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Nonlinear Realizations And Effective Lagrangian Densities For Nonlinear σ-Models

by

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Dedicated to my family and friends.

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ABSTRACT

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Nonlinear realizations of the groups $SU(N)$, $SO(m)$ and $SO(t,s)$ are analysed, described by the coset spaces $\frac{SU(N)}{SU(N-1)\otimes U(1)}$, $\frac{SO(m)}{SO(m-1)}$, $\frac{SO(1,m-1)}{SO(1,m-2)}$ and $\frac{SO(m)}{SO(m-2)\otimes SO(2)}$. The analysis consists of determining the transformation properties of the Goldstone Bosons, constructing the most general possible Lagrangian for the realizations, and as a result identifying the coset space metric. We view the λ matrices of $SU(N)$ as being the basis of an (N^2-1) dimensional real vector space, and from this we learn how to construct the basis of a Cartan Subspace associated with a vector. This results in a mathematical structure which allows us to find expressions for coset representative elements used in the analysis. This structure is not only relevant to $SU(N)$ breaking models, but may also be used to find results in $SO(m)$ and $SO(1,m-1)$ breaking models.

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Preface

No claim to originality is made for the content of the Introduction, or Chapter 1, which were compiled using a variety of sources.

The first part of Chapter 2 is based on a paper by L. Michel and L. A. Radicati entitled 'The geometry of the octet'. Original work starts towards the end of the Chapter, and continues throughout Chapters 3, 4 and 5. It is mainly the *approach* to the analysis of this subject which is original, since some of the subsequent work contains models which have been studied at length before. I have therefore tried to reference, where appropriate, as many of these models as possible.

Some of the work found in Appendix A has been submitted as a paper entitled 'How orbits of $SU(N)$ can describe rotations in $SO(6)$' to the Journal of Physics A (authors K. J. Barnes, J. Hamilton-Charlton and T. R. Lawrence); the relevant ideas of this paper have been tailored to suit the work found in this thesis.

The adjoint representation linear operator work, found in the first two sections of Appendix B, comes from the paper by L. Michel and L. A. Radicati. The last three sections of this appendix are original.

The work found in Appendix C is a collection of well known results from many sources; including the paper 'How orbits of $SU(N)$ can describe rotations in $SO(6)$'.

The second part of Appendix D is original work, and is an extension to the well established results contained in the first part of the appendix.

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I would like to thank my supervisor, Prof. K. J. Barnes, for suggesting such an enjoyable, and challenging, area of research. His guidance, with respect to some of the more seemingly impenetrable aspects of the mathematics, has proved to be invaluable.

I would also like to thank Dr. T. R. Lawrence with whom I spent the first two years of this Ph.D. working; before he graduated. I enjoyed many hours of discussing the concepts of this work with him, and also enjoyed having someone around in times of great anxiety when, seemingly, an impasse had been reached!

My thanks must also go to Dr. J. A. Vickers of the Southampton University Mathematics Department for his 'brief treatise' on complex manifolds.

In general, I would like to thank the past and present members of the Southampton High Energy Physics Theory group for the sociable working environment which made my time at Southampton so enjoyable.

Finally, my sincere thanks go to my parents for giving me the support I needed to make this work possible, and to other relatives and friends who have, on occasion, been subjected to my ramblings about the concepts of this work which are now the contents of this thesis; I thank them all for listening to me with glazed expressions.

Introduction.

Symmetries in physical theories, and Spontaneous Symmetry Breaking.

To understand the ideas of symmetry breaking in a physical theory, it is important that we first introduce the nature of symmetries with respect to a Lagrangian framework. We will begin by introducing some very simple models, which demonstrate the power of the Lagrangian formulation, and then discuss the nature of symmetries and how they are important. Once we have done this we will be able to explain the ideas behind spontaneous symmetry breaking, and where the mathematical framework of this thesis has come from.

A Lagrangian formulation of classical particle mechanics requires that a Lagrangian, $L(q_i, \dot{q}_i) = T - V$, be constructed out of generalized coordinates, q_i, and generalized velocities, $\dot{q}_i$. Here T is the kinetic energy associated with the system, and V is the potential of the system. Hamilton's principle, the principle of least action, leads to the Euler-Lagrange equations :-

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right] = \frac{\partial L}{\partial q_i} \tag{1}$$

which give the equations of motion. Similar ideas may also be used in the Lagrangian formulation of a relativistic quantum field theory. If this time a Lagrangian density,

$\mathcal{L}(\phi, \partial_\mu \phi) = T - V$, is constructed out of fields, $\phi(x^\mu)$, and field gradients, $\partial_\mu \phi$, then when the principle of least action is applied (which is now a more complicated idea because we are dealing with functions of space-time coordinates) we find the Euler-Lagrange equations :-

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] = \frac{\partial \mathcal{L}}{\partial \phi} \tag{2}$$

which give the equations of motion for the field, ϕ. To give some examples :-

- A particle is classically idealized as a point of mass, m. Now if this particle moves in a region where the potential is given by $V(x, y, z)$ then the Lagrangian is :-

  $$L = \tfrac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z) \tag{3}$$

  The Euler-Lagrange equation for the first coordinate, x, yields :-

  $$\begin{aligned} m\,\ddot{x} &= -\frac{\partial V}{\partial x} \\ &= F_x \end{aligned}$$

  with similar results for the y and z coordinates. Collectively we may write :-

  $$\underline{F} = m\ddot{\underline{r}} \tag{4}$$

  which is Newton's Second Law of motion.

- The simplest example in a (quantum) field theory is the Lagrangian density describing a relativistic, free (non-interacting), real scalar field ϕ :-

  $$\mathcal{L} = \tfrac{1}{2}\partial_\mu \phi\, \partial^\mu \phi - \tfrac{1}{2} m^2 \phi^2 \tag{5}$$

  The Euler-Lagrange equation for ϕ is :-

  $$\begin{aligned} \partial_\mu(\partial^\mu \phi) &= -m^2 \phi \\ (\Box + m^2)\,\phi &= 0 \end{aligned} \tag{6}$$

and this is the Klein-Gordon equation which is used to describe uncharged particles. We note that the Schrödinger equation is the non-relativistic approximation to the Klein-Gordon equation.

- The simplest extension to the real scalar field is the free complex scalar field and the Lagrangian density :-

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \phi^* \partial^\mu \phi - \tfrac{1}{2}m^2 \phi^* \phi \tag{7}$$

where $\phi = \phi_1 + i\phi_2$. In this case ϕ and ϕ^* are now regarded as independent fields and this is the Lagrangian density for free charged particles; we will soon explain the importance of charge. This time we will find two equations of motion :-

$$(\Box + m^2)\phi \equiv 0 \tag{8}$$

$$(\Box + m^2)\phi^* \equiv 0 \tag{9}$$

which are the Klein-Gordon equations for the two complex fields. Notice that we may also rewrite the Lagrangian density explicitly in terms of the two real components of the complex field :-

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \phi_i \partial^\mu \phi_i - \tfrac{1}{2}m^2 \phi_i \phi_i \tag{10}$$

with $i = 1, 2$ and therefore obtain similar Klein-Gordon equations of motion for ϕ_1 and ϕ_2. It is obvious that further generalisations are possible if we let the range of the index i increase.

- A theory concerning (charged) particles (and anti-particles) with spin is produced using spinor fields. The simplest Lagrangian density is :-

$$\begin{aligned}
\mathcal{L} &= i\overline{\psi}\Gamma^\mu \overleftrightarrow{\partial}_\mu \psi - m\overline{\psi}\psi \\
&= i\overline{\psi}\overleftrightarrow{\not{\partial}}\psi - m\overline{\psi}\psi \\
&= \frac{i}{2}(\overline{\psi}\overrightarrow{\not{\partial}}\psi - \overline{\psi}\overleftarrow{\not{\partial}}\psi) - m\overline{\psi}\psi
\end{aligned} \tag{11}$$

where ψ is a one column matrix, and therefore $\overline{\psi}$ has one row. Also note that, in this thesis, the notation Γ^{μ} is used for the gamma matrices of the $SO(t,s)$ groups, and γ^{A} for the $SO(m)$ groups. This time we must use the Euler-Lagrange equations :-

$$\overrightarrow{\partial}_{\mu}\left[\frac{\partial\mathcal{L}}{\partial(\overline{\psi}\overleftarrow{\partial}_{\mu})}\right] = \frac{\partial\mathcal{L}}{\partial\overline{\psi}} \qquad\qquad \left[\frac{\partial\mathcal{L}}{\partial(\overrightarrow{\partial}_{\mu}\psi)}\right]\overleftarrow{\partial}_{\mu} = \frac{\partial\mathcal{L}}{\partial\psi}$$

These yield the equations of motion :-

$$(i\overrightarrow{\not{\partial}}-m)\psi = 0 \tag{12}$$

$$\overline{\psi}(i\overleftarrow{\not{\partial}}+m) = 0 \tag{13}$$

which are the Dirac equations of motion for ψ and $\overline{\psi}$. We note that these equations of motion may also be derived from the density :-

$$\mathcal{L} = i\overline{\psi}\overrightarrow{\not{\partial}}\psi - m\overline{\psi}\psi \tag{14}$$

The only difference between this and equation (11) is a total divergence which does not change the action; so the physics remains the same.

Thus, in a Lagrangian formulation of a (field) theory, the principle of least action leads to the Euler-Lagrange equations which act on a Lagrangian (density) function. If this Lagrangian function is written properly then the laws of physics, which govern the behaviour of the system, are automatically encoded within the framework. Now, the variational principle also has another consequence, and this is where the idea of symmetries comes in. In both the classical and quantum field theory cases, the action is invariant with respect to transformations of the coordinates (fields) and velocities (field gradients). This means that there will be one or more conserved quantities, i.e. combinations of coordinates (fields) and velocities (field gradients) which are invariant under the transformations. We say that the system posesses a symmetry or, with more conserved quantities, a set of symmetries. This subject was formally investigated in 1918 and is the concern of Noether's theorem [1]. In a field theory context the theorem states that the invariance of the action, under transformation of the fields and field gradients, leads to a conserved (divergenceless) current, J^{μ}_{ν} :-

$$J^{\mu}_{\nu} = \frac{\partial \mathfrak{L}}{\partial(\partial_{\mu}\phi)}\Phi_{\nu} - \theta^{\mu}_{\kappa}X^{\kappa}_{\nu} \tag{15}$$

with $\partial_{\mu}J^{\mu}_{\nu} = 0$. We note that, in the definition of the current, we have used the energy-momentum tensor :-

$$\theta^{\mu}_{\nu} = \frac{\partial \mathfrak{L}}{\partial(\partial_{\mu}\phi)}\partial_{\nu}\phi - \delta^{\mu}_{\nu}\mathfrak{L} \tag{16}$$

and the quantities Φ_{ν} and X^{κ}_{ν} relate to the transformations of ϕ and x^{μ} :-

$$\begin{aligned} \Delta\phi &= \Phi_{\mu}\delta\omega^{\mu} \\ \Delta x^{\mu} &= X^{\mu}_{\nu}\delta\omega^{\nu} \end{aligned}$$

for the infinitesimal transformation parameter $\delta\omega^{\nu}$. The current gives rise to a conserved (time independent) charge, Q_{ν}, which is defined by :-

$$Q_{\nu} = \int_V J^0_{\nu}\, d^3x \tag{17}$$

The integration is taken to be over a spacelike hypersurface where $t = \text{const}$; i.e. over the 3-volume V. Conservation of Q_{ν} follows because :-

$$\frac{dQ_{\nu}}{dt} = 0$$

Now, when $\Phi_{\mu} = 0$ and $X^{\mu}_{\nu} \neq 0$, Noether's theorem tells us that energy-momentum and angular momentum are conserved (for spatial translations and rotations respectively). It is true that energy and momentum are conserved for any system whose Lagrangian (density) is not explicitly dependent on x^{μ}, whereas conservation of angular momentum requires θ^{μ}_{ν} to be symmetric; if it is not then we may define a canonical energy-momentum tensor, $T^{\mu\nu}$, which is. In contrast, any additional conserved quantities which a system may possess (like electric charge, isospin, strangeness ...) require $\Phi_\mu \neq 0$, i.e. the fields themselves must be transformed; which implies that they must have more than one component. For the real scalar (Poincaré invariant) field theory above, ϕ has only one real component and so the theory represented by the respective Lagrangian density is subject to energy, momentum and angular momentum conservation only.

However, for the cases of the complex scalar fields and spinor fields, the field components may be transformed into one another. The transformations are produced by matrices which are elements of Lie groups, and the simplest example is the transformation of a complex scalar field by a $U(1)$ group element :-

$$u \in U(1): \quad \phi \;\mapsto\; \phi' \;=\; u\phi \tag{18}$$

where $u \equiv e^{i\Theta}$ is just a complex number. Under this transformation the Lagrangian density, equation (7), is invariant. The components of ϕ are found to transform :-

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{19}$$

So this field transformation is just a rotation in the internal space of ϕ; not a space-time transformation. In this case Noether's theorem gives the divergenceless current :-

$$J^\mu \;\propto\; i\phi^* \overleftrightarrow{\partial^\mu}\, \phi \tag{20}$$

When $\Theta = \Theta(x^\mu)$ the group element produces a *local* transformation of ϕ. As a result, a gauge field must be introduced into $\mathcal{L}$ to guarantee invariance under the local transformation, and in this case we will eventually find a divergenceless 'covariant' current :-

$$\mathcal{J}^\mu \;\propto\; ie\phi^* \overleftrightarrow{\mathcal{D}^\mu}\, \phi \tag{21}$$

where $\mathcal{D}^\mu \phi$ is the covariant derivative of ϕ. The corresponding conserved quantity is electric charge. Thus, conservation of electric charge emerges when we require $\mathcal{L}$ to be invariant under local $U(1)$ gauge transformations; electric charge is a locally conserved quantity. On the other hand isospin, strangeness and so on result from invariance under *global* (space-time independent) transformations by elements of other Lie groups. For the Dirac Lagrangian density :-

$$\mathcal{L} = i\overline{\psi}\overset{\leftrightarrow}{\not\partial}\psi - m\overline{\psi}\psi$$

we may use, in the absence of spacetime transformations (i.e. $X^\mu_\nu = 0$), the Noether current relation :-

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\overrightarrow{\partial}_\mu\psi)}\Phi^\psi + \Phi^{\overline{\psi}}\frac{\partial\mathcal{L}}{\partial(\overline{\psi}\overleftarrow{\partial}_\mu)} \tag{22}$$

In the simplest case where ψ is transformed by a $u \in U(1)$, we have $\Phi^\psi = -i\psi$ and $\Phi^{\overline{\psi}} = i\overline{\psi}$. So we find the divergenceless current :-

$$\begin{aligned} J^\mu &= \overline{\psi}\Gamma^\mu\psi \\ \partial_\mu J^\mu &\equiv 0 \end{aligned} \tag{23}$$

The roots of the work found in this thesis lie in the 1960s. It was understood that some (quantum mechanical) systems possess a property which was distinctly different from the simple systems we have looked at so far. Two examples are the superconductor and the ferromagnet; we will briefly discuss the latter. If we define a general Hamiltonian density by :-

$$\mathcal{H} \equiv \pi_i\dot{\phi}_i - \mathcal{L}$$

where $\pi_i \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}_i}$ is the momentum field canonically conjugate to ϕ_i, then $\mathcal{H}$ is a scalar and is therefore invariant under rotations. For a ferromagnet, the contributions to $\mathcal{H}$ come from the spin-spin interactions between the atoms in the sample and above the ferromagnetic transition temperature the spins are randomly aligned. However, below the ferromagnetic transition temperature the *ground state* is not rotationally invariant because the spins (within a domain) become aligned and we have the situation of spontaneous magnetisation in a particular direction. The actual direction of spontaneous magnetisation is 'chosen' randomly; all the other possible ground state configurations may be reached from a particular one by rotation. Thus, the ground state configuration of the system does not display the full symmetry of the Hamiltonian. We say that the full symmetry of the Hamiltonian (and Lagrangian) density is hidden, or spontaneously broken. It is important to remember that the full symmetry of the system is still there; it is just that, even though any direction of magnetisation is equally good, the necessity of associating one direction with the ground state has hidden it.

In 1961 Goldstone published a paper [2] which looked at this situation with respect to a Lagrangian density for an interacting scalar field theory. The Lagrangian density is constructed out of a scalar field, or scalar field multiplet, and to achieve the required property the scalar field(s) is(are) thought of as having an imaginary mass. He found that if the Lagrangian density has a discrete symmetry (as is the case for a single real scalar field), then the ground state configuration will be discrete. In the case of a Lagrangian constructed from a scalar field multiplet, he found that since $\mathcal{L}$ possesses a continuous symmetry, then the ground state is comprised of a continuous set of degenerate configurations (vacua). For example, the Lagrangian density :-

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - \left[m^2 \phi^* \phi + \lambda(\phi^* \phi)^2\right] \tag{24}$$

is invariant under the $U(1)$ field transformations $\phi \mapsto \phi' = u\phi$, see equation (19), and this situation has already been discussed for a free field theory where $\lambda = 0$. Since $\phi \equiv \phi_1 + i\phi_2$ the potential may be rewritten in terms of these real components :-

$$V = m^2(\phi_1^2 + \phi_2^2) + \lambda(\phi_1^2 + \phi_2^2)^2 \tag{25}$$

To find the minimum of V we calculate :-

$$\left.\begin{aligned} \frac{\partial V}{\partial \phi^*} &= (m^2 + 2\lambda\phi^*\phi)\phi \\ \frac{\partial V}{\partial \phi} &= (m^2 + 2\lambda\phi^*\phi)\phi^* \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} \frac{\partial V}{\partial \phi_1} &= 2[m^2 + 2\lambda(\phi_1^2 + \phi_2^2)]\phi_1 \\ \frac{\partial V}{\partial \phi_2} &= 2[m^2 + 2\lambda(\phi_1^2 + \phi_2^2)]\phi_2 \end{aligned}\right. \tag{26}$$

From these we see that, when $m^2 > 0$, the minimum of V is at $\phi = 0$; which is equivalent to the condition $\phi_1 = \phi_2 = 0$. However, if $m^2 < 0$ then V is minimised by the condition :-

$$\phi^*\phi \;=\; (\phi_1^2 + \phi_2^2) \;=\; -\frac{m^2}{2\lambda} \;=\; a^2 \tag{27}$$

i.e. the minimum of V is where $\phi_1^2 + \phi_2^2 = a^2$. In this case the potential has the following form (we also show a vertical slice of V through the origin) :-

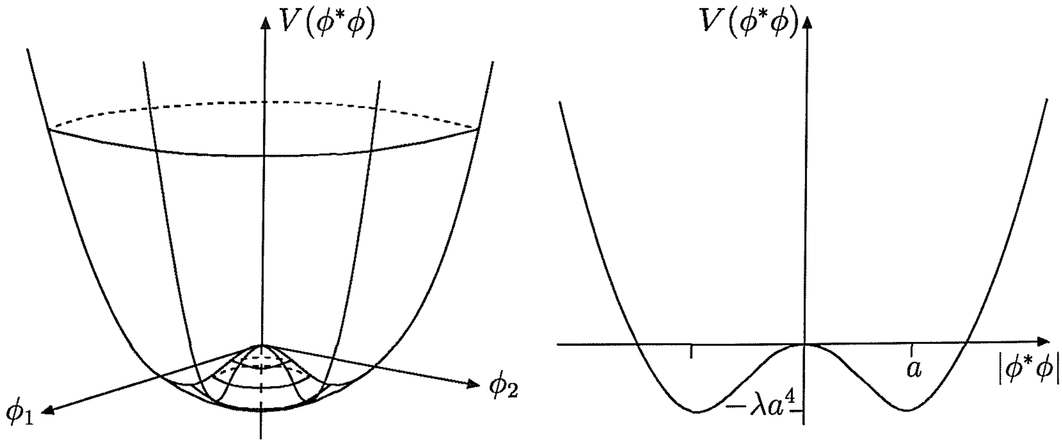

Figure 1: The $U(1) \sim SO(2)$ invariant potential.

To obtain the physical fields for the theory it is necessary to redefine the scalar fields such that one of the minima represents the ground/vacuum state of the system. This hides part of the symmetry enjoyed by $\mathcal{L}$ (or similarly $\mathcal{H}$). For this example, if we redefine ϕ_2 as :-

$$\phi_2 \;\rightsquigarrow\; \chi + a \tag{28}$$

then the potential may be rewritten :-

$$V = 4\lambda a^2\chi^2 + \lambda[(\phi_1^2 + \chi^2)^2 + 4\chi a(\phi_1^2 + \chi^2)] - \lambda a^4 \tag{29}$$

Thus we see that ϕ_1 has become a *massless* field, and the χ field now has a mass squared of $m_\chi^2 = 4\lambda a^2$. This is an important feature of Goldstone's method. In the context of breaking invariance under symmetry group G down to a subgroup H, it is more useful to construct a Lagrangian density (with a similar form as the one above) out of three scalar fields. This time we will have $G = SO(3) \sim SU(2)$ and $H = SO(2) \sim U(1)$. In this case we could write the potential part of the Lagrangian density as :-

$$V = \frac{m^2}{2}(\phi_1^2 + \phi_2^2 + \phi_3^2) + \lambda(\phi_1^2 + \phi_2^2 + \phi_3^2)^2 \tag{30}$$

This time, when we differentiate V with respect to the scalar fields, we find :-

$$\frac{\partial V}{\partial \phi_i} = (m^2 + 4\lambda\phi_k\phi_k)\phi_i \tag{31}$$

so when $M^2 < 0$, the minimum of the potential lies on a sphere of radius a because, to minimize V, we need :-

$$\phi_k\phi_k = -\frac{m^2}{4\lambda} = a^2 \tag{32}$$

If we now say that the vacuum configuration points in the direction of ϕ_3 then transforming the scalar field multiplet about the 3^{rd} (internal) direction (an $H = SO(2)$ transformation) will leave the vacuum invariant. Whereas transforming about the other two directions will rotate the initial vacuum configuration onto one of the other degenerate vacua. So if we now redefine ϕ_3 as $\phi_3 = \chi + a$ then we find :-

$$V = 4\lambda a^2\chi^2 + \lambda[(\phi_1^2 + \phi_2^2 + \chi^2)^2 + 4\chi a(\phi_1^2 + \phi_2^2 + \chi^2)] - \lambda a^4 \tag{33}$$

This time both ϕ_1 and ϕ_2 have become *massless* fields and now the χ field has a mass squared value of $m_\chi^2 = 8\lambda a^2$. In the following year (1962) Goldstone, together with

Salam and Weinberg, published [3] in which these results were restated in a more general form. They found that whenever a Lagrangian is invariant under a continuous symmetry group but the ground state configuration (vacuum) is not, there will be spinless, zero mass fields present; these are known as Goldstone boson fields. Both of these papers [2, 3] were purely theoretical in nature. A more detailed discussion of Goldstone's Theorem may be found in any good textbook on quantum field theories, such as [4, 5, 6].

In contrast, a year earlier, Gell-Mann and Lévy had published [7] which concerned pion decays (in a system of pions and nucleons). This was classic phenomenology. The nucleons were made to transform as a representation of $SO(4)$, whereas the pions were taken to only transform as a representation of its $SO(3)$ vector subgroup. A fourth scalar field called σ' was introduced such that σ' and π formed a multiplet of $SO(4)$. The σ' field was then eliminated from the Lagrangian density by using the condition :-

$$\pi^2 + \sigma'^2 = C^2$$

which constrains the modulus of σ'. This gives the relation $\sigma' = -\sqrt{C^2 - \pi^2}$. Thus, wherever σ' had previously appeared in the Lagrangian density, it was now replaced by this *nonlinear* function of π. This model therefore became known as the 'nonlinear sigma model'. This has since become a generic name for the theories found in this thesis.

During the 1960's the ideas of Gell-Mann and Lévy became increasingly popular. Much of the research was phenomenological in nature, and most of the focus was placed on the calculations for specific chiral groups (like $SU(2) \otimes SU(2)$ or $SU(3) \otimes SU(3)$). However, in 1969, the geometry of nonlinear realizations was studied by Isham [8]. This paper employed a Killing vector method in the examination of Goldstone boson transformations, and also introduced the concept of a Goldstone boson manifold metric. Later on that year Callan, Coleman, Wess and Zumino also used a geometrical approach to produce theories which agreed with (had similar properties to) the phenomenological models [9, 10]. They showed that, given any Lie group, G, and any Lie subgroup, H, it was possible to construct a general Lagrangian density in which the invariance of $\mathcal{L}$ would be produced by linear transformations of fields and field covariant derivatives under transformations of H; i.e. the fields and covariant derivatives would form a linear representation of H. The invariance of $\mathcal{L}$ under transformations of the rest of the internal symmetry group G would be realized nonlinearly; i.e. under these transformations the fields and covariant derivatives would form a nonlinear realization of G. We note that (phenomenological) fields which exhibit this behaviour are known as standard coordinates (since they are the coordinates of a manifold on which the group acts), and the transformations properties they exhibit are known as a standard realization of G. (A good summary of Callan, Coleman, Wess and Zumino's work is also to be found in [11], since their work is also applicable to theories where supersymmetry is broken.)

The work of Salam and Strathdee [12] proved that if a nonlinear realization were obtained from a (linear) representation of a group, then the vacuum could not be invariant under the whole group of transformations. Furthermore, they showed that if one were to demand that a system's Lagrangian density be invariant under (internal) transformations of a Lie group G and its vacuum be invariant under a subgroup H, then the method of Callan, Coleman, Wess and Zumino [9, 10] was the way to produce general couplings between the Goldstone fields and any other (matter) fields in the theory. In this way, a Lagrangian density constructed out of standard fields, forming a nonlinear realization of G which reduces to a linear representation of H, was just the *effective* Lagrangian of a theory resulting from the spontaneous breaking of a symmetry (or symmetries) of $\mathcal{L}$.

Finally, the line of chiral symmetry breaking research culminated in the work of Barnes, Dondi and Sarkar [13]. They used a projection operator method, in the framework of [9, 10, 8], to construct a general effective Lagrangian density for theories where chiral $SU_L(N) \otimes SU_R(N)$ invariance is broken to invariance under $SU_V(N)$, the vector subgroup.

The structure of this thesis.

This thesis was inspired by three specific models :-

1. When $SU(2)$ invariance is broken to $U(1)$,
2. when $SO(1,4)$ invariance is broken to $SO(1,3)$, and
3. when $SO(6)$ invariance is broken to $SO(4) \otimes SO(2)$.

The first two models were simple enough to analyse, and results were readily found; these models are contained in chapter 3. However the third model seemed impossible to analyse. The problem lay in the fact that the mathematical framework relies on manipulating a quantity called the coset representative element, which is the exponential of a linear sum of the coset generators (group elements are the exponentials of a linear sum of *all* the generators of group transformations). For the first two models the relevant coset representative elements were easy to calculate to all orders (as required by theory), whereas the coset representative element of the third model seemed impossible to find. Therefore, it became necessary to look at the exponents in a new way; which would make the process of exponentiation much easier. The exponents were understood in terms of the work of Michel and Radicati [17] and, as a result, took on a geometrical meaning; in this thesis the linear sum of coset generators is known as the *coset vector*. With this new understanding of coset vectors it was clear that a whole series of models could be studied at the same time. Thus, the calculations (and results) which appear in this thesis are kept in a general form, and therefore apply to a whole series of models simultaneously.

The mathematical difference between each model, in a section/chapter, is provided by the index ranges because, as we go up through a series of models, the relevant coset vectors have an ever increasing number of components. The difference between each model, in a series, lies in their physical interpretation. For example, the CP2 model (which results when $SU(2)$ invariance is broken to $U(1)$) has two charged pions (π^+ and π^-) as the Goldstone bosons, whereas the CP4 model (which results when $SU(3)$ invariance is broken to $SU(2) \otimes U(1)$) has four Kaons (K^+, K^0, K^- and $\overline{K^0}$) as the Goldstone bosons. We note that the CP2 and CP4 models will appear in different chapters (3 and 4 respectively) because the mathematics associated with the CP4 model is much more complicated than that of the CP2 model; in fact, the results of the CP2 model bear more resemblance to the two sets of models which make up the rest of chapter 3 (which is why these models have been grouped together). Nevertheless, the general results in chapter 4 which concern the CP2(N-1) series of models yield, when $N = 2$, the results for CP2 as one would expect. The only *specific* model which is calculated in this thesis is that of CP2. For the more mathematically inclined readers we note that, *strictly* speaking, true CP2(N-1) models are achieved by rewriting the coset space coordinates as 'stereographic coordinates'; by forming complex combinations and then making antipodean identifications. This is briefly discussed in the concluding chapter where the scalar part of the $SU(2)$ breaking to $U(1)$ model (with S^2 Goldstone boson manifold) is rewritten as a CP2 model. This transformation does not affect the Kählerian properties of the Goldstone boson manifold.

Our choice of models which appear in this thesis has been guided by a theorem of Borel [14] which states that, for a group G and a subgroup H, the manifold associated with the coset space $(\frac{G}{H})$ will be Kähler if the centralizer of H in G, denoted $C_H(G)$, is toroidal. This means either :-

1. H is a $U(1)$ group, or a product of commuting $U(1)$ groups, i.e. :-
   $H = U(1)$ or $H = U(1) \otimes U(1) \otimes \cdots$

2. H has a commuting $U(1)$ group, or a product of commuting $U(1)$ groups, i.e. :-
   $H = H_1 \otimes U(1)$ or $H = H_1 \otimes U(1) \otimes U(1) \otimes \cdots$

The same is also true if, in the above, we replace $U(1)$ by $SO(2)$ or $SO(1,1)$; because of the homomorphism between $U(1)$ and $SO(2)$. The importance of Borel's Theorem is that any theory with this property may be extended to include $\mathcal{N} = 1$ Supersymmetry [1]. Therefore, we have mainly chosen to study models which contain a Kähler Goldstone boson manifold; which may be identified with the coset space $(\frac{G}{H})$.

In this thesis we will analyse various nonlinear realizations. The analysis will consist of determining the transformation properties of the Goldstone fields using the Killing vector method, and then constructing invariant quantities from fields, and covariant derivatives, which may be used to form the Effective Lagrangian for the theory.

Chapter 1 begins by introducing group elements, subgroup elements and the coset representative element. The rest of the chapter focuses on the methods of Callan, Coleman, Wess and Zumino [9, 10]. We show how the notion of a symmetric space allows the use of an isomorphic mapping of the group generators (an outer involutive automorphism) which helps in the study of field transformations. We then use the first order Killing vector method, introduced by Isham [8], to study the Goldstone field transformations. Next, we show how the covariant derivatives are formed, for the Goldstone fields and the matter fields, and how they transform too. Finally we show how an effective Lagrangian density for the theory may be constructed; consisting

[1]This assumes that the fields are defined in a normal four dimensional spacetime. If the fields, however, are defined in a two dimensional spacetime then a theory with a Kähler $\frac{G}{H}$ will admit $\mathcal{N} = 2$ Extended Supersymmetries.

of two parts. Firstly, the Goldstone boson part is shown to contain a metric, also introduced by Isham [8], which is associated with the Goldstone boson manifold. We show that this metric may also be constructed from the Killing vector components. Finally, we show how the matter field part of the Lagrangian density is written.

Chapter 2 introduces the mathematics of real vector spaces and Cartan Subspaces, studied by Michel and Radicati [17]. We then tailor, and extend, their methods to suit the necessities of this thesis. We show how to write (some of) the basis elements of the Cartan subspace associated with a vector x, denoted $\mathcal{C}_x$. This tells us how we may express the coset vectors which, in turn, helps us to find the coset representative element. (This element is a fundamental quantity used in the mathematical framework of the preceeding chapter.) We find that this process is possible when we understand the geometric implications of the *characteristic* equation of a vector. It is also clear that this language may be used for $SU(N)$ breaking *and* $SO(m)$ breaking theories; because the mathematics has no regard for the physical theory we wish to investigate. We end the chapter by showing how to calculate the exponentials of various important types of vector. Thus, this chapter sets up the mathematical formalism adopted in the last three chapters.

In chapter 3 we look at the (sets of) theories which may be studied when the relevant (normalized) coset vectors square to the identity element. We begin with the theory which arises when $SU(2)$ invariance is broken to $U(1)$, we then look at the theories which arise when $SO(m)$ invariance is broken to $SO(m-1)$, and finally we look at the theories which arise when $SO(1,m-1)$ invariance is broken to $SO(1,m-2)$. In each case we find the Killing vector components which describe the Goldstone field transformations, we find the covariant derivatives for the Goldstone fields and matter fields and then we construct the Goldstone part of the effective Lagrangian densities. We also check the form of the Goldstone boson manifold using the Killing vector components. We note here that the *specific* nonlinear realization of $\frac{SU(2)}{U(1)}$ is studied, and the effective Lagrangian density for the theory found, in [15, 18]. Also, the *specific* nonlinear realization of $\frac{SO(1,4)}{SO(1,3)}$ is studied, and the effective Lagrangian density for the theory found, in [15] only. As far as we know the general sets of models have not been studied before; just specific examples. We see that this chapter contains three models which have a Kähler Goldstone boson manifold. These are the theories which arise when $SU(2)$ invariance is broken to $U(1)$, when $SO(3)$ invariance is broken to $SO(2)$, and when $SO(1,2)$ invariance is broken to $SO(1,1)$.

In chapter 4 we look at CP2(N-1) models which result when chiral $SU(N)$ invariance is broken to $SU(N) \otimes U(1)$. We show how these models may be embedded within the framework of general chiral symmetry breaking models [13]; i.e. the models which arise when $SU(N)_L \otimes SU(N)_R$ invariance is broken to $SU(N)_V$, the vector subgroup. We also show which models from chapter 3 are contained in, or are relevant to, this chapter. Again, we find the Killing vector components which describe the Goldstone field transformations, we find the covariant derivatives for the Goldstone fields and the matter fields and then we construct the Goldstone part of the effective Lagrangian densities. We also check the form of the Goldstone boson manifolds using the Killing vector components for each model. We note that all the CP2(N-1) models have a Kähler Goldstone boson manifold.

In chapter 5, the final chapter of results, we look at three models with the same structure. These are the theories which arise when $SO(m)$ invariance is broken to $SO(m-2) \otimes SO(2)$ for $m = 4, 5, 6$. Again, we find the Killing vector components which describe the Goldstone field transformations, we find the covariant derivatives for the Goldstone fields and the matter fields and then we construct the Goldstone part of the effective Lagrangian densities. We also check the form of the Goldstone boson manifolds using the Killing vector components for each model. We note that the final model where $SO(6)$ invariance is broken to $SO(4) \otimes SO(2)$ also appears in [15]. However, unlike in [15], we are able to find the Killing vector components for the Goldstone field transformations and therefore check the form of the Goldstone boson manifold. This work was omitted from [15] because the method he used in the analysis was based on the projection operator method of [13]; which yielded answers which were not as transparent (or as easy to manipulate), as the results which we have found [2]. We note that all the models in this chapter have a Kähler Goldstone boson manifold.

This thesis also contains four appendices, which we will now briefly discuss. The appendices are fairly extensive because we felt it was important to maintain the flow of ideas and results in the main body of the thesis. Appendix A is designed as supplementary work to chapter 2 and contains the more relevant ideas found in [16]; which was a geometric examination of the homomorphism between the groups $SU(4)$ and $SO(6)$. In this appendix we explicitly calculate the diagonal r and q_r-vectors of the real vector spaces $\Re^8$ and $\Re^{15}$. With these explicit forms in mind (and remembering the use of the characteristic equation in the idea of rotating vectors around the real vector spaces) it is easier, for example, to see how to form group elements, or commuting subgroup elements. We also get a better 'feel' of the form of the results in the thesis; in particular, because we do not resort to a projection operator method from the start (see [13, 15]) which in effect hides the coset vector structure, we are able to keep vector-like quantities (associated with the coset and subgroup spaces) in our expressions.

Appendix B looks at some relevant adjoint representation operator relations for $SU(N)$; discussed in [17]. We then go on to calculate the form of the adjoint representation projection operators which appear in the calculations in this thesis. For example, all

[2]The projection operator method of [13] is particularly useful for studying generic coset models (where general G invariance is broken to H), whereas if we choose to study certain sets of models which have simpler coset vectors then the methods and ideas found in this thesis are preferable.

results for the Killing vectors and Goldstone boson manifold metrics are phrased with respect to these adjoint projection operators. This is especially helpful when we use the Killing vectors to reconstruct the Goldstone boson manifold metric; this procedure forming a doublecheck for the metric result.

Appendix C discusses the Weyl representation of the generators of $SO(m)$ and $SO(t,s)$ groups; the generators being constructed from a set of gamma matrices for the groups. We look at the homomorphism between $SO(6)$ and $SU(4)$, and also discuss some useful subgroups of $SO(6)$ which appear in this thesis; this discussion being phrased with respect to the r and q_r-vector framework.

Lastly, appendix D shows how the results for field gradients, and vector length gradients, are calculated. These results are just quoted throughout this thesis as they are substituted into the end of the various calculations. For example in chapter 4, the coset vector in an $SU(N)$ breaking to $SU(N-1)\otimes U(1)$ model may be written as :-

$$\begin{aligned} x &= \phi^a \lambda_a \\ &= \phi n^a \lambda_a \end{aligned}$$

where ϕ is the length of the vector, and $n^a\lambda_a$ is a *unit* vector which points in the direction of x. It turns out that quantities like $\partial_\mu n^a$ and $\partial_\mu \phi$ will appear in our calculations. However, we find that their explicit forms are only required towards the end of each calculation, and this saves on the ammount we actually need to write. As a result, the equations are also easier to read because we are able to use an index free notation throughout. So appendix D shows how to calculate quantities like these for all the coset vectors which appear in this thesis.

Chapter 1

Effective Lagrangians for nonlinear σ-models.

The aim of this chapter is to introduce the important results found in [8, 9, 10], with regards to constructing, and analysing, the effective Lagrangians of nonlinear σ-models. We will, however, begin by giving simple definitions of the quantities used to build the effective Lagrangians. Here the definitions are deliberately kept simple because it is only necessary, at the moment, to introduce ideas which will later be developed, and also to introduce the notation used in calculating the various quantities in the theory. Once this is done we can then go on to describe the mathematical framework.

1.1 The exponential quantities in this thesis.

The construction of an effective Lagrangian density relies on the manipulation of a quantity known as the coset representative element; essentially, it is transformed by the elements of Lie groups. Both the coset representative element and the Lie group elements are in the form of exponentials, which we will now introduce. The following definitions simply illustrate the difference in *notation* between the groups considered in the following chapters; more rigorous definitions will be introduced where appropriate.

1.1.1 Lie group elements.

For a Lie group, G, we may write elements of the group, $g \in G$:-

$$g \equiv e^{-ix} \tag{1.1}$$

If the dimension (total number of generators) of a Lie group is denoted dim(G) then x is a linear combination of dim(G) generators. These generators are traceless hermitian matrices and, because they are linearly independent, we are able to think of them as forming the *basis* of a real vector space of dimension $dim(G)$, which we denote $\Re^{dim(G)}$. Real vector spaces are examined in detail in the next chapter. Thus, we may think of x as being a general vector of $\Re^{dim(G)}$; and we will refer to this general vector as the group vector. An important property of group elements is given by :-

$$g_1 \cdot g_2 \Leftrightarrow g_3$$

Thus, on one hand, if we multiply any two group elements together then we just end up with another group element and, on the other, a group element may be rewritten as the product of two other elements.

Group elements, and all subsequent quantities of this form, are defined by the power series expansion of the exponential :-

$$e^{-ix} \equiv \mathbf{1}_{[N]} - ix - \frac{1}{2!}x^2 + \frac{i}{3!}x^3 + \frac{1}{4!}x^4 - \frac{i}{5!}x^5 - \frac{1}{6!}x^6 + \cdots$$

where the subscript N implies we are working with $N \times N$ matrices. We may rearrange this expression and write :-

$$e^{-ix} \equiv \mathbf{1}_{[N]} + \left(-\frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + - \cdots\right) - i\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - + \cdots\right)$$

which is rather suggestive because, if $N = 1$ and $x = \Theta$ is just a real number, then this expansion is :-

$$
\begin{aligned}
e^{-i\Theta} &\equiv 1 + (\cos\Theta\ - 1) - i\sin\Theta \\
&= \cos\Theta\ - i\sin\Theta
\end{aligned}
$$

This is a very familiar result. In this case though, because $N = 1$ implies that the generator cannot be traceless, it is called an element of the group of unitary ($g^\dagger \equiv g^{-1}$) 1×1 matrices, denoted $U(1)$. In this example the element is said to be in the defining representation as it is just a complex *number*. We remark that the odd function, $\sin\Theta$, is the imaginary part of the complex number; that is, it is accompanied by the imaginary number i. We make this remark because, even when x is a matrix, the (odd) sine functions are always preceeded by the imaginary number i; the (even) cosine functions are always contained in the rest of the expansion of the exponential.

When $N \geq 2$, the traceless nature of the generators leads to group elements with unit determinant, $det\, g = 1$, and thus we will be dealing with elements of $SU(N)$ (the group of Special Unitary $N \times N$ matrices) and $SO(m)$ (the group of Special Orthogonal $m \times m$ matrices). The *size* of the matrix, x, involved in the exponent depends on the group we are considering; and also on the representation we are using for the generators of the group elements. How x is *mathematically* expressed depends on the group we are using :-

1. The group of Special Unitary matrices (with $N \geq 2$) is denoted as $G = SU(N)$, and has a dimension of $dim(G) = (N^2 - 1)$. We write the group vector x :-

   $$x \equiv \theta^I T_I \qquad\qquad \text{for } I = 1, 2, \ldots, (N^2 - 1).$$

   where the θ^I are $(N^2 - 1)$ real parameters. The T_I are defined by a relation known as the Lie Algebra :-

   $$[T_I, T_J] = i f_{IJK} T_K$$

where $I, J, K = 1, 2, \ldots, (N^2 - 1)$. The meaning of this relation will be discussed further in section 1.1.4.

In the *Defining* representation the generators are defined by $T_I \equiv \frac{1}{2}\lambda_I$, and are $N \times N$ traceless Hermitian matrices. In this thesis we use the λ_I which therefore obey :-

$$[\lambda_I, \lambda_J] = 2if_{IJK}\lambda_K$$

In the *Adjoint* representation the generators, T_I, are $(N^2-1) \times (N^2-1)$ traceless Hermitian matrices, which are defined using the structure constants :-

$$(T_K)_{IJ} \equiv -if_{IJK} \qquad \text{for } I, J, K = 1, 2, \ldots, (N^2 - 1).$$

where this is an expression for the matrix components of the generators. This is possible because the structure constants of $SU(N)$ form a representation of the Lie algebra, i.e. they too obey the commutation relations above.

2. The group of Special Orthogonal matrices is denoted $G = SO(m)$ with $m \geq 2$, and has a dimension $dim(G) = \frac{1}{2}m(m-1)$. The group vector x is written :-

$$x \equiv \omega^{AB}T_{AB} \qquad \text{for } A, B = 1, 2, \ldots, m.$$

where the $\omega^{AB}(= -\omega^{BA})$ are $\frac{1}{2}m(m-1)$ real parameters. We will be dealing with $SO(m)$ groups in the *Weyl* representation where the $T_{AB} \equiv \frac{1}{2}\sigma_{AB}$, and we construct the $\sigma_{AB}(= -\sigma_{BA})$ from a set of m gamma matrices, γ_A. Both the σ-matrix construction, and the form of the commutation relations, are discussed in Appendix C, from page 191.

3. For $G = SO(t, s)$, with $t + s = m \geq 2$, we write x :-

$$x \equiv \omega^{\mu\nu}T_{\mu\nu}$$

where the index ranges depend on the values of t and s. Again, the $\omega^{\mu\nu}(= -\omega^{\nu\mu})$ are $\frac{1}{2}m(m-1)$ real parameters. In the *Weyl* representation the $T_{\mu\nu} \equiv \frac{1}{2}\Sigma_{\mu\nu}$, and we construct the $\Sigma_{\mu\nu}(= -\Sigma_{\nu\mu})$ from a set of m gamma matrices, Γ_{μ}; these are derived from the m gamma matrices of $SO(m)$, the γ^A.

For m (or $t+s$) $= 2k, 2k+1$ with integer k (≥ 1), the generators of $SO(m)$ and $SO(t,s)$ are $2^k \times 2^k$ matrices; this thesis only concerns models where $t = 1$ and therefore $s = (m-1)$. In chapter 3 we will look at models where $G = SO(m)$ invariance will be broken to $H = SO(m-1)$, and $G = SO(1, m-1)$ invariance will be broken to $H = SO(1, m-2)$. In these cases, we must obviously have $m \geq 3$ to define the group, G, which can be broken to a normal subgroup, $H \leq G$. In chapter 5 we will look at other possibilities.

1.1.2 Subgroup elements.

For a subgroup of G, denoted H, we may write elements of the subgroup, $h \in H$:-

$$h \equiv e^{-ix} \tag{1.2}$$

where x is now a linear combination of the dim(H) generators which generate the subgroup element. Thus we may think of the vector, x, as one which lies in a subspace of $\Re^{dim(G)}$. This subspace is $\Re^{dim(H)}$. We will refer to this restricted vector (in the sense that it lies in a *subspace* of $\Re^{dim(G)}$) as the subgroup vector, and the subspace of $\Re^{dim(G)}$, in which it lies, as the subgroup subspace. In this thesis we will mainly be considering models where G invariance is spontaneously broken to a subgroup $H \leq G$ where H is of the same *rank* as G; any exceptions will be noted. The rank of a (sub)group is defined as the maximum number of generators, or equivalently (sub)group elements, which will mutually commute.

1.1.3 Cosets and the coset representative.

A left coset, denoted $(\frac{G}{H})$, is formed by acting on the subgroup, H, from the left with a general group element :-

$$\begin{aligned} gH &\equiv g_c' hH \\ &\equiv LH \end{aligned} \tag{1.3}$$

where, in the notation of $SU(N)$, we define the subgroup as $H \equiv \{e^{-i\theta^E T_E} \; \forall \; \theta^E\}$, and because of the properties of group elements, we have written $g = g_c' h$. We define the coset representative element, L as :-

$$\begin{aligned} L &\equiv g_c' \\ &= e^{-ix} \end{aligned} \tag{1.4}$$

where x is a linear combination of $dim(G) - dim(H)$ generators. These generators are generators of G in the orthogonal complement of H. Thus, the vector x is one which lies in a coset (sub)space, $\Re^{dim(G)-dim(H)}$, and we will refer to it as a coset vector. It is the coset vector parameters/coordinates which identify a particular coset, and a different set of parameters will define a different coset; hence L being called the coset representative element. In physical applications the coset parameters are functions of x^μ, the spacetime coordinates, and are known as interpolating fields; for the $SU(N)$ breaking models in this thesis they are denoted ϕ^a where a runs over the coset indices. They are related to the Goldstone Boson fields, the M^a, that arise in a theory when we spontaneously break G invariance down to H. The relation is as follows. In terms of the interpolating fields the coset vector has a length of ϕ, we then make a specific reparameterization and write $\phi = \phi(M) = M + O(M^2)$ where M is the length of the Goldstone boson coset vector. So when we write the Goldstone vector as $M^a\lambda_a$ it points in the same direction as $\phi^a\lambda_a$ and their lengths are arbitrarily related. We note that this is not the most general reparameterization.

It is simple to demonstrate how many Goldstone fields there will be in a model which incorporates the phenomenon of spontaneous symmetry breaking. In such models the vacuum state is not invariant under all transformations of a group G (it is transformed to another vacuum state); it is only invariant under transformations of a subgroup, H, of G. So if we take as an example a scalar field theory, then under the field transformations $g \in G: \ \phi \mapsto \phi' = g\phi$ we know that the potential in the Lagrangian density is invariant, i.e. $V(\phi) = V(\phi')$. So if we Taylor expand the potential about its *minimum* value then we find :-

$$\begin{aligned} V(\phi_0) &= V(g\phi_0) \\ &= V(\phi_0) + \left(\frac{\partial V}{\partial \phi^I}\right)_{\phi_0} \delta\phi_0^I + \frac{1}{2!}\left(\frac{\partial^2 V}{\partial \phi^I \partial \phi^J}\right)_{\phi_0} \delta\phi_0^I \delta\phi_0^J + \cdots \end{aligned} \tag{1.5}$$

The first order term is zero because we are expanding about the vacuum value and so, for small variations, we must have the condition :-

$$\left(\frac{\partial^2 V}{\partial \phi^I \partial \phi^J}\right)_{\phi_0} \delta\phi_0^I \, \delta\phi_0^J \equiv 0 \tag{1.6}$$

Firstly, when $g = h \in H$ we find :-

$$\begin{aligned} \phi_0^{I'} &= (\Gamma_h)^I_J \, \phi_0^J \\ &= \phi_0^I \end{aligned}$$

where Γ_h is a (matrix) representation of the subgroup element, $h \in H$. Therefore $\delta\phi_0^I \equiv 0$ and equation (1.6) is automatically satisfied. However, when $g \notin H$ then we find :-

$$\begin{aligned} \phi_0^{I\,'} &= (\Gamma_g)^I_J \, \phi_0^J \\ &= \phi_0^I + \delta\phi_0^I \end{aligned}$$

To first order in the transformation parameters we have $(\Gamma_g)^I_J = \delta^I_J - i\omega^a (T_a)^I_J$ and so we find :-

$$\delta\phi_0^I = -i\omega^a (T_a)^I_J \, \phi_0^J \neq 0 \tag{1.7}$$

Therefore, in this case, equation (1.6) is satisfied when :-

$$\left(\frac{\partial^2 V}{\partial\phi^I \partial\phi^J}\right)_{\phi_0} \equiv 0 \tag{1.8}$$

and this implies that $\delta\phi_0^I = -i\omega^a(T_a)^I_J\phi_0^J$ are massless fields. We notice that the quantity $\delta\phi_0^I$ is always a function of a number of fields equal to the number of coset space parameters, ω^a. Therefore there is a one-to-one mapping between the space of massless (Goldstone) fields and the space of coset parameters. If we assign particular values to each of the ω^a then this is equivalent to assigning particular amplitudes to the Goldstone fields. In particular, if we set all the ω^a to zero, then all the Goldstone field amplitudes also become zero. Thus, the isomorphic mapping maps the origin of one space onto the origin of the other; which may be interpreted as changing coordinates, in a patch, from coset space parameters to Goldstone fields.

At this point it should be understood that, when it comes to calculating explicit results in the physical theory, it is far simpler to deal with the $SU(N)$ groups because we have less indices to worry about. However we will see in the next chapter that, using the λ-matrices of $SU(N)$, we are able to develop a mathematical structure which allows us to describe coset vectors in an *index free* way; and we may apply this method to the coset vectors associated with Special Orthogonal group breaking models, which effectively removes this complication.

1.1.4 Structure of the Lie algebra for $G = SU(N)$.

Here we will discuss the structure of the Lie algebras with respect to $G = SU(N)$ breaking to $H = SU(N-1) \otimes U(1)$, with G and H being of the same rank. We will construct similar results for $SO(m)$ and $SO(t,s)$ breaking calculations in Appendix C. This work is to be found in sections C 2 and C 3 which start on page 195.

- The generators of $SU(N)$, the T_I, obey a set of commutation relations which we call the Lie Algebra of $SU(N)$:-

$$[T_I, T_J] \equiv i f_{IJK} T_K$$

where $I, J, K = 1, 2, \ldots, (N^2 - 1)$ and the f_{IJK}, which are called structure constants, are totally antisymmetric under the interchange of any two (neighbouring) indices; i.e. $f_{IJK} = -f_{IKJ} = f_{KIJ}$. In this thesis we use a set of λ-matrices, which are related to the generators by $T_I \equiv \frac{1}{2}\lambda_I$. So the λ_I obey the commutation relation :-

$$[\lambda_I, \lambda_J] \equiv 2i f_{IJK} \lambda_K$$

To give an example, the group $SU(2)$ has, in the defining representation, a Lie algebra :-

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$$

where $i, j, k = 1, 2, 3$. The ε_{ijk}, with $\varepsilon_{123} \equiv 1$, are the structure constants of $SU(2)$. Notice that the λ_i have been rewritten as σ_i and this is because the σ_i are the three Pauli spin matrices which obey the Lie algebra. Any three traceless Hermitian matrices which obey the above relation may be used as the generators of $SU(2)$ group elements and, as it turns out, they will be related to the Pauli spin matrices by a unitary similarity transformation :-

$$\sigma_i' = u^\dagger \sigma_i u$$

In the next chapter we will examine the geometric meaning, and the consequences, of this relation for all $SU(N)$; when we define the λ-matrices as the basis of $\Re^{N^2-1}$.

- For the subgroup, H, of G we must have the Lie algebra :-

$$[\lambda_E, \lambda_F] \equiv 2if_{EFG}\lambda_G$$

where the $E, F, G = 1, 2, \ldots, (N-1)^2 - 1$ and $N^2 - 1$ are the set of indices which label the subgroup generators.

- For the coset G/H we have :-

$$\begin{aligned}[\lambda_a, \lambda_b] &\equiv 2if_{abI}\lambda_I \\ &= 2if_{abc}\lambda_c + 2if_{abE}\lambda_E\end{aligned}$$

where the I are the (N^2-1) group indices, the $a, b, c = (N-1)^2, \ldots, N^2 - 2$ are coset indices, and the E are the subgroup indices. In our work we will be dealing with *symmetric* spaces where the $f_{abc} \equiv 0\ \forall\ a, b$ and c. Thus we have :-

$$[\lambda_a, \lambda_b] \equiv 2if_{abE}\lambda_E$$

We note that symmetric spaces allow an isomorphic mapping of the coset space generators ($\lambda_a \mapsto -\lambda_a$), known as an Outer Involutive Automorphism, which maps G onto itself; this useful property will be discussed in the next section.

- So between subgroup and coset we have :-

$$[\lambda_a, \lambda_E] \equiv 2if_{aEb}\lambda_b$$

We note that the Lie algebra exhibits a $\mathbb{Z}_2$ grading structure.

1.2 Goldstone Boson transformations.

We will now look at the mathematical structure used to find the transformations of the Goldstone bosons in the theory. The method of construction centres around transforming the coset, LH, with a *global* group element, $g \in G$, from the left. However, as we will soon see, the properties of group elements allow us to ignore the H part of the coset because it is invariant; so we may restrict our attention to the coset representative element, L. When we transform L from the left with a global $g \in G$ we find :-

$$\begin{aligned} gL &\equiv g g'_c \\ &= g'' \\ &= g''_c h \\ &= L'h \end{aligned} \tag{1.9}$$

where the properties of group elements have again been used. This is an important equation because it is also used in the construction of the rest of the theory. We see that the $h \in H$ will have no effect on the H in the coset because $h : \; H \mapsto H' = H$. This is why we have ignored it. Since we are using (interpolating or Goldstone) *fields* to parameterize L, it is clear that h is a *local* transformation.

- Now if $g = h \in H$ then we may write :-

$$\begin{aligned} gL &\equiv hL \\ &= hL\left(h^{-1}h\right) \\ &= L'h \end{aligned}$$

  and so we immediately see that :-

$$L' \equiv hLh^{-1} \tag{1.10}$$

  This is a linear transformation of the coset parameters (Goldstone Bosons); in the next chapter we will see that a relation of this form tells us that they transform in the Adjoint/Vector representation. We will find, for some models, that we are able to calculate this transformation to all orders if we consider certain subgroup elements.

- If $g = c \notin H$ then we can only make these transformations more explicit if the coset space is a *symmetric* space. If it is, then it admits an automorphism which maps the coset space basis :-

$$\begin{aligned} \lambda_a &\mapsto -\lambda_a && \text{for } SU(N) \text{ breaking models,} \\ \sigma_{a\Delta} &\mapsto -\sigma_{a\Delta} && \text{for } SO(m) \text{ breaking models, and} \\ \Sigma_{\alpha\Delta} &\mapsto -\Sigma_{\alpha\Delta} && \text{for } SO(t,s) \text{ breaking models.} \end{aligned}$$

The subgroup generators, which form the subspace basis, are invariant under the automorphism, and so looking back to Section 1.1.4 (on page 27), and forward to Appendix C (on page 195), we see that the structure relations of G remain unchanged. Note also that the index ranges will be properly defined where appropriate; but for now it is sufficient to understand that they are coset indices which run over a subset of all possible values. If the coset space is a symmetric space then we proceed by inverting equation (1.9) and then applying the automorphism to give :-

$$Lc = h^{-1}L'$$

which we may combine with equation (1.9) to find :-

$$\left(L'\right)^2 \equiv cL^2c \tag{1.11}$$

which is a nonlinear transformation of the Goldstone Bosons.

In general we have :-

$$\left(L'\right)^2 \equiv gL^2A\left(g^{-1}\right) \tag{1.12}$$

where $A\left(g^{-1}\right)$ has been used to denote the result of applying the automporhism to the inverse of an element g. For a transformation produced by an element of the subgroup, that is $g = h \in H$, this relation reduces to :-

$$
\begin{aligned}
(L')^2 &\equiv hL^2h^{-1} \\
&= hLh^{-1}hLh^{-1}
\end{aligned}
$$

This implies that $L' = hLh^{-1}$ which is just equation (1.10), as desired.

1.2.1 Analysis to first order using Killing vectors.

For both the linear and nonlinear cases we may also find quantities known as Killing vectors. They contain information about the generators of the transformations of the Goldstone Bosons because we are working with equation (1.12) to first order. In an $SU(N)$ notation the Goldstone Bosons transform :-

$$
\begin{aligned}
M^a \mapsto M^{a'} &= M^a + \delta M^a \\
&= M^a + \theta^E \mathbf{K}_E^a + \theta^b \mathbf{K}_b^a \qquad (1.13)
\end{aligned}
$$

where the $\mathbf{K}_E^a$ are the linear Killing vector components, and the $\mathbf{K}_b^a$ are the nonlinear Killing vector components. The θ^E and θ^a are the subgroup element and (coset) element transformation parameters respectively. We are able to find the Killing vector components using the coset representative, L. If we understand that a transformed coset representative element written in terms of the original Goldstone fields has the same form as the original coset representative written in terms of transformed fields, i.e. $L'(M) = L(M')$ and use the notation of differentiation with respect to Goldstone fields $L_{,a} \equiv \partial L/\partial M^a$, then we may write :-

$$
\begin{aligned}
L'(M) &= L(M) + \delta L \\
&= L(M) + L_{,a}\delta M^a
\end{aligned}
$$

and if we use equation (1.13) we find :-

- For the linear transformation :-

$$L' = L + L_{,a}\theta^E \mathbf{K}_E^a \tag{1.14}$$

where $L' \Rightarrow L'(M)$ and $L \Rightarrow L(M)$. If we expand the LHS to first order in the transformation parameters we find :-

$$\begin{aligned} L' &= hLh^{-1} \\ &= L - \frac{i}{2}\left[\theta^E \lambda_E, L\right] \end{aligned} \tag{1.15}$$

and so, to find the $\mathbf{K}_E^a$ we must solve :-

$$\left[\theta^E \lambda_E, L\right] = 2i\, L_{,a}\theta^E \mathbf{K}_E^a$$

If the subgroup transformation parameters, the θ^E, are removed from the calculation we have :-

$$[\lambda_E, L] = 2i\, L_{,a}\mathbf{K}_E^a \tag{1.16}$$

- For the nonlinear transformation :-

$$(L')^2 = L^2 + L^2_{,a}\theta^b \mathbf{K}_b^a \tag{1.17}$$

Notice the use of squared terms; because we are considering a 'coset' of transformations. If we use the the same approach as before, then we find that to calculate the $\mathbf{K}_b^a$ we must solve :-

$$\left\{\theta^b \lambda_b, L^2\right\} = 2i\, L^2_{,a}\theta^b \mathbf{K}_b^a$$

When the coset transformation parameters, the θ^b, are removed from the calculation we have :-

$$\left\{\lambda_b, L^2\right\} = 2i\, L^2_{,a}\mathbf{K}_b^a \tag{1.18}$$

Lastly, from [8], we understand that not only do these Killing vectors tell us about the transformations of the nonlinear realizations, but they may also be used to check the form of the Goldstone boson part of the Lagrangian which we will build. This will be explained in section 1.4.1a.

1.3 Covariant derivatives for the Goldstone bosons and matter fields.

To construct an Effective Lagrangian we must find Covariant derivatives for the fields in the theory. We know that there will be Goldstone boson fields and we have just looked at a method of finding their transformation properties. Now, even though Goldstone's theorem does not require any other fields in the theory, we will introduce a set of matter fields that interact with the Goldstone bosons in a natural way. We will then show how to find covariant derivatives for both the Goldstone fields and the (standard) matter fields.

1.3.1 The form of the Standard field covariant derivative.

The fields in the unbroken theory, which we may call Φ, transform as a linear representation of G :-

$$g \in G : \quad \Phi \mapsto \Phi' = g\Phi$$

Using the Φ we may define a set of *matter* fields, also called *Standard* fields :-

$$\psi \equiv L^{-1}\Phi \tag{1.19}$$

The standard fields have the property that they transform :-

$$
\begin{aligned}
g: \quad \psi \mapsto \psi' &= L'^{-1}\Phi' \\
&= (hL^{-1}g^{-1})(g\Phi) \\
&= hL^{-1}\Phi \\
&= h\psi
\end{aligned}
\tag{1.20}
$$

where, in the second line, we have used equation (1.9). For the moment we note that the Pauli adjoint spinor, $\bar{\psi}$, transforms as :-

$$
g: \overline{\psi} \mapsto \overline{\psi}' = \overline{\psi} h^{-1} \tag{1.21}
$$

1.3.1a Transformation of $\partial_\mu \psi$.

We now ask how $\partial_\mu \psi$ transforms by using the above relation. We see :-

$$
g: \partial_\mu \psi \mapsto \partial_\mu \psi' = h\partial_\mu \psi + (\partial_\mu h)\,\psi
$$

and so $\partial_\mu \psi$ is not a Covariant derivative; its transformation is different to that in equation (1.20). What we need is $\mathcal{D}_\mu = \partial_\mu + X_\mu$ such that $\mathcal{D}_\mu \psi \mapsto h\mathcal{D}_\mu \psi$. Using this new form we now have as our transformation:-

$$
\begin{aligned}
g: (\partial_\mu + X_\mu)\,\psi \mapsto [(\partial_\mu + X_\mu)\,\psi]' &= \left(\partial_\mu + X'_\mu\right)\psi' \\
&= \left(\partial_\mu + X'_\mu\right) h\psi \\
&= (\partial_\mu h)\,\psi + h\partial_\mu \psi + X'_\mu h\psi
\end{aligned}
$$

which must equal $h\,(\partial_\mu + X_\mu)\,\psi$, and so we must have :-

$$
hX_\mu \psi = (\partial_\mu h)\,\psi + X'_\mu h\psi
$$

This tells us how the quantity X_μ transforms. We see :-

$$
\begin{aligned}
X'_\mu &= hX_\mu h^{-1} - (\partial_\mu h)\,h^{-1} \\
&= hX_\mu h^{-1} + h\partial_\mu h^{-1}
\end{aligned}
\tag{1.22}
$$

We will now show how to find X_μ which will give us our covariant derivative for the standard fields, and also how to find the covariant derivative for our Goldstone fields.

1.3.2 The Covariant derivatives.

The following method will give us covariant derivatives for the Goldstone fields and Standard fields. By differentiating equation (1.9) with respect to the spacetime coordinates, x^μ, we find :-

$$g\partial_\mu L = (\partial_\mu L')\, h + L'\,(\partial_\mu h)$$

and if we now take g to be the local transformation $g = L^{-1}$ then we have :-

$$L^{-1}\partial_\mu L \equiv -\frac{i}{2}\,(a_\mu + v_\mu) \tag{1.23}$$

Under the action of $g \in G$ this transforms :-

$$\begin{aligned} g : L^{-1}\partial_\mu L \mapsto L'^{-1}\partial_\mu L' &= hL^{-1}g^{-1}\partial_\mu\left(gLh^{-1}\right) \\ &= hL^{-1}\partial_\mu\left(Lh^{-1}\right) \\ &= h\left(L^{-1}\partial_\mu L\right)h^{-1} + h\partial_\mu h^{-1} \end{aligned} \tag{1.24}$$

So we find that under the action of $g \in G$:-

$$a_\mu \mapsto ha_\mu h^{-1} \tag{1.25}$$

$$-\frac{i}{2}v_\mu \mapsto h\left(-\frac{i}{2}v_\mu\right)h^{-1} + h\partial_\mu h^{-1} \tag{1.26}$$

- Firstly, equation (1.25) tells us that we may interpret the components of a_μ as a covariant derivative for the Goldstone fields. In the next chapter we will see that this has the same form as the transformation of a vector, so the components are transforming in the adjoint representation. So our covariant derivative for the Goldstone boson fields is written in the form :-

$$a_\mu^a \equiv \mathcal{D}_\mu M^a \tag{1.27}$$

in $SU(N)$ breaking calculations, or

$$a_\mu^{iX} \equiv \mathcal{D}_\mu M^{iX} \tag{1.28}$$

in $SO(m)$, and $SO(t,s)$, breaking calculations. Note that for these models the indices will be properly defined where appropriate.

- Secondly, if we compare equation (1.26) with equation (1.22) we see that we have found what we needed to form the covariant derivative $\mathcal{D}_\mu\psi$. We thus have :-

$$\mathcal{D}_\mu\psi = (\partial_\mu - \tfrac{i}{2}v_\mu)\psi \tag{1.29}$$

Using a similar line of reasoning we may also construct a covariant derivative for $\overline{\psi}$ which we can write as :-

$$\overline{\psi}\overleftarrow{\mathcal{D}}_\mu \equiv \overline{\psi}(\overleftarrow{\partial}_\mu + \tfrac{i}{2}v_\mu) \tag{1.30}$$

Notice the change in sign of the second term. By definition, this transforms in the same way as $\overline{\psi}$:-

$$g : (\overline{\psi}\overleftarrow{\mathcal{D}}_\mu) \mapsto (\overline{\psi}\overleftarrow{\mathcal{D}}_\mu)' = (\overline{\psi}\overleftarrow{\mathcal{D}}_\mu)\, h^{-1} \tag{1.31}$$

1.4 Invariants terms and the Effective Lagrangian.

We now have all we need to construct an Effective Lagrangian for the theory. We must now find invariant terms which may be used to form the pieces of the Lagrangian. The scalar (Goldstone boson) and the spinor (matter field) parts of the effective Lagrangian density will be considered separately.

1.4.1 The Goldstone boson part of the Lagrangian.

If we look at the form of equation (1.25), we see that the quantity $\frac{1}{2}tr\, a_\mu a^\mu$ is invariant under group transformations :-

$$\begin{aligned}
\tfrac{1}{2}tr\, a_\mu a^\mu \;\mapsto\; \tfrac{1}{2}tr\, a'_\mu a'^\mu &= \tfrac{1}{2}tr\, h a_\mu h^{-1} h a^\mu h^{-1} \\
&= \tfrac{1}{2}tr\, a_\mu a^\mu \\
&= (\mathcal{D}_\mu M^a)(\mathcal{D}^\mu M^a) \\
&= g_{ab}\,(\partial_\mu M^a)\,(\partial^\mu M^b) \qquad (1.32)
\end{aligned}$$

where, in the last line, we have introduced the coset space metric, g_{ab}, of Isham [8]. In practice we use half this quantity in the Lagrangian, and when it is expanded [1] we find that the first term we get may be interpreted as a *Kinetic* term for the Goldstone bosons; the other terms being interaction terms between the Goldstone bosons. For $SO(m)$ and $SO(t,s)$ breaking we will find the following form :-

$$\tfrac{1}{2}tr\, a_\mu a^\mu = g_{iX\;jY}\left(\partial_\mu M^{iX}\right)\left(\partial^\mu M^{jY}\right) \qquad (1.33)$$

and the index ranges will be properly defined in the relevant sections. In the next chapter we will also introduce the notation $(a_\mu, a^\mu) \equiv \frac{1}{2}tr\, a_\mu a^\mu$ and we will have :-

$$(a_\mu, a^\mu) = a_\mu^a a_a^\mu$$

in the $SU(N)$ breaking models. This is simple to see because if $a_\mu = a_\mu^a \lambda_a$ then (a_μ, a^μ) contains $(\lambda_a, \lambda_b) \equiv \delta_{ab}$. For the $SO(m)$ and $SO(1, m-1)$ breaking models we will have the quantity :-

$$(a_\mu, a^\mu) = K\, a_\mu^{iX} a_{iX}^\mu$$

Notice that a constant, K, has appeared on the right hand side because $(\sigma_{iX}, \sigma_{jY})$ *and* $(\Sigma_{iX}, \Sigma_{jY})$ depend on the size of the sigma matrices (see Appendix C). However, this

[1] For the $SU(N)$ models $g_{ab} \equiv \delta_{ab} + \ldots$.

constant may be ignored because it will also appear when we physically calculate the left hand side. Therefore we will just calculate/consider $a_\mu^{iX} a_{iX}^\mu$. In the language used in the next chapter we say that the $SO(m)$ sigma matrices, σ_{AB}, are not orthonormal basis vectors; instead they are just orthogonal vectors (which may be normalized to form a set of basis vectors).

1.4.1a Using the Killing vectors to construct the metric.

We understand, from the paper by Isham [8], that we may use the Killing vector components to check the form of the Goldstone boson manifold metric. It is therefore true that, for the more complicated models considered in this thesis, this metric reconstruction may be used as a doublecheck to verify the results for the nonlinear Killing vector components (assuming the metric to be correct). In contrast, the results for the linear Killing vector components may be verified with only a small ammount of extra work, and so the relevant doublecheck will be included in the thesis; and this metric reconstruction then becomes a triplecheck for the form of the linear Killing vector components.

- For $SU(N)$ breaking models we have :-

  $$(a_\mu, a^\mu) \equiv g_{ab} \left(\partial_\mu M^a\right) \left(\partial^\mu M^b\right)$$

  and we may form the Goldstone boson manifold metric :-

  $$g_{ab} \equiv \left(\mathbf{K}_E^a \mathbf{K}_E^b + \mathbf{K}_c^a \mathbf{K}_c^b\right)^{-1} \tag{1.34}$$

  where the quantity $\left(\mathbf{K}_E^a \mathbf{K}_E^b + \mathbf{K}_c^a \mathbf{K}_c^b\right)$ is the inverse of the left hand side, and so has the property :-

  $$g_{ac} \left(\mathbf{K}_E^c \mathbf{K}_E^b + \mathbf{K}_d^c \mathbf{K}_d^b\right) = \delta_a^b$$

For these models there is no distinction between covariant and contravariant indices.

- For $SO(m)$, and $SO(t,s)$ breaking models we have :-

$$(a_\mu, a^\mu) \equiv g_{iX\ jY} \left(\partial_\mu M^{iX}\right)\left(\partial^\mu M^{jY}\right)$$

and we may form the Goldstone boson manifold metric :-

$$g_{iX\ jY} \equiv \left(\mathbf{K}^{iX}_{kl}\mathbf{K}^{kl\ jY} + \mathbf{K}^{iX}_{kZ}\mathbf{K}^{kZ\ jY}\right)^{-1} \tag{1.35}$$

where the quantity $\left(\mathbf{K}^{iX}_{kl}\mathbf{K}^{kl\ jY} + \mathbf{K}^{iX}_{kZ}\mathbf{K}^{kZ\ jY}\right)$ is the inverse of the left hand side, and so has the property :-

$$g_{iX\ kZ}\left(\mathbf{K}^{kZ}_{mn}\mathbf{K}^{mn\ jY} + \mathbf{K}^{kZ}_{lW}\mathbf{K}^{lW\ jY}\right) = \delta^{jY}_{iX}$$

For $SO(m)$ breaking calculations there is no distinction between covariant and contravariant indices, but to make things easier when we come to work on $SO(1,m-1)$ breaking models we keep the indices balanced and sum over the repeated upper and lower indices.

1.4.2 The matter field part of the Lagrangian.

For the matter fields we see that we may form two invariant terms. In a little while we will show how to construct the Pauli adjoint spinor, $\overline{\psi}$, which transforms :-

$$g:\ \overline{\psi} \mapsto \overline{\psi}' = \overline{\psi}\, h^{-1}$$

However, since we know its transformation properties, it is obvious that we may form an invariant mass term for the matter field part of the Lagrangian density because :-

$$g:\ m\overline{\psi}\psi \mapsto m\overline{\psi}h^{-1}h\psi = m\overline{\psi}\psi. \tag{1.36}$$

This quantity is also invariant under Lorentz transformations of the spacetime coordinates. Secondly, we see that we also have an invariant :-

$$g: \; \overline{\psi}\mathcal{D}_\mu\psi \mapsto \overline{\psi}h^{-1}h\mathcal{D}_\mu\psi \;\; = \;\; \overline{\psi}\mathcal{D}_\mu\psi$$

where $\mathcal{D}_\mu \Rightarrow \overrightarrow{\mathcal{D}}_\mu$. However $\overline{\psi}\mathcal{D}_\mu\psi$ is invariant only under group action. On the other hand it is a 4-vector, and so to make this piece invariant under Lorentz transformations we must use the quantity :-

$$\overline{\psi}\Gamma^\mu\mathcal{D}_\mu\psi \;\; \equiv \;\; \overline{\psi}\,\not{\mathcal{D}}\psi \tag{1.37}$$

Also, from the transformation property of the covariant derivative for the Adjoint Pauli spinor, $\overline{\psi}\,\overleftarrow{\mathcal{D}}_\mu$, we see that we also have an invariant :-

$$g: \; \overline{\psi}\,\overleftarrow{\not{\mathcal{D}}}\,\psi \mapsto \overline{\psi}\,\overleftarrow{\not{\mathcal{D}}}\,h^{-1}h\psi \;\; = \;\; \overline{\psi}\,\overleftarrow{\not{\mathcal{D}}}\,\psi \tag{1.38}$$

which is also Lorentz invariant. Therefore, together with $\overline{\psi}\,\not{\mathcal{D}}\psi$, we may write an invariant term :-

$$\overline{\psi}\,\overleftrightarrow{\not{\mathcal{D}}}\,\psi \;\; \equiv \;\; \frac{1}{2}(\overline{\psi}\,\overrightarrow{\not{\mathcal{D}}}\,\psi - \overline{\psi}\,\overleftarrow{\not{\mathcal{D}}}\,\psi) \tag{1.39}$$

1.4.2a Constructing $\overline{\psi}$, the Pauli Adjoint of the spinor ψ.

The Lagrangian has been constructed such that it is invariant under Lorentz transformations of the spacetime coordinates. In constructing the matter part of the Lagrangian we have used the Pauli adjoint spinor, $\overline{\psi}$, which is defined :-

$$\overline{\psi} \;\; \equiv \;\; \psi^\dagger A \tag{1.40}$$

We will now show how to find the matrix A. Firstly, the Dirac adjoint of a matrix, X, is defined by :-

$$(\overline{\Psi}X\Phi)^\dagger \;\; \equiv \;\; \overline{\Phi}\,\overline{X}\,\Psi \tag{1.41}$$

Using the definition of the Pauli adjoint spinor, the left hand side of this is :-

$$
\begin{aligned}
(\overline{\Psi}X\Phi)^{\dagger} &= (\Psi^{\dagger}AX\Phi)^{\dagger} \\
&= \Phi^{\dagger}X^{\dagger}A^{\dagger}\Psi \\
&= \Phi^{\dagger}AA^{-1}X^{\dagger}A^{\dagger}\Psi \\
&= \overline{\Phi}A^{-1}X^{\dagger}A^{\dagger}\Psi
\end{aligned}
$$

Therefore, we see that we need to satisfy :-

$$
\overline{X} \equiv A^{-1}X^{\dagger}A^{\dagger}
$$

For simplicity, if we now let A be an Hermitian matrix, i.e. $A^{\dagger} \equiv A$, then this relation may be written :-

$$
\overline{X} \equiv A^{-1}X^{\dagger}A \tag{1.42}
$$

Now, in the theory, we would like $\overline{\psi}\,\Gamma^{\mu}\psi \equiv J^{\mu} = (\rho, \underline{J})$ to be a divergenceless ($\partial_{\mu}J^{\mu} = 0$) Noether current which will lead to a charge operator :-

$$
\begin{aligned}
Q &\sim \int J^{0}d^{3}x \\
&\sim \int \psi^{\dagger}\psi d^{3}x
\end{aligned}
$$

which is the conserved quantity. So if we now have $X = \Gamma^{\mu}$ then we need to find the Hermitian matrix A such that $\overline{\Gamma^{\mu}} \equiv \Gamma^{\mu}$ (which will give the four-current, J^{μ}, real components). A set of gamma matrices is defined in Appendix C; or rather, we define the Γ_{μ} which are just $\eta_{\mu\nu}\Gamma^{\nu}$, but this will not change the qualitative results of what follows. We will now work on equation (1.42) in parts. Firstly, since $\Gamma^{0\dagger} = \Gamma^{0}$, we have the relation :-

$$
\begin{aligned}
\overline{\Gamma^{0}} &= A^{-1}\Gamma^{0\dagger}A \\
\Gamma^{0} &= A^{-1}\Gamma^{0}A
\end{aligned}
$$

Premultiplying this with A leads to :-

$$[A, \Gamma^0] \equiv 0 \tag{1.43}$$

Secondly, since $\Gamma^{i\dagger} = -\Gamma^i$, following the same procedure as above will now produce the anticommutator :-

$$\{A, \Gamma^i\} \equiv 0 \tag{1.44}$$

The gamma matrices obey the two relations :-

$$\begin{aligned} [\Gamma_\mu, \Gamma_\nu] &\equiv 2i\,\Sigma_{\mu\nu} \\ \{\Gamma_\mu, \Gamma_\nu\} &\equiv 2\,\eta_{\mu\nu}\,\mathbf{1}_{[2^k]} \end{aligned}$$

If we now, appropriately, restrict the indices in these expressions we find :-

$$\begin{aligned} [\Gamma_\mu, \Gamma_0] &\equiv 2i\,\Sigma_{\mu 0} \\ \{\Gamma_\mu, \Gamma_i\} &\equiv 2\,\eta_{\mu i}\,\mathbf{1}_{[2^k]} \end{aligned}$$

and then, clearly, equations (1.43) and (1.44) are satisfied by the Hermitian matrix choice $A \equiv \Gamma^0$.

It is important to understand that for different $SO(t, s)$ models the matrix A will be also be different. To give an example, without justification, we find for $SO(2, 4)$ that we need to have $A \equiv \Gamma^0\Gamma^6 = -i\,\Sigma^{06}$.

1.4.3 The complete Effective Lagrangian density.

Putting the results of the last two sections together we see that we may construct an Effective Lagrangian density which is the sum of a Goldstone boson part and a matter field part. Therefore we have :-

$$\mathcal{L}_{eff} = \frac{1}{2}\,(a_\mu, a^\mu) + \overline{\psi}(i \overleftrightarrow{\not{\mathcal{D}}} - m)\psi \tag{1.45}$$

where the form of the $(a_\mu, a^\mu) \equiv \frac{1}{2} tr\, a_\mu a^\mu$ term is given

$$\begin{aligned}
(a_\mu, a^\mu) &= g_{ab} \left(\partial_\mu M^a\right) \left(\partial^\mu M^b\right) && \text{for } SU(N) \text{ breaking.} \\
(a_\mu, a^\mu) &= g_{iX\,jY} \left(\partial_\mu M^{iX}\right) \left(\partial^\mu M^{jY}\right) && \text{for } SO(m) \text{ or SO(t,s) breaking.}
\end{aligned}$$

It is usual though just to consider the Lagrangian density :-

$$\mathcal{L}_{eff} = \frac{1}{2} (a_\mu, a^\mu) + \overline{\psi}(i \not{\mathcal{D}} - m)\psi \tag{1.46}$$

where the $\mathcal{D}_\mu$ acts to the right only. This is permissible since the only difference between equation (1.45) and equation (1.46) is a total divergence; which does not change the action. Both Lagrangian densities, however, are what we would expect for a theory of this type since :-

1. The massless scalar field multiplet (Goldstone boson) part of the Lagrangian density contains, at lowest order, the kinetic term $\frac{1}{2}\partial_\mu M^a \partial^\mu M^a$. This is the usual form of a scalar field multiplet Lagrangian density kinetic term. This is also accompanied by higher order (self) interaction terms.

2. The matter field part of the density has the usual form of a Dirac-like Lagrangian density. This is accompanied by interaction terms between the matter fields and the massless scalar (Goldstone) fields; provided by the v_μ term in the covariant derivative $\mathcal{D}_\mu \psi$.

Chapter 2

The geometry of Real vector spaces.

It is clear from the construction of the physical theory that we do not ever need to find explicit expressions for general group elements. In fact we only really need to write expressions for the coset representative, and we will see that only certain subgroup elements need to be considered to understand some of the physical consequences of the theory. The approach we will use is to understand the real vector space associated with the generators of group elements. The construction is phrased with respect to $SU(N)$ group element generators, because it is based on the techniques employed by L.Michel and L.Radicati [17] in their study of $SU(3)$ and the geometry of $\Re^8$. Although the theory has been explicitly formulated for $SU(N)$ groups, we may work with $SO(m)$ groups too. We do not necessarily have to exploit any homomorphisms to do this (though we may) because the generators of $SO(m)$ groups are also traceless hermitian matrices. Therefore our framework will allow us to apply the same ideas to find the coset representative, and subgroup elements, for theories of broken $SO(m)$ group symmetries.

Group elements have been introduced in section 1.1.1, but here we will be a little more rigorous. In the *defining* representation, $SU(N)$ is the group of Special (*unimodular* $\Rightarrow$

$det\, g = 1$), Unitary ($g^\dagger = g^{-1}$), $N \times N$ matrices. General Lie group elements, $g \in G$, may be written :-

$$\begin{aligned} g &\equiv e^{-ix} \\ &= e^{-i\theta^I T_I} \\ &= e^{-i\theta^I \frac{1}{2}\lambda_I} \end{aligned} \tag{2.1}$$

The first two lines are the definition of Lie group elements, and the last line shows explicitly the quantities we will deal with; the λ_I are a set of $(N^2 - 1)$ traceless, hermitian $N \times N$ matrices, and the θ^I are a set of $(N^2 - 1)$ real parameters. The generators for $SU(N)$ are the T_I. Now the λ_I obey anticommutation and commutation relations :-

$$\{\lambda_I, \lambda_J\} = \frac{4}{N}\delta_{IJ}\,\mathbf{1}_{[N]} + 2\,d_{IJK}\lambda_K \tag{2.2}$$

$$[\lambda_I, \lambda_J] = 2i\, f_{IJK}\lambda_K \tag{2.3}$$

Where the d_{IJK} are symmetric, and the f_{IJK} antisymmetric, under interchange of any two indices. The commutation relation is called the Lie algebra of $SU(N)$ and defines the generators of the group transformations; and therefore the λ_K. So we have the product rule :-

$$\lambda_I \lambda_J = \frac{2}{N}\delta_{IJ}\,\mathbf{1}_{[N]} + (d_{IJK} + i\, f_{IJK})\,\lambda_K \tag{2.4}$$

From this we see that we have a quantity :-

$$\begin{aligned} (\lambda_I, \lambda_J) &\equiv \frac{1}{2} tr\, \lambda_I \lambda_J \\ &= \delta_{IJ} \end{aligned} \tag{2.5}$$

and so we may think of the λ_I as forming a *Basis* for an $(N^2 - 1)$ dimensional *Real Vector Space*, $\Re^{N^2-1}$; this relation being the Euclidean Scalar Product between the

Basis Vectors. So the Euclidean scalar product between two vectors, x and y, is :-

$$\begin{aligned}(x,y) &= x^I y^J (\lambda_I,\lambda_J)\\ &= x^I y^I\end{aligned}$$

which, for $N = 2$, is a very familiar expression since we are dealing with vectors in $\Re^3$. Also, when $y = x$ we have an expression for the norm of x, and we will soon see that this is an invariant under a rotation of the basis of the vector space; it is known as a matrix invariant.

2.1 The algebras of Real Vector Spaces, $\Re^{N^2-1}$.

We represent a general vector of $\Re^{N^2-1}$:-

$$\begin{aligned}x &= \frac{\theta^I}{2}\lambda_I\\ &= x^I\lambda_I\end{aligned}$$

From the product rule we may define two linearly independent algebras :-

1. The first is based on the commutator of basis vectors :-

$$[\lambda_I,\lambda_J] \equiv 2i\, f_{IJK}\lambda_K$$

which is the Lie algebra of $SU(N)$. We rewrite this :-

$$\begin{aligned}\lambda_I\wedge\lambda_J &\equiv -\frac{i}{2}[\lambda_I,\lambda_J] \qquad (2.6)\\ &= f_{IJK}\lambda_K\end{aligned}$$

For two vectors x and y we have :-

$$x\wedge y = x^I y^J f_{IJK}\lambda_K \qquad (2.7)$$

and the vector $x \wedge y$ is orthogonal to both x and y since :

$$\begin{aligned}(x \wedge y, x) &= 0 \\ (x \wedge y, y) &= 0\end{aligned}$$

For the example of the Lie algebra of $SU(2)$, where the Pauli spin matrices are used to represent the basis of a real 3 dimensional vector space, $\Re^3$, we understand that equation (2.7) is the usual vector product, or cross product, between two vectors.

2. The second algebra is a symmetric algebra based on the anticommutator between basis vectors :-

$$\{\lambda_I, \lambda_J\} \equiv \frac{4}{N}\delta_{IJ}\,\mathbf{1}_{[N]} + 2\,d_{IJK}\lambda_K$$

which we rewrite :-

$$\begin{aligned}\lambda_I \vee \lambda_J &\equiv \frac{\sqrt{N}}{2}\{\lambda_I, \lambda_J\} - \frac{2}{\sqrt{N}}(\lambda_I, \lambda_J)\ \mathbf{1}_{[N]} \\ &= \sqrt{N}\,d_{IJK}\lambda_K\end{aligned} \tag{2.8}$$

For two vectors we have :-

$$x \vee y \equiv \sqrt{N}\,x^I y^J d_{IJK}\lambda_K \tag{2.9}$$

This is a new type of vector product which is possible for $\Re^{N^2-1}$ with $N \geq 3$.

These two algebras are used to define linear operators of the adjoint, $f_x \equiv x\wedge$ and $d_x \equiv x\vee$, which transform the vector spaces themselves. This is done in Appendix B, on page 167, where we also find some relations between the adjoint operators. But for now we proceed by letting $y = x$. We find :-

$$\begin{aligned}x \wedge x &\equiv 0 \\ x \vee x &\equiv \frac{1}{\sqrt{N}}\left(Nx^2 - 2\,(x, x)\ \mathbf{1}_{[N]}\right)\end{aligned}$$

The second of these two expressions may be rearranged :-

$$x^2 = \frac{2}{N}(x,x)\ \mathbf{1}_{[N]} + \frac{1}{\sqrt{N}}\, x \vee x \tag{2.10}$$

and we see that we have an index free expression for the second power in the expansion of the group element. However, we are no further forward at the moment since we do not understand, yet, what $x \vee x$ really means. All we can say is that $x \vee x$ is a vector which *commutes* with x.

2.2 Rotating vectors of $\Re^{N^2-1}$, and the characteristic equation.

In order to continue we must understand the nature of the basis of traceless Hermitian matrices. From a purely mathematical point of view a traceless Hermitian matrix, x, obeys a characteristic equation :-

$$x^N - \gamma_2(x)x^{N-2} - \gamma_3(x)x^{N-3} - \cdots - \gamma_N(x)\,\mathbf{1}_{[N]} \equiv 0 \tag{2.11}$$

with

$$\gamma_k(x) \equiv \frac{1}{k} tr\left(x^k - \sum_{a=2}^{k-2}\gamma_a(x)x^{k-a}\right) \tag{2.12}$$

From this we see :-

$$\begin{aligned}\gamma_2(x) &\equiv \frac{1}{2} tr\, x^2 = (x,x)\\ \gamma_3(x) &\equiv \frac{1}{3} tr\, x^3\end{aligned}$$

and we have already met $\gamma_2(x)$. We note here that this characteristic equation implies that, for $N > 2$, the highest power of x appearing in the expansion of the group element is x^{N-2}; higher powers of x reduce.

We know that (traceless) hermitian matrices may be diagonalized by a Unitary similarity transformation :-

$$u: \quad x \mapsto x_D \quad = \quad u^\dagger x\, u \tag{2.13}$$

which we think of as a transformation of $\Re^{N^2-1}$ by $u \in SU(N)$; the transformation being defined as a rotation of the vector space basis. Therefore the mathematical act of diagonalizing the matrix, x, may be physically interpreted as the rotation of the vector, x, into the subspace of $\Re^{N^2-1}$ which is defined by the diagonal basis vectors. Thus, we have defined the action of the Adjoint representation since, for *any* $g \in SU(N)$, the components of x transform :-

$$\begin{aligned} g: \quad (x, \lambda_I) \mapsto (x', \lambda_I) &= \left(g^\dagger x g, \lambda_I\right) \\ x^{I\,\prime} &= \left(g^\dagger \lambda_J g, \lambda_I\right) x^J \\ &= R_{IJ}\, x^J \end{aligned}$$

where $R_{IJ} \equiv \left(g^\dagger \lambda_J g, \lambda_I\right) = \frac{1}{2} tr\left(g \lambda_I g^\dagger \lambda_J\right)$ is defined as the Adjoint representation of the group element $g \in G$, sometimes called $Ad(g)$. We see that the scalar product between two vectors, (x, y), is *invariant* under this transformation of $\Re^{N^2-1}$; therefore, quite obviously, (x, x) is too. In general :-

$$\begin{aligned} \gamma_k\left(u^\dagger x u\right) &\equiv \frac{1}{k} tr\left(u^\dagger x^k u - \sum_{a=2}^{k-2} \gamma_a\left(u^\dagger x u\right) u^\dagger x^{k-a} u\right) \\ &= \frac{1}{k} tr\left(x^k - \sum_{a=2}^{k-2} \gamma_a(x) x^{k-a}\right) \\ &= \gamma_k(x). \end{aligned}$$

Therefore the $\gamma_k(x)$ are known as matrix invariants and, for a general $N \times N$ traceless Hermitian matrix, there are $(N-1)$ of them. The characteristic equation after the basis rotation transformation becomes :-

$$\begin{aligned} u^\dagger x^N u - \gamma_2(x) u^\dagger x^{N-2} u - \gamma_3(x) u^\dagger x^{N-3} u - \cdots - \gamma_N(x)\, \mathbf{1}_{[N]} &\equiv 0 \\ x_D^N - \gamma_2(x) x_D^{N-2} - \gamma_3(x) x_D^{N-3} - \cdots - \gamma_N(x)\, \mathbf{1}_{[N]} &= 0 \end{aligned}$$

The diagonal elements of x_D are real numbers, known as eigenvalues, and the form of the equation now implies that it may be interpreted as an eigenvalue equation for x; with $(N-1)$ independent solutions. We now make a very important point, which we may state in two equivalent ways :-

- Any two traceless Hermitian matrices, x and y, with the same characteristic equation have the same eigenvalues. This means that they are similar, or related to each other via a unitary similarity transformation.

- Any two vectors, x and y, with the same characteristic equation have the same eigenvalues. This means that they are related to each other via a rotation in $\Re^{N^2-1}$; in the language of [17] we say that these two vectors define an Orbit which lies in a particular Stratum of $\Re^{N^2-1}$.

We notice that, since neither of these statements depends on G, we may not only use this property for group vectors of $SU(N)$ groups, but also for group vectors of $SO(m)$ groups. Thus, the mathematical language/notation we will use to describe the $SU(N)$ breaking models is suited to the $SO(m)$ breaking models too.

We end this section with the physical implications of the simplest characteristic equation. For $N = 2$ we have :-

$$x^2 - \gamma_2(x)\,\mathbf{1}_{[2]} \equiv 0$$

Since we are dealing with 2×2 traceless Hermitian matrices there is, effectively, only one way of diagonalizing *any* matrix x. This is because, in this case, there is only one matrix invariant, $\gamma_2(x)$, and therefore only one diagonal matrix in the $\Re^3$ basis. In geometrical terms we may say that it makes no difference in which direction *any* vector, x, actually points in $\Re^3$ since we may always rotate it around the vector space until it points in the third direction which, using the Pauli Spin matrices to represent the basis vectors of the space, is the diagonal direction. We also see this because the three Pauli

Spin matrices are similar matrices (related to each other via similarity transformations $\Leftrightarrow$ each having eigenvalues ± 1) and this means that there is only one type of vector which may be defined in $\Re^3$. The Lie algebra is a statement of the familiar vector (cross or $\wedge$) product.

2.3 Vectors in C_x, the Cartan Subspace of x.

We may use the symmetric, $\vee$, algebra to build a set of mutually commuting vectors. We have already constructed a vector which commutes with x; namely $x \vee x$ which is defined :-

$$x \vee x \equiv \tfrac{1}{\sqrt{N}} \left(N\, x^2 - 2\gamma_2(x)\, \mathbf{1}_{[N]} \right) \tag{2.14}$$

Now the Euclidean scalar product between x and $x \vee x$ is :-

$$\begin{aligned} (x, x \vee x) &\equiv \tfrac{1}{2\sqrt{N}} tr\; x \left(N\, x^2 - 2\gamma_2(x)\, \mathbf{1}_{[N]} \right) \\ &= \tfrac{N}{2\sqrt{N}} tr\; x^3 \\ &= \tfrac{3\sqrt{N}}{2}\, \gamma_3(x) \end{aligned} \tag{2.15}$$

So we find, geometrically, that the matrix invariant $\gamma_3(x)$ is related to the scalar product between x and $x \vee x$, and these two vectors lie in, and therefore define, a commuting plane. If x and $x \vee x$ are unit vectors, that is $\gamma_2(x) = \gamma_2(x \vee x) = 1$, then we find that $(x, x \vee x) = \cos\alpha$; where α is the angle between the two vectors in the commuting plane.

We find a third order expression :-

$$x \vee x \vee x \equiv N\, x^3 - 2\gamma_2(x)\, x - 3\gamma_3(x)\, \mathbf{1}_{[N]} \tag{2.16}$$

which is constructed from x and $x \vee x$, but since $x \vee y = y \vee x$ we don't need to use any brackets. We find :-

$$
\begin{aligned}
(x, x \vee x \vee x) &= 2N\gamma_4(x) + (N-2)\gamma_2(x)^2 && (2.17)\\
&= (x \vee x, x \vee x) &&\\
(x \vee x, x \vee x \vee x) &= \tfrac{5N\sqrt{N}}{2}\,\gamma_5(x) + \tfrac{\sqrt{N}}{2}(5N-12)\,\gamma_2(x)\gamma_3(x) && (2.18)
\end{aligned}
$$

A fourth order expression is :-

$$
(x \vee x) \vee (x \vee x) = \sqrt{N}\left[N\,x^4 - 2\left(2\gamma_4(x) + \gamma_2^2(x)\right)\,\mathbf{1}_{[N]}\right] - 4\gamma_2(x)\,x \vee x \qquad (2.19)
$$

These relations are getting complicated so we will not proceed any further; besides we now have all we really need to continue since the vectors we will meet are *not* general 'group' vectors of the *whole* vector space and so are easier to handle. However, we note that :-

- $x \vee x \vee x$ is *never* linearly independent from x,
- $x \vee x$ may be linearly independent from x, but only if $\gamma_3(x) = 0$. In this case we find that $x \vee x \vee x = Nx^3 - 2\gamma_2(x)x$ and $\gamma_3(x) = 0 \Leftrightarrow x^3$ is a vector.

2.4 The basis for $\mathcal{C}_x$.

We may now define a set of vectors which can form (part of) a *basis* for the Cartan, or commuting, subspace associated with a vector, x. Once we have done this we will be able to rewrite x in terms of this basis. In doing so we will be able to exponentiate x without much trouble because we will know how the basis vectors behave.

2.4.1 r-vectors.

The simplest type of vector is the r-vector [1], defined by its matrix invariants :-

$$\begin{aligned}\gamma_2(r) &\equiv 1 \\ \gamma_k(r) &\equiv 0 \qquad\qquad \forall\ 3 \le k \le N.\end{aligned}$$

Thus the characteristic equation for an r-vector is :-

$$r^N - r^{N-2} \equiv 0 \tag{2.20}$$

Since this is equivalently an eigenvalue equation, the behaviour of the r-vector under multiplication is given by :-

$$r^3 - r = 0$$

Because $\gamma_k(r) = 0 \ \ \forall\ 3 \le k \le N$ we find that the r-vector has the eigenvalues ± 1 together with $(N-2)$ zeros. The explicit form of diagonal r-vectors of $\Re^8$ and $\Re^{15}$ are given in Appendix A, which starts on page 152. Since $\gamma_2(r) \equiv 1$ the r-vector is a unit vector. Even though r vectors are defined for $N \ge 3$ we may use the notation when $N = 2$. We have already discussed that there is only one type of vector defined in $\Re^3$ and for a unit 2×2 matrix we have a characteristic equation :-

$$x^2 - \mathbf{1}_{[2]} \equiv 0$$

because in this case $\gamma_2(x) \equiv 1$. If we multiply this by x again we have :-

$$x^3 - x \equiv 0$$

and thus we see that unit vectors in $\Re^3$ exhibit the simplest possible geometrical properties of r vectors of higher dimensional real vector spaces. So, to keep notation to a minimum, we will call any unit vector in $\Re^3$ an r-vector.

[1]These are root vectors; see p 154.

2.4.2 q_r-vectors.

When $N \geq 3$ we may use the symmetric vector product and an r-vector to construct associated q_r vectors :-

$$
\begin{aligned}
r \vee r &= \tfrac{1}{\sqrt{N}} \left(N\, r^2 - 2\ \mathbf{1}_{[N]} \right) \\
&\equiv \sqrt{N-2}\, q_r
\end{aligned} \tag{2.21}
$$

where the r subscript on the vector reminds us which r vector it is associated with; this is because in $\Re^{N^2-1}$ we may define $\frac{1}{2}N(N-1)$ different r-vectors. In the definition the $\sqrt{N-2}$ ensures that the q_r is also normalized, that is $\gamma_2(q_r) \equiv 1$. This is just a special case of equation (2.14). The explicit form of diagonal q_r-vectors of $\Re^8$ and $\Re^{15}$ are given in Appendix A, which starts on page 152.

2.4.3 Relationship between r and q_r.

Using these definitions for r-vectors, and their associated q_r-vectors, we may now rewrite equations (2.15) to (2.19) with $x = r$:-

- Equation (2.15), which gives us the cosine of the angle between x and $x \vee x$ when they are normalized, may now be written :-

$$
(r, q_r) \equiv 0 \tag{2.22}
$$

which tells us they are *orthonormal* vectors in $\Re^{N^2-1}$; q_r is linearly independent from r because $\gamma_3(r) \equiv 0$. This implies that r and q_r may be used to form the *basis* of a commuting plane in the Cartan subspace. For $N = 3$ this is the basis of the whole of the Cartan subspace because $SU(3)$ is a group of rank 2.

- Equation (2.16) reduces to :-

$$
r \vee q_r \equiv \sqrt{N-2}\, r \tag{2.23}
$$

which also implies the product :-

$$
\begin{aligned}
rq_r &= q_r r \\
&= \sqrt{\tfrac{N-2}{N}}\, r
\end{aligned}
$$

- Equations (2.17) and (2.18) reduce to :-

$$
\begin{aligned}
(r,r) = \gamma_2(r) &\equiv 1 \\
(r,q_r) &\equiv 0
\end{aligned}
$$

respectively, which is nothing new.

- Lastly, Equation (2.19) reduces to :-

$$
q_r \vee q_r \equiv \tfrac{N-4}{\sqrt{N-2}}\, q_r \tag{2.24}
$$

2.5 The exponentials of vectors.

Since we will be describing coset vectors with respect to r-vectors and q_r-vectors we will now show how each of these may be exponentiated. We also show how to exponentiate vectors with a particularly simple mathematical behaviour; those which obey the relation $x \vee x = 0$.

2.5.1 Exponentiating r-vectors.

The matrix invariants of an r-vector :-

$$
\begin{aligned}
\gamma_2(r) &\equiv 1 \\
\gamma_k(r) &\equiv 0 \qquad \forall\ 3 \leq k \leq N
\end{aligned}
$$

tell us that $r^3 = r$, and therefore it is simple to exponentiate the r-vector, or any vector which is proportional to an r-vector. We have :-

$$\begin{aligned}
e^{-ia\,r} &= \mathbf{1}_{[N]} - ia\,r - \frac{1}{2!}a^2\,r^2 + \frac{i}{3!}a^3\,r^3 + \frac{1}{4!}a^4\,r^4 - \frac{i}{5!}a^5\,r^5 - + \cdots \\
&= \mathbf{1}_{[N]} + \left(-\frac{1}{2!}a^2 + \frac{1}{4!}a^4 - + \cdots\right) r^2 - i\left(a - \frac{1}{3!}a^3 + \frac{1}{5!}a^5 - + \cdots\right) r \\
&= \mathbf{1}_{[N]} + (\cos a - 1)\, r^2 - i \sin a\ r \\
&= \mathbf{1}_{[N]} + (\cos a - 1)\,\tfrac{1}{N}\left(2\,\mathbf{1}_{[N]} + \sqrt{N(N-2)}\,q_r\right) - i\sin a\ r
\end{aligned}$$

If $a = \frac{\Theta}{2}$, which is the usual value for an $SU(N)$ (sub)group or coset representative element, then :-

$$e^{-i\frac{\Theta}{2}r} = \tfrac{1}{N}\left[N + 2\left(\cos\tfrac{\Theta}{2} - 1\right)\right]\,\mathbf{1}_{[N]} + \sqrt{\tfrac{N-2}{N}}\left(\cos\tfrac{\Theta}{2} - 1\right)\,q_r - i\sin\tfrac{\Theta}{2}\,r \qquad (2.25)$$

Obviously we would have arrived at the *same* result if we had kept everything in terms of x and understood that the characteristic equation of the vector x is, in this case, $x^3 - \gamma_2(x)\,x \equiv 0$. For $N = 2$ equation (2.25) reduces to :-

$$e^{-i\frac{\Theta}{2}r} = \cos\tfrac{\Theta}{2}\,\mathbf{1}_{[2]} - i\sin\tfrac{\Theta}{2}\,r \qquad (2.26)$$

We end this section by emphasizing that it was the simplicity of the characteristic equation of the r-vector which made the expansion, and subsequent grouping of terms, this simple.

2.5.2 Exponentiating q_r-vectors.

Unlike r-vectors, we find that exponentiating q_r-vectors is a little more involved. This is because q_r-vectors have a more complicated characteristic equation. Since :-

$$\begin{aligned}
q_r \vee q_r &\equiv \tfrac{1}{\sqrt{N}}\left(N\,q_r^2 - 2\,\mathbf{1}_{[N]}\right) \\
&= \tfrac{N-4}{\sqrt{N-2}}\,q_r
\end{aligned}$$

we find that :-

$$q_r^2 \;=\; \tfrac{1}{N}\left(2\,\mathbf{1}_{[N]} + (N-4)\sqrt{\tfrac{N}{N-2}}\,q_r\right)$$

and so, on the face of it, the regrouping of terms in the expansion of the exponential will be difficult since the cubic, and higher order terms, become more and more complicated. However, there is one exception :-

- When $N = 4$ the symmetric algebra for q_r vectors is trivial, and so we see :-

  $$q_r^2 \;=\; \tfrac{1}{2}\,\mathbf{1}_{[4]}$$

  and this implies $(\sqrt{2}\,q_r)^3 = \sqrt{2}\,q_r$, which, just like the r-vector, is easy to deal with :-

  $$\begin{aligned} e^{-i\frac{\Theta}{2}q_r} &= e^{-i\frac{\Theta}{2\sqrt{2}}\sqrt{2}\,q_r} \\ &= \cos\tfrac{\Theta}{2\sqrt{2}}\;\mathbf{1}_{[4]} - i\,\sin\tfrac{\Theta}{2\sqrt{2}}\;\sqrt{2}\,q_r \end{aligned} \qquad (2.27)$$

  where $(\sqrt{2}\,q_r)^2 = \mathbf{1}_{[4]}$.

When $N \neq 4$ we must employ a new method; a method based upon the idea of projection operators. In the defining representation of $SU(N)$ we can define N projection operators P^a where $a = 1, 2, \ldots, N$. They have the following properties :-

$$P^a P^b \;\equiv\; \begin{cases} P^a & \text{for a}=\text{b} \\ 0 & \text{for a}\neq\text{b} \end{cases} \qquad (2.28)$$

$$\sum_{a=1}^{N} P^a \;\equiv\; \mathbf{1}_{[N]} \qquad (2.29)$$

$$tr\,P^a \;\equiv\; 1 \qquad\qquad \forall\, a \qquad (2.30)$$

We have already met these operators since r-vectors are just the difference between any two of them. For example if $r = \lambda_3$ then we may write $r = P^1 - P^2$. With these definitions of the projection operators, we find that the exponential of a linear sum of *any* number, k, of projection operators is just :-

$$e^{-i(a_k P^k)} \equiv e^{-ia_k} P^k \tag{2.31}$$

where we have k constants, the a_k. We could have used projection operators as soon as we defined the r-vectors, but each subsequent relation would have looked more complicated than was necessary. At this stage it is not important, but when we come to calculate the covariant derivative for the Goldstone bosons and the metric connection for the standard fields, using $2iL^{-1}\partial_\mu L$, this form is not so useful as it over complicates the calculation because it is more difficult to isolate the coset (GB covariant derivative) and subgroup (metric connection) pieces.

Since $r \vee r = \frac{1}{\sqrt{N}}\left(Nr^2 - 2\,\mathbf{1}_{[N]}\right) = \sqrt{N-2}\,q_r$ we see :-

$$\begin{aligned} e^{-iaq_r} &= e^{-i\frac{a}{\sqrt{N(N-2)}}\left(Nr^2 - 2\,\mathbf{1}_{[N]}\right)} \\ &= e^{-i\frac{a}{\sqrt{N(N-2)}}\left[(N-2)\left(P^1+P^2\right) - 2\sum_{k=3}^{N} P^k\right]} \\ &= e^{-ia\sqrt{\frac{N-2}{N}}}\left(P^1+P^2\right) + e^{i\frac{2a}{\sqrt{N(N-2)}}}\sum_{k=3}^{N} P^k \end{aligned}$$

If we now use $q_r = \frac{1}{\sqrt{N(N-2)}}\left[(N-2)\,\mathbf{1}_{[N]} - N\sum_{k=3}^{N} P^k\right]$ then, after some rearranging and setting $a = \frac{\Theta}{2}$, we find :-

$$\begin{aligned} e^{-i\frac{\Theta}{2}q_r} &= \frac{1}{N}\left(2e^{-i\frac{\Theta}{2}\sqrt{\frac{N-2}{N}}} + (N-2)e^{i\frac{\Theta}{\sqrt{N(N-2)}}}\right)\mathbf{1}_{[N]} \\ &\quad + \sqrt{\frac{N-2}{N}}\left(e^{-i\frac{\Theta}{2}\sqrt{\frac{N-2}{N}}} - e^{i\frac{\Theta}{\sqrt{N(N-2)}}}\right) q_r \end{aligned} \tag{2.32}$$

This reduces to equation (2.27) when $N = 4$.

2.5.3 Exponentiating vectors when $x \vee x \equiv 0$.

In this final section we will, for the moment, ignore all we have learned about the Cartan subspace, and the use of its basis in rewriting vectors, and look at a special type of vector which will satisfy the relation $x \vee x = 0$. If a vector has the form :-

$$x = \alpha S$$

where α is proportional to the length of x, and S has the property $S^2 \equiv \mathbf{1}_{[N]}$ then we gain two important facts about x :-

1. x must be a linear sum of *all* defining representation projection operators, and
2. this implies, because S is traceless, that x must be an *even* dimensional matrix; that is, N is an even number.

We will now establish the defining representation projection operator form of S. Firstly we write :-

$$S = a_i P^i \qquad i = 1, 2, \ldots, N.$$

where the a_i are N constants. For $S^2 = \mathbf{1}_{[N]}$ we must have $a_i = \pm 1$ and, since N must be an even number, there are $\frac{N}{2}$ projection operators with a positive coefficient and $\frac{N}{2}$ projection operators with a negative coefficient. Therefore S must be of the form :-

$$S = \sum_{i=1}^{\frac{N}{2}} P^i - \sum_{i=\frac{N}{2}+1}^{N} P^i$$

This form for S implies that $\gamma_2(S) \equiv \frac{N}{2}$. We have already met this type of vector :

- For $N = 2$ we have :-

  $$S = P^1 - P^2$$

  and this is an r vector, and

- for $N = 4$ we have :-

$$S = \left(P^1 + P^2\right) - \left(P^3 + P^4\right)$$

and this is equal to $\sqrt{2}\,q_3$. We note that if we had defined S with a different arrangement of the projection operators, then we would have ended up with a different q_r here. For $N = 4$ we have the choices $S = \pm q_k$ with $k = 1, 2, 3$.

We will now look at the behaviour of x under multiplication. Since x given by :-

$$x = \alpha S$$

we will find, using $S^2 \equiv \mathbf{1}_{[N]}$, that we obtain :-

$$\begin{aligned} x^{2k} &= \alpha^{2k}\,\mathbf{1}_{[N]} \\ x^{2k+1} &= \alpha^{2k}\,x \\ &= \alpha^{2k+1}\,S \end{aligned}$$

where $k \geq 0$; though in practice we never meet the expression x^0 because it is always written as $\mathbf{1}_{[N]}$. Therefore, with this behaviour, we will find that the exponential of a vector of this form is :-

$$e^{-ix} = \cos\alpha\;\mathbf{1}_{[N]} - i\sin\alpha\;\;S \tag{2.33}$$

We have, quite obviously, already seen this result twice before :

1. When we calculated $e^{-i\frac{\phi}{2}r}$ for $N = 2$, we found a result of this form, see equation (2.26). This is because, in this case, we only have vectors proportional to the r vector; and r vectors, for $N = 2$, have the characteristic equation $r^2 = \mathbf{1}_{[2]}$.

2. When we calculated $e^{-i\frac{\phi}{2}q_r}$ for $N = 4$, we also found a result of this form, see equation (2.27). This is because, in this case, we find $q_r^2 = \frac{1}{2}$; which implies $(\sqrt{2}q_r)^2 = \mathbf{1}_{[4]}$.

Finally, it is simple to show that these vectors satisfy $x \vee x = 0$. We have defined the vector $x \vee x$ to be :-

$$x \vee x \equiv \tfrac{1}{\sqrt{N}} \left(N\, x^2 - 2\gamma_2(x)\, \mathbf{1}_{[N]} \right)$$

If we now use $x = \alpha\, S$ then :-

$$\begin{aligned} x^2 &= \alpha^2\, \mathbf{1}_{[N]} \\ \gamma_2(x) &\equiv \tfrac{N}{2} \alpha^2 \end{aligned}$$

and so, using these two results, we find :-

$$\begin{aligned} x \vee x &\equiv \tfrac{1}{\sqrt{N}} \left(N\, \alpha^2\, \mathbf{1}_{[N]} - 2\tfrac{N}{2} \alpha^2\, \mathbf{1}_{[N]} \right) \\ &= 0 \end{aligned}$$

2.5.3a Basis of C_x when $x \vee x \equiv 0$.

We know that $SU(2)$ is a group of rank 1, and that $SO(m)$, for $m = 2k, 2k+1$, is a group of rank k. The rank 1 groups $SU(2)$, $SO(3)$ and $SO(1,2)$ all have *group* vectors which obey $x \vee x = 0$. In the next chapter we will find that coset vectors with this behaviour appear in three types of model :-

1. when $SU(2)$ invariance is broken to $U(1)$,
2. when $SO(m)$ invariance is broken to $SO(m-1)$, and
3. when $SO(1, m-1)$ invariance is broken to $SO(1, m-2)$.

In this thesis the generators of the Special Orthogonal groups will be in the Weyl representation. We will now focus on the coset vectors for $SO(m)$ breaking to $SO(m-1)$ when the rank is greater than 1; similar results may be found for the coset vectors of

$SO(1,m-1)$ breaking to $SO(1,m-2)$. For $SO(m)$ breaking to $SO(m-1)$ (with $m=2k,2k+1$) we find that the coset vector is :-

$$\begin{aligned} x &= \Omega\, S \\ &= \Omega\, n^{a\Delta}\sigma_{a\Delta} \end{aligned}$$

where $\Delta = m$ and $S^2 = \mathbf{1}_{[2^k]}$. Normally when we rewrite x, for example in terms of r vectors and/or q_r vectors, we solve the characteristic equation which x obeys and, as we have seen, this is most easily viewed as an eigenvalue equation for x; when x is diagonalized. In practice we do not diagonalize x, but in theory we know that this transformation is possible. The diagonalized coset vector is now, at least in our minds, a vector of the subgroup subspace. This is most easily seen for the coset vector (which is used to evaluate the coset representative element) for the coset $\frac{SU(2)}{U(1)}$. We have :-

$$x = \frac{\phi}{2} n^a \sigma_a$$

with $a = 1,2$. This may be diagonalized to $x_D = \frac{\phi}{2}\sigma_3$ and in constructing the coset representative element σ_3 is used to generate subgroup elements. However, for the $SO(m)$ coset vectors we are able to diagonalize to a vector in the coset subpsace too. For example in the theory of $SO(4)$ breaking to $SO(3)$ we have a coset vector :-

$$x = \Omega\, n^{i4}\sigma_{i4}$$

which may be diagonalized to 2 different vectors in $\Re^6$ of $SO(4)$:-

$$\begin{aligned} u_1 : \;\; x \mapsto x_1^D &= u_1^\dagger\, x\, u_1 \\ &= \Omega\, \sigma_{12} \\ u_2 : \;\; x \mapsto x_2^D &= u_2^\dagger\, x\, u_2 \\ &= \Omega\, \sigma_{34} \end{aligned}$$

For our purposes here we will choose the first similarity transformation and write :-

$$\begin{aligned} u: \; S \mapsto S_D &= u^{\dagger} S u \\ &= \sigma_{12} \end{aligned}$$

and this implies that $x = \Omega\, u\, \sigma_{12}\, u^{\dagger}$. For general $SO(m)$ breaking to $SO(m-1)$ both S and σ_{12} have a norm of $\gamma_2(S) = \gamma_2(\sigma_{12}) = \frac{N}{2}$ and since $x \vee x = 0$ we can say that the first direction of the basis of $\mathcal{C}_x$ points in the direction of x and is :-

$$\begin{aligned} e_x^1 &= \sqrt{\tfrac{2}{N}}\, u\, \sigma_{12}\, u^{\dagger} \\ &= \sqrt{\tfrac{2}{N}}\, S \end{aligned}$$

When $k \geq 2$ the second basis vector direction in $\mathcal{C}_x$ is :-

$$e_x^2 = \sqrt{\tfrac{2}{N}}\, u\, \sigma_{34}\, u^{\dagger}$$

In this way, we find the general basis directions of $\mathcal{C}_x$ to be :-

$$e_x^n = \sqrt{\tfrac{2}{N}}\, u\, \sigma_{(2n-1)\,(2n)}\, u^{\dagger}$$

and the maximum value of n is k. In this notation u is the Unitary similarity transformation which diagonalizes S to σ_{12}, that is, it is the transformation which rotates the vector S around the vector space, with basis vectors given by the normalized σ's, onto the vector σ_{12} which, in the Weyl representation, is a diagonal vector.

2.6 Matrix invariants and variables found in the Lagrangian.

In the previous sections we saw how to exponentiate some vectors. It is now important to bring what we have learned into line with the language found in the literature; see for example [13, 17, 19, 16]. We refer to vectors, $N \times N$ traceless Hermitian matrices, with $(N-1)$ *different* eigenvalues as being *Generic*. Thus, in contrast, q_r-vectors are obviously non-generic. When we come to exponentiate vectors, x, in the following chapters we will first rewrite them in terms of the basis vectors of $\mathcal{C}_x$ which we have studied, and then use the exponentials we have found. For example, if a vector we need to exponentiate is proportional to an r-vector then we just have the result given by equation (2.25). However if, for example, x may be written as the linear sum of two commuting r-vectors :-

$$x = a\,r_{(1)} + b\,r_{(2)}$$

then to exponentiate this vector we need to calculate :-

$$g = e^{-ia\,r_{(1)}}e^{-ib\,r_{(2)}}$$

The important point is that generic *group* vectors, the vectors whose exponentials are group elements, are described by a linear sum of *all* the basis vectors of $\mathcal{C}_x$. Thus, for a rank k group, the group vector will contain k variables in its description. A discussion of the 2 and 3-dimensional Cartan subspaces in $\Re^8$ and $\Re^{15}$ is given in Appendix A. However, there do exist generic vectors which may be described by fewer variables than one would expect. To take $N = 4$ as an example, the generic group vector may be written :-

$$x = a\,r + b\,q_r + c\,r_\perp$$

and this vector obeys the characteristic equation :-

$$x^4 - \gamma_2\,(x)\,x^2 - \gamma_3\,(x)\,x - \gamma_4\,(x)\;\mathbf{1}_{[4]} \equiv 0$$

where none of the matrix invariants, the $\gamma_k(x)$, are zero. So, in this case, since $SU(4)$ is a rank 3 group there are 3 independent matrix invariants which is reflected by the use of 3 variables in the description of x. The exponential of this vector, as shown in Appendix A, is a general $SU(4)$ group element. In contrast though, we may also have the diagonal generic vector :-

$$
\begin{aligned}
x_D &= \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & -b \end{pmatrix} \\
&= a\, r_3 + b\, r_{3\perp}
\end{aligned}
$$

However this matrix obeys :-

$$x^4 - \gamma_2\,(x)\,x^2 - \gamma_4\,(x)\;\mathbf{1}_{[4]} \;\equiv\; 0$$

There are 2 independent matrix invariants in this characteristic equation; which is reflected by the use of only 2 variables in the description of x; not 3 as we would naïvely expect for a generic vector. This vector may be used for two main purposes. In Appendix C we show that if this vector is rotated to :-

$$x = a\, r + b\, r_{\perp}$$

with $r = n^k L_k$ and $r_\perp = n_\perp^k R_k$ then we may exponentiate it to form a general $SU_L(2) \otimes SU_R(2) \sim SO(4)$ group element; so in this respect it is a generic group vector. Whereas in Chapter 5 we will use this relation to describe the coset vectors of the $\frac{SO(m)}{SO(m-2)\otimes SO(2)}$ cosets (with $m = 4, 5, 6$). Thus, we make the following statement regarding the form of Lagrangians found in this thesis.

- The coset representative element, $L = e^{-ix}$, is used to construct various objects like the Goldstone field covariant derivative a_μ, the matter field spinor ψ, and the metric connection for the matter field covariant derivative v_μ. In turn we use these quantities to build the effective Lagrangian density for the theory. For this reason, it must be true that the effective Lagrangian density will contain only as many variables as was needed to describe the coset vector, x.

Chapter 3

Theories from coset vectors which obey $x \vee x \equiv 0$.

We will find, for all of the theories presented in this chapter, that the coset vector, x, satisfies $x \vee x = 0$. This is equivalent to saying that x squares to the identity element multiplied by a constant term. It is for this reason alone that the different theories have results which are expressed in a similar way. This is also a very special result since we may perform all the calculations using the generators directly; the more sophisticated technique of understanding the Cartan subspace associated with x, denoted $\mathcal{C}_x$, by solving x's characteristic (eigenvalue) equation is not necessary.

3.1 The $SU(2)$ breaking to $U(1)$ model.

The group $SU(2)$ and the coset $\frac{SU(2)}{U(1)}$ introduce, albeit in a rather oversimplified fashion, many of the ideas required to calculate effective Lagrangians. The groups $SU(2)$ and $U(1)$ are of rank 1, which geometrically means that we can only define one type of vector, the r-vector, in the $\Re^3$ vector space, with the Pauli matrix basis. The Cartan subspace associated with a vector, $\mathcal{C}_x$, lies in the direction of x only. Thus we do not meet q_r-vectors, and in fact it is obvious, not only from the characteristic equation for 2×2 traceless, hermitian matrices :-

$$x^2 - \gamma_2(x)\,\mathbf{1}_{[2]} \;\equiv\; 0$$

but also, equivalently, from the basis matrix product rule :-

$$\sigma_i \sigma_j \;=\; \delta_{ij}\,\mathbf{1}_{[2]} + i\varepsilon_{ijk}\sigma_k$$

that we do not really have to understand the geometrical nature of $\Re^3$ at all in order for us to calculate the required pieces for the Lagrangian.

3.1.1 The coset representative, L.

If we generate the subgroup $U(1)$ of transformations using $T_3 = \frac{1}{2}\sigma_3$ then the coset vector is :-

$$\begin{aligned} x &= x^a \sigma_a \\ &= \tfrac{\phi^a}{2}\,\sigma_a \end{aligned}$$

This squares to :-

$$\begin{aligned} x^2 &= x^a x^b \left(\delta_{ab}\,\mathbf{1}_{[2]} + i\,\varepsilon_{ab3}\sigma_3\right) \\ &= x^a x^a\,\mathbf{1}_{[2]} \end{aligned}$$

So the norm of x is :-

$$\begin{aligned} (x, x) &= \gamma_2(x) \\ &= x^a x^a \\ &= \tfrac{1}{4}\phi^a \phi^a \end{aligned}$$

which is just a number. So if we define $\phi^2 \equiv \phi^a\phi^a$ then we could have written for x :-

$$\begin{aligned} x &= \tfrac{1}{2}\phi^a\sigma_a \\ &= \tfrac{\phi}{2}n^a\sigma_a \\ &= \tfrac{\phi}{2}r \end{aligned}$$

where $n^a\sigma_a = r$ is a (unit) r-vector which is scaled by the length of x, which is $\frac{\phi}{2}$. We call the quantity $n^a\sigma_a$ an r vector even though, for $N = 2$, we can only define one type of vector; because of the characteristic equation. Nevertheless, since $r^2 = \mathbf{1}_{[2]}$ we must have $r^3 = r$ and this is how r-vectors are defined in [17] for $\Re^8$. Thus the coset vector, x, squares and cubes to :-

$$\begin{aligned} x^2 &= \left(\tfrac{\phi}{2}\right)^2\mathbf{1}_{[2]} \\ x^3 &= \left(\tfrac{\phi}{2}\right)^3 r \end{aligned}$$

and so, using equation (2.26), the coset representative element is :-

$$\begin{aligned} L &\equiv e^{-i\frac{\phi}{2}r} \\ &= \cos\tfrac{\phi}{2}\,\mathbf{1}_{[2]} - i\,\sin\tfrac{\phi}{2}\,r \end{aligned} \tag{3.1}$$

with $r = n^a\sigma_a$. This agrees, as it should, with equation (2.33) with $\alpha = \frac{\phi}{2}$ and $S = r$.

3.1.2 Goldstone boson transformations.

Since the Physical theory deals with the transformation of the coset representative, to first order in the transformation parameters, by an element of the subgroup on its own, and by an element of the coset itself, we will divide this part into two sections. The Physical theory is structured in this way because for more complicated models of symmetry breaking we will only find the Killing vectors which describe the transformation of the Goldstone Bosons to first order in the transformation parameters; we do not look at the transformation to all orders. However in this case, since we are dealing with $SU(2)$ breaking to $U(1)$, we are only having to manipulate Pauli Spin matrices to find results and this is a simple excersise.

3.1.2a The transformation of L by a subgroup element.

The $U(1)$ subgroup element, h, also has the same form :-

$$h = \cos\tfrac{\Theta}{2}\,\mathbf{1}_{[2]} - i\,\sin\tfrac{\Theta}{2}\;n^3\sigma_3 \tag{3.2}$$

with $n^3 \equiv 1$. The coset representative is transformed by this subgroup element :-

$$\begin{aligned} L' &= h\,L\,h^{-1} \\ &= \cos\tfrac{\phi}{2}\,\mathbf{1}_{[2]} - i\,\sin\tfrac{\phi}{2}\,h\,(n^a\sigma_a)\;h^{-1} \end{aligned}$$

The second term contains the transformed, or rotated, r-vector :-

$$\begin{aligned} h\;(n^a\sigma_a)\;h^{-1} &= n^a\left(\cos\tfrac{\Theta}{2}\,\mathbf{1}_{[2]} - i\,\sin\tfrac{\Theta}{2}\;n^3\sigma_3\right)\sigma_a\left(\cos\tfrac{\Theta}{2}\,\mathbf{1}_{[2]} + i\,\sin\tfrac{\Theta}{2}\;n^3\sigma_3\right) \\ &= n^a\left(\cos^2\tfrac{\Theta}{2}\,\sigma_a + i\sin\tfrac{\Theta}{2}\,\cos\tfrac{\Theta}{2}\,[\sigma_a,\sigma_3] + \sin^2\tfrac{\Theta}{2}\,\sigma_3\sigma_a\sigma_3\right) \\ &\;\;\vdots \\ &= n^a\,(\cos\Theta\;\delta_{ab} - \sin\Theta\;\varepsilon_{a3b})\,\sigma_b \end{aligned}$$

Since $h(n^a\sigma_a)h^{-1} = n^{a\prime}\sigma_a$ we see that, after we relabel indices, we have :-

$$n^{a\prime} = (\cos\Theta\;\delta_{ab} - \sin\Theta\;\varepsilon_{3ab})\,n^b \tag{3.3}$$

which is an $SO(2)$ rotation in the 1-2 plane (about the 3^{rd} axis) of the coordinates, n^a. We have this form because the adjoint representation of $SU(2)$ is homomorphic to $SO(3)$, which implies that the adjoint representation of the $U(1)$ transformation is homomorphic to an $SO(2)$ transformation.

If we now write equation (3.3) to first order in the transformation parameter, Θ, then we find :-

$$n^{a'} = (\delta_{ab} - \Theta\varepsilon_{3ab})\, n^b$$

or, in terms of the Goldstone bosons :-

$$\begin{aligned} M^{a'} &= (\delta_{ab} - \Theta\varepsilon_{3ab})\, M^b \\ &= M^a - \Theta\varepsilon_{3ab}M^b \end{aligned}$$

If we compare this with equation (1.13), which gives the Killing vector components of the transformation, then we have :-

$$\begin{aligned} M^a + \theta^E \mathbf{K}_E^a &= M^a - \Theta\varepsilon_{3ab}M^b \\ &= (\delta_{ab} - i\Theta(T_3)_{ab})\ M^b \end{aligned}$$

with $(T_3)_{ab} = -i\varepsilon_{3ab}$. In this case we have the subgroup index $E = 3$, which implies that we have one subgroup parameter $\theta^3 \equiv \Theta$. So the linear Killing vector components are :-

$$\begin{aligned} \mathbf{K}_3^a &= -\varepsilon_{3ab}M^b \\ &= \varepsilon_{a3b}M^b \end{aligned} \tag{3.4}$$

This is a result which we will confirm later on.

3.1.2b The transformation of L by a coset element.

In this part we will be using L^2 in our calculation since this is how we find the nonlinear Killing vector components in the rest of the thesis. This method will not be repeated anywhere else, although in principle it could be since we have developed a method of calculating coset elements to all orders. We proceed by writing the coset representative and coset transformation elements :-

$$
\begin{aligned}
L^2 &= \cos\phi\ \mathbf{1}_{[2]} - i\sin\phi\ r \\
c &= \cos\tfrac{\Theta}{2}\ \mathbf{1}_{[2]} - i\sin\tfrac{\Theta}{2}\, t
\end{aligned}
$$

where $r = n^a\sigma_a$ with $n^a n^a \equiv 1$, and $t = t^a\sigma_a$ with $t^a t^a \equiv 1$. So :-

$$
\begin{aligned}
(L')^2 &= cL^2c^{-1} \\
&= \left(\cos\tfrac{\Theta}{2}\ \mathbf{1}_{[2]} - i\sin\tfrac{\Theta}{2}\, t\right) L^2 \left(\cos\tfrac{\Theta}{2}\ \mathbf{1}_{[2]} + i\sin\tfrac{\Theta}{2}\, t\right) \\
&\ \vdots \\
&= L^2 - \frac{i}{2}\sin\Theta\ \left\{L^2, t\right\} - \sin^2\tfrac{\Theta}{2}\, t\left\{L^2, t\right\}
\end{aligned}
$$

We calculate :-

$$
\begin{aligned}
\left\{L^2, t\right\} &= 2\cos\phi\ t - i\sin\phi\ \{r, t\} \\
&= 2\cos\phi\ t - 2i\sin\phi\ (r, t)\ \mathbf{1}_{[2]} \\
t\left\{L^2, t\right\} &= 2\cos\phi\ \mathbf{1}_{[2]} - 2i\sin\phi\ (r, t)\ t
\end{aligned}
$$

and therefore we find :-

$$
(L')^2 = -i\sin\phi\ r + \cos\phi\ c^2 - i\sin\phi\ (r, t)\left(c^2 - \mathbf{1}_{[2]}\right)\ t
$$

We find the change in the coset representative, $\delta L^2 = (L')^2 - L^2$, is :-

$$
\begin{aligned}
\delta L^2 = &\ [\cos\phi\ (\cos\Theta\ - 1) - \sin\phi\ (r, t)\sin\Theta\ ]\ \mathbf{1}_{[2]} \\
&- i\left[\cos\phi\ \sin\Theta\ + \sin\phi\ (r, t)\left(\cos\Theta\ - 1\right)\right]\ t
\end{aligned}
$$

From the Physical theory we know that δL^2 to first order in Θ is equal to $L^2_{,a}\theta^b\mathbf{K}^a_b$. Therefore to first order in Θ this quantity is :-

$$
\delta L^2 = -\sin\phi\ n^a\theta^a\,\mathbf{1}_{[2]} - i\cos\phi\ \theta^a\sigma_a \tag{3.5}
$$

We now calculate $L^2_{,a}\theta^b \mathbf{K}^a_b$. Firstly we have :-

$$L^2_{,a} = -\sin\phi\ \phi_{,a}\,\mathbf{1}_{[2]} - i\cos\phi\ \phi_{,a}\,r - i\sin\phi\ r_{,a}$$

and since $r_{,a} = \frac{1}{M}\left(\delta_{ab} - n^a n^b\right)\sigma_b$ then we find :-

$$L^2_{,a}\theta^b \mathbf{K}^a_b = -\sin\phi\ \phi_{,a}\theta^b\mathbf{K}^a_b\,\mathbf{1}_{[2]} - i\cos\phi\ \phi_{,a}\theta^b\mathbf{K}^a_b\,r - i\frac{\sin\phi}{M}\left(\delta_{ac} - n^a n^c\right)\theta^b\mathbf{K}^a_b\sigma_c$$

If we compare this with equation (3.5) then we see that $\phi_{,a}\mathbf{K}^a_b = n^b$, and so comparing this result with the expression for δL^2 we find :-

$$\cos\phi\ \theta^a\sigma_a = \cos\phi\ n^b\theta^b n^a\sigma_a + \frac{\sin\phi}{M}\left(\delta_{ac} - n^a n^c\right)\theta^b\mathbf{K}^a_b\sigma_c$$

which is simple to rearrange and then isolate $\mathbf{K}^a_b$ because $n^a\mathbf{K}^a_b = \frac{dM}{d\phi}\,n^b$. We therefore find :-

$$\mathbf{K}^a_b = M\cot\phi\ \left(\delta_{ab} - n^a n^b\right) + \frac{dM}{d\phi}n^a n^b \tag{3.6}$$

This result will confirmed in the next section.

3.1.3 Analysis to first order using Killing vectors.

This section contains the usual methods which will be employed throughout the rest of the thesis to find the Killing vector components.

3.1.3a The linear Killing vector components, $\mathbf{K}^a_3$.

To find the linear $\mathbf{K}^a_3$ components we must solve :-

$$[\sigma_3, L] = 2i\,L_{,a}\mathbf{K}^a_3$$

Firstly, we have for L :-

$$L = \cos\tfrac{\phi}{2}\ \mathbf{1}_{[2]} - i\ \sin\tfrac{\phi}{2}\,n^a\sigma_a$$

Therefore, for the left hand side, we find :-

$$[\sigma_3, L] \;=\; 2\sin\tfrac{\phi}{2}\, n^a \varepsilon_{3ab}\sigma_b \tag{3.7}$$

When we differentiate L with respect to the Goldstone fields we find :-

$$L_{,a} \;=\; -\frac{1}{2}\sin\tfrac{\phi}{2}\,\phi_{,a}\,\mathbf{1}_{[2]} - \frac{i}{2}\cos\tfrac{\phi}{2}\,\phi_{,a}n^b\sigma_b - i\sin\tfrac{\phi}{2}\,n^b_{,a}\sigma_b$$

So the right hand side is :-

$$2i\,L_{,a}\mathbf{K}^a_3 \;=\; -i\sin\tfrac{\phi}{2}\,\phi_{,a}\mathbf{K}^a_3\,\mathbf{1}_{[2]} + \cos\tfrac{\phi}{2}\,\phi_{,a}\mathbf{K}^a_3 n^b\sigma_b + 2\sin\tfrac{\phi}{2}\,n^b_{,a}\mathbf{K}^a_3\sigma_b \tag{3.8}$$

So comparing the two sides, equations (3.7) and (3.8), we see that $\phi_{,a}\mathbf{K}^a_3 \equiv 0$ and therefore :-

$$\begin{aligned} n^a\varepsilon_{3ab} &= n^b_{,a}\mathbf{K}^a_3 \\ &= \frac{1}{M}\left(\delta_{ba} - n^b n^a\right)\mathbf{K}^a_3 \\ &= \frac{1}{M}\mathbf{K}^b_3 \end{aligned}$$

So we have found the linear $\mathbf{K}^a_3$ to be :-

$$\mathbf{K}^a_3 \;=\; \varepsilon_{a3b}M^b \tag{3.9}$$

which is precisely equation (3.4) which we found earlier.

3.1.3b The nonlinear Killing vector components, $\mathbf{K}^a_b$.

This time, to find the nonlinear $\mathbf{K}^a_b$ components we must solve :-

$$\left\{\sigma_b, L^2\right\} \;=\; 2i\,L^2_{,a}\mathbf{K}^a_b$$

Firstly, we have for L^2 :-

$$L^2 \;=\; \cos\phi\;\mathbf{1}_{[2]} - i\,\sin\phi\; n^a\sigma_a$$

Therefore, for the left hand side, we find :-

$$\{\sigma_b, L^2\} \;=\; 2\cos\phi\;\sigma_b - 2i\;\sin\phi\;n^b\,\mathbf{1}_{[2]} \tag{3.10}$$

When we differentiate L^2 with respect to the Goldstone fields we find :-

$$L^2_{,a} \;=\; -\sin\phi\;\phi_{,a}\,\mathbf{1}_{[2]} - i\cos\phi\;\phi_{,a}n^b\sigma_b - i\sin\phi\;n^b_{,a}\sigma_b$$

So the right hand side is :-

$$2i\,L^2_{,a}\mathbf{K}^a_b \;=\; -2i\sin\phi\;\phi_{,a}\mathbf{K}^a_b\,\mathbf{1}_{[2]} + 2\cos\phi\;\phi_{,a}\mathbf{K}^a_b n^c\sigma_c + 2\sin\phi\;n^c_{,a}\mathbf{K}^a_b\sigma_c \tag{3.11}$$

So comparing the two sides, equations (3.10) and (3.11), we see that $\phi_{,a}\mathbf{K}^a_b \equiv n^b$ and therefore :-

$$\cos\phi\;\sigma_b \;=\; \cos\phi\;n^b n^c\sigma_c + \sin\phi\;n^c_{,a}\mathbf{K}^a_b\sigma_c$$

Removing the basis we have :-

$$\begin{aligned}\cos\phi\;\delta_{bc} &= \cos\phi\;n^b n^c + \sin\phi\;n^c_{,a}\mathbf{K}^a_b \\ &= \cos\phi\;n^b n^c + \sin\phi\;\frac{1}{M}\left(\delta_{ca} - n^c n^a\right)\mathbf{K}^a_b \\ &= \cos\phi\;n^b n^c + \frac{\sin\phi}{M}\mathbf{K}^c_b - \frac{\sin\phi}{M}n^c n^b\frac{dM}{d\phi}\end{aligned}$$

which we rearrange to find :-

$$\mathbf{K}^a_b \;=\; M\cot\phi\;\left(\delta_{ab} - n^a n^b\right) + \frac{dM}{d\phi}n^a n^b \tag{3.12}$$

which is exactly the result we found in equation (3.6) earlier.

3.1.4 Covariant derivatives and the Goldstone boson metric.

For the covariant derivatives of our theory we need to calculate $2iL^{-1}\partial_\mu L = a_\mu + v_\mu$, where a_μ is actually the Goldstone boson covariant derivative, and v_μ is proportional to the metric connection for the Standard field covariant derivative.

We have for the coset representative :-

$$L = \cos\tfrac{\phi}{2}\, \mathbf{1}_{[2]} - i \sin\tfrac{\phi}{2}\, r$$

where $r \equiv n^a \sigma_a$. We find :-

$$\partial_\mu L = -\frac{1}{2} \sin\tfrac{\phi}{2}\, \partial_\mu \phi\, \mathbf{1}_{[2]} - \frac{1}{2} \cos\tfrac{\phi}{2}\, (\partial_\mu \phi)\, r - i \sin\tfrac{\phi}{2}\, \partial_\mu r$$

Therefore we find :-

$$\begin{aligned} L^{-1}\partial_\mu L &= -\frac{1}{2} \sin\tfrac{\phi}{2}\, \cos\tfrac{\phi}{2}\, \partial_\mu \phi\, \mathbf{1}_{[2]} - \frac{i}{2} \cos^2\tfrac{\phi}{2}\, r \partial_\mu \phi - i \sin\tfrac{\phi}{2}\, \cos\tfrac{\phi}{2}\, \partial_\mu r \\ &\quad - \frac{i}{2} \sin^2\tfrac{\phi}{2}\, r \partial_\mu \phi + \frac{1}{2} \sin\tfrac{\phi}{2}\, \cos\tfrac{\phi}{2}\, r^2 \partial_\mu \phi + \sin^2\tfrac{\phi}{2}\, r \partial_\mu r \end{aligned}$$

Since $r^2 \equiv \mathbf{1}_{[2]}$ we find :-

$$\begin{aligned} L^{-1}\partial_\mu L &= -\frac{i}{2} r \partial_\mu \phi - \frac{i}{2} \sin\phi\, \partial_\mu r + \sin^2\tfrac{\phi}{2}\, r \partial_\mu r \\ 2iL^{-1}\partial_\mu L &= (r\partial_\mu \phi + \sin\phi\, \partial_\mu r) + 2i \sin^2\tfrac{\phi}{2}\, r \partial_\mu r \qquad (3.13) \\ &\equiv a_\mu + v_\mu \end{aligned}$$

Therefore, using the results in appendix D, we have the covariant derivatives :-

$$\mathcal{D}_\mu M^a = \left\{ \frac{\sin\phi}{M} (\delta_{ab} - n^a n^b) + \left(\frac{d\phi}{dM} \right) n^a n^b \right\} \partial_\mu M^b \qquad (3.14)$$

$$\mathcal{D}_\mu \psi = \left\{ \partial_\mu + \frac{i}{M^2} \sin^2\tfrac{\phi}{2}\, M^a \partial_\mu M^b \varepsilon_{ab3}\, \sigma_3 \right\} \psi \qquad (3.15)$$

For the Goldstone boson part of the Effective Lagrangian, and the metric for the Goldstone boson manifold, we refer to equation (1.32) and construct :-

$$\begin{aligned} a_\mu a^\mu &= r^2 (\partial_\mu \phi)(\partial^\mu \phi) + \sin\phi\, (\partial_\mu \phi)(r \partial^\mu r + (\partial^\mu r) r) + \sin^2\phi\, (\partial_\mu r)(\partial^\mu r) \\ &= r^2 (\partial_\mu \phi)(\partial^\mu \phi) + \sin\phi\, (\partial_\mu \phi) \partial^\mu r^2 + \sin^2\phi\, (\partial_\mu r)(\partial^\mu r) \end{aligned}$$

Since $r^2 \equiv \mathbf{1}_{[2]}$ we have :-

$$a_\mu a^\mu = (\partial_\mu \phi)(\partial^\mu \phi)\, \mathbf{1}_{[2]} + \sin^2\phi\, (\partial_\mu r)(\partial^\mu r)$$

Therefore we can see :-

$$\begin{aligned}(a_\mu, a^\mu) &\equiv \frac{1}{2}\, tr\, a_\mu a^\mu \\ &= (\partial_\mu \phi)(\partial^\mu \phi) + \sin^2\phi\ (\partial_\mu n^a)(\partial^\mu n^a)\end{aligned}$$

Again, using the results in appendix D, we have :-

$$\begin{aligned}(a_\mu, a^\mu) &= \left\{ \frac{\sin^2\phi}{M^2}\left(\delta_{ab} - n^a n^b\right) + \left(\frac{d\phi}{dM}\right)^2 n^a n^b \right\} \partial_\mu M^a \partial^\mu M^b \\ &\equiv g_{ab}\partial_\mu M^a \partial^\mu M^b\end{aligned}$$

where we may now identify the Goldstone boson manifold metric as :-

$$g_{ab} = \frac{\sin^2\phi}{M^2}\left(\delta_{ab} - n^a n^b\right) + \left(\frac{d\phi}{dM}\right)^2 n^a n^b \tag{3.16}$$

This, from [14], is the metric of a Kähler manifold; it is denoted S^2. Therefore it is possible to extend this model to include $\mathcal{N} = 1$ Supersymmety.

3.1.4a Verifying the metric result.

We may construct the metric associated with the Goldstone boson manifold from :-

$$g_{ab} = \left(\mathbf{K}_3^a \mathbf{K}_3^b + \mathbf{K}_c^a \mathbf{K}_c^b\right)^{-1} \tag{3.17}$$

To easily invert the right hand side we need it to be written in terms of adjoint representation projection operators; and in this case this is exactly what we have. The form of these projection operators is discussed in Appendix B, section B 3.1, on page 171. We have :-

$$\begin{aligned}\mathbf{K}_3^a \mathbf{K}_3^b &= \varepsilon_{a3c} M^c \varepsilon_{b3d} M^d \\ &= M^2 \varepsilon_{ac3}\varepsilon_{bdc} n^c n^d \\ &= M^2\left(\delta_a^b \delta_c^d - \delta_a^d \delta_c^b\right) n^c n^d \\ &= M^2\left(\delta_{ab} - n^a n^b\right)\end{aligned}$$

Notice that, in terms of adjoint operators, we could have just written :-

$$\begin{aligned}
\mathbf{K}_3^a \mathbf{K}_3^b &= -M^2 \,(f_r)_{a3}\,(f_r)_{3b} \\
&= -M^2 \left(f_r^2\right)_{ab} \\
&= M^2 \left(\mathbf{1}_{[3]} - r >< r\right)_{ab} \\
&\Rightarrow M^2 \left(\delta_{ab} - n^a n^b\right)
\end{aligned}$$

and to put this explicitly into adjoint projection operator terms we would use :-

$$(\mathbf{1}_{[3]})_{ab} \equiv (\mathcal{P}^{12} + \mathcal{P}^{21} + \mathcal{P}^3)_{ab}$$

where $(\mathcal{P}^3)_{ab} \equiv (r >< r)_{ab} = n^a n^b$. For the second term we have :-

$$\begin{aligned}
\mathbf{K}_c^a \mathbf{K}_c^b &= \left(M \cot\phi\; (\delta_{ac} - n^a n^c) + \frac{dM}{d\phi} n^a n^c\right) \left(M \cot\phi\; (\delta_{bc} - n^b n^c) + \frac{dM}{d\phi} n^b n^c\right) \\
&= M^2 \cot^2\!\phi\; \left(\delta_{ab} - n^a n^b\right) + \left(\frac{dM}{d\phi}\right)^2 n^a n^b
\end{aligned}$$

This was a simple step since $(\delta_{ab} - n^a n^b) n^b n^c = 0$, or in terms of the adjoint projection operators this is $(\mathcal{P}^{12} + \mathcal{P}^{21})_{ab} \mathcal{P}^3_{bc} = 0$. Therefore we find :-

$$\mathbf{K}_3^a \mathbf{K}_3^b + \mathbf{K}_c^a \mathbf{K}_c^b = \frac{M^2}{\sin^2\!\phi} \left(\delta_{ab} - n^a n^b\right) + \left(\frac{dM}{d\phi}\right)^2 n^a n^b$$

which we may simply invert to find :-

$$g_{ab} = \frac{\sin^2\!\phi}{M^2} \left(\delta_{ab} - n^a n^b\right) + \left(\frac{d\phi}{dM}\right)^2 n^a n^b \tag{3.18}$$

This is exactly the result of equation (3.16), which we found in the last section.

3.2 The $SO(m)$ breaking to $SO(m-1)$ models.

3.2.1 The coset representative, L.

These models use the coset representative for the $\frac{SO(m)}{SO(m-1)}$ cosets. The calculations in this section will respect covariant and contravariant indices; even though, for the spaces considered here, it is not strictly necessary. However, if we are rigorous here we will be able to use the resulting structure in the next section, with only a few minor modifications. For the Clifford algebra we will use :-

$$\{\gamma_A, \gamma_B\} \;=\; 2g_{AB}\,\mathbf{1}_{[2^k]}$$

and we can then reinstate $g_{AB} \equiv \delta_{AB}$ for these orthogonal group models at the end of the calculations. The sigma matrices for $SO(m)$, in the Weyl representation are defined :-

$$\sigma_{AB} \;\equiv\; -\frac{i}{2}\,[\gamma_A, \gamma_B]$$

where $A, B = 1, 2, \ldots, m$. If we generate the subgroup $SO(m-1)$ using the σ_{ab} then the coset vector we need is :-

$$x \;=\; \omega^{a\Delta}\sigma_{a\Delta}$$

where $a = 1, 2, \ldots, (m-1)$ and $\Delta = m$. So if we write $x = \Omega n^{a\Delta}\sigma_{a\Delta}$ then the square of this is :-

$$\begin{aligned} x^2 &= \Omega^2 n^{a\Delta} n_{a\Delta}\,\mathbf{1}_{[2^k]} \\ &= \Omega^2\,\mathbf{1}_{[2^k]} \end{aligned}$$

Thus $x = \Omega S$ and $x \vee x = 0$. So the coset representative element, L, may be written :-

$$\begin{aligned} L &= \cos\Omega\;\mathbf{1}_{[2^k]} - i\sin\Omega\;S \\ &= \cos\Omega\;\mathbf{1}_{[2^k]} - i\sin\Omega\;n^{a\Delta}\sigma_{a\Delta} \end{aligned} \qquad (3.19)$$

as expected from equation (2.33). In terms of an $SU(N)$ description we have :-

- For $m = \Delta = 3$ we have $k = 1$, and $n^{a\Delta}\sigma_{a\Delta}$ is an r vector,
- for $m = \Delta = 4, 5$ we have $k = 2$, and $n^{a\Delta}\sigma_{a\Delta}$ is proportional to a q_r vector, and

- for $m = \Delta = 6$ we have $k = 3$. However, since the $SO(6)$, Weyl representation, generators are in block diagonal $4 \oplus \bar{4}$ form, we just work with the top left entries which generate transformations on the left handed spinor. Thus the $n^{a\Delta}\sigma_{a\Delta}$ which we use is a 4×4 matrix and is proportional to a q_r vector.

For $SO(m)$ breaking to $SO(m-1)$ we note that *only* when m is an odd number does the subgroup have the same rank as the group; when m is even the subgroup has a rank which is one less than the group. In Appendix C, from page 191, we discuss the construction of the generators, and Lie algebras, of $SO(m)$ groups in the Weyl representation. In this scheme m may take on the two values $m = (2k), (2k+1)$ with $k \geq 1$. Since it is the odd m, i.e. $m = (2k+1)$, gamma matrices which are initially defined and used to construct generators for $SO(2k+1)$ we find that the generators of $SO(2k)$ are defined as a subset of the generators of $SO(2k+1)$ in the Weyl representation. Since the generators of $SO(2k)$ are in a block diagonal form the last gamma matrix, γ_{2k+1}, is used to construct a projection operator which will not only project out the left and right handed spinors, but also the left and right handed generators. We have used this notation ($m = (2k), (2k+1)$) because the integer k *is* the rank of the group. Therefore, to give two examples, we see :-

1. if $m = 5$, which implies $k = 2$, we are considering the spontaneous breaking of an $SO(5)$ symmetry down to an $SO(4)$ symmetry. We immediately understand from the construction that both $SO(5)$ and $SO(4)$ are groups of rank 2, whereas

2. if $m = 6$, which implies $k = 3$, we are considering the spontaneous breaking of an $SO(6)$ symmetry down to an $SO(5)$ symmetry. However, in this case, the generators of $SO(6)$ are viewed as being a subset of the generators of $SO(7)$; and both $SO(6)$ and $SO(7)$ are rank 3 groups. But, as we have seen, $SO(5)$ is a rank 2 group.

3.2.2 Goldstone boson transformations.

3.2.2a The linear Killing vector components, $\mathbf{K}_{bc}^{a\Delta}$.

To find the linear $\mathbf{K}_{bc}^{a\Delta}$ components we must solve :-

$$[\sigma_{bc}, L] \;=\; 2i\, L_{,a\Delta}\mathbf{K}_{bc}^{a\Delta}$$

The coset representative and its derivative with respect to the Goldstone fields are :-

$$\begin{aligned} L &= \cos\Omega\ \mathbf{1}_{[2^k]} - i\sin\Omega\ n^{c\Delta}\sigma_{c\Delta} \\ L_{,a\Delta} &= -\sin\Omega\ \Omega_{,a\Delta}\,\mathbf{1}_{[2^k]} - i\cos\Omega\ \Omega_{,a\Delta}n^{c\Delta}\sigma_{c\Delta} - i\sin\Omega\ n^{c\Delta}_{,a\Delta}\sigma_{c\Delta} \end{aligned}$$

For the left hand side we have :-

$$[\sigma_{bc}, L] \;=\; -i\sin\Omega\ n^{a\Delta}\,[\sigma_{bc}, \sigma_{a\Delta}]$$

It is at this point that our results in the next section, for the $SO(1, m-1)$ breaking to $SO(1, m-2)$ models, will differ in sign only from the form we will find here. For the $SO(m)$ breaking to $SO(m-1)$ models, this commutator is :-

$$[\sigma_{bc}, \sigma_{a\Delta}] \;=\; 2i\,(g_{ab}\sigma_{c\Delta} - g_{ac}\sigma_{b\Delta})$$

with $g_{ab} = \delta_{ab}$.

If we now equate the left and right hand sides of our relation to be solved we find :-

$$\begin{aligned} \sin\Omega\ n^{a\Delta}\,(g_{ab}\sigma_{c\Delta} - g_{ac}\sigma_{b\Delta}) \;&=\; -i\sin\Omega\ \Omega_{,a\Delta}\mathbf{K}_{bc}^{a\Delta}\,\mathbf{1}_{[2^k]} + \cos\Omega\ \Omega_{,a\Delta}\mathbf{K}_{bc}^{a\Delta}n^{d\Delta}\sigma_{d\Delta} \\ &\quad + \sin\Omega\ n^{d\Delta}_{,a\Delta}\mathbf{K}_{bc}^{a\Delta}\sigma_{d\Delta} \end{aligned}$$

which automatically implies $\Omega_{,a\Delta}\mathbf{K}_{bc}^{a\Delta} = 0$. If we substitute this into the above we have :-

$$n^{a\Delta}\,(g_{ab}\sigma_{c\Delta} - g_{ac}\sigma_{b\Delta}) \;=\; n^{d\Delta}_{,a\Delta}\mathbf{K}_{bc}^{a\Delta}\sigma_{d\Delta}$$

Since $tr\, \sigma_{a\Delta}\sigma_{b\Delta} = 2^k g_{ab} g_{\Delta\Delta}$ we may remove the sigma matrices to find :-

$$\begin{aligned} n^{a\Delta} g_{ab} g_{ce} g_{\Delta\Delta} - n^{a\Delta} g_{ac} g_{be} g_{\Delta\Delta} &= n^{d\Delta}_{,a\Delta} \mathbf{K}^{a\Delta}_{bc} g_{de} g_{\Delta\Delta} \\ n_{b\Delta} g_{ce} - n_{c\Delta} g_{be} &= n^{d\Delta}_{,a\Delta} \mathbf{K}^{a\Delta}_{bc} g_{de} g_{\Delta\Delta} \\ &= \frac{1}{M} \left(\delta^{d\Delta}_{a\Delta} - n^{d\Delta} n_{a\Delta} \right) g_{de} g_{\Delta\Delta} \mathbf{K}^{a\Delta}_{bc} \end{aligned}$$

The second term on the right hand side is zero so we have :-

$$M_{b\Delta} g_{ce} - M_{c\Delta} g_{be} = \delta_{e\Delta\ a\Delta} \mathbf{K}^{a\Delta}_{bc}$$

If we act on this with $g^{ef} g^{\Delta\Delta}$ then we find the solution :-

$$\mathbf{K}^{a\Delta}_{bc} = M_b{}^{\Delta} \delta^a_c - M_c{}^{\Delta} \delta^a_b \tag{3.20}$$

For $m = \Delta = 3$ we may further refine this result. In doing so we will show, more explicitly, the homomorphism between these linear Killing vector components, and the ones we found in the $SU(2)$ breaking to $U(1)$ calculation in the last section. We may write :-

$$\begin{aligned} \mathbf{K}^{a3}_{bc} &= M_b{}^3 \delta^a_c - M_c{}^3 \delta^a_b \\ &= \left(\delta^d_b \delta^a_c - \delta^d_c \delta^a_b \right) M_d{}^3 \\ &= \varepsilon^{da3} \varepsilon_{bc3} M_d{}^3 \\ &= \varepsilon_{bc3} \varepsilon^{a3d} M_d{}^3 \end{aligned}$$

Therefore we find :-

$$\begin{aligned} \mathbf{K}^{a3}_{12} &= \varepsilon^{a3b} M_b{}^3 \\ &= \varepsilon_{a3b} M^{b3} \end{aligned}$$

because there is no distinction between upper and lower indices. This is obviously similar to equation (3.9).

3.2.2b The nonlinear Killing vector components, $\mathbf{K}^{a\triangle}_{b\triangle}$.

To find the nonlinear $\mathbf{K}^{a\triangle}_{b\triangle}$ components we must solve :-

$$\left\{\sigma_{b\triangle}, L^2\right\} = 2i\, L^2_{,a\triangle}\mathbf{K}^{a\triangle}_{b\triangle}$$

This time we find the relation :-

$$\begin{aligned}\cos 2\Omega\ \sigma_{b\triangle} - i\sin 2\Omega\ n_{b\triangle}\ \mathbf{1}_{[2^k]} &= \sin 2\Omega\ n^{c\triangle}_{,a\triangle}\mathbf{K}^{a\triangle}_{b\triangle}\sigma_{c\triangle} + 2\cos 2\Omega\ \Omega_{,a\triangle}\mathbf{K}^{a\triangle}_{b\triangle}n^{c\triangle}\sigma_{c\triangle} \\ &\quad -2i\sin 2\Omega\ \Omega_{,a\triangle}\mathbf{K}^{a\triangle}_{b\triangle}\ \mathbf{1}_{[2^k]}\end{aligned}$$

which implies $2\Omega_{,a\triangle}\mathbf{K}^{a\triangle}_{b\triangle} = n_{b\triangle}$. When we substitute this in we find :-

$$\cos 2\Omega\ \sigma_{b\triangle} = \cos 2\Omega\ n_{b\triangle}n^{c\triangle}\sigma_{c\triangle} + \sin 2\Omega\ n^{c\triangle}_{,a\triangle}\mathbf{K}^{a\triangle}_{b\triangle}\sigma_{c\triangle}$$

Removing the sigma matrices and substituting in for $n^{c\triangle}_{,a\triangle}$ yields :-

$$\begin{aligned}\cos 2\Omega\ \left(g_{bd}g_{\triangle\triangle} - n_{b\triangle}n_{d\triangle}\right) &= \frac{\sin 2\Omega}{M}\left(\mathbf{K}^{c\triangle}_{b\triangle}g_{cd}g_{\triangle\triangle} - n_{d\triangle}n_{a\triangle}\mathbf{K}^{a\triangle}_{b\triangle}\right) \\ &= \frac{\sin 2\Omega}{M}\left(\mathbf{K}_{d\triangle\ b\triangle} - \frac{dM}{d2\Omega}n_{b\triangle}n_{d\triangle}\right)\end{aligned}$$

Therefore we may now rearrange this to find the solution :-

$$\mathbf{K}_{a\triangle\ b\triangle} = M\cot 2\Omega\ \left(g_{ab}g_{\triangle\triangle} - n_{a\triangle}n_{b\triangle}\right) + \frac{dM}{d2\Omega}n_{a\triangle}n_{b\triangle} \tag{3.21}$$

where, for these models, $g_{ab} \equiv \delta_{ab}$.

3.2.3 Covariant derivatives and the Goldstone boson metric.

To find the covariant derivatives for the Goldstone Bosons and the Standard fields of the theory we must find :-

$$2iL^{-1}\partial_\mu L = a_\mu + v_\mu$$

where a_μ is the covariant derivative for the Goldstone fields, and v_μ is the metric connection in the covariant derivative for the Standard fields of the theory. Now :-

$$
\begin{aligned}
L^{-1} &= \cos\Omega\ \mathbf{1}_{[2^k]} + i\sin\Omega\ S \\
\partial_\mu L &= -\sin\Omega\ \partial_\mu\Omega\,\mathbf{1}_{[2^k]} - i\cos\Omega\ S\partial_\mu\Omega - i\sin\Omega\ \partial_\mu S
\end{aligned}
$$

where $S = n^{a\Delta}\sigma_{a\Delta}$ and $S^2 = \mathbf{1}_{[2^k]}$. Because of the form of S, calculating $2iL^{-1}\partial_\mu L$ is just as simple as in the $SU(2)$ breaking to $U(1)$ model. We find :-

$$
2iL^{-1}\partial_\mu L = 2S\partial_\mu\Omega + \sin 2\Omega\ \partial_\mu S + 2i\sin^2\Omega\ S\partial_\mu S
$$

Therefore we have :-

$$
a_\mu = 2S\partial_\mu\Omega + \sin 2\Omega\ \partial_\mu S \tag{3.22}
$$

$$
v_\mu = 2i\sin^2\Omega\ S\partial_\mu S \tag{3.23}
$$

Using appendix D, we find the explicit forms for the covariant derivatives to be :-

$$
\mathcal{D}_\mu M^{a\Delta} = \left\{\frac{\sin 2\Omega}{M}(\delta^{a\Delta}_{b\Delta} - n^{a\Delta}n_{b\Delta}) + \left(\frac{d\,2\Omega}{dM}\right) n^{a\Delta}n_{b\Delta}\right\}\partial_\mu M^{b\Delta} \tag{3.24}
$$

$$
\mathcal{D}_\mu\psi = \left\{\partial_\mu + \frac{i}{M^2}\sin^2\Omega\ M^{a\Delta}\partial_\mu M^{b\Delta}\ g_{\Delta\Delta}\sigma_{ab}\right\}\psi \tag{3.25}
$$

To find the metric for the Goldstone Boson manifold we must now evaluate $a^{a\Delta}_\mu a^\mu_{a\Delta}$ which we simply find :-

$$
\begin{aligned}
a^{a\Delta}_\mu a^\mu_{a\Delta} &= \left(2n^{a\Delta}\partial_\mu\Omega + \sin 2\Omega\ \partial_\mu n^{a\Delta}\right)\left(2n_{a\Delta}\partial^\mu\Omega + \sin 2\Omega\ \partial^\mu n_{a\Delta}\right) \\
&= 4\partial_\mu\Omega\,\partial^\mu\Omega + \sin^2 2\Omega\ \partial_\mu n^{a\Delta}\partial^\mu n_{a\Delta}
\end{aligned} \tag{3.26}
$$

Using appendix D we find that this is :-

$$
a^{a\Delta}_\mu a^\mu_{a\Delta} = \left[\frac{\sin^2 2\Omega}{M^2}(g_{ab}g_{\Delta\Delta} - n_{a\Delta}n_{b\Delta}) + \left(\frac{d2\Omega}{dM}\right)^2 n_{a\Delta}n_{b\Delta}\right]\partial_\mu M^{a\Delta}\partial^\mu M^{b\Delta} \tag{3.27}
$$

and we have as our Goldstone Boson manifold metric :-

$$
g_{a\Delta\ b\Delta} = \frac{\sin^2 2\Omega}{M^2}(g_{ab}g_{\Delta\Delta} - n_{a\Delta}n_{b\Delta}) + \left(\frac{d2\Omega}{dM}\right)^2 n_{a\Delta}n_{b\Delta} \tag{3.28}
$$

where, for these models, $g_{ab} \equiv \delta_{ab}$.

We note that *only* for $m = \Delta = 3$ and $a, b = 1, 2$ is this, from [14], the metric of a Kähler manifold. That is, *only* the model arising from the spontaneous breaking of an $SO(3)$ symmetry down to an $SO(2)$ symmetry will yield a Kähler Goldstone boson manifold metric; implying that only this model can be extended to include $\mathcal{N} = 1$ Supersymmetry.

3.2.3a Verifying the metric result.

It is now possible for us to check the form of the Goldstone Boson manifold metric result using the Killing vector components. We firstly need to find :-

$$
\begin{aligned}
\mathbf{K}^{a\Delta}_{cd}\mathbf{K}^{cd\ b\Delta} &= \mathbf{K}^{a\Delta}_{cd}\mathbf{K}^{b\Delta}_{ef}g^{ec}g^{fd} \\
&= g^{ec}g^{fd}\left(M_c{}^{\Delta}\delta^a_d - M_d{}^{\Delta}\delta^a_c\right)\left(M_e{}^{\Delta}\delta^a_f - M_f{}^{\Delta}\delta^a_e\right) \\
&= \left(M_c{}^{\Delta}\delta^a_d - M_d{}^{\Delta}\delta^a_c\right)\left(M^{c\Delta}g^{bd} - M^{d\Delta}g^{bc}\right) \\
&= 2\left(M_{c\Delta}M^{c\Delta}g^{ab}g^{\Delta\Delta} - M^{a\Delta}M^{b\Delta}\right) \\
&= 2M^2\left(g^{ab}g^{\Delta\Delta} - n^{a\Delta}n^{b\Delta}\right)
\end{aligned}
$$

Secondly we need :-

$$
\begin{aligned}
\mathbf{K}^{a\Delta}_{c\Delta}\mathbf{K}^{c\Delta\ b\Delta} &= \mathbf{K}_{d\Delta\ c\Delta}\mathbf{K}^{c\Delta\ b\Delta}g^{da}g^{\Delta\Delta} \\
&= M^2\cot^2 2\Omega\ \left(g^{ab}g^{\Delta\Delta} - n^{a\Delta}n^{b\Delta}\right) + \left(\frac{dM}{d2\Omega}\right)^2 n^{a\Delta}n^{b\Delta}
\end{aligned}
$$

The inverse of the Goldstone Boson manifold metric is given by :-

$$
\begin{aligned}
\left(g_{a\Delta\ b\Delta}\right)^{-1} &= \frac{1}{2}\mathbf{K}^{a\Delta}_{cd}\mathbf{K}^{cd\ b\Delta} + \mathbf{K}^{a\Delta}_{c\Delta}\mathbf{K}^{c\Delta\ b\Delta} \\
&= \frac{M^2}{\sin^2 2\Omega}\left(g^{ab}g^{\Delta\Delta} - n^{a\Delta}n^{b\Delta}\right) + \left(\frac{dM}{d2\Omega}\right)^2 n^{a\Delta}n^{b\Delta}
\end{aligned}
$$

This is obviously the inverse of equation (3.28).

3.3 The $SO(1, m-1)$ breaking to $SO(1, m-2)$ models.

3.3.1 The coset representative, L.

These models use the coset representative for the $\frac{SO(1,m-1)}{SO(1,m-2)}$ cosets. Firstly we state that there is an Isomorphism between the groups $SO(m)$ and $SO(1, m-1)$. This means that the method and form of the results in this section mirror the method and form of the results of the last. With such similarities it is important to remember that the differences between this section and the last lie entirely in the interpretation of the models. We will now see how the mathematics of these models differs from the last section.

For these models we use modified $SO(m)$ gamma matrices which now have the Clifford algebra :-

$$\{\Gamma_\mu, \Gamma_\nu\} \;=\; 2g_{\mu\nu}\,\mathbf{1}_{[2^k]}$$

where $g_{\mu\nu} \equiv \eta_{\mu\nu}$ and $\mu, \nu = 0, 1, 2, \dots, (m-2), m$. The matrix η, used to raise and lower indices, is defined in Appendix C, on page 196. The sigma matrices for $SO(1, m-1)$, in the Weyl representation are defined :-

$$\Sigma_{\mu\nu} \;\equiv\; +\frac{i}{2}\,[\Gamma_\mu, \Gamma_\nu]$$

with $\mu, \nu = 0, 1, 2, \dots, (m-2), m$. If we generate the subgroup $SO(1, m-2)$ using $\Sigma_{\alpha\beta}$, with $\alpha, \beta = 0, 1, 2, \dots, (m-2)$, then the coset vector we need is :-

$$\begin{aligned} x &= \omega^{\alpha\Delta}\Sigma_{\alpha\Delta} \\ &= \Omega n^{\alpha\Delta}\Sigma_{\alpha\Delta} \end{aligned}$$

where we have also introduced the label $\Delta = m$.

So the square of this is :-

$$\begin{aligned} x^2 &= \Omega^2 n^{\alpha\Delta} n_{\alpha\Delta}\, \mathbf{1}_{[2^k]} \\ &= \begin{cases} +\Omega^2\, \mathbf{1}_{[2^k]} & \text{for a timelike } n^{\alpha\Delta} \\ -\Omega^2\, \mathbf{1}_{[2^k]} & \text{for a spacelike } n^{\alpha\Delta} \end{cases} \end{aligned}$$

because we no longer have a positive definite metric. For all our results we will use a timelike unit vector $n^{\alpha\Delta}$ defined by $n^{\alpha\Delta} n_{\alpha\Delta} \equiv +1$. Therefore, because $x \vee x = 0$ again, the coset representative element, L, has the same form as before :-

$$L = \cos\Omega\ \mathbf{1}_{[2^k]} - i \sin\Omega\ n^{\alpha\Delta}\Sigma_{\alpha\Delta} \tag{3.29}$$

Thus, not only are we able to use the same method as before but it is also clear, even at this stage, that the results will have the same form too.

Once more we note that *only* when m is an odd number does the subgroup have the same rank as the group; when m is even the subgroup has a rank which is one less than the group. For example :-

1. If $m = 5$ we are considering the spontaneous breaking of an $SO(1,4)$ symmetry down to an $SO(1,3)$ symmetry and $SO(1,4)$ is of rank 2 and so is $SO(1,3)$, whereas

2. if $m = 6$ we are considering the spontaneous breaking of an $SO(1,5)$ symmetry down to an $SO(1,4)$ symmetry and $SO(1,5)$ is a group of rank 3 but $SO(1,4)$ is of rank 2.

3.3.2 Goldstone boson transformations.

3.3.2a The linear Killing vector components, $\mathbf{K}^{\alpha\Delta}_{\beta\gamma}$.

To find the linear $\mathbf{K}^{\alpha\Delta}_{\beta\gamma}$ components we must solve :-

$$[\Sigma_{\beta\gamma}, L] = 2i\, L_{,\alpha\Delta} \mathbf{K}^{\alpha\Delta}_{\beta\gamma}$$

The coset representative and its derivative with respect to the Goldstone fields are :-

$$\begin{aligned} L &= \cos\Omega\ \mathbf{1}_{[2^k]} - i\sin\Omega\ n^{\gamma\Delta}\Sigma_{\gamma\Delta} \\ L_{,\alpha\Delta} &= -\sin\Omega\ \Omega_{,\alpha\Delta}\,\mathbf{1}_{[2^k]} - i\cos\Omega\ \Omega_{,\alpha\Delta}n^{\gamma\Delta}\Sigma_{\gamma\Delta} - i\sin\Omega\ n^{\gamma\Delta}_{,\alpha\Delta}\Sigma_{\gamma\Delta} \end{aligned}$$

For the left hand side we have :-

$$[\Sigma_{\beta\gamma}, L] = -i\sin\Omega\ n^{\alpha\Delta}\,[\Sigma_{\beta\gamma}, \Sigma_{\alpha\Delta}]$$

This time, for these $SO(1, m-1)$ breaking to $SO(1, m-2)$ models we will have :-

$$[\Sigma_{\beta\gamma}, \Sigma_{\alpha\Delta}] = -2i\,(g_{\alpha\beta}\Sigma_{\gamma\Delta} - g_{\alpha\gamma}\Sigma_{\beta\Delta})$$

with $g_{\alpha\beta} \equiv \eta_{\alpha\beta}$. Note the minus sign which occurs for this commutator. If we now equate the left and right hand sides of our relation to be solved we find :-

$$\begin{aligned} -\sin\Omega\ n^{\alpha\Delta}\,(g_{ab}\Sigma_{c\Delta} - g_{ac}\Sigma_{b\Delta}) &= -i\sin\Omega\ \Omega_{,\alpha\Delta}\mathbf{K}^{\alpha\Delta}_{\beta\gamma}\,\mathbf{1}_{[2^k]} + \cos\Omega\ \Omega_{,\alpha\Delta}\mathbf{K}^{\alpha\Delta}_{\beta\gamma}n^{\delta\Delta}\Sigma_{\delta\Delta} \\ &\quad + \sin\Omega\ n^{\delta\Delta}_{,\alpha\Delta}\mathbf{K}^{\alpha\Delta}_{\beta\gamma}\Sigma_{\delta\Delta} \end{aligned}$$

which automatically implies $\Omega_{,\alpha\Delta}\mathbf{K}^{\alpha\Delta}_{\beta\gamma} = 0$. If we substitute this into the above we have :-

$$-n^{\alpha\Delta}\,(g_{\alpha\beta}\Sigma_{\gamma\Delta} - g_{\alpha\gamma}\Sigma_{\beta\Delta}) = n^{\delta\Delta}_{,\alpha\Delta}\mathbf{K}^{\alpha\Delta}_{\beta\gamma}\Sigma_{\delta\Delta}$$

Since $tr\,\Sigma_{\alpha\Delta}\Sigma_{\beta\Delta} = 2^k g_{\alpha\beta}g_{\Delta\Delta}$ we may remove the sigma matrices. We multiply by $\Sigma_{\epsilon\Delta}$ and trace the expression to find :-

$$\begin{aligned} -n^{\alpha\Delta}(g_{\alpha\beta}g_{\gamma\epsilon}g_{\Delta\Delta} - g_{\alpha\gamma}g_{\beta\epsilon}g_{\Delta\Delta}) &= n^{\delta\Delta}_{,\alpha\Delta}\mathbf{K}^{\alpha\Delta}_{\beta\gamma}g_{\delta\epsilon}g_{\Delta\Delta} \\ -(n_{\beta\Delta}g_{\gamma\epsilon} - n_{\gamma\Delta}g_{\beta\epsilon}) &= n^{\delta\Delta}_{,\alpha\Delta}\mathbf{K}^{\alpha\Delta}_{\beta\gamma}g_{\delta\epsilon}g_{\Delta\Delta} \\ &= \frac{1}{M}\left(\delta^{\delta\Delta}_{\alpha\Delta} - n^{\delta\Delta}n_{\alpha\Delta}\right)g_{\delta\epsilon}g_{\Delta\Delta}\mathbf{K}^{\alpha\Delta}_{\beta\gamma} \end{aligned}$$

Again, the second term on the right hand side is zero so we have :-

$$-(M_{\beta\Delta}g_{\gamma\epsilon} - M_{\gamma\Delta}g_{\beta\epsilon}) = \delta_{\epsilon\Delta\ \alpha\Delta}\mathbf{K}^{\alpha\Delta}_{\beta\gamma}$$

If we act on this with $g^{ef}g^{\Delta\Delta}$, and relabel, then we find the solution :-

$$\mathbf{K}^{\alpha\Delta}_{\beta\gamma} = -(M_\beta{}^\Delta\delta^\alpha_\gamma - M_\gamma{}^\Delta\delta^\alpha_\beta) \tag{3.30}$$

where we have chosen to keep the minus sign, introduced by the commutator, explicit. This difference is cancelled when we come to calculate the Goldstone boson manifold metric for these models.

3.3.2b The nonlinear Killing vector components, $\mathbf{K}^{\alpha\Delta}_{\beta\Delta}$.

To find the nonlinear $\mathbf{K}^{\alpha\Delta}_{\beta\Delta}$ components we must solve :-

$$\{\Sigma_{\beta\Delta}, L^2\} = 2i\, L^2_{,\alpha\Delta}\mathbf{K}^{\alpha\Delta}_{\beta\Delta}$$

This time we find the relation :-

$$\begin{aligned}\cos 2\Omega\ \Sigma_{\beta\Delta} - i\sin 2\Omega\ n_{\beta\Delta}\,\mathbf{1}_{[2^k]} &= -2i\sin 2\Omega\ \Omega_{,\alpha\Delta}\mathbf{K}^{\alpha\Delta}_{\beta\Delta}\,\mathbf{1}_{[2^k]} + \sin 2\Omega\ n^{\gamma\Delta}_{,\alpha\Delta}\mathbf{K}^{\alpha\Delta}_{\beta\Delta}\Sigma_{\gamma\Delta} \\ &\qquad 2\cos 2\Omega\ \Omega_{,\alpha\Delta}\mathbf{K}^{\alpha\Delta}_{\beta\Delta}n^{\gamma\Delta}\Sigma_{\gamma\Delta}\end{aligned}$$

which implies $2\Omega_{,\alpha\Delta}\mathbf{K}^{\alpha\Delta}_{\beta\Delta} = n_{\beta\Delta}$. When we substitute this in we find :-

$$\cos 2\Omega\ \Sigma_{\beta\Delta} = \cos 2\Omega\ n_{\beta\Delta}n^{\gamma\Delta}\Sigma_{\gamma\Delta} + \sin 2\Omega\ n^{\gamma\Delta}_{,\alpha\Delta}\mathbf{K}^{\alpha\Delta}_{\beta\Delta}\Sigma_{\gamma\Delta}$$

Removing the sigma matrices and expanding $n^{\gamma\Delta}_{,\alpha\Delta}$ yields :-

$$\begin{aligned}\cos 2\Omega\ (g_{\beta\delta}g_{\Delta\Delta} - n_{\beta\Delta}n_{\delta\Delta}) &= \frac{\sin 2\Omega}{M}\left(\mathbf{K}^{\gamma\Delta}_{\beta\Delta}g_{\gamma\delta}g_{\Delta\Delta} - n_{\delta\Delta}n_{\alpha\Delta}\mathbf{K}^{\alpha\Delta}_{\beta\Delta}\right) \\ &= \frac{\sin 2\Omega}{M}\left(\mathbf{K}_{\delta\Delta\ \beta\Delta} - \frac{dM}{d2\Omega}n_{\beta\Delta}n_{\delta\Delta}\right)\end{aligned}$$

Therefore we may now rearrange this to find the solution :-

$$\mathbf{K}_{\alpha\Delta\ \beta\Delta} = M\cot 2\Omega\ (g_{\alpha\beta}g_{\Delta\Delta} - n_{\alpha\Delta}n_{\beta\Delta}) + \frac{dM}{d2\Omega}n_{\alpha\Delta}n_{\beta\Delta} \tag{3.31}$$

where, for these models, we have $g_{\alpha\beta} \equiv \eta_{\alpha\beta}$.

3.3.3 Covariant derivatives and the Goldstone boson metric.

To find the covariant derivatives for the Goldstone Bosons and the Standard fields of the theory we must find $2iL^{-1}\partial_\mu L = a_\mu + v_\mu$ where a_μ is the covariant derivative for the Goldstone fields, and v_μ is the metric connection in the covariant derivative for the Standard fields of the theory. Now :-

$$\begin{aligned} L^{-1} &= \cos\Omega\ \mathbf{1}_{[2^k]} + i\sin\Omega\ S \\ \partial_\mu L &= -\sin\Omega\ \partial_\mu\Omega\,\mathbf{1}_{[2^k]} - i\cos\Omega\ S\partial_\mu\Omega - i\sin\Omega\ \partial_\mu S \end{aligned}$$

where now $S = n^{\alpha\Delta}\Sigma_{\alpha\Delta}$ and, for timelike $n^{\alpha\Delta}$, we have $S^2 = \mathbf{1}_{[2^k]}$. As we found before, calculating $2iL^{-1}\partial_\mu L$ is just as simple as in the $SU(2)$ breaking to $U(1)$ model. We find :-

$$2iL^{-1}\partial_\mu L = 2S\partial_\mu\Omega + \sin 2\Omega\ \partial_\mu S + 2i\sin^2\Omega\ S\partial_\mu S$$

Therefore we have :-

$$a_\mu = 2S\partial_\mu\Omega + \sin 2\Omega\ \partial_\mu S \tag{3.32}$$

$$v_\mu = 2i\sin^2\Omega\ S\partial_\mu S \tag{3.33}$$

Using appendix D, we find the explicit forms for the covariant derivatives to be :-

$$\mathcal{D}_\mu M^{\alpha\Delta} = \left\{\frac{\sin 2\Omega}{M}(\delta^{\alpha\Delta}_{\beta\Delta} - n^{\alpha\Delta}n_{\beta\Delta}) + \left(\frac{d\,2\Omega}{dM}\right)n^{\alpha\Delta}n_{\beta\Delta}\right\}\partial_\mu M^{\beta\Delta} \tag{3.34}$$

$$\mathcal{D}_\mu\psi = \left\{\partial_\mu - \frac{i}{M^2}\sin^2\Omega\ M^{\alpha\Delta}\partial_\mu M^{\beta\Delta}\ g_{\Delta\Delta}\Sigma_{\alpha\beta}\right\}\psi \tag{3.35}$$

To find the metric for the Goldstone Boson manifold we must now evaluate $a_\mu^{\alpha\Delta}a^\mu_{\alpha\Delta}$:-

$$\begin{aligned} a_\mu^{\alpha\Delta}a^\mu_{\alpha\Delta} &= \left(2n^{\alpha\Delta}\partial_\mu\Omega + \sin 2\Omega\ \partial_\mu n^{\alpha\Delta}\right)\left(2n_{\alpha\Delta}\partial^\mu\Omega + \sin 2\Omega\ \partial^\mu n_{\alpha\Delta}\right) \\ &= 4\partial_\mu\Omega\,\partial^\mu\Omega + \sin^2 2\Omega\ \partial_\mu n^{\alpha\Delta}\partial^\mu n_{\alpha\Delta} \end{aligned}$$

We may again use appendix D to write this explicitly. We find :-

$$a_\mu^{\alpha\Delta}a^\mu_{\alpha\Delta} = \left[\frac{\sin^2 2\Omega}{M^2}(g_{\alpha\beta}g_{\Delta\Delta} - n_{\alpha\Delta}n_{\beta\Delta}) + \left(\frac{d2\Omega}{dM}\right)^2 n_{\alpha\Delta}n_{\beta\Delta}\right]\partial_\mu M^{\alpha\Delta}\partial^\mu M^{\beta\Delta} \tag{3.36}$$

and we have as our Goldstone Boson manifold metric :-

$$g_{\alpha\Delta\ \beta\Delta} = \frac{\sin^2 2\Omega}{M^2}(g_{\alpha\beta}g_{\Delta\Delta} - n_{\alpha\Delta}n_{\beta\Delta}) + \left(\frac{d2\Omega}{dM}\right)^2 n_{\alpha\Delta}n_{\beta\Delta} \tag{3.37}$$

where, for these models, $g_{\alpha\beta} \equiv \eta_{\alpha\beta}$.

We note that *only* for $m = \Delta = 3$ and $\alpha, \beta = 0, 1$ is this, from [14], the metric of a Kähler manifold. That is, *only* the model arising from the spontaneous breaking of an $SO(1,2)$ symmetry down to an $SO(1,1)$ symmetry will yield a Kähler Goldstone boson manifold metric; implying that only this model can be extended to include $\mathcal{N} = 1$ Supersymmetry.

3.3.3a Verifying the metric result.

It is now possible for us to check the form of the Goldstone Boson manifold metric result using the Killing vector components. Firstly, squaring the linear Killing vector components, we find :-

$$\begin{aligned}
\mathbf{K}^{\alpha\Delta}_{\gamma\delta}\mathbf{K}^{\gamma\delta\ \beta\Delta} &= \mathbf{K}^{\alpha\Delta}_{\gamma\delta}\mathbf{K}^{\beta\Delta}_{\epsilon\phi}g^{\epsilon\gamma}g^{\phi\delta} \\
&= g^{\epsilon\gamma}g^{\phi\delta}\left(M_\gamma{}^\Delta\delta^\alpha_\delta - M_\delta{}^\Delta\delta^\alpha_\gamma\right)\left(M_\epsilon{}^\Delta\delta^\alpha_\phi - M_\phi{}^\Delta\delta^\alpha_\epsilon\right) \\
&= \left(M_\gamma{}^\Delta\delta^\alpha_\delta - M_\delta{}^\Delta\delta^\alpha_\gamma\right)\left(M^{\gamma\Delta}g^{\beta\delta} - M^{\delta\Delta}g^{\beta\gamma}\right) \\
&= 2\left(M_{\gamma\Delta}M^{\gamma\Delta}g^{\alpha\beta}g^{\Delta\Delta} - M^{\alpha\Delta}M^{\beta\Delta}\right) \\
&= 2M^2\left(g^{\alpha\beta}g^{\Delta\Delta} - n^{\alpha\Delta}n^{\beta\Delta}\right)
\end{aligned}$$

Notice that the minus sign which appeared in the linear Killing vectors for these models is of no consequence in the calculation of the Goldstone boson manifold metric.

Secondly we need :-

$$\begin{aligned}
\mathbf{K}^{\alpha\Delta}_{\gamma\Delta}\mathbf{K}^{\gamma\Delta\ \beta\Delta} &= \mathbf{K}_{\delta\Delta\ \gamma\Delta}\mathbf{K}^{\gamma\Delta\ \beta\Delta}g^{\delta\alpha}g^{\Delta\Delta} \\
&= M^2\cot^2 2\Omega\ (g^{\alpha\beta}g^{\Delta\Delta} - n^{\alpha\Delta}n^{\beta\Delta}) + \left(\frac{dM}{d2\Omega}\right)^2 n^{\alpha\Delta}n^{\beta\Delta}
\end{aligned}$$

The inverse of the Goldstone Boson manifold metric is given by :-

$$
\begin{aligned}
(g_{\alpha\Delta\ \beta\Delta})^{-1} &= \frac{1}{2}\mathbf{K}^{\alpha\Delta}_{cd}\mathbf{K}^{cd\ b\Delta} + \mathbf{K}^{\alpha\Delta}_{\gamma\Delta}\mathbf{K}^{\gamma\Delta\ \beta\Delta} \\
&= \frac{M^2}{\sin^2 2\Omega}(g^{\alpha\beta}g^{\Delta\Delta} - n^{\alpha\Delta}n^{\beta\Delta}) + \left(\frac{dM}{d2\Omega}\right)^2 n^{\alpha\Delta}n^{\beta\Delta}
\end{aligned}
$$

This is obviously the inverse of equation (3.37).

Chapter 4

CP 2(N-1) models from coset vectors of form $x = \frac{\phi}{2} r$.

4.1 CP 2(N-1) models in Chiral symmetry breaking theories.

If we wanted to do calculations for a Chiral symmetry breaking model, where $SU_L(N) \otimes SU_R(N)$ invariance is broken to $SU_V(N)$, then we would certainly, for $N \geq 3$, need to use a method based upon projection operators right from the start; the details of this method are in [13]. However, for $N = 2$, the full $SU_L(2) \otimes SU_R(2)$ breaking to $SU_V(2)$ model may easily be studied without having to resort to this method and, in section 4.1.1, we will show that the details of this model are contained in the last chapter. In this chapter we are only going to concern ourselves with $SU(N)$ breaking to $SU(N-1) \otimes U(1)$ because this is a far simpler problem to deal with and, as we will now show, it may also be embedded into a full Chiral symmetry breaking model; where $SU_L(N) \otimes SU_R(N)$ invariance is broken to $SU_V(N)$.

$SU_L(N) \otimes SU_R(N)$ is generated by $2(N^2 - 1)$ generators, which we call L_I and R_I, where $I = 1, 2, \ldots, N^2 - 1$. The Lie algebra is :-

$$\begin{aligned} [L_I, L_J] &= 2if_{IJK}L_K \\ [R_I, R_J] &= 2if_{IJK}R_K \end{aligned}$$

and by definition $[L_I, R_J] = [R_I, L_J] \equiv 0 \quad \forall\ I, J$. If we now form linear combinations of these generators :-

$$\begin{aligned} V_I &= L_I + R_I \\ A_I &= L_I - R_I \end{aligned}$$

then it is simple to show that they obey :-

$$\begin{aligned} [V_I, V_J] &= 2if_{IJK}V_K \\ [A_I, A_J] &= 2if_{IJK}V_K \\ [A_I, V_J] &= 2if_{IJK}A_K \end{aligned}$$

The first of these commutation relations is the Lie algebra for an $SU_V(N)$, parity conserving, subgroup. If we now restrict our attention to the subset of the V_I, namely the V_E, which generate a (parity conserving) $SU(N-1) \otimes U(1)$ subgroup of $SU_V(N)$, then the subset of the A_I, the A_a, which are present in the last two commutators are $\frac{SU(N)}{SU(N-1)\otimes U(1)}$ coset directions. Explicitly :-

$$\begin{aligned} [V_E, V_F] &= 2if_{EFG}V_G \\ [A_a, A_b] &= 2if_{abE}V_E \\ [A_a, V_E] &= 2if_{aEb}A_b \end{aligned}$$

In the language of [18] we refer to the $SU(N)$ group, which is generated by the V_E and A_a, as a chiral $SU(N)$ group. Thus, in the full theory, when we have $SU_L(N) \otimes SU_R(N)$ symmetry broken to $SU_V(N)$ then, necessarily, we are able to find an embedded model where a chiral $SU(N)$ symmetry is broken to $SU(N-1) \otimes U(1)$. For simplicity we work with $SU(N)$ in the defining representation, therefore we may embed the work contained in this chapter into a full Chiral symmetry breaking model by making the identifications $\lambda_a \rightarrow A_a$ and $\lambda_E \rightarrow V_E$. This means that, for example, when we calculate the quantity $2iL^{-1}\partial_\mu L = a_\mu + v_\mu$ in the chiral $SU(N)$ breaking model and find :-

$$
\begin{aligned}
a_\mu &= \mathcal{D}_\mu M^a \lambda_a \\
v_\mu &= v_\mu^E \lambda_E
\end{aligned}
$$

then, by making the basis identifications $\lambda_a \rightarrow A_a$ and $\lambda_E \rightarrow V_E$, we may rewrite these quantities as :-

$$
\begin{aligned}
a_\mu &= \mathcal{D}_\mu M^a A_a \\
v_\mu &= v_\mu^E V_E
\end{aligned}
$$

These are contained as a subset of the $a_\mu = a_\mu^I A_I$ and $v_\mu = v_\mu^I V_I$ which result in the full chiral $SU_L(N) \otimes SU_R(N)$ breaking to $SU_V(N)$ model.

We note that in [18] we find the main results for the $SU(2)$ breaking to $U(1)$ model, together with how the model may be embedded in the full chiral $SU_L(2) \otimes SU_R(2)$ breaking to $SU_V(2)$ scheme; it is included in this thesis because it is an instructive 'toy' model. A general discussion of embedding may be found in [19], together with the specific results for the CP2 and CP4 metrics. However, these results are also extended by rewriting the corresponding Goldstone Boson part of the Lagrangian using stereographic coordinates [21] which allows, in the CP4 case, the retrieval of the Fubini-Study metric. The CP4 metric is obviously contained within this chapter too, though its form is different to that found in [19] because we use a different method to calculate it.

4.1.1 Cross-referencing three models.

In the previous chapter we considered an infinite number of models. However, the three simplest are the models resulting from :-

- an $SO(4)$ symmetry breaking to $SO(3)$,
- an $SO(3)$ symmetry breaking to $SO(2)$, and
- an $SU(2)$ symmetry breaking to $U(1)$.

As we have stated, the main results for the $SU(2)$ breaking to $U(1)$ model, together with how the model may be embedded in the full chiral $SU_L(2) \otimes SU_R(2)$ breaking to $SU_V(2)$ scheme, may be found in [18]. However, this paper does not mention the homomorphic $SO(3)$ symmetry breaking model; or its embedding into the full $SO(4)$ breaking scheme. To this end we will firstly discuss the full $SO(4)$ breaking to $SO(3)$ model and then, secondly, we will show how the $SO(3)$ (and therefore the homomorphic $SU(2)$) breaking 'toy' models may be embedded. It also seems reasonable to say that, using similar ideas, it is possible to embed the $SO(1,2)$ symmetry breaking to $SO(1,1)$ model (contained in the previous chapter) into the framework of the larger $SO(1,3)$ breaking to $SO(1,2)$ theory.

4.1.1a $SU_L(2) \otimes SU_R(2)$ breaking to $SU_V(2)$.

Firstly, we will exploit the homomorphism between the groups $SU(2) \otimes SU(2)$ and $SO(4)$. The sigma matrices, used to form $SO(4)$ group elements, are :-

$$\sigma_{ij} = \varepsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \qquad \sigma_{k4} = \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix}$$

where $i, j, k = 1, 2, 3$. We may therefore rewrite these as :-

$$\begin{aligned}
\sigma_{ij} &= \varepsilon_{ijk}(L_k + R_k) \\
&= \varepsilon_{ijk} V_k \\
\sigma_{k4} &= (L_k - R_k) \\
&= A_k
\end{aligned}$$

We now find that the V_i and A_i obey the commutation relations :-

$$\begin{aligned}
[V_i, V_j] &= 2i\varepsilon_{ijk} V_k \\
[A_i, A_j] &= 2i\varepsilon_{ijk} V_k \\
[A_i, V_j] &= 2i\varepsilon_{ijk} A_k
\end{aligned}$$

We may now rewrite the coset vector for the $\frac{SO(4)}{SO(3)}$ coset as :-

$$\begin{aligned}
x &= \omega^{k4} \sigma_{k4} \\
&= \omega^{k4} A_k
\end{aligned}$$

So it is clear that, by working out the Lagrangian for the $SO(4)$ breaking to $SO(3)$ model we have, in a homomorphic way, also worked out the Lagrangian for the full $SU_L(2) \otimes SU_R(2)$ breaking to $SU_V(2)$ model.

4.1.1b Chiral $SU(2)$ breaking to $U(1)$.

In this section we will consider the effect of restricting the V_i, from the last section, to just V_3 which yields the commutators :-

$$\begin{aligned}
[V_3, V_3] &= 0 \\
[A_a, A_b] &= 2i\varepsilon_{ab3} V_3 \\
[A_a, V_3] &= 2i\varepsilon_{a3b} A_b
\end{aligned}$$

These are the commutation relations for a chiral $SU(2)$ group generated by the set $\{A_1, A_2, V_3\}$. Notice that V_3 is being used to generate the abelian $U(1)$ subgroup, and therefore the A_a are the $\frac{SU(2)}{U(1)}$ coset space directions.

We first notice that if we were to follow the rules for building the sigma matrices for $SO(3)$, which uses the three $SO(3)$ gamma matrices (defined as the three Pauli Spin matrices), then we would have found them to be :-

$$\sigma_{ij} = \varepsilon_{ijk}\sigma_k$$

where the σ_k are the 3 Pauli spin matrices. Therefore, working out the $SO(3)$ breaking to $SO(2)$ model based on the coset vector :-

$$\begin{aligned} x &= \omega^{a3}\sigma_{a3} \\ &= \omega^{a3}\varepsilon_{a3b}\sigma_b \end{aligned}$$

with $a, b = 1, 2$, is obviously a homomorphic problem to the $SU(2)$ breaking to $U(1)$ model (the first model in this thesis) which has the coset vector :-

$$\begin{aligned} x &= \phi^a T_a \\ &= \frac{1}{2}\phi^a\sigma_a \end{aligned}$$

where $a = 1, 2$. If we now identify the σ_{a3}, or 'equivalently' the σ_a, with the $\sigma_{a4} = A_a$ (i.e. the subset of the $SO(4) \sim SU_L(2) \otimes SU_R(2)$ sigma matrices which arises when we restrict our attention to V_3 which generates a $U(1) \sim SO(2)$ subgroup), then we have embedded both these 'chiral' models into the full Chiral $SU_L(2) \otimes SU_R(2)$ breaking to $SU_V(2)$ (homomorphic to $SO(4)$ breaking to $SO(3)$) model of the previous section. For the rest of the thesis we will just use the terms '$SU(N)$', and '$SU(N-1) \otimes U(1)$'; but we will remember that the '$SU(N)$' to which we refer is really the *chiral* $SU(N)$ subgroup of $SU_L(N) \otimes SU_R(N)$.

4.2 The $SU(N)$ breaking to $SU(N-1)\otimes U(1)$ models.

4.2.1 Identifying the Goldstone bosons.

We will now show how the coset vector may be formed using the fundamental representation of $SU(N)$. We may write the coset vector :-

$$
\begin{aligned}
x &= x^a \lambda_a \\
&= (\bar{\chi}\lambda_a\chi)\ \lambda_a
\end{aligned}
$$

where χ is the fundamental representation of $SU(N)$:-

$$
\chi = \begin{pmatrix} u \\ d \\ s \\ \vdots \end{pmatrix}
$$

If this is done then we may identify the Goldstone bosons of the model. We give two examples :-

1. For $N = 2$, the Goldstone bosons are the two charged Pions, $\pi^{\pm}$, and x is :-

$$
x = \begin{pmatrix} 0 & \pi^+ \\ \pi^- & 0 \end{pmatrix}
$$

2. For $N = 3$ the Goldstone bosons are the four Kaons and x is :-

$$
x = \begin{pmatrix} 0 & 0 & K^+ \\ 0 & 0 & K^0 \\ K^- & \overline{K^0} & 0 \end{pmatrix}
$$

4.2.2 The coset representative, L.

These models use the coset representative for the $\frac{SU(N)}{SU(N-1)\otimes U(1)}$ cosets. We note that, for all these models, the subgroup of transformations, $H = SU(N-1)\otimes U(1)$, has the same rank as the full symmetry group, $G = SU(N)$; the rank being equal to $(N-1)$. In the defining representation we have, in general, a coset vector of the form :-

$$x = \left(\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & z_1^* \\ 0 & 0 & \cdots & 0 & z_2^* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & z_{N-1}^* \\ \hline z_1 & z_2 & \cdots & z_{N-1} & 0 \end{array}\right)$$

with $(N-1)$ complex 'numbers' in the last row of the matrix, and $(N-1)$ corresponding complex conjugate 'numbers' in the last column. For example $z_1^* = \phi_{(N-1)^2} - i\phi_{(N-1)^2+1}$, but we must remember that the ϕ's are *fields*. Squaring this matrix (vector) we find :-

$$x^2 = \left(\begin{array}{cccc|c} z_1^* z_1 & z_1^* z_2 & \cdots & z_1^* z_{N-1} & 0 \\ z_2^* z_1 & z_2^* z_2 & \cdots & z_2^* z_{N-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_{N-1}^* z_1 & z_{N-1}^* z_2 & \cdots & z_{N-1}^* z_{N-1} & 0 \\ \hline 0 & 0 & \cdots & 0 & \gamma_2(x) \end{array}\right)$$

where $\gamma_2(x) = \left(z_1^* z_1 + z_2^* z_2 + \cdots + z_{N-1}^* z_{N-1}\right)$. When we calculate the next power we find :-

$$x^3 = \gamma_2(x)\, x \tag{4.1}$$

With this behaviour we understand that x is proportional to an r-vector because x now has the characteristic equation :-

$$\begin{aligned} x^3 - \gamma_2(x)\, x &\equiv 0 \\ &\Rightarrow \\ x^N - \gamma_2(x)\, x^{N-2} &= 0 \end{aligned}$$

which implies that $\gamma_k(x) \equiv 0 \ \forall\ k \geq 3$. So we may write :-

$$\begin{aligned} x &\equiv \sqrt{\gamma_2(x)}\, r \\ &= \frac{\phi}{2} r \end{aligned} \tag{4.2}$$

Therefore the coset representative element, L, is :-

$$L \;=\; \tfrac{1}{N}\left(N + 2\left(\cos\tfrac{\phi}{2} - 1\right)\right) \mathbf{1}_{[N]} + \sqrt{\tfrac{N-2}{N}}\left(\cos\tfrac{\phi}{2} - 1\right)\, q_r - i\, \sin\tfrac{\phi}{2}\, r \tag{4.3}$$

The indices of the coset vector, r, have the range $(N-1)^2 \leq a \leq N^2 - 2$, and, for $N \geq 3$, the indices of the q_r-vector have the range $E = 1, 2, \ldots, (N-1)^2 - 1, N^2 - 1$.

4.2.3 Linear Goldstone boson transformations.

Since we are breaking $G = SU(N)$ invariance down to $H = SU(N-1) \otimes U(1)$, and the two subgroups of H commute, we are able to look at the effects of transforming L in two parts. Firstly we will see how L is transformed by an element of the $U(1)$ subgroup, and then we will see how it is transformed by an element of the $SU(N-1)$ subgroup.

4.2.3a Transforming L with a $U(1)$ subgroup element.

Since the coset representative element is produced by exponentiating the coset vector $x = \frac{\phi}{2} n^a \lambda_a = \frac{\phi}{2} r$ then a subgroup $U(1)$ element, u, will commute with the q_r-vector, therefore :-

$$\begin{aligned} L' &= uLu^{-1} \\ &= \tfrac{1}{N}\left(N + 2(\cos\tfrac{\phi}{2} - 1)\right) \mathbf{1}_{[N]} + \sqrt{\tfrac{N-2}{N}}\left(\cos\tfrac{\phi}{2} - 1\right)\, q_r - i\, \sin\tfrac{\phi}{2}\, uru^{-1} \end{aligned} \tag{4.4}$$

In general, for $h \in SU(N-1) \otimes U(1)$, both r and q_r will be transformed, but both will still lie in their original respective subspaces of $\Re^{N^2-1}$.

We now need the explicit form of the $U(1)$ group element. For the same reason that we needed projection operators to write the exponential of a q_r-vector we need to use projection operators here. Besides, since we wish to find results for a general $G = SU(N)$ breaking to $H = SU(N-1) \otimes U(1)$ it is also desirable to use the projection operator method. Now, the $U(1)$ group element is generated using the last λ matrix of $SU(N)$, the $(N^2-1)^{th}$ basis element of $\Re^{N^2-1}$, which has the following form :-

$$\lambda_{N^2-1} = \frac{1}{\alpha}(P_1 + P_2 + \cdots + P_{N-1} - (N-1)P_N)$$

where $\frac{1}{\alpha}$ normalizes the matrix. Since we wish $(\lambda_{N^2-1}, \lambda_{N^2-1}) = 1$ this implies :-

$$\begin{aligned} 2\alpha^2 &= (N-1)^2 + N - 1 \\ &= N(N-1) \end{aligned}$$

So we find that λ_{N^2-1} is :-

$$\lambda_{N^2-1} = \frac{1}{\sqrt{\frac{N(N-1)}{2}}}(P_1 + P_2 + \cdots + P_{N-1} - (N-1)P_N)$$

Therefore the $u \in U(1)$ is :-

$$\begin{aligned} u &= e^{-i\frac{\Theta}{2}\lambda_{N^2-1}} \\ &= e^{-i\frac{\Theta}{\sqrt{2N(N-1)}}}(P_1 + P_2 + \cdots + P_{N-1}) + e^{i\Theta\sqrt{\frac{N-1}{2N}}}\,P_N \\ &= \sqrt{\tfrac{N(N-1)}{2}}\; e^{-i\frac{\Theta}{\sqrt{2N(N-1)}}}\lambda_{N^2-1} + \left((N-1)e^{-i\frac{\Theta}{\sqrt{2N(N-1)}}} + e^{i\Theta\sqrt{\frac{N-1}{2N}}}\right)\;P_N \end{aligned}$$

So we need to work out uru^{-1} in equation (4.4). We find :-

$$\begin{aligned} r' &= uru^{-1} \\ &= \sqrt{\tfrac{N(N-1)}{2}}\left(e^{-i\Theta\sqrt{\frac{N}{2(N-1)}}}\lambda_{N^2-1}\,r\,P_N + e^{i\Theta\sqrt{\frac{N}{2(N-1)}}}P_N\,r\,\lambda_{N^2-1}\right) \end{aligned}$$

We can tidy this up a little if we use the following notation :-

$$\begin{aligned} r^{\oplus} &= \sqrt{\tfrac{N(N-1)}{2}}\lambda_{N^2-1}\,r\,P_N \\ r^{\ominus} &= \sqrt{\tfrac{N(N-1)}{2}}P_N\,r\,\lambda_{N^2-1} \\ a &= e^{-i\Theta\sqrt{\frac{N}{2(N-1)}}} \end{aligned}$$

such that $r = r^{\oplus} + r^{\ominus}$. In terms of the matrix, r, we find that $r^{\oplus}$ is just the last column of r, and $r^{\ominus}$ is the last row of r. So we have :-

$$r' = a\,r^{\oplus} + a^{\dagger}\,r^{\ominus} \tag{4.5}$$

This shows that each entry of the last column of r is modified by the $U(1)$ phase which we have called a, and each entry of the last row of r is modified by the opposite $U(1)$ phase, $a^\dagger$. Another way of saying this is that r, the Goldstone boson vector, is split into two pieces. Each piece represents one half of the the Goldstone boson multiplet states. We see that both halves transform in equal, but opposite, ways.

This is what we found in the $SU(2)$ breaking to $U(1)$ calculation, but instead of having $(N-1)$ states in each half (as we do here) we only had one Goldstone boson in each half; $r^\oplus$ being represented by the π^+, and $r^\ominus$ by the π^-.

4.2.3b Transforming L with an $SU(N-1)$ subgroup element.

Using the same notation from the last section, and considering a specific example, we may arrive at the required general result. We consider the case of $N = 3$, that is the spontaneous breaking of an $SU(3)$ symmetry down to $SU(2) \otimes U(1)$ and ask what happens to the coset vector under the $SU(2)$ subgroup transformation.

A general $u \in SU(2)$ group element has the form :-

$$u = \begin{pmatrix} a & b^* & 0 \\ -b & a^* & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore the Kaon Goldstone boson multiplet, or coset vector, is transformed :-

$$\begin{aligned}
x' &= uxu^\dagger \\
&= \begin{pmatrix} a & b^* & 0 \\ -b & a^* & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & K^+ \\ 0 & 0 & K^0 \\ K^- & \overline{K^0} & 0 \end{pmatrix} \begin{pmatrix} a^* & -b^* & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & aK^+ + b^*K^0 \\ 0 & 0 & -bK^0 + a^*K^+ \\ a^*K^- + b\overline{K^0} & a\overline{K^0} - b^*K^- & 0 \end{pmatrix} \qquad (4.6)
\end{aligned}$$

If we now consider a defining $SU(2)$ transformation, g, on a fundamental 2 representation of $SU(2)$, which we have previously called χ then we have :-

$$
\begin{aligned}
g : \chi \mapsto \chi' &= g\chi \\
&= \begin{pmatrix} a & b^* \\ -b & a^* \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} \\
&= \begin{pmatrix} au + b^*d \\ -bu + a^*d \end{pmatrix}
\end{aligned} \tag{4.7}
$$

and we also have :-

$$
\begin{aligned}
\bar{\chi} \mapsto \bar{\chi}' &= \bar{\chi} g^\dagger \\
&= \begin{pmatrix} \overline{u} & \overline{d} \end{pmatrix} \begin{pmatrix} a^* & -b^* \\ b & a \end{pmatrix} \\
&= \begin{pmatrix} a^*\overline{u} + b\overline{d} & -b^*\overline{u} + a\overline{d} \end{pmatrix}
\end{aligned} \tag{4.8}
$$

Comparing the results of (4.7) and (4.8) with the Kaon transformation (4.6) we see that we can generalize these results.

- Under a subgroup $u \in SU(N-1)$ transformation of the $SU(N-1) \otimes U(1)$ subgroup of transformations we find :-

  1. The $r^\oplus$ part of the normalized Goldstone boson coset vector, r, transforms like the fundamental representation of $SU(N-1)$, χ, i.e :-

  $$r^\oplus \mapsto r^{\oplus\prime} = u\, r^\oplus$$

  2. The $r^\ominus$ part transforms like $\bar{\chi}$, i.e :-

  $$r^\ominus \mapsto r^{\ominus\prime} = r^\ominus u^\dagger$$

Notice that $ur^\ominus = r^\ominus$ and $r^\oplus u^\dagger = r^\oplus$, so under a transformation by $u \in SU(N-1)$ we may write :-

$$
\begin{aligned}
u \,:\, r \,\mapsto\, r' &= u\, r\, u^{\dagger} \\
&= u\left(r^{\oplus} + r^{\ominus}\right) u^{\dagger} \\
&= u r^{\oplus} u^{\dagger} + u r^{\ominus} u^{\dagger} \\
&= u r^{\oplus} + r^{\ominus} u^{\dagger} \qquad (4.9)
\end{aligned}
$$

This transformation *will* have an effect on the q_r-vector in the expression for the coset representative; but since the r-vector remains in the subspace of $\Re^{N^2-1}$ where it started, the q_r-vector also remains in its orthogonal subspace where it started. The components of the q_r-vector which are transformed lie in the subspace of $\Re^{N^2-1}$ associated with the generators of the subgroup $SU(N-1)$.

4.2.4 Analysis to first order using Killing vectors.

4.2.4a The linear Killing vector components, $\mathbf{K}^a_E$.

To find the linear Killing vector components we must solve :-

$$
[\lambda_E, L] = 2i\, L_{,a} \mathbf{K}^a_E \qquad (4.10)
$$

Now the coset representative element is :-

$$
\begin{aligned}
L &= e^{-i\frac{\phi}{2} r} \\
&= \tfrac{1}{N}\left[N + 2\left(\cos\tfrac{\phi}{2} - 1\right)\right] \mathbf{1}_{[N]} + \sqrt{\tfrac{N-2}{N}}\left(\cos\tfrac{\phi}{2} - 1\right) q_r - i\sin\tfrac{\phi}{2}\, r
\end{aligned}
$$

and so the left hand side of equation (4.10) is :-

$$
[\lambda_E, L] = \sqrt{\tfrac{N-2}{N}}\left(\cos\tfrac{\phi}{2} - 1\right)[\lambda_E, q_r] - i\sin\tfrac{\phi}{2}\,[\lambda_E, r] \qquad (4.11)
$$

which, in terms of components, is :-

$$
[\lambda_E, L] = 2i\sqrt{\tfrac{N-2}{N}}\left(\cos\tfrac{\phi}{2} - 1\right) q_r^F f_{EFG}\lambda_G + 2\,\sin\tfrac{\phi}{2}\, n^a f_{Eab}\lambda_b
$$

We will now work out the right hand side of equation (4.10). Firstly we find :-

$$
\begin{aligned}
2iL_{,a}\mathbf{K}^a_E &= -\left(\tfrac{2i}{N}\sin\tfrac{\phi}{2}\,\mathbf{1}_{[N]} + i\sqrt{\tfrac{N-2}{N}}\sin\tfrac{\phi}{2}\,q_r + \cos\tfrac{\phi}{2}\,r\right)\,\phi_{,a}\mathbf{K}^a_E \\
&\quad + \left(2i\sqrt{\tfrac{N-2}{N}}\,(\cos\tfrac{\phi}{2}\,-1)\,q_{r\,,a} + 2\sin\tfrac{\phi}{2}\,r_{,a}\right)\,\mathbf{K}^a_E
\end{aligned}
$$

and since this is equal to equation (4.12) which only contains vector-like pieces, we must have the condition $\phi_{,a}\mathbf{K}^a_E \equiv 0$. Here we use the wording vector-like because the trace of equation (4.12) is zero. So if we equate what we have left we find we have two equations to solve :-

$$
q^F_r\, f_{EFG}\lambda_G = q_{r\,,a}\mathbf{K}^a_E \tag{4.12}
$$

$$
n^a f_{Eab}\lambda_b = r_{,a}\mathbf{K}^a_E \tag{4.13}
$$

We know, if the mathematics has been worked out correctly, that the solution of both these equations *should* yield the same result. We will now show that they indeed do.

1. Firstly we have :-

$$
\begin{aligned}
q^F_r\, f_{EFG} &= q^G_{r\,,a}\mathbf{K}^a_E \\
&= \sqrt{\tfrac{N}{N-2}}\left(n^a n^b d_{abG}\right)_{,c}\mathbf{K}^c_E \\
&= 2\sqrt{\tfrac{N}{N-2}}(n^a)_{,c} n^b d_{abG}\mathbf{K}^c_E \\
&= \frac{2}{M}\sqrt{\tfrac{N}{N-2}}\,(\delta_{ac} - n^a n^c)\, n^b d_{abG}\mathbf{K}^c_E \\
&= \frac{2}{M}\sqrt{\tfrac{N}{N-2}} n^b d_{cbG}\mathbf{K}^c_E
\end{aligned}
$$

Using the notation of adjoint representation operators in Appendix B, we may write this :-

$$
(f_{q_r})_{EG} = \frac{2}{M\sqrt{N-2}}\,(d_r)_{cG}\,\mathbf{K}^c_E
$$

Now, since $(f_{q_r}) = \frac{2}{\sqrt{N-2}} f_r d_r$ we find we are able to write :-

$$
M\,(f_r)_{Ec}\,(d_r)_{cG} = \mathbf{K}^c_E\,(d_r)_{cG}
$$

Thus, by inspection, we find :-

$$\begin{aligned}
\mathbf{K}_E^a &= M\,(f_r)_{Ea} \\
&= M n^b f_{Eba} \\
&= f_{aEb} M^b
\end{aligned} \tag{4.14}$$

2. Secondly, we have a simpler relation to solve. We have :-

$$\begin{aligned}
(f_r)_{Eb} &= n^b_{,a} \mathbf{K}_E^a \\
&= \frac{1}{M}\left(\delta_{ba} - n^b n^a\right) \mathbf{K}_E^a \\
&= \frac{1}{M} \mathbf{K}_E^b
\end{aligned}$$

Therefore we see :-

$$\begin{aligned}
\mathbf{K}_E^a &= M\,(f_r)_{Ea} \\
&= M n^b f_{Eba} \\
&= f_{aEb} M^b
\end{aligned} \tag{4.15}$$

The two results are the same. Since this is true we could have, for example, just solved the second set of relations and then substituted the result into the first relation to show that both sides were equal. This is the doublecheck we will use for the nonlinear case, since it turns out that one of the two relations we end up with is extremely hard to solve.

We end this section by taking the simple case of when $N = 2$. In this case we have $f_{ijk} \equiv \varepsilon_{ijk}$, and therefore :-

$$\mathbf{K}_3^a = \varepsilon_{a3b} M^b$$

which is in agreement with the result of equation (3.4), for the subgroup $U(1)$ transformation of the coset vector.

4.2.4b The nonlinear Killing vector components, $\mathbf{K}^a_b$.

To find the nonlinear Killing vector components we must solve :-

$$\{\lambda_b, L^2\} = 2i\, L^2_{,a}\mathbf{K}^a_b$$

This time we have an anticommutator, which we find to be :-

$$\begin{aligned}\{\lambda_b, L^2\} &= \tfrac{2}{N}\left(N + 2(\cos\phi\ -1)\right)\lambda_b + 2\sqrt{\tfrac{N-2}{N}}(\cos\phi\ -1)q^E_r d_{bEc}\lambda_c \\ &\quad -\tfrac{4}{N}i\sin\phi\ n^b\,\mathbf{1}_{[N]} - 2i\sin\phi\ n^a d_{baE}\lambda_E \end{aligned} \tag{4.16}$$

and we find $2iL^2_{,a}\mathbf{K}^a_b$ to be :-

$$\begin{aligned}2iL^2_{,a}\mathbf{K}^a_b &= -\tfrac{4}{N}i\sin\phi\ \phi_{,a}\mathbf{K}^a_b\,\mathbf{1}_{[N]} - 2i\sqrt{\tfrac{N-2}{N}}\sin\phi\ \phi_{,a}\mathbf{K}^a_b q_r + 2i\cos\phi\ \phi_{,a}\mathbf{K}^a_b\, r \\ &\quad +2i\sqrt{\tfrac{N-2}{N}}(\cos\phi\ -1)\, q_{r\,,a}\mathbf{K}^a_b + 2\sin\phi\ r_{,a}\,\mathbf{K}^a_b \end{aligned} \tag{4.17}$$

this time, when we compare the two, we find that $\phi_{,a}\mathbf{K}^a_b \equiv \frac{d\phi}{dM}n^a\mathbf{K}^a_b = n^b$. Yet again we have two orthogonal relations.

1. The first is associated with the directions of the q_r-vector :-

$$\sin\phi\ n^a d_{baE} = \sqrt{\tfrac{N-2}{N}}\left(\sin\phi\ n^b q^E_r - (\cos\phi\ -1)\, q^E_{r\,,a}\mathbf{K}^a_b\right)$$

Since $q^E_{r\,,a} = 2\sqrt{\frac{N}{N-2}}\,n^c(n^d_{,a})\,d_{cdE}$ then we find :-

$$q^E_{r\,,a}\mathbf{K}^a_b = \tfrac{2}{M}\left(\tfrac{1}{\sqrt{N-2}}\,(d_r)_{aE}\,\mathbf{K}^a_b - \tfrac{dM}{d\phi}n^b q^E_r\right)$$

where we understand that $n^b q^E_r \Rightarrow (r >\!< q_r)_{bE}$. Therefore, in terms of the adjoint operators, the relation we must solve is :-

$$\begin{aligned}\sin\phi\ (d_r)_{bE} &= \sqrt{N-2}\left(\sin\phi\ + \tfrac{2}{M}\tfrac{dM}{d\phi}(\cos\phi\ -1)\right)(r >\!< q_r)_{bE} \\ &\quad -\tfrac{2}{M}(\cos\phi\ -1)\,(d_r)_{aE}\,\mathbf{K}^a_b \end{aligned} \tag{4.18}$$

However this relation is very hard to solve; because simplifying $(d_r)_{aE}\mathbf{K}^a_b$ will not be easy. So it is fortunate that we have a choice of two relations to work with.

2. The second relation, associated with the directions of the r-vector, is :-

$$\begin{aligned}\cos\phi\; n^b n^c + \sin\phi\; \mathbf{K}^a_b n^c_{,a} &= \tfrac{1}{N}\left(N + 2(\cos\phi\; - 1)\right)\delta_{bc} \\ &\quad + \sqrt{\tfrac{N-2}{N}}(\cos\phi\; - 1)\, q^E_r d_{bEc}\end{aligned} \tag{4.19}$$

This relation is much simpler to solve for $\mathbf{K}^a_b$ because we know :-

$$\begin{aligned}n^c_{,a}\mathbf{K}^a_b &= \tfrac{1}{M}\left(\delta_{ca} - n^c n^a\right)\mathbf{K}^a_b \\ &= \tfrac{1}{M}\left(\mathbf{K}^c_b - \tfrac{dM}{d\phi}\, n^c n^b\right)\end{aligned}$$

Therefore if we substitute this in then, after some simple rearranging, we find :-

$$\begin{aligned}\mathbf{K}^a_b &= \tfrac{M}{N\sin\phi}\left(N + 2(\cos\phi\; - 1)\right)\delta_{ab} + \tfrac{M\sqrt{N-2}}{N\sin\phi}(\cos\phi\; - 1)\left(d_{q_r}\right)_{ab} \\ &\quad - M\cot\phi\; n^a n^b + \frac{dM}{d\phi} n^a n^b\end{aligned} \tag{4.20}$$

To check this result we may substitute it into our first relation, equation (4.18). However, this is not strictly necessary as we will soon be using these $\mathbf{K}^a_b$ to construct the Goldstone boson manifold metric; this verification is just as good. For completeness though we perform the substitution in Appendix B, on page 182. In the next section we will also find an expression which allows us to write d_{q_r} in terms of adjoint representation projection operators; the details of this are given in Appendix B, section B 4.2. We will do this just before we construct the Goldstone boson manifold metric. But for now we see that for $N = 2$ this equation reduces to :-

$$\mathbf{K}^a_b = M\cot\phi\;\left(\delta_{ab} - n^a n^b\right) + \frac{dM}{d\phi} n^a n^b$$

which is the same as equation (3.6) in the $SU(2)$ breaking to $U(1)$ calculation.

4.2.5 Covariant derivatives and the Goldstone boson metric.

For all these models we have :-

$$L = \tfrac{1}{N}\left(N+2\left(\cos\tfrac{\phi}{2}-1\right)\right)\mathbf{1}_{[N]} + \sqrt{\tfrac{N-2}{N}}\left(\cos\tfrac{\phi}{2}-1\right) q_r - i\sin\tfrac{\phi}{2}\, r$$

and therefore we find :-

$$\begin{aligned}
L^{-1} &= \tfrac{1}{N}\left(N+2\left(\cos\tfrac{\phi}{2}-1\right)\right)\mathbf{1}_{[N]} + \sqrt{\tfrac{N-2}{N}}\left(\cos\tfrac{\phi}{2}-1\right) q_r + i\sin\tfrac{\phi}{2}\, r \\
\partial_\mu L &= -\tfrac{1}{N}\sin\tfrac{\phi}{2}\,\partial_\mu\phi\,\mathbf{1}_{[N]} - \sqrt{\tfrac{N-2}{N}}\sin\tfrac{\phi}{2}\,(\partial_\mu\phi)\,q_r + \sqrt{\tfrac{N-2}{N}}\left(\cos\tfrac{\phi}{2}-1\right)\partial_\mu q_r \\
&\quad -\tfrac{i}{2}\cos\tfrac{\phi}{2}\,(\partial_\mu\phi)\,r - i\sin\tfrac{\phi}{2}\,\partial_\mu r
\end{aligned}$$

To find the Goldstone boson covariant derivative and the metric connection for the matter field covariant derivative we calculate $2iL^{-1}\partial_\mu L = a_\mu + v_\mu$. We find :-

$$\begin{aligned}
a_\mu &= r\partial_\mu\phi + \tfrac{2}{N}\left[N+2\left(\cos\tfrac{\phi}{2}-1\right)\right]\sin\tfrac{\phi}{2}\,\partial_\mu r \\
&\quad +2\sqrt{\tfrac{N-2}{N}}\left(\cos\tfrac{\phi}{2}-1\right)\sin\tfrac{\phi}{2}\,\left(q_r\partial_\mu r - r\partial_\mu q_r\right)
\end{aligned}$$

which we will simplify using the relation :-

$$q_r\partial_\mu r - r\partial_\mu q_r = \sqrt{\tfrac{N-2}{N}}\,\partial_\mu r - \{r, \partial_\mu q_r\}$$

So the results for a_μ and $-\frac{i}{2}v_\mu$ are :-

$$a_\mu = r\partial_\mu\phi + \sin\phi\,\partial_\mu r - 2\sqrt{\tfrac{N-2}{N}}\left(\cos\tfrac{\phi}{2}-1\right)\sin\tfrac{\phi}{2}\,\{r,\partial_\mu q_r\} \tag{4.21}$$

$$\begin{aligned}
-\frac{i}{2}v_\mu &= \sqrt{\tfrac{N-2}{N}}\left(\cos\tfrac{\phi}{2}-1\right)\left[1+\tfrac{2}{N}\left(\cos\tfrac{\phi}{2}-1\right)\right]\partial_\mu q_r + \tfrac{N-2}{N}\left(\cos\tfrac{\phi}{2}-1\right)^2 q_r\partial_\mu q_r \\
&\quad + \sin^2\tfrac{\phi}{2}\, r\partial_\mu r
\end{aligned} \tag{4.22}$$

In the next section we will give the explicit form of the Goldstone boson covariant derivative $a_\mu^a \equiv \mathcal{D}_\mu M^a$; we will not do the same for the matter field covariant derivative $\mathcal{D}_\mu\psi \equiv (\partial_\mu - \frac{i}{2}v_\mu)\psi$ because it's form will be extremely untidy.

We will now construct the Lagrangian for the Goldstone bosons. In doing so we will find a relation which will help us to vastly simplify the whole calculation (this is examined in appendix B), and will also allow us to easily check the result for the Goldstone boson manifold metric in the next section. We proceed by calculating $(a_\mu, a^\mu) = \frac{1}{2} tr\, a_\mu a^\mu$ using the form of a_μ above. This calculation is slightly simplified because, obviously, the terms $(r, \partial_\mu r)$ are zero. We find :-

$$\begin{aligned}
(a_\mu, a^\mu) &= \partial_\mu\phi\partial^\mu\phi + \sin^2\phi\ \partial_\mu n^a \partial^\mu n^a \\
&\quad -\sqrt{\tfrac{N-2}{N}} \sin\tfrac{\phi}{2}\ (\cos\tfrac{\phi}{2} - 1) \sin\phi\ tr\left(\partial_\mu r \{r, \partial^\mu q_r\} + \{r, \partial_\mu q_r\} \partial^\mu r\right) \\
&\quad +4\tfrac{N-2}{N} \sin^2\tfrac{\phi}{2}\ (\cos\tfrac{\phi}{2} - 1)^2 tr\left(\{r, \partial_\mu q_r\} \{r, \partial^\mu q_r\}\right)
\end{aligned} \tag{4.23}$$

It is the first trace expression which holds the key to simplification of the calculation as a whole. This is discussed in Appendix B, section B 4.2 on page 185. But for now we will proceed. We find :-

$$\begin{aligned}
tr\left(\partial_\mu r \{r, \partial^\mu q_r\} + \{r, \partial_\mu q_r\} \partial^\mu r\right) &= 2\, tr\ \left(r \partial_\mu q_r \partial^\mu r + r \partial_\mu r \partial^\mu q_r\right) \\
&\ \ \vdots \\
&= 4\sqrt{\tfrac{N-2}{N}} \partial_\mu q_r^E \partial^\mu q_r^E
\end{aligned}$$

For the final trace expression we find :-

$$tr\left(\{r, \partial_\mu q_r\} \{r, \partial^\mu q_r\}\right) = 2 \partial_\mu q_r^E \partial^\mu q_r^E$$

where we have used relations like $r\partial_\mu q_r = \partial_\mu(r q_r) - (\partial_\mu r) q_r$, and we have then calculated $tr\, q_r \partial_\mu q_r \partial^\mu q_r = \frac{N-4}{\sqrt{N(N-2)}} \partial_\mu q_r^E \partial^\mu q_r^E$. After we substitute these in we find, after a little rearranging, that we end up with :-

$$(a_\mu, a^\mu) = \partial_\mu\phi\partial^\mu\phi + \sin^2\phi\ \partial_\mu n^a \partial^\mu n^a + \tfrac{N-2}{N} (\cos\phi\ - 1)^2\, \partial_\mu q_r^E \partial^\mu q_r^E \tag{4.24}$$

We may now write this result explicitly using the results :-

$$\begin{aligned}
\partial_\mu\phi\partial^\mu\phi &= \left(\tfrac{d\phi}{dM}\right)^2 n^a n^b \partial_\mu M^a \partial^\mu M^b \\
\partial_\mu n^a \partial^\mu n^a &= \tfrac{1}{M^2}\left(\delta_{ab} - n^a n^b\right) \partial_\mu M^a \partial^\mu M^b \\
\partial_\mu q_r^E \partial^\mu q_r^E &= \tfrac{4}{M^2(N-2)} \left((d_r^2)_{ab} - (N-2) n^a n^b\right) \partial_\mu M^a \partial^\mu M^b
\end{aligned}$$

which we substitute in to find :-

$$
\begin{aligned}
(a_\mu, a^\mu) &= \left(\tfrac{d\phi}{dM}\right)^2 n^a n^b \partial_\mu M^a \partial^\mu M^b + \tfrac{\sin^2\phi}{M^2}\left(\delta_{ab} - n^a n^b\right)\partial_\mu M^a \partial^\mu M^b \\
&\quad + \tfrac{4(\cos\phi\,-1)^2}{N M^2}\left((d_r^2)_{ab} - (N-2) n^a n^b\right)\partial_\mu M^a \partial^\mu M^b \\
&= g_{ab}\partial_\mu M^a \partial^\mu M^b
\end{aligned}
$$

where we have for the Goldstone boson manifold metric :-

$$
g_{ab} = \left(\tfrac{d\phi}{dM}\right)^2 n^a n^b + \tfrac{\sin^2\phi}{M^2}\left(\delta_{ab} - n^a n^b\right) + \tfrac{4(\cos\phi\,-1)^2}{N M^2}\left((d_r^2)_{ab} - (N-2) n^a n^b\right) \qquad (4.25)
$$

This will be further simplified in the next section.

4.2.5a Verifying the metric result.

Before we verify the metric result we will use the ideas in Appendix B, section B 4.2 on page 185, to simplify the form of a_μ. We will then find the form of the Goldstone boson manifold metric again. This is not strictly necessary, since we could use results found in appendix B to simplify equation (4.25), but it will at least be quick because we will be using Adjoint representation projection operators. We will then use the Killing vectors to verify the metric result as usual.

We start by rewriting a_μ in terms of adjoint representation projection operators. We have found :-

$$
a_\mu = r\partial_\mu\phi + \sin\phi\,\partial_\mu r - 2\sqrt{\tfrac{N-2}{N}}\left(\cos\tfrac{\phi}{2} - 1\right)\sin\tfrac{\phi}{2}\,\{r, \partial_\mu q_r\}
$$

The anticommutator may now be rewritten :-

$$
\begin{aligned}
\sqrt{\tfrac{N-2}{N}}\{r, \partial_\mu q_r\} &= 2\tfrac{\sqrt{N-2}}{N}\, r \vee \partial_\mu q_r \\
&= 2\sqrt{\tfrac{N-2}{N}}\, n^a \partial_\mu q_r^E d_{aEb}\lambda_b \\
&\;\;\vdots \\
&= \tfrac{4}{N}\tfrac{1}{M}\left((d_r^2)_{ab} - (N-2) n^a n^b\right)\partial_\mu M^a \lambda_b \\
&= \tfrac{1}{M}\left(\mathcal{P}_{f_q^2}\right)_{ab}\partial_\mu M^a \lambda_b
\end{aligned}
$$

where the last step has used the result found in Appendix B, section B4.2 on page 185. Since $(a_\mu, a^\mu) = a^a_\mu a^\mu_a$ we just need an expression for a^a_μ which we see is :-

$$a^a_\mu = \left[(\tfrac{d\phi}{dM})\, r{>}{<}r + \tfrac{\sin\phi}{M} \left(\mathbf{1}_{[N^2-1]} - r{>}{<}r \right) - \tfrac{2}{M} (\cos\phi - 1) \sin\tfrac{\phi}{2}\, \mathcal{P}_{f^2_q} \right]_{ab} \partial_\mu M^b$$

and since $(\mathbf{1}_{[N^2-1]})_{ab} = (\mathcal{P}^1 + \mathcal{P}^2 + r{>}{<}r + \mathcal{P}_{f^2_q})_{ab}$ we find :-

$$a^a_\mu = \left[\tfrac{\sin\phi}{M} (\mathcal{P}^1 + \mathcal{P}^2) + (\tfrac{d\phi}{dM})\, r{>}{<}r + \tfrac{2}{M} \sin\tfrac{\phi}{2}\, \mathcal{P}_{f^2_q} \right]_{ab} \partial_\mu M^b \tag{4.26}$$

This, as promised, is the explicit form of the Goldstone boson covariant derivative. It is now very simple to form the metric because we are dealing with an expression which is completely in terms of adjoint representation projection operators. We have :-

$$(a_\mu, a^\mu) = g_{ab} \partial_\mu M^a \partial^\mu M^b$$

and we find, without much trouble, that :-

$$g_{ab} = \tfrac{\sin^2\phi}{M^2} (\mathcal{P}^1 + \mathcal{P}^2)_{ab} + \left(\tfrac{d\phi}{dM}\right)^2 (r{>}{<}r)_{ab} + \tfrac{2}{M^2} (1 - \cos\phi\,)(\mathcal{P}_{f^2_q})_{ab} \tag{4.27}$$

We also arrive at this result if we just simplify equation (4.25), which we found at the end of the last section, by substituting in the projection operator result for $\left((d^2_r)_{ab} - (N-2) n^a n^b\right)$ which we used to simplify a_μ above. We note that these, from [14], are the metrics of Kähler manifolds. This tells us that it is possible to extend all these models to include $\mathcal{N} = 1$ Supersymmetry.

To verify that the result for the metric is correct, we will now use the relationship between the metric and the Killing vectors :-

$$g_{ab} \equiv \left(\mathbf{K}^a_E \mathbf{K}^b_E + \mathbf{K}^a_c \mathbf{K}^b_c \right)^{-1}$$

In terms of linear operators, and projection operators, of the adjoint representation we have found, equations (4.15) and (4.20), the Killing vector components :-

$$
\begin{aligned}
\mathbf{K}_E^a &= M\,(f_r)_{Ea} \\
\mathbf{K}_b^a &= \tfrac{M}{\sin\phi}\tfrac{1}{N}\,(N + 2(\cos\phi\ - 1))\,\delta_{ab} + \tfrac{M}{\sin\phi}\tfrac{\sqrt{N-2}}{N}(\cos\phi\ - 1)\,(d_{q_r})_{ab} \\
&\quad -M\cot\phi\; n^a n^b + \frac{dM}{d\phi} n^a n^b \\
&\ \ \vdots \\
&= M\cot\phi\;(\mathcal{P}^1 + \mathcal{P}^2)_{ab} + \left(\tfrac{d\phi}{dM}\right)(r{>}{<}r)_{ab} + \frac{M(\cos\phi\ + 1)}{2\sin\phi}(\mathcal{P}_{f_q^2})_{ab}
\end{aligned}
$$

Where we have used equation (B.38) to write d_{q_r} in terms of the adjoint representation projection operators. Thus we find :-

$$
\begin{aligned}
\mathbf{K}_E^a\mathbf{K}_E^b &= -M^2(f_r^2)_{ab} \\
&= M^2\left(\mathcal{P}^1 + \mathcal{P}^2 + \tfrac{1}{4}\mathcal{P}_{f_q^2}\right)_{ab} \\
\mathbf{K}_c^a\mathbf{K}_c^b &= M^2\cot^2\phi\;(\mathcal{P}^1 + \mathcal{P}^2)_{ab} + \left(\tfrac{d\phi}{dM}\right)^2(r{>}{<}r)_{ab} + \tfrac{M^2(\cos\phi\ +1)^2}{4\sin^2\phi}(\mathcal{P}_{f_q^2})_{ab}
\end{aligned}
$$

When we add these two together we finally arrive at :-

$$
\mathbf{K}_E^a\mathbf{K}_E^b + \mathbf{K}_c^a\mathbf{K}_c^b = \tfrac{M^2}{\sin^2\phi}(\mathcal{P}^1 + \mathcal{P}^2)_{ab} + \left(\tfrac{dM}{d\phi}\right)^2(r{>}{<}r)_{ab} + \tfrac{M^2}{2(1-\cos\phi)}(\mathcal{P}_{f_q^2})_{ab} \tag{4.28}
$$

which is obviously the inverse of the Kähler manifold metrics, equation (4.27) above.

Chapter 5

The $\frac{SO(m)}{SO(m-2)\otimes SO(2)}$ coset models, for $m = 4, 5$ and 6.

The models in this chapter concern the breaking of an $SO(m)$ invariance down to $SO(m-2) \otimes SO(2)$. Appendix C, from page 191, details the construction of the generators of $SO(m)$ in the Weyl representation; from which we see that the following models result from the manipulation of 4×4 matrices. Thus, all may be worked out using the language of the λ-matrices of $SU(4)$. To put it another way, we are just exploiting the homomorphisms between :-

- $SO(6)$ and $SU(4)$,
- $SO(4)$ and $SU(2) \otimes SU(2)$, and
- $SO(2)$ and $U(1)$.

We note that, for all these models, the subgroup of transformations, $H = SO(m-2) \otimes SO(2)$, has the same rank as the full symmetry group, $G = SO(m)$. When $m = 4, 5$ the rank is 2, and when $m = 6$ the rank is 3.

For the three models in this chapter, the coset vector is written as $x = \omega^{aX}\sigma_{aX}$ with $a = 1, 2, \ldots, m-2$ and $X = m-1, m$. We will now show that these three coset vectors obey the same characteristic equation. This implies that we are able to describe them in the same way, thus allowing us to manipulate a single coset vector expression, just as we did for all of the CP2(N-1) models. However, unlike the coset vectors of the last chapter, the coset vectors in these models have very different forms; which means that the fact that they obey the same characteristic equation is not at all obvious. However, as we will soon see, even though all three coset vectors have the same form and mathematical behaviour, the coset vector of the first model (when $SO(4)$ invariance is broken to $SO(2) \otimes SO(2)$) has a special form. This means that, even though the results in this chapter concern all three models, they may be simplified for the case where $m = 4$. We will give the results for this model in section 5.2 *before* we go on to calculate the more complicated results for the other two models (when $m = 5, 6$).

The method/idea of rewriting the coset vectors in terms of the 'equivalent' $SU(4)$ view is *essential*, because trying to work out higher and higher powers of $x = \omega^{aX}\sigma_{aX}$ is an exceptionally difficult task. For example, using the gamma matrices of $SO(m)$, we find :-

$$
\begin{aligned}
x^2 &= \omega^{aX}\omega^{bY}\sigma_{aX}\sigma_{bY} \\
&= -\omega^{aX}\omega^{bY}\gamma_a\gamma_X\gamma_b\gamma_Y \\
&= \omega^{aX}\omega^{bY}\gamma_a\gamma_b\gamma_X\gamma_Y \\
&= \omega^{aX}\omega^{bY}\left(\delta_{ab}\,\mathbf{1}_{[2^k]} + i\sigma_{ab}\right)\left(\delta_{XY}\,\mathbf{1}_{[2^k]} + i\sigma_{XY}\right) \\
&= \omega^{aX}\omega^{aX}\,\mathbf{1}_{[2^k]} - \omega^{aX}\omega^{bY}\sigma_{ab}\sigma_{XY}
\end{aligned}
$$

which doesn't seem too bad, apart from the fact that the second term is a little clumsy. The problems really start when we write the cubic and quartic powers of x; we just end up with a nasty jumble of terms and indices and it is very difficult to spot any patterns emerging; so we cannot easily regroup terms in the expansion of L. So we will continue with the reconstruction of our view of the problem.

5.1 The coset vectors, and expressions for L.

5.1.1 The coset vector for the $\frac{SO(4)}{SO(2)\otimes SO(2)}$ coset.

This model is the theory resulting from the spontaneous breaking of an $SO(4)$ global symmetry down to an $SO(2)\otimes SO(2)$ symmetry. To all intents and purposes this model, mathematically at least, looks like two commuting copies of the $SU(2)$ breaking to $U(1)$ model [1] already worked out in Chapter 3; as we will now see.

The $\frac{SO(4)}{SO(2)\otimes SO(2)}$ coset vector, $x = \omega^{aX}\sigma_{aX}$, is :-

$$\omega^{aX}\sigma_{aX} = \begin{pmatrix} 0 & A^* & 0 & 0 \\ A & 0 & 0 & 0 \\ 0 & 0 & 0 & B^* \\ 0 & 0 & B & 0 \end{pmatrix}$$

For this matrix we therefore find that :-

1. the top left entries may be written as $x_L^p L_p$ (p=1,2) with :-

$$x_L^1 = \omega^{23} + \omega^{14}$$
$$x_L^2 = \omega^{31} + \omega^{24}$$

   which implies $A = x_L^1 + ix_L^2$,

2. and the bottom right entries may be written as $x_R^p R_p$ with :-

$$x_R^1 = \omega^{23} - \omega^{14}$$
$$x_R^2 = \omega^{31} - \omega^{24}$$

[1] This means that the Goldstone boson manifold will be $S^2 \otimes S^2$.

which implies $B = x_R^1 + ix_R^2$

Since we have this form, it is immediately apparent that $tr\, x^3 = 0$. There are 4 Goldstone bosons in this theory. The coset indices, in terms of the λ-matrix basis of $\Re^{15}$, are $\alpha = 1, 2, 13, 14$. The coset generators are :-

$$\begin{aligned}
\sigma_{31} &= (\lambda_2 + \lambda_{14}) &\equiv& \ (L_2 + R_2) \\
\sigma_{23} &= (\lambda_1 + \lambda_{13}) &\equiv& \ (L_1 + R_1) \\
\sigma_{14} &= (\lambda_1 - \lambda_{13}) &\equiv& \ (L_1 - R_1) \\
\sigma_{24} &= (\lambda_2 - \lambda_{14}) &\equiv& \ (L_2 - R_2)
\end{aligned}$$

5.1.2 The coset vector for the $\frac{SO(5)}{SO(3)\otimes SO(2)}$ coset.

This model is the theory resulting from the spontaneous breaking of an $SO(5)$ global symmetry down to an $SO(3) \otimes SO(2)$ symmetry. The $\frac{SO(5)}{SO(3)\otimes SO(2)}$ coset vector, $\omega^{iX}\sigma_{iX}$, is :-

$$\omega^{iX}\sigma_{iX} = \begin{pmatrix} \omega^{i4}\sigma_{i4} & -\omega^{i5}\sigma_{i5} \\ -\omega^{i5}\sigma_{i5} & -\omega^{i4}\sigma_{i4} \end{pmatrix}$$

This time it is a little more involved to see that $tr\, x^3 = 0$. First, we find the second power of x to be :-

$$x^2 = \omega^{iX}\omega^{iX}\, \mathbf{1}_{[4]} + 2i\omega^{i4}\omega^{j5}\varepsilon_{ijk} \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}$$

The cube of x is then :-

$$x^3 = \omega^{iX}\omega^{iX}\, x - 2\omega^{i4}\omega^{j5}\varepsilon_{ijk}\varepsilon_{klm} \begin{pmatrix} \omega^{l5}\sigma_m & \omega^{l4}\sigma_m \\ \omega^{l4}\sigma_m & -\omega^{l5}\sigma_m \end{pmatrix}$$

and this obviously is a vector; has a trace of zero. When x is rewritten as $x^\alpha\lambda_\alpha$ the coset indices are $\alpha = 1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 13, 14, 15$ but the coset generators of the orthogonal symmetry breaking model are just linear combinations of pairs of the $\Re^{15}$ basis vectors :-

$$\begin{aligned}
\sigma_{14} &= (\lambda_1 - \lambda_{13}) \\
\sigma_{24} &= (\lambda_2 - \lambda_{14}) \\
\sigma_{34} &= (r_3 - r_3^{\perp}) \\
\sigma_{15} &= -(\lambda_6 + \lambda_9) \\
\sigma_{25} &= (\lambda_7 - \lambda_{10}) \\
\sigma_{35} &= -(\lambda_4 + \lambda_{11})
\end{aligned}$$

where $r_3 = \lambda_3$ and $r_3^{\perp} = -\frac{1}{\sqrt{3}}\lambda_8 + \sqrt{\frac{2}{3}}\lambda_{15}$. This is why there are 6 Goldstone bosons in this model.

5.1.3 The coset vector for the $\frac{SO(6)}{SO(4)\otimes SO(2)}$ coset.

This model is the theory resulting from the spontaneous breaking of an $SO(6)$ global symmetry down to an $SO(4)\otimes SO(2)$ symmetry. The $\frac{SO(6)}{SO(4)\otimes SO(2)}$ coset vector, $\omega^{aX}\sigma_{aX}$, is :-

$$\omega^{aX}\sigma_{aX} = \begin{pmatrix} 0 & 0 & A^* & C^* \\ 0 & 0 & B^* & D^* \\ A & B & 0 & 0 \\ C & D & 0 & 0 \end{pmatrix}$$

where we have for example $A^* = x^4 - ix^5$. In terms of the $\frac{SO(6)}{SO(4)\otimes SO(2)}$ coset parameters the components of A^* are :-

$$\begin{aligned}
x^4 &= -\frac{1}{2}(\omega^{35} - \omega^{46}) \\
x^5 &= -\frac{1}{2}(\omega^{36} + \omega^{45})
\end{aligned}$$

and so $\omega^{aX}\sigma_{aX} \Rightarrow x^{\alpha}\lambda_{\alpha}$ with the coset indices given by $\alpha = 4, 5, 6, 7, 9, 10, 11, 12$. By inspection we see that $tr\, x^3 = 0$. There are 8 Goldstone bosons in this model.

For all three coset vectors in this chapter we see that the quantity x^3 is itself a vector. This implies, from section 2.3, that $x \vee x$ and x are linearly independent; they are orthogonal vectors.

5.1.4 Description of x and the Coset representative, $L \equiv e^{-ix}$.

The coset vectors, x, must all obey a characteristic equation, $\Phi(x)$, of the form :-

$$x^4 - \gamma_2(x)x^2 - \gamma_4(x)\,\mathbf{1}_{[4]} \;\equiv\; 0 \tag{5.1}$$

because this is the most general characteristic equation allowed for 4×4 traceless, Hermitian matrices with $tr\, x^3 = 3\gamma_3(x) = 0$. We may represent the eigenvalue equation implied by this characteristic equation graphically :-

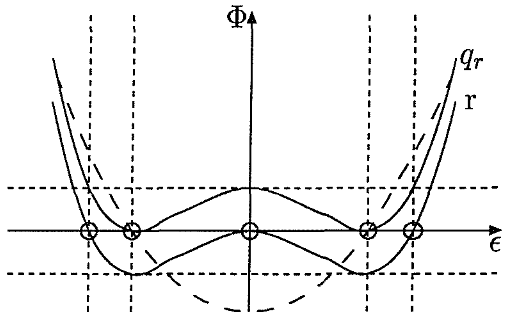

Figure 5.1: The r and q_r vector eigenvalue equations.

The eigenvalues, ϵ_1 to ϵ_4, for the vectors lie on the line $\Phi = 0$, or, as indicated, the ϵ-axis. The vertical dashed lines lie at $\epsilon = \pm\frac{1}{\sqrt{2}}, \pm 1$, and the horizontal ones at $\Phi = \pm\frac{1}{4}$. Notice that the q_r-vector's eigenvalues also satisfy a quadratic curve (because $q_r \vee q_r \equiv 0$).

The eigenvalue equations for r and q_r-vectors, shown above, are symmetric about the origin, as is the eigenvalue equation for our coset vector, x. So in order for our vector to have *real* eigenvalues it must be true that, once normalized, its eigenvalue equation will 'lie between' the r and q_r equations. The form of the characteristic equation for x makes its eigenvalues simple to find. We have :-

$$x^4 - \gamma_2(x)x^2 - \gamma_4(x)\,\mathbf{1}_{[4]} = 0$$

So let $y = x^2$. This gives :-

$$\begin{aligned} y^2 - \gamma_2(x)y - \gamma_4(x) &= 0 \\ &\rightsquigarrow \\ y &= \tfrac{1}{2}\left(\gamma_2(x) \pm \sqrt{4\gamma_4(x) + \gamma_2(x)^2}\right) \end{aligned}$$

Thus the eigenvalues of our coset vectors, x, are simply :-

$$\epsilon = \pm\sqrt{\frac{1}{2}\left(\gamma_2(x) \pm \sqrt{4\gamma_4(x) + \gamma_2(x)^2}\right)}$$

or, more clearly :-

$$\epsilon = \begin{cases} \pm\alpha = \pm\sqrt{\tfrac{1}{2}\left(\gamma_2(x) + \sqrt{4\gamma_4(x) + \gamma_2(x)^2}\right)} \\ \pm\beta = \pm\sqrt{\tfrac{1}{2}\left(\gamma_2(x) - \sqrt{4\gamma_4(x) + \gamma_2(x)^2}\right)} \end{cases}$$

So, if we were to diagonalize x, we could write these eigenvalues :-

$$x_D = \frac{a}{2}r_3 + \frac{b}{2}r_{3\perp}$$

where we have chosen to use r_3 and $r_{3\perp}$. It really doesn't matter which pair we use; we could have used r_1 and $r_{1\perp}$ or r_2 and $r_{2\perp}$. We have also used conventional values, $\alpha = \frac{a}{2}$ and $\beta = \frac{b}{2}$, for the lengths of the two commuting, orthogonal vectors which make up x. In terms of diagonal projection operators, x_D can be written :-

$$x_D = \frac{a}{2}\left(P^1 - P^2\right) + \frac{b}{2}\left(P^3 - P^4\right)$$

or explicitly as a matrix :-

$$x_D = \begin{pmatrix} +\alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & +\beta & 0 \\ 0 & 0 & 0 & -\beta \end{pmatrix}$$

Therefore, when we rotate x_D back onto x, we find that the coset representative is simply :-

$$L = e^{-i\frac{a}{2}}P^1 + e^{+i\frac{a}{2}}P^2 + e^{-i\frac{b}{2}}P^3 + e^{+i\frac{b}{2}}P^4 \tag{5.2}$$

where the projection operators are obviously no longer diagonal. However, we will not use L in this form. Instead we will write L in terms of the Cartan subspace basis $\{r, r_\perp, q_r\}$ and the identity element as usual. This is a very simple task because when we rotate x_D out of $\mathcal{C}_D$ then we just have :-

$$\begin{aligned} x_D &\mapsto x \\ \frac{a}{2}r_3 + \frac{b}{2}r_{3\perp} &\mapsto \frac{a}{2}r + \frac{b}{2}r_\perp \end{aligned}$$

and still $rr_\perp = r_\perp r \equiv 0$. So we find :-

$$\begin{aligned} L &= e^{-i(\frac{a}{2}r + \frac{b}{2}r_\perp)} \\ &= e^{-i\frac{a}{2}r}e^{-i\frac{b}{2}r_\perp} \\ &= \mathbf{1}_{[4]} + \left((\cos\tfrac{a}{2} - 1)\,r^2 + (\cos\tfrac{b}{2} - 1)\,r_\perp^2\right) - i\,(\sin\tfrac{a}{2}\;r + \sin\tfrac{b}{2}\;r_\perp) \\ &= \tfrac{1}{2}\,(\cos\tfrac{a}{2} + \cos\tfrac{b}{2})\;\mathbf{1}_{[4]} + \tfrac{1}{\sqrt{2}}\,(\cos\tfrac{a}{2} - \cos\tfrac{b}{2})\;q_r - i\,(\sin\tfrac{a}{2}\;r + \sin\tfrac{b}{2}\;r_\perp) \end{aligned} \tag{5.3}$$

This is a simple result to find. Firstly, we can just expand the first line because calculating powers of x is simple; we quickly arrive at the third line. Secondly, we may use independent results for $e^{-i\frac{a}{2}r}$ and $e^{-i\frac{b}{2}r_\perp}$, given by equation (2.25), and multiply them together; this splitting of the exponential is possible because $[r, r_\perp] \equiv 0$. This second method is a little more involved because we need to remember how r vectors and q_r vectors multiply together; still, the result is the same. We would have also arrived at equation (5.3) if we had started off with equation (5.2) and substituted in relations for the projection operators; provided by $r \equiv (P^1 - P^2)$ and $r_\perp \equiv (P^3 - P^4)$. These are :-

$$\begin{aligned} P^1 &= \tfrac{1}{2}(r^2 + r) \\ P^2 &= \tfrac{1}{2}(r^2 - r) \\ P^3 &= \tfrac{1}{2}(r_\perp^2 + r_\perp) \\ P^4 &= \tfrac{1}{2}(r_\perp^2 - r_\perp) \end{aligned}$$

We then substitute $2r^2 - \mathbf{1}_{[4]} = \sqrt{2}\, q_r$ and $2r_\perp^2 - \mathbf{1}_{[4]} = -\sqrt{2}\, q_r$ and rearrange; notice that we have also used the specific $SU(4)$ relation $q_{r_\perp} = -q_r$ to simplify the result.

It is important to understand that, when $N \geq 5$, we find that $q_{r_\perp} \neq -q_r$ and so the coset representative element, in these cases, will be in a more complicated form :-

$$\begin{aligned} L &= \left(\tfrac{N-4}{N}\right) \mathbf{1}_{[N]} + \tfrac{2}{N}\left(\cos\tfrac{a}{2} + \cos\tfrac{b}{2}\right) \mathbf{1}_{[N]} - i\left(\sin\tfrac{a}{2}\, r + \sin\tfrac{b}{2}\, r_\perp\right) \\ &\quad + \sqrt{\tfrac{N-2}{N}}\left[\left(\cos\tfrac{a}{2} - 1\right) q_r + \left(\cos\tfrac{b}{2} - 1\right) q_{r_\perp}\right] \end{aligned} \tag{5.4}$$

If we use $N = 5$ in this equation, then the resulting Coset representative element may be used to find the Effective Lagrangian when $SU(5)$ is broken to $SU(3) \otimes SU(2) \otimes U(1)$; this will be discussed in the concluding chapter.

We will also briefly discuss a further generalization to this result; cases where the coset vector is a linear sum of any allowed number of commuting *orthonormal* r-vectors. In this case the coset representative element is :-

$$L = e^{-i \sum_{k=1}^{\alpha} \frac{x^{(k)}}{2} r_{(k)}} \tag{5.5}$$

This expression is valid for $\frac{N}{2} \geq \alpha \geq 1$, where α is the total number of commuting orthonormal r-vectors used to describe the coset vector. Thus, the index k is used to distinguish between the different r-vectors. We find that this works out to be :-

$$\begin{aligned} L &= \prod_{k=1}^{\alpha} e^{-i\frac{x^{(k)}}{2} r_{(k)}} \\ &= \left(\tfrac{N-2\alpha}{N}\right) \mathbf{1}_{[N]} + \tfrac{2}{N} \sum_{k=1}^{\alpha} \cos\tfrac{x^{(k)}}{2}\, \mathbf{1}_{[N]} - i \sum_{k=1}^{\alpha} \sin\tfrac{x^{(k)}}{2}\, r_{(k)} \\ &\quad + \sqrt{\tfrac{N-2}{N}} \sum_{k=1}^{\alpha} (\cos\tfrac{x^{(k)}}{2} - 1)\, q_{(k)} \end{aligned} \tag{5.6}$$

and the relationship between the r-vectors is defined :-

$$\begin{aligned} r_{(i)} r_{(j)} &= \delta_{ijk} r_{(k)}^2 \\ r_{(k)}^2 &= \tfrac{1}{N}(2\,\mathbf{1}_{[N]} + \sqrt{N(N-2)}\, q_{(k)}) \end{aligned}$$

with $\delta_{ijk} \equiv 1$ when $i = j = k$.

We end this section by noting some important results contained within equation (5.6). We find :-

1. When $\alpha = 1$ and $N = 2$ the implied coset vector is $\frac{x^{(1)}}{2} r_{(1)}$ and in chapter 3 we wrote this as $\frac{\phi}{2} r$. We find that the coset representative element of equation (5.6) reduces to equation (3.1).

2. When $\alpha = 1$ and $N \geq 2$ the implied coset vector is $\frac{x^{(1)}}{2} r_{(1)}$ and in chapter 4 we wrote this as $\frac{\phi}{2} r$. The coset representative element of equation (5.6) reduces to equation (4.3).

3. Lastly, when $\alpha = 2$ and $N \geq 4$ the implied coset vector is $(\frac{x^{(1)}}{2} r_{(1)} + \frac{x^{(2)}}{2} r_{(2)})$ and in this chapter it is written as $(\frac{a}{2} r + \frac{b}{2} r_{\perp})$. We find that the coset representative element of equation (5.6) reduces to equation (5.4). When $N = 4$ this may be further reduced to the form of equation (5.3) because $q_{r_{\perp}} = -q_r$.

5.2 Simple results when $SO(4)$ invariance is broken to $SO(2) \otimes SO(2)$.

The results we have found in this chapter are valid for the three models considered. However for the first model, when $SO(4)$ invariance is broken to $SO(2) \otimes SO(2)$, the form of the coset vector allows the results to be simplified. This is because the coset vector is :-

$$\omega^{aX} \sigma_{aX} = \begin{pmatrix} 0 & A^* & 0 & 0 \\ A & 0 & 0 & 0 \\ 0 & 0 & 0 & B^* \\ 0 & 0 & B & 0 \end{pmatrix} = \frac{a}{2} r + \frac{b}{2} r_\perp$$

and only in this case do both r and $r_\perp$ square to *diagonal* quantities :-

$$r^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \equiv \mathbf{1}_L \quad \text{and} \quad r_\perp^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv \mathbf{1}_R$$

where still $r^2 + r_\perp^2 = \mathbf{1}_L + \mathbf{1}_R = \mathbf{1}_{[4]}$ as required; which implies that the associated q_r-vector is diagonal. This is similar to the fact that the r-vector, of $\Re^8$, given by $r = n^k \lambda_k$ (for $k = 1, 2, 3$) has the associated *diagonal* q_r-vector which is $q_r \equiv n^k n^k \lambda_8 = \lambda_8$. Thus, we will find that $\partial_\mu q_r \equiv 0$. Now because the r-vectors lie in non-interacting commuting spaces we may now write L as :-

$$\begin{aligned} L &= \mathbf{1}_{[4]} + \left((\cos\tfrac{a}{2} - 1)\, r^2 + (\cos\tfrac{b}{2} - 1)\, r_\perp^2 \right) - i \left(\sin\tfrac{a}{2}\, r + \sin\tfrac{b}{2}\, r_\perp \right) \\ &= \cos\tfrac{a}{2}\, \mathbf{1}_L - i \sin\tfrac{a}{2}\, r + \cos\tfrac{b}{2}\, \mathbf{1}_R - i \sin\tfrac{b}{2}\, r_\perp \end{aligned}$$

So, in effect, we may now ignore the q_r-vector altogether. To see the results for this model in the calculations which appear in the rest of this chapter remember that $\partial_\mu q_r = 0$ and, when evaluating $2iL^{-1}\partial_\mu L$, terms like $r \partial_\mu r_\perp$ are zero too.

We also see that r^2 and $r_\perp^2$ will act as projection operators :-

$$
\begin{aligned}
r^2 \equiv \mathbf{1}_L &= \frac{1}{2}(\mathbf{1}_{[4]} + \gamma_5) = P_L \\
r_\perp^2 \equiv \mathbf{1}_R &= \frac{1}{2}(\mathbf{1}_{[4]} - \gamma_5) = P_R \\
P_L + P_R &\equiv \mathbf{1}_{[4]}
\end{aligned}
$$

These projection operators will project out the left and right pieces from L, so we can work on 'the two halves' independently (both of which are like the first coset model, namely $\frac{SU(2)}{U(1)}$, in this thesis). Equivalently, the projection operators will project out the left and right 2-spheres from the coset space; which we now denote $S_L^2 \otimes S_R^2$. These projection operators are also defined in Appendix C.

Therefore, when we calculate $2iL^{-1}\partial_\mu L$, we will find :-

$$
2iL^{-1}\partial_\mu L = (P_L + P_R)(a_\mu + v_\mu) = (a_\mu^L + v_\mu^L) + (a_\mu^R + v_\mu^R)
$$

where $a_\mu^L \equiv P_L\, a_\mu$ and $v_\mu^L \equiv P_L\, v_\mu$. Obviously P_R has projected out the other two quantities. So we find :-

$$
\begin{aligned}
(a_\mu^L)^\alpha &= \left(\tfrac{\sin a}{M}[\delta^L_{\alpha\beta} - n^\alpha n^\beta] + \tfrac{da}{dM} n^\alpha n^\beta\right) \partial_\mu M^\beta \\
(v_\mu^L)^3 &= \tfrac{2}{M} \sin^2\tfrac{a}{2}\, n^\alpha \varepsilon_{\alpha 3\beta} \partial_\mu M^\beta \\
(a_\mu^R)^\sigma &= \left(\tfrac{\sin b}{M_\perp}[\delta^R_{\sigma\tau} - n_\perp^\sigma n_\perp^\tau] + \tfrac{db}{dM_\perp} n_\perp^\sigma n_\perp^\tau\right) \partial_\mu M_\perp^\tau \\
(v_\mu^R)^3 &= \tfrac{2}{M_\perp} \sin^2\tfrac{b}{2}\, n_\perp^\sigma \varepsilon_{\sigma 3\tau} \partial_\mu M_\perp^\tau
\end{aligned}
$$

where $\alpha, \beta = 1, 2$ and $\sigma, \tau = 1, 2$. To use these index values we must redefine the $SU_R(2)$ components $M^{13} \rightsquigarrow M_\perp^1$ and $M^{14} \rightsquigarrow M_\perp^2$; note that we cannot form objects with a mix of these two sets of indices.

The Goldstone covariant derivatives are given above, and the matter field covariant derivatives are :-

$$\mathcal{D}^L_\mu \psi = \left\{ \partial_\mu + \frac{i}{M^2} \sin^2\tfrac{a}{2}\, M^\alpha \partial_\mu M^\beta \varepsilon_{\alpha\beta 3}\, L_3 \right\} \psi \tag{5.7}$$

$$\mathcal{D}^R_\mu \chi = \left\{ \partial_\mu + \frac{i}{M_\perp^2} \sin^2\tfrac{b}{2}\, M_\perp^\sigma \partial_\mu M_\perp^\tau \varepsilon_{\sigma\tau 3}\, R_3 \right\} \chi \tag{5.8}$$

We also find the Killing vector components to be :-

$$\begin{aligned}
(\mathbf{K}_L)^\alpha_3 &= \varepsilon_{\alpha 3\beta} M^\beta \\
(\mathbf{K}_L)^\alpha_\beta &= M \cot a\; (\delta^L_{\alpha\beta} - n^\alpha n^\beta) + \tfrac{dM}{da} n^\alpha n^\beta \\
(\mathbf{K}_R)^\sigma_3 &= \varepsilon_{\sigma 3\tau} M_\perp^\tau \\
(\mathbf{K}_R)^\sigma_\tau &= M_\perp \cot b\; (\delta^R_{\sigma\tau} - n_\perp^\sigma n_\perp^\tau) + \tfrac{dM_\perp}{db} n_\perp^\sigma n_\perp^\tau
\end{aligned}$$

and, finally, we have the Goldstone boson part of the effective Lagrangian density :-

$$\tfrac{1}{2}\, g_{ab}\, \partial_\mu \mathcal{M}^a\, \partial^\mu \mathcal{M}^b = \tfrac{1}{2}\, g_{\alpha\beta}\, \partial_\mu M^\alpha\, \partial^\mu M^\beta + \tfrac{1}{2}\, g_{\sigma\tau}\, \partial_\mu M_\perp^\sigma\, \partial^\mu M_\perp^\tau$$

The two S^2 manifolds have metrics like equation (3.16). From the work of Galperin, Ivanov, Ogievetsky and Sokatchev [22] we understand that the Goldstone boson manifold $S^2_L \otimes S^2_R$ is Hyper-Kähler; and so this model (with fields defined in a 4-dimensional spacetime) will admit $\mathcal{N} = 2$ extended supersymmetry. This is the *simplest* example of a Hyper-Kähler manifold since [22] also suggests that if *any* coset space $(\frac{G}{H})$ is Kähler then $(\frac{G}{H}) \otimes S^2$ will be Hyper-Kähler; and $(\frac{G}{H}) = (\frac{SU(2)}{U(1)}) = S^2$ is the simplest Kähler manifold. We will now find the results for the other two models; when $m = 5, 6$.

5.3 Goldstone boson transformations.

5.3.1 The linear Killing vector components, $\mathbf{K}^a_E$.

To find the linear $\mathbf{K}^a_E$ we must, as usual, solve :-

$$[\lambda_E, L] \equiv 2iL_{,b}\mathbf{K}^b_E$$

We have found the coset representative element to be :-

$$L = \tfrac{1}{2}\left(\cos\tfrac{a}{2} + \cos\tfrac{b}{2}\right) \mathbf{1}_{[4]} + \tfrac{1}{\sqrt{2}}\left(\cos\tfrac{a}{2} - \cos\tfrac{b}{2}\right) q_r - i\left(\sin\tfrac{a}{2}\, r + \sin\tfrac{b}{2}\, r_\perp\right)$$

and so the left hand side is simply :-

$$[\lambda_E, L] = \tfrac{1}{\sqrt{2}}\left(\cos\tfrac{a}{2} - \cos\tfrac{b}{2}\right)[\lambda_E, q_r] - i\sin\tfrac{a}{2}\,[\lambda_E, r] - i\sin\tfrac{b}{2}\,[\lambda_E, r_\perp] \tag{5.9}$$

Next, for the right hand side, we calculate :-

$$\begin{aligned} L_{,b} &= -\tfrac{1}{4}\left(\sin\tfrac{a}{2}\, a_{,b} + \sin\tfrac{b}{2}\, b_{,b}\right) \mathbf{1}_{[4]} - \tfrac{1}{2\sqrt{2}}\left(\sin\tfrac{a}{2}\, a_{,b} - \sin\tfrac{b}{2}\, b_{,b}\right) q_r \\ &\quad + \tfrac{1}{\sqrt{2}}\left(\cos\tfrac{a}{2} - \cos\tfrac{b}{2}\right) q_{r,b} - \tfrac{i}{2}\cos\tfrac{a}{2}\, a_{,b}\, r - i\sin\tfrac{a}{2}\, r_{,b} \\ &\quad - \tfrac{i}{2}\cos\tfrac{b}{2}\, b_{,b}\, r_\perp - i\sin\tfrac{b}{2}\, r_{\perp,b} \end{aligned}$$

and so, for the right hand side, we find :-

$$\begin{aligned} 2iL_{,b}\mathbf{K}_E^b &= -\tfrac{i}{2}\left(\sin\tfrac{a}{2}\, a_{,b} + \sin\tfrac{b}{2}\, b_{,b}\right)\mathbf{K}_E^b\, \mathbf{1}_{[4]} - \tfrac{i}{\sqrt{2}}\left(\sin\tfrac{a}{2}\, a_{,b} - \sin\tfrac{b}{2}\, b_{,b}\right)\mathbf{K}_E^b\, q_r \\ &\quad + i\sqrt{2}\left(\cos\tfrac{a}{2} - \cos\tfrac{b}{2}\right) q_{r,b}\mathbf{K}_E^b + \cos\tfrac{a}{2}\, a_{,b}\mathbf{K}_E^b\, r + 2\sin\tfrac{a}{2}\, r_{,b}\mathbf{K}_E^b \\ &\quad + \cos\tfrac{b}{2}\, b_{,b}\, r_\perp \mathbf{K}_E^b + 2\sin\tfrac{b}{2}\, r_{\perp,b}\mathbf{K}_E^b \end{aligned} \tag{5.10}$$

Since equations (5.9) and (5.10) are equal, we immediately see that the pieces in front of $\mathbf{1}_{[4]}$ on the right hand side yield the results :-

$$\begin{aligned} a_{,b}\mathbf{K}_E^b &\equiv 0 \\ b_{,b}\mathbf{K}_E^b &\equiv 0 \end{aligned}$$

which we may then use to vastly simplify the right hand side to :-

$$2iL_{,b}\mathbf{K}_E^b = +i\sqrt{2}\left(\cos\tfrac{a}{2} - \cos\tfrac{b}{2}\right) q_{r,b}\mathbf{K}_E^b + 2\sin\tfrac{a}{2}\, r_{,b}\mathbf{K}_E^b + 2\sin\tfrac{b}{2}\, r_{\perp,b}\mathbf{K}_E^b \tag{5.11}$$

Once again, because of the indices involved, we have two separate relations which we may solve for the $\mathbf{K}_E^a$. Firstly we have the relation :-

$$
\begin{aligned}
\tfrac{1}{\sqrt{2}}\left(\cos\tfrac{a}{2} - \cos\tfrac{b}{2}\right)[\lambda_E, q_r] &= i\sqrt{2}\left(\cos\tfrac{a}{2} - \cos\tfrac{b}{2}\right) q_{r\,,b}\mathbf{K}_E^b \\
q_r^F f_{EFG} &= q_{r\,,b}^G \mathbf{K}_E^b \\
(f_{q_r})_{EG} &= q_{r\,,b}^G \mathbf{K}_E^b
\end{aligned}
\tag{5.12}
$$

and secondly we have :-

$$
\begin{aligned}
-i\sin\tfrac{a}{2}\,[\lambda_E, r] - i\sin\tfrac{b}{2}\,[\lambda_E, r_\perp] &= 2\sin\tfrac{a}{2}\, r_{,b}\mathbf{K}_E^b + 2\sin\tfrac{b}{2}\, r_{\perp\,,b}\mathbf{K}_E^b \\
\sin\tfrac{a}{2}\, n^b f_{Eba} + \sin\tfrac{b}{2}\, n_\perp^b f_{Eba} &= \sin\tfrac{a}{2}\, n_{,b}^a \mathbf{K}_E^b + \sin\tfrac{b}{2}\, n_{\perp\,,b}^a \mathbf{K}_E^b \\
\sin\tfrac{a}{2}\,(f_r)_{Ea} + \sin\tfrac{b}{2}\,(f_{r_\perp})_{Ea} &= \sin\tfrac{a}{2}\, n_{,b}^a \mathbf{K}_E^b + \sin\tfrac{b}{2}\, n_{\perp\,,b}^a \mathbf{K}_E^b
\end{aligned}
\tag{5.13}
$$

It is far more straightforward to work with equation (5.12) as we will now show. If we write the coset vector in terms of the Goldstone fields :-

$$
\begin{aligned}
x &\equiv \mathcal{M}^a \lambda_a \\
&= M\, n^a \lambda_a + M_\perp\, n_\perp^a \lambda_a
\end{aligned}
$$

then, in terms of the Goldstone fields, the q_r-vector components may be written :-

$$
q_r^G = \tfrac{\sqrt{2}}{(M^2 - M_\perp^2)} \mathcal{M}^a \mathcal{M}^b d_{abG}
$$

If we now differentiate these components with respect to the Goldstone fields and multiply by the Killing vector components then the relation, equation (5.12), is now :-

$$
\begin{aligned}
(f_{q_r})_{EG} &= -\tfrac{2\sqrt{2}}{(M^2 - M_\perp^2)^2}\left(M\, n^c - M_\perp\, n_\perp^c\right) \mathcal{M}^a \mathcal{M}^b d_{abG} \mathbf{K}_E^c + \tfrac{2\sqrt{2}}{(M^2 - M_\perp^2)} \mathcal{M}\, m^a d_{acG} \mathbf{K}_E^c \\
&= \tfrac{\sqrt{2}}{(M^2 - M_\perp^2)} \mathcal{M}\, (d_m)_{cG}\, \mathbf{K}_E^c
\end{aligned}
$$

because the first term on the right hand side is identically zero. We point out that we have used the unit vector $m \equiv \frac{M}{\mathcal{M}}\, r + \frac{M_\perp}{\mathcal{M}}\, r_\perp$ and so we may write :-

$$
(f_{q_r})_{EG} = \tfrac{\sqrt{2}}{(M^2 - M_\perp^2)} \left(M\, d_r + M_\perp\, d_{r_\perp}\right)_{Gc} \mathbf{K}_E^c
\tag{5.14}
$$

Now, for $N = 4$, we have the adjoint operator relations :-

$$\begin{aligned}
d_r f_r &= \tfrac{1}{\sqrt{2}} f_{q_r} \\
d_{r_\perp} f_{r_\perp} &= -\tfrac{1}{\sqrt{2}} f_{q_r} \\
d_r f_{r_\perp} + d_{r_\perp} f_r &= 0 \\
f_r d_{r_\perp} + f_{r_\perp} d_r &= 0
\end{aligned}$$

and if we analyse the right hand side of equation (5.14), by parts, using linear Killing vector components with the form $\mathbf{K}^c_E = -M\,(f_r)_{cE} - M_\perp\,(f_{r_\perp})_{cE}$ then we will find :-

$$\begin{aligned}
M\,(d_r)_{Gc}\,\mathbf{K}^c_E &= M\,(d_r)_{Gc}\,(-M\,f_r - M_\perp f_{r_\perp})_{cE} \\
&= -M^2\,(d_r f_r)_{GE} - MM_\perp\,(d_r f_{r_\perp})_{GE} \\
&= -\tfrac{M^2}{\sqrt{2}}\,(f_{q_r})_{GE} - MM_\perp\,(d_r f_{r_\perp})_{GE}
\end{aligned}$$

and, in the same way, we will also find the result :-

$$M_\perp\,(d_{r_\perp})_{Gc}\,\mathbf{K}^c_E = \tfrac{M_\perp^2}{\sqrt{2}}\,(f_{q_r})_{GE} - MM_\perp\,(d_{r_\perp} f_r)_{GE}$$

Adding these two relations together gives us :-

$$\begin{aligned}
(M\,d_r + M_\perp\,d_{r_\perp})_{Gc}\,\mathbf{K}^c_E &= -\tfrac{M^2}{\sqrt{2}}\,(f_{q_r})_{GE} - MM_\perp\,(d_r f_{r_\perp})_{GE} \\
&\quad +\tfrac{M_\perp^2}{\sqrt{2}}\,(f_{q_r})_{GE} - MM_\perp\,(d_{r_\perp} f_r)_{GE} \\
&= -\tfrac{1}{\sqrt{2}}\left(M^2 - M_\perp^2\right)(f_{q_r})_{GE} \\
&= \tfrac{1}{\sqrt{2}}\left(M^2 - M_\perp^2\right)(f_{q_r})_{EG}
\end{aligned}$$

So if we now use this result in equation (5.14) then we have :-

$$\begin{aligned}
(f_{q_r})_{EG} &= \tfrac{\sqrt{2}}{(M^2 - M_\perp^2)}\,(M\,d_r + M_\perp\,d_{r_\perp})_{Gc}\,\mathbf{K}^c_E \\
&= \tfrac{\sqrt{2}}{(M^2 - M_\perp^2)}\,\tfrac{(M^2 - M_\perp^2)}{\sqrt{2}}\,(f_{q_r})_{EG} \\
&= (f_{q_r})_{EG}
\end{aligned}$$

This tells us that our 'guessed' form for the linear Killing vector components was correct, and so we now know that we have the definite answer :-

$$\begin{aligned}
\mathbf{K}^a_E &= -M\left(f_r\right)_{aE} - M_\perp\left(f_{r\perp}\right)_{aE} \\
&= f_{aEb}M^b + f_{aEb}M^b_\perp \\
&= f_{aEb}\left(M^b + M^b_\perp\right) \\
&= f_{aEb}\mathcal{M}^b
\end{aligned} \tag{5.15}$$

We will now use this result to verify the second of the two relations, equation (5.13), which we could solve for the linear Killing vector components. We had :-

$$\sin\tfrac{a}{2}\left(f_r\right)_{Ea} + \sin\tfrac{b}{2}\left(f_{r\perp}\right)_{Ea} = \sin\tfrac{a}{2}\, n^a_{,b}\mathbf{K}^b_E + \sin\tfrac{b}{2}\, n^a_{\perp,b}\mathbf{K}^b_E$$

We will now substitute into the right hand side of this, and eventually obtain the left hand side. Using equations (D.22) and (D.23) from Appendix D and our result for $\mathbf{K}^a_E$ we will work on the right hand side in pieces. Firstly, we find :-

$$\begin{aligned}
M\, n^a_{,b}\mathbf{K}^b_E &= \tfrac{1}{2}\mathbf{K}^a_E + \tfrac{\mathcal{M}^2}{(M^2-M^2_\perp)}\left(\tfrac{1}{2}d^2_m + \tfrac{1}{4}d_{m\vee m}\right)_{ab}\mathbf{K}^b_E \\
&\;\;\vdots \\
&= -\tfrac{1}{2}\left(Mf_r + M_\perp f_{r\perp}\right)_{aE} - \tfrac{1}{2}\left(Mf_r - M_\perp f_{r\perp}\right)_{aE} \\
&= -M\left(f_r\right)_{aE} \\
n^a_{,b}\mathbf{K}^b_E &= \left(f_r\right)_{Ea}
\end{aligned}$$

where, in the first line, terms in $n^a_{,b}$ which will produce a zero when we form $n^a_{,b}\mathbf{K}^b_E$ have been omitted. Secondly, using the same method, we may also find the result :-

$$n^a_{\perp,b}\mathbf{K}^b_E = \left(f_{r\perp}\right)_{Ea}$$

It is now simple to see that substituting these two results into the right hand side of equation (5.13) will give us :-

$$\sin\tfrac{a}{2}\, n^a_{,b}\mathbf{K}^b_E + \sin\tfrac{b}{2}\, n^a_{\perp,b}\mathbf{K}^b_E \;=\; \sin\tfrac{a}{2}\ (f_r)_{Ea} + \sin\tfrac{b}{2}\ (f_{r_\perp})_{Ea}$$

This is just the relation we wanted to verify. Therefore we are sure that the form of the Killing vector components is correct. This ends the doublecheck on the result for the linear Killing vector components.

5.3.2 The nonlinear Killing vector components, $\mathbf{K}^a_b$.

In this section we will, as usual, find two relations which we may solve to find the nonlinear Killing vector components. However, this time we will only solve one of them; we will use the reconstruction of the Goldstone boson manifold metric as our second check as to whether the result for the nonlinear Killing vector components is correct; this will preserve the flow of the calculations. We will also see that, to find the Killing vector components, we will need to use a slightly different approach than the one used in previous chapters. As usual, we must solve :-

$$\{\lambda_b, L^2\} \;\equiv\; 2i\, L^2_{,a}\mathbf{K}^a_b \tag{5.16}$$

to find the nonlinear $\mathbf{K}^a_b$ components. We use the square of the coset representative element :-

$$L^2 \;=\; \tfrac{1}{2}(\cos a\ + \cos b\ )\ \mathbf{1}_{[4]} + \tfrac{1}{\sqrt{2}}(\cos a\ - \cos b\ )\ q_r - i\sin a\ r - i\sin b\ r_\perp$$

which, when we differentiate with respect to the Goldstone boson fields, gives us :-

$$\begin{aligned} L^2_{,a} \;=\; & -\tfrac{1}{2}(\sin a\ a_{,a} + \sin b\ b_{,a})\ \mathbf{1}_{[4]} - \tfrac{1}{\sqrt{2}}(\sin a\ a_{,a} - \sin b\ b_{,a})\ q_r \\ & + \tfrac{1}{\sqrt{2}}(\cos a\ - \cos b\ )\ q_{r,a} - i\cos a\ a_{,a}\, r - i\sin a\ r_{,a} - i\cos b\ b_{,a} r_\perp \\ & - i\sin b\ r_{\perp,a} \end{aligned}$$

So the left hand side of equation (5.16) is :-

$$\begin{aligned}
\left\{\lambda_b, L^2\right\} &= \tfrac{1}{2}\left(\cos a\ +\cos b\ \right)\left\{\lambda_b,\ \mathbf{1}_{[4]}\right\}+\tfrac{1}{\sqrt{2}}\left(\cos a\ -\cos b\ \right)\left\{\lambda_b, q_r\right\} \\
&\quad -i\sin a\ \left\{\lambda_b, r\right\}-i\sin b\ \left\{\lambda_b, r_\perp\right\} \\
&= -i\left(\sin a\ n^b+\sin b\ n_\perp^b\right)\mathbf{1}_{[4]}-2i\left(\sin a\ n^c d_{bcE}+\sin b\ n_\perp^c d_{bcE}\right)\lambda_E \\
&\quad +\left[\left(\cos a\ +\cos b\ \right)\delta_{ba}+\sqrt{2}\left(\cos a\ -\cos b\ \right)q_r^E d_{bEa}\right]\lambda_a
\end{aligned}$$

where we have written out the anticommutators explicitly and then regrouped the resulting terms. The right hand side is just :-

$$\begin{aligned}
2iL^2_{,a}\mathbf{K}_b^a &= -i\left(\sin a\ a_{,a}+\sin b\ b_{,a}\right)\mathbf{K}_b^a\,\mathbf{1}_{[4]}-i\sqrt{2}\left(\sin a\ a_{,a}+\sin b\ b_{,a}\right)\mathbf{K}_b^a q_r \\
&\quad +i\sqrt{2}\left(\cos a\ -\cos b\ \right)q_{r\,,a}\mathbf{K}_b^a+2\cos a\ a_{,a}\mathbf{K}_b^a\, r+2\sin a\ r_{,a}\mathbf{K}_b^a \\
&\quad +2\cos b\ b_{,a}\mathbf{K}_b^a\, r_\perp+2\sin b\ r_{\perp\,,a}\mathbf{K}_b^a
\end{aligned}$$

When we compare these last two equations we obviously have three relations. The first relation concerns the components in front of the identity elements :-

$$\sin a\ a_{,a}\mathbf{K}_b^a+\sin b\ b_{,a}\mathbf{K}_b^a = \sin a\ n^b+\sin b\ n_\perp^b$$

and this has two implications :-

$$a_{,a}\mathbf{K}_b^a \equiv n^b \tag{5.17}$$

$$b_{,a}\mathbf{K}_b^a \equiv n_\perp^b \tag{5.18}$$

The last two, vector-like, orthogonal relations are :-

$$\begin{aligned}
&\begin{array}{c} i\sqrt{2}\left(\cos a\ -\cos b\ \right)q_{r\,,a}\mathbf{K}_b^a \\ -i\sqrt{2}\left(\sin a\ a_{,a}-\sin b\ b_{,a}\right)\mathbf{K}_b^a\, q_r \end{array} = -2i\left(\sin a\ n^c+\sin b\ n_\perp^c\right)d_{bcE}\lambda_E \\
&\begin{array}{c} 2\left(\cos a\ a_{,a}\, r+\cos b\ b_{,a}\, r_\perp\right)\mathbf{K}_b^a \\ +\ 2\left(\sin a\ r_{,a}+\sin b\ r_{\perp,a}\right)\mathbf{K}_b^a \end{array} = \left[\begin{array}{l} \left(\cos a\ +\cos b\ \right)\delta_{ba} \\ +\sqrt{2}\left(\cos a\ -\cos b\ \right)q_r^E d_{bEa} \end{array}\right]\lambda_a
\end{aligned}$$

which, after removing the basis vectors and using the results implied by the first relation, reduce to :-

$$\begin{aligned} i\sqrt{2}\,(\cos a\ -\cos b\ )\,q^{E}_{r,c}\mathbf{K}^{c}_{b} & \\ -i\sqrt{2}\left(\sin a\ n^{b}-\sin b\ n^{b}_{\perp}\right)\,q^{E}_{r} &= -2i\,(\sin a\ n^{c}+\sin b\ n^{c}_{\perp})\ d_{bcE} \end{aligned} \tag{5.19}$$

$$\begin{aligned} 2\left(\cos a\ n^{b}n^{a}+\cos b\ n^{b}_{\perp}n^{a}_{\perp}\right) & \quad (\cos a\ +\cos b\ )\,\delta_{ba} \\ +\ 2\left(\sin a\ n^{a}_{,c}+\sin b\ n^{a}_{\perp,c}\right)\mathbf{K}^{c}_{b} &= +\sqrt{2}\,(\cos a\ -\cos b\ )\,q^{E}_{r}d_{bEa} \end{aligned} \tag{5.20}$$

We will not try to solve equation (5.19) because, the q_r-vector components are :-

$$q^{E}_{r} = \frac{\sqrt{2}}{(M^{2}-M^{2}_{\perp})}\mathcal{M}^{a}\mathcal{M}^{b}d_{abE}$$

and if we differentiate them with respect to the Goldstone boson fields, then we find :-

$$\begin{aligned} q^{E}_{r,c} &= \frac{2\sqrt{2}}{(M^{2}-M^{2}_{\perp})}\mathcal{M}\,m^{a}d_{acE}-\frac{2\sqrt{2}}{(M^{2}-M^{2}_{\perp})^{2}}\left(M\,n^{c}-M_{\perp}\,n^{c}_{\perp}\right)\mathcal{M}^{a}\mathcal{M}^{b}d_{abE} \\ &= \frac{2\sqrt{2}}{(M^{2}-M^{2}_{\perp})}\left[\frac{M}{2}\left(d_{r}\right)_{cE}+\frac{M_{\perp}}{2}\left(d_{r_{\perp}}\right)_{cE}-\frac{1}{\sqrt{2}}\left(M\,n^{c}-M_{\perp}\,n^{c}_{\perp}\right)\,q^{E}_{r}\right] \end{aligned}$$

Substituting this result into equation (5.19) gives :-

$$\begin{aligned} \sin a\ (d_{r})_{bE}+\sin b\ (d_{r_{\perp}})_{bE} &= \sqrt{2}\,(\sin a\ n^{b}-\sin b\ n^{b}_{\perp})\,q^{E}_{r} \\ &\quad +\frac{2\sqrt{2}(\cos a\ -\ \cos b\ )}{(M^{2}-M^{2}_{\perp})}(M\,n^{c}-M_{\perp}\,n^{c}_{\perp})\mathbf{K}^{c}_{b}\,q^{E}_{r} \\ &\quad -\frac{4(\cos a\ -\ \cos b\ )}{(M^{2}-M^{2}_{\perp})}\left[\frac{M}{2}(d_{r})_{cE}+\frac{M_{\perp}}{2}(d_{r_{\perp}})_{cE}\right]\mathbf{K}^{c}_{b} \end{aligned}$$

which will be difficult to solve. This is similar to the corresponding relation in the last chapter, see equation (4.18), which we couldn't directly solve either.

So we will instead solve the final relation given by equation (5.20). Firstly, we will rewrite it :-

$$\begin{aligned} 2(\cos a\ r><r+\cos b\ r_{\perp}><r_{\perp}) & \quad \cos a\ (\mathbf{1}_{[15]}+\tfrac{1}{\sqrt{2}}\,d_{q_{r}})_{ab} \\ +\ 2(\sin a\ n^{a}_{,c}+\sin b\ n^{a}_{\perp,c})\mathbf{K}^{c}_{b} &= +\ \cos b\ (\mathbf{1}_{[15]}-\tfrac{1}{\sqrt{2}}\,d_{q_{r}})_{ab} \end{aligned}$$

Now, in Appendix B on page 186 we are able to find the adjoint projection operator results :-

$$
\begin{aligned}
\left(\mathbf{1}_{[15]}+\tfrac{1}{\sqrt{2}}\,d_{q_r}\right)_{ab} &= 2\left(\mathcal{P}^{12}+\mathcal{P}^{21}+r{>}{<}r\right)_{ab}+\left(\mathcal{P}_{f_q^2}\right)_{ab} \\
\left(\mathbf{1}_{[15]}-\tfrac{1}{\sqrt{2}}\,d_{q_r}\right)_{ab} &= 2\left(\mathcal{P}^{34}+\mathcal{P}^{43}+r_\perp{>}{<}r_\perp\right)_{ab}+\left(\mathcal{P}_{f_q^2}\right)_{ab}
\end{aligned}
$$

which, when substituted, will simplify the relation to :-

$$
\begin{aligned}
\sin a\;\, n^a_{,c}\mathbf{K}^c_b+\sin b\;\, n^a_{\perp,c}\mathbf{K}^c_b &= \cos a\;\,\left(\mathcal{P}^{12}+\mathcal{P}^{21}\right)_{ab}+\cos b\;\,\left(\mathcal{P}^{34}+\mathcal{P}^{43}\right)_{ab} \\
&\quad +\tfrac{1}{2}\left(\cos a\;+\cos b\;\right)(\mathcal{P}_{f_q^2})_{ab}
\end{aligned} \tag{5.21}
$$

We are now in a position to find the nonlinear $\mathbf{K}^a_b$. Unfortunately, unlike in the previous chapters, we do not have a simple substitution of the field differentials which will allow us to simply rearrange the resulting expression to reveal the nonlinear $\mathbf{K}^a_b$. This is because the coset vector is now a linear sum of two commuting orthonormal r-vectors, namely r and $r_\perp$, and this means that we no longer have, for example, the relation $n^a_{,b}=\frac{1}{M}(\delta_{ab}-n^a n^b)$ which would allow such a simple solution. Instead we must now adopt a different approach; we note that this new method could have been adopted in the previous chapters, and we would have arrived at the same results.

In Appendix B we find the differentials, with respect to the Goldstone fields, of the two r-vector components are given by equations (D.24) and (D.25) which are :-

$$
\begin{aligned}
Mn^a_{,c} &= \tfrac{1}{(M^2-M_\perp^2)}\left[(M^2-M_\perp^2)(\mathcal{P}^{12}+\mathcal{P}^{21})_{ac}+M^2(\mathcal{P}_{f_q^2})_{ac}+MM_\perp(4f_rf_{r_\perp})_{ac}\right] \\
M_\perp n^a_{\perp,c} &= \tfrac{1}{(M^2-M_\perp^2)}\left[(M^2-M_\perp^2)(\mathcal{P}^{34}+\mathcal{P}^{43})_{ac}-M_\perp^2(\mathcal{P}_{f_q^2})_{ac}-MM_\perp(4f_rf_{r_\perp})_{ac}\right]
\end{aligned}
$$

where, in the above, we have used the linear combinations of two 'new' adjoint representation projection operators :-

$$
\begin{aligned}
4f_rf_{r_\perp} &\equiv \mathcal{P}^{\oplus}_{f_q^2}-\mathcal{P}^{\ominus}_{f_q^2} \\
\mathcal{P}_{f_q^2} &\equiv \mathcal{P}^{\oplus}_{f_q^2}+\mathcal{P}^{\ominus}_{f_q^2}
\end{aligned}
$$

Thus, it is obvious that both sides of equation (5.21) are now in terms of adjoint representation projection operators; and it is this important property which defines our new method. If we now operate on our relation with appropriate (combinations of) adjoint representation projection operators, then we will isolate the corresponding pieces from the nonlinear $\mathbf{K}^a_b$; once we have used all possible projection operators available to us, we will have all the pieces we require to reconstruct $\mathbf{K}^a_b$. For example, if the nonlinear $\mathbf{K}^a_b$ has the form :-

$$\mathbf{K}^a_b = A(\mathcal{P}^{12}+\mathcal{P}^{21})_{ab} + A_i(\mathcal{P}^i)_{ab}$$

where the last term represents a linear sum of *all* the other possible adjoint representation projection operators, and we act on it with $(\mathcal{P}^{12}+\mathcal{P}^{21})$ then we find :-

$$(\mathcal{P}^{12}+\mathcal{P}^{21})_{ac}\mathbf{K}^c_b = A(\mathcal{P}^{12}+\mathcal{P}^{21})_{ab}$$

and, in this way, we have managed to isolate the first term of $\mathbf{K}^a_b$.

So, bearing in mind the adjoint representation projection operators present in the right hand side of equation (5.21), if we work on the results for $n^a_{,c}$ then we see that the only non-zero quantities which may be formed in this way are :-

$$\begin{aligned}
(\mathcal{P}^{12}+\mathcal{P}^{21})_{da}Mn^a_{,c} &= (\mathcal{P}^{12}+\mathcal{P}^{21})_{dc} \\
(\mathcal{P}_{f_q^2})_{da}Mn^a_{,c} &= \tfrac{M^2}{M^2-M_\perp^2}(\mathcal{P}_{f_q^2})_{dc} + \tfrac{MM_\perp}{M^2-M_\perp^2}(4f_rf_{r_\perp})_{dc} \\
(4f_rf_{r_\perp})_{da}Mn^a_{,c} &= \tfrac{M^2}{M^2-M_\perp^2}(4f_rf_{r_\perp})_{dc} + \tfrac{MM_\perp}{M^2-M_\perp^2}(\mathcal{P}_{f_q^2})_{dc}
\end{aligned}$$

Similarly, working on the $n^a_{\perp,c}$ gives the nontrivial results :-

$$\begin{aligned}
(\mathcal{P}^{34}+\mathcal{P}^{43})_{da}M_\perp n^a_{\perp,c} &= (\mathcal{P}^{34}+\mathcal{P}^{43})_{dc} \\
(\mathcal{P}_{f_q^2})_{da}M_\perp n^a_{\perp,c} &= -\tfrac{M_\perp^2}{M^2-M_\perp^2}(\mathcal{P}_{f_q^2})_{dc} - \tfrac{MM_\perp}{M^2-M_\perp^2}(4f_rf_{r_\perp})_{dc} \\
(4f_rf_{r_\perp})_{da}M_\perp n^a_{\perp,c} &= -\tfrac{M_\perp^2}{M^2-M_\perp^2}(4f_rf_{r_\perp})_{dc} - \tfrac{MM_\perp}{M^2-M_\perp^2}(\mathcal{P}_{f_q^2})_{dc}
\end{aligned}$$

We will now start to build $\mathbf{K}^a_b$. The simplest pieces of $\mathbf{K}^a_b$ to find are given by the actions of $r{><}r$ and $r_\perp{><}r_\perp$ on the $\mathbf{K}^a_b$; which are implied by equations (5.17) and (5.18). These give :-

$$\begin{aligned}(r{><}r)_{ac}\,\mathbf{K}^c_b &\equiv \left(\tfrac{dM}{da}\right)(r{><}r)_{ab}\\ (r_\perp{><}r_\perp)_{ac}\,\mathbf{K}^c_b &\equiv \left(\tfrac{dM_\perp}{db}\right)(r_\perp{><}r_\perp)_{ab}\end{aligned}$$

So we immediately have the partial result :-

$$\mathbf{K}^a_b = \left(\tfrac{dM}{da}\right)(r{><}r)_{ab} + \left(\tfrac{dM_\perp}{db}\right)(r_\perp{><}r_\perp)_{ab} + \cdots$$

Using the six relations above, we will find the contributions made by the other adjoint representation projection operators. If we act on equation (5.21) with $(\mathcal{P}^{12}+\mathcal{P}^{21})$ then we find :-

$$(\mathcal{P}^{12}+\mathcal{P}^{21})_{ac}\,\mathbf{K}^c_b = M\cot a\,(\mathcal{P}^{12}+\mathcal{P}^{21})_{ab}$$

Using $\mathcal{P}_{f_q^2}$ leads to :-

$$\begin{aligned}&\sin a\left(\tfrac{M}{M^2-M_\perp^2}(\mathcal{P}_{f_q^2})_{dc} + \tfrac{M_\perp}{M^2-M_\perp^2}(4f_rf_{r_\perp})_{dc}\right)\mathbf{K}^c_b\\ &+\sin b\left(-\tfrac{M_\perp}{M^2-M_\perp^2}(\mathcal{P}_{f_q^2})_{dc} - \tfrac{M}{M^2-M_\perp^2}(4f_rf_{r_\perp})_{dc}\right)\mathbf{K}^c_b \quad = \tfrac{1}{2}(\cos a + \cos b)(\mathcal{P}_{f_q^2})_{db}\end{aligned}$$

Using $4f_rf_{r_\perp}$ gives us :-

$$\begin{aligned}&\sin a\left(\tfrac{M}{M^2-M_\perp^2}(4f_rf_{r_\perp})_{dc} + \tfrac{M_\perp}{M^2-M_\perp^2}(\mathcal{P}_{f_q^2})_{dc}\right)\mathbf{K}^c_b\\ &+\sin b\left(-\tfrac{M_\perp}{M^2-M_\perp^2}(4f_rf_{r_\perp})_{dc} - \tfrac{M}{M^2-M_\perp^2}(\mathcal{P}_{f_q^2})_{dc}\right)\mathbf{K}^c_b \quad = \tfrac{1}{2}(\cos a + \cos b)(4f_rf_{r_\perp})_{db}\end{aligned}$$

Lastly, if we use $(\mathcal{P}^{34}+\mathcal{P}^{43})$ then we find :-

$$(\mathcal{P}^{34}+\mathcal{P}^{43})_{ac}\,\mathbf{K}^c_b = M_\perp\cot b\,(\mathcal{P}^{34}+\mathcal{P}^{43})_{ab}$$

The first, and the last, of these four results immediately give us two more pieces for the nonlinear Killing vector result; we now have :-

$$\begin{aligned}\mathbf{K}^a_b = {}& M\cot a\,(\mathcal{P}^{12}+\mathcal{P}^{21})_{ab} + \left(\tfrac{dM}{da}\right)(r{><}r)_{ab}\\ &+M_\perp\cot b\,(\mathcal{P}^{34}+\mathcal{P}^{43})_{ab} + \left(\tfrac{dM_\perp}{db}\right)(r_\perp{><}r_\perp)_{ab} + \cdots\end{aligned}$$

There are only two projection operator terms left to sort out, and we find these using the second and third results found above. The point at which we stop the following analysis is solely governed by our choice of projection operators used to express the result for $\mathbf{K}^a_b$. Firstly, if we add the two results together then we find :-

$$\frac{1}{M-M_\perp}(\sin a \; - \sin b \;)(\mathcal{P}_{f_q^2} + 4f_r f_{r_\perp})_{ac}\mathbf{K}^c_b \;\; = \;\; \tfrac{1}{2}(\cos a \; + \cos b \;)(\mathcal{P}_{f_q^2} + 4f_r f_{r_\perp})_{ab}$$

If we now subtract the third result from the second, then we find :-

$$\frac{1}{M+M_\perp}(\sin a \; + \sin b \;)(\mathcal{P}_{f_q^2} - 4f_r f_{r_\perp})_{ac}\mathbf{K}^c_b \;\; = \;\; \tfrac{1}{2}(\cos a \; + \cos b \;)(\mathcal{P}_{f_q^2} - 4f_r f_{r_\perp})_{ab}$$

If we wanted to express $\mathbf{K}^a_b$ using the adjoint projection operators $\mathcal{P}^{\oplus}_{f_q^2}$ and $\mathcal{P}^{\ominus}_{f_q^2}$ then these last two relations are the appropriate results. However, if we add them together then we find :-

$$\begin{aligned}
(\mathcal{P}_{f_q^2})_{ac}\mathbf{K}^c_b \;\; &= \;\; \frac{\cos a \; + \cos b}{2(\sin^2 a \; - \sin^2 b \;)}(M \sin a \; - M_\perp \sin b \;)(\mathcal{P}_{f_q^2})_{ab} \\
&\quad -\frac{\cos a \; + \cos b}{2(\sin^2 a \; - \sin^2 b \;)}(M_\perp \sin a \; - M \sin b \;)(4f_r f_{r_\perp})_{ab} \\
&= \;\; -\frac{1}{2(\cos a \; + \cos b \;)}(M \sin a \; - M_\perp \sin b \;)(\mathcal{P}_{f_q^2})_{ab} \\
&\quad +\frac{1}{2(\cos a \; + \cos b \;)}(M_\perp \sin a \; - M \sin b \;)(4f_r f_{r_\perp})_{ab}
\end{aligned}$$

and we understand that this single expression is also a complete result. We see this because, if the last part of the unknown $\mathbf{K}^a_b$ is expressed in terms of coefficients multiplied by the adjoint representation projection operator combinations $\mathcal{P}_{f_q^2}$ and $4f_r f_{r_\perp}$, then using $\mathcal{P}_{f_q^2}$ in this way has just isolated the above pieces from $\mathbf{K}^a_b$. Therefore, using $\mathcal{P}^{\oplus}_{f_q^2}$ and $\mathcal{P}^{\ominus}_{f_q^2}$, we have the result :-

$$\begin{aligned}
\mathbf{K}^a_b \;\; &= \;\; M \cot a \; (\mathcal{P}^{12} + \mathcal{P}^{21})_{ab} + \left(\tfrac{dM}{da}\right)(r\!>\!<\!r)_{ab} + M_\perp \cot b \; (\mathcal{P}^{34} + \mathcal{P}^{43})_{ab} \\
&\quad + \left(\tfrac{dM_\perp}{db}\right)(r_\perp\!>\!<\!r_\perp)_{ab} + \frac{(M-M_\perp)(\cos a \; + \cos b \;)}{2(\sin a \; - \sin b \;)}(\mathcal{P}^{\oplus}_{f_q^2})_{ab} \\
&\quad + \frac{(M+M_\perp)(\cos a \; + \cos b \;)}{2(\sin a \; + \sin b \;)}(\mathcal{P}^{\ominus}_{f_q^2})_{ab} \qquad (5.22)
\end{aligned}$$

or, using $\mathcal{P}_{f_q^2}$ and $4f_r f_{r_\perp}$, we find an equivalent expression :-

$$\begin{aligned}
\mathbf{K}^a_b &= M\cot a\ (\mathcal{P}^{12}+\mathcal{P}^{21})_{ab} + \left(\tfrac{dM}{da}\right)(r><r)_{ab} + M_\perp \cot b\ (\mathcal{P}^{34}+\mathcal{P}^{43})_{ab} \\
&+ \left(\tfrac{dM_\perp}{db}\right)(r_\perp><r_\perp)_{ab} - \tfrac{1}{2(\cos a\ +\cos b\ )}(M\sin a\ - M_\perp \sin b\ )(\mathcal{P}_{f_q^2})_{ab} \\
&+ \tfrac{1}{2(\cos a\ +\cos b\ )}(M_\perp \sin a\ - M\sin b\ )(4f_r f_{r_\perp})_{ab} \qquad (5.23)
\end{aligned}$$

5.4 Covariant derivatives and the Goldstone boson metric.

We have found an expression for the coset representative element, L :-

$$L = \tfrac{1}{2}(\cos\tfrac{a}{2} + \cos\tfrac{b}{2})\ \mathbf{1}_{[4]} + \tfrac{1}{\sqrt{2}}(\cos\tfrac{a}{2} - \cos\tfrac{b}{2})\ q_r - i\,(\sin\tfrac{a}{2}\ r + \sin\tfrac{b}{2}\ r_\perp)$$

Thus we find :-

$$\begin{aligned}
L^{-1} &= \tfrac{1}{2}(\cos\tfrac{a}{2} + \cos\tfrac{b}{2})\ \mathbf{1}_{[4]} + \tfrac{1}{\sqrt{2}}(\cos\tfrac{a}{2} - \cos\tfrac{b}{2})\ q_r + i\,(\sin\tfrac{a}{2}\ r + \sin\tfrac{b}{2}\ r_\perp) \\
\partial_\mu L &= -\tfrac{1}{4}(\sin\tfrac{a}{2}\ \partial_\mu a + \sin\tfrac{b}{2}\ \partial_\mu b)\ \mathbf{1}_{[4]} - \tfrac{1}{2\sqrt{2}}(\sin\tfrac{a}{2}\ \partial_\mu a - \sin\tfrac{b}{2}\ \partial_\mu b)\ q_r \\
&+ \tfrac{1}{\sqrt{2}}(\cos\tfrac{a}{2} - \cos\tfrac{b}{2})\ \partial_\mu q_r - \tfrac{i}{2}(\cos\tfrac{a}{2}\,\partial_\mu a\, r + \cos\tfrac{b}{2}\,\partial_\mu b\, r_\perp) \\
&- i\,(\sin\tfrac{a}{2}\ \partial_\mu r + \sin\tfrac{b}{2}\ \partial_\mu r_\perp)
\end{aligned}$$

This time it takes a little longer to find $2iL^{-1}\partial_\mu L = a_\mu + v_\mu$, but the calculation is straightforward. We find for our Goldstone boson covariant derivative :-

$$\begin{aligned}
a_\mu &= r\partial_\mu a + r_\perp \partial_\mu b + (\cos\tfrac{a}{2} + \cos\tfrac{b}{2})(\sin\tfrac{a}{2}\,\partial_\mu r + \sin\tfrac{a}{2}\,\partial_\mu r_\perp) \\
&\sqrt{2}(\cos\tfrac{a}{2} - \cos\tfrac{b}{2})[\sin\tfrac{a}{2}\ (q_r\partial_\mu r - r\partial_\mu q_r) + \sin\tfrac{b}{2}\ (q_r\partial_\mu r_\perp - r_\perp\partial_\mu q_r)]
\end{aligned}$$

which, as before, we may simplify to :-

$$\begin{aligned}
a_\mu &= r\partial_\mu a + r_\perp\partial_\mu b + \sin a\ \partial_\mu r + \sin b\ \partial_\mu r_\perp \\
&- \sqrt{2}(\cos\tfrac{a}{2} - \cos\tfrac{b}{2})[\sin\tfrac{a}{2}\ \{r, \partial_\mu q_r\} + \sin\tfrac{b}{2}\ \{r_\perp, \partial_\mu q_r\}] \qquad (5.24)
\end{aligned}$$

We also find the metric connection for the Standard field covariant derivative to be :-

$$
\begin{aligned}
v_\mu &= \tfrac{i}{\sqrt{2}}\left(\cos^2\tfrac{a}{2} - \cos^2\tfrac{b}{2}\right)\partial_\mu q_r + i\left(\cos\tfrac{a}{2} - \cos\tfrac{b}{2}\right)^2 q_r \partial_\mu q_r \\
&\quad 2i\left[\sin^2\tfrac{a}{2}\, r\partial_\mu r + \sin^2\tfrac{b}{2}\, r_\perp \partial_\mu r_\perp + \sin\tfrac{a}{2}\,\sin\tfrac{b}{2}\left(r\partial_\mu r_\perp + r_\perp \partial_\mu r\right)\right]
\end{aligned} \tag{5.25}
$$

The Lagrangian density for the Goldstone bosons is found by constructing the quantity $(a_\mu, a^\mu) = \frac{1}{2} tr\, a_\mu a^\mu$. If we were to calculate the quantity $a_\mu a^\mu$ then we would end up with many terms. To simplify our task we will use the fact that we have trivial results for $(r, \partial_\mu r_\perp)$, $(r, \{r, \partial_\mu q_r\})$, $(r, \{r_\perp, \partial_\mu q_r\})$, $(r_\perp, \partial_\mu r)$, $(r_\perp, \{r, \partial_\mu q_r\})$, $(r_\perp, \{r_\perp, \partial_\mu q_r\})$, $(r, \partial_\mu r)$ and $(r_\perp, \partial_\mu r_\perp)$. So, writing $a^a_\mu a^\mu_a$ as best we can we find :-

$$
\begin{aligned}
a^a_\mu a^\mu_a &= \partial_\mu a\, \partial^\mu a + \partial_\mu b\, \partial^\mu b + \sin^2 a\ \partial_\mu n^a \partial^\mu n^a \\
&\quad + \sin^2 b\ \partial_\mu n^a_\perp \partial^\mu n^a_\perp + 2\sin a\ \sin b\ \partial_\mu n^a \partial^\mu n^a_\perp \\
&\quad -2\sqrt{2}(\cos\tfrac{a}{2} - \cos\tfrac{b}{2})\sin a\left[\sin\tfrac{a}{2}\ (\partial_\mu r, \{r, \partial^\mu q_r\}) + \sin\tfrac{b}{2}\ (\partial_\mu r, \{r_\perp, \partial^\mu q_r\})\right] \\
&\quad -2\sqrt{2}(\cos\tfrac{a}{2} - \cos\tfrac{b}{2})\sin b\left[\sin\tfrac{a}{2}\ (\partial_\mu r_\perp, \{r, \partial^\mu q_r\}) + \sin\tfrac{b}{2}\ (\partial_\mu r_\perp, \{r_\perp, \partial^\mu q_r\})\right] \\
&\quad +2(\cos\tfrac{a}{2} - \cos\tfrac{b}{2})^2 \left[\begin{array}{l} \sin^2\tfrac{a}{2}\ (\{r, \partial_\mu q_r\}, \{r, \partial^\mu q_r\}) \\ + \sin^2\tfrac{b}{2}\ (\{r_\perp, \partial_\mu q_r\}, \{r_\perp, \partial^\mu q_r\}) \\ +2\sin\tfrac{a}{2}\ \sin\tfrac{b}{2}\ (\{r, \partial_\mu q_r\}, \{r_\perp, \partial^\mu q_r\}) \end{array}\right]
\end{aligned}
$$

We now need to work on the scalar product terms in the square brackets; writing them in terms of fields, field gradients and the d_{IJK}. It takes a little time, but we find the results :-

$$
\begin{aligned}
(\partial_\mu r, \{r, \partial^\mu q_r\}) &= \tfrac{1}{\sqrt{2}} \partial_\mu q^E_r \partial^\mu q^E_r \\
(\partial_\mu r, \{r_\perp, \partial^\mu q_r\}) &= 2 n^a_\perp \partial_\mu n^b \partial^\mu q^E_r d_{abE} \\
&= -2 n^a \partial_\mu n^b_\perp \partial^\mu q^E_r d_{abE} \\
(\partial_\mu r_\perp, \{r, \partial^\mu q_r\}) &= 2 n^a \partial_\mu n^b_\perp \partial^\mu q^E_r d_{abE}
\end{aligned}
$$

$$
\begin{aligned}
(\partial_\mu r_\perp, \{r_\perp, \partial^\mu q_r\}) &= -\tfrac{1}{\sqrt{2}}\partial_\mu q_r^E \partial^\mu q_r^E \\
(\{r, \partial_\mu q_r\}, \{r, \partial^\mu q_r\}) &= \partial_\mu q_r^E \partial^\mu q_r^E \\
(\{r_\perp, \partial_\mu q_r\}, \{r_\perp, \partial^\mu q_r\}) &= \partial_\mu q_r^E \partial^\mu q_r^E \\
(\{r, \partial_\mu q_r\}, \{r_\perp, \partial^\mu q_r\}) &= -2\sqrt{2} n^a \partial_\mu n_\perp^b \partial^\mu q_r^E d_{abE}
\end{aligned}
$$

So, after we substitute these in, we eventually find :-

$$
\begin{aligned}
a_\mu^a a_a^\mu &= \partial_\mu a\, \partial^\mu a + \partial_\mu b\, \partial^\mu b + \sin^2 a\ \partial_\mu n^a \partial^\mu n^a + \sin^2 b\ \partial_\mu n_\perp^a \partial^\mu n_\perp^a \\
&\quad + 2\sin a\ \sin b\ \partial_\mu n^a \partial^\mu n_\perp^a + \tfrac{1}{2}(\cos a - \cos b)^2 \partial_\mu q_r^E \partial^\mu q_r^E \\
&= g_{ab}\partial_\mu \mathcal{M}^a\, \partial^\mu \mathcal{M}^b
\end{aligned} \tag{5.26}
$$

At this point we will digress, slightly, by restricting the parameters a and b in these primary results for a_μ and $a_\mu^a a_a^\mu$ given in equations (5.24) and (5.26). What we are about to do has no physical significance within the theory; it will be purely a mathematical excersise which will give a simple 'check' for the results we have found. Of course, the proper check for the form of $a_\mu^a a_a^\mu$ will be achieved, as usual, with the reconstruction of the Goldstone boson manifold metric using the Killing vector components. However, we continue for the moment by firstly imposing the condition $a = b$ and then see what this means. Secondly, we will impose the two (separate) conditions $a = 0$ and then $b = 0$.

- For $a = b$ we find that the primary a_μ and $a_\mu^a a_a^\mu$ results reduce to :-

$$
\begin{aligned}
a_\mu &= (r + r_\perp)\partial_\mu a + \sin a\ \partial_\mu (r + r_\perp) \\
a_\mu^a a_a^\mu &= 2\partial_\mu a\, \partial^\mu a + \sin^2 a\ \partial_\mu (n^a + n_\perp^a) \partial^\mu (n^a + n_\perp^a)
\end{aligned}
$$

When $N = 4$ we know that $r_3 + r_{3\perp} = \sqrt{2}\, q_2$. So if we let $a = 2\Omega$ then these expressions become :-

$$\begin{aligned}
a_\mu &= \sqrt{2}\, q_2 \partial_\mu a + \sqrt{2}\,\sin a\ \partial_\mu q_2 \\
&= S\partial_\mu a + \sin a\ \partial_\mu S \\
&= 2S\partial_\mu \Omega + \sin 2\Omega\ \partial_\mu S \\
\tfrac{1}{2}\, a^a_\mu a^\mu_a &= \partial_\mu a\, \partial^\mu a + \sin^2 a\ \partial_\mu q^a_2 \partial^\mu q^a_2 \\
&= 4\partial_\mu \Omega \partial^\mu \Omega + \sin^2 2\Omega\ \partial_\mu n^{a\Delta} \partial^\mu n_{a\Delta}
\end{aligned}$$

These have the same form as the results given by equations (3.22) and (3.26) on page 84, in the section on $SO(m)$ breaking to $SO(m-1)$ where $m = 4, 5, 6$. The results have the same form because, for these particular $SO(m)$ breaking models, the coset vector is proportional to a q_r-vector; and this is also what we have when $a = b$; it just happens to be a different q_r-vector with a different number of components.

- For $a = 0$ we find that the primary results for a_μ and $a^a_\mu a^\mu_a$ reduce to :-

$$\begin{aligned}
a_\mu &= r_\perp \partial_\mu b + \sin b\ \partial_\mu r_\perp + \sqrt{2}(\cos\tfrac{b}{2} - 1)\sin\tfrac{b}{2}\ \{r_\perp, \partial_\mu q_r\} \\
a^a_\mu a^\mu_a &= \partial_\mu b\, \partial^\mu b + \sin^2 b\ \partial_\mu n^a_\perp \partial^\mu n^a_\perp + \tfrac{1}{2}(\cos b\ - 1)^2 \partial_\mu q^E_r \partial^\mu q^E_r
\end{aligned}$$

and when $b = 0$ we find that the primary a_μ and $a^a_\mu a^\mu_a$ reduce to :-

$$\begin{aligned}
a_\mu &= r\partial_\mu a + \sin a\ \partial_\mu r - \sqrt{2}(\cos\tfrac{a}{2} - 1)\sin\tfrac{a}{2}\ \{r, \partial_\mu q_r\} \\
a^a_\mu a^\mu_a &= \partial_\mu a\, \partial^\mu a + \sin^2 a\ \partial_\mu n^a \partial^\mu n^a + \tfrac{1}{2}(\cos a\ - 1)^2 \partial_\mu q^E_r \partial^\mu q^E_r
\end{aligned}$$

Note that $\{r_\perp, \partial_\mu q_r\} = -\{r, \partial_\mu q_r\}$ which accounts for the difference in sign of the last term of a_μ. Now, in both these cases, we have a coset vector which is proportional to an r-vector. Therefore we expect these results to be found in the previous chapter, and indeed they are. On page 110 we found equation (4.21) which, for $N = 4$, is :-

$$a_\mu = r\partial_\mu\phi + \sin\phi\,\partial_\mu r - \sqrt{2}\,(\cos\tfrac{\phi}{2}\,-1)\sin\tfrac{\phi}{2}\,\{r, \partial_\mu q_r\}$$

and on page 111 we found equation (4.24) which, again for $N = 4$, is :-

$$(a_\mu, a^\mu) = \partial_\mu\phi\partial^\mu\phi + \sin^2\!\phi\,\partial_\mu n^a\partial^\mu n^a + \tfrac{1}{2}\,(\cos\phi\,-1)^2\,\partial_\mu q_r^E\partial^\mu q_r^E$$

This seems to suggest that the forms of a_μ and $a_\mu^a a_a^\mu$ are correct.

We will now rewrite equation (5.26) in terms of the adjoint representation projection operators. The first two terms are simple to deal with and we find :-

$$\begin{aligned} \partial_\mu a\,\partial^\mu a &= \left(\tfrac{da}{dM}\right)^2 (r{><}r)_{ab}\,\partial_\mu M^a\partial^\mu M^b \\ \partial_\mu b\,\partial^\mu b &= \left(\tfrac{db}{dM_\perp}\right)^2 (r_\perp{><}r_\perp)_{ab}\,\partial_\mu M_\perp^a\partial^\mu M_\perp^b \end{aligned}$$

For the other terms we notice that, since $(a_\mu, a^\mu) = g_{ab}\,\partial_\mu\mathcal{M}^a\,\partial^\mu\mathcal{M}^b$, we are able to remove the Goldstone field gradients from the expressions in Appendix D and therefore find the rest of g_{ab} directly. Thus, the relevant expressions we need, in the order that they appear in equation (5.26), are :-

$$\begin{aligned} n^c_{,a}n^c_{,b} &= \tfrac{1}{M^2}(\mathcal{P}^{12}+\mathcal{P}^{21})_{ab} + \tfrac{\mathcal{M}^2}{(M^2-M_\perp^2)^2}(\mathcal{P}_{f_q^2})_{ab} + 2\tfrac{MM_\perp}{(M^2-M_\perp^2)^2}(4f_r f_{r_\perp})_{ab} \\ n^c_{\perp,a}n^c_{\perp,b} &= \tfrac{1}{M_\perp^2}(\mathcal{P}^{34}+\mathcal{P}^{43})_{ab} + \tfrac{\mathcal{M}^2}{(M^2-M_\perp^2)^2}(\mathcal{P}_{f_q^2})_{ab} + 2\tfrac{MM_\perp}{(M^2-M_\perp^2)^2}(4f_r f_{r_\perp})_{ab} \\ n^c_{,a}n^c_{\perp,b} &= -\tfrac{1}{(M^2-M_\perp^2)^2}(2MM_\perp\mathcal{P}_{f_q^2} + 4\mathcal{M}^2 f_r f_{r_\perp})_{ab} \\ q^E_{r,a}q^E_{r,b} &= \tfrac{2}{(M^2-M_\perp^2)^2}(\mathcal{M}^2\mathcal{P}_{f_q^2} + 2MM_\perp 4f_r f_{r_\perp})_{ab} \end{aligned}$$

Substituting in all these results leads to :-

$$\begin{aligned}
g_{ab} &= \tfrac{\sin^2 a}{M^2}(\mathcal{P}^{12}+\mathcal{P}^{21})_{ab} + \left(\tfrac{da}{dM}\right)^2 (r><r)_{ab} \\
&+\tfrac{\sin^2 b}{M_\perp^2}(\mathcal{P}^{34}+\mathcal{P}^{43})_{ab} + \left(\tfrac{db}{dM_\perp}\right)^2 (r_\perp><r_\perp)_{ab} \\
&+\tfrac{1}{(M^2-M_\perp^2)^2}\left[2\mathcal{M}^2(1-\cos a\ \cos b\ ) - 4MM_\perp \sin a\ \sin b\ \right](\mathcal{P}_{f_q^2})_{ab} \\
&+\tfrac{2}{(M^2-M_\perp^2)^2}\left[2MM_\perp(1-\cos a\ \cos b\ ) - \mathcal{M}^2 \sin a\ \sin b\ \right](4f_r f_{r_\perp})_{ab} \qquad (5.27)
\end{aligned}$$

and equivalently, this may also be written as :-

$$\begin{aligned}
g_{ab} &= \tfrac{\sin^2 a}{M^2}(\mathcal{P}^{12}+\mathcal{P}^{21})_{ab} + \left(\tfrac{da}{dM}\right)^2 (r><r)_{ab} \\
&+\tfrac{\sin^2 b}{M_\perp^2}(\mathcal{P}^{34}+\mathcal{P}^{43})_{ab} + \left(\tfrac{db}{dM_\perp}\right)^2 (r_\perp><r_\perp)_{ab} \\
&+\tfrac{1}{(M-M_\perp)^2}\left[(\sin a\ -\sin b\ )^2 + (\cos a\ -\cos b\ )^2\right](\mathcal{P}^{\oplus}_{f_q^2})_{ab} \\
&+\tfrac{1}{(M+M_\perp)^2}\left[(\sin a\ +\sin b\ )^2 + (\cos a\ -\cos b\ )^2\right](\mathcal{P}^{\ominus}_{f_q^2})_{ab} \qquad (5.28)
\end{aligned}$$

5.4.1 Verifying the metric result.

To verify that the result for the metric is correct, we will now use the relationship between the metric and the Killing vectors :-

$$g_{ab} \equiv \left(\mathbf{K}^a_E\mathbf{K}^b_E + \mathbf{K}^a_c\mathbf{K}^b_c\right)^{-1}$$

However, this relation implies :-

$$g_{ac}(\mathbf{K}^E_c\mathbf{K}^E_b + \mathbf{K}^d_c\mathbf{K}^d_b) \equiv (\mathbf{1}_{[15]})_{ab} \qquad (5.29)$$

which is simpler to use; the right hand side being defined by :-

$$\begin{aligned}
(\mathbf{1}_{[15]})_{ab} &= (\mathcal{P}^{12}+\mathcal{P}^{21}+r><r+\mathcal{P}_{f_q^2}+\mathcal{P}^{34}+\mathcal{P}^{43}+r_\perp><r_\perp)_{ab} \\
&= (\mathcal{P}^{12}+\mathcal{P}^{21}+r><r+(\mathcal{P}^{\oplus}_{f_q^2})_{ab}+\mathcal{P}^{34}+\mathcal{P}^{43}+r_\perp><r_\perp+\mathcal{P}^{\ominus}_{f_q^2})_{ab}
\end{aligned}$$

If we now use the $\mathbf{K}_E^a$ and $\mathbf{K}_b^a$ in the form given in equations (5.15) and (5.22) then we find :-

$$
\begin{aligned}
\mathbf{K}_a^E \mathbf{K}_b^E &= M^2(\mathcal{P}^{12}+\mathcal{P}^{21})_{ab} + M_\perp^2(\mathcal{P}^{34}+\mathcal{P}^{43})_{ab} + \tfrac{(M-M_\perp)^2}{4}(\mathcal{P}^{\oplus}_{f_q^2})_{ab} + \tfrac{(M+M_\perp)^2}{4}(\mathcal{P}^{\ominus}_{f_q^2})_{ab} \\
\mathbf{K}_a^c \mathbf{K}_b^c &= M^2 \cot^2 a\ (\mathcal{P}^{12}+\mathcal{P}^{21})_{ab} + \left(\tfrac{dM}{da}\right)^2 (r><r)_{ab} + M_\perp^2 \cot^2 b\ (\mathcal{P}^{34}+\mathcal{P}^{43})_{ab} \\
&+ \left(\tfrac{dM_\perp}{db}\right)^2 (r_\perp><r_\perp)_{ab} + \left(\tfrac{(M-M_\perp)(\cos a+\cos b)}{2(\sin a-\sin b)}\right)^2 (\mathcal{P}^{\oplus}_{f_q^2})_{ab} \\
&+ \left(\tfrac{(M+M_\perp)(\cos a+\cos b)}{2(\sin a+\sin b)}\right)^2 (\mathcal{P}^{\ominus}_{f_q^2})_{ab}
\end{aligned}
$$

When we add these together we obtain :-

$$
\begin{aligned}
\mathbf{K}_a^E \mathbf{K}_b^E + \mathbf{K}_a^c \mathbf{K}_b^c &= \tfrac{M^2}{\sin^2 a}(\mathcal{P}^{12}+\mathcal{P}^{21})_{ab} + \left(\tfrac{dM}{da}\right)^2 (r><r)_{ab} + \tfrac{M_\perp^2}{\sin^2 b}(\mathcal{P}^{34}+\mathcal{P}^{43})_{ab} \\
&+ \left(\tfrac{dM_\perp}{db}\right)^2 (r_\perp><r_\perp)_{ab} + \left(\tfrac{M-M_\perp}{2}\right)^2 \left[1+\left(\tfrac{\cos a+\cos b}{\sin a-\sin b}\right)^2\right] (\mathcal{P}^{\oplus}_{f_q^2})_{ab} \\
&+ \left(\tfrac{M+M_\perp}{2}\right)^2 \left[1+\left(\tfrac{\cos a+\cos b}{\sin a+\sin b}\right)^2\right] (\mathcal{P}^{\ominus}_{f_q^2})_{ab}
\end{aligned}
$$

The first four quantities, on the right hand side of this expression, are obviously the inverses of the first four quantities on the right hand side of equation (5.28) and, with a small ammount of work, it is simple to then show that the last two quantities, of the above relation, are the inverses of the last two quantities in equation (5.28). Thus equation (5.29) will be satisfied. Therefore, equation (5.28) is the correct result for the Goldstone boson manifold metrics of the three models considered in this chapter. As previously discussed, when $m = 4$, the manifold is $S_L^2 \otimes S_R^2$ and is hyper-Kähler; which allows the theory to be extended to include $\mathcal{N} = 2$ extended Supersymmetry. When $m = 5, 6$ we just have the metrics of Kähler manifolds, and it is therefore possible to extend these models to include $\mathcal{N} = 1$ Supersymmetry only.

Chapter 6

Conclusions.

In this chapter we summarize our main findings and take a brief look at potential avenues for further research.

We have seen how the mathematical techniques in [8, 9, 10], of evaluating effective Lagrangian densities and studying field transformations, are based upon the manipulations of an exponential quantity known as the coset representative element, L. Therefore, to get anywhere with many physical models we may wish to consider, it was evident that we needed a mathematical framework to help us calculate L. By extending the work in [17] and applying it to this problem, we have ended up with an index free notation (which also supplies a geometrical understanding) and this helps us describe the coset vectors; which allows us to calculate the coset representative element [1]. Then, because the coset vectors are written in terms of vectors with a well defined behaviour, we find that the mathematical behaviour of any other vectors constructed from the coset vectors is also understood. For example, if L is the exponential of an r-vector then we know that, when it is explicitly calculated, it will contain the

[1]In this thesis we have not considered *general* $SU(N)$ coset models as they require a projection operator method from the outset. This method is used in [13, 15]

the identity element, the r-vector itself and the commuting vector q_r (all terms being preceeded by coefficients). So when we calculate $2iL^{-1}\partial_\mu L = a_\mu + v_\mu$ we will find terms involving $r\partial_\mu r$, $r\partial_\mu q_r$, $q_r\partial_\mu r$ and so on; and we immediately know which pieces are part of a_μ and which are part of v_μ. We have also seen how the symmetric algebra (used to build commuting vectors) also helps us to find field differentials like $n^a_{,b}$ for the models in chapter 5 (see the second part of Appendix D).

In this thesis we have looked at many models of spontaneous symmetry breaking :-

- when $SU(2)$ invariance is broken to $U(1)$,
- when $SO(m)$ invariance is broken to $SO(m-1)$ for all $m \geq 3$,
- when $SO(1, m-1)$ invariance is broken to $SO(1, m-2)$ for all $m \geq 3$,
- when $SU(N)$ invariance is broken to $SU(N-1) \otimes U(1)$ for all $N \geq 3$, and
- when $SO(m)$ invariance is broken to $SO(m-2) \otimes SO(2)$ for $m = 4, 5, 6$ only.

For all of these models we found the linear Killing vector components and the nonlinear Killing vector components which describe the Goldstone field transformations. We also found the covariant derivatives, for the Goldstone fields and the matter fields of the theory, which are used in constructing the effective Lagrangian density. Lastly, we verified the form of the Goldstone boson manifold metric (contained in the scalar part of the density) by reconstructing it using the Killing vector components.

All the models where the subgroup contains a commuting $U(1) \sim SO(2)$ group have a Kähler Goldstone boson manifold, and therefore admit $\mathcal{N} = 1$ supersymmetry; the manifold in the $SO(4)$ breaking to $SO(2) \otimes SO(2)$ model is Hyper-Kähler and so admits $\mathcal{N} = 2$ extended supersymmetry. We know this because [22] tells us that *any* Kähler $(\frac{G}{H})$ can be made Hyper-Kähler by forming $(\frac{G}{H}) \otimes S^2$; and $\frac{SO(4)}{SO(2)\otimes SO(2)} \sim \frac{SU(2)\otimes SU(2)}{U(1)\otimes U(1)}$ is

$S^2 \otimes S^2$ which is the simplest example. All this assumes that fields are defined in a 4-dimensional spacetime. If the spacetime is 2-dimensional then a Kähler Goldstone boson manifold implies that the model will admit $\mathcal{N} = 2$ extended supersymmetry, and a Hyper-Kähler manifold implies that the model will admit $\mathcal{N} = 4$ extended supersymmetry.

To incorporate supersymmetry we pair up the even number of Goldstone boson manifold coordinates into complex combinations. Then the functions of the invariants of the fields are restricted, leading to a basis of stereographic coordinates. A supersymmetric version of the theory is then given by replacing these coordinates with chiral superfields. This method was introduced by Zumino [21]. In [18] we find the resulting metric for CP2, and in [19] the CP2 and CP4 metrics are investigated. We will briefly show the procedure for the CP2 case. If we write $z = M^1 + iM^2$, which implies $\bar{z}z = M^a M^a$, then the $\frac{SU(2)}{U(1)}$ Goldstone boson manifold metric :-

$$\mathcal{L} = \frac{1}{2}\left(\frac{\sin^2\phi}{M^2}[\delta_{ab} - n^a n^b] + \left[\frac{d\phi}{dM}\right]^2 n^a n^b\right) \partial_\mu M^a \, \partial^\mu M^b$$

may be written :-

$$\mathcal{L} = \frac{\sin^2\phi}{2M^2}\, \partial_\mu \bar{z}\, \partial^\mu z + \frac{1}{2M^2}\left(\left[\frac{d\phi}{dM}\right]^2 - \frac{\sin^2\phi}{M^2}\right)(z\partial_\mu \bar{z})\,(\bar{z}\partial^\mu z)$$

The final terms are removed by the condition $\frac{d\phi}{dM} = \frac{\sin\phi}{M}$ which is solved by integration. We find :-

$$\begin{aligned} \int \frac{1}{M}\, dM &= \int \operatorname{cosec}\phi \, d\phi \\ \ln M &= \ln \tan\tfrac{\phi}{2} + \ln c \\ \frac{M}{c} &= \tan\tfrac{\phi}{2} \end{aligned}$$

We use simple trigonometric relations to show that $\frac{\sin^2\phi}{M^2} = \frac{4c^2}{(c^2+M^2)^2}$ and so using stereographic coordinates we find :-

$$\mathcal{L}_2 = \frac{2c^2}{(c^2 + \bar{z}z)^2}\,\partial_\mu \bar{z}\,\partial^\mu z$$

Another fine discussion of this topic, including a section on complex manifolds and the extension to a supersymmetric theory, may be found in [20]. Therefore it would be good to add supersymmetry to those models in this thesis which will allow it.

We will now discuss a new series of models which can be investigated using the mathematical structure of this thesis. The series concerns the situation where $SU(N)$ invariance is broken to $SU(N-2) \otimes SU(2) \otimes U(1)$; these models being associated with the $\frac{SU(N)}{SU(N-2)\otimes SU(2)\otimes U(1)}$ cosets. If we write $N = 2k, (2k+1)$ then we find that the coset vector, for any particular model in the series, will be a linear sum of k orthogonal, commuting, r-vectors. We note that the coset representative elements for this series of models is given by equation (5.6). In the same way that the CP2 model (associated with the $\frac{SU(2)}{U(1)}$ coset) may be regarded as the first of the CP2(N-1) series of models (associated with the $\frac{SU(N)}{SU(N-1)\otimes U(1)}$ cosets), we see that the CP4 model (associated with the $\frac{SU(3)}{SU(2)\otimes U(1)}$ coset) is the first of this series. This is simple to see when we consider the characteristic equations of the coset vectors of the series; all of which have trivial odd matrix invariants (i.e. $\gamma_{2n+1}(x) \equiv 0$ for all integer $n \geq 1$). The calculation of the details of this series, although possible, would not be easy; and the desire to do so would come from a mathematical interest only, as there is no particular physical reason for wanting to find results for all of the possible models. So it would be best if we restricted our attention to the first three models in the series. The results for the first two models are already contained in this thesis; in chapters 4 and 5. The third model, which results when $SU(5)$ invariance is broken to $SU(3) \otimes SU(2) \otimes U(1)$, is of interest since $SU(3) \otimes SU(2) \otimes U(1)$ is not only of the same rank as $SU(5)$, but this model will also admit $\mathcal{N} = 1$ supersymmetry; which means that the three coupling constants (of the strong, weak and electromagnetic interactions) in the theory will converge at a high enough energy $\sim 10^{14}$ GeV. In chapter 5 we gave the appropriate expression for the coset representative for this model, see equation (5.4). This model would be interesting to investigate and, superficially, only differs from the $\frac{SU(4)}{SU(2)\otimes SU(2)\otimes U(1)}$ model because the coset representative, given by equation (5.4), now contains the vector $q_{r_\perp}$. This will increase the complexity/length of the calculations but this is no problem; it would just require a little extra thought. The results in the second part of Appendix D will also need to be changed because we would now need to accomodate the effects of having to include $q_{r_\perp}$. For the rôle of $SU(5)$ in this grand unified theory see, for example, the work by Georgi and Glashow [23].

We would also like to see if the results of Chapter 5, in some way, could represent the models where $SO(2, m-2)$ invariance broken to $SO(1, m-3) \otimes SO(1,1)$ with $m = 4, 5, 6$; in the same way that, in Chapter 3, the results for the models where $SO(1, m-1)$ invariance is broken to $SO(1, m-2)$ looked like the results for the models where $SO(m)$ invariance is broken to $SO(m-1)$. If it is possible then it would be useful in the Maldecena conjecture [24] which states that Type IIB string theory on an $AdS_5 \otimes S^5$ background (which has the isometry group $SO(2,4) \otimes SO(6)$) is dual to an $N = 4$ conformally invariant field theory in a Minkowski spacetime. Since neither $SO(2,4)$, nor $SO(6)$, are observed in nature, these symmetries must be broken at low energies; and in Chapter 5 we looked at one possible model where $SO(6)$ invariance is broken to $SO(4) \otimes SO(2)$. If a correspondence between the results of the $\frac{SO(m)}{SO(m-2)\otimes SO(2)}$ coset models and the $\frac{SO(2,m-2)}{SO(1,m-3)\otimes SO(1,1)}$ coset models is possible then, when $m = 6$, we would have the model where $SO(2,4)$ invariance is broken to $SO(1,3) \otimes SO(1,1)$ too.

Lastly, in this thesis we have only considered invariance under *global* transformations of G broken to global transformations of H. However, it is possible to consider the breaking of *local* gauge transformations too, and this topic is also briefly discussed in [10]. A set of gauge fields χ^a_μ and σ^E_μ are then introduced which are associated with the coset and subgroup generators respectively. The gauge field transformation law is taken to be :-

$$\chi'_\mu + \sigma'_\mu = g(\chi_\mu + \sigma_\mu)g^{-1} - f^{-1}(\partial_\mu g)g^{-1}$$

where f is a constant which gives the strength of the universal coupling of the gauge fields to all other fields. We would now find that :-

$$L^{-1}(\partial_\mu + f(\chi_\mu + \sigma_\mu))L = \frac{-i}{2}(a_\mu + v_\mu)$$

and the covariant derivatives are given by :-

$$\begin{aligned} \mathcal{D}_\mu M^a &= a^a_\mu \\ \mathcal{D}_\mu \psi &= (\partial_\mu - \frac{i}{2}v_\mu)\psi \end{aligned}$$

So the effective Lagrangian density, for this new locally invariant theory, will be :-

$$\mathcal{L}_{local} = \frac{1}{2}(a_\mu, a^\mu) + \bar{\psi}(i\not{\mathcal{D}} - m)\psi$$

just like the Lagrangian densities we have found in this thesis.

Appendix A

Cartan subspaces in $\Re^8$ and $\Re^{15}$.

A 1 The Gell-Mann λ-matrix basis of $\Re^8$.

The group $SU(3)$, in the defining representation, is generated by 8 generators, $T_I = \frac{1}{2}\lambda_A$. In the Gell-Mann basis the λ's are :-

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

They have the property :-

$$\begin{aligned} (\lambda_I, \lambda_J) \equiv \frac{1}{2} tr\, \lambda_I \lambda_J &= \delta_{IJ} \\ \gamma_2(\lambda_I) \equiv \frac{1}{2} tr\, \lambda_I^2 &= 1 \qquad\qquad \forall\ I = 1, 2, \ldots, 8. \end{aligned}$$

Therefore they are orthogonal and normalized, or orthonormal, vectors; which means they form a basis; they are the basis vectors of $\Re^8$.

We see that $[\lambda_i, \lambda_8] \equiv 2i\, f_{i8K}\lambda_K = 0$ for all $i = 1, 2, 3$ and $K = 1, 2, \ldots, 8$. Therefore the three λ_i may be used to generate an isospin $SU(2)$ subgroup (of $SU(3)$) with a Lie algebra :-

$$\begin{aligned}
[\lambda_i, \lambda_j] &\equiv 2i\, f_{ijk}\lambda_k \\
&= 2i\, \varepsilon_{ijk}\lambda_k \qquad\qquad \forall\ i, j, k = 1, 2, 3.
\end{aligned}$$

and this isospin subgroup will commute with a 'hypercharge' $U(1)$ subgroup generated using λ_8 alone. Together they form the maximal subgroup $SU(2) \otimes U(1)$. The λ_i form the familiar basis for the $\Re^3$ subspace of $\Re^8$. The eigenvalues of the λ_i are the same, so they are similar matrices, i.e. they are related to eachother by a rotation. For the moment we will just focus on the diagonal matrices.

A 1.1 The Cartan Subspace basis of $\Re^8$.

For the diagonal λ-matrices we see :-

$$\begin{aligned}
[\lambda_3, \lambda_8] &\equiv 0 \\
(\lambda_3, \lambda_8) &\equiv \frac{1}{2}\, tr\, \lambda_3\lambda_8 \\
&= 0 \qquad\qquad \text{because } \lambda_3 \text{ and } \lambda_8 \text{ are part of the } \Re^8 \textit{ basis.}
\end{aligned}$$

So we say that λ_3 and λ_8 form the basis for a *commuting* space called the Cartan subspace, which in this case is diagonal, so we call it $\mathcal{C}_D$. In general the (diagonal) Cartan subspace has a dimension equal to the rank of the group; and $SU(N)$ is a group of rank $(N-1)$.

Now, the diagonal *generators* of $SU(3)$, namely T_3 and T_8, may be used to construct three weight vectors [1] :-

$$\begin{aligned} w_1 &= \left(\frac{1}{2}\,,\ \frac{1}{2\sqrt{3}}\right) \Rightarrow \frac{1}{2}\lambda_3 + \frac{1}{2\sqrt{3}}\lambda_8 \\ w_2 &= \left(-\frac{1}{2}\,,\ \frac{1}{2\sqrt{3}}\right) \Rightarrow -\frac{1}{2}\lambda_3 + \frac{1}{2\sqrt{3}}\lambda_8 \\ w_3 &= \left(0\,,\ -\frac{1}{\sqrt{3}}\right) \Rightarrow -\frac{1}{\sqrt{3}}\lambda_8 \end{aligned}$$

For $SU(N)$ there will be N of these. The r vectors are the *root* vectors, constructed by taking the differences between these weight vectors :-

$$\begin{aligned} r_1 &= w_2 - w_3 &= -\frac{1}{2}\lambda_3 + \frac{\sqrt{3}}{2}\lambda_8 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ r_2 &= w_3 - w_1 &= -\frac{1}{2}\lambda_3 - \frac{\sqrt{3}}{2}\lambda_8 &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ r_3 &= w_1 - w_2 &= \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

and, in general for $\Re^{N^2-1}$ there are $\frac{1}{2}N(N-1)$ of them. The r vectors have the characteristic equation $r^N - r^{N-2} \equiv 0$ which reduces to $r^3 - r = 0 \ \ \forall\, N$. This reduced equation is not the characteristic eigenvalue equation, but it still describes the *behaviour* of the matrix if, for example, we wanted to use it to generate a group element.

To construct vectors which commute with these r-vectors we use the symmetric vector product relation :-

$$r \vee r \ \equiv\ \tfrac{1}{\sqrt{N}}\left(Nr^2 - 2\,\mathbf{1}_{[N]}\right) \ =\ \sqrt{N-2}\,q_r$$

Thus when $N = 3$ we have :-

$$q_r \ =\ \frac{1}{\sqrt{3}}\left(3r^2 - 2\,\mathbf{1}_{[3]}\right) \ =\ \sqrt{3}\,r^I r^J d_{IJK}\lambda_K$$

[1]The components of the weight vectors are the simultaneous eigenvalues of the generators.

Using the second of the relations we find for r_3 :-

$$q_3 = \sqrt{3}\, d_{33K}\, \lambda_K \quad \text{for K}=1,2,\ldots,8.$$

For $SU(2)$ we know that $d_{ijk} \equiv 0 \;\; \forall \;\; i,j,k = 1,2,3$. Therefore we must have :-

$$q_3 = \sqrt{3}\, d_{338}\, \lambda_8$$

We know that $d_{338} = \frac{1}{\sqrt{3}}$ because $(q,q) \equiv 1$ and $(\lambda_8, \lambda_8) \equiv 1$. Thus we have :-

$$q_3 \equiv \lambda_8$$

This was a rather long winded way of finding q_3 since we could have just used the explicit form of r_3 in $r \vee r = q_r$:-

$$\begin{aligned} r_3 \vee r_3 &= \frac{1}{\sqrt{3}} \left(3\, r_3^2 - 2\, \mathbf{1}_{[3]}\right) \\ &= \frac{1}{\sqrt{3}} \left[\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \right] \\ &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{aligned}$$

but our first method was a little more rigorous.

We will now find the q_r-vectors associated with r_1 and r_2. We have a choice of methods to follow. Firstly, we may use the explicit forms of the vectors as we have just done for r_3 above; this is the simplest method. Secondly, we may rewrite them in terms of the unit vectors r_3 and q_3 :-

$$\begin{aligned} r_1 &= -\frac{1}{2} r_3 + \frac{\sqrt{3}}{2} q_3 \\ r_2 &= -\frac{1}{2} r_3 - \frac{\sqrt{3}}{2} q_3 \end{aligned}$$

and then when we calculate $(r \vee r)$ we just need the $SU(3)$ relations $r_3 \vee r_3 = q_3$, $r_3 \vee q_3 = r_3$ and $q_3 \vee q_3 = -q_3$. So to find q_1 we calculate :-

$$
\begin{aligned}
q_1 &= r_1 \vee r_1 \\
&= \left(-\frac{1}{2}r_3 + \frac{\sqrt{3}}{2}q_3\right) \vee \left(-\frac{1}{2}r_3 + \frac{\sqrt{3}}{2}q_3\right) \\
&= \frac{1}{4}r_3 \vee r_3 - \frac{\sqrt{3}}{2}r_3 \vee q_3 + \frac{3}{4}q_3 \vee q_3 \\
&= \frac{1}{4}q_3 - \frac{\sqrt{3}}{2}r_3 - \frac{3}{4}q_3 \\
&= -\frac{\sqrt{3}}{2}r_3 - \frac{1}{2}q_3 \\
&= \frac{1}{\sqrt{3}}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

and, using the same method, we also find q_2 to be :-

$$
q_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

So it is now possible to put all these diagonal vectors together, and draw the Cartan subspace which, for $\Re^8$, is a commuting plane :-

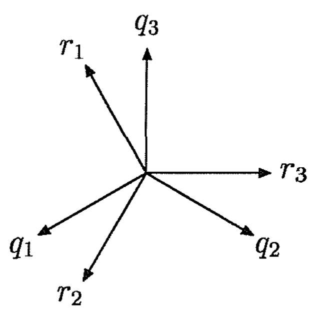

Figure A.1: The 2-dimensional Cartan subspace of $\Re^8$.

We see that we have three choices for the basis of C_D. All choices are equally good, since they are all related by a rotation :-

$$\begin{aligned} r \mapsto r' &= u\, r\, u^{\dagger} \\ q_r \mapsto q_r' &= u\, q_r\, u^{\dagger} \end{aligned}$$

If, for example, we have for u :-

$$u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

then the basis of $\Re^8$ is transformed :-

$$\begin{aligned} \lambda_1 \mapsto \lambda_1' &= \lambda_4 & \qquad \lambda_5 \mapsto \lambda_5' &= \lambda_7 \\ \lambda_2 \mapsto \lambda_2' &= \lambda_5 & \qquad \lambda_6 \mapsto \lambda_6' &= \lambda_1 \\ \lambda_3 \mapsto \lambda_3' &= r_2 & \qquad \lambda_7 \mapsto \lambda_7' &= \lambda_2 \\ \lambda_4 \mapsto \lambda_4' &= \lambda_6 & \qquad \lambda_8 \mapsto \lambda_8' &= q_2 \end{aligned}$$

and we are now using r_2 and q_2 for the basis. We also find :-

$$\begin{aligned} r_1 \mapsto r_1' &= r_3 \\ q_1 \mapsto q_1' &= q_3 \\ r_2 \mapsto r_2' &= r_1 \\ q_2 \mapsto q_2' &= q_1 \end{aligned}$$

but, as before, these last four quantities are not linearly independent, and may be rewritten, this time, in terms of the new basis vectors r_2 and q_2.

So finally, we summarize by saying that if we want a basis for $\Re^8$, then we must pick eight matrices from :-

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad r_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad q_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad r_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad q_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with the property that not only should they all be orthonormal, but two should also *commute.* Therefore we are free to pick all the matrices from the first two columns, and any of the three $r - q_r$ pairs in the last two columns as a basis for the Cartan Subspace, $\mathcal{C}_D$. Note :-

1. If we choose r_3 and q_3 as the diagonal Cartan subspace basis then the isospin $SU(2)$ subgroup is generated by λ_1, λ_2 and r_3, and the commuting $U(1)$ hypercharge subgroup is generated by q_3, or

2. if we choose r_2 and q_2 as the diagonal Cartan subspace basis then the isospin $SU(2)$ subgroup is generated by λ_4, λ_5 and r_2, and the commuting $U(1)$ hypercharge subgroup is generated by q_2, or

3. if we choose r_1 and q_1 as the diagonal Cartan subspace basis then the isospin $SU(2)$ subgroup is generated by λ_6, λ_7 and r_1, and the commuting $U(1)$ hypercharge subgroup is generated by q_1.

A 1.2 Rewriting a general group vector of $\Re^8$.

Since a general vector, x, may be diagonalized to :-

$$x_D = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$$

where $C = -(A+B)$ for a traceless matrix, we may rewrite this :-

$$\begin{aligned} x_D &= -Ar_2 + Br_1 \\ &= -A\left(-\tfrac{1}{2}r_3 - \tfrac{\sqrt{3}}{2}q_3\right) + B\left(-\tfrac{1}{2}r_3 + \tfrac{\sqrt{3}}{2}q_3\right) \\ &= a\,r_3 + b\,q_3 \end{aligned}$$

with $a = \frac{A-B}{2}$ and $b = \frac{\sqrt{3}(A+B)}{2}$. Therefore, we *must* find that our original general vector is :-

$$x = a\,r + b\,q_r$$

where the r and q_r are the rotated versions of r_3 and q_3. We see that $\gamma_2(x) \equiv a^2 + b^2$ because the r and q_r are orthonormal. If we now wish to know what r and q_r actually are in terms of our original vector x and $x \vee x$, then we simply need to form :-

$$x \vee x = 2ab\,r + \left(a^2 - b^2\right) q_r$$

and then we may now solve these last two equations to find :-

$$\begin{aligned} r &= \frac{1}{a\left(a^2 - 3b^2\right)}\left(\left[a^2 - b^2\right] x - b\,x \vee x\right) \\ q_r &= \frac{1}{a^2 - 3b^2}\left(-2b\,x + x \vee x\right) \end{aligned}$$

with $x = x^I \lambda_I$ and $x \vee x = \sqrt{3}\,x^I x^J d_{IJK} \lambda_K$, and the two quantities a and b are related to the eigenvalues of x by construction.

A 1.3 The form of $SU(3)$ group elements.

It is now simple to write general group elements for $SU(3)$. We have :-

$$g \;=\; e^{-ix} \;=\; e^{-ia\,r}e^{-ib\,q_r}$$

where for $N\;=\;3$ we have :-

$$\begin{aligned}
e^{-ia\,r} &= \tfrac{2}{3}\left(\cos a+1\right)\,\mathbf{1}_{[3]}+\tfrac{1}{\sqrt{3}}\left(\cos a-1\right)\,q_r-i\sin a\;r\\
e^{-ib\,q_r} &= \tfrac{1}{3}\left(2e^{-i\frac{b}{\sqrt{3}}}+e^{i\frac{2b}{\sqrt{3}}}\right)\,\mathbf{1}_{[3]}+\tfrac{1}{\sqrt{3}}\left(e^{-i\frac{b}{\sqrt{3}}}-e^{i\frac{2b}{\sqrt{3}}}\right)\,q_r
\end{aligned}$$

A 2 The λ-matrix basis of $\Re^{15}$.

The group $SU(4)$, in the defining representation, is generated by 15 generators, $T_I = \frac{1}{2}\lambda_I$. The λ's are :-

$$\lambda_1=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\quad
\lambda_2=\begin{pmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\quad
\lambda_3=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\quad
\lambda_8=\tfrac{1}{\sqrt{3}}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-2&0\\0&0&0&0\end{pmatrix}\quad
\lambda_{15}=\tfrac{1}{\sqrt{6}}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-3\end{pmatrix}$$

$$\lambda_4=\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}\quad
\lambda_5=\begin{pmatrix}0&0&-i&0\\0&0&0&0\\i&0&0&0\\0&0&0&0\end{pmatrix}$$

$$\lambda_6=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}\quad
\lambda_7=\begin{pmatrix}0&0&0&0\\0&0&-i&0\\0&i&0&0\\0&0&0&0\end{pmatrix}$$

$$\lambda_9=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}\quad
\lambda_{10}=\begin{pmatrix}0&0&0&-i\\0&0&0&0\\0&0&0&0\\i&0&0&0\end{pmatrix}$$

$$\lambda_{11}=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0\end{pmatrix}\quad
\lambda_{12}=\begin{pmatrix}0&0&0&0\\0&0&0&-i\\0&0&0&0\\0&i&0&0\end{pmatrix}$$

$$\lambda_{13}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\quad
\lambda_{14}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-i\\0&0&i&0\end{pmatrix}$$

This time we have the property :-

$$\begin{aligned}
(\lambda_I, \lambda_J) \equiv \frac{1}{2} tr\, \lambda_I \lambda_J &= \delta_{IJ} \\
\gamma_2(\lambda_I) \equiv \frac{1}{2} tr\, \lambda_I^2 &= 1 \qquad\qquad\qquad \forall\ I = 1, 2, \ldots, 15.
\end{aligned}$$

Therefore they are orthogonal and normalized, or orthonormal, vectors; which means they form a basis; they are the basis vectors of $\Re^{15}$.

We see that $[\lambda_i, \lambda_{15}] \equiv 2i\, f_{i\,15\,K} \lambda_K = 0$ for all $i = 1, 2, \ldots, 8$ and $K = 1, 2, \ldots, 15$. Therefore the eight λ_i may be used to generate an $SU(3)$ subgroup (of $SU(4)$) with a Lie algebra :-

$$[\lambda_i, \lambda_j] \equiv 2i\, f_{ijk} \lambda_k \qquad\qquad \forall\ i, j, k = 1, 2, \ldots, 8.$$

and this $SU(3)$ subgroup will commute with a $U(1)$ subgroup generated using λ_{15} alone. Together they form the maximal subgroup $SU(3) \otimes U(1)$.

A 2.1 The Cartan Subspace basis of $\Re^{15}$.

The diagonal commuting λ's are λ_3, λ_8 and λ_{15}; which is one possible choice for the basis of the diagonal Cartan subspace. However we will continue as before and construct the weight vectors :-

$$\begin{aligned}
w_1 &= \frac{1}{2}\lambda_3 + \frac{1}{2\sqrt{3}}\lambda_8 + \frac{1}{2\sqrt{6}}\lambda_{15} \\
w_2 &= -\frac{1}{2}\lambda_3 + \frac{1}{2\sqrt{3}}\lambda_8 + \frac{1}{2\sqrt{6}}\lambda_{15} \\
w_3 &= -\frac{1}{\sqrt{3}}\lambda_8 + \frac{1}{2\sqrt{6}}\lambda_{15} \\
w_4 &= -\frac{\sqrt{3}}{2\sqrt{2}}\lambda_{15}
\end{aligned}$$

This time we can construct six r-vectors. The first three are :-

$$w_2 - w_3 = -\frac{1}{2}\lambda_3 + \frac{\sqrt{3}}{2}\lambda_8 = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & -1 & \\ & & & 0 \end{pmatrix} = r_1$$

$$w_3 - w_1 = -\frac{1}{2}\lambda_3 - \frac{\sqrt{3}}{2}\lambda_8 = \begin{pmatrix} -1 & & & \\ & 0 & & \\ & & 1 & \\ & & & 0 \end{pmatrix} = r_2$$

$$w_1 - w_2 = \lambda_3 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} = r_3$$

which are an extension of the three r-vectors of $\Re^8$, and :-

$$w_1 - w_4 = \frac{1}{2}\lambda_3 + \frac{1}{2\sqrt{3}}\lambda_8 + \sqrt{\frac{2}{3}}\lambda_{15} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix} = r_1^{\perp}$$

$$w_2 - w_4 = -\frac{1}{2}\lambda_3 + \frac{1}{2\sqrt{3}}\lambda_8 + \sqrt{\frac{2}{3}}\lambda_{15} = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & -1 \end{pmatrix} = r_2^{\perp}$$

$$w_3 - w_4 = -\frac{1}{\sqrt{3}}\lambda_8 + \sqrt{\frac{2}{3}}\lambda_{15} = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} = r_3^{\perp}$$

We have chosen to use the notation of the last three r-vectors for a simple reason, which we will now show by giving an example :-

- The notation $r_3^{\perp}$ immediately tells us that this vector is perpendicular to r_3, i.e. $\left(r_3, r_3^{\perp}\right) \equiv 0$. It is not only orthonormal and commuting, but it also has the property such that $r_3 r_3^{\perp} = 0$ (and obviously $r_3^{\perp} r_3 = 0$) too. If we had not used this notation then we should have called this vector r_6, for example, and on the face of it we would have six vectors, $r_1 \ldots r_6$. We would therefore have to remember more relationships between them. The notation therefore helps in understanding the structure, and geometry, of the spaces containing these vectors.

We may now construct the q_r-vectors associated with these r-vectors. To do this we use the $SU(4)$ relation $r \vee r = 2r^2 - \mathbf{1}_{[4]} = \sqrt{2}\, q_r$, and do this for r_3 first :-

$$\begin{aligned} r_3^2 &= \tfrac{1}{2}\mathbf{1}_{[4]} + d_{338}\,\lambda_8 + d_{3\,3\,15}\,\lambda_{15} \\ &= \tfrac{1}{2}\mathbf{1}_{[4]} + \tfrac{1}{\sqrt{3}}\lambda_8 + d_{3\,3\,15}\,\lambda_{15} \end{aligned}$$

and if we rearrange this we find :-

$$\frac{1}{6}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -3 \end{pmatrix} = d_{3\,3\,15}\lambda_{15}$$

and so $d_{3\,3\,15} = \frac{1}{\sqrt{6}}$. Therefore :-

$$\begin{aligned} q_3 &= \sqrt{2}\left(\tfrac{1}{\sqrt{3}}\lambda_8 + \tfrac{1}{\sqrt{6}}\lambda_{15}\right) \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \end{aligned}$$

Notice that this is a long winded method because we found $d_{3\,3\,15}$ first. We could have been much quicker and just used the $r \vee r$ expression with the explicit form of the r vector :-

$$\begin{aligned} q_3 &= \frac{1}{\sqrt{2}}\left(2r_3^2 - \mathbf{1}_{[4]}\right) \\ &= \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 2 & & & \\ & 2 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} - \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\right) \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \end{aligned}$$

For the other q_r vectors we find :-

$$q_1 = \frac{1}{\sqrt{2}}(r_1 \vee r_1) = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

$$q_2 = \frac{1}{\sqrt{2}}(r_2 \vee r_2) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

and we find that $r_\perp \vee r_\perp = \sqrt{2}\, q_{r_\perp} = -\sqrt{2}\, q_r$. Therefore we have the relations :-

$$\begin{aligned}
(r, r_\perp) &= 0 && \text{because of the definition of } r_\perp. \\
(r, q_r) &= 0 && \text{by definition; because } \gamma_3\,(r) \equiv 0. \\
(r_\perp, q_r) &= 0 && \text{because } q_{r_\perp} \equiv -q_r. \\
(q_i, q_j) &= \delta_{ij} && \text{for i,j}=1,2,3.
\end{aligned}$$

and we are free to choose as our basis for $\mathcal{C}_D$:-

1. either the set of three q_r-vectors $\{q_1, q_2, q_3\}$, or

2. the set of vectors $\{r, r_\perp, q_r\}$. There are three of these basis sets to choose from; $\{r_1, r_1^\perp, q_1\}$, $\{r_2, r_2^\perp, q_2\}$ and $\{r_3, r_3^\perp, q_3\}$. However these three are all similar choices, related by a rotation. This is the same idea as we found for the Cartan subspace of $SU(3)$; but instead of having a 2-dimensional Cartan plane in $\Re^8$, we now we have a 3-dimensional Cartan subspace in $\Re^{15}$. We prefer to use the basis $\{r, r_\perp, q_r\}$ in calculations because these are always orthonormal commuting vectors for all N; whereas the three q-vectors are only orthonormal for $N = 4$.

Therefore the 3-dimensional Cartan subspace of $\Re^{15}$ may be pictured :-

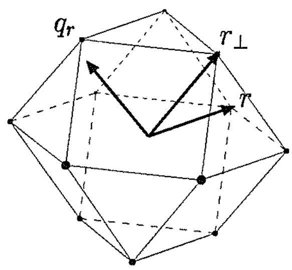

Figure A.2: The 3-dimensional Cartan subspace of $\Re^{15}$.

We note that the r_i and $r_i^\perp$ (together with $-r_i$ and $-r_i^\perp$) point to the 12 vertices, and the q_i together with the $-q_i$ point out through the 6 square faces. (The edges have been drawn in to help visualize the structure of the space).

Finally, we emphasize that when $N \geq 4$ we may *always* define the set of three vectors $\{r, r_\perp, q_r\}$ which are commuting *and* orthonormal. Thus, when $N = 4$ these three form the basis for the whole of the Cartan subspace, and when $N > 4$ they are three, of a possible $(N-1)$, orthonormal directions in the Cartan subspace.

A 2.2 Rewriting a general group vector of $\Re^{15}$.

Since a general vector, x, may be diagonalized to :-

$$x_D = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & D \end{pmatrix}$$

where $D = -(A+B+C)$ for a traceless matrix, we may rewrite this :-

$$\begin{aligned} x_D &= A r_{1\perp} + B r_{2\perp} + C r_{3\perp} \\ &\vdots \\ &= a\, r_3 + b\, r_{3\perp} + c\, q_3 \end{aligned}$$

where $a = \frac{A-B}{2}$, $b = \frac{A+B-2C}{2}$ and $c = \frac{A+B}{\sqrt{2}}$.

Therefore, we *must* find that our original general vector is :-

$$x = a\, r + b\, r_\perp + c\, q_r$$

where the r, $r_\perp$ and q_r are the rotated versions of r_3, $r_{3\perp}$ and q_3. We see that $\gamma_2(x) \equiv a^2 + b^2 + c^2$ because the r, $r_\perp$ and q_r are orthonormal. To find the values of a, b and c for a specific x we need to solve the eigenvalue equation of x.

A 2.3 The form of $SU(4)$ group elements.

It is now simple to write general group elements for $SU(4)$. We have :-

$$
\begin{aligned}
g &= e^{-ix} \\
&= e^{-i(a\,r + b r_\perp + c\,q_r)} \\
&= e^{-ia\,r} e^{-ib\,r_\perp} e^{-ic\,q_r}
\end{aligned}
$$

where :-

$$
\begin{aligned}
e^{-ia\,r} &= \tfrac{1}{2}(\cos a + 1)\ \mathbf{1}_{[4]} + \tfrac{1}{\sqrt{2}}(\cos a - 1)\, q_r - i \sin a\ r \\
e^{-ib\,r_\perp} &= \tfrac{1}{2}(\cos b + 1)\ \mathbf{1}_{[4]} - \tfrac{1}{\sqrt{2}}(\cos b - 1)\, q_r - i \sin b\ r_\perp \\
e^{-ic\,q_r} &= \cos\tfrac{c}{\sqrt{2}}\ \mathbf{1}_{[4]} - i\ \sin\tfrac{c}{\sqrt{2}}\ \sqrt{2}\, q_r
\end{aligned}
$$

To find the specific forms of $r, r_\perp$ and q_r we would need to construct $x \vee x$ and $x \vee x \vee x$ and then solve three simultaneous matrix equations. We will not do this here because it will take too long; besides, we don't explicitly use $SU(4)$ group elements in this thesis. (We only did this explicitly in the $SU(3)$ section because we just had two equations to solve, and this was easy to do.)

Appendix B

Adjoint representation operators.

B 1 Defining the linear operators f_x and d_x.

Both f_x and d_x are $SU(N)$ adjoint representation linear operators which transform the $(N^2 - 1)$-dimensional real vector spaces denoted $\Re^{N^2-1}$. We will now define them. Firstly, $f_x \equiv x\wedge$ is a linear operator which acts on a vector y :-

$$\begin{aligned} f_x : \; y \mapsto y' &= f_x y \\ &= x \wedge y \\ &= x^I y^J f_{IJK} \lambda_K \end{aligned}$$

We may write f_x as an adjoint representation operator if we take the Euclidean Scalar Product of the transformed vector y' with the basis vectors. This action takes us from the defining representation into the adjoint representation. So we find :-

$$\begin{aligned} (y', \lambda_K) &= ((x \wedge y), \lambda_K) \\ \left(y^K\right)' &= x^I y^J f_{IJK} \\ &= (f_x)_{KJ}\, y^J \end{aligned}$$

Therefore we have :-

$$(f_x)_{IJ} \equiv x^K f_{IKJ} \tag{B.1}$$

The second operator, $d_x \equiv x\vee$ is a linear operator which acts on a vector y :-

$$\begin{aligned} d_x : y \mapsto y' &= d_x y \\ &= x \vee y \\ &= \sqrt{N}\, x^I y^J d_{IJK} \lambda_K \end{aligned}$$

In the same way as for f_x, we may write d_x as an adjoint operator using the Euclidean Scalar Product :-

$$\begin{aligned} (y', \lambda_K) &= ((x \vee y), \lambda_K) \\ \left(y^K\right)' &= \sqrt{N}\, x^I y^J d_{IJK} \\ &= \sqrt{N}\, x^I d_{IJK} y^J \\ &= \sqrt{N}\, x^I d_{KIJ} y^J \\ &= (d_x)_{KJ}\, y^J \end{aligned}$$

Therefore we have :-

$$(d_x)_{IJ} \equiv \sqrt{N}\, x^K d_{IKJ} \tag{B.2}$$

B 2 Some relationships between the f's and d's.

We now write some relations between the f's and the d's :-

- f_x is a derivation of the Lie algebra, so we have :-

$$f_x (y \wedge z) = (f_x y) \wedge z + y \wedge (f_x z)$$

This is the Jacobi Identity. Written more explicitly we have :-

$$x \wedge (y \wedge z) + y \wedge (z \wedge x) + z \wedge (x \wedge y) \;=\; 0$$

and, if we isolate the linear adjoint operators which act on z, this is :-

$$[f_x, f_y] \;=\; f_{x \wedge y} \tag{B.3}$$

- f_x is also a derivation of the Symmetric algebra, so we have :-

$$f_x\,(y \vee z) \;=\; (f_x y) \vee z + y \vee (f_x z)$$

When rearranged we find the linear operator commutator :-

$$[f_x, d_y] \;=\; d_{x \wedge y} \tag{B.4}$$

- We also find :-

$$\begin{aligned} f_x(y \vee z) + f_y(z \vee x) + f_z(x \vee y) &= (f_x y) \vee z + y \vee (f_x z) + (f_y z) \vee x \\ &\quad + z \vee (f_y x) + (f_z x) \vee y + x \vee (f_z y) \\ &\equiv 0 \end{aligned}$$

so isolating the linear operators yields :-

$$f_x d_y + f_y d_x \;=\; f_{x \vee y} \tag{B.5}$$

If we transpose this relation then we find :-

$$d_x f_y + d_y f_x \;=\; f_{x \vee y} \tag{B.6}$$

Notice that when $y = x$ we find :-

$$[f_x, d_x] \;=\; 0 \tag{B.7}$$

$$\{f_x, d_x\} \;=\; 2 f_x d_x \;=\; f_{x \vee x} \tag{B.8}$$

- The associator of the $\vee$ algebra is :-

$$x \vee (y \vee z) - (x \vee y) \vee z \;=\; N\, y \wedge (x \wedge z) - 2x\,(y,z) + 2z\,(x,y)$$

or, making the operators more obvious we have :-

$$(d_x d_y - d_{x\vee y}) \cdot z \;=\; (N\, f_y f_x + 2\,(x,y)) \cdot z - 2\,(y,z)\,x$$

If we define the operator $x \!>\!\!<\! y$ by its action $(x \!>\!\!<\! y) \cdot z = x\,(y,z)$ then the last equation is now :-

$$d_x d_y - d_{x\vee y} \;=\; N\, f_y f_x + 2\,(x,y)\; \mathbf{1}_{[N^2-1]} - 2\, x \!>\!\!<\! y \tag{B.9}$$

and its transpose is :-

$$d_y d_x - d_{x\vee y} \;=\; N\, f_x f_y + 2\,(y,x)\; \mathbf{1}_{[N^2-1]} - 2\, y \!>\!\!<\! x \tag{B.10}$$

and we see that when $y = x$ they both reduce to :-

$$d_x^2 - N\, f_x^2 \;=\; d_{x\vee x} + 2\,(x,x)\left(\mathbf{1}_{[N^2-1]} - \mathcal{P}_x\right) \tag{B.11}$$

where $\mathcal{P}_x \equiv \gamma_2(x)^{-1} x \!>\!\!<\! x$. We will meet this projection operator, for *unit* vectors, later; where we will retain the notation $x \!>\!\!<\! x$. For reference we may form two new relations :-

$$[d_x, d_y] + N\,[f_x, f_y] \;=\; -2\,(x \!>\!\!<\! y - y \!>\!\!<\! x) \tag{B.12}$$

$$\begin{aligned} \{d_x, d_y\} - 2d_{x\vee y} \;=&\; N\,\{f_x, f_y\} + 4\,(x,y)\; \mathbf{1}_{[N^2-1]} \\ &-2\,(x \!>\!\!<\! y + y \!>\!\!<\! x) \end{aligned} \tag{B.13}$$

B 3 Adjoint representation Projection Operators.

B 3.1 The adjoint representation of $SU(2)$.

It is simplest, and also very informative, to start with the group $SU(2)$. In the defining representation an r-vector is given by :-

$$r \;=\; n^k\sigma_k \;=\; P^1 - P^2$$

where we have :-

$$\begin{aligned} P^1 &\equiv \frac{1}{2}\left(\mathbf{1}_{[2]} + r\right) \\ P^2 &\equiv \frac{1}{2}\left(\mathbf{1}_{[2]} - r\right) \\ P^1 + P^2 &\equiv \mathbf{1}_{[2]} \end{aligned}$$

To form adjoint representation projection operators we note that :-

$$\begin{aligned} (P^1+P^2)\sigma_i(P^1+P^2)\sigma_j &= P^1\sigma_iP^2\sigma_j + P^2\sigma_iP^1\sigma_j + P^1\sigma_iP^1\sigma_j + P^2\sigma_iP^2\sigma_j \\ &= \sigma_i\sigma_j \end{aligned}$$

So if we take half the trace of this expression, i.e. we form (σ_i, σ_j), we find :-

$$\delta_{ij} \;=\; \frac{1}{2}(P^1\sigma_iP^2\sigma_j) + \frac{1}{2}(P^2\sigma_iP^1\sigma_j) + \frac{1}{2}(P^1\sigma_iP^1\sigma_j + P^2\sigma_iP^2\sigma_j) \tag{B.14}$$

The left hand side is now the sum of *three* projection operators, so we define :-

$$\begin{aligned} (\mathcal{P}^{12})_{ij} &= \frac{1}{2}(P^1\sigma_iP^2\sigma_j) \\ (\mathcal{P}^{21})_{ij} &= \frac{1}{2}(P^2\sigma_iP^1\sigma_j) \\ (\mathcal{P}^{3})_{ij} &= \frac{1}{2}(P^1\sigma_iP^1\sigma_j + P^2\sigma_iP^2\sigma_j) \end{aligned}$$

Therefore equation (B.14) is now just :-

$$(\mathbf{1}_{[3]})_{ij} \;=\; (\mathcal{P}^{12})_{ij} + (\mathcal{P}^{21})_{ij} + (\mathcal{P}^{3})_{ij} \tag{B.15}$$

where we may simply calculate the adjoint projection operator matrix components to be :-

$$(\mathcal{P}^{12})_{ij} = \frac{1}{2}(\delta_{ij} - n^i n^j) + \frac{i}{2} n^k \varepsilon_{kij} \tag{B.16}$$

$$(\mathcal{P}^{21})_{ij} = \frac{1}{2}(\delta_{ij} - n^i n^j) - \frac{i}{2} n^k \varepsilon_{kij} \tag{B.17}$$

$$(\mathcal{P}^{3})_{ij} = n^i n^j \tag{B.18}$$

We may now write these in terms of the adjoint operators introduced at the beginning of this Appendix. We will do this in a fairly rigorous way. Firstly we may write :-

$$\begin{aligned}(\mathcal{P}^{12} - \mathcal{P}^{21})_{ij} &= i n^k \varepsilon_{kij} \\ &= -i(f_r)_{ij}\end{aligned}$$

Now, if we square this we get :-

$$\begin{aligned}(\mathcal{P}^{12} + \mathcal{P}^{21})_{ik} &= -(f_r^2)_{ik} \\ &\equiv (\mathbf{1}_{[3]} - r><r)_{ik}\end{aligned}$$

where we have used equation (B.11) for $N = 2$ and $x = r$. So we see that, in terms of the adjoint representation operators, the adjoint representation projection operators may be written :-

$$(\mathcal{P}^{12})_{ij} = \frac{1}{2}(\mathbf{1}_{[3]} - \mathcal{P}^3)_{ij} - \frac{i}{2}(f_r)_{ik} \tag{B.19}$$

$$(\mathcal{P}^{21})_{ij} = \frac{1}{2}(\mathbf{1}_{[3]} - \mathcal{P}^3)_{ij} + \frac{i}{2}(f_r)_{ik} \tag{B.20}$$

$$(\mathcal{P}^{3})_{ij} = (r><r)_{ij} \tag{B.21}$$

where $\mathcal{P}^{12} + \mathcal{P}^{21} + \mathcal{P}^3 = \mathbf{1}_{[3]}$.

B 3.2 Some more adjoint projection operators of $SU(N)$.

We start by restating the fact that the relations between r and q_r-vectors is the same no matter what the specific form the r-vector, or its associated q_r-vector, actually take. Therefore, for the purposes of this section, we will define the r-vector we will be using as the diagonal λ_3 of the Gell-Mann basis of $\Re^8$; the associated q_r-vector being defined by the relation $r \vee r \equiv q_r$, which in this case is λ_8. Now in exactly the same way that $r{>}{<}r$ is an adjoint representation projection operator, we find that $q_r{>}{<}q_r$ is one too. However since we have more than three adjoint projection operators we understand that the quantity $(\mathbf{1}_{[N^2-1]} - r{>}{<}r)$ now needs to be further refined since it must contain $q_r{>}{<}q_r$. We will now explicitly write the adjoint operators of $SU(3)$, with respect to r and q_r vectors. We find :-

$$f_r = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad d_r = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We may now either use our knowledge of the antisymmetric and symmetric structure constants for $SU(3)$, or use the relations :-

$$\begin{aligned} d_r^2 - 3f_r^2 &= d_{q_r} + 2\,\mathbf{1}_{[8]} - 2r{>}{<}r \\ 2f_r d_r &= f_{q_r} \end{aligned}$$

to calculate d_{q_r} and f_{q_r}. We find them to be :-

$$f_{q_r} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad d_{q_r} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Lastly, for this example where $r = \lambda_3$ and $N = 3$, we find :-

$$r\!><\!r \;=\; \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \quad q_r\!><\!q_r \;=\; \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}$$

In practice we will keep the notation of the last section and write $\mathcal{P}^3 \;=\; r\!><\!r$. We also note here that in our calculations where we need to use the language of adjoint projection operators to simplify expressions we do not encounter the projection operator $q_r\!><\!q_r$. This is because the matrix indices we are concerned with in expressions like the Goldstone boson manifold metric are coset indices, and $(q_r\!><\!q_r)_{ab} \equiv 0$. Therefore we will not give it another name.

B 3.2a The projection operator combinations $(\mathcal{P}^{12} + \mathcal{P}^{21})$ and $(\mathcal{P}^{34} + \mathcal{P}^{43})$.

We will now use a similar approach to that found on page 171, where we looked at the adjoint of $SU(2)$, to re-calculate the first two adjoint representation projection operators for general N. As before, if we define :-

$$P^1 - P^2 \;=\; r$$

then this time we have :-

$$\begin{aligned} P^1 + P^2 \;&=\; r^2 \\ &=\; \tfrac{1}{N}(2\,\mathbf{1}_{[N]} + \sqrt{N(N-2)}\,q_r) \end{aligned}$$

where $P^1 = \frac{1}{2}(r^2 + r)$ and $P^2 = \frac{1}{2}(r^2 - r)$. With the adjoint projection operators $\mathcal{P}^{12}$ and $\mathcal{P}^{21}$ defined as before :-

$$\begin{aligned}(\mathcal{P}^{12})_{IJ} &= \tfrac{1}{2}\, tr\, P^1 \lambda_I P^2 \lambda_J \\ (\mathcal{P}^{21})_{IJ} &= \tfrac{1}{2}\, tr\, P^2 \lambda_I P^1 \lambda_J\end{aligned}$$

then we find we have the combinations :-

$$(\mathcal{P}^{12} + \mathcal{P}^{21})_{IJ} = \tfrac{1}{4}\, tr\, (r^2 \lambda_I r^2 \lambda_J - r \lambda_I r \lambda_J) \tag{B.22}$$

$$(\mathcal{P}^{12} - \mathcal{P}^{21})_{IJ} = \tfrac{1}{4}\, tr\, (r \lambda_I r^2 \lambda_J - r^2 \lambda_I r \lambda_J) \tag{B.23}$$

For the first expression we find :-

$$(\mathcal{P}^{12} + \mathcal{P}^{21})_{IJ} = \tfrac{2}{N^2} \delta_{IJ} + \tfrac{2\sqrt{N-2}}{N^2} (d_{q_r})_{IJ} + \tfrac{1}{4}\, tr\, \left[\left(\tfrac{N-2}{N}\right) q_r \lambda_I q_r \lambda_J - r \lambda_I r \lambda_J \right]$$

Now, using the general result :-

$$tr\, (x \lambda_I x \lambda_J) = \tfrac{2}{N} \left(2N\, f_x^2 + d_{x \vee x} + 2\,(x,x)\; \mathbf{1}_{[N^2-1]} \right)_{IJ}$$

and substituting in for $x = q_r$ and also for $x = r$ we eventually find :-

$$\begin{aligned}\mathcal{P}^{12} + \mathcal{P}^{21} &= \tfrac{N-2}{N} f_{q_r}^2 - f_r^2 \\ &= -\tfrac{1}{4} \mathcal{P}_{f_q^2} - f_r^2 \end{aligned} \tag{B.24}$$

This quantity is used in our calculations; and we will soon examine $\mathcal{P}_{f_q^2}$. For now, we may calculate when $N = 3$ and $r = \lambda_3$:-

$$-f_r^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and compare this with equation (B.24). Usually, in our calculations we will use :-

$$-N f_r^2 = N(\mathcal{P}^{12} + \mathcal{P}^{21}) + \tfrac{N}{4}\mathcal{P}_{f_q^2} \tag{B.25}$$

We see that, for $N = 2$, we have :-

$$\begin{aligned} \mathcal{P}^{12} + \mathcal{P}^{21} &= -f_r^2 \\ &\equiv \mathbf{1}_{[3]} - r >< r \end{aligned}$$

which is obviously correct because there are no q_r-vectors and $\mathbf{1}_{[3]} = \mathcal{P}^{12} + \mathcal{P}^{21} + r >< r$. Now, even though we do not meet the quantity $\mathcal{P}^{12} - \mathcal{P}^{21}$ in our calculations, to complete this section we will give the result :-

$$\mathcal{P}^{12} - \mathcal{P}^{21} = -\tfrac{2i}{N}\left(f_r - \tfrac{\sqrt{N-2}}{2}(d_r f_{q_r} - d_{q_r} f_r)\right)$$

and it is possible to show, using this equation, that $(\mathcal{P}^{12} - \mathcal{P}^{21})^2 = \mathcal{P}^{12} + \mathcal{P}^{21}$ as is obviously required.

In chapter 5 we also need an expression for the adjoint representation projection operator combination $(\mathcal{P}^{34} + \mathcal{P}^{43})$. This is because, to describe the coset vector, we not only need to use r but also an orthonormal, commuting r-vector called $r_\perp \equiv P^3 - P^4$. The method used to calculate $(\mathcal{P}^{12} + \mathcal{P}^{21})$ now yields :-

$$\mathcal{P}^{34} + \mathcal{P}^{43} = \tfrac{1}{2} f_{q_r}^2 - f_{r_\perp}^2 \tag{B.26}$$

where we have substituted in $N = 4$ and used the fact that, for $N = 4$, we have the relation $q_{r_\perp} = -q_r$.

B 3.2b The adjoint projection operator $\mathcal{P}_{f_q^2}$.

We know from equation (B.6) that $2f_x d_x = f_{x\vee x}$, and equation (B.7) tells us they commute, so when $x = q_r$ we have :-

$$
\begin{aligned}
2d_{q_r} f_{q_r} &= f_{q_r \vee q_r} \\
&= \tfrac{N-4}{\sqrt{N-2}} f_{q_r}
\end{aligned}
$$

If we now premultiply this by d_{q_r} and use equation (B.11) with $x = q_r$ then we find after a little rearranging that :-

$$-\tfrac{4(N-2)}{N} f_{q_r}^3 = f_{q_r}$$

This relation implies that :-

$$\left(-\tfrac{4(N-2)}{N} f_{q_r}^2\right)^2 = -\tfrac{4(N-2)}{N} f_{q_r}^2$$

Therefore this quantity has projection operator qualities, and so we write :-

$$\mathcal{P}_{f_q^2} \equiv -\tfrac{4(N-2)}{N} f_{q_r}^2 \tag{B.27}$$

For the explicit examples given earlier where $N = 3$ and $r = \lambda_3$ we find :-

$$
\mathcal{P}_{f_q^2} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

Obviously this is the sum of 4 projection operators. However they are difiicult to isolate, so we will not try to.

When $N = 4$, specifically in chapter 5, things are a little different because $\mathcal{P}_{f_q^2}$ may be, more readily, split up into a linear sum of two other projection operators; these we will call $\mathcal{P}^{\oplus}_{f_q^2}$ and $\mathcal{P}^{\ominus}_{f_q^2}$. To introduce this idea we first recall two results from the last section, namely :-

$$\begin{aligned}
\mathcal{P}^{12} + \mathcal{P}^{21} &= -\tfrac{1}{4}\mathcal{P}_{f_q^2} - f_r^2 \\
\mathcal{P}^{34} + \mathcal{P}^{43} &= -\tfrac{1}{4}\mathcal{P}_{f_q^2} - f_{r_\perp}^2
\end{aligned}$$

From these relations we notice :-

$$f_r^2 f_{r_\perp}^2 = \tfrac{1}{16}\mathcal{P}_{f_q^2}$$

and since $[f_r, f_{r_\perp}] = 0$ we have :-

$$(4f_r f_{r_\perp})(4f_r f_{r_\perp}) = \mathcal{P}_{f_q^2}$$

To analyse this we will start by constructing the adjoint matrices f_r and $f_{r_\perp}$ which will, independently, give us the form of $\mathcal{P}_{f_q^2}$. We will then find out what the product $4f_r f_{r_\perp}$ looks like. Using, for simplicity, the r-vector $r = \lambda_3$ we have the relevant structure constants :-

$$\begin{aligned}
f_{1\,2\,3} &= 1 \\
f_{3\,4\,5} = f_{3\,9\,10} &= \frac{1}{2} \\
f_{3\,6\,7} = f_{3\,11\,12} &= -\frac{1}{2}
\end{aligned}$$

and from these it is simple to construct the adjoint matrix operator f_r because :-

$$\begin{aligned}
(f_r)_{IJ} &\equiv f_{I\,3\,J} \\
&= -f_{3\,I\,J}
\end{aligned}$$

Therefore, we find the adjoint operator to be :-

$$
f_r \;=\; \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

Therefore we have :-

$$
-f_r^2 \;=\; \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
= \mathcal{P}^{12} + \mathcal{P}^{21} + \tfrac{1}{4}\mathcal{P}_{f_q^2}
$$

Similarly, for $r_\perp = -\frac{1}{\sqrt{3}}\lambda_8 + \sqrt{\frac{2}{3}}\lambda_{15}$, using $f_{4\,5\,8} = f_{6\,7\,8} = \sqrt{\frac{3}{2}}$, $f_{8\,9\,10} = f_{8\,11\,12} = \frac{1}{2\sqrt{3}}$,

$f_{8\,13\,14} = -\frac{1}{\sqrt{3}}$ and $f_{9\,10\,15} = f_{11\,12\,15} = f_{13\,14\,15} = -\sqrt{\frac{2}{3}}$, we find :-

$$f_{r_\perp} \quad = \quad \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

This matrix, when squared, gives the result :-

$$-f_{r_\perp}^2 \quad = \quad \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$= \quad \mathcal{P}^{34} + \mathcal{P}^{43} + \tfrac{1}{4}\mathcal{P}_{f_q^2}$$

Therefore, using the results for $-f_r^2$ and $-f_{r_\perp}^2$ we see that it is easy to isolate the adjoint projection operator $\mathcal{P}_{f_q^2}$ and also the projection operator combinations $(\mathcal{P}^{12}+\mathcal{P}^{21})$ and $(\mathcal{P}^{34}+\mathcal{P}^{43})$.

We may also use f_r and $f_{r_\perp}$ to find the quantity :-

$$
\begin{aligned}
4f_r f_{r_\perp} &= \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}\\
&\equiv \mathcal{P}^{\oplus}_{f_q^2} - \mathcal{P}^{\ominus}_{f_q^2}
\end{aligned}
$$

where $\mathcal{P}^{\oplus}_{f_q^2}$ is made from the four outer (positive) components, and $\mathcal{P}^{\ominus}_{f_q^2}$ from the four inner (negative) ones. We notice that this is an adjoint vector quantity because $tr\,(4f_r f_{r_\perp}) \equiv 0$. Thus, a vector quantity may be described by the difference of two projection operators in the defining representation and also in the adjoint representation too. Also note :-

$$
\begin{aligned}
4f_r f_{r_\perp} &= (f_r + f_{r_\perp})^2 - (f_r - f_{r_\perp})^2\\
&= (\sqrt{2}f_{q2})^2 - (\sqrt{2}f_{q1})^2\\
&= \mathcal{P}_{f_{q1}^2} - \mathcal{P}_{f_{q2}^2}
\end{aligned}
$$

where we may find, using f_r and $f_{r\perp}$, the results :-

$$\begin{aligned}\mathcal{P}_{f_{q1}^2} &= (\mathcal{P}^{12}+\mathcal{P}^{21})+\mathcal{P}^{\oplus}_{f_q^2}+(\mathcal{P}^{34}+\mathcal{P}^{43})\\ \mathcal{P}_{f_{q2}^2} &= (\mathcal{P}^{12}+\mathcal{P}^{21})+\mathcal{P}^{\ominus}_{f_q^2}+(\mathcal{P}^{34}+\mathcal{P}^{43})\end{aligned}$$

B 4 Additional work for Chapter 4.

B 4.1 Checking the $\mathbf{K}^a_b$ result.

If we substitute the result for $\mathbf{K}^a_b$, equation (4.20), into the relation we couldn't directly solve, equation (4.18), then we find, after a little bit of work, that we can get as far as writing :-

$$\begin{aligned}\tfrac{1}{\sqrt{N-2}}\sin^2\phi\ (d_r)_{bE} &= (\cos\phi\ -1)^2 n^b q_r^E - \tfrac{2}{N}(\cos\phi\ -1)^2 (d_r)_{aE}(d_{q_r})_{ab}\\ &\quad -\tfrac{2}{\sqrt{N(N-2)}}(\cos\phi\ -1)(N+2(\cos\phi\ -1))(d_r)_{bE}\end{aligned} \tag{B.28}$$

It is the second term which causes us problems. It may be written either :-

$$\begin{aligned}(d_r)_{aE}(d_{q_r})_{ab} &= (d_r)_{Ea}(d_{q_r})_{ab}\\ &= (d_r d_{q_r})_{Eb}\end{aligned}$$

or it may be written :-

$$\begin{aligned}(d_r)_{aE}(d_{q_r})_{ab} &= (d_{q_r})_{ba}(d_r)_{aE}\\ &= (d_{q_r} d_r)_{bE}\end{aligned}$$

The relations between d_r and d_{q_r} which we may work out do not help us here. We have for example :-

$$\begin{aligned}d_r d_{q_r} - N f_{q_r} f_r &= \sqrt{N-2}\, d_r - 2r >< q_r\\ d_{q_r} d_r - N f_r f_{q_r} &= \sqrt{N-2}\, d_r - 2q_r >< r\end{aligned}$$

and we also know that $[f_r, f_{q_r}] \equiv 0$. But this seems to be as far as we can go; at least at this level of enquiry. However, if we assume that our calculation is correct to this point, then we *must* have the relation :-

$$(d_{q_r} d_r)_{bE} = \tfrac{N-4}{2\sqrt{N-2}} (d_r)_{bE} + \tfrac{N}{2} (r >< q_r)_{bE} \tag{B.29}$$

where $(r >< q_r)_{bE} \Rightarrow n^b q_r^E$, in order that both sides of equation (B.28) balance. In general this relation is also difficult to confirm, but we will now show that it is correct for $N = 3$, i.e. using the adjoint operators for $SU(3)$.

Now in [17] we find that the two equations (B.12) and (B.13) become :-

$$[d_{q_r}, d_r] = -2\,(q_r >< r - r >< q_r) \tag{B.30}$$

$$\begin{aligned} \{d_{q_r}, d_r\} &= \{f_{q_r}, f_r\} \\ &= -d_r + q_r >< r + r >< q_r \end{aligned} \tag{B.31}$$

which we will now verify. We calculate :-

$$d_r d_{q_r} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{4} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad d_{q_r} d_r = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{4} & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So we see that we may write these results :-

$$d_r d_{q_r} = -\frac{1}{2} d_r - \frac{1}{2} r >< q_r + \frac{3}{2} q_r >< r \tag{B.32}$$

$$d_{q_r} d_r = -\frac{1}{2} d_r + \frac{3}{2} r >< q_r - \frac{1}{2} q_r >< r \tag{B.33}$$

Therefore from these we find exactly the results of equations (B.30) and (B.31) given in [17].

In checking the nonlinear Killing vector components, back on page 183, we were forced to assume the relation :-

$$(d_{q_r} d_r)_{bE} = \tfrac{N-4}{2\sqrt{N-2}} (d_r)_{bE} + \tfrac{N}{2} (r><q_r)_{bE}$$

which, for $N = 3$ is :-

$$\begin{aligned} (d_{q_r} d_r)_{bE} &= -\tfrac{1}{2} (d_r)_{bE} + \tfrac{3}{2} (r><q_r)_{bE} \\ &= -\tfrac{1}{2} (d_r)_{bE} + \tfrac{3}{2} n^b q_r^E \end{aligned} \tag{B.34}$$

If we write equation (B.33) in terms of the operator components we have :-

$$(d_{q_r} d_r)_{IJ} = -\frac{1}{2} (d_r)_{IJ} + \frac{3}{2} n^I q_r^J - \frac{1}{2} q_r^I n^J$$

But this is for general r and q_r-vectors which lie in unspecified directions. If we now restrict the index $I \rightsquigarrow a$ (allowing it to run over the r-vector indices) and the index $J \rightsquigarrow E$ (therefore allowing it to run over the q_r-vector indices), then the last term in the above expression vanishes and we are left with equation (B.34). We also note :-

$$(d_{q_r} d_r)_{bE} = (d_r d_{q_r})_{Eb}$$

and had we considered the right hand side's form instead then, by using the same line of reasoning as above, we would have arrived at the same result. Now that both sides of equation (B.28) balance, we can say that the form of $\mathbf{K}^a_b$ is correct; because we have managed to verify the relation which we couldn't directly solve.

B 4.2 Simplification of the CP 2(N-1) metric, g_{ab}, and $(d_{q_r})_{ab}$.

This section applies to part of the method in section 4.2.5 on page 110. We found that we had to calculate $tr(r\partial_\mu q_r \partial^\mu r + r\partial_\mu r \partial^\mu q_r)$. We may write this in two ways :-

$$\begin{aligned} tr(r\partial_\mu q_r \partial^\mu r + r\partial_\mu r \partial^\mu q_r) &= tr(\partial_\mu r^2 \partial^\mu q_r) \\ &= \sqrt{\tfrac{N-2}{N}} tr(\partial_\mu q_r \partial^\mu q_r) \\ &= 2\sqrt{\tfrac{N-2}{N}} \partial_\mu q_r^E \partial^\mu q_r^E \end{aligned} \tag{B.35}$$

which uses the cyclic property of the trace, or we can write :-

$$\begin{aligned} tr(r\partial_\mu q_r \partial^\mu r + r\partial_\mu r \partial^\mu q_r) &= tr(2\partial_\mu(rq)\partial^\mu r - 2q_r\partial_\mu r \partial^\mu r) \\ &= 4\sqrt{\tfrac{N-2}{N}} \partial_\mu n^a \partial^\mu n^a - 2\, tr\, q_r \partial_\mu r \partial^\mu r \end{aligned} \tag{B.36}$$

where, again, we have used the cyclic property of the trace in conjunction with the two relations $r\partial_\mu q_r = \partial_\mu(rq_r) - (\partial_\mu r)\, q_r$ and $(\partial_\mu q_r)\, r = \partial_\mu(q_r r) - q_r \partial_\mu r$. If we now explicitly calculate both equations, and then equate the two, we eventually find the relation :-

$$2\left(d_r^2 - (N-2) r >< r\right)_{ab} = \left((N-2)\, \mathbf{1}_{[N^2-1]} - \sqrt{N-2}\, d_{q_r}\right)_{ab}$$

N.B. we have removed $\partial_\mu M^a \partial^\mu M^b$ from both sides. We rearrange this to give :-

$$(d_{q_r})_{ab} = (N-2)\, \delta_{ab} - 2(d_r^2)_{ab} + 2(N-2)\, n^a n^b \tag{B.37}$$

and if we now use the relation $d_r^2 - N f_r^2 = \sqrt{N-2}\, d_{q_r} + 2(\mathbf{1}_{[N^2-1]} - r >< r)$, then we find that we have :-

$$3\sqrt{N-2}\,(d_{q_r})_{ab} = (N-6)\, \delta_{ab} - 2N(f_r^2)_{ab} + 2N\, n^a n^b$$

For these models the coset indices of the identity may be written :-

$$(\mathbf{1}_{[N^2-1]})_{ab} \Rightarrow \delta_{ab} = (\mathcal{P}^{12} + \mathcal{P}^{21} + r >< r + \mathcal{P}_{f_q^2})_{ab}$$

since these are the only projection operators which may be constructed with non-zero entries for these components. If we also use our relation $\frac{N-2}{N} f_{q_r}^2 - f_r^2 = \mathcal{P}^{12} + \mathcal{P}^{21}$, then we find :-

$$(d_{q_r})_{ab} \;=\; \sqrt{N-2}\left(\mathcal{P}^{12} + \mathcal{P}^{21} + r >< r\right)_{ab} + \tfrac{N-4}{2\sqrt{N-2}} (\mathcal{P}_{f_q^2})_{ab} \tag{B.38}$$

This expression is used in section 4.2.5a to simplify the form of the nonlinear Killing vector components in the CP2(N-1) models. If we now substitute this into equation (B.37) then we find :-

$$(d_r^2 - (N-2) r >< r)_{ab} \;=\; \tfrac{N}{4} (\mathcal{P}_{f_q^2})_{ab} \tag{B.39}$$

which we will use to simplify the form of the Goldstone boson manifold metric; and also the form of a_μ.

B 5 Additional work for Chapter 5.

B 5.1 Simplification of $(\mathbf{1}_{[15]} \pm \frac{1}{\sqrt{2}} d_{q_r})_{ab}$.

The results of this section are used in the nonlinear Killing vector calculation in Chapter 5, on page 135, where we find that we need to simplify the quantity $(\mathbf{1}_{[15]} \pm \frac{1}{\sqrt{2}} d_{q_r})_{ab}$. We will first work out which adjoint representation projection operators are present in d_{q_r}. To do this we consider, when $x = q_r$, the coset index matrix elements of equation (B.11) :-

$$(d_{q_r}^2 - 4 f_{q_r}^2)_{ab} \;\equiv\; 2(\mathbf{1}_{[15]})_{ab}$$

We have this result because, for $N = 4$, we know that $q_r \vee q_r \equiv 0$ and, as usual, we also have the relation $(q_r >< q_r)_{ab} \equiv 0$. Since $\mathcal{P}_{f_q^2} \equiv -2 f_{q_r}^2$ we may write this :-

$$(d_{q_r}^2 + 2\mathcal{P}_{f_q^2})_{ab} \;\equiv\; 2(\mathbf{1}_{[15]})_{ab}$$

Thus we may write $d^2_{q_r}$ in terms of adjoint representation projection operators :-

$$
\begin{aligned}
(d^2_{q_r})_{ab} &= 2(\,\mathbf{1}_{[15]} - \mathcal{P}_{f^2_q})_{ab} \\
&= 2(\mathcal{P}^{12} + \mathcal{P}^{21} + r{>}{<}r + \mathcal{P}^{34} + \mathcal{P}^{43} + r_\perp{>}{<}r_\perp)_{ab} \qquad \text{(B.40)}
\end{aligned}
$$

This implies that $\sqrt{2}\,d_{q_r}$ is of the form :-

$$
\sqrt{2}(d_{q_r})_{ab} = 2\left(\pm(\mathcal{P}^{12})_{ab} \pm (\mathcal{P}^{21})_{ab} \pm (r{>}{<}r)_{ab} \pm (\mathcal{P}^{34})_{ab} \pm (\mathcal{P}^{43})_{ab} \pm (r_\perp{>}{<}r_\perp)_{ab}\right)
$$

Therefore we now need to find the sign in front of each projection operator. To help us do this we will construct the explicit form of d_r^2 by finding the form of d_r. As usual, to keep the analysis as simple as possible, we will use $r = \lambda_3$ and therefore we have $(d_r)_{IJ} = 2d_{3IJ}$.

We find the adjoint matrix d_r to be :-

$$
d_r = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&\frac{2}{\sqrt{3}}&0&0&0&0&0&0&\sqrt{\frac{2}{3}}\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&0&\frac{2}{\sqrt{3}}&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\sqrt{\frac{2}{3}}&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
$$

The coset index matrix elements of equation (B.11), when $x = r$ and $N = 4$, are :-

$$
(d^2_r)_{ab} - 4(f^2_r)_{ab} = \sqrt{2}(d_{q_r})_{ab} + 2(\,\mathbf{1}_{[15]})_{ab} - 2(r{>}{<}r)_{ab}
$$

Simplifying the f_r^2 term, using equation (B.25) with $N = 4$, gives us :-

$$
\begin{aligned}
(d_r^2)_{ab} &= \sqrt{2}(d_{q_r})_{ab} - 2(\mathcal{P}^{12} + \mathcal{P}^{21})_{ab} + (\mathcal{P}_{f_q^2})_{ab} \\
&\quad +2(\mathcal{P}^{34} + \mathcal{P}^{43} + r_\perp >< r_\perp)_{ab}
\end{aligned} \tag{B.41}
$$

Now, calculating d_r^2 will allow us to 'guess' the form of $\sqrt{2}d_{q_r}$ since, when we substitute it into the right hand side, it will be obvious if our guess is correct. The 'guess' will be fairly intuitive if we use our knowledge of the form, and behaviour, of r and $r_\perp$. Specifically, we notice how the d_{q_r} term occurs in each of the two operator relations :-

$$
\begin{aligned}
(d_r^2)_{ab} - 4(f_r^2)_{ab} &= \sqrt{2}(d_{q_r})_{ab} + 2(\mathbf{1}_{[15]})_{ab} - 2(r >< r)_{ab} \\
(d_{r_\perp}^2)_{ab} - 4(f_{r_\perp}^2)_{ab} &= -\sqrt{2}(d_{q_r})_{ab} + 2(\mathbf{1}_{[15]})_{ab} - 2(r_\perp >< r_\perp)_{ab}
\end{aligned}
$$

So firstly, we find d_r^2 to be :-

$$
d_r^2 = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&2&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&\frac{4}{3}&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{2}{3}
\end{pmatrix}
$$

If we guess that $\sqrt{2}\,d_{q_r}$ has the form :-

$$
\sqrt{2}(d_{q_r})_{ab} = 2(\mathcal{P}^{12} + \mathcal{P}^{21} + r >< r + \mathcal{P}^{34} + \mathcal{P}^{43} + r_\perp >< r_\perp)_{ab}
$$

then equation (B.41) becomes :-

$$(d_r^2)_{ab} = 2(r><r)_{ab} + (\mathcal{P}_{f_q^2})_{ab} + 4(\mathcal{P}^{34} + \mathcal{P}^{43} + r_\perp><r_\perp)_{ab}$$

and this is clearly wrong because there should be no contribution from $(\mathcal{P}^{34} + \mathcal{P}^{43})$ in d_r^2. If we now guess that $\sqrt{2}\, d_{q_r}$ is of the form :-

$$\sqrt{2}(d_{q_r})_{ab} = 2(\mathcal{P}^{12} + \mathcal{P}^{21} + r><r)_{ab} - 2(\mathcal{P}^{34} + \mathcal{P}^{43} + r_\perp><r_\perp)_{ab}$$

then equation (B.41) becomes :-

$$(d_r^2)_{ab} = 2(r><r)_{ab} + (\mathcal{P}_{f_q^2})_{ab}$$

which agrees with the form of the matrix d_r^2 above. It is important to realise that we may ignore the components $(d_r^2)_{8\,8}$ and $(d_r^2)_{15\,15}$ because they are provided by the adjoint projection operator $q_r><q_r$. It enters into d_r^2 via the (adjoint) identity element in the operator relation :-

$$d_r^2 - 4f_r^2 \equiv \sqrt{2}\, d_{q_r} + 2(\,\mathbf{1}_{[15]} - r><r)$$

Since $q_r><q_r$ only has the nonzero components $(q_r><q_r)_{EF}$, these components being specified by *subgroup* indices alone, we know that this operator is absent from our results. Basically, this is because we calculate quantities like the linear $\mathbf{K}^a_E$, the nonlinear $\mathbf{K}^a_b$ and the metric g_{ab}; none of which have two subgroup indices. Another way to see it is by realizing that all of the objects are constructed using the coset representative element, which is defined by the coset vector. Therefore, anything constructed using L must contain a coset index and, because of the structure of the $\wedge$ and $\vee$ algebras, we never see an object with a coset index together with two subgroup indices.

To summarize, we have found :-

$$\sqrt{2}(d_{q_r})_{ab} = 2(\mathcal{P}^{12} + \mathcal{P}^{21} + r><r)_{ab} - 2(\mathcal{P}^{34} + \mathcal{P}^{43} + r_\perp><r_\perp)_{ab} \tag{B.42}$$

Therefore, we may simplify $(\mathbf{1}_{[15]} + \frac{1}{\sqrt{2}} d_{q_r})_{ab}$ and $(\mathbf{1}_{[15]} - \frac{1}{\sqrt{2}} d_{q_r})_{ab}$ by writing them in terms of adjoint representation projection operators. We find :-

$$(\mathbf{1}_{[15]} + \tfrac{1}{\sqrt{2}} d_{q_r})_{ab} = 2(\mathcal{P}^{12} + \mathcal{P}^{21} + r >< r)_{ab} + (\mathcal{P}_{f_q^2})_{ab} \tag{B.43}$$

$$(\mathbf{1}_{[15]} + \tfrac{1}{\sqrt{2}} d_{q_r})_{ab} = 2(\mathcal{P}^{34} + \mathcal{P}^{43} + r_\perp >< r_\perp)_{ab} + (\mathcal{P}_{f_q^2})_{ab} \tag{B.44}$$

Appendix C

The Lie Algebras of $SO(m)$ and $SO(t,s)$.

C 1 The rôle of Gamma matrices.

For $SO(m)$, where $m = 2k, 2k+1$, if we can find a set of $2k+1$ Hermitian matrices obeying a *Clifford* algebra :-

$$\{\gamma_A, \gamma_B\} = 2\,\delta_{AB}\,\mathbf{1}_{[2^k]} \tag{C.1}$$

with $A, B = 1, 2, \ldots, m.$, then we may build a set of traceless, Hemitian, generators for $SO(2k+1)$:-

$$\begin{aligned} T_{AB} &\equiv -\frac{i}{4}\,[\gamma_A, \gamma_B] \\ &= \frac{1}{2}\sigma_{AB} \end{aligned} \tag{C.2}$$

which also obey the $SO(2k+1)$ Lie algebra, and are 2^k dimensional; instead of $(2k+1)$ dimensional. For $SO(2k)$ we only use the first $2k$ gamma matrices to construct the σ-matrices; the last gamma matrix may be used to construct two projection operators which project out left and the right handed spinor representation generators. We will explain this further in the next section.

We have as a product rule for the gamma matrices :-

$$\begin{aligned} 2\gamma_A\gamma_B &= 2\delta_{AB}\,\mathbf{1}_{[2^k]} + 2i\sigma_{AB} \\ \gamma_A\gamma_B &= \delta_{AB}\,\mathbf{1}_{[2^k]} + i\sigma_{AB} \end{aligned}$$

so for $A \neq B$ we have :-

$$\sigma_{AB} = -i\gamma_A\gamma_B$$

C 1.1 A set of γ matrices for $SO(2k+1)$ and $SO(2k)$.

The *Spinor* representation of the generators of $SO(3)$ is built from the gamma matrices for $SO(3)$; these are defined to be the Pauli Spin Matrices :-

$$\begin{aligned} \gamma_1^{\mathrm{SO(3)}} &\equiv \sigma_1 \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \gamma_2^{\mathrm{SO(3)}} &\equiv \sigma_2 \\ &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \gamma_3^{\mathrm{SO(3)}} &\equiv \sigma_3 \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

Conventionally, the generator for the normal $SO(2)$ subgroup of $SO(3)$ is built from the first 2 gamma matrices of $SO(3)$. In general, we may form an iterative process which produces gamma matrices, with which we may construct generators, for larger groups of Special Orthogonal transformation matrices.

For $k \geq 2$, the first $2k-1$ gamma matrices for $SO(2k+1)$ are defined :-

$$\gamma_A^{\mathrm{SO(2k+1)}} = \begin{pmatrix} 0 & i\,\gamma_A^{\mathrm{SO(2k-1)}} \\ -i\,\gamma_A^{\mathrm{SO(2k-1)}} & 0 \end{pmatrix}$$

and the last two are defined :-

$$\begin{aligned} \gamma_{2k}^{\mathrm{SO(2k+1)}} &= \begin{pmatrix} 0 & \mathbf{1}_{[2^{k-1}]} \\ \mathbf{1}_{[2^{k-1}]} & 0 \end{pmatrix} \\ \gamma_{2k+1}^{\mathrm{SO(2k+1)}} &= \begin{pmatrix} \mathbf{1}_{[2^{k-1}]} & 0 \\ 0 & -\mathbf{1}_{[2^{k-1}]} \end{pmatrix} \\ &= (-i)^m \prod_{A=1}^{2k} \gamma_A^{\mathrm{SO(2k+1)}} \end{aligned}$$

with $\gamma_1^{\mathrm{SO(1)}} \equiv (1)$. In this way we may produce an odd number, $(2k+1)$, of gamma matrices with which we may construct the generators of $SO(2k+1)$. For the group $SO(2k)$ we just use the first $2k$ gamma matrices, of the $(2k+1)$ used to construct the generators of $SO(2k+1)$, to form the generators of $SO(2k)$.

C 1.2 Block diagonal form of $SO(2k)$.

Because of the form of the gamma matrices, it is obvious that the generators of $SO(2k)$ have a block diagonal structure. In this case the last gamma matrix may be used, together with the identity element, to form 2 projection operators :-

$$\begin{aligned} P_L &\equiv \tfrac{1}{2}\left(\mathbf{1}_{[2^k]} + \gamma_{2k+1}^{\mathrm{SO(2k+1)}}\right) \\ &= \begin{pmatrix} \mathbf{1}_{[2^{k-1}]} & 0 \\ 0 & 0 \end{pmatrix} \\ P_R &\equiv \tfrac{1}{2}\left(\mathbf{1}_{[2^k]} - \gamma_{2k+1}^{\mathrm{SO(2k+1)}}\right) \\ &= \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1}_{[2^{k-1}]} \end{pmatrix} \end{aligned}$$

which will not only project out the left and right handed spinors from the fundamental Weyl spinor, but will also project out the two sets of generators which independently act on the left and right handed spinors. We may therefore just restrict our attention to the left handed transformations which act on the left handed spinor. We explicitly show this by writing a fundamental 2^k-spinor of $SO(2k)$ as :-

$$\phi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}$$

Under an $SO(2k)$ transformation in the Weyl representation, produced by an element $g \in SO(2k)$, we will find that the fundamental 2^k-spinor transforms :-

$$\begin{aligned} \phi \mapsto \phi' &= g\phi \\ &= \begin{pmatrix} g_L & 0 \\ 0 & g_R \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix} \\ &= \begin{pmatrix} g_L\psi \\ g_R\chi \end{pmatrix} \end{aligned}$$

Thus, we see that P_L will isolate the part which holds the relation :-

$$\psi \mapsto \psi' = g_L \psi$$

and P_R will isolate the part which holds the relation :-

$$\chi \mapsto \chi' = g_R \chi$$

So, for $SO(2k)$, we end up just having to work with generators which are $(2^{k-1} \times 2^{k-1})$ matrices. In this thesis we will, for $SO(6)$, work with the left handed representation of (4×4) generators which generate transformations on the left handed spinors. This will allow us to exploit the homomorphism between the groups $SO(6)$ and $SU(4)$.

C 2 The Lie algebra of $SO(m)$.

Using equations C.1 and C.2 we find the Lie algebra to be :-

$$[\sigma_{AB}, \sigma_{CD}] \equiv 2i\left[(\delta_{AC}\sigma_{BD} - \delta_{AD}\sigma_{BC}) - (\delta_{BC}\sigma_{AD} - \delta_{BD}\sigma_{AC})\right]$$

where $A, B, C, D = 1, 2, \ldots, m$. We note that $SO(m)$, where $m = 2k, 2k+1$, is a group of rank k.

1. For $SO(m)$ breaking to $SO(m-1)$ we have :-

   - The Lie algebra of the subgroup $SO(m-1)$ is :-

     $$[\sigma_{ab}, \sigma_{cd}] \equiv 2i\left[(\delta_{ac}\sigma_{bd} - \delta_{ad}\sigma_{bc}) - (\delta_{bc}\sigma_{ad} - \delta_{bd}\sigma_{ac})\right]$$

     where the $a, b, c, d = 1, 2, \ldots, m-1$.

   - The coset commutator closes onto the subgroup :-

     $$[\sigma_{a\Delta}, \sigma_{b\Delta}] \equiv 2i\delta_{\Delta\Delta}\sigma_{ab}$$

     where $\Delta = m$ only.

   - So between subgroup and coset we have :-

     $$[\sigma_{ab}, \sigma_{c\Delta}] \equiv 2i\,(\delta_{ac}\sigma_{b\Delta} - \delta_{bc}\sigma_{a\Delta})$$

2. For $SO(m)$ breaking to $SO(m-2) \otimes SO(2)$.

   - The coset commutator closes onto the subgroups :-

     $$[\sigma_{aX}, \sigma_{bY}] \equiv 2i\,(\delta_{ab}\sigma_{XY} + \delta_{XY}\sigma_{ab})$$

     For $a, b = 1, 2, \ldots, (m-2)$ and $X, Y = (m-1), m$.

   - So between subgroups and coset we have :-

     $$[\sigma_{ab}, \sigma_{cX}] \equiv 2i\,(\delta_{ac}\sigma_{bX} - \delta_{bc}\sigma_{aX})$$

     $$[\sigma_{XY}, \sigma_{aZ}] \equiv 2i\,(\delta_{XZ}\sigma_{aY} - \delta_{YZ}\sigma_{aX})$$

C 3 The Lie algebra of $SO(t,s)$.

We construct the Σ matrices of $SO(t,s)$, where $t+s = m$, out of modified gamma matrices used for $SO(m)$, where $m = 2k, 2k+1$. This time the gamma matrices obey a different *Clifford* algebra :-

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\,\eta_{\mu\nu}\,\mathbf{1}_{[2^k]} \tag{C.3}$$

with $\mu, \nu = 0, 1, 2, \ldots, m-2, m$. We also define the matrix η :-

$$\eta \equiv \mathrm{diag}(+1, \overbrace{-1, -1, \ldots, -1}^{m-1}) \qquad \text{for } SO(1, m-1)$$
$$\eta \equiv \mathrm{diag}(+1, \overbrace{-1, -1, \ldots, -1}^{m-2}, +1) \qquad \text{for } SO(2, m-2)$$

By convention, in going from $SO(m)$ to $SO(t,s)$, we always have $\gamma_{m-1}^{SO(m)} \equiv \Gamma_0$ and so :-

1. for $SO(1, m-1)$ we would multiply the γ_k (for $k = 1, 2, \ldots, (m-2), m$) by i, thus Γ_0 is Hermitian and the rest are anti-Hermitian, and
2. for $SO(2, m-2)$ we would multiply the γ_k (for $k = 1, 2, \ldots, (m-2)$) by i so they are anti-Hermitian, and have the Hermitian gamma matrices Γ_0 and $\Gamma_m \equiv \gamma_m$.

We may now build a set of traceless, Hemitian, generators for $SO(t,s)$:-

$$\begin{aligned} T_{\mu\nu} &\equiv \frac{i}{4}[\Gamma_\mu, \Gamma_\nu] \\ &= \frac{1}{2}\Sigma_{\mu\nu} \end{aligned} \tag{C.4}$$

which obey the $SO(t,s)$ Lie algebra, and are 2^k dimensional; instead of $(2k+1)$ dimensional. So we have :-

$$\begin{aligned} 2\Gamma_\mu\Gamma_\nu &= 2\eta_{\mu\nu}\,\mathbf{1}_{[2^k]} - 2i\Sigma_{\mu\nu} \\ \Gamma_\mu\Gamma_\nu &= \eta_{\mu\nu}\,\mathbf{1}_{[2^k]} + i\Sigma_{\mu\nu} \end{aligned}$$

so for $\mu \neq \nu$ we have :-

$$\Sigma_{\mu\nu} = i\Gamma_{\mu}\Gamma_{\nu}$$

1. For $SO(1, m-1)$ breaking to $SO(1, m-2)$

   - The Lie algebra of $SO(1, m-1)$ is defined :-

     $$[\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] \equiv -2i\left[(\eta_{\mu\rho}\Sigma_{\nu\sigma} - \eta_{\mu\sigma}\Sigma_{\nu\rho}) - (\eta_{\nu\rho}\Sigma_{\mu\sigma} - \eta_{\nu\sigma}\Sigma_{\mu\rho})\right]$$

     where $\mu, \nu, \rho, \sigma = 0, 1, 2, \ldots, m-2, m$.

   - For an $SO(1, m-2)$ subgroup we must have :-

     $$[\Sigma_{\alpha\beta}, \Sigma_{\gamma\delta}] \equiv -2i\left[(\eta_{\alpha\gamma}\Sigma_{\beta\delta} - \eta_{\alpha\delta}\Sigma_{\beta\gamma}) - (\eta_{\beta\gamma}\Sigma_{\alpha\delta} - \eta_{\beta\delta}\Sigma_{\alpha\gamma})\right]$$

     where the $\alpha, \beta, \gamma, \delta = 0, 1, 2, \ldots, m-2$.

   - The coset commutator closes on the subgroup :-

     $$[\Sigma_{\alpha\Delta}, \Sigma_{\beta\Delta}] \equiv -2i\eta_{\Delta\Delta}\Sigma_{\alpha\beta}$$

   - So between subgroup and coset we have :-

     $$[\Sigma_{\alpha\beta}, \Sigma_{\gamma\Delta}] \equiv -2i\left(\eta_{\alpha\gamma}\Sigma_{\beta\Delta} - \eta_{\beta\gamma}\Sigma_{\alpha\Delta}\right)$$

C 4 The Lie algebra of $SO(6)$.

After performing the required steps we find that $SO(6)$ σ-matrices, in the Spinor representation, are a direct sum of two sets of $SU(4)$ λ-matrices; namely the $4 \oplus \bar{4}$. So we use $P_L = \frac{1}{2}(\mathbf{1}_{[8]} + \gamma_7^{\mathrm{SO(7)}})$ and $P_R = \frac{1}{2}(\mathbf{1}_{[8]} - \gamma_7^{\mathrm{SO(7)}})$ to project out the two sets.

Here is the 4 which generates transformations on a left handed spinor :-

$$\sigma^{\mathrm{L}}_{ij} = \varepsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \quad \sigma^{\mathrm{L}}_{i4} = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} \quad \sigma^{\mathrm{L}}_{i5} = \begin{pmatrix} 0 & -\sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

$$\sigma^{\mathrm{L}}_{45} = \begin{pmatrix} 0 & i\,\mathbf{1}_{[2]} \\ -i\,\mathbf{1}_{[2]} & 0 \end{pmatrix} \quad \sigma^{\mathrm{L}}_{i6} = \begin{pmatrix} 0 & i\sigma_i \\ -i\sigma_i & 0 \end{pmatrix} \quad \sigma^{\mathrm{L}}_{46} = \begin{pmatrix} 0 & \mathbf{1}_{[2]} \\ \mathbf{1}_{[2]} & 0 \end{pmatrix}$$

$$\sigma^{\mathrm{L}}_{56} = \begin{pmatrix} \mathbf{1}_{[2]} & 0 \\ 0 & -\mathbf{1}_{[2]} \end{pmatrix}$$

C 4.1 Translation into the language of $SU(4)$.

The σ_{AB} of $SO(6)$ written in terms of the λ_a of $SU(4)$ are :-

$$\sigma_{ij} \;\rightsquigarrow\; \begin{cases} \sigma^{\mathrm{L}}_{12} = (r_3 + r_3^{\perp}) & \text{is the generator of } SO(2). \\ \sigma^{\mathrm{L}}_{23} = (\lambda_1 + \lambda_{13}) & \\ \sigma^{\mathrm{L}}_{31} = (\lambda_2 + \lambda_{14}) & \uparrow \text{ are generators of } SO(3). \end{cases}$$

$$\sigma_{i4} \;\rightsquigarrow\; \begin{cases} \sigma^{\mathrm{L}}_{14} = (\lambda_1 - \lambda_{13}) & \\ \sigma^{\mathrm{L}}_{24} = (\lambda_2 - \lambda_{14}) & \\ \sigma^{\mathrm{L}}_{34} = (r_3 - r_3^{\perp}) & \uparrow \text{ are generators of } SO(4). \end{cases}$$

$$\sigma_{i5} \;\rightsquigarrow\; \begin{cases} \sigma^{\mathrm{L}}_{15} = -(\lambda_6 + \lambda_9) & \\ \sigma^{\mathrm{L}}_{25} = (\lambda_7 - \lambda_{10}) & \\ \sigma^{\mathrm{L}}_{35} = -(\lambda_4 - \lambda_{11}) & \end{cases}$$

$$\sigma^{\mathrm{L}}_{45} = -(\lambda_5 + \lambda_{12}) \qquad \uparrow \text{ are generators of } SO(5).$$

$$\sigma_{i6} \;\rightsquigarrow\; \begin{cases} \sigma^{\mathrm{L}}_{16} = -(\lambda_7 + \lambda_{10}) & \\ \sigma^{\mathrm{L}}_{26} = -(\lambda_6 - \lambda_9) & \\ \sigma^{\mathrm{L}}_{36} = -(\lambda_5 - \lambda_{12}) & \end{cases}$$

$$\sigma^{\mathrm{L}}_{46} = (\lambda_4 + \lambda_{11})$$

$$\sigma^{\mathrm{L}}_{56} = \sqrt{2} q_3 \qquad \uparrow \text{ are generators of } SO(6).$$

Note that the $\uparrow$ symbol has been used to represent the phrase 'all generators in the table down to this point'. We have also used the following objects in the table to represent (a linear sum of) generators :-

$$\begin{aligned}
r_3 &= \lambda_3 \\
r_3^{\perp} &= -\frac{1}{\sqrt{3}}\lambda_8 + \sqrt{\frac{2}{3}}\lambda_{15} \\
q_3 &= \sqrt{2}\left(\frac{1}{\sqrt{3}}\lambda_8 + \frac{1}{\sqrt{6}}\lambda_{15}\right)
\end{aligned}$$

Also, from our work in Appendix A we see that we may also write :-

$$\begin{aligned}
\sigma_{12}^{\mathrm{L}} &= \sqrt{2}q_2 \\
\sigma_{34}^{\mathrm{L}} &= -\sqrt{2}q_1
\end{aligned}$$

C 4.2 The $SO(4)$ and $SO(2)$ subgroups of $SO(6)$.

- The usual $SO(4)$ subgroup, which is homomorphic to $SU(2)\otimes SU(2)$, is generated by :-

$$\sigma_{ij}^{\mathrm{L}} = \varepsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \quad \text{and} \quad \sigma_{i4}^{\mathrm{L}} = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}$$

To see the homomorphism we first construct :-

$$\begin{aligned}
V_k &= \frac{1}{2}\varepsilon_{ijk}\sigma_{ij}^{\mathrm{L}} \\
&= \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}
\end{aligned}$$

If we now relabel $\sigma_{i4}^{\mathrm{L}} \to A_i$, and then construct the six generators :-

$$\begin{aligned}
L_k &= \frac{1}{2}(V_k + A_k) \\
R_k &= \frac{1}{2}(V_k - A_k)
\end{aligned}$$

we find that they obey :-

$$
\begin{aligned}
[L_i, L_j] &= 2i\varepsilon_{ijk}L_k \\
[R_i, R_j] &= 2i\varepsilon_{ijk}R_k \\
[L_i, R_j] &= 0
\end{aligned}
$$

The first two relations tell us that L_i and R_i are the generators of two $SU(2)$ groups; the last relation tells us that these groups will commute. This means that the L_i form the basis for the (unit) group vector r and the R_i form the basis for the other (unit) group vector $r_\perp$; that is, we may write :-

$$
\begin{aligned}
r &= n^i L_i \\
r_\perp &= n^i_\perp R_i
\end{aligned}
$$

Therefore the subgroup vector, x, may be written in two ways :-

$$
x = \overbrace{\tfrac{\Omega}{2} n^{AB}\sigma_{AB}}^{SO(4)\ view} \Leftrightarrow \overbrace{\tfrac{a}{2} n^i L_i + \tfrac{b}{2} n^i_\perp R_i}^{SU(2)\otimes SU(2)\ view}
$$

and the $SO(4)$ subgroup element is e^{-ix}.

- The $SO(2)$ subgroup element, which commutes with the $SO(4)$ subgroup element already considered, is generated by :-

$$
\sigma^{\mathrm{L}}_{56} = \begin{pmatrix} \mathbf{1}_{[2]} & 0 \\ 0 & -\mathbf{1}_{[2]} \end{pmatrix}
$$

This is easy to show using the notation of $SU(4)$. As we have seen, in terms of vectors in the Cartan Subspace, the $SO(4)$ subgroup vector may be written as :-

$$
x = A\,r + B\,r_\perp
$$

Quite obviously a general orthogonal vector which commutes with x, i.e. lies in $\mathcal{C}_x$, may be written :-

$$y = C\, q_r$$

because we know that $[r, q_r] = [r_\perp, q_r] \equiv 0$. Thus, when $r \equiv n^i L_i$ and $r_\perp \equiv n^i_\perp R_i$, we find the result $\sqrt{2}\, q_r \equiv \sigma^{\mathrm{L}}_{56}$. Therefore we find that $e^{-iy} \in SO(2)$ is an element which will commute with the element $e^{-ix} \in SO(4)$. Therefore we may write $h = e^{-ix}e^{-iy} = e^{-iy}e^{-ix} \in SO(4) \otimes SO(2)$.

Appendix D

The explicit form of differentials.

In this appendix we will construct the explicit results which we find when we differentiate fields not only with respect to the Goldstone boson fields, but also with respect to spacetime coordinates. To keep the calculations in this thesis as concise as possible, the results which follow will be substituted in at the end of calculations.

D 1 When x is in terms of one vector only.

If the coset vector is written in terms of a single vector, then we find that it is simple to obtain the results of differentiation, either with respect to the Goldstone fields or with respect to the spacetime coordinates. The results in this section will apply to the models considered in Chapters 3 and 4; where the size of x is not important.

D 1.1 Differentiation by $\frac{\partial}{\partial M^a}$.

In practice we use the notation $F_{,a} \equiv \frac{\partial F}{\partial M^a}$ to represent differentiation of a quantity, F, with respect to M^a, the Goldstone boson fields. For the chiral $SU(N)$ breaking models of chapter 4 the coset vectors are written :-

$$\begin{aligned} x &= \tfrac{\phi}{2} r \\ &= \tfrac{\phi}{2} n^a \lambda_a \end{aligned}$$

This quantity, in terms of the Goldstone fields, may be written :-

$$M^a \lambda_a = M n^a \lambda_a$$

and the field components, isolated using (x, λ_a), are :-

$$M^a = M n^a \tag{D.1}$$

where M is the length of the vector $M^a \lambda_a$. The quantity M is defined by :-

$$M^a M^a = M^2 \tag{D.2}$$

because the n^a are the components of a unit vector in the direction of $M^a \lambda_a$. If we differentiate equation (D.2), with respect to the Goldstone fields, we find :-

$$\begin{aligned} (M^a M^a)_{,b} &= (M^2)_{,b} \\ 2M^b &= 2M\, M_{,b} \\ M^b &= M\, M_{,b} \end{aligned}$$

because $M^a_{,b} \Rightarrow \frac{\partial M^a}{\partial M^b} \equiv \delta_{ab}$. If we now compare this with equation (D.1) we see the identification :-

$$n^a = M_{,a} \tag{D.3}$$

If we now work on Mn^a and differentiate we get :-

$$(Mn^a)_{,b} = n^a M_{,b} + M n^a_{,b}$$

and so we see the result :-

$$n^a_{,b} = \tfrac{1}{M}(\delta_{ab} - n^a n^b) \tag{D.4}$$

In Chapter 3 the coset vectors are written as, the $SO(m)$ breaking to $SO(m-1)$ models, and the $SO(1,m-1)$ breaking to $SO(1,m-2)$ models, , the coset vectors are written as $x = \Omega S$ where :-

$$S = n^{a\Delta}\sigma_{a\Delta} \qquad \text{for the } SO(m) \text{ breaking to } SO(m-1) \text{ models, and}$$

$$S = n^{\alpha\Delta}\Sigma_{\alpha\Delta} \qquad \text{for the } SO(1,m-1) \text{ breaking to } SO(1,m-2) \text{ models.}$$

So we find the respective results :-

$$n^{a\Delta}_{,b\Delta} = \tfrac{1}{M}(\delta^{a\Delta}_{b\Delta} - n^{a\Delta}n_{b\Delta}) \tag{D.5}$$

$$n^{\alpha\Delta}_{,\beta\Delta} = \tfrac{1}{M}(\delta^{\alpha\Delta}_{\beta\Delta} - n^{\alpha\Delta}n_{\beta\Delta}) \tag{D.6}$$

with $\Delta = m$. We use the notation Δ to remind us that this is a fixed index, just a label, and it is necessary to distinguish between labels and indices which are summed over, for the latter we have used lower case letters. We note that for the $SO(m)$ breaking to $SO(m-1)$ models of chapter 3, the index a has the range $a = 1, 2, \ldots, (m-1)$ but for the $SO(1,m-1)$ breaking to $SO(1,m-2)$ models we relabel this index $a \rightsquigarrow \alpha$ to remind us that it takes in the values $\alpha = 0, 1, 2, \ldots, (m-2)$.

Lastly we encounter the differential of $\phi = \phi(M)$:-

$$\begin{aligned} \phi_{,a} &= \frac{d\phi}{dM} M_{,a} \\ &= \frac{d\phi}{dM} n^a \end{aligned}$$

and for the $SO(m)$ breaking to $SO(m-1)$ models, and the $SO(1,m-1)$ breaking to $SO(1,m-2)$ models, we find the results :-

$$\Omega_{,a\Delta} = \frac{d\Omega}{dM} n_{a\Delta} \tag{D.7}$$

$$\Omega_{,\alpha\Delta} = \frac{d\Omega}{dM} n_{\alpha\Delta} \tag{D.8}$$

D 1.2 Differentiation by $\frac{\partial}{\partial x^\mu} = \partial_\mu$.

In this thesis we also differentiate quantities with respect to the spacetime coordinates. Firstly, we differentiate the length of the vector associated with the interpolating fields. These are the fields which we initially use to describe the coset; before they are redefined to represent the Goldstone fields. We have :-

$$\begin{aligned}
\partial_\mu \phi &= \frac{d\phi}{dM}\,\partial_\mu M \\
&= \frac{d\phi}{dM}\,\partial_\mu (M^a n^a) \\
&= \frac{d\phi}{dM}\,(n^a \partial_\mu M^a + M^a \partial_\mu n^a) \\
\partial_\mu \phi &= \frac{d\phi}{dM}\, n^a \partial_\mu M^a
\end{aligned} \tag{D.9}$$

because for the second term $2M^a\partial_\mu n^a = 2Mn^a\partial_\mu n^a = M\partial_\mu(n^a n^a) = 0$. For the $SO(m)$ breaking to $SO(m-1)$ models, and the $SO(1,m-1)$ breaking to $SO(1,m-2)$ models, we find the results :-

$$\partial_\mu \Omega = \frac{d\Omega}{dM}\, n_{a\Delta}\partial_\mu M^{a\Delta} \tag{D.10}$$

$$\partial_\mu \Omega = \frac{d\Omega}{dM}\, n_{\alpha\Delta}\partial_\mu M^{\alpha\Delta} \tag{D.11}$$

We also differentiate n^a with respect to x^μ and find the result :-

$$\begin{aligned}
\partial_\mu n^a &= n^a_{,b}\partial_\mu M^b \\
&= \tfrac{1}{M}(\delta_{ab} - n^a n^b)\,\partial_\mu M^b
\end{aligned} \tag{D.12}$$

and for the $SO(m)$ breaking to $SO(m-1)$ models, and the $SO(1,m-1)$ breaking to $SO(1,m-2)$ models, we find the results :-

$$\partial_\mu n^{a\Delta} = \tfrac{1}{M}(\delta^{a\Delta}_{b\Delta} - n^{a\Delta}n_{b\Delta})\,\partial_\mu M^{b\Delta} \tag{D.13}$$

$$\partial_\mu n^{\alpha\Delta} = \tfrac{1}{M}(\delta^{\alpha\Delta}_{\beta\Delta} - n^{\alpha\Delta}n_{\beta\Delta})\,\partial_\mu M^{\beta\Delta} \tag{D.14}$$

D 2 When $x = \frac{a}{2}r + \frac{b}{2}r_\perp$ and $N = 4$.

If the coset vector is a linear sum of two commuting orthonormal r-vectors, say r and $r_\perp$, and $N = 4$ then the situation is rather different. The results in this section will apply to the models considered in Chapter 5.

D 2.1 Differentiation by $\frac{\partial}{\partial \mathcal{M}^a}$.

Firstly, we will demonstrate the difficulty of the task we now face. If we follow the ideas in the beginning of the last section, then the coset vectors in chapter 5 are written :-

$$x = \tfrac{a}{2}r + \tfrac{b}{2}r_\perp$$

This quantity, in terms of the Goldstone fields, may now be written :-

$$\begin{aligned} \mathcal{M}^a\lambda_a &= M^a\lambda_a + M_\perp^a\lambda_a \\ &= M\,n^a\lambda_a + M_\perp\,n_\perp^a\lambda_a \end{aligned} \tag{D.15}$$

Removing the basis obviously gives :-

$$\begin{aligned} \mathcal{M}^a &= M^a + M_\perp^a \\ &= M\,n^a + M_\perp\,n_\perp^a \end{aligned} \tag{D.16}$$

From this we find the norm of x to be :-

$$\begin{aligned} \mathcal{M}^a\mathcal{M}^a &= (M^a + M_\perp^a)(M^a + M_\perp^a) \\ &= M^aM^a + M_\perp^aM_\perp^a \end{aligned} \tag{D.17}$$

because $(r, r_\perp) \equiv 0$. If we now differentiate this with respect to $\mathcal{M}^a$ then we have :-

$$\begin{aligned} \mathcal{M}^a\mathcal{M}^a_{,b} &= M^aM^a_{,b} + M_\perp^aM^a_{\perp\,,b} \\ \mathcal{M}^b &= M^a(M_{,b}n^a + Mn^a_{,b}) + M_\perp^a(M_{\perp\,,b}n_\perp^a + M_\perp n^a_{\perp\,,b}) \\ &= M\,M_{,b} + M_\perp\,M_{\perp\,,b} \end{aligned}$$

If we compare this with equation (D.16) then we see :-

$$M_{,b} = n^b \tag{D.18}$$

$$M_{\perp,b} = n^b_\perp \tag{D.19}$$

Now if we differentiate equation (D.16) with respect to $\mathcal{M}^a$ then we find :-

$$\begin{aligned} \mathcal{M}^a_{,b} &= M_{,b}\, n^a + M\, n^a_{,b} + M_{\perp,b}\, n^a_\perp + M_\perp\, n^a_{\perp,b} \\ \delta_{ab} &= n^a n^b + M\, n^a_{,b} + n^a_\perp n^b_\perp + M_\perp\, n^a_{\perp,b} \end{aligned}$$

where we may write the left hand side as $\delta_{ab} = m^a m^b + \mathcal{M}\, m^a_{,b}$ with the m^a being the components of a unit vector defined by equation (D.16). So it is clear that we need to find expressions for the two quantities $n^a_{,b}$ and $n^a_{\perp,b}$. Using this sort of construction, it is not immediately clear how this may be done.

However, the problem is solved with the use of the symmetric vector product. For a coset vector, $x = \mathcal{M}^a\lambda_a = M^a\lambda_a + M^a_\perp\lambda_a$, we calculate $x \vee x$ and $x \vee x \vee x$. In terms of both descriptions of x these quantities are :-

$$\begin{aligned} x \vee x &= \sqrt{2}(M^2 - M_\perp^2)\, q_r \\ &= 2\mathcal{M}^a\mathcal{M}^b d_{abE}\lambda_E \\ x \vee x \vee x &= 2(M^2 - M_\perp^2)(M\, r - M_\perp\, r_\perp) \\ &= 4\mathcal{M}^a\mathcal{M}^b d_{abE}\mathcal{M}^c d_{Ecd}\lambda_d \end{aligned}$$

We now use these relations [1] to solve for Mr and $M_\perp r_\perp$. We find :-

$$\begin{aligned} M\, r &= \tfrac{1}{2}x + \tfrac{1}{4(M^2 - M_\perp^2)}\,(x \vee x \vee x) \\ M_\perp\, r_\perp &= \tfrac{1}{2}x - \tfrac{1}{4(M^2 - M_\perp^2)}\,(x \vee x \vee x) \end{aligned}$$

[1] These relations, and the ones which follow, will be different when $N > 4$ because, in these cases, we will find that $q_{r_\perp} \neq -q_r$.

If we write the commuting vectors on the right hand side explicitly, and remove the λ_a basis, then we find the vector components :-

$$M^d = \tfrac{1}{2}\mathcal{M}^d + \tfrac{1}{(M^2-M_\perp^2)}\mathcal{M}^a\mathcal{M}^b d_{abE}\mathcal{M}^c d_{Ecd} \tag{D.20}$$

$$M_\perp^d = \tfrac{1}{2}\mathcal{M}^d - \tfrac{1}{(M^2-M_\perp^2)}\mathcal{M}^a\mathcal{M}^b d_{abE}\mathcal{M}^c d_{Ecd} \tag{D.21}$$

We may now differentiate these two expressions with respect to the Goldstone boson fields ($\mathcal{M}^a$) without any trouble. We find :-

$$M^d_{,e} = \tfrac{1}{2}\delta_{de} + \tfrac{1}{(M^2-M_\perp^2)}\left[\mathcal{M}^2(\tfrac{1}{2}d_m^2 + \tfrac{1}{4}d_{m\vee m})_{de} - (M^d - M_\perp^d)(M^e - M_\perp^e)\right] \tag{D.22}$$

$$M^d_{\perp,e} = \tfrac{1}{2}\delta_{de} - \tfrac{1}{(M^2-M_\perp^2)}\left[\mathcal{M}^2(\tfrac{1}{2}d_m^2 + \tfrac{1}{4}d_{m\vee m})_{de} - (M^d - M_\perp^d)(M^e - M_\perp^e)\right] \tag{D.23}$$

We will use these expressions when we work out the linear $\mathbf{K}^a_E$. We may also write these in terms of adjoint representation projection operators. To do this we will use :-

$$\begin{aligned} M^d_{,e} &= n^d n^e + M\, n^d_{,e} \\ M^d_{\perp,e} &= n^d_\perp n^e_\perp + M_\perp\, n^d_{\perp,e} \end{aligned}$$

and, specific to $\Re^{15}$, the operator relations :-

$$\begin{aligned} d_m^2 - 4f_m^2 &= d_{m\vee m} + 2\,\mathbf{1}_{[15]} - 2m{>}{<}m \\ \sqrt{2}(d_{q_r})_{ab} &= 2(\mathcal{P}^{12} + \mathcal{P}^{21} + r{>}{<}r)_{ab} - 2(\mathcal{P}^{34} + \mathcal{P}^{43} + r_\perp{>}{<}r_\perp)_{ab} \end{aligned}$$

The 1^{st} line is the appropriate form of equation (B.11) for a unit vector m in $\Re^{15}$, and the 2^{nd} line is just equation (B.42); both are explained Appendix B. Since we also know that $\mathcal{M}f_m = Mf_r + M_\perp f_{r_\perp}$ we may use these in equations (D.22) and (D.23) and rearrange to eventually find :-

$$M\,n^a_{,b} = \tfrac{1}{(M^2-M_\perp^2)}\left[(M^2 - M_\perp^2)(\mathcal{P}^{12} + \mathcal{P}^{21}) + M^2\,\mathcal{P}_{f_q^2} + MM_\perp(4f_r f_{r_\perp})\right]_{ab} \tag{D.24}$$

$$M_\perp n^a_{\perp,b} = \tfrac{1}{(M^2-M_\perp^2)}\left[(M^2 - M_\perp^2)(\mathcal{P}^{34} + \mathcal{P}^{43}) - M_\perp^2\,\mathcal{P}_{f_q^2} - MM_\perp(4f_r f_{r_\perp})\right]_{ab} \tag{D.25}$$

When we work out the nonlinear $\mathbf{K}^a_b$ we will use these expressions.

D 2.2 Differentiation by $\frac{\partial}{\partial x^\mu} = \partial_\mu$.

The remaining differentials are easy to find. In terms of the interpolating fields, the coset vector $x = \frac{\phi}{2} m^a \lambda_a$ has been written :-

$$x = \tfrac{a}{2} r + \tfrac{b}{2} r_\perp$$

In terms of the Goldstone fields, the coset vector $x = \frac{\mathcal{M}}{2} m^a \lambda_a$ has been written :-

$$x = M r + M_\perp r_\perp$$

If we think about the properties of the r-vectors we are using, then we see that we must have $a \equiv a(M)$ and $b \equiv b(M_\perp)$. This is also supported by the fact that, in the two parameterizations, the matrix invariant $\gamma_2(x)$ is :-

$$\begin{aligned} \left(\frac{\phi}{2}\right)^2 &= \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 \\ \mathcal{M}^2 &= M^2 + M_\perp^2 \end{aligned}$$

Therefore we have :-

$$\begin{aligned} \partial_\mu a &= \frac{da}{d\mathcal{M}} \partial_\mu \mathcal{M} \\ &= \frac{da}{dM} \partial_\mu M \\ &= \frac{da}{dM} n^a \partial_\mu M^a \end{aligned} \tag{D.26}$$

Similarly we find :-

$$\partial_\mu b = \frac{db}{dM_\perp} n_\perp^a \partial_\mu M_\perp^a \tag{D.27}$$

We also have the field gradients $\partial_\mu n^a$ and $\partial_\mu n_\perp^a$ in our calculations. These we find to be :-

$$\partial_\mu n^a = n^a_{,b} \partial_\mu \mathcal{M}^b \tag{D.28}$$

$$\partial_\mu n_\perp^a = n^a_{\perp,b} \partial_\mu \mathcal{M}^b \tag{D.29}$$

where $n^a_{,b}$ and $n^a_{\perp,b}$ are given by equations (D.24) and (D.25).

Even though Chapter 5 concerns the spontaneous breaking of $SO(m)$ symmetries down to $SO(m-2) \otimes SO(2)$ for $m = 4, 5$ and 6, it is phrased entirely in terms of the corresponding $SU(4)$ objects; i.e. vector components have one index not two. The calculations are performed by manipulating defining representation objects like the coset vector, so when it comes to looking at the transformation of vector components (or even just isolating components), we automatically end up with expressions which are phrased with respect to the components of adjoint representation objects like f_r, d_{q_r} or projection operators. Therefore, we will not rewrite the results which we have found in an $SO(6)$ notation. We will, though, remember that what we really wanted to find is actually homomorphic to what we have found; but this does not change the mathematical result.

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