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THEORY OF ITERATIVE LEARNING CONTROL SYSTEM INTERCONNECTION

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ABSTRACT

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In this research, we study an interconnection of two general iterative learning control (ILC) system. This topic is a new topic in the area of ILC. To develop this theory, a new robustness notion describing the effect of disturbance and initial condition in ILC system is introduced. This new notion is called iterative input to state stability (iterative ISS), which is formulated following the idea of input to state stability. Based on this notion, the cascade and feedback interconnection of ILC system is considered. It is proved that bounded disturbance bounded state property and disturbance asymptotic gain property hold for feedback and cascade system interconnection, provided each subsystem is iterative ISS. In order to see the applicability of the theory of ILC system interconnection, we perform a case study. In particular, we choose a certain class of nonlinear adaptive Lyapunov based ILC and investigate whether iterative ISS is a property of this class of control system. If iterative ISS can be derived from this particular adaptive nonlinear ILC then the general ILC system theory interconnection is applicable to this class of nonlinear adaptive ILC. The result shows that nonlinear adaptive ILC does not have the property of iterative ISS although under restrictive disturbance and initial condition set. As a recommendation for future work, we discuss a possibility to achieve the iterative ISS property of this class of nonlinear adaptive ILC.

Contents

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Chapter 1

Introduction

1.1 General Principle of ILC

Iterative learning control (ILC for short) considers systems that repetitively perform the same task with a view to sequentially improving accuracy. Examples of this notion can be found for example in the papers of Arimoto et al. ([2],[3]), Furuta and Yamakita ([10]), Padieu and Su ([28]), Owens ([24],[25]), Lee and Lee ([21]), Moore ([22]), and Bucheit et al. ([6]) as well as the general area of trajectory following in robotics. The specified task is regarded as the tracking of a given reference signal $r(t)$ or output trajectory for an operation on a specified time interval $0 \leqslant t \leqslant T$. It is important to note that feedback control cannot, by its nature, achieve this exactly as a non-zero error is required to activate the feedback mechanism. The objective of ILC is to use the repetitive nature of a process in order to improve progessively the accuracy with which the operation is achieved by changing the control input iteratively from trial to trial.

Improvements in performance correspond intuitively to reductions (pointwise, peak or energy) in the difference between the reference signal and the actual output of the system signal in a trial. Improving performance is the objective of the control strategy and this can only be achieved by using available data from the process in an effective manner. As the ILC process is iterative, this means, that signal and measurements from the previous trials are the natural choice of data for use in the construction of control inputs for the present trial. The control system is said to '*learn*' by remembering the effectiveness of previously tried inputs and using information from their success or failure to construct a new trial control input functions. In contrast to adaptive scheme, ILC does not attempt to identify explicitly the plants but changes (or adapts) only the control input. This 'adaptation' or updating takes place after each trial, and not after each time step as in adaptive control.

The technical difficulty of ILC lies in two-dimensionality (in the mathematical sense) of the overall system ([25]) and the need for consequence changes in methods of analysis and thinking, including the ideas of causality and stability. The two dimensions are the trial index k (discrete) and the elapsed time t (continuous or discrete) during a trial. It is obviously desirable to have notions of stability with respect to both dimensions in a precisely defined sense (see Rogers and Owens ([29]) for some related ideas in the theory of repetitive dynamical systems). Whilst stability in the t direction has the simple and standard interpretation, stability in the k direction is taken to be equivalent to converge of the ILC algorithm in a precisely defined sense. As the different notions of causality, stability and convergence places ILC outside the traditional realm of control theory, it is important to study it as a subject area in its own right.

ILC was originally introduced by Arimoto ([2],[3]) who presented an algorithm that generated the new trial control input by adding a 'correction' term to the control input of the previous trial. This control increment was calculated using the previous trial tracking error data. They also derived convergence conditions for this algorithm in terms of th state-space matrices of the plant. ILC has since then been further explored using similar techniques and ideas but still underdeveloped. Various update algorithms and corresponding invariant or time-variant, linear or nonlinear and especially the particular problems of mechanical systems, can be seen, for example, in robotic manipulators. Robotic is a particularly important application area of ILC. A textbook about ILC ([22]) includes a literature survey up to 1992. A significant distinction is whether linear or nonlinear systems are considered.

In this thesis, the problem of interconnection of two ILC systems is investigated. In particular, we focus our study on ILC system which is robust to disturbance and the initial resetting error. The following problem consider: suppose we have two robust ILC systems, is the interconnection of these two ILC systems is also robust? To tackle this problem, we define precisely the desired robustness property of an ILC system. We utilise the idea of input to state stability, which is introduced by Sontag (see e.g. [35]) to formulate the robustness property of an ILC system. The possibility of interconnection of two ILC systems for having this robustness property is investigated. In section 1.4 the input to state stability theory is reviewed and in section 1.5 we will present the motivation of considering this problem. In the subsequent chapters, the interpretation of input to state stability theory into ILC context to formulate a new robustness property of ILC system is considered. The theory of ILC system interconnection and a case study will be discussed afterward.

1.2 Rigorous Theoretical Approach to ILC

1.2.1 Motivational Example

Before we discuss some theories of ILC design, we need to put ILC design problem into a sensible mathematical formulation. To understand how it is formulated, we begin with a simple example. Suppose we require to control the following system [3]:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{1.1}$$

with the output:

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{1.2}$$

Suppose the reference signal to be tracked is $y_d(t) = 12t^2(1-t)$. The objective is to track the reference signal $y(t)$. Following [3], this can be done by controlling the system iteratively and update the control input in each iteration as follows:

Iteration 1: set $u(t) = y_d(t)$. Calculate the output $y(t)$ from equation (1.1) and (1.2). Calculate the error: $e(t) = y_d(t) - y(t)$. Set $u_1(t) = u(t)$, $y_1(t) = y(t)$ and $e_1(t) = e(t)$.

Iteration 2: compute $u(t)$ as follows:

$$u(t) = u_1 + \frac{d}{dt}e_1(t) \tag{1.3}$$

Calculate $y(t)$ from equation (1.1) and (1.2). Calculate the error: $e(t) = y_d(t) - y(t)$. Set $u_2(t) = u(t)$, $y_2(t) = y(t)$ and $e_2(t) = e(t)$.
Iteration 3: compute $u(t)$ as follows:

$$u(t) = u_2 + \frac{d}{dt}e_2(t) \tag{1.4}$$

Calculate $y(t)$ from equation (1.1) and (1.2). Calculate the error: $e(t) = y_d(t) - y(t)$. Set $u_3(t) = u(t)$, $y_3(t) = y(t)$ and $e_3(t) = e(t)$. This algorithm is repeated until $y_d(t) - y_k(t) = 0$ for each t, where k is the number of iterations which have been executed. The number of iteration k can be finite or '∞' (which means the system keeps approaching the desired behaviour as iteration goes).

After approximately 10 times of calculation, it is shown in [3] by simulation that the magnitude of error $|y(t) - y_d(t)|$ approaches zero. Hence, after repeated applications of $u(t)$, the system converges to the desired control objective. Denote the number of iteration by k, referring to the equation (1.3) and (1.4) the relation between input of iteration k and $k+1$ can be written as follows:

$$u_{k+1} = u_k + \frac{d}{dt}y_d - \frac{d}{dt}y_k \tag{1.5}$$

1.2.2 One Dimensional Representation

To study many aspects of ILC deeply, we need to formulate a general (nonlinear) ILC problem in a rigorous theoretical framework. Consider again the system (1.1)-(1.2). By applying ILC, in each iteration we get different state and output trajectory. Hence, at the k-th iteration, the system (1.1) and (1.2) should be written as follows:

$$\frac{d}{dt}\begin{bmatrix} x_{1k} \\ x_{2k} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x_{1k} \\ x_{2k} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k \tag{1.6}$$

$$y_k = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x_{1k} \\ x_{2k} \end{bmatrix} \tag{1.7}$$

with the ILC law:

$$u_{k+1} = u_k + \Gamma(\dot{y}_d - \dot{y}_k) \tag{1.8}$$

Now we put the ILC problem in a general setting. Consider a system which is described in the nonlinear continuous time state space form. As in the equation (1.6)-(1.7) we put an index k in each variable to capture the system behaviour in every iteration. The system representation is written as follows :

$$\begin{aligned} \dot{x}_k(t) &= f(x_k(t), u_k(t), d_k(t)) \\ y_k(t) &= h(x_k(t), \eta_k(t)) \end{aligned} \tag{1.9}$$

In this work, we call the representation of (1.9) as *iterative system*. It is to differentiate with the usual state space representation, which does not include the index k. The index k is taken to be the element of the set of natural number i.e. $k \in \mathbb{N}$. The state x_k belongs to a state space $\mathcal{X}$, the output y_k belongs to an output space $\mathcal{Y}$, u_k belongs to an admissible input space $\mathcal{U}$, f and h are the system dynamics belong to space $\mathcal{F}$ and $\mathcal{H}$, respectively (the space $\mathcal{F}$ and $\mathcal{H}$ are typically taken to be a function space). These dynamics f (and also h) is unknown in the sense that we only know some property of this dynamics such as the bound of f (which can be constant or a known function) and the increment property of f (the Lipschitz condition).

The ILC problem can be formulated as follows. Suppose $y_d \in \mathcal{Y}$ is the reference signal to be tracked then the control objective is to find the sequence of input $\{u_k\}_{k\geqslant 1}$ so that y_k asymptotically 'track' the reference signal y_d i.e.:

$$\limsup_{k\to\infty} \|y_k - y_d\| = 0 \tag{1.10}$$

or

$$\limsup_{k\to\infty} \|y_k - y_d\| \leqslant \epsilon \tag{1.11}$$

where $\|\cdot\|$ it is a suitable defined norm on $\mathcal{Y}$ and ϵ is a 'small' positive number, which is usually specified by the designer. In some cases such as $\{\|y_k - y_r\|\}_{k\geqslant 1}$ decreases monotonically, we can replace 'lim sup' with just 'lim'. There are various approaches to find $\{u_k\}_{k\geqslant 1}$ in the literature as will be discussed in the following section.

Beside forcing the output to track the desired reference output, it is also required that the input behaves well along the iteration and over the iterations. For example the input is typically required to be bounded/uniformly bounded for each $t \in [0, T]$ and for each $k \in \mathbb{N}$. It may also require the input to converge to a desired input, etc.

The iterative system representation (1.9) is a non-standard state space representation. The control problem of achieving the convergence in the sense of (1.10) or (1.11) is also a non-standard control problem. In here, there are two kinds of information propagation that need to be considered. The information which propagates along the finite time axis and the information which is transmitted over finite time axis. Hence, a notion such as boundedness and uniformity need to be established on both direction.

As we can see, ILC involves the system behaviour over iterations as well as over a finite time (along iteration). Hence, a rigorous approach to solve ILC problem is by formulating the problem into 2-dimensional system theory. However, the theory of 2 dimensional system for nonlinear system is not well established. Therefore, we use a one dimensional nonlinear system theory to formulate and solve the ILC problem.

1.3 Theory of ILC Design: Linear System

In the literature, there are some theories have been proposed to solve ILC problem for linear and nonlinear system. The surveys [23] is among good expositions of this subject.

In this section we discuss the solution of ILC problem if (1.9) is linear. In general, theory of ILC for linear system is less complicated than nonlinear system. Hence, to grasp a better understanding about ILC theory we begin with linear system first. Consider the linear system:

$$\dot{x}_k = Ax_k + Bu_k \tag{1.12}$$

with the pair (A, B) is controllable. The problem is to design an ILC algorithm for this system. The following ILC is proposed in the literature:

1.3.1 D-type ILC

D-type ILC is the first ILC algorithm appears in the literature. It was proposed by Arimoto ([3]). The original task is to make the tracking performance of robot manipulator is better. It has the following form:

$$u_{k+1} = u_k + \Gamma \frac{d}{dt}(y_k - y_d) \tag{1.13}$$

The system (robot manipulator) to be controlled is a linear system (1.12). The objective is to make the error converge in the sense of λ-norm i.e:

$$\lim_{k\to\infty} ||e_k||_\lambda = \lim_{k\to\infty} ||y_k - y_{ref}||_\lambda = 0 \tag{1.14}$$

In the works of Arimoto ([3]), it has been proved that the D-type ILC above achieves a convergence in the sense of λ-norm (see the definition of λ-norm in the appendix A) with a condition that $CB \neq 0$. The details of the theorem and proofs can be seen in [3]. Based on this result, some researchers such as [13] and [14] try to use D-type ILC to nonlinear system. We will discuss this in the next section.

1.3.2 P-type ILC

The D-type ILC which is originally suggested by Arimoto uses a derivative in the control law (1.13). In many implementation the use of derivative needs to be avoided. Arimoto ([5],[4], see also [12] and [31]) propose ILC without using derivative as follows:

$$u_{k+1}(t) = (1-\alpha)u_k(t) + \alpha u_0 + \phi e_k(t), \quad 0 < \alpha < 1 \tag{1.15}$$

Where $\alpha > 0$ is a forgetting factor and u_0 is the initial input, which can be taken as an arbitrary continuous function or number and used to enhance the performance. If the ILC (1.15) above is applied to linear system (1.12) then the convergence in λ norm as in (1.14) is obtained ([5], [12], [31]). For nonlinear system, as pointed out by Saab([31]), the convergence can be achieved. However, there are many strict technical assumptions need to be satisfied. We will discuss about this further in the next section.

1.3.3 Universal Adaptive ILC

Consider the following linear system as in (1.12). Supposed it can be stabilised by the so-called universal adaptive controller proposed by Owens ([26],[27]):

$$u(t) = -sgn(CB)K(t)y(t), \qquad \dot{K}(t) = cy^2(t), \; c > 0, \quad K(0) = K_0 \tag{1.16}$$

and has the properties that for all values of x_0 and for all choices of K_0,

$$\lim_{t\to\infty} y(t) = 0, \quad \text{and} \quad \lim_{t\to\infty} K(t) = K_\infty(x_0, K_0) < \infty \tag{1.17}$$

The proposed adaptive ILC algorithm:

$$u_{k+1}(t) = u_k(t) + sgn(CB)[(K_{k+1}e_{k+1})(t) + (F_{k+1}e_k)(t)], \quad 0 \leqslant t \leqslant T \tag{1.18}$$

with:

$$\begin{aligned} e_k &= y_k - y_d \\ K_{k+1} &= K_k + c||e_k||^2 \end{aligned}$$

and F_k is taken to be a constant F for each $k \in \mathbb{N}$.

Assumption 1.3.1. *The system (1.12) is single input and single output*

Theorem 1.3.1. *[27] Suppose that the plant satisfies assumption (1.3.1) and the adaptive ILC algorithm described above is applied with an arbitrary choice of input $u_0 \in L_2[0,T]$, generating an initial error $e_0 \in L_2[0,T]$. Suppose also that the reference signal r can be generated exactly by an input $u_\infty \in L_2[0,T]$. Under these conditions we have:*

1. *The tracking error converges to zero in the sense of $L_2[0,T]$*
2. *The monotonically increasing adaptive feedback gain parameter sequence $\{K_k\}_{k\geqslant 0}$ converges to a limit gain $K_\infty < \infty$*
3. *There exists a gain K^* such that whenever $K_k > K^*$, the error norm $\{||e_k||\}_{k\geqslant k^*}$ is strictly monotonically decreasing*
4. *If the plant (1.12) is also minimum-phase, then it is possible to choose K^* to be independent of trial length T.*

1.4 Theory of ILC Design: Nonlinear Affine System

In this section, we generalise the theory of ILC from previous section to nonlinear systems. We consider the following time invariant nonlinear system:

$$\begin{aligned} \dot{x} &= f(x) + B(x)u + w, \;\; x(0) = \delta \\ y &= g(x) + v \end{aligned} \tag{1.19}$$

To make the result hold generally we take the following: state and output signals are all taken to be continuous functions:

$$x \in C([0,T], \mathbb{R}^n), \quad y \in C([0,T], \mathbb{R}^p)$$

The control signal is a mapping of $u : [0,T] \to \mathbb{R}^m$ which is integrable along finite interval $[0,T]$. This signal is not necessary to be continuous since we may also consider a discontinuous control. However, it requires to be integrable since we measure 'the size' of u using $L^p[0,T]$-norm, $1 \leqslant p < \infty$ as well as $L^\infty[0,T]$-norm.

The system dynamic, the input dynamic and the output dynamic are taken to be the following mappings:

$$f : \mathbb{R}^n \to \mathbb{R}^n, \;\; g : \mathbb{R}^n \to \mathbb{R}^p, \;\; B : \mathbb{R}^n \to \mathbb{R}^{n\times m}$$

The disturbance terms are taken as follows:

$$w : [0,T] \to \mathbb{R}^n, \text{ and } v : [0,T] \to \mathbb{R}^p \tag{1.20}$$

In addition w and v belong to $L^2[0,T] \cap L^\infty[0,T]$ respectively, which means the disturbances is required to have a finite energy and bounded.

Finally the initial condition (initialisation error) δ is taken to be a vector in $\mathbb{R}^n$.

1.4.1 D-type ILC

Algorithm

In this subsection we discuss some works which uses the D-type ILC as in the linear case (section 1.3) to nonlinear system. Denote x_k, y_k, δ_k, w_k, v_k the state, output, initial condition (initialisation error), and disturbances at pass k, respectively. The form of the controller is the same as in (1.13), but it has a nonlinear learning operator to handle the system nonlinearity. It is expressed as follows:

$$u_{k+1} = u_k + \mathcal{L}(x_k, t)\frac{d}{dt}(y_k - y_d) \tag{1.21}$$

or the variant of it such as :

$$u_{k+1} = \alpha u_0 + (1-\alpha)u_k + \mathcal{L}(x_k, t)\frac{d}{dt}(y_k - y_d) \tag{1.22}$$

with the mapping $\mathcal{L} : \mathbb{R}^n \times [0,T] \to \mathbb{R}^{m\times p}$ and $\mathcal{L}$ belongs to $L^\infty[0,T]$ and need to satisfy a certain condition (see equation (1.24) below. Similarly in the linear case the convergence is achieved in the sense of λ-norm as follows:

$$\lim_{k\to\infty} ||y_k - y_d||_\lambda \leqslant \epsilon \tag{1.23}$$

where $\epsilon \geqslant 0$ which depends on the size of disturbance w_k, v_k and δ_k.

To attain the convergence as in (1.23) then some additional assumptions are imposed to the system (1.19)-(1.20) as follows

1. the mapping from input to state and from state to output is one to one
2. the dynamics f, B in (1.19) and (1.20) along with the derivative of g: $\frac{\partial g}{\partial t}$ and $\frac{\partial g}{\partial x_k}$ posses a *global Lipschitz condition.* It is defined as follows: there exist positive constants k_1, k_2, k_3 and k_4 such that for any t_1 and t_2 in $[0,T]$, we have

$$\begin{aligned}
||f(x_k(t_2)) - f(x_k(t_1))||_{L^\infty[0,T]} &\leqslant k_1||x_k(t_2) - x_k(t_1)||_{L^\infty[0,T]} \\
||B(x_k(t_2)) - B(x_k(t_1))||_{L^\infty[0,T]} &\leqslant k_2||x_k(t_2) - x_k(t_1)||_{L^\infty[0,T]} \\
||\frac{\partial g}{\partial t}(x_k(t_2)) - \frac{\partial g}{\partial t}(x_k(t_1))||_{L^\infty[0,T]} &\leqslant k_3||x_k(t_2) - x_k(t_1)||_{L^\infty[0,T]} \\
||\frac{\partial g}{\partial x_k}(x_k(t_2)) - \frac{\partial g}{\partial x_k}(x_k(t_1))||_{L^\infty[0,T]} &\leqslant k_4||x_k(t_2) - x_k(t_1)||_{L^\infty[0,T]}
\end{aligned}$$

Furthermore, there is also a condition need to be satisfied by the controller (1.21) and (1.22) as follows:

$$||I - \mathcal{L}(g(x_k), t)\cdot g_x(x_k)\cdot B(x_k)||_{L^\infty[0,T]} \leqslant \rho < 1, \quad \forall(x;t) \in \mathbb{R}^n \times [0,T] \tag{1.24}$$

The assumption 1 is a technical assumption which guarantee that for any input there will be a corresponding output trajectory. It implies that for each reference trajectory there exists a unique input u_d. Or in other word, there is a unique input u_d which produce the reference trajectory y_d. The assumption 2 states that the system has a linear growth. Hence, by this assumption, we can expect that since the D-type ILC (1.21) works in linear system then it must also be able to achieve a convergence for nonlinear system having linear growth. The condition on $\mathcal{L}$ as in (1.24) is a nonlinear analogy with linear case, which is a technical condition to achieve the convergence.

First of all, we consider the most simple case where $w_k = 0$, $v_k = 0$, and $\delta_k = 0$. For this case, Hauser ([13]) has proven that the D-type ILC (1.21) achieves convergence as stated in the following theorem:

Theorem 1.4.1. *[13] Consider the system (1.19)-(1.20). Let $x_d(0)$ and $y_d(t)$, $t \in [0,T]$ be given with $y_d(\cdot)$ be a realizable trajectory. Let $x_k(0) = x_d(0)$ for all k. Suppose $\mathcal{L}$ satisfies inequality (1.24), then the learning operator given by equation (1.21) will generate a sequence of inputs, $u_k(t)$, $t \in [0,T]$, uniformly in t. Furthermore, the sequences, $x_k(t), y_k(t), t \in [0,T]$, generated by these controls are such that:*

$$\begin{aligned}
\limsup_{k\to\infty} \|u_k - u_d\|_\lambda &= 0 \\
\limsup_{k\to\infty} \|x_k - x_d\|_\lambda &= 0 \\
\limsup_{k\to\infty} \|y_k - y_d\|_\lambda &= 0
\end{aligned}$$

If disturbance occurs then the convergence of D-type ILC will depend on the size of the disturbance. This is reasonable since the disturbance will affect the performance of the controller. The work of Heinzinger and co-authors ([14])proved that under the presence of bounded disturbance, the ILC (1.21)-(1.22) will achieve the bounded convergence as in (1.23). The disturbance that they consider is bounded by λ-norm as follows:

$$\|w_k\|_{L^\infty[0,T]} \leqslant b_w, \quad v_k = 0, \text{ and } |x_k(0)| \leqslant b_{x0}$$

This result is stated in the following theorem:

Theorem 1.4.2. *[14] Let the system described as (1.19)-(1.20) satisfy the above assumptions with the ILC as in (1.21) and (1.22). Given the desired output trajectory $y_d(\cdot)$ and an initial state $x_d(0)$, which are achievable, if the condition (1.24) is satisfied then there exist positive constants $a_1, a_2, a_3, b_1, b_2, b_3, c_1$ such that:*

$$\begin{aligned}
\limsup_{k\to\infty} \|u_k - u_d\|_\lambda &\leqslant \left(\frac{1}{1-\bar{\rho}}\right)(a_1|x_k(0) - x_d(0)| + a_2 b_w + a_3\|u_0 - u_d(0)\|_\lambda) \\
\limsup_{k\to\infty} \|x_k - x_d\|_\lambda &\leqslant b_1\left(\frac{1}{1-\bar{\rho}}\right)(a_1|x_k(0) - x_d(0)| + a_2 b_w + a_3\|u_0 - u_d(0)\|_\lambda) + b_2|x_k(0)| \\
\limsup_{k\to\infty} \|y_k - y_d\|_\lambda &\leqslant c_1\left(\left(\frac{1}{1-\bar{\rho}}\right)(a_1|x_k(0) - x_d(0)| + a_2 b_w + a_3\|u_0 - u_d(0)\|_\lambda) + b_2|x_k(0)|\right)
\end{aligned}$$

The other work on D-type ILC is the work of Chen ([7]) which tries to eliminate the disturbance caused by the initialisation error. The system which is considered is still the nonlinear system (1.19)-(1.20) along with the technical assumptions 1 and 2. The ILC is as in the equation (1.21), but it is modified to be of the form:

$$u_{k+1}(t) = u_k(t) + \bar{\mathcal{L}}(t)\frac{d}{dt}(y_k(t) - y_d(t)) \tag{1.25}$$

where $\bar{\mathcal{L}}$ is a mapping of $[0,T] \to \mathbb{R}^{m\times p}$ and belongs to $L^\infty[0,T]$. This modification means that $\bar{\mathcal{L}}$ is uniform in every pass on $[0,T]$, which is different with $\mathcal{L}$ that depends on k, if viewed from the domain $[0,T]$. In addition of the controller (1.25) they propose the following updating algorithm:

$$x_{k+1}(0) = x_k(0) + B(0)\bar{\mathcal{L}}(0)e_k(0) \tag{1.26}$$

The objective that can be achieved is still a bounded convergence in λ-norm as in (1.23) but the ϵ is independence with the initial condition error.

For this purpose some more assumptions need to be put on the system (1.19)-(1.20) in conjunctions with the assumptions 1 and 2 above:

- the input dynamic B as a function of $t \in [0,T]$ is uniform in every pass: $\forall k \in \mathbb{N},\ B : [0,T] \to \mathbb{R}^m$

- the output dynamic h is linear in state and uniform in every pass as a function $t \in [0,T]$, that is: $\forall k \in \mathbb{N}, \quad C : [0,T] \rightarrow \mathbb{R}^{p\times n}$
- with regard to the disturbances, it is only required to have a finite increment for two consecutive passes as follows:

$$||w_k(t) - w_{k+1}(t)||_\lambda \leqslant b_w, \quad ||v_k(t) - v_{k+1}(t)||_\lambda \leqslant b_v \tag{1.27}$$

  with $b_w > 0$ and $b_v > 0$. The consequence of this assumption is that it is more preferable to have a uniform level of disturbances in every pass (although they may be large). It is not desirable if there is a 'jump' of disturbance in a certain iteration.

Thus, the condition that must be satisfied by $\bar{\mathcal{L}}$ needs to be modified as follows:

$$||I - \mathcal{L}(t)B(t)C(t)|| < 1 \tag{1.28}$$

In short, the system considered by Chen and coworkers is a nonlinear system with input-output dynamics which change with time but not by iteration. This is also the case for the D-type ILC that they use (1.25), where the learning gain is taken to be time varying but uniform in every pass. Furthermore, the disturbance is allowed to be time varying but it is also desired to be uniform in every pass (or at least do not change so much from pass to pass). Hence, some system components, the ILC learning gain and the disturbance are expected to be uniform/almost uniform in every pass. Since there is no much variations from pass to pass then we can expect the ILC (1.25) and initial condition (1.26) will result in convergence as in (1.23). Chen and coworkers proved this result formally in the following theorem:

Theorem 1.4.3. *[7] For the nonlinear uncertain system (1.19)-(1.20), given the desired trajectory $y_d(t)$ over the fixed time interval $[0,T]$, by using the ILC updating law as in (1.25) and the initial state learning as (1.26), if the condition as in (1.28) is satisfied, then the λ-norm of the output tracking error is bounded. Particularly, for sufficiently large λ there exists positive constants b_C, k_f, b_ϕ and b_{BL} such that:*

$$\lim_{k\to\infty} ||e_k(t)||_\lambda \leqslant \frac{2b_v + 2b_C b_w T + O_3(\lambda^{-1})}{1 - \rho - O_4(\lambda^{-1})}, \quad \forall t \in [0,T] \tag{1.29}$$

where

$$\begin{aligned} O_3(\lambda^{-1}) &= \frac{b_c b_w k_f T O(\lambda^{-1})}{1 - k_f O(\lambda^{-1})} \\ O_4(\lambda^{-1}) &= b_C b_\phi O(\lambda^{-1}) + \frac{b_C k_f O(\lambda^{-1})(b_\phi O(\lambda^{-1}) + b_{BL})}{1 - k_f O(\lambda^{-1})} \end{aligned}$$

Clearly, from the two inequalities above the convergence does not depend on the size of initialisation error anymore. It only depends on the bound of disturbance w_k and v_k in the sense of λ-norm.

Discussion

In this discussion, we present some drawbacks of using D-type ILC:

Global Lipschitz condition. This condition is the main shortcoming of this approach. As it can be seen, that to require the nonlinearity to have global Lipschitz condition, restricts the system as a polynomial of order no more than 1. This is a strong restriction, because even for the following simple nonlinear system:

$$\dot{x}_k = x_k^2 + u_k \tag{1.30}$$

it does not satisfy the Lipschitz condition since the dynamic is of polynomial of degree 2.

Convergence in λ-norm. Naturally the convergence is measured in $L^{\infty}[0,T]$ (uniform convergence) or in $L^2[0,T]$ (associated to finite energy convergence). However, as we have seen, the D-type ILC can obtain a convergence only in the sense of λ-norm. We will see that one may not get a uniform convergence although the convergence in λ-norm is achieved. Consider again equation (1.23):

$$\limsup_{k\to\infty} ||y_k - y_d||_\lambda = 0$$

which is equivalent with:

$$\limsup_{k\to\infty} e^{-\lambda T}||y_k - y_d||_{L^\infty} = 0$$

Since λ is taken to be large then $e^{-\lambda T}||y_k - y_d||_{L^\infty} \leqslant ||y_k - y_d||_{L^\infty}$, for each $k \in \mathbb{N}$. It means that the convergence in λ-norm is faster then in $L^\infty[0,T]$ norm. It implies that if the convergence in λ-norm is achieved then the uniform convergence is not necessarily achieved. Since the uniform convergence measure the upper bound of signal/error, we may not get zero tacking error although we get zero convergence error in λ-norm.

1.4.2 P-type ILC

P-type ILC is an ILC which does not require differentiation operator. As we know a numerical differentiation can cause a large error especially if there is a noise in the measurement. It can make the system very prone to instability. P-type ILC is proposed to avoid this situation.

Consider the nonlinear system as in (1.19)-(1.20). Denote $\mathcal{L}$ as a (possibly nonlinear) operator. The feedforward P-type ILC takes a form as follows:

$$u_{k+1} = u_k + \mathcal{L}(y_k - y_d) \tag{1.31}$$

as suggested in the work of Saab [31]. The goal of the ILC (1.31) is to achieve:

$$\begin{aligned}\lim_{k\to\infty} ||y_k - y_d|| &\leqslant \epsilon_1 \\ \lim_{k\to\infty} ||\dot{y}_k - \dot{y}_d|| &\leqslant \epsilon_2\end{aligned}$$

where ϵ_1 and ϵ_2 are positive constants and the norm can be $L^2[0,T]$-norm, $L^\infty[0,T]$-norm or λ-norm. This goal is more ambitious compared to D-type ILC in the previous section, since the convergence is not restricted in λ-norm . However, there are some additional technical assumptions to achieve this convergence:

1. the output dynamic h is taken to be linear in state as follows:

$$h(x) = Cx \tag{1.32}$$

   so that the output equation (1.20) becomes:

$$y(t) = Cx(t) + v \tag{1.33}$$

2. All the disturbances are bounded in λ-norm.

3. The coupling matrix B satisfies the following condition: it has a full column rank and $B^T = PC$ where P_k is positive definite matrix and CC^TP^T is symmetric.

4. The operator B and $\frac{dB}{dt}$ are bounded on $\mathbb{R} \times [0,T]$, $\forall k \in \mathbb{N}$.

5. Let $r = x - x_d$. Let f_k, B_k and r_k be f, B and r at pass k. The following inequality is satisfied for each $k \in \mathbb{N}$ on $[0,T]$:

$$\langle f_k - f(x_d) + (B_k - B(x_d))u_d - \lambda r_k, B_k^{+T} C r_k \rangle \leqslant \epsilon_1$$

where the notion $\langle \cdot, \cdot \rangle$ denote the inner product (see the definition in the appendix).

All the above technical assumptions are needed to establish the boundedness of the input error $u_k - u_d$, in the sense of its inner product, with respect to the size of disturbance v_k and w_k. This construction will not be put in here, for brevity. the details can be found in the work of Saab ([31]). Once the bound of input error is established then the bound of $||y_k - y_d||$ as $k \to \infty$ can be obtained.

Note that, different with D-type ILC and the other ILC algorithms, the ILC (1.31) does not explicitly require the system dynamic f to have a global Lipschitz condition or satisfy the convergence inequality condition as in the equation (1.24). However, those assumptions listed above are not easy to check in general, and it is not guaranteed that they are weaker than Lipschitz condition and condition (1.24). Hence, the assumptions are not more general than the assumptions on (1.21) as in the previous subsection.

It should be noted that, however, all the above assumptions is verified in the case of robotics manipulator [5].

Theorem 1.4.4. *[31] Suppose the nonlinear system (1.19)-(1.20) satisfies the additional assumptions above. Then the learning operator given by (1.31) will generate a sequence of inputs $u_k(t)$, $t \in [0,T]$, such that $||u_k - u_d||$ is bounded with $||x_k(t) - x_d(t)||$ and $||\dot{x}_k(t) - \dot{x}_d(t)||$ are bounded for each $k \in \mathbb{N}$ on $[0,T]$ as follows:*

$$\begin{aligned} ||x_k - x_d|| &\leqslant ||x_k(0) - x_d(0)||e^{Lt} + \int_0^t e^{L(t-\tau)} \left(||B(x_d)\delta u_k|| + ||\eta_k||\right) d\tau, \\ ||\dot{x}_k - \dot{x}_d|| &\leqslant L||x_k - x_d|| + ||B(x_d)\delta u_k|| + ||\eta_k|| + ||B(x_d)u_d|| + ||B_k - B(x_d)|| ||u_k|| \end{aligned}$$

Since y_k are related to x_k as in (1.20) then y_k converges to y_d.

1.4.3 Adaptive Nonlinear ILC

Another approach of designing ILC for nonlinear ILC is to utilise an adaptive nonlinear feedback control as an ILC. This approach is found on the work of French and Rogers ([9]). Consider the nonlinear system (1.19) and (1.20). Now if it is assumed that the nonlinear dynamics f can be represented as a linear combination of a known function and unknown parameter. It is also assumed that this combination follow a certain structure i.e: chain of integrator, parametrict strict feedback or output feedback system.

The approach is to design nonlinear adaptive control to the system (1.19)-(1.20). This design is based on the structure of the system as mentioned above. The adaptive nonlinear ILC simply consists of the designed nonlinear adaptive control and the initial parameter estimate update. The initial parameter estimate update is as follows:

$$\begin{aligned} \hat{\theta}_k(0) &= \hat{\theta}_{k-1}(T), \quad k \geqslant 2 \\ \hat{\theta}_1(0) &= 0 \end{aligned}$$

and the nonlinear adaptive control depends on the structure of the system (chain of integrator, parametric strict feedback and output feedback). In this subsection we only consider the system with the chain of integrator, which is the most simple structural assumptions. At each k-th iteration, the system of chain of integrator is written as follows:

$$\dot{x}_k(t) = Ax_k(t) + B(\theta^T\phi(x_k(t)) + u_k(t)), \quad x_k(0) = \delta_k \tag{1.34}$$

with

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The proposed nonlinear adaptive ILC has a form of simple nonlinear adaptive control as follows:

$$u_k(t) = -\hat{\theta}_k^T(t)\phi(x_k(t)) - a^T x_k(t) \tag{1.35}$$

$$\dot{\hat{\theta}}_k(t) = \alpha x_k^T(t) b\phi(x_k(t)), \; \hat{\theta}_1(0) = 0, \; \hat{\theta}_k(0) = \hat{\theta}_{k-1}(T) \tag{1.36}$$

Define the following cost criterion:

$$\mathcal{P}_k = ||y_k - B(y_d, \epsilon)||^2_{L^2[0,T]} \tag{1.37}$$

with $B(y_d, \epsilon)$ is a ball with the centre of y_d and the radius of $\epsilon > 0$. The objective is to have the cost criterion converges to zero:

$$\lim_{k\to\infty} ||y_k - y_d||_{L^2[0,T]} = 0 \tag{1.38}$$

Moreover system signal, control signal and parameter estimate are demanded to be bounded in term of L^∞:

$$||x_k||_{L^\infty} \leqslant \epsilon_1, \;\; ||u_k||_{L^\infty} \leqslant \epsilon_2, \text{ and } ||\hat{\theta}_k||_{L^\infty} \leqslant \epsilon_3 \tag{1.39}$$

Since the ILC uses the nonlinear feedback control then we can expect the boundedness along the pass. Moreover the parameter estimate is updated in each pass so that the system output converges to the reference trajectory. The above ILC results in a convergence in $L^2[0,T]$ sense. It is stated formally in the following proposition [9]:

Proposition 1.4.1. *Let the system is given by equation (1.34). Then the ILC design given by equation (1.35) and (1.36) achieve the control objective (1.38) and (1.39).*

The more details of this adaptive nonlinear ILC will be explained in chapter 4.

1.4.4 Nonlinear Non-Minimum Phase System

In this section we consider an ILC that can be used to control nonlinear non-minimum phase system which is proposed by Gosh and Paden ([11]). Consider the system as in the equation (1.19)-(1.20) with $v_k = 0$. Suppose it is also :

- stable in the first approximation and input to state stable
- has unstable hyperbolic zero dynamics

The proposed iterative learning controller is as follows:

$$u_{k+1}(t) = \mathcal{T}(u_k(t) + \delta\bar{u}_k(t)) \tag{1.40}$$

$$\delta\bar{u}_k(t) = \left(\bar{C}\bar{b}\right)^{-1}\left[\delta\bar{y}_k(t) + \delta\dot{\bar{y}}_k(t) - (\bar{C} + \bar{C}\bar{A})\delta\bar{x}_k(t)\right] \tag{1.41}$$

$$\delta\dot{\bar{x}}_k(t) = \left(\bar{A} - \bar{b}(\bar{C}\bar{b})^{-1}\right)\delta\bar{x}_k(t) + \left[\delta\bar{y}_k(t) + \delta\dot{\bar{y}}_k(t) - (\bar{C}\bar{A})\delta\bar{x}_k(t)\right], \ \bar{x}_k(\pm\infty) = 0 \tag{1.42}$$

The truncating operator $\mathcal{T}$ is defined as such that for any measurable function $\tau : \mathbb{R} \to \mathbb{R}^s$:

$$\begin{aligned} \mathcal{T}(\tau(t)) &= \tau(t), \forall t \in [0, T] \\ &= 0, \text{otherwise} \end{aligned}$$

The truncating operator $\mathcal{T}$ is used since the control input $\delta\bar{u}_k(t)$, which is obtained from equations (1.41) and (1.42), hold for $t \in (-\infty, +\infty)$. In iterative learning control environment we need the system to be operated in finite time length and not for the whole time horizon.

The notation $\delta\bar{u}_k$, $\delta\bar{y}_k$, $\delta\bar{x}_k$ are defined as:

$$\delta\bar{u}_k \equiv u_d - u_k, \quad \delta\bar{x}_k \equiv x_d - x_k \ \text{ and } \ \delta\bar{y}_k \equiv y_d - y_k \tag{1.43}$$

where, u_d is the desired control input, x_d the desired state and y_d the desired output, respectively. The matrices $\bar{A}$, $\bar{b}$, and $\bar{C}$ are found by linearising the nonlinear system (1.19)-(1.20) respectively.

Although the matrix $\bar{A} - \bar{b}(\bar{C}\bar{b})^{-1}$ is unstable because of the non-minimum phase assumption, the boundary condition in equation (1.42) guarantees the solution $\delta\bar{x}_k$ to be in $L^1 \cap L^\infty \cap C^0$, as stated in [11] (see also [30]). However, it involves the knowledge of $\delta\bar{x}_k$ at $t = +\infty$ and $t = -\infty$, which means it is a *non-causal* solution [30]. In other word we have to know the value of the 'future' behaviour of $\delta\bar{x}_k$ in order to calculate the 'present' value of $\delta\bar{x}_k$.

The following theorem states the convergence result of ILC (1.40)-(1.42) applied to the nonlinear system (1.19)-(1.20):

Theorem 1.4.5. *[11] Consider the nonlinear system (1.19)-(1.20). If assumptions above hold then the above ILC algorithm (1.40)-(1.42) produces the sequence of inputs which converges to $u^* \in \mathbb{R}^p$ if there are no input disturbances ($w_k = 0$, for each $k \in \mathbb{N}$) and no initialisation error. If w_k is bounded, u_k converges to $B(u^*, r)$ depends continuously on the bound on the disturbance w_k. If there exists a $u_d \in L_\infty \cap C_0[0, T]$ with $P(u_d) = y_d$, then u_k converges to the desired input solution u_d.*

This kind of ILC, apart from its capability to handle a nonlinear non-minimum phase system, has some shortcomings:

- the very obvious one is the method required the exact knowledge of nonlinear system (1.19)-(1.20), in order to obtain the controller and the zero dynamics. This is of course, make it less attractive, since one motivation of using ILC is to handle the unknown system.
- secondly, the fact that the trajectory is non-causal make it difficult in implementation.
- thirdly, the requirement the system to be input to state stable and stable in the first approximation might be too strong.
- the other shortcoming is similar with the D-type ILC, that is the use of error derivative which make the system more sensitive of output noise.

Based on these facts, this type of ILC can be good in theoretical point of view, but may not be desirable in practical purpose.

As a final remark, although this algorithm is used for non-minimum phase, it still can be used to control minimum-phase system. In this case, it is not necessary to use the non-causal stable inversion and the result will be similar with the D-type ILC in section 1.4.1.

1.4.5 The other types of ILC

There are some other various types of ILC algorithm proposed in the literature. The survey paper such as [23] provides the list of the relevance literature. In case of nonlinear system, many of algorithms are based on the existing D-type and P-type ILC. Some other algorithms still use the restrictive assumptions like Lipschitz condition and convergence measure similar to λ-norm. These works can be seen in the paper of, among them, [8], [16], [19], [37], [40],[39], [38], [41]. All of these algorithms have almost the same principle. Hence, it is not necessary to include the full lists of those kinds of algorithm . The interested readers can consult e.g. the survey [23] and references therein for further reading.

1.5 Input to State Stability Theory

1.5.1 Motivations and Definition

In this section, a brief introduction of input to state stability (ISS for short) is presented. Most of the material in this section is taken from [15] and [35]. To understand the concept of input to state stability, consider the nonlinear system as follows:

$$\dot{x} = f(x,u), \quad x(0) = x_0 \tag{1.44}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, in which $f(0,0) = 0$ and $f(x,u)$ is locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m$. The control input $u : [0,\infty) \to \mathbb{R}^m$ is taken to be any piecewise continuous bounded function. Suppose we assume that this system is asymptotically stable for $u = 0$. A natural question is to ask whether such a system has:

- the 'converging input converging state (CICS)' property:

$$\lim_{t\to\infty} u(t) = 0 \Rightarrow \lim_{t\to\infty} x(t) = 0 \tag{1.45}$$

- the 'bounded input bounded state property (BIBS)'.

$$u \in L^\infty \Rightarrow x \in L^\infty \tag{1.46}$$

If the system (1.44) above is a stable linear system:

$$\dot{x} = Ax + Bu \tag{1.47}$$

then both implications above are always true, since the solution satisfies:

$$|x(t)| \leqslant \beta(t)|x(0)| + \gamma||u||_{L^\infty} \tag{1.48}$$

with:

$$\beta(t) = ||e^{At}|| \to 0 \;\text{ and }\; \gamma = ||B|| \int_0^\infty ||e^{As}||ds \tag{1.49}$$

It is clear from the above equation that 'converging input converging state' and 'bounded input and bounded state' property hold. However, in general, asymptotic stability for nonlinear system does not imply CICS and BIBS. Consider the following example of nonlinear system [33]:

$$\dot{x} = -x + (x^2 + 1)u, \quad x(0) = 1 \tag{1.50}$$

which is asymptotically stable if $u = 0$. Now take the bounded control:

$$u(t) = 1 \tag{1.51}$$

The system will be:

$$\dot{x} = -x + x^2 + 1, \quad x(0) = 1 \tag{1.52}$$

which results in unbounded solution, since the quadratic term 'dominates' the $-x$ term. Hence the BIBS cannot be achieved in general.

It is therefore necessary to characterise a class of nonlinear systems, the solution of which like the solution of the linear system (1.47), where BIBS and CICS hold. Therefore it is natural to require the solution of nonlinear system (1.44) to behave like the solution of a linear system (1.12). Hence, one would like to have the solution of nonlinear system (1.44) is a generalisation of (1.48) as follows:

$$|x(t)| \leqslant \beta(|x(0)|, t) + \gamma(\|u\|_{L^\infty}) \tag{1.53}$$

with β is a function of class $\mathcal{KL}$, and γ is a class $\mathcal{K}/\mathcal{K}^\infty$ function (see the definition of class $\mathcal{K}$ and $\mathcal{K}^\infty$ function in the appendix) . Thus, if the solution nonlinear system (1.44) follows the above inequality then we call the nonlinear system (1.44) to be input to state stable or ISS for short. Formally, ISS is stated as follows:

Definition 1.5.1. *[15] The nonlinear system (1.44) is said to be input-to-state stable if there exists a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ, such that, for any $u \in L_\infty$ and any $x^0 \in \mathbb{R}^n$, the response of $x(t)$ of (1.44) in the initial state $x(0) = x^0$ satisfies:*

$$\|x(t)\| \leqslant \beta(\|x^0\|, t) + \gamma(\|u\|_\infty) \tag{1.54}$$

for all $t \geqslant 0$.

The definition of ISS above means that the state $x(t)$ of nonlinear system (1.44) does not decrease to zero as $t \to \infty$. Instead, we are interested in the case in which $x(t)$ is bounded, and the bound on the state is related to the bound of input through a (possibly nonlinear) gain function γ.

1.5.2 Some Characterisations of ISS

The notion of input-to state stability provides a number of alternative (equivalent) characterizations. In the following, we provide some characterizations of ISS. All of the statements are stated without proof. The proof can be seen for instance in [15].

We start by observing that:

$$\lim_{t\to\infty} \beta\left(\|x^0\|, t\right) = 0 \tag{1.55}$$

so in input to state stable system, the response $x(t)$ to any input $u \in L_\infty$ satisfies:

$$\limsup_{t\to\infty} \|x(t)\| \leqslant \gamma\left(\|u\|_\infty\right) \tag{1.56}$$

The following is the characterization of (1.56) which only involves the behaviour of $\|u\|_\infty$ for large t.

Lemma 1.5.1. *[15] Property (1.56) is equivalent to the property:*

$$\limsup_{t\to\infty} \|x(t)\| \leqslant \gamma(\limsup_{t\to\infty} \|u\|) \tag{1.57}$$

The property (1.57) is called $\mathcal{K}$-asymptotic gain ($\mathcal{K}$-AG for short) [35]. Now consider inequality (1.54). Observe that, for each $t \geqslant 0$,

$$\beta(\|x^0\|, t) \leqslant \beta(\|x^0\|, 0)$$

where $\beta(\cdot, 0)$ is a class $\mathcal{K}$ function. By inequality (1.54) the response of $x(t)$ is bounded and, in particular,

$$\|x\|_\infty \leqslant \max\left\{\gamma_0(\|x^0\|), \gamma(\|u\|_\infty)\right\} \tag{1.58}$$

for some class $\mathcal{K}$ function γ_0 and γ. This property is called bounded input bounded state (BIBS for short).

The following theorem is one among useful characterization of input to state stability:

Theorem 1.5.1. *[15] System (1.44) is ISS if and only if there exists class $\mathcal{K}$ functions γ_0 and γ such that, for each $u \in L_\infty$ and any $x^0 \in \mathbb{R}^n$, the response $x(t)$ in the initial state $x(0) = x^0$ satisfies:*

$$\begin{aligned} \|x\|_\infty &\leqslant \max\left\{\gamma_0(\|x^0\|), \gamma(\|u\|)\right\} \\ \limsup_{t\to\infty} \|x(t)\| &\leqslant \gamma(\limsup_{t\to\infty} \|u(t)\|) \end{aligned}$$

The proof of this theorem is very lengthy and it is beyond the scope of this thesis. The interested reader can refer to [35] for the details of the proof.

These two properties, BIBS and $\mathcal{K}$-AG is a boundedness and asymptotic property of nonlinear system. Hence, if a nonlinear system satisfies both of these properties then it can be expected that it is ISS, as justified in the above theorem. In chapter 5, we will formulate the boundedness and asymptotic property, called bounded disturbance bounded state and disturbance asymptotic gain, which is an analogy of BIBS and $\mathcal{K}$-AG, for ILC system.

The other important characterisation is the Lyapunov characterisation of ISS. It states that an equivalence between the property of ISS and the existence of a Lyapunov like function, which is called ISS Lyapunov function. It is defined as follows:

Definition 1.5.2 (ISS-Lyapunov function [15]). *A C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ is called an ISS-Lyapunov function for system (1.44) if there exists class $\mathcal{K}^\infty$ functions $\underline{\alpha}$, $\bar{\alpha}$, α, and a class $\mathcal{K}$ function χ such that:*

$$\underline{\alpha}(\|x\|) \leqslant V(x) \leqslant \bar{\alpha}(\|x\|), \forall x \in \mathbb{R}^n \tag{1.59}$$

and

$$\|x\| \geqslant \chi(\|u\|) \Rightarrow \frac{\partial V}{\partial x} f(x, u) \leqslant -\alpha(\|x\|), \forall x \in \mathbb{R}^n \tag{1.60}$$

Theorem 1.5.2 (Lyapunov characterisation of ISS [15]). *The system is input to state stable if and only if it has the ISS-Lyapunov function.*

This theorem is very useful in proving many important characterisation of ISS, for example in proving theorem (1.5.1). Furthermore, it plays an important role for control design based on ISS, see for instance [18].

The concept of input to state stability has attracted attention many researchers. There are many theories and applications are developed based on this notion. The discussion of all aspects of ISS is beyond this thesis. The interested reader can consult [34] for a good survey in this area.

1.5.3 Nonlinear System Interconnection Theory

Consider the following cascade interconnection nonlinear system:

$$\dot{x}_1 = f(x_1, x_2) \tag{1.61}$$

$$\dot{x}_2 = g(x_2, u) \tag{1.62}$$

in which $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $f(0,0) = 0$, $g(0,0) = 0$, and $f(x_1, x_2)$, $g(x_2, u)$ are locally Lipschitz, and the feedback interconnection nonlinear system:

$$\dot{x}_1 = f_1(x_1, x_2) \tag{1.63}$$

$$\dot{x}_2 = f_2(x_1, x_2, u) \tag{1.64}$$

in which $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $f_1(0,0) = 0$, $f_2(0,0,0) = 0$.

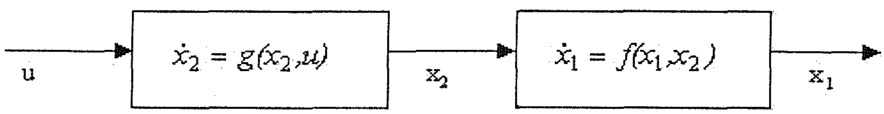

Figure 1.1: Cascade Interconnection

Theorem 1.5.3 (Cascade Interconnection Theory [34]). *Suppose the system (1.61) viewed as a system with state x_1 and input x_2 is input-to-state stable and that the system (1.62) viewed as a system with state x_2 and input u is input-to-state stable as well. Then the system (1.61) and (1.62) with the state $x = (x_1, x_2)$ and input u is input-to-state stable.*

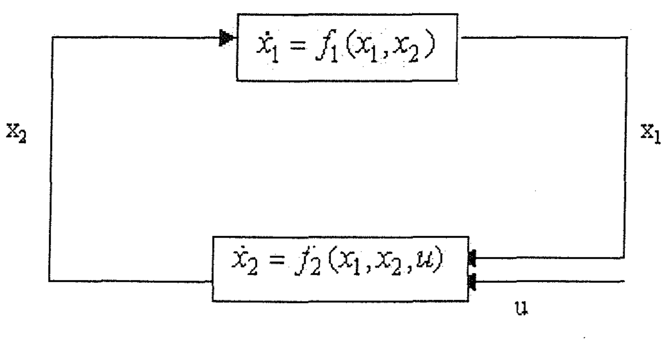

Figure 1.2: Feedback Interconnection

Theorem 1.5.4 (Feedback Interconnection Theory [15]). *Suppose the system (1.63) viewed as a system with state x_1 and input x_2 is input-to-state stable, and that the system (1.64) viewed as a system with state x_2 and inputs x_1 and u is input-to-state stable as well. If the condition:*

$$\gamma_1(\gamma_2(r)) < r, \forall r > 0 \tag{1.65}$$

holds, then the system (1.63) and (1.64) viewed as a system with state $x = (x_1, x_2)$ *and input* u *is input-to-state stable. In particular, the class* $\mathcal{K}$ *functions:*

$$\begin{aligned}\gamma_0(r) &= \max\{2\gamma_{01}(r), 2\gamma_{02}(r), 2\gamma_1 \circ \gamma_{02}(r), 2\gamma_2 \circ \gamma_{01}(r)\} \\ \gamma(r) &= \max\{2\gamma_1 \circ \gamma_u(r), 2\gamma_u(r)\}\end{aligned}$$

are such that response $x(t)$ *to any input* $u(\cdot) \in L^m_\infty$ *is bounded and:*

$$\begin{aligned}\|x\|_{L^\infty} &\leqslant \max\{\gamma_0(|x(0)|), \gamma(\|u\|_{L^\infty})\} \\ \limsup_{t\to\infty} \|x(t)\|_\infty &\leqslant \gamma(\limsup_{t\to\infty} \|u(t)\|)\end{aligned}$$

1.6 General Theory of ILC System Interconnection

The theory of nonlinear system interconnection has received an increasing attention in the area nonlinear control theory. This theory helps to overcome the problem of stability of a complicated system. Theory of nonlinear system interconnection can be found in standard nonlinear control theory literature such as [17], [15], [18], [32], [34], [36] and many others.

Following the successful of utilising this theory, then there is an interest to establish an analogue theory in ILC context. This research is an attempt to investigate that possibility. To the best of our knowledge this topic is never explored before in any ILC literature. We hope that this theory will have many applications to solve ILC problems similar to that in nonlinear control.

The theory of ILC system interconnection is to find a condition which guarantees an ILC system interconnection to be well-behaved. This condition should include the behaviour of each connected ILC. We formulate this problem by using the theory of nonlinear system connection. First of all, we formulate a new notion of robust ILC system with respect to the disturbance by using input to state stability theory (ISS). We call this notion as iterative input to state stability (iterative ISS). Using this notion we develop the theory of ILC system interconnection which is an analogue to nonlinear system interconnection theory developed from ISS.

As it has been discussed previously, ISS is a notion of stability of a system under the influence of input/disturbance and initial condition. In ILC, the initial condition also plays a role, for instance it also affects the convergence (see [9] for an example). Hence, the effect of initial condition and disturbance need to be taken into account. The ISS theory is a good approach to formulate a property of an ILC system which is influenced by the initial condition and disturbance.

We can see the analogy of ISS nonlinear system with robust ILC system. If a nonlinear system has an ISS property then the stability can be maintained whenever the initial condition and the input are bounded. In other word, the unboundedness of initial condition and control input may lead the system to be unstable. This fact has an analogue with ILC. A convergence of an ILC may be influenced by the presence of the disturbance and initial condition (as seen in some algorithms in the previous section). Hence, a concept of a robust ILC system can be formulated similarly to the ISS formulation of stability of nonlinear system.

Consider the one dimensional iterative system representation:

$$\dot{x}_k = f(x_k, u_k, d_k) \tag{1.66}$$

The desirable solution would be the state of the system converges to zero if there are no disturbances.

If there are disturbances then the size of the disturbance my affect this convergence. Therefore, the property as CICS and BIBS nonlinear system should be carried to iterative system.

As we have known from ILC, a good ILC has some (or all) of the following properties:

1. If there are no disturbances then the error/state will converge

2. Convergent disturbance give a convergent state

3. Bounded disturbance gives a bounded state

On the other hand, for nonlinear system, the notion of ISS gives a property such as: asymptotic stability (without the presence of input/disturbance), BIBS and CICS. This is an analogue with the above three properties of ILC system. Hence, this suggest us to formulate a similar notion like ISS in ILC context. Subsequently we can use this notion, to develop the theory of ILC system interconnection.

In the next chapter, we will introduce this new notion. This notion is formulated in a similar manner with ISS. It uses class $\mathcal{K}/\mathcal{K}^{\infty}$ function, to measure the disturbance and initial condition. However, we use class $\bar{\mathcal{K}}\mathcal{L}$ function instead of class $\mathcal{KL}$ function (see the definition of class $\mathcal{K}$, $\mathcal{K}^{\infty}$, and $\bar{\mathcal{K}}$ in the appendix). This is due to the fact that some ILC's have a weak convergence criteria [41]. By this notion some properties with regard to asymptotic and boundedness of ILC are derived. Using these properties we establish some theorems for ILC system interconnection.

1.7 Outline of this Thesis

- **Chapter 1**: Introduction.
  We present a brief introduction of ILC, example and theoretical framework we use in this work. We also discuss some literature review concerning the theory of ILC design. The motivation of this work is also presented.

- **Chapter 2**: Iterative ISS and System Interconnection.
  This chapter presents our contribution of this research. We divide this section in two main sections in this chapter, which are:

  - *ISS Modeling of Robust ILC System.*
    In this subsection, we formulate a desired property of robust ILC system. It follows the similar principle of ISS. The formulation is expressed in term of class $\bar{\mathcal{K}}\mathcal{L}$ function and class $\mathcal{K}$ function. This notion is called iterative input to state stability, or iterative ISS. Some properties related to boundedness and asymptotic property of a system possessing iterative ISS are also derived. These properties later are useful in proving some theorems regarding ILC system interconnection theory.

  - *General Theory of ILC System Interconnection.*
    In this section we consider the interconnection of two ILC systems. There are two kinds of interconnection: cascade interconnection and feedback interconnection. We develop theories stating that the iterative ISS of each subsystem is sufficient to guarantee some boundedness and asymptotic property of the interconnection.

- **Chapter 3**: Case Study: The Nonlinear Adaptive ILC.
  This chapter also presents our contribution of this research. We investigate to what extend the

adaptive nonlinear control proposed by French and Rogers [9] posses iterative ISS. We obtain a result that the adaptive nonlinear ILC does not have iterative ISS property even when the disturbance is restricted.

- **Chapter 4**: Conclusion and Recommendation for Further Work.
- **Appendix**: Notation and definition

Chapter 2

Iterative ISS of System Interconnection

The goal of considering the system interconnection is to have a basis of developing an ILC design method for more complicated nonlinear repetitive system. This idea emerged from the fact that a 'large' system can be seen as an interconnection of some 'smaller' subsystems.

We consider two classes of system interconnection: cascade interconnection and feedback interconnection. These classes of system interconnection proved to have useful properties called bounded disturbance bounded state and disturbance asymptotic gain. These two properties intuitively imply iterative ISS. It should be stated, however, in this work the iterative ISS property of the class of system interconnection still cannot be proved formally.

The main contributions in this chapter are:

- introduce the formal definition of iterative ISS for ILC based on the idea of ISS
- introducing some new boundedness and asymptotic properties of ILC system
- proving iterative ISS is sufficient to obtain those properties
- deriving bounded disturbance bounded state and disturbance asymptotic gain property for cascade interconnection system
- deriving bounded disturbance bounded state and disturbance asymptotic gain property for feedback interconnection system

2.1 System formulation and iterative ISS definition

First of all, we need to formulate the class of system under consideration. Let $k \in \mathbb{N}$ denote the pass number. Let $x_k \in C([0,T], \mathbb{R}^n)$. Let the control input $u_k : [0,T] \to \mathbb{R}^u$ belong to an admissible input set $\mathcal{U}_k$, which is usually taken to be $L^\infty[0,T]$. Consider the following iterative system:

$$\dot{x}_k = f(x_k, u_k, d_k), \quad x_k(0) = \delta_k \tag{2.1}$$

Suppose $d_k : [0,T] \to \mathbb{R}^d$ belongs to an admissible set of disturbance $\mathcal{D}_k$, and the initial condition sequence $\{\delta_k\}_{k \geqslant 1}$ belongs to a certain admissible initial condition set $\mathcal{I}$. The set $\mathcal{D}_k$ usually is taken to be $L_p[0,T]$, $1 \leqslant p \leqslant \infty$ whereas the set $\mathcal{I}$ is taken to be a sequence space such as l^p-space. It is necessary to define $\mathcal{D}_k$ and $\mathcal{I}$ properly, since the successful of an ILC depends on the type of the disturbance/initial condition allowed. The dynamic f is taken to be $f \in C(\mathbb{R}^n \times \mathcal{U}_k \times \mathbb{R}^d, \mathbb{R}^n)$.

Let $\mathcal{M}$ be a suitable memory space (memory is used to 'memorise' information from the previous pass), which can be taken as $L_p[0,T]$ space or even an Euclidean space. The memory $m_k \in \mathcal{M}$, is formulated as follows:

$$m_k = F(\mathcal{K}_{k-1}) \tag{2.2}$$

for a suitably defined operator F and for a set:

$$\mathcal{K}_k = \{x_k, u_k, \hat{\theta}_k\} \tag{2.3}$$

where $\hat{\theta}_k : [0,T] \to \mathbb{R}^n$ is the internal state of the controller.

Now we formulate the ILC system. Given $m_k \in \mathcal{M}$, there is an operator χ with:

$$\chi \ : \ C([0,T], \mathbb{R}^n) \times \mathcal{M} \to \mathcal{U}_k \tag{2.4}$$

so that the following controller:

$$u_k = \chi(x_k, m_k) \tag{2.5}$$

make the 'closed-loop' system:

$$\begin{aligned} \dot{x}_k &= f(x_k, \chi(x_k, m_k), d_k), \ \ x_k(0) = \delta_k \\ &= \bar{f}(x_k, m_k, d_k), \ \ x_k(0) = \delta_k \end{aligned} \tag{2.6}$$

with the memory equation as above:

$$m_k = F(\mathcal{K}_{k-1}) \tag{2.7}$$

fulfill the objective:

$$\limsup_{k\to\infty} \|x_k\| \leqslant \limsup_{k\to\infty} \mathcal{N}\left(\|d_k\|, \|\{\delta_i\}_{i\geqslant 1}\|_{l^p[1,k]}\right) \tag{2.8}$$

where the map $\mathcal{N}$ is a smooth mapping. In many ILC algorithm, the map $\mathcal{N}$ is linear combination of $\|d_k\|$ and $\|\{\delta_i\}_{i\geqslant 1}\|_{l^p[1,k]}$. It should be noted that when there is no disturbance ILC system need to fulfill the following objective:

$$\limsup_{k\to\infty} \|x_k\| \leqslant \mathcal{N}(0,0) < \infty \tag{2.9}$$

There are two points need to be pointed out with regards to this set up:

- we consider the system state x_k rather than the output. It is in contrast with many ILC algorithm, which is constructed for the system output. However, at this stage, it is more feasible to work with state rather than with the output directly.
- the reference signal is not included in this set up i.e it is assumed to be zero. Hence, we consider the task of ILC to stabilise an iterative system rather than for tracking.

The following is the formal definition of iterative ISS:

Definition 2.1.1. *The system (2.6)-(2.7) has the iterative ISS property if there exists a class $\bar{\mathcal{K}}\mathcal{L}$ function β, class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) functions γ_1 and γ_2 such that we have:*

$$\| x_k \| \leqslant \beta(\| x_1 \|, k) + \gamma_1\left(\max_{1\leqslant i\leqslant k} \| d_i' \|\right) + \gamma_2\left(\|\{\delta_i\}_{i\geqslant 1}\|_{l^p[1,k]}\right) \tag{2.10}$$

for each $x_1 \in C([0,T], \mathbb{R}^n)$, for each measurable $d_k \in \mathcal{D}_k$, for each $\{\delta_k\}_{k\geqslant 1} \in \mathcal{I}$, and for each $k \in \mathbb{N}$.

Note that β is a class $\bar{\mathcal{K}}\mathcal{L}$ function, not class $\mathcal{K}\mathcal{L}$. If the first argument is zero, and the second argument is fixed then $\beta(0,\cdot) \geqslant 0$ (see definition of class $\bar{\mathcal{K}}$ function in the appendix). The norm for the initial condition δ_k is taken to be the norm of the space where the sequence $\{\delta_k\}_{k\geqslant 1}$ is defined. For x_k, x_1 and d'_k the norm can be taken to be $L_q[0,T]$, $1 \leqslant q \leqslant \infty$ norm as well as λ-norm which is commonly used in ILC literature.

The definition 2.1.1 gives an estimate of the size of the signal in each iteration. The class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) γ_1 and γ_2 represent 'general gains' which quantify how large is the influence of the disturbances (class $\bar{\mathcal{K}}$ can be thought of representing the gain with offset). The definition (2.1.1) clearly gives understanding that if the system has iterative ISS property then, provided the system signal is bounded in the first pass, small disturbances do not affect the stability of the system. As it is seen in the definition 2.1.1 we are including two kinds of disturbances. The first kind is due to disturbance, which is represented as d_k and the other kind is the initial condition δ_k. The inclusion of initial condition is the distinctive feature of iterative ISS compared to the conventional ISS and it is formulated with a summation. It is reasonable to think that we cannot expect the behaviour in the current pass is suddenly better (though it has a perfect resetting) if we have a high resetting error at the previous pass. Hence the resetting error from the previous passes also contribute to 'disturb' the current pass, which gives an accumulation effect to the current pass - represented as a summation from the first pass up to the current pass.

2.2 Some Properties Related to Iterative ISS

Consider an ILC system:

$$\begin{aligned} \dot{x}_k &= f(x_k, m_k, d_k), \quad x_k(0) = \delta_k \\ m_k &= F(\mathcal{K}_{k-1}) \end{aligned} \tag{2.11}$$

with a suitably defined operator F and $\mathcal{K}_k$ is as defined in (2.3). Let $\mathcal{M}$ be a set of measurable and integrable function. Let d_k belong to admissible disturbance set $\mathcal{D}_k \subseteq \mathcal{M}$ and $\{\delta_k\}_{k\geqslant 1} \in l^p$. The set l^p has included all initialisation error which is bounded in every pass either decreasing or non decreasing initial condition. It is not necessary to consider the increasing initial condition since it may lead to instability. Let the system state $x_k \in ([0,T], \mathbb{R}^n)$.

The following properties are the boundedness and asymptotic property of an ILC system with respect to disturbance and initial condition:

- In the following definition, we introduce a property of an ILC system which we call *pass shifting property*. This definition represents the behaviour if an ILC system is 'shifted' to a certain iteration number then the behaviour in the later iteration is influenced by the disturbance, initial condition and signal from which the system is shifted.

  Definition 2.2.1 (Pass shifting property). *The ILC system (2.11) is said to have a pass shifting property if for each $p \in \mathbb{N}$, there exists a class $\bar{\mathcal{K}}\mathcal{L}$ function β, class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) functions γ_1 and γ_2 such that the following estimate holds:*

  $$\|x_{k+\lambda}\| \leqslant \beta(\|x_{1+\lambda}\|, k) + \gamma_1\left(\max_{1+\lambda\leqslant i\leqslant k+\lambda} \|d_i\|\right) + \gamma_2\left(\|\{\delta_i\}_{i\geqslant 1}\|_{l^p[1+\lambda,k+\lambda]}\right) \tag{2.12}$$

  for each $k \in \mathbb{N}$, for each $\lambda \in \mathbb{N}\cup\{0\}$, for each $x_{1+\lambda} \in C([0,T],\mathbb{R}^n)$, for each $d_k \in \mathcal{D}_k$ and for each $\{\delta_k\}_{k\geqslant 1} \in l^p$.

•

Definition 2.2.2 (Disturbance Asymptotic gain (DAG for short)). *The ILC system (2.11) is said to have a disturbance asymptotic gain property, if for each $p \in \mathbb{N}$, there exists class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) functions γ_1 and γ_2 such that the following holds:*

$$\limsup_{k\to\infty} ||x_k|| \leqslant \max\left\{\gamma_1\left(\limsup_{k\to\infty} ||d_k||\right), \gamma_2\left(\limsup_{k\to\infty} |\delta_k|^p\right)\right\} \tag{2.13}$$

for each $k \in \mathbb{N}$, for each $d_k \in \mathcal{D}_k$ and for each $\{\delta_k\}_{k\geqslant 1} \in l^p$.

As can be found in ISS theory, we have a $\mathcal{K}$-asymptotic gain property, stating that for any ISS system, if we take the time increases then the effect of initial condition will decrease, so the system is mainly influenced by the input/disturbance. In the ILC context, this is represented by *disturbance asymptotic gain* property. In this definition, we describe the situation where the effect of the signal at the first pass decreases as iteration increases. In other word, at a large iteration the system is mainly influenced by the disturbance and initial condition only.

Note that if we take $\{\delta_k\}_{k\in\mathbb{N}} \in l_p$ then as $k \to \infty$ then $\delta_k \to 0$. We state this fact in the following remark:

Remark 2.2.1 (Remark on disturbance asymptotic gain property). *Suppose $1 \leqslant p < \infty$ we have:*

$$\limsup_{k\to\infty} ||x_k|| \leqslant \gamma_1\left(\limsup_{k\to\infty} ||d_k||\right) \tag{2.14}$$

•

Definition 2.2.3 (Bounded disturbance bounded state (BDBS for short)). *The ILC system (2.11) is said to have a bounded disturbances bounded state (BDBS) property if for each $p \in \mathbb{N}$, there exists a class $\bar{\mathcal{K}}$ (respectively, $\bar{\mathcal{K}}^\infty$) function $\tilde{\gamma}_0$, class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) functions $\tilde{\gamma}_1$,$\tilde{\gamma}_2$ such that the following inequality holds:*

$$\max_{1\leqslant i\leqslant k} ||x_i|| \leqslant \max\left\{\tilde{\gamma}_0\left(||x_1||\right), \tilde{\gamma}_1\left(\max_{1\leqslant i\leqslant k} ||d_i||\right), \tilde{\gamma}_2\left(||\{\delta_i\}_{i\geqslant 1}||_{l^p[1,k]}\right)\right\} \tag{2.15}$$

for each $k \in \mathbb{N}$, for each $x_1 \in C\left([0,T], \mathbb{R}^n\right)$, for each $d_k \in \mathcal{D}_k$, and for each $\{\delta_k\}_{k\geqslant 1} \in l^p$.

As can be observed directly, the BDBS property is analogue with the bounded input bounded state property in nonlinear system.

The following lemmas relate the aforementioned properties with iterative ISS:

Lemma 2.2.1 (Iterative ISS equivalence with pass shifting property). *The ILC system (2.11) is iterative ISS if and only if it has a pass shifting property.*

Proof. Necessity. Suppose the system is iterative ISS. Then there exists β, γ_1 and γ_2 such that for each $k \in \mathbb{N}$, for each $x_1 \in C([0,T], \mathbb{R}^n)$, for each $d_k \in \mathcal{D}_k$, and for each $\{\delta_k\}_{k\geqslant 1} \in l^p$ we have:

$$||x_k|| \leqslant \beta(||x_1||, k) + \gamma_1\left(\max_{1\leqslant i\leqslant k} ||d_i||\right) + \gamma_2\left(||\{\delta_i\}_{i\geqslant 1}||_{l^p[1,k]}\right) \tag{2.16}$$

Hence:

$$||x_{k+\lambda}|| \leqslant \beta(||x_1||, k+\lambda) + \gamma_1\left(\max_{1\leqslant i\leqslant k+\lambda} ||d_i||\right) + \gamma_2\left(\left(\sum_{i=1}^{k+\lambda} |\delta_i|^p\right)^{1/p}\right) \tag{2.17}$$

Consider the shifted disturbance $\tilde{d}_k = d_{k+\lambda}$ belonging to $\mathcal{D}_k \subseteq \mathcal{M}$ and the shifted initial condition $\{\tilde{\delta}_k\}_{k\geqslant 1} = \{\delta_{k+\lambda}\}_{k\geqslant 1}$ belonging to l_p. According to the (2.11) then the response of the system is $\tilde{x}_k = x_{k+\lambda}$. Since the system is iterative ISS then, there exists a class $\bar{\mathcal{KL}}$ function $\tilde{\beta}$, and a class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ such that for each $k \in \mathbb{N}$:

$$||\tilde{x}_k|| \leqslant \tilde{\beta}(||\tilde{x}_1||, k) + \tilde{\gamma}_1\left(\max_{1\leqslant i\leqslant k} ||\tilde{d}_i||\right) + \tilde{\gamma}_2\left(\left(\sum_{i=1}^{k+} |\tilde{\delta}_i|^p\right)^{1/p}\right)$$

with $\tilde{x}_1$ is the shifted state at the first iteration. Hence, we obtain:

$$||x_{k+\lambda}|| \leqslant \tilde{\beta}(||x_{1+\lambda}||, k) + \tilde{\gamma}_1\left(\max_{1+\lambda\leqslant i\leqslant k+\lambda} ||d_i||\right) + \tilde{\gamma}_2\left(\left(\sum_{i=1+\lambda}^{k+\lambda} |\delta_i|^p\right)^{1/p}\right)$$

Since the above inequality holds for each $k \in \mathbb{N}$, and for any d_k, $\{\delta_k\}_{k\geqslant 1}$, $x_{1+\lambda}$ belonging to their domain then the system has pass shifting property.

Sufficiency. Suppose the system has a pass-shifting property. Then there exists a $\bar{\mathcal{KL}}$ function β, a $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) function γ_1 and γ_2 such that for each $k \in \mathbb{N}$, for each $x_{1+\lambda}$, for each $d_k \in \mathcal{D}_k$, and for each $\{\delta_k\}_{k\geqslant 1} \in l^p$, the inequality (2.12) holds. Since it holds for every $\lambda \in \mathbb{N}\cup\{0\}$, then choose $\lambda = 0$. Then the inequality (2.12) becomes inequality (2.18) which means the system is iterative ISS. □

Lemma 2.2.2 (Iterative ISS implies DAG). *Let the system (2.11) be iterative ISS. Suppose* $1 \leqslant p < \infty$. *Then the system has disturbance asymptotic gain property.*

Proof. Suppose the system is iterative ISS. Then for each $p \in \mathbb{N}$, there exists $\beta \in \bar{\mathcal{KL}}$, γ_1 and γ_2 which are in $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) such that for each $k \in \mathbb{N}$, for each $x_1 \in C([0,T], \mathbb{R}^n)$, for each $d_k \in \mathcal{D}_k$, and for each $\{\delta_k\}_{k\geq 1} \in l^p$, the following holds:

$$||x_k|| \leqslant \beta(||x_1||, k) + \gamma_1\left(\max_{1\leqslant i\leqslant k} ||d_i||\right) + \gamma_2\left(||\{\delta_i\}_{i\geqslant 1}||_{l^p[1,k]}\right) \tag{2.18}$$

Taking $k \to \infty$ we have:

$$\limsup_{k\to\infty} ||x_k|| \leqslant \gamma_1\left(\max_{1\leqslant k\leqslant\infty} ||d_k||\right) + \gamma_2\left(||\{\delta_k\}_{k\geqslant 1}||_{l^p}\right) \tag{2.19}$$

Now, we look at the term containing d_k in inequality (2.19). Define $s = \limsup_{k\to\infty} ||d_k||$. Then $\forall\epsilon > 0, \exists\tau_1 > 0$ such that $||d_k|| \leqslant s + \epsilon$, $\forall k > \tau_1$. Then, $||d_{k+\tau_1}|| \leqslant s + \epsilon$, $\forall k \in \mathbb{N}$, which implies:

$$\gamma_1\left(\max_{1\leqslant k\leqslant\infty} ||d_{k+\tau_1}||\right) \leqslant \gamma_1(s+\epsilon) \tag{2.20}$$

Now observe that the term which contains δ_k. For each $\tau_2 > 0$, we have that:

$$||\delta_k||_{l^p} = \sum_{k=1}^{\infty} |\delta_k|^p = \sum_{k=1}^{\tau_2 - 1} |\delta_k|^p + \sum_{k=\tau_2}^{\infty} |\delta_k|^p$$

The last term in right hand side goes to zero as $\tau_2 \to \infty$ since the 'tail' of the convergence infinite series tends to zero. Then, $\forall\epsilon > 0$, $\exists\tau_3 > 0$, such that $\sum_{k=\tau_3}^{\infty} |\delta_k|^p \leqslant \epsilon$, or $(\sum_{k=1}^{\infty} |\delta_{k+\tau_3}|^p)^{\frac{1}{p}} \leqslant \epsilon^{\frac{1}{p}}$.

Now, take $l = \max\{\tau_3, \tau_1\}$. Define $x_{k+l} \equiv \tilde{x}_k$, then its associate disturbances are d_{k+l} and δ_{k+l}. Thus, by *pass shifting property* :

$$\limsup_{k\to\infty} ||x_k|| = \limsup_{k\to\infty} ||\tilde{x}_k|| \leqslant \limsup_{k\to\infty} \tilde{\beta}(||x_{1+l}||, k) + \tilde{\gamma}_1\left(\max_{1+l\leqslant i\leqslant k+l} ||d_i||\right) + \tilde{\gamma}_2\left(\left(\sum_{i=1+l}^{k+l} |\delta_i|^p\right)^{1/p}\right)$$

$$
\begin{aligned}
&= \limsup_{k\to\infty} \tilde{\beta}(\|x_{1+l}\|, k) + \tilde{\gamma}_1 \left(\max_{1\leqslant i\leqslant k} \|d_{i+l}\| \right) + \tilde{\gamma}_2 \left(\left(\sum_{i=1}^{k} |\delta_{i+l}|^p \right)^{1/p} \right) \\
&\leqslant \tilde{\gamma}_1 \left(\max_{1\leqslant k\leqslant\infty} \|\tilde{d}_k\| \right) + \tilde{\gamma}_2 \left(\epsilon^{\frac{1}{p}} \right)
\end{aligned}
$$

for a class $\bar{\mathcal{K}}\mathcal{L}$ function $\tilde{\beta}$, and class $\mathcal{K}$ functions $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$, which gives:

$$
\limsup_{k\to\infty} \|x_k\| \leqslant \tilde{\gamma}_1(s+\epsilon) + \tilde{\gamma}_2 \left(\epsilon^{\frac{1}{p}} \right) \tag{2.21}
$$

Since $\epsilon > 0$ can be any number, we can take $\epsilon \to 0$, which yields:

$$
\limsup_{k\to\infty} \|x_k\| \leqslant \tilde{\gamma}_1 \left(\limsup_{k\to\infty} \|d_k\| \right)
$$

which is the desired result. □

Lemma 2.2.3 (Iterative ISS implies BDBS property). *Let the system (2.11) be iterative ISS. Then the system has bounded disturbance bounded state property.*

Proof. The iterative ISS property of ILC system (2.11) implies:

$$
\begin{aligned}
\max_{1\leqslant i\leqslant k} \|x_i\| &\leqslant \max_{1\leqslant i\leqslant k} \beta(\|x_1\|, i) + \max_{1\leqslant i\leqslant k} \gamma_1 \left(\max_{1\leqslant j\leqslant i} \|d_j\| \right) + \max_{1\leqslant i\leqslant k} \gamma_2 \left(\|\{\delta_j\}_{j\geqslant 1}\|_{l^p[1,i]} \right) \\
&\leqslant \beta(\|x_1\|, 1) + \gamma_1 \left(\max_{1\leqslant i\leqslant k} \|d_i\| \right) + \gamma_2 \left(\|\{\delta_i\}_{i\geqslant 1}\|_{l^p[1,k]} \right)
\end{aligned}
$$

Since $\beta(\cdot, 1)$ is in $\bar{\mathcal{K}}$ (respectively, $\bar{\mathcal{K}}^\infty$) then we can define a class $\bar{\mathcal{K}}$ (respectively, $\bar{\mathcal{K}}^\infty$) function $\gamma_0(\cdot)$ such that: $\gamma_0 \equiv \beta(\cdot, 1)$. This will give us:

$$
\begin{aligned}
\max_{1\leqslant i\leqslant k} \|x_i\| &\leqslant \gamma_0(\|x_1\|) + \gamma_1 \left(\max_{1\leqslant i\leqslant k} \|d_i\| \right) + \gamma_2 \left(\|\{\delta_i\}_{i\geqslant 1}\|_{l^p[1,k]} \right) \\
&\leqslant \max \left\{ 2\gamma_0(\|x_1\|), 4\gamma_1 \left(\max_{1\leqslant i\leqslant k} \|d_i\| \right), 4\gamma_2 \left(\|\{\delta_i\}_{i\geqslant 1}\|_{l^p[1,k]} \right) \right\}
\end{aligned}
$$

Define $\tilde{\gamma}_0 \equiv 2\gamma_0$, $\tilde{\gamma}_1 \equiv 4\gamma_1$, dan $\tilde{\gamma}_2 \equiv 4\gamma_2$, we have the inequality (2.15) for each $x_1 \in C([0,T], \mathbb{R}^n)$, for each $d_k \in \mathcal{D}_k$ and for each $\{\delta_k\}_{k\geqslant 1} \in l^p$, which completes the proof. □

In nonlinear control theory we know that BIBS+$\mathcal{K}$-AG implies ISS. Hence, since BDBS and DAG analog BIBS and $\mathcal{K}$-AG then intuitively BDBS+DAG also implies iterative ISS. Nevertheless, it can be argued informally that BDBS+DAG is a condition for iterative ISS as follows: BDBS is the property of boundedness of the first iteration signal (x_1), disturbance (d_k), and initial condition (δ_k) in every iteration. DAG stated that if the iteration increases then the remaining influence only from the disturbance and the initial condition. Hence if an ILC system satisfies both property it means it is bounded by x_1, d_k and δ_k for every pass and the effect of x_1 decreases as iteration increases. This is the iterative ISS property. We state the conclusion of this intuitive reasoning in the following conjecture:

Conjecture 2.2.1 (BDBS and DAG imply iterative ISS). *If the system (2.11) is BDBS and DAG then it is iterative ISS.*

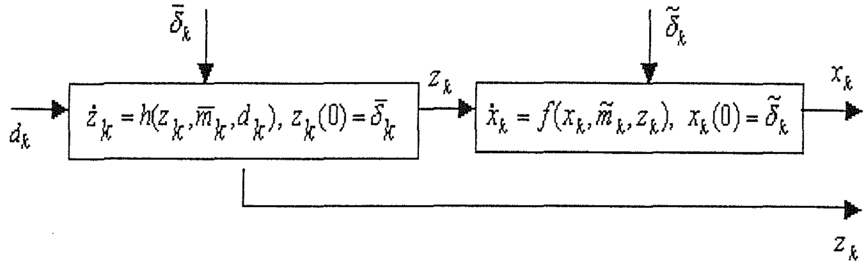

Figure 2.1: Cascade Interconnection

2.3 Cascade Interconnection

In this section, we consider the cascade interconnection of two iterative ISS system. We apply the results from the previous section to show that this type of system interconnection imply BDBS and DAG provided provided each of its subsystem is iterative ISS. In particular, we take the following system specification: let the state of the systems be

$$x_k \in C\left([0,T], \mathbb{R}^n\right), \quad z_k \in C\left([0,T], \mathbb{R}^s\right)$$

Let the system memories be $\tilde{m}_k \in \mathcal{M}$, and $\bar{m}_k \in \mathcal{M}$. We assume the initial conditions/initialisation errors decreases that is:

$$\{\tilde{\delta}_k\}_{k \geqslant 1} \in l^p, \quad \{\bar{\delta}_k\}_{k \geqslant 1} \in l^p$$

with $1 \leqslant p < \infty$. By this choice of initial condition we can relate the iterative ISS with DAG using lemma 2.2.2.

Let the disturbance $d_k : [0,T] \to R^s$ belongs to an admissible disturbance set $\mathcal{D}_k$. Denote $f \in C\left(\mathbb{R}^n \times \mathcal{M} \times \mathbb{R}^n, \mathbb{R}^n\right)$ and $h \in C\left(\mathbb{R}^s \times \mathcal{M} \times \mathbb{R}^s, \mathbb{R}^s\right)$ the nonlinear dynamic of the systems. Consider the following *cascade* interconnection of two ILC system:

$$\begin{aligned} \dot{x}_k &= f(x_k, \tilde{m}_k, z_k),\ x_k(0) = \tilde{\delta}_k \\ \tilde{m}_k &= \tilde{F}(\mathcal{K}_{k-1}) \end{aligned} \tag{2.22}$$

$$\begin{aligned} \dot{z}_k &= h(z_k, \bar{m}_k, d_k),\ z_k(0) = \bar{\delta}_k \\ \bar{m}_k &= \bar{F}(\mathcal{K}_{k-1}) \end{aligned} \tag{2.23}$$

for suitably defined operator $\tilde{F}$ and $\bar{F}$. The signal $\mathcal{K}_k$ is as defined in (2.3).

Assumption 2.3.1. *Consider the system:*

$$\begin{aligned} \dot{x}_k &= f(x_k, \tilde{m}_k, z_k),\ x_k(0) = \tilde{\delta}_k \\ \tilde{m}_k &= \tilde{F}(\mathcal{K}_{k-1}) \end{aligned} \tag{2.24}$$

viewed with state x_k and disturbance z_k, is iterative ISS.

Assumption 2.3.2. *Consider the system:*

$$\begin{aligned} \dot{z}_k &= h(z_k, \bar{m}_k, d_k),\ z_k(0) = \bar{\delta}_k \\ \bar{m}_k &= \bar{F}(\mathcal{K}_{k-1}) \end{aligned} \tag{2.25}$$

viewed with state z_k and disturbance d_k, is also iterative ISS.

Theorem 2.3.1. *If the assumptions (2.3.1) and (2.3.2) are satisfied then the cascade interconnection system (2.22)-(2.23) is BDBS and DAG, i.e. there exists a class $\bar{\mathcal{K}}$ (respectively, $\bar{\mathcal{K}}^\infty$) function $\bar{\gamma}_1$ and class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) functions γ_1, $\tilde{\gamma}_2$, $\tilde{\gamma}_3$ such that the following inequalities hold:*

$$\max_{1\leqslant i\leqslant k}\left\|\begin{bmatrix} x_k \\ z_k \end{bmatrix}\right\| \leqslant \max\left\{\tilde{\gamma}_0\left(\left\|\begin{bmatrix} x_1 \\ z_1 \end{bmatrix}\right\|\right), \tilde{\gamma}_1\left(\max_{1\leqslant i\leqslant k}\|d_i\|\right), \tilde{\gamma}_2\left(\left\|\left\{\begin{bmatrix} \tilde{\delta}_i \\ \bar{\delta}_i \end{bmatrix}\right\}_{i\geqslant 1}\right\|_{l^p[1,k]}\right)\right\} \tag{2.26}$$

and

$$\limsup_{k\to\infty}\left\|\begin{bmatrix} x_k \\ z_k \end{bmatrix}\right\| \leqslant \gamma_1\left(\limsup_{k\to\infty}(\|d_k\|)\right) \tag{2.27}$$

for each: $x_1 \in C([0,T],\mathbb{R}^n)$, $z_1 \in C([0,T],\mathbb{R}^s)$, $d_k \in \mathcal{D}_k$, $\{\tilde{\delta}_k\}_{k\geqslant 1} \in l^p$, $\{\bar{\delta}_k\}_{k\geqslant 1} \in l^p$, $k \in \mathbb{N}$.

Before proving the theorem 2.3.1, first we will provide a lemma connecting the iterative ISS property of each subsystem to DAG property for each subsystem. Following definition 2.2.2 and remark 2.2.1, the subsystem (2.24) which consists the external disturbance z_k has DAG property if there exists $\tilde{\gamma}_1^a$ of class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) such that for each $k \in \mathbb{N}$, for each $z_k \in C([0,T],\mathbb{R}^s)$, it holds that:

$$\limsup_{k\to\infty}\|x_k\| \leq \tilde{\gamma}_1^a\left(\limsup_{k\to\infty}\|z_k\|\right) \tag{2.28}$$

Similarly, the subsystem (2.25) consisting d_k as the disturbance, is DAG if there exists $\tilde{\gamma}_1^b$ of class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) such that for each $k \in \mathbb{N}$, for each $d_k \in \mathcal{D}_k$ it holds that:

$$\limsup_{k\to\infty}\|z_k\| \leq \tilde{\gamma}_1^b\left(\limsup_{k\to\infty}\|d_k\|\right) \tag{2.29}$$

Lemma 2.3.1. *If the subsystem (2.24) (respectively, subsystem (2.25)) is iterative ISS then it has DAG property.*

Proof. The proof is a simple application of lemma 2.2.2. For subsystem (2.24), since z_k as the disturbance then inequality (2.28) follows directly. For subsystem (2.25), by lemma 2.2.2 it is clear that iterative ISS implies DAG. □

Proof of Theorem 2.3.1

First, we prove that the cascade interconnection system (2.22)-(2.23) is BDBS. Let k, x_1, z_1, d_k, $\{\delta_k\}_{k\geqslant 1}$ be arbitrary in their domain. Since the subsystem (2.24) is iterative ISS then we have the following:

$$\max_{1\leq i\leq k}\|x_i\| \leq \beta_1(\|x_1\|,1) + \gamma_1^a\left(\max_{1\leq i\leq k}\|z_i\|\right) + \gamma_2^a\left(\|\{\tilde{\delta}_i\}_{i\geqslant 1}\|_{l^p[1,k]}\right) \tag{2.30}$$

Choose $\bar{\gamma}_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be such that $\bar{\gamma}_1(\cdot) = \beta_1(\cdot,1)$. Clearly $\bar{\gamma}_1$ is a $\bar{\mathcal{K}}$ (respectively, $\bar{\mathcal{K}}^\infty$). Hence,

$$\max_{1\leq i\leq k}\|x_i\| \leq \bar{\gamma}_1(\|x_1\|) + \gamma_1^a\left(\max_{1\leq i\leq k}\|z_i\|\right) + \gamma_2^a\left(\|\{\tilde{\delta}_i\}_{i\geqslant 1}\|_{l^p[1,k]}\right) \tag{2.31}$$

By following the same way, the iterative ISS property of subsystem (2.25) also implies that there exists $\bar{\gamma}_2$ of class $\bar{\mathcal{K}}$ (respectively, $\bar{\mathcal{K}}^\infty$) such that:

$$\max_{1\leq i\leq k}\|z_i\| \leq \bar{\gamma}_2(\|z_1\|) + \gamma_1^b\left(\max_{1\leq i\leq k}\|d_i\|\right) + \gamma_2^b\left(\|\{\bar{\delta}_i\}_{i\geqslant 1}\|_{l^p[1,k]}\right) \tag{2.32}$$

Substitute (2.32) to (2.31), we have:

$$\max_{1\leq i\leq k}\|x_i\| \leq \bar{\gamma}_1(\|x_1\|) + \gamma_2^a\left(\|\tilde{\delta}_k\|_{l^p[1,k]}\right) + \gamma_1^a\left(\bar{\gamma}_2(\|z_1\|) + \gamma_1^b\left(\max_{1\leq i\leq k}\|d_i\|\right) + \gamma_2^b\left(\|\{\bar{\delta}_i\}_{i\geqslant 1}\|_{l^p[1,k]}\right)\right) \tag{2.33}$$

Using the semi-triangle inequality two times we get:

$$\begin{aligned}\max_{1\leq i\leq k}\|x_i\| &\leq \bar{\gamma}_1(\|x_1\|) + \gamma_1^a(2\bar{\gamma}_2(\|z_1\|)) + \gamma_1^a\left(4\gamma_1^b\left(\max_{1\leq i\leq k}\|d_i\|\right)\right) + \gamma_2^a\left(\|\{\tilde{\delta}_i\}_{i\geqslant 1}\|_{l^p[1,k]}\right)\\ &+ \gamma_1^a\left(4\gamma_2^a\left(\|\{\bar{\delta}_i\}_{i\geqslant 1}\|_{l^p[1,k]}\right)\right)\end{aligned}$$

Using fact 7 in the appendix we have:

$$\max_{1\leq i\leq k}\left\|\begin{bmatrix} x_k \\ z_k \end{bmatrix}\right\| \leq \max_{1\leq i\leq k}\|x_i\| + \max_{1\leq i\leq k}\|z_i\|$$

And by fact 6 in the appendix we get:

$$\begin{aligned}\max_{1\leq i\leq k}\left\|\begin{bmatrix} x_k \\ z_k \end{bmatrix}\right\| &\leqslant \bar{\gamma}_1\left(\left\|\begin{bmatrix} x_1 \\ z_1 \end{bmatrix}\right\|\right) + \bar{\gamma}_2\left(\left\|\begin{bmatrix} x_1 \\ z_1 \end{bmatrix}\right\|\right) + \gamma_1^a\left(2\bar{\gamma}_2\left(\left\|\begin{bmatrix} x_1 \\ z_1 \end{bmatrix}\right\|\right)\right)\\ &+ \gamma_1^b\left(\max_{1\leq i\leq k}\left\|\begin{bmatrix} x_1 \\ z_1 \end{bmatrix}\right\|\right) + \gamma_1^a\left(4\gamma_1^b\left(\max_{1\leq i\leq k}\|d_i\|\right)\right) + \gamma_2^a\left(\left\|\left\{\begin{bmatrix} \tilde{\delta}_i \\ \bar{\delta}_i \end{bmatrix}\right\}_{i\geqslant 1}\right\|\right)\\ &+ \gamma_1^a\left(4\gamma_2^a\left(\left\|\left\{\begin{bmatrix} \tilde{\delta}_i \\ \bar{\delta}_i \end{bmatrix}\right\}_{i\geqslant 1}\right\|_{l^p[1,k]}\right)\right) + \gamma_2^b\left(\left(\left\|\left\{\begin{bmatrix} \tilde{\delta}_i \\ \bar{\delta}_i \end{bmatrix}\right\}_{i\geqslant 1}\right\|_{l^p[1,k]}\right)\right)\end{aligned}$$

Now define the following functions:

$$\begin{aligned}\gamma_0^d(\cdot) &\equiv 2\left(\bar{\gamma}_1(\cdot) + \bar{\gamma}_2(\cdot) + \gamma_1^a(2\bar{\gamma}_2(\cdot))\right)\\ \gamma_1^d(\cdot) &\equiv 4\left(\gamma_1^b(\cdot) + \gamma_1^a(4\gamma_1^b(\cdot))\right)\\ \gamma_2^d(\cdot) &\equiv 4\left(\gamma_2^a(\cdot) + \gamma_1^a(4\gamma_2^a(\cdot)) + \gamma_2^b(\cdot)\right)\end{aligned}$$

By fact 1 and fact 2, $\gamma_0^d \in \bar{\mathcal{K}}$ (respectively, $\bar{\mathcal{K}}^\infty$), γ_1^d and γ_2^d are functions of class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$).

Thus, the inequality (2.34) becomes:

$$\max_{1\leq i\leq k}\left\|\begin{bmatrix} x_k \\ z_k \end{bmatrix}\right\| \leqslant \frac{1}{2}\gamma_0^d\left(\left\|\begin{bmatrix} x_1 \\ z_1 \end{bmatrix}\right\|\right) + \frac{1}{4}\gamma_1^d\left(\max_{1\leq i\leq k}\|d_i\|\right) + \frac{1}{4}\gamma_2^d\left(\left\|\left\{\begin{bmatrix} \tilde{\delta}_i \\ \bar{\delta}_i \end{bmatrix}\right\}_{i\geqslant 1}\right\|_{l^p[1,k]}\right)$$

Hence:

$$\max_{1\leq i\leq k}\left\|\begin{bmatrix} x_k \\ z_k \end{bmatrix}\right\| \leqslant \max\left\{\gamma_1^d\left(\left\|\begin{bmatrix} x_1 \\ z_1 \end{bmatrix}\right\|\right), \gamma_2^d\left(\max_{1\leq i\leq k}\|d_i\|\right), \gamma_3^d\left(\left\|\left\{\begin{bmatrix} \tilde{\delta}_i \\ \bar{\delta}_i \end{bmatrix}\right\}_{i\geqslant 1}\right\|\right)\right\}$$

Since x_1, z_1, d_k, $\tilde{\delta}_k$, and $\bar{\delta}_k$ are arbitrary then it holds for each: x_1, z_1, d_k, $\tilde{\delta}_k$, and $\bar{\delta}_k$.

Now we prove that the cascade interconnection system (2.22)-(2.23) is DAG. Since the subsystem (2.24) is iterative ISS then according to lemma 2.3.1, subsystem (2.24) has disturbance asymptotic gain property. Hence there exists $\tilde{\delta}_1^a$ such that inequality (2.56) and (2.57) hold for each $z_k \in C([0,T],\mathbb{R}^s)$ and for each $d_k \in \mathcal{D}_k$.

Substitute (2.57) to (2.56) we have:

$$\limsup_{k\to\infty} \|x_k\| \leq \tilde{\gamma}_1^a\left(\tilde{\gamma}_1^b\left(\limsup_{k\to\infty} \|d_k\|\right)\right) \tag{2.34}$$

Using the the fact 1 in the appendix, we have:

$$\limsup_{k\to\infty} \left\| \begin{bmatrix} x_k \\ z_k \end{bmatrix} \right\| \leqslant \tilde{\gamma}_1^a\left(\tilde{\gamma}_1^b\left(\limsup_{k\to\infty} \|d_k\|\right)\right) + \tilde{\gamma}_1^b\left(\limsup_{k\to\infty} \|d_k\|\right) \tag{2.35}$$

Define the function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that:

$$\alpha(\cdot) = \tilde{\gamma}_1^a\left(\tilde{\gamma}_1^b(\cdot)\right) + \tilde{\gamma}_1^b(\cdot)$$

By fact 2 and fact 3, α is a class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) function.

Substitute α to inequality (2.35), we obtain:

$$\limsup_{k\to\infty} \left\| \begin{bmatrix} x_k \\ z_k \end{bmatrix} \right\| \leqslant \alpha\left(\limsup_{k\to\infty} \|d_k\|\right) \tag{2.36}$$

for each $d_k \in \mathbb{R}^s$ and for each $k \in \mathbb{N}$, which complete the proof. □

Conjecture 2.3.1. *If the assumptions 2.3.1 and 2.3.2 hold then the cascade interconnection system (2.22)-(2.23) is iterative ISS.*

2.4 Feedback interconnection

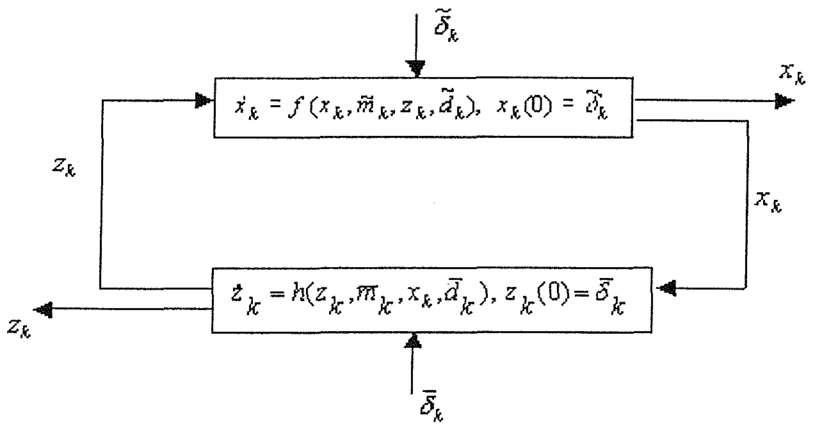

Figure 2.2: Feedback Interconnection

In this section, we discuss the iterative ISS property of feedback interconnection ILC system. The system under consideration is almost similar with the previous section, the difference only we have an additional disturbance and which is an interconnection of state in each of subsystems. We restate again the system specification for clarity. Let for each $k \in \mathbb{N}$:

$$x_k \in C\left([0,T], \mathbb{R}^n\right), \quad z_k \in C\left([0,T], \mathbb{R}^s\right)$$

be the state of the system. The memory is taken to be $\tilde{m}_k \in \mathcal{M}$ and $\bar{m}_k \in \mathcal{M}$.

The external disturbances $\tilde{d}_k : [0,T] \to \mathbb{R}^n$ and $\bar{d}_k : [0,T] \to \mathbb{R}^s$ are measurable functions. The initial conditions are taken to be:

$$\{\tilde{\delta}_k\}_{k\geqslant 1} \in l^p, \quad \{\bar{\delta}_k\}_{k\geqslant 1} \in l^p$$

$f \in C(\mathbb{R}^n \times \mathcal{M} \times \mathbb{R}^s \times \mathbb{R}^n, \mathbb{R}^n)$ and $h \in C(\mathbb{R}^s \times \mathcal{M} \times \mathbb{R}^n \times \mathbb{R}^s, \mathbb{R}^s)$, for each $k \in \mathbb{N}$. We assume that $1 \leqslant p < \infty$. Consider the following interconnected ILC system:

$$\begin{aligned} \dot{x}_k &= f(x_k, \tilde{m}_k, z_k, \tilde{d}_k), \quad x_k(0) = \tilde{\delta}_k \\ \tilde{m}_k &= \tilde{F}(\mathcal{K}_{k-1}) \end{aligned} \tag{2.37}$$

$$\begin{aligned} \dot{z}_k &= h(z_k, \bar{m}_k, x_k, \bar{d}_k), \quad z_k(0) = \bar{\delta}_k \\ \bar{m}_k &= \bar{F}(\mathcal{K}_{k-1}) \end{aligned} \tag{2.38}$$

Consider the system:

$$\begin{aligned} \dot{x}_k &= f(x_k, \tilde{m}_k, z_k, \tilde{d}_k), \quad x_k(0) = \tilde{\delta}_k \\ \tilde{m}_k &= \tilde{F}(\mathcal{K}_{k-1}) \end{aligned} \tag{2.39}$$

viewed as a system with state x_k and disturbances z_k and $\tilde{d}_k$. The system is iterative ISS if for all $p \in \mathbb{N}$ there exists a class $\bar{\mathcal{KL}}$ function β, class $\mathcal{K}$ (respectively , $\mathcal{K}^\infty$) functions $\gamma_1^a, \gamma_2^a, \gamma_3^a$ the following holds:

$$||x_k|| \leqslant \beta(||x_1||, k) + \gamma_1^a \left(\max_{1\leqslant i\leqslant k} ||d_i|| \right) + \gamma_2^a \left(\max_{1\leqslant i\leqslant k} ||z_i|| \right) + \gamma_3^a \left(||\{\tilde{\delta}_i\}_{i\geqslant 1}||_{l^p[1,k]} \right) \tag{2.40}$$

for each $k \in \mathbb{N}$, for all $x_1 \in C([0,T], \mathbb{R}^n)$, for each $d_k \in \mathcal{D}_k$, for each $\{\tilde{\delta}_k\}_{k\geqslant 1} \in l^p$.

Assumption 2.4.1. *The system (2.39) is iterative ISS.*

Now consider the second subsystem:

$$\begin{aligned} \dot{z}_k &= h(z_k, \bar{m}_k, x_k, \bar{d}_k), \quad z_k(0) = \bar{\delta}_k \\ \bar{m}_k &= \bar{F}(\mathcal{K}_{k-1}) \end{aligned} \tag{2.41}$$

viewed as a system with state z_k and disturbances x_k and $\bar{d}_k$. The system is iterative ISS if there exists a class $\bar{\mathcal{KL}}$ function β, class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) functions $\gamma_1^b, \gamma_2^b, \gamma_3^b$ such that:

$$||z_k|| \leqslant \beta(||z_1||, k) + \gamma_1^b \left(\max_{1\leqslant i\leqslant k} ||d_i|| \right) + \gamma_2^b \left(\max_{1\leqslant i\leqslant k} ||x_i|| \right) + \gamma_3^b \left(||\{\tilde{\delta}_i\}_{i\geqslant 1}||_{l^p[1,k]} \right) \tag{2.42}$$

for each $k \in \mathbb{N}$, for each $z_1 \in C([0,T], \mathbb{R}^n)$, for each $\tilde{d}_k \in \mathcal{D}_k$, for each $\{\bar{\delta}_k\}_{k\geqslant 1} \in l^p$.

Assumption 2.4.2. *The system (2.41) is iterative ISS.*

Assumption 2.4.3. *For each $s \in \mathbb{R}$, and for ϵ such that $0 < \epsilon < 1$ the following gain composition condition holds:*

$$\gamma_2^a \circ 2\gamma_2^b(s) \leqslant \epsilon s \tag{2.43}$$

and,

$$\gamma_2^b \circ 2\gamma_2^a(s) \leqslant \epsilon s \tag{2.44}$$

Theorem 2.4.1. *Suppose the assumptions 2.4.1, 2.4.2 and 2.4.3 hold then the feedback interconnection system (2.37)-(2.38) is BDBS and DAG.*

Proof of Theorem 2.4.1

The proof of theorem 2.4.1 is divided into two. The first part is to prove BDBS property of feedback interconnection (2.37)-(2.38) and the second part is DAG property of (2.37)-(2.38). The following is the first part of the proof:

Proof of BDBS property of feedback interconnection The iterative ISS property of subsystem (2.39) says that for each $p \in \mathbb{N}$, there exists $\beta_1 \in \bar{\mathcal{K}}\mathcal{L}$, $\gamma_1^a, \gamma_2^a, \gamma_3^a$ all are of class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) such that for each $x_1 \in C([0,T], \mathbb{R}^n)$, for each $\tilde{d}_k \in \mathcal{D}_k$, for each $\{\tilde{\delta}_k\}_{k\geqslant 1} \in l^p$, and for each $k \in \mathbb{N}$, we have:

$$\begin{aligned}\max_{1\leqslant i\leqslant k} \|x_i\| &\leqslant \max_{1\leqslant i\leqslant k} \beta_1(\|x_1\|, k) + \gamma_1^a\left(\max_{1\leqslant i\leqslant k} \|\tilde{d}_i\|\right) + \gamma_2^a\left(\max_{1\leqslant i\leqslant k} \|z_i\|\right) + \gamma_3^a\left(\|\{\tilde{\delta}_i\}_{i\geqslant 1}\|_{l^p[1,k]}\right)\\ &\leqslant \beta_1(\|x_1\|, 1) + \gamma_1^a\left(\max_{1\leqslant i\leqslant k} \|\tilde{d}_i\|\right) + \gamma_2^a\left(\max_{1\leqslant i\leqslant k} \|z_i\|\right) + \gamma_3^a\left(\|\{\tilde{\delta}_i\}_{i\geqslant 1}\|_{l^p[1,k]}\right)\end{aligned}$$

Clearly that $\beta_1(\cdot, 1)$ is a function of class $\bar{\mathcal{K}}$ (respectively, $\bar{\mathcal{K}}^\infty$). Let $\gamma_0^a(\cdot) \equiv \beta_1(\cdot, 1)$, we get:

$$\max_{1\leqslant i\leqslant k} \|x_i\| \leqslant \max_{1\leqslant i\leqslant k} \gamma_0^a(\|x_1\|) + \gamma_1^a\left(\max_{1\leqslant i\leqslant k} \|\tilde{d}_i\|\right) + \gamma_2^a\left(\max_{1\leqslant i\leqslant k} \|z_i\|\right) + \gamma_3^a\left(\|\{\tilde{\delta}_i\}_{i\geqslant 1}\|_{l^p[1,k]}\right) \tag{2.45}$$

Analog with subsystem (2.39), since the system (2.41) is iterative ISS then we have:

$$\max_{1\leqslant i\leqslant k} \|z_i\| \leqslant \gamma_0^b(\|z_1\|) + \gamma_1^b\left(\max_{1\leqslant i\leqslant k} \|\tilde{d}_i\|\right) + \gamma_2^b\left(\max_{1\leqslant i\leqslant k} \|x_i\|\right) + \gamma_3^b\left(\|\{\tilde{\delta}_i\}_{i\geqslant 1}\|_{l^p[1,k]}\right) \tag{2.46}$$

Substitute inequality (2.46) to inequality (2.45), using the semi-triangle inequality three times, and then re-arranging we have:

$$\begin{aligned}\max_{1\leqslant i\leqslant k} \|x_i\| &\leqslant \gamma_0^a(\|x_1\|) + \gamma_2^a \circ 4\gamma_0^b(\|z_1\|) + \gamma_1^a\left(\max_{1\leqslant i\leqslant k} \|\tilde{d}_i\|\right) + \gamma_2^a \circ 8\gamma_1^b\left(\max_{1\leqslant i\leqslant k} \|\tilde{d}_i\|\right)\\ &+ \gamma_2^a \circ 2\gamma_2^b\left(\max_{1\leqslant i\leqslant k} \|x_i\|\right) + \gamma_3^a\left(\|\{\tilde{\delta}_k\}_{i\geqslant 1}\|_{l^p[1,k]}\right) + \gamma_2^a \circ 8\gamma_3^b\left(\|\{\tilde{\delta}_i\}_{i\geqslant 1}\|_{l^p[1,k]}\right)\end{aligned} \tag{2.47}$$

Substitute inequality (2.45) to inequality (2.46), using the semi-triangle inequality three times, and then re-arranging we have:

$$\begin{aligned}\max_{1\leqslant i\leqslant k} \|z_i\| &\leqslant \gamma_0^b(\|z_1\|) + \gamma_2^b \circ 4\gamma_0^a(\|x_1\|) + \gamma_1^b\left(\max_{1\leqslant i\leqslant k} \|\tilde{d}_i\|\right) + \gamma_2^b \circ 8\gamma_1^a\left(\max_{1\leqslant i\leqslant k} \|\tilde{d}_i\|\right)\\ &+ \gamma_2^b \circ 2\gamma_2^a\left(\max_{1\leqslant i\leqslant k} \|z_i\|\right) + \gamma_3^b\left(\|\{\tilde{\delta}_i\}_{i\geqslant 1}\|_{l^p[1,k]}\right) + \gamma_2^b \circ 8\gamma_3^a\left(\|\{\tilde{\delta}_i\}_{i\geqslant 1}\|_{l^p[1,k]}\right)\end{aligned} \tag{2.48}$$

Using the condition (2.43) and then rearranging, the inequality (2.47) becomes:

$$\begin{aligned}(1-\epsilon)\max_{1\leqslant i\leqslant k} \|x_i\| &\leqslant \gamma_0^a(\|x_1\|) + \gamma_2^a \circ 4\gamma_0^b(\|z_1\|) + \gamma_1^a\left(\max_{1\leqslant i\leqslant k} \|\tilde{d}_i\|\right)\\ &+ \gamma_2^a \circ 8\gamma_1^b\left(\max_{1\leqslant i\leqslant k} \|\tilde{d}_i\|\right) + \gamma_3^a\left(\|\{\tilde{\delta}_i\}_{i\geqslant 1}\|_{l^p[1,k]}\right)\\ &+ \gamma_2^a \circ 8\gamma_3^b\left(\|\{\tilde{\delta}_i\}_{i\geqslant 1}\|_{l^p[1,k]}\right)\end{aligned} \tag{2.49}$$

Using the condition (2.44) and then rearranging, the inequality (2.48) becomes:

$$\begin{aligned}(1-\epsilon)\max_{1\leqslant i\leqslant k} \|z_i\| &\leqslant \gamma_0^b(\|z_1\|) + \gamma_2^b \circ 4\gamma_0^a(\|x_1\|) + \gamma_1^b\left(\max_{1\leqslant i\leqslant k} \|\tilde{d}_i\|\right)\\ &+ \gamma_2^b \circ 8\gamma_1^a\left(\max_{1\leqslant i\leqslant k} \|\tilde{d}_i\|\right) + \gamma_3^b\left(\|\{\tilde{\delta}_i\}_{i\geqslant 1}\|_{l^p[1,k]}\right)\\ &+ \gamma_2^b \circ 8\gamma_3^a\left(\|\{\tilde{\delta}_i\}_{i\geqslant 1}\|_{l^p[1,k]}\right)\end{aligned} \tag{2.50}$$

Adding the inequality (2.49) and (2.50) we get:

$$
\begin{aligned}
& (1-\epsilon)\left(\max_{1\leqslant i\leqslant k}\|x_i\|+\max_{1\leqslant i\leqslant k}\|z_i\|\right)\leqslant \gamma_2^b\circ 4\gamma_0^a(\|x_1\|)+\gamma_0^b(\|z_1\|)+\gamma_2^a\circ 4\gamma_0^b(\|z_1\|)+\gamma_0^a(\|x_1\|)\\
+ & \gamma_1^a\left(\max_{1\leqslant i\leqslant k}\|\tilde{d}_i\|\right)+\gamma_2^b\circ 8\gamma_1^a\left(\max_{1\leqslant i\leqslant k}\|\tilde{d}_i\|\right)+\gamma_3^a\left(\|\{\tilde{\delta}_i\}_{i\geqslant 1}\|_{l^p[1,k]}\right)+\gamma_2^b\circ 8\gamma_3^a\left(\|\{\tilde{\delta}_i\}_{i\geqslant 1}\|_{l^p[1,k]}\right)\\
+ & \gamma_2^a\left(\max_{1\leqslant i\leqslant k}\|\bar{d}_i\|\right)+\gamma_2^a\circ 8\gamma_1^b\left(\max_{1\leqslant i\leqslant k}\|\bar{d}_i\|\right)+\gamma_3^b\left(\|\{\bar{\delta}_i\}_{i\geqslant 1}\|_{l^p[1,k]}\right)+\gamma_2^a\circ 8\gamma_3^b\left(\|\{\bar{\delta}_i\}_{i\geqslant 1}\|_{l^p[1,k]}\right)
\end{aligned}
$$

Using the fact 6 and 7 we get:

$$
\begin{aligned}
& (1-\epsilon)\max_{1\leqslant i\leqslant k}\left\|\begin{bmatrix} x_i\\ z_i\end{bmatrix}\right\|\leqslant \gamma_2^b\circ 4\gamma_0^a\left(\left\|\begin{bmatrix} x_1\\ z_1\end{bmatrix}\right\|\right)+\gamma_0^b\left(\left\|\begin{bmatrix} x_1\\ z_1\end{bmatrix}\right\|\right)\\
+ & \gamma_2^a\circ 4\gamma_0^b\left(\left\|\begin{bmatrix} x_1\\ z_1\end{bmatrix}\right\|\right)+\gamma_0^a\left(\left\|\begin{bmatrix} x_1\\ z_1\end{bmatrix}\right\|\right)+\gamma_1^a\left(\max_{1\leqslant i\leqslant k}\left\|\begin{bmatrix} \tilde{d}_i\\ \bar{d}_i\end{bmatrix}\right\|\right)+\gamma_2^b\circ 8\gamma_1^a\left(\max_{1\leqslant i\leqslant k}\left\|\begin{bmatrix} \tilde{d}_i\\ \bar{d}_i\end{bmatrix}\right\|\right)\\
+ & \gamma_3^a\left(\left\|\left\{\begin{bmatrix} \tilde{\delta}_i\\ \bar{\delta}_i\end{bmatrix}\right\}_{i\geqslant 1}\right\|_{l^p[1,k]}\right)+\gamma_2^b\circ 8\gamma_3^a\left(\left\|\left\{\begin{bmatrix} \tilde{\delta}_i\\ \bar{\delta}_i\end{bmatrix}\right\}_{i\geqslant 1}\right\|_{l^p[1,k]}\right)+\gamma_2^a\left(\max_{1\leqslant i\leqslant k}\left\|\begin{bmatrix} \tilde{d}_i\\ \bar{d}_i\end{bmatrix}\right\|\right)\\
+ & \gamma_2^a\circ 8\gamma_1^b\left(\max_{1\leqslant i\leqslant k}\left\|\begin{bmatrix} \tilde{d}_i\\ \bar{d}_i\end{bmatrix}\right\|\right)+\gamma_3^b\left(\left\|\left\{\begin{bmatrix} \tilde{\delta}_i\\ \bar{\delta}_i\end{bmatrix}\right\}_{i\geqslant 1}\right\|_{l^p[1,k]}\right)+\gamma_2^a\circ 8\gamma_3^b\left(\left\|\left\{\begin{bmatrix} \tilde{\delta}_i\\ \bar{\delta}_i\end{bmatrix}\right\}_{i\geqslant 1}\right\|_{l^p[1,k]}\right)
\end{aligned}
$$

Define the following functions:

$$
\begin{aligned}
\gamma_1^d(\cdot) &\equiv \frac{2}{1-\epsilon}\gamma_2^b\left(4\gamma_0^a(\cdot)\right)+\gamma_0^b(\cdot)+\gamma_2^a\circ 4\gamma_0^b(\cdot)+\gamma_0^a(\cdot) && (2.51)\\
\gamma_2^d(\cdot) &\equiv \frac{4}{1-\epsilon}\gamma_1^a(\cdot)+\gamma_2^a(\cdot)+\gamma_2^a\circ 8\gamma_1^b(\cdot)+\gamma_2^b\circ 8\gamma_1^a(\cdot) && (2.52)\\
\gamma_3^d(\cdot) &\equiv \frac{4}{1-\epsilon}\gamma_3^a(\cdot)+\gamma_3^b(\cdot)+\gamma_2^a\circ 8\gamma_3^b(\cdot)+\gamma_2^b\circ 8\gamma_3^a(\cdot) && (2.53)
\end{aligned}
$$

by the fact 2 and fact 3, $\gamma_1^d \in \bar{\mathcal{K}}$ (respectively, $\bar{\mathcal{K}}^\infty$) , γ_2^d and γ_3^d are class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$ functions.

The above inequality becomes:

$$
\max_{1\leqslant i\leqslant k}\left\|\begin{bmatrix} x_i\\ z_i\end{bmatrix}\right\|\leqslant \frac{1}{2}\gamma_1^d\left(\left\|\begin{bmatrix} x_1\\ z_1\end{bmatrix}\right\|\right)+\frac{1}{4}\gamma_2^d\left(\max_{1\leqslant i\leqslant k}\left\|\begin{bmatrix} \tilde{d}_i\\ \bar{d}_i\end{bmatrix}\right\|\right)+\frac{1}{4}\gamma_3^d\left(\left\|\left\{\begin{bmatrix} \tilde{\delta}_i\\ \bar{\delta}_i\end{bmatrix}\right\}_{i\geqslant 1}\right\|_{l^p[1,k]}\right) \qquad (2.54)
$$

or,

$$
\max_{1\leqslant i\leqslant k}\left\|\begin{bmatrix} x_i\\ z_i\end{bmatrix}\right\|\leqslant \max\left\{\gamma_1^d\left(\left\|\begin{bmatrix} x_1\\ z_1\end{bmatrix}\right\|\right),\gamma_2^d\left(\max_{1\leqslant i\leqslant k}\left\|\begin{bmatrix} \tilde{d}_i\\ \bar{d}_i\end{bmatrix}\right\|\right),\gamma_3^d\left(\left\|\left\{\begin{bmatrix} \tilde{\delta}_i\\ \bar{\delta}_i\end{bmatrix}\right\}_{i\geqslant 1}\right\|_{l^p[1,k]}\right)\right\} \qquad (2.55)
$$

since k, x_1, z_1, $\tilde{d}_k$, $\bar{d}_k$, $\tilde{\delta}_k$ and $\bar{\delta}_k$ are arbitrary then the above inequality holds for each: k, x_1, z_1, $\tilde{d}_k$, $\bar{d}_k$, $\tilde{\delta}_k$ and $\bar{\delta}_k$ in the set where they belong. □

Now we will prove that the iterative ISS of subsystems (2.39) and (2.41) imply DAG property of system (2.37)-(2.38). Before go to the proof we will formulate the DAG property of each subsystem and then prove that those property exist for each subsystem. The proof of DAG property of feedback interconnection follows subsequently.

Since we assume $1 \leqslant p < \infty$, then according to the remark 2.2.1, the subsystem (2.39) has DAG property if there exists γ_1^a, γ_2^a are of class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) such that for each $k \in \mathbb{N}$, for each $z_k \in C([0,T], \mathbb{R}^s)$, and for each $\tilde{d}_k \in \mathcal{D}_k$ it holds that:

$$\limsup_{k\to\infty} ||x_k|| \leqslant \gamma_1^a \left(\limsup_{k\to\infty} ||\tilde{d}_k|| \right) + \gamma_2^a \left(\limsup_{k\to\infty} ||z_k|| \right) \tag{2.56}$$

Similarly, the subsystem (2.41) is DAG if there exists γ_1^b and γ_2^b all are of class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) such that for each $k \in \mathbb{N}$, for each $x_k \in C([0,T], \mathbb{R}^n)$, and for each $\tilde{d}_k \in \mathcal{D}_k$ it holds that:

$$\limsup_{k\to\infty} ||z_k|| \leqslant \gamma_1^b \left(\limsup_{k\to\infty} ||\tilde{d}_k|| \right) + \gamma_2^b \left(\limsup_{k\to\infty} ||x_k|| \right) \tag{2.57}$$

The following lemma is used in the proof of the next main theorem.

Lemma 2.4.1. *If the ILC system (2.39) (respectively, (2.41)) is iterative ISS then it has DAG property.*

Proof. We only prove that the iterative ISS property implies DAG property for ILC system (2.39); for ILC system (2.41) the proof of iterative ISS property implies DAG property follows similarly. Now, suppose the system (2.39) is iterative ISS. Let k, x_1, d_k, z_k, $\tilde{\delta}_k$ be arbitrary in the set where they belong. The following inequality holds:

$$||x_k|| \leqslant \beta(||x_1||, k) + \gamma_1^a \left(\max_{1\leqslant i\leqslant k} ||d_i|| + ||z_i|| \right) + \gamma_2^a \left(\max_{1\leqslant i\leqslant k} ||d_i|| + ||z_i|| \right) + \gamma_3^a \left(|| \left\{ \tilde{\delta}_i \right\}_{i\geqslant 1} ||_{l^p[1,k]} \right) \tag{2.58}$$

Let a class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) function γ_4^a be:

$$\gamma_4^a(v) = \max \left\{ 2\gamma_1^a(v), 2\gamma_2^a(v) \right\} \tag{2.59}$$

for each $v \in \mathbb{R}_{\geqslant 0}$. We get:

$$||x_k|| \leqslant \beta(||x_1||, k) + \gamma_4^a \left(\max_{1\leqslant i\leqslant k} ||d_i|| + ||z_i|| \right) + \gamma_3^a \left(|| \left\{ \tilde{\delta}_i \right\}_{i\geqslant 1} ||_{l^p[1,k]} \right) \tag{2.60}$$

Using lemma 2.2.2 we obtain:

$$\limsup_{k\to\infty} ||x_k|| \leqslant \gamma_4^a \left(\limsup_{k\to\infty} ||d_k|| + ||z_k|| \right) \tag{2.61}$$

Using the semi triangle inequality we have:

$$\limsup_{k\to\infty} ||x_k|| \leqslant \gamma_4^a \left(2 \limsup_{k\to\infty} ||d_k|| \right) + \gamma_4^a \left(2 \limsup_{k\to\infty} ||z_k|| \right)$$

Define $\tilde{\gamma}^a(\cdot) = \gamma_4^a(2\cdot)$, we have:

$$\limsup_{k\to\infty} ||x_k|| \leqslant \tilde{\gamma}^a \left(\limsup_{k\to\infty} ||d_k|| \right) + \tilde{\gamma}^a \left(\limsup_{k\to\infty} ||z_k|| \right)$$

which completes the proof. Using the same way we can also prove that the system (2.41) is DAG. □

Proof of DAG property of feedback interconnection. We will prove that iterative ISS of subsystem (2.39) and (2.41) imply DAG. Since ILC system (2.39) and (2.41) are iterative ISS then according to lemma 2.4.1, they have DAG property. Substitute inequality (2.57) to inequality (2.56), and then use semi triangle inequality we obtain:

$$\begin{aligned} \limsup_{k\to\infty} ||x_k|| \quad &\leqslant \quad \gamma_1^a \left(\limsup_{k\to\infty} ||\tilde{d}_k|| \right) + \gamma_2^a \circ 2\gamma_1^b \left(\limsup_{k\to\infty} ||\tilde{d}_k|| \right) \\ &+ \quad \gamma_2^a \circ 2\gamma_2^b \left(\limsup_{k\to\infty} ||x_k|| \right) \end{aligned} \tag{2.62}$$

Substitute inequality (2.56) to inequality (2.57) and then use the semi triangle inequality, we obtain

$$\begin{aligned}\limsup_{k\to\infty}\|z_k\| &\leqslant \gamma_1^b\left(\limsup_{k\to\infty}\|\bar{d}_k\|\right)+\gamma_2^b\circ 2\gamma_1^a\left(\limsup_{k\to\infty}\|\tilde{d}_k\|\right)\\ &+ \gamma_2^b\circ 2\gamma_2^a\left(\limsup_{k\to\infty}\|z_k\|\right)\end{aligned}\tag{2.63}$$

Adding inequality (2.62) with (2.63) and then use the condition (2.43) and (2.44) we have:

$$\begin{aligned}\limsup_{k\to\infty}\|x_k\|+\limsup_{k\to\infty}\|z_k\| &\leqslant \gamma_1^a\left(\limsup_{k\to\infty}\|\tilde{d}_k\|\right)+\gamma_1^b\left(\limsup_{k\to\infty}\|\bar{d}_k\|\right)\\ &+ \gamma_2^b\circ 2\gamma_1^a\left(\limsup_{k\to\infty}\|\tilde{d}_k\|\right)\\ &+ \epsilon\left(\limsup_{k\to\infty}\|x_k\|\right)+\epsilon\left(\limsup_{k\to\infty}\|z_k\|\right)\end{aligned}$$

Rearranging we get:

$$\begin{aligned}(1-\epsilon)\left(\limsup_{k\to\infty}\|x_k\|+\limsup_{k\to\infty}\|z_k\|\right) &\leqslant \gamma_1^a\left(\limsup_{k\to\infty}\|\tilde{d}_k\|\right)+\gamma_1^b\left(\limsup_{k\to\infty}\|\bar{d}_k\|\right)\\ &+ \gamma_2^b\circ 2\gamma_1^a\left(\limsup_{k\to\infty}\|\tilde{d}_k\|\right)+\gamma_2^a\circ 2\gamma_1^b\left(\limsup_{k\to\infty}\|\bar{d}_k\|\right)\end{aligned}$$

Using the fact 6 and the fact 7, we obtain:

$$\begin{aligned}(1-\epsilon)\limsup_{k\to\infty}\left\|\begin{bmatrix}x_k\\ z_k\end{bmatrix}\right\| &\leqslant \gamma_1^a\left(\limsup_{k\to\infty}\left\|\begin{bmatrix}\tilde{d}_k\\ \bar{d}_k\end{bmatrix}\right\|\right)+\gamma_1^b\left(\limsup_{k\to\infty}\left\|\begin{bmatrix}\tilde{d}_k\\ \bar{d}_k\end{bmatrix}\right\|\right)\\ &+ \gamma_2^b\circ 2\gamma_1^a\left(\limsup_{k\to\infty}\left\|\begin{bmatrix}\tilde{d}_k\\ \bar{d}_k\end{bmatrix}\right\|\right)+\gamma_2^a\circ 2\gamma_1^b\left(\limsup_{k\to\infty}\left\|\begin{bmatrix}\tilde{d}_k\\ \bar{d}_k\end{bmatrix}\right\|\right)\end{aligned}$$

Define a function $\alpha:\mathbb{R}_{\geqslant 0}\to\mathbb{R}_{\geqslant 0}$ such that:

$$\alpha(\cdot)=\frac{1}{1-\epsilon}\left(\gamma_1^a(\cdot)+\gamma_1^b(\cdot)+\gamma_2^b\circ 2\gamma_1^a(\cdot)+\gamma_2^a\circ\gamma_1^b(\cdot)\right)\tag{2.64}$$

By fact 3, α is a function of class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$).

Thus,

$$\limsup_{k\to\infty}\left\|\begin{bmatrix}x_k\\ z_k\end{bmatrix}\right\|\leqslant\alpha\left(\limsup_{k\to\infty}\left\|\begin{bmatrix}\tilde{d}_k\\ \bar{d}_k\end{bmatrix}\right\|\right)\tag{2.65}$$

which says that the system (2.37)-(2.38) has DAG property. □

Conjecture 2.4.1. *The feedback interconnection system (2.37)-(2.38) is iterative ISS if the assumptions (2.4.1), (2.4.2) and (2.4.3) hold.*

2.5 Summary

In this chapter the new robustness property of ILC system called iterative ISS is introduced. The properties related to this notion are derived. By using these properties the theory of ILC system interconnection is developed. It is proved that for the cascade and feedback interconnection, iterative ISS of each subsystem implies BDBS and DAG of the system interconnection.

Chapter 3

Nonlinear Adaptive ILC: a Case Study

The main contributions in this chapter is a case study the applicability of iterative ISS to a class of nonlinear adaptive ILC. Particularly, we study the possibility to derive the iterative ISS property of adaptive iterative learning control which is proposed by [9]. Denote $k \in \mathbb{N}$ the pass number, following [9] we consider the nonlinear iterative system which has an *integrator chain structure* as follows:

$$\dot{x}_k(t) = Ax_k(t) + B(\theta^T \phi(x_k(t)) + u_k(t) + d'_k(t)), \quad x_k(0) = \delta_k \tag{3.1}$$

The state x_k is taken to be a continuous function i.e $x_k \in C([0,T], \mathbb{R}^n)$; $\theta \in \mathbb{R}^n$ is the vector of an unknown system parameter; the control input function is $u_k : [0,T] \to \mathbb{R}$; the nonlinear dynamic is $\phi \in C(\mathbb{R}^n, \mathbb{R}^n)$; the disturbance $d'_k : [0,T] \to \mathbb{R}$ is taken to be a measurable function and matrix $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times 1}$ are as follow:

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

We take the initial condition error $\{\delta_k\}_{k\geqslant 1} \in l_1$ i.e. we assume the influence of initialisation error decreases as the iteration increases. As it has been proved in [9] that this kind of initialisation error does not destabilise the nonlinear adaptive ILC if the external disturbance d'_k is not present. *Throughout this section, we write $x_k(t)$ as x_k, $\hat{\theta}_k(t)$ as $\hat{\theta}_k$ and $d'_k(t)$ as d'_k, unless in the situations where the argument of t needs to be explicitly written to clarify the meaning.*

Consider the following performance cost:

$$\mathcal{P}_k = \| x_k \|^2_{L^2[0,T]} \tag{3.2}$$

The objective of the nonlinear adaptive ILC is to make $\lim_{k\to\infty} \mathcal{P}_k = 0$ while keeping the $\|u_k\|_{L^\infty[0,T]}$ to be bounded. Following [9] the following adaptive iterative learning control is used:

- let $a \in \mathbb{R}^n$ be such that $A_* = A - Ba^T$ is Hurwitz

- let $Q \in \mathbb{R}^{n\times n}$ and $\tilde{P} \in \mathbb{R}^{n\times n}$ be the real positive definite, symmetric matrix, respectively, satisfying the following Lyapunov equation:

$$A_*^T \tilde{P} + \tilde{P} A_* = -Q$$

- let

$$b = (\tilde{P}^T + \tilde{P})B$$

- let the adaptation gain be $\alpha \in \mathbb{R}_{>0}$.
- let $\hat{\theta}_k \in C([0,T], \mathbb{R}^n)$ denote the parameter estimate.

Consider the following nonlinear adaptive ILC design/ILC input law as follows:

$$\begin{aligned} u_k &= -\hat{\theta}_k^T(t)\phi(x_k(t)) - a^T x_k(t) \\ \dot{\hat{\theta}}_k(t) &= \alpha x_k^T(t) b \phi(x_k(t)),\ \hat{\theta}_1(0) = 0,\ \hat{\theta}_k(0) = \hat{\theta}_{k-1}(T) \end{aligned} \tag{3.3}$$

Applying the control law (3.3) to the system (3.1) we get the closed-loop system:

$$\dot{x}_k(t) = A_* x_k(t) + B((\theta - \hat{\theta}_k(t))^T \phi(x_k(t)) + d_k'(t)), \quad x_k(0) = \delta_k \tag{3.4}$$

$$\dot{\hat{\theta}}_k(t) = \alpha x_k^T(t) b \phi(x_k(t)),\ \hat{\theta}_1(0) = 0,\ \hat{\theta}_k(0) = \hat{\theta}_{k-1}(T) \tag{3.5}$$

To guarantee the adaptive ILC fulfill its objective we require the disturbance decrease as the iteration increases. Hence, the following set is taken to be the admissible set of disturbance d_k':

$$\mathcal{D}_k = \left\{ d \in \mathcal{M} \ | \ |d(t)| \leqslant \frac{\lambda(Q)}{|b|}\sqrt{\frac{x_k^T(t)x_k(t)}{2}}, \forall t \in [0,T] \right\} \tag{3.6}$$

Similarly we also require the initial condition $\{\delta_k\}_{k\geqslant 1} \in l^1$, to be monotonically decreasing. The following set is the admissible set of initial condition sequence $\{\delta_k\}_{k\geqslant 1}$:

$$\mathcal{I} = \left\{ \{\delta_k\}_{k\geqslant 1} \in l^1 \ | \ |\delta_{k+1}| \leqslant |\delta_k|, k \geqslant 2, \forall k \in \mathbb{N} \right\} \tag{3.7}$$

If the disturbance and initial condition error decrease over the pass, we can expect the convergence can be achieved. The reason is the disturbance and initial condition will deter the ILC system to converge. If the effect of these disturbances decrease then it will not prevent the system to converge. Hence, it is sufficient to have the disturbance to decay if we want to get the convergent learning.

We formulate the iterative ISS of the closed-loop system (3.4) as follows: the closed-loop system (3.4)-(3.5) has iterative ISS property if there exists a class $\bar{\mathcal{K}}\mathcal{L}$ function β, class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) functions γ_1 and γ_2 such that we have the following hold:

$$\| x_k \|_{L^2[0,T]} \leqslant \beta(\| x_1 \|_{L^2[0,T]}, k) + \gamma_1\left(\max_{1\leqslant i\leqslant k} \| d_i' \|_{L^2[0,T]}\right) + \gamma_2\left(\|\delta_k\|_{l^2[1,k]}\right) \tag{3.8}$$

for each $x_1 \in C([0,T], \mathbb{R}^n)$, for each $d_k \in \mathcal{D}_k$, for each $\{\delta_k\}_{k\geqslant 1} \in \mathcal{I}$ and for each $k \in \mathbb{N}$.

Theorem 3.0.1. *Consider the closed-loop system (3.4)-(3.5). Then there exists $\Gamma : \mathbb{R}_{\geqslant 0} \times \mathbb{N} \to \mathbb{R}_{\geqslant 0}$, γ_1 and γ_2 both of them are of class $\mathcal{K}/\mathcal{K}^\infty$ functions, such that the following estimate holds:*

$$\|x_k\|_{L^2[0,T]} \leqslant \Gamma(\|x_1\|_{L^2[0,T]}, k) + \gamma_1\left(\max_{1\leqslant i\leqslant k} \|d_i'\|_{L^2[0,T]}\right) + \gamma_2\left(\|\delta_k\|_{l^2[1,k]}\right) \tag{3.9}$$

with $\Gamma(\|x_1\|_{L^2[0,T]}, k) \to 0$ as $k \to \infty$ and $\Gamma(0,k) = 0$, for each $k \in \mathbb{N}$.

To facilitate the proof of theorem 3.0.1, we consider the following positive definite function V which is taken from [9]:

$$V(x_k, \pi_k) = f(x_k) + g(\pi_k) \tag{3.10}$$

where

$$f(x_k) \equiv x_k^T \tilde{P} x_k \tag{3.11}$$
$$g(\pi_k) \equiv \frac{1}{2\alpha}\pi_k^T \pi_k = \frac{1}{2\alpha}(\theta - \hat{\theta}_k)^T(\theta - \hat{\theta}_k) \tag{3.12}$$

with $\pi_k \equiv (\theta - \hat{\theta}_k)$.

The idea of the proof is similar with [9]. To establish the proof the theorem 3.0.1, we do the following steps: first we prove that V decreases along the pass (lemma 3.0.1), and then using this lemma we prove x_k and $\theta - \hat{\theta}_k$ are uniformly bounded (lemma 3.0.2) and $\mathcal{P}_k$ converges to zero (lemma 3.0.3). Using lemma 3.0.2 we obtain the uniform boundedness of $\dot{x}_k$ (lemma 3.0.4). By using the boundedness of $\dot{x}_k$ we show that x_k is uniformly continuous along the interval $[0, T]$ and in every pass (lemma 3.0.5). Using the uniform continuity of x_k and the convergence of $\mathcal{P}_k$ we prove that x_k uniformly converges to zero (lemma 3.0.6). Using the uniform convergence of x_k, the convergence of $|g(\pi_k(T)) - g(\pi_{k-1}(T))|$ is established (lemma 3.0.7). Finally using lemma 3.0.7 we prove the theorem 3.0.1.

Lemma 3.0.1. *Consider the closed-loop system (3.4)-(3.5). Suppose $d'_k \in \mathcal{D}_k$ and $\{\delta_k\}_{k\geqslant 1} \in \mathcal{I}$. Consider the positive definite function V as in equation (3.10). Then $V(x_k(t), \pi_k(t))$ decreases along the interval $[0, T]$, $\forall k \in \mathbb{N}$.*

Proof.

$$\begin{aligned}\dot{V}(x_k(t), \pi_k(t)) &= \dot{V}(x_k, \theta - \hat{\theta}_k)\\ &= \left(Ax_k + B\left(\phi(x_k) + u_k + d'_k\right)\right)^T \tilde{P} x_k + x_k^T \tilde{P}\left(Ax_k + B\left(\phi(x_k) + u_k + d'_k\right)\right)\\ &\quad -\frac{1}{2\alpha}\dot{\hat{\theta}}(\theta - \hat{\theta}) - \frac{1}{2\alpha}(\theta - \hat{\theta})^T\dot{\hat{\theta}}\end{aligned}$$

Substituting the control law (3.3) and adaptation law (3.5) and rearranging we have:

$$\dot{V}(x_k, \theta - \hat{\theta}_k) = x_k^T\left(A_*^T\tilde{P} + \tilde{P}A_*\right)x_k + 2x_k^T\tilde{P}B\left((\theta - \hat{\theta}_k)^T\phi(x_k)\right) + 2x_k^T\tilde{P}Bd'_k - (x_k^T\phi(x_k)b)^T(\theta - \hat{\theta}_k)$$

Note that $b = (\tilde{P}^T + \tilde{P})B = 2\tilde{P}B$, since $\tilde{P}$ is a symmetric matrix. Using this we have:

$$\dot{V}(x_k, \theta - \hat{\theta}_k) = x_k^T\left(A_*^T\tilde{P} + \tilde{P}A_*\right)x_k + x_k^T b\left((\theta - \hat{\theta}_k)^T\phi(x_k)\right) + x_k^T bd'_k - (x_k^T\phi(x_k)b)^T(\theta - \hat{\theta}_k)$$

Since the second and fourth terms are canceled and also $A_*^T\tilde{P} + \tilde{P}A_* = -Q$ then the above equation is simplified to:

$$\dot{V}(x_k, \pi_k) = -x_k^T Q x_k + x_k^T bd'_k \tag{3.13}$$

Since Q is a real positive definite symmetric matrix then $\underline{\lambda}(Q) > 0$. Note that the following fact is used: $x_k^T Q x_k \geqslant \underline{\lambda}(Q)x_k^T x_k$, for every real positive definite symmetric matrix Q. Hence:

$$\begin{aligned}\dot{V}(x_k, \pi_k) &= -x_k^T Q x_k + x_k^T bd'_k\\ &\leqslant -\underline{\lambda}(Q)x_k^T x_k + x_k^T bd'_k\\ &\leqslant -\frac{1}{2}\underline{\lambda}(Q)x_k^T x_k - \frac{1}{2}\underline{\lambda}(Q)x_k^T x_k + x_k^T bd'_k\\ &= -\frac{1}{2}\underline{\lambda}(Q)x_k^T x_k + \frac{1}{2}\underline{\lambda}(Q)\left(-x_k^T x_k + x_k^T b\frac{d'_k}{\frac{1}{2}\underline{\lambda}(Q)}\right)\end{aligned}$$

Using Young's inequality: $c^T e - c^T c \leqslant \frac{1}{4}e^T e$, $\forall c \in \mathbb{R}^n$, $\forall e \in \mathbb{R}^n$ and make the identification $c = x_k$ and $e = \frac{bd'_k}{\frac{1}{2}\underline{\lambda}(Q)}$, the above inequality now becomes:

$$\dot{V}(x_k, \pi_k) \leqslant -\frac{1}{2}\underline{\lambda}(Q)x_k^T x_k + \frac{|b|^2|d'_k|^2}{2\underline{\lambda}(Q)} \tag{3.14}$$

From the assumption, we obtain:

$$\begin{aligned}\dot{V}(x_k, \pi_k) &\leqslant -\frac{1}{2}\underline{\lambda}(Q)x_k^T x_k + \frac{|b|^2|d'_k|^2}{2\underline{\lambda}(Q)} \\ &\leqslant -\frac{1}{2}\underline{\lambda}(Q)x_k^T x_k + \frac{1}{4}\underline{\lambda}(Q)x_k^T x_k \\ &\leqslant -\frac{1}{4}\underline{\lambda}(Q)x_k^T x_k\end{aligned}$$

which completes the proof. □

Lemma 3.0.2. *Consider the closed-loop system (3.4)-(3.5). Suppose $d'_k \in \mathcal{D}_k$ and $\{\delta_k\}_{k\geqslant 1} \in \mathcal{I}$. Then there exists M_1 and M_2 with $0 < M_1 < \infty$, $0 < M_2 < \infty$ such that for each $k \in \mathbb{N}$:*

$$\begin{aligned}\| x_k \|_{L^\infty[0,T]} &\leqslant M_1 \\ \| \theta - \hat{\theta}_k \|_{L^\infty[0,T]} &\leqslant M_2\end{aligned}$$

Proof. Consider the closed-loop system (3.4)-(3.5). First, we prove that $\| x_k \|_{L^\infty[0,T]} \leqslant M_1$, for each $k \in \mathbb{N}$. Consider V as in the equation (3.10). Choose:

$$M_1 = \sqrt{\frac{1 + V(x_1(0), \pi_1(0)) + \sum_{k=1}^{\infty} x_k^T(0)x_k(0)}{\underline{\lambda}(\tilde{P})}}$$

By assumption $\{x_k(0)\}_{k\geqslant 1} \in l_1$ we have: $0 < M_1 < \infty$.

$$\begin{aligned}M_1^2 &= \frac{1 + V(x_1(0), \pi_1(0)) + \sum_{k=1}^{\infty} x_k^T(0)x_k(0)}{\underline{\lambda}(\tilde{P})} \\ \underline{\lambda}(\tilde{P})M_1^2 &= 1 + V(x_1(0), \pi_1(0)) + \sum_{k=1}^{\infty} x_k^T(0)x_k(0)\end{aligned}$$

Let k be arbitrary in $\mathbb{N}$ and t be arbitrary in $[0, T]$. Since $d'_k \in \mathcal{D}_k$, then by lemma 3.0.1, V decreases. Hence according to the proof of proposition 2 in [9], we have:

$$\begin{aligned}V(x_k(t), \pi_k(t)) &\leqslant 1 + V(x_1(0), \pi_1(0)) + \sum_{j=1}^{k} x_j^T(0)x_j(0), \\ &\leqslant 1 + V(x_1(0), \pi_1(0)) + \sum_{k=1}^{\infty} x_k^T(0)x_k(0)\end{aligned} \tag{3.15}$$

Hence,

$$\underline{\lambda}(\tilde{P})M_1^2 \geqslant V(x_k(t), \pi_k(t))$$

Using the definition of V:

$$x_k^T \tilde{P} x_k \leqslant x_k^T \tilde{P} x_k + \frac{1}{2\alpha}(\theta - \hat{\theta}_k)^T(\theta - \hat{\theta}_k) = V(x_k, \pi_k) \leqslant M_1^2 \underline{\lambda}(\tilde{P})$$

Note that $\forall t \in [0, T]$, $\forall k \in \mathbb{N}$,

$$\underline{\lambda}(\tilde{P})x_k^T x_k \leqslant x_k^T \tilde{P} x_k$$

Since $k \in \mathbb{N}$ and $t \in [0,T]$ are chosen arbitrary, hence for each $k \in \mathbb{N}$ and for each $t \in [0,T]$ we have:

$$
\begin{aligned}
\underline{\lambda}(\tilde{P}) x_k^T x_k &\leqslant M_1^2 \underline{\lambda}(\tilde{P}) \\
\sqrt{x_k^T x_k} &\leqslant M_1 \\
\| x_k \|_{L^\infty} &\leqslant M_1
\end{aligned}
$$

Now we prove that $\| \theta - \hat{\theta}_k \|_{L^\infty} \leqslant M_2, \forall k \in \mathbb{N}$. Let k be arbitrary on $\mathbb{N}$, and t be arbitrary on $[0,T]$. Choose:

$$
M_2 = \sqrt{2\alpha \left(1 + V(x_1(0), \pi_1(0)) + \sum_{k=1}^{\infty} x_k^T(0) x_k(0) \right)}
$$

Clearly $0 < M_2 < \infty$. We have,

$$
\begin{aligned}
M_2^2 &= 2\alpha \left(1 + V(x_1(0), \pi_1(0)) + \sum_{k=1}^{\infty} x_k^T(0) x_k(0) \right) \\
\frac{M_2^2}{2\alpha} &= 1 + V(x_1(0), \pi_1(0)) + \sum_{k=1}^{\infty} x_k^T(0) x_k(0)
\end{aligned}
$$

Using inequality (3.15), the above equation becomes:

$$
\frac{M_2^2}{2\alpha} \geqslant V(x_k(t), \pi_k(t))
$$

Using the definition of V we obtain:

$$
\begin{aligned}
\frac{1}{2\alpha} (\theta - \hat{\theta}_k)^T (\theta - \hat{\theta}_k) &\leqslant x_k^T \tilde{P} x_k + \frac{1}{2\alpha} (\theta - \hat{\theta}_k)^T (\theta - \hat{\theta}_k) \\
&= V(x_k, \pi_k) \leqslant \frac{M_2^2}{2\alpha}
\end{aligned}
$$

Since $k \in \mathbb{N}$ and $t \in [0,T]$ are arbitrary, then for each $k \in \mathbb{N}$ and for each $t \in [0,T]$, we get:

$$
\begin{aligned}
\sqrt{(\theta - \hat{\theta}_k(t))^T (\theta - \hat{\theta}_k(t))} &\leqslant M_2, \forall t \in [0,T] \\
\| \theta - \hat{\theta}_k \|_{L^\infty} &\leqslant M_2
\end{aligned}
$$

□

Lemma 3.0.3. *Consider the closed-loop system (3.4)-(3.5). Let $d_k' \in \mathcal{D}_k$ and $\{\delta_k\}_{k \geqslant 1} \in \mathcal{I}$. Then:*

$$
\mathcal{P}_k \to 0 \quad as \quad k \to \infty \tag{3.16}
$$

Proof. Consider the closed-loop system (3.4)-(3.5). Consider V as in the equation (3.10). Since $d_k \in \mathcal{D}_k$, it follows from lemma 3.0.1 that:

$$
\dot{V}(x_k, \pi_k) \leqslant -\frac{1}{4} \underline{\lambda}(Q) x_k^T x_k, \quad \forall k \in \mathbb{N}, \quad \forall t \in [0,T] \tag{3.17}
$$

Rearranging we have:

$$
x_k^T x_k \leqslant \left(\frac{4}{\underline{\lambda}(Q)} \right) \left(-\dot{V}(x_k, \pi_k) \right) \tag{3.18}
$$

Take the integral from 0 to T and then followed by the summation from 1 to K, $K \geqslant 1$, $K \in \mathbb{N}$ we obtain:

$$
\begin{aligned}
\sum_{k=1}^{K} \int_0^T x_k^T x_k dt &\leqslant \frac{4}{\underline{\lambda}(Q)} \left(\sum_{k=1}^{K} \int_0^T -\dot{V}(x_k, \pi_k) dt \right) \\
&= \frac{4}{\underline{\lambda}(Q)} \sum_{k=1}^{K} \left(V(x_k(0), \pi_k(0)) - V(x_k(T), \pi_k(T)) \right)
\end{aligned} \tag{3.19}
$$

Consider the the term $\sum_{k=1}^{K}(V(x_k(0),\pi_k(0)) - V(x_k(T),\pi_k(T)))$. Using

$$V(x_k(0),\pi_k(0)) = f(x_k(0)) + g(\pi_k(0)) = f(x_k(0)) + g(\pi_{k-1}(T))$$

the following holds:

$$\sum_{k=1}^{K}(V(x_k(0),\pi_k(0)) - V(x_k(T),\pi_k(T)) = g(\pi_1(0)) + \sum_{k=1}^{K} f(x_k(0)) \tag{3.20}$$

Let $K \to \infty$ and use the assumption that $\{x_k(0)\}_{k\geqslant 1} \in l_1$, we have the second term in RHS is finite. Define $\mathcal{W} = \sum_{k=1}^{\infty} f(x_k(0))$. We have:

$$\sum_{k=1}^{\infty}(V(x_k(0),\pi_k(0)) - V(x_k(T),\pi_k(T)) = g(\pi_1(0)) + \mathcal{W}$$

The inequality (3.19) becomes:

$$\sum_{k=1}^{\infty}\int_0^T x_k^T x_k dt \leqslant \frac{4}{\underline{\lambda}(Q)}\sum_{k=1}^{\infty}(V(x_k(0),\pi_k(0)) - V(x_k(T),\pi_k(T)) = \frac{4}{\underline{\lambda}(Q)}(g(\pi_1(0)) + \mathcal{W}) \tag{3.21}$$

Now, from the definition of $\mathcal{P}_k$ and (3.21), we have:

$$\begin{aligned}\sum_{k=1}^{\infty}\mathcal{P}_k &= \sum_{k=1}^{\infty}\int_0^T x_k^T x_k dt \\ &\leqslant \frac{4}{\underline{\lambda}(Q)}(V(x_k(0),\pi_k(0)) - V(x_k(T),\pi_k(T))) \\ &< \infty\end{aligned}$$

with $g(\pi_1(0))$, $\mathcal{W}$ and $\underline{\lambda}(Q)$ are positive constants.

Since $\mathcal{P}_k \geqslant 0$, $\forall k \in \mathbb{N}$, thus $\mathcal{P}_k \to 0$ as $k \to \infty$. This completes the proof. □

The following lemma is established using the first two lemmas.

Lemma 3.0.4. *Consider the closed-loop system (3.4)-(3.5). Suppose $d_k' \in \mathcal{D}_k$ and $\{\delta_k\}_{k\geqslant 1} \in \mathcal{I}$. Then there exists M, $0 < M < \infty$, such that for each $k \in \mathbb{N}$, $\| \dot{x}_k \|_{L^\infty[0,T]} \leqslant M$.*

Proof. Since $d_k' \in \mathcal{D}_k$, by lemma 3.0.2, $\exists M_1$ such that $\|x_k\|_{L^\infty[0,T]} \leqslant M_1$ for each $k \in \mathbb{N}$, hence there exists $0 < \mathcal{W} < \infty$ such that by (3.6):

$$|d_k'| \leqslant \mathcal{W}, \quad \forall k \in \mathbb{N}, \quad \forall t \in [0,T]$$

From lemma 3.0.2, there exists $M_2 > 0$ such that $|\theta - \hat{\theta}_k| \leqslant M_2$. Consider the nonlinearity $\phi(\cdot)$. Since $\phi \in C(\mathbb{R}^n, \mathbb{R}^n)$ and $|x_k| \leqslant M_1$ then there exists M_3, $0 \leqslant M_3 < \infty$, such that:

$$|\phi(x_k)| \leqslant \sup_{|x_k|\leqslant M_1} |\phi(x_k)| \leqslant M_3 \tag{3.22}$$

Now choose:

$$M = |A_*|M_1 + |B|M_2M_3 + |B|\mathcal{W} + 1 \tag{3.23}$$

Clearly $0 < M < \infty$. Consider the equation (3.4). Taking the modulus on both sides, applying triangle and Cauchy-Schwarz inequality we get:

$$|\dot{x}_k| \leqslant |A_*||x_k| + |B||\theta - \hat{\theta}_k||\phi(x_k)| + |B||d_k'| \tag{3.24}$$

Hence, by (3.23), we obtain the following:

$$\begin{aligned} |\dot{x}_k| &\leqslant M < \infty \\ \| \dot{x}_k \|_{L^\infty} &\leqslant M \end{aligned}$$

Since k and t were arbitrary then the above inequality hold for each $k \in \mathbb{N}$ and $t \in [0, T]$, which completes the proof. □

The next lemma provides the uniform continuity along the pass and in the pass axis.

Lemma 3.0.5. *Consider the closed-loop system (3.4)-(3.5). Suppose* $d'_k \in \mathcal{D}_k$ *and* $\{\delta_k\}_{k \geqslant 1} \in \mathcal{I}$. *Then for each* $\epsilon > 0$, *there exists* $\delta > 0$ *such that for each* $k \in \mathbb{N}$, *for each* $t_1, t_2 \in [0, T]$:

$$|t_2 - t_1| \leqslant \delta \Rightarrow |x_k(t_2) - x_k(t_1)| \leqslant \epsilon \tag{3.25}$$

Proof. Let ϵ be arbitrary positive number, we need to choose δ_ϵ so that $\forall k \in \mathbb{N}$, $|x_k(t_2) - x_k(t_1)| \leqslant \epsilon$, $\forall t_1, t_2 \in [0, T]$.

Following lemma 3.0.4, there is an M, with $0 < M < \infty$ such that $\| \dot{x}_k \|_{L^\infty} \leqslant M$. Hence:

$$\| \dot{x}_k \|_{L^\infty} |t_2 - t_1| \leqslant M |t_2 - t_1|, \quad \forall k \in \mathbb{N}, \quad \forall t_1, t_2 \in [0, T] \tag{3.26}$$

Let:

$$\delta_\epsilon = \frac{\epsilon}{M}$$

Clearly $\delta_\epsilon > 0$ since $\epsilon > 0$ and $M > 0$. Hence for each $t_1, t_2 \in [0, T]$:

$$|t_2 - t_1| \leqslant \frac{\epsilon}{M}$$

Using (3.26), we obtain:

$$\| \dot{x}_k \|_{L^\infty} |t_2 - t_1| \leqslant M |t_2 - t_1| \leqslant \epsilon, \quad \forall k \in \mathbb{N} \quad \forall t_1, t_2 \in [0, T] \tag{3.27}$$

Let k be arbitrary in $\mathbb{N}$ and t_1 and t_2 be arbitrary in $[0, T]$. Following the mean value theorem ([1], [20]), for each $a \in \mathbb{R}^n$, $\exists c$ with $t_1 < c < t_2$ such that :

$$a^T \{x_k(t_2) - x_k(t_1)\} = a^T \{\dot{x}_k(c)(t_2 - t_1)\} \tag{3.28}$$

(c depends on both a and k) Consider the left hand side of (3.28). Take the case $x_k(t_2) = x_k(t_1)$. In this case, LHS is zero which makes the RHS is also zero. Hence, the inequality (3.26) is trivially satisfied. So let assume $x_k(t_2) - x_k(t_1) \neq 0$. Let:

$$a = \frac{x_k(t_2) - x_k(t_1)}{|x_k(t_2) - x_k(t_1)|} \tag{3.29}$$

Clearly $|a| = 1$. By this choice of a, the equation (3.28) becomes:

$$|x_k(t_2) - x_k(t_1)| = \frac{x_k(t_2) - x_k(t_1)}{|x_k(t_2) - x_k(t_1)|} \{\dot{x}_k(c)(t_2 - t_1)\}, \quad \forall t_1, t_2 \in [0, T] \tag{3.30}$$

and also with the help of Cauchy-Schwarz inequality, we obtain, $\forall k \in \mathbb{N}$, $\forall t_1, t_2 \in [0, T]$:

$$\begin{aligned} |x_k(t_2) - x_k(t_1)| &\leqslant \left| \frac{x_k(t_2) - x_k(t_1)}{|x_k(t_2) - x_k(t_1)|} \right| |\dot{x}_k(c)| |t_2 - t_1| \\ &\leqslant |\dot{x}_k(c)| |t_2 - t_1| \end{aligned} \tag{3.31}$$

Consider RHS of (3.31). Since the point c is always in $[0,T]$ then $|\dot{x}_k(c)| \leqslant \| \dot{x}_k \|_{L^\infty}$. Using (3.26) we have:

$$|\dot{x}_k(c)||t_2 - t_1| \leqslant \| \dot{x}_k \|_{L^\infty} |t_2 - t_1| \leqslant \epsilon \tag{3.32}$$

Since k, t_1, and t_2 were arbitrary then the above inequality holds for each $k \in \mathbb{N}$ and for each $t_1, t_2 \in [0,T]$. Hence, using (3.32) and inequality (3.31) becomes:

$$|x_k(t_2) - x_k(t_1)| \leqslant \epsilon, \quad \forall k \in \mathbb{N}, \quad \forall t_1, t_2 \in [0,T]$$

which completes the proof. □

The next lemma establishes the uniform convergence if the iteration increases as a consequence of the property developed in the previous lemmas.

Lemma 3.0.6. *Consider the closed-loop system (3.4)-(3.5). Suppose $d'_k \in \mathcal{D}_k$ and $\{\delta_k\}_{k\geqslant 1} \in \mathcal{I}$. Then $|x_k| \to 0$ as $k \to \infty$ uniformly on $[0,T]$.*

Proof. Consider the closed-loop system (3.4)-(3.5). From lemma 3.0.3 we have $\mathcal{P}_k \to 0$, as $k \to \infty$. We will show that this implies $|x_k| \to 0$, as $k \to \infty$ uniformly on the closed interval $[0,T]$. We prove this lemma by contradiction, that is suppose: $|x_k| \nrightarrow 0$ as $k \to \infty$ then $\mathcal{P}_k \nrightarrow 0$, as $k \to \infty$ uniformly under the given assumption.

Since $|x_k| \nrightarrow 0$ uniformly, we have there exists $\epsilon > 0$, such that for each $\bar{K} > 0$, there exists $k > \bar{K}$ and there exists $\tilde{t}_k \in [0,T]$ with:

$$|x_k(\tilde{t}_k)| \geqslant \epsilon \tag{3.33}$$

We will construct an interval such that $|x_k(\cdot)| > 0$ along that interval, at k where (3.33) holds.

From lemma 3.0.5 it follows that, given $\tilde{\epsilon} > 0$, there exists $\delta > 0$, such that for each $k \in \mathbb{N}$, for each $\bar{t} \in [0,T]$, for each $t \in [0,T]$, for $|t - \bar{t}| \leqslant \delta$ we have:

$$\big|\ |x_k(t)| - |x_k(\bar{t})|\ \big| \leqslant |x_k(t) - x_k(\bar{t})| \leqslant \tilde{\epsilon} \tag{3.34}$$

Choose $\tilde{\epsilon} = \frac{1}{4}\epsilon$. Choose k to be arbitrary in $\mathbb{N}$, t and $\bar{t}$ to be arbitrary in $[0,T]$ then for $|t - \bar{t}| \leqslant \delta_\epsilon$ we have:

$$\begin{aligned} \big|\ |x_k(t)| - |x_k(\bar{t})|\ \big| &\leqslant \frac{1}{4}\epsilon \\ -\frac{1}{4}\epsilon + |x_k(\bar{t})| \leqslant |x_k(t)| &\leqslant |x_k(\bar{t})| + \frac{1}{4}\epsilon \end{aligned} \tag{3.35}$$

Note that since:

$$\begin{aligned} |t - \bar{t}| &\leqslant \delta_\epsilon \\ \Leftrightarrow \bar{t} - \delta_\epsilon \leqslant t &\leqslant \bar{t} + \delta_\epsilon \end{aligned}$$

hence (3.35) holds if:

$$t \in [0,T] \cap [\bar{t} - \delta_\epsilon, \bar{t} + \delta_\epsilon]$$

Since k was arbitrary in $\mathbb{N}$ and $\bar{t}$ was arbitrary in $[0,T]$, we can consider a specific k and t where a certain property holds. Now consider at $k > \bar{K}$ and let $\bar{t} = \tilde{t}_k \in [0,T]$ so that (3.33) holds. Then we have if $t \in \left[\tilde{t}_k - \delta_\epsilon, \tilde{t}_k + \delta_\epsilon\right] \cap [0,T]$, inequality (3.35) becomes:

$$-\frac{1}{4}\epsilon + |x_k(\tilde{t}_k)| \leqslant |x_k(t)| \leqslant \frac{1}{4}\epsilon + |x_k(\tilde{t}_k)|$$

Using (3.33), we get :

$$|x_k(t)| \geqslant -\frac{1}{4}\epsilon + \epsilon = \frac{3}{4}\epsilon, \tag{3.36}$$

Define $[a, b] \equiv [\tilde{t}_k - \delta_\epsilon, \tilde{t}_k + \delta_\epsilon] \cap [0, T]$ with:

$$a = \max\{0, \tilde{t}_k - \delta_\epsilon\} \tag{3.37}$$

$$b = \min\{T, \tilde{t}_k + \delta_\epsilon\} \tag{3.38}$$

We will show that $b - a$ is always greater than zero. From equation (3.37) and (3.38):

$$b - a = \min\{T, \tilde{t}_k + \delta_\epsilon\} - \max\{0, \tilde{t}_k - \delta_\epsilon\} \tag{3.39}$$

If $\max\{0, \tilde{t}_k - \delta_\epsilon\} = 0$ then:

$$b - a = \min\{T, \tilde{t}_k + \delta_\epsilon\} > 0$$

since $\delta_\epsilon > 0$ and $T > 0$.

Suppose $\max\{0, \tilde{t}_k - \delta_\epsilon\} = \tilde{t}_k - \delta_\epsilon$ then we have, either:

$$b - a = \tilde{t}_k + \delta_\epsilon - (\tilde{t}_k - \delta_\epsilon) = 2\delta_\epsilon > 0$$

or

$$b - a = T - \tilde{t}_k + \delta_\epsilon > 0$$

since $0 \leqslant \tilde{t}_k \leqslant T$ and $\delta_\epsilon > 0$. Hence $b - a$ is always greater than zero.

Now consider inequality (3.36). We can conclude:

$$\exists \epsilon > 0, \text{ such that } \forall \bar{K} > 0, \exists k > \bar{K}, \text{ with } |x_k(t)| \geqslant \frac{3}{4}\epsilon \tag{3.40}$$

since t was chosen arbitrary in $[0, T]$, then the above inequality holds for each $t \in [a, b]$. It means $\exists \epsilon > 0$, such that $\forall \bar{K} > 0$, $\exists k > \bar{K}$ with:

$$\int_a^b |x_k(t)|^2 dt \geqslant \int_a^b \left(\frac{3}{4}\epsilon\right)^2 dt = \frac{9}{16}\epsilon^2(b - a) > 0$$

Since $\mathcal{P}_k = \| x_k \|^2_{L^2[0,T]} \geqslant \int_a^b |x_k(t)|^2 > 0$, $\forall k \in \mathbb{N}$ we can conclude that: $\exists \epsilon > 0$, such that $\forall \bar{K} > 0$, $\exists k > \bar{K}$ with $\mathcal{P}_k > 0$. Thus, $\mathcal{P}_k \nrightarrow 0$ as $k \to \infty$. In other word, if $\mathcal{P}_k \to 0$, as $k \to \infty$ then $|x_k(t)| \to 0$ uniformly on $[0, T]$ as $k \to \infty$. This complete the proof. □

The next lemma is our main goal. It is developed using the conclusion from the previous lemma.

Lemma 3.0.7. *Consider the closed-loop system (3.4)-(3.5). Let $d_k' \in \mathcal{D}_k$ and $\{\delta_k\}_{k \geqslant 1} \in \mathcal{I}$. Then:*

$$\lim_{k \to \infty} |g(\pi_k(T)) - g(\pi_{k-1}(T))| = 0 \tag{3.41}$$

Proof. Consider the closed-loop system (3.4)-(3.5). According to the lemma 3.0.6 then:

$$|x_k(t)| \to 0, k \to \infty$$

uniformly on $[0, T]$. We will prove that $|x_k(t)| \to 0$, $k \to \infty$ uniformly on $[0, T]$ implies $|g(\pi_k(T)) - g(\pi_{k-1}(T))| \to 0$ as $k \to \infty$.

From the definition of uniformly convergence [1] we have:

$$\forall \epsilon > 0, \exists \bar{K} > 0, \text{such that} \forall k > \bar{K}, |x_k(t)| < \epsilon, \text{ for each } t \in [0, T] \tag{3.42}$$

Let t be arbitrary in $[0,T]$ and let k be such that (3.42) holds. From the proof of lemma 3.0.4, we know that there exists $\bar{C}$, $0 < \bar{C} < \infty$, $|\phi(x_k(t))| \leqslant \bar{C}$ (see inequality (3.22)). Let α and b be as in the parameter estimate equation (3.5). Then the following inequality holds:

$$|\alpha| \; |b|\bar{C}|x_k(t)| \geqslant |\dot{\hat{\theta}}_k(t)| \tag{3.43}$$

By lemma 3.0.2 there exists M_2, $0 < M_2 < \infty$ such that for each $k \in \mathbb{N}$, for each $t \in [0,T]$, $|\theta - \hat{\theta}_k(t)| \leqslant M_2$. Hence, multiply both sides of (3.43) with M_2T, we get:

$$\begin{aligned} M_2T|\alpha||b|\bar{C}|x_k(t)| &\geqslant M_2T|\dot{\hat{\theta}}_k(t)| \\ &\geqslant |\theta - \hat{\theta}_k(t)||\dot{\hat{\theta}}_k(t)|T \\ &= |\hat{\theta}_k(t) - \theta||\dot{\hat{\theta}}_k(t)|T \\ &\geqslant |(\hat{\theta}_k(t) - \theta)^T\dot{\hat{\theta}}_k(t)|T \end{aligned} \tag{3.44}$$

Since $\left|(\theta - \hat{\theta}_k(t))^T\dot{\hat{\theta}}_k(t)\right|$ is continuous and it is defined on compact interval $[0,T]$ then there exists a global maximum on $t \in [0,T]$. Since t were arbitrary then we can choose t on $[0,T]$ such that $\left|(\theta - \hat{\theta}_k(t))^T\dot{\hat{\theta}}_k(t)\right|$ maximum. Hence, the inequality (3.44) becomes:

$$M_2T|\alpha||b|\bar{C}|x_k(t)| \geqslant \max_{t\in[0,T]} \left|(\hat{\theta}_k(t) - \theta)^T\dot{\hat{\theta}}_k(t)\right| T \tag{3.45}$$

Now consider the positive definite function g as defined in (3.12). By chain rule we have:

$$\frac{d}{dt}g(\theta - \hat{\theta}_k(t)) = -\frac{2}{2\alpha}(\theta - \hat{\theta}_k(t))^T\dot{\hat{\theta}}_k(t) = \frac{1}{\alpha}(\hat{\theta}_k(t) - \theta)^T\dot{\hat{\theta}}_k(t) \tag{3.46}$$

Substitute (3.46) to the inequality (3.45), we have:

$$\begin{aligned} M_2T|\alpha||b|\bar{C}|x_k(t)| &\geqslant \max_{t\in[0,T]} \left|\alpha\frac{d}{dt}g(\theta - \hat{\theta}_k(t))\right| T \\ &\geqslant \left|\alpha\int_0^T \frac{d}{dt}g(\theta - \hat{\theta}_k(t))dt\right| \\ &= \alpha\left|\Big(g(\theta - \hat{\theta}_k(T)) - g(\theta - \hat{\theta}_k(0))\Big)\right| \end{aligned} \tag{3.47}$$

By using $\theta_k(0) = \theta_{k-1}(T)$ and (3.42), we get:

$$0 \leqslant \left|g(\theta - \hat{\theta}_k(T)) - g(\theta - \hat{\theta}_{k-1}(T))\right| \leqslant M_2T|b|\bar{C}|x_k(t)| < M_2T|b|\bar{C}\epsilon \tag{3.48}$$

Define: $\tilde{\epsilon} = M_2T|b|\bar{C}\epsilon$. Since ϵ can be any positive number and k is chosen such that (3.42) holds then we can conclude that for each $\tilde{\epsilon} > 0$, there exists $\bar{K} > 0$, such that for each $k > \bar{K}$ we have:

$$\left|g(\theta - \hat{\theta}_k(T)) - g(\theta - \hat{\theta}_{k-1}(T))\right| < \tilde{\epsilon}$$

which means that $\left|g(\theta - \hat{\theta}_k(T)) - g(\theta - \hat{\theta}_{k-1}(T))\right| \to \infty$ as $k \to \infty$, as required. □

Proof of theorem 3.0.1 Consider the closed-loop system (3.4)-(3.5). By lemma 3.0.7 we have that:

$$\lim_{k\to\infty} |g(\pi_k(T)) - g(\pi_{k-1}(T))| = 0$$

Let k be arbitrary in $\mathbb{N}$. Choose $\Lambda(k) = \frac{2}{\sqrt{\underline{\lambda}(Q)}}\sqrt{|g(\pi_k(T)) - g(\pi_{k-1}(T))|}$. Clearly that $\Lambda(k) \geqslant 0$ and $\Lambda(k) \to 0$ as $k \to \infty$ so that Λ is a class $\mathcal{L}$ function. Choose:

$$\gamma_1(s) = \sqrt{s}, \;\; \forall s \in \mathbb{R}_{\geqslant 0} \tag{3.49}$$

And also choose:

$$\gamma_2(s) = \frac{2s}{\sqrt{\underline{\lambda}(Q)}}\sqrt{\bar{\lambda}(\tilde{P})}, \quad \forall s \in \mathbb{R}_{\geqslant 0} \tag{3.50}$$

Observe that, γ_1 and γ_2 are always positive, zero at $s = 0$, and increase as s increases. Therefore, γ_1 and γ_2 are class $\mathcal{K}$ functions. Hence for each $d'_k \in \mathcal{D}_k$, for each $\{\delta_k\}_{k\geqslant 1} \in l_1$, we have:

$$\begin{aligned}
&\Lambda(k) + \gamma_1\left(\max_{1\leqslant i\leqslant k} \| d'_i \|\right) + \gamma_2\left(\|\delta_k\|_{l^2[1,k]}\right) \\
= \quad & \frac{2}{\sqrt{\underline{\lambda}(Q)}}\sqrt{|g(\pi_k(T)) - g(\pi_{k-1}(T))|} + \sqrt{\max_{1\leqslant i\leqslant k} \| d'_i \|} + \frac{2\|x_i(0)\|_{l^2[1,k]}}{\sqrt{\underline{\lambda}(Q)}}\sqrt{\bar{\lambda}(\tilde{P})} \\
\geqslant \quad & \sqrt{\frac{4}{\underline{\lambda}(Q)}|g(\pi_k(T)) - g(\pi_{k-1}(T))| + \max_{1\leqslant i\leqslant k} \| d'_i \|_{L^2[0,T]} + \frac{4}{\underline{\lambda}(Q)}\bar{\lambda}(\tilde{P})\|x_i(0)\|^2_{l^2[1,k]}}
\end{aligned} \tag{3.51}$$

Now consider V as in equation (3.10). It follows from lemma 3.0.1, $\forall t \in [0,T]$:

$$\dot{V}(x_k, \pi_k) \leqslant -\frac{1}{4}\underline{\lambda}(Q)x_k^T x_k \tag{3.52}$$

Rearranging and then take the integral from 0 to T, we have:

$$\begin{aligned}
\int_0^T x_k^T x_k dt &\leqslant \frac{4}{\underline{\lambda}(Q)}\left(-\int_0^T \dot{V}(x_k(t), \pi_k(t))dt\right) \\
\| x_k \|^2_{L^2[0,T]} &\leqslant \frac{4}{\underline{\lambda}(Q)}\left(V(x_k(0), \pi_k(0)) - V(x_k(T), \pi_k(T)\right)
\end{aligned} \tag{3.53}$$

Note that, by using the definition of V as in equation (3.10), we get:

$$\begin{aligned}
V(x_k(0), \pi_k(0)) - V(x_k(T), \pi_k(T)) &= x_k^T(0)\tilde{P}x_k(0) + g(\pi_k(T)) - x_k^T(T)\tilde{P}x_k(T) - g(\pi_{k-1}(T)) \\
&\leqslant \bar{\lambda}(\tilde{P})x_k^T(0)x_k(0) + |g(\pi_k(T)) - g(\pi_{k-1}(T))|
\end{aligned}$$

By using the above inequality, the inequality (3.53) becomes:

$$\begin{aligned}
\| x_k \|^2_{L^2[0,T]} &\leqslant \frac{4}{\underline{\lambda}(Q)}\left(\bar{\lambda}(\tilde{P})x_k^T(0)x_k(0) + |g(\pi_k(T)) - g(\pi_{k-1}(T))|\right), \quad \forall k \in \mathbb{N} \\
\| x_k \|_{L^2[0,T]} &\leqslant \sqrt{\frac{4}{\underline{\lambda}(Q)}\bar{\lambda}(\tilde{P})x_k^T(0)x_k(0) + \frac{4}{\underline{\lambda}(Q)}|g(\pi_k(T)) - g(\pi_{k-1}(T))|}, \quad \forall k \in \mathbb{N}
\end{aligned} \tag{3.54}$$

Using inequality (3.51), inequality (3.54) becomes:

$$\begin{aligned}
\| x_k \|_{L^2[0,T]} &\leqslant \sqrt{\frac{4}{\underline{\lambda}(Q)}|g(\pi_k(T)) - g(\pi_{k-1}(T))| + \max_{1\leqslant i\leqslant k} \| d'_i \|_{L^2[0,T]} + \frac{4\bar{\lambda}(\tilde{P})}{\underline{\lambda}(Q)}\|x_i(0)\|^2_{l^2[1,k]}} \\
&\leqslant \Lambda(k) + \gamma_1\left(\max_{1\leqslant i\leqslant k} \| d'_i \|_{L^2[0,T]}\right) + \gamma_2\left(\|\delta_k\|_{l^2[1,k]}\right)
\end{aligned}$$

To complete the proof, let $x_1 \in C([0,T], \mathbb{R}^n)$ be a response of closed-loop system (3.4)-(3.5) at $k = 1$, then define:

$$\Gamma(\|x_1\|_{L^2[0,T]}, k) = \Lambda(k) + e^{-k}\|x_1\|_{L^2[0,T]} \tag{3.55}$$

Note that since $|g(\pi_k(T)) - g(\pi_{k-1}(T))|$ not only depends on k but also depends on $x_1(0)$ then so does $\Lambda(k)$. We will examine the value of $\Lambda(k)$ when $\|x_1\|_{L^2[0,T]} = 0$. From equation (3.47) we have :

$$|\alpha\left(g(\pi_k(T)) - g(\pi_{k-1}(T))\right)| \leqslant M_2 T|\alpha||b|\bar{C}|x_k(t)|, \quad \forall t \in [0,T] \tag{3.56}$$

which implies:

$$|\alpha\left(g(\pi_k(T)) - g(\pi_{k-1}(T))\right)| \leqslant M_2 T|\alpha||b|\bar{C}|x_k(0)| \leqslant M_2 T|\alpha||b|\bar{C}|x_1(0)| \tag{3.57}$$

By the assumption of $\{\delta_k\}_{k\geqslant 1} \in \mathcal{I}$, it follows that if $||x_1||_{L^2[0,T]} = 0$ and hence $x_1(0) = 0$:

$$|\alpha\left(g(\pi_k(T)) - g(\pi_{k-1}(T))\right)| = 0 \tag{3.58}$$

Hence $\Lambda(k) = 0$ when $||x_1||_{L^2[0,T]} = 0$ which means $\Gamma(0,k) = 0$. Moreover, it follows from (3.57) and (3.58) that $|g(\pi_k(T)) - g(\pi_{k-1}(T))|$ is continuous at $x_1(0) = 0$ which means it is also continuous at $||x_1||_{L^2[0,T]} = 0$.

Thus, we have :

$$||x_k||_{L^2[0,T]} \leqslant \Gamma(||x_1||_{L^2[0,T]}, k) + \gamma_1\left(\max_{1\leqslant i\leqslant k} ||d_i'||_{L^2[0,T]}\right) + \gamma_2\left(||\delta_k||_{l^2[1,k]}\right) \tag{3.59}$$

with $\Gamma(||x_1||_{L^2[0,T]}, k) \to 0$ as $k \to \infty$ and $\Gamma(0,k) = 0$. Since k was taken to be arbitrary in $\mathbb{N}$ then the above inequality holds for each $k \in \mathbb{N}$, which completes the proof. □

From the theorem 3.0.1, we have seen that a class of adaptive nonlinear ILC proposed by [9], in general, may have iterative ISS property although we still cannot prove it up to this point. As we have seen, the function Γ is not a class $\bar{\mathcal{K}}\mathcal{L}$ function. It is only a class $\mathcal{L}$ function. Hence, more restriction is needed so that the property of iterative ISS can be fulfilled.

Although we impose a restriction on $\mathcal{D}_k$ as in (3.6) and $\mathcal{I}$ as in (3.7), we still can apply the theorem 2.3.1 (cascade connection) and theorem 2.4.1 (feedback connection). It is because, those theorems require $d_k' \in \mathcal{D}_k$ and $\delta_k \in l_p$. Since $\mathcal{I} \in l_p$ the condition is already fulfilled.

Chapter 4

Conclusion and Future Works

4.1 Conclusion

In this work, key elements of a theory of nonlinear ILC system interconnection has been developed. We consider two types of interconnection: feedback interconnection and cascade interconnection. An overview of the results obtained is given:

- It has been shown that the idea of input to state stability theory can be used to formulate a robustness property for the ILC system. The formulation of this property involves an estimate in term of class $\bar{\mathcal{K}}\mathcal{L}$ functions and class $\mathcal{K}$ functions. We call this robustness property *iterative ISS*. This property shows that:

  - in an ILC system, the convergent behaviour depends on the behaviour of disturbance and initial condition. The system may diverge if the disturbance and initial condition tend to infinity.
  - without the presence of disturbance and initial condition, the convergence only depends on the bound of the signal in the first pass

  Note that, the initial condition enters the iterative ISS formulation (inequality (3.8)) as a summation starting from the first pass to the current pass. Hence, if the initial condition does not decrease over the iterations, then by this formulation, the ILC system may not converge to the desired behaviour. In many ILC systems, this may not be the case. In fact there are few algorithms allowing the initial condition to be bounded (not necessarily decrease) over iterations, but still can achieve a convergence ([7], [14], [31]).

- We have shown that for any ILC system to possess iterative ISS property, some boundedness and asymptotic properties can be derived. We term these properties bounded disturbance and bounded state (BDBS) and disturbance asymptotic gain (DAG) respectively. By these properties, every iterative ISS ILC system is bounded by the disturbance and initial condition. Moreover, as the iterations continue, the influence of the first pass signal decreases. The only thing that can influence the system is the disturbance and the initial condition. Therefore, it is not surprising that iterative ISS is sufficient to guarantee that BDBS and DAG hold. It has been shown these properties are essential to development of the ILC system interconnection theory.

The assumption that needs to be satisfied is that the initial condition belongs to an l_p-space (it decreases over iterations). It can be achieved, for example if we utilise a special controller to drive the initial condition. Nevertheless, in many practical circumstance, we cannot always have this scheme. For example there are many ILC system which work well (achieve convergence) although the initial condition is only known to be bounded (e.g. [7], [14], [31]). Hence, the assumption that $\delta_k \in l_p$ is a strong assumption for an ILC system to have BDBS and DAG property.

- The theory of ILC system interconnection has been developed based on iterative ISS as follows: it is assumed that the iterative ISS as a desirable property of any ILC system. Hence the theory of ILC system interconnection is to show that whether the ILC system interconnection is iterative ISS if each subsystem is iterative ISS. We have shown that if each subsystem is iterative ISS then the BDBS and DAG properties are achieved in both types of interconnection.

  This result can be considered to be a very significant achievement towards iterative ISS property. The most desirable result is to have the ILC system interconnection to be iterative ISS if given each subsystem is iterative ISS.

- To apply the theory of ILC system interconnection to a concrete ILC problem, then it is necessary to check the applicability of the iterative ISS property in an ILC system. If an ILC system is iterative ISS then the theory of system interconnection will apply. For this purpose, we choose the nonlinear adaptive ILC proposed by French and Rogers [9]. We assumed that the disturbance and initial condition to decrease over iterations. We can only show that this class of adaptive nonlinear ILC is bounded by class $\mathcal{K}$ functions of disturbance and initial condition, and with a class $\mathcal{L}$ function. Whereas to satisfy iterative ISS property, the bound on class $\bar{\mathcal{K}}\mathcal{L}$ function of initial (first) pass and iteration need to be obtained. Hence, the iterative ISS property is a strong property to be satisfied by the nonlinear adaptive ILC.

4.2 Future Works

This research has identified new open problems which should be investigated in further research:

- For the ILC system interconnection (both feedback and feedforward), the iterative ISS has not been proved to be satisfied. It requires a statement stating that BDBS and DAG imply iterative ISS (in this work, we can only state this implication in a conjecture). Further work is required to fill in this gap. A suggestion is to use the idea from the converse theorem of Lyapunov ISS ([36]).

- The ILC system interconnection theory has been developed by assuming that the initial condition belongs to l_p. The future work need to weaken this assumption since the assumption that the initial condition belong to l_p may be too strong. Usually the initial condition is taken to belong to l_∞. One way to deal with this problem is to redefine the iterative ISS so as not to depend on the sum of the initial conditions. Instead, we only need the bound of initial condition, in terms of l_∞, in the new iterative ISS formulation.

- In the theory of ILC system interconnection, the reference signal is considered to be zero. This is an ideal case. In many ILC systems, the reference signal is not zero and the task of ILC is to track a reference signal. It is recommended for the future work, the reference signal is included.

- The iterative ISS property of nonlinear adaptive ILC still cannot be derived although the disturbance and initial condition are taken to be restrictive. However, we believe that, the iterative ISS can be obtained for a special case of a chain of integrators. Hence, in the future work, we suggest to restrict the nonlinear dynamic term ϕ (in equation (4.1)). For this case the iterative ISS property should follow.

Appendix A

Mathematical Notations and Definitions

A.1 Norm

- For any vector $\jmath \in \mathbb{R}^n$, the symbol $|\ \jmath\ |$ represents the Euclidean norm of $\jmath$ i.e. $|\ \jmath\ |= \sqrt{\jmath^T \jmath}$. If $\jmath$ is a scalar then $|\ \jmath\ |$ represents the absolute value (modulus).
- For any real vector valued function $\ell : \mathbb{R} \to \mathbb{R}^n$, the L^2 norm of ℓ is written as $\|\ \ell\ \|_{L^2[0,T]}$, which is defined as follows:

$$\|\ \ell\ \|_{L^2[0,T]} = \left(\int_0^T |\ell(\tau)|^2 d\tau \right)^{\frac{1}{2}}$$

- The L^∞ norm of ℓ over $[0, T]$ is written as $\|\ \ell\ \|_{L^\infty}$ which is defined as follows:

$$\|\ \ell\ \|_{L^\infty[0,T]} = \sup_{t \in [0,T]} |\ell(t)|$$

- For any real sequence $\{\xi_k\}_{k \geqslant 1}$, the l^p-norm, $1 \leqslant p \leqslant \infty$ is defined as:

$$\|\{\xi_i\}_{i \geqslant 1}\|_{l^p[1,k]} = \left(\sum_{i=1}^{k} |\xi_i|^p \right)^{1/p}, 1 \leqslant p < \infty$$

and

$$\|\{\xi_i\}_{i \geqslant 1}\|_{l^p[1,k]} = \sup_{1 \leqslant i \leqslant k} |\xi_i|, \quad p = \infty$$

- If k is ∞, then we write $\|\{\xi_k\}_{k \geqslant 1}\|_{l^p}$ instead of $\|\{\xi_k\}_{k \geqslant 1}\|_{l^p[1,k]}$ which is defined as:

$$\|\{\xi_i\}_{i \geqslant 1}\|_{l^p} = \left(\sum_{i=1}^{\infty} |\{\xi_i\}_{k \geqslant 1}|^p \right)^{1/p}$$

- Let ℓ be functions from the interval $[0, T]$ to $\mathbb{R}^n$. Then the λ-norm of ℓ at $t \in [0, T]$ denote by $\|\ell\|_\lambda$, which is defined as:

$$\|\ell\|_\lambda = \sup_{t \in [0,T]} e^{-\lambda t} |\ell(t)|$$

where λ is a positive constant.

- Let ℓ_1 and ℓ_2 be functions from the interval $[0,T]$ to $\mathbb{R}^n$. Then the innerproduct of ℓ_1 and ℓ_2 at $t \in [0,T]$ denote by $\langle \ell_1(t), \ell_2(t) \rangle$, which is defined as:

$$\langle \ell_1(t), \ell_2(t) \rangle = \int_0^t \exp -\lambda\tau \ell_1^T(\tau)\ell_2(\tau)d\tau$$

  where λ is a positive constant.

A.2 Function

- the notation $C(\mathfrak{M}_1, \mathfrak{M}_2)$ denotes the space of continuous functions from the set $\mathfrak{M}_1$ to the set $\mathfrak{M}_2$.
- we say a function $\mathfrak{P}$ is measurable if *there exists a sequence of step functions that converge pointwisely to $\mathfrak{P}$ almost everywhere* ([18]).
- the function $\gamma : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a class $\mathcal{K}$ function if it is positive definite and strictly increasing. If in addition $\lim_{s\to\infty} \gamma(s) = \infty$ then it is said to be class $\mathcal{K}^\infty$ function.
- if the function $\gamma : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is strictly increasing, but $\gamma(0) \geqslant 0$ then it is a class semi-$\mathcal{K}$ (respectively, semi-$\mathcal{K}^\infty$) function, which is denoted by $\bar{\mathcal{K}}$ (respectively, $\bar{\mathcal{K}}^\infty$)
- we say $\Lambda : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ is a class $\mathcal{L}$ function if $\lim_{s\to\infty} \Lambda(s) = 0$.
- the function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ is a $\mathcal{KL}$ if it is a class $\mathcal{K}$ function in the first argument and a class $\mathcal{L}$ in the second argument. It is a class $\bar{\mathcal{K}}\mathcal{L}$ if class $\bar{\mathcal{K}}$ in the first argument and a class $\mathcal{L}$ in the second argument.
- Semi-triangle inequality:

  Fact 1. *Let $\gamma \in \mathcal{K}$ (respectively, $\mathcal{K}^\infty$) then we have:*

$$\gamma(a + b) \leqslant \gamma(2a) + \gamma(2b)$$

- Composition of two class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) functions:

  Fact 2. *Composition of two class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) functions is again class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) function.*

- Addition of two class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) functions:

  Fact 3. *Addition of any class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) function with the same argument is again class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) function.*

- The fact 1, fact 2 and fact 3 are also valid for every class $\bar{\mathcal{K}}$ (respectively, $\bar{\mathcal{K}}^\infty$) function.
- Composition of a class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) function with a class $\bar{\mathcal{K}}$ (respectively, $\bar{\mathcal{K}}^\infty$) function:

  Fact 4. *Suppose γ_1 is a class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) function and γ_2 is a class $\bar{\mathcal{K}}$ (respectively, $\bar{\mathcal{K}}^\infty$) function then $\gamma_1 \circ \gamma_2$ and $\gamma_2 \circ \gamma_1$ is a class $\bar{\mathcal{K}}$ (respectively, $\bar{\mathcal{K}}^\infty$) function.*

- Addition of a class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) function with a class $\bar{\mathcal{K}}$ (respectively, $\bar{\mathcal{K}}^\infty$) function:

Fact 5. *Suppose γ_1 is a class $\mathcal{K}$ (respectively, $\mathcal{K}^\infty$) function and γ_2 is a class $\bar{\mathcal{K}}$ (respectively, $\bar{\mathcal{K}}^\infty$) function then $\gamma_1 + \gamma_2$ is a class $\bar{\mathcal{K}}$ (respectively, $\bar{\mathcal{K}}^\infty$) function.*

•

Fact 6. *For each f_1 and f_2 in $\mathcal{M}$ (respectively, $\mathbb{R}^n$) the following holds:*

$$\left\| \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\| \geqslant \|f_1\|, \quad \left\| \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\| \geqslant \|f_2\| \tag{A.1}$$

where the norm $\|\cdot\|$ is taken to be L_p-norm over $[0,T]$ /λ-norm (respectively, Euclidean norm).

- Triangle inequality:

Fact 7. *Suppose f_1 and f_2 in $\mathcal{M}$ (respectively, $\mathbb{R}^n$). Then the following holds:*

$$\left\| \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\| \leqslant \|f_1\| + \|f_2\| \tag{A.2}$$

where the norm $\|\cdot\|$ is taken to be L_p-norm over $[0,T]$/λ-norm (respectively, Euclidean norm).

A.3 Other Mathematical Notations

- we use $\mathbb{N}$ for the set of natural number $\{1, 2, \ldots\}$.
- the symbol $\{\omega_k\}_{k\geqslant 1}$ denotes the sequence of ω_1, ω_2, ..., ω_k where $\omega_k \in \mathbb{R}^n$ and $k \in \mathbb{N}$.
- For any real positive definite symmetric matrix X denote $\bar{\lambda}(X)$ as a maximum eigenvalue and $\underline{\lambda}(X)$ as a minimum eigenvalue of matrix X.

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