is known as fine-tuning of the parameters.

If one instead decides to impose the boundary conditions at the physical point $\Lambda_R = 0$, if the theory includes massless particles, one has to impose non-vanishing subtraction points. This causes a mix of relevant and irrelevant vertices in the consistency conditions spoiling their power.

Once all the details of this procedure have been set up, one is left with a fine-tuning equation. If the equation is solvable, the symmetry (the gauge symmetry in our case)

is implemented at the quantum level and no anomalies appear. A more detailed description of the above methods can be found in the literature (see *e.g.* [45]). The main problem now is that the equation mentioned above is usually difficult to solve, even at the first non trivial order in perturbation theory.

There are successful attempts of avoiding the task to solve the fine-tuning equation by fixing proper boundary conditions to the RG equation (see Bonini *et al.* in [43]), but we will not discuss them here.

Instead of doing so, since all these difficulties come from the incompatibility between gauge invariance and the division of high and low momenta, we try to follow the second approach mentioned in the previous section. Following the lead of T.R.Morris we will try here to describe first, a possible way to regularise gauge theories without breaking their symmetry, and then how to generalise the RG method in order to preserve the symmetry in the flowing effective action. This will be done through the construction of a generalised flow equation, gauge invariant itself, capable to describe a gauge invariant flowing effective action. Before we start, it is worth having an overview of other efforts towards a gauge invariant regulator.

3.2 Higher derivatives and P-V fields

The first step is to regularise the action in such a way as to preserve its symmetry. As we have seen in the previous chapters, in a simple case such as the scalar theory, there are many possible choices of regularising the action, involving the introduction of cutoff functions. These function have the rôle to cut the high modes in the loop integrals in order to make them finite. There is a wide choice for the cutoff function which can be chosen to be either a step function (sharp cutoff) or a smooth one as long as it preserves the symmetry of the scalar action (for example for a single scalar field this involves the request of being even in the fields). One possible choice, as can

be found in Chapter 2, is to introduce in the kinetic term a function in the derivatives (which in the momentum space is a function of the momenta).

When it is the case of gauge invariance, this last requirement causes troubles for the reason described in the previous sections. Following the example set by the scalar case, the first attempt towards this goal, was to introduce as a cutoff, a function in the covariant derivative, rather than in the ordinary ones. The method starts from the observation that a kinetic-like term (quadratic in the fields) containing higher derivatives modifies the propagators, conferring them a better behaviour at high momentum. A term like this,

$$\partial_\mu \phi \, \partial^\mu \phi \quad \rightarrow \quad \partial_\mu \phi \, c^{-1} \left(-\partial^2 / \Lambda^2 \right) \partial^\mu \phi, \tag{3.7}$$

substituted in the Lagrangian in place of the usual kinetic term, gives, in the momentum space, a correction to the propagator which amounts to change the ordinary one (e.g. in the massless scalar field) as

$$\frac{1}{p^2} \rightarrow \frac{c(p^2/\Lambda^2)}{p^2} \tag{3.8}$$

The new propagator certainly leads to convergent momentum integrals for a suitable choice of the function appearing in (3.7) (for example if c^{-1} is chosen to be a polynomial for a certain choice of its degree). The idea, for a scalar field is as simple as that, and the physical information is restored as $\Lambda \rightarrow \infty$: at finite Λ all loop diagrams (responsible for divergences) are finite and the calculations are made at this point. The physical quantities can be calculated with a proper renormalisation condition and sending the scale Λ to infinity gives a finite answer for them.

3.2.1 Covariantisation

This does not work for gauge theories. As we said before, a term like the one in (3.7) would break the invariance. One way, as we cited before, to deal with the problem is to break the invariance and restore it when the scale is sent to infinity. Nevertheless, since we want to follow the other path and write a manifestly covariant Exact Renormalisation Group (ERG), our bare action must be gauge invariantly regulated. A first attempt embracing this philosophy was to introduce a cutoff function in the covariant derivative, as we mentioned in the previous section, rather than in the ordinary ones: instead of the term (3.7) we write

$$c\left(-\frac{\partial^2}{\Lambda^2}\right) \to c\left(-\frac{\nabla^2}{\Lambda^2}\right) \tag{3.9}$$

This is known as the higher *covariant derivatives* regularisation [15, 21]. It is known that this method cannot work by itself since it creates a new problem: when the higher derivatives are covariantised, divergences at one loop are still present due to further interactions coming in with them. One way out is to introduce by hand massive (mass of order cutoff Λ) fields with opposite spin-statistic (the so called $Pauli - Villars$ fields) capable of cancelling these 1 loop divergencies. Due to their statistic, they provide a "-" sign in loops as it is shown in fig.3.1 At high momenta, in fact, when

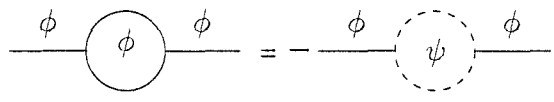

Figure 3.1: Pauli-Villars field cancel out residual 1-loop divergencies

the integrals at one-loop diverge, the propagators and the interactions of the two different fields have the same behaviour and due to the sign difference, they cancel each other. Once the calculation is carried out with finite integrals and the renormalisation conditions have been applied, sending the cutoff to infinity would eventually restore

the physics, since these fields decouple from the physical ones in this limit. Higher covariant derivatives and Pauli-Villars (PV) fields combined together provide a good scheme to regulate gauge theories, as is considered in [15]. This regularisation scheme still creates problems though. First of all when these PV fields came in external lines there were divergences that even if discarded assuming them non-physical, caused overlapping divergences at higher loops containing these diagrams [17]. Moreover, even though this problem was solved by Bakeyev and Slavnov in [14], the method was not straightforwardly applicable to the RG equation approach.

A first attempt of overcoming this problem was presented in [6, 7], where a gauge invariant flow equation for a free Yang-Mills (YM) theory, regulated with higher covariant derivatives and PV fields, has already been studied and the 1-loop β-function for $SU(N)$ YM at $N = \infty$, has been calculated for the first time without fixing the gauge. The work was based on insisting that the regularisation respected the flow, adding higher order interactions for the PV fields (instead of adding them just as mass terms), and with the aid of an auxiliary scalar field. The regularisation was only valid for 1-loop diagrams and at $N = \infty$ and it could not allow to perform calculations beyond this order. On the way of doing this, it appeared clear that all the right content of fields was contained in a bigger group, called $SU(N|N)$, which in its bosonic sector, contains SU(N)⊗SU(N)⊗U(1). Through a Higgs-type mechanism of spontaneous symmetry breaking through an auxiliary scalar field, one of the two $SU(N)$ sectors results in a YM theory gauge invariantly regulated by a naturally combined action of higher covariant derivatives and PV fields. This group and its application for the present purposes will be better described in the next section.

3.3 Regulating via $SU(N|N)$ Gauge Theory

3.3.1 $SU(N|N)$ superalgebra

Since we will have to deal with a $SU(N|N)$ gauge theory, it is worth spending few words on $SU(N|N)$ group and related algebra. $SU(N|N)$ is a graded Lie group, whose elements U an be represented in the exponentiated form as:

$$\mathcal{U} = \exp(i\mathcal{H}) \tag{3.10}$$

The set of elements $\mathcal{H}$ belong to the corresponding Lie superalgebra $SU(N|N)$. An element of the superalgebra, can be represented with a $2N \times 2N$ Hermitian matrix $\mathcal{H}$:

$$\mathcal{H} = \begin{pmatrix} H_N^1 & \theta \\ \theta^\dagger & H_N^2 \end{pmatrix} \in SU(N|N) \tag{3.11}$$

The two H_N^i are Hermitian $N \times N$ matrices whose elements are bosonic complex numbers (commuting i.e. ordinary $\mathbb{C}$), and θ is an $N \times N$ matrix filled up with anticommuting fermionic (Grassmann) numbers. A matrix such as the one described in (3.11) belongs to the algebra $SU(N|N)$ if it satisfies the additional requirement of being "supertraceless":

$$\mathrm{str}\mathcal{H} = \mathrm{trH}_N^1 - \mathrm{trH}_N^2 = 0 \tag{3.12}$$

The *Supertrace*, defined in (3.12), is the natural replacement of the trace for ordinary matrices. It is in fact cyclically invariant because it compensates the sign picked up by commuting the Grassmann components:

$$\mathrm{str}XY = \mathrm{str}YX \tag{3.13}$$

where X and Y are two general supermatrices. In this way the supertrace of commutators vanishes, and makes it invariant under the adjoint action of the group. Once the matrix σ_3 is defined:

$$\sigma_3 = \begin{pmatrix} \mathbb{1}_N & 0 \\ 0 & -\mathbb{1}_N \end{pmatrix} \tag{3.14}$$

where $\mathbb{1}$ is the $N \times N$ identity matrix, the supertrace of a matrix $\mathcal{H}$ can be rewritten in terms of it as

$$\mathrm{str}(\mathcal{H}) = \mathrm{tr}(\sigma_3 \mathcal{H}). \tag{3.15}$$

The request of being supertraceless for elements of $SU(N|N)$ is the natural extension of the request on the elements of the ordinary $SU(N)$ algebra: it guarantees that $\mathcal{U}$ in eq.(3.10) has unit superdeterminant. The supertraceful matrix σ_3 generates a $U(1)$ group absent from $SU(N|N)$. This $U(1)$ group though is not orthogonal to $SU(N|N)$ because being S_α a generic generator of $SU(N|N)$, $\mathrm{str}(\sigma_3 S_\alpha)$ can be non zero in the case of the identity. Moreover, even though σ_3 commutes with all the bosonic generators of $SU(N|N)$, it does not commute with all the fermionic ones (unlike the case of $SU(N)$ with the $U(1)$ generated by the traceful identity). This confers to $SU(N|N)$ a different character, which will be used in the symmetry breaking mechanism described later. The bosonic subalgebra of $SU(N|N)$ is, as we have anticipated, $SU(N)_1 \times SU(N)_2 \times U(1)$ the latter being the subgroup generated by the unity matrix (which, since supertraceless, belongs to the algebra).

We will consider the generators to be Hermitian matrices with complex number entries. The superalgebra will be then defined through a set of commutation and anticommutation rules (the Grassmann character will be carried by the coefficients). Let us consider an element of the Lie algebra as a linear combination of the generators:

$$H = S_\alpha \; H^\alpha \tag{3.16}$$

where the S_α's are the generators:

$$S_\alpha = \begin{cases} \mathbb{1}_{2N} & \alpha = 0 \\ B_a & \alpha = 1, \ldots, 2N^2 - 2 \\ F_a & \alpha = 2N^2 - 1, \ldots, 4N^2 - 2 \end{cases} \tag{3.17}$$

$\mathbb{1}_{2N}$ is the $2N \times 2N$ identity matrix, B_a are the $2N^2 - 2$ block diagonal traceless and supertraceless generators (along the directions of the bosonic components) and the F_α's are the $2N^2$ off-diagonal generators (fermionic components). The commutation and anticommutation rules which define the algebra are:

$$\begin{aligned} &1)\ [B_a, B_b] = \mathcal{F}_{ab}{}^c\ B_c \\ &2)\ [B_a, F_b] = \mathcal{G}_{ab}{}^c\ F_c \\ &3)\ \{F_a, F_b\} = \mathcal{D}_{ab}{}^c\ B_c + \mathcal{H}_{ab}\ \mathbb{1} \\ &4)\ [\mathbb{1}, *] = 0; \\ &\quad \text{(where ' } * \text{ ' stands for any element)} \end{aligned} \tag{3.18}$$

All the generators are matrices with commuting numbers as entries, being the Grassmann character carried by the parameters. Here, $\mathcal{F}_{ab}{}^c$, $\mathcal{G}_{ab}{}^c$, $\mathcal{D}_{ab}{}^c$ and $\mathcal{H}_{ab}$ are coefficients which define the algebra $SU(N|N)$. Since one can get anything from first principles by using the fact that the generators S_α span the space of Hermitian matrices, it is not important to specify them here. However to be more clear, a specific choice of a basis is considered in Appendix A, in order to write the relations of eq. (3.18) all in terms of the structure constants of $SU(N)$, f and d.

It is useful for future reference to list also the commutation and aticommutation relations of the generators of $SU(N|N)$ with the generator of the $U(1)$, σ_3 defined in

eq.(3.14):

$$\begin{aligned} &5)\ [\sigma_3, B_a] = \{\sigma_3, F_a\} = 0 \\ &6)\ [\sigma_3, F_a] = \mathcal{G}_{3a}{}^{b}\ F_b \end{aligned} \tag{3.19}$$

First let us split the generators as $S_\alpha \equiv (\mathbb{1}, T_A)$[1]. It is now useful to define the Killing super-metric as:

$$h_{\alpha\beta} = 2\ \mathrm{str}(S_\alpha S_\beta) \tag{3.20}$$

$h_{\alpha\beta}$ is symmetric when either index is bosonic and antisymmetric when they are both fermionic:

$$h_{\alpha\beta} = (-)^{f(\alpha)f(\beta)} h_{\beta\alpha} \tag{3.21}$$

where $f(\alpha)$ is 0 if the index is bosonic and 1 if it is fermionic. The normalisation of the generators is defined via the following form of the metric (where all elements not indicated are zero):

$$h_{\alpha\beta} = \left(\begin{array}{c|c|c|c} 0 & & & \\ \hline & \begin{array}{ccc} 1 & & \\ & 1 & \\ & & \ddots \end{array} & & \\ \hline & & \begin{array}{ccc} -1 & & \\ & -1 & \\ & & \ddots \end{array} & \\ \hline & & & \begin{array}{cccc} 0 & i & & \\ -i & 0 & & \\ & & 0 & i \\ & & -i & 0 \end{array} \ddots \end{array}\right) \tag{3.22}$$

$U(1)$ $SU_1(N)$ $SU_2(N)$ Fermionic

[1]The T_A are the traceless and supertraceless generators and span the same space of matrices as B_a and F_a

This super-metric has no inverse due to the presence of $\mathbb{1}$ in the generators (which gives a column and row of zeros in the matrix above). It is possible to define $SU(N|N)$ consistently with or without the identity matrix, changing the definition of the commutators (see [22]). Here the definition including it will be considered. However, we will see that the gauge theory constructed on this group will decouple the component in this direction. Specialising to just the space including the T^A generators, we can consider the Killing super-metric in this subspace, which is invertible and defined as:

$$g_{AB} = 2\mathrm{str}(T_A T_B) = h_{AB} \tag{3.23}$$

its inverse is defined by

$$g_{AB} g^{BC} = g^{CB} g_{BA} = \delta_A^C \tag{3.24}$$

g_{AB} can be used to lower or raise indices as in

$$X^A = X_B g^{AB} \tag{3.25}$$

Since the ordering of the indices of the super-metric is important (see eq.(3.21)), it is worth commenting that in (3.25) the sum is on the second index and in general:

$$X^A = X_B g^{AB} \neq X_B g^{BA} = (-)^{f(A)f(B)} X_B g^{AB} \tag{3.26}$$

For the generators we have the dual relation given by:

$$T^A = T_B g^{BA} \tag{3.27}$$

Another useful relation which holds for the generators of $SU(N|N)$ is finally the completeness relation:

$$(T^A)^i{}_j (T_A)^k{}_l = \frac{1}{2}\delta^i{}_l (\sigma_3)^k{}_j - \frac{1}{4N}\left[\delta^i{}_j (\sigma_3)^k{}_l + (\sigma_3)^i{}_j \delta^k{}_l\right] \tag{3.28}$$

(see App. B for a derivation.) This is most usefully cast in the following forms

$$\mathrm{str}(XT_A)\,\mathrm{str}(T^AY) = \frac{1}{2}\,\mathrm{str}(XY) - \frac{1}{4N}\left[\mathrm{tr}X\,\mathrm{str}Y + \mathrm{str}X\,\mathrm{tr}Y\right], \tag{3.29}$$

$$\mathrm{str}(T_AXT^AY) = \frac{1}{2}\,\mathrm{str}X\,\mathrm{str}Y - \frac{1}{4N}\mathrm{tr}(XY + YX), \tag{3.30}$$

for arbitrary supermatrices X and Y. Let us now consider the adjoint representation of the group. An element of it can be written as:

$$\mathcal{M} = \mathcal{M}^\alpha S_\alpha = \mathcal{M}^0\ \mathbb{1} + \mathcal{M}^a_B\ B_a + \mathcal{M}^a_F\ F_a \tag{3.31}$$

where the S_α's are $4N^2 - 1$ $(2N \times 2N)$ matrices of $SU(N|N)$. An element of the adjoint transforms, under an infinitesimal transformation of the group $\omega = \omega^A T_A = \omega^a_B\ B_a + \omega^a_F\ F_a$ as follows:

$$\delta\mathcal{M} = -i[\mathcal{M}, \omega]. \tag{3.32}$$

In components, given the commutation and anticommutation rules of the group, it has the following form:

$$\begin{cases} i\delta\mathcal{M}^c_B = \mathcal{M}^a_B\omega^b_B\ \mathcal{F}_{ab}{}^c + \mathcal{M}^b_F\omega^a_F\ \mathcal{D}_{ab}{}^c \\ i\delta\mathcal{M}^c_F = \mathcal{M}^a_B\omega^b_F\ \mathcal{G}_{ab}{}^c + \mathcal{M}^a_F\omega^b_B\ \mathcal{G}_{ba}{}^c \\ i\delta\mathcal{M}^0 = \mathcal{M}^a_F\omega^b_B\ \mathcal{H}_{ab} \end{cases} \tag{3.33}$$

For future reference it can be useful to consider also the $2N \otimes \overline{2N}$ representation[2]. An element of it can be represented as:

$$\mathcal{C} = \mathcal{C}^\alpha S_\alpha + \mathcal{C}^3\sigma_3 \tag{3.34}$$

[2]Unlike $SU(N)$, the group $SU(N|N)$ is indecomposable, thus this representation is not the Adjoint$\oplus$ the singlet

for it also the transformation will be of the form of eq.(3.32) and in components is:

$$\begin{cases} i\delta\mathcal{C}_B^c = \mathcal{C}_B^a\omega_B^b\ \mathcal{F}_{ab}{}^c + \mathcal{C}_F^b\omega_F^a\ \mathcal{D}_{ab}{}^c \\ i\delta\mathcal{C}_F^c = \mathcal{C}_B^a\omega_F^b\ \mathcal{G}_{ab}{}^c + \mathcal{C}_F^a\omega_B^b\ \mathcal{G}_{ba}{}^c + \mathcal{C}^3\omega_F^a\ \mathcal{G}_{3a}{}^c \\ i\delta\mathcal{C}^0 = \mathcal{C}_F^a\omega_B^b\ \mathcal{H}_{ab} \\ i\delta\mathcal{C}^3 = 0 \end{cases} \tag{3.35}$$

The last line shows how the component along σ_3 does not transform under $SU(N|N)$[3].

Having described here the main properties of this graded group and of its Lie algebra, it is now possible to move onto the description of the regularisation scheme adopted making use of it.

3.3.2 Regularisation: Gauge group and Higgs-type mechanism

Instead of working just with the $SU(N)$ gauge field, which we write as $A_\mu^1(x) \equiv A_{a\mu}^1\tau_1^a$, where τ_1^a are the $SU(N)$ generators orthonormalised to $\mathrm{tr}(\tau_1^a\tau_1^b) = \delta^{ab}/2$, we embed it in a $SU(N|N)$ supergauge field [33, 34]:

$$\mathcal{A}_\mu = \mathcal{A}_\mu^0\mathbb{1} + \begin{pmatrix} A_\mu^1 & B_\mu \\ \bar{B}_\mu & A_\mu^2 \end{pmatrix}. \tag{3.36}$$

Here we have written $\mathcal{A}$ as an element of the $SU(N|N)$ Lie superalgebra, using the defining representation, *i.e.* as a supermatrix with bosonic block diagonal terms A^i and fermionic block off-diagonals B and $\bar{B}$, together with the central term $\mathcal{A}^0\mathbb{1}$. As required by $SU(N|N)$, the supermatrix (and thus also $\mathcal{A}$) is supertraceless, *i.e.* $\mathrm{tr}A^1 - \mathrm{tr}A^2 = 0$. This excludes in particular σ_3, defined in eq.(3.14), from the Lie

[3]However, σ_3 is not a singlet of $SU(N|N)$ though (as $\mathbb{1}$ is for $SU(N)$), since it takes part in the transformations of the other components

algebra. From now on we will write simply $\sigma = \sigma_3$. The supermatrix is in addition also traceless, the trace having been parametrised by $\mathcal{A}^0$. Equivalently, as we have seen in the previous section, we can introduce a complete set of traceless and supertraceless generators T_A (normalised as in eq. (3.22)) and thus expand $\mathcal{A}$ as

$$\mathcal{A}_\mu = \mathcal{A}^0_\mu \mathbb{1} + \mathcal{A}^A_\mu T_A. \tag{3.37}$$

The B fields are wrong statistics gauge fields. They will be given a mass of order the cutoff Λ. The supergroup $SU(N|N)$ has $SU(N) \times SU(N) \times U(1)$ as its bosonic subgroup. $A^2_\mu(x) \equiv A^2_{a\mu}\tau^a_2$ is the gauge field for the second $SU(N)$, and $\mathcal{A}^0$ is the $U(1)$ connection. Interactions are built via commutators, using the covariant derivative:

$$\nabla_\mu = \partial_\mu - i\mathcal{A}_\mu, \tag{3.38}$$

The coupling constant g does not appear in the definition of the covariant derivative, as it usually does, because it is considered scaled out. This rescaling of the fields is proved to be useful and a more detailed discussion about this issue is presented in section 4.4.1 Thus the superfield strength is given by $\mathcal{F}_{\mu\nu} = i[\nabla_\mu, \nabla_\nu]$. The kinetic term will be regularised by higher derivatives which thus take the form:

$$\text{str } \mathcal{F}_{\mu\nu} \left(\frac{\nabla}{\Lambda}\right)^n \cdot \mathcal{F}_{\mu\nu}, \tag{3.39}$$

(where the dot means ∇ acts by commutation. In practice we will add the higher derivatives as a power series with coefficients determined by the cutoff function c). The supertrace, which, from the discussion around (3.13), is necessary to ensure $SU(N|N)$ invariance, forces the kinetic term for A^2 to have wrong sign action, leading to negative norms in its Fock space [34].

As can be seen from eq. (3.37), $\mathcal{A}^0$ does not appear in the kinetic term. Providing the interactions can be written as str($\mathcal{A} \times$ commutators), $\mathcal{A}^0$ will not appear anywhere in

the action. More generally we will need to impose its non-appearance as a constraint, since otherwise $\mathcal{A}^0$ has interactions but no kinetic term and thus acts as a Lagrange multiplier resulting in a non-linear constraint on the theory, which does not look promising for its use as a regularisation method for the original $SU(N)$ Yang-Mills.

On the other hand, if the constraint is satisfied, $\mathcal{A}^0$ is then protected from appearing by a local "no-$\mathcal{A}^0$" shift symmetry: $\delta\mathcal{A}^0_\mu(x) = \Lambda_\mu(x)$, which implies in particular that $\mathcal{A}^0$ has no degrees of freedom. Together with supergauge invariance the theory is then invariant under

$$\delta\mathcal{A}_\mu = \nabla_\mu \cdot \omega + \Lambda_\mu \mathbb{1}. \tag{3.40}$$

The effect of the no-$\mathcal{A}^0$ symmetry is to dynamically define the gauge group as the quotient $SU'(N|N) = SU(N|N)/U(1)$, in which Lie group elements are identified modulo addition of an arbitrary multiple of $\mathbb{1}$.

An alternative and equivalent formulation [34] is to pick coset representatives, which can for example be taken to be traceless, so that $\mathcal{A}^0$ is set to zero, and thus discarded. (This is the strategy used in ref. [28] to define a $SU'(N|N)$ sigma model. Incidentally this paper contains arguments for finiteness of these models which are similar to those given for $SU(N|N)$ gauge theory in [34].[4]) In this reduced representation, eq. (3.40) is replaced by Bars' solution [22]:

$$\delta\mathcal{A}_\mu = [\nabla_\mu, \omega]^* \equiv [\nabla_\mu, \omega] - \frac{\mathbb{1}}{2N}\mathrm{tr}[\nabla_\mu, \omega]. \tag{3.41}$$

The *bracket replaces the commutator as a representation of the Lie product so in particular $\mathcal{F}_{\mu\nu} = i[\nabla_\mu, \nabla_\nu]^*$ [34].

The lowest dimension interaction that violates no-$\mathcal{A}^0$ symmetry contains four superfield strengths, for example:

$$\mathrm{str}\ (\mathcal{F}_{\mu\nu})^2(\mathcal{F}_{\lambda\sigma})^2\,. \tag{3.42}$$

[4]We thank Hugh Osborn for drawing our attention to this paper

Such terms are not invariant under the 'Bars*' eq. (3.41), either. Since eq. (3.42) is already irrelevant, no-$\mathcal{A}^0$ symmetry is automatic for the conventional supergauge invariant bare action of ref. [34]. Here there is no such bare action, and interactions are generated by a largely unspecified exact RG, so we need to impose no-$\mathcal{A}^0$ as an extra constraint.

We introduce a superscalar field

$$\mathcal{C} = \begin{pmatrix} C^1 & D \\ \bar{D} & C^2 \end{pmatrix} \tag{3.43}$$

in the fundamental $\otimes$ its complex conjugate representation, equivalently as a matrix in the defining representation of $U(N|N)$ [34]. Under supergauge transformations

$$\delta\mathcal{C} = -i\,[\mathcal{C}, \omega]. \tag{3.44}$$

In the Bars* representation we do not replace this by a *bracket, since commutators are necessary for powers of $\mathcal{C}$ (appearing in its potential) to transform covariantly [34]. However, as in ref. [34], since working with the full cosets seems more elegant, we will employ eq. (3.40) and the full representation in this thesis.

We will arrange for $\mathcal{C}$ to develop a vacuum expectation value along the σ direction through an appropriate Higgs-type potential, so that classically $< \mathcal{C} > = \Lambda\sigma$.[5] This spontaneously breaks $SU(N|N)$ down to its $SU(N)\times SU(N)\times U(1)$ bosonic subgroup and provides the fermionic fields B and D with masses of order Λ. In usual unitary gauge interpretation, D is the would be Goldstone mode eaten by B. However, since we will not gauge fix, they instead gauge transform into each other and propagate as a composite unit (see Appendix D). The reason why the fermionic components of the $\mathcal{A}$ super-gauge field (the B's) get a mass, is the Higgs mechanism, being them the

[5] Later however we will use an unconventional normalisation for $\mathcal{C}$.

fields along the broken gauge generators. The D field (fermionic component of the super-Higgs field $\mathcal{C}$), being the component of the Higgs along the direction where the symmetry is broken, is the Goldstone boson and its kinetic term vanishes at $p = 0$, as it is stated by the choice made in eq.(5.15). However, since the fields B and D are coupled, if we diagonalise their kinetic terms, we can indeed notice, that the diagonalised mass matrix describes B and D as two massive particles with masses of order Λ. Moreover since the physical mass of a particle corresponds to the pole in its propagator written in the Minkowski space, D can be regarded as a massive field, since its zero point wine (*i.e.* effective propagator, see App. D) does not have a massless pole, as one can notice from eq. (D.18) (the only field which does is the bosonic gauge field A). In fact, the coupled fields B and D, have decoupled effective propagators or, in other words, the coupled two-point functions BB and DD and the cross term kinetic term $BD\sigma$, have uncoupled inverses in the transverse space.

Finally, we arrange for the remaining 'Higgs' fields C^i also to have masses of order Λ. This is done here by the choice made in eq.(5.14). The two point C vertex is chosen to be non vanishing at $p = 0$, and the coefficient is chosen positive ($\lambda > 0$), so that it is a mass term.

All the information that was encoded in the regularisation scheme for the gauge invariant effective action of ref. [34], will be here mainly contained in the choice of the two-point functions, eqs.(5.12)-(5.16)

In ref. [34], it was proved by conventional methods that if the kinetic term of $\mathcal{A}$ is supplied with covariant higher derivatives (parametrised by the cutoff function c) enhancing its high momentum behaviour by a factor $c^{-1} \sim p^{2r}/\Lambda^{2r}$, and the kinetic term of $\mathcal{C}$ has its high momentum behaviour similarly enhanced by $\tilde{c}^{-1} \sim p^{2\tilde{r}}/\Lambda^{2\tilde{r}}$, then providing

$$r - \tilde{r} > 1 \quad \text{and} \quad \tilde{r} > 1, \tag{3.45}$$

all amplitudes are ultraviolet finite to all orders of perturbation theory. Since the underlying theory is renormalisable, the Appelquist-Carazzone theorem implies that at energies much lower than the cutoff Λ, the remaining massless fields A^1 and A^2 decouple. In this way, this framework was used as a regularisation of the original $SU(N)$ Yang-Mills theory carried by A^1.

In brief, the reasons for the above facts are as follows. Providing eqs. eq. (3.45) hold, all divergences are superficially regularised by the covariant higher derivatives, except for some 'remainders' of one-loop graphs with only $\mathcal{A}$ fields as external legs and only four or less of these legs. These remainders form a symmetric phase contribution, in the sense that the superficially divergent interactions between $\mathcal{C}$ and $\mathcal{A}$ are just those that come from $\mathcal{C}$'s covariant higher derivative kinetic term, whilst all terms containing a σ from the breaking are already ultraviolet finite by power counting. For three or less external $\mathcal{A}$ legs the remainders vanish by the supertrace mechanism: the fact that in the unbroken theory, the resultant terms contain $\mathrm{str}\mathcal{A} = 0$ or $\mathrm{str}\mathbb{1} = 0$. By manifest gauge invariance, the four point $\mathcal{A}$ remainder is then actually totally transverse, which implies that it is already finite by power counting.

The decoupling of A^1 and A^2 follows from the unbroken local $SU(N) \times SU(N)$ invariance since the lowest dimension effective interaction

$$\frac{1}{\Lambda^4} \mathrm{tr} \left(F^1_{\mu\nu}\right)^2 \mathrm{tr} \left(F^2_{\mu\nu}\right)^2 \tag{3.46}$$

is already irrelevant [34, 19].

Actually, there are a number of differences between the treatment we give here and that of ref. [34]. Since ref. [34] followed a conventional treatment, gauge fixing and ghosts were introduced, with a corresponding higher derivative regularisation for them; longitudinal parts of the four point $\mathcal{A}$ vertex were then related to ghost vertices using the Lee Zinn-Justin identities, which were separately proved to be finite. Also,

a specific form of bare action and covariantisation was chosen.

Here we do not fix the gauge and the regularisation scheme is much more general. As well as not specifying the covariantisation or the bare action (see below) there is anyway much more freedom in introducing interactions via the flow equation. We shall not here supply a rigorous proof that up to appropriate restrictions, the flow equation is finite. Since we never have to specify the details, we only need to *assume* that this is true for at least one choice. However, we take care that the scheme as described above is qualitatively correctly implemented. Where we do have to explicitly compare terms we can use eq. (3.45) as a guide, although it should be borne in mind that cutoff functions with non-power law asymptotics, for example exponential, could also be used.[6] In practice, it is easy to see at one loop that the high energy cancellations are occurring as expected.

In this scheme, higher covariant derivative and P-V fields come out naturally combined together and $SU(N|N)$ is proved to be a finite theory [34]. This results in a $SU(N)$ gauge invariantly regulated theory suitable for a RG flow equation approach. The purpose of the present thesis is to check the consistency of this statement, writing a flow equation for the theory and calculating universal quantities such as the β function at one loop as a check.

[6]The proof given in ref. [34] could also be easily extended to these cases.

Chapter 4

$SU(N|N)$ flow equation

In order to be able to construct a flow equation for $SU(N|N)$, we have to recall a set of properties that such an equation must have, in order to lead the right physical interpretation. Some of these requirements are more general, and are related to the structure which a flow equation must describe. Others are due to the symmetries which must be preserved through the flow, in the present case $SU(N|N)$ gauge symmetry.

Before we continue it is also necessary to add some more preliminary comments. As we have mentioned already, throughout this thesis we work in Euclidean space of dimension D. We could formulate everything directly in dimension $D = 4$ as in [36], even though strictly speaking the limit $D \to 4$ is necessary to rigorously define the $SU(N|N)$ regularisation [34]. However, here we want to show, for the calculation of terms such as the one-loop β function in $SU(N)$ Yang-Mills, we do not need to pay attention to this subtlety, and we will then keep a general dimension D until the very end.

4.1 Necessary properties of the exact RG and their interpretation

The extra fields we have added form a necessary part of the regularisation structure. We gain an interpretation of these fields at the effective level by imagining integrating out the heavy fields B, C and D at some scale Λ. The result is an effective action containing only the unbroken gauge fields A^i, but it is not finite. In particular, the one-loop determinant formed from integrating out the heavy fields is necessarily divergent: the divergences are there to cancel those left by the one-loop hole in the remaining covariant higher derivative regularisation [16] of the $SU(N) \times SU(N)$ Yang Mills theory, in a similar way to that done explicitly in gauge invariant Pauli-Villars regularisation [15].

A gauge invariant exact RG description of gauge theory thus requires not only an effective action but a separate measure term, here provided by the above functional determinant. The measure term is not itself finite, but can be represented by a finite addition to the effective action, after introducing auxiliary fields (here B, C and D).

Whilst this interpretation is reasonable, similarly to the scalar field case, we need to be sure that we are still only representing the original quantum field theory (here $SU(N)$ Yang-Mills). In the previous chapter this was ensured by asking the "seed action" vertices neither to lead to UV divergent integrals nor to have IR divergences (Taylor expandable to all orders). In the present case, this demand is especially pertinent in (but not restricted to) the case where there are extra regulator fields, particularly here A^2 which remains massless and in this effective description only decouples at momenta much less than Λ. More generally, even if there are only physical fields in the effective action, we need to be sure that locality, an important property of quantum field theory [20, 21], is properly incorporated.[1] Note that Λ is intended to

[1] otherwise non-physical effects or other propagating fields, could be hidden in the vertices.

be set at the energy scales of interest, which is why it makes sense to use the exact RG and solve for the effective action directly in renormalised terms, see *e.g.* [9]. Indeed, to extract the physics (*e.g.* correlation functions *etc.*) we will even want to take $\Lambda \to 0$ eventually ([9]).

These demands are fulfilled implicitly through the $\Lambda \to \infty$ limit, providing some very general requirements on the exact RG are implemented, as we now explain.

Firstly, we require that all parts of the flow equation can be expanded in external momenta to any order, so that the solutions S can also be required to have an all orders derivative expansion [6, 7, 9].[2] This 'quasilocality' requirement [6] is equivalent to the fundamental requirement of the Wilsonian RG that Kadanoff blocking take place only over a localised patch [1], *i.e.* here that each RG step, $\Lambda \mapsto \Lambda - \delta\Lambda$, be free from infrared singularities.

The flow equation is written only in terms of renormalised quantities at scale Λ. In fact, we require that the only explicit scale parameter that appears in the equations is the effective cutoff Λ. Again this is so that the same can be required of S where it implements the concept of self-similar flow [29]. Here this amounts to a non-perturbative statement of renormalisability, *i.e.* existence of a continuum limit, equivalent to the requirement that S lie on a renormalised trajectory [9]. This is clearer if we first scale to dimensionless quantities using the appropriate powers of Λ. Then, S is required to have no dependence on Λ at all except through its dependence on the running coupling(s) $g(\Lambda)$ [9].

Note that the $\Lambda \to \infty$ end of the renormalised trajectory, *i.e.* the perfect action [30] in the neighbourhood of the ultraviolet fixed point at $\Lambda = \infty$, amounts to our choice of bare action. Its precise form is not determined beforehand but as a result of solution of the exact RG, but it is constrained by choices in the flow equation. Since these

[2]Sharp cutoff realisations [3] are more subtle [8, 10, 12] and will not be discussed here.

choices are however here to a large extent unmade, we deal with an infinite class of perfect bare actions.

Moreover, we require that the flow of the Boltzmann measure $\exp(-S)$ is a total functional derivative, as we discussed below eq. (2.13). As we have seen, importantly, this ensures that the partition function $\mathcal{Z} = \int \mathcal{D}\phi \exp(-S)$, and hence the physics derived from it, is invariant under the RG flow. Since we will solve the exact RG approximately, but by controlled expansion in a small quantity, this property is left undisturbed. Therefore we may use different scales Λ at our convenience to interpret the computation.

For example, although locality is obscured in the Wilsonian effective action at any finite Λ, it is important to recognise that invariance of $\mathcal{Z}$ together with the existence of a derivative expansion and self-similar flow (*viz.* that the only explicit scale be Λ), ensure that locality is implemented, since it is then an automatic property of the effective action as $\Lambda \to \infty$.

Similarly, it is as $\Lambda \to \infty$ that we confirm from the Wilsonian effective action that we are describing $SU(N)$ Yang-Mills theory: B, C and D really are infinitely massive, and in spacetime dimension four or less, A^2 is guaranteed decoupled by the Appelquist-Carazzone theorem and eq. (3.46). In general strong quantum corrections might alter either of these properties. Thus in general we would need to add appropriate sources to the $\Lambda \to \infty$ action; compute the partition function by computing the $\Lambda \to 0$ limit of $\exp(-S)$; and finally explicitly test these properties by computing appropriate correlators. (This is the most general way to extract the results for physical quantities from S.) However since g is perturbative at high energies (indeed $g \to 0$ as $\Lambda \to \infty$), we can be sure that the above deductions about the regulator fields, drawn at the perturbative level, are not destroyed by quantum corrections.

As already mentioned, we require that an ultraviolet regularisation at Λ, is imple-

mented so that the right hand side of the flow equation makes sense. Note that this ensures that all further quantum corrections to S (computed by solving for the flow at scales less than Λ) are cutoff (smoothly) at Λ. Since momentum modes $p > \Lambda$ were fully contributing to the initial $\Lambda \to \infty$ partition function, and since $\mathcal{Z}$ is invariant under the flow, we can be sure that their effect has been incorporated S. In other words we can be sure that our final requirement on the flow, namely that it corresponds to integrating out momentum modes, has been incorporated.

(In refs. [5, 6], a possible further requirement on the flow equation, called "ultralocality" was discussed, replacing the usual notion of locality, although it was not clear that it was necessary however. We have seen here that the usual concept of locality is recovered providing the existence of a derivative expansion, invariance of $\mathcal{Z}$, and self-similar flow, are implemented. Furthermore the successful calculations of ref. [35] and here, confirm that the restriction of 'ultralocality' is unnecessary since they do not assume it.)

4.2 Supergauge invariance and functional derivatives

The requirements we have mentioned in the previous section are necessary for a general flow equation. However, since we are dealing with a particular theory we have to consider some additional ones. The peculiarities of $SU(N|N)$, in fact, affect functional derivatives with respect to $\mathcal{A}$ and lead to some constraints on the form of the exact RG if the flow equation is to be invariant under supergauge transformations.

As in refs. [7, 34], it is convenient to define the functional derivatives of $\mathcal{C}$ and $\mathcal{A}$ so as to extract the dual from under the supertrace. For an unconstrained field such as

$\mathcal{C}$ we simply have [7, 34]:

$$\frac{\delta}{\delta\mathcal{C}} := \begin{pmatrix} \delta/\delta C^1 & -\delta/\delta\bar{D} \\ \delta/\delta D & -\delta/\delta C^2 \end{pmatrix}, \tag{4.1}$$

or in components

$$\frac{\delta}{\delta\mathcal{C}}^{i}_{\ j} := \frac{\delta}{\delta\mathcal{C}^{k}_{\ i}}\sigma^{k}_{\ j}. \tag{4.2}$$

Under supergauge transformations eq. (3.44), the functional derivative transforms as one would hope:

$$\delta\left(\frac{\delta}{\delta\mathcal{C}}\right) = -i\left[\frac{\delta}{\delta\mathcal{C}}, \omega\right]. \tag{4.3}$$

Such a derivative[3] has the properties of 'supersowing' [7]:

$$\frac{\partial}{\partial\mathcal{C}}\,\mathrm{str}\,\mathcal{C}Y = Y \quad \Longrightarrow \quad \mathrm{str}X\frac{\partial}{\partial\mathcal{C}}\,\mathrm{str}\,\mathcal{C}Y = \mathrm{str}XY, \tag{4.4}$$

and 'supersplitting' [7]:

$$\mathrm{str}\frac{\partial}{\partial\mathcal{C}}X\mathcal{C}Y = \mathrm{str}X\mathrm{str}Y, \tag{4.5}$$

i.e. of sowing two supertraces together, and splitting one supertrace into two, where X and Y are arbitrary supermatrices. These two properties come directly from the completeness relation for the generators of the group $U(N/N)$ (see eq. (3.28) and below, for the case of $SU(N|N)$ without $\mathbb{1}$).

(*N.B.* it is a helpful trick to contract in arbitrary supermatrices at intermediate stages of the calculation: it allows index-free calculations in the $SU(N|N)$ algebra and more importantly means that we can permute overall bosonic structures past each rather than have to carry intermediate minus signs from fermionic parts of supermatrices anticommuted through each other. Its efficacy will be seen in examples later. It also leads as we will show, to efficient diagrammatic techniques. The arbitrary superma-

[3]for simplicity, written with partial derivatives, to neglect the irrelevant spatial dependence

trices can always be stripped off at the end, if necessary.)

Since $\mathcal{A}$ is constrained to be supertraceless, its dual under the supertrace $\mathrm{str}\mathcal{J}_\mu\mathcal{A}_\mu$ has without loss of generality no $\mathbb{1}\!\mathrm{l}$ component: only

$$\mathcal{J}_\mu - \frac{\mathbb{1}\!\mathrm{l}}{2N}\mathrm{tr}\mathcal{J}_\mu \tag{4.6}$$

really couples. The natural construction for the $\mathcal{A}$ functional derivative from eq. (3.37) [34]:

$$\frac{\delta}{\delta\mathcal{A}_\mu} := 2T_A\frac{\delta}{\delta\mathcal{A}_{A\,\mu}} + \frac{\sigma}{2N}\frac{\delta}{\delta\mathcal{A}^0_\mu} \tag{4.7}$$

pulls out precisely this combination. However from eq. (3.40) and the completeness relations for the T_A (3.28), under supergauge transformations

$$\begin{aligned}\delta\left(\frac{\delta}{\delta\mathcal{A}_\mu}\right) &= -i\left[\frac{\delta}{\delta\mathcal{A}_\mu},\omega\right] + \frac{i\mathbb{1}\!\mathrm{l}}{2N}\mathrm{tr}\left[\frac{\delta}{\delta\mathcal{A}_\mu},\omega\right] \\ &= -i\left[\frac{\delta}{\delta\mathcal{A}_\mu},\omega\right]^*.\end{aligned} \tag{4.8}$$

The correction is to be expected since it ensures that $\delta/\delta\mathcal{A}$ remains traceless, but the fact that $\delta/\delta\mathcal{A}$ does not transform homogeneously means that supergauge invariance is destroyed unless $\delta/\delta\mathcal{A}$ is contracted under the supertrace into something that is supertraceless (in which case the correction term vanishes). This is an extra constraint on the form of the flow equation.

[As an alternative one might try defining $\delta/\delta\mathcal{A}$ as only the $2T_A\delta/\delta\mathcal{A}_A$ term in eq. (4.7), however one can show from eq. (3.40) that this does not transform into itself but into the full functional derivative given in eq. (4.7). It works however in the Bars* representation, where the transformation again takes the form eq. (4.8).]

Similarly there are corrections to eq. (4.4) and eq. (4.5) that arise because the deriva-

tive is constrained:[4]

$$\mathrm{str}X\frac{\partial}{\partial\mathcal{A}}\,\mathrm{str}\,\mathcal{A}Y = \mathrm{str}XY - \frac{1}{2N}\,\mathrm{str}X\mathrm{tr}Y \tag{4.9}$$

as expected from eq. (4.6), and

$$\mathrm{str}\frac{\partial}{\partial\mathcal{A}}X\mathcal{A}Y = \mathrm{str}X\mathrm{str}Y - \frac{1}{2N}\,\mathrm{tr}\,YX. \tag{4.10}$$

These come directly from the completeness relation for $SU(N|N)$ and are a way to rephrase Eqs. (3.29) and (3.30), from the previous chapter. Since these corrections contain $\mathrm{tr}Z \equiv \mathrm{str}\,\sigma Z$ (where Z is some supermatrix), they similarly violate $SU(N|N)$ invariance. As we discuss in sec. 4.5.1, they also effectively disappear with the above constraint that $\delta/\delta\mathcal{A}$ is contracted into something supertraceless. (This is obvious in eq. (4.9) where thus $\mathrm{str}X = 0$.)

In this way the supersplitting and supersowing rules actually become exact for both fields, even at finite N (compare [6, 7]). As we will see, this leads to a very efficient diagrammatic technique incorporated into the Feynman diagrams, for evaluating the gauge algebra, analogous to the 't Hooft double line notation [31] and utilised earlier [5, 6, 7], but here applying even at finite N.

4.3 Covariantisation

Since we want to build a flow equation which is invariant under supergauge transformations, we need to have covariant generalisations of the momentum space kernels appearing in other ERG equations' formulations *e.g.* in the scalar field case described in the first chapter. In that case they were present in the flow equation as a result

[4] ignoring the spacetime index and spatial dependence

of the regularisation scheme, which there did not have any particular invariance-preserving prescriptions, being just the first derivatives of the cutoff function. In the present formulation, making use of the freedom allowed by scheme independence, the flow equation will be written incorporating covariantised versions of these objects, which we are going to describe in the present section. These covariantised momentum space kernels, will be then related back to the gauge invariant regularisation scheme of [33, 34]. Their introduction will involve more terms in the flow equation, but will insure it describes a gauge invariant flowing effective action.

Given some momentum space kernel $W_p \equiv W(p,\Lambda)$ as the one defined in eq. (2.10) and below, we define a general covariantisation of any such kernel (the 'wine' [6, 7]) via the supergauge invariant:

$$
\begin{aligned}
& u\,\{W\}_{\mathcal{A}}\,v = \qquad\qquad (4.11)\\
& \sum_{m,n=0}^{\infty} \int d^D x\, d^D y\, d^D x_1 \cdots d^D x_n\, d^D y_1 \cdots d^D y_m\, W_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m}(x_1,\cdots,x_n;y_1,\cdots,y_m;x,y) \\
& \qquad\qquad \mathrm{str}\,[\,u(x)\,\mathcal{A}_{\mu_1}(x_1)\cdots\mathcal{A}_{\mu_n}(x_n)\,v(y)\,\mathcal{A}_{\nu_1}(y_1)\cdots\mathcal{A}_{\nu_m}(y_m)\,]\,, \\
& \quad = \int d^D x\; d^D y\; [\sigma_3 u(x)]^{l\;\;i}_{\mathbf{x}l}\{W\}^{k}_{j\mathbf{y}} v(y)^{j}_{k}
\end{aligned}
$$

where u and v are any two supermatrix representations, and with the symbol W is introduced the *wine* (Wilson-line) as in [6, 7], the Wilson line implementing the parallel transport between the two representations (this will be seen more clearly in (4.17)). A graphical representation of it is shown in fig. 4.1. As we can notice from eq.(4.11), the wine is expandable in fields. Its expansion in terms of $\mathcal{A}$ fields is

$$
\begin{aligned}
{}^{i}_{\mathbf{x}l}\{W\}^{k}_{j\mathbf{y}} = \; & (-)^h \sum_{n,m=0}^{\infty} \int d^D x_1 \cdots d^D x_n\, d^D y_1 \cdots d^D y_m\, W_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m}(x_i;y_j;x,y) \\
& [\mathcal{A}_{\mu_1}(x_1)\cdots\mathcal{A}_{\mu_n}(x_n)]^{i}{}_{j}[\mathcal{A}_{\nu_1}(y_1)\cdots\mathcal{A}_{\nu_m}(y_m)]^{k}_{l} \qquad (4.12)
\end{aligned}
$$

where $h = f(\alpha)\sum_i f(\alpha_i)$, f is defined below eq. (3.21) and where the indices α and

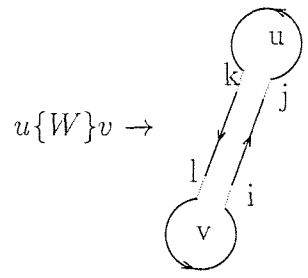

Figure 4.1: Parallel transport between two matrix representation through the wine, eq.(4.11). There is no explicit representation of the σ_3 because, it is incorporated in the closed line which defines already a supertrace.

α_i refer to those in the expansions: $v = v^\alpha S_\alpha$ and $\mathcal{A}^{\alpha_i}_{\nu_i} S_{\alpha_i}$.

A graphic representation of this expansion is shown in fig.4.2. In order to explain

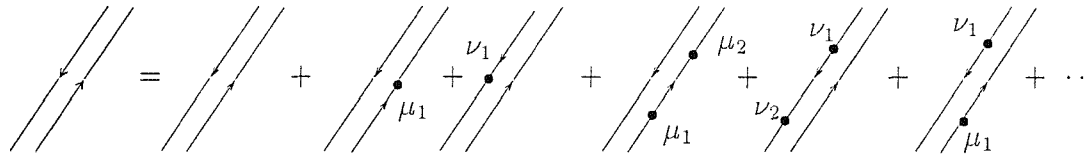

Figure 4.2: Wine expansion. The blobs represent $\mathcal{A}$ fields.

better the graphical notation, the first of the terms with one blob is represented in formula by

$$\int d^D x_1 \underbrace{W_{\mu_1,}(x_1;;x,y)}_{W_{\mu_1}(x_1;x,y)} [\mathcal{A}_{\mu_1}(x_1)]^i{}_j. \tag{4.13}$$

The Feynman rules in the momentum space for a general wine vertex are explained in fig.4.3.

Without loss of generality we may insist that $\{W\}_{\mathcal{A}}$ satisfies $u\{W\}_{\mathcal{A}} v \equiv v\{W\}_{\mathcal{A}} u$. We write the $m = 0$ vertices (where there is no second product of gauge fields), more compactly as

$$W_{\mu_1 \cdots \mu_n}(x_1, \cdots, x_n; x, y) \equiv W_{\mu_1 \cdots \mu_n,}(x_1, \cdots, x_n;; x, y), \tag{4.14}$$

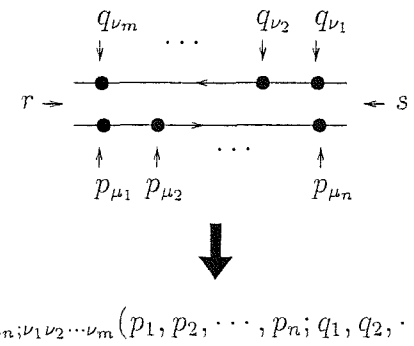

Figure 4.3: Feynman rules for wine vertices

while the $m = n = 0$ term is just the original kernel appearing in eq. (2.10) and below, *i.e.*

$$W_{,}(;;x,y) \equiv W_{xy}. \tag{4.15}$$

We leave the covariantization general, up to certain restrictions. One of these is already encoded into eq. (4.11), namely that there is just a single supertrace in eq. (4.11), involving just two ordered products of supergauge fields. Another is that we require that the covariantization satisfy coincident line identities [6] which in particular imply that if $v(y) = \mathbb{1}g(y)$ for all y, *i.e.* is in the scalar representation of the gauge group, then the covariantization collapses to

$$u\left\{W\right\}_{\mathcal{A}} v = (\operatorname{str} u) \cdot W \cdot g. \tag{4.16}$$

As shown in ref. [7], the coincident line identities are equivalent to the requirement that the gauge fields in eq. (4.11) all act by commutation. This requirement is necessary to ensure no-$\mathcal{A}^0$ remains valid and to ensure that $\delta/\delta\mathcal{A}$ is indeed contracted into something supertraceless. It is this that we need rather than the identities themselves, which are used occasionally, only to collect terms in the calculation.

Again, although we will not use it explicitly, let us remark that these constraints are

solved by the following general covariantization [6, 7]:

$$u\,\{W\}_{\mathcal{A}}\,v = \iint d^Dx\,d^Dy \int \mathcal{D}_W \ell_{xy}\ \mathrm{str}\,u(x)\,\Phi[\ell_{xy}]\,v(y)\,\Phi^{-1}[\ell_{xy}], \tag{4.17}$$

where

$$\Phi[\ell_{xy}] = P\exp -i\int_{\ell_{xy}} dz^\mu \mathcal{A}_\mu(z) \tag{4.18}$$

is a path ordered exponential integral, *i.e.* a Wilson line, and the appearance of $\Phi^{-1}[\ell_{xy}]$ means that we traverse backwards along another coincident Wilson line. The covariantization is determined by the measure $\mathcal{D}_W$ over configurations of the curves ℓ_{xy} and is so far left unspecified except for its normalisation:

$$\int \mathcal{D}_W \ell_{xy}\ 1 = W_{xy}, \tag{4.19}$$

as follows from eq. (4.11) and eq. (4.15). It is easy to see that eq. (4.17) indeed does satisfy eq. (4.23).

Finally, we will require that the covariantization satisfies

$$\frac{\delta}{\delta \mathcal{A}_\mu}\{W\}_{\mathcal{A}} = 0, \tag{4.20}$$

(where the previous is understood contracted on a supermatrix X independent of $\mathcal{A}$) *i.e.* that there be no diagrams in which the wine bites its own tail [5, 6, 7]. This leads to identities for the W vertices which again we do not need in practice: as we will confirm, such terms do not in any case contribute to the one-loop β function. However wine-biting-their-tail diagrams do appear in general to lead to some improperly regularised terms and so some restriction is needed for consistency. We can use the representation eq. (4.17) to see that sensible solutions to eq. (4.20) do exist. For example we can simply insist that ℓ_{xy} is a straight Wilson line, and more generally that the measure $\mathcal{D}_W$ has no support on curves ℓ_{xy} that cross the points x or y. The end points need defining carefully so that they only touch x and y after a limit has been taken [5].

However since we never specify the covariantization, we only need to assume that such a thing exists. In the calculation we just use eq. (4.20) and thus just forbid all wine-biting-their-tail diagrams.

4.3.1 Decoration with $\mathcal{C}$

Making use of the freedom we have on the choice of various parts of the flow equation, given by scheme independence, and since it will prove convenient for later purposes, we allow having occurences of $\mathcal{C}$ also on the Wilson lines (with the obvious corresponding extension of fig. 4.2) although we can limit their appearance to attachments at either end of ℓ_{xy}. Throughout all this thesis, as in [36], they will furthermore act only via commutation at both ends. Precisely, we extend the definition eq. (4.11) so that

$$u\{W\}v = u\,\{W\}_{\mathcal{A}}\,v - \frac{1}{4}\mathcal{C}\cdot u\,\{W_m\}_{\mathcal{A}}\,\mathcal{C}\cdot v. \tag{4.21}$$

where $W_m(p,\Lambda)$ is some new kernel. This is represented graphically in fig. 4.4, where the $\mathcal{C}$ fields are drawn by a white circle. In the expansion we now have vertices that

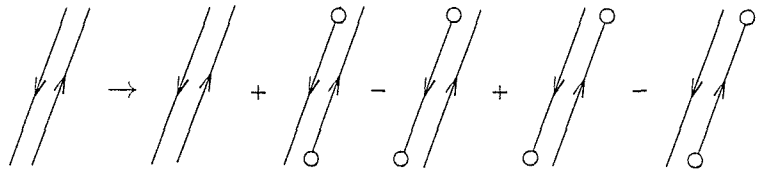

Figure 4.4: Wines decorated with $\mathcal{C}$ fields, represented with white blobs. Each of the lines have an expansion in $\mathcal{A}$ fields as the one of fig. 4.2.

come from both $\mathcal{A}$ and $\mathcal{C}$. Typically in this case u and v will actually correspond to functional differentials, with respect to, say, Z_1 and Z_1, and it will also be helpful to keep track of their flavours. by including them as labels in the naming convention for

the kernel, *viz.* as $W_{(m)}^{Z_1 Z_2}$. The notation we will use in general is

$$\frac{\delta}{\delta Z_1^c}\{W^{Z_1 Z_2}\}\frac{\delta}{\delta Z_2^c} = \tag{4.22}$$

$$\sum_{m,n=0}^{\infty}\int d^D x\, d^D y\, d^D x_1 \cdots d^D x_n\, d^D y_1 \cdots d^D y_m\, W_{a_1 \cdots a_n,\ b_1 \cdots b_m}^{X_1 \cdots X_n, Y_1 \cdots Y_m, Z_1 Z_2}(x_1, \cdots, x_n; y_1, \cdots, y_m; x, y)$$

$$\mathrm{str}\left[\frac{\delta}{\delta Z_1^c(x)} X_1^{a_1}(x_1) \cdots X_n^{a_n}(x_n)\frac{\delta}{\delta Z_2^c(y)} Y_1^{b_1}(y_1) \cdots Y_m^{b_m}(y_m)\right],$$

where the superfields X_i, Y_i and Z_i, are $\mathcal{A}$ or $\mathcal{C}$, and the indices $a_i = \mu_i$, $b_i = \nu_i$ and $c = \gamma$ in the case that the corresponding field is $\mathcal{A}$ and null if the field is $\mathcal{C}$. In fact, as a consequence of the restricted structure eq. (4.21), the $X_2, \cdots, X_{n-1}$ and $Y_2, \cdots, Y_{m-1}$ must be $\mathcal{A}$s if they appear at all.

We can still insist without loss of generality that $u\{W\}v \equiv v\{W\}u$, and use the shorthand eq. (4.14), where now we keep track of flavour labels as in eq. (4.22) however. It is still the case that with no fields on the wine, the original W kernel is recovered as in eq. (4.15). The commutator structure in eq. (4.21) ensures that eq. (4.16) holds for the full wine also:

$$u\,\{W\}v = (\mathrm{str}\, u)\cdot W \cdot g. \tag{4.23}$$

Finally, the $\mathcal{C}$s as further 'decorations' of the covariantized kernels are required to partake in the restriction described below eq. (4.20), so this equation extends to

$$\frac{\delta}{\delta \mathcal{A}_\mu}\{W\} = \frac{\delta}{\delta \mathcal{C}}\{W\} = 0. \tag{4.24}$$

(In fact by $X = \mathbb{1}$ in eq. (4.5), the contribution from differentiating the leftmost $\mathcal{C}$ vanishes in any case.)

4.4 Superfield expansion

Let us consider first the effective (flowing) action S. We can expand it in powers of the fields bearing in mind it must be an invariant under the group $SU(N|N)$. The most general one, is a linear combination of product of supertraces of fields:

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \frac{1}{s_n} \int d^D x_1 \cdots d^D x_n \, S^{X_1 \cdots X_1}_{a_1 \cdots a_n}(x_1, \cdots, x_n) \, \mathrm{str}\, X_1^{a_1}(x_1) \cdots X_n^{a_n}(x_n) \\ &+ \frac{1}{2!} \sum_{m,n=1}^{\infty} \frac{1}{s_n s_m} \int d^D x_1 \cdots d^D x_n \, d^D y_1 \cdots d^D y_m \, S^{X_1 \cdots X_n, Y_1 \cdots Y_m}_{a_1 \cdots a_n,\ b_1 \cdots b_m}(x_1, \cdots, x_n; y_1, \cdots, y_m) \\ &\qquad \mathrm{str}\, X_1^{a_1}(x_1) \cdots X_n^{a_n}(x_n) \, \mathrm{str}\, Y_1^{b_1}(y_1) \cdots Y_m^{b_m}(y_m) \\ &+ \cdots, \end{aligned} \tag{4.25}$$

where again the $X_i^{a_i}$ are $\mathcal{A}_{\mu_i}$ or $\mathcal{C}$, and $Y_j^{b_j}$ are $\mathcal{A}_{\nu_j}$ or $\mathcal{C}$. (Note that throughout this thesis we discard the vacuum energy.) Only one cyclic ordering of each list $X_1 \cdots X_n$, $Y_1 \cdots Y_m$ appears in the sum. Furthermore, if either list is invariant under some nontrivial cyclic permutations, then s_n (s_m) is the order of the cyclic subgroup, otherwise $s_n = 1$ ($s_m = 1$). (For example, in the terms where every $X_i^{a_i}$ is a $\mathcal{C}$, $s_n = n$.) The expansion can be represented diagrammatically, where a thick closed line stands for a supertrace, as in fig. 4.5 and each blob represents a field in it (fig.

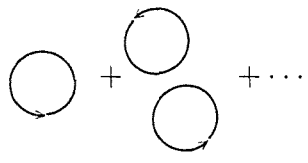

Figure 4.5: Action's expansion in product of supertraces

4.6). In a somewhat similar way to eq. (4.17) and eq. (4.21), these closed lines can be interpreted as decorated Wilson loops [6, 7].

When we spontaneously break the fermionic invariance by shifting $\mathcal{C}$ in the σ direction,

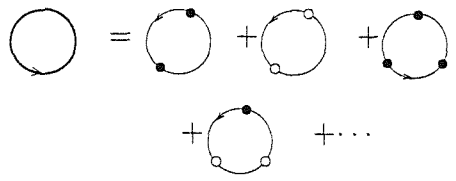

Figure 4.6: Each supertrace in the action expansion is a sum of supertraces of fields

it will prove to be better to work separately with the bosonic and fermionic parts of the superfields. Thus we write in the broken phase

$$\mathcal{A}_\mu = A_\mu + B_\mu, \qquad \text{and} \qquad \mathcal{C} = C + D + \sigma. \tag{4.26}$$

where A and C are the block diagonals, and B and D are the block off-diagonals in eqs. eq. (3.36) and eq. (3.43) respectively. (We will see in the sec. 4.5 that $\mathcal{C}$'s effective vacuum expectation value is just σ.)

Thus in the broken phase we will expand as in eq. (4.25), but the flavours X and Y are set to A, B, C or D. There will also be occurences of σ. However since σ commutes with A and C, and anticommutes with B and D, to define the expansion we can take the convention that we (anti)commute all such occurences to the far right in the supertrace. Upon using $\sigma^2 = \mathbb{1}$, we are then left with terms with either one σ at the end or none at all. Since σ has no position dependence, we put the flavour label in the superscript, but we omit the corresponding position label. Clearly, since the broken fields can still be cyclically permuted by (anti)commutation through σ, we also omit it from the determination of the symmetry factor, *i.e.* s_n is equal to the order of the cyclic permutation subgroup of the fields X_i, ignoring the σ (if present). Finally note that each supertrace term must separately hold only totally bosonic combinations since if $X_1 \cdots X_n$ (or $X_1 \cdots X_n \sigma$) is fermionic, it is block off-diagonal and has vanishing supertrace.

Similarly, in eq. (4.22), in the broken phase, X, Y and Z will be A, B, C or D. Note

that Z_1 can be the opposite statistic partner from Z_2. Since it is a single supertrace, again each contribution in eq. (4.22) is overall bosonic however. Single occurences of σ can also appear at the ends of the Wilson lines, after taking into account that these can also (anti)commute through the Z functional derivatives.

Finally, the momentum space vertices are written as

$$S^{X_1\cdots X_n}_{a_1\cdots a_n}(p_1,\cdots,p_n)\ (2\pi)^D\delta(\sum_{i=1}^{n}p_i) = \int d^Dx_1\cdots d^Dx_n\, \mathrm{e}^{-i\sum_i x_i\cdot p_i} S'^{X_1\cdots X_n}_{a_1\cdots a_n}(x_1,\cdots,x_n), \tag{4.27}$$

where all momenta are taken pointing into the vertex, and similarly for all the other vertices including eq. (4.22). We use the short hand $S^{XY}_{ab}(p) \equiv S^{XY}_{ab}(p,-p)$ and $S^{XY\sigma}_{ab}(p) \equiv S^{XY\sigma}_{ab}(p,-p)$ for action two-point vertices.

We will see later many examples. See also ref. [7].

4.4.1 Rescaling g

As in the case of the scalar field, in order to put the coupling constant in front of the action, we want to rescale the field as:

$$\mathcal{A} = \frac{1}{g}\tilde{\mathcal{A}}. \tag{4.28}$$

In this way, as in the previous case, the Boltzman factor in the partition function becomes:

$$e^{-S/\hbar} \to e^{-S/\hbar g^2}, \tag{4.29}$$

and the loop ($\hbar$) expansion conicides with the coupling expansion. In the present case, though, this rescaling give us a further nice feature. To explain this consider just the $SU(N)$ gauge field A_1, with covariant derivative $\nabla^1_\mu = \partial_\mu - igA^1_\mu$. If we

consider a gauge transformation on it,

$$\delta A^1_\mu = -\frac{1}{g}[\nabla^1_\mu, \omega] = \frac{1}{g}(\partial_\mu \omega - ig[A^1_\mu, \omega]) \tag{4.30}$$

If we now consider the rescaled field of eq. (4.28), and we perform a gauge transformation on it:

$$\delta \tilde{A}^1_\mu = \partial_\mu \omega - i[\tilde{A}^1_\mu, \omega] \tag{4.31}$$

If we now suppose $\tilde{A}^1$ runs as: $\tilde{A}^1_\mu = Z^{1/2}_{\tilde{A}^1} \tilde{A}^{1\ R}_\mu$, eq. (4.31), would become:

$$Z^{1/2}_{\tilde{A}^1} \tilde{A}^{1\ R}_\mu = \partial_\mu \omega - i Z^{1/2}_{\tilde{A}^1} [\tilde{A}^{1\ R}_\mu, \omega] \tag{4.32}$$

For gauge invariance to be preserved, $Z^{1/2}_{\tilde{A}^1}$ must be equal to one and this ensures that the rescaled field does not renormalise. In order to extract the one-loop beta function, it will then be enough to evaluate the one-loop two-point equation, and not also higher points as in the scalar field case (see sec. 2.3), since here $\gamma = 0$. The only quantity that renormalises is now g itself and the renormalisation condition is set by eq. (4.37).

4.5 A manifestly $SU(N|N)$ gauge invariant ERG

Our strategy is to write down a manifestly supergauge invariant flow equation, obeying the rules outlined in the previous sections, and then spontaneously break it. Defining $\Sigma_g = g^2 S - 2\hat{S}$, we simply set

$$\Lambda \partial_\Lambda S = -a_0[S, \Sigma_g] + a_1[\Sigma_g], \tag{4.33}$$

where

$$a_0[S, \Sigma_g] = \frac{1}{2} \frac{\delta S}{\delta \mathcal{A}_\mu} \{\dot{\Delta}^{\mathcal{A}\mathcal{A}}\} \frac{\delta \Sigma_g}{\delta \mathcal{A}_\mu} + \frac{1}{2} \frac{\delta S}{\delta \mathcal{C}} \{\dot{\Delta}^{\mathcal{C}\mathcal{C}}\} \frac{\delta \Sigma_g}{\delta \mathcal{C}}, \tag{4.34}$$

and

$$a_1[\Sigma_g] = \frac{1}{2}\frac{\delta}{\delta \mathcal{A}_\mu}\{\dot{\Delta}^{\mathcal{A}\mathcal{A}}\}\frac{\delta \Sigma_g}{\delta \mathcal{A}_\mu} + \frac{1}{2}\frac{\delta}{\delta \mathcal{C}}\{\dot{\Delta}^{\mathcal{C}\mathcal{C}}\}\frac{\delta \Sigma_g}{\delta \mathcal{C}}. \tag{4.35}$$

where the notation for the wines is explained below eq.(4.22). Here instead of indicating the kernels with W, we used the symbol $\dot{\Delta}$, because, as we will see, the integrated kernels, play here the rôle of effective propagators. More precisely they will be the inverse of the corresponding two point functions, in the transverse space (see App. D). Eq.(4.33) can be represented diagrammatically in fig. 4.7, appearing later. In the rest of this section we explain the meaning of the various components, at the same time developing some of the properties of this exact RG.

The definition of Σ_g and the form of the flow equation eq. (4.33) are the same as in refs. [6, 7]. In contrast to ref. [7] however, the exact RG is very simple in conception. The basic structure is inherited from the Wilson exact RG [1, 2, 5]: the bilinear functional -a_0 generates the classical corrections, whilst the linear functional a_1 generates quantum corrections (compare with eq. (2.9)). As in refs. [6, 7], a_1 has exactly the same structure as a_0 except that the leftmost functional derivatives differentiate everything to their right. Consequently we have

$$\Lambda\partial_\Lambda \, \mathrm{e}^{-S} = \sum_{X_i=\mathcal{A},\mathcal{C}} \frac{\delta}{\delta X_i}\left[\{\dot{\Delta}^{X_i X_i}\}\frac{\delta \Sigma_g}{\delta X_i}\mathrm{e}^{-S}\right], \tag{4.36}$$

(similarly to eq.(2.13)) which shows that the condition for the Boltzman measure to be a total functional derivative, is fulfilled.

As before, $g(\Lambda)$ is the renormalised coupling of the $SU(N)$ Yang-Mills theory carried by A^1. It is defined through the renormalization condition:

$$S[\mathcal{A} = A^1, C = \bar{C}] = \frac{1}{2g^2}\,\mathrm{tr}\int d^D x \, \left(F^1_{\mu\nu}\right)^2 + \cdots, \tag{4.37}$$

After the rescaling of g the previous is the only condition to be set, g being now the only quantity that runs (see sec. 4.4.1). The ellipsis in eq. (4.37), stands for

higher dimension operators and the vacuum energy, and $\bar{\mathcal{C}}$ is the effective vacuum expectation value defined so as to minimise the effective potential $V(\mathcal{C})$ in S:

$$\left.\frac{\partial V}{\partial \mathcal{C}}\right|_{\mathcal{C}=\bar{\mathcal{C}}} = 0. \tag{4.38}$$

$\bar{\mathcal{C}}$ is spacetime independent and generically contains terms proportional to σ and $\mathbb{1}$ (this is explained at the end of this section). We will see later that for our purposes we can simply set $\bar{\mathcal{C}} = \sigma$.

The strategy now to get the 1-loop β-function, will be the same as in [6, 7] and consists in expanding the flow equation eq. (4.33) in loop ($\hbar$ powers), which, at this point after having rescaled g, amounts in a coupling expansion[5]. Expanding S first

$$S = \frac{1}{g^2} S_0 + S_1 + g^2 S_2 + \cdots, \tag{4.39}$$

where S_0 is the classical effective action, S_1 the one-loop correction, and so on. Substituting this expansion in eq. (4.33), we see that the β function must also take the standard form

$$\beta := \Lambda \frac{\partial g}{\partial \Lambda} = \beta_1 g^3 + \beta_2 g^5 + \cdots. \tag{4.40}$$

From eq. (4.39) and eq. (4.40), we obtain the loopwise expansion of eq. (4.33):

$$\Lambda \frac{\partial}{\partial \Lambda} S_0 = -a_0[S_0, S_0 - 2\hat{S}], \tag{4.41}$$

$$\Lambda \frac{\partial}{\partial \Lambda} S_1 = 2\beta_1 S_0 - 2a_0[S_0 - \hat{S}, S_1] + a_1[S_0 - 2\hat{S}], \tag{4.42}$$

$$\Lambda \frac{\partial}{\partial \Lambda} S_2 = 2\beta_2 S_0 - 2a_0[S_0 - \hat{S}, S_2] - a_0[S_1, S_1] + a_1[S_1], \tag{4.43}$$

etc. From the second, we will try to get β_1. Actually, we will find it convenient to add some simple quantum corrections to the supergauge invariant seed action $\hat{S}$, giving it a g dependence (as we outline below). We also need to take account of the

[5]The redefinition of $\mathcal{A}$ described in section 4.4.1 led to this result

flow of g_2, the coupling for the second $SU(N)$ carried by A_2. However, neither of these complications have an effect on the one-loop β function computation, so will be largely ignored here.

$\hat{S}$ is used to determine the form of the classical effective kinetic terms and the kernels $\dot{\Delta}(p,\Lambda)$. It therefore has to incorporate the covariant higher derivative regularisation and allow the spontaneous symmetry breaking we require. Unlike previously [5, 6, 7], we will see that we otherwise leave it almost entirely unspecified. The kernels $\dot{\Delta}$ are determined by the requirement that after spontaneous symmetry breaking, the two-point vertices of the classical effective action S_0 and $\hat{S}$ can be set equal (see section 5.1). As previously [5, 6, 7], this is imposed as a useful technical device, since it allows classical vertices to be immediately solved in terms of already known quantities. It also means that the integral of the kernels defined via

$$\Lambda\partial_\Lambda\Delta = -\dot{\Delta} \tag{4.44}$$

will play a closely similar rôle to that of propagators, in particular being the inverse of these two-point vertices up to gauge transformations (see Appendix D).

The $\mathcal{C}$ commutator terms in eq. (4.21), yield σ commutators on spontaneous symmetry breaking. Since σ commutes with A and C but anticommutes with B and D, $\Delta_m^{\mathcal{A}\mathcal{A}}$ and $\Delta_m^{\mathcal{C}\mathcal{C}}$ allow for the addition of spontaneous mass creation for B and D whilst still keeping the two-point vertices of $\hat{S}$ and S_0 equal. The appearance of the $\mathcal{C}$ commutator on both sides allows us to insist that $\mathcal{C} \leftrightarrow -\mathcal{C}$ is an invariance of the symmetric phase. The form (4.34,4.35) preserves charge conjugation symmetry $\mathcal{C} \mapsto \mathcal{C}^T$, $\mathcal{A} \mapsto -\mathcal{A}^T$ (using the definition of the supermatrix transpose in ref. [34]. Note that here the transformation for $\mathcal{C}$ is determined by the fact that its vacuum expectation value is even under charge conjugation.)

From eq. (4.3) and sec. 4.3.1, it is trivial to see that the $\delta/\delta\mathcal{C}$ terms are supergauge

invariant. Under a supergauge transformation we have by eq. (4.8) and eq. (4.23),

$$\delta\left(\frac{\delta S}{\delta\mathcal{A}_\mu}\{\dot{\Delta}^{\mathcal{A}\mathcal{A}}\}\frac{\delta\Sigma_g}{\delta\mathcal{A}_\mu}\right) = \frac{i}{2N}\,\mathrm{tr}\left[\frac{\delta S}{\delta\mathcal{A}_\mu},\omega\right]\cdot\dot{\Delta}^{\mathcal{A}\mathcal{A}}\cdot\,\mathrm{str}\frac{\delta\Sigma_g}{\delta\mathcal{A}_\mu} + (S\leftrightarrow\Sigma_g), \tag{4.45}$$

where $S\leftrightarrow\Sigma_g$ stands for the same term with S and Σ_g interchanged. But by eq. (4.7) and no-$\mathcal{A}^0$,

$$\mathrm{str}\frac{\delta\Sigma_g}{\delta\mathcal{A}_\mu} = \frac{\delta\Sigma_g}{\delta\mathcal{A}^0_\mu} = 0, \tag{4.46}$$

similarly for S, and thus the tree level terms are supergauge invariant. Similarly, the quantum terms are $SU(N|N)$ gauge invariant, since

$$\delta\left(\frac{\delta}{\delta\mathcal{A}_\mu}\{\dot{\Delta}^{\mathcal{A}\mathcal{A}}\}\frac{\delta}{\delta\mathcal{A}_\mu}\Sigma_g\right) = \frac{i}{N}\,\mathrm{tr}\left[\frac{\delta}{\delta\mathcal{A}_\mu},\omega\right]\cdot\dot{\Delta}^{\mathcal{A}\mathcal{A}}\cdot\,\mathrm{str}\frac{\delta\Sigma_g}{\delta\mathcal{A}_\mu} = 0. \tag{4.47}$$

This completes the proof that the exact RG is supergauge invariant!

Note that there is no point in incorporating longitudinal terms into the exact RG (as was done in ref. [7]) because here the manifest supergauge invariance means that they can be exchanged for $\mathcal{C}$ commutators:

$$\nabla_\mu\cdot\frac{\delta S}{\delta\mathcal{A}_\mu} = i\mathcal{C}\cdot\frac{\delta S}{\delta\mathcal{C}} \tag{4.48}$$

(as holds for any supergauge invariant functional) and thus absorbed into the $\dot{\Delta}^{\mathcal{C}\mathcal{C}}_m$ term.

It is important for the working of the $SU(N|N)$ regularisation that the effective scale of spontaneous symmetry breaking is tied to the higher derivative regularisation scale, which thus both flow with Λ. This is not the typical situation, but can be arranged to happen here by constraining $\hat{S}$ appropriately. However, as we now show, the constraint is straightforward only if we take $\mathcal{C}$ to be dimensionless in eq. (4.33) – eq. (4.35).

Contracting an arbitrary supermatrix X into eq. (4.38) (for convenience, *c.f.* sec. 4.2) and differentiating with respect to Λ, we have:

$$\left[\mathrm{str}\,\frac{\partial\bar{\mathcal{C}}}{\partial\Lambda}\frac{\partial}{\partial\mathcal{C}}\,\mathrm{str}\,X\frac{\partial V}{\partial\mathcal{C}}+\mathrm{str}\,X\frac{\partial}{\partial\mathcal{C}}\frac{\partial V}{\partial\Lambda}\right]_{\mathcal{C}=\bar{\mathcal{C}}}=0. \tag{4.49}$$

We can compute the flow $\partial V/\partial\Lambda$ by setting $\mathcal{A}=0$ and $\mathcal{C}=\bar{\mathcal{C}}$ in eq. (4.33). Taking the classical limit $V\to V_0$, we find that the resulting equation simplifies dramatically. Using eqs. (4.41), (4.34), (4.38), (4.11), (4.15), the fact that vertices in the actions with only one $\mathcal{A}_\mu$, vanish at zero momentum (by Lorentz invariance), and

$$[\bar{\mathcal{C}},\frac{\partial\hat{V}}{\partial\mathcal{C}}]\Big|_{\mathcal{C}=\bar{\mathcal{C}}}=0, \tag{4.50}$$

which follows from global $SU(N|N)$ invariance (where $\hat{V}$ is the potential in $\hat{S}$), we get

$$\mathrm{str}\left[\left(\Lambda\frac{\partial\bar{\mathcal{C}}}{\partial\Lambda}+\dot{\Delta}^{\mathcal{C}\mathcal{C}}(0,\Lambda)\frac{\partial\hat{V}}{\partial\mathcal{C}}\right)\frac{\partial}{\partial\mathcal{C}}\,\mathrm{str}\,X\frac{\partial V_0}{\partial\mathcal{C}}\right]_{\mathcal{C}=\bar{\mathcal{C}}}=0. \tag{4.51}$$

With $\mathcal{C}$ dimensionless, we can and will insist that the classical vacuum expectation value $\bar{\mathcal{C}}=\sigma$. eq. eq. (4.51) is then satisfied if and only if[6] $\hat{V}$ also has a minimum at $\mathcal{C}=\sigma$. This is delightful since it ensures that at the classical level at least, neither action has one-point $\mathcal{C}$ vertices in the broken phase. We will thus impose

$$\frac{\partial\hat{V}}{\partial\mathcal{C}}\Big|_{\mathcal{C}=\sigma}=0 \tag{4.52}$$

as a constraint on $\hat{S}$.

Had we not taken $\mathcal{C}$ to be dimensionless, we would have had to require that $\bar{\mathcal{C}}$ depend on Λ, in order that the effective breaking scale flows with Λ. Since X is general, eq. (4.51) would then imply that $\hat{V}$ *cannot* have a minimum also at $\mathcal{C}=\bar{\mathcal{C}}$. Further

[6] We will see that the requirement that C has a mass in the broken phase forces $\dot{\Delta}^{\mathcal{C}\mathcal{C}}(0,\Lambda)\neq 0$.

analysis shows that $\hat{V}$ is then forced to violate $\mathcal{C} \leftrightarrow -\mathcal{C}$ symmetry in the symmetric phase.

Although conventionally $\mathcal{C}$ would have dimension one, for these reasons we will take it to be dimensionless from now on. (It is intriguing that the conclusion that $\mathcal{C}$ [actually C] must be dimensionless was reached for very different reasons in refs. [5, 7] which are no longer necessarily applicable, now that eq. (4.48) is a symmetry.)

At the quantum level, $\bar{\mathcal{C}} = \sigma$ can be expected to receive loop corrections. Since $SU(N) \times SU(N)$ invariance is left unbroken, these corrections can only be proportional to σ or $\mathbb{1}$. Corrections proportional to the latter do not affect the breaking (but presumably through eq. (4.37) give important contributions at higher loops), however corrections proportional to σ would result, through eq. (4.48), in broken gauge invariance identities that explicitly involve g and thus mix different loop orders. We can avoid this by again using the freedom in our choice of $\hat{S}'$ to design things appropriately. We can constrain the appearance of $\hat{V}$ one-point vertices in the broken phase

$$v^{C}\,\mathrm{str}\,\mathcal{C} \;+ v^{C\sigma}\,\mathrm{str}\,\mathcal{C}\sigma \tag{4.53}$$

by imposing $\bar{\mathcal{C}} = \sigma$ as a renormalization condition. Each v is then a non-vanishing function of g, but from the analysis above, only from one-loop onwards:

$$v^{C}(g) = v_1^{C}\,g^2 + v_2^{C}\,g^4 + \cdots \qquad \text{and} \qquad v^{C\sigma}(g) = v_1^{C\sigma}\,g^2 + v_2^{C\sigma}\,g^4 + \cdots . \tag{4.54}$$

We will see that these corrections in fact are already too high an order to affect the one-loop β function calculation.

4.5.1 Supersowing and supersplitting in the $\mathcal{A}$ sector

The inherent supersymmetry has a remarkable effect on the gauge algebra: one can replace the usual manipulation of structure constants and reduction to Casimirs, which becomes increasingly involved at higher loops, by simple steps eq. (4.4) and eq. (4.5) which always either just sow together supertraces or split them open. These have an immediate diagrammatic interpretation. The apparent violations present in eq. (4.9) and eq. (4.10) must somehow disappear since they would violate even global $SU(N|N)$. We first prove that this indeed the case.

For the case where the action contains just a single supertrace, which will turn out to be all we need here, we could adapt the proof given in sec. 6.2 of ref. [34]. However, in preparation for future work, we will give a more sophisticated proof which is applicable when working with multiple supertrace contributions. Indeed we will see that there is then one special case, where the corrections in (4.9,4.10) do survive, and result in a simple supergauge invariant correction.

The corrections present in (4.9,4.10) arise because $\mathcal{A}$ is constrained to be supertraceless. To compare their effect to the unconstrained case (4.4,4.5), we momentarily 'lift' $\mathcal{A}$ to a full superfield $\mathcal{A}^e$ by adding a σ part:

$$\mathcal{A}_\mu \mapsto \mathcal{A}^e_\mu := \mathcal{A}_\mu + \sigma \mathcal{A}^\sigma_\mu. \tag{4.55}$$

$\mathcal{A}^\sigma_\mu$ is taken arbitrary so the map is not at all unique. We similarly extend all functionals of $\mathcal{A}$ to the full space, simply by replacing $\mathcal{A}$ with $\mathcal{A}^e$, *e.g.*

$$S^e[\mathcal{A}^e, C] := S[\mathcal{A} \mapsto \mathcal{A}^e, C]. \tag{4.56}$$

Again, this is a not unique procedure, as can be seen for example in the fact that $\mathrm{str}\,\mathcal{A}$ vanishes, but the promoted functional $\mathrm{str}\,\mathcal{A}^e$ does not. We also introduce the

projection back onto the supertraceless space:

$$\pi \mathcal{A}^e_\mu = \mathcal{A}_\mu, \quad \pi S^e = S, \quad etc. \,, \tag{4.57}$$

which of course is unique. Functional derivatives with respect to $\mathcal{A}^e$ can be written as

$$\frac{\delta}{\delta \mathcal{A}^e_\mu} = \frac{\delta}{\delta \mathcal{A}_\mu} + \frac{1\!\!1}{2N} \frac{\delta}{\delta \mathcal{A}^\sigma_\mu}, \tag{4.58}$$

using eq. (4.7), or equivalently defined as in eq. (4.2). $\delta/\delta\mathcal{A}^e$ thus satisfies the exact supersowing and supersplitting relations (4.4,4.5). In the extended space, the constrained derivative eq. (4.7) can now be written in terms of an unconstrained derivative:

$$\frac{\delta}{\delta \mathcal{A}_\mu} = \frac{\delta}{\delta \mathcal{A}^e_\mu} - \frac{1\!\!1}{2N} \mathrm{tr} \frac{\delta}{\delta \mathcal{A}^e_\mu}. \tag{4.59}$$

Of course π and $\delta/\delta\mathcal{A}^\sigma$ do not commute, however

$$\frac{\delta S}{\delta \mathcal{A}_\mu} \{\dot{\Delta}^{\mathcal{A}\mathcal{A}}\} \frac{\delta \Sigma_g}{\delta \mathcal{A}_\mu} = \pi \left\{ \frac{\delta S^e}{\delta \mathcal{A}_\mu} \{\dot{\Delta}^{\mathcal{A}\mathcal{A}}\}^e \frac{\delta \Sigma^e_g}{\delta \mathcal{A}_\mu} \right\}, \tag{4.60}$$

since $\mathcal{A}^\sigma$ is not differentiated on the right hand side. Substituting eq. (4.59) or eq. (4.58), and using eq. (4.23) and eq. (4.7),

the term in big curly braces becomes

$$\frac{\delta S^e}{\delta \mathcal{A}^e_\mu} \{\dot{\Delta}^{\mathcal{A}\mathcal{A}}\}^e \frac{\delta \Sigma^e_g}{\delta \mathcal{A}^e_\mu} - \frac{1}{2N} \frac{\delta S^e}{\delta \mathcal{A}^0_\mu} \cdot \dot{\Delta}^{\mathcal{A}\mathcal{A}} \cdot \frac{\delta \Sigma^e_g}{\delta \mathcal{A}^\sigma_\mu} - \frac{1}{2N} \frac{\delta \Sigma^e_g}{\delta \mathcal{A}^0_\mu} \cdot \dot{\Delta}^{\mathcal{A}\mathcal{A}} \cdot \frac{\delta S^e}{\delta \mathcal{A}^\sigma_\mu}. \tag{4.61}$$

Now, as we explain below, no-$\mathcal{A}^0$ symmetry is violated in the extended space. However the $\mathcal{A}^0$ derivatives in eq. (4.61) do vanish after the projection. Thus eq. (4.60) becomes

$$\frac{\delta S}{\delta \mathcal{A}_\mu} \{\dot{\Delta}^{\mathcal{A}\mathcal{A}}\} \frac{\delta \Sigma_g}{\delta \mathcal{A}_\mu} = \pi \left\{ \frac{\delta S^e}{\delta \mathcal{A}^e_\mu} \{\dot{\Delta}^{\mathcal{A}\mathcal{A}}\}^e \frac{\delta \Sigma^e_g}{\delta \mathcal{A}^e_\mu} \right\}, \tag{4.62}$$

which says precisely that the corrections in eq. (4.9) can be ignored: exactly the same

result is obtained if exact supersowing is used.

However, performing the same analysis on the corresponding quantum term in eq. (4.35), we get a correction to exact supersplitting, consisting of an attachment of the (zero-point) kernel $\dot{\Delta}^{\mathcal{A}\mathcal{A}}(p,\Lambda)$ to two $\mathcal{A}$ points in Σ_g:

$$\frac{\delta}{\delta\mathcal{A}_\mu}\{\dot{\Delta}^{\mathcal{A}\mathcal{A}}\}\frac{\delta\Sigma_g}{\delta\mathcal{A}_\mu} = \pi\left\{\frac{\delta}{\delta\mathcal{A}^e_\mu}\{\dot{\Delta}^{\mathcal{A}\mathcal{A}}\}^e\frac{\delta\Sigma^e_g}{\delta\mathcal{A}^e_\mu}\right\} - \frac{1}{N}\pi\frac{\delta}{\delta\mathcal{A}^\sigma_\mu}\cdot\dot{\Delta}^{\mathcal{A}\mathcal{A}}\cdot\frac{\delta\Sigma_g}{\delta\mathcal{A}^0_\mu}. \tag{4.63}$$

To understand when this correction is non-vanishing, we need briefly to analyse the consequences of no-$\mathcal{A}^0$ symmetry in more detail. Considering the transformation[7] $\delta\mathcal{A}_\mu = \lambda_\mu\mathbb{1}\!\mathbb{1}$ in eq. (4.25), we see that the result must vanish either via the supergroup algebra because the corresponding vertex contains a factor str$\mathcal{A}\mathcal{A}$, thus generating str$\mathcal{A} = 0$ (but str$\mathcal{A}^e \neq 0$ in the extended space), or because a non-trivial constraint exists on the corresponding vertex function. (This is simply that the sum over all possible valid placings of $\mathcal{A}^0$s associated position and Lorentz argument inside a vertex function leaving other arguments alone, yields zero.) This non-trivial constraint then causes the coefficient to vanish whether or not the remaining supergauge fields are extended by $\mathcal{A}^\sigma\sigma$. Thus the correction in eq. (4.63) vanishes in all cases except where the zero-point $\dot{\Delta}^{\mathcal{A}\mathcal{A}}$ kernel attaches each end to a str$\mathcal{A}\mathcal{A}$ factor. Comparing the result to the computation assuming exact supersplitting, *i.e.* the first term in eq. (4.63), we see that instead of getting a supergroup factor $(\text{str}\mathbb{1}\!\mathbb{1})^2 = 0$ we get $-\frac{1}{N}\text{str}\sigma$ *i.e.* a supergroup factor of -2.

(Note that in deriving this rule we have assumed that vertices in Σ_g with factors str$\mathcal{A}$ have been set to zero from the beginning [as would follow immediately from the $SU(N|N)$ group theory]. If for some reason this was not done then the first term in eq. (4.63) can get a non-zero computation from the kernel attaching to this str$\mathcal{A} = 2N\mathcal{A}^\sigma$ point. However it then also appears in the correction with precisely

[7] there are higher order constraints from separating out higher powers of $\mathcal{A}^0$ but from eq. (4.63) we only need the first order

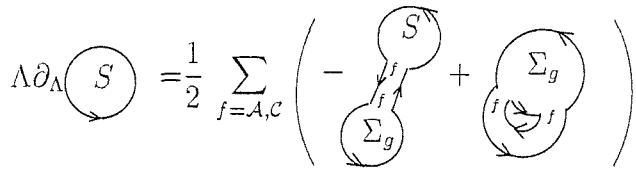

Figure 4.7: Graphical representation of the exact RG, when S and $\hat{S}$ contain only single supertraces.

equal and opposite coefficient.)

This supergroup factor should have been expected since the algebra part of the attachment of a zero-point kernel to a two-point vertex simply counts the number of bosonic degrees of freedom in the algebra minus the number of fermionic degrees of freedom. There are N^2 fermionic such terms in B, but only N^2-2 in A, since both $\mathcal{A}^\sigma$ and, by no-$\mathcal{A}^0$ symmetry, $\mathcal{A}^0$, are missing.

Since the correction in eq. (4.63) is non-vanishing only when using up a separate $\mathrm{str}\mathcal{A}\mathcal{A}$ factor, it is clear that the result is still supergauge invariant in the remaining external superfields. Furthermore in the present case where we will be able to work with actions with only a single supertrace, the entire effect of the correction is a just vacuum energy contribution, which from now on we ignore.

4.5.2 Diagrammatic interpretation

$\mathcal{A}$ thus also effectively satisfies the exact supersowing and supersplitting relations eq. (4.4) and eq. (4.5). By using these equations when the covariantized kernels eq. (4.22) act on the actions eq. (4.25), and comparing the result to the diagrammatic interpretation of the covariantized kernels and actions, fig. 4.2 and figs. 4.5,4.6, it is clear that the exact RG is given diagrammatically as in fig. 4.7. Here we have specialized to the case of our interest, where S and $\hat{S}$ can be assumed to have only

a single supertrace. (The extension to the more general contributions fig. 4.5 is obvious.) Expanding the thick lines (representing any number of fields) into a power series in the fields, we translate the figure into individual Feynman diagrams, whose Feynman rules are given by the momentum space versions of the vertices in eq. (4.22) and eq. (4.25) (without the symmetry factors).[8] The points representing individual fields and their associated momenta and Lorentz indices, appear in all places on a composite loop with equal weight, whilst respecting the cyclic order. Of course if one of the corresponding vertices does not appear in the expansions eq. (4.22) and eq. (4.25), the corresponding Feynman rule is zero.

It can be seen from fig. 4.7 that the tree level corrections preserve the assumption that there is only a single supertrace in S, but that each quantum correction results in an extra supertrace factor. Thus in general S has terms with any number of supertraces, and already a minimum of a product of two supertraces at one-loop. However for the computation of the β function, we need only look at contributions to the AA two-point vertex (see eq. (4.37) and later, or refs. [5, 6, 7]). Since A is both traceless and supertraceless, to get a non-vanishing answer both As must lie in the same supertrace, leaving the other one empty of fields. In this way, S effectively contains only a single supertrace to the order in which we are working.

4.5.3 After spontaneous breaking

We substitute $\mathcal{C} \mapsto \mathcal{C} + \sigma$, and from now on work in the spontaneously broken phase. Working with fields appropriate for the remaining $SU(N) \times SU(N)$ symmetry, we break $\mathcal{A}$ and $\mathcal{C}$ down to their bosonic and fermionic parts A, B, C and D as in eq. (4.26).

[8] This part of the analysis is the same as in ref. [7], except that here we make explicit the factor of $1/2$ from eq. (4.34) and eq. (4.35), in fig. 4.7 and the Feynman diagrams, and the factor of $1/\Lambda^2$ is now incorporated in the definition of the kernels $\dot{\Delta}$.

The diagrammatic interpretation is still the same, except that we now have the four flavours to scatter around the composite loops, and appearances of σ, which can be simplified as explained in sec. 4.4. In addition, we must recall the corrections to supersplitting and supersowing arising from differentiating only partial supermatrices [7]. These lead to further appearances of σ which are easily computed by expressing the partial supermatrices in terms of full supermatrices via the projectors $\mathrm{d}_\pm$ onto the block (off)diagonal components

$$\mathrm{d}_\pm X = \frac{1}{2}(X \pm \sigma X \sigma), \tag{4.64}$$

(hence $C = \mathrm{d}_+\mathcal{C}$, $D = \mathrm{d}_-\mathcal{C}$, *etc.*). Diagrammatically this simply amounts to corrections involving a pair of σs inserted either side of the attachment as in fig. 4.8 [7].

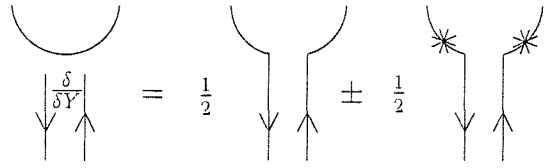

Figure 4.8: Feynman diagram representation of attachment via a partial supermatrix.

For tree-level type attachments as in eq. (4.4), the corrections merely ensure that the coefficient supermatrices (X and Y) have the appropriate statistics to make each supertrace term totally bosonic (*c.f.* sec. 4.4), but this has already been taken into account in the Feynman rules. Thus these corrections have no effect at tree level [7].

Since the classical action S_0 (similarly $\hat{S}$) has only a single supertrace and respects $\mathcal{C} \leftrightarrow -\mathcal{C}$ invariance in the symmetric phase (*c.f.* sec. 4.5), upon spontaneous breaking we have the 'theory space' symmetry

$$C \leftrightarrow -C,$$

$$
\begin{aligned}
D &\leftrightarrow -D, \\
\sigma &\leftrightarrow -\sigma.
\end{aligned}
\tag{4.65}
$$

The single supertrace part of the one-loop effective action S_1 has a single supertrace because it also has a supertrace void of fields (*c.f.* sec. 4.5.2). In order for this not to vanish it must 'trap' a σ (so that we get $\mathrm{str}\,\sigma = 2N$ rather than $\mathrm{str}\,\mathbb{1} = 0$). Therefore, the non-trivial supertrace has one less σ (mod two) and is thus odd under the symmetry eq. (4.65).

These observations, which can be easily extended to multiple loops and supertraces, are useful in limiting the possible vertices.

Chapter 5

One-loop β-function

5.1 The one-loop equation

To start with the calculation of the first coefficient of the β-function, we can now consider eq. (4.42) which we rewrite here in its extended form:

$$\Lambda\frac{\partial}{\partial\Lambda}S_1 \;=\; 2\beta_1 S_0 - \sum_{X_i}\left[\frac{\delta(S_0-\hat{S})}{\delta X_i}\{\dot{\Delta}^{X_iX_i}\}\frac{S_1}{\delta X_i}+\frac{1}{2}\frac{\delta}{\delta X_i}\{\dot{\Delta}^{X_iX_i}\}\frac{\delta\Sigma_0}{\delta X_i}\right] \quad (5.1)$$

Where X_i can be either $\mathcal{A}_\mu$ or $\mathcal{C}$. Before we continue, this is the right time to point out that, unlike the formulation described in [36], for the rest of the calculation here[1], the wines with $\mathcal{C}$ will not be incorporated with the "undecorated" ones. This was only done in section 4.3.1 and 4.5 in order to write the properties of the wines and the flow equation more compactly. To get to the new formulation, recall first that from eq. (4.21), we can recast a_0 and a_1 appearing in the flow equation eq. (4.33) as

[1] Unless stated differently

follows:

$$a_0[S,\Sigma_g] = \frac{1}{2}\sum_{X_i}\left[\frac{\delta S}{\delta X_i}\{\dot{\Delta}^{X_iX_i}\}\frac{\delta\Sigma_g}{\delta X_i} + \frac{1}{4}\frac{\delta S}{\delta X_i}\cdot\mathcal{C}\{\dot{\Delta}_m^{X_iX_i}\}\frac{\delta\Sigma_g}{\delta X_i}\cdot\mathcal{C}\right], \tag{5.2}$$

and

$$a_1[\Sigma_g] = \frac{1}{2}\sum_{X_i}\left[\frac{\delta}{\delta X_i}\{\dot{\Delta}^{X_iX_i}\}\frac{\delta\Sigma_g}{\delta X_i} + \frac{1}{4}[\frac{\delta}{\delta X_i}\cdot\mathcal{C}\{\dot{\Delta}_m^{X_iX_i}\}\frac{\delta\Sigma_g}{\delta X_i}\cdot\mathcal{C}\right]. \tag{5.3}$$

The wine vertices will then be considered from now on in a different basis. For the kernels:

$$\dot{\Delta}^{AA} = \dot{\Delta}^{\mathcal{A}\mathcal{A}} \quad \dot{\Delta}^{BB} = \dot{\Delta}^{\mathcal{A}\mathcal{A}} + \dot{\Delta}_m^{\mathcal{A}\mathcal{A}} \tag{5.4}$$

$$\dot{\Delta}^{CC} = \dot{\Delta}^{\mathcal{C}\mathcal{C}} \quad \dot{\Delta}^{DD} = \dot{\Delta}^{\mathcal{C}\mathcal{C}} + \dot{\Delta}_m^{\mathcal{C}\mathcal{C}} \tag{5.5}$$

We are ready now to proceed. In order to extract β_1 from eq. (5.1), let us first consider the renormalisation condition for the coupling g. After the redefinition of the gauge field $\mathcal{A}_\mu$ (see section 2.2) we have already mentioned that the only quantity which renormalises is now the gauge coupling itself. We can then set the renormalisation condition asking for g to be, in the expansion of $S(\Lambda)$, the coefficient of the quadratic term in the two-point function for the bosonic gauge field A at order p^2 (following [7]), in order to have the physical β-function for $SU(N)$ Yang-Mills. In formulae it is expressed in eq. (4.37). That expression is non-perturbative. At any order, this means;

$$S^{AA}_{\mu\nu} + S^{AA\sigma}_{\mu\nu} = \frac{2}{g^2(\Lambda)}\,\Box_{\mu\nu}(p) + \mathcal{O}(p^3) \tag{5.6}$$

where we define $\Box_{\mu\nu}(p) = p^2\delta_{\mu\nu} - p_\mu p_\nu$, and

$$S^{AA}_{\mu\nu} = \frac{1}{g^2}S^{(0)AA}_{\mu\nu} + S^{(1)AA}_{\mu\nu} + \cdots \tag{5.7}$$

$$\begin{aligned} S^{AA\sigma}_{\mu\nu} &= \frac{1}{g^2}S^{(0)AA\sigma}_{\mu\nu} + S^{(1)AA\sigma}_{\mu\nu} + \cdots \\ &= S^{(1)AA\sigma}_{\mu\nu} + \cdots \end{aligned} \tag{5.8}$$

but at tree level we will see we already have:

$$S^{(0)AA}_{\mu\nu} = 2\,\Box_{\mu\nu}(p) + \mathcal{O}(p^3) \tag{5.9}$$

The condition is already fulfilled, so:

$$\frac{1}{g^2} S^{(0)AA}_{\mu\nu} + \sum_{n=1} g^{2n-2}(S^{(n)AA}_{\mu\nu} + S^{(n)AA\sigma}_{\mu\nu}) = \frac{1}{g^2}\, S^{(0)AA}_{\mu\nu}\Big|_{p^2} + \mathcal{O}(p^3) \tag{5.10}$$

This means that as for the scalar field case, considering the one-loop equation eq. (5.1) if we take the combination $S^{AA}_{\mu\nu} + S^{AA\sigma}_{\mu\nu}$ in the expansion of S and its order p^2, we have a great simplification due to the term on the LHS which now vanishes. Moreover, we can simplify even further eq. (5.1) if we take into account the freedom we still have on the choice of $\hat{S}$. As we have mentioned in the previous chapter, if we choose it so that all its two point functions are equal to the two point effective action at tree level, the classical term in eq. (5.1) is zero (for the component in the expansion we are looking for). In formulae:

$$\hat{S}^{X_1X_2}_{a_1\,a_2} = S^{X_1X_2}_{a_1\,a_2} \tag{5.11}$$

where X_i can be any field and a_i is a Lorentz index if X_i is a gauge field and nothing otherwise. This is not a big restriction and it is the same request we set in the scalar field case. It just amounts, as we pointed out in section 2.3 for the scalar field, to asking the two-point effective tree level vertices, to flow keeping the same functional dependence upon Λ. It is an arbitrary choice which is worth taking since it greatly simplify the calculation. These two-point vertices can be almost uniquely determined via dimensional analysis, gauge invariance (in the next section the Ward identities relating them are considered) and recalling they must be derived from $SU(N|N)$ theory used as a regulator. We will briefly discuss their form here.

The first one, the AA vertex, turns out to be the most general transverse function of p^2 of dimension Λ^2. The third, the CC vertex, must be a general function of p^2, of

dimension 4, which is not zero at $p = 0$. Since we want to interpret this term as a mass for C (of order the cutoff Λ, in order for the regularisation to work), we have also to choose its coefficient positive. The BB vertex, does not have to be transverse, and it is written as a combination of a transverse term and a non transverse one. The former is chosen equal to the AA one, in order for the regularisation to work (so to have the right cancellations of the propagators at high momenta). The latter has a form, constrained in order to produce through Ward identities the last two in the list ($BD\sigma$ first, and from it DD). The DD one is required to have the momentum dependence of C's for the regularisation to work, and it must vanish at $p = 0$, since it's the kinetic term of a Goldstone (massless) field. Finally $BD\sigma$ is obtained by dimensional analysis and through Ward identities from the BB vertex. A list of two-point vertices with such requirements, follows below:

$$\hat{S}^{AA}_{\mu\nu}(p) = 2\frac{\Box_{\mu\nu}}{c_p} \equiv \hat{S}_{\mu\nu} \tag{5.12}$$

$$\hat{S}^{BB}_{\mu\nu}(p) = \hat{S}_{\mu\nu} + 4\frac{\Lambda^2}{\tilde{c}_p}\delta_{\mu\nu} \tag{5.13}$$

$$\hat{S}^{CC}(p) = \frac{\Lambda^2 p^2}{\tilde{c}_p} + 2\lambda\Lambda^4 \tag{5.14}$$

$$\hat{S}^{DD}(p) = \frac{\Lambda^2 p^2}{\tilde{c}_p} \tag{5.15}$$

$$\hat{S}^{BD\sigma}_{\mu}(p) = -2\frac{\Lambda^2 p_\mu}{\tilde{c}_p} \tag{5.16}$$

where $\Box_{\mu\nu}$ was defined below eq. (5.6), c and $\tilde{c}$ are general functions of $x = p^2/\Lambda^2$ and λ is a constant. The only necessary requirement on c is $c_0 = 1$, from the normalisation condition and eq. (5.9), as for the scalar field. We will also require $\tilde{c}_0 = 1$ and, as we have mentioned earlier, in order for C to have a mass, we choose $\lambda > 0$. In principle we could have chosen two pairs of different functions, c and $\hat{c}$ for the A and the B and $\tilde{c}$ and $\hat{\tilde{c}}$ for the C and the D vertices. In fact, even though they come from the same supermultiplets, when one considers the broken phase, the two pairs of two-point function, can pick different contributions due to their different statistics. Although

the most general requirements would then be that the functions chosen must have the same behaviour as $p/\Lambda \to \infty$, we decide to choose them equal.

The form of $\hat{S}$ is then still left quite general. Apart from having to preserve all the symmetries of the theory, and the request expressed by eq. (5.11) on its two-point vertices, we just add as for the scalar field case, the restriction on its higher vertices to be Taylor expandable to any order and to keep UV finite all the integral in which they appear.

Considering again eq. (5.1) we are then left with:

$$a_1[\Sigma^{(0)}]^{AA}_{\mu\nu} + a_1[\Sigma^{(0)}]^{AA\sigma}_{\mu\nu} = -4\beta_1 \,\Box_{\mu\nu} + \mathcal{O}(p^3) \tag{5.17}$$

where $\Sigma^{(0)X_1X_2}_{a_1\,a_2} = S'^{(0)X_1X_2}_{a_1\,a_2} - 2\hat{S}^{X_1X_2}_{a_1\,a_2}$. Because of (4.65) though, the 1-loop vertices must contain a σ, so eq. (5.17) becomes finally

$$a_1[\Sigma^{(0)}]^{AA\sigma}_{\mu\nu} = -4\beta_1 \,\Box_{\mu\nu} + \mathcal{O}(p^3) \tag{5.18}$$

Computing all the diagram contributing to the vertex $S'^{AA\sigma}_{\mu\nu}$ at one loop, and considering the form of the flow equation, we can find the following expression for eq. (5.18):

$$\begin{aligned}
-4\beta_1 \,\Box_{\mu\nu}(p^2) \;=\; & \\
& 2N \int_k \Big\{ \dot{\Delta}^{AA}_k \Sigma^{AAAA}_{\alpha\,\alpha\,\mu\,\nu}(-k,k,-p,p) + \dot{\Delta}^{A,AA}_{\mu}(p;k-p,-k)\Sigma^{AAA}_{\alpha\,\alpha\,\nu}(p-k,k,-p) \\
& + \dot{\Delta}^{AA,AA}_{\mu\,\nu}(p,-p;k,-k)\Sigma^{AA}_{\alpha\,\alpha}(k) \\
& - \dot{\Delta}^{BB}_k \Sigma^{BBAA}_{\alpha\,\alpha\,\mu\,\nu}(-k,k,p,-p) - \dot{\Delta}^{A,BB}_{\mu}(p;k-p,-k)\Sigma^{BBA}_{\alpha\,\alpha\,\nu}(p-k,k,-p) \\
& - \dot{\Delta}^{AA,BB}_{\mu\,\nu}(p,-p;k,-k)\Sigma^{BB}_{\alpha\,\alpha}(k) \\
& + \dot{\Delta}^{CC}_k \Sigma^{CCAA}_{\mu\,\nu}(-k,k,p,-p) + \dot{\Delta}^{A,CC}_{\mu}(p;k-p,-k)\Sigma^{CCA}_{\nu}(p-k,k,-p) \\
& + \dot{\Delta}^{AA,CC}_{\mu\,\nu}(p,-p;k,-k)\Sigma^{CC}(k)
\end{aligned}$$

$$-\dot{\Delta}_k^{DD}\Sigma^{DDAA}_{\mu\nu}(-k,k,p,-p) - \dot{\Delta}_\mu^{A,DD}(p;k-p,-k)\Sigma^{DDA}_{\nu}(p-k,k,-p)$$
$$-\dot{\Delta}_{\mu\nu}^{AA,DD}(p,-p;k,-k)\Sigma^{DD}(k)\Big\}\Big|_{p^2} \tag{5.19}$$

where $\int_k = \int \frac{d^D k}{(2\pi)^D}$ and all the Σ's are at tree level but the subscript "0" has been omitted. What we have to find at this stage are the vertices at tree level shown in Tab.5.1. The last two lines list the vertices needed in order to calculate the oth-

4-point	3-point	2-point
$S^{(0)AAAA}_{\mu\nu\rho\sigma}(p,q,r,s)$	$S^{(0)AAA}_{\mu\nu\rho}(p,q,r)$	$S^{(0)AA}_{\mu\nu}(p)$
$S^{(0)BBAA}_{\mu\nu\rho\sigma}(p,q,r,s)$	$S^{(0)BBA}_{\mu\nu\rho}(p,q,r)$	$S^{(0)BB}_{\mu\nu}(p)$
$S^{(0)CCAA}_{\mu\nu}(p,q,r,s)$	$S^{(0)CCA}_{\mu}(p,q,r)$	$S^{(0)CC}(p)$
$S^{(0)DDAA}_{\mu\nu}(p,q,r,s)$	$S^{(0)DDA}_{\mu}(p,q,r)$	$S^{(0)DD}(p)$
	$S^{(0)ABD\sigma}_{\mu\nu}(p,q,r)$	$S^{(0)BD\sigma}_{\mu}(p)$
	$S^{B(0)AD\sigma}_{\mu\nu}(p,q,r)$	

Table 5.1: Tree level vertices needed to calculate β at one loop

ers. The two three-point vertices in the second part of Table 5.1 are related via a symmetry of the theory, which will be discussed in the next section, called "Charge Conjugation". In particular: $S^{ABD\sigma}_{\mu\nu}(p,q,r) = -S^{BAD\sigma}_{\nu\mu}(q,p,r)$.

5.2 Symmetries

Before we continue studying the equations of the vertices needed to perform our calculation, it is worth spending some words about the symmetries of the theory we are considering.

The main symmetry involved here is, of course, supergauge invariance under $SU(N|N)$ transformation. This invariance must hold for the action we want to describe, and

it is preserved along its flow due to the properties of the flow equation and the way it has been constructed. Expanding the action in fields and imposing it is invariant under applying to them the transformations expressed by (3.40,3.44), it is possible to get constraint on the vertices, *i.e.* Ward identities. These identities are expressed by eqs. (5.29), (5.30) and (5.31), while a pictorial representation is given in fig. 5.1. This symmetry is of great importance and preserving it in the flowing action was the

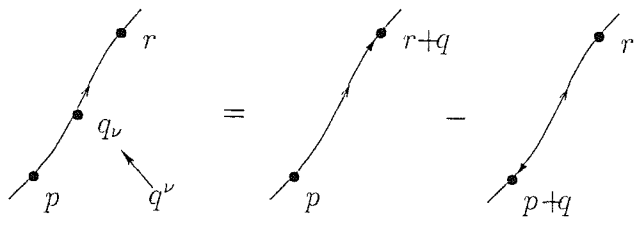

Figure 5.1: Graphical representation of gauge invariance identities.

purpose of the whole thesis. It has been used through the various steps of the calculation of β at one-loop as a check, helping us to be sure it was correctly performed. Since the supergauge theory will have to be spontaneously broken, in order to the regularisation to work, it is convenient to find an expansion for the action in terms of a different basis, *i.e.* the shifted fields. Imposing the corresponding transformations to hold, it is possible then to find the Ward identities for the vertices in the broken phase, which is the basis that will be used in the calculation. This issue will be addressed in section 5.2.1.

Another relevant symmetry involved here, and which follows directly from the construction of the action expanded in supertraces, is the cyclicity. This property ensures that vertices can be set equal if their arguments (*i.e.* momenta and indices), are related by cyclic permutations. An example of this property is expressed below for the three-point vertex of pure A:

$$S^{AAA}_{\mu\nu\rho}(p,q,r) = S^{AAA}_{\nu\rho\mu}(q,r,p) = S^{AAA}_{\rho\mu\nu}(r,p,q) \tag{5.20}$$

Cyclicity is automatically incorporated in the diagrammatic representation and it is one of the reasons that makes it so powerful. As an example, all the three vertices of eq. (5.20) are represented by the same diagram in fig. 5.2

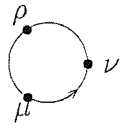

Figure 5.2: Diagram representing the three vertices of eq. (5.20)

A further symmetry, which we have already mentioned in the previous section is "Charge Conjugation" (CC). This invariance has to hold since the final goal is to describe a $SU(N)$ Yang-Mills theory, which has this symmetry, and it is imposed by asking the action to be invariant under the transformation:

$$\mathcal{A}_\mu \rightarrow -\mathcal{A}_\mu^T \tag{5.21}$$

$$\mathcal{C} \rightarrow \mathcal{C}^T \tag{5.22}$$

This requirement sets as well relations on the vertices, and in particular sets equal those with inverted set of arguments, up to a minus sign to the power of the number of gauge fields. Namely:

$$S^{X_1\cdots X_n}_{a_1\cdots a_n}(p_1,\ldots,p_n) = (-)^r S^{X_n\cdots X_1}_{a_n\cdots a_1}(p_n,\ldots,p_1) \tag{5.23}$$

where r is the number of gauge fields in $X_1,\ldots,X_n$. This property is easily expressed in diagrammatic form, as can be seen in fig. 5.3. Through CC symmetry, it has been possible to rule out some vertices, which might have appeared in the action otherwise, and might have caused trouble. One of them is the two-point vertex $AC\sigma$, which, assuming the action invariant for charge conjugation, is identically zero (it also vanishes for other reasons, such as gauge invariance and Lorentz invariance). This

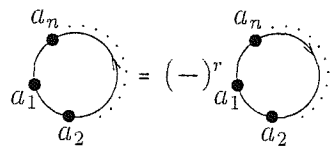

Figure 5.3: Diagrammatic representation of CC invariance

fact will lead the two equations for AA and CC to be decoupled, while those for BB and DD will be coupled through and with the equation for the non-zero vertex $BD\sigma$. Other important properties can be extracted imposing this symmetry , also on wine vertices. One of them, widely used throughout the whole thesis, relates one-point wine vertices and reads (in the case of a pure gauge wine vertex):

$$\dot{\Delta}_{\mu}^{A,AA}(p;q,r) = -\dot{\Delta}_{\mu}^{A,AA}(p;r,q) \tag{5.24}$$

The last invariance we will describe in this section and which was already discussed in the previous chapter, is the no-$\mathcal{A}^0$ symmetry. As it was already discussed there, this symmetry has to be imposed for the action as we start considering the supergauge field $\mathcal{A}_\mu$ containing the identity matrix in its expansion on the generators of $SU(N|N)$, if we do not want its presence to create a non linear constraint on the theory. Its requirement reflects on the vertices through a set of constraints, obtained imposing the transformation

$$\delta\mathcal{A}_{\mu}^{0} = \Lambda_{\mu}(x), \tag{5.25}$$

to be an invariance for the expanded action. Given for example the four-point pure A vertex, we can imagine for example that one of them were an $\mathcal{A}_{\mu}^{0}$. In this case invariance for eq. (5.25) would give the following relation:

$$S'^{AAAA}_{\mu\nu\rho\sigma}(p,q,r,s) + S'^{AAAA}_{\mu\rho\sigma\nu}(p,r,s,q) + S'^{AAAA}_{\mu\rho\nu\sigma}(p,r,q,s) = 0 \tag{5.26}$$

If two of them were $\mathcal{A}^0_\mu$, the symmetry would give us a different constraint:

$$S'^{AAAA}_{\mu\nu\rho\sigma}(p,q,r,s) + S'^{AAAA}_{\mu\rho\sigma\nu}(p,r,s,q) + S'^{AAAA}_{\mu\rho\nu\sigma}(p,r,q,s) +$$
$$S'^{AAAA}_{\mu\sigma\nu\rho}(p,s,q,r) + S'^{AAAA}_{\mu\sigma\rho\nu}(p,s,r,q) + S'^{AAAA}_{\mu\nu\sigma\rho}(p,q,s,r) = 0 \qquad (5.27)$$

This second relation, since must hold together with the first one give us in particular:

$$S'^{AAAA}_{\mu\sigma\rho\nu}(p,s,r,q) + S'^{AAAA}_{\mu\nu\sigma\rho}(p,q,s,r) + S'^{AAAA}_{\mu\sigma\nu\rho}(p,s,q,r) = 0 \qquad (5.28)$$

(5.26,5.28) must hold for the four-point pure A vertices, for S to be invariant under no-$\mathcal{A}^0$ symmetry. Their diagrammatic form is expressed in fig. 5.4. As can be seen more clearly from fig. 5.4, the two previous equations are the same constraint through Charge Conjugation invariance. If three or four of them were $\mathcal{A}^0$'s we would not get any constraint since, we would be left respecively with $\mathrm{str}(\mathcal{A}_\mu)$ and $\mathrm{str}(\mathbb{1})$ which are already zero.

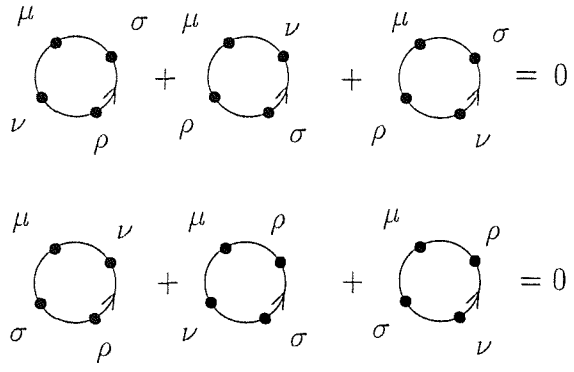

Figure 5.4: Diagrammatic representation of no-$\mathcal{A}^0$ symmetry for the four-point pure A vertex.

Relations like these can be found for all possible vertices. This symmetry will be particularly useful in the calculation for the second coefficient of the β-function, as can be found in [37]. As far as the present work is concerned, this symmetry played a crucial rôle together with CC invariance, in order to rule out the vertex $AAC\sigma$. As

we will see in the last chapter, this allowed us to treat the three different sectors A, C and the sector B and D[2], in a similar way.

Other important consequences of this invariance, relevant to the present work were already described in section 3.3.2 and will not be repeated here.

5.2.1 Ward identities in the broken phase

As we have already mentioned earlier in this chapter, supergauge invariance is the most relevant symmetry throughout the present work, and since we will be working in the broken phase of the theory, it is worth exploring the Ward identities in this regime. According to the split fields notation which has just been introduced in eq. (4.26), they read:

$$q^{\nu}U^{\cdots XAY\cdots}_{\cdots a\,\nu\,b\cdots}(\cdots p,q,r,\cdots) = U^{\cdots XY\cdots}_{\cdots a\,b\cdots}(\cdots p,q+r,\cdots) - U^{\cdots XY\cdots}_{\cdots a\,b\cdots}(\cdots p+q,r,\cdots) \tag{5.29}$$

$$\begin{aligned} q^{\nu}U^{\cdots XBY\cdots}_{\cdots a\,\nu\,b\cdots}(\cdots p,q,r,\cdots) &= \pm U^{\cdots X\hat{Y}\cdots}_{\cdots a\,b\cdots}(\cdots p,q+r,\cdots) \mp U^{\cdots\hat{X}Y\cdots}_{\cdots a\,b\cdots}(\cdots p+q,r,\cdots) \\ &\quad +2U^{\cdots XD\sigma Y\cdots}_{\cdots a\quad b\cdots}(\cdots p,q,r,\cdots) \end{aligned} \tag{5.30}$$

The notation of Eqs.(5.29) and (5.30) needs a brief explanation. U represent any vertex either from the expansion of the effective action or a wine vertex. X and Y are generic fields and the indices a and b, respectively referring to X and Y, are either Lorentz indices or nothing, according to the scalar or vectorial nature of X and Y. The "$\wedge$" on the fields in the RHS of eq.(5.30) indicates a change in the spin-statistic nature of the fields, *i.e.* $\hat{A}_\mu = B_\mu$, $\hat{B}_\mu = A_\mu$ and so on. Finally, the signs in the first line of eq.(5.29) are $+$ and $-$ if there is no σ between X and B and Y and B, and

[2] B and D will be considered together, for reasons which will become clear later

opposite sign in the other case.

We write separately the identities for wine vertices where the point is at the end of the line:

$$p_1^{\mu_1} W^{AX_2\cdots X_n,Y_1\cdots Y_m,Z_1Z_2}_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m}(p_1,\ldots,p_n;q_1,\ldots,q_m;r,s) = \tag{5.31}$$
$$W^{X_2\cdots X_n,Y_1\cdots Y_m Z_1Z_2}_{\mu_2\cdots\mu_n,\nu_1\cdots\nu_m}(p_1+p_2,p_3,\ldots,p_n;q_1,\ldots,q_m;r,s)$$
$$-W^{X_2\cdots X_n,Y_1\cdots Y_m Z_1Z_2}_{\mu_2\cdots\mu_n,\nu_1\cdots\nu_m}(p_2,\ldots,p_n;q_1,\ldots,q_m;r+p_1,s)$$

If the field on the wine hit by $p_1^{\mu_1}$ were a B we would have had a relation similar to eq. (5.30). Similar identities would be obtained hitting the vertex with momenta $p_n^{\mu_n}$, $q_n^{\nu_1}$ and $q_m^{\nu_m}$ as is clear from fig. 5.1.

We can now see them in some particular examples. For the two-point vertices in the effective action expansion (the same applies to the $\hat{S}$ vertices), for example, we have:

$$p^\mu S'^{AA}_{\mu\,\nu}(p) = 0 \tag{5.32}$$
$$p^\mu S'^{BB}_{\mu\,\nu}(p) = -2S'^{BD\sigma}_{\nu}(p) \tag{5.33}$$
$$p^\mu S'^{BD\sigma}_{\mu}(p) = -S'^{DD}(p) \tag{5.34}$$

The first one, simply states that in the A sector the gauge invariance is not broken, and the A propagator is still transverse. The last two, relate vertices in the broken sector. Another observation which can be done is that, since the seed action $\hat{S}$ must be gauge invariant itself, (5.29) and (5.30) must apply to its vertices. It is easy to check for the two-point ones, since we have written explicitly the expressions for their identities in (5.32-5.34). It is straightforward to see that (5.32) is true for (5.12) since it is proportional to the transverse tensor $\Box_{\mu\nu}(p)$. Moreover, we can see that,

applying p^μ to (5.13) and comparing with (5.16), we get

$$p^\mu \hat{S}^{BB}_{\mu\nu}(p) = p^\mu \left(\hat{S}^{AA}_{\mu\nu}(p) + 4\frac{\Lambda^2}{\tilde{c}_p}\delta_{\mu\nu} \right) = 4\frac{\Lambda^2}{\tilde{c}_p} p_\nu = -2\hat{S}^{BD\sigma}_{\nu}(p) \tag{5.35}$$

and applying it to (5.16) and comparing with (5.15):

$$p^\mu \hat{S}^{BD\sigma}_{\mu}(p) = p^\mu \left(-2\frac{\Lambda^2 p_\mu}{\tilde{c}_p} \right) = -2\frac{\Lambda^2 p^2}{\tilde{c}_p} = -2\hat{S}^{DD}(p) \tag{5.36}$$

Another check which can be done is on the two-point function equations (5.38) and (5.40)-(5.42). If everything is consistent the first should give zero if contracted with p^μ and the others should be connected by Ward identities. This is indeed the case, as it can be verified applying (5.29) and (5.30) to them (the same of course should apply for the higher point equations). Relations (5.29) and (5.30) will be useful for calculations and other checks in the present section and in the next chapter.

5.3 Tree level vertices

After having discussed the importance of the symmetries of the flowing effective action and their consequences on its vertices, we can now concentrate on the equations which we need to extract the β-function at one-loop from eq. (5.1). They are all listed in Tab.5.1. Since they are all tree level vertices, let us consider first the tree level equation (4.41) which we rewrite here in its extended form:

$$\Lambda \frac{\partial S_0}{\partial \Lambda} = \sum_{f=\mathcal{A},\mathcal{C}} \frac{1}{2} \frac{\delta S_0}{\delta f} \{\dot{\Delta}^{ff}\} \frac{\delta \Sigma_0}{\delta f} \tag{5.37}$$

where just in order to have a more compact equation the incorporated $\mathcal{C}$-wines notation has been used. The previous equation is shown diagrammatically in fig. 5.5.

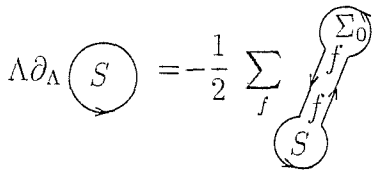

Figure 5.5: Tree level equation; f can be either an A, B, C or D field

We can now specify eq. (5.37) to the vertices of Tab 5.1.

5.3.1 Two point tree level vertices and kernels

We will start studying the equations for the two-point vertices. The kernels (zero-point wine vertices) in the basis we are working (the ones listed in eq. (5.4)), will be determined here through the request of the seed action two-point vertices to equal the tree level ones. They are represented diagrammatically in fig. 5.6.

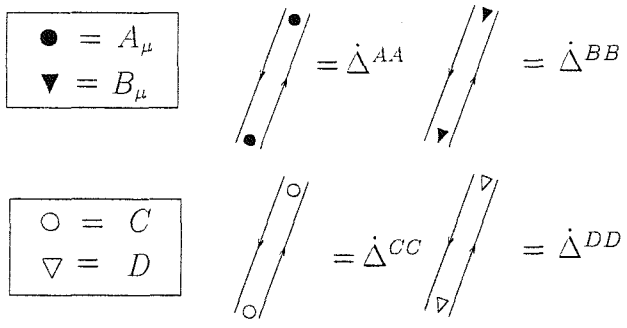

Figure 5.6: Graphical representation of 0-point wines

The first step is now, once the equations for the two-point vertices are written, to assume the request expressed in eq. (5.11). Since the form of the vertices of the two point is set by the argument described in section 5.1, this gives rise to a set of equations for the kernels (as we have mentioned earlier). Because of the extra terms in the flow equation that we have introduced through the "decoration" of the wines, we

have enough freedom to impose these constraint and the set of equations has actually a solution that will be evaluated later in the present section.

The first two-point tree level equation to be studied here, is going to be the two A's. In the graphical representation, it is shown in fig. 5.7.

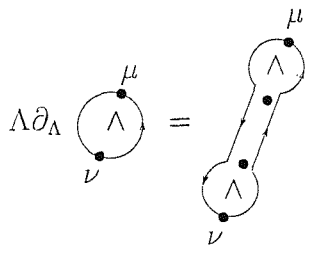

Figure 5.7: AA tree level equation

The equation in formulae reads:

$$\Lambda\partial_\Lambda \hat{S}^{AA}_{\mu\nu}(p) = \hat{S}^{AA}_{\mu\alpha}(p)\ \dot{\Delta}^{AA}_{p}\hat{S}^{AA}_{\alpha\nu}(p) \tag{5.38}$$

where $\hat{S}^{AA}$ has the expression of eq. (5.12). Before we solve the equation for $\dot{\Delta}^{AA}$ we are going to list them all.

The next one to be considered is then the two C's equation. Diagrammatically it is represented in fig. 5.8.

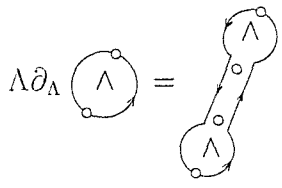

Figure 5.8: CC tree level equation

While in formulae we have:

$$\Lambda\partial_\Lambda \hat{S}^{CC}(p) = \hat{S}^{CC}(p)\ \dot{\Delta}_p^{CC}\ \hat{S}^{CC}(p) \tag{5.39}$$

As one can notice, these first two equations are decoupled, and one can solve for $\dot{\Delta}^{AA}$ and $\dot{\Delta}^{CC}$. As we will shortly see, the last three equations, for BB, $BD\sigma$ and DD vertices, will be coupled. The way they are connected, allows to write all of them in one, introducing a compact notation for this sector, which will be introduced in the next chapter for the gauge invariant calculation. We will refer to this sector as the $BD\sigma$ sector. Here they will be kept separate.

The first of the three is the equation for the vertex BB. In fig. 5.9 it is displayed in its diagrammatic notation.

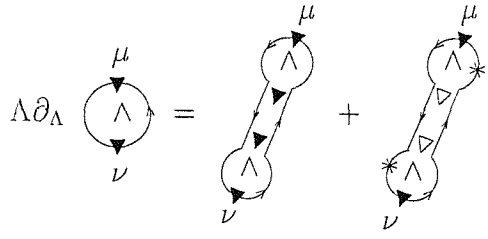

Figure 5.9: BB tree level equation; the "star" represents a σ.

Its analytical form is the following:

$$\Lambda\partial_\Lambda \hat{S}^{BB}_{\mu\,\nu}(p) = \hat{S}^{BB}_{\mu\,\alpha}(p)\ \dot{\Delta}_p^{BB} \hat{S}^{BB}_{\alpha\,\nu}(p) + \hat{S}^{BD\sigma}_{\mu}(p)\ \dot{\Delta}_p^{DD}\ \hat{S}^{BD\sigma}_{\nu}(p) \tag{5.40}$$

where for the only non-symmetric vertex in the change $p \leftrightarrow -p$, namely $\hat{S}^{BD\sigma}_{\mu}$, the argument $+p$ refers to the momentum of the first field, and: $S^{BD\sigma}_{\nu}(-p,p) = S^{D\sigma B}_{\nu}(p,-p) = S^{D\sigma B}_{\nu}(p) = -S^{BD\sigma}_{\nu}(p)$. The sign in the second term is then determined by an extra minus sign, from the fact that one of the σ's, in order to "hit" the other one and give the same supertrace as the LHS ($\mathrm{str}B_\mu B_\nu$), must pass through a fermionic

field. This result is more clear from its graphical representation of fig. 5.9.

The equation for the $BD\sigma$ vertex is represented graphically in Fig.5.10.

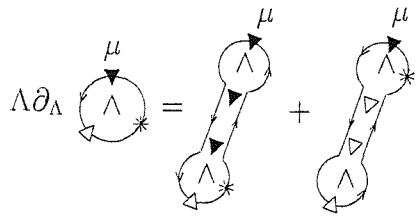

Figure 5.10: $BD\sigma$ tree level equation

In formulae is:

$$\Lambda\partial_\Lambda \hat{S}^{BD\sigma}_{\mu}(p) = \hat{S}^{BB}_{\mu\alpha}(p)\ \dot{\Delta}^{BB}_{p}\ \hat{S}^{BD\sigma}_{\alpha}(p) + \hat{S}^{BD\sigma}_{\mu}(p)\ \dot{\Delta}^{DD}_{p}\ \hat{S}^{DD}(p) \tag{5.41}$$

Finally, the equation for the last two-point vertex, DD, is represented graphically in Fig.5.11 and in formulae in:

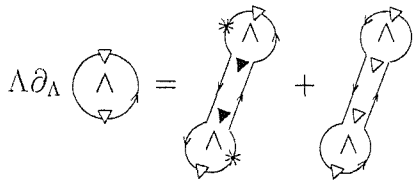

Figure 5.11: DD tree level equation

$$\Lambda\partial_\Lambda \hat{S}^{DD}(p) = \hat{S}^{BD\sigma}_{\alpha}(p)\ \dot{\Delta}^{BB}_{p}\ \hat{S}^{BD\sigma}_{\alpha}(p) + \hat{S}^{DD}(p)\ \dot{\Delta}^{DD}_{p}\ \hat{S}^{DD}(p) \tag{5.42}$$

We are now ready to extract the four kernels from these five equations. The first two equations give us two conditions to determine the two A's and two C's kernels and the last three for the B and D ones. As one might think the set of three coupled equations

is not overconstrained because they are related to each other via Ward identities. It is in fact possible to check that starting from the equation for the two B's vertex, one can obtain the other two via contracting respectively once or twice with a momentum p^μ (respectively $-p^\nu$)[3]. They set then two conditions: one is given by the equation for the transverse part of $\hat{S}^{BB}_{\mu\nu}$, the other one by the equation for its non transverse part, or by one of the last two equations related via gauge invariance. What we find are function of $x = p^2/\Lambda^2$, written in terms of the two functions c and $\tilde{c}$ introduced in Eqs.(5.12)-(5.16). They will be the momentum space kernels, which will need to be covariantised as described in section 4.3. Solving the two-point equations for the kernels, we finally get:

$$\dot{\Delta}^{AA}_p = \frac{1}{\Lambda^2} c' \tag{5.43}$$

$$\dot{\Delta}^{BB}_p = \frac{1}{\Lambda^2} \left(\frac{x c \tilde{c}}{x\tilde{c} + 2c} \right)' \tag{5.44}$$

$$\dot{\Delta}^{CC}_p = \frac{1}{\Lambda^4 x} \left(\frac{x^2 \tilde{c}}{x + 2\lambda c} \right)' \tag{5.45}$$

$$\dot{\Delta}^{DD}_p = \frac{1}{\Lambda^4 x} \left(\frac{2x^2 \tilde{c}^2}{x\tilde{c} + 2c} \right)' \tag{5.46}$$

where c and $\tilde{c}$ are intended as functions of x and the prime stands for the derivative with respect to this argument.

We just spend a few words on how the equations may be solved. For the two decoupled ones, it was just a question of substituting the two-point vertices and solve for the kernels. For the coupled ones, a possible way was to contract the equation for the BB vertex with the combination $\Box_{\mu\nu}$. In such a way, in eq. (5.40) we are left with the first member, containing only $\dot{\Delta}^{BB}$. It is then possible to solve directly for that kernel, and substituting it in the equation for the two D's vertex, get the final kernel $\dot{\Delta}^{DD}$.

[3]This and others were some of the checks made possible by gauge invariance, that let us keep under control the calculation

It is now time to move to the higher point vertex equations.

5.3.2 Three and four point tree level vertices

The equations we finally have to find are the ones for the four-points and three points vertices which are needed to extract the 1-loop β-function (plus the three point needed to find the four point). Some of these vertices are directly needed to be substituted in the 1-loop equation. Others will have to be used to calculate the latter.

An important remark must be done at this point. Although we would expect these tree level vertices to be finite, as will be discussed in the next two sections, this is not true in general (for example if $\hat{S}$ is the one chosen in [34]). However, this problem can be fixed with an appropriate shift of the hatted vertices. This will be shown for an explicit example in the last section of this chapter. We will then have to add a further request on the seed action, which is to ensure there are no classical divergences.

While writing down the equations for these higher point vertices, we will need the covariantised kernels described in section 4.3. This did not happen for two-point vertices, because had we considered a field on the wines, we would have been left with one point vertices, which are ruled out from this theory. From three point vertices onwards, we can instead consistently construct supertraces of three fields, having one of them (or more) on the wines *i.e.* coming from the expansion in fields of the covariantisation of the kernels. In fig. 5.12 these one-point wines can be found in their graphical representation. Even though $\dot{\Delta}_\mu^{A,AA}(p;q,r)$, $\dot{\Delta}_\mu^{B,AB}(p;q,r)$ and $\dot{\Delta}_\mu^{B,BA}(p;q,r)$ are all equal to $1/\Lambda^2 c'_\mu(p;q,r)$ (since they all come from the covariantisation of $\dot{\Delta}^{AA} = c'/\Lambda^2$) they will be indicated in the equations as they appear in fig. 5.12, to make more clear the link between their analytic and diagrammatic form. The same applies to the three covariantisations of $\dot{\Delta}^{CC}$, namely $\dot{\Delta}_\mu^{A,CC}(p;q,r)$, $\dot{\Delta}_\mu^{B,CD}(p;q,r)$ and $\dot{\Delta}_\mu^{B,DC}(p;q,r)$. One could also check that $\dot{\Delta}^{\sigma D,BA}(p;q,r) = \dot{\Delta}_{m\ q}^{AA}/2$

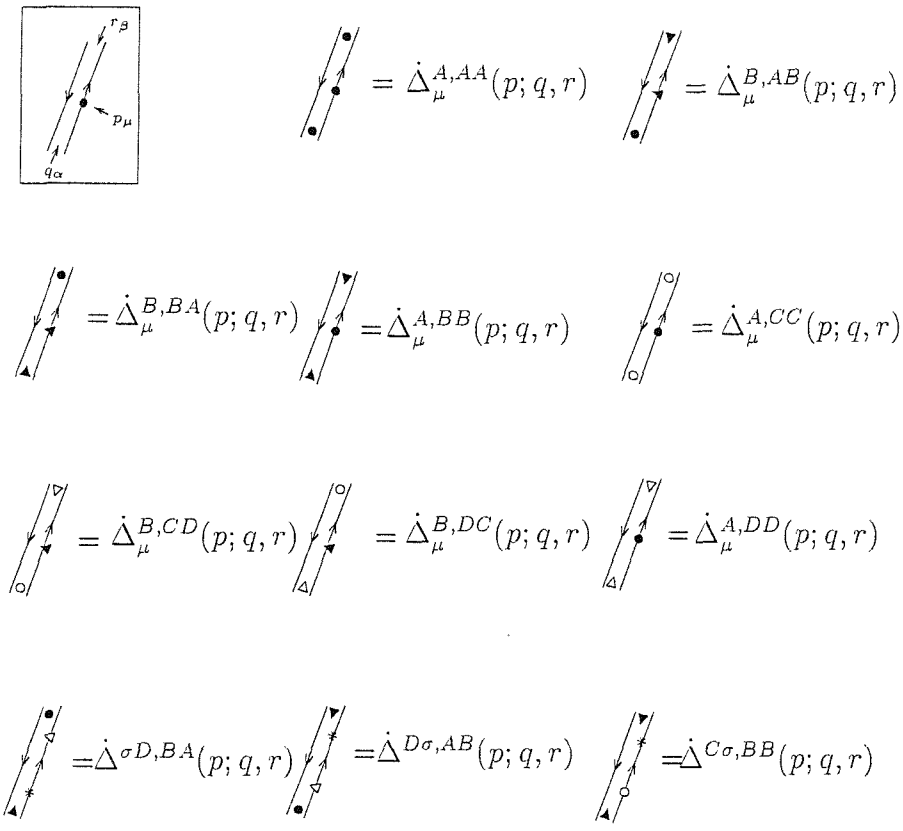

Figure 5.12: Graphical representation of the 1-point Wines

and $\dot{\Delta}^{C\sigma,BB}(p;q,r) = (\dot{\Delta}^{\mathcal{A}\mathcal{A}}_{m\ q} + \dot{\Delta}^{\mathcal{A}\mathcal{A}}_{m\ r})/2$, from eq. (5.2).

We can start now with the first three point vertex we want to consider which is the three A's vertex. Its equation is shown in the diagrammatic representation in Fig.5.13.

In formulae is eq. (5.47). The other 3-point vertex equations follow it[4].

$$S^{AAA}_{\mu\nu\rho}(p,q,r) = -\int_\Lambda^\infty \frac{d\Lambda_1}{\Lambda_1}\left(\hat{S}^{AAA}_{\mu\nu\alpha}(p,q,r)\dot{\Delta}^{AA}_r + \hat{S}^{AA}_{\mu\alpha}(p)\dot{\Delta}^{A,AA}_\nu(q;p,r)\right)\hat{S}^{AA}_{\alpha\rho}(r)$$
$$+2(p_\rho\delta_{\mu\nu} - p_\nu\delta_{\rho\mu}) + \text{cycles}$$

[4]If not stated otherwise, in all the formulae, we will drop the label (0) and S will be intended as S_0, tree level action

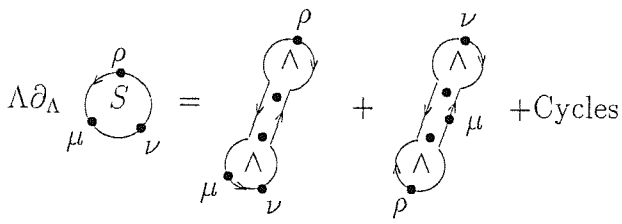

Figure 5.13: Diagrammatic representation of the three A's vertex's equation

(5.47)

$$\begin{aligned}
S'^{BBA}_{\mu\nu\rho}(p,q,r) = -\int_\Lambda^\infty \frac{d\Lambda_1}{\Lambda_1}\Big\{ & \hat{S}^{BBA}_{\mu\nu\alpha}(p,q,r)\dot{\Delta}^{AA}_r \hat{S}^{AA}_{\alpha\rho}(r) + \hat{S}^{BBA}_{\alpha\nu\rho}(p,q,r)\dot{\Delta}^{BB}_p \hat{S}^{BB}_{\alpha\mu}(p) \\
& + \hat{S}^{BBA}_{\mu\alpha\rho}(p,q,r)\dot{\Delta}^{BB}_q \hat{S}^{BB}_{\alpha\nu}(q) + \hat{S}^{BB}_{\mu\alpha}\dot{\Delta}^{A,BB}_\rho(r;q,p)\hat{S}^{BB}_{\alpha\nu}(q) \\
& + \hat{S}^{BB}_{\mu\alpha}(p)\dot{\Delta}^{B,BA}_\nu(q;p,r)\hat{S}^{AA}_{\alpha\rho}(r) + \hat{S}^{BB}_{\rho\alpha}(r)\dot{\Delta}^{B,AB}_\mu(p;q,r)\hat{S}^{AA}_{\alpha\nu}(q) \\
& - \hat{S}^{BAD\sigma}_{\nu\rho}(q,r)\dot{\Delta}^{DD}_p \hat{S}^{BD\sigma}_{\mu}(p) - \hat{S}^{ABD\sigma}_{\rho\mu}(r,p)\dot{\Delta}^{DD}_q \hat{S}^{BD\sigma}_{\nu}(q) \\
& - \hat{S}^{BD\sigma}_{\mu}(p)\dot{\Delta}^{A,DD}_\rho(r;q,p)\hat{S}^{BD\sigma}_{\nu}(q)\Big\} + \text{Int.Const.}
\end{aligned}$$

(5.48)

$$\begin{aligned}
S'^{ACC}_{\mu}(p,q,r) = -\int_\Lambda^\infty \frac{d\Lambda_1}{\Lambda_1}\Big\{ & \hat{S}^{ACC}_{\alpha}(p,q,r)\dot{\Delta}^{AA}_p \hat{S}^{AA}_{\alpha\mu}(p) + \hat{S}^{ACC}_{\mu}(p,q,r)\dot{\Delta}^{CC}_r \hat{S}^{CC}(r) \\
& + \hat{S}^{ACC}_{\mu}(p,q,r)\dot{\Delta}^{CC}_q \hat{S}^{CC}(q) + \hat{S}^{CC}(r)\dot{\Delta}^{A,CC}_\mu(p;r,q)\hat{S}^{CC}(q)\Big\} + I.C.
\end{aligned}$$

(5.49)

$$\begin{aligned}
S'^{ADD}_{\mu}(p,q,r) = -\int_\Lambda^\infty \frac{d\Lambda_1}{\Lambda_1}\Big\{ & \hat{S}^{AA}_{\mu\alpha}(p)\dot{\Delta}^{AA}_p \hat{S}'^{ADD}_{\alpha}(p,q,r) + \hat{S}^{BAD\sigma}_{\alpha\mu}(r,p,q)\dot{\Delta}^{BB}_r \hat{S}^{BD\sigma}_{\alpha}(r) \\
& + \hat{S}^{ABD\sigma}_{\mu\alpha}(p,q,r)\dot{\Delta}^{BB}_q \hat{S}^{BD\sigma}_{\alpha}(q) - \hat{S}^{BD\sigma}_{\alpha}(r)\dot{\Delta}^{A,BB}_\mu(p;r,q)\hat{S}^{BD\sigma}_{\alpha}(q) \\
& - \hat{S}^{AA}_{\mu\alpha}(p)\dot{\Delta}^{D\sigma,AB}(q;p,r)\hat{S}^{BD\sigma}_{\alpha}(r) + \hat{S}^{BD\sigma}_{\alpha}(q)\dot{\Delta}^{D\sigma,BA}(r;q,p)\hat{S}^{AA}_{\alpha\mu}(p) \\
& + \hat{S}^{ADD}_{\mu}(p,q,r)\dot{\Delta}^{DD}_r \hat{S}^{DD}(r) + \hat{S}^{ADD}_{\mu}(p,q,r)\dot{\Delta}^{DD}_q \hat{S}^{DD}(q) \\
& + \hat{S}^{DD}(r)\dot{\Delta}^{A,DD}_\mu(p;r,q)\hat{S}^{DD}(q)\Big\} + I.C.
\end{aligned}$$

(5.50)

$$S^{ABD\sigma}_{\mu\nu}(p,q,r) = -\int_\Lambda^\infty \frac{d\Lambda_1}{\Lambda_1}\Big\{-\hat{S}^{BBA}_{\nu\alpha\mu}(q,r,p)\dot{\Delta}^{BB}_r\hat{S}^{BD\sigma}_{\alpha}(r) + \hat{S}^{ABD\sigma}_{\alpha\nu}(p,q,r)\dot{\Delta}^{AA}_p\hat{S}^{AA}_{\alpha\mu}(p)$$
$$+\hat{S}^{ABD\sigma}_{\mu\alpha}(p,q,r)\dot{\Delta}^{BB}_q\hat{S}^{BB}_{\alpha\nu}(q) - \hat{S}^{AA}_{\mu\alpha}(p)\dot{\Delta}^{B,AB}_{\nu}(q;p,r)\hat{S}^{BD\sigma}_{\alpha}(r)$$
$$-\hat{S}^{BB}_{\nu\alpha}(q)\dot{\Delta}^{A,BB}_{\mu}(p;r,q)\hat{S}^{BD\sigma}_{\alpha}(r) + \frac{1}{2}\hat{S}^{AA}_{\mu\alpha}(p)\dot{\Delta}^{D\sigma,BA}(r;q,p)\hat{S}^{BB}_{\alpha\nu}(q)$$
$$+\hat{S}^{ABD\sigma}_{\mu\nu}(p,q,r)\dot{\Delta}^{DD}_r\hat{S}^{DD}(r) + \hat{S}^{ADD}_{\mu}(p,q,r)\dot{\Delta}^{DD}_q\hat{S}^{BD\sigma}_{\nu}(q)$$
$$+\hat{S}^{BD\sigma}_{\nu}(q)\dot{\Delta}^{A,DD}_{\mu}(p;r,q)\hat{S}^{DD}(r)\Big\} + I.C.$$

(5.51)

The four point needed are four: AAAA, AABB, AADD, AACC. The equation for the

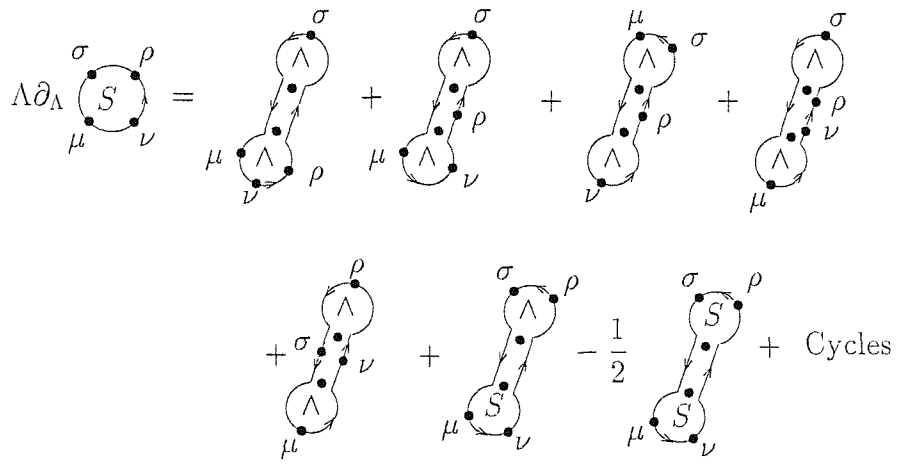

Figure 5.14: Diagrammatic representation of the four A's vertex's equation

4-A's vertex is shown in its diagrammatic representation in Fig.5.14 and in formulae:

$$S^{AAAA}_{\mu\nu\rho\sigma}(p,q,r,s) = -\int_\Lambda^\infty \frac{d\Lambda_1}{\Lambda_1^3}\Big\{\hat{S}^{AAAA}_{\mu\nu\rho\alpha}(p,q,r,s)\dot{\Delta}^{AA}_s\hat{S}^{AA}_{\alpha\sigma}(s)$$
$$-\frac{1}{2}S^{AAA}_{\mu\nu\alpha}(p,q,r+s)\dot{\Delta}^{AA}_{p+q}S^{AAA}_{\alpha\rho\sigma}(p+q,r,s)$$
$$+\frac{1}{2}\hat{S}^{AAA}_{\mu\nu\alpha}(p,q,r+s)\dot{\Delta}^{AA}_{p+q}S^{AAA}_{\alpha\rho\sigma}(p+q,r,s)$$

$$+\frac{1}{2}S^{AAA}_{\mu\nu\alpha}(p,q,r+s)\dot{\Delta}^{AA}_{p+q}\hat{S}^{AAA}_{\alpha\rho\sigma}(p+q,r,s)$$
$$+\hat{S}^{AAA}_{\mu\nu\alpha}(p,q,r+s)\dot{\Delta}^{A,AA}_{\rho}(r;p+q,s)\hat{S}^{AA}_{\alpha\sigma}(s)$$
$$+\hat{S}^{AAA}_{\sigma\mu\alpha}(s,p,q+r)\dot{\Delta}^{A,AA}_{p}(r;q,p+s)\hat{S}^{AA}_{\alpha\nu}(q)$$
$$+\hat{S}^{AA}_{\mu\alpha}(p)\dot{\Delta}^{AA,AA}_{\nu\rho}(q,r;p,s)\hat{S}^{AA}_{\alpha\sigma}(s)$$
$$+\frac{1}{2}\hat{S}^{AA}_{\mu\alpha}(p)\dot{\Delta}^{A,A,AA}_{\nu,\sigma}(q;s;p,r)\hat{S}^{AA}_{\alpha\rho}(r)+\text{cycles}\Big\}+I.C. \tag{5.52}$$

In the previous equations, I.C. stands for "Integration Constant" and was not written explicitly here but in the three A's equation (5.47). For the other vertices, the divergent parts of the integrating constants will be discussed in the next two sections. Their finite parts are given in Appendix F.

The equations for the four-point vertices left out here, are listed in Appendix C

5.3.3 Enforcing universality of β_1 (and β_2)

Before we discuss the finiteness of the tree level vertices from the previous section, let us review in this context, the standard argument for why we should expect to get the same value for β_1, and indeed β_2, in the β function eq. (4.40) as in other methods, despite the fact that our renormalisation scheme for $g(\Lambda)$ differs from that of the corresponding coupling $\tilde{g}(\mu \mapsto \Lambda)$ defined by these other methods.

In principle we can extract from eq. (4.39), by computing quantum corrections, the value of the other coupling as a function of ours, and thus match the two couplings perturbatively:

$$1/\tilde{g}^2 = 1/g^2 + \gamma + O(g^2), \tag{5.53}$$

where the classical agreement is guaranteed by the standard normalisations of the

fields and kinetic term in eq. (4.37), after scaling g back to its usual position, and γ is a one-loop matching coefficient. Differentiating with respect to Λ and using eq. (4.40), the corresponding β function for $\tilde{g}$, and eq. (5.53), we have

$$\tilde{\beta}_1 + \tilde{\beta}_2\, g^2 = \beta_1 + \beta_2\, g^2 + \Lambda\partial_\Lambda\gamma + O(g^4). \tag{5.54}$$

Since γ is dimensionless, it cannot depend upon Λ, there being no other scale to form the necessary dimensionless combination. Thus $\Lambda\partial_\Lambda\gamma = 0$ in eq. (5.54), and we immediately recover the standard facts that $\tilde{\beta}_1 = \beta_1$ and $\tilde{\beta}_2 = \beta_2$.

Clearly this argument fails if some other scale has been introduced, for example the standard arbitrary finite physical scale μ, or if other running couplings get introduced. (After solving for their flows, *i.e.* solving their corresponding β functions, this becomes equivalent to the first failure since by dimensional transmutation a new finite physical scale has been introduced.) Importantly, $\Lambda\partial_\Lambda\gamma$ can then have an $O(g^2)$ one-loop contribution or in extreme cases even a tree-level $O(g^0)$ contribution. From eq. (5.54) one sees that a one-loop contribution to the running of γ destroys β_2 agreement, whilst a tree-level running would even modify β_1.

As we will see shortly, a generic $\hat{S}$, including the simple form used for the bare action in ref. [34], can lead to such tree-level corrections. Fortunately, there is also an infinite class of seed actions that cannot. As with the earlier constraints discussed, since we never specify $\hat{S}$, it is not the solution that matters, only knowing that one exists.

To get agreement with the standard β function at the two-loop level, one needs to confirm that there are no further couplings hidden, that run at one loop, and to take into account contributions from $g_2(\Lambda)$. This can be done [36].

Even with a non-vanishing $\Lambda\partial_\Lambda\gamma$, one could still recover the usual β function coefficients, by defining a standard low energy –or infrared– coupling $\tilde{g}(\mu)$ at some scale $\mu < \Lambda$, this coupling being distinguished from the 'ultraviolet' coupling $g(\Lambda)$ in the

effective action S_Λ [40, 41]. We want to avoid this because the introduction of μ would destroy, or at least obscure, the power and elegance of self-similarity (*c.f.* sec. 4.1).

5.3.4 Finiteness at tree level

As we have anticipated earlier in this chapter, it is now time to prove that it is possible to have all the integrals for the tree level vertices, UV regulated for a specific choice of $\hat{S}$. One should not usually expect divergences at the classical level, but the incorporation of Pauli-Villars fields directly into an exact RG, can cause them. In this section we do not intend to give a proof, but we just want to show that all the vertices we have to deal with in order to perform our calculation (two, three and four point tree level vertices), are convergent for an appropriate choice of the seed action. We start recalling the two main requirements on $\hat{S}$, namely to be supergauge invariant, and to have the two-point vertices of the form of Eqs.(5.12)-(5.16). A possible choice in the unbroken phase is:

$$\begin{aligned}\hat{S} &= \frac{1}{2}\mathcal{F}^{\mu\nu}\{c^{-1}\}\mathcal{F}_{\mu\nu} + \frac{1}{2}\nabla^\mu \cdot \mathcal{C}\{\tilde{c}^{-1}\}\nabla_\mu \cdot \mathcal{C} + \\ &+ \frac{\lambda}{4}\mathrm{str}\left(\mathcal{C}^2 - \Lambda^2\right)^2 \end{aligned} \tag{5.55}$$

Now, once the $\mathcal{C}$ field has been shifted ($SU(N|N)$ broken phase), and it has been redefined in order to be dimensionless ($\mathcal{C} \to \Lambda\mathcal{C}$), the seed action chosen here has the following form (the bare action in ref. [34]):

$$\begin{aligned}\hat{S} &= \frac{1}{2}\mathcal{F}_{\mu\nu}\{c^{-1}\}\mathcal{F}_{\mu\nu} + \frac{\Lambda^2}{2}\nabla_\mu \cdot \mathcal{C}\{\tilde{c}^{-1}\}\nabla_\mu \cdot \mathcal{C} - \frac{\Lambda^2}{2}[\mathcal{A}_\mu, \sigma]\{\tilde{c}^{-1}\}[\mathcal{A}_\mu, \sigma] \\ &-i\Lambda^2[\mathcal{A}_\mu, \sigma]\{\tilde{c}^{-1}\}\nabla_\mu \cdot \mathcal{C} + \frac{\lambda}{4}\Lambda^4\mathrm{str}\int_x(\{\sigma, \mathcal{C}\} + \mathcal{C})^2 \end{aligned} \tag{5.56}$$

We can, at this point split the fields in the diagonal and off-diagonal (bosonic + fermionic) components:

$$\mathcal{A}_\mu = A_\mu + B_\mu \tag{5.57}$$

$$\mathcal{C} = C + D \tag{5.58}$$

Since $[A_\mu, \sigma] = 0$ and $[B_\mu, \sigma] = 2B_\mu\sigma$, our vertices simplify because now they have either one σ (which for convention we decide to put at the end), or no σ's in them. The seed action in terms of split fields is:

$$\begin{aligned}\hat{S} &= \frac{1}{2}\mathcal{F}_{\mu\nu}\{c^{-1}\}\mathcal{F}_{\mu\nu} + \frac{\Lambda^2}{2}\nabla_\mu \cdot \mathcal{C}\{\tilde{c}^{-1}\}\nabla_\mu \cdot \mathcal{C} - 2\Lambda^2 B_\mu\sigma\{\tilde{c}^{-1}\}B_\mu\sigma \\ &\quad -2i\Lambda^2 B_\mu\sigma\{\tilde{c}^{-1}\}\nabla_\mu \cdot \mathcal{C} + \frac{\lambda}{4}\Lambda^4 \mathrm{str}\int_x (C^2 + D^2 + CD + DC + 2C\sigma)^2\end{aligned} \tag{5.59}$$

One can easily check that the two-point vertices are exactly the one listed in section 5.1. The higher vertices (three and four-point), which are a possible covariantisation of the two-point ones, are:

- Three point:

$$\begin{aligned}\hat{S}^{AAA}_{\mu\nu\rho}(p,q,r) &= \frac{2}{c_p}(p_\rho\delta_{\mu\nu} - p_\nu\delta_{\mu\rho}) \\ &\quad +2c_\nu^{-1}(q;p,r)(p_\rho r_\mu - p\cdot r\delta_{\rho\mu}) + \text{ cycles} \\ &\equiv \hat{S}_{\mu\nu\rho}\end{aligned} \tag{5.60}$$

$$\hat{S}^{BBA}_{\mu\nu\rho}(p,q,r) = \hat{S}_{\mu\nu\rho} + 4\Lambda^2\tilde{c}_\rho^{-1}(r;q,p)\delta_{\mu\nu} \tag{5.61}$$

$$\begin{aligned}\hat{S}^{ACC}_{\mu}(p,q,r) &= \Lambda^2\left(\frac{q_\mu}{\tilde{c}_q} - \frac{r_\mu}{\tilde{c}_r} - q\cdot r\tilde{c}_\mu^{-1}(p;r,q)\right) \\ &= \hat{S}^{ADD}_{\mu}(p,q,r)\end{aligned} \tag{5.62}$$

$$\hat{S}^{BAD\sigma}_{\mu\nu}(p,q,r) = 2\Lambda^2(\tilde{c}_p^{-1}\delta_{\mu\nu} - r_\mu\tilde{c}_\nu^{-1}(q;r,p))$$

$$= -\hat{S}'^{ABD\sigma}_{\nu\mu}(q,p,r) \tag{5.63}$$

- Four point:

$$\begin{aligned}
\hat{S}'^{AAAA}_{\mu\nu\rho\lambda}(p,q,r,s) &= \frac{1}{c_{p+q}}(\delta_{\lambda\mu}\delta_{\rho\nu}-\delta_{\rho\mu}\delta_{\nu\lambda})+2\tilde{c}^{-1}_{\nu}(q;p,r+s)(p_\lambda\delta_{\rho\mu}-p_\rho\delta_{\lambda\mu}) \\
&\quad +2c^{-1}_{\lambda}(s;p,r+q)(p_\nu\delta_{\mu\rho}-p_\rho\delta_{\mu\nu}) \\
&\quad +2c^{-1}_{\nu\rho}(q,r;p,s)(p_\lambda s_\mu - p\cdot s\delta_{\lambda\mu}) \\
&\quad c^{-1}_{\nu,\lambda}(q;s;p,r)(p_\rho r_\mu - p\cdot r\delta_{\rho\mu}) + \text{ cycles} \equiv \hat{S}_{\mu\nu\rho\lambda} \qquad (5.64)
\end{aligned}$$

$$\hat{S}'^{AABB}_{\mu\nu\rho\lambda}(p,q,r,s) = \hat{S}_{\mu\nu\rho\lambda} + 4\Lambda^2\delta_{\rho\lambda}\tilde{c}^{-1}_{\mu\nu}(p,q;r,s) \tag{5.65}$$

$$\begin{aligned}
\hat{S}'^{AADD}_{\mu\nu}(p,q,r,s) &= \Lambda^2(\tilde{c}^{-1}_{p+s}\delta_{\mu\nu}+s_\nu\tilde{c}^{-1}_{\mu}(p;q+r,s)+r_\mu\tilde{c}^{-1}_{\nu}(q;p+s,r) \\
&\quad -r\cdot s\ \tilde{c}^{-1}_{\mu\nu}(p,q;s,r)) \qquad (5.66)
\end{aligned}$$

$$\hat{S}'^{AACC}_{\mu\nu}(p,q,r,s) = \hat{S}'^{AADD}_{\mu\nu}(p,q,r,s) \tag{5.67}$$

$$\hat{S}'^{AABD\sigma}_{\mu\nu\rho}(p,q,r,s) = 2\Lambda^2(s_\rho\tilde{c}^{-1}_{\mu\nu}(p,q;s,r)+\delta_{\mu\nu}\tilde{c}^{-1}_{\nu}(q;r,p+s)) \tag{5.68}$$

$$\hat{S}'^{BAAD\sigma}_{\mu\nu\rho}(p,q,r,s) = \hat{S}'^{AABD\sigma}_{\rho\nu\mu}(r,q,p,s) \tag{5.69}$$

One could check that indeed all the Ward identities are satisfied, by contracting the vertices with the proper momenta.

In order to control the divergences, we have first to regulate the integrals with a cutoff Λ_0, to make the divergences explicit. We can then substitute the seed action vertices in the three-point tree-level integrals, and, then in the four-point ones, and see if there are any logarithmic or power-like divergences.

What we expect is to find that many divergences are connected via Ward identities, some of them related back to two point vertices. Since they present only power divergences, we are not worried about those. These are canceled by term that must be present for gauge invariance, in the action at tree level when $\Lambda = \Lambda_0$, leaving the

constant which are listed in Appendix F. More worrying are possible logarithmic divergencies of the form $\alpha \ln \frac{\Lambda_0}{\Lambda}$. Terms of this kind would have to be transverse since in the two-point vertices such divergences are not present. To cure those, we would have to add marginal operators to the action at Λ_0, proportional to $\ln \frac{\Lambda_0}{\mu}$, μ being another finite scale (this way was the one followed in [6, 7]). Although this is a possible solution, introducing a new scale would result in a loss of self-similarity, as it was discussed in the previous section, and causing problems through the calculation carried on in the next chapter, which relies on the fact that we are dealing with only one finite scale Λ. We will then see that a possible way out is to redefine the vertices of the seed action in order to tune to zero these logarithmic divergences, without fixing a new scale. In other words, it seems reasonable to infer that it is possible to choose a seed action which keeps all the integrals in which it appears UV finite .

The first vertex to be considered is the three-point pure A vertex. It is easy to see that it does not present any divergence as we would expect. The corresponding two-point vertex is in fact finite and, by dimensions and Lorentz invariance, a transverse term in three different momenta cannot be constructed, which would carry a divergent factor of Λ_0.

Let us analyse the other divergences, dividing them in two sets. The first set includes the two vertices AAC and $AACC$. Their divergences are:

$$S_\mu^{ACC}(p,q,r)\Big|_{DIV} = \alpha_\mu(p,q,r) \ln \frac{\Lambda_0}{\Lambda} + \beta_\mu(p,q,r)\Lambda_0^2 \tag{5.70}$$

where

$$\alpha_\mu(p,q,r) = -4(c_0' + 2\tilde{c}_0')r_\alpha \, \Box_{\alpha\mu}(p) \tag{5.71}$$

$$\beta_\mu(p,q,r) = 2(q-r)_\mu \tag{5.72}$$

and

$$S'^{AACC}_{\mu\nu}(p,q,r,s)\Big|_{DIV} = 4\ln\frac{\Lambda_0}{\Lambda}(c'_0+2\tilde{c}'_0)(r_\mu s_\nu - r_\nu s_\mu + p_\nu s_\mu + q_\mu r_\nu - (p\cdot s + q\cdot r)\delta_{\mu\nu}) \quad (5.73)$$

$$+4\ln\frac{\Lambda_0}{\Lambda}(c'_0-2\tilde{c}'_0)(p_\nu q_\mu - p\cdot q\delta_{\mu\nu}) \quad (5.74)$$

$$+2\Lambda_0^2\delta_{\mu\nu} \quad (5.75)$$

respectively. For the first one, the three-point ACC, one can notice that the quadratic divergence is exactly cancelled by a term already contained in $S'^{ACC}_{\mu bare}$ and the logarithmic one is transverse in the A-field momentum, p_μ, as expected. For the four-point one, the same can be said about the power divergences, but there are two independent logarithmic divergent terms. We will see that one of the two, namely (5.73), can be cancelled via the same counterterm that cancels the one in (5.70), in which it can be transformed via a Ward identity. For the other one, transverse in both the A momenta, p and q, we will have to add an independent term to the seed action.

The second set of vertices includes ABB, ADD, $ABD\sigma$, $AABB$ and $AADD$. Their divergences are:

- Three point:

$$S^{BBA}_{\mu\nu\rho}(p,q,r)\Big|_{DIV} = 16\ln\frac{\Lambda_0}{\Lambda}(r_\mu\delta_{\nu\rho} - r_\nu\delta_{\rho\mu}) \quad (5.76)$$

$$S'^{ADD}_{\mu}(p,q,r)\Big|_{DIV} = 12\Lambda_0^2(q-r)_\mu - 4\ln\frac{\Lambda_0}{\Lambda}r_\alpha\,\Box_{\alpha\mu}(p) \quad (5.77)$$

$$S'^{ABD\sigma}_{\mu\nu}(p,q,r)\Big|_{DIV} = -4\Lambda_0^2\delta_{\mu\nu} + 8\ln\frac{\Lambda_0}{\Lambda}(p\cdot r\delta_{\mu\nu} - p_{nu}r_\mu) \quad (5.78)$$

- Four point:

$$S'^{AABB}_{\mu\nu\rho\sigma}(p,q,r,s)\Big|_{DIV} = -16\ln\frac{\Lambda_0}{\Lambda}(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}) \quad (5.79)$$

$$S^{AADD}_{\mu\nu}(p,q,r,s)\Big|_{DIV} = 4\ln\frac{\Lambda_0}{\Lambda}(r_\mu s_\nu - r_\nu s_\mu + p_\nu s_\mu + q_\mu r_\nu - (p\cdot s + q\cdot r)\delta_{\mu\nu}) + 2\Lambda_0^2\delta_{\mu\nu} \quad (5.80)$$

Here the situation is simpler: all the three-point divergences have the logarithmic part which is transverse in the corresponding A momentum, as one can expect, and all the four point have their logarithmic divergences, falling in those of their corresponding coefficients related by gauge transformation. In this case it will then be possible to cancel all of them with the addition of only one counterterm.

Let us start with the former set of vertices. Following [6, 7], we can add to the tree-level action at Λ_0 the terms:

$$S_0^1 = -\frac{i}{16}\gamma^{ACC}\mathrm{str}\{\mathcal{C},\nabla_\mu\cdot\mathcal{C}\}\mathcal{F}_{\mu\nu}\{\mathcal{C},\nabla_\nu\cdot\mathcal{C}\} \quad (5.81)$$

$$+\frac{\gamma^{AACC}}{4}\mathrm{str}[\mathcal{C}^2,\mathcal{F}_{\mu\nu}]^2 \quad (5.82)$$

It can be shown that both terms (5.81) and (5.82) are $SU(N|N)$ invariant and no-$\mathcal{A}^0$ symmetric. Moreover, choosing the two constants as follows:

$$\gamma^{ACC} = 16(c_0' + 2\tilde{c}_0')\ln\frac{\Lambda_0}{\mu} \quad (5.83)$$

$$\gamma^{AACC} = 2(c_0' - 2\tilde{c}_0')\ln\frac{\Lambda_0}{\mu}, \quad (5.84)$$

one can check that all the logarithmic divergences in eq.(5.70) and (5.73,5.74) are cancelled.

For the second set of terms, it is possible to show that adding to the tree level action at Λ_0 the following counterterm[5]

$$S_0^2 = -\frac{i}{16}\gamma^{BBA}\mathrm{str}[\mathcal{C},(\nabla_\mu\cdot\mathcal{C})]\mathcal{F}_{\mu\nu}[\mathcal{C},(\nabla_\nu\cdot\mathcal{C})], \quad (5.85)$$

[5] $SU(N|N)$ invariant and no-$\mathcal{A}^0$ symmetric as well.

can cancel all the logarithmic divergences of Eqs.(5.76)-(5.80), for the following choice of the constant:

$$\gamma^{BBA} = 16 \ln \frac{\Lambda_0}{\mu} \tag{5.86}$$

As we have mentioned in the previous section, though, even if this procedure cures all the divergences at tree level, it introduces a new scale μ. In the next chapter, we will see that throughout the calculation, we will often rely on the fact that there is only one finite scale Λ. This argument is no longer true if we regularise the tree level vertices by modifying the integration constant in the way we just mentioned. Fortunately, it is possible to overcome this problem. Instead of adding terms to the tree level action at Λ_0, we can in fact add terms to the seed action which appears inside the integrals, without introducing any new scale and, thus, making sure logarithmic divergences do not appear. In other words, we can choose an $\hat{S}$ which does not produce any logarithmic divergences, leaving the power divergences unchanged which must appear by gauge invariance. In the specific case we are considering, for example, it is enough to add to the seed action, the same terms we wanted to add to the bare action (5.81), (5.82) and (5.85), with different coefficients. If one chooses here coefficients (similarly defined) to be:

$$\begin{aligned} \hat{\gamma}^{BBA} &= -4 \\ \hat{\gamma}^{ACC} &= (c_0' + 2\tilde{c}_0')/2 \\ \hat{\gamma}^{AACC} &= (c_0' - 2\tilde{c}_0')/2 \end{aligned} \tag{5.87}$$

it is easy to check[6] that all the logarithmic divergences disappear.

A way to see this issue from a more general perspective, as can be found in [36], is to realise as we have already mentioned earlier, that the problem of classical divergences is associated to the Pauli-Villars sector. These terms have a classical divergent

[6]This check was done with a script in FORM.

action as $\Lambda \to \infty$, having a divergent mass, causing logarithmic divergences along the marginal directions. The solution which avoids the introduction of a new scale, is precisely to shift the $\hat{S}$ vertices along those directions, in order to tune the logarithmic divergences to zero. If the shift for each vertex is, in our particular case, a coefficient as the ones listed in (5.87) times the structure of the divergence of the corresponding vertex, then the potential logarithmic divergent term is removed from the integrand. Since the structure of the classical flow equations eq. (4.41) is such that the flow of each vertex $S_{0\,a_1\cdots a_n}^{X_1\cdots X_n}$ has the corresponding $\hat{S}'^{X_1\cdots X_n}_{a_1\cdots a_n}$ as its highest-point $\hat{S}$ contribution, contracted with kernels and with the appropriate two point vertices (*viz.* $\dot{\Delta}^{XX}\hat{S}^{XX}$ where $X = A, C, B$ or D) [6, 7], and since these $\dot{\Delta}^{XX}\hat{S}^{XX}$ terms are non-vanishing at zero momentum precisely when X is a massive Pauli-Villars field, it follows that we can always remove the divergence associated with these marginal directions by tuning $\hat{S}'^{X_1\cdots X_n}_{a_1\cdots a_n}$ in the same direction.

Once this last check has been performed it is now possible to move onto the gauge invariant calculation of the $SU(N)$ Yang-Mills β-function at one loop. This calculation was mostly inspired by the scalar field case, which gave us all the necessary hints and uncovered the whole machinery that made it possible: above all, the use of the flow equations to eliminate the hatted vertices in favour of the effective ones, and the integrated wine technique. This will all be explained in detail in the next chapter.

Chapter 6

Gauge Invariant calculation

Let us first consider the equation for β_1 (5.19). This can be expressed in diagram, as it is represented in fig.6.1. f in the figure, can represent any of the fields present

$$-4\beta_1 \ \Box_{\mu\nu}(p) + O(p^3) = 2 \sum_{\substack{f = A, B \\ C, D}}$$

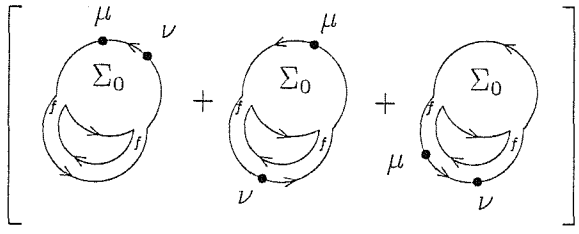

Figure 6.1: Graphical representation of the equation for β_1

in the theory: A, B, C or D. The content of fields and the symmetries, allow to separate the problem in three different sectors: the A and C ones, which are possible to study independently and the $B - D$ which can be considered together. The reason we must study the B and D sectors together, comes from the fact that unlike the A and the C ones, their equations appear coupled trough the vertex $BD\sigma$. In the $A - C$ case, due to the fact that $AC\sigma$ is not allowed (see section 5.2), this mix does not occur and the study of the two sectors can be carried out separately. Moreover, it is possible to avoid the study of a complicated set of coupled equations, recognising that

all of them can be recast into only one equation via the introduction of a compact notation which will be introduced in the next section. The crucial step in order to obtain a set of only one equation for each generalised vertex is the introduction of a doublet field $F = (B_\mu, D\sigma)$. This will also allow to have equations of the same form of the corresponding ones in the A and C sectors. This further observation allows us to write all the equations for each vertex in term of an even more generalised field multiplet, f, containing also the A and C fields. Of course this is not a fundamental choice as it is for the $BD\sigma$ sector, because the former two being decoupled fields, it will just amount to having to deal with block diagonal matrices, but it will help indeed to keep the calculation neater.

In the next section, the compact notation for the $BD\sigma$ sector will be introduced as well as the corresponding equations and Ward identities. In the second section, the actual calculation will be carried out in the 5-fields notation as far as it possible and it will be finished by splitting down to components in the final part. Finally in the last section, will be considered the potentially non-universal terms proportional to $\dot{\Delta}_0^{AA}$ and it will be shown that, as one can expect, they do not contribute to the final result.

6.1 Compact notation for the $BD\sigma$ sector

As can be recognised from Eqs.(5.40)-(5.42) it is possible to combine them together introducing a compact notation as follows. Let us first define the matrices:

$$S^{FF}_{MN}(p) = \begin{pmatrix} S^{BB}_{\mu\nu}(p) & S^{BD\sigma}_{\mu}(p) \\ S^{D\sigma B}_{\nu}(p) & S^{D\sigma D\sigma}(p) \end{pmatrix} \tag{6.1}$$

$$\dot{\Delta}^{FF}_{MN}(p) = \begin{pmatrix} \dot{\Delta}^{BB}_{p}\delta_{\mu\nu} & 0 \\ 0 & -\dot{\Delta}^{DD}_{p} \end{pmatrix} \tag{6.2}$$

where F refers to the doublet $(B, D\sigma)$ and in the eq.(6.1), the momenta are referred to the first field. Making use of Eqs.(6.1) and (6.2) we can now rewrite Eqs.(5.40)-(5.42) in the following compact form:

$$\Lambda\partial_\Lambda S^{FF}_{MN}(p) = S^{FF}_{ML}(p)\dot{\Delta}^{FF}_{LS}(p)S^{FF}_{SN}(p) \tag{6.3}$$

Equation (6.3) has now exactly the same form as the two-point A and C vertices of Eqs. (5.38,5.39). Extending the idea it is possible to rewrite all the equations for the three and four point vertices in the $BD\sigma$ sector. First of all we have to group together the three point vertices in the following tensor representation

$$S^{AFF}_{\mu RS}(p,q,r) = \begin{pmatrix} S^{ABB}_{\mu\rho\sigma} & S^{ABD\sigma}_{\mu\rho} \\ S^{AD\sigma B}_{\mu\quad\sigma} & S^{AD\sigma D\sigma}_{\mu} \end{pmatrix} \tag{6.4}$$

and similarly for the four point ones:

$$S^{AAFF}_{\mu\nu RS}(p,q,r,s) = \begin{pmatrix} S^{AABB}_{\mu\nu\rho\sigma} & S^{AABD\sigma}_{\mu\nu\rho} \\ S^{AAD\sigma B}_{\mu\nu\quad\sigma} & S^{AAD\sigma D\sigma}_{\mu\nu} \end{pmatrix} \tag{6.5}$$

(where, in both, the momentum dependence on the right hand side is omitted since it is the same); then a one point wine must be defined as:

$$\dot{\Delta}^{A,FF}_{\mu RS}(p;q,r) = \begin{pmatrix} \dot{\Delta}^{A,BB}_{\mu}(p;q,r)\delta_{\rho\sigma} & 0 \\ 0 & -\dot{\Delta}^{A,DD}_{\mu}(p;q,r) \end{pmatrix} \tag{6.6}$$

and its natural two point extension:

$$\dot{\Delta}^{AA,FF}_{\mu\nu RS}(p,q;r,s) = \begin{pmatrix} \dot{\Delta}^{AA,BB}_{\mu\nu}(p,q;r,s)\delta_{\rho\sigma} & 0 \\ 0 & -\dot{\Delta}^{AA,DD}_{\mu\nu}(p,q;r,s) \end{pmatrix} \tag{6.7}$$

Finally in order to be able to write all the three point equations of the $BD\sigma$ sector in a compact form, it is necessary to define the following two objects:

$$\dot{\Delta}_R^{F,AB}(p;q,r) = \begin{pmatrix} \dot{\Delta}_\rho^{B,AB}(p;q,r) \\ \dot{\Delta}^{D\sigma,AB}(p;q,r) \end{pmatrix} \tag{6.8}$$

$$\dot{\Delta}_R^{F,BA}(p;q,r) = \begin{pmatrix} \dot{\Delta}_\rho^{B,BA}(p;q,r) \\ \dot{\Delta}^{D\sigma,BA}(p;q,r) \end{pmatrix} \tag{6.9}$$

These are one point wine vertices which have either a B or a D field. All the usual wine vertices rules and properties apply to them. Their two point extensions (covariantisations) with an A field is then:

$$\dot{\Delta}_{\mu R}^{AF,AB}(p,q;r,s) = \begin{pmatrix} \dot{\Delta}_{\mu\rho}^{AB,AB}(p,q;r,s) \\ \dot{\Delta}_\mu^{AD\sigma,AB}(p,q;r,s) \end{pmatrix} \tag{6.10}$$

$$\dot{\Delta}_{\mu R}^{AF,BA}(p,q;r,s) = \begin{pmatrix} \dot{\Delta}_{\mu\rho}^{AB,BA}(p,q;r,s) \\ \dot{\Delta}_\mu^{AD\sigma,BA}(p,q;r,s) \end{pmatrix} \tag{6.11}$$

where $\dot{\Delta}^{AD\sigma,BA}(p,q;r,s) = \dot{\Delta}_m^{AA}{}_\mu(p;r,q+s)/2$. These are two point wine vertices, one of whose field is an A and the other one can be either a B or a D. Also in this case, all the usual rules and properties for wine vertices are valid. It is now possible with this notation to write the equations for the three point functions (5.48), (5.50) and (5.51) collected together in the following form:

$$\begin{aligned} \Lambda\partial_\Lambda \dot{S}_{\mu RS}^{AFF}(p,q,r) \;=\; & \hat{S}_{\mu\alpha}^{AA}(p)\dot{\Delta}_p^{AA}\dot{S}_{\alpha RS}^{AFF}(p,q,r) + \dot{S}_{\mu TS}^{AFF}(p,q,r)\dot{\Delta}_{TU}^{FF}(q)\hat{S}_{UR}^{FF}(-q) \\ & + \hat{S}_{ST}^{FF}(r)\dot{\Delta}_{TU}^{FF}(r)\dot{S}_{\mu RU}^{AFF}(p,q,r) \\ & + \hat{S}_{ST}^{FF}(r)\dot{\Delta}_{\mu TU}^{A,FF}(p;r,q)\hat{S}_{UR}^{FF}(-q) \\ & + \hat{S}_{\mu\alpha}^{AA}(p)\dot{\Delta}_R^{F,AB}(q;p,r)\hat{S}_{\alpha S}^{BF}(-r) \\ & + \hat{S}_{\mu\alpha}^{AA}(p)\dot{\Delta}_S^{F,BA}(r;q,p)\hat{S}_{\alpha R}^{BF}(q) \end{aligned} \tag{6.12}$$

With this formalism all the four point equation of the $BD\sigma$ sector can be as well recast in only one compact equation that reads:

$$
\begin{aligned}
&\Lambda\partial_\Lambda S'^{AAFF}_{\mu\nu RS}(p,q,r,s) = \hat{S}'^{AAFF}_{\mu\nu RT}(p,q,r,s)\dot{\Delta}^{FF}_{TU}(s)\hat{S}^{FF}_{US}(-s) + \hat{S}'^{AAFF}_{\mu\nu TS}(p,q,r,s)\dot{\Delta}^{FF}_{TU}(r)\hat{S}^{FF}_{UR}(-r)\\
&+\hat{S}'^{AAFF}_{\mu\alpha RS}(p,q,r,s)\dot{\Delta}^{AA}_{q}\hat{S}^{AA}_{\alpha\nu}(q) + \hat{S}'^{AAFF}_{\alpha\nu RS}(p,q,r,s)\dot{\Delta}^{AA}_{p}\hat{S}'^{AA}_{\alpha\mu}(p)\\
&-S'^{AAA}_{\mu\nu\alpha}(p,q,r+s)\dot{\Delta}^{AA}_{p+q}S'^{AFF}_{\alpha RS}(p+q,r,s) + \hat{S}'^{AAA}_{\mu\nu\alpha}(p,q,r+s)\dot{\Delta}^{AA}_{p+q}S'^{AFF}_{\alpha RS}(p+q,r,s)\\
&+S'^{AAA}_{\mu\nu\alpha}(p,q,r+s)\dot{\Delta}^{AA}_{p+q}\hat{S}'^{AFF}_{\alpha RS}(p+q,r,s) - S'^{AFF}_{\mu TS}(p,q+r,s)\dot{\Delta}^{FF}_{TU}(q+r)S'^{AFF}_{\nu RU}(q,r,p+s)\\
&+\hat{S}'^{AFF}_{\mu TS}(p,q+r,s)\dot{\Delta}^{FF}_{TU}(q+r)S'^{AFF}_{\nu RU}(q,r,p+s)\\
&+S'^{AFF}_{\mu TS}(p,q+r,s)\dot{\Delta}^{FF}_{TU}(q+r)\hat{S}'^{AFF}_{\nu RU}(q,r,p+s)\\
&+\hat{S}'^{AFF}_{\alpha RS}(p+q,r,s)\dot{\Delta}^{A,AA}_{\mu}(p;r+s,q)\hat{S}'^{AA}_{\alpha\nu}(q) + \hat{S}'^{AFB}_{\nu R\alpha}(q,r,s+p)\dot{\Delta}^{F,BA}_{S}(s;q+r,p)\hat{S}'^{AA}_{\alpha\mu}(p)\\
&+\hat{S}'^{AAA}_{\mu\nu\alpha}(p,q,r+s)\dot{\Delta}^{F,AB}_{R}(r;p+q,s)\hat{S}'^{BF}_{\alpha S}(-s) + \hat{S}'^{AFF}_{\mu TS}(p,q+r,s)\dot{\Delta}^{A,FF}_{\nu TU}q;p+s,r)\hat{S}^{FF}_{UR}(-r)\\
&+\hat{S}'^{AA}_{\mu\alpha}(p)\dot{\Delta}^{A,AA}_{\nu}(q;p,r+s)\hat{S}'^{AFF}_{\alpha RS}(p+q,r,s) + \hat{S}^{FF}_{ST}(s)\dot{\Delta}^{A,FF}_{\mu TU}(p;s,q+r)\hat{S}'^{AFF}_{\nu RU}(q,r,p+s)\\
&+\hat{S}'^{fB}_{R\alpha}(r)\dot{\Delta}^{F,BA}_{S}(s;r,p+q)\hat{S}'^{AAA}_{\alpha\mu\nu}(r+s,p,q) + \hat{S}'^{AA}_{\nu\alpha}(q)\dot{\Delta}^{F,AB}_{R}(r;q,s+p)\hat{S}'^{ABF}_{\mu\alpha S}(p,q+r,s)\\
&+\hat{S}'^{AA}_{\mu\alpha}(p)\dot{\Delta}^{AF,AB}_{\nu R}(q,r;p,s)\hat{S}'^{BF}_{\alpha S}(-s) + \hat{S}^{FF}_{ST}(s)\dot{\Delta}^{AA,FF}_{\mu\nu\ TU}(p,q;s,r)\hat{S}^{FF}_{UR}(-r)\\
&+\hat{S}'^{FB}_{R\alpha}(r)\dot{\Delta}^{FA,BA}_{S\mu}(s,p;r,q)\hat{S}'^{AA}_{\alpha\nu}(q) + \hat{S}'^{AA}_{\nu\alpha}(q)\dot{\Delta}^{FF,AA}_{RS}(r,s;q,p)\hat{S}'^{AA}_{\alpha\mu}(p)\\
&+\hat{S}'^{FB}_{S\alpha}(s)\dot{\Delta}^{A,F,BA}_{\mu,R}(p;r;s,q)\hat{S}'^{AA}_{\alpha\nu}(q) + \hat{S}'^{AA}_{\mu\alpha}(p)\dot{\Delta}^{A,F,AB}_{\nu,S}(q;s;p,r)\hat{S}'^{FB}_{R\alpha}(r) \qquad (6.13)
\end{aligned}
$$

In this form it is easy to notice a similarity with the A (and C) sector. In fact the Ward identities obtained acting on the A momenta of the vertices defined in Eqs.(6.5) and (6.4) are the same as the ones for the pure A case. We can also recognise generalised Ward identities for the remaining momenta, carrying the indices in capital letters, which show that gauge invariance is fully preserved in this sector. Considering the B and D sectors separately, as we can recall from the discussion carried out in section 5.2.1, we had instead to work with broken Ward identities. Moreover eq. (6.12) has the same form of the equation for the three A's vertex (5.47) and of the three-point vertex ACC of eq. (5.49), with A (C) replaced by F, and eq. (6.13) has the same of the equations for the four A's vertex (5.52) and the four-point vertex $AACC$ (C.2). This

observation will allow us in the next section to use an even more compact notation, with the introduction of the label f, representing all the fields of the theory.

Before we start this analysis in detail, let us first define some more elements which will be useful later, the two *generalised momenta*:

$$k_R = (k_\rho, -2) \tag{6.14}$$

$$k'_R = (f_k k_\rho/\Lambda^2, -g_k), \tag{6.15}$$

where $f_k \equiv f(x = \frac{k^2}{\Lambda^2}) = \frac{\tilde{c}(x)}{x\tilde{c}(x)+2c(x)}$ and $g_k \equiv g(x = \frac{k^2}{\Lambda^2}) = \frac{c(x)}{x\tilde{c}(x)+2c(x)}$. We can now recognise that the former of the two generalised momenta acts on the vertices as though it was a standard gauge transformation. On the four point vertices, for example:

$$k_R S^{AAFF}_{\mu\nu RS}(p,-p,k,-k) = S^{AA\vec{F}}_{\mu\nu S}(p,-p,0) - S^{ABF}_{\mu\nu S}(p,k-p,-k) \tag{6.16}$$

$$(-k)_S S^{AAFF}_{\mu\nu RS}(p,-p,k,-k) = S^{BAF}_{\mu\nu R}(p-k,-p,k) - S^{AA\overleftarrow{F}}_{\mu\nu R}(p,-p,0) \tag{6.17}$$

and on the three point ones:

$$k_R S^{AFF}_{\mu RS}(p,k,-p-k) = S^{A\vec{F}}_{\mu S}(p) - S^{BF}_{\mu S}(p+k) \tag{6.18}$$

$$(-k)_S S^{AFF}_{\mu RS}(p,-p+k,-k) = S^{BF}_{\mu R}(p-k) - S^{A\overleftarrow{F}}_{\mu R}(p) \tag{6.19}$$

where $\vec{F} = (A, C\sigma)$ and $\overleftarrow{F} = (A, -C\sigma)$. The action of the generalised momentum (6.14) on the two point vertices is finally a further gauge invariance statement since, as for the pure A case a generalised transversality is underlined:

$$k_R S^{FF}_{RS}(k,-k) = 0 \tag{6.20}$$

$$(-k)_S S^{FF}_{RS}(k,-k) = 0 \tag{6.21}$$

It is now time to move to the actual calculation.

6.2 Calculation

Due to the notation introduced in the previous section, comparing the set of equations for the 2,3, and 4-point vertices in the $BD\sigma$ sector with the corresponding equations in the A and C sectors, it is possible to notice some similarities. The label F, which in the previous section was introduced to represent the field doublet $(B, D\sigma)$ and which simplified drastically the set of equations, could be replaced by A or C and introducing some new Feynman rules for the new wine vertices, from (6.3), (6.12) and (6.13) we can get the corresponding equations for the other two sectors. It is possible then to introduce a field multiplet $f = (A, C, B, D\sigma)$, to represent all the equations, which could then be specified for each sector with the right wine rules. As it was mentioned at the beginning of this section, this further grouping of fields is not necessary since the A and C sectors are decoupled, but since the calculation to be done is similar in all the sectors, it is worth doing it in this notation to avoid repetitions and specifying the components only at a later stage. In this new notation, (5.19) gets the form:

$$\begin{aligned} -4\beta_1 \,\Box_{\mu\nu}(p^2) + O(p^3) = 2\int_k \quad & \sum_{f=A,C,F} (-)^{s_f} \Big\{ \dot{\Delta}^{ff}_{SR}(k) \Sigma^{ffAA}_{RS\mu\nu}(-k,k,p,-p) \\ & + \dot{\Delta}^{A,ff}_{\mu SR}(p;k-p,-k) \Sigma^{ffA}_{RS\nu}(p-k,k,-p) \\ & + \dot{\Delta}^{AA,ff}_{\mu\nu SR}(p,-p;k,-k) \Sigma^{ff}_{RS}(k) \Big\} \end{aligned} \tag{6.22}$$

where $s_A = s_C = 0$ and $s_F = 1$. The Σ's here are $S_0 - 2\hat{S}$, the equations for the S_0's are now (6.3), (6.12) and (6.13), with F replaced by the new f. The wine vertices are defined in the previous section, when $f = F$ and for A and C are the ones listed in sections 5.3.1 and 5.3.2.

First of all, once we have defined the zero point wine (kernel) $\dot{\Delta}^{ff}_{RS}(p)$, we can define

its integrated form, as it was anticipated in section 4.5 (see also App.D for details):

$$\dot{\Delta}^{ff}_{SR}(p) = -\Lambda\partial_\Lambda \Delta^{ff}_{SR}(p) \tag{6.23}$$

being now $\Delta^{ff}_{SR}(p)$ the *integrated wine*. Let us now consider the two point equation for the generalised field multiplet f:

$$\Lambda\partial_\Lambda S^{ff}_{MN}(p) = S^{ff}_{ML}(p)\dot{\Delta}^{ff}_{LS}(p)S^{ff}_{SN}(p) \tag{6.24}$$

We can recognise the following relation (App.D):

$$S^{ff}_{RS}(k)\Delta^{ff}_{ST}(k) = \delta_{RT} - B^{ff}_{RT}(k) \tag{6.25}$$

Let us now consider the equation for β_1 (6.22). This is expressed in diagram in fig.6.1. Following the steps of the the scalar field case, shown in section 2.3 and more extensively in [35], we will try to use the flow equations of the effective vertices in order to eliminate the $\hat{S}$ ones. Consider then the first line on the RHS in (6.22):

$$\dot{\Delta}^{ff}_{SR}(k)\Sigma^{ffAA}_{RS\mu\nu}(-k,k,p,-p) = \dot{\Delta}^{ff}_{SR}(k)\left[S^{ffAA}_{RS\mu\nu} - 2\hat{S}^{ffAA}_{RS\mu\nu}\right] \tag{6.26}$$

Let us take only the effective vertex term (the first one in the previous equation) and recalling eq.(6.23) we can write it as:

$$\begin{aligned}
\dot{\Delta}^{ff}_{SR}(k)S^{ffAA}_{RS\mu\nu}(-k,k,p,-p) &= -(\Lambda\partial_\Lambda\Delta^{ff}_{SR}(k))S^{ffAA}_{RS\mu\nu}(-k,k,p,-p) \\
&= -\Lambda\partial_\Lambda\left[\Delta^{ff}_{SR}(k)S^{ffAA}_{RS\mu\nu}(-k,k,p,-p)\right] \\
&\quad +\Delta^{ff}_{SR}(k)\Lambda\partial_\Lambda S^{ffAA}_{RS\mu\nu}(-k,k,p,-p)
\end{aligned} \tag{6.27}$$

This is represented diagrammatically in fig.6.2. We can now use the equation for the effective vertex $ffAA$ at tree-level, substituting it into the previous one. The

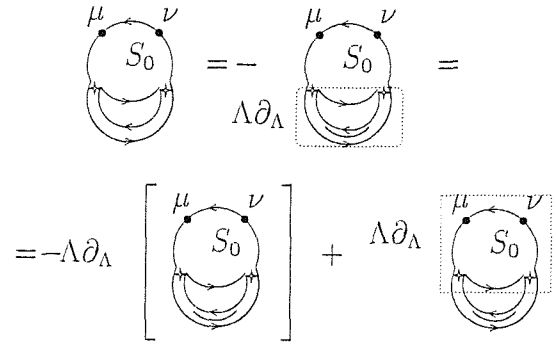

Figure 6.2: Graphical representation of the method used to eliminate $\hat{S}$ in the equation for β_1. The stars represent the field multiplet f

graphical representation of the equation for the effective vertex $ffAA$ is illustrated in fig.6.3

Substituting this equation into (6.27), which correspond to substitute all the diagrams of fig.6.3 in fig.6.2, leads to the following equation:

$$\dot{\Delta}^{ff}_{SR}(k)S^{ffAA}_{RS\mu\nu}(-k,k,p,-p) = -\Lambda\partial_\Lambda\left[\Delta^{ff}_{SR}(k)S^{ffAA}_{RS\mu\nu}(-k,k,p,-p)\right] \quad (0)$$

$$+\Delta^{ff}_{SR}(k)\Big\{\hat{S}^{ffAA}_{RS\mu\alpha}(k,-k,p,-p)\dot{\Delta}^{AA}_{p}\hat{S}^{AA}_{\alpha\,\nu}(p) \quad (1)$$

$$+\hat{S}^{ffAA}_{RS\alpha\nu}(k,-k,p,-p)\dot{\Delta}^{AA}_{p}\hat{S}^{AA}_{\alpha\,\mu}(p) \quad (2)$$

$$+\hat{S}^{ffAA}_{RT\mu\nu}(k,-k,p,-p)\dot{\Delta}^{ff}_{TU}(k)\hat{S}^{ff}_{US}(-k) \quad (3)$$

$$+\hat{S}^{ffAA}_{TS\mu\nu}(k,-k,p,-p)\dot{\Delta}^{ff}_{TU}(k)\hat{S}^{ff}_{UR}(k) \quad (4)$$

$$-S^{ffA}_{TS\mu}(k-p,-k,p)\dot{\Delta}^{ff}_{TU}(k-p)S^{Aff}_{\nu RU}(-p,k,p-k) \quad (5)$$

$$+S^{ffA}_{TS\mu}(k-p,-k,p)\dot{\Delta}^{ff}_{TU}(k-p)\hat{S}^{Aff}_{\nu RU}(-p,k,p-k) \quad (6)$$

$$+\hat{S}^{ffA}_{TS\mu}(k-p,-k,p)\dot{\Delta}^{ff}_{TU}(k-p)S^{Aff}_{\nu RU}(-p,k,p-k) \quad (7)$$

$$-S^{AAA}_{\mu\nu\alpha}(p,-p,0)\dot{\Delta}^{AA}_{0}S^{Aff}_{\alpha RS}(0,k,-k) \quad (8)$$

$$+S^{AAA}_{\mu\nu\alpha}(p,-p,0)\dot{\Delta}^{AA}_{0}\hat{S}^{Aff}_{\alpha RS}(0,k,-k) \quad (9)$$

$$+\hat{S}^{AAA}_{\mu\nu\alpha}(p,-p,0)\dot{\Delta}^{AA}_{0}S^{Aff}_{\alpha RS}(0,k,-k) \quad (10)$$

$$+\hat{S}^{ffA}_{RS\alpha}(k,-k,0)\dot{\Delta}^{A,AA}_{\mu}(p;0,-p)\hat{S}^{AA}_{\alpha\,\nu}(p) \quad (11)$$

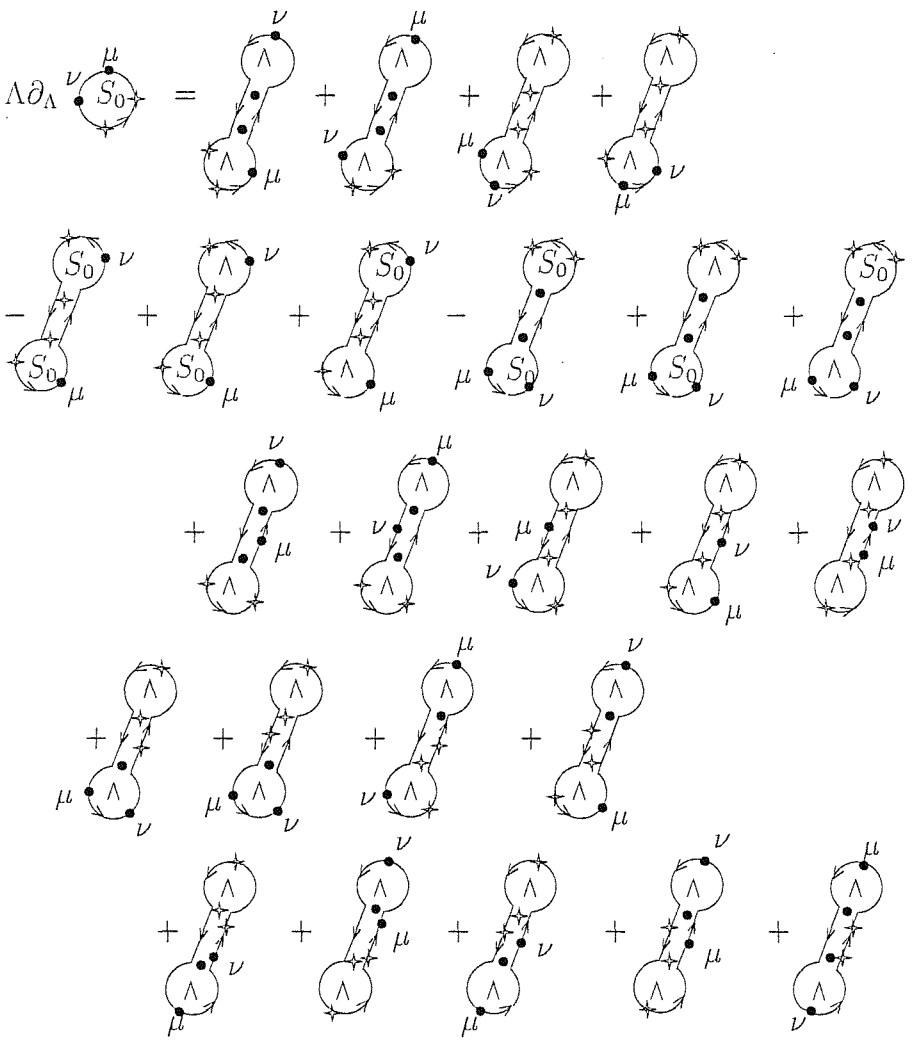

Figure 6.3: Graphical representation of the flow of the tree level vertex AAff

$$+\hat{S}'^{AA}_{\mu\alpha}(p)\dot{\Delta}^{A,AA}_{\nu}(-p;p,0)\hat{S}'^{Aff}_{\alpha RS}(0,k,-k) \qquad (12)$$

$$+\hat{S}'^{ff}_{ST}(-k)\dot{\Delta}^{A,ff}_{\mu TU}(p;-k,k-p)\hat{S}'^{Aff}_{\nu RU}(-p,k,p-k) \qquad (13)$$

$$+\hat{S}'^{Aff}_{\mu TS}(p,k-p,-k)\dot{\Delta}^{A,ff}_{\nu TU}(-p;p-k,k)\hat{S}'^{ff}_{UR}(-k) \qquad (14)$$

$$+\hat{S}'^{ff}_{ST}(-k)\dot{\Delta}^{AA,ff}_{\mu\nu\ TU}(p,-p;-k,k)\hat{S}'^{ff}_{UR}(-k) \qquad (15)$$

$$+\hat{S}'^{AAA}_{\mu\nu\alpha}(p,-p,0)\dot{\Delta}^{f,AB}_{R}(k;0,-k)\hat{S}'^{Bf}_{\alpha S}(k) \qquad (16)$$

$$+\hat{S}'^{fB}_{R\alpha}(k)\dot{\Delta}^{f,BA}_{S}(-k;k,0)\hat{S}'^{AAA}_{\alpha\,\mu\,\nu}(0,p,-p) \qquad (17)$$

$$+\hat{S}'^{AfB}_{\nu R\,\alpha}(-p,k,p-k)\dot{\Delta}^{f,BA}_{S}(-k;k-p,p)\hat{S}'^{AA}_{\alpha\,\mu}(p) \qquad (18)$$

$$+\dot{\hat{S}}^{AA}_{\nu\,\alpha}(p)\dot{\Delta}^{f,AB}_{R}(k;-p,p-k)\hat{S}^{BfA}_{\alpha S\,\mu}(k-p,-k,p) \quad (19)$$

$$+\dot{\hat{S}}^{AA}_{\mu\,\alpha}(p)\dot{\Delta}^{Af,AB}_{\nu R}(-p,k;p,-k)\hat{S}^{Bf}_{\alpha S}(k) \quad (20)$$

$$+\dot{\hat{S}}^{fB}_{R\alpha}(k)\dot{\Delta}^{fA,BA}_{S\mu}(-k,p;k,-p)\hat{S}^{AA}_{\alpha\,\nu}(p) \quad (21)$$

$$+\dot{\hat{S}}^{AA}_{\mu\,\alpha}(p)\dot{\Delta}^{A,f,AB}_{\nu,S}(-p;-k;p,k)\hat{S}^{fB}_{R\alpha}(-k) \quad (22)$$

$$+\dot{\hat{S}}^{fB}_{S\alpha}(-k)\dot{\Delta}^{A,f,BA}_{\mu,R}(p;k;-k,-p)\hat{S}^{AA}_{\alpha\,\nu}(p) \quad (23)$$

$$+\dot{\hat{S}}^{AA}_{\nu\,\alpha}(p)\dot{\Delta}^{ff,AA}_{RS}(k,-k;-p,p)\hat{S}^{AA}_{\alpha\,\mu}(p)\Big\} \quad (24)$$

(6.28)

The terms inserted follow the order of fig.6.3, and the momenta are specialised to the ones needed in the present case. Since we want only the order p^2 on the RHS of

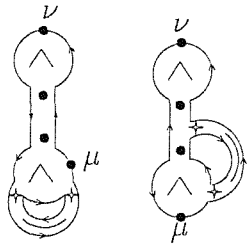

Figure 6.4: Examples of potentially universal diagram

eq.(6.22), the previous equation can be greatly simplified. Noticing that the two point A vertex is already order momentum squared, in all the terms in which it appears carrying momentum p, we can set all the other p dependences to zero. This will cause many terms to be either not contributing to the wanted order of p, or, due to gauge invariance, to have simplified expressions. Terms of this kind, depending only on seed action two-point vertices and their associated zero-point kernels (integrated or otherwise), will be addressed as *potentially universal*: since the seed action two-point vertices and the kernels derived from them, are the only things that we have explicitly prescribed, for the result to be universal, it must be that we can reduce everything to such potentially universal terms or to total Λ derivatives as in eq. (6.27). In turn, potentially universal terms must, and do, collect into total k derivatives, whose

boundary terms on integration, are universal as a result of restrictions on the large momentum behaviour, *e.g.* eq. (3.45), and the renormalization condition eq. (4.37). (Actually, since $\dot{\Delta}_0^{AA} \propto c_0'$, by eq. (5.43), and $1/c_0'$ is never produced, terms such as eq. (6.31) are universal only because they combine to give boundary terms that vanish, as it is proved in section 6.3). Examples of these terms are presented in the diagrammatic form in fig.6.4. As it can be noted, in fact, the upper blobs are two A's point vertices with momentum p.

Before starting to consider any of the terms of eq.(6.28), it is important to point out some relations, due to gauge invariance, for vertices evaluated at special momenta:

$$S'^{Aff}_{\mu RS}(0,k,-k) = -S'^{Aff}_{\mu RS}(0,-k,k) = \partial^k_\mu S^{ff}_{RS}(k) \tag{6.29}$$

$$S'^{AAff}_{\mu\nu RS}(0,0,k,-k) + S'^{AAff}_{\nu\mu RS}(0,0,k,-k) = \partial^k_\mu \partial^k_\nu \dot{\Delta}^{ff}_{RS} \tag{6.30}$$

(The second one, is symmetric in $k \to -k$). For the derivation of these equations see Appendix E. It is now time to perform the analysis of eq.(6.28) term by term. First of all we notice that the first two terms of eq.(6.28), have a factor containing the two point A vertex, evaluated at momentum p. At order p^2 it becomes then proportional to $\dot{\Delta}_0^{AA}$:

$$2\dot{\Delta}_0^{AA}\, \Box_{\nu\alpha}(p) \Delta^{ff}_{SR}(k) \partial^k_\mu \partial^k_\alpha S^{ff}_{RS}(k) \tag{6.31}$$

(the extra factor 2 is for using $\mu \leftrightarrow \nu$ invariance and eq.(6.30) has been used). A detailed study of this terms is left to section6.3. There is now another group of terms

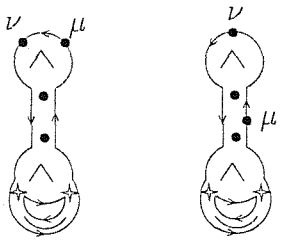

Figure 6.5: These diagrams do not contribute to the order p^2

to be considered together, which are those diagrams not contributing to the order p^2. Two examples of this type are represented diagrammatically in fig.6.5. As one can notice, in fact, one of them has the factor: $S^{AAA}_{\mu\nu\alpha}(p,-p,0)$ (the left one in fig.6.5). Making use of eq.(6.29), it is possible to see that this term becomes:

$$S^{AAA}_{\mu\nu\alpha}(p,-p,0) = \partial^p_\alpha S^{AA}_{\mu\nu}(p) = \partial^p_\alpha \left(\frac{2\,\Box_{\mu\nu}(p)}{c_p} \right) \tag{6.32}$$

which is of order odd in p and can not contribute to the wanted order.The terms ruled out by this observation are (8)-(10) and (16)-(17). We can now make a further reduction of the terms. Among the group of the potentially universal the terms (11) and (12) in eq.(6.28) are left with $\dot{\Delta}^{A,AA}_{\mu}(0;0,0)$ (the right one in fig.6.5) which is zero (clearly by Lorentz invariance). Finally, term (24) has two factors $S^{AA}_{\mu\alpha}(p)$ therefore it is of order p^4. We are now ready to proceed evaluating the terms left which will eventually give the β-function at 1-loop.

We still have to evaluate (3)-(7), (13)-(15) and (18)-(23) of eq.(6.28). Let us then start with (3) and (4):

$$\begin{aligned}\Delta^{ff}_{SR}(k) \Big\{ &\hat{S}^{ffAA}_{RT\mu\nu}(k,-k,p,-p)\dot{\Delta}^{ff}_{TU}(k)\hat{S}^{ff}_{US}(-k) \\ &+\hat{S}^{ffAA}_{TS\mu\nu}(k,-k,p,-p)\dot{\Delta}^{ff}_{TU}(k)\hat{S}^{ff}_{UR}(k)\Big\}\end{aligned} \tag{6.33}$$

Making use of eq.(6.25) it is possible to write the previous equation as:

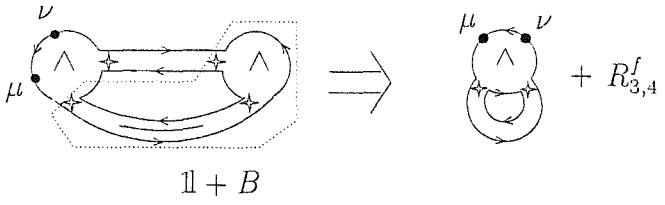

Figure 6.6: Diagrammatical representation of the mechanism responsible of the cancellation of the first $\hat{S}$-term of fig.6.1 (first line of eq.(6.22))

$$2\hat{S}^{ffAA}_{TR\mu\nu}(k,-k,p,-p)\dot{\Delta}^{ff}_{RT}(k) + R^{f}_{3,4} \tag{6.34}$$

since the wine vertices are diagonal. Where $R_{3,4}$ is the remainder, since the contraction between the two point vertex and the integrated wine, eq.(6.25), does not give just $\mathbb{1}$, and it will be considered later. For now, let us compare the expressions in eq.(6.34) and the second term in eq.(6.26). As it is shown diagrammatically in fig.6.6, the two terms (3) and (4) get the same form and opposite sign of the hatted term of fig.6.1 up to the rest $R^{f}_{3,4}$, so they cancel out.

The next terms which will be considered are (13) and (14) of eq.(6.28):

$$\begin{aligned}\Delta^{ff}_{SR}(k)\Big\{&\hat{S}^{ff}_{ST}(-k)\dot{\Delta}^{A,ff}_{\mu TU}(p;-k,k-p)\hat{S}^{Aff}_{\nu RU}(-p,k,p-k)\\ &+\hat{S}^{Aff}_{\mu TS}(p,k-p,-k)\dot{\Delta}^{A,ff}_{\nu TU}(-p;p-k,k)\hat{S}^{ff}_{UR}(-k)\Big\}\end{aligned} \tag{6.35}$$

Applying also in this case eq.(6.25) one can recast the equation above in the following

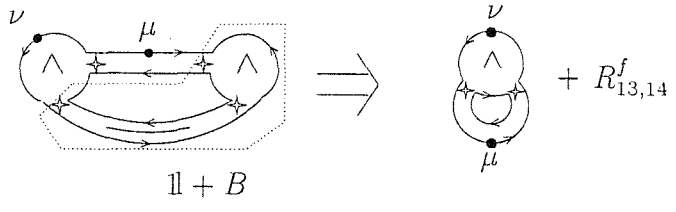

Figure 6.7: Diagrammatical representation of the mechanism responsible of the cancellation of the second $\hat{S}$-term of fig.6.1 (second line of eq.(6.22))

form:

$$2\dot{\Delta}^{A,ff}_{\mu TU}(p;-k,k-p)\hat{S}^{Aff}_{\nu TU}(-p,k,p-k) + R^{f}_{13,14} \tag{6.36}$$

Here we made use of the fact that $\Delta^{ff}_{RS}(k)$ is diagonal in RS and is an even function of k and we used the $\mu \leftrightarrow \nu$, $p \leftrightarrow -p$ symmetry. We also have considered the fact that the above term had to be integrated over k and used the translation invariance for the integrated variable. It is possible also here to notice that the first term in Eq(6.36) exactly cancels the term containing $\hat{S}$ in the second line of eq.(6.22). This

is diagrammatically expressed in fig.6.7. The two terms (13) and (14) of eq.(6.28) cancel the three-point hatted vertex from eq.6.22) and what is left is just $R^f_{13,14}$ which will be considered later. We will now move to term (15) of eq.(6.28). Its expression is the following:

$$\Delta^{ff}_{SR}(k)\hat{S}^{ff}_{ST}(-k)\dot{\Delta}^{AA,ff}_{\mu\nu TU}(p,-p;-k,k)\hat{S}^{ff}_{UR}(-k) \tag{6.37}$$

Using the usual (6.25) relation, we get:

$$\dot{\Delta}^{AA,ff}_{\mu\,\nu\,RU}(p,-p;-k,k)\hat{S}^{ff}_{UR}(-k) + R^f_{15} \tag{6.38}$$

In this case it is already clear that the first term in eq.(6.38) cancels the term

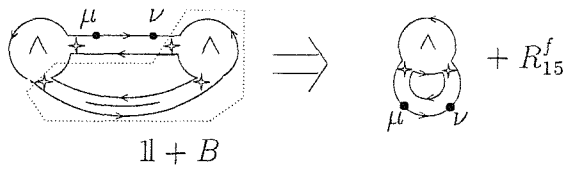

Figure 6.8: Diagrammatical representation of the mechanism which causes the cancellation of the third $\hat{S}$-term of fig.6.1 (third line of eq.(6.22))

containing $\hat{S}$ in the third line of eq.(6.22). The mechanism is again described by the diagrams of fig.6.8. What is left of term (15) from eq.(6.28) after the cancellation is the usual reminder, which will be indicated with R^f_{15} and considered later. We now move to study the terms from (5) to (7) of eq.(6.28). Also here the method adopted involves the use of the flow equations for the effective vertices. Let us consider (5):

$$-\Delta^{ff}_{SR}(k)S^{ffA}_{TS\mu}(k-p,-k,p)\dot{\Delta}^{ff}_{TU}(k-p)S^{Aff}_{\nu RU}(-p,k,p-k) \tag{6.39}$$

As we can see from fig.6.9, eq.(6.39) can be rewritten as:

$$\frac{1}{2}\Lambda\partial_\Lambda\left[\Delta^{ff}_{SR}(k)\Delta^{ff}_{TU}(k-p)S^{ffA}_{TS\mu}(k-p,-k,p)S^{Aff}_{\nu RU}(-p,k,p-k)\right]$$

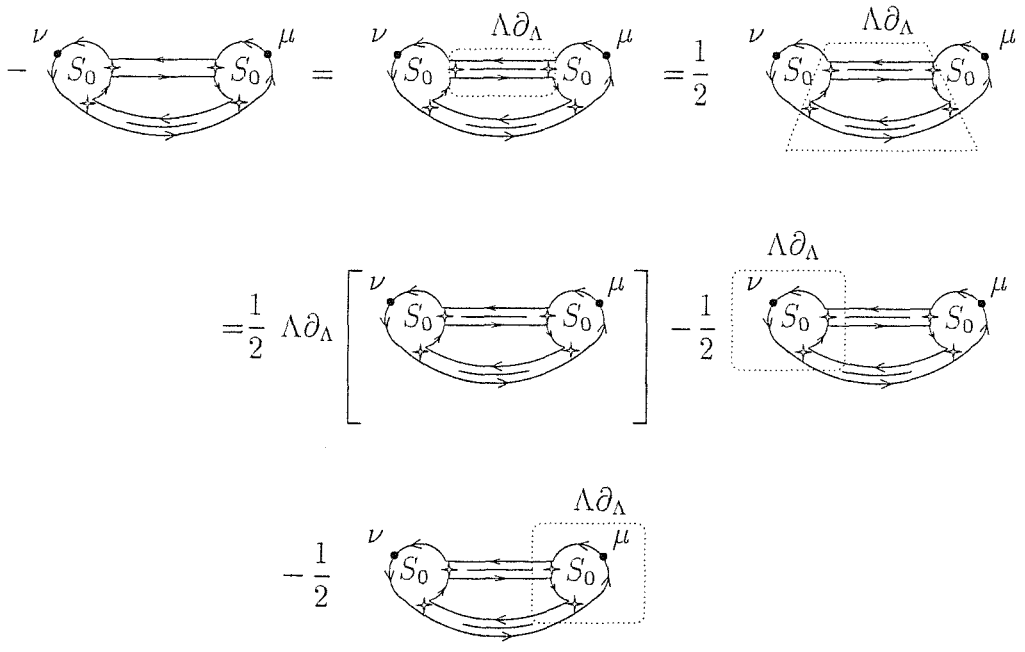

Figure 6.9: Method used for the diagrams (5)-(7) of equation (6.28). The flow equation for the 3-point Aff is now needed

$$-\frac{1}{2}\Delta^{ff}_{SR}(k)\Delta^{ff}_{TU}(k-p)S^{Aff}_{\nu RU}(-p,k,p-k)\Lambda\partial_\Lambda S^{ffA}_{TS\mu}(k-p,-k,p)+p_\mu\leftrightarrow p_\nu \quad (6.40)$$

The first term will be considered later, together with (0) in eq.(6.28), while the second term needs now the introduction of the equation for the three point function ffA. This equation is expressed in diagrams in fig.6.10. If now one substitutes the terms of fig.6.10 in fig.6.9 there would be terms such as those represented in figs.6.11, 6.12 and 6.14. Making use of the inverse relation eq.(6.25) many cancellations occur. To start with, as one can notice from fig.6.11, term (6) is cancelled when the last two terms of the first line in fig.6.10 are substituted in fig.6.9. In the same way (7) is canceled when the $\mu \leftrightarrow \nu$ term of fig.6.9 is considered. Moreover, the effective three point vertex term (the last left in eq.(6.22)) is canceled when the first term on the second line of fig.6.10 is substituted in fig.6.9 (as it is explained diagrammatically in fig.6.12) and when the same is done for its analogous in the swop $\mu \leftrightarrow \nu$. This can be checked writing them in formulae using the rules for wines, effective and hatted vertices. A

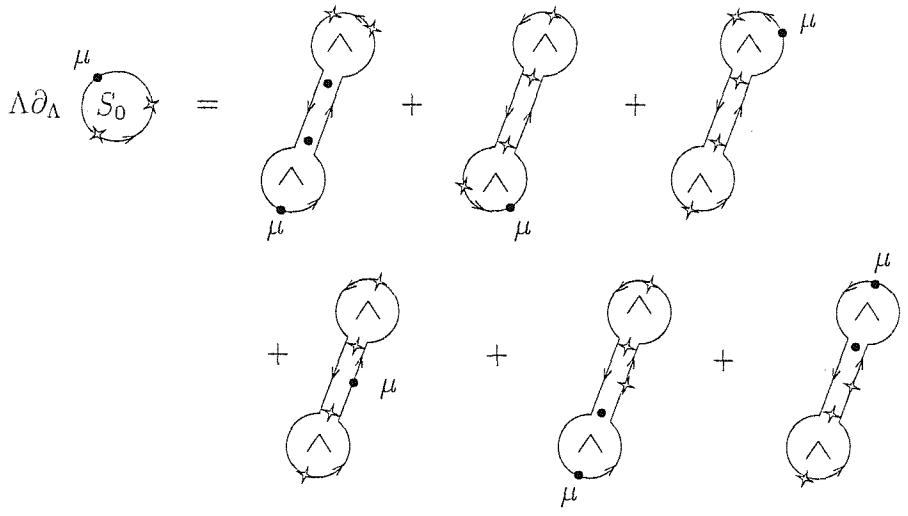

Figure 6.10: Flow equation for the three point vertex Aff

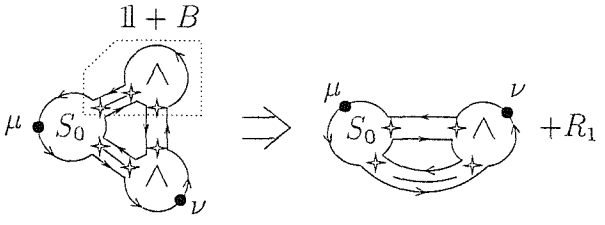

Figure 6.11: Term arising from (5) in eq.(6.28), once the equation for Aff is used, canceling (6)

further cancellation occurs due to (5), and it is the one described in fig.6.13. Due to the inverse relation (6.25), the last two terms of the three point vertex Aff when substituted in eq.(6.40), cancel exactly terms (18) and (19) of eq.(6.28). What we are left with, after the analysis of term (5), are then three reminders from the three cancellations which occurred, which will be considered later on with all the others, and the term described in diagrams in fig.6.14. This term comes from the first term of the equation for Aff. Its analytical form comes directly from the diagram in fig.6.14. Since it contains the two point A vertex evaluated at momentum p, which as we noted previously has already order p^2, once the other momenta p are set to zero to get only

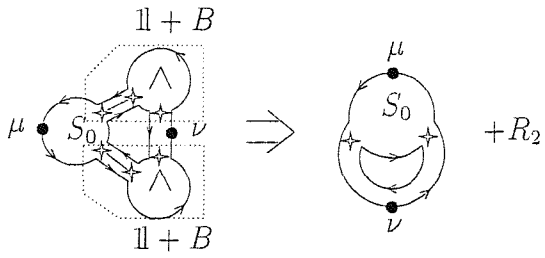

Figure 6.12: Term arising from (5) in eq.(6.28), once the equation for Aff is used, canceling the three point effective action term in eq.(6.22)

the wanted order in p we get:

$$-2\dot{\Delta}_0^{AA}\,\Box_{\nu\alpha}(p)\Delta^{ff}_{RU}(k)\Delta^{ff}_{ST}(k)\partial^k_\mu S^{ff}_{RS}(k)\partial^k_\alpha S^{ff}_{TU}(k) \tag{6.41}$$

(the extra factor of 2 comes from Lorentz invariance, considering it together with the analogous $\mu \leftrightarrow \nu$ term and eq.(6.29) has been used). This is another term proportional to $\dot{\Delta}_0^{AA}$ and will be considered separately in section 6.3. What we are

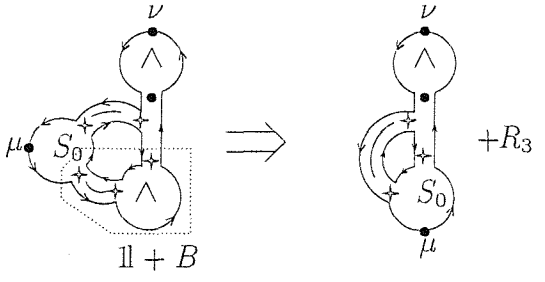

Figure 6.13: Term arising from (5) in eq.(6.28), once the equation for Aff is used, canceling (18) and (19) in the same equation.

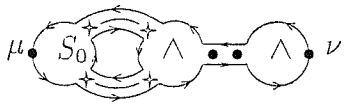

Figure 6.14: Potentially universal terms left from (5) of eq.(6.28)

left with from eq.(6.28) are then the terms (20)-(23), the six remainders and the two total derivative terms. Let us consider (20)-(23) first. Since they all are proportional to the two-point A vertex at momentum p, using the inverse relation (6.25), they can be grouped together as follows:

$$4\,\Box_{\nu\alpha}(p)\left[B^{ff}_{\alpha R}(-k)\dot{\Delta}^{Af,BA}_{\mu R}(0,k;-k,0)-\dot{\Delta}^{Af,fA}_{\mu R,R\alpha}(0,k;-k,0)\right] \tag{6.42}$$

To get eq. (6.42), we have also used a consequence of the coincident lines identity, which relates wine vertices, with fields on different lines; in particular, for our purposes, we have considered:

$$\dot{\Delta}^{A,f,BA}_{\mu,R}(p;q;r,s)=-\dot{\Delta}^{Af,BA}_{\mu R}(p,q;r,s)-\dot{\Delta}^{fA,BA}_{R\mu}(q,p;r,s) \tag{6.43}$$

It is also important to notice, that the second term in eq. (6.42), will not appear when f is specified to the C sector, since there is no $\dot{\Delta}^{AC}$ kernel. For the A and $F=(B,D\sigma)$ sectors, it will be respectively $\dot{\Delta}^{AA,AA}_{\mu\alpha}(0,k;-k,0)$ and $\dot{\Delta}^{AB,BA}_{\mu\alpha}(0,k;-k,0)=\dot{\Delta}^{AA,AA}_{\mu\alpha}(0,k;-k,0)$ (again because there is no $\dot{\Delta}^{AC}$ kernel). These contributions, as it can be proved, cancel between the A and F sectors, being of opposite sign. However, since they represent Wines biting their tail diagrams, which are excluded by eqs. (4.20) and (4.24), they will not be included in the rest of the calculation. We are now ready to summarise and rewrite eq.(6.22) as:

$$\begin{aligned}
-4\beta_1\,\Box_{\mu\nu}(p)=N\int_k\Big\{\Lambda\partial_\Lambda\Big[&\Delta^{ff}_{SR}(k)\Delta^{ff}_{TU}(k-p)S^{ffA}_{TS\mu}(k-p,-k,p)S^{Aff}_{\nu RU}(-p,k,p-k)\\
&-2\Delta^{ff}_{SR}(k)S^{ffAA}_{RS\mu\nu}(-k,k,p,-p)\Big]_{p^2} \quad (1)\\
-2\dot{\Delta}^{ff}_{TU}(k)&\left[\hat{S}^{ffAA}_{RT\mu\nu}(k,-k,p,-p)B^{ff}_{UR}(k)+\hat{S}^{ffAA}_{TS\mu\nu}(k,-k,p,-p)B^{ff}_{US}(-k)\right] \quad (2)\\
&-2\dot{\Delta}^{A,ff}_{\mu TU}(p;-k,k-p)\hat{S}^{Aff}_{\nu RU}(-p,k,p-k)B^{ff}_{TR}(k)\\
&-2\dot{\Delta}^{A,ff}_{\nu TU}(-p;p-k,k)\hat{S}^{Aff}_{\mu TR}(p,k-p,-k)B^{ff}_{UR}(-k) \quad (3)\\
&-2\dot{\Delta}^{AA,ff}_{\mu\nu\ TU}(p,-p;-k,k)\hat{S}^{ff}_{UR}(-k)B^{ff}_{TR}(k) \quad (4)\\
+2S^{Aff}_{\nu RU}(-p,k,p-k)&\left[\hat{S}^{Aff}_{\mu MS}(p,k-p,-k)\Delta^{ff}_{SR}(k)\dot{\Delta}^{ff}_{MN}(k-p)B^{ff}_{NU}(p-k)+\right.
\end{aligned}$$

$$+\hat{S}^{Aff}_{\mu TM}(p,k-p,-k)\Delta^{ff}_{TU}(p-k)\dot{\Delta}^{ff}_{MN}(k)B^{ff}_{NR}(k)\Big] \quad (5)$$

$$+2S^{Aff}_{\mu RT}(p,k-p,-k)\Big[\dot{\Delta}^{A,ff}_{\nu RU}(-p;p-k,k)B^{ff}_{UT}(-k)+\dot{\Delta}^{A,ff}_{\nu UT}(-p;p-k,k)B^{ff}_{UR}(k-p)$$

$$-\dot{\Delta}^{A,ff}_{\nu VU}(-p;p-k,k)B^{ff}_{UT}(-k)B^{ff}_{VR}(k-p)\Big] \quad (6)$$

$$+8\,\Box_{\nu\alpha}(p)\dot{\Delta}^{f,Af}_{U,\alpha V}(k;0,-k)B^{ff}_{VR}(k)\partial^k_\mu\hat{S}^{ff}_{RT}(k)\Delta^{ff}_{TU}(k) \quad (7)$$

$$+8\,\Box_{\nu\alpha}(p)B^{ff}_{VR}(k)\dot{\Delta}^{Af,fA}_{\mu R,V\alpha}(0,k;-k,0) \quad (8)$$

$$+4\dot{\Delta}^{AA}_0\,\Box_{\nu\alpha}(p)\Big[\Delta^{ff}_{SR}(k)\partial^k_\mu\partial^k_\alpha S^{ff}_{RS}(k)-\Delta^{ff}_{RU}(k)\Delta^{ff}_{ST}(k)\partial^k_\mu S^{ff}_{RS}(k)\partial^k_\alpha S^{ff}_{TU}(k)\Big]\Big\} \quad (9)$$

(6.44)

6.2.1 The A sector

Let us start, with the first contribution to be evaluated in this sector which is $R^f_{3,4}$. The general expression is represented by term (2) in eq. (6.44). Recalling the form of $B^{ff}_{RS}(k)$ in the pure A case from eq.(D.4), we can rewrite this term as follows:

$$R^A_{3,4}=-4\dot{\Delta}^{AA}(k)\frac{k_\sigma k_\rho}{k^2}\hat{S}^{AAAA}_{\rho\sigma\mu\nu}(k,-k,p,-p). \quad (6.45)$$

Making use of the usual Ward identities for the pure gauge field, and recalling eq.(6.29), we can write the previous equation as:

$$R^A_{3,4}=-\frac{4\dot{\Delta}^{AA}(k)}{k^2}\left(k_\sigma\partial^p_\sigma S^{AA}_{\mu\nu}(p)+S^{AA}_{\mu\nu}(p-k)-S^{AA}_{\mu\nu}(p)\right) \quad (6.46)$$

Since the first term in the above equation has only odd powers of p, it does not contribute to the order p^2 and it will not be taken into account for the rest of the calculation. We are then left with:

$$R^A_{3,4}\Big|_{p^2}=-\frac{4\dot{\Delta}^{AA}(k)}{k^2}\left[S^{AA}_{\mu\nu}(p-k)\Big|_{p^2}-2\,\Box_{\mu\nu}(p)\right] \quad (6.47)$$

The next term to be considered is now $R^f_{13,14}$ (term (3) in eq. (6.44)) which in this sector gets the following form:

$$\begin{aligned} R^A_{13,14} &= -2\dot{\Delta}^{A,AA}_{\mu}(p;-k,k-p)\frac{k_\rho k_\sigma}{k^2}\hat{S}'^{AAA}_{\nu\,\rho\,\sigma}(-p,k,p-k) \\ &\quad -2\dot{\Delta}^{A,AA}_{\nu}(-p;p-k,k)\frac{(-k)_\rho(-k)_\sigma}{k^2}\hat{S}'^{AAA}_{\mu\,\rho\,\sigma}(p,k-p,-k) \end{aligned} \tag{6.48}$$

Applying the Ward identities and making use of the symmetry $\mu \leftrightarrow \nu$, it is possible to get the final expression (at order p^2):

$$R^A_{13,14}\Big|_{p^2} = \frac{4k_\rho}{k^2}\left[\dot{\Delta}^{A,AA}_{\nu}(-p;p-k,k)\hat{S}'^{AA}_{\mu\,\rho}(p-k)\Big|_{p^2} - 2\partial^k_\mu\dot{\Delta}^{AA}(k)\,\Box_{\nu\rho}(p)\right] \tag{6.49}$$

We now have to analyse term (3) of eq. (6.44), which is R^f_{15}. Its expression in the sector A is:

$$R^A_{15} = \dot{\Delta}^{AA,AA}_{\mu\,\nu}(p,-p;,-k,k)\frac{k_\alpha k_\beta}{k^2}S'^{AA}_{\alpha\beta}(k) = 0, \tag{6.50}$$

by gauge invariance.

We have now to evaluate the last three reminders (5)-(7) of eq.(6.44). They have in the A sector the following form:

$$R^A_1 = -\frac{16\dot{\Delta}^{AA}(k)}{k^2}\,\Box_{\nu\alpha}(p)\left(\delta_{\alpha\mu}-\frac{k_\alpha k_\mu}{k^2}\right) + 4\frac{\dot{\Delta}^{AA}(k)}{k^2}\,\hat{S}'^{AA}_{\mu\,\nu}(p-k)\Big|_{p^2} \tag{6.51}$$

$$R^A_2 = \frac{4k_\rho}{k^2}\left[-\dot{\Delta}^{A,AA}_{\nu}(-p;k,p-k)\,\hat{S}'^{AA}_{\mu\,\rho}(p-k)\Big|_{p^2} + \partial^k_\nu\dot{\Delta}^{AA}(k)\,\Box_{\mu\rho}(p)\right] \tag{6.52}$$

$$R^A_3 = \frac{-8k_\alpha}{k^2}\,\Box_{\nu\alpha}(p)\left[\dot{\Delta}^{A,AA}_{\mu}(k;0,-k) + \frac{k_\mu}{k^2}(\dot{\Delta}^{AA}(k)-\dot{\Delta}^{AA}_0)\right] = 0 \tag{6.53}$$

where Ward identities and the inverse relation (6.25) have been used, and the order p^2 taken. Before starting with the total derivative term (1) and the one proportional to $\dot{\Delta}^{AA}_0$ (9), let us consider term (8) in eq.(6.44). In this sector, it takes the form:

$$8\,\Box_{\nu\alpha}(p)\left[\frac{k_\alpha}{k^2}\partial^k_\mu\dot{\Delta}^{AA}(k) - \frac{\mathrm{k}_\mu k_\alpha}{k^4}\dot{\Delta}^{AA}(k)\right] \tag{6.54}$$

It is possible to notice now that adding up all these contributions, the non-universal terms cancel out and that many other cancellations occur. Collecting the terms which are left, recalling the expression of the A-kernel from eq. (5.43) and considering the integral over k in (6.44), after the average over the k components is taken, we finally get:

$$-\frac{8N}{\Lambda^4}\,\Box_{\mu\nu}(p)\int\frac{d^Dk}{(2\pi)^D}\,\left[\left(1-\frac{1}{D}\right)\frac{\Lambda^2 c_k'}{k^2}-\frac{1}{D}c_k''\right] \tag{6.55}$$

Setting now $x = k^2/\Lambda^2$, and with some algebra, we find the previous equation becomes:

$$\begin{aligned}2N\Omega_D\,\Box_{\mu\nu}(p)\int_0^\infty dx\,\left\{\left[\left(\frac{4}{D}-3\right)c(x)\,x^{D/2-2}+\frac{2}{D}c'(x)\,x^{D/2-1}\right]'\right.\\ \left.-\frac{(D-4)(4-3D)}{D}\,\frac{x^{D/2-3}}{2}c(x)\right\}\end{aligned} \tag{6.56}$$

where Ω_D is the integration over the angles divided by $(2\pi)^D$. (For the other calculations, we will not specify this last step, but we will just write the result in $D = 4$.) Since this integral is regular in D around $D = 4$, we can specify to that value and get a total derivative:

$$\begin{aligned}N\Omega_4\,\Box_{\mu\nu}(p)\int_0^\infty dx\,[-4c(x)+x\;c'(x)]'\\ =4N\Omega_4\,\Box_{\mu\nu}(p)\end{aligned} \tag{6.57}$$

In eq. (6.55) we did not include the total derivative term and the ones proportional to $\dot{\Delta}_0^{AA}$. The total contribution from the A sector to this latter set of terms is:

$$4\dot{\Delta}_0^{AA}\,\Box_{\mu\alpha}(p)\left[\Delta^{AA}(k)\partial_\alpha^k\partial_\nu^k S^{AA}_{\beta\beta}(k)-\left(\Delta^{AA}(k)\right)^2\partial_\alpha^k S^{AA}_{\lambda\rho}(k)\partial_\nu^k S^{AA}_{\lambda\rho}(k)\right] \tag{6.58}$$

which is term (9) of eq. (6.44) in this sector, and:

$$8\dot{\Delta}_0^{AA}\,\Box_{\mu\alpha}(p)\frac{k_\nu k_\alpha}{k^4} \tag{6.59}$$

a further term of this kind to be added to the previous one, which comes from

eqs.(6.52) and (6.54). These terms will be considered later on in section 6.3.

It is now time to consider (1) of eq. (6.44), *i.e.* the total derivative term. After substituting in it the expression for the integrated A-kernel from eq. (D.16), a first observation we can make is that if the term was UV and IR regulated we could pull the Λ-derivative out of the integral, as in the scalar field case, and the result would be identically zero. The integrand is in fact dimensionless and once the integration is carried out the result would be a Λ-derivative of a scale independent quantity. Since the UV finiteness is ensured by the regularisation, the non zero contribution of this term, if there is one, must come from its IR poles. We want then to study the $k \to 0$ behaviour of the following term:

$$\frac{c_k}{2k^2}\left[\frac{c_{p-k}}{2(p-k)^2}S^{AAA}_{\mu\rho\sigma}(p,k-p,-k)S^{AAA}_{\nu\sigma\rho}(-p,k,p-k) - 2S^{AAAA}_{\mu\nu\alpha\alpha}(p,-p,-k,k)\right] \tag{6.60}$$

and take its order p^2. Eq.(6.60) has to be integrated in d^Dk, and the second term of the two does not contribute at all since its integral is IR convergent. The first one instead has to be studied. The first step is to expand the factor that might carry extra poles (the second integrated wine) up to the order p^2:

$$\left.\frac{c_{p-k}}{(p-k)^2}\right|_{p^2} = \frac{c_k}{k^2} - 2\frac{c_k}{k^4}k\cdot p - \frac{c_k}{k^4}\left(p^2 - 4\frac{(k\cdot p)^2}{k^2}\right) + \frac{c'_k}{k^2\Lambda^2}\left(p^2 - 4\frac{(k\cdot p)^2}{k^2}\right) \tag{6.61}$$

The last term cannot contribute because the product of the two three point vertices gives at least as k^2, and thus results in an integrable term. We are then left with three terms, a p^0 term, a term linear in p and a term of order p^2. The crucial observation to handle this total derivative term is now that even though it is not potentially universal, its IR contribution happens to be, as we will describe. Let us take for example the p^0 term in the expansion (6.61). In $D = 4$, substituting it in eq. (6.60), we can see that the integral is IR convergent unless in both of the three point vertices appearing in the product, we take the order zero in k. This means that using eq. (6.29)

we get a potentially universal term. If we now consider the term linear in p, we must get another power of p and this can come from either of the two three-point vertices in the product. This means that the other one must be taken at order p^0 and this makes it potentially universal. Moreover, in $D = 4$, the first three point function must be taken at order zero in k since otherwise is at least linear in k and the term would be integrable. This makes the term potentially universal. Finally, there is the term of eq. (6.61) which is already of order p^2. It is obvious in this latter case that we do not need any more powers of p and the three point functions must be taken at $p = 0$. The whole total derivative term is then potentially universal in its IR-divergent regime. The only contribution from expression (6.60) takes the following form:

$$\frac{c_k^2}{2k^4}\left[\partial_\sigma^p S^{AA}_{\mu\rho}(p)\partial_\sigma^p S^{AA}_{\nu\rho}(p) - \frac{4}{k^2}(k\cdot p)\partial_\mu^k S^{AA}_{\rho\sigma}(k)\partial_\sigma^p S^{AA}_{\nu\rho}(p) - \frac{1}{k^2}\left(p^2 - 4\frac{(k\cdot p)^2}{k^2}\right)\partial_\mu^k S^{AA}_{\rho\sigma}(k)\partial_\nu^k S^{AA}_{\rho\sigma}(k)\right] \tag{6.62}$$

Considering now the expression for the two point function $2\,\Box_{\mu\nu}/c_p$, taking the order p^2 of the previous equation and the average on the k momenta, we get (in $D = 4$ and recalling the factor N in (6.44)):

$$N\Omega_4\left(\frac{19}{6}p^2\delta_{\mu\nu} - \frac{11}{3}p_\mu p_\nu\right) \tag{6.63}$$

where the following identity has effectively been used:

$$\lim_{D\to 4^+}\Lambda\partial_\Lambda\int\frac{d^Dk}{(2\pi)^D\,k^4} = \lim_{D\to 4^+}\frac{\Omega_D}{D-4}\Lambda\partial_\Lambda\Lambda^{D-4} = \Omega_4 \tag{6.64}$$

(Here, only the IR divergent part is shown). This was the contribution from the total derivative term of the A sector. In order to get the total contribution to the β function at one loop from this sector we have to sum to this, the result obtained from the rest in eq. (6.57). As one can notice this partial result is not transverse as it would be expected (the LHS of eq. (6.44) is transverse and so the RHS ought to

be). However, the contribution from the $BD\sigma$ sector will contribute with another non transverse term and give a transverse final result.

6.2.2 The C sector

In the present sector there is no contribution expected from terms (2)-(7) in eq. (6.44), since as can be seen from eq.(D.15) $B^{CC}(k) = 0$. What we have is then:

$$R^C_{3,4} = R^C_{13,14} = R^C_{15} = R^C_1 = R^C_2 = R^C_3 = 0 \tag{6.65}$$

Moreover term (8) from eq. (6.44) is zero too, since it comes from (20)-(23) in eq. (6.22) and these two point wines are zero when $f = C$ (as was pointed out above eq. (6.44)). As far as this sector is concerned, the only contributions are from (1) and (9) of eq. (6.44), the total derivative term and the one proportional to $\dot{\Delta}_0^{AA}$. Let us analyse the former first.The analysis to be carried out here is similar to that one for the A sector. We will be looking for IR poles. The total derivative term in this sector has the following form:

$$\Delta^{CC}(k)\left[\Delta^{CC}(k-p)S'^{ACC}_{\mu}(p,k-p,-k)S'^{ACC}_{\nu}(-p,k,p-k) - 2S'^{AACC}_{\mu\nu}(p,-p,-k,k)\right] \tag{6.66}$$

From the explicit expression of Δ^{CC}, *e.g.* eq. (6.106), one can notice that it is regular as $k \to 0$, as its derivatives are. The expression in eq. (6.66) is then not only UV regular (after adding in the other sectors), but also IR regular, and the Λ-derivative can be pulled out of the integral giving a vanishing contribution.

To finish the analysis of this sector we are then left with the set of terms proportional

to $\dot{\Delta}_0^{AA}$. Here we will just write term (9) of eq. (6.44) specialised in the sector C:

$$4\dot{\Delta}_0^{AA}\,\Box_{\nu\alpha}(p)\left[\Delta^{CC}(k)\partial_\mu^k\partial_\alpha^k S^{CC}(k)-(\Delta^{CC}(k))^2\partial_\mu^k S^{CC}(k)\partial_\alpha^k S^{CC}(k)\right] \tag{6.67}$$

This term will be considered in section 6.3.

6.2.3 The $BD\sigma$ sector

Let us finish this analysis considering the last sector. The first term that has to be evaluated is again $R^f_{3,4}$. In this particular case, from the expression of $B^{FF}_{RS}(k)$ of eq.(D.5) and recalling that there is an extra minus sign for all the terms when $f=F$ (as one can see comparing eqs.(5.19) and (6.22)), we can write it as:

$$R^F_{3,4}=2\dot{\Delta}^{FF}_{TU}(k)\left[k'_U k_R \hat{S}'^{FFAA}_{RT\,\mu\nu}(k,-k,p,-p)+(-k')_U(-k)_S\hat{S}'^{FFAA}_{TS\,\mu\nu}(k,-k,p,-p)\right] \tag{6.68}$$

where k and k' are the generalised momenta of eqs.(6.14) and (6.15). It is now possible to see that, in this case, one can apply the generalised Ward identities described in section 6.1, namely Eqs.(6.16)-(6.21). Due to a similar argument to the one used in the pure A field, the wanted expression at order p^2 is:

$$R^F_{3,4}\Big|_{p^2}=-4\dot{\Delta}^{FF}_{TU}(k)k'_U\;\hat{S}'^{ABF}_{\mu\nu T}(p,k-p,-k)\Big|_{p^2} \tag{6.69}$$

the components will be specified once the other terms are considered. The next term which has to be studied is then $R^f_{13,14}$. Its general expression is term (3) of eq. (6.44) and in the present sector, gets the form:

$$\begin{aligned} R^F_{13,14} &= 2\dot{\Delta}^{A,FF}_{\mu TU}(p;-k,k-p)k'_T k_R\hat{S}'^{AFF}_{\nu RU}(-p,k,p-k) \\ &= 2\dot{\Delta}^{A,FF}_{\nu TU}(-p;p-k,k)(-k)'_U(-k)_R\hat{S}'^{AFF}_{\mu TR}(p,k-p,-k) \end{aligned} \tag{6.70}$$

Making use of the generalised Ward identities, using the properties of the wine vertices and with some algebra, it is possible to recognise the final result to be:

$$R^F_{13,14}\Big|_{p^2} = 4k'_U\ \dot{\Delta}^{A,FF}_{\mu TU}(p;k-p,-k)\hat{S}'^{BF}_{\nu T}(k-p)\Big|_{p^2} + 8\ \Box_{\mu\alpha}(p)k'_\alpha\dot{\partial}^k_\nu\dot{\Delta}^{BB}(k) \qquad (6.71)$$

Also in this case we keep for now the compact notation before we consider all the other terms, in order to see all the simplifications which occur already at this level. The last term of this kind to consider is R^f_{15}, which in the $BD\sigma$ sector gets the form:

$$R^F_{15} = 2\dot{\Delta}^{AA,FF}_{\mu\nu TU}(p,-p;-k,k)k'_T k_R\hat{S}'^{FF}_{UR}(-k,k) \qquad (6.72)$$

Due to the generalised Ward identity (6.21) this is identically zero as in the case of the pure A.

The last three remainders in the $BD\sigma$ sector have the following form:

$$\begin{aligned} R^F_1\Big|_{p^2} = -8\ \Box_{\mu\alpha}(p)\Big[&\dot{\Delta}^{BB}(k)k'_\alpha k'_\nu + \dot{\Delta}^{FF}_{MN}(k)\Delta^{BB}(k)k'_N\partial^k_\nu\hat{S}'^{BF}_{\alpha M}(k)\Big] \\ &+4\dot{\Delta}^{FF}_{SN}(k)k'_N(k-p)'_\nu\hat{S}'^{BF}_{\mu S}(k)\Big|_{p^2} \\ &+4\dot{\Delta}^{FF}_{MN}(k)k'_N\hat{S}'^{ABF}_{\mu\nu M}(p,k-p,-k)\Big|_{p^2} \end{aligned} \qquad (6.73)$$

$$\begin{aligned} R^F_2\Big|_{p^2} = -4\ \Box_{\mu\alpha}(p)\Big[&2k'_\alpha\partial^k_\nu\dot{\Delta}^{BB}(k) + k_\alpha k'_U(-k')_V\partial^k_\nu\dot{\Delta}^{FF}_{UV}(k)\Big] \\ &-4\dot{\Delta}^{A,FF}_{\mu RU}(p;k-p,-k)k'_U\hat{S}'^{BF}_{\nu R}(k-p)\Big|_{p^2} \end{aligned} \qquad (6.74)$$

$$R^F_3\Big|_{p^2} = -8k'_\alpha\ \Box_{\mu\alpha}(p)\Big[\dot{\Delta}^{B,BA}_\nu(k;-k,0) + k'_\nu\left(\dot{\Delta}^{AA}_0 - \dot{\Delta}^{BB}(k)\right)\Big] \qquad (6.75)$$

From the two terms (8) of eq. (6.44), and recalling the comment just above eq.(6.44), we have this final expression:

$$-8\ \Box_{\mu\alpha}(p)k'_\alpha\left(\partial^k_\nu\dot{\Delta}^{BB}(k) - \dot{\Delta}^{B,BA}_\nu(k;-k,0)\right) \qquad (6.76)$$

It is possible already to notice many cancellations, in particular of the non-potentially universal terms of eqs. (6.69) and (6.71) respectively due to R_1^F and R_2^F, plus many others as in the A sector. Unlike from the pure gauge field case there is a residual term left, from R_1^F, which is not clearly transverse. As we will analyse it, it will be clear that it is not transverse and in fact will restore the transversality of the pure gauge result. The other terms left are the total derivative, the ones proportional to $\dot{\Delta}_0^{AA}$ and all the ones which are not cancelled between the equations above but are already potentially universal. Let us consider the latter group of terms first. After specifying the components and grouping them together it is possible integrating by parts and with some algebra to recognise that in $D = 4$, we are left with a total derivative, which becomes a surface term to be evaluated between the boundaries:

$$- \Omega_4 \, \Box_{\mu\nu}(p) \; x^2 f^2(x)[xc'(x) - 4c(x)]\Big|_0^\infty \tag{6.77}$$

where $x = k^2/\Lambda^2$, and the two functions $f(x)$ and $g(x)$ were defined below eq. (6.15). Considering again the behaviour of c and $\tilde{c}$ at the boundaries, it is possible to recognise that this surface term does not contribute.

It is now time to consider the non-transverse term mentioned earlier:

$$4\dot{\Delta}_{SN}^{FF}(k)k'_N(k-p)'_\nu \hat{S}_{\mu S}^{BF}(k)\Big|_{p^2} \tag{6.78}$$

Specifying the components, taking the order p^2 and with some algebra it is possible to show it gets the following form:

$$\frac{16N}{D(D+2)}\Lambda^{D-4}\Omega_D(p^2\delta_{\mu\nu} + 2p_\mu p_\nu)\int_0^\infty dx \; g'(x)\left(x^{D/2+1}f'(x)\right)' \tag{6.79}$$

where f and g are the two functions defined above. Integrating by parts and making use of the properties of c and $\tilde{c}$ at the boundaries, we get to the following final result

for this term (in $D = 4$):

$$\frac{N}{6}\mathcal{Q}_4(p^2\delta_{\mu\nu} + 2p_\mu p_\nu) \tag{6.80}$$

As one can notice, this term is not transverse, as it was announced, but we will see that it is the right combination which added to the contribution (6.63), makes the total result transverse.

We have now to consider the total derivative term in this section, term (1) in eq. (6.44). Expanding in components to have a clearer view, its expression in this sector is:

$$\begin{aligned}
&2\Delta^{BB}(k)S^{AABB}_{\mu\nu\alpha\alpha}(p,-p,k,-k) + 2\Delta^{DD}(k)S^{AADD}_{\mu\nu}(p,-p,k,-k)\\
&+2\Delta^{BB}(k)\Delta^{DD}(p-k)S^{ABD\sigma}_{\nu\rho}(-p,k,p-k)S^{BAD\sigma}_{\rho\mu}(-k,p,k-p)\\
&-\Delta^{BB}(k)\Delta^{BB}(p-k)S^{ABB}_{\nu\rho\sigma}(-p,k,p-k)S^{ABB}_{\mu\sigma\rho}(p,k-p,-k)\\
&-\Delta^{DD}(k)\Delta^{DD}(p-k)S^{ADD}_{\nu}(-p,k,p-k)S^{ADD}_{\mu}(p,k-p,-k)
\end{aligned} \tag{6.81}$$

In this case, as in the case of the C sector, we can notice that due to the regular behaviour of $\Delta^{BB}(k)$ and $\Delta^{DD}(k)$ and their derivatives as $k \to 0$ (see eqs.(6.112) and (6.113)), there are no IR poles and therefore no contribution comes from this term either.

Before we add up all the contribution collected so far, we have finally to recognise the terms proportional to $\dot{\Delta}_0^{AA}$ from this sector. The first one is the same in common with all the three sectors, and is term (9) of eq. (6.44). In this sector it gets the following form:

$$-4\dot{\Delta}_0^{AA}\,\Box_{\nu\alpha}(p)\left[\Delta^{FF}_{SR}(k)\partial^k_\mu\partial^k_\alpha S^{FF}_{RS}(k) - \Delta^{FF}_{RU}(k)\Delta^{FF}_{ST}(k)\partial^k_\mu S^{FF}_{RS}(k)\partial^k_\alpha S^{FF}_{TU}(k)\right] \tag{6.82}$$

There are in this case two more contributions to this set of terms. The first one comes from R_3^F (eq. (6.75)) and the second from the third term of eq. (6.76). Their

expression is:

$$-8\dot{\Delta}_0^{AA}\,\Box_{\mu\alpha}(p)k'_\alpha k'_\nu \tag{6.83}$$

These terms will be treated with all the ones proportional to c'_0, from the other sectors in the next section. In particular it will be proved that there is no contribution from them. As we would expect in fact, a contribution from those terms would make the result dependent upon the choice of the cutoff.

We have now all the contribution to the β-function at one loop, from eq. (6.44) and we are ready to sum them up. First of all, adding (6.63), from the total derivative term of the pure gauge sector, and (6.80) from the remainder R_1^F of the $BD\sigma$ sector (which is the only contribution from this sector), we get:

$$\frac{N}{6}\Omega_4(p^2\delta_{\mu\nu}+2p_\mu p_\nu)+\Omega_4 N\left(\frac{19}{6}p^2\delta_{\mu\nu}-\frac{11}{3}p_\mu p_\nu\right)=\frac{10}{3}N\Omega_4\,\Box_{\mu\nu}(p) \tag{6.84}$$

This makes the RHS of eq. (6.44) transverse just as it is on the LHS. Adding finally (6.57), from the remainders of the A sector and substituting everything in eq. (6.44):

$$-4\beta_1\,\Box_{\mu\nu}(p)=\frac{22}{3}N\Omega_4\,\Box_{\mu\nu}(p),\ \ i.e.\ \ \beta_1=-\frac{11}{3}\frac{N}{16\pi^2} \tag{6.85}$$

Which is the β-function at one loop for $SU(N)$ Yang-Mills, evaluated with the use of a gauge invariant flow equation.

6.3 $\dot{\Delta}_0^{AA}$-terms

In all the three sectors it is possible to recognise terms which are proportional to $\dot{\Delta}_0^{AA}$. Since a final answer dependent on the cutoff function at zero momentum would make the result dependent on its choice, it must be possible to collect all of them and recognise a surface term in any dimension D, vanishing at the boundary. That

is indeed the case, as we will prove in this section.

Let us first consider the class of these terms, coming from the pure A sector. They are:

$$\begin{aligned} &4\dot{\Delta}_0^{AA}\,\Box_{\mu\alpha}(p)\left[\Delta^{AA}(k)\partial_\alpha^k\partial_\nu^k S'^{AA}_{\beta\beta}(k)-\left(\Delta^{AA}(k)\right)^2\partial_\alpha^k S'^{AA}_{\lambda\rho}(k)\partial_\nu^k S'^{AA}_{\lambda\rho}(k)\right] \quad (1)\\ &8\dot{\Delta}_0^{AA}\,\Box_{\mu\alpha}(p)\frac{k_\alpha k_\nu}{k^4} \quad (2)\end{aligned} \tag{6.86}$$

Let us now write the corresponding ones from the C sector:

$$4\dot{\Delta}_0^{AA}\,\Box_{\mu\alpha}(p)\left[\Delta^{CC}(k)\partial_\alpha^k\partial_\nu^k S^{CC}(k)-\left(\Delta^{CC}(k)\right)^2\partial_\alpha^k S^{CC}(k)\partial_\nu^k S^{CC}(k)\right] \tag{6.87}$$

Finally the ones from the F sector are:

$$\begin{aligned} &-4\dot{\Delta}_0^{AA}\,\Box_{\mu\alpha}(p)\left[\Delta^{FF}_{RS}(k)\partial_\alpha^k\partial_\nu^k S^{FF}_{RS}(k)-\Delta^{FF}_{TU}(k)\Delta^{FF}_{RS}(k)\partial_\alpha^k S^{FF}_{RU}(k)\partial_\nu^k S^{FF}_{TS}(k)\right] \quad (1)\\ &-8\dot{\Delta}_0^{AA}\,\Box_{\mu\alpha}(p)k'_\alpha k'_\nu \quad (2)\end{aligned} \tag{6.88}$$

Let us consider the first term of each group. Modulo a factor $4c'_0\,\Box_{\mu\alpha}(p)$, we can indicate them with (term (9) of eq. (6.44) after some relabelling):

$$\Delta^{ff}_{RS}(k)\partial_\alpha^k\partial_\nu^k S^{ff}_{RS}(k)-\Delta^{ff}_{TU}(k)\Delta^{ff}_{RS}(k)\partial_\alpha^k S^{ff}_{RU}(k)\partial_\nu^k S^{ff}_{TS}(k) \tag{6.89}$$

where f can represent either A, C or F. If we now integrate the first term by parts, the term in brackets becomes;

$$\begin{aligned} &\partial_\nu^k\left[\Delta^{ff}_{RS}(k)\partial_\alpha^k S^{ff}_{RS}(k)\right]-\left(\partial_\nu^k\Delta^{ff}_{RS}(k)\partial_\alpha^k S^{ff}_{RS}(k)\right)\\ &-\Delta^{ff}_{TU}(k)\Delta^{ff}_{RS}(k)\partial_\alpha^k S^{ff}_{RU}(k)\partial_\nu^k S^{ff}_{TS}(k)\end{aligned} \tag{6.90}$$

Remembering the relation between the two point functions and their integrated zero point wines, eq.(6.25), the last two terms of the three of eq.(6.90) can be rearranged

as follows:

$$\Delta^{ff}_{TU}(k)\left[\partial^k_\nu B^{ff}_{US}(-k)\right]\partial^k_\alpha S^{ff}_{TS}(k) - \partial^k_\nu \Delta^{ff}_{RS}(k) B^{ff}_{RT}(k)\partial^k_\alpha S^{ff}_{TS}(k) \tag{6.91}$$

We now remember that:

$$B^{ff}_{RT}(k) S^{ff}_{TS}(k) = 0 \tag{6.92}$$

We use this to transfer ∂_α in the second term to B_{RT}, and integrate by parts the ∂_ν onto S_{TS}, similarly to above. Redefining the indices, changing $k \to -k$ and using the $\alpha \leftrightarrow \nu$ symmetry (which automatically has, since it is just a function of k), we can rewrite eq.(6.89) as:

$$\partial^k_\nu \left[\Delta^{ff}_{RS}(k)\partial^k_\alpha S^{ff}_{RS}(k)\right] - (\partial^k_\nu B^{ff}_{RT}(k))(\partial^k_\alpha B^{ff}_{TR}(k)) \tag{6.93}$$

Before continuing, we can simplify further the first term of the equation above. We can in fact notice via a dimensional analysis that the expression in square brackets can be written as:

$$\frac{2k_\alpha}{\Lambda^2} F^f(k^2/\Lambda^2) \tag{6.94}$$

The full term with all the coefficients as it appears in the equation for β_1 has to be integrated in $d^D k$ and it has then the following form:

$$\frac{4c'_0}{\Lambda^4} N \, \Box_{\mu\alpha}(p) \int \frac{d^D k}{(2\pi)^D} \partial^k_\nu [2k_\alpha F^f(k^2/\Lambda^2)], \tag{6.95}$$

(recalling that $\dot{\Delta}^{AA}_0 = c'_0/\Lambda^2$). Performing the derivative, taking the average on the k components and defining $x = k^2/\Lambda^2$ we finally find the expression:

$$\frac{16c'_0}{D} N \, \Box_{\mu\nu}(p) \Omega_D \Lambda^{D-4} \int_0^\infty dx \left[x^{D/2} F^f(x)\right]' \tag{6.96}$$

which allows us to extract from the first term of eq.(6.93) the surface term. We can now specify for each different sector these expressions in order to evaluate the

contribution of this set of terms to the one loop β-function.

6.3.1 A sector

In this sector, we recall from eq.(D.4) that:

$$B^{AA}_{\mu\nu}(p) = \frac{p_\mu p_\nu}{p^2} \tag{6.97}$$

and eq.(6.93) becomes:

$$\partial^k_\nu \left[\Delta^{AA}(k)\partial^k_\alpha S^{AA}_{\beta\beta}(k)\right] - \left(\partial^k_\nu \frac{k_\rho k_\lambda}{k^2}\right)\left(\partial^k_\alpha \frac{k_\lambda k_\rho}{k^2}\right) \tag{6.98}$$

The second term evaluates to:

$$-\frac{2}{k^2}\left(\delta_{\nu\alpha} - \frac{k_\nu k_\alpha}{k^2}\right) \tag{6.99}$$

Remembering now the factor $\frac{4c'_0}{\Lambda^2}\,\Box_{\mu\alpha}(p)$ in front of it, we can collect it together with (2) in eq.(6.86), to get:

$$-8\frac{c'_0}{\Lambda^2}\,\Box_{\mu\alpha}(p)\partial_\alpha \frac{k_\nu}{k^2} \tag{6.100}$$

This is a total derivative term and it will be considered later. The last term proportional to c'_0 in the A sector, yet to be considered, is another total derivative which we are going to evaluate now. First of all we recall that:

$$\Delta^{AA}(k) = \frac{1}{2\Lambda^2}\frac{c(x)}{x} \tag{6.101}$$

$$S^{AA}_{\beta\beta}(k) = 2(D-1)\Lambda^2 \frac{x}{c(x)} \tag{6.102}$$

where $x = k^2/\Lambda^2$. In this case then we have:

$$F^A(x) = (D-1)\left(\ln\frac{x}{c(x)}\right)' \tag{6.103}$$

In order to find the contribution of this term we must evaluate at the boundary the quantity:

$$x^{D/2}\left(\ln\frac{x}{c(x)}\right)'\Big|_0^\infty \tag{6.104}$$

with $c(x)\to 1$ as $x\to 0$ and $c(x)\to 0$ as $x\to\infty$. This term is regular in the infrared and it does not contribute at $x=0$, but it has a divergent ultraviolet behaviour. As we shall see this infinity will be fixed when we take into account the corresponding term coming from the $BD\sigma$ sector.

6.3.2 C sector

In this sector as we can see from eq.(D.15), $B^{CC}(k)=0$, and the only possible contribution from this sector to the c_0' terms comes from the total derivative. In this case this term takes the form:

$$\partial_\nu^k\left[\Delta^{CC}(k)\partial_\alpha^k(\Delta^{CC}(k))^{-1}\right] \tag{6.105}$$

where (here $x=k^2/\Lambda^2$):

$$\Delta^{CC}(k)=\frac{1}{2\Lambda^4}\,\frac{\tilde{c}(x)}{x+2\lambda\tilde{c}(x)} \tag{6.106}$$

referring to the notation of eq.(6.96), we can recognise:

$$F^C(x)=\left[\ln\left(\frac{x}{\tilde{c}(x)}+2\lambda\right)\right]' \tag{6.107}$$

The term we have to evaluate at the boundaries is now:

$$x^{D/2}\left[\ln\left(\frac{x}{\tilde{c}(x)}+2\lambda\right)\right]'\Big|_0^\infty \tag{6.108}$$

Also in this case, due to the properties of $\tilde{c}(x)$ at 0 and ∞ (similar to those of $c(x)$), there is no contribution at $x=0$ and a divergence at $x\to\infty$. Nevertheless, the

corresponding term of the $BD\sigma$ sector will cure also this problem.

6.3.3 $BD\sigma$ sector

In the present sector, from eqs.(D.3) and (D.6), we can see there are two types of remainder, depending on the order the two point vertex and the integrated zero point wine are placed:

$$\begin{aligned} A^{FF}_{RS}(k) &= (-k)'_R(-k)_S \\ B^{FF}_{RS}(k) &= k'_R k_S \end{aligned} \tag{6.109}$$

Nevertheless, we can always express everything in term of one of the two, since relation (D.8) holds between them. The second term of eq.(6.93) can then be written in terms of $B^{FF}_{RS}(k)$ and considering the explicit expression of the generalised momenta of eqs.(6.14) and (6.15), we get

$$-(\partial^k_\nu B^{FF}_{RT}(k))(\partial^k_\alpha B^{FF}_{TR}(k)) = -2k'_\alpha k'_\nu - 2\partial^k_\alpha k'_\nu \tag{6.110}$$

Adding the right factors in front, the first term on the RHS, cancels exactly (2) in eq.(6.88). The second one on the RHS, once it is considered with all its factors and in the $d^D k$ integral, is of the form of the one in eq.(6.95). It is therefore a total derivative and it has to be considered together with the first term of eq.(6.93). Let us first consider the latter. Once we split in components we can apply the same arguments of the two previous sections and it gets the following form:

$$\partial^k_\nu \left[\Delta^{BB}(k)\partial^k_\alpha S^{BB}_{\beta\beta}(k) + \Delta^{DD}(k)\partial^k_\alpha S^{DD}(k)\right] \tag{6.111}$$

Let us now consider (see Appendix D):

$$\Delta^{BB}(k) = \frac{1}{2\Lambda^2}\frac{\tilde{c}(x)c(x)}{x\tilde{c}(x)+2c} \tag{6.112}$$

$$\Delta^{DD}(k) = \frac{1}{\Lambda^4}\frac{\tilde{c}^2(x)}{x\tilde{c}(x)+2c}, \tag{6.113}$$

($x = k^2/\Lambda^2$ here), and

$$S^{BB}_{\beta\beta}(k) = 2\Lambda^2\left((D-1)\frac{x}{c(x)}+\frac{2D}{\tilde{c}(x)}\right) \tag{6.114}$$

$$S^{DD}(k) = \Lambda^4\frac{x}{\tilde{c}(x)} \tag{6.115}$$

In this last case it is possible to see that we can write:

$$F^B(x) = (D-1)\left[\ln\left(\frac{x}{c(x)}+\frac{2}{\tilde{c}(x)}\right)\right]' - \frac{2c(x)\tilde{c}(x)}{x\tilde{c}(x)+2c}\left(\frac{1}{\tilde{c}(x)}\right)' \tag{6.116}$$

$$F^D(x) = \left[\ln\left(\frac{x}{\tilde{c}(x)}+\frac{2c(x)}{\tilde{c}^2(x)}\right)\right]' - \frac{2\tilde{c}^2(x)}{x\tilde{c}(x)+2c}\left(\frac{c(x)}{\tilde{c}^2(x)}\right)' \tag{6.117}$$

What we have to calculate is now:

$$x^{D/2}F^F(x)\Big|_0^\infty \tag{6.118}$$

It is possible to notice that the second terms of both eqs.(6.116) and (6.117) do not give contribution either at $x = 0$ or when $x \to \infty$. On the contrary, even if the first terms of the two equations do not give a contribution for $x = 0$ they do give one for $x \to \infty$, but it cancels the divergent contribution of respectively the two terms of eqs.(6.104) and (6.116). In order to finish the check of the contributions to the 1-loop β function of the terms proportional to c_0', we just have now to evaluate the second term in the RHS of eq.(6.110), with the right factor in front, and the term in

eq. (6.100). They can be collected together into the form:

$$
\begin{aligned}
-\frac{8c_0'}{\Lambda^2}N\,\Box_{\mu\alpha}(p)\int\frac{d^Dk}{(2\pi)^D}\,\partial_\alpha^k\left(\frac{k_\nu}{k^2}-k_\nu'\right) &= -\frac{8c_0'}{\Lambda^2}N\,\Box_{\mu\alpha}(p)\int\frac{d^Dk}{(2\pi)^D}\,\partial_\alpha^k\left(\frac{2c}{x(x\tilde{c}+2c)}\,k_\nu\right)\\
&= -\frac{16c_0'}{D}N\Omega_D\Lambda^{D-4}\,\Box_{\mu\nu}\left[x^{D/2}\frac{2c}{x(x\tilde{c}+2c)}\right]_0^\infty\\
&= 0 \qquad (6.119)
\end{aligned}
$$

Since neither of these last two terms give any contribution, we can conclude that the final result will be independent from c_0'. This result does not surprise us and we were expecting it from the beginning, since the β function at one loop is a universal quantity, independent from the choice of the cutoff function.

Chapter 7

Conclusions

To summarise the work that has been developed in this thesis, we can say that we started by revisiting an exact RG flow equation for a massless scalar field theory as it was described by Polchinski in [2]. Making use of the ideas about scheme independence introduced in [23], we considered a generalisation of it, and computed the effective mass and wavefunction renormalisation at one loop. Combining these results with the flow equation for the one-loop four-point vertex at zero momenta, we calculated the first coefficient of the β-function, obtaining the expected universal answer expressed in eq. (2.41).

This result, achieved with a Polchinski-type flow equation with a general kernel is a proof of universality for β at one loop for the massless scalar field, beyond the independence of the choice of the cutoff function. The totally generic form of the kernel, in fact, includes not only a general form of the cutoff, but also the introduction in the game of an auxiliary action, the *seed action*, which can contain all sorts of extra vertices compatible with the symmetry. (In the case of Polchinski it is recognised to coincide with the kinetic term only). The presence of the seed action can complicate the calculation as long as the scalar field is concerned, but the freedom on its choice

(with just some mild constraint on its vertices) turns out to be useful for the gauge case.

The independence of universal quantities, such as β at one loop, upon the unphysical vertices of the seed action, introduced by hand, come out as an expected result from the calculation performed for the scalar field case, but the way these extra parameters were eliminated in favour of the physical ones, was crucial to set up a powerful method to deal with these generalised flow equations. It was through the calculation in the scalar case that it becomes clear the need to use the flow equations, in order to eliminate the seed action vertices in favour of the effective ones.

Since the main aim of this work was to set up a flow equation for a $SU(N)$ Yang-Mills gauge theory which preserved the symmetry, in the second part of the thesis, we started by revisiting a scheme in which this theory was regularised in a gauge invariant way considering it as a sector of a bigger graded group, known as $SU(N|N)$, which is broken spontaneously (see [34]). This was the first step towards our goal, since with a gauge invariant regularised action, we could then build a flow equation with the right features in order to preserve this symmetry through the flow. With the wide choice given by scheme independence, it was possible to write a flow equation for $SU(N|N)$, see eq. (4.33), which was supergauge invariant, via the introduction of the covariantisation of the kernels, described in section 3.2.1. The extra choice on the seed action vertices comes out to be another crucial point, since it was through the properties we could set on them, that the calculation was simplified. Moreover, the finiteness at tree level could be also ensured, without the introduction of an extra scale, and without loosing the selfsimilar flow property, not only elegant but crucial for our calculation.

In the last part of the thesis, we could eventually perform a check on the flow equation just built, evaluating the first coefficient of the β-function for the physical $SU(N)$ Yang-Mills sector of $SU(N|N)$.

The method used for the scalar field case was implemented and adapted to the present case, with the extra aid of the graphical interpretation. The power of the diagrams, representing the vertices can be appreciated through the whole calculation, as they could often be used instead of specifying their analytical expressions. The symmetries present in the effective action allowed also to use, for most part of the calculation, a compact notation, in which all the fields present in the theory (the basis used in the broken phase), were collected in a supermultiplet f and all the equations could be written in terms of it. This greatly simplified the task and avoided repetitions.

The expected universal answer for the first coefficient of the β-function was eventually calculated for the $SU(N)$ Yang-Mills sector of $SU(N|N)$ as it is expressed in eq. (6.85). This resulted not only in a check of universality for this quantity beyond the change of the cutoff, as for the scalar case, but also represented the first time the finite N value was extracted in a gauge invariant way.

We are now trying to adapt this method, to perform other gauge invariant calculations making use of this flow equation *e.g.* second coefficient of the β-function. Even though we made many progresses and all this machinery seems to be not just exploitable in the relatively simple calculation of β_1, we will not develop this further analysis in this thesis and leave this discussion to future references (see [37]).

Appendix A

Basis for $SU(N|N)$ algebra, and (anti)commutation relations

In order to write more explicitly the commutation and anticommutation relations for the $SU(N|N)$ algebra, we have to choose a basis. We want to choose the one that allows us to write all the relations of the algebra in terms of the structure constants of $SU(N)$. A possible choice is then the following:

$$\begin{aligned}
\tau_\alpha^{(1)} &= \begin{pmatrix} \tau_a & 0 \\ 0 & \tau_a \end{pmatrix} \quad \tau_\alpha^{(2)} = \begin{pmatrix} \tau_a & 0 \\ 0 & -\tau_a \end{pmatrix} \\
s_\alpha^{(1)} &= \begin{pmatrix} 0 & \tau_a \\ \tau_a & 0 \end{pmatrix} \quad s_\alpha^{(2)} = \begin{pmatrix} 0 & -i\tau_a \\ i\tau_a & 0 \end{pmatrix} \\
s_0^{(1)} &= \begin{pmatrix} 0 & \mathbb{1}_N \\ \mathbb{1}_N & 0 \end{pmatrix} \quad s_0^{(2)} = \begin{pmatrix} 0 & -i\mathbb{1}_N \\ i\mathbb{1}_N & 0 \end{pmatrix}
\end{aligned} \tag{A.1}$$

The index α runs here from 1 to N^2-1, and they are $4N^2-2$ $(2N \times 2N)$ traceless and supertraceless matrices T_A, which, together with $\mathbb{1}_{2N}$ form a possible basis $\{S_\alpha\}$ of generators for $SU(N|N)$. The two in the first line are the direct product of the

τ_a's, generators of $SU(N)$, with $\mathbb{1}_2$ and the Pauli matrix σ_3 respectively, and the last two lines are the τ_a's and $\mathbb{1}_N$ in direct product with the Pauli matrices σ_1 and σ_2 respectively. Relating this basis with the matrices introduced in section 3.3.1, we have: $B_a = \{\tau_\alpha^{(1)}, \tau_\alpha^{(2)}\}$ and $F_a = \{s_\alpha^{(1)}, s_\alpha^{(2)}, s_0^{(1)}, s_0^{(2)}\}$.

In terms of the basis defined in eq.(A.1), we can rewrite the first relation of eq.(3.18) as:

$$[\tau_\alpha^{(i)}, \tau_\beta^{(i)}] = i\, f_{\alpha\beta}{}^{\gamma}\, \tau_\gamma^{(1)} \tag{A.2}$$

$$[\tau_\alpha^{(1)}, \tau_\beta^{(2)}] = i\, f_{\alpha\beta}{}^{\gamma}\, \tau_\gamma^{(2)} \tag{A.3}$$

The second can be rewritten:

$$[\tau_\alpha^{(1)}, s_\beta^{(i)}] = i\, f_{\alpha\beta}{}^{\gamma}\, s_\gamma^{(i)} \tag{A.4}$$

$$[\tau_\alpha^{(2)}, s_\beta^{(1,2)}] = \pm i\, d_{\alpha\beta}{}^{\gamma}\, s_\gamma^{(2,1)} \pm i\, \delta_{\alpha\beta}\, s_0^{(2,1)} \tag{A.5}$$

$$[\tau_\alpha^{(i)}, s_0^{(j)}] = 0 \tag{A.6}$$

Finally the third:

$$\{s_\alpha^{(1)}, s_\beta^{(2)}\} = f_{\alpha\beta}{}^{\gamma}\, \tau_\gamma^{(2)} \tag{A.7}$$

$$\{s_\alpha^{(i)}, s_\beta^{(i)}\} = d_{\alpha\beta}{}^{\gamma}\, \tau_\gamma^{(1)} + \delta_{\alpha\beta}\, \mathbb{1}_{2N} \tag{A.8}$$

$$\{s_\alpha^{(i)}, s_0^{(j)}\} = 2\, \delta_{ij}\, \tau_\alpha^{(1)} \tag{A.9}$$

$$\{s_0^{(i)}, s_0^{(j)}\} = 2\delta_{ij}\, \mathbb{1}_{2N} \tag{A.10}$$

In all the equations above, f's and d's are indeed the antisymmetric and symmetric structure constants of $SU(N)$.

Appendix B

Completeness relation for $SU(N|N)$

In order to derive the completeness relation for the T_A generators of $SU(N|N)$, let us first write a generic non constrained supermatrix, expanded on the generators of $U(N|N)$, in this form:

$$X = X_A T^A + X_0 \mathbb{1} + X_3 \sigma_3 \tag{B.1}$$

Recalling now that:

$$X_A = 2 \ \mathrm{str} T_A X \tag{B.2}$$

and:

$$X_0 = \frac{1}{2N} \mathrm{str} \sigma_3 X \tag{B.3}$$

$$X_3 = \frac{1}{2N} \mathrm{str} \mathbb{1} X \tag{B.4}$$

Then we can rewrite eq. (B.1) as:

$$X = 2 \ T^A \mathrm{str} T_A X + \frac{1}{2N} \left(\mathbb{1} \ \mathrm{str} \sigma_3 X + \sigma_3 \ \mathrm{str} \mathbb{1} X \right) \tag{B.5}$$

Writing both sides in components:

$$\delta^l{}_k\delta^i{}_m X^k{}_i = 2(T_A)^l{}_m(\sigma_3)^i{}_j(T^A)^j{}_k X^k{}_i + \frac{1}{2N}\delta^l{}_m(\sigma_3)^i{}_j(\sigma_3)^j{}_k X^k{}_i + \frac{1}{2N}(\sigma_3)^l{}_m(\sigma_3)^i{}_j\delta^j{}_k X^k{}_i \tag{B.6}$$

Considering the previous equation must be valid for any $X^k{}_i$, and rearranging the $SU(N|N)$ generators (without $\mathbb{1}$), on the LHS, we get:

$$(T_A)^l{}_m(\sigma_3)^i{}_j(T^A)^j{}_k = \frac{1}{2}\delta^l{}_k\delta^i{}_m - \frac{1}{4N}\left[\delta^l{}_m(\sigma_3)^i{}_j(\sigma_3)^j{}_k + (\sigma_3)^l{}_m(\sigma_3)^i{}_j\delta^j{}_k\right] \tag{B.7}$$

If now we multiply both sides by $(\sigma_3)^n{}_i$, considering that:

$$(\sigma_3)^n{}_i(\sigma_3)^i{}_j = \delta^n{}_j \tag{B.8}$$

and the fact that the elements of the generators are all ordinary commuting (bosonic) numbers we get the completeness relation of eq.(3.28).

Appendix C

Four point equations at tree level

The four point tree level vertices whose equations were not explicitly written in section 5.3.2, can be found here:

$$
\begin{aligned}
S^{AABB}_{\mu\nu\rho\sigma}(p,q,r,s) = -\int_\Lambda^\infty \frac{d\Lambda_1}{\Lambda_1}\Big\{ & \hat{S}^{AABB}_{\mu\nu\rho\alpha}(p,q,r,s)\dot{\Delta}^{BB}_{s}\hat{S}'^{BB}_{\alpha\sigma}(s) + \hat{S}'^{AABB}_{\mu\nu\alpha\sigma}(p,q,r,s)\dot{\Delta}^{BB}_{r}\hat{S}'^{BB}_{\alpha\rho}(r) \\
& + \hat{S}'^{AABB}_{\mu\alpha\rho\sigma}(p,q,r,s)\dot{\Delta}^{AA}_{q}\hat{S}'^{AA}_{\alpha\nu}(q) + \hat{S}'^{AABB}_{\alpha\nu\rho\sigma}(p,q,r,s)\dot{\Delta}^{AA}_{p}\hat{S}'^{AA}_{\alpha\mu}(p) \\
& - S'^{AAA}_{\mu\nu\alpha}(p,q,r+s)\dot{\Delta}^{AA}_{p+q}S'^{BBA}_{\rho\sigma\alpha}(r,s,p+q) + \hat{S}'^{AAA}_{\mu\nu\alpha}(p,q,r+s)\dot{\Delta}^{AA}_{p+q}S'^{BBA}_{\rho\sigma\alpha}(r,s,p+q) \\
& + S'^{AAA}_{\mu\nu\alpha}(p,q,r+s)\dot{\Delta}^{AA}_{p+q}\hat{S}'^{BBA}_{\rho\sigma\alpha}(r,s,p+q) - S'^{BBA}_{\alpha\sigma\mu}(q+r,s,p)\dot{\Delta}^{BB}_{p+s}S'^{BBA}_{\rho\alpha\nu}(r,p+s,q) \\
& + \hat{S}'^{BBA}_{\alpha\sigma\mu}(q+r,s,p)\dot{\Delta}^{BB}_{p+s}S'^{BBA}_{\rho\alpha\nu}(r,p+s,q) + S'^{BBA}_{\alpha\sigma\mu}(q+r,s,p)\dot{\Delta}^{BB}_{p+s}\hat{S}'^{BBA}_{\rho\alpha\nu}(r,p+s,q) \\
& - \hat{S}'^{AABD\sigma}_{\mu\nu\rho}(p,q,r,s)\dot{\Delta}^{DD}_{s}\hat{S}'^{BD\sigma}_{\sigma}(s) - \hat{S}'^{BAAD\sigma}_{\sigma\mu\nu}(s,p,q,r)\dot{\Delta}^{DD}_{r}\hat{S}'^{BD\sigma}_{\rho}(r) \\
& + S'^{BAD\sigma}_{\sigma\mu}(s,p,q+r)\dot{\Delta}^{DD}_{p+s}S'^{ABD\sigma}_{\nu\rho}(q,r,p+s) - \hat{S}'^{BAD\sigma}_{\sigma\mu}(s,p,q+r)\dot{\Delta}^{DD}_{p+s}S'^{ABD\sigma}_{\nu\rho}(q,r,p+s) \\
& - S'^{BAD\sigma}_{\sigma\mu}(s,p,q+r)\dot{\Delta}^{DD}_{p+s}\hat{S}'^{ABD\sigma}_{\nu\rho}(q,r,p+s) + \hat{S}'^{BBA}_{\alpha\sigma\mu}(r+q,s,p)\dot{\Delta}^{B,AB}_{\rho}(r;q,p+s)\hat{S}'^{AA}_{\alpha\nu}(q) \\
& + \hat{S}'^{BBA}_{\rho\alpha\nu}(r,s+p,q)\dot{\Delta}^{B,BA}_{\sigma}(s;q+r,p)\hat{S}'^{AA}_{\alpha\mu}(p) + \hat{S}'^{AAA}_{\mu\nu\alpha}(p,q,r+s)\dot{\Delta}^{B,BA}_{\sigma}(s;r,p+q)\hat{S}'^{BB}_{\alpha\rho}(r) \\
& + \hat{S}'^{AAA}_{\mu\nu\alpha}(p,q,r+s)\dot{\Delta}^{B,AB}_{\rho}(r;p+q,s)\hat{S}'^{BB}_{\alpha\sigma}(s) + \hat{S}'^{BBA}_{\rho\alpha\nu}(r,p+s,q)\dot{\Delta}^{A,BB}_{\mu}(p;s,q+r)\hat{S}'^{BB}_{\alpha\sigma}(s)
\end{aligned}
$$

$$+\hat{S}^{BBA}_{\rho\sigma\alpha}(r,s,p+q)\dot{\Delta}^{A,AA}_{\mu}(p;r+s,q)\hat{S}^{AA}_{\alpha\nu}(q)+\hat{S}^{BBA}_{\rho\sigma\alpha}(r,s,p+q)\dot{\Delta}^{A,AA}_{\nu}(q;p,r+s)\hat{S}^{AA}_{\alpha\mu}(p)$$

$$+\hat{S}^{BBA}_{\alpha\sigma\mu}(q+r,s,p)\dot{\Delta}^{A,BB}_{\nu}(q;s+p,r)\hat{S}^{BB}_{\alpha\rho}(r)+\hat{S}^{AA}_{\mu\alpha}(p)\dot{\Delta}^{AB,AB}_{\nu\rho}(q,r;p,s)\hat{S}^{BB}_{\alpha\sigma}(s)$$

$$+\hat{S}^{AA}_{\nu\alpha}(q)\dot{\Delta}^{BB,AA}_{\rho\sigma}(r,s;q,p)\hat{S}^{AA}_{\alpha\mu}(p)+\hat{S}^{BB}_{\rho\alpha}(r)\dot{\Delta}^{BA,BA}_{\sigma\mu}(s,p;r,q)\hat{S}^{AA}_{\alpha\nu}(q)$$

$$+\hat{S}^{BB}_{\rho\alpha}(r)\dot{\Delta}^{AA,BB}_{\mu\nu}(p,q;s,r)\hat{S}^{BB}_{\alpha\sigma}(s)+\hat{S}^{BB}_{\sigma\alpha}(s)\dot{\Delta}^{A,B,BA}_{\mu\rho}(p;r;s,q)\hat{S}^{AA}_{\alpha\nu}(q)$$

$$+\hat{S}^{AA}_{\mu\alpha}(p)\dot{\Delta}^{A,B,AB}_{\nu,\sigma}(q;s;p,r)\hat{S}^{BB}_{\alpha\rho}(r)-\hat{S}^{ABD\sigma}_{\nu\rho}(q,r,p+s)\dot{\Delta}^{A,DD}_{\mu}(p;s,q+r)\hat{S}^{BD\sigma}_{\sigma}(s)$$

$$-\hat{S}^{BAD\sigma}_{\sigma\mu}(s,p,q+r)\dot{\Delta}^{A,DD}_{\nu}(q;p+s,r)\hat{S}^{BD\sigma}_{\rho}(r)-\hat{S}^{BD\sigma}_{\rho}(r)\dot{\Delta}^{AA,DD}_{\mu\nu}(p,q;s,r)\hat{S}^{BD\sigma}_{\sigma}(s)\Big\}+I.C. \tag{C.1}$$

$$S'^{AACC}_{\mu\nu}(p,q,r,s)=-\int_{\Lambda}^{\infty}\frac{d\Lambda_1}{\Lambda_1}\Big\{\hat{S}'^{AACC}_{\alpha\nu}(p,q,r,s)\dot{\Delta}^{AA}_{p}\hat{S}'^{AA}_{\alpha\mu}(p)+\hat{S}'^{AACC}_{\mu\alpha}(p,q,r,s)\dot{\Delta}^{AA}_{q}\hat{S}'^{AA}_{\alpha\nu}(q)$$

$$-S'^{AAA}_{\mu\nu\alpha}(p,q,r+s)\dot{\Delta}^{AA}_{p+q}S'^{ACC}_{\alpha}(p+q,r,s)+\hat{S}'^{AAA}_{\mu\nu\alpha}(p,q,r+s)\dot{\Delta}^{AA}_{p+q}S'^{ACC}_{\alpha}(p+q,r,s)$$

$$+S'^{AAA}_{\mu\nu\alpha}(p,q,r+s)\dot{\Delta}^{AA}_{p+q}\hat{S}'^{ACC}_{\alpha}(p+q,r,s)+\hat{S}'^{AA}_{\nu\alpha}(q)\dot{\Delta}^{A,AA}_{\mu}(p;r+s,q)\hat{S}'^{ACC}_{\alpha}(p+q,r,s)$$

$$+\hat{S}'^{AA}_{\mu\alpha}(p)\dot{\Delta}^{A,AA}_{\nu}(q;p,r+s)\hat{S}'^{ACC}_{\alpha}(p+q,r,s)+\hat{S}'^{AACC}_{\mu\nu}(p,q,r,s)\dot{\Delta}^{CC}_{s}\hat{S}'^{CC}(s)$$

$$+\hat{S}'^{AACC}_{\mu\nu}(p,q,r,s)\dot{\Delta}^{CC}_{r}\hat{S}'^{CC}(r)-S'^{ACC}_{\mu}(p,q+r,s)\dot{\Delta}^{CC}_{p+s}S'^{ACC}_{\nu}(q,r,p+s)$$

$$+\hat{S}'^{ACC}_{\mu}(p,q+r,s)\dot{\Delta}^{CC}_{p+s}S'^{ACC}_{\nu}(q,r,p+s)+S'^{ACC}_{\mu}(p,q+r,s)\dot{\Delta}^{CC}_{p+s}\hat{S}'^{ACC}_{\nu}(q,r,p+s)$$

$$+\hat{S}'^{ACC}_{\nu}(q,r,p+s)\dot{\Delta}^{A,CC}_{\mu}(p;s,q+r)\hat{S}'^{CC}(s)+\hat{S}'^{ACC}_{\mu}(p,q+r,s)\dot{\Delta}^{A,CC}_{\nu}(q;p+s,r)\hat{S}'^{CC}(r)$$

$$+\hat{S}'^{AA}_{\mu\alpha}(p)\dot{\Delta}^{CC,AA}(r,s;q,p)\hat{S}'^{AA}_{\nu\alpha}(q)+\hat{S}'^{CC}(s)\dot{\Delta}^{AA,CC}_{\mu\nu}(p,q;s,r)\hat{S}'^{CC}(r)\Big\}+I.C. \tag{C.2}$$

$$S'^{AADD}_{\mu\nu}(p,q,r,s)=-\int_{\Lambda}^{\infty}\frac{d\Lambda_1}{\Lambda_1}\Big\{\hat{S}'^{AADD}_{\mu\alpha}(p,q,r,s)\dot{\Delta}^{AA}_{q}\hat{S}'^{AA}_{\alpha\nu}(q)+\hat{S}'^{AADD}_{\alpha\nu}(p,q,r,s)\dot{\Delta}^{AA}_{p}\hat{S}'^{AA}_{\alpha\mu}(p)$$

$$+\hat{S}'^{BAAD\sigma}_{\alpha\mu\nu}(s,p,q,r)\dot{\Delta}^{BB}_{s}\hat{S}'^{BD\sigma}_{\alpha}(s)+\hat{S}'^{AABD\sigma}_{\mu\nu\alpha}(p,q,r,s)\dot{\Delta}^{BB}_{r}\hat{S}'^{BD\sigma}_{\alpha}(r)$$

$$-S'^{AAA}_{\mu\nu\alpha}(p,q,r+s)\dot{\Delta}^{AA}_{p+q}S'^{ADD}_{\alpha}(p+q,r,s)+\hat{S}'^{AAA}_{\mu\nu\alpha}(p,q,r+s)\dot{\Delta}^{AA}_{p+q}S'^{ADD}_{\alpha}(p+q,r,s)$$

$$+S'^{AAA}_{\mu\nu\alpha}(p,q,r+s)\dot{\Delta}^{AA}_{p+q}\hat{S}'^{ADD}_{\alpha}(p+q,r,s)+S'^{ABD\sigma}_{\mu\alpha}(p,q+r,s)\dot{\Delta}^{BB}_{q+r}S'^{BAD\sigma}_{\alpha\nu}(p+s,q,r)$$

$$-\hat{S}'^{ABD\sigma}_{\mu\alpha}(p,q+r,s)\dot{\Delta}^{BB}_{q+r}S'^{BAD\sigma}_{\alpha\nu}(p+s,q,r)-S'^{ABD\sigma}_{\mu\alpha}(p,q+r,s)\dot{\Delta}^{BB}_{q+r}\hat{S}'^{BAD\sigma}_{\alpha\nu}(p+s,q,r)$$

$$+\hat{S}^{BD\sigma}_{\alpha}(s)\dot{\Delta}^{A,BB}_{\mu}(p;s,q+r)\hat{S}^{BAD\sigma}_{\alpha\nu}(p+s,q,r)+\hat{S}'^{ADD}_{\alpha}(p+q,r,s)\dot{\Delta}^{A,AA}_{\mu}(p;r+s,q)\hat{S}'^{AA}_{\alpha\nu}(q)$$

$$+\hat{S}'^{AA}_{\mu\alpha}(p)\dot{\Delta}^{A,AA}_{\nu}(q;p,r+s)\hat{S}'^{ADD}_{\alpha}(p+q,r,s)+\hat{S}^{ABD\sigma}_{\mu\alpha}(p,q+r,s)\dot{\Delta}^{A,BB}_{\nu}(q;p+s,r)\hat{S}^{BD\sigma}_{\alpha}(r)$$

$$-\hat{S}^{BD\sigma}_{\alpha}(s)\dot{\Delta}^{AA,BB}_{\mu\nu}(p,q;s,r)\hat{S}^{BD\sigma}_{\alpha}(r)+\hat{S}'^{AADD}_{\mu\nu}(p,q,r,s)\dot{\Delta}^{DD}_{s}\hat{S}'^{DD}(s)$$

$$\hat{S}'^{AADD}_{\mu\nu}(p,q,r,s)\dot{\Delta}^{DD}_{r}\hat{S}'^{DD}(r)-S'^{ADD}_{\mu}(p,q+r,s)\dot{\Delta}^{DD}_{q+r}S'^{ADD}_{\nu}(q,r,p+s)$$

$$+\hat{S}^{ADD}_{\mu}(p,q+r,s)\dot{\Delta}^{DD}_{q+r}S'^{ADD}_{\nu}(q,r,p+s)+S'^{ADD}_{\mu}(p,q+r,s)\dot{\Delta}^{DD}_{q+r}\hat{S}'^{ADD}_{\nu}(q,r,p+s)$$

$$+\hat{S}'^{DD}(s)\dot{\Delta}^{A,DD}_{\mu}(p;s,q+r)\hat{S}'^{ADD}_{\nu}(q,r,p+s)+\hat{S}^{ADD}_{\mu}(p,q+r,s)\dot{\Delta}^{A,DD}_{\nu}(q;p+s,r)\hat{S}'^{DD}(r)$$

$$+\hat{S}'^{AA}_{\nu\alpha}(q)\dot{\Delta}^{\sigma D,AB}(q;p,r+s)\hat{S}^{ABD\sigma}_{\mu\alpha}(p,q+r,s)-\hat{S}'^{AAA}_{\mu\nu\alpha}(p,q,r+s)\dot{\Delta}^{D\sigma,AB}(r;p+q,s)\hat{S}^{BD\sigma}_{\alpha}(s)$$

$$-\hat{S}^{BAD\sigma}_{\alpha\nu}(p+s,q,r)\dot{\Delta}^{D\sigma,BA}(s;q+r,p)\hat{S}'^{AA}_{\alpha\mu}(p)+\hat{S}^{BD\sigma}_{\alpha}(r)\dot{\Delta}^{D\sigma,BA}(s;r,p+q)\hat{S}'^{AAA}_{\mu\nu\alpha}(p,q,r+s)$$

$$+\hat{S}'^{AA}_{\mu\alpha}(p)\dot{\Delta}^{A,D\sigma,AB}_{\nu}(q;s;p,r)\hat{S}^{BD\sigma}_{\alpha}(s)+\hat{S}'^{AA}_{\nu\alpha}(q)\dot{\Delta}^{A,D\sigma,BA}_{\mu}(p;r;s,q)\hat{S}^{BD\sigma}_{\alpha}(s)$$

$$+\hat{S}'^{AA}_{\mu\alpha}(p)\dot{\Delta}^{AD\sigma,AB}_{\nu}(q,r;p,s)\hat{S}^{BD\sigma}_{\alpha}(s)+\hat{S}^{BD\sigma}_{\alpha}(r)\dot{\Delta}^{D\sigma A,BA}_{\mu}(s,p;r,q)\hat{S}'^{AA}_{\alpha\nu}(q)$$

$$+\hat{S}'^{AA}_{\mu\alpha}(p)\dot{\Delta}^{D\sigma D\sigma,AA}(r,s;q,p)\hat{S}'^{AA}_{\nu\alpha}(q)+\hat{S}^{DD}(s)\dot{\Delta}^{AA,DD}_{\mu\nu}(p,q;s,r)\hat{S}^{DD}(r)\Big\}+I.C.$$

(C

The integration constants, not specified in most of the previous equations, are the bare action vertices. As mentioned in section 5.3.2, although the request on the seed action vertices to keep UV finiteness, ensures there are not classical divergencies, we must at least prove this is possible for a particular $\hat{S}$. This check is done in section 5.3.2.

Appendix D

Integrated wines

First of all, once we have defined the zero point wine $\dot{\Delta}^{ff}_{RS}(p)$, we can define its integrated form. We can define

$$\dot{\Delta}^{ff}_{RS}(p) = -\Lambda\partial_\Lambda \Delta^{ff}_{RS}(p) \tag{D.1}$$

$\Delta^{ff}_{RS}(p)$ now being the *integrated wine*. Let us now consider the two point equation for the generalised field multiplet f:

$$\Lambda\partial_\Lambda S^{ff}_{MN}(p) = S^{ff}_{ML}(p)\dot{\Delta}^{ff}_{LS}(p)S^{ff}_{SN}(p) \tag{D.2}$$

It turns out that:

$$S^{ff}_{RS}(k)\Delta^{ff}_{ST}(k) = \delta_{RT} - B^{ff}_{RT}(k) \tag{D.3}$$

This relation will be crucial throughout the entire calculation that will follow. In the case of the A field, in particular we have:

$$B^{AA}_{\alpha\beta}(k) = \frac{k_\alpha k_\beta}{k^2} \tag{D.4}$$

In the C sector $B^{CC}(k) = 0$ and finally in the $BD\sigma$ one we have:

$$B^{FF}_{RT}(k) = k'_R k_T \tag{D.5}$$

where k and k' are the generalised momenta of Eq.(6.14) and (6.15). There is another relation which holds in this sector if we place the two point vertex with the integrated wine the other way round:

$$\Delta^{FF}_{RS}(k) S^{FF}_{ST}(k) = \delta_{RT} - A^{FF}_{RT}(k) \tag{D.6}$$

where:

$$A^{FF}_{RT}(k) = (-k)_R(-k')_T \tag{D.7}$$

but it can be seen that there is a relation between them which will allow us to make use of only one of the two, namely:

$$A^{FF}_{SR}(-k) = B^{FF}_{RS}(k) \tag{D.8}$$

For the C sector the derivation of these relations is much easier and it is similar to the scalar field case in the ϕ^4 theory considered in section 2.3. Since the two point vertex is here invertible, and there is no remainder ($B^{CC}(k) = 0$), it is easy to present the full derivation of the previous equations for the present case.

In the C-sector, we remind from eq. (5.45) that the zero-point wine has the following form:

$$\dot{\Delta}^{CC}(p) = \frac{1}{\Lambda^4 x}\left(\frac{2x^2\tilde{c}_p}{x + 2\lambda\tilde{c}_p}\right)' \tag{D.9}$$

where $x = p^2/\Lambda^2$ in this case. Now, recalling that in this notation:

$$\partial_x = -\frac{1}{2x}\Lambda\partial_\Lambda, \tag{D.10}$$

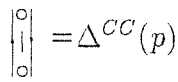

Figure D.1: Graphical representation of the integrated 0-point wine for the C sector

we can write the following identity:

$$\dot{\Delta}^{CC}(p) = -\Lambda\partial_\Lambda \Delta^{CC}(p) \tag{D.11}$$

where:

$$\Delta^{CC}(p) = \frac{1}{\Lambda^4}\frac{\tilde{c}_p}{x + 2\lambda\tilde{c}_p} \tag{D.12}$$

This is the integrated 0-point wine for the C sector, which is represented in fig.D.1. Recalling now the equation for S^{CC}:

$$\Lambda\partial_\Lambda S^{CC}(p) = S^{CC}(p)\dot{\Delta}^{CC}(p)S^{CC}(p) \tag{D.13}$$

and since $S^{CC}(p)$ is invertible, we can rewrite it as:

$$-\Lambda\partial_\Lambda (S^{CC}(p))^{-1} = \dot{\Delta}^{CC}(p) = -\Lambda\partial_\Lambda \Delta^{CC}(p) \tag{D.14}$$

Since at $\Lambda \to \infty$ we have $(S^{CC}(p))^{-1} - \Delta^{CC}(p) \to 0$ (choosing the integration constant here and later, so that the 'effective propagator' vanishes as $p \to \infty$), we see we must have:

$$(S^{CC}(p))^{-1} = \Delta^{CC}(p) \tag{D.15}$$

which we can indeed see explicitly from (5.14) and (D.12).

We represent the integrated wine as in fig. 5.6, but with a line down its spine, and thus eq. (D.15) is represented diagrammatically as in fig.D.2. Similarly to eq. (D.12),

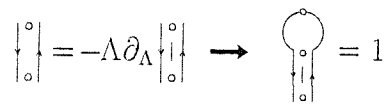

Figure D.2: The integrated wine in the case of the bosonic component of the scalar field is the inverse of the two-point vertex

one can find all the other integrated kernels. They are listed below:

$$\Delta^{AA}(p) = \frac{1}{2p^2} c_p \tag{D.16}$$

$$\Delta^{BB}(p) = \frac{1}{2\Lambda^2} \frac{\tilde{c}c}{x\tilde{c} + 2c} \tag{D.17}$$

$$\Delta^{DD}(p) = \frac{1}{\Lambda^4} \frac{\tilde{c}^2}{x\tilde{c} + 2c} \tag{D.18}$$

where c and $\tilde{c}$ are intended as functions of $x = p^2/\Lambda^2$. Let us spend few words about the previous equations. The two A's integrated kernel, in eq. (D.16), despite its similarity to a regularised Feynman propagator, has no gauge fixing. Indeed this 'effective propagator' is the inverse of the classical AA kinetic term only in the transverse space, as one can see from (D.3,D.4). Since in practice Δ^{AA} will be connected to an A point on some other vertex, the remainder term above will simply generate gauge transformations via eq. (5.29). This observation proves crucial in the 'magic' of the calculation.

The B and D integrated kernels are described in (D.17,D.18). Note that despite the classical D kinetic term being that of a massless (Goldstone) field, the D effective propagator like that of C and B (but unlike A) has no massless pole. Of course this is nothing but the Higgs mechanism, arising here from the B and D two-point vertices being intimately related via (5.40,5.42) (the $BD\sigma$ vertex being non zero). Similarly to the above reasoning, the pair of effective propagators (D.17,D.18), would form the inverse of the *matrix* of these fermionic two-point vertices (see eq. (6.1)), if this matrix was invertible. It is not, for the same reason that these flows are

necessarily entangled: B and $D\sigma$ rotate into each other under the broken supergauge transformations eq. (5.30).

Appendix E

Special momenta

In the present appendix is contained a derivation of Eqs.(6.29) and (6.30), plus some comments about the behaviour of the vertices at some particular momenta. The symmetry which governs the theory is gauge symmetry and also in this case, will give us a hint on how to derive the expressions mentioned above. Let us consider first the three point vertex $S^{Aff}_{\mu RS}$ at momenta $(0, k, -k)$. We can imagine to consider it evaluated at momenta $(\epsilon, k-\epsilon, -k)$ and applying the Ward identity with momentum ϵ_μ, we get:

$$\epsilon_\mu S^{Aff}_{\mu RS}(\epsilon, k-\epsilon, -k) = S^{ff}_{RS}(k) - S^{ff}_{RS}(k-\epsilon) \tag{E.1}$$

Expanding in ϵ both sides of the equation and taking order linear in ϵ, we get precisely the first equality of eq.(6.29). For the second one, we can repeat the procedure the same way.

If we now want to have an espression for the four point vertex $S^{AAff}_{\mu\nu RS}$ evaluated at momenta $(0, 0, k, -k)$, we can as well consider it instead at momenta $(\epsilon, -\epsilon, k, -k)$ and applying twice the Ward identity, with ϵ_μ first and then ϵ_ν, we get:

$$\epsilon_\mu \epsilon_\nu S^{AAff}_{\mu\nu RS}(\epsilon, -\epsilon, k, -k) = \epsilon_\mu \partial^k_\mu S^{ff}_{RS}(k) - S^{ff}_{RS}(k) + S^{ff}_{RS}(k-\epsilon) \tag{E.2}$$

In this case, if we expand the equation in ϵ and take the order ϵ^2, what we get is eq.(6.30).

Other vertices at particular momenta can be evaluated in similar ways.

Appendix F

Bare action vertices (finite part)

This is a list of finite part of the 4-point and 3-point vertices of the bare action, when Λ_0 is sent to infinity that must be added to the equations in section 5.3.2. They represent the finite part of the integration constants, necessary in order to have gauge invariant tree level vertices.

Four-point:

$$\hat{S}^{AAAA}_{\mu\nu\rho\sigma}(p,q,r,s)\Big|_{\Lambda_0\to\infty} = 2\delta_{\mu\nu}\delta_{\rho\sigma} - 4\delta_{\mu\rho}\delta_{\nu\sigma} + 2\delta_{\mu\sigma}\delta_{\nu\rho} \tag{F.1}$$

$$\hat{S}^{AABB}_{\mu\nu\rho\sigma}(p,q,r,s)\Big|_{\Lambda_0\to\infty} = 2\delta_{\mu\nu}\delta_{\rho\sigma} - 4\delta_{\mu\rho}\delta_{\nu\sigma} + 2\delta_{\mu\sigma}\delta_{\nu\rho} - 4\tilde{c}'_0\delta_{\mu\nu}\delta_{\rho\sigma} \tag{F.2}$$

$$\begin{aligned}\hat{S}^{AADD}_{\mu\nu}(p,q,r,s)\Big|_{\Lambda_0\to\infty} = {} & \tilde{c}'_0\left(p_\nu r_\mu + q_\mu s_\nu - r_\mu r_\nu + 2r_\mu s_\nu - s_\mu s_\nu\right.\\ & \left.-\delta_{\mu\nu}p^2 - 2\delta_{\mu\nu}p\cdot s + \delta_{\mu\nu}r\cdot s - \delta_{\mu\nu}s^2\right)\end{aligned} \tag{F.3}$$

$$\begin{aligned}\hat{S}^{AACC}_{\mu\nu}(p,q,r,s)\Big|_{\Lambda_0\to\infty} = {} & \tilde{c}'_0\left(p_\nu r_\mu + q_\mu s_\nu - r_\mu r_\nu + 2r_\mu s_\nu - s_\mu s_\nu\right.\\ & \left.-\delta_{\mu\nu}p^2 - 2\delta_{\mu\nu}p\cdot s + \delta_{\mu\nu}r\cdot s - \delta_{\mu\nu}s^2\right)\end{aligned} \tag{F.4}$$

$$\hat{S}^{AABDS}_{\mu\nu\rho}(p,q,r,s)\Big|_{\Lambda_0\to\infty} = -2\tilde{c}'_0\left(\delta_{\mu\nu}s_\rho + \delta_{\mu\rho}p_\nu - \delta_{\mu\rho}r_\nu + \delta_{\mu\rho}s_\nu\right) \tag{F.5}$$

$$\hat{S}^{BAADS}_{\mu\nu\rho}(p,q,r,s)\Big|_{\Lambda_0\to\infty} = 2\tilde{c}_0'\left(\delta_{\mu\rho}p_\nu - \delta_{\mu\rho}r_\nu - \delta_{\mu\rho}s_\nu - \delta_{\nu\rho}s_\mu\right) \quad \text{(F.6)}$$

Three-point:

$$\hat{S}^{BADS}_{\mu\nu}(p,q,r)\Big|_{\Lambda_0\to\infty} = 2\tilde{c}_0'\left(p_\nu r_\mu - r_\mu r_\nu - \delta_{\mu\nu}p^2\right) \quad \text{(F.7)}$$

$$\hat{S}^{ABDS}_{\mu\nu}(p,q,r)\Big|_{\Lambda_0\to\infty} = -2\tilde{c}_0'\left(q_\mu r_\nu - r_\mu r_\nu - \delta_{\mu\nu}q^2\right) \quad \text{(F.8)}$$

$$\begin{aligned}\hat{S}^{AAA}_{\mu\nu\rho}(p.q.r)\Big|_{\Lambda_0\to\infty} &= 2\delta_{\mu\nu}p_\rho - 2\delta_{\mu\nu}q_\rho - 2\delta_{\mu\rho}p_\nu + 2\delta_{\mu\rho}r_\nu + 2\delta_{\nu\rho}q_\mu \\ &\quad -2\delta_{\nu\rho}r_\mu \quad \text{(F.9)}\end{aligned}$$

$$\begin{aligned}\hat{S}^{BBA}_{\mu\nu\rho}(p,q,r)\Big|_{\Lambda_0\to\infty} &= 2\delta_{\mu\nu}p_\rho - 2\delta_{\mu\nu}q_\rho - 2\delta_{\mu\rho}p_\nu + 2\delta_{\mu\rho}r_\nu + 2\delta_{\nu\rho}q_\mu \\ &\quad -2\delta_{\nu\rho}r_\mu - 4\tilde{c}_0'(\delta_{\mu\nu}p_\rho - \delta_{\mu\nu}q_\rho) \quad \text{(F.10)}\end{aligned}$$

$$\hat{S}^{ADD}_{\mu}(p,q,r)\Big|_{\Lambda_0\to\infty} = -\tilde{c}_0'(q_\mu q^2 - q_\mu q\cdot r + r_\mu q\cdot r - r_\mu r^2) \quad \text{(F.11)}$$

$$\hat{S}^{ACC}_{\mu}(p,q,r)\Big|_{\Lambda_0\to\infty} = -\tilde{c}_0'(q_\mu q^2 - q_\mu q\cdot r + r_\mu q\cdot r - r_\mu r^2) \quad \text{(F.12)}$$

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