UNIVERSITY OF SOUTHAMPTON

On Field Theory from Gravity Duals

by

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Dedicated to my parents

'What we need is imagination. We need to find a new view of the world.'

- *Richard Feynman*

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ABSTRACT

FACULTY OF SCIENCE

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Doctor of Philosophy

On Field Theory from Gravity Duals

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We review strings and branes in general, and then introduce the AdS/CFT Correspondence. The original work begins with an examination of the geometry for $\mathcal{N}=4$ on moduli space. We find a neat prescription for the encoding of the gravity solution in terms of the dual gauge theory. We next try to extend this to the $\mathcal{N}=2^*$ scenario, but encounter problems due to the gravity solution giving unexpected renormalization. Then we consider the correspondence applied to two field theories off their moduli spaces. We encounter unexpected problems with $\mathcal{N}=2^*$ again, but are successful in interpreting the Leigh-Strassler case. Finally, we apply the AdS/CFT correspondence to examine $\mathcal{N}=4$ super Yang-Mills at finite $U(1)_R$ charge density, using the supergravity backgrounds around spinning D3 branes. We complete the interpretation of the field theory duals of these backgrounds by interpreting the non-supersymmetric naked singularity class of the solutions. We find that these naked spinning D-brane distributions describe the coulomb branch at finite density. At a critical density a phase transition occurs to a spinning black brane representing the deconfined phase where the higgs vevs have evaporated. We also extend our analysis to include finite temperature. We perform a free energy calculation to determine the phase diagram of the coulomb branch at finite density and temperature.

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Preface

Original work appears in Chapters 4, 5, 6 and 7. All work was done in collaboration with Dr. Nick Evans. James Babington collaborated with the research in Chapters 4 and 5. Dr. Clifford Johnson and Dr. Michela Petrini were collaborators on the Leigh-Strassler part of Chapter 6.

The work in Chapters 4 and 5 has been previously published in:

J. Babington, N. Evans and J. Hockings, "Secrets of the metric in N = 4 and N = 2* geometries," JHEP **0107**, 034 (2001) [arXiv:hep-th/0105235].

The research content of Chapter 6 has not been previously published.

The work in Chapter 7 has been previously published in:

N. Evans and J. Hockings, "N = 4 super Yang Mills at finite density: The naked truth," JHEP **0207**, 070 (2002) [arXiv:hep-th/0205082].

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show it very often!

Chapter 1

Introduction

1.1 Motivation

All of fundamental physics, except gravity, is described by gauge field theory. It is difficult to do calculations in these field theories not using perturbation theory [3]. Perturbative calculation is only possible when the strength, or coupling, of the force being considered is small. This is not a practical problem for electromagnetism and the weak nuclear force, since their couplings are small at low energies. The problem really lies in dealing with Quantum Chromodynamics (QCD) [4, 3]. QCD is the gauge theory believed to describe the strong nuclear force, that is, the force which binds together atomic nuclei. The coupling in QCD gets weaker at shorter distances, which is known as asymptotic freedom. In this regime a few of the lowest order terms in perturbation theory form a good approximation. At longer range, that is, lower energy however, the QCD coupling become large, so perturbation theory is not applicable - calculations cannot be made at real world energies for the force which binds together atomic nuclei. There are, of course, ways to do non-perturbative field theory, e.g. lattice field theory, but to find a better way of understanding gauge theories at strong coupling would certainly be a very significant achievement.

One of the reasons that the AdS/CFT correspondence [15, 16, 17] has become such a vigourous area of research is its potential for providing such understanding of gauge theories. Because the AdS/CFT correspondence is a duality between a strongly coupled gauge theory and a weakly coupled gravity background, one could hope to find a gravity theory dual to the gauge theory one is interested in, perform a perturbative calculation on the gravity side and then interpret the result in terms of the field theory. As yet, no-one has managed to find such a gravity dual for a gauge theory that is truely thought to describe the physical world. The work in this thesis is to add to the effort to find such phenomenological theories. The AdS/CFT correspondence is not a proven duality, so any evidence for its truth in the particular cases examined in this thesis also adds further weight to the credibility of this extremely popular theory.

1.2 Style and Outline

Some effort has been made to make this work clearer to the less expert reader. There is supposed to be a reasonable amount of more general introductory material to put the more technical or specialist work in context, plus references to further background introductory material. There has been an attempt to not be too terse, and not to leave out too many 'obvious' but important points.

The following two chapters set out background material to the research. Chapter 2 introduces strings, branes, supergravity and some related field theory. Chapter 3 gives an introductory account of the AdS/CFT correspondence, with the essential preliminaries on $\mathcal{N} = 4$ super Yang-Mills theory and anti-de-Sitter space. The first original research comes in Chapter 4, with an examination of the moduli space of $\mathcal{N} = 4$ SYM. We interpret the dual supergravity solution in terms of this field theory, finding a simple prescription for the encoding of the field theory operators in the supergravity solution. In Chapter 5 we try to extend the work of Chapter 4 to the so-called $\mathcal{N} = 2^*$ scenario. Our interpretations are hampered by unexpected renormalization appearing from the supergravity solution.

Chapter 6 covers our attempts to interpret the off moduli space regions of two theories. The first of these is the $\mathcal{N} = 2^*$ case, where we again had unexpected problems. We were unable to satisfactorily interpret the supergravity dual in terms of the field theory. The second case was the Leigh-Strassler theory. Here we were able to write the supergravity solution in terms of the field theory, as expected.

The research content is completed in Chapter 7, where we apply the AdS/CFT correspondence to examine $\mathcal{N} = 4$ super Yang-Mills at finite $U(1)_R$ charge density, using the supergravity backgrounds around spinning D3 branes. We complete the interpretation of the field theory duals of these backgrounds by interpreting the non-supersymmetric naked singularity class of the solutions. We found that these naked spinning D-brane distributions describe the coulomb branch at finite density. At a critical density a phase transition occurs to a spinning black brane representing the deconfined phase where the higgs vevs have evaporated. We also extended our analysis to include finite temperature. We performed a free energy calculation to determine the phase diagram of the coulomb branch at finite density and temperature.

The thesis is completed with a conclusions section, and appendices on conformal symmetry, $\mathcal{N} = 4$ super Yang-Mills multiplets and the $\mathcal{N} = 2^*$ field theory.

Chapter 2

Strings and Branes

This chapter covers the basics of strings and branes, with some emphasis on the areas most relevant to later chapters of this thesis. The standard basic texts on string theory are those by Green, Schwarz and Witten [1] and Polchinski [2]. The first text dedicated to D-branes has recently been written by Johnson [7]. Reviews of the AdS/CFT correspondence often contain good introductory material on D-branes, for example [13] and [51]. These books and reviews should be consulted by the reader seeking more detail, and for more complete references to original papers.

String theory is a very compelling idea. All of the previous wonderful successes in particle theory [4, 3] have come through field theories dealing in infinitessimal point particles, so the great, but simple, step of attributing a spatial extent to the fundamental 'particle' of the theory is in itself very exciting to the eye of the uninitiated. But string theory is, of course, such a popular subject for more than just being a nice idea. It holds enormous promise for solving great problems in physics, and as it becomes more and more studied it is proving to have incredibly rich and deep structure. Indeed advances in research seem to be leading beyond just string theory; maybe the fundamental theory is really 'M-theory' [10], and perhaps branes should be viewed as the fundamental objects, not strings.

The greatest promise of string theory is that it can provide a true unification of the four fundamental forces seen in Nature. It is *relatively* easy to come up with a theory which unifies the quantum field theories of the electroweak and strong forces [3]. The problem lies with gravity [5] and in particular the non-renormalizability of quantum gravity [2]. The cause of this problem can be seen from considering graviton corrections to the free propagation of two particles. The ratio of the two-graviton exchange to the zero-graviton exchange amplitudes diverges in the high energy limit, and the divergence becomes more severe the more gravitons are added. These divergences occur when all the graviton vertices are coincident.

Now it may be that this divergence arises from the expansion that is done in the powers of the interaction and everything would be fine if an exact method of calculation could be found, but there is one other alternative and it is a historically fruitful line of attack. Quantum gravity may be the low energy limit of some higher theory, so then the problem is simply that one is trying to apply a theory to an energy region where it is not valid. The full higher theory can provide a softening of the divergence by spreading out the interaction region.

However, some very considerable problems arise in trying to 'smear out' a quantum field theory interaction. In particular, if we want to preserve Lorentz invariance then any spreading out of the interaction in space will mean spreading it in time as well, giving a violation of causality or unitarity. In fact such problems are so considerable that no one has managed, in a consistent way, to cut off the divergences of a quantum field theory gravitational interaction by such a smearing. Instead it has been realized that to have an extended object as the elementary particle is a very natural way to have a 'spread out' interaction. String theory is in fact the only consistent quantum theory of gravity known. This provides great incentive to study string theory but, as we will see, the discovery of the AdS/CFT correspondence [15, 16, 17] means string theory has become of great interest (again) to those interested in strong coupling and confinement in field theory [3].

2.1 String Theory

A point particle traces out a world-line as it moves in spacetime. If the invariant interval of this world-line is ds and the particle has mass m, then its action is

$$S = -m \int ds \tag{2.1}$$

- proportional to the invariant length of the world-line. A string on the other hand sweeps out a two dimensional world-sheet as it moves in spacetime. The point particle case then suggests that the string action should be proportional to the area of the world-sheet.

We parametrize the world-sheet with the two parameters τ, which runs from $-\infty$ to $+\infty$, and σ, which runs from 0 to l. We then use the functions $X^\mu(\tau, \sigma)$, $\mu = 0, ..., D-1$, to map the world-sheet into the D-dimensional physical spacetime. We need the induced metric h_{ab} which is given by

$$h_{ab} = \partial_a X^\mu \partial_b X_\mu \tag{2.2}$$

where the indices a, b run over (τ,σ), and we use here a $(- + ++)$ signature metric, $\eta_{\mu\nu}$. The area of the world-sheet is given by the square root of the determinant of this metric, so we arrive at the Nambu-Goto action,

$$S_{NG} = -T \int_M d\tau d\sigma (-\det\, h_{ab})^{1/2} \tag{2.3}$$

where M denotes the world-sheet and T is a constant of proportionality which makes the action dimensionless, by having dimension $(\text{mass})^2$. T is actually the tension of the string. The Regge slope parameter, α', is often used instead of the tension. These two parameters are related by

$$T = \frac{1}{2\pi\alpha'} \tag{2.4}$$

The square root of a determinant form of the Nambu-Goto action tends to make it awkward to use. One can introduce an independent world-sheet metric, $\gamma_{ab}(\tau,\sigma)$, to find the more practically useful Polyakov action,

$$S_P[X,\gamma] = -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma (-\gamma)^{1/2}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu \tag{2.5}$$

where $\gamma = \det\gamma_{ab}$. S_p has the following three symmetries:

1. D-dimensional Poincaré invariance:

$$\begin{aligned} X'^\mu(\tau,\sigma) &= \Lambda^\mu_\nu X^\nu(\tau,\sigma) + a^\mu \\ \gamma'_{ab}(\tau,\sigma) &= \gamma_{ab}(\tau,\sigma) \end{aligned} \tag{2.6}$$

2. General coordinate transformation, or diffeomorphism (diff) invariance:

$$\begin{aligned} X'^\mu(\tau',\sigma') &= X^\mu(\tau,\sigma) \\ \frac{\partial\sigma'^c}{\partial\sigma^a}\frac{\partial\sigma'^c}{\partial\sigma^a}\gamma'_{cd}(\tau',\sigma') &= \gamma_{ab}(\tau,\sigma) \end{aligned} \tag{2.7}$$

for some new coordinates $\sigma'^a(\tau,\sigma)$

3. Two-dimensional Weyl invariance:

$$\begin{aligned} X'^\mu(\tau,\sigma) &= X^\mu(\tau,\sigma) \\ \gamma'_{ab}(\tau,\sigma) &= \exp(2\omega(\tau,\sigma))\gamma_{ab}(\tau,\sigma) \end{aligned} \tag{2.8}$$

for arbitrary $\omega(\tau,\sigma)$.

The energy-momentum tensor is defined via a variation of the action as

$$\begin{aligned} T^{ab}(\tau,\sigma) &= -4\pi(-\gamma)^{-1/2}\frac{\delta}{\delta\gamma_{ab}}S_P \\ &= -\frac{1}{\alpha'}(\partial^a X^\mu \partial^b X_\mu - \frac{1}{2}\gamma^{ab}\partial_c X^\mu \partial^c X_\mu) \end{aligned} \tag{2.9}$$

Diff invariance requires this to be conserved, i.e. $\nabla_a T^{ab} = 0$. Also, the Weyl invariance of S_P means that

$$\gamma_{ab}\frac{\delta}{\delta\gamma_{ab}}S_P = 0 \qquad \Rightarrow \qquad T^a_a = 0 \tag{2.10}$$

The equation of motion from the variation of the action with respect to γ_{ab} is

$$T_{ab} = 0 \tag{2.11}$$

The variation with respect to X^μ gives

$$\begin{aligned} \delta S_P &= \frac{1}{2\pi\alpha'}\int_{-\infty}^{\infty} d\tau \int_0^l d\sigma(-\gamma)^{1/2}\delta X^\mu \nabla^2 X_\mu \\ &\quad -\frac{1}{2\pi\alpha'}\int_{-\infty}^{\infty} d\tau(-\gamma)^{1/2}\delta X^\mu \partial^\sigma X_\mu|_{\sigma=0}^{\sigma=l} \end{aligned} \tag{2.12}$$

The first term gives the wave equation

$$\partial_a[(-\gamma)^{1/2}\gamma^{ab}\partial_b X^\mu] = (-\gamma)^{1/2}\nabla^2 X^\mu = 0 \tag{2.13}$$

There are two, and only two, ways in which the second, surface term can be made to vanish if we want D-dimensional Poincaré invariance and the equations of motion to hold. The first of these are *Neumann* boundary conditions on X^μ,

$$\partial^\sigma X^\mu(\tau,0) = \partial^\sigma X^\mu(\tau,l) = 0 \tag{2.14}$$

or

$$n^a \partial_a X_\mu = 0 \text{ on } \partial M \tag{2.15}$$

where n^a is normal to the boundary ∂M. These describe the ends of an open string which can move freely in spacetime. The second are *periodic* boundary conditions,

$$X^\mu(\tau, l) = X^\mu(\tau, 0), \qquad \partial^\sigma X^\mu(\tau, l) = \partial^\sigma X^\mu(\tau, 0),$$
$$\gamma_{ab}(\tau, l) = \gamma_{ab}(\tau, 0) \tag{2.16}$$

These describe strings without a boundary. The endpoints are joined to form a closed loop; the closed string.

We will now fix the gauge to light-cone gauge. This will allow us to find the critical dimension, but hides the covariance of the theory. We begin by defining light-cone coordinates in spacetime:

$$x^\pm = 2^{-1/2}(x^0 \pm x^1), \qquad x^i, i = 2, ..., D-1 \tag{2.17}$$

The lower case x^μ are the spacetime coordinates and $X^\mu(\sigma, \tau)$ are the associated world-sheet fields. The metric is then

$$ds^2 = -2dx^+ dx^- + dx^{i^2} \tag{2.18}$$

Now setting the world-sheet parameter τ at each point of the world-sheet to be equal to the spacetime coordinate x^+ means that x^+ plays the role of time and p^- is the energy. The longitudinal variables x^- and p^+ and the transverse x^i and p^i are the spatial coordinates and momenta.

Now consider first the open string case, with the coordinate region $-\infty < \tau < \infty$ and $0 \leq \sigma \leq l$. We choose the gauge by choosing light-cone gauge for the world-sheet time coordinate and imposing two conditions on the metric:

$$X^+ = \tau, \tag{2.19}$$

$$\partial_\sigma \gamma_{\sigma\sigma} = 0, \tag{2.20}$$

$$\mathrm{det}\gamma_{ab} = -1 \tag{2.21}$$

The gauge condition (2.21) can be solved for $\gamma_{\tau\tau}(\tau,\sigma)$, and since $\gamma_{\sigma\sigma}$ is independent of σ, the independent degrees of freedom of the metric are $\gamma_{\sigma\sigma}(\tau)$ and $\gamma_{\sigma\tau}(\tau,\sigma)$. The inverse metric is then given by

$$\begin{bmatrix} \gamma^{\tau\tau} & \gamma^{\tau\sigma} \\ \gamma^{\sigma\tau} & \gamma^{\sigma\sigma} \end{bmatrix} = \begin{bmatrix} -\gamma_{\sigma\sigma}(\tau) & \gamma_{\tau\sigma}(\tau,\sigma) \\ \gamma_{\tau\sigma}(\tau,\sigma) & \gamma_{\sigma\sigma}^{-1}(1-\gamma_{\tau\sigma}^2(\tau,\sigma)) \end{bmatrix} \tag{2.22}$$

The Polyakov lagrangian becomes

$$L = -\frac{1}{4\pi\alpha'}\int_0^l d\sigma \Big[\gamma_{\sigma\sigma}(2\partial_\tau X^- - \partial_\tau X^i \partial_\tau X^i) \tag{2.23}$$

$$-2\gamma_{\sigma\tau}(\partial_\sigma Y^- - \partial_\tau X^i\partial_\sigma X^i) + \gamma_{\sigma\sigma}^{-1}(1-\gamma_{\tau\sigma}^2)\partial_\sigma X^i \partial_\sigma X^i\Big] \tag{2.24}$$

where

$$x^-(\tau) = \frac{1}{l}\int_0^l d\sigma X^-(\tau,\sigma), \tag{2.25}$$

$$Y^-(\tau,\sigma) = X^-(\tau,\sigma)x^-(\tau) \tag{2.26}$$

$x^-(\tau)$ is the mean value of X^- at a given τ. $Y^-(\tau,\sigma)$ has a mean value of zero. It is nondynamical since since it does not appear in any terms with time derivatives. It is a Lagrange multiplier which constrains $\partial_\sigma\gamma_{\tau\sigma}$ to vanish. The open string boundary condition (2.14) becomes in this gauge

$$\gamma_{\tau\sigma}\partial_\tau X^\mu - \gamma_{\tau\tau}\partial_\sigma X^\mu = 0 \qquad \text{at } \sigma = 0, l \tag{2.27}$$

For $\mu = +$ we have

$$\gamma_{\tau\sigma} = 0 \qquad \text{at } \sigma = 0, l \tag{2.28}$$

and $\gamma_{\tau\sigma}$ vanishes everywhere since $\partial_\sigma \gamma_{\tau\sigma}$. For $\mu = i$ the boundary condition is

$$\partial_\sigma X^i = 0 \qquad \text{at } \sigma = 0, l \tag{2.29}$$

Taking the lagrange multiplier Y^- into account and imposing the gauge conditions, the system reduces to the variables $x^-(\tau)$, $\gamma_{\sigma\sigma}(\tau)$ and fields $X^i(\tau,\sigma)$ with lagrangian

$$L = -\frac{l}{2\pi\alpha'}\gamma_{\sigma\sigma}\partial_\tau x^- + \frac{1}{4\pi\alpha'}\int d\sigma \left(\gamma_{\sigma\sigma}\partial_\tau X^i \partial_\tau X^i - \gamma_{\sigma\sigma}^{-1}\partial_\sigma X^i \partial_\sigma X^i\right) \tag{2.30}$$

The momentum conjugate of x^- is

$$p_- = -p^+ = \frac{\partial L}{\partial(\partial_\tau x^-)} = -\frac{l}{2\pi\alpha'}\gamma_{\sigma\sigma} \tag{2.31}$$

$\gamma_{\sigma\sigma}$ is a momentum and not a cooordinate. The momentum density conjugate to $X^i(\tau,\sigma)$ is

$$\Pi^i = \frac{\delta L}{\delta(\partial_\tau X^i)} = \frac{1}{2\pi\alpha'}\gamma_{\sigma\sigma}\partial_\tau X^i = \frac{p+}{l}\partial_\tau X^i \tag{2.32}$$

The Hamiltonian is

$$\begin{aligned} H &= p_- \partial_\tau x^- - L + \int_0^l d\sigma \Pi_i \partial_\tau X^i \\ &= \frac{l}{4\pi\alpha' p^+}\int_0^l d\sigma \left(2\pi\alpha' \Pi^i \Pi^i + \frac{1}{2\pi\alpha'}\partial_\sigma X^i \partial_\sigma X^i\right) \end{aligned} \tag{2.33}$$

This describes $D-2$ free fields X^i, with p^+ a conserved quantity. The equations of motion are

$$\partial_\tau X^- = \frac{\partial H}{\partial p_-} = \frac{H}{p^+}, \qquad \partial_\tau p^+ = \frac{\partial H}{\partial x^-} = 0, \tag{2.34}$$

$$\partial_\tau X^i = \frac{\delta H}{\delta \Pi^i} = 2\pi\alpha' c\Pi^i, \qquad \partial_\tau \Pi^i = -\frac{\delta H}{\delta X^i} = \frac{c}{2\pi\alpha'}\partial_\sigma^2 X^i \tag{2.35}$$

which implies the wave equation

$$\partial_\tau^2 X^i = c^2 \partial_\sigma^2 X^i \tag{2.36}$$

with velocity $c = l/2\pi\alpha' p^+$. We can can set $c = 1$ by choosing the coordinate length of each string to be proportional to p^+. p^+ is positive and conserved, so the total string length is also conserved.

It is useful to expand in normal modes the free wave equation satisfied by the transverse coordinates. The general solution to this wave equation with the boundary condition (2.29) is

$$X^i(\tau,\sigma) = x^i + \frac{p^i}{p^+}\tau + i(2\alpha')^{1/2} \sum_{n=-\infty, n\neq 0}^{\infty} \frac{1}{n}\alpha_n^i \exp\left(-\frac{\pi i n c \tau}{l}\right) \cos\frac{\pi n \sigma}{l} \tag{2.37}$$

For reality of X^i, $\alpha_{-n}^i = (\alpha_n^i)^\dagger$, and the following centre of mass variables have been defined, giving the average postion and the total momentum:

$$x^i(\tau) = \frac{1}{l}\int_0^l d\sigma X^i(\tau,\sigma), \tag{2.38}$$

$$p^i(\tau) = \int_0^l d\sigma \Pi^i(\tau,\sigma) = \frac{p^+}{l}\int_0^l d\sigma \partial_\tau X^i(\tau,\sigma) \tag{2.39}$$

$X^\pm$ also satisfy the free wave equation; this is trivial for X^+, and only a short calculation is needed for X^-.

The theory can now be quantized by imposing the equal time canonical commutation relations:

$$[x^-, p^+] = i\eta^{-+} = -i, \tag{2.40}$$

$$\left[X^i(\sigma), \Pi^j(\sigma')\right] = i\delta^{ij}\delta(\sigma - \sigma') \tag{2.41}$$

All other commutators between independent variables vanish. The Fourier component relations are

$$[x^i, p^j] = i\delta^{ij}, \tag{2.42}$$

$$\left[\alpha^i_m, \alpha^j_n\right] = m\delta^{ij}\delta_{m,-n} \tag{2.43}$$

The modes satisfy a harmonic oscillator algebra, for each m and i, with non-standard normalization

$$\alpha^i_m \sim m^{1/2}a, \qquad \alpha^i_{-m} \sim m^{1/2}a^\dagger, \qquad m > 0 \tag{2.44}$$

Define the state $|0;k\rangle$, where $k = (k^+, k^i)$, as being annihilated by the lowering operators and being an eigenstate of the centre-of-mass momenta,

$$p^+|0;k\rangle = k^+|0;k\rangle, \qquad p^i|0;k\rangle = k^i|0;k\rangle, \tag{2.45}$$

$$\alpha^i_m|0;k\rangle = 0, \qquad m > 0 \tag{2.46}$$

Raising operators can then be applied to $|0;k\rangle$ to form a general state,

$$|N;k\rangle = \left[\prod_{i=2}^{D-1}\prod_{n=1}^{\infty}\frac{(\alpha^i_{-n})^{N_{in}}}{(n^{N_{in}}N_{in}!)^{1/2}}\right]|0;k\rangle \tag{2.47}$$

After inserting the mode expansion (2.37) into the Hamiltonian (2.33) we get

$$H = \frac{p^ip^i}{2p^+} + \frac{1}{2p^+\alpha'}\left(\sum_{n=1}^{\infty}\alpha^i_{-n}\alpha^i_n + A\right) \tag{2.48}$$

The order of operators in the Hamiltonian is ambiguous; here we have chosen to put the lowering operators on the right and raising operators on the left and introduced the unknown constant A from the commutators. This constant can be determined carefully by the following procedure. One has to check the Lorentz invariance of the theory by finding the generators of Lorentz transformations $M^{\mu\nu}$ and then checking that they have the correct algebra with p^μ and each other. This only works out when $A = -1$ and $D = 26$. This would take a long time, but we can give here a brief plausibility argument.

We start by asserting that for a free field, the Hamiltonian's operator ordering constant is a result of summing the zero-point energies of each oscillator mode $\frac{1}{2}\omega$ for a bosonic field like X^μ. For H this gives

$$A = \frac{D-2}{2}\sum_{n=1}^{\infty} n \tag{2.49}$$

where the $D-2$ factor comes from summing over transverse directions. This sum can be evaluated by regulating the theory, then preserving Lorentz invariance in the renormalization, giving the strange result,

$$\sum_{n=1}^{\infty} n \to -\frac{1}{12} \tag{2.50}$$

Some motivation for this can be seen as follows. Insert the smooth cut-off factor

$$\exp(-\epsilon\gamma_{\sigma\sigma}^{-1/2}|k_\sigma|) \tag{2.51}$$

into the sum, with $k_\sigma = n\pi/l$, and invariance under σ reparameterizations is achieved by the $\gamma_{\sigma\sigma}^{-1/2}$. Then we get

$$A \quad \to \quad \frac{D-2}{2}\sum_{n-1}^{\infty} n \, \exp\left[-\epsilon n(\pi/2p^+\alpha' l)^{1/2}\right]$$

$$= \frac{D-2}{2}\left(\frac{2lp^+\alpha'}{\epsilon^2\pi} - \frac{1}{12} + O(\epsilon)\right) \qquad (2.52)$$

It turns out that Weyl invariance requires that the first and third terms cancel, leaving only the cut-off independent second term,

$$A = \frac{2-D}{24} \qquad (2.53)$$

Since $p^- = H$, we get

$$m^2 = 2p^+H - p^ip^i = \frac{1}{\alpha'}\left(N + \frac{2-D}{24}\right) \qquad (2.54)$$

where N is the *level*

$$N = \sum_{i=2}^{D-1}\sum_{n=1}^{\infty} nN_{in}, \qquad (2.55)$$

so the mass of a state is determined by its level of excitation. The lowest excited state is given by

$$\alpha^i_{-1}|0;k\rangle, \qquad m^2 = \frac{26-D}{24\alpha'} \qquad (2.56)$$

Now consider spin, first for massive particles. One may use the rest frame $p^\mu = (m, 0, ..., 0)$, where the internal states form a representation of the spatial representation group $SO(D-1)$. There is no rest frame for massless particles, so choose the frame $p^\mu = (E, E, 0, ..., 0)$. Here the internal states form a representation of $SO(D-2)$. So, in D dimensions, a massive vector particle has $D-1$ spin states, and a massless one needs only $D-2$ states. We have just found the $D-2$ states α^i_{-1} at the first level, so they must be massless and

$$A = -1, \qquad D = 26 \qquad (2.57)$$

So, although the classical theory is Lorentz-invariant for any D, this symmetry only survives quantization in $D = 26$.

We will now discuss the closed strings, but only very briefly, since the light-cone treatment is quite similar to that for open strings. The first differnce to note is that the imposition of the gauge condition (2.21) for the closed string still leaves the coordinate freedom

$$\sigma' = \sigma + s(\tau) \mathrm{mod}\ l \tag{2.58}$$

since the point $\sigma = 0$ can be chosen anywhere along the string. The further gauge condition

$$\gamma_{\tau\sigma}(\tau, 0) = 0 \tag{2.59}$$

reduces this to

$$\sigma' = \sigma + s \ \mathrm{mod}\ l \tag{2.60}$$

We will come back to this shortly.

The lagrangian, canonical momenta, Hamiltonian and equation of motion are exactly as for the open string. The general periodic solution to the equation of motion is

$$\begin{aligned} X^i(\tau, \sigma) = x^i + \frac{p^i}{p^+}\tau + i\left(\frac{\alpha'}{2}\right)^{1/2} \\ \times \sum_{n=-\infty, n\neq 0}^{\infty} \left(\frac{\alpha_n^i}{n}\exp\left[-\frac{2\pi i n(\sigma + c\tau)}{l}\right] + \frac{\tilde{\alpha}_n^i}{n}\exp\left[-\frac{2\pi i n(\sigma - c\tau)}{l}\right]\right) \end{aligned} \tag{2.61}$$

There are now two independent sets of oscillators, α_n^i and $\tilde{\alpha}_n^i$, representing left- and right-moving wave along the string. Again, the independent degrees of freedom are the transverse oscillators and the transverse and longitudinal centre-of-mass variables,

$$\alpha_n^i,\ \tilde{\alpha}_n^i,\ x^i,\ p^i,\ x^-,\ p^+ \tag{2.62}$$

with canonical commutators

$$\left[x^-,p^+\right] = -i, \tag{2.63}$$

$$\left[x^i,p^j\right] = i\delta^{ij}, \tag{2.64}$$

$$\left[\alpha_m^i,\alpha_n^j\right] = m\delta^{ij}\delta_{m,-n}, \tag{2.65}$$

$$\left[\tilde{\alpha}_m^i,\tilde{\alpha}_n^j\right] = m\delta^{ij}\delta_{m,-n} \tag{2.66}$$

Defining the state $|0,0;k\rangle$ with centre-of-mass momentum k^μ and annihilated by $\alpha_m^i s$ and $\tilde{\alpha}_m^i s$ for $m > 0$, then the general state is

$$|N,\tilde{N};k\rangle = \left[\prod_{i=2}^{D-1}\prod_{n=1}^{\infty}\frac{(\alpha_{-n}^i)^{N_{in}}(\tilde{\alpha}_{-n}^i)^{\tilde{N}_{in}}}{(n^{N_{in}}N_{in}!n^{\tilde{N}_{in}}\tilde{N}_{in}!)^{1/2}}\right]|0,0;k\rangle \tag{2.67}$$

The mass formula is

$$\begin{aligned} m^2 &= 2p^+H - p^ip^i \\ &= \frac{2}{\alpha'}\left[\sum_{n=1}^{\infty}(\alpha_{-n}^i\alpha_n^i + \tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i) + A + \tilde{A}\right] \\ &= \frac{2}{\alpha'}(N+\tilde{N}+A+\tilde{A}) \end{aligned} \tag{2.68}$$

where both the level and zero point constant have been split into left- and right-moving parts. Summing the zero-point energies again gives

$$A = \tilde{A} = \frac{2-D}{24} \tag{2.69}$$

The physical states are those which are gauge-invariant, but we still have the

σ-translations. These translations are generated by

$$\begin{aligned} P &= -\int_0^l d\sigma \Pi^i \partial_\sigma X^i \\ &= -\frac{2\pi}{l}\left[\sum_{n=1}^{\infty}((\alpha_{-n}^i \alpha_n^i + \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i) + A + \tilde{A}\right] \\ &= -\frac{2\pi}{l}(N - \tilde{N}) \end{aligned} \tag{2.70}$$

Therefore states must satisfy

$$N = \tilde{N} \tag{2.71}$$

The first excited states are

$$\alpha_{-1}^i \tilde{\alpha}_{-1}^j |0, 0; k\rangle, \qquad m^2 = \frac{26 - D}{6\alpha'} \tag{2.72}$$

Just as the open string case, these states do not fill out a representation of $SO(D-2)$, and we find that this level must be massless, so

$$A = \tilde{A} = -1, \qquad D = 26 \tag{2.73}$$

We have found Lorentz invariant quantized bosonic theory in 26 dimensions for both the open and closed strings. We have not examined the problem that there exists a tachyon in both these scenarios. This is one reason why we will now introduce superstrings.

2.1.1 Superstrings

Two major problems with bosonic string theory are that it contains tachyons and that it does not contain fermions. Both of these problems can be solved by introducing a world-sheet supersymmetry [56] that relates the spacetime coordinates to new fermionic partners. These are two-component world-sheet spinors. Physically, they correspond to internal degrees of freedom which can propagate freely along the string. In light-cone gauge, it can be shown that, as in the bosonic case, the requirement of a Lorentz invariant quantized theory specifies the number of dimensions, but now $D = 10$ rather than $D = 26$. A generic spinor in ten dimensions has 32 components, but we are going to assume that we start with Majorana-Weyl spinors. The Majorana condition makes them real, and the Weyl condition sets half equal to zero, leaving 16 real components. The light-cone gauge condition halves this to only eight real components. The only symmetry manifest in the light-cone gauge is the rotational invariance of the eight transverse dimensions. The eight surviving components of the original spinor form an eight-dimensional spinor representation of the transverse $SO(8)$ group, in fact its spin(8) covering group.

The light-cone gauge action can be written (with explicit string tension) [1]

$$S = -\frac{1}{2}\int d^2\sigma (T\partial_\alpha X^i \partial^\alpha X^i - \frac{i}{\pi}\bar{S}^a \rho^\alpha \partial_\alpha S^a) \tag{2.74}$$

Here S^a is the world-sheet spinor for the eight components that survive the restriction to light-cone gauge. It can be thought of as being composed of two components S^{1a} and S^{2a} which may be regarded as one-component Majorana-Weyl world-sheet spinors for right- and left-moving degrees of freedom, repectively. ρ^α are two-dimensional Dirac matrices.

We have already seen analysis for bosonic coordinates, so we will now just consider the quantization of the fermionic ones. The S^{Aa} coordinates have canonical

[1]We are using here the conventions and notation of Chapter 5 of [1], which should make it easy for the interested reader to find a more rigourous and detailed treatment.

anticommutation relations

$$\{S^{Aa}(\sigma,\tau), S^{Bb}(\sigma',\tau)\} = \pi\delta^{ab}\delta^{AB}\delta(\sigma-\sigma') \tag{2.75}$$

We now need to specify boundary conditions. For open strings the appropriate choices are

$$S^{1a}(0,\tau) = S^{2a}(0,\tau) \tag{2.76}$$

$$S^{1a}(\pi,\tau) = S^{2a}(\pi,\tau) \tag{2.77}$$

The mode expansions for the open string that follow from (2.74) are

$$S^{1a}(\sigma,\tau) = \frac{1}{\sqrt{2}}\sum_{-\infty}^{\infty} S_n^a e^{-in(\tau-\sigma)} \tag{2.78}$$

$$S^{2a}(\sigma,\tau) = \frac{1}{\sqrt{2}}\sum_{-\infty}^{\infty} S_n^a e^{-in(\tau+\sigma)} \tag{2.79}$$

From the reality of these coordinates we get

$$S_{-m}^a = (S_m^a)^\dagger \tag{2.80}$$

and in terms of the expansion coefficients, the canonical commutation relations become

$$\{S_m^a, S_n^b\} = \delta^{ab}\delta_{m+n} \tag{2.81}$$

Now for the closed strings. Their only boundary condition is periodicity:

$$S^{Aa}(\sigma,\tau) = S^{Aa}(\sigma+\pi,\tau) \tag{2.82}$$

and the mode expansion becomes

$$S^{1a}(\sigma,\tau) = \sum S_n^a e^{-2in(\tau-\sigma)} \tag{2.83}$$

$$S^{2a}(\sigma,\tau) = \sum \tilde{S}_n^a e^{-2in(\tau+\sigma)} \tag{2.84}$$

with independent right and left moving modes.

From a consideration of the super-Poincaré algebra in spin(8) notation, one can arrive at the light-cone hamiltionian,

$$H = \frac{1}{2p^+}\left((p^i)^2 + 2N\right) \tag{2.85}$$

where

$$N = \sum_{m=1}^{\infty}\left(\alpha_{-m}^i \alpha_m^i + mS_{-m}^a S_m^a\right) \tag{2.86}$$

Note that because there is a cancellation of zero-point energies of the α and S modes, normal ordering is trivial here. The mass-shell condition is just $H = p^-$.

Consider the massless spectrum and note that since there are no tachyons here, this is the ground state. In the spin(8) description, the ground state represents the algebra $\{S_0^a, S_0^b\} = \delta^{ab}$. It can then be shown that the ground states form an $\mathbf{8_v} + \mathbf{8_c}$, that is eight Bose states in the $\mathbf{8_v}$ representation of spin(8) and eight Fermi states in the $\mathbf{8_c}$ representation. Use $|\phi_0\rangle$ to denote this 16-dimensional multiplet of massless ground states. One can obtain the excited (massive) open-string states by applying α_{-n}^i and S_{-n}^a excitations to the ground state $|\phi_0\rangle$. The physical states at the first excited level are

$$\alpha_{-1}^i|\phi_0\rangle, \qquad S_{-1}^a|\phi_0\rangle \tag{2.87}$$

which describe 128 bosonic and 128 fermionic modes. These fit into spin(9) multiplets, the representations being **44** + **84** for the bosons, and a **128** 'spin 3/2' for the fermions.

For closed strings, one set of modes is needed for the right-movers and another for the left-movers. The massless states are described by the direct product of states $|\phi_0\rangle \times |\tilde{\phi}_0\rangle$. There are two distinct cases here since the two original Majorana-Weyl spinors can have either the same or opposite chirality. When they are opposite there can be no symmetrization of the two factors, so the massless multiplet must have $16 \times 16 = 256$ modes. The spin(8) content is the tensor product of two super Yang-Mills multiplets of opposite chirality,

$$\begin{aligned} (\mathbf{8}_\mathrm{v} + \mathbf{8}_\mathrm{c}) \otimes (\mathbf{8}_\mathrm{v} + \mathbf{8}_\mathrm{s}) &= (\mathbf{1} + \mathbf{28} + \mathbf{35}_\mathrm{v} + \mathbf{8}_\mathrm{v} + \mathbf{56}_\mathrm{v})_B \\ &\quad + (\mathbf{8}_\mathrm{s} + \mathbf{8}_\mathrm{c} + \mathbf{56}_\mathrm{s} + \mathbf{56}_\mathrm{c})_F \end{aligned} \tag{2.88}$$

The subscripts B and F indicate bosonic and fermionic states, respectively. This is the particle content of $D = 10$ type IIA supergravity.

When both spinors have the same chirality and no other restrictions are imposed there are still 256 modes. Now the spin(8) content of the massless multiplet is formed by the product of two super Yang-Mills multiplets of the same chirality,

$$\begin{aligned} (\mathbf{8}_\mathrm{v} + \mathbf{8}_\mathrm{c}) \otimes (\mathbf{8}_\mathrm{v} + \mathbf{8}_\mathrm{c}) &= (\mathbf{1} + \mathbf{28} + \mathbf{35}_\mathrm{v} + \mathbf{1} + \mathbf{28} + \mathbf{35}_\mathrm{c})_B \\ &\quad + (\mathbf{8}_\mathrm{s} + \mathbf{8}_\mathrm{s} + \mathbf{56}_\mathrm{s} + \mathbf{56}_\mathrm{c})_F \end{aligned} \tag{2.89}$$

We now have the $D = 10$ chiral type IIB supergravity particle content. We can also consider the same-chirality case with an additional symmetrization condition that only terms invariant under the interchange of $|\phi_0\rangle$ and $|\tilde{\phi}_0\rangle$ are allowed. This means a graded symmetrization in the tensor product of the super Yang-Mills multiplets,

$$[(\mathbf{8}_\mathrm{v} + \mathbf{8}_\mathrm{c}) \otimes (\mathbf{8}_\mathrm{v} + \mathbf{8}_\mathrm{s})]_{graded\ sym} = (\mathbf{8}_\mathrm{v} + \mathbf{8}_\mathrm{v})_{sym} + (\mathbf{8}_\mathrm{v} + \mathbf{8}_\mathrm{c}) + (\mathbf{8}_\mathrm{c} + \mathbf{8}_\mathrm{c})_{antisym}$$

$$= \quad (\mathbf{1} + \mathbf{28} + \mathbf{35}_\mathrm{v})_B + \mathbf{8}_\mathrm{s} + \mathbf{56}_\mathrm{s})_F \tag{2.90}$$

This is the particle content of $D = 10$ chiral type I supergravity.

The condition that a fermionic field on the world-sheet be periodic is the *Ramond* (R) boundary condition. The *Neveu-Schwarz* (NS) boundary condition on the other hand, is when the fermionic field on the world-sheet is anti-periodic. Since the closed string can have any combination of these for its left- and right-moving modes, there are four different closed string sectors, known as NS-NS, NS-R, R-NS and R-R.

2.2 D-Branes

D-branes were discovered by examining the T-duality of string theory [2]. Since a superstring theory must live in ten dimensions but we live in only four we must do something with the six extra dimensions. Consider compactifying one of these extra dimensions on a circle of radius R. If we make this radius very small, we will not be able to see this extra dimension, solving the problem. This is the principle of compactification, but we will now discuss some remarkable subtleties. First of all, consider a point particle with a compact dimension. When the radius of the compact dimension goes to zero, the excitation energy for a particle in this direction diverges, so this dimension can be ignored as we just suggested.

The picture is changed dramatically for a closed string because a closed string can be wound on the compact dimension. The mass spectrum can be calculated [2],

$$M^2 = \frac{n^2}{R^2} + \frac{\omega^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2) \tag{2.91}$$

where n and ω are integers. The third term is the normal result without a compactification - N and $\tilde{N}$ are the number operators

$$N = \sum_{n=1}^{\infty} \alpha_{-n}.\alpha_n, \qquad \tilde{N} = \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}.\tilde{\alpha}_n \tag{2.92}$$

This spectrum is unchanged by the interchange of variables,

$$\begin{aligned} \omega &\leftrightarrow n \\ R &\leftrightarrow \frac{\alpha'}{R} \end{aligned} \tag{2.93}$$

This means that there is a good string description for when R is much smaller than the string length in terms of a theory with R much greater than the string length. As $R \to 0$ the momentum modes become infinitely massive, as in the particle case, but here the ω modes with $R \to \alpha'/R$ replace them in a sense, so that the theory still sees the original number of dimensions.

What about open strings in this scenario? An open string itself cannot wind on the compact dimension, so they must experience the dimensional reduction, like the particle. However, open string theories generate closed strings at loop level, so there must always be closed strings which see the full number of dimensions. Also, sections of an open string away from the end-points are just like sections of closed string, so it is only the end-points of the open strings that are restricted to the lower dimensions. We can T-dualize in more directions, increasing the number of dimensions in which the open string end-points are restricted. The picture that emerges is open strings with their end-points restricted to lower dimensional hyperplanes, or n-branes, while the closed strings propagate throughout the higher dimensional space. See Figure (2.2). It is Dirichlet boundary conditions which restrict the open strings to the n-branes, hence the name Dirichlet-branes.

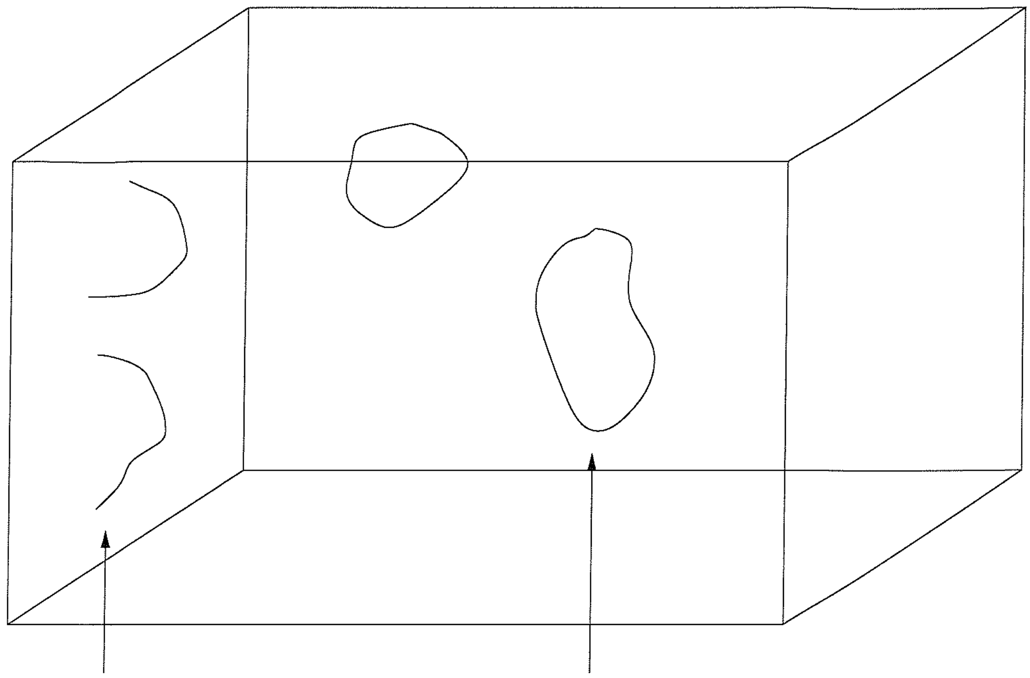

Figure 2.1: Ends of open strings are restricted to D-brane while closed strings propagate in the bulk

2.3 Field Theory on D-branes

Global symmetries can be introduced into string world-sheet theory. The only natural place to put global charges is at the ends of open strings. These are called Chan-Paton degrees of freedom [12]. If we have N such charges and either end of a string can have a different charge i and j, then there can be N^2 different strings (see Figure 2.2). A string wavefunction may then be decomposed as

$$|k;a\rangle = \sum_{i,j=1}^{N} |k;ij\rangle \lambda_{ij}^{a} \tag{2.94}$$

where the basis of $N \times N$ matrices, λ_{ij}^{a}, are called Chan-Paton factors. All open string vertex operators also have such factors. Consider the diagram for four-point scattering of open strings shown in Figure 2.3. The Chan-Paton state must be the same all along each edge of the world-sheet, i.e., each dashed, arrowed line, which

Figure 2.2: An open string with Chan-Paton degrees of freedom.

gives a trace of the product of Chan-Paton factors,

$$\lambda^1_{ij}\lambda^2_{jk}\lambda^3_{kl}\lambda^4_{li} = \mathrm{Tr}(\lambda^1\lambda^2\lambda^3\lambda^4) \tag{2.95}$$

Similar traces arise in all open string amplitudes, and they are invariant under the $U(N)$ transformation

$$\lambda^a \to U\lambda^a U^{-1} \tag{2.96}$$

Amplitudes can be calculated in this enriched string theory and one can then look for a spacetime action which reproduces these amplitudes to first order. This effective action is a $U(N)$ gauge theory. What has happened is that the $U(N)$ global world-sheet symmetry has been promoted to a local spacetime symmetry. [1, 2]

If we follow through the T-dualizing procedure of the previous section, the ends of these strings will again be restricted to D-branes. The plural D-branes is used because each of the N charges is associated with a different D-brane. The important feature that these Chan-Paton factors have introduced is that the low energy ($\alpha' \to 0$ limit) effective theory of these open strings is $U(N)$ gauge theory in the $p+1$ dimensions of the D-branes. We have the nice intuitive picture of the ends of open strings which are points in the D-brane world-volume describing a point particle $U(N)$ gauge theory. If we have D3-branes, that is, D-branes with a world-volume of three spatial plus one time dimensions, then we have a gauge field theory

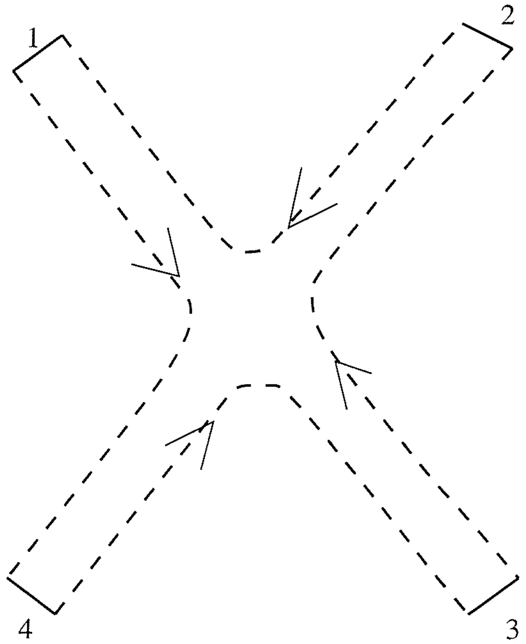

Figure 2.3: Four-point scattering of open strings.

of the correct dimensions for describing our observed low-energy world[2].

As we are dealing with a theory of gravity the branes must be allowed to move in the bulk space, since gravity waves distort everything. One can imagine moving one of the branes off in the transverse space, leaving $N-1$ branes still coincident. This will mean that strings with one end having the Chan-Paton charge of the single brane and the other end carrying a charge of one of the other D-branes will be stretched between the single brane and the other $N-1$ branes (see Figure 2.4). It is clear that we will then have the low energy theories of $U(1)$ gauge theory on the single brane and $U(N-1)$ on the $N-1$ branes. In fact this a geometric description of the Higgs mechanism [3] since there are scalar fields in the gauge theory which represent the position of the D-branes, and moving the single brane away corresponds to giving the relevant scalars vevs. This corresponds to the strings being stretched to get masses

[2]We will see at the start of the next chapter how the $\mathcal{N}=4$ super Yang-Mills field theory in four dimensions can be derived from a dimensional reduction of $d=10$ $\mathcal{N}=1$ SYM.

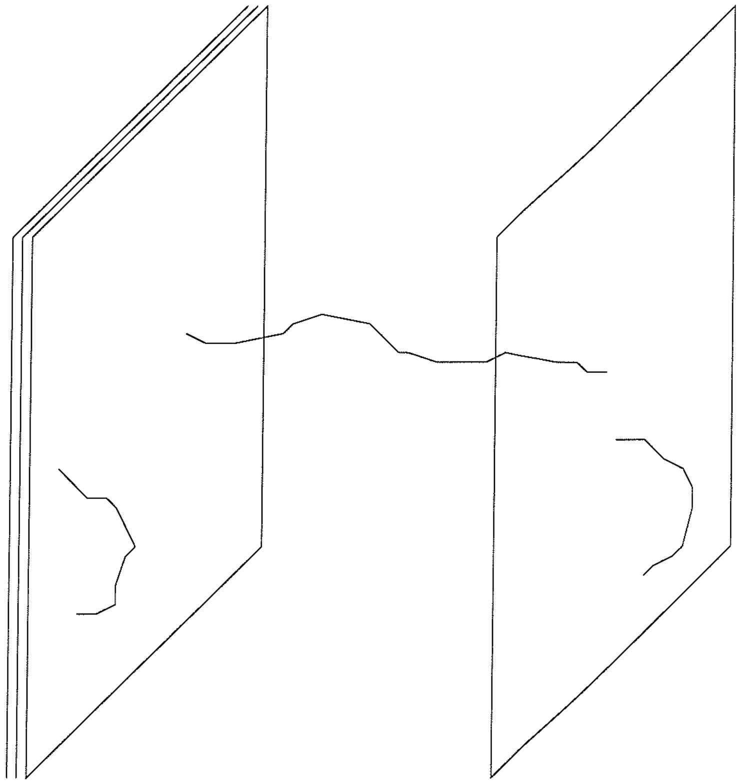

Figure 2.4: A geometrical picture for the Higgs mechanism with D-branes

[(tension)×(distance stretched)=mass]. Generically, all N branes may be separated to give the breaking $U(N) \to U(1)^N$.

2.4 The Dirac-Born-Infeld Action

To do the D-brane calculations in our research we need an action for them, so we will now sketch out the form of the Dirac-Born-Infeld action [7]. We saw earlier that the simple Nambu-Goto action for a bosonic string is basically just the area of the string world-sheet. This principle extends to higher dimensional world-volumes, so we should expect the Dp-brane action to contain the square-root of a determinant which contains an induced metric on the world-volume, G_{ab}. We want to describe a brane with a gauge field living on it and a perturbative expansion of the string theory indicates the ansatz of adding a term $2\pi\alpha' F_{ab}$ in to the determinant. We have also seen that in general there is an antisymmetric background tensor field $B_{\mu\nu}$, so there must be an induced antisymmetric field B_{ab} in there too. The final field we need to include is the dilaton. We are dealing with open string tree level physics, so we should have a factor of g_s^{-1}, that is $e^{-\Phi}$. Putting all this together, we get the Dirac-Born-Infeld action for Dp-branes,

$$S_p = -T_p \int d^{p+1}\xi e^{-\Phi} \det{}^{1/2}(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}) \qquad (2.97)$$

2.5 From Superstrings to Supergravity

Although the full conjecture of the AdS/CFT Correspondence concerns Type IIB string theory [1, 2] (see following chapter on the AdS/CFT Correspondence), my research has principally only involved dealing with its low energy Type IIB supergravity description. It was shown previously that a string theory contains a finite number of massless states plus an infinite tower of massive excitations. These massive states are characterized by the Regge slope α', or equivalently the string tension. To produce a gravitational interaction with the observed Newtonian strength, this fundamental parameter must be of order the Planck mass, $M_P = 10^{19}$GeV, assuming the compactification radius corresponds to the string scale and the string coupling is of order one. At energies much lower than this, it should be possible to find a very good approximation to the full theory which does not involve the massive states. M_P is very large, so such an effective theory should be valid for phenomenology at any practical energies. So we want an effective action for just the massless states.

If we represent the finite number of massless fields by ϕ_0 and the infinite number of massive fields by ϕ_H, then the string theory should be able to be described by some classical, or a quantum effective, action $S(\phi_0, \phi_H)$. Of course it would be great to find such an exact classical action and learn directly from it, but this has proved too difficult to do as yet. From this action one could integrate out the massive fields to leave a low energy action involving only the massles fields, ϕ_0:

$$e^{iS_{eff}(\phi_0)} \sim \int D\phi_H e^{iS(\phi_0,\phi_H)} \tag{2.98}$$

At the classical level, this amounts to eliminating the massive fields from the equations of motion to leave only equations of the massless fields. This is just the tree level computation of the path integral. In the quantum theory the massive fields of course still come into the low energy theory, through the massive loop diagrams in the rest of the path integral. $S_{eff}(\phi_0)$ can take the form of a power series in $\hbar$, with the power of $\hbar$ corresponding to the number of ϕ_H loops. The classical elimination of ϕ_H is just the first, $(\hbar)^0$ term.

We still do not have a satisfactory $S(\phi_0, \phi_H)$ from which to derive, as just described, a low energy action for the massless fields of string theory. What has been done though is to just construct a classical action for the massless fields which reproduces the string S-matrix elements. For example, Type IIB supergravity was constructed to reproduce the results from Type IIB string theory. IIB supergravity is the low energy effective theory of IIB string theory, but it does not simply fall out of the string theory when one takes a certain limit in the equations, as say taking the low velocity limit of special relativity produces the equations of Newtonian mechanics.

The field and particle content of Type IIB supergravity is [13]

$$
\begin{array}{ccc}
G_{\mu\nu} & 35_B & \text{metric - graviton} \\
C + i\Phi & 2_B & \text{axion - dilaton} \\
B_{\mu\nu} + iA_{2\mu\nu} & 56_B & \text{rank 2 antisymmetric} \\
A^+_{4\mu\nu\rho\sigma} & 35_B & \text{antisymmetric rank 4} \\
\psi^I_{\mu\nu},\ I = 1,2 & 112_F & \text{Majorana-Weyl gravitinos} \\
\lambda^I_\alpha,\ I = 1,2 & 16_F & \text{Majorana-Weyl dilatinos}
\end{array}
\tag{2.99}
$$

The action for Type IIB may be written as [3]

$$
\begin{aligned}
S_{IIB} &= \frac{1}{4\kappa_B^2}\int \sqrt{G}e^{-2\Phi}(2R_G + 8\partial_\mu\Phi\partial^\mu\Phi - |H_3|^2) \\
&\quad -\frac{1}{4\kappa_B^2}\left[\sqrt{G}(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2) + A_4^+ \wedge H_3 \wedge F_3\right] \\
&\quad +\text{fermions}
\end{aligned}
\tag{2.100}
$$

where the field strengths are defined by

$$
\begin{aligned}
F_1 = dC, \qquad H_3 = dB, \qquad F_3 = dA_2 \\
F_5 = dA_4^+, \qquad \tilde{F}_3 = F_3 - CH_3 \\
\tilde{F}_5 = F_5 - \frac{1}{2}A_2 \wedge H_3 + \frac{1}{2}B \wedge F_3
\end{aligned}
\tag{2.101}
$$

A_4^+ has a self-dual field stength, so we also require the supplementary condition $*\tilde{F}_5 = \tilde{F}_5$. κ_B is the Newton constant in ten dimensions. The first line of the action comes from the NS-NS sector of the string theory and the second line comes from the RR sector. The fermion part of the action is rarely written out explictly in the literature, because it is rather large and is set to zero in finding a supersymmetric vacuum anyway.

[3] We use exterior differential notation for anti-symmetric tensor fields, e.g. for a rank 3 tensor $A_3 \equiv 1/3! A_{3\mu\nu\rho} dx^\mu dx^\nu dx^\rho$, with field strength $F_4 \equiv dA_3$.

It is not evident from looking at (2.100), but Type IIB supergravity has an $SU(1,1) \sim SL(2,\mathbf{R})$ non-compact symmetry. This symmetry can be revealed by a change of coordinates. To do this, swap the *string metric* $G_{\mu\nu}$ that has been used so far for the *Einstein metric* $G_{E\mu\nu}$, defined by

$$G_{E\mu\nu} \equiv e^{-\Phi/2} G_{\mu\nu} \tag{2.102}$$

and introduce the complex fields

$$\tau \equiv C + ie^{-\Phi}, \qquad G_3 \equiv \frac{F_3 - \tau H_3}{\sqrt{\mathrm{Im}\tau}} \tag{2.103}$$

The action then takes the form,

$$\begin{aligned} S_{IIB} &= \frac{1}{4\kappa_B^2} \int \sqrt{G_E} \left(2R_{G_E} - \frac{\partial_\mu \bar{\tau} \partial^\mu \tau}{(\mathrm{Im}\tau)^2} - \frac{1}{2}|F_1|^2 - |G_3|^2 - \frac{1}{2}|\tilde{F}_5|^2 \right) \\ &\quad - \frac{1}{4i\kappa_B^2} \int A_4 \wedge \bar{G}_3 \wedge G_3 \end{aligned} \tag{2.104}$$

The metric and A_4^+ fields are invariant under the $SU(1,1) \sim SL(2,\mathbf{R})$ symmetry. τ, the combined dilaton-axion field undergoes the Möbius transformation,

$$\tau \to \tau' = \frac{a\bar{\tau} + d}{c\tau + d}, \qquad ad - bc = 1, \;\; a, b, c, d \in \mathbf{R} \tag{2.105}$$

Lastly, the linear transformation associated with this Möbius transformation acts to rotate the $B_{\mu\nu}$ and $A_{2\mu\nu}$ fields into one another. This may be written concisely in terms of the complex 3-form field G_3:

$$G_3 \to G_3' = \frac{c\bar{\tau} + d}{|c\tau + d|} G_3 \tag{2.106}$$

2.6 Branes in Supergravity

We have discovered the existence of Dp-branes in string theory, and also that 10-dimensional supergravity is the low-energy effective theory of the superstring theory, so one should be able to find a supergravity description for the Dp-branes of the string theory. In fact, p-brane solutions of Type IIA/B supergravity had been discovered before the D-branes of Type IIA/B string theory had been found.

A general $(p+1)$-form,

$$A_{p+1} \equiv \frac{1}{(p+1)!} A_{\mu_1 \dots \mu_{p+1}} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{p+1}} \tag{2.107}$$

couples naturally to geometrical objects σ_{p+1} with spacetime dimension $p+1$ since the diffeomorphism invariant action,

$$S_{p+1} = T_{p+1} \int_{\sigma_{p+1}} A_{p+1} \tag{2.108}$$

may be constructed. This is just a generalisation of the standard gauge coupling to a point particle [3]. This action is invariant under the rank p abelian gauge transformations $\rho_p(x)$,

$$A_{p+1} \to A_{p+1} + d\rho_p \tag{2.109}$$

since S_{p+1} transforms with a total derivative. A_{p+1} has a gauge invariant field strength F_{p+2} with conserved flux. A supergravity solution with non-trivial A_{p+1} is called a p-brane due to the dimension of its spacetime geometry.

Every A_{p+1} gauge field has a 'magnetic' dual $(D-3-p)$-form, $A^{\mathrm{magn}}_{D-3-p}$, which is Hodge dual [14, 2] (denoted by a preceding *) to A_{p+1},

$$dA^{\mathrm{magn}}_{D-3-p} \equiv *dA_{p+1} \tag{2.110}$$

Brane Name	Associated Field	Magnetic Dual Brane
D(-1) instanton	$A_0 = C + ie^{-i\Phi}$	D7
F1 string	$B_{\mu\nu}$	NS5
D1 string	$A_{2\mu\nu}$	D5
D3 brane	$A^+_{4\mu\nu\rho\sigma}$	D3

Table 2.1: Type IIB supergravity branes.

This field couples to a $(D-4-p)$-brane, which constitutes a magnetic dual to the original 'electric' p-brane. This is a simple generalisation of standard electro-magnetic duality in (3+1) dimensions, where a field strength $F^{\mu\nu}$ has a dual $\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} \equiv^* (F^{\alpha\beta})$. The ten dimensional case is similar, but the ϵ-tensor has ten indices. Table (2.1) summarises the electric branes and their magnetic duals in Type IIB supergravity. All these fields are charged in the R-R sector and are associated with D-branes, except $B_{\mu\nu}$. This a NS-NS field associated with the fundamental string, F1, whose magnetic dual is something called the NS5-brane [7].

We now look at the supergravity solutions for the branes we have been discussing. The $(p+1)$-dimensional flat hypersurface of a p-brane has the Poincaré invariance group $\mathbf{R}^{p+1} \times SO(1,p)$. This leaves a $(D-p-1)$-dimension transverse space, and solutions can always be found with the associated $SO(D-p-1)$ maximal rotation symmetry. So p-branes in 10D supergravity are solutions with symmetry group $\mathbf{R}^{p+1} \times SO(1,p) \times SO(D-p-1)$. For example, $D3$-branes have the symmetry group $\mathbf{R}^4 \times SO(1,3) \times SO(9-p)$.

We shall denote the coordinates parallel to the brane as x^μ, $\mu = 0,1,...,p$ and coordinates perpendicular to the brane as $y^u = x^{p+u}$, $u = 1,2,...,D-p-1$. In the $p+1$ dimensions parallel to the brane the metric must be a rescaling of the Minkowski flat metric due to Poincaré invariance. The transverse directions must be a rescaling of the Euclidean metric because of their rotational invariance. Also, these metric rescaling functions should not depend on x^μ. If one builds these three restrictions into an Ansatz and substitutes it into the IIB field equations, one find that the solutions may be written in terms of a single function H thus [13],

$$Dp: \quad ds^2 = H(\vec{y})^{-1/2} dx^\mu dx_\mu + H(\vec{y})^{1/2} d\vec{y}^2, \quad e^\Phi = H(\vec{y})^{(3-p)/4} \tag{2.111}$$

$$NS5: \quad ds^2 = dx^\mu dx_\mu + H(\vec{y}) d\vec{y}^2, \quad e^{2\Phi} = H(\vec{y}) \tag{2.112}$$

where the Dp metric is in the string frame and H must be harmonic in y.

These metrics should tend to flat spacetime as $y \to \infty$. Using this and the assumption of $SO(D-p-1)$ maximal rotational symmetry in the transverse dimensions, the most general solution is parametrised by a single scale factor, L, so that H is

$$H(y) = 1 + \frac{L^{D-p-3}}{y^{D-p-3}} \tag{2.113}$$

Obviously, to make the second term of (2.113) dimensionless L must have dimensions of length, and since the only dimensionful parameter of the theory is α', which has dimension (length)^2, L must be proportional to $\alpha'^{1/2}$. L may also depend on the dimensionless string coupling, g_s. Although we are dealing with supergravity, we are interpreting it here as a low energy effective theory of string theory. This string theory parameter comes in through the 10D Newton constant, κ_B, in the IIB SUGRA action (2.100). An important example is the solution for N coincident branes, for which $L^{D-p-3} = N\rho_p$, and then for Dp-branes, $\rho_p = g_s(4\pi)^{(5-p)/2}\Gamma((7-p)/2)(\alpha')^{(D-p-3)/2}$. If one relaxes the condition on rotational invariance in the transverse space, then the general solution extends to

$$H(\vec{y}) = 1 + \sum_{I=1}^{N} \frac{C_I}{|\vec{y} - \vec{y}_I|^{D-3-p}}, \qquad C_I = N_I \rho_p, \ N_I \in \mathbf{N} \tag{2.114}$$

for an array of N points $\vec{y}$. This is a multicentre solution; the branes are found at multiple points spread out in the transverse space [19].

We now examine the dependence of the various brane solutions of Type IIB string theory on the string coupling g_s, in particular, the dependence of ρ_p. Remember that the string coupling is given by $g_s = e^{\phi}$, where $\phi = \langle\Phi\rangle$. Now look at the field equation for IIB (from the IIB action (2.100), but without derivative terms in the dilaton and axion for simplicity):

$$R_{\mu\nu} = \frac{1}{4}H_{\mu\rho\sigma}H_{\nu}^{\rho\sigma} + e^{2\Phi}\left(F_{1\mu}F_{1\nu} + \frac{1}{4}\tilde{F}_{3\mu\sigma\rho}\tilde{F}_{3\nu}^{\rho\sigma} + \frac{1}{24}\tilde{F}^{+}_{5\mu\rho\sigma\tau\upsilon}\tilde{F}_{5\nu}^{+\rho\sigma\tau\upsilon}\right) \tag{2.115}$$

Now recall that the fundamental string F1 and the NS5 brane have non-vanishing $H_{\mu\rho\sigma}$ fields, but vanishing RR fields R_i. So these brane solutions do not involve g_s, therefore their ρ_p is independent of g_s. In contrast, the D-brane solutions have $H_{\mu\rho\sigma} = 0$, but at least one of the F_i, RR antisymmetric fields is non-zero. These solutions have explicit string coupling dependence, and so $\rho_p \sim g_s$.

2.7 D3-Branes

The special case of D3-branes is of particular interest for a number of reasons:

- It has 4-dimensional Poincaré invariance on its worldvolume.
- It has constant axion and dilaton fields.
- It is regular at $y = 0$.
- It is self-dual.

D3-branes are also extremely important in the context of the AdS/CFT Correspondence and come into all of the research content of this thesis. They are therefore worthy of a fuller discussion. The D3 solution is given by [13]:

$$g_s = e^{\phi}, \qquad C \text{ constant},$$

$$B_{\mu\nu} = A_{2\mu\nu} = 0$$
$$ds^2 = H(y)^{-1/2} dx^\mu dx_\mu + H(y)^{1/2}(dy^2 + y^2 d\Omega_5^2)$$
$$F^+_{5\mu\nu\rho\sigma\tau} = \epsilon_{\mu\nu\rho\sigma\tau\upsilon} \partial^\upsilon H \qquad (2.116)$$

where $\epsilon_{\mu\nu\rho\sigma\tau\upsilon}$ is the volume element transverse to the D3-brane. The solution for $N = \sum_I N_I$ parallel D3-branes placed in groups of N_I at position $\vec{y}_I$ in the transverse space is

$$H(\vec{y}) = 1 + \sum_{I=1}^{N} \frac{4\pi g_s N_I (\alpha')^2}{|\vec{y} - \vec{y}_I|^4} \qquad (2.117)$$

c.f. (2.114) and the expression for ρ_p just before it.

Now examine how the scales involved in the D3-brane solution relate to the coupling constant. The Planck length is defined by $l_P^2 = \alpha'$ and it is related to the radius L of the D3-brane solution to string theory by $L^4 = 4\pi g_s N l_P^4$. When $g_s N \ll 1$, L is negligible in comparison to the string length l_P, in which case the supergravity approximation is not a valid approximation for the full string solution. However, in this regime $g_s \ll 1$, so that string perturbation theory may be trusted and the power of conformal field theory methods may be used to examine the D3-brane. In the other case of $g_s N \gg 1$, L will be much greater than l_P, and then the full string theory is open to the supergravity approximation. In this instance it is possible to take $g_s \ll 1$ if N is large enough, then both supergravity and string perturbation theory are simultaneously applicable. Note that in general, when $p \neq 3$ the dilaton is not constant and so the strength of the coupling depends on the distance to the brane.

D3-branes can be viewed as a two-parameter family of solutions, those parameters being the string coupling, g_s, and the instanton angle $\theta_I = 2\pi C$. Alternatively, these two can be combined to form the single complex parameter $\tau = C + ie^{-i\phi}$. All these solutions are in a single orbit of the $SU(1,1) \sim SL(2,\mathbf{R})$ symmetry of Type

IIB supergravity since it acts transitively on τ. The situation is altered in superstring theory since the range of θ_I is quantized so the identification $\theta_I \sim \theta_I + 2\pi$ can be made, which implies $\tau \sim \tau + 1$. This means that the allowed Möbius transformations are elements of the $SL(2, \mathbf{Z})$ subgroup of $SL(2, \mathbf{R})$, where $a, b, c, d \in \mathbf{Z}$. String solutions are mapped into equivalent solutions by these transforms, and so any string theories which are defined on D3 backgrounds which are related by this $SL(2, \mathbf{Z})$ duality are all equivalent.

2.8 Brane Probing

We will now describe the very useful technique of brane probing [6]. We have used brane probing repeatedly in our research as a tool in interpreting supergravity backgrounds in terms of their field theory AdS/CFT duals (see later). In particular, it leads to the unique coordinates appropriate to the field theory dual.

In a brane probe one considers introducing a single probe Dp-brane into the supergravity background created by a large number, N of Dp-branes. N must be large so that $g_s N$ is large, the curvatures are small and we may trust our supergravity solutions. Of course, we want to take g_s small rather than large so we don't have strongly coupled strings. The single Dp-brane can be taken to move in the background without altering it. In brane probing it is sensible to use what is known as "static gauge". This is where spacetime Lorentz invariance and world-volume reparametrizations are used to align the world-volume coordinates, ξ^a, of the brane with the spacetime coordinates so that

$$\begin{aligned} \xi^0 &= x^0 = t; \\ \xi^i = x^i; &\qquad i = 1, ..., p, \\ \xi^m = \xi^m(t); &\qquad m = p+1, ..., 9 \end{aligned} \tag{2.118}$$

The branes have p common directions in which the background fields will have no structure, so the problem is just as a particle moving in the $9-p$ transverse directions. The action for the probe is the Dirac-Born-Infeld action (2.97) plus a coupling to the 4-form, $C_{(p+1)}$,

$$S_p = -T_p \int d^{p+1}\xi e^{-\Phi} \det{}^{1/2}(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}) + \mu_p \int C_{(p+1)} \tag{2.119}$$

where G_{ab} is the pull-back metric,

$$[G]_{ab} = \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\nu}{\partial \xi^b} G_{\mu\nu} \tag{2.120}$$

To perform a brane probe we substitute in all the required terms from the configuration we want to probe and take the probe to be slow moving, i.e. that the velocity defined by $v_m \equiv dx^m/d\xi^0$ is small enough that only terms up to quadratic order in v need be considered in the expansion of the determinant.

The resulting action can tell us a number of important things. The first thing one might look for is how many directions have no potential. Remember that there is a $U(N)$ $(p+1)$-dimensional gauge theory on the N Dp-branes. This theory has a family of $(9-p)$ adjoint scalars, Φ^m. The crucial point is that these fields correspond to the positions of the branes transverse to their world-volumes (c.f. the geometrical Higgs mechanism in Section 2.3). This means that giving a vev to one of these fields corresponds to moving a brane to the equivalent position in the transverse space. If there is no potential to this movement of the brane then there is a "moduli space" of inequivalent vacua in the field theory. This means, for example, that if one has a supergravity solution which is supposed to correspond to $\mathcal{N} = 4$ SYM (see next chapter), then the brane probe had better reveal the required six dimensional moduli space.

Chapter 3

The AdS/CFT Correspondence

This chapter is an exposition of the AdS/CFT Correspondence, with some emphasis on aspects particularly relevant to the later research chapters. $\mathcal{N}=4$ SYM and AdS space are introduced as preliminaries to the Correspondence. For more on the AdS/CFT Correspondence there are many reviews available, for example [50, 13, 51, 52, 53, 54, 55]. To the beginner we might recommend [50], followed by the first half of [13].

3.1 $\mathcal{N}=4$ Super Yang-Mills

The CFT of the AdS/CFT Correspondence is $\mathcal{N}=4$ Super Yang-Mills (SYM) in four dimensions, so we will describe this theory here. The lagrangian for $\mathcal{N}=4$ Super Yang-Mills is unique and may be written as [13],

$$\mathcal{L} = \mathrm{Tr}\left(-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta_I}{8\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} - \sum_a i\bar{\lambda}^a\bar{\sigma}^\mu D_\mu\lambda_a - \sum_i D_\mu X^i D^\mu X^i \right.$$
$$\left. + \sum_{a,b,i} gC_i^{ab}\lambda_a[X^i,\lambda_b] + \sum_{a,b,i} g\bar{C}_{iab}\bar{\lambda}^a[X_i,\bar{\lambda}^b] + \frac{g^2}{2}\sum_{i,j}[X^i,X^j]^2\right) \tag{3.1}$$

This lagrangian can be derived from a dimensional reduction of $d = 10$ $\mathcal{N} = 1$ SYM,

$$\mathcal{L} = -\frac{1}{4g_{YM}^2}\mathrm{Tr}[F_{MN}F^{MN}] - \frac{i}{2}\mathrm{Tr}[\bar{\lambda}\Gamma^M D_M\lambda] \tag{3.2}$$

where λ is a Majorana-Weyl **16** spinor of $SO(1,9)$. The reduction decomposes this symmetry by

$$SO(1,9) \to SO(1,3) \times SO(6) \tag{3.3}$$

which means

$$\mathbf{16} = (2,4) + (\bar{2},\bar{4}) \tag{3.4}$$

The ten dimensional gauge field becomes a 4d gauge field plus six scalar fields:

$$\begin{aligned} A_M = (A_\mu, X_a), \qquad & M = (\mu, a), \\ \mu = 0,1,2,3, \qquad & a = 4,...,9 \end{aligned} \tag{3.5}$$

The four dimensional lagrangian then comes by assuming that the fields depend on x^μ only. The ten dimensional theory we started with is the low energy effective action for open superstrings and we have reduced that to the four dimensional effective theory when the open strings are restricted to end on a D3-brane.

$\mathcal{N} = 4$ is the maximum supersymmetry one can have in four dimensions and have only particles of spin ≤ 1 [56]. There are 16 supercharges that transform as four spinors $(Q^a_\alpha, \bar{Q}^a_{\dot{\beta}})$, $a = 1,2,3,4$, where Q_α, $\bar{Q}_{\dot{\beta}}$ are Weyl spinors. An $SU(4)$ rotation of these four spinors is an automorphism of the supersymmetry algebra, so the theory has an $SU(4)$ R-symmetry. The field content is an '$\mathcal{N}$=4 gauge multiplet', $(A_\mu\ \lambda^a_\alpha\ X^i)$. A_μ is a vector field in the adjoint representation of $SU(N)$, and is a singlet under $SO(6)$. The X^a are six real scalars in the **6** vector representation of

$SO(6)$, which transform in the adjoint of $SU(N)$. The λ^a_α are four Weyl fermions in the adjoint of $SU(N)$ and in a **4** spinor representation of $SO(6)$ (or the fundamental of $SU(4)$).

$\mathcal{N} = 4$ $U(N)$ SYM in $3+1$ dimensions is a conformal (see Appendix A.1) theory, which means that its beta function is zero to all orders. The symmetries of a conformal theory are made up of Poincaré symmetry (Lorentz transformations $L_{\mu\nu}$ and translations P^μ) plus dilatations D and 'special conformal transformations' K^μ, which go together to form an $SO(2,4) \sim SU(2,2)$. In fact, $\mathcal{N} = 4$ $U(N)$ is even more symmetric because the conformal symmetry fits together with the other global symmetries to form a superalgebra. The other constituent symmetries of this superalgebra are:

- The R-symmetry $SO(6)_R \sim SU(4)_R$ generated by T^A, $A = 1, ..., 15$
- The Poincaré supersymmetries as mentioned above, generated by the supercharges Q^a_α and their complex conjugates $\bar{Q}_{\dot{\alpha}a}$, $a = 1, ..., 4$.
- *Conformal supersymmetries* generated by supercharges $S_{\alpha a}$ and their complex conjugates $\bar{S}^a_{\dot{\alpha}}$. These are the commutators of the Poincaré supersymmetries and the special conformal transformations, K_μ. Since these two things are symmetries their commutator must also be a symmetry.

The two bosonic subalgebras $SO(2,4)$ and $SU(4)_R$ commute, while the supercharges Q^a_α and $\bar{S}^a_{\dot{\alpha}}$ transform in the **4** of $SU(4)_R$, and their conjugates $\bar{Q}_{\dot{\alpha}a}$ and $S_{\alpha a}$ in the $\mathbf{4}^*$. So all these generators fit into a superalgebra in the form

$$\begin{pmatrix} P_\mu \; K_\mu \; L_{\mu\nu} \; D & Q^a_\alpha \; \bar{S}^a_{\dot{\alpha}} \\ \bar{Q}_{\dot{\alpha}a} \; S_{\alpha a} & T^A \end{pmatrix} \tag{3.6}$$

and the global continuous symmetry group of $\mathcal{N} = 4$ SYM is in fact the supergroup $SU(2,2|4)$ [57].

The dynamical behaviour of $\mathcal{N} = 4$ SYM can be deduced from the potential energy term,

$$-g^2 \sum_{i,j} \int \mathrm{Tr}[X^i, X^j]^2 \tag{3.7}$$

Since the generators of the gauge algebra, t_r^a, are Hermitian, $\mathrm{Tr}[t_r^a, t_r^b]$ is positive definite. Therefore each sum in this potential is positive or zero. So when the potential is zero we have a minimum, corresponding to an $\mathcal{N} = 4$ supersymmetric ground state. The condition for this ground state

$$[X^i, X^j] = 0, \qquad i, j = 1, ..., 6 \tag{3.8}$$

has two different classes of solution:

- $\langle X^i \rangle = 0$ for all $i = 1, ..., 6$ which is called the *superconformal phase* since the superconformal symmetry $SU(2,2|4)$ is unbroken.
- $\langle X^i \rangle \neq 0$ for at least one i, called the *spontaneously broken or Coulomb phase.* The superconformal symmetry is spontaneously broken since the non-zero vev $\langle X^i \rangle$ sets a scale. The generic symmetry breaking is $\mathcal{G} \to U(1)^r$ where $r = \mathrm{rank}\mathcal{G}$ where the low energy theory will look like r copies of $\mathcal{N} = 4$ $U(1)$.

3.2 Anti-de-Sitter Space

Anti-de-Sitter (AdS) space is a maximally symmetric spacetime with constant negative curvature. $(p+2)$ dimensional AdS space is the hyperboloid

$$x_0^2 + x_{p+2}^2 - \sum_{i=1}^{p+1} x_i^2 = R^2 \tag{3.9}$$

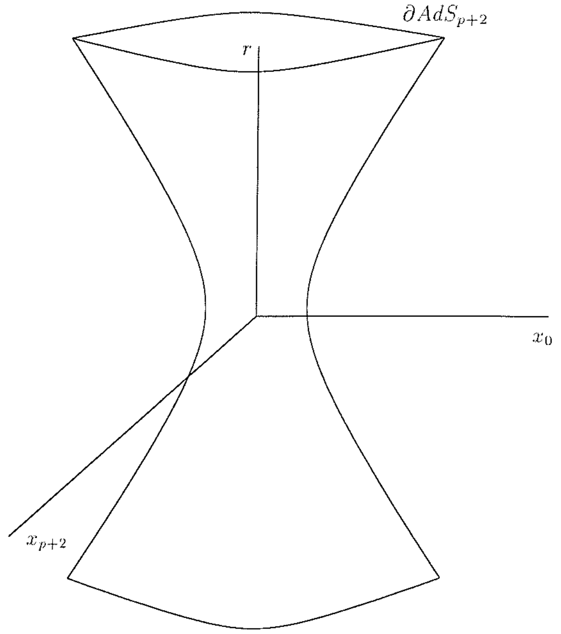

Figure 3.1: (Minkowskian) Anti-de-Sitter Space.

where the x's are coordinates of $\mathbf{R}^{p+3}$ with metric

$$ds^2 = -dx_0^2 - dx_{p+2}^2 + \sum_{i=1}^{p+1} dx_i^2 \tag{3.10}$$

One can see that the topology of this manifold is that of a cylinder $S^1 \times \mathbf{R}$ times a sphere S^p. The boundary is therefore $\partial AdS_{p+2} = S^1 \times S^p$. A sketch of AdS space is shown in Figure 3.1, in which $r^2 = x_1^2 + ... + x_{p+1}^2$. This is obviously a good representation to spot the $SO(2, p+1)$ isometry of AdS_{p+2}, but some coordinate changes can be used to extract other properties of this space.

The first of these is achieved by substituting

$$R^2 z^{-1} = x_{p+1} + x_{p+2}, \qquad v = -x_{p+1} + x_{p+2}, \qquad z_n = \frac{z x_n}{R} \tag{3.11}$$

into (3.10) to obtain the *Poincaré metric* of AdS,

$$ds^2 = \frac{R^2}{z^2}(dz^2 - dz_0^2 + dz_1^2 + ... + dz_p^2) \tag{3.12}$$

Then using a coordinate $u = 1/z$ one can write the AdS metric as

$$ds^2 = R^2u^2(-dz_0^2 + dz_1^2 + ... + dz_p^2) + R^2\frac{du^2}{u^2} \tag{3.13}$$

In this form we can make a very important visualization of AdS space. The terms inside the brackets of (3.13) are just a Minkowski space. In addition we have a single u-direction, so we can think of AdS as a continuum of Minkowski spaces 'stacked' in the radial u-direction. The u^2 factors act as a 'warp factor' between the Minkowski slices. By writing $u = e^y$ we get

$$ds^2 = R^2(dy^2 + e^{2y}dz^\mu dz_\mu) \tag{3.14}$$

in which e^{2y} is the warp factor.

3.3 The AdS/CFT Correspondence

We are now ready to put forward the argument for Maldacena's Conjecture [15]. We consider type IIB superstring theory in the presence of N D3-branes and find two different descriptions for a certain limit of this configuration.

In this background string theory has closed string excitations which propagate in the bulk and open string excitations, attached to the D3-branes. At energies much less than the string scale $1/l_s$, there are only massless excitations and the effective action is

$$I = \int d^{10}x \mathcal{L}_{IIB} + \int d^4x \mathcal{L}_{brane} \tag{3.15}$$

Here $\mathcal{L}_{IIB}$ is the effective action of IIB string theory, containing the supergravity action plus higher derivative terms and $\mathcal{L}_{brane}$ is the low energy theory on the brane. If we now take the low energy $\alpha' = l_s^2 \to 0$ limit then the gravitational coupling goes to zero,

$$8\pi G_{10} = \kappa^2 \sim g^2\alpha'^2 \to 0 \tag{3.16}$$

So in this limit the gravitational and higher derivative interaction terms disappear. We saw in the previous chapter that $\mathcal{L}_{brane}$ becomes $U(N)$ SYM in 3+1 dimensions in the $\alpha' \to 0$ limit, so what we are left with after taking this limit is $\mathcal{N} = 4$ $U(N)$ SYM with free gravity in the bulk.

Now turn to the supergravity description for this D3-brane configuration. Such a solution may be written as

$$ds^2 = f^{-1/2}(r)[-dt^2 + dx_1^2 + dx_2^2 + dx_3^2] + f^{1/2}(r)(dr^2 + r^2 d\Omega_5^2) \tag{3.17}$$

with

$$F_{0123r} = \partial_r f^{-1}, \qquad e^{2\phi} = g^2 = \text{const} \tag{3.18}$$

where

$$f(r) = 1 + \frac{\alpha'^2 R^4}{r^4}, \tag{3.19}$$

Clearly, when $r^2 \gg \alpha' R^2$ this spacetime looks like flat $\mathbf{R}^{10}$. $r^2 < \alpha' R^2$ (which means close to the branes) is often called the 'throat' region. This label is motivated by the

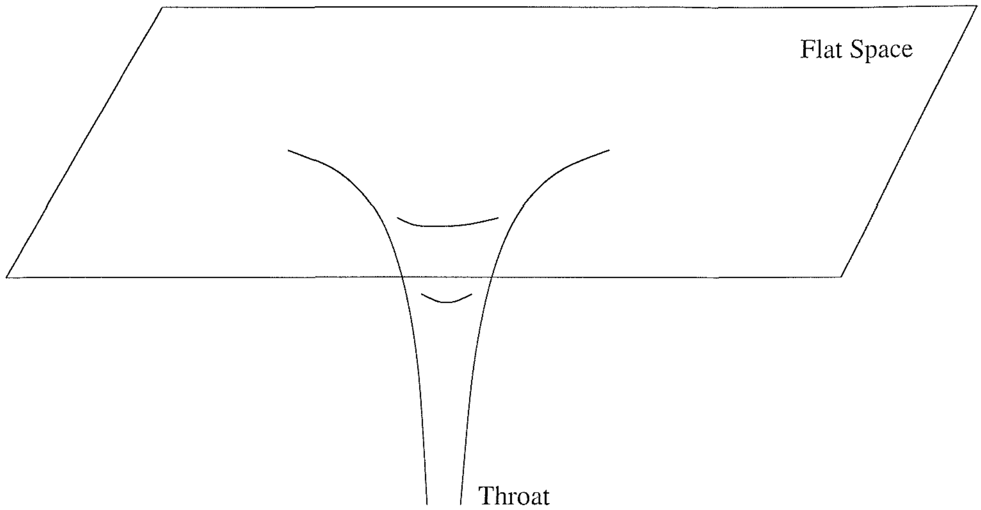

Figure 3.2: A picture representing the spacetime for N coincident D3 branes, showing the flat region and the throat, where the space looks like $AdS_5 \times S^5$

visualization of the space shown in Figure 3.2.

This background has two types of low-energy excitations; those close to the horizon at $r = 0$, and massless particles in the bulk. The large redshift near the horizon means that an excitation of energy E in that region is seen from infinity to have an energy,

$$E_\infty = f^{-1/4} E \sim \frac{r}{\alpha'}(E\sqrt{\alpha'}) \tag{3.20}$$

Now if we take the $\alpha' \to 0$ limit as in the previous description, keeping $r/\alpha' \equiv u$ constant (so $r \to 0$), then an observer at infinity measures finite energies for string excitations of level $n = \alpha' E^2$. Applying this limit to the metric (3.17) gives,

$$ds^2 = \alpha' \left[\frac{u^2}{R^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + R^2 \frac{du^2}{u^2} + R^2 d\Omega_5^2 \right] \tag{3.21}$$

which is just $AdS_5 \times S^5$; we have superstring theory on $AdS_5 \times S^5$. Maldacena spotted that there are two different descriptions of the same configuration and conjectured that they must be equivalent. The statement of the AdS/CFT Correspondence is that the following two theories are dual,

- Type IIB superstring theory on $AdS_5 \times S^5$, with equal radius L for both AdS_5 and S^5, with integer flux $N = \int_{S^5} F_5^+$ for the 5-form F_5^+, and with string coupling g_s.
- $\mathcal{N} = 4$ SYM in 4 dimensions, with gauge group $SU(N)$ and Yang-Mills coupling g_{YM}, in its superconformal phase.

with the following identifications between the parameters of the string and gauge theories,

$$g_s = g_{YM}^2, \qquad L^4 = 4\pi g_s N(\alpha')^2, \qquad \langle C \rangle = \theta_I \tag{3.22}$$

where $\langle C \rangle$ is the axion expectation value.

This statement applies for all values of N and $g_s = g_{YM}^2$, but the quantization of string theory on a general curved manifold, including $AdS_5 \times S^5$ is an unsolved problem. It is possible though to take different limits of this so-called 'strong form' of the conjecture which are still highly non-trivial but are much more open to study.

3.3.1 The 't Hooft Limit

Keeping the 't Hooft coupling [58], $\lambda \equiv g_{YM}^2 N = g_s N$ fixed while taking $N \to \infty$ constitutes the 't Hooft limit. This limit of Yang-Mills field theory yields a perturbative topological expansion in Feynman diagrams. On the AdS side of the Correspondence, the string coupling may be written in terms of the 't Hooft coupling as $g_s = \lambda/N$. Therefore the string coupling goes small in the 't Hooft limit, giving weak coupling string perturbation theory. In this limit the conjecture becomes a correspondence between classical string theory and the large N limit of gauge theories.

3.3.2 The Large 't Hooft Coupling Limit

λ is the only parameter left after taking the 't Hooft limit. The perturbative limit of quantum field theory is when $\lambda \ll 1$, however, $\lambda \gg 1$ is the natural limit to take on the AdS side, as we will now show. An expansion of the effective action in powers of α' takes the form [13]

$$\mathcal{L} = a_1 \alpha' R + a_2 (\alpha')^2 R^2 + a_3 (\alpha')^3 R^3 + ... \tag{3.23}$$

with R the Riemann tensor. We are concerned with the throat region, where distance scales are set by the AdS radius L. Therefore, the Riemann tensor is of the scale

$$R \sim \frac{1}{L^2} = \frac{(g_s N)^{-1/2}}{\alpha'} = \frac{\lambda^{-1/2}}{\alpha'} \tag{3.24}$$

so the expansion of the effective action in powers of α' becomes an expansion in powers of $\lambda^{-1/2}$:

$$\mathcal{L} = a_1 \lambda^{-1/2} + a_2 \lambda^{-1} + a_3 (\alpha')^3 \lambda^{-3/2} + ... \tag{3.25}$$

In the large λ limit, the conjecture becomes an equivalence between this expansion of string theory in $\lambda^{-1/2}$ and the strong coupling limit of Yang-Mills field theory. The three forms of the AdS/CFT conjecture we have just covered are summarised in Table 3.1.

• $\mathcal{N}=4$ SYM • $g_s = g_{YM}^2$	$\Leftrightarrow$	• Type IIB string theory on $AdS_5 \times S^5$ • $L^4 = 4\pi g_s N \alpha'^2$
• 't Hooft limit of $\mathcal{N}=4$ SYM ($\lambda = g_{YM}^2 N$ fixed, $N \to \infty$) • $1/N$ expansion	$\Leftrightarrow$	• Classical Type IIB string theory on $AdS_5 \times S^5$ • g_s string loop expansion
• Large λ limit of $\mathcal{N}=4$ SYM (for $N \to \infty$) • $\lambda^{-1/2}$ expansion	$\Leftrightarrow$	• Classical Type IIB supergravity on on $AdS_5 \times S^5$ • α' expansion

Table 3.1: The AdS/CFT Correspondence's three forms, in decreasing strength order.

3.4 Mapping Global Symmetries

The most obvious thing to check after proposing the duality is that the global unbroken symmetries of the two theories are identical:

1) $SU(2,2|4)$

We saw earlier that the continous global symmetry of $\mathcal{N} = 4$ SYM in its conformal phase is $SU(2,2|4)$, with $SU(2,2) \times SU(4)_R \sim SO(2,4) \times SO(6)_R$ the maximal bosonic subgroup. This is the isometry group of $AdS_5 \times S^5$; $SO(2,4)$ for the AdS_5 and $SO(6)_R$ for the S^5. On the AdS side the supergroup is completed because only 16 of the 32 Poincaré supersymmetries are preserved in the full D3-brane geometry, and taking the AdS limit introduces an additional 16 conformal supersymmetries.

2) $SL(2,Z)$

Type IIB string theory has an $SL(2,Z)$ S-duality (c.f. the discussions of IIB SUGRA and D3-branes in the previous chapter); if we define

$$\tau = \frac{A}{2\pi} + ie^{-\phi} = \tau_1 + i\tau_2 \tag{3.26}$$

then

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad ad - bc = 1, \qquad a,b,c,d \in Z \tag{3.27}$$

is the $SL(2, Z)$ symmetry. $\mathcal{N} = 4$ SYM is invariant under the similar transformation,

$$\tau = \frac{\theta_I}{2\pi} + i\frac{4\pi}{g_{YM}^2} \tag{3.28}$$

This is the Montonen-Olive or S-duality symmetry [59]. It should be pointed out that this S-duality is a good symmetry only for the strongest form of the Correspondence. It is spoilt by taking the 't Hooft limit; if one then takes $\theta_I = 0$, then the S-duality maps $g_{YM} \to 1/g_{YM}$ and so $\lambda \to N^2/\lambda$, which is supposed to be fixed in the 't Hooft limit.

3.5 Mapping Representations and Correlators

We have shown that the global symmetries on either side of the correspondence match. The next thing to show is that the representations of $SU(2, 2|4)$ also coincide.

Single colour trace operators are important objects since they can be used to form any higher trace operators, via the operator product expansion. It is natural to identify these single trace operators in the SYM with the single particle states (or canonical fields) on the AdS side [15, 60]. Bound states of these one particle states will then correspond to the multiple trace states.[1]

To find the contents of irreducible representations of $SU(2,2|4)$ on the AdS side one can use fields φ on the $AdS_5 \times S^5$ to describe all Type IIB massless supergravity and massive string degrees of freedom. We write the metric as

$$ds^2 = g^{AdS}_{\mu\nu} dz^\mu dz^\nu + g^{S}_{\mu\nu} dy^u dy^v \tag{3.29}$$

where we have introduced coordinates z^μ, $\mu = 0, ..., 4$ for the AdS_5 and y^u, $u = 1, ..5$ for the S^5. φ is then a function of z and y and can be decomposed as a series on S^5,

$$\varphi(z, y) = \sum_{\Delta=0}^{\infty} \varphi_\Delta(z) Y_\Delta(y) \tag{3.30}$$

where Y_Δ are a basis of spherical harmonics on S^5. In the case of scalars, for example, the Y_Δ are labelled by the rank Δ of the totally symmetric traceless representations of $SO(6)$. The momentum mode of a field on circle contributes to the mass of that field; fields compactified on S^5 get a mass contribution in the same way. A relation between the mass and scaling dimension for the scalar case may be derived in the following manner [61, 62, 63]. If we assume that away from the bulk interaction region, the bulk fields are free asymptotically, then the free field satisfies $(\Box_{AdS} + m^2_\Delta)\phi_{r\to\infty} = 0$ for scalars. Writing the AdS_5 metric as

$$ds^2 = e^{2r/L} dx^\mu dx_\mu + dr^2 \tag{3.31}$$

[1]Single and higher trace operators are reviewed in Appendix A.2, along with other background material relevant to this section, including descendant states and $\mathcal{N} = 4$ chiral or BPS multiplets.

the scalar field equation is

$$\phi'' + \frac{4}{L}\phi' = m_\Delta^2 \phi \tag{3.32}$$

This has solution

$$\phi = \mathcal{A}e^{-\Delta r/L} + \mathcal{B}e^{-(4-\Delta)r/L} \tag{3.33}$$

with

$$m_\Delta^2 = \Delta(\Delta - 4) \tag{3.34}$$

Relations between mass and scaling dimension for other spins may be similarly found:

$$\begin{aligned} \text{spin } 1/2,\ 3/2 \qquad & |m| = \Delta - 2 \\ \text{p-form} \qquad & m^2 = (\Delta - p)(\Delta + p - 4) \\ \text{spin } 2 \qquad & m^2 = \Delta(\Delta - 4) \end{aligned} \tag{3.35}$$

The full mapping of $SU(2,2|4)$ representations in the correspondence is summarized in Table 3.2. The mapping of descendant states can be worked out explicitly. A particular case which is relevant for the research chapters is that for the SYM operator, $\mathcal{O}_k \sim \mathrm{tr}X^k$, $k \geq 2$. This corresponds to the supergravity fields h_α^α and $A_{4\alpha\beta\gamma\delta}$. Its dimension is k and its $SO(2,4) \times U(1)_Y \times SU(4)_R$ quantum numbers are spin $(0,0)$, $Y = 0$ and $(0,k,0)$. Its lowest representations are the **20'**, **50** and **105**.

We now need to take the final step in making the proposal of duality complete by setting out how correlators are to be mapped between the two theories. At the AdS boundary, $u = \infty$, the string fields are general functions of x^μ. These fields act as sources for operators in the Yang-Mills field theory. The prescription for the field

Type IIB String Theory	$\mathcal{N}=4$ SYM
Supergravity Excitations 1/2 BPS, spin$\leq$2	Chiral primary + Descendants $\mathcal{O}_2$=tr$X^{\{i}X^{j\}}$ + desc.
Supergravity Kaluza-Klein 1/2 BPS, spin$\leq$2	Chiral primary + Descendants $\mathcal{O}_\Delta$=tr$X^{\{i_1}\cdots X^{i_\Delta\}}$ + desc.
Type IIB massive string modes non-chiral, long multiplets	Non-chiral operators, dimensions $\sim\lambda^{1/4}$ e.g. Konishi trX^iX^i
Multiparticle states	products of operators at different points $\mathcal{O}_{\Delta_1}(x_1)...\mathcal{O}_{\Delta_n}(x_n)$
Bound states	product of operators at same point $\mathcal{O}_{\Delta_1}(\mathrm{x})...\mathcal{O}_{\Delta_n}(\mathrm{x})$

Table 3.2: Mapping string and supergravity states to SYM operators.

theory correlator functions is

$$\begin{aligned}\langle\exp\left[\int d^4x\phi_0(x)\mathcal{O}\right]\rangle &= Z_{string}(\phi_0(x)) \\ &\cong \exp[-I_{sugra}(\phi_0)]\end{aligned} \tag{3.36}$$

where

$$\phi_0(x)=\phi(x,u)|_{u=\infty} \tag{3.37}$$

This sets out explicitly that each field propagating in the AdS corresponds to an operator in the CFT.

At this point it can be seen how to generalize the AdS/CFT correspondence to general string vacua on $AdS_5\times X_5$, by taking Equation (3.36) to define the conformal field theory correlators via the string partition function on the said more general space. Just this sort of generalization arises in the following research chapters. These backgrounds tend not to preserve any supersymmetry and therefore define non-supersymmetric four dimensional conformal field theories. The set-up can, in fact, be generalized further by using any 5d asptotically AdS space, Y, in place of the AdS_5. Any ten-dimensional string solution that looks like $Y\times X$ at infinity can be considered, but unless there is a D-brane interpretation available to enable an understanding of the low energy theory, it is difficult to find the dual field theory.

3.6 Renormalization Group Flow in AdS/CFT

An important feature of the duality is how Renormalization Group (RG) flow [3] in the field theory is incorporated in the AdS side. Consider again the $\mathcal{N} = 4$'s conformal symmetry $SO(2,4)$; this includes the dilatations

$$x \to e^{\alpha} x, \qquad \phi \to e^{-\alpha}\phi \tag{3.38}$$

which leave the massless scalar action,

$$\int d^4x \partial^{\mu}\phi \partial_{\mu}\phi \tag{3.39}$$

invariant. The supergravity theory, having mass scales associated with the Kaluza-Klein states on the S^5, is not conformal and this symmetry is instead realized as a pure spacetime symmetry. In particular, for the x-transformation in (3.38) to be a symmetry of the AdS_5 metric

$$ds^2 = \frac{du^2}{u^2} + u^2 dx^{\mu} dx_{\mu} \tag{3.40}$$

u must transform as

$$u \to e^{-\alpha} u \tag{3.41}$$

So the duality implies that under dilatations of the field theory, the radial u direction of AdS transforms as a mass dimension. The natural identification to make is that motion in the radial direction of AdS corresponds to RG flow in the field theory.

The field theory may live on any of the Minkowski slices of AdS_5 but the action on each, defined by the supergravity vevs, is that of the field theory at a different RG scale. Maldacena's original proposal of duality concerned $\mathcal{N} = 4$ at its moduli space origin, for which there are no sources turned on and so there are no supergravity field vevs. The supergravity is the same on every Minkowski slice, because $\mathcal{N} = 4$ SYM is conformal. In the following two research chapters we will be investigating extensions of the correspondence to describe field theories with more interesting RG flows.

Chapter 4

The $\mathcal{N} = 4$ Geometry

This covers the first half of the paper 'Secrets of the Metric in $\mathcal{N} = 4$ and $\mathcal{N} = 2^*$ Geometries' (Babington, Evans and Hockings [8])

4.1 Introduction

Maldacena's original duality proposal concerns $\mathcal{N} = 4$ at the origin of its moduli space, so an obvious step in testing how generic such dualities are is to try to extend to the whole of the $\mathcal{N} = 4$ moduli space. This has been done by deducing the gravity duals both from D3-brane constructions [19, 35] and from deformed 5d gravity solutions [34] lifted to 10d solutions [46]. These two solutions have been shown to match. We have examined this connection from the field theory side. We performed a brane probe on the 10d lift supergravity solution and showed that this provides an easy method of finding the natural coordinates of the field theory. In these coordinates, a prescription for the encoding of the metric in terms of field theory operators is manifest. While the 5d solutions describe only part of the full moduli space, the full set of 10d supergravity solutions needed to cover the whole moduli space can be deduced from this encoding prescription.

This explicitly realises the expectation that because the two theories are dual they should simply be reparametrisations of the same "solution", and if the complete solution to some field theory is known then its gravity dual should be uniquely determined.

Before launching into the research content in the next section it is a good idea to give a discussion of multicentre D3-brane solutions, bringing together a number of points covered in Chapter 2. The multicentre solution describing N parallel D3-branes spread out at multiple points in the transverse space is given by

$$ds^2 = H(\vec{y})^{-1/2} dx^\mu dx_\mu + H(\vec{y})^{1/2} d\vec{y}^2, \quad e^\Phi = \text{constant} \tag{4.1}$$

where

$$H(\vec{y}) = 1 + \sum_{I=1}^{N} \frac{C_I}{|\vec{y} - \vec{y}_I|^4}, \qquad C_I = N_I \rho_3, \ N_I \in \mathbf{N} \tag{4.2}$$

and $\rho_3 = 4\pi g_s \alpha'^2$, c.f. Equations (2.111) and (2.114). Open strings are stretched between the D3-branes in the transverse space. If all N of these D3-branes are coincident then Chan-Paton factors on the ends of the open strings give rise to a $U(N)$ 4-dimensional field theory on the D3-branes' worldvolumes (which has $\mathcal{N} = 4$ supersymmetry and is in fact superconformal). This theory contains a family of six adjoint scalars, Φ^m. If one of the D3-branes is moved away from the other $N-1$ then the field theory on the worldvolumes is broken to $U(N-1) \times U(1)$, with the $U(N-1)$ living on the $N-1$ branes and the $U(1)$ living on the single brane. The position of the single brane in the six tranverse dimensions is given by the vevs of the six scalars. This vev is then the minimal length for the open strings stretched between the branes. A massive 'W-boson' is formed from this broken gauge group. Its mass is given by the tension of the stretched string multiplied by the distance stretched, and it is in the bi-

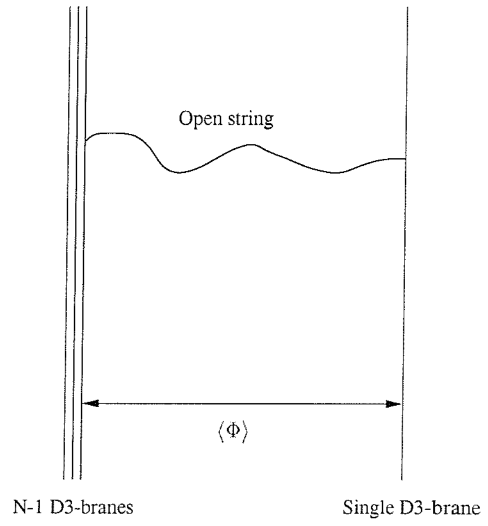

Figure 4.1: Separating a single D3-brane from $N-1$ D3-branes.

fundamental representation of $U(N-1) \times U(1)$. This provides a geometrical version of the Higgs mechanism. Figure 4.1 provides an illustration, where the vertical lines represent the D3-branes and the horizontal direction represents the six dimensions transverse to the D3-branes, c.f. Figure 2.4. Moving all the D3-branes to different points in the transverse space (the general multicentre solution) provides the generic breaking of the field theory symmetry, $U(N) \to U(1)^N$. If they are separated out in k groups, instead of singly, the $U(N)$ will be broken to $U(N_1) \times ... \times U(N_i) \times ... \times U(N_k)$, where $\sum_{i=1}^{k} N_i = N$, and the W-bosons will be in the bi-fundamental representations of $U(N_i) \times U(N_j)$.

4.2 The gravity dual of $\mathcal{N} = 4$ on moduli space

We wish to examine the six dimensional moduli space of $\mathcal{N} = 4$ SYM by using gravity duals derived from 5d supergravity. This moduli space is parametrised by the six scalars, ϕ, so we want to have a non-zero vev for the scalar operator $\mathrm{tr}\phi^2$.

This operator is a symmetric traceless 6×6 matrix which transforms in the **20** of the global $SU(4)_R$ symmetry. The AdS/CFT correspondence tells us that the scalar, α, in the 5D truncation of IIB supergravity on $AdS_5 \times S^5$ acts as the source for $\mathrm{tr}\phi^2$. So to study $\mathcal{N} = 4$ SYM with a scalar vev switched on, one should look for solutions of the 5d supergravity equations of motion with non-zero α. The task of getting these equations of motion is actually quite hard work as the scalars live in the coset $E_6/USp(8)$. Happily, these matters are discussed in [34], so we can just work from the final results here.

We shall consider the example of turning on $\mathrm{tr}\phi^2 = \mathrm{diag}(1,1,1,1,-2,-2)$. The corresponding supergravity scalar was identified in [34]. Now, of course, in the supergravity theory the metric is dynamical and one cannot consider the scalar vev in isolation. The metric may be parametrized as

$$ds^2 = e^{2A(r)} dx_{//}^2 - dr^2 \tag{4.3}$$

where $x_{//}$ are the coordinates of the Minkowski space slices of the AdS space, r is the radial direction, and in the AdS limit $A(r) = r/L$, L being the AdS radius. The equations of motion preserving sixteen supercharges are first order,

$$\frac{\partial \rho}{\partial r} = \frac{1}{3L}\left(\frac{1}{\rho} - \rho^5\right), \qquad \frac{\partial A}{\partial r} = \frac{2}{3L}\left(\frac{1}{\rho^2} + \frac{\rho^4}{2}\right) \tag{4.4}$$

where $\rho = e^{\alpha}$ and the fermionic shifts vanish. One can solve these equations in the $\rho - A$ plane using

$$\frac{\partial \rho}{\partial A} = \frac{1}{2}\left(\frac{\rho - \rho^7}{1 + \rho^6/2}\right) \tag{4.5}$$

with the result

$$e^{2A} = \frac{l^2}{L^2}\frac{\rho^4}{\rho^6 - 1} \tag{4.6}$$

l^2/L^2 is a constant of integration. At this point we already have a solution of the supergravity with the α scalar turned on, but it is not yet clear what this can tell us about the dual gauge theory.

The first step is to use the lift of this solution back to ten dimensions [34, 46],

$$ds^2 = \frac{X^{1/2}}{\rho} e^{2A(r)} dx_{//}^2 - \frac{X^{1/2}}{\rho}\left(dr^2 + \frac{L^2}{\rho^2}\left[d\theta^2 + \frac{\sin^2\theta}{X} d\phi^2 + \frac{\rho^6 \cos^2\theta}{X} \Omega_3^2\right]\right) \quad (4.7)$$

where $d\Omega_3^2$ is the metric on a 3-sphere and

$$X \equiv \cos^2\theta + \rho^6 \sin^2\theta \quad (4.8)$$

A consistent solution also requires a non-zero C_4 potential of the form

$$C_4 = \frac{e^{4A} X}{g_s \rho^2} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \quad (4.9)$$

This metric has been shown [34] to be equivalent to the near horizon limit of a multi-centre D3-brane distribution solution. It is still far from clear how this relates to the field theory, but now we are in ten dimensions we will be able to use the tool of brane probing which will give us the unique set of coordinates in which the field theory duality is manifest.

We now perform the brane probe by substituting (4.7)-(4.9) into the Dirac-Born-Infeld action,

$$S_{probe} = -\tau_3 \int_{\mathcal{M}_4} d^4x \det\left[G_{ab}^{(E)} + 2\pi\alpha' e^{-\Phi/2} F_{ab}\right]^{1/2} + \mu_3 \int_{\mathcal{M}_4} C_4 \quad (4.10)$$

with the result

$$S = -\frac{\mu_3}{2g_s}\int_{\mathcal{M}_4} d^4x \left[\frac{X e^{2A}}{\rho^2}\dot{r}^2 + \frac{L^2 e^{2A}}{\rho^4}\left(X\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2 + \rho^6 \cos^2\theta\, \dot{\Omega}_3^2\right)\right] \quad (4.11)$$

The first thing to spot is that there is no potential for the motion of the probe in the six transverse directions, which corresponds to a six dimensional moduli space for the field theory scalars. This is as expected, since $\mathcal{N}=4$ SYM has six scalars with a potential of the form $\mathrm{tr}[\phi,\phi]^2$, which vanishes when one takes commuting vevs.

These kinetic terms correspond to kinetic terms for the field theory scalars. In $\mathcal{N}=4$ SYM (in $\mathcal{N}=1$ notation) these are $1/8\pi \mathrm{Im}(\tau\Phi^\dagger\Phi)|_D$, so the coefficients of these kinetic terms are the gauge coupling. The $\mathcal{N}=4$ theory is conformal, so the probe should see a flat metric on moduli space. This is not manifest as it stands, but one can make the probe metric flat with the coordinate change

$$(r,\theta)\to(u,\alpha) \tag{4.12}$$

such that

$$u^2\cos^2\alpha = L^2 e^{2A}\rho^2\cos^2\theta, \qquad u^2\sin^2\alpha = L^2\frac{e^{2A}}{\rho^4}\sin^2\theta \tag{4.13}$$

In these coordinates the action becomes

$$S=-\frac{\mu_3}{2g_s}\int_{\mathcal{M}_4} d^4x\left[\dot{u}^2+u^2\left(\dot{\alpha}^2\ +\sin^2\alpha\ \dot{\phi}^2+\cos^2\alpha\ \dot{\Omega}_3^2\right)\right] \tag{4.14}$$

We now have the unique set of coordinates in which the probe displays the conformal property and the gravity solution should now have a clear field theory interpretation. Applying this coordinate change to the full metric gives

$$ds^2=\left(\frac{\rho^2}{Xe^{4A}}\right)^{-1/2}dx_{//}^2+\left(\frac{\rho^2}{Xe^{4A}}\right)^{1/2}\sum_{i=1}^{6}(du_i)^2 \tag{4.15}$$

which is of the standard form

$$ds^2=H^{-1/2}dx_{//}^2+H^{1/2}\sum_{i=1}^{6}du_i^2, \qquad C_4=\frac{1}{Hg_s}dx^0\wedge dx^1\wedge dx^2\wedge dx^3 \tag{4.16}$$

Using the 5d solution (4.6), we can extract from the coordinate transformations (4.13) a quadratic for ρ^6 in the new coordinates[1],

$$\frac{u^2}{l^2}\sin^2\rho^{12}+\left(\frac{u^2}{l^2}\cos^2\alpha-\frac{u^2}{l^2}\sin^2\alpha-1\right)\rho^6-\frac{u^2}{l^2}\cos^2\alpha=0 \tag{4.19}$$

Taking the large u limit to connect with the field theory and solving for ρ^6 yields

$$\rho^6=1+\frac{l^2}{u^2}+\left(\frac{l^2}{u^2}\right)^2(1-\sin^2\alpha)+\left(\frac{l^2}{u^2}\right)^3(1-3\sin^2\alpha+2\sin^4\alpha) \\ +\mathcal{O}\left(\frac{L^4l^6}{u^{10}}\right) \tag{4.20}$$

and using this to calculate H from (4.15) gives

$$H(u)=\frac{L^4}{u^4}\left(1+\frac{l^2}{u^2}(3\sin^2\alpha-1)+\frac{l^4}{u^4}(1-8\sin^2\alpha+10\sin^4\alpha)\right)+\mathcal{O}\left(\frac{L^4l^6}{u^{10}}\right) \tag{4.21}$$

Finally we have arrived at the point were we should be able to interpret the metric in terms of field theory operators. Firstly u has the scaling dimension of mass, so the scaling dimension of the coefficient of each term can be read off. Secondly it turns out that each term has a unique spherical harmonic in it; the angular dependence of the $1/u^6$ term is the **20** of $SU(4)_R$ spherical harmonic, that of the $1/u^8$ term is the **50** of $SU(4)_R$ spherical harmonic, etc. So we see that the Nth coefficient has the dimension and symmetry properties of trϕ^n, plus the operators are not renormalized - each spherical harmonic appears only once. This leads one to the general form

[1] It has been shown [34] that in these coordinates $H(u)$ can be written as a multi-centre solution with a D3 density, σ,

$$H(u)=\int d^6x\;\sigma(x)\frac{L^4}{|\vec{u}-\vec{x}|^4} \tag{4.17}$$

and here σ is a uniform density 2d disk in the $\theta=\pi/2$ plane

$$\sigma(x)=\frac{1}{\pi l^2}\theta(l^2-x^2) \tag{4.18}$$

for H,

$$H(u) = \frac{L^4}{u^4}\left(1 + \sum_n \frac{\mathrm{tr}\phi^n}{u^n} Y_n\right) \tag{4.22}$$

So it turns out that there is a very nice interpretation of the supergravity solution in terms of the field theory. The obvious next step is to see if similar results can be obtained in more complicated RG flow theories, so in the next chapter we will examine the $\mathcal{N} = 2^*$ theory.

Before moving on there are a couple of points to be noted. The first is that in the 5d supergravity theory, only a vev for the dimension two operator, $\mathrm{tr}\phi^2$, was introduced, but the 10d lifted solution contains vevs for higher dimension operators. The original would be recovered in a truncation back to five dimensions. As shown by (4.21), the 5d supergravity gives specific relations between the operators, whilst the six dimensional moduli space of the field theory tells us that they should be arbitrary. Indeed, this expansion but with arbitrary coefficients has been shown to solve the supergravity field equations [19]. This really had to be the case as it was already known that the multi-centre configurations solve the field equations for arbitrary D3-brane distributions.

The second point of interest is in further appreciating the power of the brane probing proceedure. The solutions above can be derived from a brane probe alone. To find the metric dual to a point on the moduli space of $\mathcal{N} = 4$ one could start with an arbitrary 10d metric. Then if we impose the fact that a probe brane should see both a six dimensional moduli space and a conformal coupling, the metric must be of the form of (4.16). Taking this as an ansatz, the supergravity equations reduce to the six dimensional transverse flat space laplacian [47],

$$\Delta_6 H(u) = 0 \tag{4.23}$$

and this has the multi-centre solutions. Again, as should be possible for two truly dual theories, we see the supergravity being uniquely determined from the field theory.

4.3 Summary

Our goal in this chapter was to discover how the field theory operators of $\mathcal{N} = 4$ on moduli space are encoded in their gravity dual solutions. We were successful in doing this and found a very nice prescripton for said encoding. This invites one to see if similar success can be achieved in a more complicated theory, which is the aim of the next chapter.

Chapter 5

The $\mathcal{N} = 2^*$ Geometry

This covers the second half of the paper 'Secrets of the Metric in $\mathcal{N} = 4$ and $\mathcal{N} = 2^*$ Geometries' (Babington, Evans and Hockings [8])

5.1 Introduction

After finding the prescription for the encoding of the field theory operators in the $\mathcal{N} = 4$ case in the previous chapter, it is natural to hope that this prescription is more generic. In this chapter we investigate this by examining the gravity dual of the $\mathcal{N} = 2^*$ theory, that is, the $\mathcal{N} = 4$ theory with a mass term added which breaks the the supersymmetry to $\mathcal{N} = 2$ at low energy. This is a more interesting field theory in that it has more interesting RG flow properties.

Such solutions have been found by introducing relevant perturbations in the 5d supergravity theory in [32, 44, 45] and lifted to 10d by Pilch and Warner [46] and also in [44]. It is far from clear how to interpret these 10d results in terms of the gauge theory, but the connections were elucidated via the power of brane probing in [27, 28]. The metric was found to describe a 2d moduli space, as required for an $\mathcal{N} = 2$ theory. The function for the gauge coupling was also determined and, once cast in the suitable $\mathcal{N} = 2$ coordinates [27], it is as expected from field theory. Of the set of solutions describing the moduli space there is a particularly interesting one corresponding to a singular point where in the IR the gauge coupling diverges. This is an example of the enhancon mechanism [33], when there are points in the space where a probe brane's tension falls to zero.

When we write the moduli space metric in the appropriate field theory coordinates it takes the form of a single function, as in the $\mathcal{N} = 4$ case, multiplied by the gauge coupling function. We then applied our $\mathcal{N} = 4$ presciption to this function to read off gauge theory operators. It has been shown that the only RG flow in the field theory is in the gauge coupling [48], but the supergravity solution seems to also have renormalisation of the scalar operators. Also there is a logarithmic renormalisation in the far UV which prevents a return to the $\mathcal{N} = 4$ form.

To re-enforce this discrepancy, we applied the method of [27] to derive the D3-brane distribution as a function of the moduli space coordinates from the expected field theory gauge coupling, and the supergravity form of the coupling. This allows us to arrive at the distributions for all the 5d supergravity lifts and from these we can work out the expected scalar operators; again these differ from the function in the metric. We presume that there is some hidden subtlety that renders the prescription inaccurate for the more complicated $\mathcal{N} = 2^*$ theory.

5.2 The gravity dual of $\mathcal{N} = 2^*$

As mentioned above, the 5d supergravity with the required deformations turned on had previously been studied in [32, 44, 45]. Two scalars must be turned on in the supergravity; one describes the mass term and the other any vev given to the two remaining scalar fields. Some interpretation of these 5d solutions in terms of the dual field theory had been achieved, but much was still unclear. In [46, 44] these solutions have been lifted to 10d. The result may be summarised:

$$\begin{aligned} ds^2 &= \Omega^2(e^{2A}dx_{//}^2 + dr^2) \\ &+\frac{L^2\Omega^2}{\rho^2}\left(\frac{d\theta^2}{c}+\rho^6\cos^2\theta\left(\frac{\sigma_3^2}{cX_2}+\frac{\sigma_1^2+\sigma_2^2}{X_1}\right)+\frac{\sin^2\theta}{X_2}d\phi^2\right) \end{aligned} \tag{5.1}$$

$$\begin{aligned} \Omega^2 &= \frac{(cX_1X_2)^{1/4}}{\rho}, & c &= \cosh 2m \\ X_1 &= \cos^2\theta + \rho^6 c\sin^2\theta, & X_2 &= c\cos^2\theta + \rho^6\sin^2\theta \\ & & C_4 &= \frac{e^{4A}X_1}{g_s\rho^2}dx^0\wedge dx^1\wedge dx^2\wedge dx^3 \end{aligned} \tag{5.2}$$

with

$$\begin{aligned} \sigma_1 &= \frac{1}{2}(\cos\alpha d\psi + \sin\alpha\sin\beta d\beta) \\ \sigma_2 &= \frac{1}{2}(-\sin\alpha d\psi + \cos\alpha\sin\psi d\beta) \\ \sigma_3 &= \frac{1}{2}(d\alpha + \cos\psi d\beta) \end{aligned} \tag{5.3}$$

Also, the dilaton is non-trivial, and writing the complex scalar $\lambda = C_0 + ie^{-\Phi}$ we have

$$\lambda = i\left(\frac{1-B}{1+B}\right), \qquad B = \left(\frac{b^{1/4}-b^{-1/4}}{b^{1/4}+b^{-1/4}}\right), \qquad b = \cosh(2m)\frac{X_1}{X_2} \tag{5.4}$$

m, A and $\rho = e^{\alpha}$ are the supergravity fields determined by the 5d supergravity equations of motion:

$$\begin{aligned} \frac{\partial\alpha}{\partial r} &= \frac{1}{3L}\left(\frac{1}{\rho^2} - \rho^4\cosh(2m)\right) \\ \frac{\partial A}{\partial r} &= \frac{2}{3L}\left(\frac{1}{\rho^2} + \frac{1}{2}\rho^4\cosh(2m)\right) \end{aligned}$$

$$\frac{\partial m}{\partial r} = -\frac{1}{2L}\rho^4 \sinh(2m) \tag{5.5}$$

which are solved by

$$\begin{aligned} e^A &= k\frac{\rho^2}{\sinh(2m)} \\ \rho^6 &= \cosh(2m) + \sinh^2(2m)\left(\gamma + \log\left[\frac{\sinh m}{\cosh m}\right]\right) \end{aligned} \tag{5.6}$$

A general solution also has non-zero 2-forms [46], but we are interested here in just the $\theta = \pi/2$ plane, in which these vanish.

To see why this plane is the interesting region, again the ubiquitous brane probe was used to get in touch with the field theory. The authors of [28] and [27] noted that on substitution of the above 10d solution into the DBI action, the potential is zero in the $\theta = \pi/2$ plane. This is the moduli space for the brane motion, matching the two dimensional moduli space of the $\mathcal{N} = 2^*$ field theory, with its two real scalar fields. A probe brane postioned off this moduli space corresponds to giving a vev to a massive scalar field, which is neither supersymmetric nor a vacuum of the theory. Since we know of no field theory results with such vevs turned on, we would have nothing to compare any results from a the dual theory to. So we examine only the moduli space plane.

The $U(1)$ field theory from the moduli space of this brane probe is given by

$$\mathcal{L} = \frac{1}{2}\left(\rho^4 \cosh(2m) e^{2A}\dot{r}^2 + \frac{L^2\rho^4\cosh(2m)e^{2A}}{\rho^8}\dot{\phi}^2\right) + \frac{1}{4}\tau_3(2\pi\alpha')^2 e^{-\Phi}F^{\mu\nu}F_{\mu\nu} \tag{5.7}$$

The relation to $\mathcal{N} = 2^*$ is not yet clear; we need to find the correct coordinates to make the duality manifest. There are two separate changes that must be made. Firstly, the two kinetic terms for the scalar fields must be the same. Secondly, there must be a common coefficient for these scalar kinetic pieces and the gauge kinetic term, given by the running coupling of the gauge theory, $1/g^2_{YM}(r)$. The change of coordinates for this first step is

$$v = L\sqrt{\frac{\cosh 2m + 1}{\cosh 2m - 1}} \tag{5.8}$$

such that

$$\frac{\partial v}{\partial r} = \frac{\rho^4}{L} v \tag{5.9}$$

giving

$$\mathcal{L} = \frac{1}{2} \frac{k^2 L^2 \cosh 2m}{\sinh^2 2m \; v^2} (\dot{v}^2 + v^2 \dot{\phi}^2) \tag{5.10}$$

for the scalar kinetic terms.

Before performing the second step to reach the $\mathcal{N} = 2^*$ form, this is a good point to make some observations about these solutions. The solutions depend on the two constants k and γ corresponding to the mass term and the scalar vev respectively [27]. As in previous work [32, 45, 28, 27], since we have no physical interpretation for $\gamma > 0$, we consider only $\gamma \leq 0$. Although v-ϕ are not the coordinates that exibit the duality, they do show the SO(2) symmetry in (5.10). Varing γ changes the postion of the divergence of the metric when $\rho \to 0$. Using (5.6) and (5.8) one can get an expression for γ in terms of the radius l,

$$\gamma = -\frac{l^2}{4L^2} + \frac{L^2}{4l^2} + \ln \frac{l}{L} \tag{5.11}$$

This divergence is interpreted as signaling the presence of a D3-brane source, so larger negative γ means a larger disc of branes, corresponding to larger vevs in the field theory. With $\gamma = 0$ the metric is well-behaved right down to the radius $v = l = L$, at which point $\cosh 2m \to \infty$ and so the scalar kinetic terms' coefficient goes to zero. This is the enhancon locus where the tension of the branes goes to zero (the coupling diverges in the field theory). Previous work [33] tells us to discard the solutions inside this radius. For all the other metrics ($\gamma < 0$), $\rho \to 0$ at a larger radius, before the coefficient of the scalar kinetic terms can diverge, so the enhancon point cannot be reached.

We now return to finding the $\mathcal{N} = 2^*$ form for the solution. In [27] it was noted that as the coefficient of the $F_{\mu\nu}^2$ is

$$e^{-\Phi} = \frac{c}{g_s|\cos\phi + ic\sin\phi|^2} \tag{5.12}$$

the theory is not in an $\mathcal{N} = 2$ form and the connection to the gauge coupling cannot yet be made. The following conformal transformation must be made in the v-ϕ plane which equates the gauge and scalar kinetic pieces [27].

$$Y = \frac{kL}{2}\left(\frac{V}{L} + \frac{L}{V}\right) \tag{5.13}$$

$V = ve^{i\phi}$, $Y = ye^{i\eta}$ are complex parameters on this plane. We then have the low energy theory in the required form,

$$\mathcal{L} = \frac{1}{g_{YM}^2(Y)}|\dot{Y}|^2 + \mathrm{Im}\left(\tau(F^{\mu\nu}F_{\mu\nu} + iF^{\mu\nu}\tilde{F}_{\mu\nu})\right) \tag{5.14}$$

where $4\pi/g_{YM}^2(Y) = \mathrm{Im}\tau$ and

$$\tau = \frac{i}{g_s}\sqrt{\frac{Y^2}{Y^2 - k^2L^2}} \tag{5.15}$$

The background in these coordinates is given by

$$\begin{aligned} ds^2 &= \frac{1}{g_{YM}}(H^{-1/2}dx_{//}^2 + H^{1/2}dy^2) \\ C_4 &= \frac{g_{YM}^2}{Hg_s}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \end{aligned}$$

$$\tau_3(2\pi\alpha')^2 e^{-\Phi} = \frac{1}{g_{YM}^2} \tag{5.16}$$

with[1]

$$g_{YM}^2 H = \frac{\sinh^4 2m}{k^4\rho^{12}\cosh 2m} \tag{5.17}$$

We have now arrived at the unique coordinates to show an $\mathcal{N} = 2$ supersymmetric theory from the brane probe of the $\theta = \pi/2$ plane. It is in these coordinates that one should be able to interpret the rest of the metric in terms of field theory operators, as we were able to do in the $\mathcal{N} = 4$ case. Here the process is a bit more complicated, and, as we will see, the results somewhat puzzling.

We see that the metric on the moduli space, in the physical coordinates has two functions in it; the Yang-Mills coupling and H, which remains to be interpreted. As in the previous chapter, to make contact with the low energy field theory we need to expand H at large radius in these new coordinates. To do this we need to note the slightly unusual nature of the final coordinate transformation we made, (5.13). The coordinates V are in fact a double cover of the Y space as the circle $v = L$ maps to the real line of length $2kL$ and the interior of this circle is mapped to the points exterior to the line in Y-space. Since the probe in the v coordinates cannot go through the enhancon, the region $v < 1$ should be ignored. Now, from (5.13), we can expand v at large Y,

$$v = \frac{2y}{k} - \frac{k\cos 2\eta}{2y} + \frac{k^3}{32y^3}(1 - 5\cos 4\eta) + \ldots \tag{5.18}$$

[1]It may be noted that the brane probe result is not affectd by a rescaling of H by an arbitrary power of the Yang-Mills couplings, so the form of H is not fixed uniquely. This cannot help with the discrepancies we will discover below, since the coupling (5.15) contains no logarithms.

Then , using (5.6), (5.13) and (5.17), we arrive at our final expansion for H:

$$
\begin{aligned}
H = \frac{L^4k^4}{16y^4} + \frac{L^6k^6}{64y^6}&\left(-2+2\frac{l^2}{L^2}-2\frac{L^2}{l^2}+8\ln\left(\frac{y}{l}\right)+6\cos(2\eta)\right)\\
&+\frac{L^8k^8}{2^8y^8}\left[3\left(1-\frac{l^2}{L^2}+\frac{L^2}{l^2}+4\ln\left(\frac{y}{L}\right)-2\cos 2\eta\right)^2\right.\\
&\qquad+2\cos 2\eta\left(-2+2\frac{l^2}{L^2}-2\frac{L^2}{l^2}+8\ln\left(\frac{y}{l}\right)\right)\\
&\left.+\left(3+2\frac{l^2}{L^2}-2\frac{L^2}{l^2}-8\ln\left(\frac{y}{L}\right)-8\cos 2\eta+14\cos 4\eta\right)\right]+\ldots
\end{aligned}
\tag{5.19}
$$

We can use the same process of examining symmetry properties as in the $\mathcal{N}=4$ case. y has mass dimension 1, and the angular dependence in η can be identified as $SO(2)$ harmonics $\cos(n\eta)$ with $U(1)$ charge n. This invites the interpretation of the coefficient of $\cos(n\eta)$ as $\mathrm{tr}\phi^n$ (ϕ being the massless, two component scalar field), which should be associated with a factor of $y^{(n+4)}$. The uncharged terms would correspond to $\mathrm{tr}|\phi|^n$, which would also be associated with a factor of $y^{(n+4)}$. Observing the $\cos 2\eta$ term at order $1/y^8$ shows that there mixed operators in the form of products of these two types of operator.

Unfortunately there are also log terms in (5.19), which are completely unexpected and really spoil our previous interpretations. One would expect the $\mathcal{N}=2^*$ theory to revert to $\mathcal{N}=4$ as $l\to\infty$, but the log terms cannot be ignored in this limit, and this solution seems to be giving logarithmic renormalization in the UV. In the large l limit the leading terms in l do match the $\mathcal{N}=4$ expansion (4.21), but this does not contain any log terms. Later on we will venture some possible explanations and solutions for the evident disparity. In the following subsection we will emphasize the discrepancy by calculating the D3-brane distributions implied by the gauge coupling and finding the expansion of H that follows from them.

5.2.1 D3 Distributions

In finding the D3-brane distribution we generalize work done in [27]. They find the line distribution in Y-space which arises in the special case $\gamma = 0$, we find the result for all γ. The starting point is to assume the standard one loop renormalized expression for the prepotential for $\mathcal{N} = 2^*$ which completely determines the low energy effective action,

$$\mathcal{F} = \frac{i}{8\pi}\left[\sum_{i\neq j}(a_i - a_j)^2 \ln\left(\frac{(a_i - a_j)^2}{\mu^2}\right) - \sum_{i\neq j}(a_i - a_j + m)^2 \ln\left(\frac{(a_i - a_j + m)^2}{\mu^2}\right)\right] \tag{5.20}$$

The field theory is reviewed in [27], and in Appendix A.3. Since in the supergravity dual for $\mathcal{N} = 2^*$ the scalar vevs should be large compared to the mass term one can arrive at [27],

$$\frac{\partial^2 \mathcal{F}}{\partial \phi^2} = \tau(Y) = \frac{i}{g_s} + \frac{i}{2\pi}\int \sigma \frac{m^2}{(Y - a)^2} d^2 a \tag{5.21}$$

$d^2 a$ represents an element of Y-space and σ is the density of vevs/D3-branes. Matching this with the supergravity result requires the identification $m = k^2\pi/L^2$ [27]. We can now find the distributions that reproduce the supergravity result for τ. This is not practical in Y-space since there is no spherical symmetry, but in V-space the distributions are circular from the low radius cut-off at $v = L$ out to l. Pleasingly, a simple expression may be found for the density for any γ, or equivalently l, by re-writing (5.21) in V-space using (5.13) and

$$\sigma_v v dv d\phi = \sigma_y y dy d\eta \tag{5.22}$$

Expanding said result for large y, and using an expansion in powers of $1/v$ for σ_v we have

$$\sigma_v = \frac{1}{\pi(l^2 - L^4/l^2)}\left(1 + \frac{L^4}{v^4} - 2L^2\frac{\cos(2\phi)}{v^2}\right) \tag{5.23}$$

It can be shown that this can reproduce the supergravity result (5.15) to all orders of expansion.

If we take $\gamma = 0$ and integrate this density over vdv from $v = L$ to l, then take the limit $l \to L$ we find that the number of D3-branes is given by

$$N_{D3} = \frac{1}{\pi}\int_0^{\pi}(1 - \cos 2\theta)d\theta \tag{5.24}$$

and changing variables to $y = kL\cos\theta$ gives the line density

$$\sigma_y = \frac{2}{m^2}\sqrt{k^2 - y^2} \tag{5.25}$$

These are exactly the special case results of [27].

We can now use (4.17) to calculate H. As $\mathcal{N} = 2^*$ only has renormalization in its gauge coupling, the $\mathcal{N} = 4$ expression for H evaluated in the $\theta = \pi/2$ plane should contain the full set of operators. So rescaling y to $2y/k$, changing variables to V-space via

$$y\cos\eta = \frac{kL}{2}\left(v + \frac{1}{v}\right)\cos\phi, \qquad y\sin\eta = \frac{kL}{2}\left(\frac{1}{v} - v\right)\sin\phi$$
$$y^2 = \frac{k^2L^2}{4}\left(v^2 + \frac{1}{v^2} + 2\cos\phi - 2\sin\phi\right) \tag{5.26}$$

doing the integration, expanding at large Y, and restricting to the $\theta = \pi/2$, we have

$$H = \frac{L^4k^4}{16y^4} + \frac{L^6k^6}{64y^6}\left(\frac{2l^2}{L^2} + \frac{2L^2}{l^2} + 6\cos(2\eta)\right)$$

$$+\frac{k^8L^8}{2^8y^8}\left(3\left(\frac{l^4}{L^4}+4+\frac{L^4}{l^4}\right)+16\frac{l^2}{L^4}\left(1+\frac{L^4}{l^4}\right)\cos 2\eta+20\cos 4\eta\right) \tag{5.27}$$

This does not match (5.19). The supergravity is showing logarithmic renormalization that is not present in this field theory prediction.

5.2.2 Explaining the Discrepancy

We can think of two possible reasons for the discrepancy between our supergravity result and the result expected from field theory. They are both complications arising from using 5d supergravity. The first is that we may have inadvertently introduced more than just a mass term into the field theory. As we pointed out in the last chapter with Eq. (4.21) in the $\mathcal{N}=4$ case, if one introduces a dimension 2 operator in the 5d supergravity and then lifts the solution to 10d, a whole tower of higher dimensional operators results. If the same thing is happening here then the $\mathcal{N}=2^*$ solutions may be encoding an unknown tower of deformations as well as the field theory scalar vevs we were looking for.

The other possibility concerns the transformation from the V coordinates to the physical Y coordinates. The 5d solution came in the V coordinates, but these are a double cover of the Y coordinates and we excised the solution inside $v=L$. It is possible that this was throwing away some internal structure which is projected to large y in the Y coordinates. This could mean that there are D3-branes throughout the whole space in the physical coordinates!

5.3 Discussion and Summary

Our attempt to extend the work of the previous chapter on the moduli space of $\mathcal{N}=4$ has run into difficulties at the next simplest case of $\mathcal{N}=2^*$. We were not able to satisfactorialy interpret our gravity solution in terms of the field theory it is supposed to be dual to, because of the extra renormalization it seems to indicate. However the simple form of the metric on moduli space (5.16) strongly encourages the application of the prescription which we carried through, and we believe the philosophy of this work is sound. The difficulties we have encountered might turn out to be useful and interesting in future work constructing and interpreting such dualities.

Chapter 6

Off Moduli Space

This Chapter presents previously unpublished work done in collaboration with Dr. Nick Evans. Dr. Clifford Johnson and Dr. Michela Petrini were also collaborators on the Leigh-Strassler section.

6.1 Introduction

So far, in the previous two chapters, we have examined field theories only on their moduli spaces. We have seen that the AdS/CFT duality seems to hold on these moduli spaces. The duality should of course extend off the moduli space of a field theory, so we have made some efforts to verify this. Our attempts to tackle general cases have been hampered by messy algebra, so we present here a treatment of only a subspace of the full moduli space for a couple of field theories. First we consider a scenario which has been studied by Leigh and Strassler [68], in which a mass term is given to only one of the chiral superfields which gives an $\mathcal{N} = 1$ theory in the IR, rather than $\mathcal{N} = 2$. This is followed by the $\mathcal{N} = 2^*$ theory we have just been looking at in the previous chapter.

6.2 Leigh-Strassler Off Moduli Space

6.2.1 The Leigh-Strassler Flow

We consider a scenario in which a mass term is introduced in the field theory for only one of the chiral superfields (as opposed to two for $\mathcal{N}=2^*$). We are concerned with the results of [69], which describe an RG flow to a large N version of the Leigh-Strassler point [68]. This dual field theory has only $\mathcal{N}=1$ supersymmetry. Our starting point is the ten dimensional metric calculated in [69], which may be written as

$$ds_{10}^2 = \Omega^2 ds_{1,4}^2 + ds_5^2 \tag{6.1}$$

where

$$ds_{1,4}^2 = e^{2A(r)}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + dr^2 \tag{6.2}$$

and

$$\begin{aligned} ds_5^2 &= L^2 \frac{\Omega^2}{\rho^2 \cosh^2\chi}\left[d\theta^2 + \rho^6\cos^2\theta\left(\frac{\cosh\chi}{\bar{X}_2}\sigma_3^2 + \frac{\sigma_1^2+\sigma_2^2}{\bar{X}_1}\right)\right. \\ &\quad \left. + \frac{\bar{X}_2\cosh\chi\sin^2\theta}{\bar{X}_1^2}\left(d\phi + \frac{\rho^6\sinh\chi\tanh\chi\cos^2\theta}{\bar{X}_2}\sigma_3\right)^2\right] \end{aligned} \tag{6.3}$$

with

$$\begin{aligned} \Omega^2 &= \frac{\bar{X}_1^{1/2}\cosh\chi}{\rho} \\ \bar{X}_1 &= \cos^2\theta + \rho^6\sin\theta \\ \bar{X}_2 &= \mathrm{sech}\chi\cos^2\theta + \rho^6\cosh\chi\sin^2\theta \end{aligned} \tag{6.4}$$

$\rho(r) \equiv e^{\alpha(r)}$ and χ are supergravity scalars which couple to particular operators in the dual gauge theory. ρ, χ and $A(r)$ obey

$$\begin{aligned}
\frac{d\rho}{dr} &= \frac{1}{6L}\rho^2\frac{\partial W}{\partial \rho} = \frac{1}{6L}\left(\frac{\rho^6(\cosh(2\chi) - 3 + \cosh(2\chi) + 1}{\rho}\right) \\
\frac{d\chi}{dr} &= \frac{1}{L}\frac{\partial W}{\partial \chi} = \frac{1}{2L}\left(\frac{(\rho^6 - 2)\sinh(2\chi)}{\rho^2}\right) \\
\frac{dA}{dr} &= -\frac{2}{3L}W = -\frac{1}{6L\rho^2}(\cosh(2\chi)(\rho^6 - 2) - (3\rho^6 + 2))
\end{aligned} \tag{6.5}$$

W is the supergravity superpotential,

$$W = \frac{1}{4}\rho^4(\cosh 2\chi - 3) - \frac{1}{2\rho^2}(\cosh 2\chi + 1) \tag{6.6}$$

There is also the R-R four-form potential to which the D3-branes naturally couple:

$$C_4 = -\frac{4}{g_s}w(r,\theta)dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \tag{6.7}$$

where

$$w(r,\theta) = \frac{e^{4A}}{8\rho^2}[\rho^6 \sin^2\theta(\cosh(2\chi) - 3) - \cos^2\theta(1 + \cosh(2\chi))] \tag{6.8}$$

This background was probed with a D3-brane and its moduli space examined in [49]. Its Kähler structure was also investigated in [29]. The effective lagrangian for this slowly moving probe is [49]

$$\mathcal{L} \equiv T - V = \frac{\tau_3}{2}\Omega^2 e^{2A} G_{mn}\dot{y}^m\dot{y}^n - \tau_3 \sin^2\theta e^{4A}\rho^4(\cosh(2\chi) - 1) \tag{6.9}$$

Here the coordinates transverse to the brane are $\dot{y}^m = (r, \phi, \theta, \varphi_1, \varphi_2, \varphi_3)$ and G_{mn} are the Einstein frame metric components. This descibes a field theory with a four dimensional moduli space (when $\theta = 0$) [49]. Before examining the off moduli region, we give some more detail about this gauge theory.

6.2.2 The Gauge Theory

We are dealing with a specific mass deformation the $\mathcal{N}=4$ theory to $\mathcal{N}=1$. This deformation can be described in an explicitly $\mathcal{N}=1$ supersymmetric way using the superfield formalism [49, 70]. In $\mathcal{N}=1$ terms there is a vector multiplet (A_μ, λ_4) and three chiral multiplets. Each of these contains one of the remaining fermions and a complex scalar, thus, $\Phi_k \equiv (\lambda_k, \phi_k)$, where $\phi_k = X_{2k-1} + iX_{2k}$, $k = 1, 2, 3$. They have the superpotential

$$W = h\mathrm{Tr}(\Phi_3[\Phi_1, \Phi_2]) \tag{6.10}$$

h is related to g_{YM} through the superconformal symmetry. The mass deformation we have been talking of is introduced by

$$\mathcal{L}_{ft} \to \mathcal{L}_{ft} + \int d^2\theta \frac{1}{2} m \Phi_3^2 + \text{h.c.} \tag{6.11}$$

so that the superpotential is now

$$W = h\mathrm{Tr}(\Phi_3[\Phi_1, \Phi_2]) + \frac{1}{2} m \mathrm{Tr}(\Phi_3^2) \tag{6.12}$$

In a flow to a scale below this mass the supersymmetry is broken to $\mathcal{N}=1$. This field theory then has "matter" multiplets in two "flavours", Φ_1 and Φ_2, that transform in the adjoint of $SU(N)$. The $\mathcal{N}=4$ $SU(4) \simeq SO(6)$ R-symmetry breaks to $SU(2)_F \times U(1)_R$. The 'R' signifies the $\mathcal{N}=1$ R-symmetry and the 'F' denotes the flavour symmetry which gives the matter multiplet doublet. This $SU(2) \times U(1)$ is manifest in the 10d metric (6.1) as the squashed S^3 in the σ_i coordinates of (6.3).

We are introducing a small, but relevant mass perturbation to the UV and flowing to the IR. In the supergravity this means switching on particular scalar fields, which asymptote to zero. Also, the supergravity equations of motion require a back reaction on the geometry, deforming the spacetime, as given by $A(r)$, etc. For more details see [49, 70] and the refences therein.

In the far infra-red the massive scalar Φ_3 can be integrated out, giving the quartic superpotantial

$$W = \frac{h^2}{4m}\mathrm{Tr}([\Phi_1, \Phi_2]^2) \tag{6.13}$$

This is actually a marginal operator of the theory. Variation of its coefficient defines a fixed[1] line of theories [68]. There is a line of fixed points in the space of couplings, so the value of the coupling changes along this line without breaking scale invariance. The line does not in fact pass through the origin, so it is not certain to exist, but Leigh and Strassler conjectured that this deformed theory goes to a conformal fixed point on this line of marginal couplings. Adding this deformation does not break the conformal invariance, because of its exactly marginal coupling. The $\mathcal{N} = 1$ supersymmetry combines with this conformal symmetry to make the infra-red, 'Leigh-Strassler' theory superconformal, with supergroup $SU(2,2|1)$. The presence of this marginal operator also means that we have an interacting conformal theory. The moduli space here is determined by $[\phi_1, \phi_2] = 0$, with ϕ_i the lowest components of Φ_i.

6.2.3 Off the Moduli Space

We now examine a particular part of the off moduli space region, namely that obtained by setting $\sin\theta = 1$ and ignoring the $\varphi_1, \varphi_2, \varphi_3$ coordinates. From (6.9) one then gets

$$T = \cosh^2(2\chi)e^{2A}\rho^4\dot{r}^2 + \frac{\cosh^2(2\chi)e^{2A}}{\rho^4}\dot{\phi}^2 \tag{6.14}$$

[1]'Fixed' refers to the β-functions for the gauge coupling, g, and the couplings in the superpotential being zero. Under an RG flow a theory will stay at such a fixed point in the space of couplings; the couplings do not run and the theory is scale invariant. These beta functions can be calculated [71] exactly using non-renormalization theorems.

for the kinetic piece of the lagrangian, and

$$V = 2e^{4A}\rho^4(\cosh^2(2\chi) - 1) \tag{6.15}$$

for the potential piece. But we know it should be possible to write this field theory in terms of a Kähler potential, K [56]. Specifically, there should be a change of radial coordinate $r \to v$ such that the kinetic piece assumes the canonical form

$$f(v)(\dot{v}^2 + v^2\dot{\phi}^2) \tag{6.16}$$

where $f(v) = K''$, with $'$ representing differentiation with respect to the chiral superfields. Also, the potential should be given by

$$V = v^2/K'' \tag{6.17}$$

We will now find an explicit form for v and check that it is consistent.

Start with the kinetic piece. From equating the coefficients of $\dot{\phi}^2$ in (6.14) and (6.16) we get

$$f(v) = \frac{\cosh^2(2\chi)e^{2A}}{\rho^4 v^2} \tag{6.18}$$

And from equating the $\dot{r}^2$ and $\dot{v}^2$ terms,

$$\left(\frac{dv}{dr}\right)^2 = \frac{\rho^4 e^{2A}\cosh^2(2\chi)}{f(v)} \tag{6.19}$$

Now turn to the potential and equate (6.15) and (6.17). Having done this one can then use (6.18) to get the following expression for v:

$$v^4 = 2e^{6A}\cosh^2(2\chi)(\cosh^2(2\chi) - 1) \tag{6.20}$$

The consistency of this can be checked. From (6.19) and (6.18) we get

$$\frac{dv}{dr} = \rho^4 v \tag{6.21}$$

Using the flow equations (6.5), one can show that (6.20) is indeed consistent with (6.21). We have shown that the supergravity dual does indeed provide a good description for at least part of the Leigh-Strassler field theory's moduli space.

6.3 $\mathcal{N} = 2^*$ Off Moduli Space

We now take the 10d metric for $\mathcal{N} = 2^*$ (5.1) and set $\cos\theta = 1$, and $\psi, \beta =$ constant, so that $\sigma_1 = 0$, $\sigma_2 = 0$ and $\sigma_3 = \frac{1}{2}d\alpha^2$. Then a brane probe in the (r, α) directions yields lagrangian kinetic piece:

$$T = \frac{e^{2A}\cosh 2m}{2\rho^2}\dot{r}^2 + \frac{L^2 e^{2A}\rho^2}{8\cosh 2m}\dot{\alpha}^2 \tag{6.22}$$

and potential:

$$V = \frac{e^{4A}}{\rho^2}(\cosh 2m \;\; - 1) \tag{6.23}$$

But we know that it should be possible to write the field theory in the form

$$T = f(v)(\dot{v}^2 + v^2\dot{\alpha}^2) \tag{6.24}$$

with again $f(v) = K''$, where K is the Kähler potential. Equating the $\dot{\alpha}^2$ terms in (6.22) and (6.24) then gives

$$f(v)v^2 = \frac{L^2 e^{2A}\rho^2}{8\cosh 2m} \tag{6.25}$$

Equating the other kinetic terms produces

$$\left(\frac{dv}{dr}\right)^2 = \frac{e^{2A}\cosh 2m}{2\rho^2 f(v)} \tag{6.26}$$

then using (6.25) gives

$$\frac{dv}{dr} = \frac{2v\cosh 2m}{L\rho^2} \tag{6.27}$$

If the potential is simply that of $\mathcal{N} = 2^*$ (i.e. superpotential $W = m\Phi^2$), then it should also be determined by the Kähler potential, as

$$V = \frac{v^2}{K''} \tag{6.28}$$

Then, using (6.25),

$$V = \frac{8v^4\cosh 2m}{L^2 e^{2A}\rho^2} \tag{6.29}$$

Equating (6.23) and (6.29) gives

$$v^4 = \frac{L^2 e^{6A}(\cosh 2m - 1)}{8\cosh 2m} \tag{6.30}$$

We can then differentiate this expression with respect to r, using the flow equations for A and m to check that it is consistent with (6.27). Carrying this out shows that the two expressions are in fact *not* consistent. We have uncovered another puzzling result from examining the $\mathcal{N} = 2^*$ scenario, to go with that of the previous chapter. We have been unable to satisfactorily interpret this supergravity solution in terms of the field theory it is supposed to be dual to. We have found here a particular problem with trying to interpret the off moduli space region.

6.4 Summary

We set out to verify that the supposed supergravity dual of the Leigh-Strassler field theory does give a good desciption of the theory off its moduli space. Practical difficulties prevented us finding a description for the whole off moduli space region, but for the part of the off moduli region which we considered, we did find a good description for the supergravity in terms of the dual field theory.

For the $\mathcal{N} = 2^*$ case however, even for the restricted region we have examined here, we were unable to interpret the supergravity solution in terms of the field theory as should have been possible. Along with the results of the previous chapter, we believe that this shows some deficiency in our understanding of this particular scenario. (Rather than indicating some problem with the AdS/CFT correspondence itself!) Hopefully future work will resolve the difficulties.

Chapter 7

$\mathcal{N} = 4$ Super Yang Mills at Finite Density

This covers the paper '$\mathcal{N} = 4$ Super Yang Mills at Finite Density: the Naked Truth', Nick Evans and James Hockings [9].

7.1 Introduction

In this chapter we will again use the AdS/CFT Correspondence to study a field theory non-perturbatively, this time $\mathcal{N} = 4$ super Yang Mills at finite $U(1)_R$ charge density [15, 16, 17]. The gravity dual to this field theory is the background to a stack of spinning D3 branes [18, 19, 20].

A set of metrics have been found to describe spinning D3 branes by lifting five dimensional charged black hole solutions to 10d [18]. These ten dimensional solutions split into two classes. The first are rotating black branes which have been much studied in the literature [18, 19, 20, 22]. They have been interpreted as being dual to the high density and high temperature phase of the $\mathcal{N}=4$ gauge theory. The phase structure for the $\mathcal{N}=4$ theory at the origin of moduli space was worked out in [20, 21]. The other class of solutions are nakedly singular metrics. Their supersymmetric limit has been shown to correspond to disc distribution, multi-centre D3 brane solutions [19]. We will recount this work and use brane probing, in the well established manner [27, 28, 8, 29], to find the coordinate system in which the duality is manifest. The main goal though is to find an interpretation for the non-supersymmetric members of the class. They have many of the properties of the rotating black branes, so it seemed likely that they described spinning disc distributions that correspond to the gauge theory Coulomb branch at finite density. This is what we in fact found.

As density is increased, there is a critical point at which a horizon develops, signaling a switch from the non-supersymmetric naked solutions to the zero temperature black brane solutions. We interpret this as the high density deconfinement transition of the Coulomb branch of the gauge theory, above which the scalar vevs evaporate. The parameters of the model have completely different meanings on either side of the transition. We then extend our analysis to include finite temperature as well as density, finding the phase diagram for the Coulomb branch. This is done by compactifying the time dimension and comparing the free energy of the spinning D3 brane distribution background with a black brane geometry having the same temperature and density.

It had been realized previously [22] that the black brane backgrounds considered here are unstable for large spin or density. We show this using a brane probe (note that some related configurations have been probed in [30]). Evans and Petrini [31] had shown the perturbative instability of the gauge theory at zero temperature and finite density, as the chemical potential destabilizes the scalar potential giving a runaway vacuum. The naive scalar instability of the theory is clear when one notes that a probe in pure $AdS_5 \times S^5$ feels zero potential, so there is no force to support any rotational motion. In any attempt at rotation, the angular momentum of the probe will simply send it off to infinity, corresponding to a runaway scalar vev in the field theory. Our analysis in this chapter confirms this instability non-perturbatively. This instability does not prevent us discovering the physics of the finite density phase transition.

7.2 Thermodynamics of Schwarzchild Black Holes in AdS_5

We start by reviewing what happens when finite temperature is introduced into an AdS_5 Schwarzchild solution. This was done for general AdS_{n+1} (in a box) by Witten [38]. Temperature, T, is introduced by compactifying the time-like direction with period $\beta = 1/T$. The metric for a Schwarzchild black hole in $AdS_5 \times S^5$ can be written as

$$ds^2 = -Kdt^2 + r^2 dx_{//}^2 + K^{-1}dr^2 + d\Omega_5^2 \tag{7.1}$$

where $K = r^2 - k/r^2$. Setting $k = 0$ gives AdS, without a black hole. There is also a 4-form,

$$B_4 = -r^4 dt \wedge d^3x \tag{7.2}$$

which gives the 5-form field strength $G_5 = dB_4$.

We now examine the thermodynamics by taking the action difference of the black hole and no-black hole cases, following Hawking and Page [40]. The appropriate action is

$$I = -\frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}\left(R + \frac{1}{480}G_5'^2\right) - \frac{1}{\kappa^2}\int d^9x\sqrt{-h_9}\overline{K} \tag{7.3}$$

The second integral is a surface term where

$$\overline{K} = G^{\mu\nu} K_{\mu\nu}, \qquad K_{\mu\nu} = \frac{1}{2\sqrt{G_{rr}}} \frac{\partial}{\partial r} G_{\mu\nu}, \qquad h_{(9)} = \det G_{\mu\nu}, \qquad \mu, \nu \neq r \quad (7.4)$$

There is a subtlety that must be dealt with. As described in [40, 38], to allow comparison of the two spacetimes the period of the time integral of the naked case, $\tilde{\beta}$, must be set to match the geometry of the hypersurface at large R in the two cases. To achieve this we require

$$\tilde{\beta} = \beta \frac{\sqrt{G_{tt}}}{\sqrt{\tilde{G}_{tt}}} \qquad (7.5)$$

which here becomes

$$\tilde{\beta} = \frac{\beta K^{1/2}}{r} \qquad (7.6)$$

The curvature R is zero in both cases, so we are left with the G_5 and surface pieces to deal with. First, calculating the surface term contribution gives

$$\begin{aligned} \Delta I_{surface} &= -\frac{\text{Vol}(S_5)Vol(3)}{\kappa^2} \left[\frac{1}{2} \beta K r^3 \left(K' K^{-1} + \frac{6}{r} \right) - \frac{1}{2} \tilde{\beta} r^5 \left(\frac{2r}{r^2} + \frac{6}{r} \right) \right] \\ &= -\frac{\text{Vol}(S_5)Vol(3)\beta}{\kappa^2} \left[K' r^3 + 6 K r^2 - 8 K^{1/2} r^3 \right] \qquad (7.7) \end{aligned}$$

where $\text{Vol}(S_5)$ is the volume of the five-sphere and Vol(3) is the volume of the spatial part of the branes (both common to the two geometries). But (7.7) vanishes in the $r \to \infty$ limit, so only the G_5 piece remains. Calculating this gives

$$\begin{aligned} \Delta I_{G_5} &= \frac{16}{\kappa^2} \text{Vol}(S_5)\text{Vol}(3)\beta \left(\beta \left[\frac{r^4}{4} \right]_{r_h}^{\infty} - \tilde{\beta} \left[\frac{r^4}{4} \right]_0^{\infty} \right) \\ &= \frac{4}{\kappa^2} \text{Vol}(S_5)\text{Vol}(3)\beta \left(\frac{1}{2} k - r_h^4 \right) \qquad (7.8) \end{aligned}$$

where r_h is the horizon radius of the black hole.

We now need to make an interpretation in terms of temperature. The temperature of a black hole is given by

$$2\pi T = \frac{1}{\sqrt{g_{rr}}} \frac{d}{dr} \sqrt{g_{tt}} \bigg|_{r=r_h} \tag{7.9}$$

where r_h is the horizon radius. r_h is when $K = 0$, and therefore $r_h^4 = k$. Plugging this into (7.8) gives

$$\Delta I = -\frac{2}{\kappa^2} \mathrm{Vol}(S_5)\mathrm{Vol}(3)\beta r_h^4 \tag{7.10}$$

but (7.9) gives

$$\pi T = r_h \tag{7.11}$$

so

$$\Delta I = -\frac{2}{\kappa^2} \mathrm{Vol}(S_5)\mathrm{Vol}(3)\beta \pi^4 T^4 \tag{7.12}$$

So in fact a black hole will form as soon as a finite temperature is introduced. (This is essentially because a scale has been introduced where none existed previously.) We will see that this is not the case when we introduce rotation into the scenario. Next we bring in rotation, but with zero temperature.

7.3 Introducing Spin/Finite Density

First of all, why should a spinning supergravity solution provide the AdS/CFT dual of a field theory with finite charge density? Consider a general lorentz boost to a rotating frame,

$$\Lambda^{\mu}{}_{\nu}: \quad \phi \to \phi + \omega t \tag{7.13}$$

Applying such a transformation to a diagonal metric can rotate the time-time component into some off-diagonal t-ϕ components,

$$\Lambda^{t}{}_{\phi}: \quad G^{tt} \to G^{t\phi} \tag{7.14}$$

which after Kaluza-Klein compactification on the S^5 becomes a gauge field for the $SU(4)_R$ symmetry. On the CFT side of the Correspondence, this field becomes a source which is in fact a chemical potential that puts the field theory at finite density. A term of the form

$$\bar{\lambda}\mu\gamma^0\lambda \tag{7.15}$$

is introduced into the Lagrangian, where λ represents a fermion field and μ is the chemical potential.

Our practical starting point is a background found by Cvetic et al [18] from the lift of five dimensional charged black hole solutions. It is the near horizon limit of a rotating D3 brane configuration.

$$\begin{aligned} ds^2_{10} &= \sqrt{\tilde{\Delta}}\left[-(H_1H_2H_3)^{-2/3} f dt^2 + (H_1H_2H_3)^{1/3}\left(f^{-1}dr^2 + \frac{r^2}{L^2}dx^2_{//}\right)\right] \\ &\quad + \frac{L^2}{\sqrt{\tilde{\Delta}}}\sum_{i=1}^{3} X_i^{-1}\left(d\mu_i^2 + \mu_i^2\left(d\phi_i + gA^i dt\right)^2\right) \end{aligned} \tag{7.16}$$

where the μ_i are three direction cosines and

$$f = -\frac{\mu}{r^2} + \frac{r^2}{L^2}H_1H_2H_3, \qquad \frac{1}{L^2} = \frac{1}{\sqrt{2m}\sinh\alpha}, \qquad \mu = \frac{2m}{L^2} \tag{7.17}$$

$$A^i = \frac{L(1 - H_i^{-1})}{l_i \sinh\alpha}, \qquad H_i = 1 + \frac{l_i^2}{r^2} \tag{7.18}$$

$$\tilde{\Delta} = (H_1 H_2 H_3)^{1/3} \sum_i \frac{\mu_i^2}{H_i}, \qquad X_i = H_i^{-1}(H_1 H_2 H_3)^{1/3} \tag{7.19}$$

$$B_{(4)} = -\frac{r^4}{L^4} H_1 H_2 H_3 \sum_i \frac{\mu_i^2}{H_i} dt \wedge d^3x + \frac{1}{\sinh\alpha} (\sum_i l_i \mu_i^2 d\phi_i) \wedge d^3x \tag{7.20}$$

At large r the solution asymptotes to $AdS_5 \times S^5$ with the AdS radius L. We will keep L fixed in the following analysis. The solution then has four free parameters, the l_i and μ (or equivalently m or α). In the five dimensional black hole solutions the l_i are rotation parameters and μ the temperature. After the lift to ten dimensions we expect these parameters to become those that control the rotation in the three distinct $U(1)$ planes of the S^5, and the temperature. The horizon radius, r_h is found where the function $f = 0$, giving

$$r_H^4 H_1(r_H) H_2(r_H) H_3(r_H) = \mu L^2 \tag{7.21}$$

Then plugging our ten dimensional solutions into the formula for the temperature of a black hole (7.9) yields

$$4\pi T = \frac{2}{L^2} r_H (H_1 H_2 H_3)^{1/2} \left(2 - \frac{1}{L\mu^{1/2}(H_1 H_2 H_3)^{1/2}} (l_1^2 H_2 H_3 + l_2^2 H_1 H_3 + l_3^2 H_1 H_2)\right) \tag{7.22}$$

We can solve these equations for a number of special cases to find the value of μ that corresponds to $T = 0$. For example when a single l_i is non-zero $T = 0$ corresponds to $\mu = 0$, for two equal non-zero l_i $T = 0$ corresponds to $\mu = l^4/L^2$ and when there all three l_i are equal $T = 0$ corresponds to $\mu = 27 l^4/4L^2$. For μ greater than or equal to these μ_c values, the solutions have a singularity, originating in the

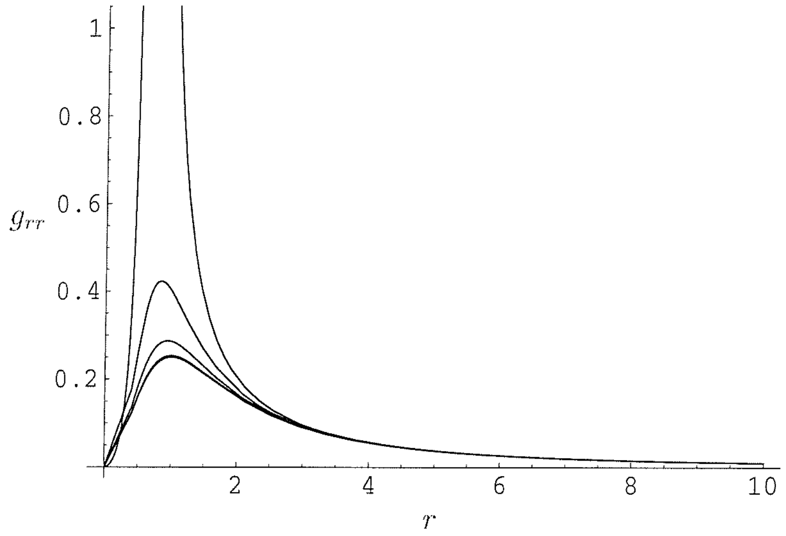

Figure 7.1: The metric component g_{rr} plotted as a function of radial position for varying μ up to $\mu = \mu_c$ at fixed $l_1 = l_2 = l_3 = l$ showing the development of a horizon.

f function, which corresponds to the horizon of the black hole. As μ increases the black hole temperature increases. However, for $0 < \mu < \mu_c$ the solutions do not have a horizon but have a naked singularity at $r = 0$. We show this in the plots of Figure 7.1 where g_{rr} is plotted against r at varying μ for the case when all three l_i are equal.

The black hole/brane solutions are closely related to those analysed in [20, 21] to describe the behaviour of $\mathcal{N}$=4 super Yang Mills at finite temperature and density. In [20, 21] the three l_i were taken equal and the variant of the above metric where the Minkowski space slices of AdS are compactified was considered[1] . The parameters l_i control the rotation speed of the black hole or the chemical potential in the field theory. The parameter μ controls the temperature of the black hole, or in the dual field theory, with $\mu = \mu_c$ corresponding to $T = 0$. Following [20, 21] these black

[1]Putting the gauge theory on an S^3 brings another scale into the problem that increases the area of the thermodynamic temperature vs density plane where the confined phase survives [20]. On R^3 the phase transition to the deconfined phase is induced as soon as a temperature or chemical potential is introduced.

hole solutions should be interpreted as gravity duals of the field theory at the origin of moduli space across the full temperature and density plane (the origin of the $T-\rho$ plane is described by the usual AdS/CFT correspondence). The behaviour of a Wilson loop [38] in these backgrounds show that at finite chemical potential and temperature the theory lives in a distinct (deconfined) phase from the (confined) theory at the origin.

7.3.1 Spinning Discs

We now turn to the nakedly singular solutions, when $\mu < \mu_c$. Recent work [32, 42] has shown that such naked singlarities are not necessarily unphysical, as would previously have been declared. They may now be interpreted as signaling the presence of extended objects, such as D-branes, in the space. The simplest such backgrounds are the multi-centre solutions [19, 34, 35] that describe distributions of D-branes dual to the Coulomb branch of the $\mathcal{N}=4$ gauge theory. The supersymmetric, $\mu \to 0$ limit of the particular backgrounds we study here had in fact already been identified as multi-centre solutions [19]. We will now start with that analysis, before turning μ on and seeing what happens.

One must be careful in taking the $\mu \to 0$ limit to remember to keep L fixed which also requires $\alpha \to \infty$. The background becomes

$$\begin{aligned} ds^2_{10} &= \sqrt{\tilde{\Delta}}\left[(H_1H_2H_3)^{1/3}\frac{r^2}{L^2}(-dt^2+dx^2_{//})+\frac{L^2(H_1H_2H_3)^{-2/3}}{r^2}dr^2\right] \\ &\quad +\frac{L^2}{\sqrt{\tilde{\Delta}}}\sum_{i=1}^{3}X_i^{-1}\left(d\mu_i^2+\mu_i^2\,(d\phi_i)^2\right) \end{aligned} \tag{7.23}$$

$$B_4 = -\frac{r^4}{L^4}H_1H_2H_3\sum_i\frac{\mu_i^2}{H_i}dt\wedge dx^3 \tag{7.24}$$

Note that the one form potential vanishes in this limit leaving a non-rotating solution. The difficulty with interpreting backgrounds as duals of gauge theory is though the familiar problem of finding the coordinates appropriate to the duality. We once again apply the very useful tool of brane probing, allowing us to use field theory intuition to find the correct coordinates [27, 28, 8, 29]. Thus we place a slow moving D3 brane in the above background through the Born-Infeld action

$$S = -\frac{\tau_3}{g_s}\int d^4\xi\sqrt{-\det g_{ab}} - \mu_3\int B_4 \tag{7.25}$$

where $\tau_3 = \mu_3 g_s^{-1}$ and g_{ab} is the pull back of the background to the world sheet. We find the action

$$\mathcal{L} = \frac{1}{2}\left(\sum_i \frac{\mu_i^2}{H_i^2}\dot{r}^2 + \sum_i \frac{1}{2} r^2 H_i(\dot{\mu}_i^2 + \mu_i^2\dot{\phi}_i^2)\right) \tag{7.26}$$

There is no potential obstructing motion of the probe in the six dimensional transverse space giving another strong hint that the theory is indeed the pure $\mathcal{N}$=4 theory. In the coordinates appropriate to the duality we expect a canonical kinetic term for the six scalar fields on the probe suggesting we try the new coordinates

$$w^2\tilde{\mu}_i^2 = (r^2 + l_i^2)\mu_i^2 \tag{7.27}$$

which render the $\dot{\phi}^2$ terms canonical. It follows that

$$w^2 = \sum_i (r^2 + l_i^2)\mu_i^2 \tag{7.28}$$

These are the coordinates identified in [19] that convert the metric to the familiar form of a multi-centre solution. They transform the probe action so that it has a flat metric and leave the spacetime background in the form

$$ds_{10}^2 = H_D^{-1/2} dx_{//}^2 + H_D^{1/2} dw^2 \tag{7.29}$$

with

$$B_4 = -H_D^{-1} dt \wedge dx^3 \tag{7.30}$$

We may find the form of H_D from the g_{xx} component of the metric using the coordinate transformation in (7.27). For example, for a single l_i switched on we find

$$H_D^{-1} = \frac{1}{L^4}(w^2 - l^2\mu_1^2)^2\left(\mu_1^2 + H\mu_2^2 + H\mu_3^2\right), \qquad H = 1 + \frac{l^2}{w^2 - l^2\mu_1^2} \tag{7.31}$$

where

$$\mu_1^2 = \frac{w^2}{w^2 + l^2(1-\mu_1^2)}\tilde{\mu}_1^2, \qquad \mu_{2/3}^2 = \frac{w^2}{w^2 - l^2\mu_1^2}\tilde{\mu}_{2/3}^2 \tag{7.32}$$

and thus

$$\mu_1^2 = \frac{(w^2 + l^2) \pm \sqrt{(w^2 + l^2)^2 - 4l^2 w^2 \tilde{\mu}_1^2}}{2l^2} \tag{7.33}$$

This result is unenlightening, except that if we look in the ϕ_1 plane at $w = l$ by setting $\tilde{\mu}_1 = 1, \tilde{\mu}_{2/3} = 0$ which corresponds to $\mu_1 = 1, \mu_{2/3} = 0$ at $w = l$ and we find $H_D = 0$. The metric in this case is singular at $w = l$, or in the original coordinates, at $r = 0$. The singularity corresponds to the position of the D3 brane distribution responsible for the background - it is a disc in the ϕ_1 plane at $w = l$.

Similar manipulations for the case with two equal l_i give

$$\begin{aligned} H_D^{-1} &= \frac{1}{L^4}(w^2 - l^2(\mu_1^2 + \mu_2^2))^2\left(\mu_1^2 + \mu_2^2 + H\mu_3^2\right), \\ H &= 1 + \frac{l^2}{w^2 - l^2(\mu_1^2 + \mu_2^2)} \end{aligned} \tag{7.34}$$

Again looking at $w = l$ and setting $\tilde{\mu}_3 = 0$ ($\mu_3 = 0$) so $\tilde{\mu_1} + \tilde{\mu_2} = 1$ ($\mu_1 + \mu_2 = 1$) we find singularities in the four dimensional space described by the ϕ_1 and ϕ_2 planes corresponding to a spherical D3 distribution in that space. The case with three equal l_i gives the much simpler result

$$H_D^{-1} = w^4/L^4 \tag{7.35}$$

Here the distribution is an S^5 at $w = l$ as can be deduced from the fact that the r coordinates only extend to $r = 0$ or $w = l$, or by following the deformation of one of the above singular distribution as l_3 is switched on. Note that the S^5 distribution does not show up as singularities in H_D because it is an $SO(6)$ singlet and hence does not appear in the supergravity because it is not an operator in a short multiplet. The space is $AdS_5 \times S^5$ truncated at $w = l$.

Now we have identified the physical coordinates for $\mu = 0$ we can consider turning μ back on, for a fixed distribution, i.e. with the l_i fixed. Turning on μ introduces spin, corresponding to finite density in the field theory. This can be seen by observing the metrics at large w ($\simeq$ r) where they look like AdS with a gauge potential

$$A_i \simeq \frac{l_i \mu^{1/2}}{r^2} \tag{7.36}$$

In this limit the solution may be treated as five dimensional and one may calculate the charge density in the interior. In this way we may deduce a charge density in the dual field theory associated with each of the three $U(1)_R$ subgroups of $SU(4)_R$ which are proportional to $l_i \mu^{1/2}$. It seems reasonable to conclude that we are looking at a spinning version of the disc distribution.

We must however check that we have not introduced any other unwanted deformations of the theory. Let us first look at the middle example, where two of the l_i are switched on with equal values. Firstly, note that switching on μ does not change the g_{xx} component of the metric. The singularity locus in this component stays at the same place. Also the four-form $dt \wedge dw^3$ is not changed, so the number of D3 branes in the interior is unchanged. The final observation is that μ introduces no angular dependence in the ϕ_1 or ϕ_2 plane. The combination of these facts show that μ does not change the angularly constant in the ϕ_1 and ϕ_2 planes, distribution of D3 branes at $w = l$.

Thus the metrics with $\mu < \mu_c$ seem to naturally describe spinning versions of the multi-centre solution corresponding to the dual $\mathcal{N}$=4 theory being on its coulomb branch with a chemical potential. In fact it is clear that these metrics must describe such configurations because they are the unique solutions of the field equations with the symmetries of these systems. This sharing of symmetries between the black hole solutions and rotating D-brane distributions explains why the two sets of solutions are naturally intertwined. It should be noted here that it has not as yet been checked that these solutions are 'sensible', in the sense of not containing negative densities of branes, see Gubser, et al [].

7.3.2 Finite Density Phase Transition

It is interesting that, for a fixed distribution (fixed l_i), we cannot increase the chemical potential to infinity and maintain a rotating distribution form for the solution - at $\mu = \mu_c$ there is a transition to a black brane and we loose all information about the interior structure. In the field theory at this critical density apparently knowledge of the scalar vevs is lost. Note that there is a sharp change in the interpretation of the parameters of the solution. When the interior is naked the solution must provide information about the interior structure which it does through the parameter l and then μ plays the role of rotation speed. Above the critical μ there is a black brane and knowledge of the interior structure is lost and so l switches to describing the rotation and μ describes the newly available parameter, temperature.

In the field theory dual we must be seeing the *finite density transition of the coulomb branch* where the scalar potential is forced to favour zero vevs. When the chemical potential is much less than the scalar vevs, the vevs will be unaffected, but when the chemical potential is much larger the theory should look like the deconfined phase at the moduli space origin. The scale of the transition should be set by the size of the vevs, i.e. the l_i and we have indeed seen this since $\mu_c \sim l^4$. Above the critical density the spacetime is a black hole, a phase that has been identified with the deconfined phase of the field theory, as expected for the phase when the scalar vevs evaporate.

If we begin with a black brane metric with $\mu = \mu_c$ $(T = 0)$ and want to decrease the chemical potential we now realize there are two possibilities in the field theory. If the theory has small or zero vev it will remain in the deconfined phase as we decrease the density, else, if the theory has a large vev, then as we decrease the density below that vev the system should undergo a transition to the coulomb phase. It's now clear that the dual background elegantly offers us both of these choices! We can decrease the density in two ways - either we keep $\mu = \mu_c$ and decrease l in which case we retain a black brane configuration corresponding to the first case in the field theory, or we can keep l fixed and decrease μ in which case we obtain a spinning multi-centre solution describing the coulomb phase.

The solutions with the three l_i equal fit this story equally well except that there is no singularity to monitor the position of the D3 branes as μ is switched on. Again by considering deformations of other singular configurations it is clear that the interpretation is the same as that just given. The metrics with a single l_i switched on, however, do not show this behaviour. In fact as we saw above the condition for a $T = 0$ black brane is precisely $\mu = 0$ where the solution becomes a supersymmetric non-rotating disc distribution. For some reason these metrics do not provide us with *any* description of the rotating zero temperature states. Presumably this is just a failure of the completeness of these solutions rather than anything more subtle and we would expect similar behaviour on that part of the coulomb branch if only we had the appropriate metrics.

We note that this transition from the coulomb phase to the deconfined phase is also apparent in the similar solutions in which the Minkowski space slices of AdS are compactified [20, 21]. Recently Myers and Tafjord [43] have argued that the nakedly singular metrics in that case correspond to distributions of giant gravitons. Again though, above some critical angular momentum the solutions shift to black hole solutions showing that at high enough density the giant gravitons are forced to evaporate leaving a deconfined phase.

7.3.3 Stability

The black branes from the class of geometries we examine here have been studied by many authors [22, 30] and they have found them to be unstable for large densities. This instability fits with an analysis of the dual gauge theory [31]. If, at zero temperature, a chemical potential is introduced in to the $\mathcal{N} = 4$ gauge theory at tree level via a vev for the temporal component of a spurious gauge field then, as the scalars are in the $\mathbf{6}$ of $SU(4)_R$, there is a contribution to the scalar potential,

$$\Delta\mathcal{L} = |D^\mu\phi| \to A_0^2|\phi|^2 \tag{7.37}$$

This is a negative mass term for the scalar, which destabilizes the moduli space of the theory, giving a runaway potential. This is the same phenomena that occurs if one tries to rotate a D3 probe in AdS space. Rotational motion cannot be supported as there is no potential in the transverse space (as we found in (7.26)), so the brane will go straight to the edge of moduli space, corresponding to the runaway scalar vev. Quantum effects could stabilize the potential, so this argument is somewhat naive. In contrast, the backgrounds to spinning branes we have been examining are a complete dual description of the field theory with a chemical potential, so we may truely determine their stability by calculating the potential for a probe of them. Consider the example of the case where the three l_i are set equal. Expanding the resulting probe potential for small $A_0 = A_i(w = l) = \mu^{1/2}/l$ gives

$$V = \frac{w^4}{L^4}\left(1 - \sqrt{1 - \frac{L^2 A_0^2 l^2}{w^4}}\right) \simeq \frac{1}{2L^2}A_0^2 l^2 + \frac{1}{8}\frac{A_0^4 l^4}{w^4} + ... \tag{7.38}$$

The probe is forced to infinity by the potential (we plot the full expression in Figure 7.2). We deduce that the whole configuration is indeed unstable since any of the D3s in the distribution can be considered as the probe - they all want to run away to infinity. Remarkably, these backgrounds have though allowed us to explore the finite density behaviour of the coulomb branch of the theory ignoring this instability.

7.4 Thermodynamics of the Coulomb Branch

We will now extend our analysis to include finite temperature along with finite density and deduce the phase diagram for a point on the coulomb branch, *à la* Hawking-Page phase transition [40]. The procedure is just as in Section 7.2. Temperature, T, is introduced by compactifying the time-like direction with period $\beta = 1/T$. To simplify the calculation we choose the points on the coulomb branch where the global $SU(4)_R$ symmetry is preserved, which are the distributions where the D3 branes live on an S^5. This means taking the naked solutions above with all three l_i equal and fixed. These solutions exist up to μ_c and above we identified them with an S^5 distribution of D3 branes spinning equally in the three transverse planes. The chemical potential for these geometries is given by $\mu^{1/2}l$. Below $\mu_c^{1/2}l$, the geometries above also contain

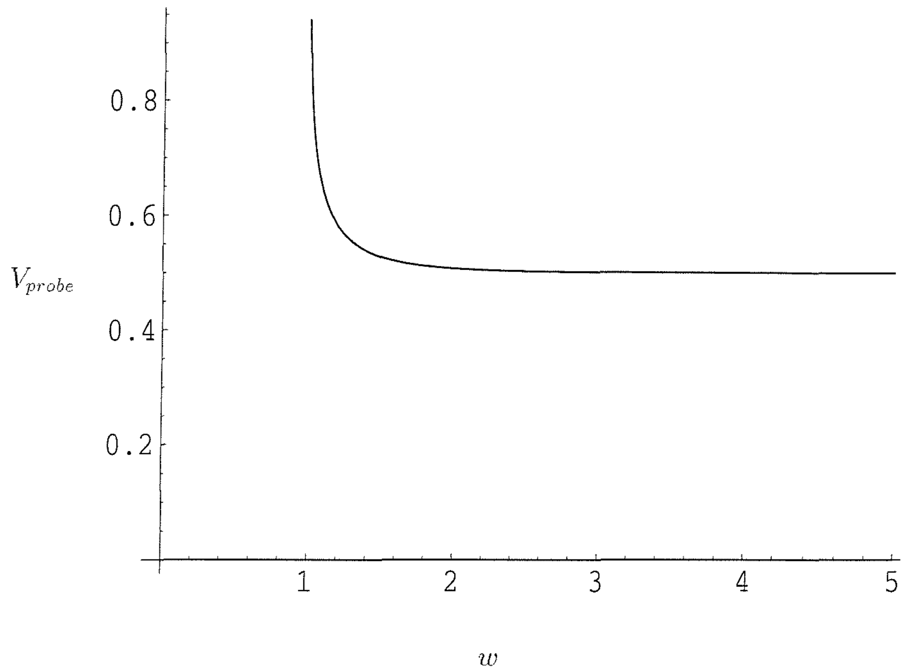

Figure 7.2: The form of the probe potential as a function of radial distance in the spinning D3 background.

black brane solutions with the same temperature and chemical potential. To find which type of solution is energetically prefered we need to calculate the free energy difference. The action is (7.3) still. Similar to previously, to allow comparison of the two spacetimes the period of the time integral of the naked case, $\tilde{\beta}$, must be set to match the geometry of the hypersurface at large R in the two cases. This is again achieved by requiring (7.5).

Calculation shows that the curvature, $R = 0$, leaving us with just the five-form and surface pieces. We use subscripts on the μ and l parameters to distinguish the naked and black brane cases; a '1' subscript denotes the black hole and a '2' the naked geometry. Carrying out the rest of the calculation we find the action difference

$$I = I_1 - I_2 = \frac{1}{\kappa^2}\mathrm{Vol}(S_5)\mathrm{Vol}(3)\beta\left(2l_2^4 - 2l_1^4 - \mu_2 + \mu_1 - 2r_h^4 - 4r_h^2 l_1^2\right) \qquad (7.39)$$

where $\mathrm{Vol}(S_5)$ is the volume of the deformed five-sphere, $\mathrm{Vol}(3)$ is the volume of the spatial part of the branes (both common to the two geometries) and r_h is the horizon radius of the black hole. Note that setting the l's and μ's to zero returns to the origin of the moduli space, so the previous result of (7.10) should be recovered, and it is. (7.39) is a function of temperature T and density ρ, just as (7.12) is a function of T, but we could not solve for the parameters of the current solution in terms of T and ρ. We had to resort here to the use of Maple/Mathematica.

This action calculation reveals the phase transition between the two geometries as a function of temperature and density that we were looking for. The form of the phase diagram is plotted in Figure 7.3. At low temperatures and densities the naked solution is preferred, and at high temperature and densitity the black brane solution is preferred. The transition is essentially governed by the black brane radius becoming larger than the distribution size. This is as expected since these are the only two scales in the problem. One may note that the transition point on the zero temperature axis is slightly below the $\mu_c = 27/4$ value we determined earlier. This shows that although one can have zero temperature naked solutions up to μ_c, they are thermodynamically disfavoured above this lower value of μ. There is no thermodynamic reason why this should not be true - the phase diagram still matches expectations, but it does make a precise interpretation unclear.

In the dual field theory at low temperature and density the solution describes a point on the coulomb branch with scalar vevs. At high temperature and density there is a transition to a deconfined phase without scalar vevs. The transition is brought about if the temperature or chemical potential is of the order of the scalar vevs. The result of the supergravity calculation matches field theory expectations.

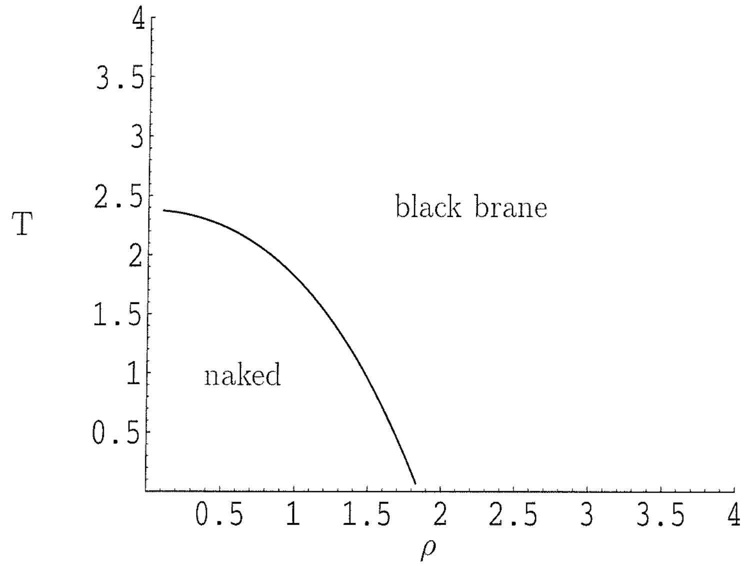

Figure 7.3: The temperature-density plane, showing the critical line inside which one has the naked solutions (confined), and beyond it the black holes (deconfined)

7.5 Summary

We have used the ten dimensional backgrounds around spinning D3 branes to study $\mathcal{N} = 4$ super Yang Mills theory at finite $U(1)_R$ charge density. We have completed the interpretation of the field theory duals of these backgrounds by interpreting the non-supersymmetric naked singularity class of the solutions. We found that these naked spinning D-brane distributions describe the coulomb branch at finite density. At a critical density a phase transition occurs to a spinning black brane representing the deconfined phase where the higgs vevs have evaporated. We also extended our analysis to include finite temperature. We performed a free energy calculation to determine the phase diagram of the coulomb branch at finite density and temperature.

Chapter 8

Conclusions

Following the generous amount of general introductory material on strings, branes, supergravity and field theory, the research work has been set out. This started in Chapter 4 with the considerations of the $\mathcal{N} = 4$ geometry. Specifically, we considered the extension of Maldacena's original duality proposal to the full moduli space of $\mathcal{N} = 4$ super Yang-Mills theory. We set out to discover how the field theory operators of $\mathcal{N} = 4$ on moduli space are encoded in their gravity dual solutions. We were successful in doing this and in fact found the encoding prescription to take a nice simple form.

Following this, in Chapter 5 we tried to extend our success to a case with a less symmetric field theory, namely the so-called $\mathcal{N} = 2^*$ theory, hoping that our reasoning in the $\mathcal{N} = 4$ theory would prove to be more generic. Unfortunately, we encountered problems because the gravity theory seemed to be giving extra renormalization that should not be present in the gauge theory. We offered some possble explanation for the evident discrepancy. We were however encouraged by the simple form of the metric on moduli space and its support for the philosophy of our approach.

Since the AdS/CFT correspondence is supposed to include the off moduli space region of the field theories, we investigated this explicitly in Chapter 6. Although we again encountered unexpected problems in the $\mathcal{N} = 2^*$ case, we did have success with the Leigh-Strassler scenario. Albeit only for a subspace of the full off moduli theory, we did indeed find a good description of the supergravity solution in terms of the dual field theory.

Finally, we applied the AdS/CFT correspondence to examine $\mathcal{N} = 4$ super Yang-Mills at finite $U(1)_R$ charge density, using the supergravity backgrounds around spinning D3 branes. We have completed the interpretation of the field theory duals of these backgrounds by interpreting the non-supersymmetric naked singularity class of the solutions. We found that these naked spinning D-brane distributions describe the coulomb branch at finite density. At a critical density a phase transition occurs to a spinning black brane representing the deconfined phase where the higgs vevs have evaporated. We also extended our analysis to include finite temperature. We performed a free energy calculation to determine the phase diagram of the coulomb branch at finite density and temperature.

This research has been concerned with examining extensions to the Maldacena conjecture, with an emphasis on the potential to learn about gauge theories. Overall the effort has been successful. Looking at the bigger picture, these small steps have added to the weight of evidence that the AdS/CFT correspondence is a very general and powerful principle with enormous potential for describing the physical world. We are edging towards a better understanding of strongly coupled field theories and confinement. We can also see hope for a successful quantized theory of gravity, perhaps even see hope for the grandest aims of theoretical physics, to understand basically all of physics as we know it! We can dream.

Appendix A

A.1 Conformal Symmetry

A conformal transformation is a diffeomorphism that preserves the metric up to an overall scale factor [13],

$$G'_{\mu\nu}(x) = w(x)G_{\mu\nu}(x) \qquad or \qquad \delta_v G_{\mu\nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu = wG_{\mu\nu} \tag{A.1}$$

It brings together Poincaré and scale transformations. In flat Minkowski space (of dimension d ≥ 3) its generators are the Lorentz generators $M_{\mu\nu}$, the Poincaré translations P_μ, plus dilatations D: $x^\mu \to \lambda x^\mu$ and the special conformal transformations K_μ: $x_\mu \to 2c_\rho x^\rho x_\mu - x_\rho x^\rho c_\mu$. The conformal group is isomorphic to $SO(d,2)$ with the identification [50]

$$\begin{aligned} \mathcal{M}_{\mu\nu} &= M_{\mu\nu}, \qquad \mathcal{M}_{d\mu} = \frac{1}{2}(P_\mu - K_\mu), \\ \mathcal{M}_{\mu(d+1)} &= \frac{1}{2}(P_\mu + K_\mu), \qquad \mathcal{M}_{d(d+1)} = D \end{aligned} \tag{A.2}$$

A field theory operator φ's scaling dimension Δ is determined by its transformation under the coordinate scaling:

$$D: \ x^\mu \to \lambda x^\mu \qquad \varphi(x) \to \varphi'(x) = \lambda^\Delta \varphi(\lambda x) \tag{A.3}$$

A.2 $\mathcal{N} = 4$ SYM Multiplets

A.2.1 Superconformal Multiplets of Local Operators

The unrenormalized dimensions of the canonical fields X^i, λ_a and A_μ are 1, 3/2 and 1, respectively. The gauge covariant objects X^i, λ_a, $F^\pm_{\mu\nu}$ and D_μ (the covariant derivative) are used to construct gauge invariant operators. Their dimensions are:

$$[X^i] = [D_\mu] = 1, \qquad [F^\pm_{\mu\nu}] = 2, \qquad [\lambda_a] = \frac{3}{2} \tag{A.4}$$

Ignoring renormalization effects of composite operators, all operator dimensions are therefore positive and there are a finite number of operators with dimension below a given number.

Now consider the conformal supercharges S. They have dimension $-1/2$, so repeated application to an operator of definite dimension must sooner or later produce 0 (to avoid violating unitarity with a negative dimension operator). This leads to the introduction of a *superconformal primary operator* $\mathcal{O}$, defined as

$$[S, \mathcal{O}]_\pm = 0, \qquad \mathcal{O} \neq 0 \tag{A.5}$$

An equivalent definition is as the lowest dimension operator in a given superconformal multiplet or representation. Note that a *conformal primary operator* is defined through its annihilation by the special conformal generators K^μ and are only a subset of the superconformal primaries.

A *superconformal descendant operator* $\mathcal{O}$ of some other (well-defined local polynomial gauge invariant) operator $\mathcal{O}'$ is when

$$\mathcal{O} = [Q, \mathcal{O}']_{\pm} \tag{A.6}$$

Their dimensions are related by $\Delta_{\mathcal{O}} = \Delta_{\mathcal{O}'} + 1/2$. It follows from this that in any irreducible superconformal multiplet there is a single operator of lowest dimension - the superconformal primary, from which all other operators 'descend'.

The fact that a superconformal primary operator is not the Q-commutator of another operator can be used to get explicit forms of the superconformal primary operators in $\mathcal{N} = 4$ SYM. The schematic Q-transforms of the canonical fields can be written

$$\begin{aligned} \{Q, \lambda\} = F^{+} + [X, X], \qquad & [Q, X] = \lambda \\ \{Q, \bar{\lambda}\} = DX, \qquad & [Q, F] = D\lambda \end{aligned} \tag{A.7}$$

A local polynomial could not be primary if it contained any of the elements on the right-hand side of these structure relations. This severely limits the constituents of superconformal primary operators; they are gauge invariant scalars involving only X in a symmetrized way.

What is left splits into *single trace* and more complicated *multiple trace* operators. Single trace operators take the form:

$$\mathrm{str}\left(X^{i_1} X^{i_2} ... X^{i_n}\right) \tag{A.8}$$

Here i_j, $j = 1, ...n$ are $SO(6)_R$ fundamental representation indices and 'str' is the symmetrized trace over the gauge algebra (so this operator is symmetric in $SO(6)_R$

indices). $\mathrm{tr}X^i = 0$, leaving

$$\begin{aligned} \sum_i \mathrm{tr}X^i X^i &\sim \text{ Konishi multiplet} \\ \mathrm{tr}X^{\{i}X^{j\}} &\sim \text{ supergravity multiplet} \end{aligned} \tag{A.9}$$

to constitute the simplest operators, where $\{ij\}$ means the traceless part only. The multiple trace operators are formed from products of the single trace operators. Taking tensor products of totally symmetric representations leads to (partially) anti-symmetrized representations of $SO(6)_R$. There is a one-to-one map between the chiral primary operators and unitary superconformal multiplets, which allows a labelling of all state and operator multiplets in terms of the superconformal chiral primary operators.

A.2.2 Chiral or BPS Multiplets of Operators

The quantum numbers of the bosonic subgroup of $SU(2,2|4)$ are good labels for the unitary representations of this superconformal algebra, thus,

$$\begin{array}{ccccc} SO(1,3) & \times & SO(1,1) & \times & SU(4)_R \\ (s_+, s_-) & & \Delta & & [r_1, r_2, r_3] \end{array} \tag{A.10}$$

$s_\pm$ are positive or zero half-integers, Δ is the positive or zero dimension and $[r_1, r_2, r_3]$ are the Dynkin labels from the $SU(4)_R$. Instead of Dynkin labels, the $SU(4)_R$ representations can be labelled by their dimensions, given in terms of $\bar{r}_i \equiv r_i + 1$ by

$$\dim[r_1, r_2, r_3] = \frac{1}{12}\bar{r}_1\bar{r}_2\bar{r}_3(\bar{r}_1 + \bar{r}_2)(\bar{r}_2 + \bar{r}_3)(\bar{r}_1 + \bar{r}_2 + \bar{r}_3) \tag{A.11}$$

Complex conjugation is given by $[r_1, r_2, r_3]^* = [r_3, r_2, r_1]$.

The dimensions Δ are bounded below, in unitary representations, by the spin and $SU(4)_R$ quantum numbers. To show this, it is only necessary to consider the primary operators, since they have the lowest dimension in a given multiplet. We have seen above that these operators are scalars, so the bounds will in fact only involve the $SU(4)_R$ quantum numbers. It can be shown [64, 66, 67] that there are four distinct series:

- $\Delta = r_1 + r_2 + r_3$
- $\Delta = \frac{3}{2}r_1 + r_2 + \frac{1}{2}r_3 \geq 2 + \frac{1}{2}r_1 + r_2 + \frac{3}{2}r_3$ this requires $r_1 \geq r_3 + 2$
- $\Delta = \frac{1}{2}r_1 + r_2 + \frac{3}{2}r_3 \geq 2 + \frac{3}{2}r_1 + r_2 + \frac{1}{2}r_3$ this requires $r_3 \geq r_1 + 2$
- $\Delta \geq \text{Max}\left[2 + \frac{3}{2}r_1 + r_2 + \frac{1}{2}r_3;\ 2 + \frac{1}{2}r_1 + r_2 + \frac{3}{2}r_3\right]$

The second and third are complex conjugates of each other.

The first three cases correspond to discrete series of representations, where at least one supercharge Q commutes with the primary operator. These are shortened representations called *chiral multiplets* or *BPS multiplets* (BPS from analogy with Poincaré supersymmetry). The dimension of these representations is unrenormalized (cannot get quantum corrections) due to this shortening. The fourth case corresponds to continuous series of representations, called non-BPS, where no supercharges Q commute with the primary operator. The properties of the various BPS and non-BPS multiplets are set out in Table A.1 [13] (in which #Q is the number of Poincaré supercharges that leave the primary invariant).

A.3 $\mathcal{N} = 2^*$ Field Theory

As mentioned in the main text, the $\mathcal{N} = 2^*$ field theory is a mass deformation of $\mathcal{N} = 4$ $SU(N)$ Yang-Mills theory [27]. We first review its content in terms of four

Operator Type	#Q	Spin Range	$SU(4)_R$ Primary	Dimension Δ
identity	16	0	$[0,0,0]$	0
1/2 BPS	8	2	$[0,k,0]$, $k \geq 2$	k
1/4 BPS	4	3	$[l,k,l]$, $k \geq 2$	$k+2l$
1/8 BPS	2	7/2	$[l,k,l+2m]$, $l \geq 1$	$k+2l+3m,\ m \geq 1$
non-BPS	0	4	any	unprotected

Table A.1: Properties of BPS and non-BPS multiplets

dimensional $\mathcal{N} = 1$ supersymmetry. It has a vector multiplet V and an adjoint chiral superfield Φ which go together to form an $\mathcal{N} = 2$ vector multiplet. It has two further adjoint chiral multiplets Q and $\tilde{Q}$ that form an $\mathcal{N} = 2$ hypermultiplet. There are the standard gauge-invariant kinetic terms for these fields, but there are additional interactions and hypermultiplet mass terms, which can be summarized by the following superpotential:

$$W = \frac{2\sqrt{2}}{g_{YM}^2}\mathrm{tr}([Q,\tilde{Q}]\Phi) + \frac{m}{g_{YM}^2}(\mathrm{tr}Q^2 + \mathrm{tr}\tilde{Q}^2) \tag{A.12}$$

There is a Coulomb vacua moduli space which is parametrized by expectation values of the adjoint scalar,

$$\Phi = \mathrm{diag}(a_1, a_2, ..., a_N), \qquad \sum_i a_i = 0 \tag{A.13}$$

in the Cartan subalgebra of the gauge group. The low energy effective action $\mathcal{L}$ for the $N-1$ $U(1)$ $\mathcal{N} = 2$ vector multiplets can be completely determined in terms of the prepotential $\mathcal{F} \equiv \mathcal{F}(\tau, m; \{a_i\})$ where the gauge coupling $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g_{YM}^2}$ and m is the hypermultiplet mass. The prepotential is defined as

$$\mathcal{F} = \frac{1}{2}\tau\phi^2 \tag{A.14}$$

so that

$$\frac{\partial^2 \mathcal{F}}{\partial \phi^2} = \tau \tag{A.15}$$

We have

$$\begin{aligned} 8\pi\mathcal{L} &= -g_{i\bar{j}}\left(D_\mu a^i D^\mu \bar{a}^{\bar{j}} + i\bar{\psi}\bar{\sigma}^\mu D_\mu \psi^i\right) \\ &\quad + \mathrm{Re}\left\{\tau_{ij}\left(\frac{i}{2}F^i_{\mu\nu}F^{j\mu\nu} + \frac{1}{2}F^i_{\mu\nu}\tilde{F}^{j\mu\nu} - 2\bar{\lambda}^i\bar{\sigma}^\mu D_\mu \lambda^j\right)\right\} \end{aligned} \tag{A.16}$$

where

$$\tau_{ij} = \frac{\partial^2 \mathcal{F}}{\partial a^i \partial a^j}, \qquad g_{i\bar{j}} = \mathrm{Im}[\tau_{ij}] \tag{A.17}$$

and ψ^i and λ^i are the fermionic superpartners of the scalars and gauge bosons respectively. The Levi-Civita connnection Γ^i_{jk} of the scalar metric forms the covariant derivative,

$$D_\mu a^i = \partial_\mu a^i + \Gamma^i_{jk} a^j \partial_\mu a^k \tag{A.18}$$

The full quantum prepotential can be written as a sum of classical and perturbative and non-pertubative quantum corrections:

$$\mathcal{F} = \mathcal{F}_{class} + \mathcal{F}_{pert} + \mathcal{F}_{non-pert} \tag{A.19}$$

The classical part is given by

$$\mathcal{F}_{class} = \frac{1}{2}\tau \sum_i a_i^2 \tag{A.20}$$

and the non-perturabtive part is generated by instantons. The perturbative piece quoted in Chapter 5, in (5.20),

$$\mathcal{F} = \frac{i}{8\pi}\left[\sum_{i\neq j}(a_i - a_j)^2 \ln\left(\frac{(a_i - a_j)^2}{\mu^2}\right) - \sum_{i\neq j}(a_i - a_j + m)^2 \ln\left(\frac{(a_i - a_j + m)^2}{\mu^2}\right)\right] \tag{A.21}$$

is one-loop exact [73, 72] and comes from standard quantum field theory calculation.

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