UNIVERSITY OF SOUTHAMPTON

The process of behavioural, representational and conceptual change in young children's strategies when solving arithmetic tasks

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Thesis submitted for the qualification of
Doctor of Philosophy

Faculty of Social Sciences
Research and Graduate School of Education

December 2002

To my mother, and to the loving memory of my father.

UNIVERSITY OF SOUTHAMPTON

ABSTRACT

FACULTY OF SOCIAL SCIENCES

RESEARCH AND GRADUATE SCHOOL OF EDUCATION

Doctor of Philosophy

THE PROCESS OF BEHAVIOURAL, REPRESENTATIONAL AND CONCEPTUAL CHANGE IN YOUNG CHILDREN'S STRATEGIES WHEN SOLVING ARITHMETIC TASKS

by Chronoula Voutsina

This study is situated in the context of projects which, in the field of arithmetic, explore the process of change in young children's thinking and strategies within problem situations. In particular, the study aims at exploring the pathway of changes that occur in 5-6 year old children's problem solving strategies when they are engaged in solving a specific form of additive task. It is hypothesised that higher conceptualisation and control of the employed strategy and of the factors involved in the task, develops *after* the achievement of an efficient solution. Previous research has shown that group work involving discussion and reflection upon the solution process are effective practices towards this direction. For this study, Karmiloff-Smith's model of *Representational Redescription* (RR) provides the theoretical and methodological framework, and is used as a basis for the analysis of the changes observed in the behavioural level and of those inferred at the representational and conceptual level.

The study focuses on a number of cases. Changes in children's strategies and their progressive movement from procedural success to higher conceptualisation, and control of the employed strategies, are studied in a micro-developmental level. This takes place in the course of a sequence of sessions during which children are individually involved in solving a specific form of additive tasks. The micro-developmental method of data collection and analysis is combined with the clinical method of interviewing.

The study shows that children move beyond success, and introduce qualitative changes and modifications to their successful strategies. These changes indicate the movement from success-oriented behaviour to an organisation-oriented phase in problem solving during which children, as problem solvers, acquire better control and increasing conscious access to knowledge which is present in their cognitive system; i.e. knowledge that they already have. The RR model is proved to be a valuable tool for the exploration and analysis of the post-success behaviours which were identified in the particular arithmetic, problem solving situation. However, the study reveals points of diversion between the data and the predictions of the RR model, and indicates aspects of the model that need to be possibly modified so that the model becomes more flexible and applicable in the particular domain.

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Preamble

The aim of this study is to explore the ways by which 5-6 year-old children organise different pieces of knowledge to develop strategies for solving a specific arithmetical task and furthermore the way by which children move beyond their efficient and successful strategies to the development of understanding.

Karmiloff-Smith (1992) claims that "...certain types of change take place after the child is successful (i.e. already producing the correct linguistic output, or already having consistently reached a problem solving goal)" (p. 25). The idea of 'success based' cognitive change is based on the theoretical model of *Representational Redescription* (RR) which describes the movement from implicit information embedded in an efficient problem solving procedure, to rendering the knowledge progressively more flexible and explicit. Within the context of problem solving, the notion of knowledge redescription and explicitation has been studied in spatial, physics, linguistic and notational, but not in mathematical tasks (Karmiloff-Smith, 1984).

For the purposes of this study, the model of *Representational Redescription* is used both as theoretical as well as methodological framework, in an attempt to study the ways by which further reflection and work on a successful solution can induce (or not) evolutionary changes in the solution process of an arithmetical problem, and also in the level of student's understanding and control regarding the employed solution technique. Particularly, the purpose is to explore how and when qualitative changes such as the ones that the RR model postulates, occur in the problem solving techniques that 5-6 year olds employ in order to solve a specific form of additive task.

The research questions that this study seeks to answer are presented in chapter 1. Here, two of the research questions are outlined so that the reader has a first sense of what this project is about:

* Do children evolve and develop their successful problem solving approach while engaged in a problem solving situation repeatedly? If yes, what is the process, and types of change that occur in children's problem solving approach in the 'post-success elaboration' phase?

* Are the behaviours observed, and the introduced behavioural modifications, consistent with the behaviours that the RR model describes when accounting for the post-success development of children's problem solving approaches?

This thesis is organised as follows:

In Chapter 1 an overview of the research problem is presented, accompanied by the research questions that this study seeks to answer.

Chapter 2 presents a literature review of the main theoretical positions and research work across the area of mathematics education and psychology regarding the key-notions and issues that this study addresses.

Chapter 3 discusses the model which provides the theoretical as well as methodological basis for the study.

Chapter 4 is devoted to methodological issues, and presents the design of the study.

In Chapter 5, a small-scale pilot study which was carried out and aimed at testing the feasibility of the project, and the appropriateness of the methodological approach for the purposes of the study, is presented and discussed.

Chapter 6 opens the presentation of the main study. The chapter discusses issues which drove the analysis of the data and presents the form of presentation of the data and of their analysis.

Chapters 7 and 8 are devoted to the presentation and discussion of one group of cases correspondingly. In the framework of each of the two chapters the individual profile of each case is presented and discussed.

Chapter 9 presents a general discussion of the cases. This discussion, following the data analysis, raises specific issues which are found to be in agreement or in conflict, or be uncovered by the explanatory framework that the RR model provides.

Chapter 10, presents the concluding remarks of the thesis.

The last part of the thesis is devoted to appendices. The appendices present specific segments of each child's engagement with the task/s, and of transcribed dialogue between the interviewer and each child, including examples of the children's problem solving behaviour that is discussed in chapters 7 and 8.

Acknowledgements

I would like to thank Dr. Keith Jones for his guidance and support during my PhD years. I am also grateful to Dr. Ann Hirst for reading my thesis and making valuable suggestions.

I thank Professor Gary Davis for advising me at the early stages of this project, and Professor Konstantinos Ravanis for the discussions we had at the beginning of this research work. I am grateful to Claire Mooney for making all the necessary arrangements so that I could carry out my fieldwork.

I wish to acknowledge the Economic and Social Research Council for funding, partially, this project these past three years.

This project would have not been possible without the children's participation. I thank them for sharing with me their thoughts and explorations. I am most grateful to the Head teachers and classroom teachers of the two Primary schools in which the field work took place, for the interest they showed to my work, and their collaboration. My thanks also go to the members of staff in the two schools where the field work took place. They all made me feel welcomed and offered their help whenever I needed it.

Many thanks to my friends and colleagues Corina Hathaway and Julie-Ann Edwards for all the discussions we had and for sharing a warm and inspiring working environment with me.

This work would have never been initiated or completed without the love and support that my mother offered me in every step of the way. I cannot express in words my gratitude to her. Also, I am grateful to my brother for providing me with all the facilities I needed to complete the writing of my thesis at the final stage.

Finally, I would like to thank my husband for his patience and love. He gave me encouragement during stressful times. I cannot thank him enough for all the help he gave me, in so many different ways, to complete this work.

Chapter 1 Introduction

1.1. Overview

In this chapter the grounds for the theoretical and educational importance of the research aim that this study focuses on are given. The chapter is organised in three sections. In the first section the research problem is presented. The presentation of theoretical positions, drawn from the area of mathematics education, aims at providing a justifiable basis for the significance of the particular issue under study. A summary of the hypotheses and assumptions that the study sets out to examine, is presented in the second section. The chapter concludes with the presentation of the particular research questions that the study seeks to address.

1.2. The research problem

Arithmetic forms the major part of mathematics education for young children. This study was prompted by a simple observation-reflection while working in a reception class (pupils aged 5) during the primary mathematics lesson. Students who produce the correct answer or solution to a problem seem to be rarely (or never) asked to justify their solution, and explain how they produced the specific result. Students who solve a problem correctly seem to be rarely (or never) prompted to work further, and elaborate their problem solving strategies. Although this observation, and subsequent reflection occurred while working in a particular classroom, it can be observed in many other classrooms. Teachers often do no more than tick correct answers while they ask for justifications, explanations and further work when errors occur. This is quite understandable as it is not at all clear that further learning can be initiated from, and be advanced beyond a correct answer.

The argument that this study aims to develop is that, in the context of an arithmetical problem situation, once a strategy is developed and successfully applied, there is still space for further elaboration and control. It is hypothesised that deeper understanding and conceptualisation of a problem situation, and of the factors involved, develops in the course of children's engagement in a problem situation before *and after* the achievement of an efficient solution.

This study aims at documenting and analysing children's movement from procedural success to higher *conceptualisation* and *understanding* of the procedures employed in the solution of an arithmetical task. The objective is to study this movement as it takes place in the context of solving a specific form of arithmetical task that children engage with, during a sequence of experimental sessions. The assumption being made is that, in the course of this iterative process, children build a deeper understanding of the situation, and of their solution strategies, gradually, through a sequence of *changes* that occur both at the procedural as well as the conceptual level.

The importance of studying these changes has two aspects. First, the practical aspect of teaching. If qualitative changes in the solution procedure do actually take place, even after a successful solution has been achieved, then it could be argued that a correct solution should not be considered and treated by the teaching practice as the end-point of thinking and reasoning, but also as the starting point of such a procedure. Second, the study of changes in children's strategies, that is changes which denote a transition from procedural success to conceptual understanding, has a great theoretical importance. The different forms of knowledge that underlie mathematical learning constitute one of the core issues, which have been widely discussed both by mathematics educators and cognitive psychologists. The distinction between *procedural* and *conceptual* knowledge and their interrelationship, that is the relationship between one's ability to perform a task and to understand the task as well as the reasons of the action's effectiveness, constitutes the object of a long debate in mathematics education. This thesis aims at contributing to the understanding of the interplay between the different forms of arithmetical knowledge in the context of a problem solving situation. Certain theoretical positions that ascribe notable importance to the issue are presented herein.

1.2.1. Conceptual and procedural knowledge in mathematics

Hiebert and Lefevre (1986), define conceptual knowledge as a network in which linking relationships are established between different pieces of information. It is particularly stressed that an isolated piece of information cannot be considered as a unit of conceptual knowledge unless it is related to other pieces of information. Conceptual knowledge is thus a product of a linking process, which may create relationships between pieces of information which are already stored in memory or between existing knowledge and information that is newly learned. Procedural knowledge, as defined by Hiebert and Lefevre (1986), consists of two parts. One part involves knowledge of the formal mathematical language which:

"...includes familiarity with the symbols used to represent mathematical ideas and an awareness of the syntactic rules for writing symbols in a acceptable form. For example, those who possess this aspect of procedural knowledge would recognise that the expression 3.5 ÷ =2.71 is syntactically acceptable (although they may not know the 'answer') and that 6+= 2 is not acceptable." (p. 6).

The other part of procedural knowledge involves knowledge of algorithms and rules for solving mathematical problems and it is characterised as "step-by-step" execution of procedures in a "predetermined linear sequence" (p. 6). Hiebert and Lefevre also underline that an important distinction must be made between two different kinds of procedures; there are procedures which operate upon written symbols (e.g. 5, +, =), and there are procedures which operate on concrete objects, visual diagrams, mental images or other objects that are "non-symbolic" in the sense that they do not belong to the conventional system of mathematical symbols. Procedures on concrete objects rather than on written mathematical symbols are most often used by pre-school children or by children in the early years of schooling.

Another form of knowledge that is involved mainly in arithmetic but is less commonly referred to as distinct from procedural knowledge, is *factual* knowledge. Factual knowledge consists of memorised information of arithmetic relations among numbers, or numerical associations (e.g. 3+3=6, 5×6=30). Memorised number combinations that are retrieved directly from memory are referred to as *number facts*. Retrieval of number facts is a procedure that involves a rather mechanical, automatic and rapid access to memory. Improvement of factual knowledge together with procedural knowledge are considered as the end product of arithmetical learning and development (Bisanz & Lefevre, 1990; Maclellan, 1995). For Baroody and Ginsburg (1986), both procedural and factual knowledge constitute parts of the *mechanical* knowledge, as opposed to *meaningful* (conceptual) knowledge which is defined as the "semantic knowledge with implicit or explicit knowledge of concepts or principles" (p. 75).

Although, in certain cases, knowledge of number associations (factual) and knowledge of rules and algorithms (procedural) are differentiated, the term *procedural* is more widely employed by researchers when referring generally to mechanical and automatic knowledge that underlies arithmetic routines. The terms *declarative* and *procedural* knowledge have also been adopted by mathematics educators to account for the knowledge of "knowing that" and the knowledge of "knowing how" correspondingly (Silver 1987a; Vergnaud, 1992; English & Halford, 1995).

Regardless of the term used, knowledge that grows from the building of relationships either between already acquired pieces of information or between old and newly learned information is associated with *understanding*, whereas knowledge of automatic and mechanical execution of procedures and operations is associated with *skill*. Success based on procedural skill and understanding were differentiated by Piaget (1978) according to whom understanding an action means to understand why it works; understand the reasons for which a certain action is successful. In his own words, "success means having enough understanding of a situation to attain the requisite ends in action, and understanding is successful mastery in thought of the same situation to the point of being able to solve the problem of the 'how' and the 'why' of the connections observed and applied in action." (p. 218).

It must be noted that 'action' in Piaget's sense has the meaning of physical actions on concrete objects and understanding means *conceptual* understanding which allows the explanation of the reasons underlying success and also anticipation of why certain actions would or would not be successful. Piaget argues that it is such a conceptual basis that enables and guides the development of plans in problem solving. In a series of tasks, Piaget explored the relation between success and understanding and the movement from 'practical success' which allows effective utilisation of things, to the 'conceptual comprehension' which brings out the reason of things (for more information see Piaget, 1978, p. 222).

In mathematics education, the distinction between *doing* and *understanding* was pointed out by Skemp (1971, 1978) who described and discussed the difference between *relational* and *instrumental* understanding in mathematics. The assumption that underlies this differentiation is that the building of relational understanding is necessarily the product of the linking process on the basis of which relationships between units of knowledge are recognised and constructed. On the other hand, it is acknowledged that procedures can be executed without understanding and therefore successful performance does not always indicate understanding of the reasons that support procedural success (Skemp, 1978; Hiebert & Lefevre, 1986).

1.2.2. The relationship between conceptual and procedural knowledge

In the past, procedural and conceptual knowledge had been treated as distinct. In recent times, it is recognised that a clear-cut definition and distinction between the two forms of knowledge is difficult to be made as not all mathematical knowledge can be described and classified as either conceptual or procedural. Despite the scepticism regarding a clear dichotomy between the two forms of knowledge, the distinction between them can be helpful in the clarification of another important aspect; that of the differentiation and relation between *knowledge* and *understanding*. Sierpinska (1994), in her analysis of understanding in mathematics, criticises Piaget's approach to the notion because in his work the relationships between "skill and knowledge", "action and thought", "doing and knowing", "action and conceptualisation", are referred to as one and the same thing (p. 104). In Sierpinska's view, all understanding should not be reduced to knowledge, rather, it should be seen as "an achievement which requires a long process involving acts of tentative understanding, reasonings, corrections, shifts of attention, etc." (p. 24).

Although the distinction between the two forms of knowledge provides a useful basis for discussions concerning certain aspects of mathematical learning, the study of the *relationship* between elements of conceptual and procedural knowledge has been considered as far more fruitful by researchers in mathematics education. Particularly in *problem solving*, elements of different forms of knowledge must be dynamically linked for a solution to be produced. Experiences with problem-situations are considered as fundamental to the way in which concepts and problem solving processes and strategies are built up or acquired. Therefore, mathematical problem solving situations have been seen as an appropriate area for the study of the possible links between the two forms of knowledge (Silver, 1986; Siemon, 1992).

The study of relationships between the two forms of knowledge in the framework of mathematical problem solving has mainly focused on the advantages that are gained when procedural knowledge is grounded on a conceptual basis; a connection which allows understanding of the procedures and why they work. In this sense, it is assumed that one of the purposes of conceptual knowledge is to support procedural knowledge (Carpenter, 1986). Research work that has focused on studying this aspect of the relationship between the two forms of knowledge, has provided detailed analyses and models of children's flexibility in choosing procedures to solve arithmetical word problems (e.g. Riley et al., 1983; Briars & Larkin, 1984). Word problems involve one-step translations from words to mathematical sentences (Lester, 1983, p. 233), and the models that have been developed account for the way in which conceptual understanding of the relations described in the text of the problem influences the choice of the appropriate arithmetical operation and subsequently problem solving performance.

The observations that have been made strengthen the position according to which skill alone can only be applied restrictedly in the context in which it is practised. On the contrary, when skill is supported by conceptual understanding, problem solving techniques can be flexibly adapted and transferred to novel situations, and students' ability to reason meaningfully with and about mathematical concepts and principles is enhanced (Skemp, 1978; Schoenfeld, 1985a Greeno, 1988; Booker et al., 1997).

On the other hand, the position that conceptual knowledge is based on procedural knowledge is not so commonly supported and therefore it is much less explored. Silver (1986) argues that the study of students' perceptions of problem similarity provides a clear example of conceptual knowledge that is depended on procedural knowledge. An analysis of students' decision of whether two or more problems are mathematically related has shown that most students' decisions were based on procedural knowledge, since problems were considered as mathematically related in so far as the same procedure could be used to solve them. Baroody and Ginsburg (1986), also support the idea that in certain cases young children's conceptual development in arithmetic is motivated by procedural knowledge. Moreover, the authors suggest that the development of conceptual knowledge and understanding of relations and principles is not always necessary for the acquisition of procedures.

By the repeated application of procedures children often notice regularities in the procedural routines. Noticing regularities, reflecting and abstracting from them a general structure, makes procedural knowledge meaningful and in this case it can be argued that the development of conceptual knowledge is fostered by procedural knowledge rather than vice versa. Perceiving underlying rules in the repeated application of procedures has also been stressed by Ginsburg (1982) as a means by which understanding of mathematical principles and structures can be built. Perceiving and understanding the underlying structures in mathematics is not necessary for producing the right answers and applying procedures correctly. However, failing to perceive can result in rigid, inflexible and mechanical execution of procedures.

The above mentioned positions make clear that the relation between conceptual and procedural knowledge is not simple. It is made apparent that the two forms of knowledge are related in a cyclical rather than a linear way, in the sense that the interplay between them entails benefits in both levels, the conceptual and the procedural. However, most of the research on this relationship in the context of problem solving situations, stresses how much conceptual knowledge and understanding guides the selection and application of solution procedures which, related to conceptual knowledge acquire stability, effectiveness and transferability to new situations. One of the reasons for this asymmetrical focus lies in the fact that procedural knowledge is related to students' action, which constitutes a part of their overt behaviour in a problem-situation. Thus, procedural knowledge as opposed to conceptual, is easier to observe and evaluate since possible deficiencies are more evident. Conceptual knowledge on the other hand can only be inferred by individuals' procedural behaviour (Carpenter, 1986).

Although deficiencies and errors are usually associated with lack of conceptual understanding, success can be underlain by both lack or presence of such a conceptual base. It is assumed that when procedural knowledge and also procedural success in a problem solving situation are not supported by conceptual understanding, they lead to rigid, inflexible and in certain cases automatic behaviour (Ginsburg, 1982; Ohlsson & Rees, 1991). The possibility for inflexible procedural success to be underlain by lack of conceptual understanding, as well as the, not as much explored, idea that the development and motivation of conceptual understanding can be grounded on successful procedural applications, are issues that the model which provides the theoretical framework of the study addresses. The main postulations of the model are presented in the following section.

1.3. The Representational Redescription model

Within the framework of developmental psychology, Karmiloff-Smith (1992) has proposed a model to account for development and learning based on the assumption that inflexible procedural behaviour lies on knowledge which is *implicit*; i.e. knowledge which is not available as manipulable data. Karmiloff-Smith argues that after procedural success certain types of cognitive change may take place and that in this process of change, implicit information embedded in an efficient problem-solving procedure progressively becomes more explicit, manipulable and flexible. In the framework of the model of *Representational Redescription*, development and learning are seen as complementary directions. It is postulated that both learning and development involve a process of "proceduralisation" which renders behaviour automatic and inflexible, and a process of "explicitation" during which knowledge which underlies the execution of procedures becomes more explicit, leading consequently to more flexible procedural behaviour. Both processes that are presumably involved in development and learning, are viewed as relevant to cognitive change. In Karmiloff-Smith's developmental model, the issue of cognitive change is addressed in terms of *representational explicitation* which applies in a variety of cognitive domains, including mathematics.

Karmiloff-Smith (1992) presents evidence of knowledge explicitation by exploring qualitative changes that take place both macro-developmentaly (i.e. over a period of years) and micro-developmentaly (i.e. within the boundaries of an experimental session). Her exploration covers a variety of domains such as language, science and psychology. In the case of mathematics she discusses and explains, on the basis of the RR model, the *macro-developmental* changes that occur concerning issues like number acquisition, number notation and counting. Within the context of problem solving, however, the idea of knowledge explicitation has been studied in physics, spatial, linguistic and notational, but not in mathematical tasks (Karmiloff-Smith, 1984).

This study aims at approaching and analysing changes in children's strategies within an arithmetical task, on the basis of the premises and explanations that the RR model introduces. The model of representational redescription, and the implied process of gradual explicitation, will be used to account for changes that children introduce to their strategies in an arithmetical context.

1.4. Summary

In the framework of this study, children's developed strategies are seen as a product of the combination of different pieces and forms of knowledge (conceptual, factual, procedural). As children keep working upon and beyond their successful procedural applications their developed strategies may become the object of reflection, and subsequently the object of understanding.

The need for elaboration of a given solution *after success* has been addressed by mathematics educators, within the theoretical framework of social constructivism. This theoretical perspective will be presented in more detail, in the following chapter. Here, it suffices to say that research that has been carried out within this particular framework has not explored the issue of elaboration after success on a psychological level. Rather, research work that follows the socio-constructivist paradigm has used the classroom as a unit of analysis, and has shown the effectiveness of group-work and classroom discourse in promoting students' conceptual change and understanding (e.g. Cobb et al., 1991, 1997; Yackel, 1996).

One assumption that underlies this study is the following. Group work on problems, discussion and argumentation on different ways of solving a task are certainly beneficial. However, it may be the case that not all students benefit to the same extent by group work and whole-class discussions. Reflecting and internally working upon their own problem solving approach, and knowledge that supports a developed, efficient strategy can be another source of learning and conceptual change for children.

Children's building of a deeper understanding and acquisition of better control of the factors involved in a problem situation are studied in the course of micro-developmental changes that take place as children engage in working and reflecting upon their successful solution strategies.

Conceptual understanding as it is built in the "micro-context" of an arithmetical task is approached and studied on the basis of the idea of *knowledge redescription* (Karmiloff-Smith, 1992). Therefore, higher conceptualisation of one's problem solving strategy is seen as the product of redescription of prior *implicit* (or less explicit) knowledge to higher levels of explicitness. This process of knowledge redescription is studied as it takes place while children engage themselves in problem solving and work upon and beyond their successful procedural applications.

Children's movement from procedural success to deeper understanding and control of a situation, is studied during their engagement in solving an additive task with multiple steps. The task that will be used is a computational problem which requires the production of families of number combinations. In particular, the task requires children to produce all the possible number bonds that result in a specific number each time (e.g. find all the possible number bonds to make 9, or 12, etc.).

It is hypothesised that, in the context of this particular task, children initially approach each step of the task separately, employing different pieces of procedural, factual as well as conceptual knowledge. This kind of approach may well be successful. However, if appropriate motivation is given to children to keep working on the task, eventually, the different pieces of knowledge will be organised in a strategy applied consistently for every step, to the whole of the task.

It is argued that by keeping working on their successful strategy children will eventually step back and notice important regularities that underlie the strategy and the numerical relations involved in the task. Understanding of these relations develops in the course of applying and elaborating the successful strategy by introducing qualitative modifications. The development of such an understanding is seen as the product of the process of *gradual explicitation* of knowledge. Explicitation of the forms of knowledge that children evoke to develop strategies, may lead to a higher conceptualisation of the rationale and potential of the developed strategy; lead to the acquisition of a better control of the situation.

The aforementioned assumptions and hypotheses underlie the study's prospective argument. Their presentation serves in justifying the choice of the specific theoretical model that will be used as a tool of exploration and analysis. Two issues that the previously mentioned assumptions denote need to be highlighted. First, this study sets out to explore the path of qualitative changes that children introduce *after* the development of a successful solution. Documentation and analysis of changes that occur *after* procedural success are put into focus, in particular. Second, there is a particular interest to study changes and modifications in children's strategies that arise as a result of children's reflection, and work upon their own approach to the task. That is, changes which result from each child's internal process of elaborating the form, and different pieces of knowledge that support his or her own strategy.

On this basis, the model of representational redescription is considered as appropriate to provide the theoretical framework for this study, for the following reasons:
First, the RR model supports the idea of *'success based'* cognitive change, in the sense that success is viewed as a possible source of cognitive change.
Second, the model provides an explanatory basis for changes observed in the behavioural level, and also for changes inferred at the representational, and conceptual level.

1.5. The research questions

The study, initiated from the preceding hypotheses, seeks an answer for the following questions:

* Do children evolve and develop their successful problem solving approach while engaged in a problem solving situation repeatedly? If yes, what are the process, and types of change that occur in children's problem solving approach in the 'post-success elaboration' phase?
* Do the procedural/behavioural changes indicate qualitative changes at the conceptual level as well?
* Are the behaviours observed, and the introduced behavioural modifications, consistent with the behaviours that the RR model describes when accounting for the post-success development of children's problem solving approaches?

* Can the qualitatively different levels of knowledge representation that the RR model describes, be assigned to the different types of arithmetical knowledge (either procedural, declarative or conceptual) that underlie children's methods and strategies when solving the particular task?

To answer these questions, qualitative changes at both the behavioural (procedural) as well as conceptual level are studied. Evidence of behavioural changes is sought by observing children's overt behaviour in the problem solving situation. By the changes observed at the behavioural level, conceptual changes are inferred. The basis that will be used for such an inference is the one provided by the model of *Representational Redescription* (Karmiloff-Smith, 1992).

In the following chapter, theoretical views and research on the notions of procedural and conceptual knowledge in mathematics education is reviewed. In particular, the chapter presents theoretical ideas and research work which:

a. tackle the issue of children's procedural knowledge in arithmetic in connection with the development of children's strategies in problem solving, and
b. focus on the notion of conceptual understanding as one aspect of conceptual knowledge.

Chapter 2 Procedural and conceptual aspects of children's mathematics problem solving

2.1. Overview

The aim of the study is to explore changes that children introduce to their employed problem solving strategies. It is hypothesised that such changes reflect a movement in the use of different pieces and forms of knowledge that children call upon, in the course of solving an arithmetical task. Thus, in this chapter, certain theoretical positions which address the issue of the different forms of knowledge that are thought to underlie mathematical knowledge and the development of problem solving strategies, are presented. The chapter is organised in three sections. The first section is devoted to the presentation of theoretical ideas and research work which examine children's procedural knowledge in mathematics, in relation to the development of problem solving strategies and procedures in additive tasks. In the second section, the aspect of conceptual knowledge that is discussed is *understanding*. Thereafter, a review of theoretical approaches that address the issue of conceptual understanding as this is related to mathematical problem solving, in particular, is presented. In the final section, the main issues that the theoretical review reveals which are of importance for the conceptual structuring of this study, are summarised.

2.2. Problem solving strategies and procedures: the case of addition

The development of children's procedural knowledge, applications and strategies has been studied in the context of problem solving. In this study, the focus is on changes that children introduce to their strategies in the context of an additive task. In this section, after considering cognitive views regarding the term *strategy* and its various uses, research on the development of children's strategies in additive problems, and the main aspects of this development, is presented.

2.2.1. Children's strategies: cognitive views

In cognitive development, children's goal-directed behaviour has been explored by means of studying the strategies that children invent and apply in problem solving situations. Among researchers, different definitions have been employed to account for *strategies* and the term *strategy* has been used in various, different ways. There is a general agreement that strategies are goal oriented operations employed to facilitate task performance (Harnishfeger & Bjorklund, 1990; Fayol, 1994). However, two particular aspects and characteristics of children's strategies; their deliberate implementation and possible availability for conscious evaluation, constitute the object of long discussions and contradictions (Harnishfeger & Bjorklund, 1990).

According to the traditional view, children's behaviour can be considered as *strategic* if it is organised and directed toward a goal. Intention of achieving a goal is sufficient to define strategies, and aspects of behaviour such as planning and conscious awareness are not involved in the definition (Bjorklund & Harnishfeger, 1990). On the contrary, Bisanz and Lefevre (1990), argue that strategies should be differentiated from any procedures or class of operations used to accomplish a task. Therefore, a definition of strategic behaviour should account for the ability to make decisions when more than one options are available, and also for the ability to adapt one's decisions and actions in a flexible way. In these terms, a *strategy* is defined as "a procedure that is invoked in a flexible, goal-oriented manner and that influences the selection and implementation of subsequent procedures" (p. 236). Thus, in Bisanz's and LeFevre's view, different procedures may be involved in the execution of a strategy but in their view these procedures constitute the products of a strategy, they are not equated with the strategy itself.

Even though selection and planning of the action before it takes place, are involved in this definition, consciousness is not being referred to as a essential aspect of strategies. Bisanz and LeFevre, in their attempt to define strategies, do not deny that individuals can be consciously aware of some strategies. However, it is argued that on certain occasions strategies "can become increasingly automatic and, presumably, less conscious with extensive practice" (ibid., p. 239). Like Bisanz and LeFevre, Siegler and Jenkins (1989) do not stipulate that a strategy must be consciously formulated or produced by a conscious choice. In this sense strategies are differentiated from *plans* which are usually conceived as involving consciousness. However, they also differentiate *strategies* from *procedures* in the sense that they define strategies as "goal-directed" but also "nonobligatory". The "nonobligatory" aspect of strategies in this definition, differentiates strategies from procedures which may represent the only way to achieve a goal (Siegler & Jenkins, 1989, p. 11). On the other hand, there are views according to which strategies are complicated processes and subject to consciousness. Thus, according to these points of view, automatic and effortless processes can not be considered as strategies.

Acceptance of this exclusive definition entails that the term *strategy* can not be used to account for all the different forms of procedures that may be employed to facilitate task performance (Bjorklund et al., 1990).

Although a consensus concerning what can and what cannot be considered as strategy seems difficult to establish, researchers agree generally that strategies are fundamentally characterised by their ability to develop. Recent research on children's strategies has particularly focused on how problem solving strategies and their use change and develop as children grow older. The study of developmental differences in strategy use has followed two directions; the different strategies that children of different ages use are investigated in terms of *inter-individual differences*, whereas the variety of strategies that a single child uses in similar tasks, in different contexts are investigated in terms of *intra-individual differences* (Harnishfeger & Bjorklund, 1990).

Strategies and their development are studied in consideration with the domain of knowledge and context in which they are applied. Studies focused on the development of children's arithmetical knowledge investigate the development of children's strategies in different areas of arithmetic such as mental arithmetic performance and word problems. Arithmetic is a domain in which the term strategy has been used in a rather loose way. The terms *procedures*, *methods*, *strategies*, have been employed almost interchangeably by researchers who study the development of children's different approaches to arithmetical problem situations. Furthermore, these terms are employed in a broad sense and include the deliberate as well as unconscious and automatic procedures (Ashcraft 1990). In the context of this study the focus is on strategies that children employ to solve an additive task. Research on the development of children's problem solving strategies and procedures in additive tasks is presented in the following section.

2.2.2. Children's strategies in additive tasks

The strategies that children of school age use to solve problems which involve addition have been mainly studied in the context of word problems. Even though Carpenter et al. (1981) suggest that comparisons between the strategies that children apply in word problems and the strategies applied for the solution of number sentences should be made cautiously, Gray (1991), argues that the basic categories of strategies that children apply for number calculations have been proved consistent with those employed by children in solving basic computational problems.

Hiebert, et al. (1982) and Carpenter and Moser (1983) identified the following addition strategies that young children use to solve simple word problems transformable to a number sentence of the form: $m + n = t$, where m is the smaller of the two addends:

* *Counting all*: to solve the problem, the child counts, starting from 1 and ending with t. For example, to solve a simple addition problem such as 3 + 4 the child counts 1,2,3,4,5,6,7.
* *Counting on from the first (smaller) number*: the child counts, starting with m or $m + 1$, and ending with t, the counting sequence involves n increments. In the case of the previous problem, the child will repeat the first number (3) and count starting from that number (3,4,5,6,7) or from the number exactly after that (4,5,6,7).
* *Counting on from the larger number*: the child counts starting with n or $n + 1$, and ending with t, the counting sequence involves m increments. For the same problem, the child will proceed as in the previous example but starting with 4. Thus, less counting will be required.[2.1]
* *Known fact*: the child solves the problem by recalling from memory the particular number fact. Usually the "doubles" are the more easily memorised and recalled facts; e.g. 2 + 2, 3 +3.
* *Derived fact*: the child recalls from memory a known number fact and uses it to calculate the one that is unknown. For example 6 + 7 is 13, because 6 + 6 is 12 and 13 is 1 more. [2.2]

Children's addition strategies as identified by Carpenter and Moser can be classified in two categories: *counting* and *non-counting* strategies. The non-counting strategies include the known facts and derived facts. Cobb and Steffe (1988) differentiate between the following three types of strategies for deriving facts from known ones: The "addend-increasing" strategy, the "addend-decreasing" strategy, and the "compensation" strategy.

a) *The addend-increasing strategy*: One of the addends is decomposed into two parts. The sum of one of these parts is found. Finally, the remaining part is added to the partial sum. For example, "He himself noted his error with 9+6 and immediately corrected it by solving from 9+5: thus, '9 and 5 is 14, and one is 15" (the authors quote this example from Brownell, 1928, p.125). [2.3]

[2.1] This strategy has been called the *min* strategy by certain researchers (e.g. Groen & Parkman, 1972; Ascraft 1982; Siegler & Jenkins 1989).
[2.2] Siegler (1987, 1996) refers to this strategy as the *decomposition* strategy.
[2.3] Brownell, W.A. (1928). *The Development of Children's Number Ideas in the Primary Grades.* Chicago: University of Chicago Press.

b) The *addend-decreasing* strategy: One of the addends of a sum is increased. The sum of the increased addend and the other addend is then found. Finally, this sum is decreased by the amount of increase. For example "He first gave 14 for 9+6. He then solves thus, '9 and 8 is 17, less 1 is 16, less 1 is 15'" (the authors cite this example from Brownell, 1928, p.30).
c) The *compensation* strategy: One addend is increased and the other is decreased by the same amount. The sum of the resulting two addends is then found. For example, "He solved 7+3 as '6 and 4 is 10, so 7 and 3 is 10'" (the authors cite this example from Brownell, 1928, p.128).

(Steffe & Cobb, 1988, pp. p. 252-253)

A further differentiation among counting strategies that has been identified, distinguishes counting procedures that children employ mentally, from counting procedures that are supported by *concrete aids*. Concrete aids include fingers, cubes, marks on a piece of paper or any other physical objects that children may use to model the numerical relationships involved in the problem (Houlihan & Ginsburg, 1981; Baroody & Ginsburg, 1986).

Counting procedures are seen as critical to the development of addition. Until children can use the retrieval of addition facts, a strategy which is considered as more mature and sophisticated, counting strategies are employed to allow children to calculate sums (Maclellan, 1995). Thompson (1995) suggests that even when children have developed and acquired a wide variety of arithmetic skills, counting continues to be an important part of their problem solving repertoire, and that counting is a skill that children always combine with other existing or newly acquired skills, facts, and knowledge.

Counting strategies develop and, in this developing process children progressively abandon previously applied counting procedures for more sophisticated ones. The change and development of children's counting procedures over time in the context of additive problems has been widely studied and the strategies employed have been classified in different cognitive hierarchy levels. Counting all strategies has been classified in the lowest cognitive level (Fuson, 1982; Baroody & Gannon, 1984). It is observed that counting all strategies tends to be followed developmentally by counting on strategies. Research focused on the transition from counting all to counting on strategies has provided detailed analysis of the intermediate stages of development as children employ the various forms of counting on and counting all procedures (Carpenter & Moser, 1983, 1984; Secada et al., 1983; Steffe & Cobb, 1988; Fuson, 1982, 1992).

Counting procedures which are not supported by concrete aids are considered as the product of *abstraction* which allows children to focus on the counting sequence itself rather than on each of the quantities involved. Greater flexibility and efficiency are assumed to be the results of this abstraction and therefore counting strategies without concrete aids are considered as more sophisticated (Carpenter & Moser, 1983). On the other hand, counting on strategies are also considered as more sophisticated than counting all strategies because it is suggested that counting on procedures depend on understanding of basic arithmetic principles such as the relation between cardinality and counting (Fuson, 1982).

The category of non-counting strategies that children employ to solve simple addition problems comprises the use of known and derived facts. Ashcraft (1982) argues that number fact efficiency develops following a shift from relying on procedural knowledge of counting to relying on declarative knowledge (a stored network of number facts). In his analysis, Ashcraft studied the chronometric characteristics of strategies based on children's procedural knowledge which included counting methods, in comparison to strategies based on declarative retrieval, i.e. retrieval of number facts from memory. Ashcraft concluded that procedural processes are much more slow than fact retrieval, however the majority of first graders relied on the slow counting procedures. A notable increase of competency in number fact retrieval is observed at the third grade; an observation which presumably indicates the developmental nature of the shift to relying on declarative knowledge.

Baroody (1983) argues that Ashcraft's model underestimates the role of procedural knowledge and proposes instead an alternative explanation for the development of number fact efficiency based on the argument that the development of procedural knowledge also plays a crucial role in the efficient production of number facts. In Baroody's view, number fact efficiency involves a developmental shift from slow counting procedures to a reliance on automatic "principled" procedural knowledge (p. 227). The notion of "principled" procedural knowledge implies that number facts are not stored individually in a retrieval network. Rather, the generation of basic number facts depends on stored rules and principles of arithmetic. "Principled" procedural knowledge develops from exploiting already internalised regularities and relationships and thus the need for one to learn and store numerous individual number combinations is eliminated. Therefore, "using stored procedures rules or principles to quickly construct a range of combinations is cognitively more economical than relying exclusively on a network of individually stored facts" (Baroody, 1985, p. 89).

During school years, or even before, children have the opportunity to observe rules and principles that underlie numerical relationships. For example, addition combinations arranged in a table produce various patterns, including N + 1 progressions (e.g. 1 + 0 = 1, 1 + 1 = 2, 1 +

2 = 3, e.t.c.). This example, given by Baroody et al. (1983), is indicative of a mathematical structure underlying numerical relationships. It is argued that mathematical principles are abstracted from observation, internalisation and use of mathematical structures. When such principles and rules are observed, abstracted and stored in memory, different *families* of number combinations which are underlain by different principles can be very easily generated (Baroody et al., 1983; Baroody & Ginsburg, 1986).

In opposition to the chronometric explanations for number fact efficiency, Baroody's alternative model suggests that practice is important provided that it offers opportunities to children to discover new relationships, routinise known facts and relationships and make them *explicit*, so that harder problems can be processed (Baroody, 1985). Moreover, though practice may play a role in discovering relations and routinising knowledge, the development of basic number-combination facility basically depends on internalising mathematical relationships; that is on the growth of the structural network underlying general mathematical knowledge (Baroody & Ginsburg, 1986, p. 92).

In the framework of the model developed by Ashcraft, number facts are mentally represented in memory as a printed table and thus their generation is seen as a *reproductive* process. On the other hand, Baroody's and Gannon's (1984) suggestion that number facts are not necessarily stored as such but instead, they can be generated by stored rules and principles, implies that efficiency in number combinations may also include *reconstructive* processes which involve relations as well as facts; procedural as well as conceptual knowledge.

The idea that knowledge of facts does not involve the simple 'storage' and 'retrieval' but a coherent system of relationships is also supported by Kamii (1985) who, from a Piagetian perspective, argued that a fact is always a reconstruction that children make "through their own mental actions of putting numbers into relationships" (p. 72). Kamii introduced the notion of *mental regrouping* which she defined as a way of producing new knowledge in relation to what is already known. In the case of factual knowledge, the notion of *mental regrouping* entails that facts are stored and remembered as relationships. The notion of *mental regrouping* seems very close to Baroody's notion of *families* of number facts, which in Kamii's view as well facilitates memory retrieval "by facilitating the construction of a network of relationships" (ibid., p. 78).

Baroody's model however, does not exclude the possibility for certain combinations in a family of number combinations to be stored in the form of a specific association *and* represented by a rule. Baroody (1985) gives the following example; 1 + 0 is a number combination which belongs in the (N + 0 and 0 + N) family of combinations ("zero addition family"). This specific combination (1 + 0) may be stored as the fact 1 + 0 = 1 and represented by the (N+0=N and 0+N=N) rule. It is assumed that less familiar members of the family (e.g. 0+9867) would probably be represented only by the rule (Baroody, 1985, p. 91).

Baroody's suggestion that certain members of a family of combinations can be stored in the form of factual representation or as a rule implies that number combinations may be stored in the form of *multiple* representations. Also, Baroody (1985) complements the argument made in previous papers regarding the reconstructive processes involved in the generation of number facts as opposed to reproductive processes, by proposing that both processes may work together: "knowledge of rules, procedures or principles and specific numeral associations may, in various degrees interact to generate number combinations" (ibid, p. 92).

The identification of various different strategies that children use in additive tasks arises the issue of whether these strategies form a cognitive hierarchy, and whether children of different age use different strategies (and if so, which) when dealing with addition problems. Research on these issues is presented in the following section.

2.2.3. Cognitive hierarchy, and variability among addition strategies

The issue of cognitive hierarchy concerning the strategies that children employ in additive tasks is approached with scepticism by Gray (1991) who suggests that the documented research cannot sufficiently resolve the issue of whether the identified addition strategies (e.g. the various counting procedures and use of facts) form a conceptual hierarchy. Gray argues that factors related to the specific instruction that children receive surely play a crucial role in the development of children's ability to use the various counting strategies. He also suggests that for number fact competency, a position within a hierarchy is difficult to identify since this strategy may be used with or without evidence of understanding: "It may only provide evidence of routine and a good memory" (p. 554).

Moreover, research which suggests that the shift to the application of economy in effort procedures (e.g. counting on) depends on understanding of basic arithmetical principles (Fuson, 1982; Hiebert et al., 1982) have been questioned. A series of studies has focused on the development of economical addition strategies in relation to the principle of commutativity of addition according to which "the order in which terms are added does not affect the sum" (Baroody & Gannon, 1984, p. 321). Conclusions drawn from these studies indicate that children may discover commutativity by informal means, and the invention of labour-saving addition strategies does not necessarily imply, and does not depend on, conceptual understanding and appreciation of the principle (Baroody, et al. 1983; Baroody & Gannon, 1984; Baroody, 1987).

Although the issue of cognitive hierarchy among addition strategies has been questioned, strategy *variability* in additive problems has been recognised, as an important aspect of young children's problem solving behaviour in additive tasks. Research in the field of mathematics education, some indicative examples of which were reviewed in the previous paragraphs, has shown that children of a given age use a variety of strategies when dealing with arithmetic, additive problems (e.g. Carpenter & Moser, 1982; Fuson 1982; Baroody, 1987). These findings come in opposition to earlier research in mathematics education which tended to depict children of a particular age as consistently using a specific addition strategy. For example, Groen and Parkman's (1972) *min* model. This model was developed on the base of a chronometric study. The assumption that supported the model was that the size of the smaller addend was a predictor of first grader's solution times on simple addition problems. Groen and Parkman postulated that children of this age solved such problems by using consistently the min strategy, i.e. by counting up from the larger addend. This view was later supported by Ashcraft's (1982) chronometric study.

Siegler, in a series of studies, has argued against the idea of 'stage' theories in mathematics thinking development, according to which, children of a given age are said to use a specific strategy when dealing with additive problems. Siegler (1987) presented data from a study with kindergarteners, first and second graders solving simple addition problems. In these problems the smaller addend was not bigger than 10, the larger addend ranged from 4 to 15, and the sum from 5 to 23. Consideration of solution times replicated the results drawn from earlier chronometric studies, i.e. that first and second graders consistently used the *min* strategy to add. However, children's verbal reports in this study, indicated the use of diverse strategies. The *min* strategy was only one of five different approaches that children reported using. Further analysis showed that the *min model* was a good predictor of solution times only on trials where children had reported the use of the *min* strategy. In contrast, the *min model* was not a good predictor of solution times when the use of the other strategies was reported. In an even earlier study, Siegler and Robinson (1982) showed that the majority of pre-schoolers (4 to 5 years old) solved addition problems with addends which ranged from 1 to 5, by using at least three addition strategies. The *counting fingers* strategy involved putting up the fingers on one hand then on the other, and then count the two sets of fingers. The *fingers* strategy involved putting up fingers as in the *counting fingers* strategy but giving no evidence of counting. Finally, when using the *counting* strategy children counted aloud without using any visible reference. Siegler and Shrager (1984) further examined pre-schoolers' (4-5 year olds) strategy choices in simple additive problems of two addends. When sums were no greater than 10, results drawn from this study closely paralleled those of the Siegler and Robinson (1982) study. However, it was shown that with sums of 11 or 12, where the usual strategies were unlikely to work, children adopted or invented new strategies.

In summary, the aforementioned studies, indicate the following pattern of development across kindergarten, first and second grade children's additive strategies, for the types of problems where both addends are between 5 and 10: Use of retrieval increases from kindergarten to first grade and even more from first to second grade. Counting all and guessing decline sharply in first and second grade. The *min* strategy becomes more frequent by first grade and declines by second grade. Use of decomposition (i.e. derived facts) increases steadily but it is not very frequent (Siegler, 1996).

This series of studies showed that even very young children do not use invariably a single strategy for solving addition problems, rather, they use multiple, diverse strategies. What is more, children's strategies develop and change during childhood. Development does not only involve changes in the mix of existing strategies, but also construction of new ones and abandonment of old ones. These studies showed that *variability* is a crucial aspect of young children's strategies in additive tasks; it is even evident within a single trial, between children of a given age, as well as within individual children, i.e. within an individual solving the same problem on two occasions close in time (Siegler & Shrager, 1984). The issue that is raised is how children choose which addition strategy to use in order to solve a given task, and on which grounds they construct and discover new addition strategies. The following section presents research that has addressed the issue of children's strategy choice and discovery.

2.2.4. Strategy choice and discovery in additive tasks

The importance of studying children's *choice* of strategy in the context of addition problems was underlined by Siegler and Robinson (1982) who argued that it is not sufficient to build a model of how children may apply a particular strategy, it is also necessary to account for how they choose between alternative strategies. In the case of word problems, certain models account for children's strategy choice based on the assumption that strategy choice depends on children's understanding and representation of the actions and relationships described in the problems. A more detailed discussion of these models is presented in the following section because of their premises regarding children's way of understanding problem situations.

Gray (1991) suggests that children's choice of strategy can be seen as a matter of preference. The author explains the idea of a possible "preferential hierarchy" in the following way:

"If a child's preferred way of solving one of the numerical problems is to remember the answer (known fact) then the preferred, and most efficient alternative if the fact is not known, will be to use other known facts to derive the answer. Should either of these two strategies fail, the child will then need to resort to the next preference which will involve counting" (p. 554).

However, in the framework of Siegler and Shrager's (1984) associative model children's choice was determined by the strength of associations between the pairs of numbers that were to be added. The strength of the associations was determined by the correct and incorrect answers that children gave in a separate experiment, where they were asked to say what they thought to be the correct answer for the addition of each combination of numbers, without using fingers or counting. The model's basic assumption was that people associate whatever answer they state, correct or incorrect, with the problem on which they state it. Siegler and Shrager argued that the developed associative model could adequately predict children's strategy choice in the simple addition that they worked with. Three factors seemed to influence children's choices: Difficulty of the problem and execution of a back-up strategy, frequency of exposure to the problem, and related knowledge.[2.4] In a later discussion of the Siegler and Shrager study, Siegler (1996) argued that children's adaptive decisions did not derive from explicit, rational, metacognitive analysis. The harder the problem, the more often children who participated in that study, relied on back-up strategies (i.e. strategies other than retrieval). However, children's explanations regarding the difficulty of the problems were only moderately correlated with either the objective difficulty of the problem or with their strategy choices. This showed that children could exhibit adaptive behaviour without having explicit knowledge of the problem

[2.4] For more detail on the model and its computer simulation see Siegler & Shrager (1984) and Siegler & Jenkins (1989).

difficulty. Therefore, Siegler (1996) argues that adaptive choices are relative to the choices available to the individual, and can be produced through application of either implicit or explicit knowledge.

Houlihan and Ginsburg (1981) have also suggested that the first and second graders actually select their counting strategies in additive problems adaptively, and according to the familiarity and size of the problem's addends. As development proceeds and children have to deal with more complex problems, they tend to create non-counting based *invented* strategies. For Houlihan and Ginsburg "invented strategies" are the result of a combination of what children know and thus are indicative of children's understanding of addition (p. 95).

The issue of strategy development, and construction has been addressed by Steffe and Cobb (1988) who, following the constructivist paradigm, created a constructivist model of first graders' counting, addition and subtraction strategies as developed and changed within a longitudinal teaching experiment. The invention and construction of new strategies in additive problems in particular, and within the context of a *microgenetic* study was studied by Siegler and Jenkins (1989) who focused not on how children choose among strategies that they already know, but on how children add new additive strategies to their repertoire. In this later study, Siegler and Jenkins examined 4-5 year-olds' discoveries of the *min* strategy for adding numbers, and the way in which children generalised the strategy in other problems. This type of study involved following the strategy discovery process from before the discovery until it has been generalised to other problems. Children who participated in the study were selected through a 'pre-test' process, the purpose of which was to identify children who did not yet know the *min* strategy. The main phase of the experiment involved the repeated presentation of twenty computational ("non tie", p. 54) problems with addends from 1 to 5 until all of the children discovered the strategy. Subsequently, a set of *challenge* problems was presented. The problems involved one addend greater than 10 (low 20s), and one addend between 1 and 4 (e.g. 23+1, 2+21). In the latter weeks of the experiment a mix of problems was presented to children (e.g. problems as easy as 3+1 and as difficult as 22+4). Children in this study were found to use strategies that previous studies of young children's addition have described (e.g. Siegler & Robinson, 1982; Fuson, 1982; Baroody & Ginsburg, 1986).

However, this study revealed the use of a strategy that was not described previously as a distinct strategy. The *short sum* strategy as described by Siegler and Jenkins is the equivalent of the *count all* strategy as described by Carpenter and Moser (1983) (see section 2.2.2. of the thesis). In the Siegler and Jenkins (1989) study this strategy was differentiated from the *sum* strategy which, on a problem such as 4+3, involved "counting '1, 2, 3, 4.....1, 2, 3.....1, 2, 3, 4, 5, 6, 7'" (Siegler & Jenkins, 1989, p. 59). This differentiation was made because it was considered that the *short sum* strategy played an important, mediating role to the discovery of the *min* strategy. It was emphasised that children were not always aware of their discovery, and they varied greatly in the degree of insight that accompanied their new strategy. Regarding strategy generalisation Siegler and Jenkins reported that most children rarely used the min strategy to new (of the same type) problems, in the period immediately after they discovered it:

"... a new strategy may generate answers accurately and efficiently on a problem, but if an already-known strategy can solve that problem equally accurately and even more efficiently, the new strategy will rarely be used on the problem. Similarly, a strategy may be used often on a class of problems not because it works particularly well on them but because no other strategy works well at all."
(Siegler & Jenkins, 1989, p. 99).

Strategy variability and strategy choice constitute the primary focus of the *overlapping waves theory*. This theory was developed by Siegler (1996), on the basis of Siegler and his colleagues' findings over the series of their studies. The theory is based on the following premises:

* Children typically use a variety of strategies over prolonged periods of time.
* Experience brings changes on the existing strategies, as well as introduction of more advanced approaches.
* Acquisition of new strategies involves a mix of conscious and unconscious processes.

Concerning the third point, in particular, Siegler (2000) notes:

"In at least some cases, new strategies are constructed on an unconscious level before people are aware of doing anything different than they had done previously; behavioral indices show that new approaches are being used, although verbal reports of use of the new strategy lag slightly behind. Thus, discovery is not exclusively a metacognitive process, not is it exclusively an associative process. Both types of processes are crucial."
(Siegler, 2000, p. 29).

The issue of conscious or unconscious strategy discovery was addressed by Siegler and Stern (1998). A microgenetic methodological approach was applied on an intensive trial-by-trial basis for eight sessions. It was believed that this approach would reveal the role of consciousness in children's strategy discoveries, as well as give information on whether discoveries are made abruptly or gradually. Thirty-one children aged 8-9 participated in the study. Their strategy discoveries were studied on an inversion problem such as 28+36-36. "Inversion is the principle that adding and subtracting the same number leaves the result unchanged" (p. 377). Knowledge of this principle allows solution of inversion problems by the "*shortcut strategy*": ignore the number that is both added and subtracted. For Siegler and Stern, solution of the problem by applying the shortcut strategy requires a small insight, "... the recognition that, on this type of arithmetic problem, answers can be obtained without executing the arithmetic operations" (p. 377). Use of the strategy does not only require knowledge of the principle but also choosing to use it on a particular task. The microgenetic design was applied together with two different measures: the consideration of solution times in each trial was considered as a measure of unconscious, *implicit* discovery of the shortcut strategy, whereas children's verbal report was considered as a measure of conscious, *explicit* discovery. Children were randomly assigned to two groups. One group was exposed to blocked problems (i.e. inversion problems 100%), whereas the other group was exposed to mixed problems (i.e. 50% inversion problems-50% standard problems).

Siegler and Stern's data showed that strategy discoveries can be unconscious. The generation of very fast solution times was considered as indicative of use of the shortcut before children explicitly reported using it. Unconscious (i.e. non-reportable) use of the shortcut strategy preceded conscious recognition of its use. The first use of the unconscious shortcut strategy was considered as representing a qualitative change in the solution procedure. In the last, eighth session, the researchers presented to children a wide range of problems including ones that superficially resembled the inversion problems but did not allow the same solution. This was expected to allow researchers to examine both the degree of appropriate generalisation of the shortcut strategy to unfamiliar types of problems to which it was applicable, and the degree of inappropriate generalisation to problems on which it was not applicable. In both groups of children, the rate of appropriate generalisation was low. Moreover, it was shown that the blocked problems condition produced greater use, and more frequent generalisation of the shortcut strategy to new problems in which it was applicable, but it also produced greater inappropriate generalisation of the strategy. Another finding of the study, which was considered to be the most puzzling, was that children in both groups continued the use of their initial computational strategy on about one third of trials throughout the seven sessions. The researcher give two possible explanations of this persistence: Children either forgot the shortcut during the week which separated one session form the next one, or "...the persistence of well-established strategies is a basic characteristic of human cognition that occurs even without week-long gaps between sessions" (Siegler & Stern, 1998, p. 395). The authors provide further support for this second explanation by quoting findings from other studies, like for example Karmiloff-Smith (1992) and Alibali and Goldin-Meadow (1993). Both these studies report findings similar to Siegler and Stern's regarding the issue of conscious / unconscious strategy discovery, but in contexts other than arithmetic. The broader framework of these findings will be discussed in detail in the following chapter.

2.2.5. Summary

The theoretical positions and research studies that are reviewed in this section bring to light certain aspects of children's strategies, as these are developed and used in the context of addition problems. It must be noted once again that in mathematics education, the term strategy is being used in a rather loose way, and the constraints that are posed by certain psychological positions regarding the use of the term do not constitute a particular object of analysis and discussion when children's strategies in arithmetic tasks are explored. Hence, children's ways of approaching an addition problem either by relying on their procedural or factual knowledge are considered and explored in terms of 'strategies' or 'methods' or even 'procedures'.

In this study, the term strategy is also used in a broad sense. Because the task in the context of which the micro-development of a strategy is studied involves multiple steps, the term *strategy* is used to refer to the unified technique that children will possibly apply for the whole of the task. When different steps are approached in a different way, the term *method* is employed. This differentiation does not imply though any particular stance regarding the definition of what constitutes a strategy. Rather, different terms are used for reasons of clarity and also because the development of a strategy for the whole of the task, after the initial application of different separate methods for each step, constitutes a significant shift in children's problem solving behaviour; a shift that this study particularly focuses on.

2.3. Conceptual understanding in mathematical problem solving

The notion of *understanding* in mathematics has received considerable research attention. As an object of study in mathematics education, understanding has been approached from numerous different perspectives, and in the framework of each, different definitions and qualities have been attributed to the term.

Most of the research devoted to the notion of understanding in mathematics education focuses on students' understanding in the context of mathematical situations such as word problems (Polya, 1945/1990; Schoenfeld, 1985a, 1985b; Verschaffel & De Corte, 1997); students' understanding of mathematical concepts and operations (Resnick, 1983; Herscovics et al., 1987; Bergeron et al., 1987; Greeno, 1991); and also students' informal understanding of mathematics and its link with the formal mathematical knowledge as this is established through school instruction and students' initiation to the use of symbols and formal computational rules (Ginsburg, 1977/1982; Hiebert, 1984; Bryant, 1997).

In this section, a selective review of the approaches that relate conceptual understanding and mathematical problem solving is presented. The focus is moved onto those theoretical positions that are considered as possible contributors to the clarification of understanding as this is viewed in the framework of this study.

2.3.1. Understanding as a process of establishing relationships between mathematical concepts and ideas

Understanding is the notion on which Skemp (1971/1986) grounded his distinction between two kinds of mathematical learning; 'habit learning, or rote memorising', and 'intelligent learning', that is, learning which involves understanding (p. 15). In a later article (Skemp, 1978), the author underlined that 'intelligent learning' involves 'relational understanding' which he defined as "knowing both what to do and why", as opposed to 'instrumental understanding' which was described as "rules without reasons" (p. 9). The author strongly criticised school instruction which encourages instrumental instead of relational mathematics. Although both practices can generate successful performances, two highly different types of learning, in quality and power, underlie these attainments.

Learning based on instrumental understanding shows its limiting character especially in a problem solving situation where the problem solver learns to follow and simply apply a specific sequence of memorised rules or solution steps without identifying any connection between them. In this case, students can only apply their solution plan "from particular starting points (the data) to required finishing points (the answers to the questions)" (ibid, p. 14). Thus, the application of a problem solving method is highly restricted to specific situations

On the contrary, a powerful advantage of relational understanding in problem solving is that it makes possible the connection of a successful problem solving solution with the reasons for this effectiveness. Skemp argues that such a connection, enhances the adaptability of a problem solving method to new problem-situations. The reason for this is that relational understanding is grounded on knowledge of not just isolated mathematical ideas, but of mathematical relationships out of which the problem solver can deduce rules or procedures when facing a novel problem-situation (Skemp, 1979). Moreover, the identification of relationships between the successive steps towards the solution and the final goal is enabled, giving to the problem solver the power to develop numerous alternative plans "...getting from any starting point within his schema to any finishing point" (Skemp, 1978, p. 14).

The power of relational understanding is embedded in the construction of "schemas". The term "schema" originates from Piaget's "scheme theory" which emerged as an antipode to behaviourism. In Piaget's words, "... no knowledge is based on perceptions alone, for these are always directed and accompanied by schemes of action. Knowledge, therefore, proceeds from action, and all action that is repeated or generalised through application to new objects engenders by this very fact a 'scheme', that is, a kind of practical concept." (Piaget, 1980, p. 23-24). Piaget's notion of scheme and its integration as such, in mathematics education by the constructivist paradigm will be discussed further on. At this point, it should be stressed that the piagetian scheme is strongly connected with action; it is an "action-scheme" that guides cognition and is tightly related to behaviour (Marshall, 1995).

Skemp introduced the term in mathematics education defining it though in a different way. 'Schemas' in Skemp's sense, are conceptual structures, that is networks of concepts suitably connected. Schemas are constructed on the basis of the individual's experiences and allow adaptable behaviour in different situations. Skemp (1971/1986), describes the following functions of a schema: "...it integrates existing knowledge, it acts as a tool for future learning and it makes possible understanding" (p. 37). Schemas, viewed as conceptual networks, are considered to have the ability of embodying relationships between concepts, rules or procedures and thus underlying the form of understanding which Skemp names "relational".
It is underlined that relational understanding can only be evolved by students' reflective activity; i.e. a goal-directed mental activity that guides the construction and improvement of schemas and plans which subsequently govern the individual's behaviour in the process of achieving his or her goals (Skemp, 1979, p. 44).

2.3.2. Understanding and the construction of "viable" knowledge: the constructivist paradigm

Constructivism constitutes a major research paradigm in mathematics education. By the various forms that have been attributed to constructivism, different theoretical positions emerge. The compatibility between these positions and their implications for the research as well as educational practice has been studied extensively (Ernest, 1994a, 1996; Confrey, 1995; Hendry, 1996). Here, only those aspects of the constructivist paradigm that address the issue of conceptual understanding in mathematics will be presented.

Radical constructivism has its roots in Piaget's epistemology. Two of the major assumptions that underlie this theoretical position are the following:

1. Knowledge is a result of the learner's constructive activity rather than of passive reception of information (Von Glasersfeld, 1991).
2. Knowing is an adaptive activity that organises the subject's experiential world; it does not depict or represent an independent reality (Von Glasersfeld, 1991, 1995a).

From the first principle it follows that learners necessarily construct their own mathematical reality. Steffe (1983) stresses though that it should not be assumed that children who appear to use the same mathematical method to resolve a situation, have the same underlying knowledge. Extending Skemp's distinction between instrumental and relational understanding and using the Piagetian terms, Steffe argues that children's methods may be "instrumental" or "operative" (p. 110).

Operative is the term that Piaget used to characterise mathematical knowledge; that is knowledge that involves mental operations such as relating, coordinating and abstracting, as opposed to *figurative* knowledge which organises sensory-motor experience and thus involves elements of sensation, or motor action, or representations of such elements (Von Glasersfeld, 1995b). Operative knowledge is not associative retrieval of a particular answer but rather knowledge of what to do in order to produce an answer. The notion accounts for the ability to carry out and monitor certain activities, but most importantly it goes beyond successful performance of activities and describes a form of knowledge that presupposes student's understanding of the reasons that underlie successful performance. As such, operative knowledge is constructed by the individual's "reflection"; i.e. the ability of the mind to observe its own operations (definition given by Locke and cited in Von Glasersfeld, 1987a, p. 11), and its constructive nature is better demonstrated when the subject is facing new situations (Von Glasersfeld, 1987a).

The term *understanding* as used here, refers to conceptual understanding, that is understanding of the conceptual relationships between mathematical ideas. In this sense, and for this reason, conceptual understanding is considered as a fundamental requirement for successful performance in novel problem-situations, i.e. other than those which constitute objects of explicit instruction. Therefore, it is this form of understanding that is considered as being "the only learning that is ultimately worthwhile" (Von Glasersfeld, 1995b, p.382).

Conceptual understanding is tightly connected with the construction of operative knowledge and is, therefore, fostered by students' reflective activity (Von Glasersfeld, 1995a; Pace, 1987). It is argued that conceptual understanding grows by constructing relationships between different pieces of information, or between existing knowledge and new information (Hiebert & Lefevre, 1986). In this sense, the notion of *conceptual understanding*, as this is viewed in the constructivist framework, is similar to the idea of *relational understanding* as this was defined and described by Skemp, who also stressed the importance of students' reflective activity as means by which this type of understanding and knowledge is evolved. In the constructivist view, reflection and building of understanding are viewed as activities that learners have to carry out by themselves and which are guided by the mechanism of *reflective abstraction.* Following the Piagetian tradition, reflective abstraction for the constructivists is strongly connected to action. For Piaget (1970), knowing an object means acting on it and transforming it: "Knowledge then, is a system of transformations that become progressively adequate" (p. 15).

In this sense, new knowledge is constructed through the changes and transformations that the subject's action (i.e. goal-oriented activity) bears on the relation between subject and object. Effective and successful actions are abstracted. Piaget argued for different levels of abstraction: *empirical abstraction* is derived from the subject's reflection on the properties of objects whereas *reflective abstraction* proceeds from the subject's actions and operations on the objects, and it is the mechanism that drives the construction of *logico-mathematical knowledge* (Piaget, 1980). In this process, subjects construct better knowledge (though never perfect) of the object, but also better knowledge of their own actions and thought processes.

Recurrent restructuring of an individual's actions or thought operation system drives the movements towards ever more viable knowledge (Sinclair, 1987). Piaget's *scheme theory* accounts for this recursive building of viable knowledge by identifying conceptual structures (*schemes*) that guide the cognising subject's actions and assimilate new experiences. The guidance of behaviour by action-schemes is called "regulation" and it is defined as a process of feeding back the results of a behavioural act to the behavioural scheme that controls it

(Damerow, 1996). *Assimilation* is the process in the course of which activation of an existent scheme allows anticipation and recognition of a situation. In such a case, an existent scheme "fits" the subject's experienced situation, and thus it is retained. In other cases, a scheme may need to change and evolve in response to a new experience. This is the process of *accommodation*; the schema needs to adapt before assimilating the new situation. Assimilation and accommodation are two complementary processes that drive the subject's ever ending pursue of *equilibration* between itself and its experienced world (Marshall, 1995).

In Piaget's scheme theory, reflective abstraction is the instrument of accommodation and it is triggered by *perturbation* that is a failure in the mechanism of assimilation. Accommodation, that is modification of a conceptual structure in response to a perturbation is considered as necessary for cognitive development to occur. (Confrey, 1994a; Steffe & Wiegel, 1996). Perturbation caused by interaction with the environment is differentiated from perturbation that is caused by the subject itself. Von Glasersfeld (1995b), argues that this differentiation was acknowledged in the first place by Piaget, but it has been overlooked by his critics. Perturbation caused by an unexpected relation to, or interaction with, the experienced environment is referred to as "maladaptation". On the other hand, perturbation may be caused by problems that are triggered by the subject itself and are not imposed by someone else. It is this kind of perturbation that triggers reflective abstraction because "...unless a problem is seen and felt to be a problem by the student, it is unlikely to trigger reflective abstraction" (ibid, p.378). Perturbation caused by the divergent responses or actions of the participants in a specific situation has been viewed as particularly important in cognitive development and is referred to, under the term of socio-cognitive conflict. (Doise & Mugny, 1984). Cognitive conflict is considered by Steffe and Wiegel (1996), not as a synonym of perturbation but only as one type of perturbation that may occur. Hence, they define perturbation as referring to "...any disturbance in the components of an interactive system created through the functioning of the system. A perturbing element can activate or disequilibrate a system at rest or a system in a dynamic equilibrium" (ibid, p. 491). In this sense, perturbation can include cognitive conflict but it is not identical to it. Perturbation and action to resolve the perturbation are internalised through the process of reflective abstraction. The repeated sequence of perturbation-action-reflective abstraction until the action is stabilised, results in the construction of the structure that constitutes a scheme (Confrey, 1995). Radical constructivism views mathematical concepts and operations as products of such a sequence of reflective abstractions. Von Glasersfeld (1987a) argues that the value of schemes lies in their experiential adequacy, and their viability as means for solving problems. Among these, is the never ending problem of consistent organisation of knowledge, which he calls understanding. The need for a construction of knowledge which is structured, organised, and rich in relationships, and which also provides cognitive anchor points so that new concepts can be integrated by maintaining, at the same time, the cognitive continuity and relevance, leads to the idea of "relational" (for Skemp) or "conceptual" (for the constructivists) understanding.

Skemp's model of the construction of *relational schemes* is only one of the several models that have been developed to describe various aspects of the notion. Herscovics and Bergeron (Herscovics et al., 1987; Bergeron et al., 1987) suggested a constructivist model to account for the construction of *conceptual schemes.* After several refinements of the initial model, Herscovics (1996) suggests a "two tiered model" for the analysis of conceptual schemes in elementary mathematics. Any fundamental mathematical notion (e.g. number, arithmetic operations, fractions) can be viewed as a different conceptual scheme, which is defined as a network of related and organised in a hierarchical way knowledge, together with all the problem situations in which it can be used (p. 352).

Another model of the growth of mathematical understanding, which is constructivist in its roots, is the model developed by Pirie and Kieren (1989). This model was developed in reaction to Sierpinska's (1994) question of whether understanding is an act, a process, or a way of knowing. Pirie and Kieren (1992) view the notion of understanding in mathematics as a dynamic, organic whole which entails phases of growth as well as phases of retrogression. The model describes eight potential levels, or modes, of personal growth of understanding with regard to a particular topic[2.5] . Pirie and Kieren (1989, 1992) give the following schematic representation of the model:

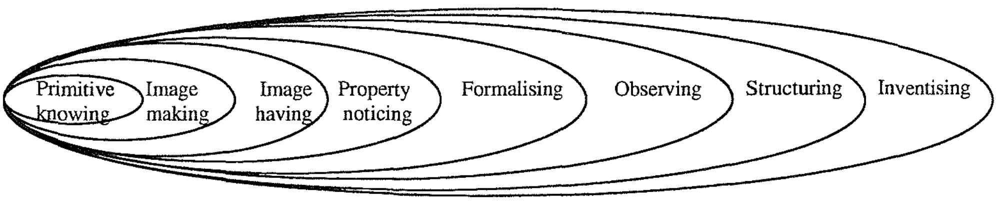


Figure 2.1.
The levels of personal growth of understanding as described by Pirie and Kieren (1989, 1992).

The important feature of the model is that each of these levels has embedded in it all the other inner levels and is itself embedded in all the outer levels. Pirie and Kieren (1992) note:

"We see growth as represented by a particular form of back-and-forth movement among levels and it is thus that we characterise understanding as a dynamic organising process. The most critical feature of our theory then is that of **folding back.** When faced with a question or circumstance at any level of

[2.5] Detailed accounts of the levels in the model can be found in Pirie and Kieren (1989).

understanding activity, which is not immediately resolvable, we argue that one can *fold back* to any inner level of understanding activity in order to extend one's current, inadequate understanding." (Pirie & Kieren, 1992, p. 248).

This suggests that the process of extending one's understanding is not seen simply as a process of generalisation of one's activity at a given level, nor as a process of reflectively abstracting one's understanding to an outer level. Rather, extending one's understanding is seen as a process of folding back to inner level knowledge and recursively *reconstruct* it in order to further extend outer level understanding (Pirie & Kieren, 1992). In this sense, and as following work by Kieren and Pirie (1994) and Lyndon and Pirie (1998) demonstrates, mathematical understanding is always under construction.

The aforementioned model is fundamentally constructivist in the sense that it suggests that the growth of mathematical understanding is a process of personal building, re-organisation and re-structuring of one's knowledge in situations of difficulty, or insufficient previous understanding. Pirie and Kieren (1992) emphasise that this process of re-organisation can be prompted by the appropriate *teaching intervention* which should *validate* students' current understanding, *provoke* its extension to new cases, and *invoke* obstacles to students' understanding in order to prompt folding back to inner levels.

On similar grounds, radical constructivism conceives the generation and growth of understanding as the goal of *teaching*. In this sense, teaching practices that aim at fostering understanding should always be directed by the following principles:

1. "There is no understanding without reflection, and reflection is an activity students have to carry out themselves."
2. "Although reflective abstraction always begins on the basis of some sort of sensory motor experience and action, it is not caused by it. Therefore, no programme of specific action and manipulation of concrete materials could guarantee students' abstracting. An activity, in the case of one student, may trigger reflective abstraction but may not do the same for another student."

(Von Glasersfeld, 1995b, p. 382).

However, Von Glasersfeld (1991) reckons that leading students to discuss, for example, their view and approach to a problem solving situation, is considered as the teaching practice that mainly provides opportunities for students to reflect on their activity and develop different and even more viable conceptual structures. Especially in the framework of *social constructivism*, the theoretical position which is built on the Vygotskian theory of mind and which particularly stresses meaningful linguistic interaction and its account of knowing (Ernest, 1996), extensive research has focused on the mathematics class as a unit of analysis, and has demonstrated the importance of class discussion, in which mathematical activity, problem solving in particular, becomes an explicit topic of conversation. Classroom discourse provides the students with the opportunity to argue, explain and justify their methods in an attempt to resolve the disagreements and conflicts that emerge. This type of discourse, the "reflective discourse", has been shown to be particularly effective in promoting students' reflection and thus fostering conceptual change and deeper understanding (Bauersfeld, 1988, 1995; Wheatley, 1992; Cobb et al., 1991; Cobb et al., 1997; Wood, 1999; Yackel et al., 1991; Yackel et al., 1996).

In particular, Wheatley (1992) emphasises the need for mathematics learning which focuses on problem solving activities, and for instruction which has reflection as a primary component. It is argued that the learning environment should encourage students to devise their own methods, and take their own activity as an object of thought and discussion. In this way, initial primitive methods may become a valuable basis for the construction of other mathematical relationships. Wheatley (1992) notes:

"It is not enough for students to complete tasks; we must encourage students to reflect on their activity. For example, being asked to justify a method of solution will often promote reflection.... Finally carefully selected tasks can cause perturbation which results in reflection" (p. 535).

Similarly, Yackel et al. (1991, 1996) argue for classroom cultures in the context of which '*sociomathematical norms*' are established. 'Sociomathematical norms' are normative aspects of mathematical discussions that are specific to students' mathematical activity. In these settings, additional learning opportunities arise because students have the opportunity to discuss, explain, and justify solutions they have given to a problem, in comparison to solutions given by their peers. Students' solutions as well as explanations become the object of reflection. It is stressed that such a reflective activity has the potential to contribute to children's mathematical learning significantly, because it contributes to the development of *deeper understanding of a given solution* as well as of what constitutes explanation.

As Cobb et al. (1997) note, research in the framework of social constructivism focuses on the communal activity of *collective* reflection as this occurs in classroom or small-group settings that encourage reflective discourse and argumentation. It is stressed that although reflective discourse supports individual reflection on, and *reorganisation* of, prior activity, it does not cause it, determine it, or generate it. Cobb et al. (1997) discuss the relationship between classroom discourse and mathematical development, and advance the view that it is the individual child who has to do the reflecting and reorganising while participating in, and contributing to, the development of the discourse. It is posited that children's development cannot be accounted for directly in terms of their participation in reflective discourse. Rather, research which focuses on children's reflective activity in classroom settings needs to be accompanied by an analytical approach "...that is fine-grained enough to account for qualitative differences in individual children's thinking even as they participate in the same collective activities" (p. 272). Thereafter, it is argued that an indirect linkage between social and psychological processes is needed so that research can account for differences in individual children's mathematical activity.

The aforementioned positions study conceptual change and understanding in relation to the psychological process of reflection on one's action as advanced in the wider framework of constructivism. The need for reflection and elaboration *after* a solution has been given is stressed. However, this issue is mainly studied in the context of classroom discourse. In this setting, *conceptual change*, *understanding*, and *reorganisation* of one's activity are mainly viewed as the product of a process of resolving conflicts, and recognising conceptual differences between solutions given by different students to the same problem. Even though this framework of research focuses on classroom settings, the need for analytical approaches that address the issue of individual mathematical development, even within such collective settings, is acknowledged. This study aims at exploring the issue of *conceptual understanding, reorganisation, and after success elaboration* of children's mathematical activity on a rather psychological level. In the following section, theoretical views that focus on the psychological aspects of conceptual understanding are presented.

2.3.3. Understanding and the development of mental models

Within the framework of the theory of mental models, understanding has a central role and is considered as one of the most essential acquisitions of children. This is because autonomous cognitive behaviour, ability to acquire new knowledge and deal with new situations are conceived as highly connected to the notion of understanding (Johnson-Laird, 1983; Halford, 1993). In this sense, understanding is viewed as a prerequisite for the construction of mental models which strongly influence and guide the development of problem solving strategies and cognitive skills. Even though it is acknowledged that understanding does not underlie all skilled performance, it is argued that strategies and skills that are developed in the basis of mental models are notably powerful because of attributes such as their flexibility and generality which makes them adaptable to new situations, and transferable from one context to another (English & Halford 1995).

Halford (1993)[2.6], discusses the notion of understanding as referring either to a concept, or a situation. Thus, understanding of a concept is defined as "having an internal, cognitive representation or mental model that reflects the structure of that concept. The representation defines the workspace for problem solving and decision making with respect to the concept." (Halford, 1993, p. 7). On the other hand, understanding a problem-situation is defined as having a mental model of the task. In this case, the notion of understanding entails the presence of an internal representation that matches the structure of the task (Halford, 1982). Representations and mental models provide a basis for understanding in so far they have a degree of *"generality"* which allows them to be transferable to new situations, and *"generativity"* which provides a basis for predictions and inferences to be made. This is why understanding based on mental models is conceived as a guide for the development of problem solving strategies and skills and consequently, the ability to develop appropriate strategies in the context of a task is considered as a criterion of understanding. Finally, it is argued that understanding enables the organisation of knowledge because on its basis, consistent relations between representations can be identified (Halford, 1993, p. 8).

From this particular perspective, understanding is viewed as strongly related to the notion of *mental representation*. Internal representations of knowledge is a central notion in the context of this study as it constitutes a critical explanatory tool for the model that provides the theoretical framework. This notion needs to be further explored also because all of the theoretical views that were presented in the section devoted to conceptual understanding, approach the issue by referring to a certain form of internal structure of knowledge organisation. The approaches presented employ different terms (schemes, schemas, mental models) to refer to structures which organise one's knowledge. Irrespectively of the term used the underlying idea is that conceptual understanding is built on the basis of internal, mental representations which organise one's knowledge base.

Internal knowledge representations constitute a highly controversial notion. The following two subsections open a parenthesis in which the main aspects of the notion of *representation* which are addressed in the field of cognitive developmental psychology, and in the field of mathematics education and psychology, are briefly presented.

[2.6] Halford (1993), presents a detailed description of the basic processes entailed in understanding and their role in cognitive development. This section is informed mainly by Halford's work because his account of understanding as this is viewed in the theory of mental models is particularly (even though not entirely) related to the mathematical context.

A. Knowledge representations: brief presentation of the main aspects of the notion that the cognitive developmental science addresses

The term *representation* has been used in two senses in the field of cognitive developmental psychology. One sense refers to representation as the use of symbols, i.e. words, or other artifacts that people use to represent and refer to some aspect of the world or some aspect of their knowledge of the world. This sense, refers to external systems of representation in contrast to *internal* systems, i.e. *mental* representations. In this second sense, the term representation refers to knowledge and the way it is internally, mentally organised (Mandler, 1983; Pratt & Garton, 1993). It should be emphasised that in this study the term 'representation' is used in the sense of knowledge and the way it is *internally* stored. This is why aspects of this sense of the term will be mainly discussed in this, as well as the following, subsection.

Internal representations of knowledge refer both to what is known, i.e. whether a certain piece of information is represented, and also how this knowledge is internally structured, i.e. how it is represented (Mandler, 1983; Flavell, 1985). Although the notion of mental representation is accepted as a crucial feature of cognitive functioning, it also constitutes a topic of an extended debate. Researchers in the field of cognitive developmental psychology argue regarding the value of mental representations in cognitive and developmental science as well as regarding certain postulated properties of the notion (Müller, Socol & Overton, 1998; Markman & Dietrich, 2000). Various different aspects of mental representations have been studied, and several distinctions have been made regarding the nature of mental representations. Except from the distinction between *external* and *internal* systems of representation, in the literature, distinctions have been made regarding the extent to which representations are related to the objects or events they represent. Perner (1991) refers to the relationship between an object, (or concept, or event) and its representations as the "*representing relationship*" (p. 15-16). Differences concerning the representing relationship have been addressed in terms of the arbitrariness and abstractness of this relationship. Piaget (1976) posits that during the sensorimotor stage, knowledge consists only of perceptions and actions. He does not consider this knowledge to be 'mental' representation because of the absence of a symbolic representational system. For Piaget conceptual thought is inextricably connected to the development of symbolic representations and therefore he does not ascribe conceptual knowledge to children at this stage (younger than 1,5 years). At the end of the sensorimotor stage Piaget (1966) posits a fundamental change: the first preconcepts develop and are represented in the form of images. Images are considered to be the first mental symbols. It is posited that their acquisition characterises preoperational thought (2-7 years). Imaginal representations are considered to be the major form of representations in early childhood that take children one step away from the concrete and the physical to the realm of mental imagery. Bruner (1967, 1973) has formulated a similar hypothesis. Representations are considered to be the means by which one conserves past experience and translates this experience into a model of the world. Bruner has suggested three ways in which this can happen: through actions, images and symbols.

"A representation of the world or of some segment of one's experience has several interesting features. For one thing it is in some medium. We may represent some events by the actions they require, by some form of picture, or in words or other symbols" (Bruner, 1973, p. 316).

Therefore, Bruner talks about three types of mental representations which are considered to grow in this specific sequence: the enactive, the iconic and the symbolic. It is posited that enactive representations is a mode of representing past experience through the appropriate motor response. Enactive representations are based "...upon the learning of responses and forms of habituation" (Bruner, 1967, p. 11). Iconic representations are related to mental imagery. They are governed by visual or other sensory organisation. Finally, symbolic representations translate experience into language. Symbols (words) are arbitrary in nature in the sense that there is no analogy between the symbol and the object it represents.

Several aspects of Piaget's and Bruner's theoretical formulation on imagery have been under attack (e.g. Fodor, 1975; Kosslyn, 1978). Recent theories of imagery distinguish between two types of mental representations: the *picture-like* and the *language-like*. Picture-like representations are considered to be analogue and iconic, and have referentially isomorphic properties. They include pictures, diagrams or drawings. Language-like representations, on the other hand, are considered to be non-analogue, non-iconic, referentially arbitrary and propositional. They include natural human languages and formal systems as mathematical symbols (Johnson-Laird, 1983; Paivio, 1986; Miller, 1993). The existence and properties of different modes of representation constitute an issue of strong arguments among the different sides of mental imagery theory.

Probably the most generally agreed distinction regarding aspects of representation is the distinction between *declarative* and *procedural* aspects of knowledge. Declarative knowledge refers to *knowing that*, i.e. to knowledge of facts. For example, knowledge of concepts or events. Procedural knowledge refers to *knowing how*, i.e. knowledge of how to do things. For example, drive a car or add two numbers (Stevenson, 1993; Pratt & Garton, 1993; Meadows, 1996). *Accessibility* is the main factor that defines the distinction between procedural and declarative knowledge representations. It is commonly argued that procedural knowledge is *implicit*. This entails that a piece of information embedded within a given procedure, may not be easily accessed and used when relevant to another situation. Implicit knowledge is considered to be verbally inaccessible knowledge. On the other hand, declarative knowledge is most often equated with conceptual knowledge and it is considered to be *explicit*. That is declarative knowledge is considered to be accessible to awareness (Dorfman, et al., 1996; Underwood & Bright 1996; Meadows, 1996). Mandler (1983, 1988) argues that it is not clear whether conceptual knowledge must be considered as declarative in nature, and procedural knowledge as being generally inaccessible to awareness. In her own words:

"Most schemata governing our understanding of various concepts seem to be a mixture of procedural and declarative knowledge. Procedures, of course, do not involve only the control of actions; they can consist of rules telling us what to do next in a situation or how to go about finding out something." (Mandler, 1983, p. 425).

The author recognises that it is fair to equate, at least partially, between conceptual and declarative (i.e. explicit) knowledge, because conceptual knowledge has the potential of being brought to conscious awareness. She emphasises, however, that it should be acknowledged that concepts, even if they are formed by conscious processes, are not always accessible: i.e. they are not always in conscious awareness (Mandler, 1988). Without confining the notion of procedural and declarative knowledge representations to an exclusively implicit or explicit nature, correspondingly, Mandler (1983) discusses the possibility for one kind of knowledge to turn into the other:

"... after following a procedure to find an answer to a question, the answer may be separately stored so that it is not necessary in the future to carry out the procedure to access it. Running through a routine to locate or generate a piece of information may result in that information forming part of a declarative knowledge system".
(Mandler 1983, p. 424).

The issue of whether procedural knowledge can become explicit and be represented in a declarative form is the subject of considerable debate. Underwood and Bright (1996) distinguish between two opposite sides in cognitive developmental science: Those who argue that knowledge acquired is explicit and has initially a declarative form, and those who argue that knowledge acquisition may proceed from procedural to declarative.

The distinction between procedural and declarative knowledge and the accessibility of each of these two types of knowledge representations is of central importance in the field of mathematics education and psychology as well. This issue in connection with the notion of conceptual understanding is discussed in the following section.

B. Knowledge representations: main aspects of the notion that are discussed in the field of mathematics education and psychology

As with *understanding*, the term *representation* has been widely used in various different contexts, in mathematics education, and various definitions have been produced to account for the meaning of the term. The engagement in mathematical activities entails moving across both internal and external systems of knowledge representations (Janvier 1987b). Representations as *external* notational systems, i.e. observable configurations such as words, symbols or graphs (Goldin & Kaput, 1996), have been studied at length (e.g. Dufour-Janvier, et al., 1987; Janvier, 1987a; Kaput, 1987; Kaput, 1991; Meira, 1992).

The notion of representations as *internal* systems of knowledge organisation has also been widely discussed and various views have been expressed regarding the nature of the notion. Some of these views are selectively presented herein. Within the theory of mental models the term *representation* refers to a cognitive representation that encompasses an *internal* representing structure which must be in high correspondence to the structure of a segment of the environment. A representation entails mapping the internal representing elements or relations to the represented elements, functions or transformations in the environment and may consist of propositions or images. *Mental models* on the other hand, are not viewed as simply one kind of representation. Rather, they are defined as constructs that can have different types of representations as components, and which "are active while solving a particular problem and provide the workspace for inference and mental operations" (Halford, 1993, p. 25).

Goldin (1987) and Goldin and Kaput (1996), use the term internal representation in a way which initially seems similar to Halford's view. Internal representations are thus viewed as referring to *possible* (highlighted here) mental configurations of learners or problem solvers. However, the authors underline that they use the term of internal representation as a theoretical construct devised by the observer on the basis of an observed behaviour that may be verbal, or mathematical (hence the use of the word *possible*). The term, as they use it, does not refer to a direct object of mental activity and thus it is intended to be clearly distinguished from positions that use the term to make "ontological assumptions about the 'mind' " (ibid., p. 400). It is considered that learning depends on the inferred constructs that Goldin (1992) names "cognitive representational systems" and conceptual understanding always (highlighted in the original) does not involve only one but many types of representational systems (p. 254).

In this sense, this particular approach does not seem so close to Halford's view according to which understanding necessarily entails representations that must and *do* (highlighted here) reflect the individual's knowledge and experience. Representations, can be retrieved from memory, and must be able to constrain actions, strategies, and procedures used in problem solving (Halford. 1993, p. 24).

Davis and Maher (1990), who favour the cognitive science approach to mathematics education, created a model to account for the way in which problem solvers think when engaged in a mathematical situation. In their model, *mental representation* constitutes the central notion. The model describes the following steps through which one must "cycle" repeatedly before attaining the result of a problem:

1. Build a mental representation for the input data; i.e. a representation of the situation.
2. Search in memory and retrieve or construct a representation of relevant knowledge.
3. Construct mappings between the *data* representation and the *knowledge* representation.
4. Check these mappings and constructions for their correctness and if they are satisfactory,
5. Use technical devices and other information associated with the knowledge representation to solve the problem.

(Davis & Maher, 1990, p. 65).

For Davis (1992, 1996), one cannot think about a problem without having some mental representation of it, and cannot evoke and use a piece of knowledge without some representation of this knowledge. The representation of a problem situation sometimes, though rarely, can be retrieved from memory. However, engagement in novel problem situations entails a gradual building of the problem representation which once built, enables the retrieval and construction of a representation of the relevant knowledge required to solve the problem. Mental representations can be analogies with something already familiar and are conceived as 'tools that we can think with' and can be drawn from the experience with the environment; this includes experience on symbols as well as experience on concrete materials and objects which materialise the meaning of symbols (Davis, 1992; 1997).

Children's understanding in arithmetical word problems, by way of constructing mental representations, was modelled by Greeno (1980) who based his analysis on the distinction between two aspects of understanding arithmetic: the linguistic and the conceptual. *Linguistic understanding* refers to the semantic aspect of language in word problems and allows the construction of procedural and declarative representations of arithmetical expressions. Procedural representations are used in computations that are required to answer, for example, addition or subtraction questions, whereas declarative or relational representations allow the understanding of the existent relations between the components in an addition or subtraction sentence. *Conceptual understanding*, on the other hand, refers to the connection between the relations expressed in formal mathematical language and the relations expressed verbally in the text of the problem. This is a type of understanding that allows the selection of the appropriate operation for solving a word problem.

Extending Greeno's work, Riley, et al. (1983) developed a computer-based model that accounts for the way in which children's representation of the situation described in additive word problems influence their performance. The authors argued against an all-or-none view of children's understanding and strongly emphasised the relationship between the conceptual knowledge that is required to understand the problem and the procedural knowledge which allows carrying out problem-solving procedures and strategies. Thus, understanding was defined as "a process of representing problem information or solution components in coherent relational networks constructed on the basis of general conceptual knowledge" (p. 156). On this basis, differences in performance between younger and older children on the same problem were attributed to differences in children's representation of the problem, that is to differences related to the conceptual understanding that is required for the situation described in word problems to be related to the appropriate problem solving strategy.

Briars and Larkin (1984) also presented a computer-implemented model to account for children's ability to solve addition and subtraction word problems. In opposition to Riley et al., their model was not based on the assumption that children construct specific schemata based on their understanding of the semantic relationships in the problems. Rather, Briars and Larkin based the construction of their model on the assumption that children naturally understand and build rich representations of problems by acting upon them.

The construction of a model, or representation of a problem situation was also addressed by Carpenter et al. (1988, 1993) who argued that *modelling*, as a way of understanding the action or relationships described in arithmetical word problems, is one of the most fundamental children's problem solving processes.

The term *representation* lies at the core of each of the above mentioned models of problem solving behaviour in mathematics. However, this widely discussed and employed term is used in each model from a slightly different perspective. For Goldin, and Kaput representations are clearly viewed as constructs created by the researchers who wish to explore mathematical thinking and infer the individual's mental configurations and processes involved in mathematical problem solving. Representations in this sense are considered as conventional constructs not as ontological, existent elements of thought. On the other hand, Davis and Maher base their model on the constructivist assumption that new knowledge is necessarily constructed from the individual's previous experience and from the old, already acquired knowledge. Thus, the building of problem representation and knowledge representation is presented as a process that depends highly on experience with the environment. In this sense, building of problem representation, use of relevant knowledge, and in consequence problem solving performance, are assumed to be enhanced by the manipulation of concrete objects and engagement in familiar and relevant situations that can even belong in a context other than the mathematical.

Finally, Greeno's model and its further elaboration by Riley, et al. seems to be underlain by the same as Davis and Maher's idea that one needs a representation of the problem situation ('problem schema', for Greeno) and a separate representation of relevant knowledge required to develop a solution for the problem ('action schema' in Greeno's view). However, Greeno and Riley, et al., do not connect the idea of problem and knowledge representation to the individual's experience in a context other than the mathematical. Rather, when talking about word problem representations, they refer to representations which are constructed on the basis of the information given in the problem text. In this sense, problem representations reflect the relations between quantities and concepts involved in a word problem as these are identified by the problem solver. It is on the basis of such a process of representing relations involved in the problem that understanding is built, and relevant knowledge is accessed, so that the appropriate mathematical operation can be subsequently chosen and applied.

Greeno's ideas on representations and their value in the building of conceptual understanding that enhances problem solving behaviour in arithmetical word problems seems to be close to Halford's view of understanding as a process of constructing cognitive representations that correspond to relations in the environment; in the case of mathematics 'concepts', 'symbols' and 'mathematical problems' are part of the environment. As mentioned before, in the case of a problem, understanding means having a representation that is "structurally isomorphic" to the problem, and in the case of a concept, understanding means having a representation structurally isomorphic to the concept. Moreover, a dynamic rather than a static character is attributed to representations. The recoding and reorganisation of the information that representations encode is considered as an important component of understanding (Halford, 1993).

Halford emphasises that the idea of structural correspondence between representations and the aspects of the environment that they represent, does not imply in any way that representations are 'pictures in the head' which constitute mental copies of the world. Rather, cognitive representations as a theoretical concept, imply that there exist cognitive processes that can be mapped with consistency and correspondence into aspects of the environment. Consistency and correspondence as properties of cognitive representations have a relative rather than absolute character. It is stressed though that even if discrepancies between representations and the environment can be acknowledged, "...it is essential that most of our stored representations of the environment represent it validly, and it would be dangerously maladaptive for it to be otherwise. ...in general, our cognitive processes must, and do, represent the environment correctly" (Halford, 1993, p. 26-27).

The representational view of mind is strongly criticised by radical constructivists who argue that a conception of knowledge that assumes a "match" between the individual's cognitive structures and the object of knowledge that these structures are supposed to represent, is problematic (Von Glasersfeld, 1987b, 1990). The position against the representational view of mind is grounded on the second of the two fundamental epistemological premises of constructivism according to which all knowledge, including mathematical, is actively constructed by the subject's cognizing activity which is considered to be instrumental and adaptive leading to the *subjective* construction of the knower's experiential world. On this basis, the existence of an objective reality is not denied, however the presumed subjective nature of any act of knowing renders any knowledge of the independent objective reality impossible. That is why the theory of knowledge as this is built by radical constructivists accounts for the *viability* and *fit* of knowledge to experience and not for the *match* between knowledge and reality (Kilpatrick, 1987). Thus, the use of the term *representation* and its instructional implications are rejected in so far the term is used as a conceptual structure that is assumed to be 'isomorphic' with a part of the objective 'real' world (Cobb et al., 1992). However, two possible meanings of the term are accepted by radical constructivists and these are the following: first the Piagetian use of the term which refers to the "re-presentation" of an experience from memory, and second the use of the term which refers to "graphic or symbolic structures" on the basis of which mental operations can be carried out (von Glasersfeld, 1996, p. 308). It must be noted that, the acceptance of the second principle which entails the absolute rejection of any realism is not widely accepted by all of those who consider themselves as constructivists in the sense that they accept the first principle which accounts for the constructive nature of knowledge. Among these, are Davis and Maher whose model was previously described.

For those in the broader field of mathematics education and psychology who accept the notion of representation as explanatory tool, imagery, and the distinction between procedural and declarative knowledge representations are two of the main issues of interest. The role of imagery in mathematical reasoning, i.e. the symbolic, iconic, analogue, or propositional nature of students' mental representations, has been addressed by researchers like for example Dehaene and Cohen (1994), Tall (1995), Gray and Pitta (1996), Thompson (1996). The aforementioned aspect of the nature of mental representations does not constitute an issue of study in this research project. Rather, the focus is on the distinction between procedural and declarative knowledge representations, and their interplay in the development of children's strategies.

According to Halford (1993), mental representations depend on domain knowledge, even though they are not always drawn from the same domain as the problem to which they are applied. The domain knowledge can be *declarative* (knowing that) or *procedural* (knowing how). Declarative and procedural knowledge representations can be implicit or explicit. *Implicit representations* are considered to be unconscious, not accessible to strategic cognitive operations, and not modifiable without external input. Implicit knowledge is usually identified with ability to perform a task without being able to explain the performance or modify it. For English and Halford (1995) a child who can add three digit numbers without having knowledge of place value would be an example of implicit knowledge. The child has the skill to perform the task but he/she cannot modify it or relate it to other mathematical performance. *Explicit representations* on the other hand, are accessible and modifiable without external input. In this case, modification can take place by "the operation of strategic cognitive processes (e.g. changing hypotheses)" (Halford, 1993, p. 240). Explicit representations are considered to be possibly accessible to consciousness. Explicit knowledge can be explained, can be related to other cognitive processes, and can be modified without additional experience. Explicit knowledge is considered to be more under the control of the performer. The difference between explicit and implicit knowledge is depicted as follows:

"The difference between implicit and explicit knowledge corresponds to the difference between being subject to rules and being able to make rules. Implicit knowledge is sufficient for performance that is consistent with rules, even though the person might not be aware of the rules per se. An example is knowledge of grammar in one's native language. Another way of expressing the difference is that implicit knowledge is ability to perform the task, but explicit knowledge is required for autonomous modification of one's own performance" (English & Halford, 1995, p. 24).

For Halford (1993), maybe the most important criterion is that explicit representations can be operated on by strategic cognitive processes. Thus, they can be related to other representations, can be organised into systems, and can be used to guide the development of strategies and skills. There are no clear data to make conscious accessibility of representations a definite criterion of explicit knowledge: there is probably some knowledge that can be influenced by cognitive processes but that is not conscious. Also the verbal representation is another non defining attribute of explicitness. The author explains why:

"Especially when dealing with young children, it is not clear that they could give a verbal account of all knowledge that was explicit in the sense that it was cognitive accessible. The three criteria, cognitively accessible, consciously accessible, and verbally stateable, are related but it is not clear at the present time that the relationship is close enough for all three to be defining criteria of explicit knowledge." (Halford, 1993, p. 240).

With this argument, Halford leaves open the issue of cognitive, and verbal accessibility of all explicit knowledge. This issue is discussed, in more detail, in the following chapter. It constitutes the central issue of interest that the model which supports this study theoretically, addresses, and attempts to provide answers for.

2.4. Summary

This chapter was organised in two main sections. In the first section, the literature reviewed addressed issues regarding young children's strategies in additive tasks. It was shown that a definite and clear definition regarding what can, or cannot, be considered as a strategy is difficult to establish. In the field of mathematics education and psychology, researchers employ the term in a rather loose way, and tend to define it in the context of their particular studies. This is a choice made in the context of this study as well. An outline of what is considered as a strategy has been already given. A more detailed definition of the term, as used in this study, is given in Chapter 6, after the consideration of theoretical issues that stem from the presentation of the model that provides the theoretical framework for this project.

Another issue that has been addressed is the issue of strategy variability and strategy discovery. It was shown that young children use a variety of diverse strategies to solve additive problems.

Strategy variability has been evident not only between children of different ages, but also between children of the same age, and even within individual children when solving the same type of problem within trials close in time. Children's strategies develop. This development does not only involve changes in the choice and mixture of additive strategies that children use, but also the construction of new strategies. The discovery and construction of new strategies by individual children, within a specific number of trials on a certain type of problems, as well as the generalisation of new strategies in new problems of the same type, has been studied in a series of research projects carried out by Siegler and his colleagues. Their microgenetic exploration (Siegler & Jenkins, 1989), in particular, brought to light very interesting aspects of the process of strategy discovery and change in additive tasks. Siegler and Jenkins' study revealed behavioural precursors of strategy discovery, and transition strategies after which a new approach is discovered. Siegler and his colleagues' series of studies also addressed issues related to the conceptual basis upon discovery took place. Siegler and Shrager, (1984) showed that children make adaptive choices among alternative strategies. It was argued, though, that these choices were relative to problem characteristics and difficulty, but were not always accompanied by explicit knowledge of these aspects of the problem. In the Siegler and Jenkins (1989) study on strategy discovery, it was again indicated that children were not always aware of their discovery, and they varied greatly in the degree of insight that accompanied their new strategy. The issue of conscious or unconscious strategy discovery in additive problems was also addressed by Siegler and Stern (1998). In that study the analysis focused on whether children could report verbally the *use* of the newly discovered strategy.

In the second section of this chapter, the review of literature related to conceptual understanding revealed that all the theoretical approaches which address this issue, despite their contrast as to what they view as a basis for the development of understanding, reach a point of agreement which is the acknowledgement that conceptual understanding is one of the most important acquisitions. The main reason of its importance lies in the fact that understanding enables the development of problem solving strategies that can be flexible, modifiable, adaptable, and generalisable to new situations. Moreover, conceptual understanding makes possible the connection of a successful problem solving solution with the reasons of this effectiveness. The need for reflection, elaboration and further work on one's solution has been addressed by research on classroom and collective settings (see section 2.3.2.). In these settings elaboration of one's solution to a problem, and the development of conceptual understanding have been seen as the product of the process of resolving conflicts, and justifying a solution in the context of classroom discourse. This study aims at exploring the issue of elaboration of one's solution after success, and subsequent possible conceptual changes on a rather psychological level.

Most of the theoretical positions that are presented and address the psychological aspects of conceptual understanding, recognise that conceptual understanding is built on the basis of the individual's internal, mental representations of knowledge. The distinction that is of importance in this study is the differentiation between *procedural*, *declarative* and *conceptual* aspects of knowledge representations. It was shown that, in the field of cognitive developmental psychology, declarative knowledge is most often equated with conceptual knowledge which is generally considered as being explicit, i.e. accessible to awareness. Procedural knowledge representations, on the other hand, are commonly considered as being implicit, i.e. non-accessible to awareness. However, Mandler (1983) argues against these rough equations, and leaves open the issue of possible existence of accessible as well as inaccessible (i.e. explicit as well as implicit) procedural and declarative knowledge representations. The issue of accessibility of knowledge, either procedural or declarative, is also an open issue in the field of mathematics education and psychology (Halford, 1993; English & Halford, 1995). Procedural and declarative knowledge representations are not seen restrictedly as either being implicit or explicit, correspondingly. Accessibility or inaccessibility is assigned to both types of knowledge representations. However, conceptual understanding (know *why*) seems to be viewed as another aspect of knowledge, separate of the procedural (know *how*), and declarative (know *that*). *Knowing why* seems to be equated with explicit knowledge representations.

Given the recognised importance of the conceptual aspect of knowledge in the mathematics domain, the gap that this study attempts to fill, is to delineate separately the procedural (behavioural), as well as conceptual facet of children's newly introduced strategies, and their possible development within the context of an additive problem. The literature review showed that children's discoveries, and construction of new strategies need to be studied in relation to more than one aspect of knowledge. Siegler and his colleagues' series of studies have mainly explored the issue of implicit/explicit strategy use and discovery at the behavioural-procedural level (e.g. precursors of strategy discovery, verbal reportability of strategy *use*). It seems that the possibility for different levels of explicitness to be ascribed to each of the different aspects of mathematical knowledge (i.e. the procedural and the conceptual) as they interplay in the course of solving an additive problem, and constructing new strategies, is an issue that has not been clearly addressed.

The following chapter presents a theoretical position derived from the area of developmental psychology which discusses the possibility for different levels of knowledge accessibility and explicitness to exist and develop. This theoretical idea is incorporated in a model which suggests a plausible process towards the construction and development of understanding, and knowledge representations that sustain creative, and flexible problem solving behaviour.

Chapter 3 Theoretical framework: the Representational Redescription model

3.1. Overview

In this chapter the ideas, on the basis of which the theoretical framework of this study is built, are presented. Karmiloff-Smith (1992) in her book *"Beyond Modularity"* discusses certain theoretical issues concerning knowledge, development and learning, as these are conceived by two major schools in cognitive development: Fodor's nativism and Piaget's constructivism. An outline of Karmiloff-Smith's critical commentary on both approaches is followed by the presentation of a new theory of developmental change. This theoretical position, which Karmiloff-Smith calls *Representational Redescription*, situated midway between nativism and constructivism, suggests that development and learning, are two complementary directions, both relevant to cognitive change. Hence, a developmental perspective on cognitive science is essential, if one seeks to understand human cognition. Karmiloff-Smith's theoretical model has received support from a number other researchers (including Donald, 1994; Ohlsen, 1994; Spensley, 1997) but, currently, is under-researched in mathematics education. As concluded in chapter 1, this theoretical model is judged as appropriate for the present study of the evolution of young children's problem-solving capability in mathematics.

This chapter is organised as follows: in the first section the main ideas that Karmiloff-Smith opposes her theory to, are briefly presented. In the second section, the model which incorporates the new theory of developmental change is described, followed by a brief presentation of empirical data. The third section is devoted to presenting Karmiloff-Smith's view on how her model can provide an account for cognitive changes that occur within the mathematics domain. Subsequently, critical theoretical comments and data from empirical testing that certain aspects of the theoretical model have been subjected to are presented. The chapter closes with a summary of the main aspects and issues that the review of the theoretical framework revealed and this study focuses on.

3.2. Premises on development and knowledge acquisition

3.2.1. Nativism versus constructivism

One of the strongest debates in the area of cognitive development is that between those who argue for the innate character of cognition as opposed to views which embrace the idea that knowledge is constructed by the individual who actively interacts with the physical and socio-cultural environment.

Nativist positions emphasise the existence of innate constraints which drive the acquisition of knowledge and which model development as the maturation of knowledge structures which are not sensitive to environmental inputs, and which are qualitatively different across domains (Keil, 1990). The view that knowledge is domain specific implies that different types of knowledge are confined into different domains. In this sense, knowledge of language can be referred to as a domain distinct from that of mathematics or physics. Fodor (1983), refers to the idea of domain specific knowledge as 'faculty psychology' (p. 1) and defines it as the view according to which mental life can only be explained if different kinds of psychological mechanisms (faculties) are to be postulated.

Karmiloff-Smith (1992) critically discusses both views on domain specificity and domain generality of knowledge and proposes a reconciliation. The discussion is initiated with a critical review of Fodor's (1983) ideas as these are expressed in his book *The Modularity of Mind.* In his book, Fodor supports the *modularity thesis* according to which the mind consists of modular cognitive systems which "are domain specific, innately specified, hardwired, autonomous, and not assembled" (p. 37). 'Modules' or 'input systems' are the terms that are used to refer to these systems. In contrast with the rigid domain-specificity of input systems, the central system of the mind is considered to be domain *in-specific* and hence non-modular. In the central system, what the mind knows and believes is stored. Information that is delivered by modules in the form of a common representational format[3.1] is combined with information coming from other domains. In this sense, the domain-general mechanisms of the central system cross the domains that modules establish, and exploit the information which is delivered by the different domain-specific input systems.

[3.1] Fodor hypothesises that there is a formalised internal language in which the mind carries out symbolic reasoning. This "Language of Thought" (LOT) is not a spoken language but rather a representational.

In contrast to nativist ideas, Piaget's constructivism argues that development is undifferentiated across different domains and there are general learning mechanisms which sustain cognitive action and which are found at the various levels of thought (Piaget, 1970, 1980). For Piaget, innate factors influence cognitive functioning only in three ways: through the hereditary transmission of physical structures (e.g. the human nervous system), the inherited automatic behavioural reactions (i.e. reflexes), and the inherited functions of *organisation* and *adaptation* (see e.g. Flavell 1963; Ginsburg & Opper, 1969).

It must be underlined particularly that Piaget considers organisation and adaptation as two general principles or tendencies of the individual. In this view, and in contrast to Fodor's position, it is these general *tendencies* that the individual inherits and not any particular cognitive reactions. Also, these principles and the processes involved are considered to be domain-general and any changes that occur during development are considered to take place more or less simultaneously across different domains. (Ginsburg & Opper, 1969).

3.2.2. The process of modularization

Karmiloff-Smith views nativism and constructivism as two complementary directions and proposes a developmental perspective which accepts both views on domain specificity and domain generality. From this perspective, young children are seen as active constructors of their own cognition. However, this constructive process is assumed to involve both domain specific as well as domain general processes.

Within the general framework of Karmiloff-Smith's position on knowledge acquisition and development, it is posited that knowledge is acquired in the following ways. First, knowledge can be innately specified through evolutionary processes. It is hypothesised that innate predispositions may be specified either in detail or as "skeletal", only basic, domain specific predispositions. These merely "skeletal" predispositions involve attention biases sensitive to particular inputs of information and a number of "principled predispositions" which drive the processing of these inputs. In the former case of highly specified innate predispositions, the role of the environment is to act simply as a trigger. In the latter case of certain very basic predispositions, the interaction between the individual and the environment (physical and socio-cultural) is considered as highly influential and not merely as a trigger (Karmiloff-Smith, 1992, p. 15). Environmental data also play a critical role and are taken into account when the individual fails to reach a goal, while representations of new knowledge can also be acquired directly from external sources, i.e. the physical and socio-cultural environment in the form of a direct linguistic statement (Karmiloff-Smith, 1994).

As well as knowledge changes which have an external, environmental source, Karmiloff-Smith (ibid.) argues that there are also internal sources of change. It is postulated that there are two complementary, internal processes which drive cognitive developmental changes. One of these internal sources of knowledge change is the aforementioned process of *modularization* which progressively encapsulates knowledge and makes it less accessible and manipulable by other parts of the cognitive system resulting in inflexible and automatic behaviour (Karmiloff-Smith 1994). The idea of modularization as opposed to the idea of a pre-specified modularity leaves space for an important role to be assigned to information coming from the external as well as internal environment of the individual. As Karmiloff-Smith has put it:

"The brain is not prestructured with ready-made representations which are simply triggered by environmental stimuli; it is channelled to progressively develop representations via interaction with both the external environment and its own internal environment"

(Karmiloff-Smith, 1994, p. 697-698).

Karmiloff-Smith (1992) defines a *domain* as "a set of representations sustaining a specific area of knowledge: language, number, physics, and so forth." (p. 6). These broad domains of knowledge are differentiated from *microdomains* which are considered as subsets within each of the broad domains (e.g. number within the domain of mathematics). The term *representation*, even though it is not explicitly defined, is used to denote something internal to the child's mind (Karmiloff-Smith, 1990, footnote p. 58), some internal way of coding information (personal communication). Furthermore a *module* is defined as an "information-processing unit" which encapsulates knowledge of a particular domain and the computations on it, resulting in automatic and inflexible behaviour (Karmiloff-Smith, 1994, p. 696)

Karmiloff-Smith underlines that the idea of a process of modularization instead of a pre-specified modularity is only a speculation. This speculation though, as she notes, indicates her theoretical stance according to which certain modifications in both views on domain specificity and domain generality need to be made since it does not seem reasonable to assume that development is wholly domain specific or domain general (Karmiloff-Smith, 1992).

3.2.3. The process of Representational Redescription

Karmiloff-Smith (1992) suggests that knowledge encapsulation, which results in automatic and inflexible behaviour, is followed by a process of progressive knowledge *explicitation.* Knowledge explicitation involves a process of internal exploitation of already existing knowledge and constitutes the most important, in the author's view, way of acquiring new knowledge. It is argued that implicit information which is already stored in a certain form of internal representations and is embedded in special-purpose procedures, is subject to an iterative process of *redescription.* During this process, knowledge which is stored in certain representational formats is being re-represented.

The hypothesised process of *representational redescription* is considered as a fundamental aspect of human development and it is defined as "a process by which implicit information *in* the mind subsequently becomes explicit knowledge *to* the mind, first within a domain and sometimes across domains" (Karmiloff-Smith, 1992, p. 18). It is presumed that this process of redescription occurs spontaneously and it is driven by an internal process of representational change after which elements of the same knowledge are re-represented at higher levels of abstraction (Karmiloff-Smith, 1990). The idea of an endogenous, spontaneously driven process implies that cognitive change can occur not only in the framework of the organism's interaction with the environment and necessarily after external prompting, but *also* as the product of system-internal dynamics (Karmiloff-Smith, 1993, 1994).

Another type of knowledge change via representational redescription involves *explicit theory change* which is defined as the "conscious construction and exploration of analogies, thought experiments and real experiments". It is speculated that this type of knowledge change which is considered as a particular characteristic of older children and adults, can only take place on the basis of the previous form of redescription which turns implicit information into explicit knowledge (Karmiloff-Smith, 1994, p. 698).

To explain the idea of representational redescription, Karmiloff-Smith gives the following illustrative example of the learning pathway of a pianist:

"There is a first period during which a sequence of separate notes is laboriously practised. The beginning pianist pays conscious attention to particular notes. There is a second period during which chunks of several notes are played together as blocks, until finally the whole piece can be played more or less automatically. In other words the sequence gradually becomes proceduralized. It is something like this that I call "reaching behavioural mastery". But the automaticity is constrained by the fact that the learner can neither start at the middle of the piece nor play variations of the theme. The performance is generated, I hypothesise, by procedural representations which are simply run off in their entirety. There is little flexibility. At the best, in a third period, the learner is able to play the whole piece softer, louder, slower, or faster. The pianist's "knowledge" is embedded in the procedural representations sustaining the execution. But most learners do not stop there. During a forth period, the learner can interrupt the piece and start, say, the third bar without having to go back to the beginning and repeat the entire procedure from the outset".

(Karmiloff-Smith, 1994, p. 699)

In Karmiloff-Smith's view, the reason for the behavioural changes that occur in the last period of the pianist's learning pathway lies in the process of representational redescription in the course of which procedural knowledge, which is automatised and inflexible is being re-represented in a representational format which renders knowledge explicit, flexible, and available as manipulable data. Knowledge becomes progressively more explicit via a redescriptive process which renders information that already exists in "independently functioning, special purpose representations", available to other parts of the cognitive system. This happens mainly within a domain but can also occur across different domains allowing the establishment of intra-domain and inter-domain relationships (Karmiloff-Smith, 1994, p. 700). Karmiloff-Smith argues that in the framework of a learning or problem solving situation, the end-product of this reiterative movement from implicit knowledge embedded in efficient procedural behaviour to knowledge which progressively becomes more explicit, is "representational flexibility and control which allows for creativity". It is also underlined that at the end of this process procedural skill is not lost. The learner, or problem solver, is able to invoke his or her automatic skill when this is necessary and can also make use of explicit knowledge which allows flexible and creative behaviour (Karmiloff-Smith, 1992, p. 16).

The process of representational redescription is considered as a domain-general process in the sense that it is a process which operates throughout development. Karmiloff-Smith underlines that "domain-general" here does not imply simultaneous changes across different domains. In the author's view, representational redescription is a domain-general process in the sense that the process is the same within each domain although "it is affected by the form and the level of explicitness of the representations supporting particular domain-specific knowledge at a given time" (Karmiloff-Smith, 1992, p. 18). Also, it is underlined that although the hypothesised process of redescription is mainly considered as an endogenous and spontaneously driven process, it is not denied that in certain cases the process can also be triggered by external factors.

In this section, Karmiloff-Smith's theoretical positions about development and knowledge acquisition were presented in consideration with certain aspects of theoretical ideas expressed in the framework of two well known schools in cognitive developmental psychology; nativism and Piaget's constructivism. Representational redescription, the process which in Karmiloff-Smith's view drives development and learning, will be considered in more detail in the following section, where the model that encompasses all the different aspects and implications of the postulated process will be described.

3.3. The RR model

Representational redescription, as previously discussed, is considered as a process relevant to cognitive change on the basis of which development is viewed as occurring through the iterative movement from implicit information which drives efficient procedural though inflexible behaviour, to the progressive explicitation of knowledge.

Eventually, through this redescriptive process, conscious access to knowledge and children's theory building emerges. "This is precisely what I think development is about: Children are not satisfied with success in learning to talk or to solve problems; they want to understand how they do these things. And in seeking such understanding, they become little theorists" (Karmiloff-Smith, 1992, p. 17). For Karmiloff-Smith (1984), it is this continuous search for understanding and control over one's external environment and internal representations that cognitive change emanates from. It is not failure nor reasons of economy that constitute the primary motivations for cognitive change to emerge.

A developmental model which is called *the RR model* was built to describe the process of change towards the acquisition of such understanding and control. The RR model is process-oriented and accounts for children's movement beyond their successful procedural behaviour to working on their internal representations *for themselves* as this being a problem itself (Karmiloff-Smith, 1981).

3.3.1. Phases and levels

The RR model is a *recurrent 3-phase model.* Karmiloff-Smith stresses particularly that, in contrast to stage models (e.g. Piaget's model of cognitive development) which describe the way that children of a particular age think, the RR model is not age-related and does not assume domain-general changes. Stage models refer to similarity of structure and hence consider changes in children's thinking and changes in children's age as a one to one relation (Siegler, 1996). Furthermore, stage-models describe changes as these occur across the different domains of development. In contrast, Karmiloff-Smith's model, focuses on the process rather than the structure. Hence, changes are considered to be *recurrent* and not as occurring simultaneously across the entire cognitive system. This is because, as mentioned before, representational redescription is viewed as a general process which, however, occurs repeatedly *within each* domain. Therefore, the choice for the term "phase" to be used is based on the assumption that children, when involved in a problem solving situation, pass through the same three phases within the various microdomains and also across different domains. Karmiloff-Smith (1984), illustrates this view in the following way: "the same child may be at phase 1 in one physics problem, at phase 3 in another physics problem, and at phase 2 in a spatial problem. ...Thus, at any age, the child will be simultaneously at phase 1 for a certain number of problems, at phase 2 for others and at phase 3 for yet others; and, faced with any new acquisition problem, the child will pass through the same three phases" (p. 41).

"Levels" on the other hand, is the term that is used to designate the different formats, in which the internal representations that sustain these recurrent phases are re-represented, during the process of representational redescription.

After having specified the meaning of the terms that constitute the scaffold of the RR model, a detailed description of the recurrent phases will be presented as these apply to development and children's problem solving in particular.

Phase 1

During phase 1 which is named the *"procedural phase"*, it is the information from the external environment that the child mainly focuses on. At this phase, the external environment primarily controls the child's behaviour which is generated by adaptation to external stimuli. By focusing on data coming from the external environment, the child creates "representational adjunctions". Karmiloff-Smith speculates that representational adjunctions are added and stored domain-specifically and independently from the existing representations. They are not brought in relation with, and do not alter, the already existing knowledge (Karmiloff-Smith, 1992, p. 18). At the end point of this phase consistent successful performance is achieved and this is what Karmiloff-Smith calls "behavioural mastery" (ibid, p. 19).

The notion of *"behavioural mastery"* implies that successful performance can be generated by a series of representations which are independently stored and are not yet consistently linked into a system. However, success as performed at this phase and sustained by the particular type of representations, is not considered as the end point of development in a particular micro-domain. Karmiloff-Smith (1992) stresses the fact that further on, in phase 3, performance will also be successful. At that phase though, the same behavioural output that is also observed in phase 1 will be sustained by very different representations. Thus, there is a need to differentiate between *behavioural change* and *representational change*. The distinction implies that similarities in behaviour can be only superficial since identical behaviour can be grounded on a less or more coherent system of internal representations (Karmiloff-Smith, 1991, 1992).

In a problem solving situation the behaviour of children who are in phase 1 is "success-oriented". The child reaches procedural success by adapting both to positive and negative external feedback. However, the child's adaptations constitute separate "behavioural units". These units of behaviour are not brought in relation one to another. Rather, each of these units consists of efficient and automatic procedures which "have the status of "cognitive tools" for successfully reaching a goal". These procedures need to be run in their entirety, again, for each part of the problem (Karmiloff-Smith, 1984, p. 43).

Phase 2

Phase 2 which follows is named as *"metaprocedural phase"*. At this phase, children's internal representations are brought into the focus at the expense of information coming from the external environment. In phase 2 behaviour is considered as generated by "top-down control mechanisms". The term is used to describe behaviours which ignore environmental information and at the same time impose internal representations on the external environment (ibid, p. 43). Karmiloff-Smith (1992) underlines that overlooking environmental data at this phase, may cause errors and inflexibilities. However, the regression that may be observed at the behavioural level does not coincide with regression at the representational level. This U-shaped pattern of *behavioural regression* is only temporary and is considered as a sign of *representational progression*. The reason for this is that ignorance and violation of environmental information is viewed as a consequence of an overall organisation of the internal representations which takes place at the same time (Karmiloff-Smith, 1985, 1991, 1992).

In a problem solving situation, the internal organisation that takes place during this phase is being manifested as follows: children's behaviour is guided and constrained by children's implicit ideas about factors involved in a given task. Karmiloff-Smith (1974, 1984) uses the term *"theory-in-action"* to denote these implicit ideas. The notion of "theory-in-action" refers to the way by which one explains a lot of data to oneself. This particular explanation guides one's thinking and is subsequently imposed on the data. This means that children, in order to maintain their developed theories, simplify or even disregard negative feedback i.e. data that should be considered as counterexamples (Karmiloff-Smith, 1999, personal communication).

The notion of "theory-in-action" is illustrated in a study that Karmiloff-Smith and Inhelder (1974) carried out to explore children's developmental progression and spontaneous organising activity in goal-oriented tasks. Here, it suffices to say that "theories-in action" are considered as rigid ideas that are not necessarily accessible to the child explicitly. In the case of a physics task, for example, the child's underlying "theory" could be interpreted by the experimenter as a naïve law of physics (Karmiloff-Smith, 1984).

The reason for not being adaptive to negative feedback is that in phase 2 as opposed to phase 1, children approach what was previously viewed as a sequence of isolated problems, as a whole. On the basis of their implicit theories-in-action, children "generate a simplified procedure that allows them to have control and to link the previously isolated and juxtaposed procedures into a single representational framework". At this phase, children's behaviour is "organisation-oriented" and the generated procedures do not function as tools but rather as "cognitive units of attention" (ibid, p. 43).

As a result of children's attempt to organise and connect the procedures involved in the problem, there is a deterioration (in comparison to phase 1) in their successful performance. This deterioration in successful behaviour is only superficial as it is compensated by the profit gained from children's movement beyond procedural success to theory building. It is this movement that results in the internal representational organisation into a single format leading to the generation of a unified, single approach for all the parts of the problem.

Phase 3

The third phase is termed *"conceptual phase"*. During this phase the interaction between external data and internal representations is regulated and balanced as a result of the search for both internal and external control. In a problem solving situation, the balance achieved is manifested in children's behaviour which is now guided by both environmental data and internal representations. Although children's behaviour in this phase is successful and seems identical to the behavioural output at phase 1, the similarity is only superficial. Representations that sustain children's behaviour in the third phase are richer and more coherent. "Phase 3 is the result of the reorganisational processes at work in phase 2 which, once consolidated, can take environmental feedback into account without the overall organisation being jeopardised" (Karmiloff-Smith, 1984, p. 44).

Accepting the hypothesis of a process of knowledge redescription entails that the aforementioned developmental phases are sustained by different formats of internal representations of knowledge. In the framework of the RR model, it is argued that there are at least four levels at which knowledge is represented and re-represented.. Karmiloff-Smith (1992) names these levels Implicit (I), Explicit-1 (E1), Explicit-2 (E2), and Explicit-3 (E3). These levels of knowledge redescription, like phases, are considered as part of the cyclical process of knowledge explicitation that takes place repeatedly within each micro-domain and not as age-related stages of developmental change. It is postulated that different representational formats correspond to different levels.

Implicit (I)

At this level, representations are in the form of procedures for responding to environmental data. It is argued that the following factors constrain representational adjunctions that are formed at this level:

* Information is encoded in procedural form.
* The procedure-like encodings are sequentially specified.
* New representations are stored independently.
* Level-I representations are bracketed, and hence no intra-domain or inter-domain representational links can yet be formed.

(Karmiloff-Smith, 1992, p. 20).

Karmiloff-Smith (1986, 1992) explains that in the case of two procedures with common informational components, the commonality is not represented explicitly in the child's mind at this level. A procedure can only be run in its entirety and its components cannot be accessed and operated separately. "It takes developmental time and representational redescription for component parts to become accessible to potential intra-domain links, a process which ultimately leads to inter-representational flexibility and creative problem-solving capacities" (Karmiloff-Smith, 1994, p. 700).

The sequentially specified internal representations, allow for little intra-representational and inter-representational flexibility. This means that there is no ability for introducing changes into the representations and also for establishing links between a new representation and representations from other domains (Karmiloff-Smith, 1990).

Because the potential representational links and the information stored remain implicit, the generated behaviour is inflexible. This is because only specific inputs are computed in a preferential way although effective responses are rapidly generated. It is postulated that level (I) is followed by levels E1, E2 and E3 which constitute the recurrent process of representational redescription (Karmiloff-Smith, 1992).

Explicit-1 (E1)

Implicit information embedded in level I representations is redesribed in level E1 in the same representational code (spatial, temporal, linguistic) as in level I. However, E1 representations, as the product of the redescriptive process which involves higher-level abstractions, are simpler and lose informational details while at the same time, they become less special purpose, more cognitively flexible, and thus transportable to other goals. This is because, during this primary explicitation, potential relationships between representations and procedural components are explicitly defined. It is upon these explicitly defined representational links, that the child starts building theories and make analogies (Karmiloff-Smith, 1986, 1992).

Explicitly represented representations give rise to children's ability to "introduce violations to their data-driven, veridical descriptions of the world-violations which allow, for instance, for pretend play, false belief, and the use of counterfactuals" (Karmiloff-Smith, 1994, p. 701). It is also underlined that the implicitly defined, level I, representations are still available to children for use when, for certain goals, speed and automaticity is required. Finally, it is particularly stressed that, although E1 representations are available as manipulable data, they are not necessarily available to conscious access and verbal report. Only at the subsequent levels is the conscious access and verbal report made possible (Karmiloff-Smith, 1992).

Explicit-2 (E2)

It is speculated that at level E2 representations are available to conscious access but not to verbal report. Karmiloff-Smith (1992) argues that in contrast to some theorists' belief that consciousness entails the ability for verbal report, the RR model posits that representations of this level are consciously available but are not verbally reportable because they are still in a similar code as the E1 representations of which they are redescriptions. "...for example, E1 spatial representations are recoded into consciously accessible E2 *spatial* representations. We often draw diagrams of problems that we cannot verbalise" (ibid, p. 22). It is argued that the end-product of this sequence of redescriptions is the existence of multiple levels of detail and explicitness at which the same knowledge is represented.

Explicit-3 (E3)

At E3 level, knowledge is re-coded into a common representational format "a cross-system code" which is hypothesised to be very close to natural language allowing translation into a communicable form and thus verbalisation (Karmiloff-Smith, 1994, p. 701). It is argued that knowledge which is learned from verbal interaction with others may be stored directly at this level being represented in linguistic code. However, this form of knowledge is not yet linked to similar knowledge which is stored in other codes. Karmiloff-Smith (1992) explains this as follows: "Often linguistic knowledge (e.g. mathematical principles governing subtraction) does not constrain nonlinguistic knowledge (e.g. an algorithm used for actually doing subtraction) until both have been redescribed into a similar format so that inter-representational constraints can operate" (p. 23).

Although the RR model argues for four levels at which knowledge is redescribed, in the empirical examples which are given by Karmiloff-Smith (1992), only three levels are distinguished: I, E1, and E2/3. The author explains that levels E2 and E3 have not been distinguished for the following reason. Although, in her view, both levels involve conscious accessibility, no research has, so far, focused on the E2 level where conscious access of knowledge is possible but there is no verbal report. Karmiloff-Smith underlines that E2 needs to be tested empirically. However, she still does not exclude the possibility for spatial, kinaesthetic or other non-linguistically encoded representations to be available to conscious access.

3.3.2. Some relevant considerations

Before closing this section devoted to the RR model, certain ideas that are of particular significance and are connected with the model will be summarised and highlighted.

Karmiloff-Smith (1994) particularly underlines the notion of *multiple encoding*; that is the fact that the end-product of the sequence of redescriptions is "the existence in the mind of multiple representations of similar knowledge at different levels of detail and explicitness". In Karmiloff-Smith's view, the possibility of multiple encoding is important because it indicates that the human mind is not economically driven but constitutes a "very redundant store of knowledge and processes" (p. 701).

Furthermore, for Karmiloff-Smith (1986, 1992) the differentiation between distinct levels of representational redescription and explicitation is an important distinction for the understanding of development and cognitive change as this occurs *over time*. In her view, two-level dichotomies that are used in the developmental literature such as implicit/explicit, procedural/declarative, unconscious/conscious, controlled/automatic are insufficient to capture the complex nature of the processes leading to conscious access. The notion of *representational change over time*, that Karmiloff-Smith particularly focuses on, entails a developmental perspective on cognitive science which allows for the recognition of more than two kinds of knowledge representation than a dichotomy implicit/explicit would suggest (Karmiloff-Smith, 1993). It is thus strongly argued that "levels exist between implicitly stored procedural information an verbally statable declarative knowledge. It is particularly via a developmental perspective that one can pinpoint this multiplicity of levels of representational formats" (Karmiloff-Smith, 1992, p. 22).

What is also particularly stressed is the distinction that Karmiloff-Smith draws between the process of representational redescription and its realisation in the framework of the RR model. The process of knowledge redescription involves the re-coding of knowledge from one representational code into another (for example, a spatial representation can be re-coded into a linguistic code). The RR model posits that the process of representational redescription takes place at four hierarchically different levels, each one of these being a more "condensed" version of the previous. Karmiloff-Smith (1992) argues that even if empirical data refute the hierarchy of these levels, postulated by the RR model, the process of representational redescription will remain unchallenged. In the figure that follows, alternative ways of modelling the process of representational redescription are suggested.

I format
↓
E1 format
↓
E2 format
↓
E3 format

I format
↓
E1 format
E2 format E3 format

I format
E1 format E2 format E3 format

Figure 3.1
Possible models of RR (Karmiloff-Smith, 1992, p. 24)

The alternative option that is illustrated in figure 3.1 is that redescription of level I representations can directly lead to either E1, E2, or E3 format. This implies that knowledge can be redescribed directly into a linguistic code without the mediating step of the E1 level as the RR model originally suggests.

Furthermore, in the framework of the RR model, reaching behavioural mastery, that is reaching a stable state of success, is a prerequisite for subsequent representational change to occur. Karmiloff-Smith (1992) argues that even if this hypothesis were shown to be invalid, the process of representational redescription would still remain unchallenged because the RR model argues for three recurrent phases that lead to behavioural mastery and beyond. *Within* phases, it is failure (i.e. negative feedback) which plays an important role for representational change to occur leading progressively to behavioural mastery. However, success (i.e. positive feedback) is posited to be essential for representational redescription to occur leading to the transition *between* phases.

Karmiloff-Smith clarifies that what she wishes to argue for, is a "success-based view of cognitive change" when learning theories (including Piaget's) account for change in the learner's behaviour on the basis of cognitive conflict and failure to reach a goal. "Rather, for the RR model certain types of change take place *after* the child is successful (i.e. already producing the correct linguistic output, or already consistently reached a problem-solving goal)" (Karmiloff-Smith, 1992, p. 25). The author particularly stresses that she does not deny the importance of cognitive conflict in the generation of certain types of cognitive change. Also, representational redescription is not suggested as the only source of change. What is suggested is an *additional* source of cognitive change: this being the internal system stability (the learner having reached behavioural mastery) which generates the redescription of knowledge representations. "Representational redescription is a process of 'appropriating' stable states to extract the information they contain, which can then be used more flexibly for other purposes" (Karmiloff-Smith, 1994, p. 701)

In fact, Karmiloff-Smith (1994), responding to theorists'[3.2] suggestions that behavioural mastery may emerge as a consequence of a first process of representational redescription,

[3.2] Vinter & Perruchet (1994), Goldin-Meadow & Alibali (1994).

admits that research situated in the connectionist[3.3] theory of learning, has analysed hidden units during learning and has shown that representational change can take place before it is observable at the output. For Karmiloff-Smith, this suggests a way in which the RR model may be modified in that full behavioural mastery may not be the absolute requirement for representational change to occur; "that is, representational change may start prior to *overt* behavioural mastery". However, the author clarifies that her strategy was to look specifically beyond behavioural mastery, a point where most of other studies stop (Karmiloff-Smith, 1994, p. 704).

It must also be clarified and stressed that Karmiloff-Smith, by naming phase 2 the "metaprocedural phase", does not restrict the notion of *meta* to conscious accessibility, conscious reflection and verbalisation. *Metaprocesses* are considered as an essential component of acquisition but, in the framework of the RR model, they should not be confused with *meta*cognition which implies conscious verbalisation and explanation on the part of the child (Karmiloff-Smith, 1984). In phase 2, metaprocedural processes re-write the implicit procedural representations of phase 1 in explicit form in order to operate on them. Karmiloff-Smith (1984) argues that metaprocesses are generated on the basis of success and are not directly available to consciousness. Conscious awareness results from further re-writing and representational redescription from E1 to E2/E3 form. The term *meta*procedural is used to indicate a different, a *"meta"* level at which children *"work on"* their existing representations, reanalyse and reorganise them with no necessary conscious awareness (p. 85).

Karmiloff-Smith (1986) explains that the common format in which representations in phase 3 are re-written is a cross-system code which is assumed to be abstract and less constrained by spatial, temporal and causal constraints which are intrinsic in most other representational codes. This abstract code is "more amenable to linguistic encoding than other codes which explains why ultimately metacognitive knowledge is frequently available to verbally statable form" (p. 105) In this sense, metacognitive (i.e. conscious) awareness is not considered as a prerequisite but the end-product of representational explicitation which establishes representational links during phase 3. Thus, consciousness is regarded as the highest level developmentally of representational redescription.

[3.3] Karmiloff-Smith (1992) explores the possibility for the RR model to be implemented as a connectionist computer simulation. Some of the research works that are quoted are the following; Rumelhart and McClelland (1986); Clark (1989); Klahr (1992), [full reference included in the thesis' references].

In this section, the RR model was described and the ideas which provide the theoretical basis for the rationale and design of this study were presented and discussed. The following section is devoted to the presentation of empirical data from children's problem solving as these were explored and analysed in the light of the aforementioned theoretical ideas.

3.4. Empirical data: an example

Karmiloff-Smith (1979, 1984, 1986) has explored children's problem solving and theory building in a series of experimental situations, within various different domains such as language, science, drawing and so forth. Her exploration takes place on the basis of the process-oriented theoretical and methodological framework that the RR model establishes. This section presents an outline of her study of changes in children's representations while developing external memory devices in the context of a problem solving situation. Certain methodological issues which arise and are considered as relevant to the design and rationale of this study are highlighted.

The following example has been selected on the basis of its methodological design and aim. It must be remembered that this study aims at exploring changes that occur in children's strategies at a micro-developmental level, i.e. changes that take place in the framework of a single, specific task, during a sequence of sessions. The example that follows is only one of the experiments that Karmiloff-Smith has designed and carried out focusing on the process of cognitive change as this occurs micro-developmentally and is inferred by the change in children's problem solving behaviour.

Data from the problem of creating external memory devices

Karmiloff-Smith (1986) notes that the 3-phase cycle of changes is detectable in micro-developmental tasks provided a problem solving task is well within a subject's cognitive capacity. It is underlined that the task needs to be well within subjects' competency because time and space has to be given for representational and behavioural changes to occur within the boundaries of an experimental session. A notational task which involved the creation of an external memory device was designed to test whether the concept of metaprocedural processes can be extended to micro-developmental change.

63 children of the 7-11 age group were chosen to participate in the experiment. Previous experimentation with the same age group had shown that a variety of notational systems was

within the children's competence. The task was as follows: A 12-meter roll of wrapping paper on which a route from a house to a hospital was drawn, was shown to children. Along the route, there were 20 bifurcation points at which one branch of the route was leading to a cul-de-sac while the other permitted the child to continue towards the hospital. Some of the bifurcation points were marked with topographical or figural indices such as trees, little men, etc. (fig. 3.2).

The task for the children was to drive a patient in a small ambulance from the house at the beginning of the route to the hospital. As children were driving the ambulance, the experimenter was unrolling the paper, and was rolling up again the already covered parts of the route. During the first run the "patient" was not in the ambulance so if children chose a cul-de-sac they were allowed to backtrack. During this run children were encouraged to *mark* something down on a piece of paper: something that would help them avoid cul-de-sacs during the runs when the patient would have to be driven to the hospital. So the task for the children was to create a notational system that could be used as an external memory device to help them remember which way to turn. The experimenter made no suggestions about the form of notation.

A typology of the forms of notation that children used was created, and included five types of notational systems ranging from more concrete to more abstract forms. It is stressed that examples of all the types were found at all ages (for more information see Karmiloff-Smith, 1979, 1984). Figure 3.3 illustrates the types of notational systems that children generated and used; a) figural reproduction, b) figural schematization, c) analogical abstraction, d) nonanalogical abstraction, e) linguistic (Karmiloff-Smith, 1979, p. 99).

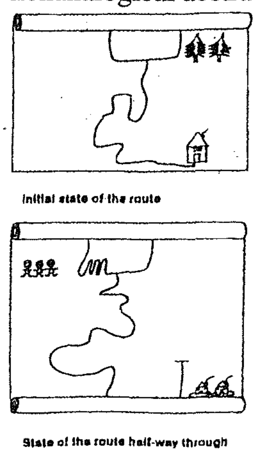


Figure 3.2
The 'map' task

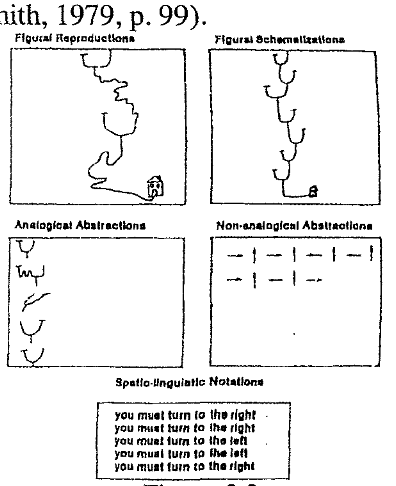


Figure 3.3
Typology of children's forms of notation
(Karmiloff-Smith, 1984, pp. 62-63)

The main premise of the RR model is, that it is not only failure via exogenous factors but internal stability and endogenous constraints that primarily motivate representational change (Karmiloff-Smith, 1992). Karmiloff-Smith indicates two possible changes in the task: one invoked by an *exogenous* cause, and one by an *endogenous* cause. The following example illustrates the difference:

"Imagine that you have chosen to write instructions indicating figural indices (e.g. "take the branch where the man is standing", "don't take the side with the pond") or to draw the figural indices next to the bifurcations. You will be forced to change such a system on encountering a bifurcation bare of any such indices. In such a case, your change would be generated by an *exogenous* cause." (ibid., p. 152).

On the other hand it is highlighted that, as in language where children generate certain changes under no apparent external pressure, in the map task as well, children spontaneously introduce modifications in their notational system even when the already generated one contains all the adequate information to succeed in the task.

"One must then invoke *endogenous* causes, because failure or inconsistency of the notational system cannot be adduced to explain the changed behaviour." (ibid., p. 152).

Karmiloff-Smith's analysis for this task was focused on the endogenously provoked modifications that children spontaneously introduced into their notational system during the experimental session. Karmiloff-Smith's interpretation of the many other micro-developmental changes of this nature that were revealed, is that addition of redundant, explicit information after the original successful system has become consolidated, indicates that the child is working on her representations "in an organisation-oriented fashion, explicitly representing the meaning oppositions rather than merely generating a success-oriented procedure for reaching the goal" (Karmiloff-Smith, 1984, p. 64). It is exactly this movement beyond the initial success-oriented process to an organisation-oriented one that constitutes indication of *metaprocedural* behaviour, as this is conceived in the framework of the RR model.

"Metaprocedural behaviour thus, acts as a control mechanism in the striving for this delicate balance of information content and information processing effort. It also functions as a control mechanism for the interprocedural organisation of what was previously a plethora of juxtaposed, unconnected procedures yielding superficially similar behaviour."
(Karmiloff-Smith, 1979, p. 115).

It must be stressed that Karmiloff-Smith does not argue that change is *solely* due to endogenous factors. Simply, endogenous causes of change constitute her particular interest of work. The importance of exogenous factors is not denied. But, it is argued that when change is provoked by exogenous causes, subsequently endogenously provoked representational change must still take place. This is because representational redescription and change are not equivalent to mere addition of new representations on the basis of external stimuli (Karmiloff-Smith, 1990).

The aforementioned experiment was presented as an example of an exploration focusing on the micro-developmental changes that occur in a problem solving procedure, within the time and space of an experimental session. It also illustrates certain theoretical issues that were presented in the previous section. The particular experiment is only one, of many others, that Karmiloff-Smith has carried out using the same method and focusing on changes of this nature. None of these experiments though, focus on the domain of mathematics.

However, in her book, Karmiloff-Smith (1992) refers to previous research on number conservation and counting. The discussion on the different theoretical views is followed by the presentation of her perspective on how early mathematical knowledge is built, *macro*-developmentally, in the light of the RR model. Karmiloff-Smith's position regarding the development and construction of children's theories with respect to *number* will be presented in the following section.

3.5. RR and mathematics

As mentioned before, there is no empirical data which, in the light of the RR model, focus on the micro-developmental changes that may occur within a mathematical problem-solving situation. However, in her book, Karmiloff-Smith (1992) devotes a chapter to the discussion of controversial theoretical views regarding the development and construction of early mathematical knowledge. The issues that are particularly discussed are number conservation and the conceptual aspects of counting. These issues, do not constitute part of the particular research interest of this study. Thus, in this section, Karmiloff-Smith's discussion and perspective will be presented briefly. However, a presentation, even brief, of the way in which the RR model can account for the development of early mathematical knowledge is considered as necessary. This is because Karmiloff-Smith offers only limited indication and evidence, so far, of how the model can be used as an explanatory tool within the mathematical domain.

Karmiloff-Smith (1992) discusses and presents research that challenges Piaget's view that all aspects of number are part of the domain-general cognitive development and are constructed as a result of the actions that form sensory motor intelligence. Challenges to the Piagetian account also focus on the results coming from Piaget's experiment on number conservation which revealed that children younger than 5 years old fail to conserve number[3.4].

Among the different research works that are presented and discussed, Karmiloff-Smith considers Gelman and Gallistel's (1978) work as the most, theoretically, serious challenge to the Piagetian view of early number acquisition. From a nativist perspective, Gelman and Gallistel (1978) argue that children's early learning about number and counting is highly constrained by innately specified number and counting-relevant principles. Gelman and Meck (1986) and Gelman (1990) reported data which support the argument for the existence of innately specified principles and attention biases that presumably constrain early number-learning and early counting. Karmiloff-Smith (1992) notes that even if one accepts the argument, the issue of why children whose counting procedures successfully embodies number-relevant principles fail to succeed in the number-conservation task, is still left open. The author quotes Gelman and Gallistel's explanation of children's failure in conservation tasks according to which, children fail in the task because "they lack an *explicit* understanding of the principle of one-to-one correspondence by which conservation of non-specified values is achieved" (ibid., p. 104). Karmiloff-Smith argues that, in this explanation, the notion of *explicit* needs further clarification and proposes an alternative view.

In the light of the RR model, it is suggested that "innately specified principles are never directly available, but are embedded in procedures for interacting with the environment. "Clearly nothing in the external environment will directly inform the child. The RR model postulates that the movement to algebraic concepts involves a focus on the child's *internal* representations." (Karmiloff-Smith, 1992, p. 109). Furthermore, it is argued that one-to-one correspondence is an implicit principle embedded in successful counting procedures. It needs to be redescribed and represented in a format independent of the procedural encoding. Representations which result from the redescription of the level-I counting procedures to E1 representations can then be used for unspecified quantities.

[3.4] For Piaget (1970) number, as a logical mathematical structure, is a synthesis of relationships of order and also class inclusion. "The conservation characteristic takes the form of the notion of the permanence of an object." (p. 43). In this sense, "a set or collection is only conceivable if it remains unchanged irrespective of the changes occurring in the relationship between the elements. For instance, the permutations of the elements in a given set do not change its value" (Piaget & Szeminska, 1995, p. 300).

Karmiloff-Smith (1992) seems to agree with Gelman and Gallistel's position that number-relevant information is available very early to young children. However, Karmiloff-Smith argues that this information is implicitly embedded in procedures for processing environmental data. Subsequently, components of this information and number-relevant principles become explicitly defined and available in the E1 representational format. Mathematical knowledge is then subsequently built on the basis of such redescriptions. Karmiloff-Smith stresses that "however rich the innate specifications turn out to be, ... it is clear that we must focus on *the representational status* of such knowledge in order to understand the nature of the subsequent development." (Karmiloff-Smith, 1992, p. 115).

3.6. Theoretical comments and empirical testing of the RR model

Karmiloff-Smith's theoretical ideas focus on the development of two different notions: the gradual process of modularisation, and the process of representational redescription. The idea of gradual modularisation implies that modular-like, i.e. specialised cognitive structures are the product of a gradual developmental process: they do not constitute the starting point of development. The notion of representational redescription, accounts for the growing flexibility of the human cognitive system: a development which is posited to take place in parallel with the growing specialisation (i.e. modularisation). The two notions are interrelated, and their theoretical development and introduction have had a great impact in the field of cognitive developmental psychology. Both these postulated aspects of cognitive development have been the focus of multiple discussions and, as theoretical constructs, have been the object of either positive comments or skepticism, in relation to different domains of theoretical interest (e.g. language acquisition, children's 'theory of mind', memory). Although the two notions are interrelated, this study focuses on the idea of representational redescription. This is why, in this section, comments and views, developed by researchers and theoreticians in the field of cognitive developmental psychology, which address certain strong points of the idea of representational redescription as well as certain shortcomings, are presented. Among the various aspects of the model of representational redescription, those which are of particular interest in the context of this study, have been selected to be discussed in this section. Namely, the issue of endogenously or exogeneously motivated change, the issue of the specific, fixed sequence of changes and phases that the RR model postulates, and the implicit / explicit procedural / declarative distinctions related to knowledge representations. This section is separated in two parts. The first part is devoted to the presentation of theoretical comments as related to the aforementioned aspects of the RR model.[3.5] The second part is devoted to the presentation of empirical studies which have aimed at evaluating certain aspects of the model of representational redescription.

3.6.1. Theoretical comments

Endogenously or exogenously motivated change: does the RR have any educational implications?

Several theoreticians have generally welcomed the theoretical construct of representational redescription, but have argued that the role of sociocultural influences and social experience have been under emphasised, while the internally driven nature of representational change has been overestimated (e.g. Bodor & Pléh, 1994; Campbell, 1994; Olson, 1994). Karmiloff-Smith (1994) answers these criticisms by clarifying that she does not deny that external, socio-cultural factors generate representational change. Her argument is that externally driven factors (such as failure of a procedure to fulfil a goal, or communicative pressures from others) are not the *only* factor that contributes to representational change. The author states that "development should not be seen in 'either/or' terms. There are many different processes by which we learn, some of which are spontaneous and endogenously generated." (p. 739). However, as Clark and Karmiloff-Smith (1993) have postulated, in the main, the RR process is considered as being spontaneously generated from within:
"... although the RR process is indeed sometimes generated by externally provoked stimuli, it is a predominantly endogenous process" (p. 503).

In a more recent commentary of this aspect of the RR model, Gellatly (1997a, 1997b) argues that similar to Piaget's "individualistic" theoretical approach, Karmiloff-Smith's notion of representational redescription considers children as isolated thinkers and investigators of their worlds, rather than members of a specific social, and educational context (p. 37). The author emphasises that his intention is not to question the undoubted importance of representational redescription in cognitive development, but to argue that conceptual change owes far more to exogenous factors than is acknowledged by Karmiloff-Smith. Gellatly challenges Karmiloff-

[3.5] In Behavioral and Brain Sciences, 1994, vol. 17 (4), Karmiloff-Smith was invited to publish a Précis of *Beyond Modularity* for an open peer commentary. This commentary continued in Behavioral and Brain Sciences, 1997, vol. 20 (2). This section is informed, at great extent, by comments on the RR theory that theoreticians from the field of cognitive and developmental psychology published in the aforementioned volumes.

Smith's (1984, 1992) assertions regarding children's experience of block balancing by posing the following question:

> "How do we know that when a child develops more elaborate concepts of 'weight' and 'middle' her so doing is endogenously provoked rather than socially mediated? Karmiloff-Smith represents the child's experience of block balancing as isolated and individualistic: but a child may well have had instruction from an adult or peer outside the laboratory.... Furthermore, it would be interesting to know if children, in, say hunter-gatherer societies, also develop the belief that things balance at their geometrical centres."

(Gellatly, 1997a, p. 37)

In response to this criticism, Spencer and Karmiloff-Smith (1997) argue that Gellatly seems to confound the *content* of knowledge and its *source*, be it social or physical environment, and the *process* by which this knowledge is integrated into children's minds.

> "To think that verbally provided information is immediately stored as such and usable to solve relevant problems at any age is to take a very empiricist, non-developmental view of the child's ontogeny." (p. 52)

Spencer and Karmiloff-Smith insist on the idea that there exist internal processes by which children integrate into their minds information from various sources. Even though the *content* of knowledge may vary cross culturally, similar *processes* may be used by children to make distinctions which are relevant to their socio-cultural products, and subsequently *integrate* this knowledge into their minds. This should not draw attention away from the necessary interaction between social, physical, and internal cognitive environments.

Other theoreticians have welcomed Karmiloff-Smith's analysis of the internal sources of cognitive transformations (e.g. Deloache & Brown, 1987; Freeman, 1994; English & Halford, 1995). In particular, Estes (1994) recognises that although representational redescription is described as a primarily endogenously and spontaneously driven process, Karmiloff-Smith's contribution in making the process of knowledge explicitation explicit, and providing insight into the cognitive, representational changes that take place when a new area of learning is mastered, might have profound implications for the educational practice, which could try to actively "push" level I or E1 knowledge representations towards higher levels of explicitness (p.716).

Implicit / explicit knowledge representations

Kuhn (1994) agrees with Karmiloff-Smith's conception of a continuum between implicit and explicit knowing, and expresses her view that Karmiloff-Smith's most important contribution is the emphasis on the gradual explicitation of implicit knowledge; a major aspect of cognitive development which has been overlooked by research in the area of cognitive science. However, Kuhn notes that Karmiloff-Smith needs to elaborate more the metacognitive aspects of the reflective processes and the developmental process that she calls 'explicitation'. De Gelder (1994), on the other hand, argues against the assumed link between implicit and explicit representations. The alternative view that she supports, and for which she reckons there is ample empirical evidence, is that implicit and explicit knowledge representations exist and operate independently. However, de Gelder discusses the issue of dissociation between implicit and explicit knowledge by referring to data from adult studies. Karmiloff-Smith (1994) points this out and states that the fact that something may end up to be dissociated in adults does not tell us how it started off in development. Campbel (1994) argues that Karmiloff-Smith's account of the transition from implicit knowledge to E1, E2, and E3 levels of knowledge representation supplants the usually incautious discourse about 'implicit', 'explicit' knowledge. The RR model is considered as being far more precise than Piaget's discussion of reflective abstraction. However, the commentator emphasises that Karmiloff-Smith needs to develop a more adequate conception of the representation (p. 711). A similar view is expressed by Scholnick (1994) who considers that the RR model contributes in re-describing Piaget's theory regarding these aspects of knowledge, but also argues that Karmiloff-Smith needs to specify further what is a representation.

Carassa and Tirassa (1994) make the following, very interesting comment regarding the implicit/explicit distinction in relation to the procedural or declarative nature of knowledge representations. They posit that Karmiloff-Smith's account of the differentiation between implicit procedural and explicit declarative knowledge should take under consideration a further distinction: the distinction between procedural representations, and the possibly declarative representations of procedures.

"The former are actually encapsulated, similar to what RR posits... On the contrary, representations of procedures, if expressed declaratively, can be more easily disassembled into basic steps. Thus, decomposition requires no translation into different codes." (p. 711)

Also, for Bodor and Pléh (1994) it is not clear whether in the framework of the RR model level-I representations are considered as being equivalent to the 'knowing how' type of knowledge or to the 'knowing that' (p. 709). Karmiloff-Smith (1994) answers to the aforementioned comments by stating that these comments fall back into the dichotomy between implicit and explicit knowledge, while the RR model attempts to show that there are multiple levels of 'knowing that' that a simple dichotomy cannot capture. As for the 'knowing how', she states that her discussion on the different uses of the term 'procedures' in *Beyond Modularity* (1992) takes under consideration the distinction that Carassa and Tirassa (1994) draw attention to. On the basis of empirical data coming from the domain of drawing in particular, Karmiloff-Smith (1992) has acknowledged that procedural representations may be less confined by sequential constraints than her original definition predicted. According to her original definition, procedural representations were seen as being compiled procedures, sequentially represented and constrained; "...an unanalysable whole that is run in its entirety, with the components no longer accessible." (Karmiloff-Smith, 1994, p. 162). The data that show that the sequential constraint of procedural representations is weaker than Karmiloff-Smith predicted, at least in the domain of drawing, are discussed in the section devoted to the empirical testing of the RR model, together with Karmiloff-Smith's explanatory account of these new data.

Another issue that several commentators have drawn attention to, is the difficulty of clearly identifying and distinguishing levels, as well as the lack of data that can possibly support the hypothesised non-verbal explicit knowledge representations (i.e. E2 level) (e.g. Ohlsson, 1994; Shultz, 1994). Karmiloff-Smith (1994) considers that studies on children's gestures, which have been carried out by Goldin-Meadow and Alibali, provide exciting new evidence that support the multiple level distinctions that the RR model postulates, and show how explicit representations can exist prior to verbal ones. This series of studies together with others that support empirically the idea of multiple levels of explicitness, and in particular the operation of E2 level representations are presented in the section devoted to studies that have attempted to evaluate empirically certain aspects of the RR model.

Phases of knowledge explicitation: the issue of mechanism and sequence

The idea that explicitness, as a quality of knowledge, is not an all-or-none affair, but is progressively developed in phases resulting to representational flexibility and control, has been positively viewed by theoreticians. Estes (1994) considers the RR model as

"...the most detailed and plausible existing model for the development of what is arguably our most important and quintessentially human characteristic-the ability to make what we already know in some form increasingly explicit and conscious, and consequently more meaningful, flexible, and useful" (p. 715)

It has been acknowledged that most of the contemporary models do not go beyond the Implicit level of knowledge representation as the RR model does. Cognitive science needs to take a more 'developmental' view of cognitive phenomena, and the RR approach has been viewed as making a timely and important contribution towards this direction (Graham, 1994; Hampson, 1994; Kuhn, 1994; Shultz, 1994; Bodor & Pléh, 1994).

However, one aspect of the RR model that is vague is the mechanism that makes representational redescription possible (Dartnall, 1994; Zelazo, 1994). Karmiloff-Smith (1994) agrees, and makes a call for theoretical suggestions that can provide an answer to that. Certain theoreticians suggest that the study of the developmental period before behavioural mastery may contribute to the discussion of mechanism (e.g. Goldin-Meadow & Alibali, 1994; Vinter & Perruchet, 1994). In particular, Vinter and Perruchet (1994) suggest that behavioural mastery may emerge as a consequence of a first RR process, and may be grounded on prior explicit representations more often than Karmiloff-Smith assumes. Goldin-Meadow and Alibali (1994) argue for the un-blocking of the theory of representational redescription from behavioural mastery. They suggest that behavioural mastery is not necessary for redescription to occur. Goldin-Meadow and Alibali base this argument on empirical data from the domain of mathematical reasoning which have shown that children are able to express in speech or gesture beliefs about a given task, before they have mastered it (e.g. Alibali & Goldin-Meadow, 1993). According to the authors, this implies that "redescription has already gone on" (Goldin-Meadow & Alibali, 1994, p. 718).

A more radical view concerning the concept of behavioural success is expressed by Spensley (1997) who finds 'behavioural mastery' a rather problematic aspect of the RR model. Spensley stresses that Karmilof-Smith's RR model is the *only* (highlighted in the original) model of cognitive flexibility in the literature (p. 355). However, she reckons that development after success requires the operation of a completely different developmental mechanism following behavioural mastery from that which precedes it. She argues that cognitive flexibility is independent of behavioural mastery, and even suggests that "dropping the concept of behavioural success would allow representational redescription to develop into a more generally applicable developmental theory" (p. 355). However, because dropping the notion of success leads to theoretical problems with the phases and levels of the RR model, Spensley (1997) and Spensley and Taylor (1999) suggest a new model, the *Recursive Re-Representation (3Rs) model*, which evolved from the RR theory and attempts to overcome certain theoretical problems. The empirical data on which the newly suggested model is grounded come from the domain of drawing and will be discussed in the section devoted to the empirical testing of the RR theory.

These suggestions put under question the fixed order and sequence in which representational changes presumably take place. However, Karmiloff-Smith (1992, p. 24) has already suggested various possible models of RR (see section 3.3.2. of the thesis). It has also been emphasised that her argument is that behavioural success is one source of change, but not the only one. The occurrence of representational redescription before behavioural mastery would necessitate the modification of the format of the model, but it would not affect the concept of the process of representational redescription. It is also suggested that different solutions regarding this issue may apply to different domains (Karmiloff-Smith, 1992, 1994). Karmiloff-Smith's comments on the evidence that the Alibali and Goldin-Meadow (1993), and Spensley and Taylor (1997) studies provide, are presented in the following section together with the brief discussion of these research projects.

Even though *behavioural mastery* and its essential, or not, role in the mechanism that generates representational redescription, as well its space and place in the RR model, have been the object of dispute, it has also been acknowledged that the notion of post-success conceptual change may have a great educational effect. The pedagogical implications of knowledge redescription have been stressed by Goldin-Meadow and Alibali (1994) who underline the educational importance and contribution of the RR in understanding the process of learning after behavioural success. Similarly, Ohlsson (1994) considers that this notion constitutes Karmiloff-Smith's most important contribution. He refers to conceptual change after behavioural mastery and explains its possible educational implication as follows:

"This idea has not received much attention recently, although classical learning research from Ebbinghaus onwards established that *overlearning*, that is, continued practice after behavioural mastery, has strong effects on forgetting. Perhaps the need for postmastery representational change is the reason why educational programs that aim for behavioural mastery tend to have poor results in terms of conceptual understanding?"

(Ohlsson, 1994, p. 724).

However, Ohlsson emphasises that the mechanism of the RR hypothesis needs to be further specified, for the explanatory potential of the RR model to operate.

Closing this section devoted to critical theoretical comments upon the theory of representational redescription, it should be stressed that this was a selective presentation of theoretical views related to only certain aspects of the RR theory that are of particular interest and importance in he framework of this study. The RR model and its broader theoretical basis involve multiple ther aspects which have had a great impact since their theoretical introduction, and have been scussed in detail by theoreticians in the field of cognitive and developmental psychology. In 'eral instances, Karmiloff-Smith has emphasised that the RR hypothesis was intended as a nework, rather than a complete theory for exploring and explaining possible generalities in lopmental change across different domains. This framework is still open to concrete estions and further empirical testing that could shed light on its unspecified elements.

3.6.2. Empirical testing

The need for further empirical justification of the RR model has been acknowledged by Karmiloff-Smith (1992) as well as by theoreticians who consider that the model incorporates interesting and useful ideas which deserve careful conceptual analysis and thorough empirical grounding (e.g. Donald, 1994; Ohlsson, 1994; Spensley, 1997). In this section, certain studies which have tested empirically specific aspects of the model, and Karmiloff-Smith's postulations are presented. Most of these studies have focused on validating specific aspects of the RR model in domains like children's drawing and children's understanding of balance. Findings from a series of studies that have focused on children's changing gestures when explaining their problem solving behaviour are discussed. Some of these findings are derived from explorations using a mathematical task, and seem to provide interesting evidence regarding certain aspects of the RR. Finally, the terms employed in the RR framework have been used to explain, a posteriori, the derived data of a study that has taken place in the domain of early mathematics and focused on the development of mathematical notions.

Zhi, et al. (1997) and Spensley and Taylor (1999) tested Karmiloff-Smith's claim that at an early stage of skill acquisition, knowledge is internally represented in an implicit, procedural format which is sequentially constrained. Karmiloff-Smith (1990) supported this idea on evidence drawn from the domain of drawing by claiming that young children's drawings are produced by implicit, inflexible procedural representations.

In their study, Zhi, et al. attempted to replicate Karmiloff-Smith's findings in the task of 'drawing a man with two heads'. 3-9 year old children participated in the study which gave results that challenge certain aspects of Karmiloff-Smith's analysis. On the basis of their data Zhi, et al. argue that flexibility in drawing occurs at younger ages than Karmiloff-Smith originally claimed. They show that the specific task, in particular, turns out to provide only limited support for Karmiloff-Smith's position regarding initial, inflexible, and sequentially constrained knowledge. The explanation that the researchers provide is that possibly, the early flexibility that their study reveals is due to the fact that the particular task involves drawing of the human figure which is a highly practised drawing schema. Therefore, the internal representations underlying human figure drawings by many 3-4-year-olds have already attained Explicit (E1) status. Even though Zhi, et al. challenge Karmiloff-Smith's findings in the context of the particular task, they acknowledge that other studies of children's attempts to extend their drawings have shown inflexibility such as the one that characterises implicitly represented knowledge (level-I).

Similarly, Spensley and Taylor's (1999) findings in a study which used variations of the same drawing task, gave no evidence to support Karmiloff-Smith's hypothesis that young children are executing inflexible procedures when they are drawing a man. The types of modifications that children introduce to their drawings, and which were identified by Karmiloff-Smith, are replicated in this study. However, the recent data challenge Karmiloff-Smith's hypothesis for initial rigid, inaccessible and sequentially constrained procedures by which early drawings are generated. On the basis of this evidence, Spensley and Taylor (1999) consider the concept of behavioural mastery as problematic in the domain of drawing, and suggest a new model (the *3Rs* model) which drops the fundamental distinction between pre- and post success processes and postulates the operation of the exactly same mechanism for leading *to* and *beyond* behavioural mastery.

Karmiloff-Smith (1997, 1999) acknowledges the existence of recent data from the domain of drawing which show that flexibility occurs earlier than she originally claimed. Concerning the concept of behavioural mastery she notes that her more recent studies on children with learning difficulties have also shown that behavioural mastery is not sufficient for representational change. However, she answers to Spensley and Taylor's radical suggestions by emphasising that dropping the constraint of behavioural mastery does not change in any fundamental way the notion of representational redescription. The aim of the RR theory is to challenge the prevalent focus on negative feedback for learning, and show how positive feedback can generate change as well. In contrast to Spensley and Taylor's model which according to Karmiloff-Smith stresses quantitative change, the argument of the RR framework remains that both negative and positive feedback are essential, but play qualitatively different roles at different moments:

"I believe that external negative feedback plays a crucial role in reaching behavioural mastery (as ill-defined as that notion may be) in humans and other intelligent species. But it is internal positive feedback, in my view, that drives human development to higher levels of cognitive flexibility. Being able to work on stable, internal representations stored in long-term memory allows the human to derive novelty from within, and not rely solely on information in the external environment"

(Karmiloff-Smith, 1999, p. 327).

Other recent studies aimed at validating the levels of explicitness outlined in the RR model by studying children's performance on a balance task. Pine and Messer (1999, 2000) assessed and classified the representations of a large sample of 4-9 year old children on a balance-beam task according to levels of representation derived from Karmiloff-Smith's RR model. A detailed account of the criteria of this classification can be found in Pine and Messer (1999). Here an outline of the main findings of the study are presented. The majority of children (72%) were assigned to levels derived from the RR model. It was considered that the classification of behaviour according to various levels of explicitness rather than as dichotomous measures of success or failure, revealed important aspects of children's problem solving behaviour. The data of this study showed, consistently with the idea in the RR model that success for some children involves implicit representations inaccessible to verbalisations, whereas for other children involves explicit representations which children can access to explain their behaviour.

Level E1 was the main feature of many children's behaviour. For Pine and Messer (1999) "this has considerable implications for teaching science since it suggests a stage when the child's own naïve theory may render it resistant to teaching or instruction" (p. 24). However, a substantial number of children who were at Level E1 were also able to explain their centre theory which drove their problem solving behaviour. On the basis of this indication, Pine and Messer suggest that level E1 may in fact be two levels, consisting first of a non-verbal and then of a verbal representation. The researchers stress the difficulty to find behavioural correlates for the level E2 of knowledge representations, since Karmiloff-Smith does not offer an operational definition of this level. However, in the Pine and Messer (1999) study certain children were found to be able to make predictions without being able to give verbal justifications. The researchers consider that this problem solving behaviour could be an indication of a level of knowledge representation similar to that described in the RR model as level E2. A rather high percentage of the sample (28%) did not fall into any of the categories derived from the RR model. This is viewed as an indication that may be further levels which are absent from the model and require identification and explanation. Finally, there were children who were at level E3 but whose explanations could be described as strategic rather than conceptual: they described their own actions rather than the effects of these actions. Pine and Messer viewed this type of behaviour as a possible "transitional precursor to level E3" and full conceptual understanding (p. 25).

In a more recent study, Pine and Messer (2000) discussed the positive effects of interpersonal explanation, again in the domain of balance, in relation to Karmiloff-Smith's model. The particular focus of that study was whether generating interpersonal, verbal explanations can foster the transition from implicit knowledge to explicit understanding. The study suggests that a combination of social (discussion-verbalisation) and task experience is needed to bring about conceptual change. Children's verbalisable theories were shown to be more easy to change. This indication is considered as being in accordance with Karmiloff-Smith's suggestion that verbalisations may lead to reduction of procedural rigidity. This study, as the previous one, provides evidence that supports the idea that E1 level of representations encompasses two distinct levels: a verbalisable and a non-verbalisable. Pine and Messer's study suggest that the verbalisable E1 level (which they name Abstraction Verbal level) appears to be the level at which it is relatively easy for teaching to make an impact and bring about cognitive change.

Empirical support for the E2 level of knowledge representation which Karmiloff-smith (1992) characterised as non operational due to lack of empirical evidence, is provided by a series of studies on gesture-speech mismatch (e.g. Church, R.B., & Goldin-Meadow, S., 1986; Goldin-Meadow, et al. 1993; Alibali & Goldin-Meadow, 1993). These studies have shown that very often children, when asked to explain a concept, convey a different procedure in speech and a different procedure in gesture. Results from these studies showed that in the case of problem solving, children who exhibited a mismatch between gesture and speech were children who considered multiple hypotheses not only when explaining how they solved a problem, but also when they were actually solving the problem. Alibali and Goldin-Meadow (1993) and Goldin-Meadow, et al. (1993) based on evidence drawn from solving a task on mathematical equivalence, suggest that children (9-10 year olds) who convey a different procedure in speech, and a different procedure in gesture are in a "transitional knowledge state" which is described as a state at which multiple hypotheses and beliefs are simultaneously activated. After examining the whole repertoire of procedures and responses that children produced, Alibali and Goldin-Meadow (1993), suggest that children have a number of representations that are accessible only to one modality, primarily accessible to gesture.

The accessibility of a larger number of representations by gesture and not speech at the transitional state, suggests that children in this state "... have a an implicit understanding of a larger and more correct set of representations that they can explicitly articulate in speech" (Goldin-Meadow, et al., 1993, p. 290). With further tests, and using a recognition technique[3.6] the same researchers found that procedures produced in gesture only reflect an implicit awareness of that procedure. Implicit knowledge is usually defined as knowledge of how to perform a task correctly without being able to articulate this knowledge.

[3.6] For more information on this technique see for example Goldin-Meadow, et al. (1993).

The researchers make clear that knowledge which is exhibited in gesture but not in speech differs somehow from knowledge that is broadly recognised as being implicit because "...children can often demonstrate an awareness of a correct procedure in gesture before they can perform correctly on the task" (ibid., p. 291). This is considered as akin to the notion of redescription proposed by Karmiloff-Smith (1986, 1992) which accounts for the movement from a state in which knowledge is implicitly grasped to one in which it is grasped consciously and explicitly.

Finally, a study carried out by Fluck and Henderson (1996) examined the development of counting and cardinality in English nursery pupils (3,5-4,5 year olds). Data from this study indicated a notable developmental discrepancy between procedural and conceptual knowledge of counting: children do not understand the significance of counting for some time after they have learned how to count objects. Children who could reliably count did not show any understanding of cardinality. On discussing, a posteriori, this piece of evidence, Fluck and Henderson refer to the fundamental idea that supports the RR model, and use its terms to explain, the observed discrepancy. They conclude that "... the individual components of the counting procedure such as the final word and the one-to one correspondence become separately accessible, but until this happens the child cannot make the connection between counting and cardinality despite being able to count proficiently" (p. 515).

This section presented studies which have set out to validate specific aspects and features of the RR model, and studies which have employed and borrowed terms from the RR model to explain their data. The aim of this study is situated midway. The study does not set out to validate specific aspects of the RR by replicating previous findings. Rather, the aim is to explore and document the observed changes in children's problem solving successful strategies in arithmetic tasks, and study whether children's after-success changes in this particular domain can be explained by the model's postulations. The focus is on the interplay of different aspects of arithmetic knowledge and their micro-development in the context of a task. In this process, or as an outcome of this process, behaviours that confirm or contradict the model's predictions will probably be revealed. Therefore, validation of certain aspects of the RR model may be one outcome of this study but it is not the main purpose nor the initiating goal.

3.7. Summary

This chapter presented the theoretical ideas which provide the explanatory framework for the purposes of this study. The innovative and exciting aspect of these ideas is that children go beyond behavioural success. Given the interest of this study on that phase of children's behaviour in the context of arithmetic, i.e. on what happens *after success* in children's thinking and strategies, it is considered that the RR model can be used as an insightful guide for the exploration. The review of the theoretical ideas that support the RR model revealed certain issues which need to be taken under consideration.

It was shown that certain aspects of Karmiloff-Smith's theoretical hypotheses are still vague and need further specification. This makes the task of applying the model as an explanatory tool particularly difficult. Especially because this is attempted in a domain where the model's predictions have been applied in a limited extent, theoretically as well as empirically. Previous experimentation carried out by Karmiloff-Smith has shown that the model's premises can provide an account for macro- and micro-developmental changes in domains such as language, science, drawing and notation. Studies carried out by other researchers have confirmed but also challenged specific aspects of the model, and have suggested possible modifications. In particular, the qualities of the implicit level I behaviour and underlying representations as described by Karmiloff-Smith have been questioned in certain domains (e.g. drawing). Most of the empirical evidence for RR hypothesis relates to the transition from level-I to level-E1. The possibility for the E1 level to have a non-verbalisable but also a verbalisable aspect has been addressed. The difficulty that the non-operational character of the E2 level imposes has been stressed out, not only theoretically but also in an empirical context. Certain researchers have provided evidence that could put the E2 level into operation. It is particularly interesting that one of these studies (Alibali & Goldin-Meadow, 1993) provides such evidence in the context of a mathematical task. Finally, the possibility for two different aspects of the E3 level, a strategic and a conceptual, has been suggested.

Karmiloff-Smith (1994) notes that different solutions out of such difficulties, and answers to such issues may be found in different domains. It must be underlined once again, that the RR model is a developmental model that has not, so far, been used as an explanatory and analytical tool for the *micro-developmental* changes that occur in children's problem solving in arithmetic. Furthermore, most of the studies that have addressed, empirically, aspects of the model, have done so in a different methodological framework: differences between children who cover a wide range of age have been studied, large samples have been used, and the focus has been on particular *types* of changes that children introduce to their behaviour, not on the *process* of transition from one type of change to the other.

It is the task of this study to explore whether this new theoretical approach to cognitive change can account for the types as well as the process of change that occurs in children's behaviour in the micro-context of an arithmetical problem solving situation. Issues that the critical, theoretical review of the model and findings from empirical studies revealed are taken under consideration in the present exploration. These issues will be discussed again after the presentation of evidence that this study has to offer.

Chapter 4 Methodological framework

4.1. Overview

After having outlined the theoretical basis upon which the study is situated, this chapter discusses methodological issues in consideration of which the study has been designed. Two methodological approaches drive the collection and analysis of the data. The micro-developmental method and the clinical method of interviewing will be combined and used as methodological tools of exploration. The chapter is organised as follows. The first section presents general theoretical issues concerning the micro-developmental method. In the following section, methodological considerations regarding Karmiloff-Smith's applied method for the study of micro-developmental changes will be highlighted. General theoretical characteristics of the clinical method of interviewing will be presented in the third section. The chapter closes with a section devoted to the presentation of the design of the study.

4.2. The micro-developmental method

4.2.1. General characteristics

Developmental research has, so far, mainly focused on analyses of cognitive changes that occur macro-developmentally, that is over an extended period of time. The value and the rationale of applying *the micro-genetic* or *micro-developmental* method of research when cognitive change is put into focus have been long acknowledged.[4.1] This type of research has been proven to be a promising approach when the focus is on the *process* rather than the products of cognitive change. This is because, within the micro-developmental approach, change is studied moment-by-moment, during a short time, i.e. a number of experimental sessions over weeks or months (Kuhn, 1995; Siegler, 1996; Miller & Coyle, 1999).

Within the Soviet theory of activity, it has been long emphasised that any mental function must be studied in a *process* of development. Vygotsky (1978) criticises psychological

[4.1] The term "micro-genetic" originates from Vygotsky's genetic method of analysis, while the term "micro-developmental" has its roots in the Piagetian tradition. Although these two schools in developmental psychology agree on little else, in this case, the difference in terminology does not imply any difference in meaning. Here, the term micro-developmental is preferred for reasons of consistency with the employed term within the RR model.

research which has mainly studied concepts and skills "only after they have become fossilized" (p. 68). Vygotsky argues for the importance of studying and analysing the *process* of change, rather than examining procedures which are developmentally complete.

Wertsch and Stone (1978), and Wertsch (1985) underline that the central claim in Vygotsky's *genetic method* is that human mental processes can be understood only by considering and examining the origins of these processes, and the transitions that lead up to their later form. In Vygotsky's genetic analysis the notion of *microgenesis* is considered in the context of psychological, experimental procedures. According to Wertsch and Hickmann (1987), in such a context the investigator has the opportunity to observe how individuals become familiar with a skill, concept or strategy within a limited observational session. This is how the researcher becomes aware of "the microgenetic processes involved in the formation and execution of a psychological process" (Wertsch, 1985, p. 54). Further more, Wertsch (1985) points out two basic types of microgenesis that Vygotsky recognised:

"The first type of microgenesis identified by Vygotsky concerns the short-term formation of a psychological process. The study of this domain requires observations of subjects' repeated trials in a task setting... The second type of microgenesis is the unfolding of an individual perceptual or conceptual act, often in the course of milliseconds" (p. 55).

Within the framework of the *contextual event approach* which is also Vygotskian oriented, Rogoff (1982) argues that in order to understand human cognition and development, it is necessary to study the particular context or situation because cognitive activities are considered as structured events which integrate person and context.

Central to this approach is the use of the event or activity as the unit of analysis. The contextual event approach integrates change and development as part of the event. The focus on events and activities implies that processes of transformation and change are inherent to the phenomenon observed and constitute the focus of study. In consequence, *process analysis* rather than correctness or incorrectness of outcome, is stressed. The contextual approach puts the micro-developmental method into focus and stresses its pertinence for examining processes of development.

"Microgenetic studies examine the qualitative and quantitative transformations in a person's skill over the course of an event, such as learning a game, solving a problem, or gaining expertise in a particular realm."
(ibid., p. 156).

Rogoff also addresses the issue of *generalizability* of data generated by micro-developmental studies. She clarifies that, although the analysis is relevant to single events, the approach does not preclude the application of the same micro-analysis to a number of similar facts. When studies involve only few cases, the researchers balance the small sample with more intensive analysis. Intensive analysis aims at the examination of all relevant pieces of data provided in the event.

"This may be contrasted with the standard methods used with large samples, in which most of the variation observed is regarded as random and relegated to the error term, requiring large numbers of subjects to observe an effect. The error term generally accounts for the greatest proportion of variance in such studies. In small-n analysis, an attempt is made to account for more of the observed behavior" (ibid., p. 152).

Moreover, Rogoff (1982) notes that the *reliability* of the micro-analysis is assured, because the researchers do not apply firm coding systems as in conventional research. Consequently, the reported data are not abstracted far from the observed event. In micro-analysis, researchers provide explicit evidence for their interpretations and also give excerpts of raw transcripts to the reader to confirm the validity of the suggested interpretation.

Siegler (1996) presented a detailed account of micro-developmental methods. The following sections are mainly informed by Siegler and his colleagues' account of the microdevelopmental method of data collection and analysis, because they revitalised the approach by applying it mainly in the domain of arithmetic. Siegler's (1996) main consideration is, that in most studies of cognitive development, the occurring changes are inferred by comparing behaviour before and after the change. According to the author's view, this indirect method cannot depict changes in children's thinking that do not follow the most imaginable route.

Siegler refers to Karmiloff-Smith's study on the modifications that children introduce to their (adequate) notational systems and argues, that the short-lived regressions that the experiment revealed would not have been detected without a close examination of the performance as it was changing. For Siegler, information about the endpoints of change is useful but cannot substitute the detailed examination of changing competence that micro-developmental methods of research allow. Progress in understanding the mechanisms that produce change in cognitive development requires the production of detailed data about particular changes in the course of their occurrence and evolution.

The micro-developmental method is considered as particularly appropriate in studying changes while they are occurring. Siegler and Crowley (1991) highlight the following key properties that define the micro-developmental approach:

a) Observations of individual children span the entire period of rapid change in the domain of interest. That is, from the beginning of the change to the time at which it reaches a relative stable state.
b) The density of observations is high relative to the rate of change of the phenomenon.
c) Observed behaviour is subjected to intensive trial-by-trial analysis with the goal of inferring the processes that give rise to both quantitative and qualitative aspects of change.

(Siegler and Crowley, 1991, p. 606).

The characteristic that mainly differentiates this approach from most longitudinal developmental methods which sample the thinking of children at different ages, is the density of observations during the period of change. It is this particular aspect of the method that allows the temporal resolution that is needed to inform the understanding of change processes. This is because, data resulting from densely sampling changes provide information about what actually happens during the rapid periods of change. In this case, understanding of change processes is not limited to inferences from the performance before and after the change (Siegler, 2000).

Dense observations of behaviour are followed by intensive analysis of *both* qualitative and quantitative aspects of change. This allows the generation of differentiated descriptions of particular changes, and makes micro-developmental methods highly pertinent as a source of information about how change occurs. (Siegler and Crowley, 1991).

Siegler (1996) indicates two variants of the method that have been used by investigators. First, a task from everyday experience can be selected and used. The researcher formulates hypotheses about the types of experiences that typically provoke changes in performance and provides a higher concentration of such experiences that would otherwise occur. Second, the researcher can present a novel task and observe children's changing understanding as they interact with the task during a single session or over multiple sessions.

4.2.2. Strengths, criticisms, and inherent difficulties of the method

Micro-developmental methods are considered as highly pertinent to provide information about all aspects of change: path, rate, breadth, variability, and sources of change (Siegler 1996; Miller & Coyle, 1999). Here, the characteristics that constitute the strengths of the approach in relation to the study of change in strategy use, in particular, are briefly summarised.

* The micro-developmental method allows observation of strategy-discoveries as they are being made and thus gives valuable information about the qualitative sense of the discovery process.
* The method can indicate the conditions under which the most frequent changes take place and allows observation of short-lived, transitions strategies that emerge before the strategy of interest.
* The method can also reveal paths of development that are not imaginable. For example, regressions in thinking about a given task.
* Because micro-developmental studies involve repeated presentation of the task to the same children and the sessions are close in time, they can provide detailed information about how quickly a new strategy comes to be consistently used after the first time that it was employed in a given problem.
* Micro-developmental studies can also provide information about the extent to which children generalise a new strategy. The dense sampling of performance, both before and after the construction of a new strategy, yields data on whether a new strategy is immediately applied to different types of problems that it is useful for, or it is later extended.
* Finally, micro-developmental studies have revealed different types of experiences that can lead to change; cognitive conflict, failure as well as success.

(Siegler & Jenkins, 1989; Siegler & Crowley, 1991; Siegler, 1996)

Kuhn (1995) refers to the most common criticism to which the method has been subjected. It has been argued that close examination of change over short periods of time, may address issues of learning rather than development. Kuhn addresses the criticism by arguing that modern research has made clear that learning processes share all of the complexity, organisation, structure, and internal dynamics which were once attributed exclusively to development. Learning is now recognised to be more like development in many fundamental aspects. This is also acknowledged by Siegler (1996, 2000) who argues that a clear distinction between phenomena of development, and phenomena of learning is, in any case, very difficult to establish. Development and learning are two processes so complexly connected to each other that a clear separation of their influences is not possible. As Siegler (1996) has put it:

"...many of the most striking 'developmental' phenomena involve differences in learning at different ages.... Regardless of whether microgenetic methods are viewed as providing information about learning or about development, they allow us to compare the ways in which changes occur at different ages."
(p. 179)

Another issue that is discussed regarding the micro-developmental method is the issue of whether change that occurs in micro-developmental studies resembles change as occurs in natural settings. Miller and Coyle (1999) note that this concern is more relevant to some studies than to others. Most experiences in micro-developmental studies are very close to natural settings. For example, the repeated experience with mathematics problems (e.g. Siegler & Jenkins, 1989). Miller and Coyle argue that in the classroom, children can be (and are often) asked to solve a particular series of problems over several days, as in a micro-developmental study. Kuhn (1995) addresses the same issue by noting the following:

"...there is no independent characterisation of the natural change process against which microdevelopmental data can be compared. To the contrary, it is the microgenetic method that has been advocated as the most promising means of insight into naturally occurring change" (p. 138).

Despite the acknowledged value of the method in revealing important aspects of cognitive change, the number of studies that employ the micro-developmental method of research is very limited. This is because such studies are time consuming and have a high cost in terms of effort. In the course of repeated sessions, the researcher has to conduct individual interviews and manage to keep children's interest in the task at hand. Video-recording is usually used to capture the occurrence of shifts and changes in each individual's performance. The occurrence of such events is difficult to be anticipated for each particular case, since it varies depending on the task and the children that participate in the study (Siegler & Jenkins, 1989; Siegler, 1996).

4.2.3. Some general findings from micro-developmental studies in mathematics and other domains

Micro-developmental studies conducted within diverse domains, and by investigators with diverse theoretical predispositions, show convergence in several important findings (Kuhn, 1995; Siegler, 1997). This section highlights the four more consistent findings regarding aspects of cognitive change that micro-developmental studies have revealed in mathematics and other domains.

a) One important finding involves the inter-individual and intra-individual *variability* in problem solving behaviour. In particular, micro-developmental studies conducted in the domain of arithmetic problem solving (e.g. Siegler, 1989; Siegler & Jenkins, 1989; Alibali & Goldin-Meadow, 1993), and number conservation (e.g. Siegler 1995), have revealed a great degree of variability during the process of change. Variability in strategy use has been observed both between children of the same age, as well as within the same child when engaged in solving the same type of problem, in the context of a problem solving session, or across sessions.

b) Another consistent characteristic of cognitive change, as shown by micro-developmental studies, is that change is *gradual*. Even after children have discovered more sophisticated ways of dealing with a problem, they continue to use less sophisticated approaches as well. This has been found in the domain of problem solving (e.g. Wertsch & Hickmann, 1987; Fireman, 1996), numerical equivalence (e.g. Church & Goldin-Meadow, 1986), number conservation (e.g. Siegler, 1995), and addition problems (e.g. Siegler & Jenkins, 1989).

c) Micro-developmental studies have revealed that *strategy discovery may follow success as well as failure*. This finding on the basis of which (among others) Karmiloff-Smith built her model of Representational Redescription has been also revealed by the micro-developmental study that Siegler and Jenkins (1989) conducted in the domain of arithmetic, and the study that Miller and Aloise-Young (1995) conducted working with preschoolers in a 'same-different' task.

d) Finally, another consistent finding is that change in strategy use, as well as the discovery of new strategies are constrained and guided by children's *existing knowledge* (Siegler & Jenkins, 1989; Siegler, 1997, 2000).

Micro-developmental studies have been critical in establishing the aforementioned aspects concerning cognitive change and how it occurs. This is why this methodological approach is viewed as a valuable and promising tool in exploring processes of change (Kuhn, 1995; Siegler, 2000). After the delineation of general theoretical aspects of the method that mainly guide the collection and analysis of data in this study, the following section discusses particular issues concerning the way in which the method was applied in Karmiloff-Smith's explorations. The presentation of these considerations provide a basis for the subsequent presentation of the design of the study which is informed by (but also differentiated, in certain aspects, from) both the formal descriptions of the method, as well as its application in the light of the RR model.

4.3. Methodological considerations on Karmiloff-Smith's micro-developmental explorations

Karmiloff-Smith (1979) had noted that, at the time, relatively little work had focused on spontaneous micro-developmental changes which occur during an experimental session. Such an exploration would contribute to understanding the *process* of cognitive change *while* it was occurring.

Representational redescription is considered as a process that pervades human macro-development. However, it is argued that its occurrence can be also established in the study of *micro-developmental changes* that occur within the bounds of an experimental session. (Karmiloff-Smith, 1984, 1992).

In a series of explorations focusing on children's problem solving, and in the light of the RR model, Karmiloff-Smith studied children's goal-oriented behaviour *in progress*. The design of each experiment and the choice of the population depended on the particular aim, the context and the domain that the exploration was focusing on. However, certain general features that characterise her methodological approach can be outlined. Karmiloff-Smith (1984) underlines that the aim that pervaded her work and differentiated it from most other developmental studies is twofold. First, she aimed to ascertain that superficially similar successful behaviour could stem from different underlying representations. Second, she aimed to demonstrate that children go beyond success in their problem solving. Her explorations were designed and carried out on the basis of very specific considerations which are described below.

The focus was not on success or failure at solving the problem. Rather, the exploration was focusing on the interplay between children's spontaneous action-sequences in *micro*-formation and the changing modes of representation underlying these sequences.

Given that Karmiloff-Smith's intention was to document and analyse the processes that underlie children's problem solving behaviour, it was considered as necessary to design tasks that would be simple for children to solve, successfully, during the limited time and space of the experimental session. Yet, the tasks should be stimulating, and challenging enough in order to keep children's interest as long as the session would last.

The process of data collection was involving individual interviews which were video-recorded. The experimenter was focusing on children's *overt behaviour*; i.e. all the actions, focus and shifts if attention, corrections, hesitations, long pauses, distractions, gross eye movements and verbal comments.

Qualitative analyses were favoured over quantitative ones. The protocols from Karmiloff-Smith's studies mainly consisted of detailed descriptions of children's actions. The analysis therefore, was focusing on the interplay between the child's sequences of action and his/her implicit theories, i.e. the underlying representations which are inferred from children's sequence of actions rather from children's verbal comments.

The importance of focusing and analysing action sequences is particularly underlined by Karmiloff-Smith (1974) who argues that "action sequences are not merely a reflection of the child's implicit theories. The very organisation and reorganisation of the actions themselves, the lengthening of their sequences, their repetition and generalised application to new situations give rise to discoveries that will regulate the theories, just as the theories have a regulating effect on the action sequences." (p. 207).

The advantage of this approach which is based on the observation and analysis of subjects' overt behaviour, rather than merely of their verbal explanations, is that it allows the avoidance of *conscious reflection techniques.* These techniques involve the analysis of "talk aloud" protocols which record children's linguistic explanations regarding the problem solving process (Silver, 1987b). Karmiloff-Smith (1984) argues that problems of subjectivity are inherent in such techniques because children do not easily talk about their problem solving behaviour. In her work, conscious introspection techniques were avoided. The experiments were designed in such a way that children would express clues of their internal representations not only verbally, but mainly in the their external behaviour. Changes in external behaviour were considered as possible indicators of children's reorganisation of internal processes (Karmiloff-Smith, 1981).

Yet, Karmiloff-Smith (1979) remarks that the problems of interpreting metaprocedural behaviour are enormous because it is highly interpretative. The child's internal representation can never be elicited but merely various external expressions of it and how they develop. Therefore, in children's problem solving, the changing functions and status that the actions have for the *child* must be put into focus and not the fact that for the *observer* actions may seem to be superficially analogous or different.

It must also be highlighted that in Karmiloff-Smith's work it was children's *spontaneous* behaviour that was put into focus. Once the goal of the task was set and explained by the experimenter, children were let to devise their own means in order to achieve the goal. Children were allowed to regulate themselves their own interaction with the materials involved in the task. The experimenter usually did not intervene.

On the basis of this methodological approach, children, in her studies, "channelled the problem solving task themselves so that their ongoing activities inherently contained evidence supporting or channelling the hypotheses developed by the experimenter. Only rarely were the hypotheses tested via the experimenter's verbal questions, and then, in an interactional fashion, according to the dynamics of the ongoing behavior, rather than in terms of a predesigned set of questions to be posed at specific points during the task" (ibid., p. 46).

Certainly, considerations based on Karmiloff-Smith's methodological applications inform the design of the study. However, within the framework of this design, the micro-developmental method of research is combined with the clinical method of interviewing. The way in which the two methods will be combined and the reasons for this combination will be discussed in the section devoted to the presentation of the study's design. Before that, general theoretical aspects and characteristics of the clinical method will be presented in the section that follows.

4.4. The clinical method of interviewing

4.4.1. General characteristics

The use of the clinical method as a research tool originates from Piaget's investigations of children's thinking. As Opper (1977) notes, Piaget needed a method that would give children the opportunity to "verbalise freely", and to researchers the opportunity to infer "the covert intellectual processes" (p. 91). Thus, he designed a similar method to the clinical interviews used in therapy, in order to explore children's thinking and reasoning. In Opper's view, the partially standardised clinical method that is used especially in the context of cognitive psychology, is a "diagnostic tool applied to reasoning in children" and its fundamental character is that "it constitutes a hypothesis-testing situation permitting the interviewer to infer rapidly a child's competence in a particular aspect of reasoning by means of observation of his performance at certain tasks" (p. 92).

The clinical method takes the form of a dialogue between the interviewer and a child (the subject of the study). The interviewer, having a guiding hypothesis, selects and presents to the child a certain task which involves a specific material, i.e. objects that are placed in front of the child. Ginsburg, et al. (1983) name this form of clinical interview which involves concrete objects and a problem to be solved, *revised* clinical interview. This type of the method is differentiated from the *verbal* form of clinical interview which involves questioning of individual children only on a verbal level.

Typically, when the clinical interview involves a specific task and material, the researcher poses verbal questions related to the situation. Children are led to predict, observe, and explain the results of the actions performed on the concrete material (Opper, 1977). The procedure aims at eliciting intellectual activities, accounting for the nature and organisation of cognitive processes, and evaluating the level of the child's cognitive competence (Ginsburg, 1981; Ginsburg et al., 1983).

The following question of the researcher is determined by the child's response. This dimension of the clinical interview which Ginsburg (1981) and Ginsburg, et al. (1983) name "contingency of questioning" is considered as being the essence of the clinical interviewing. Both, the child's verbal responses and actions upon the concrete objects that the task involves provide the researchers with information, so that they can test the initial hypothesis. If the initial hypothesis is not confirmed, the researcher can reformulate it considering the child's responses. The interviewer can also ask further questions and introduce additional material to clarify these responses.

The interviews are usually audio- and/or video-recorded for later analysis and reflection upon children's responses and actions. Video-recording the interview, includes non-verbal data (e.g. gestures, eye movements, actions upon objects) and thus gives the researchers the opportunity to review problematic parts of the session, evaluate their questioning technique, or test alternative interpretations (Rowland, 1995; Goldin, 2000).

4.4.2. Strengths and weaknesses

The above description underlines the main characteristics of the clinical method. Having the form of a dialogue, the clinical interview entails an interactive communication between the researcher and the child. In this communication process, language plays an important role. As

Hunting (1996) underlines, the role of the language as well as "the importance of clarification of meaning" are central in this method, as the researcher asks questions and poses problems each one of which is an invitation for the child to reflect and explain an action or solution (p. 5).

When the first response of the child is given, an interview can follow a number of different directions. Further tasks may be prepared or refinements of tasks, if the researcher wishes to clarify certain aspects of the child's thinking. Hunting (1983), stresses the fact that "because of the dependent relationship between the child's response and the investigator's questions, no two children will ever receive exactly the same interview. It follows that the interviews can vary greatly across subjects in any one experiment." (p.48). Thus, the clinical interview as an interactive process has a non-deterministic nature and gives to the researcher the freedom to pose a variation of the problem, re-word a question, or involve some extra material so that the child will have every opportunity "to display behavior from which mental mechanisms used in thinking about that task or solving that problem can be inferred." (Davis & Hunting, 1991, p. 209). It is precisely the fact that the clinical interview is an open-ended technique that differentiates it from other, non-clinical, less open-ended data gathering techniques, and allows the exploration of hidden structures and processes in the individual's thinking (Clement, 2000).

The use of a specific material is another important feature of the clinical method. The manipulation of physical materials, and the child's action upon them, reveals to the researcher information about the child's thinking and reasoning when the verbal responses seem to be obscure. Designing situations that involve not only verbal questions but also a concrete material to work upon, the researcher "...sets the stage in which the playing around will take place. To do so, he or she designs an experiment, or microworld, that is both conceptually rich and meaningful to the child. It can be a puzzle, a mechanical gadget, or a computer-based game" (Ackermann, 1995, p. 346). As Ackermann underlines, the *playing around* is not less important than the *talking about.* Because graphical explanations can be collected together with oral ones and clarifications can be sought where appropriate, clinical interviews are considered as a valuable tool for the exploration of the depth of conceptual understanding (Clement, 2000). The freedom that gives the researcher the opportunity to incorporate more or simpler tasks, reformulate a question, change or add extra material, constitutes one of the great advantages of the method. The clinical method as an open ended process, gives room for unexpected and insightful statements from the part of the children and as Hunting (1996) stresses, this flexibility allows the thorough exploration of children's thinking processes in addition to an account of the results of that thinking.

Another advantage of the method is the degree to which it can motivate both the researcher and the subjects of the study. The conversational and playful character of the setting makes the children relax and thus maximises their willingness to participate, talk and express their thinking. Hence, "mutual engagement" can be attained although the motives of the researcher and the children are totally different.

Ackermann (1995), discussing the nature of the clinical method brings to light another of its aspects, which is its relevance to learning. Drawing on her own experience as a researcher, Ackermann claims that the use of the technique as a means to gain insights into children's thinking, has proved to be "an ideal vehicle to foster learning", as within the interactive setting of the interview, and through exploration and argumentation, children have the opportunity to learn about the phenomenon under study. In Ackermann's words, the clinical interview, if well conducted, fosters mutual learning as a result of the interaction:

"The experimenter learns to invent and design a series of questions on the fly, and derive experiments as a way to uncover 'hidden cognitive events' in the child's mind. The child learns to explore a phenomenon, to probe and explore his ideas in a variety of ways, which happens to be a good way to stretch his views." (p. 347).

However, the clinical method has undergone several criticisms. Davis (1993) expresses his concern as to the degree to which results from clinical interviews are "artifacts" of those interviews, because of their high dependency on the interview's particular context. According to the author, the fact that those interviews' outcomes are "significantly influenced by the observations and participation of the research workers", and also that "slight changes in the format of questions or answers, or in the interviewer's intent might lead to quite different results" should be considered seriously before any attempt to generalise or make inferences of mental functioning (p.51).

Opper (1977), within the framework of a detailed discussion upon the different aspects of the clinical interview, claims that the lack of standardisation of procedural techniques may give rise to concerns about the generality and comparability of the results, however, it should not be ignored that this is the rationale of the technique: an absence of standardisation which results in the flexible structure, required to investigate children's thought processes.

4.4.3. Technicalities

In most of the studies on the clinical method and its applications in research, many different steps and procedures are suggested in order to facilitate the method's use in practice. Preparation of an interview plan is considered to be absolutely necessary in order to maintain a certain level of standardisation so that each child is initially confronted with the same task. An interview plan helps the interviewer to keep the dialogue focused on the original research direction, and also to systematise the later analysis in order to ensure the comparability of the results.

Ackermann (1995), stresses the necessity of a "mutual engagement": "Both partners need to be deeply interested in pursuing the journey together despite their initial disparity of motives." (p.346). Engaging the children in the interview setting is one of the most difficult tasks. Researchers should give time to the children to become familiar with the setting and with them before the interview commence.

Care should also be taken in the designing of the tasks. Hunting (1983), underlines that the researcher as a task developer, should design and propose tasks which give validity to the problem that is studied by engaging the children in reflection and conversation, and also tasks that are appropriate to the child's reality, i.e. tasks that "...make sense to the child as well as to the investigator" (p.49).

Attention should also be given to the fact that young children tire very easily, so the time of an interview session should be estimated in such a way that the interview will not exhaust the children but at the same time the maximum of the available information will be obtained.

Regarding the researcher's intervention during the interview, neutrality is a major factor that an interviewer should pay attention to. Although, according to Hunting (1996), it is not wrong to provide the child with information and there are benefits in seeing if and how far a child can progress with some assistance, the interviewer should encourage as much as possible children's verbal explanation, though avoiding to direct in any way their responses. As Rowland (1995, 1999) underlines, the primary goal of the clinical interview is not to teach the child but to inform the interviewer. Although it is often difficult, especially when the researcher is also a teacher, giving feedback during testing should be avoided.

With regard to the degree of intervention in clinical interviews, and particularly in research in mathematics education, Goldin (2000) recognises that the repeated questioning alone may affect the individual's problem solving processes. However, this type of intervention is part of the task environment, and a general feature in task-based clinical interviews. In the framework of this methodological approach, it is the individual's behaviour in the context of structured interventions that is observed, and aspects of the individual's internal, cognitive processes that are inferred, in the context of structured interventions as well. Goldin (2000) argues that this should not be a concern regarding the methodology. In his own words:

> "Though results cannot be interpreted as corresponding to what would have occurred in an entirely free situation, this is not a 'limitation'. It is simply a fact about human interaction, about the phenomenon under study; mathematical problem solving during discourse with another person. There is nothing in this to discredit the methodology" (p. 521).

4.4.4. The use of the clinical interview as research method in mathematics education

Ginsburg (1981) and Ginsburg et al. (1983) argue for the value of the clinical method as a research tool into children's mathematical thinking. The gathering of rich data in the form of written protocols in which the clinical method results, is considered as a valuable basis for inferences about strategies, operations or underlying cognitive processes involved in mathematical activities. Similarly, Goldin (2000) views task-based clinical interviews as a valuable tool in designing research in mathematics education; a tool which allows the collection of qualitative data that help researchers infer and describe children's conceptual understanding, and internal constructions of mathematical meaning.

The clinical research method as a tool for investigating children's mathematical learning is mainly based on the general principles of the constructivist perspective which, within the framework of mathematics education "strongly emphasises the direct observation of children engaged in mathematics activities" with the intent to explore, and understand the internal processes that rule children's mathematical thinking and reasoning (Davis & Hunting, 1991, p. 212).

Cobb and Steffe (1983), argue that children's construction of mathematical knowledge is greatly influenced by the experiences that children gain through their interaction with adults. Therefore, "the technique of the clinical interview is ideally suited to the psychological objective of investigating a sequence of steps children take when constructing a mathematical concept." (p. 84). In accordance with the former position, Hunting (1983), notes the several purposes for which the clinical method is adopted in research in mathematics education.

Among them, the following two: a) "identification and classification of behavioural strategies exhibited by children", and b) "study of short term effects on a child's performance in the course of an interview when investigator's assistance is provided." (p. 48).

Even though this study is not directly situated within a constructivist framework, the last mentioned purposes that the clinical interview serves, when applied in constructivist research, fit in with the purposes of this study. The combination of the clinical interview with the micro-developmental method, at least at the level of purposes, seems to be possible. Also, according to Rogoff (1982) such a combination is feasible because the development of cognitive activities over a brief period such as an experimental session, is consistent with the clinical method's practice and focus on the microanalysis of an experiment, or a child's response to a task. The rationale of this combination, for the purposes and the needs of this study, is discussed in the following section.

4.5. The design of the study

In the previous sections, theoretical aspects of two methodological approaches were presented, together with certain methodological issues concerning the application of the micro-developmental method in the light of the RR model, which supports theoretically this study. In this section, the design of the study is presented. Also, the way in which the aforementioned methodological approaches are combined in supporting it is discussed.

It must be remembered that this study aims at documenting changes and modifications to children's strategies while solving an additive task. The interest is focused on changes that children introduce to their strategies after the achievement of an efficient solution. Changes are studied micro-developmentally. That is within the context of a specific form of task that is presented to children during a sequence of sessions. Before proceeding to explaining and justifying the methodological choices and applications, it is essential to give information on the population and sample that the study focuses on, and also describe the setting and the task upon which the empirical part of the study is based.

4.5.1. Study focused on cases

The study follows the qualitative paradigm within which the methodological assumption is that research is an inductive process of framing a generalisation from particular cases. The generalisation predicts that what has been observed to be true in certain instances will apply more widely. Research methods which have in common the decision to focus as inquiry around an instance, are placed under the 'umbrella' term *case studies.* The case study researcher observes the characteristics of an individual unit that can be a child or a class or even a community, having as a purpose to catch the complexity and analyse intensively the behaviour of this individual with a view to establishing generalisations about the wider population to which the individual belongs (Cohen et al., 2000). Detailed descriptions of individual cases have the advantage of communicating the sense of the quality of the cognitive activity under examination (Siegler and Jenkins, 1989). Thus, the choice to work with a certain number of cases rather than with an extended sample is grounded on the need to carry out dense observation and intensive analysis of qualitative changes in each individual's performance, as the micro-developmental exploration entails.

4.5.2. Population and sample

The population that the study focuses on is 5 to 6 year-old children. That is children in year 1 of Primary. Children selected from a year 1 class of a Southern England infant school constitute the sample of the main study. Children's selection was made after a 6-week observation in the classroom during the primary mathematics lesson. During this period the observer had the opportunity to work with all groups of children and follow their progress in arithmetical activities involving addition. Twelve children were initially selected to participate in the study. The main criteria for their selection were:

* The degree of their competence in counting, and mental calculation in additive tasks.
* Their verbal articulation.
* Their facility and competence in using arithmetical notation.
* Their willingness to work with the experimenter.

Because the study focuses on children's evolving strategies during a relatively limited number of sessions, and also on changes that children introduce to their problem solving behaviour after success, children most competent in addition were selected to participate. Thus less time would be devoted to consider arithmetical misunderstandings and errors. Twelve is not the final number of cases that the study focuses on. Data collected after working with two children, out of the twelve that were initially selected, are not presented here. This is because one of the children, in four sessions, completed only some steps of the task (though correctly) but never the whole task. It is considered that this happened mainly because of the child's difficulty to stay concentrated and interested in the task until its completion, and not on the child's lack of skill. Nevertheless, the non-completion of the task in all the four sessions that the child participated in, does not correspond with what is considered as 'successful completion' of the particular task (see section 4.5.3.). The other case, data from which are also not presented, is the case of a child who was absent for a long period of time after having participated in two sessions. It was considered that the intervened time was far too long and this differentiated this case from the other ones. For these reasons these two cases of children were excluded from the sample of cases. Ten (four boys and six girls) is the final number of cases that the study focuses on, and from which data are presented and discussed in the chapters devoted to the main study.

4.5.3. Ethical considerations

Ethical considerations may arise from the early age of children who participate in the study, and especially from the intention to video-record the interview sessions. These issues were addressed by writing to children's parents asking their permission. The purposes and methods of the study were explained, and the confidentiality of the video-recorded material was assured. There were no objections or hesitation expressed by parents concerning their children's participation. Also, the children's anonymity was protected by using pseudonyms.

4.5.4. The design of the tasks

Rogoff (1982) argues that in any situation (laboratory, school, or home), aspects of context which play an important part in cognitive activity, are involved. Thus, in any situation, the researcher must attend to the contribution of the context. Rogoff also emphasises that "no situation provides a window on *true* cognition". However, it is suggested that because performance seems to be relatively domain-specific, "the task should represent a domain of specific interest rather than an arbitrary exercise assumed to tap a general level of functioning" (p. 146). In consideration of this position, a justifiable and causal connection between the choice to use a specific task and the research aim is sought.

This study aims at exploring the organisation of different pieces and forms of knowledge into a unified strategy, and subsequently documenting the evolution and reorganisation of the strategy. It is considered that an appropriate task should be one that involves multiple steps. This gives the opportunity to observe more clearly: a) the hypothesised, initial application of different methods for each of the steps and, b) the subsequent movement to the organisation of different methods and different underlying forms of knowledge, into a strategy applied for the whole task. Also, the duration of a task involving multiple steps was expected to be long enough so that changes and modifications, from one step to another, would be likely to occur and be observed even during a single trial. With this in mind, the following task was chosen to be used in the empirical part of the study.

'Cards'

Children had to produce families of number combinations. In particular, the task required from children to find *all* the possible number bonds that result in a specific number each time which is called the 'target' number (e.g. find all the possible number bonds to make 9, or 10 etc.). A pile of cards with incomplete number sentences, such as the one below, was at children's disposal. Children had to pick up one card at a time, and complete the number bond until there were no more possible ways to do so. The completed number sentences were put in a column. Children had to complete the number sentence in all the possible ways. The task was repeated with different 'target' numbers. Not all children were given the same amount of target numbers. This depended on how quick each was in solving the task, and on how quickly he/she was producing changes in his/her problem solving approach.

$$\boxed{\square + \triangle = 9}$$

Figure 4.1
Example of cards used in the 'card' task.

The size of target-numbers increased gradually. In order to successfully solve the task, a number bond could not be repeated. This means that children should not complete two number sentences in a way that a number which was used as first addendum in one sentence was also used as first addendum in another one. For example, writing $5 + 4 = 9$ two times was considered as repetition. But writing $5 + 4 = 9$ and then $4 + 5 = 9$, was not considered as repetition. Children could complete the number sentences by swapping around numbers that had been used previously.

Each number bond that children produced was considered as a step within the solution procedure. It was hypothesised that eventually children would develop a strategy for organising the number combinations, in a way that allows them to know whether all the possible number bonds have been found. One strategy that was found to be developed by most of the children (initially in the context of the pilot study and later in main study) was the 'ordering' strategy. The 'ordering' technique (i.e. the strategy of putting the number bonds in order) allows the production of all the possible number combinations without the need of counting or any type of calculation. For example, to find all the ways to make 4, one can simply complete the missing first addend in each number sentence by going forwards, and the second by going backwards or, the other way round. (as shown in figure 4.2).

0 + 4	4 + 0
1 + 3	3 + 1
2 + 2	2 + 2
3 + 1	1 + 3
4 + 0	0 + 4

Figure 4.2
Example of the possible arrangements of number bonds after the application of the 'ordering' strategy.

Children who developed the 'ordering' strategy had previously used a method for deriving number bonds from previous ones. For example if the number combination [3+6] had been previously produced, children were adding 1 on number 3 and taking away 1 from number 6 to create the [4+5] number bond which also results to the same target number. In certain cases, it was the consistent application of the 'deriving' method which transformed it to a strategy and which led to the introduction of the idea of 'ordering'. Different children developed different ways for organising the solution process employed. However, in one or the other way, the rationale which underlay the 'deriving' method and/or the 'ordering' strategy was found to be involved in every type of organisation of the solution process. This is why it was considered that it was necessary to obtain evidence regarding the conceptualisation of the 'deriving' method and the 'ordering' strategy, as well as of the flexibility and explicitness of the representations underlying their application. For this reason variations of the main task were introduced:

The 'missing numbers' task

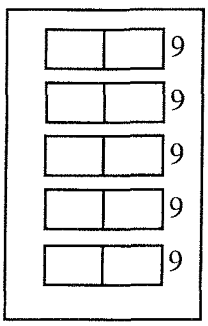
3+6=9 11+4=15
..+5=9 12+..=15

Figure 4.3
Examples of pairs of number sentences used in the 'missing numbers' task.

Pairs of incomplete number sentences, such as the sentences shown above, were presented to the children. Children were asked to fill in the missing numbers. These pairs of number sentences do not constitute part of a series of number sentences produced in a sequence as in the 'card' task. However, the arithmetical relations between the addends of the number sentences in each of these pairs is the same with the arithmetical relation between the addends of number bonds when produced in a sequence following the 'ordering' strategy in the 'card' task. This task was introduced after children had developed, as hypothesised, the 'ordering' strategy in the 'card' task. The aim was to see whether children recognise the applicability of the rationale behind the 'ordering' strategy in this situation, and can apply it to complete number bonds in a context other than the sequential production of number combinations as required in the context of the initial task.

The generalisation of any type of organisational strategy (including the 'ordering' strategy), and its application to tasks with similar goals, but different superficial characteristics, was also tested. For this aim the following tasks were introduced.

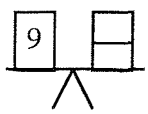
<u>'The domino'</u>



On a piece of paper, a column of blank dominoes was shown. Children were asked to put a number of dots in each part of the domino so that all together makes the target-number.

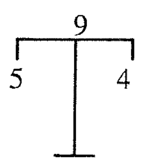
<u>Figure 4.4</u>
The 'domino' task.

<u>The 'balance'</u>



(a)

<u>Figure 4.5</u>
Balances on paper



(b)

<u>Figure 4.6</u>
Balance on cardboard

Two variations of this task were presented: (a) On a piece of paper, balances as the one shown above, were drawn. On the left side of the balance, a number was written on a block. This was the 'target' number. On the right side of the balance, there were two blocks put one on the top of the other. Children had to write a number on each of the blocks. Their sum should be equal to the number written on the left so that balance could be achieved. (b) Children had a set of plastic numbers at their disposal. A 'target' number was given to them. They had to put a number on each arm of a balance made on a big piece of cardboard, so that all together makes the 'target' number. The procedure continued until all the possible number combinations had been found.

The 'domino' and the 'balance' tasks are only superficially different from the initial, main task. The goal remains the same. Yet, the intention was to, gradually, go further and further away from the original task. Since in the framework of the RR model higher, levels of explicitness of one's knowledge representations are assumed to allow the generalisation of knowledge in similar goals, the introduction of these tasks aimed at testing children's ability to generalise the developed strategy to other problems in which it can be useful. The use of two variations of the balance task mainly served in keeping the task and the procedure interesting for children. The use of concrete material usually makes a task more appealing to them and keeps their interest longer.

Generalisation, in particular, was tested not only at the level of tasks, but also at the level of 'target' numbers. Gradually, bigger 'target' numbers were introduced. If representational explicitness, and conceptual understanding underlie children's strategy, then children should be able to easily apply the strategy for 'target' numbers that they consider as "too big". That is numbers that possibly involve too many combinations to be written. Higher conceptualisation of the rationale and the potential of the strategy was expected to enable children to affirm and anticipate that the strategy can be successful for any number; no matter how "big" a number is, the application of the strategy will eventually lead to the generation of all the possible number combinations.

4.5.5 Data collection

The tasks were presented to each child individually. Each session took place in the school, but in a separate room, other than the classroom. Sessions, which lasted for 30-45 minutes approximately, were video-recorded. Video-recording served the subsequent analysis and interpretation because it helped to retain a rich record of children's overt behaviour, that is verbal explanations but also actions, gestures, eye movements, and hesitations. Video-recording took place within children's awareness. The purpose of video-recording was explained to them.

Experience drawn from a pilot study that was carried out before the main study, showed that children did not seem to be annoyed by the presence of the camera.

The number of sessions that each child participated in was fixed. Again, it was after conducting the pilot study (which is discussed in the following chapter) that it was shown that five sessions provide the time and space that children need to develop and evolve new strategies, and engage with all the additional tasks introduced. This means that the experimenter worked with each child for five days (one session per day). The initial intention was to work with each child for four successive days, and carry out the last session a few days later in order to examine the degree of consolidation of the developed strategy. Both in the context of the pilot as well as the main study, it was soon shown that this schedule of sessions was not feasible. The main reason for this was that, in the case of the pilot as well as the main study, sessions with each child were taking place at the same time when the mathematics lesson was taking place in the classroom. The same child could not be taken out of the classroom and miss the mathematics lesson for four or five consecutive days. This explains the intervened time of one day (in most cases) between some of the sessions, particularly the last two or three sessions. Other reasons for which the initially intended schedule of sessions proved to be difficult to maintain was the absence of some children for reasons of illness, the requirement for some children to participate in specific classroom activities on specific days, or the participation of the whole classroom in out of school activities. The best of effort was made so that conducting the study would cause the least possible inconvenience to the everyday life of children in the classroom and the school. Thus, the initially planned schedule of the sessions had to change to a more flexible one. The schedule of sessions that each child participated in is presented in the profile of each case in the chapter devoted to the main study.

4.5.6. Intervention

To justify the choice of combining the micro-developmental method with the clinical interview, certain points of methodological differentiation between this study and Karmiloff-Smith's micro-developmental experimentation, need to be clarified.

Karmiloff-Smith, in her micro-developmental explorations, intended to focus specifically on changes that children introduce to their problem solving procedures spontaneously, driven by endogenous, rather than exogenous causes. Hence, in Karmiloff-Smith's experiments, the researcher's intervention was minor and children were let to interact freely with the task at hand (Karmiloff-Smith, 1979). In the case of experiments where the feedback (negative or positive) was playing a central role, Karmiloff-Smith designed tasks in which the physical material itself was providing feedback to the subjects (Karmiloff-Smith, 1974, 1984).

This study focuses both on spontaneous modifications as well as on changes that may be triggered by external causes. The reason is this: first, although the RR model's premise that verbal explanations should not be considered as the only indication of representational change and conscious control of a situation is taken into consideration, the study does not preclude the role that the interviewer's questions, and subsequently children's explanations and descriptions, may play in triggering representational change. Also, following Karmiloff-Smith's explanation on what constitutes an 'exogenous' cause (see chapter 3, section 3.4.), the study also focuses on changes in children's strategies that may be introduced out of the need to meet the requirements of the task. For example, the need to generate *all* the possible number bonds and *know* if all the possible number bonds have been found. It must be stressed and clarified what is meant here by the interviewer's intervention, and what type of feedback was given to children. In the context of this study free problem solving was encouraged in order to allow observation of children's spontaneous behaviour within the particular setting. However, the interviewer intervened in instances such as the ones described below.

The problem situation that was presented to the children involved an arithmetical additive problem. The task was such that the material itself could not give negative or positive feedback concerning the arithmetical correctness of the result. Thus, the researcher intervened to correct an arithmetical error that might occur during the solving procedure, or to reassure, if necessary, the child for the arithmetical correctness of the result.

On no occasion did the researcher intervene to provide guidance concerning a strategy that could possibly be applied. Right at the beginning of the session, it was explained to children that they could choose and apply any method they wanted in order to solve the task.

As already explained, the task involved multiple steps. Each number bond was viewed as a step in the procedure of generating all the possible number bonds that result to a specific number. During the solving procedure, children were asked to describe how they completed a step; which method they chose to apply in order to retrieve a number bond. For example, questions that were asked were: "How did you know that?" or "How did you choose these numbers?" or "What did you do to find this number bond?" Such questions were asked consistently.

Children's responses gave indication of the method applied and the rationale that supported the choice of the particular method.

When a unified strategy had been developed, the interviewer intervened to seek indications of the extent of children's understanding concerning the rationale and the effectiveness of the strategy. For example, in the 'card' task, and when/if the 'ordering' strategy had been developed, the solving procedure was frequently interrupted. Children were then asked to anticipate, without actually writing anything down, the numbers that were going to constitute the following missing combinations. In that case, the previous completed number sentence was not written so it could not help them in keeping track of the numbers. This is an example of a 'violation' of the routine, which aimed at testing whether children realise the effectiveness of the 'ordering' strategy and they keep using it in order to produce mentally the next missing combinations.

Finally, the researcher intervened to remind or further explain the requirements of the task to the children. For example, the requirement to find out all the possible number bonds, or the requirement not to write the same number bond twice. In the case that a child believed that all the number bonds had been found, and there were no other missing, the interviewer asked the child to justify this. Questions that were asked were: "How do you know that there are no more?", or "How can you tell that there is no number bond missing?" If the child insisted that all the possible number bonds were found, but actually they were not, the researcher intervened to say that there were still some number bonds to be found, and prompted the child to find them. If the child could not find any other way to complete the number sentence, the procedure would stop and another target-number would be given. Justifications at that point were important because they provided indication of whether the child was aware or not that the task was completed, and whether she or he could base this belief on the applied strategy, its potential and its efficiency.

4.5.7. The rationale for combining the two methods of research

After having presented and discussed the particularities of the design and aim of the study, the rationale for combining the micro-developmental method with the clinical interview seems to have a more justifiable basis. The need to apply the micro-developmental approach as a method of data collection and analysis is inherent in the study, since the focus is put on changes and transformations that occur within the micro-context of a task in the course of a number of sessions. On the other hand, because interaction between the child and the experimenter is not precluded by the design of the study, the application of the clinical method of interviewing seems appropriate. Especially because, aims that the clinical method serves - such as the identification and classification of children's behavioural strategies as well as the study of short term effects on a child's performance in the course of a session, (Hunting, 1983) - are certainly consistent with the aims of the micro-developmental exploration of this study.

The clinical method gives to the interviewer a high degree of freedom regarding the manipulation of the material and the re-formulation of questions. This freedom is a very important factor when working with young children. In the case of a very shy child with difficulty engaging in conversation with the researcher, which is a very usual phenomenon when working with children of such an early age, the material involved in the situation and its features may trigger the child's attention and interest, and can be used as the starting point of the conversation.

Also, at this early age, children tend to change their responses to a question very easily and frequently. When the researcher has doubts as to the stability of a particular response, the structure of the clinical method is flexible enough to allow changes or refinements of the questions or the task. Changes in the task that may even involve the introduction of additional material, can help the researcher develop a better understanding of the child's thinking about the problem under consideration.

4.5.8. Data analysis

Dey (1993) defines *analysis* as the process of "...resolving data into its constituent components, to reveal its characteristic elements and structure" (p. 30). The analytical process allows the researcher to present an account based on the re-conceptualisation of the data, rather than just rely on intuitions and impressions about the data.

In this study, the methodology for analysis follows the qualitative paradigm in which observations of sessions are the data source for the interpretation. Whereas in quantitative research the process of analysis is deductive and proceeds from theories to specific hypotheses, in qualitative research, researchers tend to analyse their data inductively; a process which involves framing a generalisation after studying particular cases and moving on to theories. Also, in qualitative research, data analysis does not constitute a distinct phase as in quantitative research. Instead, "...research design, data collection and analysis are simultaneous and continuous processes" (Bryman and Burgess, 1994). In the context of this study as well, data were interpreted in the course of their collection. It is on the basis of this initial interpretation that decisions concerning the plan of subsequent sessions with each child were taken.

The study examines children's *evolving strategies* as applied in an arithmetical task, and also children's progression to higher levels of explicit use of a certain solving technique. Thus, analysis focuses on *changes*. After reading thoroughly the transcribed protocols of each case, the instances in which a change and transformation of the solving procedure could be identified, were highlighted. Coding helped in differentiating different types of change that occurred. For example, changes in the problem solving approach, changes in efficiency and solution time, changes that occurred spontaneously while others were triggered by the investigator's intervention, and also changes that were more stable than others.

Instances in which explanation, justification, or verbal report of the activity provided by the child were identified, were also highlighted. Children's verbalisations were categorised according to what they indicated: e.g. justification, planning, anticipation, reflection, "out loud reasoning". Such verbalisations were also classified according to whether they were triggered or not. It was examined whether different types of children's verbalisations could be reasonably correlated with the occurrence or not of a qualitative change in the solving procedure. As in the case of the observation and data collection, the subsequent analysis and interpretation focused on children's *verbal and non-verbal behaviour*. Children's gestures, eye / hand movements, and hesitations provided a significant basis for interpretations and access to children's level of representations. Non-verbal behaviour in particular, is significant for the identification of transitional states one of which is the beginning of the redescription and re-organisation process (Alibali & Goldin-Meadow, 1993, pp. 516-518).

Overall, the process of analysis and interpretation focused on changes that occurred:

a) In aspects of children's problem solving activity (such as problem solving approach, efficiency, confidence, solution time)
b) In children's verbal and non-verbal behaviour (this includes verbal report and description of activity, explanations, justifications, movements and gestures)

Through the various and qualitative different changes that occurred in the aforementioned aspects, it was made feasible to follow the itinerary of each case. This itinerary of changes is described and subsequently discussed in the framework of the profile of each case that is presented in the chapter devoted to the main study.

Interpretation of changes that took place in children's solving activity (behavioural/procedural level) in connection with their overt behaviour enabled inferences on the child's representational, and conceptual explicitation. Inferences were made on the basis of the conceptual framework that the RR model provides. Steffe and Wiegel (1996) draw attention to the fact that, frequently, mathematics educators borrow and apply theories from outside of the field of mathematics education. The authors emphasise that the principles of a learning or developmental theory should not be applied naively. A misapplication of a theory to the practice of mathematics education leads to misinterpretation and distortion of the theory's main principles. The consideration of Karmiloff-Smith's model of *representational redescription* provides a valuable framework for the design and development of the study's conceptual framework. The application of a model of cognitive and developmental nature such as the RR model for the analysis of the data proceeded with the greatest caution. Certain aspects of the model were thoughtfully applied for the development of concepts that can explain the phenomena under study.

In this chapter the design of the study was presented. A pilot study, which was carried out before the main empirical study, played a significant role in the consideration and final consolidation of the aforementioned aspects of the design, as well in the construction and clarification of ideas that drove the interpretative process. The ideas that provided the basis upon which interpretations and analysis were built are presented in chapter 6; a chapter devoted to the introduction of the main study. Before that, the following chapter presents and discusses a brief example of data from the pilot study the conduct of which contributed to the clarification of practical as well as conceptual aspects of the design and the analytical process.

Chapter 5 The pilot study

5.1. Overview

A small-scale empirical work was carried out at the end of the first year of the study. This chapter presents the objective and the design of this exploratory work together with certain indicative results and findings that informed the main study with respect to methodological, theoretical and analytical aspects.

5.2. Objective

The pilot study was carried out at a period of time when the hypotheses had just been formulated, and the theoretical direction that the study would follow had been chosen. Also, certain ideas concerning the design of the empirical work had been developed. It was considered that a small-scale empirical work would provide the field for testing the hypotheses and research questions. The aim was to test that the project was feasible and also, that the theoretical model that was chosen to support the study, could provide the conceptual framework needed for the analysis of the phenomena observed.

Various methodological aspects were to be tested such as the experimenter's intervention, the possible questions and challenges, children's interaction with the particular tasks. Thus, the design of the pilot work was not particularly structured. In a relatively free manner, the interviewer experimented with various options concerning the form of questions, the presentation and order of the tasks, as well as the number of sessions that each child participated in. Also, the interventions were not following accurately, the standards and limits that are put by the design of the study, as this was presented in the previous chapter. The unstructured nature of the pilot study, gave the opportunity to consider different options concerning the aforementioned methodological aspects. On the basis of experience drawn from this exploratory work, decisions were taken and were subsequently consolidated to structure the design of the main study. In the following sections, the case of a child who participated in the pilot is briefly presented. The presentation of this case provides a basis for the discussion of the main issues that the pilot study helped in clarifying, regarding methodological aspects as well as the conceptual framework for the analysis of data.

5.3. Setting

Five children from a year-1 class of a South England infant school were selected to participate in the pilot study. A three-week observation in the classroom preceded the children's selection. The selection was based on the criteria that were described in the previous chapter (see section 4.5.2.). The sessions took place in the school, in a room other than the children's classroom. The tasks presented in the previous chapter were used. The balance task, however, was used only as a paper-and-pencil task.

5.4. Cases: an example

As explained before, the design of this exploratory work was purposefully unstructured. As a result of this, not all children participated in the same number of sessions. More time was given to children whose problem solving behaviour presented particular interest considering the aims of the study. Among the five cases, one, which provided the greatest amount of relevant data, is selected to be examined in this section. Examination of this particular case provides an indicative example of the phenomena that were expected to occur, and be observed, during the main empirical work. Interpretation and analysis of the data from this particular case is briefly presented, also as an indicative example of the analytical approach.

The case of CHRIS

Chris was 5 years 8 months old when the empirical work began. He was classified by the teacher as in the most advanced group in mathematics. Chris participated in five sessions. Four sessions took place on four successive days. The fifth session took place one week later. The reason for this was the intention to examine the degree of consolidation of his employed strategy. Each session lasted from 30 to 40 minutes. A brief summary of the most indicative points during each session will be given. Frequently, description is followed by short comments.

Key:
I: the interviewer C: Chris ( ): movements, actions [ ]: writing

First session

[6+1]
[5+2]
[4+3]

Number 7 was given as the target number in the card task. Chris produced the number combinations that are shown on the left. After writing down the first number, Chris counted using his fingers to figure out the second. After the three first number sentences that had been completed, the interviewer asked Chris how he chose which number to write first. Chris explained:

C: Cause 6 it's just next to 7.
I: And why did you choose 5 after that?
C: Cause it's 1 more.
I: And how did you choose 4 afterwards?
C: It' s 3 more.

The first number that Chris chose was the one which was closest to the target number. This choice allowed him to count less. The rationale underlying his choice was made even clearer later when he was asked to find out all the possible ways to make 8.

C: [7+1]
[0+8]
C: I've put 7 because 7 is a quick way to make 8.
Then 0 and 8... you don't do anything you just put 8.

After completing the first three or four number sentences by applying this 'economical'-in counting method, Chris started swapping around the number bonds that were already produced.

[6+1]
[5+2]
[4+3]

[3+4]
[1+6]
[2+5]

In the case where the target number was 6, Chris completed the next three number sentences as shown below. Even though Chris produced all the possible number bonds, when asked if he had finished he said: "There are more, but I don't know them". At the end of the session, it was evident that Chris was not aware of his success, or of the completion of the task.

Second session

In the second session, when Chris successfully completed the task (target number, 11), the interviewer asked him if he had finished, or if there were some number bonds missing.

C: I need the number line for this one.
I: Why do you need it?
C: Because I've done all these (shows already completed cards) and they are lots and I don't know anymore. I want the number line to help me.
(takes a number line and looks at the numbers) I've done them all.
I: How do you know?
C: Cause... (shows already completed cards and puts his finger on each number of the line) 1.. I've done 2.. I've done 3.. I've done 4, 5, 6, 7, 8, 9, 10, like that.

The need to check if all the possible number bonds were produced, made Chris think of the number line. To check, he focused his attention on the first number of each number combination. When 8 was given to him next as the 'target' number, Chris completed the number sentences as follows:

C: [7+1]
[0+8]
[8+0]
[1+7], I saw this one (he explains showing the [7+1] above)
[2+6], Because there is no 2 there (shows the already completed cards)
[3+5], Then it gets higher and higher and higher…
I: Which one is getting higher?
C: (shows first numbers in the last three number bonds)

To produce the first four number bonds, Chris used his 'economical' method, and the 'swapping' method. Then, contrary to his 'economical' method, he chose 2 to start with and counted on to find 6. He did the same for the next number bond. Subsequently, Chris noticed a regularity that attracted his interest. However, his attention was partially focused on the first number of the number bonds. The interviewer prompted him to de-focus and notice what happened with the second numbers as well.

C: It goes 7,…6, 5 (shows from the top to the bottom). Oh, 5, 6, 7 (shows from the bottom to the top). The lowest numbers go down there (shows 1, 2, 3 from the top to the bottom) and then like that (shows second numbers from the bottom to the top)…like a zigzag.
4 and 4! **[4 + 4]**
I: How did you think of that now?
C: I don't know. (looks at the cards and without probing, starts counting) 1, 2, 3, 4 (shows first numbers from the top to the bottom).
4, 5, 6, 7 (shows second numbers from the bottom to the top). They go backwards!
I: That's very good. Why do you think this happens?
C: I don't know.

After the interviewer's prompting, Chris noticed the regularity in the second numbers of the last three number bonds as well. Numbers in the position of the first addendum were in ascending order by steps of one, and numbers in the position of second addendum were in descending order also by steps of one. Chris quickly produced the next number bond which followed the regularity. But when asked to explain how he found this new combination of numbers, he did not give an explanation.

[0+8]
[1+7]
[2+6]
[3+5]
[4+4]
[5+3]
[6+2]
[7+1]
[8+0]

Chris produced the next two number bonds by applying the 'ordering' strategy. He explained: "I have the number line in my head". At the end, Chris put all the cards in the order that is shown on the left. When asked if there were any more number bonds missing, he said that he did not know. In the following run with the task ('target' number-12), Chris did not apply the 'ordering' strategy. He went back to using his initial methods.

Third session

In the third session, 6 was given to Chris as a 'target' number, again. He applied his initial methods to find the number bonds. The number bonds that he produced were put in the order that is shown on the left. When asked if he had finished, and if all the possible combinations had been found, Chris started moving his pencil from one number to the other, from one column to the other, uttering the numbers in order.

[0+6]
[6+0]
[5+1]
[1+5]
[2+4]
[4+2]
[3+3]

C: 1, 2, 3, 4, (finds and shows these numbers in this order, looking at the second column of numbers).
5 (shows the 5 which is in the first column of numbers), 6 (shows the 6 which is in the second column of numbers).

Although Chris did not apply the strategy to solve the task, he realised the need to put the numbers in order, so that he could check if all the possible combinations had been found. However, it was evident that he had not grasped, conceptually, the rationale of his strategy, because he checked the numbers that he had used, moving from one column to the other. He did not consistently check if all the possible numbers had been used as first addends and second addends.

Chris fully applied the 'ordering' strategy in the 'balances on paper' task. When finished, Chris said right away:

5	4	3	2	1	0	6
1	2	3	4	5	6	0

C: **No more.**
I: How do you know?
C: 1, 2, 3, 4, 5, 6, 0 (shows second numbers from the top)
That's all the different that you can make.

As opposed to previous runs of the task, Chris did not put the number bonds in order after their completion. Ordering was not used just as a checking tool at the end of the solving procedure. Rather, ordering was used as a strategy for successfully solving the task.

Fourth and fifth session session

In the fourth session, Chris fully applied the 'ordering' strategy in the 'card' task, for higher 'target' numbers (11, 14). He completed the number sentences very quickly. He just looked at the previous card, and filled in the missing numbers in the next one. From time to time he went back and repeated, starting from the top, the numbers of the first column and the numbers of the second column. When asked why he put the number bonds in this order, he justified the use of the strategy as follows: "It's easier… I know when I finish". In the fifth session, 19 was given as a 'target' number in the 'card' task. Chris fully applied the 'ordering' strategy. When he first discovered the strategy and started applying it, the first addendum in the first number bond that he was writing, was always the bigger: the one closest to the target. For example, if the 'target' number was 10, the first number bond would be either 10+0, or 9+1. This time, as in the previous session as well, the first addendum of the first number bond was smaller than the second: 1+18. This came in opposition to his initial 'theory' that was guiding his method at his first runs with the task. This probably happened because Chris realised that the use of the new strategy allowed him to avoid counting altogether; it allowed him to be successful and even more economic in effort.

Chris did not apply the 'ordering' strategy in the 'domino' task. He went back to the initially applied methods. He wrote down the first number and then counted with his fingers to find the second one. Then, he swapped around number bonds that he had already produced.

5.5. Discussion

In the previous section, particular moments during Chris' interaction with the task were highlighted. The completion of the series of sessions and data collection was followed by a first attempt to approach and interpret these moments in the light of the theoretical premises that the RR model holds. That attempt is presented herein.

Initially, Chris approached the task by focusing on each step separately. Also, Chris' solution procedure can be viewed as consisting of two parts. In the first part, he applied his "theory" of economy in order to choose the first addend. His "theory" said that choosing the number which was closest to the target, would allowed him to count less, in order to find the second addend. This method was supported by two different pieces of knowledge. His "theory" could be seen as an example of *declarative* knowledge: *he knows that* with this particular choice, less effort will be needed. His initial method is also supported by the *procedural* knowledge of counting; i.e. the procedure which gives him the second missing addend. The point after which a second part in the solving procedure can be differentiated is where Chris started using the 'swapping' method. He started looking at the already completed cards and changing the numbers around. This method was supported by his declarative knowledge of the principle of commutativity. He *knows that* in addition, he can change the numbers, of a number sentence, around, and still obtain the same result.

The combination of these two methods allowed Chris to complete the task successfully. However, he was not aware of his success since he kept looking for more number bonds even after all the possible number combinations had been generated. In those initial runs, Chris focused his attention on each step of the solving procedure separately. Each one of Chris' steps in the procedure can be viewed as a separate unit of behaviour which was elaborated enough for successfully achieving its goal. This approach was followed consistently for some time. It can be considered that at this phase, and in terms of the RR model, a first level of *procedural* success had been reached.

Two significant instances followed: the need to check if the task is actually completed, made Chris think of the number line. It is at that point that, for the first time, Chris thought of the utility of *ordering*. Subsequently, Chris noticed a particular pattern, a regularity in the numbers of certain number combinations that were put in a column. Chris, followed the regularity to produce the following number bonds. Chris completed the task by applying a new strategy: the 'ordering' strategy.

An important change had occurred. Chris moved away from the initial mixture of isolated, though successful, methods and discovered a new strategy which he applied consistently for all the steps in the task.

Gradually, Chris proceeded to the consistent application of the strategy in every single run. Chris treated what was previously a sequence of isolated problems as a single one. A unified strategy was generated and applied to the whole of the task. Chris had shifted from his initial success-oriented approach to a new one which allowed him to organise the different steps of the task into a consistent whole. The above mentioned changes in Chris' employed strategy gave indications of behaviour which had gone over to a *meta-procedural* level, in the sense that, Chris was at an organisation-oriented stage where a new, simplified procedure was generated. The notion of ordering and the realisation of its utility constituted the single representational framework on the basis of which a new, simplified, and more economic strategy was generated.

Nevertheless, subsequent runs with the task and discussion with the interviewer revealed that Chris had not conceptualised the reason of the effectiveness of his strategy. He did not provide an explanation of why putting numbers that were used as first and second addendum, in that order, and in that relation (one more/one less) generated the number combinations which resulted to the required target number. It could be claimed that Chris' meta-procedural behaviour entailed a second level of success which involved the generation and application of a strategy the conceptual basis of which was not yet, fully, and explicitly represented.

5.6. Points on which the pilot study informed the design and analysis of the main empirical work.

The examination of the particular case was presented merely as an indicative example of the possible changes and modifications that children who participated in the pilot study introduced to their problem solving approach. The discussion of this case gave the opportunity to attempt a possible interpretation of the phenomena observed under the light of the theoretical framework that supports this study. In spite of the limitations in time and cases that a pilot work entails, this early exploratory work informed the design of the main empirical work and subsequent analytical process significantly.

On the level of the design and methodological approach, it was shown that:

* Five sessions were enough for changes to be introduced and the newly generated strategy to be evolved and consolidated. At the same time, with this number of sessions the micro - scope of the study was respected and maintained.
* Children seemed to have a productive, with respect to changes introduced, interaction with the tasks. They did not seem to have a difficulty in appreciating nor in achieving the goal of the tasks.
* In cases where the session lasted more than 35-40 minutes the interest of the child in the task was very difficult to be maintained. Longer sessions did not provide richer data in respect of changes in the problem solving approach or children's verbalisations.
* Encouragement of free problem solving provided very interesting data on changes that children introduced to their successful problem solving approaches. The pilot study helped in consolidating the form of questions that children should be asked so that these questions were clear to them.

On the level of the theoretical framework it was shown that:

* Qualitative changes do occur in children's successfully employed strategies within the specific setting that supports the empirical part of the study. This finding is consistent with findings from previous micro-developmental studies in arithmetic (e.g. Siegler & Jenkins, 1989).
* The form and type of changes that the RR model accounts for, seem to pertain to the changes and modifications observed in children's overall problem solving approach within the setting of an arithmetical problem solving situation such as the one developed for the purposes of this study. These changes indicated the passage from initial success-

oriented behaviour to an organisation-oriented, 'meta-procedural' phase in the RR framework during which children as problem solvers acquired a better control over the features of the task.

Finally, difficulties that were met in the attempt to interpret and analyse observations that were made during this exploratory work showed that:
There was a need to approach the particular problem solving situation by differentiating two planes that interpretation on the basis of the RR model should focus on: that of children's overall problem solving approach where organisational attempts were observed, and that of children's methods and strategies. Such a differentiation would allow the subsequent differentiation and clarification of two different aspects of the RR model as an interpretative framework: the *phases* of children's problem solving behaviour, and the *levels* of explicitness which can be ascribed to different methods and strategies that children may employ in the course of solving the task. It was shown that the plane of children's overall problem solving approach (*phases* in the RR framework) needed to be differentiated from that of the methods and knowledge that children were calling upon (*levels* of explicitness in the RR framework) for the purposes of the analytical process. Furthermore, for a detailed and in depth analysis of the particular problem solving situation, one should consider the procedural and conceptual facet of each method employed. This differentiation would serve not only the data analysis and presentation, but also the study of the two aspects of the RR model (phases and levels), as they interrelate and can be possibly ascribed to the different types of arithmetical knowledge (i.e. procedural, declarative, conceptual). The aforementioned considerations and the way they drove the analysis of data derived from the main study are discussed in detail in the following chapter.

Chapter 6 The main study: framework of analysis and form of presentation

6.1. Overview

This chapter opens the presentation of the main study. It presents the rationale which drove the analysis of the data. The requirements and particularities of the problem situation, and children's response to them were made known during the pilot study, and in the first stage of analysing the data derived from the main study. It was during this primary phase of dealing with the data that a framework of analysis started being formulated. This framework informed the process of analysis constantly. The analysis and discussion of data in the following two chapters were conducted, and are presented in accordance with this particular framework. This, introductory to the main study, chapter comprises two sections devoted to the framework of analysis, and the form into which the data are presented. The aim is to facilitate the reading and comprehension of the analysis which follows.

6.2. The planes of analysis, the grounds upon which interpretation has been built, and its inherent difficulties

Interpretation of changes in children's behaviour in a multi-step arithmetical problem such as the one used for the purposes of this study is a very difficult task. The difficulties stem not only from the model itself and its complexity, but also both from the particularity of the domain (arithmetical knowledge) in which the model has not been previously applied, and the problem that children are asked to deal with in this study. Before addressing the difficulties which stem from the theory, certain issues which apply to the particular situation in the context of which the model is applied need to be discussed. The task that children are asked to solve involves multiple steps. For the accomplishment of each step a different method can be applied. Combination of different methods to the various steps can lead to successful completion of the task. The analysis is thus focused on changes that occur in two planes of the problem situation:

A. The plane of the methods or strategies that are used when children engage themselves with the solution process. Analysis in this plane is informed by Karmiloff-Smith's description of levels of knowledge explicitness as outlined in the RR model.

B. The plane of the overall problem solving approach. Changes that occur in this plane are related to the overall representation of the task and are discussed on the basis of the problem solving phases that Karmiloff-Smith describes in her 1984 article "Children's problem solving".

The two planes of analysis are presented below in a schematic way:

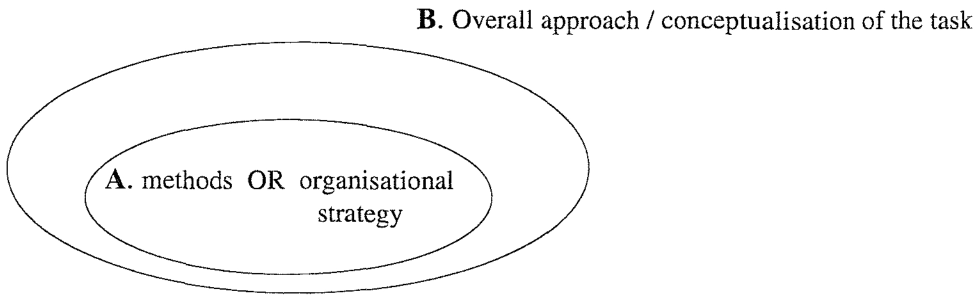


Figure 6.1
Planes of analysis

In **plane A** the methods that each child used for completing the task in the initial runs are identified together with the source of knowledge that the child calls upon to apply each of these methods. In this study, and in the context of the particular task, children have been found to use and combine four different methods in their initial encounters with the task:

* Certain number bonds are recalled directly from memory. As discussed in chapter 2, fact retrieval is broadly considered as being supported by children's *factual/declarative* knowledge (Ashcraft, 1982) even though alternative views have been expressed. For example, Baroody (1983, 1985) argues that the generation of number facts may be supported by *principled procedural* knowledge which involves the production of number facts on the basis of stored and subsequently abstracted rules such as N+0=N/0+N=N. The use of "principled procedural knowledge" to generate number facts is not considered to be necessarily slower than retrieving facts from declarative knowledge. Such rules may also be learnt as a result of explicit instruction (the experimenter witnessed explicit mentions to the 0+N=N and N+0=N rule at multiple instances during the time of classroom observation). The employment of fact retrieval by children is assumed after consideration of the solution time, that is the time that the child needed to generate a number bond (Ashcraft, 1982, 1983; Siegler 1984; Siegler & Jenkins 1989). Also, an indication of factual knowledge and direct recall from memory is often given by children's explanations of the type "I knew that" or "I remembered that", "I just thought of it".
* 'Swapping' is the method of changing the addend order of an already produced number bond in order to produce another number combination. For example, a child may have already produced the 5+4 number bond as a number combination that results to 9, and then

change the addend order of this produced number bond to produce another combination: 4+5. The employment of this method is considered as being supported by children's declarative knowledge of the principle of *commutativity*. Children *know that* in addition, the addend order can be changed without affecting the result. The use of this method necessarily entails some notion of the principle of commutativity. This principle usually constitutes an object of explicit instruction. It may therefore be learnt and directly stored as an arithmetical rule. However, Baroody (1984) argues that children can invent addition strategies that achieve computational economy and disregard addend order, without appreciating the principle of commutativity. Baroody's argument was based on a study in which 5-6 year olds participated and which showed that to solve problems of the type: 2+4, some children invented economical calculation strategies such as the COL and CAL[6.1] without explicitly appreciating or discovering commutativity. These children, according to Baroody, had a primitive notion of commutativity which he calls 'protocommutativity' and which is an extension of children's "order-indifferent tagging scheme"[6.2] (ibid. p.336). Keeping in mind Baroody's argument, cautious reasoning is needed to support any interpretation regarding the knowledge representation that underlies the use of the 'swapping' method for the production of number bonds. The fact is that the type of problem that is used in the present study is not the same as that in Baroody's study. For the present task, children did not need to calculate a sum having the two addends given. Instead, children know the sum and need to find out appropriate addends. In practice, when the children employed the method of changing the addend order, they had in front of them already produced number bonds resulting in the specific sum. When applying the 'swapping' method, the children indicated the number bond of which they changed the addend order to produce another one. It is thus considered that the application of the 'swapping' method entails the intentional, deliberate change of the addend order to produce more number bonds. In this case, this solution could not be given without appreciating the commutativity principle. This is why it is considered that, in the context of the particular task, in this study, when applying the 'swapping' method, children changed the addend order of already produced and complete number bonds because they *knew* that they can do that without affecting the result.

* Counting is the method that children used to complete a number bond after having specified the number to use as first addend. This method is supported by children's procedural as well

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[6.1] COL: counting on starting with the larger addend. CAL: counting all starting with the larger addend.

[6.2] "An order-indifferent tagging scheme implies an appreciation that elements of a set may be enumerated in any order; it does not also imply that differently ordered counts of a set will produce the same cardinal value (as an order irrelevance principle does)" (Baroody, 1984, p. 335-footnote).

as conceptual knowledge. Both aspects of counting as a method in arithmetical problem solving have been broadly studied (e.g. Greeno, Riley and Gelman, 1984; Fuson, 1988, 1992; Baroody, 1992).

* Finally, the 'deriving' method is the method that children used to derive a number bond from a known one or one that was previously produced in the context of the task. This method is considered as of particular importance in the context of the particular task and will be discussed in more detail further on.

It should be clarified that in plane A of the analysis, as this is built for the needs of this study, the methods of fact retrieval, 'swapping' and counting are not studied and discussed in terms of the RR model. The study of the procedural and conceptual aspects of these methods, as applied by the particular children, in terms of the explicit or implicit internal representations that may support them could constitute the object of more than one research project. However, it should be mentioned that the use of each of these three methods was verbally reported by all children. This is why it is considered that is safe to assume that the procedural facet of these methods, in terms of the RR model, was underlain by procedural explicit knowledge representation of E3 format. The task of assigning a level of knowledge explicitness to the conceptual facets of these methods is a task that cannot be accomplished in the framework of this study.

In contrast to the other three aforementioned methods, the 'deriving' method is put under the spectrum of analysis, and it is discussed and studied on the basis of the levels of knowledge explicitness that the RR model describes. This is because the application of this method has been found to prepare the ground on which children who participated in the study built and developed their organisational strategies. Two facets of the application of this method were studied: the procedural and the conceptual. The procedural facet of the method refers to the *know how*. The procedural application of the method involves the combination of operations that children apply on the addends of the known number bond. For example, if the 6+3 number bond has been already produced, children produce a new number bond by taking away 1 from the first addend of the known number bond ('number bond-reference') and adding 1 on the second addend of the known number bond. In this way a new number bond 5+4 is produced. The operations involved in the application of this method constitute its procedural components. For a level of explicitness to be ascribed to the procedural facet of this method, as applied by each child, these issues were put under consideration: appreciation of the operations/procedures that are combined, verbal report (or not) of these operations/procedures. It should be noted that different children used different vocabulary to describe the procedures they combined to apply the 'deriving' method. For example, some children talked in terms of choosing the number "before" and the number "after" to account for the choice of each of the addends of the new number bond after referring to each of the addends of the known number bond. Other children talked in terms of "adding" and "taking away" to account for the same procedural components of the method. The use of different vocabulary is an indication of a different type of representation that the procedures involved in the method are supported by (e.g. iconic or arithmetic). The explicitness of the representation that underlies the procedural facet of the method is ascribed not on the basis of the type of representation the child has, but on the basis of how accessible the representations of the procedural components of the method are, for modifications to be introduced and verbal reports to be given.

The conceptual facet of the method refers to the *know why*. It involves having conceptualised why the method works the way it does, why the combination of these procedures/operations results in the production of a number bond that gives the same sum as the number bond which is used as reference. Different levels of explicitness are ascribed to this conceptualisation according to the children's ability to access and work upon the fundamental idea and relationships between number bonds that support the method, as well as children's ability to report and explain the relationships that support conceptually their method.

While solving the task, children put the produced number bonds in a column: the one below the other. The consistent application of the 'deriving' method had as a result the ordered arrangement of numbers in the columns of first and second addends. In certain cases, children who noticed this pattern of numbers, subsequently abstracted it and generalised it, to develop the 'ordering' strategy. Some children developed the 'ordering' strategy being aware at the same time of the fact that the arithmetical relations involved in the 'deriving' method were integrated into the 'ordering' strategy. In these cases, the representational system which supported the application of the strategy was elaborated enough to allow this awareness. Therefore, in the framework of analysis the organisational strategy is referred to either as 'deriving' and/or 'ordering' according to the vocabulary that the child used while giving explanations. Other children developed the 'ordering' strategy after having abstracted the regularity in the arrangement of numbers, without though having an explicit representation of the underlying arithmetical relations. In those cases the strategy is referred to as 'ordering'. In either case, in analysing the data, and on the basis of the explanations that the RR model provides, a certain representational format is ascribed to the strategy that the child develops.

The aforementioned methods were identified as methods that children who participated in the study used in different steps of the solution process during the initial runs with the task. In **plane B**, the approaching of the solution procedure as including separate steps, and the employment of various, different methods in each of these steps is associated with a certain problem solving phase. Namely the 'procedural phase' (Karmiloff-Smith 1984). In plane B of analysis, and in the context of this study, children in the 'procedural phase' are considered as having an implicit representation of the features and demands of the task. They succeed in the task by applying a mixture of methods and without relating one solution step to the other. Children in this phase, most of the time, are not aware of their success nor of the completion of the task.

The movement beyond the application of various methods to the development of a strategy in plane A, marks the passage to a different phase in plane B. What is considered as a 'strategy' in the context of this study needs to be clarified. A child is considered as having or starting developing a 'strategy' when he/she starts organising the solution steps in a certain way. In the context of this study a 'strategy' is defined as the outcome of an organisation-oriented problem solving behaviour. With the implementation of a 'strategy' the solution steps are connected and related one to the other. The child has a whole view of the task and applies the same problem solving approach to the whole of the task. A strategy is a deliberately implemented, goal and organisation directed process which allows the child to make plans. It is considered as being potentially available to consciousness.

In plane A of analysis, the different, organisational, problem solving *strategies* that different children develop are considered and studied with regard to two facets: the procedural and the conceptual. Again, the two facets of a strategy, procedural and conceptual, refer to the *know how* and *know why* correspondingly. It is attempted to ascribe a certain level of explicitness to the procedural and conceptual facet of the strategy that each child develops.

In plane B, the introduction of an organisational strategy is associated with the passage to a 'metaprocedural' and subsequently (but not necessarily) 'conceptual' phase in problem solving. In the framework of this study a 'metaprocedural' phase in problem solving is associated with 'organisation-oriented' behaviour. For such a behaviour to be observed children need to redescribe their former implicit task representation to E1 format. The redescribed representation of the features and demands of the task becomes the cognitive unit of attention. It is at this point that children start working towards the development of an organisational strategy. The level of explicitness of the representation that sustains this strategy, procedurally and conceptually, may vary across the cases and it is difficult to identify. It is, however, considered that the redescription of the representation that underlies the strategy to higher levels of explicitness is related to the redescription of the task representation to E2/E3 format ('conceptual' phase). This gives to children a better control over the task, and allows explanations and justifications to be formulated.

In summary, the RR model is used in the analysis to account for:

Plane A

1. The levels of explicitness of the representations that sustain children's methods/strategy.

Plane B

2. The levels of explicitness of the representations that sustain children's view and understanding of the features and demands of the task (task representation). On the basis of this second point interpretations are built regarding point 3.
3. Changes in task representation are associated with movement to different problem solving phases as these are outlined by Karmiloff-Smith (1984).

It should be emphasised that the distinction of the two planes of the problem solving situation only serves the analytical process. In fact, the behavioural and conceptual changes that are observed in the two planes are closely connected and interact. This is the main feature of the situation which constitutes the unit of analysis in this study. From this feature stems one of the difficulties of the analytical task. Difficulties also stem from the complexity of the RR model itself.

The model incorporates and addresses issues regarding notions such as conscious/unconscious knowledge accessibility. The model states that verbal report is only made possible in the higher level of knowledge representation, that is Level Explicit 3. Thus, behaviours which are not accompanied by verbalisation need to be studied in consideration of the preceding levels of knowledge representation, that is Implicit, Explicit 1 or Explicit 2. Implicit knowledge representations have been associated with the absence of any conceptual understanding of the factors affecting notions which are inherent to the task at hand. This is the criterion on the basis of which interpretations regarding the Implicit level (I) have been formulated.

Explicit 1 (E1) representations involve an abstraction of some regularity or common feature detected in the implicit procedures. They are considered to underlie metaprocedural behaviour and be unconscious for the child. Karmiloff-Smith (1986, p. 104-105) posits that E1 representations contain explicitly defined links, they are available for children to work on but they are not directly accessible to consciousness. This is why this internal state is unobservable to the researcher. However Karmiloff-Smith has described E1 behaviour and has offered certain handles which can uphold possible interpretations. It has been argued that E1 representations can only be inferred by the U-shaped behavioural sequence, self-repairs to adequate output and the process-oriented account. In this study, interpretations regarding the redescription of knowledge representations to this level have been made on the grounds of these types of behaviour.

The idea of further redescription of knowledge representations to level Explicit 2 (E2) constitutes a speculative and tantalising idea that the RR model argues for. That is the idea of knowledge which is consciously accessible but not verbalisable. Karmiloff-Smith (1992) does not offer an operational definition of Level E2. She admits that this level is difficult to identify, and that no research has been directly focused on E2 Level (conscious access without verbal report). Direct focus on Level E2 is not the aim of this study either. However, within the framework of the analysis presented here, such an analysis attempts to explore whether behaviour which corresponds to the description of consciously accessible but non-verbalisable knowledge can be identified. It is believed that the consideration of problem solving methods and strategies in two facets, the procedural and the conceptual, may provide the grounds for identifying such behaviours. This is because the application of the model for analysis, considering these two facets of children's strategies separately, opens the possibility to identify strategies which are represented at a different level of explicitness, procedurally, and different level of explicitness, conceptually. It should be noted that Karmiloff-Smith does not differentiate levels of explicitness which apply to procedural knowledge and levels of explicitness which apply to conceptual knowledge. For example she does not talk about procedural implicit or explicit representations and conceptual/declarative implicit or explicit representations. However, the issue of implicit/explicit representations of procedural knowledge, and implicit/explicit representations of conceptual knowledge is discussed in the field of mathematics education and mathematics psychology (Halford, 1993; English, 1995). This issue and consideration drives the analytical process in this study, and it is believed that it may bring to light behaviours which may bring the E2 level into operation.

6.3. Classification of cases

The analysis of the data obtained from each case is presented in the form of individual profiles. Each profile comprises a table which presents the description of changes that each child introduced to his/her problem solving approach in the course of the five sessions. The table is followed by the discussion of changes in children's problem solving behaviour in the light of the explanations that the RR model provides. In the course of five sessions that each child participated in, all the tasks that were described in chapter 4 were proposed. However, not all the tasks are presented and discussed in each of the profiles. The analysis presents and discusses specific runs with each task in which changes of particular interest for the subsequent evolution of the child's problem solving approach took place.

In the table presented at the beginning of each profile, the reader can follow the itinerary of all the changes and modifications that each child introduced to the solution process, or to his/her verbal explanations, all along the five sessions. The discussion that follows the table in each case is driven by the points of analysis which were presented in section 6.2. Changes in plane A and plane B of the problem situation are discussed in parallel. This means that the discussions in plane A and plane B are interwoven to reflect the changes that took place in each child's problem solving. This form of presentation is designed to assist the reader in following the rationale for the analytical considerations. Within this section of discussion of each case, information in brackets shows the specific paragraphs in the relevant table that present the changes in each child's problem solving behaviour, and also to specific paragraphs in the appendices which constitute the last part of this thesis. The appendices present specific segments of each child's engagement with the task/s, and of transcribed dialogue between the interviewer and each child, including examples of the children's problem solving behaviour that is discussed.

The profiles of the ten cases are presented in two chapters. Each of the following two chapters is devoted to a group of cases. Each of the cases exhibited an individual, particular problem solving behaviour. The separation of cases in groups was done using the following major characteristic of problem solving behaviour: the phase of problem solving that the cases reached while solving the particular task.

The first chapter of analysis (chapter 7) presents five cases of children who in the course of the five sessions moved from a 'procedural' to a 'meta-procedural' phase of problem solving behaviour. That is, their initial 'success-oriented' approach to the problem changed and became 'organisation-oriented'. These children in the course of the sessions developed a specific strategy to solve the task. They exhibited 'organisation-oriented' behaviour but did not give evidence (either verbal or behavioural) of their awareness of specific aspects of the task at hand, and/or of the conceptual aspects of their employed solution strategies. The following chapter of analysis (chapter 8), presents five cases of children who moved even further, from the 'meta-procedural' phase to a 'conceptual' phase of problem solving behaviour. These children developed organisational strategies and were in position to explain and justify the success of their strategies.

It should be stressed that children were classified in each of the two groups according to the characteristics of their problem solving behaviour at the end of the five sessions. There is no evidence on the basis of which one would assume that, if more time were available, children who are classified as 'meta-procedural' problem solvers would not have moved to a 'conceptual' phase of problem solving.

Chapter 7 Cases of 'meta-procedural' problem solving behaviour

7.1. Overview

This chapter presents five cases of children, Rakhi, Henry, Isa, Leo and James, who exhibited problem solving behaviour which pertained to the 'procedural' and subsequently 'meta-procedural' phase as these are described in the framework of the RR model. All the five cases exhibited 'organisation-oriented' behaviour very early, right in the first session. However, in the course of the five sessions in which they participated, they never reached the 'conceptual' phase. The particularity in the cases of Rakhi and Henry was that they reached a point at which they had a very good control over the task and their employed strategy. This phase was very short. Soon, they lost some of the control that they had over the task after introducing a new strategy, the 'ordering' strategy. Isa and Leo were two cases of children who employed a strategy which consisted of the combination of two methods. Each of these methods was represented at different levels of explicitness procedurally and conceptually. James is a very interesting case which could constitute another chapter. It was chosen to present the case of James in this chapter because of its agreement with the main characteristic which constitutes the criterion for the presentation of cases in groups: that is the phase of problem solving behaviour that they reached. James exhibited 'meta-procedural' behaviour but had a very good control of his strategy in comparison to the other cases which are presented here. Nevertheless, he was very sparing of verbalisations.

7.2. The case of RAKHI

Rakhi was 6 years 4 months old. She was classified as in the most advanced group in mathematics in the class. Rakhi participated in five sessions. The first two sessions took place on two consecutive days. One day intervened between the second and third session. The third and fourth session took place on two consecutive days. Three days intervened between the fourth and the final session.

Table 7.2: Summary of changes that Rakhi introduced into her problem solving approach.

First session	**Second session**
First run – 'card' task: 1.1 Rakhi produced number bonds in two steps: She wrote down the first addend. She counted on to figure out the second addend. Rakhi used her fingers in both steps, except from the case of number bonds including '0'. 1.2 The mechanism that produced the second addend i.e. counting or declarative knowledge was apparent and reported. Rakhi did not explain the mechanism or criterion on the basis of which she was specifying the number to be used as first addend in each new number bond. 1.3 Rakhi's overt behaviour showed that she was aware of the completion of the task. She did not provide any justification. *Second run – 'card' task*: 1.4 Midway in the solving procedure Rakhi started considering the numbers she was using as first addends in order, so that she could find numbers that had not been used yet. She explained the criterion of her choice of numbers. 1.5 Rakhi showed that the task was complete by using the number line and considering the numbers used as first addends, in order.	*First run – 'card' task*: 2.1 Same approach as in the last run with the task in the previous session. *Second run – 'card' task*: 2.2 Rakhi kept producing number bonds in two steps. This time, the first step involved the specification of first addend following a sequential order, all along the solution process. The second step still involved counting on with fingers. 2.3 Rakhi reported the mechanism on the basis of which she was specifying the first addends and justified its use. 2.4 Rakhi was certain of the completion of the task right after the production of the last number bond. She justified her certainty with no further checking.

Third session	Fourth session
<u>*Run with 'card' task*</u>: 3.1 Rakhi gave signs of greater flexibility in employing the method of following an order for the specification of first addend. 3.2 Rakhi employed the word "order" to denote her method for the selection of first addend. Her 'a posteriori' observation made her notice that the second addends were following an order as well.	<u>*'Card' task*</u>: 4.1 On the basis of her observation which took place in the previous session, Rakhi applied 'ordering' for the production of the first *and* second addend of each new number bond. 4.2 She indicated the different kind of order that she was following for the specification of the first and second addends. She did not explain why there was a different kind of order in each of the two columns of numbers. 4.3 Rakhi applied the 'ordering' strategy all along the solution process, for different target numbers in the 'card' task. <u>*'Domino'*</u> 4.4 Rakhi used counting with fingers and 'swapping. She was aware that the task was complete but her justifications were weak.
Fifth session <u>*Run with 'card' task*</u>: 5.1 Rakhi elaborated her initial report of the procedures involved in her strategy and soon denoted the kind of order that she was following using the words "backwards" and "in order". 5.2 Rakhi did not provide explanations on *why* her strategy worked the way it did.	<u>*Similar in goal tasks*</u>: 5.3.Rakhi approached the 'domino' task in the same way as in the previous session. 5.4.In the 'balances on paper' and 'balance on cardboard' tasks Rakhi applied the mixture of methods that she had used in the initial runs with the 'card' task. She used the idea of 'ordering' only to check the use of all the possible numbers at the end or close to the end of the solution process.

Discussion

<u>*Plane A*</u>: In the first run with the 'card' task, in the first session, Rakhi produced number bonds by combining two different mechanisms (Table 7.2-para 1.1, 1.2). The available data do not allow one to make any justifiable interpretation regarding the knowledge representation which sustained the choice of first addend in the first run with the task (Appendix 7.2-para 1.1). <u>*Plane B*</u>: In this first run, the production of all the number bonds was underlain by the same rationale: all the number bonds were produced by a 'two-step' process. However, Rakhi approached each step of the solution process, that is the production of each number bond, as a separate problem. She did not seem to refer to previously completed steps until the point that quite a lot of number bonds had been produced, and she needed to find the number/s that

was/were missing. Her overt behaviour showed that, probably, at that point Rakhi used the previously completed steps of the solution process as reference for the identification of the missing number/s. Rakhi's overt behaviour showed that she was aware of the completion of the task (Appendix 7.2-para 1.2, 1.3). However, she did not give any reply to the interviewer's relevant questions. It is believed that in this first run with the 'card' task, Rakhi gave signs of a problem solving approach which had the characteristics of the **'procedural'** phase. Very soon, in the second run with the task in the first session, Rakhi introduced interesting modifications in her approach to the task (Table 7.2-para 1.3).

In *plane A*, the change concerned a process of explicitation that the criterion on the basis of which Rakhi produced the first addend of each new number bond seemed to be subjected to. It is believed that the criterion for the specification of first addend was represented in an explicit **E3** format. Rakhi's solution process was accompanied by verbalisations which made her reasoning explicit (Appendix 7.2-para 1.2). Rakhi's first steps towards a systematic way in which she considered the numbers used in the course of the solution process, made her be certain for her success after the completion of the task (Table 7.2-para 1.4). She justified her certainty on the basis of the same rationale which drove the choice of numbers: she used the number line and showed that all the possible numbers had been used as first addends (Appendix 7.2-para 1.3). In *plane B*, the systematisation in the choice and use of numbers signalled the passage to a **'meta-procedural'** phase of problem solving. This systematisation entailed reference to number combinations which had been already produced, and to numbers which had been already used.

Plane A: In the second session, Rakhi extended the idea of 'ordering' which drove this primary systematisation, and developed a *unified strategy* (Table 7.2-para 2.2, 2.3). Rakhi reported the mechanism on the basis of which she was specifying the first addends and justified its use (Appendix 7.2-para 2.1, 2.2). The use of counting was apparent and reported, as well as the use of declarative knowledge. The consistent combination of the aforementioned mechanisms for the production of number bonds all along the solution process, constituted a strategy the ***procedural and conceptual facet*** of which is considered as being represented in an explicit **E3** format.

In *plane B*, the development and consistent application of this strategy could be considered as an indication of a 'conceptual' phase in problem solving. However, the following sessions showed that she was not aware of all the aspects of the task after the application of the 'ordering' strategy for the specification of the first addend. For example, she was not aware of the 'ordered' sequence of numbers in the column of second addends, which was the consequence of applying 'ordering' for the specification of the first addend of each new number bond. Rakhi was in control of only certain aspects of the task. The awareness of these aspects of course allowed her to be successful and be in position to give verbal explanations and justifications. However, this 'partial' awareness is the reason why it is believed that the 'conceptual' phase should not be assigned to Rakhi's problem solving approach. As it was shown in the third session, this stable phase of control was very short, and it is considered as part of Rakhi's 'meta-procedural' approach to the problem. Soon, Rakhi discovered aspects of which she was not aware and a new cycle of **'meta-procedural'** work upon the knowledge representations which underlay the 'ordering' strategy opened. In the third session, Rakhi's 'meta-procedural' work upon her strategy of applying 'ordering' to produce the first addends became evident by the changes in vocabulary which rendered her strategy explicit, and actions which were organisation-oriented and quite flexible (Appendix 7.2-para 3.1, 3.2). Still, she could not provide answers for certain aspects of the task after the application of her strategy. Her observation that the second addends were following a specific order as well (Appendix 7.2-para 3.2), was triggered by the interviewer's questions. The changes that took place in plane A in the following session were grounded on that observation (Table 7.2-para 4.1).

Plane A: The ***procedural facet*** of the 'ordering' strategy which, in the fourth session, was introduced and applied for the specification of the first *and* second addend of each new number bond (Table 7.2-para 4.1; Appendix 7.2-para 4.1), was explicit and reportable (**E3** format). Moreover, in the fifth session Rakhi elaborated the verbal report of the procedures involved into her strategy (Appendix 7.2-para 5.1). The ***conceptual facet*** of the strategy though is considered as being represented in an explicit **E1** format: the new strategy seemed to be represented in a format which was explicit enough for Rakhi to access it and work upon it. However, Rakhi did not seem to be in position to explain why her strategy worked the way it did (Appendix 7.2-para 5.2). The development of the 'ordering' strategy seemed to be the result of Rakhi's observation and subsequent abstraction of a certain regularity. However, up to the final session, Rakhi did not seem to have conceptualised explicitly that the kind of order that she was following for the specification of the first addends was related to the order that she was following for the specification of the second addends, and the other way round. Also, there were signs of rigidity in the way Rakhi was applying the 'ordering' strategy. Rakhi only applied the 'ordering' by following an ascending order in the column of first addends and a descending order in the column of second addends. She did not seem to have conceptualised the reversibility of the strategy (Appendix 7.2-para 5.2). However, the 'ordering' strategy was generalised and applied in every run with the 'card' task and for big target numbers. The strategy was not applied in tasks with similar goals (Table 7.2-para 4.4, 5.3, 5.4).

With the introduction of the 'ordering' strategy for the specification of the first as well as the second addend of each number bond, Rakhi seemed to lose some of the control that she had over the aspects of the task and of the solution process. The 'ordering' strategy allowed her to solve the task rapidly, avoid any type of calculation, and be certain of her success. However, Rakhi did not seem to have built an understanding of all the aspects of her strategy. There was regression observed not at the level of performance, but on the level of the control that Rakhi had over the conceptual aspects of the strategy that she applied. It was not made possible to find in the framework of the RR model an explanation or prediction for this kind of regression: i.e. regression related to the degree of control that the problem solver had over the aspects of his/her strategy. This type of regression was not accompanied by unsuccessful efforts in solving the task.

7.3. The case of HENRY

Henry was 6 years 6 months old. He was classified as in the second most advanced group in mathematics in the class. He participated in five sessions. The first four sessions took place in four successive days. Three days intervened between the fourth and fifth session.

Table 7.3: Summary of changes that Henry introduced into his problem solving approach.

First session <ins>*First two runs with 'card' task*</ins>: 1.1 Henry produced number bonds by calling upon his declarative/factual knowledge, and by applying counting and 'swapping'. 1.2 Henry appeared to be aware of the completion of the task. He explained that he checked the use of all the possible numbers as first addends. He did not report a systematic way of checking the use of these numbers. <ins>*Third run with 'card' task*</ins>: 1.3 Henry increased the use of 'swapping' as a method for the production of new number bonds. 1.4 'Swapping' drove the emergence of a new checking method. Henry's focus only on this aspect of the task deteriorated his performance.	**Second session** <ins>*First run – 'card' task*</ins>: 2.1 Henry applied a mixture of methods similar to the ones used in the previous session. Previously employed checking methods were combined to develop a new, thorough, although redundant method of checking. <ins>*Second run – 'card' task*</ins>: 2.2 'Instant swapping' of each number bond produced emerged as a new strategy for producing number bonds and organising the solution process. 2.3 Henry applied the 'deriving' method for the production of isolated number bonds. <ins>*Third run – 'card' task*</ins>: 2.4 The 'deriving' method was consistently used for the production of the first number bond of each pair, all along the solution process. Henry reported the mechanism underlying the production of the first as well as second addend.
Third session <ins>*First run – 'card' task*</ins>: 3.1 Henry returned to the use of counting. The 'deriving' method was only employed in the case of external pressure for rapid solution times. <ins>*Second run – 'card' task*</ins>: 3.2 Henry showed that he acknowledged the reversibility of the 'deriving' method. 3.3 'Ordering' emerged as a new strategy for producing number bonds <ins>*Third run – 'card' task*</ins>: 3.4 Henry did not provide explanations on why the 'ordering' strategy worked the way it did. 3.5 The 'deriving' method and the 'ordering' strategy were not applied in the 'domino' task.	**Fourth session** <ins>*'Card' task*</ins>: 4.1 Henry reported and named the sequence of numbers that he was following while applying the 'ordering' strategy. 4.2 Henry did not provide an explanation of the different kind of order that first and second addends were following. 4.3 Henry applied the 'ordering' strategy to produce number bonds for 'big' target numbers. 4.4 In the 'balances on paper' and 'balance on cardboard' tasks Henry applied the mixture of methods he had applied initially in the 'card' task.

Fifth session	
'Card' task: 5.1 Henry reported his choice not to apply the 'ordering' strategy in this run, but acknowledged its advantages. 5.2 Henry did not consider using as reference a previous number bond in order to produce a new one, in a situation out of the framework of the 'ordering' strategy. 5.3 He applied the 'ordering' strategy to produce, verbally, all the number bonds for big target numbers.	*Similar in goal tasks*: 5.4.Henry found difficult the application of the 'deriving' method in the 'missing numbers' task. 5.5.He was successful in similar tasks without applying the 'more/less' rationale nor the 'ordering' strategy.

Discussion

Plane A: In the first session, the procedural facet of all the methods that Henry employed (Table 7.3-para 1.1, 1.3) is considered as being sustained by knowledge representations of E3 format. However, in *plane B*, the main feature of his initial, general problem solving approach was that the production of each number bond was considered as a separate step within the solution process. Henry produced every new number bond by drawing from different sources of knowledge, and employing different methods. Previously produced number bonds were used as reference only in the cases that Henry used 'swapping'. In this sense, Henry, as a problem solver at that particular point, is considered as being in the '**procedural phase**': there were no attempts for organising the isolated solution steps into a consistent problem solving strategy (Appendix 7.3-para 1.1).

Plane A: Rich changes were observed in the methods of checking that Henry employed, in the first three runs with the 'card' task. The initially applied checking method (Table 7.3-para 1.2; Appendix 7.3-para 1.2), as reported, did not seem to be systematic but it seemed to contain all the necessary information for Henry to know whether the task had been completed. However, Henry went on to introduce a new checking method which, was followed by a phase of limited control over the task, rigid behaviour and overlooking of negative feedback (Table 7.3-para 1.4; Appendix 7.3-para 1.3). In the second session, Henry went on to combine the two checking methods and develop a new, reliable, but redundant method of checking (Appendix 7.3-para 2.1).

In plane B, this cycle of introduced changes after the employment of a sufficient and successful method, suggests the activation of a redescribing process which the representation of the task, its features and demands, was subjected to. Henry ultimately regained control of the task. That phase was underlain by a redescribed task representation on the basis of which Henry introduced a new strategy for the organisation of the solution process. 'Instant swapping' emerged as an organisational strategy (Table 7.3-para 2.2; Appendix 7.3-para 2.2). The redescribing process that was initiated on the plane of checking affected Henry's representation of the task and subsequently his production methods and problem solving approach. The deterioration of Henry's performance together with his first attempts to organise his problem solving approach are considered as indications of a '**meta-procedural**' phase. Henry was viewing the solution process as a whole, in the context of which a consistent strategy started being developed.

Plane A: The 'instant swapping' strategy involved the consistent combination of two methods: the newly introduced 'deriving' method and 'swapping' (Table 7.3-para 2.3, 2.4; Appendix 7.3-para 2.3). The application of 'instant swapping' was reported as part of a plan, and Henry justified its use. This is why the ***procedural as well as conceptual facet*** of this organisational strategy is considered as being represented in an explicit **E3** format. The ***procedural facet*** of the 'deriving' method appeared to be explicit enough for Henry to report the operations involved (**E3** format). On the other hand, the rationale of the arithmetical relations underlying the operations involved in the method did not appear to be represented in a format which allowed verbalisation. Henry never referred to the more/less arithmetical relation between the numbers-reference and the numbers produced to explain the maintenance of the sum. Nevertheless, the ***conceptual facet*** of the method seemed to be explicitly and consciously accessible given that Henry generalised and applied the method in several runs, with different target numbers. Also, he shifted to its application whenever he was pushed for rapid solution times (Table 7.3-para 3.1). Furthermore, Henry showed that he was aware of the reversibility of the method. This suggests that conceptualisation of the *why* and *how* the method worked was underlain by knowledge represented in an explicit **E2** format.

After the consecutive application of the 'deriving' method Henry noticed the pattern of numbers and extended it with the development of the 'ordering' strategy (Table 7.3-para 3.3; Appendix 7.3-para 3.1, 3.2). At the procedural level, Henry mastered the 'ordering' strategy (Appendix 7.3-para 4.1, 4.2). This is why the ***procedural facet*** of the strategy is considered as being represented in **E3** format. The 'more/less' rationale which underlay the 'deriving' method is actually integrated into the 'ordering' strategy. However, Henry did not seem to be aware of this integration. He never referred to the arithmetical rationale of 'more/less' which underlay the 'ordering' strategy as well as the 'deriving' method (Appendix 7.3-para 4.1). However, the 'ordering' strategy was generalised to big target numbers (Appendix 7.3-para 4.2). It is quite interesting that, after the emergence of the idea of 'ordering' and its procedural mastery, Henry did not seem to be in position to call upon the knowledge representation which underlay the 'deriving' method, in the context of the 'card' task or in other problem situations. After the introduction of the 'ordering' strategy, Henry did not seem to be in position to call upon the knowledge representation that used to support his previous extensive use of the 'deriving' method (Appendix 7.3-para 5.1, 5.2).

At that phase of Henry's problem solving approach, the 'ordering' strategy was procedurally mastered and was easily employed to produce (even verbally) number bonds for any target number, no matter how big it was (Appendix 7.3-para 4.2). However, Henry remained focused on the ascending and descending order of numbers and somehow dismissed that this was the consequence of the 'more/less' arithmetical relation between the numbers. Being focused on the main feature of the 'ordering' strategy, Henry was talking in terms of "going like 0, 1, 2, 3, 4....", or "going upwards" and "going downwards" instead of "more" and "less". For Henry, the knowledge representation which underlay the 'more/less' rationale of the 'deriving' method did not seem to be explicitly linked, or associated with the knowledge representation which underlay the 'ordering' strategy. At that point, Henry gave signs of rigidity in his problem solving behaviour with the partial and insisting focus on one particular aspect of the 'ordering' strategy. Because of this rigidity and the regression in the application of the 'more/less' rationale, the ***conceptual facet of the 'ordering' strategy*** is considered as being underlain by knowledge representation of **E1** format. The 'ordering' strategy was applied as an automatic procedure, and was not associated with any other previously employed knowledge representation within the task. Also the 'ordering' strategy and the 'deriving' method were never applied in tasks with similar goals (Table 7.3-para 3.5, 4.5, 5.4, 5.5; Appendix 7.3-para 5.2). However, it is not believed that the 'ordering' strategy was underlain by procedure-like representations: i.e. it is not believed that knowledge represented in the I format can be associated with the 'ordering' strategy. This is because the 'ordering' strategy was the product of abstraction of a certain feature of the task and Henry was in position to generalise the application of the strategy in runs with big target numbers.

In *plane B*, the emergence of the 'ordering' as an organisational strategy followed the development of another organisational strategy (the 'instant swapping'). With the application of both strategies Henry gave signs of **'meta-procedural'** problem solving behaviour. His problem solving approach was 'organisation-oriented' but it was also inflexible, driven by Henry's strong, partial focus on one specific aspect of the task, and of his strategy. The 'ordering' strategy, was the product of observation, and abstraction of certain features of the task, namely a number pattern. However, it developed the characteristics of an *automatic*, and rigid procedure. Henry was not aware of the arithmetical relationships which sustained the creation of the pattern.

In terms of the RR model, the 'ordering' strategy seemed to be sustained by a new representation which was introduced and needed to undergo further explicitation before it becomes associated with existing knowledge representations, and before it becomes available for more flexible use. The emergence of this strategy dominated Henry's approach and view over the task having as a consequence the fading (at the procedural-utilisation level) of a previously employed, elaborated method, the 'deriving' method. It is believed that this 'fading' was only temporary. It is likely that, if more time was available, Henry would work 'meta-procedurally' upon the idea of 'ordering' opening a new cycle of knowledge redescription, followed by the balanced conceptual coexistence of representations which sustained his previous, as well as his latest developments in approaching the particular task.

7.4. The case of ISA

Isa was 6 years 6 months old. She was classified as in the most advanced group in mathematics. Isa participated in five sessions. Four of these sessions took place on four consecutive days. The fifth session took place nine days after the fourth.

Table 7.4: Summary of changes that Isa introduced into her problem solving approach.

First session	**Second session**
<u>*First run – 'card' task*</u>: 1.1 Isa mainly used fact retrieval and 'swapping'. One number bond was derived from a previous one. 1.2 Isa did not provide a report of the mechanism/s involved in the 'deriving' method. 1.3 She did not justify her belief that the task was complete. <u>*Second run – 'card' task*</u>: 1.4 Isa produced number bonds in pairs. Fact retrieval and 'instant swapping' were mainly used for the production of each pair. 1.5 One number bond was derived form a previous one. Isa did not report the mechanism/s involved in her 'deriving' method. 1.6 She thought that the task was because all the number bonds had been swapped.	<u>*First run – 'card' task*</u>: 2.1 Initially, Isa did not organise the solution process in the way she did at the end of the previous session. Fact retrieval was used considerably less. Isa relied much more on previous number bonds to derive from them new ones. 2.2 This time, the number bonds were put in pairs as a method of checking. The number line was also introduced as an additional tool for checking and completing number bonds. <u>*Second run – 'card' task*</u>: 2.3 The number bonds were produced in pairs right from the start and all along the solution process. The use of the number line as a production as well as a checking tool, increased substantially.

Third session	Fourth session
'Card' task: 3.1 Initially, Isa did not employ 'instant swapping', and she did not ask to use the number line. 3.2 The 'deriving' method reappeared. For the first time Isa referred to and explained the relationship between the addends of the number bond-reference and the derived number bond. 3.3 Midway in the solution process Isa started producing number bonds in pairs. Two methods were employed: the 'deriving' method and 'instant swapping'. 3.4 Isa was not immediately aware of her success. She put all the number bonds in pairs as a checking method. She did not consider the numbers in order. At the end of the 'checking' process Isa was certain of her success.	*'Card' task*: 4.1 Isa applied her strategy of producing number bonds in pairs all along the solution process. 4.2 Isa reported the use of the 'deriving' method of but she did not explain nor justify the success of this method. 4.3 Isa was aware of the completion of the task almost immediately after its completion. However, she did not justify her awareness and certainty. 4.4 Isa applied her strategy (even verbally) in runs with big target numbers (19, 20). *'Domino' task*: 4.5 Isa applied her strategy of producing number combinations in pairs. She appeared to be aware of the completion of the task but she did not justify her certainty.
Fifth session *'Card' task*: 5.1 Inflexible application of the 'deriving' method was observed. *Similar-in goal tasks*: 5.2 Isa applied her strategy of producing number bonds in pairs in a paper and pencil task with bif target numbers (e.g.100), as well as in the 'domino', the 'balances on paper' and 'balance on cardboard' tasks.	Isa's verbal reports and explanations were the same as the ones that she had given in the context of the 'card' task.

Discussion

Plane A: Fact retrieval and 'swapping' which were mainly used in the first runs with the 'card' task, are considered as being sustained by Isa's declarative/factual knowledge, that is by knowledge representations of E3 format. There was absence of any verbal explanation regarding the 'deriving' method (Table 7.4-para 1.2, 1.5). This allows one to assume that, at that particular point, the ***procedural as well as the conceptual facet*** of the method was underlain by knowledge which was not represented in a format explicit enough to be verbally accessible. In both runs with the 'card' task in the first session, it was observed that the 'deriving' method was used in isolated steps within the solution process when an economic in effort method like fact retrieval and 'swapping' could not be used (Appendix 7.4-para 1.1, 1.2). The fact that Isa called upon this method only at those instances, is considered as a sign of a conscious shift to the use of this method at specific points within the solution process. For this reasons it is believed that knowledge representation of **E2** format sustained the ***procedural facet*** of the method. At that point, since there were no complete reports of the operations involved, there were also no indications concerning the ***conceptual facet*** of the method.

<u>*Plane B*</u>: Isa applied the three different methods (fact retrieval, 'swapping' and the 'deriving' method) by considering the production of each number bond as a separate step within the solution process. This is why her problem solving approach in the first run (Appendix 7.4-para 1.1) is considered as being at the **'procedural phase'**. Isa did not give signs of any attempt to organise the solution process. Furthermore, she simply announced that the task had been completed without being in position to explain and justify her certainty (Table 7.4-para 1.3).

<u>*Plane A*</u>: A first attempt at organising the solution process was observed in the last run with 'card' task in the first session (Table 7.4-para 1.4; Appendix 7.4-para 1.2, 1.3). Isa realised that one important feature of the task was that at the end of the solution process all the number bonds had a corresponding 'other half': i.e. a number bond with the same addends in different order. Isa produced number bonds in pairs by applying 'instant swapping'. In this way she could be certain that for each number bond she had created the corresponding 'other half'. The introduction of the strategy of producing number bonds in pairs was a change that could be considered as a result of Isa's need to acquire a better control of this particular aspect of the task. Isa's previous approach was successful regarding the completion of the task, but it did not allow her to be immediately certain that all the number bonds had been swapped. Isa's explanations regarding the strategy indicate that the ***procedural as well as conceptual facet*** of this strategy (i.e. why it is applied) are considered as being sustained by knowledge representation of **E3** format (Appendix 7.4-para 1.3). However, it must be emphasised that this organisational strategy integrated the use of a method (i.e. Isa's 'deriving' method) that still did not appear to be represented in an explicit, verbalisable format. The procedural facet of this method is considered as being sustained, still, by knowledge representation of **E2** format.

In <u>*plane B*</u>, the introduction of the 'instant swapping' strategy indicates a shift to an 'organisation-oriented' behaviour. In the framework of the RR model this movement beyond the initial 'success-oriented' behaviour to an 'organisation-oriented' one constitutes an indication of Isa's passage to the **'meta-procedural'** phase. At this phase Isa started organising her behaviour on the basis of her realisation of a specific aspect of the task. 'Instant swapping' did allow her to know whether all the number bonds had been swapped (Table 7.4-para 1.6). However, this was not enough for one to know that the task had been completed. Isa seemed to be strongly focused on one particular aspect, and she based on that aspect the development of her strategy and the certainty of her success.

In the second session, and in the framework of Isa's **'meta-procedural'** work on the task, the consideration of number bonds in pairs was used as a checking method instead of a strategy of producing number bonds. This checking method was combined with the use of the number line as a tool for checking the use of all the possible numbers (Table 7.4-para 2.2; Appendix 7.4-para 2.1, 2.2). The strategy of generating number bonds in pairs reappeared as a production strategy while the use of the number line was retained as a checking tool as well as a tool of number bond production in cases where factual knowledge was not available (Table 7.4-para 2.3). The introduction of the number line indicated Isa's realisation that by considering the number bonds in pairs, she could not be certain of the completion of the task. (see also Appendix 7.4-para 3.2). This realisation was the result of a process of further explicitation that Isa's understanding regarding the requirements and conceptual aspects of the task had been subjected to.

<u>Plane A</u>: At the same time of this back and forth regarding the use of the strategy of producing number bonds in pairs, for the first time Isa appeared to be in position to give a complete report of her 'deriving' method (Table 7.4-para 3.2; Appendix 7.4-para 3.1, 4.1). For the first time, she referred to the relation between the first and second addends of the number bond-reference and the derived number bond. The knowledge representation underlying the procedural facet of the 'deriving' method seemed to have been subjected to a process of explicitation. At the ***procedural level***, the method was explicit enough for the actions involved (i.e. the procedural components of the method) to be accessed and be verbally reported. This suggests that the procedures involved were underlain by knowledge representations of **E3** format.

However, Isa was not in position yet to explain the conceptual rationale behind the method: i.e. why the combination of these specific actions and procedures led to a successful result (Table 7.4-para 4.2; Appendix 7.4-para 4.1). Furthermore, in subsequent sessions it was observed that the method was restrictedly employed when the two actions involved (i.e. choice of number 'before' and choice of number 'after') could be applied in this specific sequence: 'before/after'. Isa had not conceptualised the reversibility of the actions-components of the method. This had as an effect the rigid and inflexible use of the method. The method was not applied in situations where the conditions did not allow a number bond to be derived from a previous one by the application of this specific sequence of actions: 'before/after' (Appendix 7.4-para 5.1). Assigning a level of explicitness to the knowledge representation sustaining this method is a difficult task. At the procedural level, the method was consciously chosen to be applied and its actions-components were accessible and verbally reportable. However the conceptual rationale behind the procedural success of the method did not seem to be explicitly represented nor accessible for generalisation and application to other situations. This is why it is believed that the ***conceptual facet of the 'deriving' method*** was underlain by knowledge representations of **E1** format.

Plane B: From the fourth session onwards, Isa started applying the strategy of producing the number bonds in pairs consistently as a single, unified strategy all along the solution process (Table 7.4-para 4.1, 4.4; Appendix 7.4-para 4.1). It is noteworthy that each pair was produced by the combination of two different methods each of which was sustained by different knowledge representations of different level of explicitness and conceptualisation: i.e. the 'deriving' method and the 'swapping' method. However, these different methods were coherently combined and put together to develop a strategy for organising the solution process from the very first up to the last solution step.

Because Isa was consistently following the pattern 'before/after' when she was applying the 'deriving' method, the number bonds produced appeared in a specific order. This allowed Isa to be aware of the completion of the task immediately after the production of the last number bond without, though, being in position to justify her success (Table 7.4-para 4.3; Appendix 7.4-para 4.2). However, Isa strictly applied the strategy following a specific pattern: she was starting with the number bond that had the bigger number as first addend. This allowed her to go on with the production of number bonds following the 'before/after' sequence of actions that her 'deriving' method involved. It is considered that the limited conceptualisation and inflexible use of the 'deriving' method had as an effect the inflexible application of the overall strategy. There is a paradox: the strategy of producing number bonds in pairs was reported and Isa showed the conscious and intentional development and application of this type of organisation of the solution process. The procedural components of the strategy i.e. the procedural facet of the 'swapping' and 'deriving' method were explicit. However, conceptually, the 'deriving' method (i.e. one of the components of the overall strategy) was not represented in a high level of explicitness. The limited understanding that sustained the 'deriving' method led to inflexible application of the overall strategy. However, with the introduction and development of this strategy, Isa, as a problem solver, had passed from a **'procedural'** phase to a **'meta-procedural'**, organisation-oriented phase regarding her approach to the task. Also it is noteworthy that the inflexibility of the strategy did not hinder its application to tasks with similar goals (Appendix 7.4-para 4.3).

7.5. The case of LEO

Leo was 5 years 11 months old. He was classified as in the second most advanced group in mathematics. Leo participated in five sessions. The first three sessions took place on three consecutive days. The fourth and fifth sessions were consecutive as well but took place twelve days after the third. This was due to a prolonged period of Leo's absence from school.

Table 7.5: Summary of changes that Leo introduced into his problem solving approach.

First session	Second session
First run – 'card' task: 1.1 First part of the solution process: Leo used fact retrieval and 'swapping'. There was no verbal report or visible indication of the method used for the production of two number bonds of the first set. Second part of the solution process: Leo reported the use of 'swapping'. 1.2 Leo did not appear to be aware of the completion of the task. *Second run – 'card' task*: 1.3 First part of the solution process: Leo used counting to complete number bonds after having specified the first addend. The mechanism that drove the selection of the first addends was not reported. Second part of the solution process: Leo reported the use of 'swapping'.	*First run – 'card' task*: 2.1 First part of the solution process: Number bonds were mainly produced by a two-step process. After specifying the first addend Leo used counting to figure out the second. Leo did not explain the rationale behind the choice of the first addend. Second part: Leo reported the use of 'swapping' for the production of number bonds. 2.2 A particular, sequential pattern was observed in the column of first addends. 2.3 Leo was not aware of the completion of the task. *Second run – 'card' task*: 2.4 Leo reported the use of 'ordering'. 2.5 He did not explain the shift to the application of 'swapping'.

Third session	Fourth session
'Card' task: 3.1 Leo used 'ordering' to produce number bonds in the first part of the solution process. He reported the different kind of order he followed for the first and second addends. 3.2 This time the 'ordering' method was reversed. 3.3 Leo did not justify his certainty that the task was complete. 3.4 Leo gave the impression that he noticed the order that the second addends were following only after the interviewer's relevant question. 3.5 Leo did not give any further or clearer explanation of the rationale behind the shift from 'ordering' to 'swapping'.	*'Card' task*: 4.1 Ordering' and 'swapping', were applied in each of the two parts of the solution process correspondingly. The number bonds of the first set were produced in two steps which were characterised by the application of the same mechanism at different levels of explicitness. 4.2 Leo organised the second part of the solution process: He changed around the number bonds considering them in order. *'Domino' and 'balances on paper' tasks*: 4.3 Leo separated the solution process in two parts. 'Ordering' was not applied. However, Leo did apply 'swapping' in the second part of the solution process.
Fifth session *Two runs with 'card' task*: 5.1 Leo did not adjust his 'ordering' method and did not use it to overcome the violations introduced. 5.2 Leo attempted to explain the shift from the first part of the solution process to the second. However, up to end of his participation, he did not explain the rationale that underlay the method of 'ordering'.	5.3.Leo applied his overall organisational approach to big target numbers. 5.4.Leo did not make correct use of the rationale that underlay the 'ordering' method in the 'missing numbers' task. 5.5.He applied the two part strategy but not 'ordering' in the 'balances on paper' task. In the 'balance on cardboard' task he found difficult to apply 'swapping' due to the absence of visible reference.

Discussion

Plane A: In the first session, the use of fact retrieval in the first part of the solution process and 'swapping' in the second part of the solution process was reported. Therefore, the procedural facet of these methods is considered as being sustained by knowledge representations of E3 level. For the production method of two of the number bonds produced in the first part of the solution process, Leo did not provide a verbal explanation, and did not give any visible indication on the basis of which the use of a certain source of knowledge could be inferred.

Plane B: After the production of the initial set of number bonds, Leo shifted to the application of a single method for the production of a second set of number bonds. This shift signalled the passage to the second part of the solution process in which 'swapping' was used for the production of a whole set of new number bonds (Appendix 7.5-para 1.1). Even though 'swapping' as a method for the production of number bonds appeared to be consciously accessible and mastered procedurally, the rationale behind the shift to the application of this method did not seem to be part of a choice explicit enough to be verbally reported. At that phase of Leo's work on the task, it seemed like this shift at that particular point of the solution process, was part of a conscious, even though not reportable, plan. In terms of the RR model, Leo's problem solving approach, at that particular point, is considered as having the characteristics of an early **'meta-procedural phase'**. Leo produced the initial set of number bonds by considering each one of these as a distinct solution step and thus by applying a mixture of different methods represented at different levels of explicitness (i.e. fact retrieval and non-reported method/s). However, in the second part of the solution process Leo used previous solution steps as reference, consistently, for the production of new number bonds.

Plane A: Changes in the way Leo was approaching the first part of the solution process appeared in the second run with the task. Except from the two first number bonds which were recalled from memory, the rest of the number bonds in the first part of the solution process were produced by a 'two-step' process (Appendix 7.5-para 1.2). Leo did not give a verbal account of the mechanism or criterion on the basis of which he specified the first addend of each new number bond. The available data do not provide any indication for the level of explicitness of the criterion that supported the choice of first addend. The use of counting for the specification of the second addend was reported. The procedural facet of this method is considered as being supported by knowledge representation of E3 format. Two mechanisms underlain by knowledge represented at different levels of explicitness seemed to be combined. This combination was used consistently for the production of all the number bonds in the first part of the solution process. In the second part of the solution process all the number bonds were produced by the method of 'swapping', the procedural facet of which is considered as being represented in an Explicit E3 format.

Plane B: In the second run with the task in the first session, Leo's **'meta-procedural'** work upon the task continued. There were signs of an attempt for an amelioration of the overall organisation and movement towards a more unified approach. However, Leo's solution approach consisted of one production mechanism and a midway shift to the 'swapping' method the rationale behind which did not appear to be explicit enough to be verbally accessible and reportable. Furthermore, at the end of each run with the 'card' task, until the end of this session Leo was not aware of the completion of the task. However, the attempt for an overall organisation is considered as an indication of Leo's early steps within a **'meta-procedural'**, organisation-oriented phase.

<u>Plane A</u>: In the second session further steps towards an overall organisation were observed and some verbal explanations started being formulated. Two methods were consistently applied in each of the two parts of the solution process: 'ordering' and 'swapping' (Table 7.5-para 2.1, 2.2, 2.4; Appendix 7.5-para 2.1, 2.2). Leo seemed to master the 'ordering' method procedurally. The application of the method and the order that Leo followed for the specification of first and second addends was reported. However, Leo's report was poor and descriptive. That is he attempted to explain the method, simply by uttering the first and second addends in the order that he would follow to specify them. He did not report the procedures used, he did not name these procedures. However, Leo's choice to apply the method was conscious. There was a reason which drove the application of the method ("It's so quicker...I know."). Also, the 'ordering' method was applied consistently. Therefore it is believed that in the second session the ***procedural facet*** of the 'ordering' method was sustained by knowledge representation of **E2** format. This format allowed conscious access to knowledge but it did not allow verbal explanations. Because of the poor verbalisation that accompanied the application of the 'ordering' method, there were no indications regarding the knowledge representation which supported the conceptual facet of the method.

Soon after the introduction of the 'ordering' method Leo indicated and named the kind of order that he was following to specify the first addends (Table 7.5-para 3.1; Appendix 7.5-para 3.1). Further questioning by the interviewer made Leo notice the order of numbers in the column of second addends (Appendix 7.5-para 3.2). From that point onwards he reported the order that he was following for the specification of numbers in the column of first and second addends using the phrases ("going downwards" and "going upwards"). It is believed that it was the interviewer's prompt for reflection upon the sequence of numbers that initiated a process of further explicitation of the representations that sustained the ***procedural facet*** of the 'ordering' method to the **E3** format.

The ***conceptual facet*** of the 'ordering' method is considered as being represented in Explicit **E1** format. Leo seemed to acknowledge only specific aspects of his method. Until the end of his participation in the sessions, Leo never explained the rationale which sustained the generation of number bonds by following a descending or ascending order in the column of first addends, and an ascending or descending order, correspondingly, in the column of second addends.

However, Leo seemed to acknowledge a particular aspect of the method: reversibility (Appendix 7.5-compare para 2.2, 3.1). He seemed to be aware that the method could be applied and produce correct number bonds by following either an ascending or descending order for the first addends and a correspondingly descending or ascending order for the second addends. Leo's understanding of the reversibility of the method was inferred on the basis of Leo's procedural applications and problem solving behaviour. He never provided a verbal explanation regarding this quality of the method. However, the realisation of this particular aspect of the method is considered as an indication of knowledge represented in a format which was beyond the implicit, procedural level: i.e. a piece of knowledge which was explicit enough to be manipulable but not explicit enough to be accessible for verbal report. Moreover, 'ordering' was generalised to bigger target numbers in the context of the 'card' task. Leo had abstracted the fundamental idea of 'ordering' and was in position to apply it for any target number. However, generalisation of the method was observed only in the context of the 'card' task (Table 7.5-para 4.3, 5.5; Appendix 7.5-para 4.1, 4.2). It is particularly difficult to find in the data signs of conceptual knowledge accessible to conscious reflection even though not accessible to verbal report. This is why it is believed that knowledge representations of E1 format could be probably assigned to the conceptual facet of the method. Little omissions such as the omission of the 'add' sign in the last session (Appendix 7.5-para 5.2), and partial centrations on certain aspects of the strategy are considered as indications of Leo's 'meta-procedural' work upon knowledge represented in E1 format (Karmiloff-Smith, 1992). This belief is also supported by the fact that Leo did not adjust the fundamental idea of 'ordering' to overcome violations that the interviewer introduced in his usual practice in the context of the 'card' task (Table 7.5-para 5.1). It was made evident that Leo had conceptualised the 'ordering' method as a procedure of putting numbers in a specific sequence, in a specific order: that is from the 'target' going down to 0 or from 0 going up to the 'target'. Each of the two addends constituting a number bond was, for Leo, a link of the vertical sequence-chain of first or second addends. It is believed that Leo had not explicitly conceptualised the arithmetical rationale behind the strategy. He did not seem to apply the 'ordering' strategy on the basis of an arithmetical representation of the type 'more/less' or 'less/more' which is the arithmetical relation between the numbers he was using as first or second addends, and which resulted in their ordered disposition. This is why Leo was not in position to use the idea of ordering in situations where the whole sequence of numbers as this appeared in the context of the 'card' task was not available (e.g. 'missing numbers' task).

Plane B: Before closing this section it needs to be emphasised and clarified that Leo is considered as having passed to a 'meta-procedural' phase because of his overall attempt to organise his solution approach. The application of two unified methods in each part of the solution process constituted a strategy. This organisation-oriented approach consisted of the combination of two methods sustained by two pieces of knowledge represented at different level of explicitness. The conceptual facet of the overall problem solving strategy of combining the two methods is considered as being underlain by knowledge which was initially represented in **E2** format. Leo approached the task having a specific plan. He combined the two methods consciously and consistently in his engagement with the task. However, Leo never explained and justified his certainty that the task had been completed (Appendix 7.5-para 1.2, 3.1) Also, until the fourth session Leo did not appear to be in position to explain the shift from 'ordering' to 'swapping' (Appendix 7.5-para 2.3, 3.3).

It was only in the last session that Leo attempted to explain the rationale behind the shift to the application of 'swapping' at a particular moment during the problem solving process (Appendix 7.5-para 5.1). This is considered as an indication of a process of further explicitation that Leo's overall approach to the 'card' task was subjected to. However, in the time available, it was not made possible to obtain more evidence which could strongly support the ascription of the E3 format in the knowledge representation which sustained the conceptual facet of Leo's organisation strategy.

7.6. The case of JAMES

James was 6 years 6 months old. He was classified as in the second most advanced group in mathematics. James participated in five sessions. The first two sessions took place on two consecutive days. One day intervened between the second and the third. The third and the fourth session took place on two consecutive days. The fifth session took place two days after the fourth.

Table 7.6: Summary of changes that James introduced into his problem solving approach.

First session	**Second session**
First two runs – 'card' task: 1.1 James applied fact retrieval, counting, and 'swapping. He was not aware of the completion of the task. *Third run – 'card' task* 1.2 James introduced the strategy of producing number bonds in pairs. 1.3 The use of 'instant swapping' was reported for the production of the second number bond of each pair. Declarative/factual knowledge or a 'two-step' process was used for the production of the first number bond of each pair. 1.4 In the case of number bonds which were produced in 'two steps', James did not provide a report of the mechanism on the basis of which he was specifying the first addend. He reported the use of counting for the specification of the second addend. 1.5 James was not certain of the completion of the task. He did not employ any method for checking.	*Two runs – 'card' task*: 2.1 James introduced the 'deriving' method. This method was verbally reported and consistently used for the production of the first number bond of each pair. 2.2 James did not provide verbal explanations regarding the *why* and *how* the 'deriving' method worked as it did. However, he seemed to have grasped conceptually important aspects and qualities of the method that he was using consistently. 2.3 James did not justify his certainty that the task was complete.

Third session	Fourth session
First two runs – 'card' task: 3.1 James did not introduce any changes into his solving approach. *Third run – 'card' task*: 3.2 James showed that he acknowledged the reversibility of the 'deriving' method. 3.3 James did not recognise the repetition of a pair of number bonds. 3.4 James did not explain the rationale behind the strategy of producing number bonds in pairs. He did not justify his certainty that the task was complete.	*'Card' task*: 4.1 James applied his strategy in runs with bigger target numbers. His overt behaviour gave indications that he overcame the violations that the interviewer introduced by applying the rationale of the 'deriving' method. *Similar in goal tasks*: 4.2 James did not apply the strategy of producing the number combinations in pairs in the initial runs with the 'domino' task. 4.3 In the 'balances on paper' task James applied the strategy of producing number combinations in pairs. The 'deriving' method was not applied consistently.
Fifth session *Similar in goal tasks*: 5.1 James did not apply his strategy of producing number bonds in pairs in the 'balance on cardboard' task. He applied the 'deriving' method in the 'missing numbers' task. *'Card' task*: 5.2 James overcame violations that the interviewer introduced to his usual procedure. The use of the 'deriving' method was reported. However, James did	not explain the conceptual basis for the choice of the operation that he was applying. 5.3.James extended the fundamental rationale of 'adding 1/taking away 1' that underlay the 'deriving' method and produced number bonds by 'adding 2 / taking away 2'. 5.4.James applied the strategy of producing number bonds in pairs and the 'deriving' method to generate number combinations to make 100.

Discussion

Plane A: The procedural facet of the methods that James used in the first two runs with the card task is considered as being underlain by knowledge representations of E3 format. In the third run, and in the case of number bonds produced by a 'two-step' process, the criterion or mechanism on the basis of which James was choosing and specifying the number that was used as first addend was not reported (Table 7.6-para 1.3, 1.4). The available data do not provide any indication on the basis of which the level of explicitness of the criteria which drove this choice could be inferred. In the third run James attempted, for the first time, to organise the solution process.

Plane B: In the first two runs James' approach to the task is considered as being in a **'procedural phase'**. Each step of the solution process was considered separately and a variety of methods was employed for the attainment of the goal. Even though James was successful, he did not appear to be aware of his success. James' first attempt for organisation of the solution steps took place very soon, right in the third run with the task, in the first session (Table 7.6-para 1.2; Appendix 7.6-para 1.1). The production of number bonds in pairs is considered as a first movement to a **'meta-procedural phase'**. Nevertheless, James was still not certain for the completion of the task (Table 7.6-para 1.5; Appendix 7.6-para 1.2).

In the second session the organisational strategy of producing number bonds in pairs was applied consistently. Two methods were consistently involved in the production of each pair of number bonds: the 'deriving' method and 'instant swapping'. In _plane A_, the 'deriving' method was introduced for the production of the first number bond of each pair. The 'deriving' method was initially applied for the production of isolated number bonds within the context of the strategy of producing number bonds in pairs. Little by little the method was generalised and consistently applied for the production of the first number bond of each pair of number combinations that James produced (except from the first two pairs in the case of which the first number bond was usually the product of James' declarative knowledge). This method involved the combination of two arithmetical operations that James applied to the first and second addend of the number bond-reference in order to derive the new number bond. James was always using as number bond-reference the second number combination of the last pair of combinations he had produced. For this reason, the combination of operations was changing: James was either applying a 'take away/add' combination or an 'add/take away' combination. Because James never repeated a number bond, it is considered that the shifts from the application of one combination to the other were not made in a haphazard way. Rather, they were conscious shifts. However, James did not explain these shifts at any point during his participation to the sessions. At the ***procedural level***, the 'deriving' method seemed to be supported by knowledge representations of **E3** format. This is because James always reported the procedures and operations involved in the method, and he always indicated the number bond-reference. Also, James' explanations right from the beginning of the application of the method indicated that he had an explicit representation of the arithmetical relationship (take away 1/add 1) between the addends of the number bond-reference and the derived number bond (Appendix 7.6-para 2.1).

At the ***conceptual level*** though, this method seemed to be sustained by knowledge representations of **E2** format. James never explained why this method was working the way it did: i.e. why he had to combine these two operations, adding and taking away, to produce a new number bond. Also, he never gave an explanation for the shifts from one combination of operations to the other. Up to the last session, the rationale that sustained the 'deriving' method did not seem to be accessible for verbal report (e.g. Appendix 7.6-para 2.3). Nevertheless, James always applied the method correctly, and seemed to have conceptualised its reversibility (Appendix 7.6-para 2.1, 2.2, 3.1). This conceptualisation allowed him to shift from one combination of operations to the other and apply the method flexibly. This flexibility was also made apparent in situations where the interviewer violated James' usual practice of producing number bonds. In those cases, James applied the fundamental rationale that underlay the 'deriving' method and extended it (Appendix 7.6-para 4.1, 5.1, 5.2). This is why it is believed that the rationale behind the method was underlain by knowledge representations which were explicit, and consciously accessible, though not accessible for verbal report and explanations.

<u>*Plane B*</u>: By the third session, it was shown that the ***conceptual facet*** of James' overall strategy of producing number bonds in pairs and applying 'instant swapping' was represented in a format which was not accessible for verbalisations, that is in **E2** format. James reported the use of the two methods which constituted the strategy of producing number bonds in pairs (i.e. the 'deriving' method and 'swapping'). But he never explained the rationale behind the combination of these methods. Also, he never justified his certainty that the task was complete (e.g. Table 7.6-para 2.3, 3.4; Appendix 7.6-para 2.4).

James' strategy seemed to be mastered procedurally but it was not conceptualised explicitly enough to allow the formulation of verbal explanations. James was successful. Nevertheless, he did not seem to have constructed an explicit representation of all the aspects of the task. It is believed that the realisation that each number bond should have a corresponding 'other half' drove the development of the strategy of producing number bonds in pairs, and provided a basis for James' certainty that the task was complete. James seemed to be strongly focused on this idea and overlooked other aspects, for example the possibility for a pair of number bonds to be missing all together. Also, James' organisational strategy did not seem flexible enough to integrate number bonds produced by different knowledge representations and avoid repetitions (Table 7.6-para 3.3; Appendix 7.6-para 3.2). The strong, partial focus on one particular aspect of the task which drove James' organisational attempt is considered as evidence of a **'meta-procedural'** problem solving approach.

James did not introduce any other modifications to his organisational strategy up to the last session. The strategy of producing number bonds in pairs and the 'deriving' method were not applied in the 'domino' and 'balance on cardboard' tasks. James applied his organisational strategy in the 'balances on paper' task. However, it should be noted that in that task the strategy of producing number bonds in pairs was applied in the initial form of its development in the context of the 'card' task. That is James applied a mixture of methods and called upon various sources of knowledge for the production of the first number bond of each pair. The 'deriving' method was one of these methods, but it was not applied consistently as in the runs with the 'card' task in the last three sessions.

7.7. Summary

This chapter presented five cases of children who in the course of the five sessions in which they participated, gave signs of 'procedural' and subsequently 'meta-procedural' problem solving behaviour, while solving a specific task. Each of the five cases presented particularities in their problem solving approach. Henry was in control of the solution process by applying the strategy of producing number bonds 'in pairs'. Also, Rakhi was in very good control of her strategy of specifying the first addend following an order and specifying the second addend by counting. Both children lost their control over the task after introducing the 'ordering' strategy for the specification of the first and second addend of each new number bond. In both cases the 'ordering' strategy was introduced after children's observation of a certain pattern of numbers. This pattern was generalised. But this generalisation did not have a conceptual basis. In both cases the conceptual facet of the strategy was represented in an E1 format. As a result of this, both children's problem solving behaviour presented regression: Rakhi presented regression at the level of conceptual control over the aspects of her employed strategy and of the task, but not at the level of performance (that is she was still successful procedurally). Henry presented regression at the level of performance in one run with the 'card' task (that is he did not solve the task successfully and overlooked the negative feedback). Isa and Leo were two cases which did not present regression at any level, at any point of their encounter with the task. The itinerary of changes that they introduced to their approach presented a rather linear progress towards the 'meta-procedural' phase. In the context of this phase the combination of two methods constituted a strategy. The 'deriving' method in the case of Isa and the 'ordering' method in the case of Leo, were represented conceptually in an E1 format. This resulted in inflexible problem solving behaviour driven by partial, insisting focus on certain aspects of the task only. James' approach was also characterised by partial focus on certain aspects of his strategy and of the task. The difference in this case was that James' overt behaviour showed that he was aware and in control of most of the aspects (mainly conceptual aspects) of his strategy. These conceptual aspects were represented in an E2 format, that is they were represented in a higher level of explicitness in comparison with the rest of the cases which are presented in this chapter. They were represented though in a format which did not allow verbal explanations to be formulated.

Chapter 8 Cases of 'conceptual' problem solving behaviour

8.1. Overview

The chapter presents an analysis of five cases of children who, at the end of the series of sessions in which they participated, gave signs of 'conceptual' problem solving behaviour. Of the five cases, four passed through all the three phases in problem solving as these are outlined in the framework of the RR model before they reach the 'conceptual' phase. These children are referred to as Grace, Hazel, Elsa, Erna and Sean. Grace was a child who exhibited a very slow process of change. In the itinerary of these changes one can follow Grace's slow, and clear steps, her progressive movement to higher levels of explicitness regarding her own approach to the task. Hazel was also a case which presented clear indication of the passage from one phase to the other. Also, in the case of Hazel only in this group, there was deterioration of performance during the 'meta-procedural' phase. Elsa was a case in which one can observe clearly the progressive process of explicitation in her verbalisations. Finally, Erna and Sean gave signs of 'conceptual' problem solving behaviour very soon, in the second session. The particularity in the fifth case, the case of Sean, was that, right at the beginning of his encounter with the task, Sean exhibited organisation-oriented behaviour. He was the only one who, solving the particular task, did not give signs of all the three phases. Sean started from a 'meta-procedural' phase and moved towards a 'conceptual' phase. For each of the cases, indicative moments of each session that illustrate children's problem solving behaviour and verbalisations, are given in a corresponding appendix (see appendices 8.2, 8.3, 8.4, 8.5, 8.6). The first two of the profiles that follow (that is, the profiles of Grace and Hazel) are structured on a chronological basis. This means that changes in plane A and B are discussed session by session. The reason is that both these children exhibited a slow, clear movement from one phase to the other, session by session. The changes that were observed in the rest of the cases were not introduced session by session. Rather, in the cases of Elsa, Erna and Sean, there were sessions in which more than one change occurred, and sessions where children's behaviour was consolidated and there were no changes introduced. For this reason, the analysis of each of the cases of Elsa, Erna, and Sean, focuses on specific points of the children's problem solving behaviour which are of particular interest and which are notable for this group of cases.

8.2. The case of GRACE

Grace was 6 years 5 months old. She was classified as in the most advanced group in mathematics in the class. The first three sessions with Grace lasted for only half the time of that devoted for sessions that most of other children participated in. The reason for this was that Grace was quite timid at the beginning of her participation in the sessions. Going slow with her at the first sessions would give her the time to familiarise herself with the interviewer and the setting. The first three sessions took place on three consecutive days. Two days intervened between the third and the fourth session. The fourth and the fifth sessions took place on two consecutive days. The changes that Grace introduced in her solving approach in the course of the five sessions are presented in the table that follows.

Table 8.2: Summary of changes that Grace introduced into her problem solving approach

First session	**Second session**
<u>*One run with 'card' task:*</u> 1.1 Grace reported the use of fact retrieval and 'swapping'. 1.2 Most of the number bonds were produced by a 'two-step' process: Grace did not report the mechanism or criterion on the basis of which she was specifying the first addend of each new number bond. Counting or declarative knowledge was used for the specification of the second addend. 1.3 Grace repeated a number bond and then denied the repetition. This repetition is considered as the result of the same number bond being produced by different knowledge representations. 1.4 Grace did not appear to be aware of the completion of the task.	<u>*First run –'card' task*</u>: 2.1 The solution process was separated in two parts. In the first part Grace applied a mixture of methods. In the second part Grace reported the use of 'swapping' for the production of all the number bonds. 2.2 Grace was not aware of the completion of the task. <u>*Second run-'card' task*</u>: 2.3 Separation of the solution process in two parts. 2.4 For the first time, in the case of a number bond produced by a 'two-step' process, Grace explained the criterion on the basis of which she was specifying the first addend. <u>*Following two runs with 'card' task*</u>: 2.5 The solution process was separated in two parts: First part: Grace used mainly her declarative/factual knowledge or the 'two-step' process to produce number bonds. Second part: Grace applied 'swapping'. 2.6 Grace appeared to be certain of the completion of the task. She did not justify her certainty.

Third session	Fourth session
Run with 'card' task - target number 9 as in the first session: 3.1 Grace applied the 'two-part' strategy. She provided a clear report of her organisational strategy. 3.2 She explained at which particular moment of the solving procedure she was shifting to the application of 'swapping'. 3.3 Grace was aware of the completion of the task. She justified her success by explicitly referring to her method of checking the numbers that she had used, in order.	_First run-'card' task_: 4.1 Grace applied the 'deriving' method for the production of one number bond in the first part of the solution process. She indicated the number bond - reference. She did not report the operations that she carried out. _Two runs with 'card' task_: 4.2 First part of the solution process: Grace used the 'deriving' method for the production of all the number bonds. Starting with the number bond in which 0 was the first addend, the number bonds produced were following a specific order. She indicated the number bond - reference. She did not report the mechanism. Second part of the solution process: Grace used 'swapping'. _Three runs 'domino' task_: 4.3 Grace applied the 'two-part' strategy. However, in the first part she did not apply the 'deriving' method.
Fifth session _First run-'card' task_: 5.1 Grace formulated and provided a clear verbal explanation of the 'deriving' method. 5.2 Grace applied the rationale that sustained her 'deriving' method to complete number bonds which were not part of the sequence of number bonds that she was producing. Grace explained the solution she gave on the basis of the arithmetical relation that underlay her strategy.	_Similar in goal tasks_: 5.3 Grace applied the 'two-part' strategy. She did not apply the 'deriving' method in the 'domino' task. She did apply the method in the 'balances on paper' and 'balance on cardboard' tasks. 5.4 Grace applied the 'deriving' method to produce number bonds for big target numbers in a paper and pencil task. Also, she showed that she was aware of the reversibility of the method.

Discussion

Plane A: In the first session, most of the number bonds were produced by a 'two-step' process (Table 8.2-para 1.2). In those cases, each of the two addends was produced by a different mechanism sustained by different knowledge representations. The mechanism which produced the second addend (counting, or retrieval from memory) is considered as being supported by knowledge representations of **E3** format. The mechanism or criterion on the basis of which Grace was specifying the first addend, was not accessible for verbal report. However, the specification of the first addend was a matter of a specific choice.

The data from this session show that the rationale behind this choice was not verbally accessible to Grace, but do not allow any reasonable and justifiable interpretation regarding the level of explicitness of the rationale that drove this particular choice (see Appendix 8.2-para 1.1).

Plane B: By applying the mixture of aforementioned methods and mechanisms of choice, Grace was successful in the task. However, at the end of the first session, she was not aware of her success (Appendix 8.2-para 1.3). In the first session, Grace's problem solving approach was success-oriented. Multiple, different pieces of knowledge were activated for the goal to be attended. The fact that Grace repeated one number bond and then denied the repetition indicates the activation of different, unconnected knowledge representations at different moments during the solution process (Table 8.2-para 1.3; Appendix 8.2-para 1.2). Grace approached the production of each number bond as a separate problem in the context of her attempt to find all the possible number bonds. She did not give any evidence of an attempt to integrate each step of the solution process into a whole: a unified approach. This is why Grace's problem solving approach up to that point of her work with the task is classified as having the characteristics of the **'procedural phase'**.

Plane A: In the second session the rationale which underlay the choice of a number to be used as first addend seemed to have undergone a process of explicitation (Table 8.2-para 2.4; Appendix 8.2-para 2.2). From that point onwards, both mechanisms which sustained the production of the first and second addend in the context of the 'two-step' process, are considered as being represented in an explicit format which allowed verbal explanations to be formulated, that is in **E3** format.

The separation of the solution process in two parts (Table 8.2-para 2.1-2.6; Appendix 8.2-para 2.1, 2.2) is considered as Grace's first step towards an organisational strategy. This strategy, in its initial phase, consisted of the combination of various different methods. The procedural facet of the strategy at this initial phase of its development, is considered as being represented in a format which was accessible and which allowed Grace to describe, work upon, and elaborate this organisational idea in the following sessions with the introduction of a new method. This is why the ***procedural facet of the 'two-part' strategy*** is considered as being represented in an Explicit **E3** format. The methods that Grace applied in the context of her organisational strategy were consistently combined in more than one run with the task. Also, a consistency regarding the timing in the use of each of these methods, was observed. In the second session, Grace did not provide an explanation of this 'timing': i.e. why she was shifting to the application of 'swapping' at that particular moment. At that point, Grace did not appear to be ready to talk about the rationale behind this shift: a turning point which constituted the main aspect of her strategy of separating the solution process in two parts. The consistent combination of the aforementioned methods into a strategy which was applied in every single run from that point onwards, is considered as an indication of deliberate problem solving behaviour. Furthermore, in the following sessions Grace introduced in the context of this strategy a new method (the 'deriving' method) leaving though intact the structure of the solution process: i.e. the fundamental idea of separating the solution process in two distinct parts. This is why it is believed that the conceptual facet of the strategy was sustained by knowledge represented in a format which was consciously accessible: Grace's problem solving behaviour was deliberate, consistent and open to the introduction of new elements. However, there was absence of any verbal explanation of the rationale which supported Grace's strategy. For the aforementioned reasons the ***conceptual facet of the strategy*** up to that point is considered as being underlain by knowledge representation of **E2** format. In *plane B*, the introduction of the strategy of separating the solution process in two parts, signalled Grace's passage to a **'meta-procedural'** phase in her approach to the task. Still, the application of a mixture of methods could be observed. However, the various methods were consistently combined in the context of an organisational strategy. This combination seemed to be underlain by a specific rationale, which was not reported nor explained at the initial phase of the introduction of Grace's strategy.

Plane A: The explanations that Grace provided in the third session (Table 8.2-para 3.1, 3.3; Appendix 8.2-para 3.1) are a sign of further explicitation that Grace's organisational strategy and the rationale that sustained it had been subjected to. With the application of a strategy the procedural and conceptual facet of which were represented in **E3** format, Grace seemed to have acquired a very good control over the task. In *plane B* this could be meaning the passage to a **'conceptual phase'** in the solution approach to that particular task.

In the fourth session and in *plane A*, a further modification was observed (Table 8.2-para 4.1, 4.2; Appendix 8.2-para 4.1, 4.2). The introduction of the 'deriving' method in the context of the same organisational strategy constituted a change for which there was no apparent reason. This change in the problem solving procedure took place when Grace seemed to control all the aspects of the task adequately. Little by little, the newly introduced method developed as the main method for the production of number bonds in the first part of the solution process (Table

8.2-para 4.2; Appendix 8.2-para 4.2, 5.1-5.4). As a result of this, the solution time was notably shorter. However, at the initial phase of the application of the method Grace did not report the procedures/operations involved in the 'deriving' method or explained the rationale that sustained the method.

At the initial phase of the consistent application of the 'deriving' method the procedural facet of the method was represented in a format which was not accessible for verbal report (Table 8.2-para 4.1, 4.2). However, the ***procedural facet*** of the method was explicit enough for Grace to access it in order to develop it and generalise it in the course of two consecutive runs with the 'card' task. This is why it is believed that even at the initial phase of the introduction of the 'deriving' method, the underlying knowledge was represented in an explicit **E2** format. This allowed conscious access, elaboration and generalisation of the method to all the solution steps of the first part of the solution process. Since there were no reports of the operations involved in the method, at that point, there were no indications regarding the ***conceptual facet*** of the method. It should be noted that the consistent application of the 'deriving' method had as a consequence a new, ordered organisation of the number bonds in the first part of the solution process. Grace's solution times were remarkably shorter and her strategy was further elaborated and could be now described as a 'two-part' strategy each part of which was marked by the application of a unified, systematic method.

In the fifth session, Grace's explanations indicated that she had a good understanding of the fundamental rationale that underlay the 'deriving' method. It was shown that Grace acknowledged that the operation that she was carrying out to specify the first addends was connected to the operation that she was carrying out to specify the second addend: She was adding 1 more because she had previously taken away 1 (Appendix 8.2-para 5.1). This justification gave signs of explicit knowledge representations of **E3** format that supported the ***conceptual facet*** of the method. Grace's performance in the fifth session (Table 8.2-para 5.2-5.4) gives strong evidence of the high level of explicitness into which the 'deriving' method was represented (Appendix 8.2-para 5.1-5.4).

In <u>*plane B*</u>, what needs to be emphasised is that with the introduction of the 'deriving' method in the fourth session, and its subsequent elaboration and generalisation, Grace's approach to the task still had the characteristics of the **'conceptual phase'**. The overall 'two-parts' strategy was retained together with Grace's good control over the procedural and conceptual aspects of the task.

Regarding the tasks with similar goals, it is noteworthy that Grace applied the 'two-part' strategy in which the 'deriving' method and 'swapping' was combined, in the 'balances on paper' task. In the 'balance on cardboard' task Grace applied the 'deriving' method in the first part of the solution process and announced her intention to apply 'swapping' in the second. However she found difficult to proceed in the application of 'swapping'. The difficulty was due to the fact that in that task the number bonds produced were not available as visible reference as Grace herself explained: "Now... I need to do the changing but... I don't remember the ones." The 'two-parts' strategy was applied in the 'domino' task but its application did not include the 'deriving' method (Table 8.2-para 4.3 and 5.3). A trivial speculation concerning the application or non application of the 'deriving' method in the tasks with similar goals, could be that Grace applied the 'deriving' method only in tasks which involved numerals. There might be a certain constraint in the application of the method which had to do with Grace's ability to apply the operations involved in the method, and the fundamental rationale in a context which did not involve numbers. Of course this is only a speculation which needs further examination.

8.3. The case of HAZEL

Hazel was 6 years 2 months old. She was classified as in the second most advanced group of mathematics. Hazel participated in five sessions. Two days intervened between the first and second session. All the other four sessions took place on consecutive days. The itinerary of changes that Hazel introduced into her problem solving approach is summarised in the table below.

Table 8.3: Summary of changes that Hazel introduced into her problem solving approach

First session	**Second session**
<u>*First run – 'card' task:*</u> 1.1 Hazel used her declarative/factual knowledge, counting and 'swapping'. In cases where counting was used for the specification of the second addend, Hazel did not report the mechanism/criterion on the basis of which she specified the first addend. 1.2 Hazel justified her certainty that the task was complete by uttering the numbers she had to have used, in order. She did not make clear her way of checking the actual use of these numbers. <u>*Following three runs with the 'card' task*</u>: 1.3 Hazel made explicit her way of checking the use of all the possible numbers.	<u>*First run – 'card' task:*</u> 2.1 Hazel started organising the solution process. She reported her method of specifying the first addend of each number bond following the order of numbers in the number line. 2.2 One number bond was repeated. It is believed that this was the result of the use of two different knowledge representations for the production of the same number bond at different moments during the solution process. <u>*Second run – 'card' task:*</u> 2.3 Hazel specified the first addends following a descending order and she explained why: this was the beginning of the development of a strategy.
Third session	**Fourth session**
<u>*Run with 'card' task*</u>: 3.1 Hazel used her declarative knowledge for the production of the first two number bonds. Each of the addends of the third number bond was specified by a different knowledge representation. 3.2 Hazel applied the 'deriving' method of '1 less/1 more' to specify both the first as well as the second addend of each of the following number bonds.	<u>*First run – 'card' task*</u>: 4.1 Hazel applied the 'deriving' strategy for the production of all the number bonds (except from the first one). She reported her strategy and justified its use. <u>*Second run – 'card' task*</u>: 4.2 Hazel interrupted the production of number bonds by the 'deriving' method to swap all the number combinations that she had produced already. Then, she continued applying the 'deriving' method without realising, at first, the repetition of number bonds. A certain redundancy appeared in Hazel's way of checking the numbers in order.

Fifth session	
Run with 'card' task: 5.1 Hazel applied the 'deriving' strategy all along the solution process. She gave explanations regarding the arithmetical operations involved and the reversibility of their interrelation: she explained that the numbers in the columns of first and second addends could either follow an ascending or descending order. *Violations*: 5.2 Hazel created the missing steps to succeed in tasks that violated the sequence of number bonds that she was producing.	*Similar in goal tasks*: 5.3 Hazel 'rediscovered' the 'deriving' strategy while engaging in solving the 'domino' and the 'balances on paper' tasks in multiple runs. 5.4 Hazel applied the 'deriving' / 'ordering' strategy to find number bonds for big target numbers (e.g. 100) in a paper and pencil task.

Discussion

Plane A: In the first session Hazel used counting to complete certain number bonds after having chosen and specified the number to be used as first addend. In those cases, each of the addends of a number bond was produced by a different mechanism underlain by different knowledge representations. Whereas counting, as a mechanism which produced the second addend was reported, Hazel did not explain on the basis of which mechanism, or criterion she was specifying the first addend (Table 8.3-para 1.1; Appendix 8.3-para 1.1). As in the case of other children who used the same method for the production of number bonds (e.g. Grace), the choice of a number to be used as first addend might have been made in a haphazard way or not. In the case of Hazel as well, the available data do not allow us to make any justifiable interpretation regarding the implicit or explicit knowledge representation which sustained the choice of first addend in the first run with the task. *Plane B*: Hazel applied various methods for the production of number bonds in the first session. She approached the production of each number bond as a separate problem within the solution process and did not give any indication of an overall organisation of the production of number combinations. The lack of such an organisational attempt, and the use of various different methods at each of the solution steps indicate a problem solving approach which had the characteristics of the **'procedural'** phase.

Plane A: Right from the first run with the 'card' task, Hazel seemed to be aware of her success. She made explicit her method for checking the production of number bonds at the end of the first session (Appendix 8.3-para 1.2, 1.3). Unlike her production methods, Hazel applied a systematised 'checking' method. This systematisation in checking the use of numbers, impelled the subsequent development of Hazel's organisational strategy for the production of number bonds.

In the second session, the two mechanisms that Hazel used to specify the first and second addend of each number bond were explicit enough to be reported (Table 8.3-para 2.1, 2.3). Hazel applied 'ordering' and counting, or 'ordering' and declarative knowledge for the specification of first and second addends correspondingly. The consistent combination of these methods constituted the first steps towards the development of a strategy. The ***procedural and conceptual*** facet of the strategy at that point is considered as being represented in an explicit **E3** format. Hazel reported the combination of the two methods and justified her success (Appendix 8.3-para 2.1, 2.3). In <u>*plane B*</u>, the first steps towards the development of a strategy signalled Hazel's passage to a **'meta-procedural'** phase. Hazel introduced a sequence of modifications in her attempt to organise the solution process. Multiple representations were still activated. Hazel's focus on organising the solution steps resulted in minor errors in the course of the solution process (e.g. repetition of a number bond: Appendix 8.3-para 2.2).

<u>*Plane A*</u>: In the third session, Hazel used and combined succcessfully different knowledge representations (Appendix 8.3-para 3.1, 3.2). Hazel had a good control over the task and a good understanding of the arithmetical interrelation between the first and second addends of the number bond-reference and the number bond that she was working upon (Appendix 8.3-para 3.1, 3.2). Both the ***procedural as well as the conceptual facet*** of the 'deriving' method is considered as being sustained by explicit **E3** knowledge representations. Hazel reported the operations that she was carrying out for the specification of the each addend, and explained that "going less and more" for the specification of each addend, was required to retain the sum. It was the first time that both addends were produced by the same knowledge representation.

In the fourth session, Hazel showed again her understanding of the fact that the combination of subtracting and adding was required for the sum to be retained and conserved (Appendix 8.3-para 4.1). At that point Hazel appeared to have a very good control over the task and her strategy. However, in the following run with the 'card' task the activation of other methods, together with the 'deriving' method, had, as a result, the repetition of number bonds and the *deterioration* of Hazel's problem solving performance (Table 8.3-para 4.2, 4.3; Appendix 8.3-para 4.2).

In *plane B*, the non successful 'coexistence' of multiple methods is considered as an indication of Hazel's **'meta-procedural'** work upon the task; as an indication of a process of redescription that the representation of the task, its particular aspects and requirements were subjected to. The multiple methods were activated in the context of Hazel's attempt of organise her problem solving approach. It should be noted, that even though the different methods which were activated at that particular point seemed to be represented in high levels of explicitness procedurally and conceptually (**E3** format), Hazel seemed to have lost some of the control that she had over the task in the previous sessions. This was not observed to that extent (or even at all) in other cases.

Plane A: In the fifth session Hazel approached the solution process in a unified way. She used the 'deriving' strategy all along the solution process. There was further evidence of the high level of explicitness of the knowledge representation which supported the application of the 'deriving' strategy (Table 8.3-para 5.1, 5.2, 5.3; Appendix 8.3-para 5.1). For the first time Hazel explained her strategy using the phrases "counting down" and "counting up". This showed that she had a rich system of representations regarding her strategy. She was aware of the fact (and she was in position to explain verbally) that applying the operations of adding 1 and subtracting 1 consistently, as she did, had as a result the ascending and descending order of numbers. In *plane B*, Hazel's problem solving behaviour in the fifth session indicates a phase of consolidation of the particular problem solving approach where the procedural and conceptual facet of the strategy was represented in an explicit **E3** format. This phase of consolidation marked Hazel's passage to a **'conceptual'** phase in the framework of the particular task.

Concerning the tasks with similar goals, it took time before Hazel started applying the 'deriving' method in the 'domino' and 'balances on paper' tasks. In those tasks, Hazel seemed to pass through various phases of rediscovery of the strategy and of constructing a representation of the task. Her short itinerary through these phases resembled the itinerary of changes observed in the context of the 'card' task. Even when she started applying the 'ordering' strategy, in her verbal explanations, Hazel did not acknowledge the similarity between these tasks and the main task. She did acknowledge though that the strategy was the same. It was not made clear in what respect Hazel thought that the tasks were not similar: the goal or the superficial elements.

The case of Hazel is notable because it was the only case in this group in which the **'meta-procedural'** phase was accompanied by signs of deterioration in performance. Also, it was particularly interesting that all the changes observed in Hazel's problem solving approach were changes that involved methods and sources of methods which seemed to be represented in high levels of explicitness. This was observed all along Hazel's itinerary from the 'procedural' up to the 'conceptual' phase. There were signs of redescription on the plane of the representation of the task (plane B). There were no clear signs of redescription on the plane of the separate methods and strategy that Hazel applied (plane A).

8.4. The case of ELSA

Elsa was 5 years 9 months old. She was classified as in the most advanced group in mathematics. Elsa participated in five sessions. The first three sessions took place on three consecutive days. Three days intervened between the third and fourth session. The fourth and fifth session took place in two successive days.

Table 8.4: Summary of changes that Elsa introduced into her problem solving approach.

First session	**Second session**
<u>*First two runs – 'card' task*</u>:	<u>*First run – 'card' task*</u>:
1.1 Elsa used fact retrieval, counting, and 'swapping'. She was aware of the completion of the task, but the justifications she gave were weak. Her way of checking did not seem to be organised in any particular or systematic way.	2.1 The solution process was separated mainly in three parts. Elsa's strategy was to apply 'ordering' in the first part, and 'swapping' in the second. In the third part the two last number bonds were those which involved 0.
<u>*Third run-'card' task*</u>:	2.2 Elsa used the words "order" and "down" to name the different kind of order that first and second addends were following in the first part.
1.2 Elsa modified her approach to the task. The idea of 'ordering' both the first as well as the second addends was introduced and applied for the production of the first four number bonds.	<u>*Second run – 'card' task*</u>:
1.3 Elsa used 'swapping' and counting to produce the rest of the number bonds.	2.3 Elsa explained the shift from 'ordering' to 'swapping': she was applying the 'ordering' up to the point where all the possible numbers had been used either as first or second addends.
	2.4 Elsa gave signs of flexibility in the application of 'ordering'. She was aware of the reversibility of the method.

Third session	Fourth session
<u>*One run with 'card' task*</u>: 3.1 The solution process was separated in two parts: Elsa applied 'ordering' and 'swapping' correspondingly. 3.2 Each of the two number bonds that included 0 was integrated in one of the two parts of the solution process. Elsa explained her method of 'ordering' using the words 'next' and 'less' instead of 'order' and 'down' as she did in previous runs. <u>*First two runs with 'domino' task*</u>: 3.3 Elsa did not apply 'ordering' in the 'domino' task. She produced combinations in pairs, using counting and 'swapping'.	<u>*Three runs with big target numbers in 'card' task:*</u> 4.1 Elsa applied her strategy to bigger target numbers. She provided verbal explanations about the order of first and second addends, and applied the idea of 'ordering' to overcome violations in the sequence of produced number combinations. <u>*Two runs with 'domino' and three runs with 'balances on paper' tasks*</u>: 4.2 As in the previous session, in the 'domino' task Elsa applied the strategy of producing number bonds in pairs. 'Ordering' was not applied. In the 'balances on paper' task Elsa applied the strategy of combining 'ordering' and 'swapping' as in the 'card' task.
Fifth session <u>*One run with 'card' task*</u>: 5.1 Elsa gave detailed explanations of her method of producing number bonds in order. She extended the '1 more/1 less' rationale of 'ordering' to overcome violations that the interviewer introduced, and to complete the 'missing numbers' task. <u>*'Missing numbers task':*</u> 5.2 Elsa applied the 'next'/'less' rationale in more than one steps, and in a context other than the 'card' task and the 'ordering' method. However, she did not explain why one had to make the same number of steps 'next' and 'less' in both columns of addends.	<u>*Balance on 'cardboard'*</u>: 5.3 Elsa applied 'ordering' to produce a first set of number combinations. She could not apply 'swapping' because the already produced number bonds were not available as visible reference. She explained: "I need to change it but I don't remember the numbers". 5.4 She applied 'ordering' when she was asked to produce number bonds for big target numbers verbally, or in a paper and pencil task (e.g. 100).

Discussion

<u>*Plane A*</u>: In the first two runs with the task, the procedural facet of each of the methods that Elsa used is considered as being sustained by knowledge representations of E3 format. Elsa produced certain number bonds by a 'two-step' process. She reported the use of counting for the specification of the second addend and explained the criterion on the basis of which she was choosing the number to be used as first addend (Appendix 8.4-para 1.1). The two addends were produced by two different mechanisms supported by two different pieces of knowledge. Both

mechanisms were reported. Elsa was the only case in which, right from the beginning, the ***procedural facet*** of both mechanisms involved in the 'two-step' process is considered as being represented in an **E3** format.

Plane B: In the two initial runs with the task Elsa applied methods the procedural facet of which was sustained by knowledge representations of E3 format. However, her overall problem solving approach, at that point, is considered as being at the **'procedural phase'**. This is because Elsa applied a mixture of knowledge resources and methods the procedural facet of which seemed to be explicit to her but she did not have a unified problem solving approach in every step of the task. On the basis of an 'a posteriori' observation of the number bonds that Elsa produced, it is speculated that she used swapping when all the possible numbers had been used either as first or second addends. At that point, Elsa did not reply to the interviewer's questions regarding the moment that she decided to apply 'swapping'. Also, her certainty that the task was complete was based on a weak justification.

Plane A: The 'ordering' method was introduced in a rather abrupt way, in the sense that Elsa did not give signs of an observation of hers, or reflection which might have led to the development and introduction of this method (Table 8.4-para 1.2). The ***procedural facet*** of the 'ordering' method is considered as being sustained by knowledge representation of **E3** format (Appendix 8.4-para 1.2). However, Elsa did not give any answer when she was asked to explain the rationale that underlay her 'ordering' method: i.e. *why* putting the first and second addends in this particular order led to the creation of the right sums. Because of the absence of verbal explanation, it can be considered that the ***conceptual facet*** of the 'ordering' method might be sustained by a knowledge representation of **I, E1, or E2** format. The data available up to that point (i.e. end of first session) cannot help in clarifying that.

Plane B: The first time that 'ordering' was introduced in the solution process Elsa applied different methods for the production of two subsequent sets of number bonds (Table 8.4-para 1.2, 1.3). Still, there was no application of one, unified strategy all along the solution process. However, in that run with the 'card' task, Elsa gave the first signs of an organisation-oriented behaviour. It was the first time that the same method was applied for the production of *sets* of number bonds rather than isolated number bonds within the solution process. With the separation of the solution process in parts, Elsa gave clear signs of an attempt to organise the solution process: of an attempt to develop a strategy. However, at that point Elsa did not give any explanation regarding this particular strategy. She did not explain the shift to the application of 'swapping' which signalled the passage to the second part of the solution process. However, the strategy of separating the solution process in parts, was consistently applied in the following session and signs of evolution were made evident (Elsa started applying the same method for the production of bigger sets of number bonds). This is why in _plane A_ of the situation, the strategy of separating the solution process in parts, is considered as being applied consciously. This problem solving approach was supported by knowledge representations which were accessible for Elsa to develop it and evolve it. The absence of verbal explanations regarding the particular way of organising the production of number bonds, indicates that the ***procedural as well as conceptual facet*** of the organisational strategy were not explicit enough for explanations to be formulated. However, this organisational strategy was in the course of development and therefore it is considered as being sustained by knowledge representations of **E2** format.

In _plane B_, the development of this strategy signalled Elsa's passage to a **'meta-procedural phase'** in her problem solving behaviour. In the second session Elsa' meta-procedural work upon her strategy and the method of 'ordering' was made evident. In the second session, the solution process was separated in three parts (Appendix 8.4-para 2.1). Elsa made clear that she was applying the method of 'ordering' up to the point where all the possible numbers had been used either as first or second addends (Table 8.4-para 2.3). The moment when 'ordering' led to the use of an already used number was a 'turning point', after which Elsa realised that she could start applying 'swapping'. The shift to the method of 'swapping' at that particular moment indicates that Elsa appreciated the fact that since a number had already been used in one of the already produced number bonds she could simply change the addend order of those number bonds to create a new set of number combinations. With this explanation the rationale behind Elsa's strategy of producing sets of number bonds was made explicit. It is considered that, at that point, Elsa's strategy entailed conscious shifts in the application of different methods and pieces of knowledge. It was thus supported ***procedurally and conceptually*** by explicit **E3** knowledge representations.

Plane A: In the second session and the following two, there were changes in vocabulary whenever Elsa gave explanations regarding the arithmetical operations involved in the method of 'ordering' (Appendix 8.4-para 2.1, 3.1, 4.1). These changes are considered as an indication of a process of explicitation that the conceptual aspects of the method were subjected to. During this process it was made possible to observe the gradual elaboration of Elsa's vocabulary, and the replacement of descriptive phrases by words/phrases which denoted arithmetical relations. In the third session Elsa gave signs of flexibility in the application of 'ordering': she could either follow an ascending order in the column of first addends, or follow a descending order in the column of first addends. Also, Elsa appeared to be clear about the connection between 'ordering' and the increasing / decreasing relation of numbers in each column of first and second addends (Appendix 8.4-para 3.1). The application of the 'ordering' method in the fourth and fifth session (Table 8.4-para 4.1, 5.1 - 5.4) and, in particular, the extension of the rationale that sustained the method in contexts other than the 'card' task (e.g. Appendix 8.4-para 5.1) provides the grounds to consider that the ***procedural and the conceptual facet*** of the 'ordering' method was sustained by knowledge representations of **E3** format.

In _plane B_, it is considered that, right from the second session, the ***conceptual facet of Elsa's strategy*** was underlain by an explicit knowledge representation of **E3** format. This signalled the passage to a **'conceptual phase'** of problem solving. In the framework of this phase Elsa made further steps towards a unified solution approach and generalised this approach to big target numbers in the 'card' task (Table 8.4-para 3.1, 3.2, 4.1).

In the case of the tasks with similar goals, Elsa applied the 'two-part' strategy in the 'domino' task. However, the 'ordering' strategy was not applied. Elsa used counting and 'swapping' in each of the two parts of the solution process. On the contrary, Elsa applied 'ordering' and 'swapping' right from the first run with the 'balances on paper' task, and 'ordering' in the 'balance on cardboard' task. Elsa's explanations showed that she was aware of the fact that the 'cards', the 'balances on paper' and 'balance on cardboard' tasks had similar goals (Appendix 8.4-para 4.2). The non application of the 'ordering' method in the 'domino' task was also observed in the case of Grace. As in that case, the speculation is that the non application of the 'ordering' method in the 'domino' task may be related with the absence of numerals in that particular task: i.e. it may be relevant with the superficial elements of the task.

8.5. The case of ERNA

Erna was 6 years old. She was classified as in the second most advanced group in mathematics. Erna participated in five sessions. The first three sessions took place on three consecutive days. One day intervened between the third and fourth session. The fifth and final session took place two days after the fourth.

Table 8.5: Summary of changes that Erna introduced into her problem solving approach.

First session	**Second session**
First run – 'card' task: 1.1 Fact retrieval, counting, and 'swapping' were the main methods that Erna used to solve the task. 1.2 In cases where counting was used for the specification of the second addend, Erna explained that she specified the first addend by referring to the previous number bonds to find which number had not been used yet. 1.3 Erna appeared to be certain of the completion of the task however she did not justify her certainty. *Following two runs – 'card' task:* 1.4 Erna applied the 'deriving' method to specify the second addend and produce certain number bonds during the solution process. 1.5 Erna was aware and certain of the completion of the task. She justified her certainty.	*Two runs with 'card' task*: 2.1 Erna approached the task having a specific plan. 2.2 Erna appeared to have a good control over the features of the task. She was aware of the different kind of order that each column of addends was following and justified it. 2.3 Erna followed the "order" of numbers to specify the first addends. She specified the second addends on the basis of two pieces of knowledge: her knowledge of the principle of commutativity and her knowledge that in the column of second addends number should follow a descending order if the numbers in the column of first addends followed an ascending order.

Third session	Fourth session
First run – 'card' task: 3.1 Erna described and explained the arithmetical relation which sustained the 'ordering' strategy. 3.2 She explained and justified the need to take away in one column of addends and add in the other by relating it to the necessity to maintain the sum. *Second run – 'card' task*: 3.3 Erna adjusted her strategy procedurally to overcome the violation that the interviewer introduced to her usual practice. However, a gesture/speech mismatch appeared in Erna's explanatory attempt. 3.4 Erna created the 'missing steps' to complete number bonds that were not following her usual 'order'.	*Violations:* 4.1 Erna stopped creating the intermediate missing number bonds in order to overcome violations that the interviewer introduced. She overcame the violations by thinking in terms of the number of steps (in those cases, more than 1) that she had to add or take away. *First run – 'domino' task*: 4.2 Erna used factual knowledge, counting and 'swapping' in her first encounter with the 'domino' task. *Second run – 'domino' task*: 4.3 Erna applied the 'ordering' strategy in the 'domino' task. She commented that the two tasks, the 'domino' and the 'card' task, were the same "apart from these dots".
Fifth session 5.1 In paper and pencil tasks, Erna applied the 'ordering' strategy to big target numbers and she applied the 'add/take away' rationale to overcome violations.	*Tasks with balances*: 5.2 Erna applied the 'ordering' strategy in the 'balances on paper' and 'balance on cardboard' task and explained that the two tasks were similar to the 'card' task: "It's the same with the cards apart from… (shows the balances)".

Discussion

Plane A: The procedural facet of the methods that Erna used in the first session (Table 8.5-para 1.1, 1.2) can be classified as being sustained by knowledge representations of E3 format. In the second run with the task in the first session, the application of the 'deriving' method replaced counting (Table 8.5-para 1.3; Appendix 8.5-para 1.1). The application of this method must be supported by knowledge represented in a format which was clearly beyond the procedural level. The knowledge was explicit enough to allow Erna to apply it for the completion of number bonds for which a more 'direct' method, for example, 'swapping' or recall from memory, was not available. Procedurally, the application of the method was correct. Also, the use of the method was verbally reportable. For these reasons the ***procedural facet*** of the method is considered as being supported by explicit **E3** knowledge representations. However, Erna, at that point, did not seem to have an explicit representation of the arithmetical-relations that sustained this method. She did not seem to have built an explicit, verbalisable, conceptual understanding of 'how' and 'why' the method worked. Thus, the ***conceptual facet*** of the method could be supported by knowledge which was represented in **E1 or E2** format. Even though Erna, at that point, did not provide a verbal explanation of the arithmetical rationale behind the method, she seemed to have conscious access to the knowledge representation that underlay the method. This was shown by the way in which she worked on the fundamental idea behind the method, and developed it into a strategy in the following runs with the task. This is why it is believed that the 'deriving' method was most probably supported by **E2** knowledge representations. However, the available data do not give evidence that could strongly support this interpretation.

<u>Plane B</u>: Up to the end of the first session, Erna's overall problem solving approach, consisted of the application of a mixture of methods which were sustained by different knowledge representations of different levels of explicitness. Even though Erna was referring to previous steps of the solution process (i.e. already produced number bonds) in order to complete new combinations, the production of number bonds was not the outcome of a unified problem solving approach. Shifts to various sources of knowledge were observed. This is why Erna's problem solving approach to the task is considered as being in a **'procedural phase'**. In two runs with the 'card' task, Erna applied the 'deriving' method in a rather consistent way within the solution process, i.e. after the production of the first two or three number bonds by fact retrieval. At that point, signs of an organisational attempt were traced. Moreover, Erna showed that she had a very good conceptualisation of the features and demands of the task. This was made evident when she appeared to be aware and certain of her success (Appendix 8.5-para 1.2).

It is considered that with the consistent application of the method of deriving number bonds from previous ones at the level of 'production', and the introduction of the notion of *order* at the level of 'checking', the first steps towards an 'organisation oriented', **'meta-procedural'** behaviour were made. It is believed that the very interesting change that occurred in Erna's problem solving approach in the following session was grounded in the way that she considered the numbers in the last two runs of the first session, with the purpose of checking and justifying her success. Also, this is believed to be a change grounded in further redescription that the knowledge representations which supported the 'deriving' method were subjected to.

<u>Plane A</u>: In the second session a very important change occurred. At the beginning of the solution process Erna announced and justified her choice to apply the 'ordering' strategy for

the specification of first addends (Appendix 8.5-para 2.1). In _plane B_, the introduction of this strategy had as a result the application of a unified problem solving approach to the whole of the task. This unified approach connected one solution step with the other in a single representational format. The organisation-oriented behaviour that Erna exhibited from that point onwards, seemed to have the characteristics of a movement even beyond the **'meta-procedural'** phase. The indications which are considered as signs of Erna's movement to a **'conceptual phase'** are discussed below.

Plane A: Erna proceeded to the procedural application of the 'ordering' deliberately, with the intention to achieve her goal in an "easier" way. This intention was verbally reported. This is why the introduction of the strategy must be a conscious decision. Procedurally, the application of the strategy was correct. The strategy was not applied by Erna as an automatic compiled procedure. The operations involved were accessible and verbally reported (Table 8.5-para 2.1-2.3). Furthermore, Erna exhibited the very good control that she had over the strategy and the features of the task, by explaining and showing awareness of the fact that the operation of subtracting for the specification of second addends could be replaced by 'swapping' when the numbers in the column of number bonds started being repeated. This replacement could take place while the 'ordering' of the number bonds could still be retained and respected. (Appendix 8.5-para 2.1, 2.2). Thus, it can be considered that, ***procedurally***, the strategy was supported by explicit, verbally accessible representations of **E3** format.

Verbally accessible representations of **E3** format are considered to support the ***conceptual facet*** of Erna's strategy as well. In the second session Erna was in position to be certain of the completion of the task right after writing down the last number bond. She justified her success on the basis of the rationale which sustained her strategy and she also justified why the two columns of addends were following a different kind of order when the strategy was applied (Appendix 8.5-para 2.2). Also, it was made evident that Erna had a rich representation of the features of the task following the application of her strategy. This was made apparent by the shifts in her vocabulary. Some times she was talking and explaining the features of the solution process using words such as "up" and "down" indicating the direction of numbers in the number line. Some other times she was talking in terms of "adding" and "taking away" indicating the arithmetical relationship that sustained the different kind of order that the two column of addends were following (Appendix 8.5-para 3.1). The use of the terms 'taking away' and 'adding' and their connection to the descending and ascending order of addends, correspondingly, indicates that Erna had explicitly represented the arithmetical relation underlying the order. Moreover, Erna used the rationale behind the strategy and adjusted it appropriately to overcome violations that the interviewer introduced in the way in which the strategy was usually applied. One, particular case of strategy violation by the interviewer will be noted here. Erna adjusted her strategy procedurally to the violation of her usual 'starting point' but appeared to be a bit confused when giving explanations (Appendix 8.5-para 3.2). The subtle and partial mismatch between speech and gesture that was observed is not considered as an indication of a non explicit conceptualisation of the way the violated strategy worked. Speech and gesture did not convey different procedures thoughts, and hypotheses [8.5.1] It is considered as a type of gesture-speech conflict which reflected Erna's awareness of the need to make adjustments to her representation of the strategy. It also reflected Erna's attempt to reorganise her representation of the strategy and the task into a system which would fit the situation after the violation. The partial conflict between gesture and speech was recognised by Erna after the interviewer's intervention. Karmiloff-Smith (1985), argues that when such a recognition takes place the child is in a state where knowledge is grasped consciously and explicitly. In this state the child pulls back and reorganises his/her thinking in order to create a more efficient system of representation. Reorganisation of Erna's thinking was also made evident when she extended and generalised the fundamental rationale behind the strategy, i.e. the adding and subtracting of 1 which resulted in the ascending and descending order of the addends (Table 8.5-para 4.1; Appendix 8.5-para 4.1).

The aforementioned examples provide the grounds to consider that the 'ordering' strategy was supported ***conceptually*** by explicit **E3** representations in the sense that explanations, justifications and answers regarding the 'how' and 'why' of the strategy were given, and adaptations to violations of the usual situation were observed. Also, it was shown that the rationale which sustained the 'ordering' strategy, and the 'deriving' method, was represented explicitly enough to be abstracted, and applied in situations where Erna recognised the similarity of goal irrespectively of different superficial characteristics (Table 8.5-para 4.3, 5.1, 5.2; Appendix 8.5-para 5.1). It is speculated that the fact that Erna did not apply 'ordering' in the first run with the 'domino' task is due to the fact that she probably needed to work a bit with the task before recognising and acknowledging the similarity in goal.

[8.5.1] As in the case of the main type of gesture-speech mismatch that Goldin-Meadow et al. (1993) discuss.

8.6. The case of SEAN

Sean was 5 years 9 months old. He was classified as in the second most advanced group in mathematics. Sean participated in five sessions. The duration of all the sessions that Sean participated in was very short (around 15-20 minutes). This happened because it appeared to be rather difficult for Sean to stay concentrated on the tasks for much longer. The first three sessions took place on three consecutive days. Two days intervened between the third and fourth session. The fifth session took place two days after the fourth.

Table 8.6: Summary of changes that Sean introduced into his problem solving approach.

First session	**Second session**
<u>*First run – 'card' task*</u>: 1.1 Sean completed the task by calling upon his declarative knowledge, applying the 'deriving' method, and the method of 'swapping'. 1.2 Sean noticed the order of first and second addends and explained it by associating it with the application of the 'deriving' method. 1.3 Sean was aware of the completion of the task. He justified his certainty by considering the numbers he had used as first addends, in order. <u>*Second run – 'card' task*</u>: 1.4 Sean applied the 'deriving' method consistently: a strategy had been developed. Sean reported the arithmetical operations involved, and reported his intention to produce number bonds in "order".	<u>*First run – 'card' task*</u>: 2.1 Sean gave explanations regarding the production of the second addends. He also showed that he had a good control over several aspects of the task and his strategy. <u>*Second run – 'card' task*</u>: 2.2 Sean announced his strategy before engaging with the task. He developed a new way of checking and knowing whether all the possible number bonds had been produced.

Third session	Fourth session
<u>One run with 'card' task</u>: 3.1 Sean anticipated the number of all the possible number bonds before the completion of the task. 3.2 Sean anticipated the last number bond that he would produce. He seemed to acknowledge the reversibility of his strategy and was in position to apply it procedurally.	<u>'Card' task</u>: 4.1 Sean overcame violations in the sequence of produced number bonds that the interviewer introduced by adjusting accordingly the 'less/more' rationale. <u>Two runs with the 'domino' task</u>: 4.2 Sean applied the 'ordering' strategy right from the beginning of the solution process. However, he was reluctant in giving any verbal explanations. 4.3 He exhibited automatic problem solving behaviour in the second run with the task.
Fifth session	
<u>Similar in goal tasks</u>: 5.1 Sean applied the 'ordering' strategy in the tasks with similar goals and explained verbally the similarity in goal between the 'card' task, the 'domino' and 'balances on paper' tasks: I: Are you doing something similar to what you did with the cards? S: (nods 'yes'). I: What's the same between this and the cards? S: You have to make a number. I: What do you do to make the number? S: Put the numbers in order.	<u>Paper and pencil task with big target number</u>: 5.2 Sean generalised the 'ordering' strategy to produce number bonds that resulted to 100. To be "quick", he applied the 'ordering' strategy by counting in 10s.

Discussion

<u>Plane A</u>: In the first run with the 'card' task, one of the methods that Sean reported that he used was the 'deriving' method (Table 8.6-para 1.1). However, Sean reported only the operation that he carried out to produce the first addend of the new number bond. He did not provide a report of the operation that he carried out to produce the second addend. The verbal report of the method was not complete. However, there were verbalisations, even partial, concerning the operations involved in the application of the method (Appendix 8.6-para 1.1). This provides the grounds to infer that the ***procedural facet*** of the method, at that point, was sustained by knowledge representations which were explicit enough for Sean to access it and work upon it in order to formulate a complete verbal report as he did in the following session. This is why it is believed that even in the first session, the Explicit **E3** format can be ascribed to the knowledge representation that underlay the procedural facet of the method, even though Sean did not provide a detailed and complete, according to the interviewer's expectations, verbal report. The available data up to that point (i.e. first session) do not provide strong indications on the base

of which a certain format of knowledge representation could be related to the conceptual facet of the 'deriving' method. However, Sean's awareness regarding the consequences of the consistent application of the 'deriving' method (Table 8.6-para 1.2; Appendix 8.6-para 1.2) indicates a high level of explicitness in which conceptual aspects of the method seemed to be represented.

Plane B: Sean proceeded to the application of 'swapping' after he realised that, after a certain point, a number bond that he produced by applying the 'deriving' method, had the same addends with a number bond produced at the initial phase of the solution process, just in a different order. This shows that, Sean, right from the first run with the 'card' task, had started constructing a complex representation of the features and requirements of the task in the context of which multiple pieces of knowledge coexisted. These pieces of knowledge seemed to be harmoniously connected in the course of the solution process, and Sean seemed to control them. An example if this control is the conscious shift to the application of 'swapping'. Also, there were no repetitions observed, or other types of error. It is interesting though, that in the context of the a-posteriori explanations that Sean provided, the different pieces of knowledge did not seem to be harmoniously linked; Sean denied that he produced the last number bond by 'swapping' even though this is what he reported at the time (Appendix 8.6-para 1.1, 1.2). It is believed that this inconsistency between what Sean actually did and what he explained that he did after the completion of the task, is due to the process of redescription and further explicitation that Sean's representation of the task was subjected to. Even in the first run with the 'card' task, Sean's approach to the problem gave signs of problem solving behaviour which was *beyond* the 'procedural' phase. Except from the first number bond, Sean produced each of the rest number combinations by referring to a previous solution step; i.e. a previously produced number bond. Sean seemed to consider and treat each solution step as a link of a chain of number bonds that he had to produce. Furthermore, he was in position to justify his success (Appendix 8.6-para 1.3). Very soon, in the second run with the 'card' task, Sean applied the 'deriving' method consistently. He clearly reported his intention to produce number bonds in "order" because it was "easy" (Appendix 8.6-para 1.4). The 'ordering' strategy was developed by the consistent application of the 'deriving' method. Sean, therefore, right from the beginning of his engagement with the 'card' task, gave signs of an organisation-oriented behaviour. He gave signs of a **'meta-procedural'** problem solving approach.

Plane A: In the second session, Sean elaborated the application of the 'ordering' strategy and gave a complete verbal report of the operations involved in the production of each of the two addends of a new number bond. His awareness that after a certain point, number bonds which were produced by the 'deriving' method were number bonds which consisted of numbers which had previously used in number combinations, but in a different order was made evident (Appendix 8.6-para 2.1). In the following sessions, Sean used different vocabulary (e.g. "take away", "add", "a number after", "going in order", "going higher and lower") to report the relationship between the addends of the number bond-reference and the newly produced number bond. In this way, Sean indicated the rich, and explicit representational system which sustained the procedural as well as the conceptual facet of his strategy. Sean used different vocabulary in the course of explaining and justifying the need to combine the two operations of adding and subtracting, or "going higher and lower" to produce number bonds "in order". Sean's explanations together with his overall performance in the third and fourth session (Table 8.6-para 3.1, 3.2, 4.1) ascertain that knowledge represented in an explicit **E3** format underlay the ***conceptual facet*** of the 'ordering' strategy (Appendix 8.6-para 3.1, 4.1).

Plane B: The aforementioned developments in plane A, signalled the passage to a **'conceptual phase'** in Sean's problem solving approach. Sean showed that he was in control of several aspects of the task. He clearly explained and gave verbal evidence of his awareness that after a certain point in the solution process, the number bonds produced by 'ordering' consisted of numbers which had been previously used in number bonds produced at the beginning of the solution process, but in different order (Appendix 8.6-para 2.1). Also, after stating his view about the number of the possible number combinations (Appendix 8.6-para 2.2) and in the course of talking with the interviewer about this particular aspect of the task, Sean abstracted the rule regarding the number of the possible combinations (target number +1), generalised it, and used it in the following runs, regardless of the target number.

In the fourth and fifth session Sean gave even more convincing signs of 'conceptual' problem solving behaviour (Table 8.6-para 4.1, 5.2; Appendix 8.6-para 4.1, 5.1). Also in these sessions Sean applied the 'ordering' strategy in the tasks with similar goals right from the first run. His explanations showed that he was aware of the similarity between these tasks and the 'card' task (Table 8.6-para 5.1). In only one run with the 'domino' task, when the target number was 7, Sean gave signs of automatic problem solving behaviour (Appendix 8.6-para 4.2). In following runs with the 'domino' task Sean did not repeat this approach. There was no evidence on the basis of which one could assume that when Sean exhibited the automatic problem solving behaviour he had lost the conceptual control that he had over the features of his strategy and of the task. There was no evidence that he was not aware of the operations which resulted practically in the specific ordered arrangement. On the contrary, another indication of the high level of explicitness of the knowledge representation which sustained the procedural and conceptual facet of the 'ordering' strategy, as well as of the representation of the task was given when Sean generalised the rationale of '1 more/1 less', and he applied the 'deriving/ordering' strategy in steps of '10' (Table 8.6-para 5.2). This was also an indication of Sean's good conceptualisation of the regularities which underlay the system of decimal numeration (Appendix 8.6-para 5.1).

8.7. Summary

This chapter presented and discussed five cases of children who, in the course of the five sessions in which they participated, and while solving a specific task, gave signs problem solving behaviour which pertains to the 'conceptual' phase: that is the highest in coherence and representational richness phase of problem solving, as this is described in the framework of the RR model. Each of the five children approached the task in an individual way. The 'conceptual' problem solving behaviour was ascribed to each of these cases not on the basis of the type of strategy that they developed, but on the basis of the control that they appeared to have over the aspects of their strategy, and the aspects of the task. In these cases it was made possible to observe mainly: the progressive explicitation and evolution of the problem solving approach and of children's verbalisations, the flexible application of a strategy, and the use of rich vocabulary which indicated a rich representational system which underlay the employed strategy. Each of the five cases had particularities and raised certain issues which are going to be discussed in relation to the RR model in chapter 9.

Chapter 9 Discussion

9.1. Overview

The preceding chapters presented cases of children whose problem solving behaviour was analysed and discussed on the basis of two main aspects of the RR model: the different levels of knowledge explicitness and the different phases of problem solving. This chapter is organised in two sections each one of which presents a general discussion of issues that the analysis of the cases raised, in relation to the aforementioned two aspects of the model. The first section discusses the findings of the study concerning the plane A of analysis, that is, the levels of explicitness of the different types of knowledge that children called upon while applying their problem solving methods or strategies. The second section of this chapter discusses the findings of the study regarding the plane B of analysis. That is, children's overall approach to the task, and the characteristics of the phases of problem solving behaviour that they passed through, in the course of the five sessions.

9.2. Types of knowledge and levels of explicitness

The literature reviewed in chapter 2 showed that children's construction and use of strategies in arithmetical problem solving needs to be studied in relation to more than one aspect of mathematical knowledge. Following the analysis of data, this study supports the aforementioned view. In order to solve the particular task, children who participated in this project employed various methods and strategies sustained by different pieces of their declarative, procedural and conceptual knowledge. This is in agreement with Karmiloff-Smith's (1984, 1992) view that problem solving involves generation and activation of multiple knowledge representations.

Within the ten cases analysed here, apart from variability in the choice and use of particular methods and strategies, there was variability in the degree of control that each child had over aspects of his/her strategy. The provocative feature of the RR model is that it shows how 'doing' and 'knowing' can be dissociated. On the basis of this view, the methods and strategies that children employed were considered in two facets: the procedural and the conceptual. The procedural facet of the method refers to the correctness in use, and the accessibility of the procedural components of a method or strategy. The conceptual facet refers to the accessibility of the conceptual aspects of a method or strategy: that is the rules and rationale which make a particular strategy work and be successful. It was shown that the two facets of the same method or strategy were controlled at a different degree by different children. In contrast to views, including Karmiloff-Smith's view, which associate the quality of 'implicit' or 'explicit' with one particular type of knowledge (usually the procedural knowledge is equated with mechanic procedures and thus implicit knowledge, and the conceptual knowledge with knowledge which is explicit, see chapter 2), in this study, children's strategies are considered as encompassing both aspects of knowledge; the procedural and the conceptual. The methods and strategies that children employed were not approached as being sustained only by procedural or conceptual knowledge. It was considered that each method or strategy has two facets and it is necessary to approach and study both of them. Therefore, different levels of explicitness are ascribed to both the conceptual and procedural facet of a method or strategy. The difference in control and awareness of the aspects of a strategy indicated a different level of explicitness into which the knowledge which sustained the strategy was represented.

The analyses presented in chapters 7 and 8 show that the *procedural facet* of the isolated methods that children used at the initial phase of their encounter with the task was represented in an Explicit **E3** format. This means that the procedures involved in a specific method for producing or completing number bonds were accessible for verbal report. This happened in all the cases except for the cases of Grace and Isa when they first applied the 'deriving' method. The cases of these children are considered as belonging to a different group in relation to the problem solving approach. However, initially, both children did not seem to be in position to provide a clear report of the operations/procedures that they were carrying out while applying the 'deriving' method. At that phase, the procedural facet of the method was considered as being sustained by knowledge representation of **E2** format, since, according to indications which were presented in the context of their profiles, the procedures involved in the method seemed to be consciously accessible. Later on, in the course of the sessions, both children provided a verbal report of the procedures that they were carrying out. The knowledge representation of E2 format seemed to be redescribed to a higher level of explicitness (**E3**).

The *conceptual facet* of isolated methods that children used at the initial phase of their encounter with the task, for example counting, or 'swapping', was not studied and analysed in this study. It was the conceptual facet of the 'deriving' method which was of interest because, as explained in chapter 6, the application of this method was found to prepare the ground on which children who participated in the study built and developed their organisational strategies.

The study showed that there was variability at the level of explicitness at which the conceptual facet of the 'deriving' method was represented.

In chapter 7, amongst the cases of children who, reached a **'meta-procedural'** phase in their problem solving, three children used the 'deriving' method. In the case of Henry, the conceptual facet of the 'deriving' method was represented in an explicit **E2** format: that is, a format which allowed conscious access to the conceptual aspects of the method but no verbal report. However, when Henry introduced subsequently the 'ordering' method, he did not seem to recognise the similarity in the conceptual basis between the 'deriving' method and the 'ordering' strategy. Henry just happened to notice a pattern of numbers which he subsequently generalised without having constructed a conceptual basis for this generalisation. In this case, the conceptual facet of the 'ordering' strategy was represented in explicit **E1** format. Isa also used the 'deriving' method as part of her strategy of producing number bonds in pairs. In her case as well, the conceptual facet of the method was represented in explicit **E1** format. James also belonged to the same group but in his case, the conceptual facet of the 'deriving' method was represented in an explicit **E2** format. James integrated the 'deriving' method into a strategy (the strategy of producing number bonds in pairs) of which he had a very good control, both at the procedural as well as at the conceptual level. Two cases in this group, developed the strategy of 'ordering' without having employed before the 'deriving' method, in the course of the sessions. In both these cases (Rakhi and Leo), the conceptual facet of the 'ordering' strategy was represented in explicit **E1** format.

In chapter 8, children who reached the **'conceptual'** phase in their problem solving approach, and used the 'deriving' method, gave signs of high control over the conceptual aspects of the method. In those cases the conceptual facet of the method was represented in an Explicit **E3** format. Amongst these cases, two cases of children, before they redescribed their conceptual knowledge of the method into the **E3** format, gave signs of a knowledge representation of **E2** format: that is knowledge which is consciously accessible but not verbally accessible. These were the cases of Grace and Erna. In chapter 8, it was shown that in those cases of children who belong to this group and developed the strategy of 'ordering' following the application of the 'deriving' method, the procedural and conceptual facet of the 'ordering' strategy was represented in an explicit **E3** format. These were the cases of Sean, Elsa, Erna and Hazel. The particularity in the cases of Erna and Hazel was that before the conceptual facet of the 'ordering' strategy was redescribed to **E3** format, they seemed to pass through a short phase into which they did not provide verbal report of the conceptual aspects of the strategy. That is, they gave signs of knowledge represented of **E2** format. Elsa also passed through a short phase during which she did not have verbal access to the knowledge which sustained the conceptual facet of the 'ordering' strategy. However, in her case, the stages before she redescribed her knowledge representation to an **E3** format were not made clear.

Summarising, in chapter 7, it was shown that, in the case of children who reached the **'meta-procedural'** phase in their problem solving approach, the conceptual facet of the 'deriving' method and the 'ordering' strategy was represented in explicit **E1** format. The case of James was an exception. In his case the conceptual facet of these methodological approaches was represented in an explicit **E2** format. His case was discussed in the context of the second group because of certain other characteristics of his problem solving behaviour which apply to the general characteristics of this group, and which were discussed in chapter 7.

In chapter 8, it was shown that, in the case of children who reached the **'conceptual'** phase in their problem solving approach, the conceptual facet of the 'deriving' method, and the 'ordering' strategy, at some point, was redescribed to the highest level of explicitness, that is into **E3** format. Before this happens two of these cases gave signs of **E2** knowledge representation.

It was shown that the various levels of explicitness as these are described in the RR model can be ascribed to the procedural as well as the conceptual facet of the methods and strategies which were employed in the context of the particular problem situation. In agreement with views in the field of mathematics education and mathematics psychology (e.g. Halford, 1993; English, 1995) which emphasise that implicit *and/or* explicit representations can sustain procedural knowledge as well as conceptual knowledge, this study showed that the *know how* can be more or less explicit, as well as the *know why*.

It should be underlined that, in the context of this study, the procedural facet of any method employed was not associated with knowledge represented in implicit (**I**) format as this is defined in the framework of the RR model. The identification knowledge representation of I format was not possible given the definition that Karmiloff-Smith gives for this format of knowledge representation, and considering the requirements of the particular task used in this study. For Karmiloff-Smith (1992), knowledge representations of I format are procedure-like encodings. At this level, knowledge has the form of procedures sequentially specified, procedures which cannot be associated with any other piece of information from the same or any other domain of knowledge. None of the methods employed in the context of the particular task was simply a sequentially specified procedure, that is a sequence of procedural steps that the problem solver carried out automatically, without having access to its components, and in the absence of any link with another piece or type of knowledge. It must be underlined that the task had a certain particularity: the problem solver was not asked to carry out one particular procedure or operation for solving it. Rather, in order to solve the task, the problem solver had to *call upon any method that he/she considered as pertinent for the accomplishment of the goal*. This requires a certain degree of sensitivity to the relationship between means and goal, and entails the ability to access and choose the pertinent method from the spectrum of all the methods that are available to the problem solver to use in the context of arithmetical tasks. Since it was an issue of accessibility and choice there was no space of knowledge representations which were 'implicit' in the way that Karmiloff-Smith defines the term. The problem solvers, encountering the particular task, either recalled from memory knowledge which was directly accessible (i.e. declarative/factual knowledge), or called upon methods which involved a combination of operations. This combination of operations was necessarily underlain by some sort of conceptual knowledge (the *know why* of the combination). In this case, the issue was to identify how accessible this conceptual knowledge was, that is to identify a certain level of explicitness to this piece of conceptual knowledge.

The model of Representational Redescription was found to be a particularly useful exploratory and explanatory tool, since it postulates the existence of several levels of explicitness into which a certain piece of knowledge can be represented. However, the ascription of the various levels of explicitness to the procedural as well as the conceptual facet of a problem solving method is introduced by this study because it was considered as necessary. It was considered as necessary because it was found that explanations which give indication of explicit E3 knowledge representation were given for the procedural aspect of a strategy as well as the conceptual. This is not something for which the RR model provides an explanatory framework. However, the need to consider the different types of explanations and verbal reports that a child gives of the different aspects of a strategy, is also addressed, not in depth though, in the study of Pine and Messer (1999). In applying the RR model as exploratory tool in their research project, Pine and Messer, at some point, talk about 'strategic' and 'conceptual' explanations that children gave about their problem solving approach (see final section of chapter 3).

It is now worth addressing certain remarks to the identification and ascription of the different levels of explicitness, in the context of this study. The **E1** format of knowledge representation was identified only in relation to the conceptual facet of certain employed methods. Also, this format was identified in cases of children who exhibited 'meta-procedural' problem solving behaviour. E1 representations were not inferred by the U-shaped behavioural sequence. This was not observed almost at all in the context of this study. The E1 format was associated with behaviour which was process-oriented and organisation-oriented, but it was also driven by partial, insisting focus on one particular aspect of the procedural or conceptual facet of the method or strategy.

The most interesting point which needs to be emphasised, is that, in the context of this study, the **E2** format of knowledge representation seemed to be into operation. It is believed that the E2 format came into operation because in this study, the particular problem situation had to do with methods and strategies for which the application of different pieces and types of knowledge were activated. This study did not deal with single notions, or ideas which may be sustained by a single piece of knowledge. In this study, the problem situation that children dealt with, required the application and construction of methods and strategies for which different pieces of knowledge needed to be activated and linked. This is why it is believed that in the case of the particular problem situation, when a child developed a strategy, this strategy was the product of deliberate and conscious problem solving behaviour. However, the knowledge which supported a strategy was not always accessible for verbal report. Children were applying a method or strategy but they could not talk about the procedural (rarely) or conceptual (more frequently) facet of this strategy. In these cases the **E2** format of knowledge representations came into operation.

Regarding the **E3** level of knowledge representation which allows the formulation of verbal explanations, the analysis of data revealed a point of concern. This has to do with the quality of the verbal explanation or report that the problem solver provides. In certain cases (e.g. the case of Sean), at some point, there were verbal reports which were not complete, in the sense that they did not provide all the information regarding a particular aspect of the method or strategy. Of course this may happen simply because the child did not consider it as necessary to say anything else, or he/she was not willing at that particular point to talk. However, incomplete verbalisations may indicate something more than that. Also, in the case of Leo, there were verbalisations, however, these were poor regarding the information that they provided, and the vocabulary used indicated limited conceptualisation in comparison with verbalisations that other children provided. However, in this case as well some sort of verbalisation existed. These limitations may be related with the level of explicitness of a certain piece of knowledge or, they may simply be related with the child's ability to use the language. These are issues that need to be further explored. In the framework of the RR model there is no differentiation regarding the quality of verbalisations that the problem solver produces. It is believed that further study of this aspect should inform a possible modification of the model.

In terms of the generalisation of a method or strategy to tasks with similar goals, the RR model predicts that problem solvers who have reached the conceptual phase should be able to generalise the knowledge they used in one particular task and apply it in other problem situations with similar goals. However, in chapter 8, it was shown that within the cases of children who reached the 'conceptual' phase, only Erna and Sean directly applied the 'ordering' strategy in tasks with similar goals. Grace and Elsa applied their organisational strategy of producing number bonds in pairs in tasks with similar goals. In the context of the main task this strategy involved the application of the 'deriving' method. This method was not included in the strategy of producing number bonds in pairs when this was applied in the 'domino' task. It seemed that this had to do with the absence of numerals. But this is only a speculation which needs further exploration. Hazel was another child of this group who, in the context of tasks with similar goals, passed (rapidly) through all the phases of rediscovering and redeveloping the 'deriving' strategy that she employed in the main task. It should be noted that all the children of this group generalised their strategies to bigger target numbers, and they did this even at phases when they had not reached the highest level of conceptualisation of their employed strategy. It was in the process of applying the same strategy for other target numbers repeatedly that the process of explicitation was made evident on the level of their verbal explanations, and on the level of their overt behaviour.

In chapter 7, it was shown that within the cases of children who reached the 'meta-procedural' phase, four children, Rakhi, Henry, Leo, and James did not apply the strategy that they developed in the main task, in tasks with similar goals. Isa was the only case of child in this group who, immediately, applied her strategy of producing number bonds in pairs to solve tasks with similar goals, even though this strategy involved the application of the 'deriving' method of which Isa had limited conceptual control. In this group of cases as well, all the children generalised their strategies to bigger target numbers within the context of the main task, the 'card' task. The information that the data give about the generalisation of children's strategies is unclear. It seems that children who reached the 'conceptual' phase of problem solving behaviour tended to generalise their strategies to tasks with similar goals considerably more than children did not reach the 'conceptual' phase. However, both groups of children generalised their strategy in subsequent runs, with big target numbers, in the context of the task into which they had developed the particular strategy in the first place. This happened even though in each group of children there was a different degree of conceptualisation and control over the aspects of the employed strategy. It is believed that the particular data do not give information which can provide the basis for reliable and generalisable conclusions regarding the generalisation of children's strategies and what it may indicate regarding the conceptualisation of a strategy. The generalisation of children's strategies did not constitute an object of research per se in the context of this study. Because across the literature, there are views, including Karmiloff-Smith's view, according to which generalisation of one's strategy to other, similar goals is considered as an indication of a higher level of conceptualisation and explicitness of knowledge, it was considered that this aspect needed to be taken under account. In the framework of this study, after the development of a strategy there was generalisation of this strategy in subsequent runs of the same task with different target numbers. This *is* a specific type of generalisation and it is considered as of particular importance because it entails a significant degree of abstraction. When this generalisations occurs then this is an indication that children have moved from the concrete, specific example, to the rule. All the children who participated in this study exhibited this kind of generalisation. Sean was a particular case of a child who reflected upon and generalised, amazingly, another aspect of the task: the rule concerning the amount of possible number bonds for *any* target number. The specific type of generalisation (i.e. from the concrete to the rule within the context of the same task) occurred in every case. It should be noted that this occurred when children had passed from the 'procedural' phase to a 'meta-procedural' phase of problem solving. The 'meta-procedural' behaviour is organisation-oriented behaviour. Organisation entails understanding of certain rules regarding the task at hand, thereupon children's behaviour at this phase exhibit the aforementioned type of generalisation.

The generalisation of a strategy to other tasks with similar goals, is a different type of generalisation. This type of generalisation entails a certain way of thinking which allows recognition of the aspects of similarity between two or more tasks. The data showed that children at the 'conceptual' phase tend to exhibit more this type of generalisation. However, it is believed that further research needs to explore more, and possibly on a different methodological basis, this type of generalisation in connection with the levels of explicitness and phases of problem solving as these are described in the RR model.

9.3. Phases of problem solving behaviour

This study focused on the micro-developmental changes that occur in children's problem solving behaviour during a sequence of sessions. Because the focus was on a *sequence* of sessions and not on a single run with the task or a single session, it was possible to observe and follow children's passage from more than one phase of problem solving as these are described in the RR model. All the children, after being already successful in solving the task, produced qualitative changes at the procedural and conceptual level. These changes indicated a movement from the 'procedural' to the 'meta-procedural' phase and, for some cases, further on to the 'conceptual' phase. Each of the three phases that children passed from had the main characteristics that the RR model describes. However, certain aspects of children's behaviour within each of the phases, and while solving the particular task, did not conform to certain characteristics of the phases as described in the framework of the RR model.

For example, all the children, when first encountering the 'card' task, gave signs of 'procedural' behaviour, that is behaviour which is 'success-oriented'. Within the **'procedural'** phase, children approached each step of the solution process separately, as an isolated problem. At that point, children did not have a unified representation of the task. They did not seem to have a sense of the end-point of the process of producing number bonds. Indicative of this was the fact that in the initial runs with the task children were not aware of the completion of the task; they kept thinking and looking for new number bonds. This approach conforms with Karmiloff-Smith's (1984) account according to which, in this phase, each of the behavioural units, which are simply juxtaposed and not linked the one to the other, consists of a sequence of procedures which are isolated, well-functioning, but need to be re-computed afresh for each part of the problem. Furthermore, Karmiloff-Smith notes that by the end of this phase, the problem solving behaviour is characterised by 'behavioural mastery'; that is the consolidated use of well-functioning, automatic procedures. This is something that was not identified in the context of this study. The methods that children used in the 'procedural' phase did not consist of automatic procedures, rather, children in this phase applied a mixture of different methods the procedural facet of which and sometimes the conceptual (in the case of 'swapping' and the 'deriving' method) were represented in a certain degree of explicitness (sometimes higher sometimes lower). Children showed that they had access to the procedural components of the methods they used by giving a verbal report of the operations that they were carrying out while applying these methods. None of these methods was sustained by knowledge represented in I format, that is knowledge encoded merely in a procedure-like representation. Procedural success in this case, did not entail 'behavioural mastery', that is the application of automatic, rigid procedures. Procedural success in this case entailed the combination of methods which were clearly beyond the procedural level, as this is defined in the RR model, and the conceptual facet of which was subject to further redescription.

Therefore, the data in this study show that 'behavioural mastery', that is automatic and rigid behaviour, does not necessarily precede the passage to a 'meta-procedural' phase of problem solving behaviour. This is in agreement with views of other researchers regarding this aspect of the model on the basis of their research findings (e.g. Goldin-Meadow & Alibali, 1994), and with the revised views of Karmiloff-Smith (1994) who tends to accept the idea that 'behavioural mastery' may not by essential for redescription to occur, and for representations to become explicit and flexible. On the contrary, this study shows that it is possible for 'behavioural mastery' to occur after a sequence of redescriptions, that is after the problem solver has elaborated and explicitated at the highest level the knowledge representations which sustain his/her behaviour (see, for example, the case of Sean). It is certain though, that, as the RR model postulates, the movement to the 'meta-procedural' phase cannot occur unless the problem solver feels the need to search for, and acquire better control of the task at hand. All the children who participated in this study sought for better understanding of the task, and better control of their actions and strategies.

In all the cases children passed to a **'meta-procedural'** phase which entailed 'organisation-oriented' behaviour. Children's attempt to organise the solution process indicated the beginning of the construction of a unified approach: that is of a unified representation of the task. Different children developed different strategies: different types of organisation of the solution process. In the context of this phase there was progressive explicitation of the knowledge representations which underlay the particular organisational strategy no matter of its type. What needs to be noted here is that, according to the RR model, in this phase, children's focus on the organisational aspects of their approach, has as a result the deterioration of their performance. According to the model, children in this phase are less successful than in the previous, 'procedural' phase. There were two children whose behaviour in this phase had this particular characteristic. Henry (see section 7.3) and Hazel (see section 8.3) were the only ones who started being unsuccessful in solving the task while they had started organising their problem solving approach. In the cases of all the other children, there was a deterioration observed but not at the level of success in solving the task, but at the level of control that they had over the aspects of their newly introduced organisational strategy. Children in this phase appeared to be partially and strongly focused on one aspect of the task. It was this particular aspect which drove the organisation of their behaviour. The loss of control in this case, had to do mainly with the conceptual facet of the strategy not with the procedural. Therefore, even though children in that phase, applied a strategy of which they had limited control, they were still successful in solving the task. This finding does not necessarily confront the postulations of the RR model. Rather, it brings to light another aspect of this particular characteristic of the 'meta-procedural' phase. It shows that 'deterioration' is a characteristic of this phase of problem solving which may be detected in, and may concern, other aspects of problem solving behaviour, not only that of the performance.

Finally, certain cases of children exhibited behaviour which applies to the **'conceptual'** phase. As the RR model predicts, in this phase, children were aware of, and in control of all the aspects, the procedural and conceptual, of their employed strategies, and were in position to report and explain them. In this phase, the problem solvers applied their strategy flexibly, and could adapt the rationale which underlay the application of the particular strategy in situations which diverged from the usual practice in the context of which the strategy had been developed.

9.4 Overall comment

Closing this section and this chapter, it must be emphasised that even though the aforementioned behaviours were observed in the context of a study focused on cases and in the course of solving one particular arithmetic task, it is believed that these behaviours will also be observed in other tasks which belong to the domain of arithmetic, involve more than one solution step, and require the activation and combination of more than one piece of knowledge. The reason is that these are characteristics which apply in most arithmetic tasks. Arithmetic problem solving at this level (i.e. primary) entails the activation not only of multiple representations of the same piece of knowledge, but also of various, different pieces of knowledge. In this respect, the model of Representational Redescription needs to be further elaborated so that it develops explanations and predictions which embrace the particularities of a domain such as arithmetic. This study elucidated only certain aspects and particularities of the arithmetic problem solving the consideration of which may help with this task. Of course arithmetic problem solving is a large domain. Further research of the applicability of the RR model as explanatory tool in this domain is needed and will certainly reveal issues that this study left intact.

Issues which were raised by the analysis of the ten cases, which were discussed in this chapter and are of direct relevance to research questions that this study sought to answer, are summarised and highlighted in the following and final chapter.

Chapter 10 Conclusions

10.1. Overview

This study set out to explore children's movement from procedural success to higher *conceptualisation* and *understanding* of the procedures employed in the solution of an arithmetical task. In the framework of a micro-developmental methodology focused on cases, the objective was to study this movement as it takes place in the context of solving a specific form of arithmetical task that children engage with, during a sequence of experimental sessions. The model of Representational Redescription was considered as the appropriate exploratory and explanatory framework because it describes how behavioural, procedural success may occur in the absence of conceptual understanding. The RR model argues for 'success-based' cognitive change. The model was built on the basis of the idea that children as problem solvers, are not satisfied with their success in solving a task. They want to understand how they solved it. The model of Representational Redescription describes a way in which problem solvers gain increasing control and conscious access to knowledge which is already present in their cognitive system. As a framework of analysis the model has not been used before in arithmetic problem solving. Its use as an exploratory and explanatory tool gave answers that revealed issues of particular theoretical and educational interest. The research questions that the study sought to answer point at two main issues: the idea of 'after success' elaboration of one's problem solving approach, and the applicability of certain aspects of the RR model as an explanatory framework in the domain of arithmetic problem solving. The following two sections highlight findings of this study in relation to these two issues of interest.

10.2. Post-success elaboration

* *Do children evolve and develop their successful problem solving approach while engaged in a problem solving situation repeatedly? If yes, what is the process, and types of change that occur in children's problem solving approach in the 'post-success elaboration' phase?*
* *Do the procedural/behavioural changes indicate qualitative changes at the conceptual level as well?*

All the children who participated in the study and dealt with the particular task, right from the first run, were successful in solving it. In the course of the five sessions, all the children introduced qualitative changes into their initial, successful, problem solving approach. At a first level, children introduced modifications to their procedural applications. New methods were added or a particular method started being applied more consistently than others. In all the cases, at some point, the various, isolated methods that children used initially, were replaced by the application of an organisational, unified strategy. Progressively, changes were observed at the conceptual basis which sustained either a particular modification of approach, or the particular strategy in use. Changes at the conceptual level were made evident first by children's overt behaviour which gave signs of children's better control and flexibility in the application of their strategy, and second by the formulation and progressive elaboration of children's verbal explanations concerning the employed strategy. All the children produced very interesting and rich changes at the procedural level. Even though, across the cases there were no differences in the initial procedural approach, there were differences observed concerning the type of the procedural modification that different children introduced to their initial approach to the task. Different children developed different strategies in order to increase the control they had over the task. Even more interesting differences were observed regarding the process of conceptual change. Children who introduced the same type of modification and developed the same strategy were found to be in different levels of control of the conceptual aspects of the particular strategy. The aforementioned types of differences across the cases were identified and studied on the basis of the differential levels and types of behaviour that the model of Representational Redescription describes. The following section discusses findings of this study which relate to the application of the RR model as an exploratory and explanatory framework.

10.3. The RR model as tool of exploration in this study

* *Are the behaviours observed, and the introduced behavioural modifications, consistent with the behaviours that the RR model describes when accounting for the post-success development of children's problem solving approaches?*
* *Can the qualitatively different levels of knowledge representation that the RR model describes, be assigned to the different types of arithmetical knowledge (either procedural, declarative or conceptual) that underlie children's methods and strategies when solving the particular task?*

Post-success behaviours which are described by the RR model and designate specific phases in problem solving were observed and identified in this study. It was made possible to follow children's passage through more than one phase. Even though this study discerned the same main characteristics of behaviour in each phase of problem solving, there were certain particularities which diverge from the account of behaviour that the model provides. These particularities are highlighted below.

The **'procedural'** phase was not characterised by the application of automatic, rigid methods sustained by procedure-like representations. In this phase, children had limited understanding of the aspects of the task and approached each step of the solution process as an isolated problem space. However, the methods that children employed in this phase were well-functioning and very well controlled at the procedural level and, in certain cases, at the conceptual level as well. 'Behavioural mastery' as Karmiloff-Smith describes it was not observed. However, children went on to introduce modifications into their approach showing that procedural and conceptual change does follow success, but this success does not necessarily have the characteristics of 'behavioural mastery'.

The modifications that children introduced concerned an organisational attempt. The attempt to organise the solution steps signalled children's passage to the **'meta-procedural'** phase. The organisation-oriented behaviour is a characteristic of this phase which was identified in accordance to the predictions of the RR model. However, the model also predicts that in this phase, problem solvers are less successful in solving the task, because of their primary interest and focus in organising their solution steps. Only two cases in this study exhibited this kind of behaviour (the case of Henry, chapter 7, section 7.3, and the case of Hazel, chapter 8, section 8.3). In all the other cases the procedural facet of the strategy was very well controlled during this phase. Therefore, deterioration at the level of performance was not observed. These children were still successful in solving the task, but exhibited some loss of control over the conceptual facet of their newly developed strategies. This loss of control was related to children's partial and insisting focus on one particular aspect of their organisational strategy or of the task.

The behaviours observed at the **'conceptual'** phase conform with the description and predictions that the model provides. Children in this phase were in control of the procedural and conceptual level of their strategies. They were in position to provide verbal explanations, and adapt their strategies flexibly to situations which violated the usual practice within which the strategy was developed.

Regarding the levels of knowledge explicitness, the study showed that in an arithmetic problem situation which involves multiple steps and requires the activation of various different pieces and types of knowledge, problem solvers are successful by employing methods which have two facets: the procedural and the conceptual. In the course of children's itinerary through the different phases of problem solving, different levels of explicitness were ascribed to the knowledge that supported children's strategic choices Different formats of knowledge representations were found to underlay the procedural and conceptual facet of a method, or strategy at different moments in the course of the five sessions that children participated in. However, all the identified formats were formats of a higher or lower explicitness. The implicit I-format of knowledge representation as described in the RR model was not identified in this study. This is because none of the methods employed had the characteristics of an automatic, merely sequentially specified procedure. All the methods and strategies that children employed had a conceptual basis which was for children more or less explicit.

Most importantly, in this study the explicit speculative **E2** format of knowledge explicitness came into operation. With the consideration and approach of the two facets of each method and strategy, separately, it was made possible to identify moments at which children made a conscious choice to apply a specific strategy without being able to talk about this choice and the reasons which drove it. Also, it was possible to identify moments at which children's behaviour showed awareness of conceptual aspects of their strategies for which, though, they were not in position to provide a verbal explanation.

Finally, concerning the explicit **E3** format of knowledge representation which allows the problem solver to formulate explanations and justifications, the analysis of data raised the issue of the quality of the verbal report. The definition of the particular level of knowledge explicitness does not refer to the issue of quality. It is believed that verbalisations of different quality may indicate different levels of conceptual understanding, and therefore they probably need to be related to the existence of more than one formats of knowledge representation.

10.4. Overall theoretical and methodological comments

This study showed that the RR model provides predictions and descriptions of behaviours which, to a great extent, pertain to the behaviours observed in the context of solving an arithmetic task such as the one in this study. This confirms that the RR model can be a valuable tool for exploring and explaining micro-developmental changes in problem solving behaviour. However, this study revealed certain points of diversion between the particular data and the predictions of the model. These points are summarised herein:

In terms of the different levels of explicitness that are ascribed to knowledge which sustains the application of a method or strategy this study showed that:

* In contrast to views, including Karmiloff-Smith's view, which equate the procedural knowledge with mechanic procedures and thus implicit knowledge, and the conceptual knowledge with knowledge which is explicit (see chapter 2), children's strategies need to be considered as encompassing both aspects of knowledge; the procedural and the conceptual. The study emphasised the need to consider the procedural and the conceptual facet of each method or strategy, by showing that different levels of explicitness can be associated with each of the two facets. As a result of this view, the **E2** level of knowledge explicitness came into operation.
* In this study, the **E1** level of knowledge representations was not inferred by the U-shaped behavioural sequence (that is, deterioration of performance). Deterioration of performance was only observed in two cases. In all the other cases the E1 format was associated with behaviour which was process-oriented and organisation-oriented, but it was also driven by partial, insisting focus on one particular aspect of the procedural or conceptual facet of the method or strategy.

In terms of the different phases of problem solving behaviour, it was shown that:

* Similar to the findings of previous research (e.g. Goldin-Meadow & Alibali, 1994) 'behavioural mastery', as defined by Karmiloff-Smith (1992), was not identified as a characteristic of the 'procedural' phase of problem solving.
* Unlike the prediction of the RR model according to which children in the 'meta-procedural' phase are less successful than in the 'procedural' phase, deterioration of performance was observed only in two cases during the 'meta-procedural' phase. In all the other cases, there was a deterioration observed but not at the level of success in solving the task, but at the level of control that they had over the aspects of their newly introduced organisational strategy.
* All the children generalised their strategy in subsequent runs, with big target numbers, in the context of the task into which they had developed the particular strategy in the first place. However, not all the children who were classified as 'conceptual' problem solvers did generalise their strategies to other tasks with similar goals. This is not fully compatible with the prediction of the RR model that problem solvers who have reached the conceptual phase should be able to generalise the knowledge they used in one particular task and apply it in other problem situations with similar goals. Certain considerations regarding this issue were discussed, in detail, in chapter 9.

Even though this project focused on a specific arithmetic problem, it is believed that this problem had particularities that most of the arithmetic problem solving situations have. Therefore, it is believed that these particularities, and points of diversion that this study revealed, need to be taken under account in future research as well as in modifying the RR model and making it more flexible and applicable in the particular domain.

In particular, future research could be informed by the aforementioned issues that this study revealed in relation to the RR model, and focus on questions such as the following:

* Can the levels of knowledge explicitness that the RR model describes, be ascribed to knowledge used in a number of different arithmetical tasks?
* During a specific number of sessions, devoted to each one of different arithmetical tasks, does the same child reach the same or different phases of problem solving behaviour?
* Can behavioural inflexibility (i.e. behaviour which could have some of the characteristics of 'behavioural mastery') be observed after the problem solver has reached the 'conceptual' phase of problem solving?
* To what extent do 'conceptual' problem solvers transfer their problem solving strategies to other tasks with similar goals?
* Is it possible to identify endogenous and exogenous causes of representational change? Which would be the appropriate methodology?

The list of possible questions for future research is, of course, much longer than the list given above. The model of Representational Redescription is complicated and has numerous different aspects each of which can constitute an interesting object of exploration within the domain of arithmetic. All the aforementioned questions for research focus on micro-developmental change as did this project. The micro-developmental approach of data collection and data analysis proved to be suitable for studying the *process* of behavioural and representational change in the domain of problem solving. The micro-developmental method and the clinical method of interviewing were valuable tools for revealing subtle behaviours (verbal and non-verbal), the analysis of which constituted the essence of this study.

10.5. Educational implications

Even though educational recommendations have been beyond the focus of this research, this study was initiated by an idea and question nascent in the course of teaching, and in the course of observing the teaching practice. Therefore, the study needs to close with a remark related to the practical, educational implications. This study showed that children do move beyond success, and do introduce qualitative changes and modifications to their successful strategies. These changes indicate the passage from initial success-oriented behaviour to an organisation-oriented phase during which children, as problem solvers, acquire better control and increasing conscious access to knowledge which is present in their cognitive system; knowledge that they already have. The findings of this study support the idea that the process of Representational Redescription constitutes another way of constructing knowledge. Thereupon, teaching practice could usefully focus on success to the same extent that it focuses on failure. Better understanding and learning follows not only from failure but also from success.

In certain cases, in certain classes, success is seen as a source of learning. However, in these cases it is often the success of others which is used as tool of learning. In a group or class setting, pupils discuss and try to understand, in the presence of the teacher or not, successful solutions that their peers have given to a problem. This is certainly beneficial. But what about understanding one's own solution? Successful solutions are not always accompanied and sustained by conceptual understanding. This research demonstrates that understanding can be built by the successful problem solver him/herself through the process of Representational Redescription. This is a process which the problem solver activates while dealing with a problem situation in which he/she is successful more than once. It is a cognitive process for the activation of which the teaching practice needs to give more time. It is a cognitive process for which the teaching practice can constitute a critical trigger. This is a very interesting possibility which needs, though, to be further explored. What this study argues for, is that it is certainly worthwhile to give children the time and space they need to work upon the knowledge that supports their own successes.

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APPENDICES

Transcription Key

I: the interviewer	**R**: Rakhi	[ ]: child's writing
G: Grace	**H**: Henry	( ): child's movements
H: Hazel	**Is**: Isa	**(...)**: pause, non-answer
El: Elsa	**L**: Leo	**...** : dragging of voice, hesitation
E: Erna	**J**: James	
S: Sean		

It should be noted that following standard ethical procedures, all names are pseudonyms.

APPENDIX 7.2

THE CASE OF RAKHI

Itinerary of changes observed

First session

1.1 *Number 6 was given as the target number in the 'card' task.*

[4+2]
[5+1]
[2+4]
[3+3]
[1+5]
[0+6]
[6+0]

Rakhi completed the task after producing the number bonds which are shown on the left. In the case of the [4+2] Rakhi wrote 4 down. Then she counted on 2, using her fingers, and completed the number bond. Rakhi employed the same method for the production of the next four number bonds. After the production of the first five number combinations the interviewer asked:

I: How do you choose the numbers? Can you tell me? How do you choose the numbers that you use each time?
R: I don't know, with my fingers.
I: How do your fingers help you choose the number that you write down first?
R: (...)
I: Do they help you find the second number?
R: Yes.
I: How?
R: ... to count.

To explain the specification of the first addend, Rakhi said that she was using her fingers, without giving any further explanation of this use. However, Rakhi explained that she was using her fingers to count and find out the second addend. Rakhi completed the task after producing the two last number bonds which involved 0. Rakhi said that she "knew them".

1.2 *Number 8 was given as 'target' number next.*

[1+7]
[0+8]
[5+3]
[4+4]
[2+6]
[3+5]
[6+2]
[7+1]
[8+0]

Each of the first six number bonds was produced in two steps: Rakhi wrote down the first addend. She counted on with fingers to figure out the missing second addend. In the case of the [0+8] Rakhi did not use counting at all Rakhi reported her declarative knowledge of the rule 0+N=N. The following four number combinations were produced by the same two-step process that produced the [1+7]. After writing down the [3+5], Rakhi focused on the already completed number combinations and started whispering:

R: 1, 2, 3, 4, 5.... (looks at these as first addends. After 5 she says): 6... (looks at the cards top to bottom) 6 (says again and puts 6 fingers up. Counts the rest and writes down):
[6+2]
I: How did you choose 6 to work with now?

R: I didn't see a 6 (shows first addends).

Rakhi repeated the same procedure before producing the following number bond [7+1]. She started whispering 1, 2, 3, 4, 5, 6, 7. Saying each of these numbers, in this order, she pointed with her pencil at the card that each one of these numbers appeared as first addend. After whispering "7" and not finding a number bond with 7 as first addend, she put 7 fingers up and then 1 finger up and wrote down: [7+1]. In the same way the last number bond [8+0] was produced.

1.3 After writing down this number bond Rakhi said: "That's it." The interviewer asked:

I: How do you know that these are all the number bonds that you can find?
R: I could get the number line and then I could check all of them again.
I: (gives R a number line).
R: (points with her pencil at each of the numbers on the number line from 1 to 8. Each time she points at a number, she points at the number bond that has this number as first addend. At the end, she says '0' and shows the number bond): There is no number missing.

Consideration of the numbers used as first addends in order, was the way that Rakhi used to figure out the last set of three number bonds. Also, with the help of the number line which helped her keep track of the numbers, Rakhi used this idea of 'ordering' again, to justify her belief that the task had been completed.

Second session

2.1 <u>*Number 12 was given as a target number.*</u>

[0+12]
[1+11]
[2+10]
[3+9]
[4+8]
[5+7]
[6+6]
[7+5]
[8+4]
[9+3]
[10+2]
[11+1]
[12+0]

The first two number bonds were produced in a few seconds. After the production of the [1+11] number bond Rakhi whispered "0, 1, 2" and wrote down number 2 as first addend. She counted on using her fingers to figure out the second addend. The same procedure was followed for the production of the following number combinations. The numbers to be used as first addends seemed to be specified following an order, right from the beginning of the solving procedure. After the production of the [4+8] number bond, Rakhi whispered "5". The interviewer asked:

I: Why did you choose 5 to work with now?
R: I… (takes some time to think and then says): Because it's the… end of the line.
I: Which line?
R: This line (shows the column of first numbers top to bottom).
I: Can you show me which numbers are in this line?
R: 0, 1, 2, 3, 4 (shows these as first numbers top to bottom).
I: Oh right, so, after that comes 5? (shows below the 4 as first addend).
R: (nods 'yes').
I: Is this the way you choose the numbers?
R: (nods 'yes').
I: Why do you choose the numbers like that?

R: Because... um... you can work all the way up to 12 (moves her hand vertically from the top of the table up to the bottom, pointing at the side of first numbers).

Rakhi explained that she was choosing the number to be used as first addend in a sequential order. The specification of the second addend still involved counting on with fingers. Rakhi applied the combination of these two methods for the generation of the rest of the number bonds.

2.2 The new method of organising the selection and specification of first addend allowed Rakhi to be aware of the completion of the task right after producing the last number bond as the following extract shows:

R: **[12+0]** (puts this number bond at the bottom of the column and then puts her pencil down and folds her hands).
I: Are there any more?
R: (shakes her head to say 'no').
I: How do we know?
R: Because... uhm... 13 is next after 12 (shows 12 as first number in the last in the column number bond).
I: Can't you use it to make 12?
R: (shakes her head to say 'no') We do adding.
I: So, are these all the possible ways?
R: (nods 'yes').

Rakhi justified her certainty that the task was completed on the basis of the rationale that underlay the specification of first addend in each number bond. The next number in the sequence was number 13. Rakhi had no difficulty to explain that this number could not be used because the requirement of the task was to produce number combinations using addition.

Third session

3.1 <u>*Number 13 was given as the target number in the 'card' task.*</u>

[0+13]
[1+12]
[2+11]
[3+10]
[4+9]
[5+8]
[6+7]
[7+6]
[8+5]
[9+4]
[10+3]
[11+2]
[12+1]
[13+0]

Rakhi employed the same 'two-step' process that she employed in the previous session in order to produce number bonds. It should be noted that in this run Rakhi did not produce the [0+13] number bond at the beginning of the solving procedure. [1+12] was the first number bond that Rakhi produced. It was after the production of the [5+8] that Rakhi looked at the number combinations that she had completed up to that point, took a new card, wrote down the [0+13], and put it at the top of the column. Then she went on with the production of the rest of the number bonds. The interviewer asked:

I: Why did you put this number bond there? (shows [0+13] at the top of the column)
R: Because 0 is before 1.

At this point Rakhi showed that she had a good control over the solution process. She interrupted the solving procedure to produce the [0+13], and then she continued the number bond production from the point she had stopped at. What is also notable in this run with the 'card' task is that, before writing down the number to be used as first addend, Rakhi did not repeat all the sequence of numbers she had used as first addends up to that point, as she did in the previous session. Looking at the last produced number bond was enough for Rakhi to specify the first addend of the following number combination.

3.2 After the completion of the task, the interviewer asked:

I: Are you sure that you have used all the possible numbers?
R: Yes, yes. 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 (shows these as first addends bottom to top). I am going in order.
I: Ok. So, you put these numbers in order. Can you explain to me why you put the numbers in this order?
R: (looks at the cards. Then shows with her pencil the 0 as second number in the last number bond: 13+0 and says): 0.... and up... (talks with a bit of hesitation. Looks at the column of second numbers. Her eyes are going upwards)... will be 13... (looks and points at 13 as second addend at the top of the column) and if..... you put them up to 0 (shows column of first addends bottom to top)....
I: What happens with these two lines of numbers? (shows columns of first and second numbers).
R: (looks at the bottom of the column. Then says): Well, the squares start up there (points at the top of the column of first numbers) and the triangles start down here (shows the 0 as second number in the last number bond at the bottom of the column: 13+0).
I: Right. How come this happens?
R: (...)

For the first time, Rakhi used the word "order" to explain the method she used for the specification of each of the first addends. The interviewer's second question made her focus on the produced number bonds for a while. What Rakhi said seemed to be her out-loud thinking while observing the sequence of the first and second addends. She described the order of numbers in each column of first as well as second addends, even though, in the course of the solving procedure, the second addends were specified by counting on and not by following a specific order as in the case of first addends. Rakhi described the ascending and descending order that the first and second addends were following, correspondingly, if one considered them from the top of the column of produced number bonds. The interviewer asked Rakhi to say more:

I: For which numbers are you going in order?
R: (looks at the top of the column of cards) Uhm... (then looks at the bottom of the column of cards and says): 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0 (shows these as first numbers bottom to top).
I: So, for these numbers you are going in order (shows first numbers bottom to top).
R: (nods 'yes') and these (shows second addends bottom to top).
I: Did you want them to go like that from the beginning? Did you know that it will go like that? (shows column of second addends).
R: (...)
I: Did you know that these numbers were going in order, or did you just notice it? (shows second addends).
R: I just notice it.

Rakhi's answer shows that 'going in order', constituted a method for the specification of first addends while in the case of second addends constituted an 'a posteriori' observation. First and second addends in this run with the task were produced by different methods which were supported by different representations.

Fourth session

4.1 <u>*Number 17 was given as the target number.*</u>

[0+17]
[1+16]
[2+15]
[3+14]
[4+13]
[5+12]
[6+11]
[7+10]
[8+9]
[9+8]
[10+7]
[11+6]
[12+5]
[13+4]
[14+3]
[15+2]
[16+1]
[17+0]

Up to the production of the [11+6] number bond Rakhi was employing the two-step process that she employed in the previous session. After writing down the [11+6] Rakhi wrote down 12 as first addend, and then she looked back at the cards with the completed number bonds on. Rakhi, with her pencil, pointed at and uttered whispering each of the following second addends: "10, 9, 8, 7, 6". Then she wrote down number 5 and completed the number bond that she was currently working on: [12+5]. The interviewer asked:

I: How did you find that now?
R: (shows the 6 as second number in the 11+6 number bond).
I: How did that help you?
R: Because… 5 is… **before** 6. (she emphasises).
I: How did you know that you had to write down the number which is before that one? (shows the 6 as second addend).
R: I put 6 there (shows 6 as second addend) then I looked at the cards… then I thought… that it could have been right. Then I looked again (shows the last in the column card: 11+6) and then I thought… it was 5.

For the first time, Rakhi did not produce the second addend by counting on. She reported that considering the number she had used as second addend in the previously produced number bond made her think of the number that, as second addend, would complete the number bond she was working on. For the next number bond [13+4], Rakhi followed the same procedure. She repeated the numbers she had already used as second addends: " 10, 9, 8, 7, 6, 5", and then wrote down number 4 as second addend. Rakhi explained the production of 4 as the second addend in the following way: "I looked at the numbers down". At that point, Rakhi did not provide any further explanation for the production of the number bonds.

Fifth session

5.1 <u>*Number 19 was given as the target number in the 'card' task.*</u>

The first three number bonds that Rakhi produced are shown below. At that point the interviewer interrupted the solving procedure, put an incomplete card below the [2+17] and asked Rakhi which number bond she would produce next.

[0+19]
[1+18]
[2+17]
[..+..]

R: (looks at the completed number combinations) 3 and…. (then points at the 17 as second addend and completes the number bond) **3 and 16**.
I: After that? (shows below the incomplete card).
R: 4… and… **4 and….** (looks at the cards for some time. Then focuses on the column of

second numbers, taps her hand on the table whispering): 19, 18, 17, 16, 15...
15 (says high-voiced this time).
I: How did you find that out?
R: I go... backwards (shows second addends top to bottom), and here it goes in order (shows first addends).

This time Rakhi explained the kind of order she was following and anticipated the production of one more number bond using the words "backwards" and "in order": i.e words which indicated the underlying arithmetical relation of the numbers in each of the two columns. The interviewer asked Rakhi to go on with the written completion of the cards.

5.2 At some point during the solution process the interviewer asked:

I: Why do you think these second numbers go like that, backwards? (shows second numbers top to bottom).
R: (looks at the cards) Uhm... I don't know.
I: Can you figure out? Take some time and try to think, why do these numbers go backwards?
R: Because the squares are in front (shows the square in one of the cards), then the add is next (shows the add sign on the same card) and then the triangles are second (shows the triangle on the same card).
I: What if the triangles were first?
R: Um... I should go forwards.
I: Forwards in the triangles, and what about the squares?
R: Backwards.
I: Why would you go backwards in the squares?
R: (shrugs her shoulders).
I: How come if you go forwards for the first numbers and backwards for the second numbers you always find the right number bonds?
R: I don't know.
I: Do you need to go forwards and backwards, or you can also go forwards for both numbers?
R: Um...I am not sure. Maybe I can.

Even though Rakhi used the 'ordering' strategy to anticipate following number bonds, she did not seem to be in position to explain why her strategy worked the way it did. She did not seem to have conceptualised the arithmetical relations that underlay her strategy and had as a result the particular sequence that the first and second addends were following. Rakhi applied the 'ordering' strategy when the interviewer asked her to produce a few number bonds for bigger 'target' numbers like 32 and 100.

APPENDIX 7.3

THE CASE OF HENRY

Itinerary of changes observed

First session

1.1 *9 was given as the target number in the 'card' task.*

[9+0]

[5+4]
[8+1]
[7+2]
[6+3]
[3+6]
[2+7]
[1+8]

[0+9]
[4+5]

Henry completed the task after producing the number bonds which are shown on the left. The rapid production of the first number bond [9+0] allows to assume that it was the product of Henry's declarative/factual knowledge. The next number bond [5+4], was produced in the following way: Henry put 9 fingers up. Then he separated 5 fingers and counted the rest up to 9. The number bond produced was [5+4]. Henry used the same method to produce the following six number bonds of this set. Whenever he was asked how he found the number bonds, Henry said: "With my fingers". Henry explained that he produced the last two number bonds by "changing around" the [9+0] and [5+4] that he had produced at the beginning of the solution process.

1.2 After producing the last number bond Henry took quite a long time to think looking at the number bonds that he had produced. After a while the interviewer asked:

I: Is there another one?
H: (shakes his head to say 'no').
I: No? How do you know?
H: (...)
I: What do you look to check? Can you show me?
H: Because I've done 4 add 5, I've done 0 add 9, I've done 1 add 8 ... (repeats all the number bonds that he has made bottom to top. He gathers one by one all the cards in front of him in a pile).
I: (spreads out the cards again) Ok, these are the ones that you have found. How do you know that these are all the ways and there is no one missing?
H: I think what else... I think what else... uhm... I look at these first (shows first addends). I look at these (shows the first addend of each number bond) and see what I've done and I think if there is any... one else... and... or I do it with my fingers.

Henry explained that he checked the completion of the task by focusing only on the numbers used as first addends. This may entail an understanding of the fact that checking only the numbers used either as first or second addends provides enough information for one to know whether all the possible combinations have been produced or not. However, Henry did not report any systematic way of checking the use of these numbers.

1.3 *12 was given as a target number.*

[0+12]
[10+2]
[8+4]
[5+7]
[4+8]
[3+9]
[9+3]
[2+10]
[12+0]
[7+5]

Except from the [0+12] which was produced rapidly at the beginning of the solution process, the following number bonds were produced either by 'swapping' or by the same 'two steps' process that Henry used in the previous run.

After the production of the number bonds which are shown above, Henry started rearranging the cards. He put each number bond together with the one which had the same addends in a different order (arrangement shown on the left). When the rearrangement was done Henry said:

[12+0]
[0+12]
[10+2]
[2+10]
[8+4]
[4+8]
[5+7]
[7+5]
[3+9]
[9+3]

H: I've done them.
I: Yes. You have swapped them all, but may be there is one that you haven't found yet and you haven't swapped around. May be there is one that is missing and you haven't thought of it yet.
H: (shakes his head to say 'no') I've done them.

Even when the interviewer explicitly told Henry that these were not all the possible number bonds resulting to 12, Henry kept insisting that these were all because he had swapped them all.

Second session

2.1 *Number 8 was given as the target number in the 'card' task.*

[8+0]	[6+2]
[0+8]	[2+6]
[5+3]	
[3+5]	
[1+7]	
[7+1]	
[4+4]	

Henry employed the same methods that he used in the previous session. After writing down the last number bond Henry started putting the cards in pairs: each number bond together with the one which had the same numbers in different order (arrangement shown on the left). Then Henry unfolded and kept 1 finger up while looking at the column of completed number bonds top to bottom. Then he unfolded and kept 2 fingers up while looking again at the column of completed number bonds, and so on. He repeated this procedure up to the point that he kept 8 fingers up and looked at the column of number bonds top to bottom. Then he said:

H: I've done them all.
I: Can you explain to me how you checked?
H: (holds 1 finger up) I've done 1... (shows the pair of number bonds that have 1 as first and second addend), I've done 2... (shows the pair of number bonds that have 2 as first and second addend)
I: What do you look at? Which 2 do you look at?
H: Both of them but not at the same time, when I get to the number (shows 2s in both number bonds of the pair. Then goes on..) I've done 3... (shows 3 in both number bonds of the pair), I've done 4 (shows 4 in both number bonds of the pair), I've done 5.. I've done 6 (shows the pair of number bonds), I've done 7 (shows the pair of number bonds), and I've done 8 (shows the pair of cards).

The method of checking that Henry developed in this run appeared to be the combination of checking methods that he had used in previous runs. He put the number bonds in pairs to check if he had swapped them all. Furthermore, he checked if all the possible numbers had been used as first and second addends.

2.2 *Number 15 was given as the target number.*

[10+5]
[5+10]
[0+15]
[15+0]

Henry produced number bonds in pairs by applying 'instant swapping'. For the [10+5] he explained that he "knew it". He reported that he swapped this number combination in order to produce the [5+10]. The same explanations were given for the production of the second pair of number bonds.

[9+6]
[6+9]
[8+7]
[7+8]
[4+11]
[11+4]
[3+12]
[12+3]
[2+13]
[13+2]
[1+14]
[14+1]

The use of the 'deriving' method was reported for the production of the [9+6] and [8+7]. For the [9+6] Henry explained:

H: Because… 1 less than 10 (shows 10 as first addend) is 9… and I need 1 less that f… no, then I need more.
I: More than what?
H: More than… (takes some to think) more than 5, so I need 6.
I: Very good. But can you explain to me this: why do you need more than 5?
H: (…)
I: What if you put a number which is 1 less than 6?
H: (shakes his head to say 'no').
I: Why not?
H: (shrugs his shoulders).

Henry reported the mechanism which produced the first and second addend, but not the rationale that underlay this mechanism. He used counting and 'instant swapping' to produce the rest of the number bonds.

2.3 *9 was given as the target number next.*

[0+9]
[9+0]
[8+1]
[1+8]
[7+2]
[2+7]
[6+3]
[3+6]
[5+4]
[4+5]

After the production of the [7+2] number bond the interviewer asked:

I: How did you do that? You were so quick!
H: Because.. I needed less than 8 (shows the 8 as first addend and the 7 as first addend) and I needed more that 1 (shows the 1 as second addend and then 2 as second addend). And then I'll do the changing.
I: Why do you change it?
H: Uhm… it's a quick way.
I: "A quick way" for what?
H: Uhm… it's quicker to do them all the changing, and… if I forget one...
I: What happens if you forget one?
H: (…)

In this run, for the first time, Henry attempted to explain *why* he was applying 'instant swapping'. Even though his explanation was not very clear, it provided an indication of deliberate, and planned problem solving behaviour.

Third session

3.1 *Number 14 was given as target number.*

[10+4]
[2+12]
[3+11]
[4+10]
[5+9]

Henry did not apply 'instant swapping'. The first number bond was recalled from memory as Henry explained. The following two number bonds were produced in two steps: Specification of first addend-counting on with fingers to figure out the second addend. But for the production of the [4+10] Henry explained:

H: I did 1 less than 11 (shows the 11 in the 3+11 number bond).
I: How did you think of 4?
H: It's 1 more than… 3 (shows 3 as first addend).

It is notable that this time (in comparison to the application of the method in the previous session) the method was reversed. Henry added 1 and took away 1 to produce each of the two addends of the new number bond. Henry showed that he acknowledged the reversibility of the method. He reported the use of the 'deriving' method for the [5+9].

3.2 The number bonds which were produced up to that point are shown on the left.

[10+4]
[2+12]
[3+11]
[4+10]
[5+9]

Henry spent quite some time looking at the number bonds. Then moved the [10+4] number bond and kept it further aside. The new arrangement of number bonds is shown below.

[10+4]
[2+12]
[3+11]
[4+10]
[5+9]

The interviewer asked Henry about this rearrangement.

I: Why do you want it there? (shows the [5+9]).
H: 2, 3, 4, 5 (shows first addends top to bottom) 12, 11, 10, 9 (shows second addends top to bottom. Then he looks at the number bonds and puts his fingers on the 2 in the [2+12] number combination. Then he writes down the first addend of the next number bond) **[1]** (he looks at the cards) add… 13.
[1+13] That goes up here (puts this card above the 2+12). 1, 2, 3, 4, 5 (he repeats the first addends top to bottom).

[10+4]
[1+13]
[2+12]
[3+11]
[4+10]
[5+9]

The new arrangement of cards is shown on the left. It is believed that Henry observed, and noticed a pattern: the first addends in the last four number bonds, before the rearrangement, appeared to be in an ascending order, while the second addends in these same number bonds appeared to be in a descending order. Henry produced a new number bond ([1+13]) and put it at the top of the rearranged column of number bonds. In the process of this rearrangement, Henry made explicit his observations and subsequent decisions by talking aloud and describing his actions: "That goes up here". After completing the new number bond he repeated the sequence of first addends as if he wanted to make sure that the order was maintained.

[0+14]
[1+13]
[2+12]
[3+11]
[4+10]
[5+9]
[6+8]
[7+7]
[8+6]
[9+5]
[10+4]
[11+3]
[12+2]
[13+1]
[14+0]

Following the ascending and descending order of number in each of the two columns, Henry completed the task. After writing down the last number bond Henry said:

H: This is all.
I: How do you know?
H: Because it's up to 14.

Henry's new strategy allowed him to complete the task in minimum time comparing to the time he needed in the previous runs with the task. Also it allowed him to be aware of his success right away, after writing down the last number bond. Subsequently, number 19 was given as target number in the 'card' task. Henry applied 'ordering' all along the solution process. He did not provide explanations on why the strategy worked the way it did.

Fourth session

4.1 *Number 15 was given as the target number in the 'card' task.*

[0+15]
[1+14]
[2+13]
[3+12]
[4+11]
[5+10]
[6+9]
[7+8]
[8+7]
[9+6]
[10+5]
[11+4]
[12+3]
[13+2]
[14+1]
[15+0]

Henry completed the task by applying the 'ordering' strategy all along the solution process. At the end of this run the interviewer prompted Henry to focus on the number bonds and asked:

I: How do these numbers go? (shows column of second addends).
H: Down (repeats the second addends top to bottom).
I: And how do these numbers go? (shows column of first addends).
H: Properly. (repeats first addends top to bottom).
I: Right. Why do you think that these numbers (shows first addends) go in this order and these numbers (shows second addends) go down?
H: (…)
I: Why does this happen?
H: (…)
I: Can you explain it?
H: (shakes his head to say 'no').

This was not the first time that Henry reported the different type of order into which the first and second addends appeared after applying his 'ordering' strategy. However, he did not seem to be in position to explain why the numbers appeared in a different type of order. Henry did not give an answer to this question at any instance during this session or the following one.

4.2 Subsequently, the interviewer asked Henry if he thought that he could find all the

[0+100]
[1+99]
[2+98]
[3+97]

possible number bonds for any target number, no matter how big it may be. Henry, nodded assertively. In a paper and pencil task Henry produced the number bonds which are shown on the left in a few seconds. The interviewer asked Henry to say a few more number bonds without writing them down. Henry continued the production of number bonds verbally:

H: 4 add 96, 5 add 95, 6 add 94, 7 add 93.

Henry produced each of these number bonds verbally, with no difficulty and without needing to use the whole sequence of previous number bonds as a visible reference. He ascertained that if he had the time, he would be able to find all the possible number bonds for 100 and anticipated that the number bond that he would complete the task with, would be "100 add 0".

Fifth session

5.1 <u>*Number 19 was given as target number in the 'card' task.*</u>

Before the solution process begins, Henry said that he was going to do it "in any order". Henry applied counting and 'swapping and produced the number bonds which are shown below. He reported his choice not to apply the 'ordering' but he acknowledged that the 'ordering' strategy was better, because it allowed him to know when all the possible number bonds had been found:

[10+9]
[18+1]
[15+4]
[5+14]
[0+19]
[4+15]
[9+10]
[7+12]

H: I wanted to do it in a different... way because I... have done that 0, 1, 2, 3, 4....
I: Ok. But which way you think is better, that one or this one? (shows the produced number bonds).
H: That one.
I: Why?
H: Because I know when... it's all the ways.

After writing down the [7+12], Henry wrote down 8 as first addend: [8+...]. Then he put his fingers up to count. The interviewer asked:

I: Now, if I ask you to find the next one in the most quick and easy way. What will you do?
H: One like starting with 0, 1, 2, 3, 4, 5, 6, 7.... (shows position of first addends top to bottom)
I: Right. But now you haven't started like this. So what can you do to be quick now?
H: (...)
I: Say that you've got only these two (removes all the completed cards from the table except from the last two) and you want to find the missing number very quickly (shows the [8+...] card). What can you do?
H: (looks at the cards and thinks for quite some time. He does not answer).

Henry acknowledged that if he wanted to be quick, 'ordering' was the strategy that he should apply. However the number bonds were not in order this time. With the intention to see whether Henry would apply the 'deriving' method in a situation where 'ordering' had not been applied for the whole task, the interviewer removed all the number bonds leaving just two on the table (shown below). The number bond that Henry had to complete is shown in bold.

[9+10]	**[8+...]**
[7+12]	

Henry did not complete the number bond and did not answer when the interviewer asked him which method would be the easiest and quickest in this case. Henry did not seem to consider here the possibility of applying the 'deriving' method.

5.2 *'Missing numbers' task*

The following pair of number sentences, one of which was incomplete, were presented to Henry. The interviewer asked him to find the missing addend. Again, the intention was to see whether Henry would consider the use of the 'deriving' method:

12+3=15
13+..=15

Henry looked at the number sentences and after a short while he said:

H: 13 add 4.
I: Are you sure?
H: 13.... add 2.
I: Very good. How did you find it?
H: I thought upwards... but then I thought downwards.
I: Oh, right. Why did you think upwards at the beginning?
H: Because I thought it was... upwards.

This time Henry did consider the use of the given, complete number sentence as reference for the completion of the incomplete one. But, this was the first time that he applied the 'deriving' method incorrectly. Two more pairs of number sentences such as these were subsequently presented to Henry. To find the missing number Henry used counting both times.

APPENDIX 7.4

THE CASE OF ISA

Itinerary of changes observed

First session

1.1 *Number 9 was given as the target number.*

[0+9]
[1+8]
[9+0]
[7+2]
[6+3]
[8+1]
[5+4]
[4+5]
[3+6]
[2+7]

For the production of the number bonds which are shown in italics Isa explained that she "already knew them". This explanation together with the very short solution time suggests the use of declarative/factual knowledge. For the production of the [9+0] and of the last set of three number bonds Isa explained that she did "the other way around". She indicated which number bonds she swapped and said that she knew that she could do that in addition.

For the production of the [6+3] Isa focused her gaze on the previously produced number bond for a few seconds. She explained:

IS: 7 add 2 (shows previous number bond) and so it must be 6 add 3.
I: How did you know that it has to be 3 there? (shows position of second addend).
IS: I don't know...
I: Didn't you look at anything to help you?
IS: (shakes her head to say 'no').

Isa's verbalisation indicates that she derived the new number bonds from the previous one. However, she did not report the 'deriving' mechanism. For the production of the [5+4] number bond Isa did not provide any explanation. After writing down the last number bond Isa focused at the produced number bonds for quite a long time. Then she said: "These are all I can do". She did not provide any further explanation to the interviewer's questions regarding her certainty that she had finished.

1.2 *Subsequently, number 7 was given as target number.*

The number bonds were produced in pairs. The second number bond of each pair was produced by 'swapping' the first number bond of the pair. The first number bond of each of the first three pairs was produced very quickly. The considerably rapid solution time suggests the use of factual knowledge. When asked to explain how she found the [4+3] number bond Isa explained:

[1+6]
[6+1]
[0+7]
[7+0]
[5+2]
[2+5]
[4+3]
[3+4]

IS: Because… 5 add 2 is 7 (shows the [5+2]). It must be 4 and 3 is 7.
I: Did you look at that one (shows 2 as second addend) to help you find this one? (shows 3 as second addend).
IS: (nods 'yes').
I: How did this help you find the next one? I find this very interesting. Can you explain it to me?
IS: (…)

As in the previous run, Isa did not report the 'deriving' mechanism. For the [3+4] she reported the use of 'swapping'.

1.3 Isa took a very long time to think looking at the column of produced number bonds. To the interviewer's question if she was looking for another number combination Isa replied:

IS: I tried to think of one, but I had already done it.
I: Is there a way to know if these are all the possible ways to make 7? How can we check?
IS: (looks at the cards) …all of them really. I've done these (separates the two first cards), I've done these (separates the two second cards), I've done these (separates the next two) I've done these (separates the last two).
I: So, are these all?
IS: (nods 'yes') I think so.
I: Why do you put these two together? (shows [1+6] and [6+1]) and these two together? (shows [0+7], [7+0]).
IS: 1 and 6, 6 and 1, it's the same. 0 and 7, 7 and 0, 5 and 2, 2 and 5, 4 and 3, 3 and 4 (puts together these pairs of number bonds). I have done them all around.
I: So, why do you put them in pairs?
IS: Because these go like that because you know you have done it all around.

After writing down the last number bond Isa did not appear to be certain of the completion of the task. The interviewer prompted her to check if all the possible number bonds had been produced. Isa thought that the task was complete because all the number bonds had been swapped.

Second session

2.1 <u>*14 was given as target number in the 'card' task.*</u>

[7+7]
[13+1]
[12+2]
[14+0]
[0+14]
[2+12]
[11+3]
[10+4]
[4+10]
[1+13]
[3+11]

Isa did not produce number bonds in pairs. She explained that she "already knew" the number combinations which are shown in italics. For the production of the number bonds which are shown in bold, Isa reported that she used the [13+1] to find the [12+2], the [12+2] to find the [11+3], and the [11+3] to find the [10+4]. However, she did not explain the mechanism of production. For the rest of number bonds Isa reported the use of 'swapping'.

Isa took quite some time to think looking at the already produced number combinations. Then she started putting each number bond together with the one that had the same addends put in different order. At the end of this rearrangement all the produced number bonds were put in pairs. The 'double' [7+7] was kept at the top of this rearranged column. Isa said:

IS: I've done them all. I think I have. Any way I can't think of any more.
I: So, are these all?
IS: (...)
I: Is there something that can help you find if there are numbers that you haven't used?
IS: The number line.

After putting the number bonds in pairs to check if they had all been swapped around, Isa did not appear to be certain of whether the task had been completed or not. After further prompting by the interviewer Isa thought of the number line as a tool that could help her check if there were any numbers that had not been used. Isa looked at the number line put her finger on number 8. She looked at the number bonds. She wrote down 8 and counted on using the number line to find the second number. In this way the [8+6] and [9+5] number bonds were produced. Then she said: "Oh! I can write around these ones". The [6+8] and [5+9] number bonds were put below the [8+6] and [9+5] correspondingly to create two new pairs of number bonds. With the number line, Isa checked the use of all the possible numbers from 0 to 14 considering them in order. At the end of this process she showed all the numbers from 0 to 14 on the number line and said: "I've done all of these. There are no more".

2.2 <u>*Number 11 was given as target number in the 'card' task.*</u>

[10+1]
[1+10]
[5+6]
[6+5]
[2+9]
[9+2]
[0+11]
[11+0]
[3+8]
[8+3]
[7+4]
[4+7]

Isa applied her strategy of producing number bonds in pairs all along the solution process. 'Instant swapping' was used for the production of the second number bond of each pair. The [10+1] and [0+11] were produced by Isa's declarative/factual knowledge. For the production of all the rest first number bonds of the pairs Isa used the number line. Without following any particular order, she was picking up a number that she was using as first addend and then she was counting on, using the number line, to find the adequate second addend. When Isa started having difficulties with finding a number that had not been already used she focused on the number line again and started considering the numbers in order. She was keeping her finger on each of the numbers from 0 to 11 while looking at the cards to see if that number had been used. She also explained that she was checking if each number had been used in a square *and* in a triangle: i.e. both as first and second addend. At the end of this process Isa was certain of her success.

Third session

3.1 <u>*Number 10 was given as the target number.*</u>

Isa produced the number bonds which are shown below by calling upon her declarative/factual knowledge (shown in italics) and by applying swapping. For the production of the [8+2] the interviewer asked Isa:

[1+9]
[5+5]
[9+1]
[10+0]
[0+10]
[8+2]

I: How did you think of this one?
IS: Because... that one helped me (shows the 9+1 number bond).
I: How did that help you?
IS: 2 was after 1, and 8 before 9.

This was the first time that Isa reported her 'deriving' method by referring to the relationship between each of the addends (the first *and* the second) of the two number bonds: i.e. the number bond-reference and the derived number bond. Isa referred to the relationship between the numbers in terms of *after* and *before*.

[2+8]
[7+3]
[3+7]
[6+4]
[4+6]

Isa completed the task by producing number bonds in pairs. The 'deriving' method and 'instant swapping' were applied for the completion of the task. Isa reported the use of the 'deriving' method, as well as the mechanism involved, for the production of the number bonds which are shown in bold.

3.2 After writing down the [4+6] Isa took a new card from the pile of cards with incomplete number sentences and wrote number 3 in the position of first addend:[3+...]. She did not go on completing the number bond. She looked at the column of already produced number combinations and said: "No..., I've done all of them. I think." Isa started rearranging the number bonds and putting them in pairs: one number bond opposite to the number bond that had the same addends in different order. The arrangement is shown below.

[1+9]	[9+1]
[0+10]	[10+0]
[2+8]	[8+2]
[3+7]	[7+3]
[4+6]	[6+4]
[5+5]	

The interviewer asked Isa if these were all the possible number bonds. Isa started showing and uttering each of the first addends in both columns top to bottom. She did the same for the second addends. At the end of this process she said with certainty that there was nothing missing. It is noteworthy that this time Isa did not check the use of all the possible numbers considering them in order as she did in previous runs using the number line. However she appeared to be certain that all the possible number had been used as first as well as second addend.

Fourth session

4.1 <u>*Number 11 was given as the target number*</u>

[10+1]
[1+10]
[11+0]
[0+11]
[9+2]
[2+9]
[8+3]
[3+8]
[7+4]
[4+7]
[6+5]
[5+6]

The number bonds were produced in pairs all along the solution process. The second number bond of each pair was produced by 'instant swapping'. The number bonds which are shown in italics are considered as the product of declarative/factual knowledge. Isa reported the use of the 'deriving' method for the production of the number bonds which are shown in bold. For example, for the [9+2] Isa explained:

IS: (she shows the 10 at the 10+1 number bond)...uhm... 9 before 10.
I: Yes, 9 is before 10, how did you figure out the 2?

IS: It's after 1.

Isa explained that she used as first addend the number which, in the sequence of natural numbers, was *before* the first addend of the number bond-reference. Then she used as second addend the number which, in the sequence of natural numbers, was *after* the second addend in the number bond-reference.

After the production of the [8+3] number bond by the application of the 'deriving' method, the interviewer asked Isa why, she thought, her method worked as it did:

I: Why do you think it works that way?
IS: (...)
I: Why did you choose a number which was before that (shows 9 as first addend further above) and another one which was after that (shows 2 as second addend further above)?
IS: (...)
I: How come, if you choose numbers in this way you find another number bond that makes 11?
IS: (shrugs her shoulders).
I: Has anybody shown to you that you can do it in this way?
IS: (shakes her head to say 'no').
I: Did you find this way yourself?
IS: (nods 'yes').

Isa did not justify the success of the 'deriving' method at any instance during this or the following session.

4.2 After writing down the last number bond [5+6] Isa focused on the last pair of number bonds she had produced and after a few seconds she said:

IS: I've done all of them.
I: You seem to be sure this time. How do you know?
IS: Cause I looked at these (shows the last two number bonds) and I think I know I've done all of them.
I: What makes you think that?
IS: (...)
I: How do you know that there is nothing missing?
IS: I don't know.

For the first time Isa appeared to be certain of the completion of the task almost immediately after writing down the last number bond. However, she was not in position to explain what made her be so certain.

4.3 *Number 9 was given as target number in the 'domino' task*

[9, 0]
[0, 9]
[1, 8]
[8, 1]
[5, 4]
[4, 5]
[7, 2]
[2, 7]
[6, 3]
[3, 6]

Isa applied her strategy of producing number combinations in pairs all along the solution process. The second combination of each pair was produced by 'swapping'. Fact retrieval was mainly used for the production of the first combination of each pair. Isa reported that she used the 'deriving' method to produce the [6, 3] combination. After producing the last combination Isa took quite some time to think looking at the completed dominoes. She went on to the next empty domino and drew 3 dots on the left part of it. After looking at the completed dominoes again she said without completing the domino with the 3 dots: "No, I think these are all". When asked to explain how she knew that, Isa simply repeated the number of dots that she had put in each part of each domino top to bottom. She did not provide any further explanation.

Fifth session

5.1 <u>Number 12 was given as a target number in the 'card' task.</u>

[6+6]
[11+1]
[1+11]
[10+2]
[2+10]
[9+3]
[3+9]

The number bonds which are shown in italics were produced by Isa's declarative/factual knowledge. For the rest of the solution process Isa applied the strategy of producing number bonds in pairs. She combined the 'deriving' method and 'swapping'. 'Before/after', was the pattern that she was always following to produce the first and second addend of the new number bond correspondingly. It was considered that it would be interesting to see how flexible this method was and whether Isa would be in position to adjust her method in cases where her usual procedure was violated. The interviewer took a card from the pile of cards with incomplete number sentences resulting to 12 and put number 4 in the position of first addend: [4+...]. She asked Isa to complete the number bond. It was presumed that following Isa's usual practice, the next number bond that she would produce after the last produced pair of number combinations would be [8+4]. Isa would derive this number bond from the [9+3] by putting as first addend the number *before* 9 and as second addend the number *after* 3. Thus, the interviewer introduced a violation to Isa's usual procedure by asking her to complete a number bond in which 4 was the first addend.

Isa spent quite some time focusing her gaze on the incomplete number bond. After a while the interviewer asked:

I: Any ideas?
IS: I was counting but I... I'm going to do it again.
I: Do you think that any of these (shows already produced number bonds) might help you find the number you are looking for?
IS: (looks at the column of complete number bonds for some time then shakes her head to say 'no').
I: What about this one? (shows the [3+9]) Can this help you find the number you are looking for?
IS: (looks at the [3+9]) I don't know.

Isa did not consider the possibility of completing the number bond using as reference a previous number bond, not even when the interviewer attempted to direct her attention to the number bond that could help her find the missing the number. To use the [3+9] as reference to complete the incomplete number bond [4+...], Isa would have to *reverse* the pattern that she used to apply.

APPENDIX 7.5

THE CASE OF LEO

Itinerary of changes observed

First session

1.1 *Number 7 was given as the target number in the 'card' task.*

[0+7]
[6+1]
[2+5]
[3+4]

[5+2]
[7+0]
[4+3]
[1+6]

When the interviewer asked Leo how he found the first two number bonds Leo replied: "I thought in my head". Leo's explanation does not really say too much about the mechanism that produced these two number bonds, but the very short solution time suggests the use of factual knowledge. Leo took quite some time to think before writing down the two next number bonds [2+5] and [3+4]. He did not provide any verbal explanation or visible indication of the mechanism that helped him produce these two number bonds. The solution times can not provide a basis for any interpretation. They were not short, however the use of factual-declarative knowledge can not be excluded. Leo reported the use of 'swapping' for the production of the second set of number bonds.

After writing down the last number bond, Leo took a long time to think looking at the number bonds he had produced. The interviewer asked:

I: Is there something missing? What do you think?
L: (...)
I: Do you think that there is something missing and you can't find it, or that there is nothing missing?
L: Maybe... it's missing but I can't find it (keeps looking at the cards, he seems troubled).
I: Which numbers are you looking at to find if there is something missing?
L: (...)

Leo was not certain of the completion of the task. Furthermore, there was no indication of any particular method her used to check if there was a missing number bond.

1.2 N*umber 8 was given as target number in the 'card' task.*

[4+4]
[0+8]
[2+6]
[1+7]
[3+5]

[8+0]
[5+3]
[6+2]
[7+1]

For each of the first two combinations Leo said that he "knew it". Both these number bonds are considered as being produced by Leo's declarative/factual knowledge. For the next three number bonds Leo wrote down the first addend first and after a while he wrote down the second addend. When asked to explain how he found the [2+6] Leo explained: "I was counting in my head". The use of counting as a method for completing the number bonds was made explicit when Leo counted on whispering before completing the next two number bonds. It is noteworthy that the two addends of each of these number bonds were produced by a different mechanism. For the second set of number bonds Leo explained that he changed the addend order of the previously produced number combinations. After writing down the [7+1] Leo kept trying to find another number bond. After a while the interviewer let him know that the task had been completed.

Second session

2.1 <u>*Number 9 was given as the target number in the 'card' task.*</u>

[0+9]
[1+8]
[2+7]
[3+6]
[4+5]
[6+3]
[5+4]
[7+2]
[8+1]
[9+0]

The first number bond is considered as the product of Leo's declarative/factual knowledge. The next four number bonds were produced in two steps. Leo wrote down the first addend and then counted on to find the second addend. Again, Leo did not provide an explanation for the specification of first addend. The numbers that Leo chose to use as first addends seemed to follow a particular order: 0, 1, 2, 3, 4. This pattern may be accidental. But, if indeed Leo chose the numbers to be used as first addends following this particular order, this means that behind this choice there was a specific intention. Therefore, it could be assumed that this choice was conscious. However, Leo at that particular point did not appear to be in position to report his choice and the rationale behind it. Leo went on to the rapid production of the second set of number bonds. He reported the application of 'swapping'.

2.2 <u>*Number 8 was given as target number.*</u>

[8+0]
[1+7]
[2+6]

For the production of the first number bond the interviewer asked:

I: Why did you start with that one?
L: Because… I'm doing the first one like… 10 and 0, or 8 and 0…
I: I see. Why do you choose to do the first one like that?
L: To be quicker.

Leo justified his choice of starting the solution process with a number bond involving 0 by saying that this is a choice that allows him to be quick. The explanation and example that he gave ("like… 10 and 0, or 8 and 0") also show that this was a conscious choice and generalised practice that he followed when dealing with the task. Leo went on producing the following two number bonds. It should be noted that he applied the method in the reverse. That is he specified the first addends following an ascending order not a descending order as he had announced. The interviewer asked:

I: Oh, how did you find this so quickly?
L: I… I'm going like… this way 1 and 7, 2 and 6 (shows these number bonds in the column of cards top to bottom and goes on saying): **3 and 5** (saying each of the numbers, shows below the 2 as first addend and the 6 as second addend correspondingly).
I: And then?
L: 4 and… (thinks for a while. Looks at the cards and says): **4 and 4.**
I: And then?

L: 3… (takes some time to think): I can't think of the next one.

Leo explained the production of these number bonds considering them in the context of a particular "way" that he reported that he was following to solve the task. Following this "way" he went on producing verbally, two more number bonds (appearing in bold). When Leo was asked how he found these number bonds so quickly he said:

L: 7, 6, (shows these as second addends and then goes on saying): 5, 4, 3, 2 and then 1.
I: And how does this line go? (shows column of first addends).
L: 1, 2, 3, 4, 5, 6, 7.
I: Right. How do you know that you can do it in this way?
L: (…)
I: Has anybody told you that you can do it like that?
L: (shakes his head to say 'no').
I: Did you find it yourself?
L: (nods 'yes').
I: Very good. Why do you prefer doing it like that?
L: It's so quicker... I know.

Leo did not provide a clear and complete report of the method he followed to produce those number bonds. Instead, he uttered the numbers that he had already used, or was going to use as second and first addends. The numbers he uttered were following a specific order. Leo appeared to have a specific plan which allowed him to anticipate the numbers which were going to be used as first and second addends up to the completion of the task. Leo reported his preference to use this 'plan' by saying that he knew it was very quick.

2.3 Leo went on to the second part of the solution process. He explained that he produced the second set of number bonds by applying the method of 'swapping'. The interviewer asked:

[3+5]
[4+4]
[6+2]
[5+3]
[0+8]
[7+1]

I: Don't you want to do it any more like… 1, 2, 3…? (shows these as first addends top to bottom)
L: (shakes his head to say 'no').
I: Why not?
L: I can't think of any more going like that.
I: What do you mean you can't think of any more going like that?
L: (…)

For some unspecified reason, at a particular point of the solution process, Leo considered that he could not apply the 'ordering' method anymore. The speculation, is that he realised that the number bond he was going to produce exactly after the last number bond produced following the order, would be one with numbers that had been previously used. Therefore, Leo proceeded to the application of 'swapping'. This speculation was examined subsequently.

Third session

3.1 <u>*Number 11 was given as the target number in the 'card' task.*</u>

[0+11]
[10+1]
[9+2]
[8+3]
[7+4]
[6+5]
[1+10]
[5+6]
[4+7]
[3+8]
[2+9]
[11+0]

Leo completed the task by applying 'ordering' in the first part of the solution process and 'swapping' in the second. After producing the first number bond Leo said:

L: Now I'm going downwards 0 and 11, 10 and 1, 9 and 2.
I: And then? Can you go on?
L: 8 and 3, 7 and 4... I can't think of any more.

Leo went on with the written production of number bonds. After writing down the last number bond Leo took another incomplete card in front of him and spent some time looking at the number combinations he had produced. After a few minutes the interviewer asked:

I: Are you trying to find another one?
L: Yeah (looks at the cards again. At one point, puts 5 fingers up. Looks at the cards.
I: What are you doing to find another one?
L: I don't know. I just think of some numbers.
I: You can tell me whenever you think that there are no more.
L: I don't think there are more.

Even though Leo was not immediately aware of his success, at the end of this dialogue with the interviewer, he appeared to be more certain, in comparison with previous runs, that all the possible number bonds had been produced. However, he did not justify his certainty.

3.2 After the completion of the task the interviewer asked Leo:

I: Which number bonds did you find by going "downwards"?
L: (looks at the cards for a while) this and this and this (shows the first addend in the first six number bonds).
I: Up to this card (shows 6+5) you said you were going downwards. How do these numbers go? (shows second addends from 1 up to 5).
L: (Looks at the cards. He does not answer).
I: These are going down (shows first addends from 10 up to 6). How do these go? (shows second numbers from 1 up to 5).
L: (Looks at the cards for quite some time frowning) Upwards?
I: They are going upwards. Did you want them to go that way?
L: (hesitates) Yeah.
I: How come this line goes down (shows first addends from 10 up to 6), and this line goes up? (shows second addends from 1 up to 5).
L: (Thinks for a while). I don't know.
I: How come if you go down one line and up the other you always find the right sums? Can you figure out?
L: (shakes his head to say 'no').

Leo had no difficulty in characterising the order he followed for the first addends. However, he gave the impression that he only noticed the order that the second addends were following when the interviewer asked him. Leo seemed to be surprised after noticing the order of numbers in the column of second addends ("Upwards?"). Moreover, Leo did not appear to be in position to explain why each column of numbers (i.e. first and second addends) was following a different kind of order. Leo said that he was aware of the order that the second addends were following and wanted them to be in this order, but his overt behaviour (hesitation, surprise) casts doubt on this.

3.3 Aiming at obtaining an explanation regarding the Leo's shift to the application of 'swapping', the interviewer asked:

I: Up to this one (shows the 6+5 and moves the rest of the cards further below), you said you were going downwards (shows first numbers from the top up to 6).
L: But then I stopped doing it (shows the rest of the cards).
I: Right. Why did you stop doing it?
L: Because I couldn't think of any more going downwards (shows first numbers from the top up to 6). I just couldn't.

Once again, Leo did not provide a clear explanation of his choice to shift to the application of 'swapping' after a certain point in the solution process. A close look at the produced number bonds and the point at which the shift took place seems to uphold the previous speculation. According to that speculation, Leo shifted to the application of 'swapping' when all the possible numbers have been used, either as first or second addend, in the part of the solution process where 'ordering' was employed.

Fourth session

4.1 <u>*Number 14 was given as a target number in the 'card' task.*</u>

[14+0]
[13+1]
[12+2]
[11+3]
[10+4]
[9+5]
[8+6]
[7+7]
[6+8]
[5+9]
[4+10]
[3+11]
[2+12]
[1+13]
[0+14]

Leo applied 'ordering' in the first part of the solution process. The explanations he gave were the same as in previous runs where the 'ordering' had been applied. A change was introduced in the second part of the solution process. Leo reported the use of 'swapping'. However this time, he changed around one by one the number bonds produced in the first part, starting from the top and considering them in order. This had as a result the continuation, of the descending and ascending sequence of numbers in the columns of first and second addends correspondingly. Leo did not observe the new pattern of numbers in the second part of the solution process.

4.2 <u>*Number 8 was given as target number in the 'domino' task.*</u>

[4, 4]
[0, 8]
[3, 5]
[2, 6]
[1, 7]
[7, 1]
[8, 0]
[6, 2]
[5, 3]

Leo produced the first two combinations in a few seconds. To complete the next three dominoes, Leo drew a number of dots in the left part of the domino, and then counted on using his fingers to find the number of dots he needed to draw in the right part of the domino. After completing the last domino of this set, Leo focused his gaze on each of the combinations he had produced and then produced a new set of combinations. Leo explained that he found these ways of completing the dominoes by "swapping around" the previous ones. After writing down the last combination [5, 3] Leo said right away:

L: I don't think there are any more ways.
I: What makes you think that there are no more?
L: I've found them all.
I: How do you know?
L: Just looking at them.

Leo appeared to be certain of the completion of the task right away after completing the last domino. However, in this task as in the 'card' task he did not justify his certainty and did not give any indication of any particular method of checking.

Leo approached the *'balances on paper'* in the same way that he approached the 'domino'. In the *'balance on cardboard'* task Leo stopped the solution process after producing the number combinations which are shown below. The interviewer asked:

7 ↔ 0
6 ↔ 1
3 ↔ 4
5 ↔ 2

I: Is it difficult now?
L: (nods 'yes').
I: How come?
L: (...)

Leo did not explain why he found it difficult to go on. It is assumed that his difficulty stems from the fact that in order to apply the 'swapping' method in the second part, he needed to have the previous combinations as visible reference in order to change them around. Because of the lack of this visible reference he could not apply 'swapping' and thus found it difficult to complete the task.

Fifth session

5.1 Number 12 was given as target number in the 'card' task.

[12+0]
[11+1]
[10+2]
[9+3]

When Leo started swapping around the previously produced number bonds the interviewer asked him:

I: Why are you starting swapping around now?
L: Because I can't think of anymore... different numbers.

For the first time Leo explained that he was starting changing around the previously produced number bonds after realising that he could not use other numbers, *different* from the ones he had already used. Leo's explanation upheld previous speculation about the rationale of this shift. After the completion of the task the interviewer draw Leo's attention to the two columns of numbers, that is the first and second addends, and asked:

I: Why do you think this happens? How come these numbers are going down and these are going up (shows first and second numbers correspondingly in the number bonds produced by 'ordering')?
L: (He takes quite some time to think. He frowns) I don't know.

Leo did not provide an explanation of the rationale that underlay the 'ordering' strategy at any other instance until the end of his participation in the sessions.

5.2 Leo was asked to produce some number bonds to make 100 on a sheet of A4 paper. The interviewer wrote the first number bond: [0+100]. The aim was to see whether Leo, in contrast with his usual practice, would be able to apply the 'ordering' strategy following an ascending order in the column of first addends and a descending order in the column of second addends. This is what Leo wrote down rapidly:

1	99
2	98
3	97

He first made a column of numbers from 1 to 3. Next to this column he created another one, with numbers going from 99 to 97. At that point the interviewer interrupted him and reminded him to put the 'add' sign. Leo went on adding two more numbers in the column of first addends and two more numbers in the column of second numbers. Then, he put the 'add' sign in between. Two more number bonds were generated.

4+96
5+95

Leo had no apparent difficulty to apply the 'ordering' strategy by following a different type of order in each of the two columns of addends. It is quite interesting the fact that Leo went on to the rapid production of two columns of numbers instead of complete number bonds. The omission of the 'add' sign and production of two columns of numbers is considered as a sign of Leo's primary focus on that aspect of the strategy which had as an outcome the ordered disposition of the numbers involved.

APPENDIX 7.6

THE CASE OF JAMES

Itinerary of changes observed

First session

1.1 *Number 8 was given as target number in the 'card' task.*

[8+0]
[0+8]
[7+1]
[1+7]
[6+2]
[2+6]
[3+5]
[5+3]
[4+4]

James produced number bonds in pairs. For the production of the second number bond of each pair (shown in italics) James used and reported the method of 'instant swapping'. For the production of each of the [8+0] and [7+1] number bonds James's explanation was: "I thought of it in my head". The very short solution times allows to assume the use of factual/declarative knowledge. James produced the [6+2] and [3+5] number bonds by a two-step process. He wrote down the first addend. Then he put his hands under the table and counted before writing down the second addend. After the production of the [6+2] the interviewer asked:

I: How did you think of 6? Why did you choose 6?
J: (shrugs his shoulders) I don't know.
I: Did anything help you think of 6?
J: No.
I: Ok. And how did you figure out the 2?
J: I used my fingers.

James provided the same type of explanation for the production of the [3+5]. He did not provide any explanation of why or how he chose and specified which number to use as first addend. Before writing down the last number bond [4+4] James took quite some time to think looking at the cards with the already completed number sentences on.

J: (looks at the cards for some time) 4... (then takes his eyes away from the cards. Takes some more time and then says): 4 and 4.
[4+4]
I: Very good! How did you do that?
J: Because it's the double.
I: What gave you the idea to use, 4 in the first place? (shows 4 as first addend).
J: (...)

It is believed that this number bond was produced by a two-step process as well. James focused on the completed cards for a rather long time and then uttered the number 4. It is believed that James' explanation "it's the double" refers to the production of the second addend. After having specified 4 as first addend it was James' factual/declarative knowledge which constituted the source for the retrieval of the appropriate second addend.

1.2 After writing down the last number bond James took another card from the pile of cards with incomplete number sentences on. He focused on the column of complete number bonds for a rather long time. The interviewer asked:

I: Is there another way?
J: I don't know (shrugs his shoulders).
I: How can we be sure if there is another way or not?
J: Uhm... check?
I: Ok. What can we check?
J: (shrugs his shoulders).
I: So what are you going to do now? Is there another one or do you think that these are all the possible ways?
J: They are all.
I: How do you know that you haven't missed one?
J: Uhm... I don't know.

Even though James suggested that checking was the way to know whether a number bond was missing, he did not seem to know how he could check. The impression that he gave to the interviewer was that he was not willing at the time to do anything else in order to check.

Second session

2.1 *Number 9 was given as the target number.*

[5+4]
[4+5]
[3+6]
[6+3]
[7+2]
[2+7]
[1+8]
[8+1]
[9+0]
[0+9]

The number bonds were produced in pairs once again. James reported the use of 'instant swapping' for the production of the second number bond of each pair (shown in italics). In this run an interesting change occurred. James introduced the 'deriving' method and used it for the production of the first number bond of each pair throughout the solution process.

The first number bond that James wrote down was [5+4]. To the interviewer's question how he knew that, James replied: "I remembered 5 add 5 and I took 1 away". James' reply indicates that the [5+4] number bond was produced as a derived fact. After the production of the [3+6] number bond the interviewer asked:

I: How did you know this?
J: Uhm... Because I took away 1 again.
I: From which number?
J: That one (shows 4 as first addend in the previous number bond).
I: So you took 1 away from 4 and you wrote down 3 (shows 3 as first number in the last number bond). How did you figure out the 6?
J: Because I added 1 more.
I: To which number?
J: Number 5 (shows 5 as second addend in the previous number bond).

James explained how he derived the new number bond using as reference the number combination he had produced just before. James applied this method consistently, up to the end of the solution process, to produce the first number bond of each pair.

2.2 After James having produced the [7+2] number bond the interviewer asked him:

I: What did you do for this one?
J: Um... (looks at the cards). Um... I added 1 and I took away 1. I... I added 1 on 6 (shows 6 as first addend and then 7 as first addend in the number bond he just produced).
I: Which number did you take away from?
J: That one (shows 3 in the [6+3] number bond).

This time, in order to derive the new number bond James added 1 on the first addend and took away 1 from the second addend of the number bond he was using as reference. James seemed to acknowledge the reversibility of the method and use the 'deriving' method quite flexibly. He was always using as reference the last produced number bond, which was a number bond produced by 'instant swapping'. He would either take away from, or add on the first addend, and add on, or take away from the second addend of that number bond as appropriately.

2.3 When James was asked *why* the 'deriving' worked the way it did, he did not provide any explanation as the following extract shows.

I: That's a very good way. Has anybody explained this to you?
J: No.
I: Did you find it yourself?
J: (nods 'yes').
I: Why do you think it works that way? Why do you take away 1 (shows position of second numbers
J: I don't know.
I: What if you add 1 here and 1 there (shows position of first and second addends)?
J: (shakes his head to say 'no').
I: Why not? Why do you have to take away and add?
J: (shrugs his shoulders).

At that point, James did not appear to be in position to give verbal explanations on the *why* and *how* the method of 'deriving' number bonds was working and was successful.

2.4 After writing down the last number bond James took another card from the pile of cards with incomplete number sentences on. He focused on the number bonds he had produced for a long time. The interviewer asked:

I: How can we find which one is missing, if there is one missing?
J: I don't know.
I: Ok. It's up to you. I want you to tell me whether you think that these are all or whether you think that may be there is one more but you can't find it now.
J: I think that's all of them.
I: That's all of them. You are right. But how do you know? How can we be sure?
J: I just guessed.

James' prolonged focus on the produced number bonds can be an indication of the fact that he was using the already produced number combinations as reference to help him find a number bond that might be missing. However, he did not report this verbally. He did not report any method that he might have used for checking if the task had been completed.

Third session

3.1 Number 15 was given as target number in the 'card' task.

[10+5]
[5+10]
[15+0]
[0+15]
[1+14]
[14+1]
[13+2]
[2+13]
[12+3]
[3+12]
[11+4]
[4+11]

This time James started the solution process by writing down a number bond that he recalled directly from memory. He uttered "10 add 5" right after the interviewer told him that he was going to work with 15 as the target number. He wrote down the first number bond and swapped it around. Then he started applying the strategy of producing number bonds in pairs starting with the one that included 0. For the [1+14] he applied an 'add/take away' combination, for the [13+2] he applied a 'take away/add' combination of operations. It is speculated that given that the second addend of each pair was used as reference for the application of the 'deriving' method, this adjustment and shift from one combination to the other was taking place so that a number bond would not be repeated. However James never explained this verbally. From that point onwards, however, he continued the production of the first number bond of each pair using as reference the first number bond of the last pair.

3.2 After writing down the [4+11], that is the last number bond of the set shown above, James went on to the production of another set of number bonds which is shown below.

[10+5]
[5+10]
[9+6]
[6+9]
[7+8]
[8+7]

The first two number bonds of this set were number bonds that James had already produced at the beginning of the solution process. The interviewer asked:

I: How did you find these two?
J: I took away and added.
I: Which numbers did you take away and added on?
J: (shows [11+4] above).
I: Do you think that you may have done this number bond before? (shows lastly produced [10+5]).
J: (shakes his head to say 'no').
I: Are you sure?
J: (nods 'yes').

It is believed that this repetition of number bonds was due to the fact that the production of the [10+5] and [5+10] at the beginning was supported by a different piece and representation of knowledge than the production of the same number bonds later. The strategy of producing the number bonds in pairs was applied right from the beginning of the solution process. However, the [10+5] at the top of the column (i.e. the first number bond of the first pair produced) was a number bond that James recalled from memory as fact when the interviewer told him that 15 was the target number this time. Subsequently, he swapped this number bond around. From that point onwards, the strategy of producing the number bonds in pairs involved the 'deriving' method and 'swapping'. The production of the [10+5] as first number bond of the last pair was thus supported by a different knowledge representation than the [10+5] at the beginning of the solution process. In this case two different knowledge representations had been involved in the application of James' strategy. James dismissed and subsequently denied the repetition.

Fourth session

4.1 *Number 29 was given as target number in the 'card' task.*

[29+0]
[0+29]
[28+1]
[1+28]
[27+2]
[2+27]

After James having produced the [2+27] number bond, the interviewer took a card from the pile with incomplete number sentence on and put number 25 in the position of first addend: **[25+...]**. She asked James to complete this number bond to make 29. James looked at the last in the column number bond and then wrote down the second addend of the incomplete number bond: [25+**4**]. The interviewer asked:

I: Very good! How did you find this?
J: (...)
I: Did you count?
J: No.
I: What did you do?
J: I looked at that one (shows the 27 as first addend).
I: And then what did you do?
J: (...)
I: How did this help you? (shows the [27+2]).
J: (...)

It did not seem to be so easy for James to report and explain explicitly the procedure he followed to solve the problem. He denied the use of counting and indicated the number bond [27+2] as the one that her referred to, in order to figure out the missing addend. This indication agrees with his overt behaviour (i.e. focus on the last number bond at the bottom of the column). It is speculated that James used this number bond in order to derive the combination he was working on. This type of violation was introduced by the interviewer later on in the context of the same run. James overcame the difficulty easily. However, he did not provide explanations that were more clear or detailed than the one he gave previously.

Fifth session

5.1 *Number 16 was given as target number in the 'card' task.*

[16+0]
[0+16]
[1+15]
[15+1]
[14+2]
[2+14]

After the production of the first six number bonds, the interviewer took a card with an incomplete number sentence on and put number 12 in the position of the first addend **[12+...]**. Following James' usual strategy, a number combination with number 12 as one of its addends would constitute one step further down the solution process. The interviewer asked James to find the second addend in order to make 16. James focused on the last completed number bonds in the column and after a while he wrote down the second addend and correctly completed the number bond [12+4]. To the interviewer's question how he found the missing addend, James showed the [14+2] number bond but did not give any explanation. Also, he denied the use of counting. The completed number bond [12+4] was put at the bottom of the column with complete number bonds. The interviewer gave James another incomplete number bond to complete: **[...+11]**. Again, James focused his gaze on the bottom of the column of complete number bonds and after a few seconds he completed the number bond [5+11]. The interviewer asked:

I: How did you find this?
J: (shrugs his shoulders).
I: Come on, I am sure you can tell me. Did you count?
J: No.
I: Then how did you find it so quickly?
J: Um...
I: How did you think of 5?
J: I added on 4 (shows 4 as second addend in the previous number bond).
I: How did you know that you had to add on 4 to find the right number?
J: (...)
I: Did you look at anything that helped you know what to do to find the 5?
J: (...)

This time the interviewer insisted a bit more in her attempt to elicit a verbal explanation from James. For the first time James reported how he was overcoming this type of violations, i.e. by the use of the 'deriving' method. Even though he reported the arithmetical operation that he applied to derive the missing number and indicated the number he used as reference, he did not provide an explanation of the rationale behind this method. He did not justify the choice of arithmetical operation he used.

5.2 The completed number bond [5+11] was put at the bottom of the column of cards. James produced another number bond [11+5] by 'swapping'. The produced number combinations are shown on the left. Subsequently, the interviewer gave James another incomplete number bond to complete: **[9+...]**. James clearly focused on the last card in the column and after a few seconds he completed the number bond: [9+7] and put it at the bottom of the column. The interviewer asked him:

[16+0]
[0+16]
[1+15]
[15+1]
[14+2]
[2+14]
[12+4]
[5+11]
[11+5]

I: How did you find it?
J: Last time it was 11...add 5. I... remembered 5 and I counted on 2, to... 7.
I: Very good. Why did you do this? Why did you count on 2?
J: I don't know.
I: How did you know that you had to count on (shows the 5) just 2 to find the number that was missing? (shows 7)
J: I don't know.

This time again, James reported that he used the 'deriving' method to figure out the missing addend. He explained that he added 2 on 5, i.e. the addend of the number bond he used as reference. James adjusted the operations involved in the 'deriving' method to the situation. The number bond he needed to complete consisted of numbers with difference of 2 in relation to the number bond he wanted to use as reference. Instead of adding 1 and taking away 1, James reported that he added 2 on the number he used as reference and thus specified the missing addend correctly. Even though the operation James applied seemed to be accessible for verbal report, the same did not happen with the rationale that sustained the choice and implementation of this operation.

APPENDIX 8.2

THE CASE OF GRACE

Itinerary of changes observed

First session

1.1 *Number 9 was given as the target number in the 'card' task.*

[0+9]
[1+8]
[9+0]
[7+2]
[6+3]
[5+4]
[4+5]
[3+6]
[2+5]
[1+8]

Grace used a different method for the production of each of the sets of number bonds which are shown on the left. When asked to explain how she found each of the first two number combinations Grace answered: "I just thought of it". The commendably short time that Grace needed to produce the first two number bonds suggests that these number bonds are likely to be the product of fact retrieval. For the production of the number bonds which are shown in italics, Grace indicated which number bonds she "changed around" to produce the new ones. Before writing down the [7+2] Grace moved her head rhythmically for a few seconds. The interviewer asked:

I: Can you tell me how you figured this out? (shows the [7+2]).
G: I thought of 7 and I counted on to see when it makes 9… and then I found out that it was 2.
I: So, did you count in your head?
G: (nods 'yes').
I: How did you think of the 7?
G: Because uhm… I just thought of it.

Grace used the same 'two-step' process for the production of the [6+3] and [5+4]. She wrote down the first addend and counted on to figure out the second. Grace did not provide any explanation of how she chose the first addend of these number bonds.

1.2 The last number bond that Grace produced was [1+8]. She wrote down number 1 and then counted on using her fingers. The interviewer asked:

I: Are you sure that you haven't done that already? (shows the 1+8).
G: (…)
I: 1 add 8. Are you sure that you haven't written that already?
G: (nods 'yes' without looking at the already completed cards at all).

In fact, [1+8] was the very first number bond that Grace produced at the beginning of the solution process. When the same number bond was produced at the beginning of the solution process, the use of declarative/factual knowledge had provided a basis of interpretation. The [1+8] of the end concluded a sequence of number bonds that Grace produced by a 'two-step' process. On the contrary, the [1+8] of the beginning together with the [0+9], were the first number bonds which were retrieved as integrated addition facts from Grace's long term memory. There are two different representations that underlie the production of the [1+8] of the beginning and the [1+8] of the end. The repetition that Grace did not realise and subsequent denied, is considered as the result of these two different representations.

1.3 After producing the last number bond Grace took another card from the pile of cards with incomplete number sentences on. She kept trying to find another number bond. The interviewer asked:

I: Are you trying to find another way to make 9?
G: (nods 'yes').
I: Are you sure that there is still something missing?
G: I don't know. Maybe. (she looks at the already completed number bonds).
I: Are you doing something to check?
G: Not really.

It was shown that Grace was not aware of the completion of the task. She remained focused on the number bonds that she had already produced. After a while, the interviewer let her know that the task had been completed.

Second session

2.1 *<u>7 was given as the target number in the 'card' task.</u>*

[0+7]
[3+4]
[5+2]
[1+6]

[6+1]
[2+5]
[4+3]
[7+0]

Grace produced the number bonds which are shown on the left by applying the mixture of methods that she had used in the previous session. Grace produced the second set of number bonds rapidly. She reported the use of 'swapping' and indicated the number bond that she "changed around" to produce each of the number bonds of this set. It seemed that the solution process in this run was separated in two parts.

After producing the [7+0], Grace took another card from the pile of cards with incomplete number sentences on. She spent quite some time thinking and looking at all the number bonds that she had already produced. Although Grace had successfully completed the task she did not seem to be aware of her success. The interviewer prompted her to check whether all the possible number bonds had been produced. However, at that point, Grace did not employ any mechanism for checking the solution that she gave to the task.

2.2 *<u>Number 10 was given as a target number in the 'card' task.</u>*

Grace completed the task after producing the number bonds which are shown below.

After the production of the [7+3] number bond the interviewer asked:

I: Well done. Is there something that helped you think of 7?

[5+5]
[9+1]
[0+10]
[6+4]
[8+2]
[7+3]
[1+9]
[10+0]
[2+8]
[4+6]
[3+7]

G: (looks at the cards) Uhm.. Uhm... I looked at all the numbers and then I saw that there was no 7 there, so I thought of it.
I: I see. Which numbers did you look at?
G: Those... and those, and those... (using her index and middle finger of her right hand shows both addends in each of the number bonds).

This was the first time that Grace explained the choice of the number she used as first addend. She explained that she looked at both addends of each of the previous number bonds to find which number had not been used yet. She used counting to figure out the second addend.

Grace reported the use of 'swapping' for the production of the following set of five number bonds. She made her method explicit by indicating the number bond that she swapped around in order to produce each of the second set of number bonds. Grace did not give any answer to the interviewer's questions regarding the shift to the application of 'swapping' at that particular moment of the solution process and her certainty that the task was complete.

Third session

3.1 <u>*9 was given as a target number in the 'card' task.*</u>

[3+6]
[8+1]
[7+2]
[0+9]
[4+5]
[6+3]
[1+8]
[2+7]
[9+0]
[5+4]

In the first part of the solution process Grace produced number bonds by the 'two-step' process. The use of counting was reported for the production of the second addend of all the number bonds in the first part of the solution process except from the [0+9] for which Grace explained that she "knew it". Again, the criterion on the basis of which Grace specified the first addend of each new number bond was not reported. It should be noted that in this run, in the second part of the solution process, the number bonds that were produced in the first part of the solution process were swapped around in a systematic way: i.e. one by one starting from the first one at the top. After writing down the last number bond Grace focused at the first set of number bonds produced for a while. Then she said:

G: I can't think of any more.
I: Do you think that you have finished?
G: (nods 'yes' in an assertive way).
I: Right. Can you tell me what you did to be sure that you have finished?
G: (...)
I: You are right. These are all. But can you please explain to me how we can be sure that these are all the possible ways?
G: Uhm... Before I did those ones (shows the last five number bonds) I... I looked at... I looked at all of the numbers (shows with her pencil first five cards) and in order, to see if there.... if I was having up to 9.
I: Oh! You looked at all the numbers in order?
G: (nods 'yes').
I: Which numbers did you look at in order?

G: Those (shows with the index and middle finger of her right hand both addends in each of the five first number bonds top to bottom).
I: Did you look at both numbers?
G: Yes.
I: Both of them at the same time?
G: No.
I: Can you explain to me how you did it?
G: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 (shows these numbers in the first five number bonds).
I: Oh, right. So you looked at those (shows first five number bonds). You had up to 9. What did you do afterwards?
G: Uhm... I just needed to swap them around.

Grace formulated and provided a justification of success based on the features and potential of the employed strategy. Moreover, Grace's justification introduced a new feature into her problem solving approach: the notion of order as a tool for checking. It was after this process of checking that she proceeded into the second part of the strategy where she swapped around the already produced number bonds.

Fourth session

4.1 *Number 12 was given as a target number in the 'card' task.*

The following extract illustrates the newly introduced method that Grace used for the production of only one number bond: [2+10].

I: What did you do for this one?
G: I thought of the number 2 and then... I looked at that one (shows the 11+1 card) and then I thought of the number before 11 and then I knew that it was 10 so I wrote 10.

[6+6]
[11+1]
[0+12]
[5+7]
[2+10]
[3+9]
[4+8]
[1+11]
[12+0]
[7+5]
[10+2]
[9+3]
[8+4]

I: Very good! Did you look at 11 and then you knew that it was 10 the number you needed here? (shows 10 as second addend in the 2+10 number bond).
G: (nods 'yes').
I: How come?
G: (...)
I: How did you think of the 2 first?
G: Uhm... I looked at those ones (shows with index and middle finger of her right hand both numbers in the cards that she had completed before the last one, i.e. the 2+10) and so... uhm... I saw that I hadn't used it.
I: You said that you looked at 11. How did you know that you needed to look at 11 to help you find the 10?
G: (...)

The mechanism that underlay the production of the first addend was clearly reported. Grace gave only a partial indication of the mechanism that led to the completion of the [2+10] number bond. She did not provide an explanation of the rationale on the basis of which she used the second addend of a previous number bond (namely 11) as reference in order to complete the number bond that she was working on. The following number bonds of the first part were produced by Grace's usual method; i.e. specification of the number to be used as first addend and use of counting to figure out the adequate second addend.

4.2 *Number 11 was given as a target number in the 'card' task.*

[0+11]
[1+10]
[2+9]
[3+8]
[4+7]
[5+6]
[11+0]
[10+1]
[9+2]
[8+3]
[7+4]
[6+5]

In this run the number bonds produced in the first part of the solution process appeared to be organised in a sequential order. Furthermore, in the first part of the solution process, Grace did not give any indication of use of counting. Also, Grace's solution times for the number bonds produced in this first part were considerably shorter in comparison to her solution times in the same part of the solution process in previous runs. This time, (from the [1+10] onwards) Grace produced each new number bond, after focusing her gaze on the last completed number bond for a short moment. After the production of the [4+7] number bond, the interviewer asked:

I: You wrote down this number bond so quickly! How did you do it?
G: Uhm... I thought of the number 4 and then... (shows with her pencil the 8 in the 3+8 card)... and then I.... I... looked at that one (shows 8) to see... (...) to see which number was... it was 8 and then I... I wrote 7.
I: Very good! So, did the 8 here help you to figure out the 7?
G: (nods 'yes')
I: How?
G: (...)
I: How did you know that it had to be 7 there? How did the 8 help you?
G: (...)
I: What about the 4? What made you think of 4?
G: (...)

Grace at that point did not explicitly explain the rationale that supported her method of completing a new number bond using as reference the addends of the previously produced number combination. The consistent application of this method had as a consequence a new, ordered organisation of the number bonds in the first part of the solution process.

Fifth session

5.1 *Number 15 was given as the target number in the 'card' task.*

After the production of the first three number bonds, the interviewer asked Grace:

[0+15]
[1+14]
[2+13]

I: Very good! How did you know that it was 13? (shows the [2+13]).
G: (she looks at the cards for some time) I looked at that (shows the 1 as first number) and.... (she looks at the 2+13) and... 1 more than 1 was 2 so I decided to put that down and... (she moves her finger on the 14 as second number) and then... uhm... and then uhm.... (she keeps her finger on the 14 and thinks for some time) and then I thought of...... (she looks at the 13 as second addend in the number bond in front of her) of 1 less than 14 and it was 13 (she shows the 13 in the 2+13 card).
I: Very Good! Can you tell me how did you know that you had to do 1 less than 14?
G: Because that... (she shows the 2 in the 2+13 card) Because 2 was 1 more than 1.

For the first time Grace gave a clear report of the operations involved in the 'deriving' method. The 'deriving' method was applied for the production of all the number bonds in the first part of the solution process. That is all the number bonds except from the first one which constituted the beginning of the sequence of number bonds and the production of which was supported by the knowledge of the rule 0+N=N).

5.2 After the production of the first six number bonds, the interviewer took a card from the

[0+15]
[1+14]
[2+13]
[3+12]
[4+11]
[5+10]

pile of cards with incomplete number sentences on, and put number 7 in the position of first addend: [7+...=15]. The interviewer asked Grace to find the missing second addend. Grace focused on the last completed number bond (i.e. [5+10]). After a short while she wrote down the second addend and completed the number bond.

I: How did you find it so quickly?
G: (puts her finger on the 5 as first addend, then moves it further below and then a little bit further below) Uhmm... 7 is 1 more than 6 (shows exactly below 5 as first addend) and uhm.. there is 5 there.. and 1 less than 10 is.... (moves her finger exactly below the 10 as second addend), 1 less than 10 is... 9 and ... 7 was there (shows further down below the 5) and then I put 8 there.
I: Well done!

Grace dealt with the incomplete number bond as this being part of the sequence of number bonds that she was producing. She reconstructed the violated sequence by picturing the missing number bond-step of the solution process. On the basis of the '1 more/1 less' rationale, Grace found the second addend by using as reference the addends of the missing, intermediate number bond [6+9]. With the same approach Grace overcame successfully all the violations of this type that the interviewer introduced.

5.3 The interviewer asked Grace if she thought that she could find all the possible number

[1+99]
[2+98]
[3+97]
[4+96]

bonds for big numbers like 100 for example. Grace quickly produced the number bonds which are shown on the left on an A4 piece of paper.

The interviewer interrupted the solution process and asked:

I: Ok. Say that you go on like this. Do you think that you can find them all in this way?
G: (nods 'yes' in an assertive way).
I: How do you know that you can find them all? What are you going to do to find them all?
G: Uhmm... (shows first addends top to bottom) I am going in order and then I'll do the changing.

Grace applied her '1 more-1 less' method of deriving number bonds from previous ones which resulted in the ordered arrangement of number bonds. She reported her plan to apply the 'two-part' strategy. She justified her certainty that she could produce all the possible number bonds on the basis of the rationale and potential of her 'two-part' strategy.

5.4 When number 29 was given as a target number in the 'card' task, the first number bond

[28+1]
[27+2]
[26+3]
[25+4]

was given by the interviewer: [28+1]. Grace was asked to produce the rest of the number bonds resulting in 29. Grace usually started with the number bond in which 0 was the first addend. Therefore, she was following an ascending order in the column of first addends and a descending order in the column of second addends. The [28+1] number bond was given again as a violation of Grace's usual practice in applying her 'deriving' method. Grace flexibly adapted her method to the violation and produced the number bonds which are shown above by taking away 1 in the column of first addends, and adding 1 in the column of second addends.

APPENDIX 8.3

THE CASE OF HAZEL

Itinerary of changes observed

First session

1.1 *Number 6 was given as target number in the 'card' task.*

[3+3]
[2+4]
[1+5]
[6+0]
[0+6]
[4+2]
[5+1]

The explanation that Hazel gave for the production of the first number bond was that she "knew the double". The [3+3] is thus considered as the product of Hazel's declarative/factual knowledge. Hazel produced the number bonds which are shown in italics in two steps: she wrote down the first addend and then she counted on with fingers to find the adequate second addend. The following extract gives an example of how she explained the production of one of these number bonds: [1+5].

H: Uhm... I thought of 1 and then I counted on 5 and that makes 6.
I: Right. So what gave you the idea to use 1?
H: I just thought of 1.

Hazel did not give any particular explanation regarding the way by which she was specifying the first addend in each new number bond. Hazel produced the following two number bonds [6+0] and [0+6] very quickly. After writing down 6 as first addend, she said: "Then you only need 0 here." It is considered that Hazel called upon her declarative knowledge to complete this number bond. She reported the use of 'swapping' for the production of the [0+6].

1.2 After writing down the last number bond, Hazel took quite some time to think staying focused on her hands again that she was keeping, in a fist, in front of her. She did not say anything or do anything for a long time. The interviewer asked her:

I: Can you find another way? Can you do one more?
H: (thinks for a short while without looking at the cards. She shakes her head to say 'no').
I: No? How do you know?
H: Because I have done 1 and 2 and 3 and 4 and 5 and 6 and 0 (while saying that, she does not look at the cards).
I: Are you sure? What do you look at to know?
H: Because I count them all the way up to 6 and then... if I... if I find out that there is a missing number... but I know I have done all of them.
I: What do you mean you count them? Which numbers do you count?
H: All these numbers (shows the number bonds that she has produced).
I: How do you know that you have used all the possible numbers up to 6? Did you check?
H: (nods 'yes').
I: How? What did you check?
H: All the numbers (shows the number bonds that she has produced).
I: Which numbers?
H: All the numbers here (shows all the number bonds).

Hazel appeared to be aware of the completion of the task. She justified her certainty saying that she had used all the possible numbers up to 6. In the context of this explanation, Hazel considered the numbers she had used, in order: i.e. from 1 up to 6 (she mentioned the use of 0 at the end). Hazel did not make clear which numbers exactly she looked at to check, and which numbers she considered in order to be sure that there was no number missing.

1.3 Numbers 7, 8 and 9 were also given as target numbers in the 'card' task. In those runs Hazel's problem solving approach did not change. In the case of 8 as target number, Hazel justified her success saying: "1 is there, 2 is there, 3 is there, 4 is there, 5 is there, 6 is there, 7 is there, 8 is there, and 0 is there". Hazel showed each of these numbers in the column of first addends, in the order that she uttered them. She applied this way of checking in the following run (target number 9).

Second session

2.1 <u>*Number 8 was given again as target number in the 'card' task.*</u>

[4+4]
[1+8]
[2+6]
[3+5]
[4+4]
[5+3]
[6+2]
[7+1]
[8+0]
[0+8]

Except from the first number bond, the 'double', which was recalled as fact from memory right after the interviewer asked Hazel to find ways to make 8, all the other number bonds were created in two steps: Hazel specified and wrote down the first addend. In the case of the number combinations that included 0, Hazel used her declarative/factual knowledge to specify the adequate second addend. In the case of all the other number bonds, Hazel found the second addend by counting on with fingers. After the production of the first two number bonds and when Hazel had written down number 3 as first addend, the interviewer asked:

I: What made you think of 3 now?
H: Because it comes after 2.
I: Which number are you going to use as first number next?
H: 4.
I: So how do you choose which number to use each time?
H: I think of the number line. I did 1, 2, 3 and then 4 (she doesn't say more. She starts counting on with fingers to find the second addend) [3+5].

For the first time, Hazel explained that she was choosing the number to use as first addend by following the number line. She anticipated the first addend of the number combination that she would create next.

2.2 Hazel went on to the production of the [4+4] number bond. This was the first number bond that she had produced at the very beginning of the solution process. After the repetition of this number bond the interviewer asked:

I: Haven't you done this before?

H: No (she says that with certainty and without checking the previously produced number bonds. She takes a new card from the pile of cards with incomplete number sentences on, and goes on with the production of number bonds).

It is believed that the two number bonds were created on the basis of a different knowledge representation. The [4+4] at the beginning of the solution process was recalled as the 'double' for 8; i.e. as a number fact from memory. On the other hand, when Hazel produced the [4+4] later again she applied the 'two-step' method: i.e. specification of the first addend following the number line and counting on with fingers for the specification of the second addend. It is believed that the production of the same number bond from two different knowledge representations resulted to the repetition and Hazel's initial certainty that there was no repeated number combination.

2.3 *Number 9 was given as target number next.*

[8+1]
[7+2]
[6+3]
[5+4]
[4+5]
[3+6]
[2+7]
[1+8]
[0+9]
[9+0]

Hazel started approaching the task having a specific strategy. She applied again her method of specifying the first addend of each new number bond by following a specific order. However, the order that she followed this time was a descending order. Hazel explained the choice of first addend in each new number bond:

I: How do you choose the first number each time?
H: Well, 8 add 1 is the closest... and then... I just... counted down to find all these numbers (shows first addends).
I: What do you mean the "closest"?
H: Closest to 9.
I: Why do you start with the closest?
H: Because it's the easiest.

The choice of the "closest" as the "easiest" number bond can be explained if one thinks that Hazel was specifying the second addend of each number bond by counting on with fingers. Starting with 8 as first addend was easy because it allowed Hazel to count less to find the adequate second addend. After writing down the last number bond Hazel put it at the top of the column number bonds and said:

H: Now they are all in order.
I: What is in order?
H: 9, 8, 7, 6, 5, 4, 3, 2, 1, 0. Finished. (shows each these numbers as first addend).
I: Why do you want them in order?
H: I like it.

Once again Hazel was certain of the completion of the task and justified it by showing that all the numbers had been used as first addends.

Third session

3.1 *Number 13 was given as target number in the 'card' task.*

Hazel completed the task after producing the number bonds that are shown below.

[12+1]
[13+0]
[11+2]
[10+3]
[9+4]
[8+5]
[7+6]
[6+7]
[5+8]
[4+9]
[3+10]
[2+11]
[1+12]
[0+13]

For the production of the first number bond Hazel explained that, again, she started with the "closest". For the [13+0] she said: "This is easy, I know". It is believed that for the production of both these number bonds Hazel used her declarative knowledge. In the case of the [11+2] Hazel used her fingers to count before writing down the second addend. The interviewer asked:

I: How did you find this? (shows the [11+2].
H: Well... I looked at 13 and I've done 12 already so I went down... 1 and I found out 11 and I counted to find out how to make 13.

According to Hazel's strategy of following a descending order for the specification of first addend in each new number bond, number 12 had to be used as first addend after 13. Even though the [12+1] had been produced by a different knowledge representation, Hazel recognised that number 12 had already been used as first addend, and explained that she made 1 step down to specify the first addend of the number bond that she would create next. After the specification of 11 as first addend, Hazel reported that she counted to find the second addend. Her overt behaviour confirms this explanation.

3.2 Hazel produced the [10+3] number bond very quickly. The interviewer asked:

I: Did you know that already?
H: No, I looked at 11 and then I looked at 10 and I looked at uhm... 10 again and then I looked at 2 and then I knew 10 is 1 less... than 11 so I needed 1 more to make 13.

Hazel reported the use of the 'deriving' method. She used the words "less" and "more" to explain the arithmetical relation on the basis of which she specified the addends of the new number bond. For the production of the next three number bonds Hazel gave the same explanation that she gave for the production of the [10+3]. For the production of the [6+7] number bond Hazel reported that she changed around the [7+6].

For the production of the rest of the number bonds up to the completion of the task, Hazel reported the use of the 'deriving' method. She provided the same explanation that she had given earlier for the production of the [10+3] number bond. To justify her certainty that the task was complete, Hazel considered the numbers she had used as first addends, in order and showed that she had used all the possible ones.

Fourth session

4.1 *Numbers 15 and 14 were given as target numbers in the 'card' task.*

When the target number was 15, Hazel produced the number bonds which are shown below. After the completion of the task, Hazel was asked to explain her strategy. She said:

H: I am going less, and less, and less (shows each of the first addends) and then I am adding 1 more, and 1 more, and 1 more (shows each of the second addends).
I: Why do you add 1 more for each of these numbers?

H: Because if it's 1 less then I do 1 more to equal 15.

[15+0]
[14+1]
[13+2]
[12+3]
[11+4]
[10+5]
[9+6]
[8+7]
[7+8]
[6+9]
[5+10]
[4+11]
[3+12]
[2+13]
[1+14]
[0+15]

I: Why do you do it like that? Going less and less, and more and more?
H: Because it's easy.

Hazel showed that she had a good understanding of how the 'deriving' strategy worked. She reported the arithmetical operations which were combined in the framework of her strategy, and most importantly, Hazel explained why the two operations, taking away and adding, had to be combined for the sum to be retained and conserved. Hazel explained her choice to apply the 'deriving' strategy by saying that she found it easy.

4.2 *Number 14 was given as the target number.*

[14+0]
[13+1]
[12+2]
[11+3]
[10+4]
[9+5]
[8+6]

Hazel applied the 'deriving' method for the production of the first eight number bonds. After the production of these number bonds, Hazel focused on the completed number sentences and went on to the production of another set of number bonds after announcing the method that she was going to use:

H: I am swapping them all around now.

[0+14]
[1+13]
[2+12]
[3+11]
[4+10]
[5+9]
[6+8]

The number bonds that Hazel produced by 'swapping' are shown on the left.

After the production of these number bonds Hazel focused on the first set of number bonds that she had produced by the 'deriving' method and said:

H: Am I up to... I am up to 8 add 6 right? (shows the last number bond that she had created by the 'deriving' method).
I: Yeah you have done 8 add 6.
H: She focuses on the [8+6] number bond and she whispers) 7 add...7.
I: Very good. How did you find it?
H: I looked at 8 (shows the 8+6 number bond).
H: Yeah, and I then I looked at 6 and then I thought of 7 was 1 less than 8 so I needed 1 more... than... 6.

After producing the second set of number bonds by swapping', Hazel went back to the first set of number combinations to find another number bond by the same method. She found the last combination of the first set and used it as reference for the production of the [7+7] number bond. It was like she thought that the production of number bonds by the 'deriving' method had not been finished. She just interrupted it to produce quickly the set of number bonds by swapping, and then she went back to the 'deriving' to complete the task. It was surprising that, after the production of the [7+7] number bond, Hazel continued the production of number bonds which are shown below.

[6+8]
[5+9]
[4+10]

After the production of the [4+10] number bond Hazel said:

H: I need that rubbed out. (shows the [4+10] number bond).
I: Why?
H: Because I have already swapped it around. (looks at the cards) I swapped around 4 add 10 and 10 add 4… so… (seems like being in difficulty, but she keeps trying to find another one. She thinks for a while without looking at the cards).
3 add 11?
I: I think you've done 3. Do you think that there may be other number bonds that you done twice?
H: (…)

Hazel realised the repetition of the three last number bonds last number bonds and she removed them from the column of cards. Even though she had completed the task, she kept looking for new number bonds. In this attempt, she started creating number bonds which resulted to the wrong sum. The interviewer prompted her to check whether she had already produced all the possible number combinations.

Hazel put all the number bonds in a column all over again but this time in order, from the [14+0] down to the [0+14]. After doing this, she uttered all the numbers that she had used as first addends top to bottom. She did the same with the numbers she had used as second addends, and then said:

H: So I've done all the ways.
I: How do you know?
H: Because I put them in order and they are going down and up and I have swapped them all around. Well, not the 7 and 7.
I: Do you need to look at both columns of numbers, or if you look only at one (shows column of second addends), say the squares or the triangles you can say if these are all?
H: No.
I: No. Do you need to look at both of them?
H: Yeah, because you might have missed some and you haven't swapped them around.

For the first time, in the framework of her explanation Hazel named the different kind of order that first and second addends were following. She seemed to realise that, for the task to be complete, each of the number bonds (except from the 'double') needed to have a corresponding number bond with different addend order. On the basis of this new realisation, Hazel said that one needed to check both columns of numbers to make sure that all the number bonds had been produced. This was surprising given that in previous sessions, she seemed to realise that checking only one column of numbers was enough for one to know whether the task was complete.

Fifth session

5.1 <u>*Number 19 was given as target number in the 'card' task.*</u>

Hazel had produced the number bonds that are shown below when the interviewer interrupted the solution process. The interviewer took a card from the pile of cards with incomplete number sentences on and put number 7 in the position of first addend: [7+...]. She asked Hazel:

[0+19]
[1+18]
[2+17]
[3+16]
[4+15]
[5+14]

I: Can you find the number that is missing, to make 19?
H: (takes some time to think looking at the already completed cards: seems like being focused on the last completed card [5+14]).
If it's.... (focuses on the cards for a short moment) If it's 6 add 13... (shows below the [5+14]).
I: Yeah...
H: 7 add 12?
I: 12! Very good. How did you find it?
H: **[7+12]** Well, I looked at 5 and then I counted down to 6 (shows below 5) 6 add... 13, and then... 7 add... and then I thought 12.

Hazel succeeded in finding the missing addend by calling upon the same rationale that sustained her 'deriving' method. Hazel found the missing addend by creating the missing, intermediate step: i.e. the number bond which would precede the one she was working on, in the framework of the sequence of number bonds that she was producing by applying the 'deriving' strategy consistently. The interviewer introduced such violations to Hazel's practice in solving the task repeatedly in following runs. In all these cases, Hazel applied the same method of creating the missing steps (some times it was more than one) in order to overcome this type of violation and complete the number bond.

APPENDIX 8.4

THE CASE OF ELSA

Itinerary of changes observed

First session

1.1 *Number 6 was given as the target number in the 'card' task.*

[3+3]
[1+5]
[2+4]
[0+6]
[6+0]
[5+1]
[4+2]

Right after the interviewer asked Elsa to find all the possible ways to make 6, Elsa said "I know one, 3 and 3". This first number bond is considered as being rapidly produced by Elsa's declarative knowledge of the 'double'. For the production of the next two number bonds Elsa put up the number of fingers that she used as first addend and then she counted on using her fingers to find the adequate second addend. Before writing down the [0+6] number bond Elsa looked at the already produced number bonds. The interviewer asked her:

I: I saw you looking at the cards. What did you look at?
El: I didn't see a zero.
I: How did you think of 6? Did you count?
El: No. Because I know I'm thinking of it. 0 and 6 makes 6. I think of it and then I write it down.

After identifying 0 as a number that was missing, Elsa completed the number combination, calling upon her factual/declarative knowledge. She reported the use of 'swapping' for the production of the last three number bonds. After writing down the last number bond, Elsa said:

El: These are all because I can't change the 3 and 3 around (shows the 3+3).
I: So, are these all the ways?
El: (nods 'yes').
I: How do you know?
El: I just looked at them (says looking at the cards) and saw if it was all right.

Elsa's certainty that the task was complete was based on the fact that she had changed around all the number bonds which could give her another number combination. This is considered as a weak justification. Changing around all the number bonds that are already produced does not necessarily show that there is no number bond missing.

1.2 *Number 14 was given as a target number next.*

[1+13]
[2+12]
[3+11]

The first three number bonds were produced rapidly. The interviewer asked:

I: How did you think of that? (shows the [3+11]).
El: Well, this one is... this one is easy.. (shows the 3+11) and I put in... all the numbers in order (shows first addends top to bottom) going that way, and the others going that way (shows second addends top to bottom).
I: I see. How do you know that if you do it like that you are going to find all the possible sums?
El: (...)

I: How come these numbers go that way and the others go that way? (shows first and second addends correspondingly).
El: (...)

Elsa explained that she was specifying the first as well as the second addend of each new number bond by following a specific order. She put the [0+14] that she produced next at the top of the column of number bonds saying: "Then I have to put that one up there".

The interviewer asked her why she put the new number bond at the top of the column and not at the bottom as she did with every new number bond. Elsa replied:

[0+14]
[1+13]
[2+12]
[3+11]

El.: "Well, because I am putting all these one numbers in order" (shows first addends top to bottom).

Both addends of the following number bond [4+10] were also produced by the 'ordering' method. Elsa did not provide any further explanation regarding the different order of numbers that she was following to specify first and second addends correspondingly.

[0+14]
[1+13]
[2+12]
[3+11]
[4+10]

[10+4]
[11+3]
[12+2]
[13+1]
[14+0]

Elsa reported the use of 'swapping for the production of a second set of number bonds. After the production of the [11+3], the interviewer asked:

I: Have you stopped now putting them in order?
El: (looking at the cards, nods 'yes').
I: How come?
El: Well, because I can't think of any more going in order.
I: No?
El: (shakes her head to say 'no')

It is not clear what made Elsa think that she could not apply the method of 'ordering' up to the completion of the task. She went on to the production of a new set of number bonds.

[5+9]
[9+5]
[6+8]
[8+6]

The [5+9] and [6+8] were produced by the same 'two-step' process that Elsa used in the previous run. The [9+5] and [8+6] number bonds were produced by 'swapping'. Elsa appeared to be aware of the completion of the task after spending quite some time looking at the number combinations she had produced. She did not provide any justification of her certainty that the task was complete.

Second session

2.1 <u>*Number 11 was given as the target number.*</u>

The solution process was separated in three parts (see further below). Elsa reported the use 'ordering' for the production of the first set of number bonds. After the production of the [4+7], the interviewer asked:

I: How did you choose 4 to write down first? Why 4 now?
El: Because that's after 3 (shows 3 as first addend).
I: I see. I saw you looking at the cards (shows the produced number bonds) and thinking for a little while before writing 7 down.
El: Yes, so that I could... go down (shows second addends top to bottom).
I: I see.

[1+10]
[2+9]
[3+8]
[4+7]
[5+6]
[10+1]
[9+2]
[8+3]
[7+4]
[6+5]
[11+0]
[0+11]

El: That one (shows first addends top to bottom) is going in order but that one is just going down (shows second numbers top to bottom). And it's going 10, 9, 8, 7.

Elsa did not give any explanation when the interviewer asked her to explain how come the first addends went in 'order' and the second addends went 'down', and the sum was always 11. Elsa reported the use of 'swapping' for the production of the second set of number bonds. The two last number bonds were those that included 0. Elsa did not give any explanations or justifications of her success.

As it was shown in the next runs with the 'card' task, the 'swapping' method was applied in a rather systematic way. Elsa's practice was to swap the number bonds in a particular order starting with the number bond at the top of the column and ending with the number bond at the bottom of the column of already produced number combinations.

Third session

3.1 <u>*Number 16 was given as the target number.*</u>

[0+16]
[1+15]
[2+14]
[3+13]
[4+12]
[5+11]
[6+10]
[7+9]
[8+8]
[16+0]
[15+1]
[14+2]
[13+3]
[12+4]
[11+5]
[10+6]
[9+7]

Elsa applied 'ordering' and 'swapping'. This time the number bonds which included 0 were integrated in the two main parts of the solution process: i.e. the set of number bonds produced by 'ordering' and the set of number bonds produced by 'swapping'. For the first time, Elsa provided a full verbal report of how each new number bond was produced by 'ordering'. The following explanation was given after the production of the [4+12]:

El: I look at the... card before the other card... and I try to find which number is next... (shows below the 4 as first addend) and what's going less (shows below the 12 as second addend).
I: Right. So, for which line of numbers are you trying to find which number is 'next'?
El: This line (shows column of first numbers).
I: Right, for which line are you trying to find numbers that are 'less'?
El: This line (shows column of second addends). It doesn't go next (shows first addends) and next (shows second addends) or less (shows first addends) and less (shows second addends), they are not going the same.
I: Why don't they go the same?
El: (...)
I: Would it be correct if they were both going 'next' or if they were both going 'less'?
El: (nods 'no').
I: Why not?
El: (looking at the cards) Because when you are writing a sum out, ehm... if you put them in... order, they will go less and more and... if you put them the wrong way around... ehm... ehm... they will be.... They wont be the same but they'll be... (... pause to think. She seems confused. Looks at C and doesn't say anything else).

Elsa appeared to be clear about the connection between 'ordering' and the increasing/decreasing relation of numbers in each column of first and second addends. At the end of the solution process Elsa justified her certainty that the task was complete as follows:

El: There is nothing missing. These are all the numbers and I have swapped them all around.

I: What do you mean "these are all the numbers"?
El: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16 (shows these as first and second addends in the first nine number bonds that she has produced).

This justification made more clear Elsa's strategy: i.e. the application of 'ordering' up to the point where all the possible numbers had been used either as first as second addends, and then the shift to the 'swapping' method.

Fourth session

4.1 <u>*Number 17 was given as target number in the 'card' task.*</u>

Before Elsa started producing number bonds, the interviewer asked:

I: How are you going to find all the possible ways for 17?
El: ...Uhm... well... I just do it in order and it goes up and down... They are going the same way (shows with her pencil top to bottom) but they are not going in the same... in the same... direction. It goes up and down (moves her pencil vertically first on the left then on the right side of the table), or down and up (again, moves her pencil vertically first on the left then on the right side of the table), and then I do the changing around.

Elsa reported her plan of combining 'ordering' and 'swapping'. She was aware of the fact that each of the two columns of addends could either follow an ascending or descending order and still the right sums would be produced. After the production of the first four number combinations which are shown below, the interviewer put number 12 in the position of first addend and asked:

[17+0]
[16+1]
[15+2]
[14+3]

I: **[*12*+...=17]** Can you find the number that is missing?
El: (Takes quite some time to think, looking at the completed cards. Then she says): 5. **[12+5]**
I: Very good. How did you find this?
El: Uhm... I looked at that one (shows the 14+3) and I thought of the one that was going to be next... and then it will be 4 there (shows exactly below the 3 as second number) and then 13 there (shows exactly below the 14 as first number). And then it will be 12 there (shows one step further down from the position where she showed that the 13 would be) and 5 there (shows one step further down from the position where she showed that the 4 would be).

To find the missing addend, Elsa treated the incomplete number bond as part of the sequence of number combinations that she was producing before the interruption, and applied the same rationale that underlay the 'ordering' strategy. She visualised and created the intermediate step: i.e. the number bond [13+4], the second addend of which helped her find the missing number, and complete the number sentence. Subsequently, Elsa was asked to complete number combinations the production of which constituted two or three steps further down the last produced number bond. In all these cases Elsa applied the same method of creating and visualising the intermediate missing number bonds.

4.2 <u>*'Balances on paper' task. Numbers 6 and 7 and 8 were given as target numbers.*</u>

Elsa approached the task in the same way in all the three runs. In the case of 7 as target number, Elsa completed the task after producing the number combinations that are shown below.

1	2	3	0	4	7	6	5
6	5	4	7	3	0	1	2

After the production of the first three combinations the interviewer asked:

I: How did you find all these ways so quickly?
El: I don't count. I'm just doing it in order.
I: What do you mean you are doing it in order?
El: I going 1, 2, 3 and then it goes 6, 5, 4.
I: Are you doing the same as with the cards?
El: (nods 'yes').
I: How come? Is this the same as the cards?
El: Yes, I am thinking of ways to make 7.

Right from the first run with the 'balances on paper' task, Elsa applied the strategy she had developed in the 'card' task. She applied the 'ordering' until all the possible numbers had been used either in the upper or lower cube of the balances. Then she changed around the combinations that she had already created to complete the task.

Fifth session

5.1 <u>*'Missing numbers' task.*</u>

9+13=22	9+11=20
..+14=22	7+...=20

Elsa completed the incomplete number sentences successfully and very quickly. In the case of the second pair of number sentences, Elsa explained:

El: (she looks at the number sentences for a moment and says): 12, 13. It's 13.
I: Exactly. How did you find it? Why 13 and not 12?
El: Because they are going in two (shows the 9 and 7 as first addends).
I: I see. So since they are going in two (shows the 9 and 7) do these have to go in two again? (shows the 11 and below this the position of the missing number).
El: (nods 'yes').
I: Why do you have to make the same number of steps here and here? (shows first and second addends).
El: Because you have to.

In a context other than the 'card' task, Elsa used as reference the arithmetical relation between the first addends of the two number sentences which resulted to the same sum, to specify the addend that was missing and complete the second number sentence. Elsa seemed to be clear about which should be the arithmetical relationship between the first addends of the two number sentences and the second addends of the two number sentences. She seemed to be certain of the necessity of this relationship for the sum to be correct but at that point she did not give further explanations.

APPENDIX 8.5

THE CASE OF ERNA

Itinerary of changes observed

First session

1.1 <u>*Number 8 was given as a target number in the 'card' task.*</u>

[7+1]
[1+7]
[0+8]
[2+6]
[3+5]
[4+4]
[5+3]
[6+2]

The number bonds which are shown in italics are considered as the product of Erna's declarative/factual knowledge. After producing each of these number bonds Erna explained that she "knew it". For the production of only two number bonds, (the [2+6] and [3+5]), Erna used the 'deriving' method, for the specification of the second addend. After the production of the [2+6] the interviewer asked:

I: How did you think of that?
E: Because… I just did less than 7 (shows the [7+1] at the top).
I: Can you explain this to me? What do you mean you did 'less than 7'?
E: Because I thought of 2 and I wrote the 2 down and then I thought… but what's the other number? So I looked at 7 and I thought 1 less than 7 is 6.
I: Very good! How did you know that the other number should be the one which is 1 less than 7?
E: Because…. (…)
I: Why did you take away 1 from 7? (shows 7 as first addend in the [7+1] number bond).
E: Because that would make 7.

Erna explained that she thought of using number 2 first. Then she found the second addend by referring to the previously produced [7+1], and by thinking of the number which was 1 less than 7. At that point, Erna did not give any further explanations about her 'deriving' method. For the production of the rest of the number bonds Erna reported the use of 'swapping'.

1.2 Erna looked at all the number sentences top to bottom. While looking at the completed cards she was unfolding one by one her fingers of her right and subsequently left hand until she had 8 fingers up. She said:

E: None is missing.
I: How do you know?
E: Because… (she shows her 8 fingers) we've done all the numbers.
I: How do you know?
E: 1, 2, 3, 4, 5, 6, 7 (she shows these numbers, in this order, as first addends) 8 (she shows 8 as second addend. Then she says right away): Oh! (she takes a new card and writes down): [8+0].

Erna justified her certainty by saying that all the possible number bonds had been used. Her focus only on the first addends can be an indication of her understanding that checking if all the possible numbers have been used either as first or second addends is enough to allow one to know whether all the possible number combinations have been produced. In her attempt to justify her success, Erna realised that one number was missing from the column of first addends and could been found only in the column of second addends: i.e. number 8. She completed the task by producing the missing number bond.

Second session

2.1 <u>*Number 9 was given as a target number in the 'card' task.*</u>

Erna wrote down the first two number bonds very quickly and explained:

E: [0+9]
[1+8]
Because I wanted to... start from 0 and then 1 and go up in number order.
I: Do you want to put the numbers in order?
E: (nods 'yes').
I: Which numbers do you put in order?
E: 0, 1 (shows these as first addends) 2, 3, 4, 5, 6, 7, 8, 9... like that (while saying these numbers keeps moving her pencil downwards, on the table, following the column of first addends).
I: Why do you want to do it like that?
E: Um... (thinks for a short moment) I just do..... Because it's... it gets easier to do it.

Erna clearly explained her strategy of using as first addends the numbers from 0 up to 9 (i.e. the 'target' number) in order, and justified the choice of this strategy by saying that this was making the solution process easier. At this point, it could be said that a strategy had been developed.

2.2 <u>*11 was given as the target number subsequently.*</u>

[0+11]
[1+10]
[2+9]
[3+8]
[4+7]
[5+6]

Erna explained again that she was choosing the numbers to use as first addends, in order. The interviewer asked her how she knew which number to put as second addend to complete each of the number combinations:

I: How do you find the second numbers? (shows column of second addends top to bottom).
E: I'm going down (moves her pencil downwards).
I: You are going "down". How come this happens? Why do these numbers go "down"?
E: (she looks at the cards for a short moment and then answers) Because... you need to make 11 so they are going down (shows first numbers *bottom to top*) (...pause for a few seconds: she keeps looking at the numbers) and they are going.... up (shows second numbers *bottom to top*).
I: Is this the same kind of order?
E: No. This is taking away so they are going down (shows second addends top to bottom) and these go 1, 2, 3, 4, 5... (shows first addends top to bottom) so they are going up.

Erna explained how she was figuring out the appropriate second addend to complete each of the number bonds. She seemed to be aware and clear about the different kind of order that the first and second addends were following, and provided a clear explanation of why this was happening: i.e. because all the number bonds needed to add up to the same sum, that is 11.

Erna completed the task after producing the number bonds shown below.

[6+5]
[7+4]
[8+3]
[9+2]
[10+1]
[11+0]

After Erna writing down the [6+5] number bond the interviewer asked:

I: How did you find that one?

E: Because 6 is after 5 (shows 5 as first addend in the [5+6] above) and 5.... (shows 5 as second addend in the number bond she just completed), I swapped that around (shows 5+6 above).(Takes a new card. For the first number looks at the last completed card at the bottom. For the second number looks at the last completed card and then focuses on the [4+7] further above).

The numbers that were used as first addends in this set of number bonds were chosen and specified following the ascending order that the first addends in the previous set of number bonds were also following. However, Erna gave a different explanation about the way in which she figured out the second addend and completed each of these number combinations. To specify the second addend in each of these number bonds Erna used as reference number combinations from the previous set which included the same addends put in different order. So, two mechanisms were now involved in the production of number bonds.

She justified her certainty by explaining that all the number bonds had been produced because all the possible numbers up to 11 had been used in both columns (i.e. first and second addends). The interviewer asked Erna if anybody had ever showed to her this way of finding number combinations. Erna replied that no one had ever showed to her how to do it and that she made it up in her head.

Third session

3.1 *Number 16 was given as target number in the 'card' task.*

[0+16]
[1+15]
[2+14]
[3+13]
[4+12]
[5+11]
[6+10]
[7+9]
[8+8]
[9+7]
[10+6]
[11+5]
[12+4]
[13+3]
[14+2]
[15+1]
[16+0]

Erna quickly completed the task after producing the number bonds which are shown below. The interviewer asked Erna to describe once again the order in which she was using the numbers in both columns (i.e. first and second addends) to complete the task. Erna explained referring first to the column of second addends:

E: It's called taking away because you are taking the numbers away (moves her hand from the right to the left to show how the number go in that case, on an imaginable number line). That's why you are going down (she shows now the column of second addends top to bottom). And that's (shows columns of first addends) because it's called adding, because you are adding more numbers. So you are going up the number line (moves her hand from the left to the right to show how the numbers go in this case, on an imaginable number line).

Erna seemed to have grasped and explicitly represented the arithmetical relation her 'ordering' strategy. This was made evident by the use of the terms 'taking away' and 'adding' and their connection to the descending and ascending order of addends, correspondingly.

3.2 *Number 12 was given as the target number next.*

The interviewer gave Erna the first number bond: [12+0]. So far, when applying the 'ordering' strategy, Erna used to start the solution process with the number bond which had 0 as first addend. Thus, she was adding in the column of first addends following an ascending order and taking away in the column of second addends following a descending order. The number bond that the interviewer gave was one in which 0 was the second addend. The intention was to see whether Erna would go on with the production of number bonds using the given number bond as the starting point, and adjust the 'ordering' strategy appropriately. Erna wrote down the first two number bonds in a few seconds.

[12+0] → given number bond
[11+1]
[10+2]

The interviewer asked Erna to explain how she found these two number bonds so quickly.

E: Because these (shows second addends) because... these... they are still going down these numbers (shows column of second addends *top to bottom*) and these numbers are still going up (shows column of first addends bottom to top). Because... uhm... because... well, you did 12 (shows 12 as first number at the top) so 12... 10 11 12 (shows first numbers bottom to top) and I am still doing this in order.
I: Ok. In what kind of order are you doing this?
E: These are going up (shows column of first addends bottom to top) and these ones are going down (shows second addends *top to bottom*).

Erna used the given number bond as a starting point and did adjust the 'ordering' strategy to produce new number combinations. However, when she was asked to describe the order that she was following this time, she appeared to be a bit confused. As in the previous session, Erna used the word "up" to describe the order of the first addends. She adjusted her gesture and showed that this time the first addends were "going up" bottom to top. However, when she described the order of the second addends, she used word "down" as in previous runs but she did not adjust her gesture appropriately. She showed that the second addends were going "down" top to bottom as happened in previous runs. With the interviewer's intervention, she recognised her mistake and corrected her gesture.

Fourth session

4.1 *Number 14 was given as target number.*

[0+14]
[1+13]
[2+12]
[3+11]
[4+10]

Erna had already produced the first five number bonds. At that point, the interviewer wrote down number 7 as first addend and asked Erna to complete the number bond: **[7+...]**. Erna looked at the last completed card in the column and completed the number bond as follows: [7+7].

I: How did you find it?
E: Because it was 3 more to 7 (shows the 4 as first addend in the last completed number bond) so I took away 3 (shows 10 as second addend in the last completed number bond).

This was the first time that Erna did not create the missing, from the sequence, steps to find the missing addend. Instead, she extended the 'add 1/take away 1' rationale which sustained the 'ordering' strategy as she usually applied it and adjusted it to the situation. She talked in terms of "adding" and "taking away", only this time, she realised that she had to add and take away 3 instead of 1. She justified the number of steps she had to take away in order to find the missing second addend by relating it to the arithmetical relation between the first addend of the number bond - reference and the given first addend of the number bond she was currently working on.

Fifth session

5.1 <u>*Big target numbers in a paper and pencil task*</u>:

To the interviewer's question whether Erna thought that she could find all the possible number combinations for any number, Erna replied "Any number is very easy". The interviewer gave Erna a piece of A4 paper and asked her to find some ways to make 100. Erna wrote down the number bonds which are shown below.

[1+99] [2+98] [3+97] [4+96] [5+95] [6+94]

Then the interviewer gave Erna an incomplete number sentence and asked her to find the missing addend to complete: [8+...=100]. Erna took a very long time to think looking at what was written on the paper. After a while she said:

E: I'm trying to take away but...
I: You are trying to take away from which number?
E: (shows the 94 as second addend).
I: If you tell me how many steps back you need to take away I'll tell you the number. Just tell me how many steps back from 94 you need to go, and I'll find the number for you.
E: (looks at what is written on the paper for a while-not long-and says): Two.
I: Two steps back from 94 is 92. Well done! [8+92]
How did you know that it was two steps back?
E: Because... I thought of 6... (shows 6 as first addend) and then 8 (shows 8 as given addend in the number bond she had to work on) so it will be... two (shows at the side of second addends).

Erna reported the way by which she intended to solve the problem, that is by applying the 'add/take away' rationale. It was shown that her difficulty in figuring out the appropriate number stemmed from the size of the target number and the numbers involved. The degree of good control that Erna had over the solution process and the strategy even in the case of a big target number such as 100 was also shown by Erna's reply to the following question:

I: Ok. Which one is going to be the last one?
E: (thinks for a while without looking at the cards) The very very last one? (asking that, moves her hand downwards on the paper).
I: The "very very last one".
E: (looks at what is written on the paper for a few seconds and says): 100 and 0.

APPENDIX 8.6

THE CASE OF SEAN

Itinerary of changes observed

First session

1.1 *Number 7 was given in the 'card' task.*

[6+1]
[5+2]
[4+3]
[3+4]
[2+5]
[1+6]
[7+0]
[0+7]

For the production of the first number bond Sean said: "I know 6 and 1 makes 7". It is believed that Sean produced this number bond calling upon his declarative/factual knowledge. The [5+2] number bond was also produced very quickly. The interviewer asked:

I: Well done! How did you think of that?
S: Because this was 6 add 1... take one away (shows 5 as first addend)...and that's 2 (shows 2 as second addend).

Sean gave the same kind of explanation for the production of the [4+3] number bond. He talked aloud while producing the [3+4]:

S: Take away 1 of that (shows 4 as first addend), makes 3, and then 4. *He writes down)*: **[3+4].**

After writing down the [3+4], Sean said:

S: So, I've done the other way.
I: Can you explain this to me? What did you do?
S: Cause that was 4 (shows 4 as second addend) and that was 3 (shows 3 as first addend) and that was 4 (shows 4 as first addend in the previous number bond), and that was 3 (shows 3 as second addendum in the previous number bond).

After this realisation, Sean went on with the quick production of the next two number bonds (shown in italics). He reported the use of 'swapping'.

1.2 After writing down the [1+6], Sean took some time to think looking at the column of cards:

S: 1, 2, 3, 4, 5, 6 (shows first addends bottom to top. He looks at the cards for some more time). And that goes 1, 2, 3, 4, 5, 6 (shows second addends top to bottom).
I: Why do you think it goes like that?
S: Because I found... You know it's 6 add 1 (shows first card at the top) then take away 1 equals 5 add 2, (shows second card from the top) and take away 1 equals 4 add 3, and take away 1 equals 3 add 4, take away 1 equals 2 add 5, take away 1 equals 1 add 6.
I: Did you take away to make this number bond, the [1+6]?
S: Yes.

Sean was aware of the order that first and second addends were following. He explained the order of numbers by relating it with the application of the 'deriving' method for the production of number bonds even though the 'deriving' method was not consistently applied all along the solution process. He had reported the use of 'swapping' for the production of two number bonds (shown in italics).

1.3 After this interruption for explanations, the interviewer asked Sean whether he could find more number bonds. Sean looked at the completed number bonds again and after a short moment he said:

S: Oh! [7+0].
I: How did you think of that?
S: I didn't see 7. (Then writes down right away): [0+7]. I can't think of anymore.
I: Are these all?
S: Yeah.
I: How do you know?
S: Cause 1, 2.... 0, 1, 2, 3, 4, 5, 6, 7 (shows first addends in this order). These are all.

After identifying 7 as the number that was missing, he completed the number bond using his declarative/factual knowledge. He changed the addend order of the [7+0] and produced the last number bond [0+7]. He justified his belief that the task was complete by showing that he had used all the numbers. To show this, he considered the numbers he had used as first addends in order.

1.4 *Number 9 was given as target number in the 'card' task next.*

[8+1]
[7+2]
[6+3]
[5+4]
[4+5]
[3+6]
[2+7]
[1+8]
[0+9]
[9+0]

Sean applied the 'deriving'/ordering' strategy all along the solution process. Again, he did not explain in detail, nor reported the operation that he was carrying out for the specification of the second addend. After producing the last number bond [9+0] Sean explained that he swapped the [0+9] around. It is noteworthy that Sean put the [9+0] number bond at the top of the column. While putting the card at the top, Sean said without being asked:

S: I'm doing that in order.
I: Why do you do it in order? Is there a reason for that?
S: It's better. It's easy, because if you have to make 9, what you have to do is 9 add nothing (shows first card at the top) and take way 1 (shows 8 as first addend), and another 1 (shows 7 as first addend).
I: How do you find the second numbers?
S: You take away 1 so it has to be 2 there and 3, 4, 5, 6, 7, 8, 9 shows these as second addends top to bottom).

Sean justified the use of his strategy. However, once again, the interviewer did not elicit a clear explanation regarding the production of the second addend in each new number bond.

Second session

2.1 *Number 11 was given as target number in the 'card' task.*

Sean applied the 'ordering' strategy all along the solution process.

[11+0]
[10+1]
[9+2]
[8+3]
[7+4]
[6+5]
[5+6]
[4+7]
[3+8]
[2+9]
[1+10]
[0+11]

After the production of the [2+9] number bond, the interviewer asked:

I: How did you find that?
S: Take away 1 makes 2 add 9.
I: Which number did you take away from?
S: (shows 3 as first addend).
I: How did you find that 9 had to be the second number? (shows 9 as second addend).
S: It's after 8.
I: How did you know that you had to put the number which is after 8?
S: Cause take away 1 makes 2 and then it goes 7, 8, 9.
I: Right. Before writing the [2+9] down, I saw you looking something over there (shows further up the column of cards). What did you look at?
S: I changed it around (shows [9+2]).
I: So, how did you find this (shows the [2+9])? Did you change around the 9 add 2, or did you "take away 1"? (shows the [3+8]).
S: Both.

Sean's answer ("both") is considered as an indication of his awareness that for each number bond produced by the 'deriving' strategy, there was a corresponding number bond with different addend order. For the first time Sean explained the process on the basis of which he was specifying the second addend. It is quite interesting that, for the production of the first addend Sean was talking about the arithmetical operation that he carried out: i.e. taking away 1 from the previous first addend. On the other hand, for the production of the second addends, Sean used the term "after". He did not mention the arithmetical operation (i.e. adding 1) that he was practically carrying out while thinking of the number which was "after" the number/reference. The use of different vocabulary does not imply necessarily the existence of different representation. At that point, there was no indication that by using the term "after", Sean was not aware that this implied the operation of adding.

2.2 *Number 12 was given as target number next.*

Sean announced the completion of the task right after the production of the last number bond. The interviewer asked:

I: What makes you be so sure that these are all?
S: Because if you want to make 1, is only 1. If you want to make 2, is only 2. If you want to make 3 is only 3.
I: So now that you want to make 12...
S: Is 12 there.
I: 12 of what? Number bonds?
S: (nods 'yes').
I: Are you sure?
S: Yes (counts the number bonds. He finds them to be 13).
I: 13! How come they are 13?
S: I don't know (then almost immediately he says): Because that's a 0 (shows 0 as first addend at the last number bond, at the bottom).
I: Oh right! Is this why there are 13 number bonds and not 12?
S: Yeah.
I: So you've got 13 number bonds here. Are these all?
S: Yeah.
I: How do you know?
S: Because it's up to 12 and 0.

Sean attempted to justify his certainty that the task was complete based on the number of number bonds that he had produced in total. Sean appeared to believe that the number of all the possible number bonds that one could produce for a specific target number was equivalent to the target number. Sean counted the number bonds he had produced. They were 13: i.e. 1 more than the target. He provided the explanation himself. Sean connected this new knowledge regarding the number of possible number combinations with the fact that all the numbers between 0 and 12 had been used: a fact that made him be sure that the task was complete. Sean generalised this new knowledge to the following runs with the 'card' task and to bigger target numbers.

Third session

3.1 *Number 19 was given as target number.*

The first number bond that Sean produced was [19+0]. The interviewer asked:

I: Which is going to be the last number bond down here? (shows at the bottom of the table).
S: 0 add 19.
I: What if you started with 0 add 19? Could you do that?
S: (nods 'yes'. He produces the [0+19] and puts it at the top of the table. The interviewer puts aside the [19+0]).

[0+19]
[1+18]
[2+17]
[3+16]
[4+15]

Sean went on with the production of the number bonds that are shown on the left. It was shown that Sean was in position to apply the 'ordering' strategy in the reverse: i.e. following an ascending order in the column of first addends and a descending order in the column of second addends. After the production of the [4+15] number bond, the interviewer asked Sean:

I: How did you find the 15?
S: (takes some time to think looking at the number bonds). It goes higher and lower (shows the columns of first and second addends correspondingly).
I: Why does it go like that? Higher and lower.
S: Cause that's 0 add 19 and if you want to make 19 you have to go 1 and 1 and 1 (shows first addends) and then take away 1 makes 18, 17, 16, 15.
I: Why do you take away 1?
S: Cause if it's 1 higher you need 1 less.
I: Why is that?
S: To make 19.

Sean described the order that first and second addends were following in this run using the terms "higher" and "lower". It was shown that Sean had a good understanding of the fact that the two operations that he was carrying out in each column of addends were interrelated. The need for that was to retain the sum.

Fourth session

4.1 *19 was given as target number in the 'card' task.*

After the production of the number combinations which are shown below, the interviewer interrupted the solution process and asked Sean to complete the number sentence:

[19+0]
[18+1]
[17+2]
[16+3]

[14+...=19]. Sean put the card with the incomplete number sentence below the [16+3] and wrote down the missing second addend: [14+**5**=19]. The interviewer asked:

I: How did you figure that out so quickly?
S: Because 16 add 3, take away 2, 14 add 5.
I: Which number did you take away from?
S: 16.
I: How did you find that 5 was the missing number?
S: Because if it's 2 less then you need 2 more.

To find the missing addend, Sean treated the incomplete number sentence as part of the process of producing number bonds to make 19. Sean generalised and adjusted the fundamental idea of 1 less/1 more that sustained the 'deriving' method. He recognised that the given addend 14 was 2 less than the addend of the number bond/reference and he added 2 more on the second addend of the number bond/reference to find the missing addend and complete the number sentence. Sean was the only child who overcame this type of violation without the need to create the intermediate, missing steps: i.e. the number bonds that he would have created before the production of the [14+5] number bond. Such violations of the sequence of produced number bonds were introduced again by the interviewer. Sean was asked to complete number bonds which, following the sequence of number bonds would be produced 3, 4 or 5 steps further down the last produced number bond. In all the cases Sean adjusted the 'less/more' rationale accordingly.

4.2 *Number 7 was given as target number in the 'domino' task next.*

[**0**, 7]
[**1**, 6]
[**2**, 5]
[**3**, 4]
[**4**, 3]
[**5**, 2]
[**6**, 1]
[**7**, 0]

Sean completed the task in the following way: he drew dots only in the left part of each domino, top to bottom, starting with 0 dots and finishing with 7. Then he drew dots in the right part of each domino, bottom to top, starting with 0 dots and finishing with 7 dots. The interviewer asked Sean:

I: This time you were so quick! How did you do it?
S: I did it in order.
I: Are you sure that you have 7 dots in total in each domino?
S: (nods 'yes').
I: How do you know that you put the dots in each domino like that you are going to have 7 dots in each domino?
S: I know.
I: How did you know that you can do it like that?
S: I just know.

Sean solved the task efficiently and very quickly. He reported that he did it in order. This time though he did not create a sum in each step, he just produced in two phases and in a rather automatic way the arrangement of dots that he knew that would result to the right sums. Sean did not give any explanations. However, there is no evidence on the basis of which one could assume that Sean was not aware of the operations which resulted practically in the specific ordered arrangement.

Fifth session

5.1 *Paper and pencil task-target number 100.*

The interviewer asked:

I: Do you think that in this way (meaning the production of number bonds in order) you can find all the number combinations for any number?
S: Yeah.
I: Ok. Choose a really big number.
S: 100.
I: How many ways you think there are?
S: Uhm... 101.

The interviewer gave Sean a piece of A4 paper and asked him to find some ways to make 100. Sean produced the number bonds which are shown below.

[100+0] [90+10] [80+20] [70+30] [60+40]

The first number bond is considered as the product of Sean's declarative knowledge. The following three number bonds were produced very quickly. After the production of the [60+40] number bond the interviewer asked:

I: How did you do it so quickly?
S: I thought in my head.
I: Did you count in your head?
S: Actually I didn't. I took away **10** (shows the 60) and I put **10** more on that (shows the 40).
I: Why do you count in 10s and not in 1?
S: Because it's quicker to go up to 100.
I: If you count in 10s are you going to use all the possible numbers? Are you going to find all the possible number bonds?
S: (...)
I: In order to find all the 101 ways, do you need to count in 10s or in 1?
S: In 1 but it's too many. I don't want to do anymore.

Sean applied his 'ordering' strategy to produce number bonds which resulted to 100. Surprisingly, he applied the 'ordering' strategy counting in 10's. This shows that Sean had a good understanding of this specific regularity regarding counting in 1s and counting in 10s across the system of decimal numeration. He appeared to be aware of the fact that counting in 10s and applying the 'deriving' strategy by taking away 10 and adding 10, would not lead to the production of all the possible number bonds that make 100. Sean appreciated that counting in 10s was a quick way to do this. He seemed to be tired and unwilling to do anymore.

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