UNIVERSITY OF SOUTHAMPTON

Towards a Performance Theory of Robust Adaptive Control

by

Ahmad Sanei

A thesis submitted in partial fulfillment for the degree of Doctor of Philosophy

in the

Faculty of Engineering and Applied Science
Department of Electronics and Computer Science

June 2003

UNIVERSITY OF SOUTHAMPTON

ABSTRACT

FACULTY OF ENGINEERING AND APPLIED SCIENCE

DEPARTMENT OF ELECTRONICS AND COMPUTER SCIENCE

Doctor of Philosophy

Towards a Performance Theory of Robust Adaptive Control

by Ahmad Sanei

The main contribution of this thesis is to establish a comparative theory of robust adaptive control designs. We consider dead-zone and projection based robust adaptive controllers for scalar systems, relative degree one, minimum phase linear systems of known high frequency gain, and nonlinear systems in the form of integrator chain. We compare their performance with respect to a worst case non–singular transient cost functional penalising the $\mathcal{L}^\infty$ norm on the state, control and control derivative. If a bound on the $\mathcal{L}^\infty$ norm of the disturbance is known, it is shown that the dead-zone controller outperforms the projection controller if the a-priori information on the uncertainty level is sufficiently conservative. The complementary result shows that the projection controller is superior to the dead-zone controller when the a-priori information on the disturbance level is sufficiently conservative.

A secondary contribution is to present an alternative solution to the problems of robust adaptive control, due to right hand side discontinuity of adaptive law, by developing the so-called hysteresis dead-zone method and showing that the sliding motions are avoided and chattering effects can be mitigated.

In loving memory of my father

Contents

List of Figures

Nomenclature

$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the interval $[0, \infty)$
$\sigma(A)$	spectrum of matrix A
$\underline{\lambda}(A), \overline{\lambda}(A)$	minimum, maximum eigenvalue of A
D_-	left hand derivative
$\lim t \to \omega^-$	left hand limit $\lim_{t \to \omega, t < \omega}$
Ω^0	interior of Ω
$\partial\Omega$	boundary of Ω
$conv\, \Omega$	convex hull contained Ω
$\overline{conv}\, \Omega$	convex closure set contained Ω
$\inf_\Omega, \sup_\Omega$	infimum, supremum over Ω
$\overline{\lim}$	$\limsup\limits_{t \to \infty}$
$m(\Omega)$	Lebesgue measure of Ω
$\mathcal{C}(\Omega)$	set of continuous functions $\Omega \to \mathbb{R}$
$\mathcal{C}^k(\Omega)$	k times differentiable function over $\Omega \to \mathbb{R}$
$\lVert . \rVert_p$	p Euclidian norm
$\mathcal{L}^p(\Omega)$	set of p^{th} integrable functions over Ω
$\mathcal{L}^\infty(\Omega)$	set of essentially bounded functions over Ω
$\lVert \cdot \rVert_{\mathcal{L}^p}$	$\mathcal{L}^p$ norm
$\lVert \cdot \rVert_{\mathcal{L}^\infty}$	$\mathcal{L}^\infty$ norm
F	set valued function
$\mathcal{X}_\omega$	characteristic function, $\mathcal{X}_\omega(t) = 1$ if $t \in \omega$, $\mathcal{X}_\omega(t) = 0$ elsewhere
Q	positive definite state performance weighting matrix

P	solution to Lyapunov equation
B	vector $(0, \ldots, 1)^T \in \mathbb{R}^n$
S	smooth surface
G^-, G^+	subspaces of G partitioned by S
f^-, f^+	vector field approaching from G^-, G^+
N_s	normal to the surface S
f_N^-, f_N^+	projection of f^-, f^+ onto N_s
S^{-1}, S	tranformation matrix
$\bar{x}$	$S^{-1}x$
(y, z)	$\bar{x}$
Δ_s, Δ_u	structured, unstructured uncertainty
$\mathcal{X}$	system state space
$\mathcal{X}_0$	set of initial conditions
$\mathcal{D}$	bounded disturbance space
Δ	unknown constant parameter space
Σ	system
f	nonlinear function
x_0	initial state, $x_0 \in \mathcal{X}_0$
x	state trajectory
x_{ref}	reference signal
e	error signal $x - x_{ref}$
u	control signal
y	output signal
θ	unknown constant system parameter, $theta \in \Delta$
d	disturbance, $d \in \mathcal{D}$
$\theta_{\max}$	a-priori knowledge of parameter uncertainty level
$d_{\max}$	a-priori knowledge of disturbance level
ϕ	basis function of model
b	state weighting vector

V	Lyapunov function
Ξ	controller
$\hat{\theta}$	parameter estimator
$\hat{\delta}$	tuning function
j	general feedback law
g	general adaptive law
Ω_0, Ω_1	error set
η_0, η_1	parameters determining the size of Ω_0, Ω_1
D_{Ω_0}	dead-zone operator
D'_{Ω_0}	smooth dead-zone operator
H_{Ω_0,Ω_1}	hysteresis dead-zone operator
$\Xi_D(d_{\max})$	dead-zone controller
$\Xi_{\acute{D}}(d_{\max})$	smooth dead-zone controller
$\Xi_H(d_{\max})$	hysteresis dead-zone controller
$P_{\theta_{\max}}$	smooth function $\mathbb{R}^m \to \mathbb{R}$
Π	convex set contain θ defined by $P_{\theta_{\max}}$
$\nabla_{\hat{\theta}} P$	gradient of P
Proj	projection operator
$\Xi_P(\theta_{\max})$	projection controller
δ_θ	the most lower bound of stable system...
$\mathcal{P}$	performance measure of closed loop
$\delta(P, P1)$	the gap metric between plants $P, P1$

Acknowledgements

It is my pleasure to express my gratitude to Dr. Mark French for his encouragement and influence. I am indebted to him for bridging my way into the world of Control Theory. His advice and support from the beginning are gratefully acknowledged. Prof. Eric Rogers must be thanked for his advice and support. I also would like to thank Jason Farquhar for reading through the early drafts of this thesis and correcting my English.

I owe a debt of thanks to the Ministry of Science, Research and Technology, Islamic Republic of Iran, for financial support during my study at University of Southampton.

I am grateful to my mother for her support before and during my years at Southampton. Lastly, I would like to express my sincere appreciation to my wife for her loving support of all my activities. My daughter and my little son have been a source of continuous happiness, and deserve my deepest gratitude.

Chapter 1

Introduction

1.1 Control of Uncertain Dynamics

Control theory attempts to improve the behaviour or performance of physical systems by gathering and exploiting knowledge about the system's operation. Usually this knowledge is encoded as a descriptive mathematical model of the physical plant from which the controller design is derived. Theoretically, building an *exact* mathematical model of a physical system is possible if and only if the relations between *all* its internal components and the effect of *all* external or environmental components on the system are precisely known. Obtaining such complete system knowledge is not practical, hence the mathematical model will not *precisely* reflect a true physical system; moreover accurate mathematical models generally contain high order complex dynamic equations for which designing an appropriate controller is practically inappropriate or even impossible. So, in constructing a mathematical model, one must consider the trade-off between the simplicity of the model and the conservatism of the results.

The art of modeling is thus to gain a perspective representative view of the actual system by selecting a set of physical quantities of interest and defining the relationships between them such that they reflect the behaviour of the actual system dynamics. In one method, which is referred to as *state space* modeling, the mathematical representation of a physical system is given as a set of first order differential equations:

$$\begin{aligned} \dot{x} =& f(x, u), \qquad x(0) = x_0 \in \mathbb{R}^n, \\ y =& h(x). \end{aligned} \tag{1.1}$$

where $x(\cdot)$ is the state vector, x_0 the initial condition, $u(\cdot)$ is the control input and $y(\cdot)$ is the measured output.

Given such a mathematical representation, we are interested in designing a controller $u(\cdot)$ to achieve some objectives such as stability (convergence of state/output to some equilibria) or output tracking of some reference signal. Typically for robustness reasons, the control $u(\cdot)$ is realised as a dynamic feedback of measured variables.

¿From a system dynamics point of view, control theory can be divided into two main categories; linear and nonlinear systems, the dynamic features of which can be either depended on time or time invariant. Historically, the simplest form, Linear Time Invariant (LTI) systems, formed the foundation for analysis and control of models in the form of (1.1) i.e.

$$\begin{aligned} \dot{x} =& Ax + Bu, \qquad x(0) = x_0 \in \mathbb{R}^n, \\ y =& Cx + Du, \end{aligned} \tag{1.2}$$

where $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times l}, C \in \mathbb{R}^{m \times n}, D \in \mathbb{R}^{m \times l}$ and $u(\cdot) \in \mathbb{R}^l, y(\cdot) \in \mathbb{R}^m$. Thousands of text books and many pioneering works made LTI systems in the centre of attention for decades. However, physical systems are not generally linear time invariant. To extend the coverage of linear systems, for which analysis and control are well understood, to nonlinear systems, we often use an approximate linear model by taking a linearisation action around some operating points and reducing the dimension of the model by ignoring the additional dynamics beyond the frequency range of interest [76]. The result is a locally linear model of the corresponding nonlinear system, which can be controlled using linear techniques. A direct consequence of this approach is so-called *modeling error* which is generally defined as the difference between the true system and linearised model. Modeling error can be seen as a member of a larger set, referred to as *uncertainty* Δ, consisting of imperfect knowledge of the physical system dynamics, existence of unknown or uncertain system parameters, approximation of a complex plant by a straightforward model, etc.

A drawback of linearisation is that there are a large class of physical systems that either have strong nonlinearity (e.g. the nonlinearity $f(\cdot,\cdot)$ may have rapid nonlinear growth) or are not even linearisable. Linear approximation of the strong nonlinearities may lead to a control design which performs very poorly. Linear approximation is impossible for systems having so-called 'hard nonlinearities' such as friction, backlash, hysteresis, and other non smooth or discontinuous nonlinearities. These factors have made linearisation impotent in many applications, resulting in the development of completely different techniques for the control of classes of nonlinear systems. For a recent survey see [31].

An intuitive approach in control of nonlinear systems is to cancel nonlinear terms. As an example consider the scalar system $\Sigma(f)$ described by the following differential equation:

$$\Sigma(f):\ \dot{x} = f(x) + u \qquad x(0) = x_0 \in \mathbb{R}. \tag{1.3}$$

If $f(\cdot)$ is an *exactly known* nonlinear function, then it can be canceled by a simple control law of the form:

$$\Xi:\ u = -f(x) - x, \tag{1.4}$$

resulting in a stable closed loop. Unfortunately this simple method and some other earlier results in nonlinear control (see eg. [28] and references therein) require exact knowledge of the system dynamics (ie. the function $f(\cdot)$). In reality, however, $f(\cdot)$ is not usually known exactly. As we will see in section 3.3, neglecting uncertainties and designing controllers for the ideal case can lead to a wide range of problems ranging from performance degradation to eventual loss of stability.

Despite its importance, uncertainty had long been considered a 'side' issue. Classically, feedback control was used as an efficient tool to minimise the undesirable effect of disturbances. In 1960s, a few attempts [29, 19, 47, 48] of considering small disturbances by studying 'perturbed motion' based on Lyapunov's direct method resulted in the concepts of 'uniform asymptotic stability' and 'exponential stability'. These works gradually brought uncertainty to the core of modern control research.

Uncertainties can be classified in two main categories: structured (or parametric) uncertainties Δ_s, and unstructured uncertainties Δ_u. Loosely speaking, structured uncertainties occur when some parameters of the system may not be precisely known in advance or the parameters may be slowly time-varying. Unstructured uncertainties are mostly due to unmodeled dynamics which arise from the inability to precisely model the high frequency behaviour of any physical system [27]. In this dissertation we focus on structured uncertainty. A typical relationship between Δ_s and the plant described in (1.3) is as follows:

$$f = f_0 + \Delta_s, \tag{1.5}$$

where f_0 denotes the nominal or modelled part of the plant which can be either linear or nonlinear.

Although, by its nature, it is impossible to express the exact nature of Δ_s, some general knowledge of uncertainties can usually be obtained in terms of some known bounds. Defining the following set

$$\Delta(f_0, \mu) = \{f \mid \|f - f_0\| \leq \mu\}, \tag{1.6}$$

for some known function μ, we observe that $\Sigma(\Delta(f_0, \mu)) := \{\Sigma(f) \mid f \in \Delta(f_0, \mu)\}$ represents a family of perturbed systems centred at the nominal system $\Sigma(f_0)$. It is a common interest to design a controller $\Xi(f_0)$ such that; firstly, the corresponding closed loop $(\Sigma(f_0), \Xi(f_0))$ meets the control objective (stability, performance) for the nominal system, and secondly, $\Xi(f_0)$ has the ability to produce the required stability/performance for all members of the family $\Sigma(\Delta(f_0, \mu))$, i.e. any closed loop $(\Sigma(f), \Xi(f_0))$ for all $f \in \Delta(f_0, \mu)$ meets (up to some degree) the control objective. In order to construct such a controller, $\Xi(f_0)$ must be designed to be robust to the uncertainty Δ_s. Such a controller is called *robust* and provides a 'robustness margin' (μ) in which it can control every member of the family $\Delta(f_0, \mu)$.

Controlling uncertain dynamics began a new era in modern control theory. Numerous algorithms have been proposed, which can basically be classified into two main classes: robust control and adaptive control.

1.1.1 Robust Control

Robust control achieves its goal in the 'worst case' scenario; given $\Delta(f_0, \mu)$, find a controller Ξ which meets the control objective *for all* $f \in \Delta(f_0, \mu)$. One of the earliest approaches was *sliding mode control* (SMC), first studied in 1960's [75]. Here, f_0 is used to define a lower order, 'sliding surface' in the state space for which controllers are designed which constrain the system trajectory to lie within a neighbourhood of the sliding surface. A decision mechanism constraints the system state to the sliding surface by switching to the appropriate feedback law at any time instant. By this choice the closed loop response becomes insensitive to some uncertainties.

The first, and perhaps most severe, drawback of SMC is *chattering* due to the discontinuities introduced by the switching function. In theory, the system 'slides' on the sliding surface by 'infinitely fast' switching activity of the decision mechanism. However in practice, physical constraints prevent an infinite switching rate and high frequency chattering results. This phenomenon will be discussed in detail in section 1.2.

Another drawback of the SMC is that, in general, it only applies to the uncertain systems which satisfy the matching condition i.e. where the uncertainty and the control appear in the same equation.

Robust control has seen extensive development in the past two decades leading particularly to optimal control. Numerous text books and papers cover the topics of robust control. However, by its worst-case design philosophy, SMC requires reliable knowledge of uncertainty i.e. the design results in 'high gain' controllers if the uncertainty set description is conservative. It has been shown [10] that adaptive control outperforms robust control when the actual uncertainty level is sufficiently high and the a-priori known uncertainty level is sufficiently conservative. The focus of this thesis is adaptive control, so for more detail on robust control we refer the interested reader to the relevant books (see for example [85] and the references therein).

1.1.2 Adaptive Control

Adaptive control is a suitable choice for systems having only parametric uncertainty. In particular suppose $f = \theta\phi$ where θ is unknown constant and $\phi(\cdot)$ is a known function. It follows that the system (1.3) can be rewritten as follows:

$$\Sigma(\theta) : \dot{x} = \theta\phi + u. \tag{1.7}$$

The role of adaptive control is thus to find a suitable controller which satisfies the control objectives and is applicable for any unknown constant θ. This can be achieved by defining a parameter estimator function $\hat{\theta}(\cdot)$ in the controller and tuning it appropriately in response to changes in the dynamics of the process. We will explain this method extensively later in section 3.3 by using a simple example. The beauty of adaptive control is that no a-priori knowledge of the plant parameters is required.

Over the years there have been many attempts to define adaptive systems; in 1957 Drenick and Shahbender [5] inspired by the biological definition of *adaptation*[1], introduced the term *adaptive system* to represent a control system that monitors its own performance and adjust its parameters in the direction of better performance. By monitoring different system characteristics and taking different control actions, a large number of algorithms were proposed. This resulted in a number of definitions for adaptive systems from different points of view. These definitions were collected in the survey papers of Aseltine *et al.* (1958)[1] and Stromer (1959) [71]. However, as yet there is no universally accepted definition of adaptive systems.

Most early adaptive control designs were heuristic and focused on the performance issue, i.e how to adjust the controller parameters to minimise a performance index, without rigorous consideration of stability. In 1966 Parks [59] demonstrated that gradient-based adaptive systems, such as MIT rule-based adaptive control, could be unstable. Also he showed that an adaptive control design based on the Lyapunov method could make a class of systems globally stable. Research then concentrated on the stability issue. The first stability results appear in the late 1970s for LTI systems [39, 54, 70]. They

[1] Biological systems cope easily and efficiently with changes in their environments.

established the boundedness of all signals in the control loop and the convergence of the plant output to a desired value, providing the reference signal is bounded and the following so-called *ideal-case* assumptions hold:

(i) upper bounds on the degrees of the numerator and the denominator are known.

(ii) there are no external disturbances.

(iii) the plant is minimum-phase.

(iv) the sign of the high–frequency gain is known.

The first two assumptions are clearly not realistic since systems are usually more complicated than the mathematical models used in control design and almost all practical controllers have to take disturbances into account. Many people soon discovered that the adaptive controllers derived for the ideal-case had serious robustness problems. Egardt [6] showed that even small bounded disturbances could cause the parameter error to grow without bound, destabilising the system. Rohrs *et al.* [64] also demonstrated by simulation that other perturbations, such as time-varying parameters and unmodeled dynamics, could lead to instability. Since then the robustness problem has been a focus of research in adaptive control and led to a body of work referred to as the *robust adaptive control* theory.

1.1.3 Robust Adaptive Control

The earliest work in robust adaptive control involved the relaxation of assumption (ii), i.e. the control of plants in the presence of bounded disturbances. Two distinct approaches have been used to achieve robustness: (a) using an appropriate reference input, and (b) modifying the adaptation law.

It was realised in the 1960s [2] that certain conditions, generally referred to as persistent excitation (PE) conditions, are necessary to achieve parameter convergence. Narendra and Annaswamy [53] demonstrated that if the degree of persistent excitation is sufficiently large compared to the norm of disturbance signals, then all signals in the closed loop are bounded. Unfortunately, this method is not practical for most applications (such as output tracking) since the reference input (or desired output trajectory) is specified by the task and normally does not satisfy the PE condition.

Several approaches were proposed which modify the adaptation law to achieve robustness in the presence of bounded disturbance. The three most popular methods are:

1. Switch off the adaptation when the output/error signal is not distinguishable from the disturbances. There are several ways to determine when the estimator should be switched off. Egardt [6], Peterson and Narendra [60], Samson [66], and Praly [63] introduced a *dead-zone* in the estimator and defined a switching controller to turn the adaptation off when the estimation error is inside the dead-zone. The result is that all closed loop signals remain bounded and the system trajectory converges to the dead-zone boundary.

2. Modify the algorithm so that the parameters are *projected* into a given compact convex set containing the true parameter vector. Egardt [6], and Kreisselmeier and Narendra [33] used this method to show the global boundedness of all signals in the closed-loop system in the presence of bounded disturbance. Typically convergence of output/error is only established if the disturbance vanishes, when the output/error will tend to zero. A *smooth* version of parameter projection was introduced by Pomet and Praly [62] so as to avoid discontinuity in the control law.

3. Add a *leakage* term to the adaptation law such that the parameter estimate is driven toward a compact region containing origin when it is far from it. This method which is referred to as *σ-modification*, proposed by Ioannou and Kokotovic [25], guarantees the boundedness of closed loop signals. However, if the disturbance is not present the error no longer tends to zero and asymptotic stability is lost.

All the modification approaches, however, require some appropriate a-priori knowledge. For example, to construct the size of the dead-zone requires a-priori information about the size of disturbance. The projection method requires some a-priori knowledge on unknown constant θ to define the compact convex parameter set in such a way that the actual value of θ lies within the set. We will study the dead-zone and parameter projection modification methods extensively in this thesis.

1.2 Switching Controllers

The dead-zone and projection modification methods can be interpreted as defining a decision mechanism which chooses the right adaptation law at the right time. In fact these methods can be classified as members of the larger class of *switching controllers*. Switching phenomena were first studied from a dynamical systems point of view by Filippov[8], and from a control theory perspective by Utkin[75]. Switching controllers are widely used in the control literature. In fact, the history of switching controllers began in the 1960's when Utkin established the sliding mode control scheme.

In the area of adaptive control, switching algorithms have been used for different purposes. Attempting to relax the required a-priori knowledge of the sign of high-frequency gain motivated the idea of switching utility. Nussbaum [56] proposed a switching control scheme which did not require the sign of high frequency gain to be known a-priori. Extending the idea of Middleton *et al.* [45], Morse *et al.* [51] suggested that one way to adaptively control a wider class of process models is to use an algorithm consisting a finite family of controllers with an online switching algorithm capable of selecting between candidate controllers based on their prediction error. This idea has been applied to robust adaptive control by defining switching adaptive laws to enhance robustness of the systems, assure boundedness of some estimates and prevent certain signals from approaching undesired regions [26, 18, 54].

Due to their discrete nature, switching control schemes lead to differential equations with a r.h.s. discontinuity. This presents a number of theoretical and practical problems when dealing with such systems:

1. The ambiguity in the meaning of *solution* of such differential equations. In fact the classical Carathéodory solutions (C-solutions) defined for ordinary differential equations some times are not valid (cf. the first example in chapter 2.3.2). So we have to define the solution in some other sense. For example Filippov solutions (F-solutions) arise from considering an appropriate differential inclusion[2].

[2]Differential inclusions are generalisation of differential equation $\dot{x} = f(x)$ where $f(\cdot)$ is a set valued function.

2. Proving the *uniqueness* and *boundedness* of solutions to this system is not straightforward, indeed, the solutions may not be unique.

3. Theoretically, a controlled system can operate by switching infinitely fast between two control signals on the switching surface. However, the means of switching in real world, sensors and actuators, cannot operate instantaneously. Therefore system trajectories travel back and forth within a neighbourhood of the switching surface at high frequency, leading to the undesirable phenomenon known as *chattering*. Chattering is often harmful as it may excite unmodeled high-frequency dynamics of the system [84]. Note that having a perfect sliding motion is desirable in sliding mode control, so we try to convert chattering to a perfect sliding motion. In contrast, sliding motion and consequently chattering are undesirable in robust adaptive control and we aim to eliminate them.

4. Simulating such a system is difficult due to the stiff differential equations which are difficult to investigate numerically. Runge-Kutta is commonly used for integrating discontinuous systems as it is less sensitive to discontinuities in the r.h.s. of differential equations [84] than multi-step or extrapolation methods. However, switching at an infinite rate in sliding motion forces the fixed step-size Runge-Kutta integrator to limit its step-size resulting in consuming considerable time to simulate the behaviour of the system at the discontinuity surface and the high frequency chattering close to the switching surface which do not provide any significant information from the design point of view.

5. By definition, any adaptive control scheme contains some *on-line* learning part which is updated in real-time. The time consumption of conventional dead-zone algorithm arisen from the discontinuity of the controller is a bottleneck from the implementation point of view.

To overcome some of the above problems, there were some attempts to smooth the switching activity. So-called 'soft projection' method introduced by Pomet *et al.* [62] is a case in point. Some other efforts were made to define several switching surfaces and build a safe switching mechanism between them [61].

An alternative solution is to add a *hysteresis* effect to the switching mechanism. In the domain of adaptive control, this was first proposed by Middleton *et al.* [45]. In connection with the problem of adaptive pole placement, they designed an adaptive algorithm which had a finite number of parameter estimators in parallel and used a hysteresis switching mechanism to decide which candidate should be put in feedback with the process at any time. This prevents changing controllers too quickly. This idea was extended by Morse *et al.* [51] to a much broader class of problems. Lozano *et al.* [42] used a transformation on estimated parameters involving a form of hysteresis to avoid division by zero in the control law in their model reference adaptive control scheme. A typical requirement for MRAC algorithms to be successful for multivariable systems is to assume that the zero structure at infinity and the high frequency gain matrix are known. Waller and Goodwin [79] considerably weakened these assumptions and extended the result of Morse *et al.* to minimum phase multivariable linear systems. They built a hysteresis dead-zone into the switching mechanism so that the switching between candidate estimators occurred only when a differential index was exceeded some threshold.

In the area of robust adaptive control, the idea of hysteresis dead-zone was used by Brogliato and Neto [3] for stabilising a class of nonlinear systems. Under some restrictive assumptions, they proved the boundedness of the estimate and ultimate boundedness of the state of the system. In this thesis, proposing a rigorous proof, we extend the idea of hysteresis dead-zone for the class of nonlinear systems in the form of an integrator chain.

1.3 Performance

The ultimate goal in control theory is to design control laws which achieve *good performance* for any member of a specified class of systems. Robust and adaptive controllers attempt to achieve good performance in different ways for the class of systems which are members of their family[45].

Performance of a closed loop is measured by a cost functional of some measurable signals (state/output/input). For example if system trajectory $x(\cdot) \in \mathcal{X}$ is available for measurement, we define a *singular* cost functional

$$J : \mathcal{X} \to \mathbb{R}^+, \tag{1.8}$$

penalising the system trajectory. A *non-singular* cost functional is defined when we take control effort into account also. That is

$$J : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^+, \tag{1.9}$$

where $\mathcal{U}$ is the function space representing the input signal space.

Consider a system $\Sigma \in \mathcal{S}$ that belongs to a set of all admissible systems. The performance of a controller $\Xi \in \mathcal{C}$ is defined either in *average case* or *worst case.* For example a singular average case performance can be defined as follows:

$$\mathcal{P} : \mathcal{S} \times \mathcal{C} \to \mathbb{R}^+, \qquad \mathcal{P}(\Sigma, \Xi) = \frac{1}{T}\int_0^T J(x(t))dt, \tag{1.10}$$

for some $T > 0$. The worst case singular performance is formulated as a supremum of the cost functional over Υ , where Υ is a set which contains all parameters (e.g. initial values, uncertainty, solutions of the closed loop, etc.) that distinguish one system from another[3]:

$$\mathcal{P} : P(\mathcal{S}) \times \mathcal{C} \to \mathbb{R}^+, \qquad \mathcal{P}(\Sigma, \Xi) = \sup_{\Upsilon} J(\cdot), \tag{1.11}$$

where $P(\mathcal{S})$ is the power set of $\mathcal{S}$.

As well as the average case, two other classes of performance measure can be defined, namely asymptotic and transient performance. Roughly speaking, asymptotic performance shows the ultimate behaviour of a system, while transient performance monitors its behaviour in time. There is no specific definition for these costs and in general any measurement that satisfies above can be used as a cost function. For example, suppose the control objective is the stability of the system. Then the singular transient cost functional

$$J(x(\cdot)) = \int_0^\infty x(t)^2 dt, \tag{1.12}$$

[3]A more precise definition e.g. for $\Sigma(f)$ defined in (1.3) would be $\mathcal{P}(\Sigma(f), \Xi) = \sup_{f\in\Delta} J(\cdot)$.

can be used to penalise the response of system. As an example of the asymptotic performance, the singular cost functional

$$J(x(\cdot)) = \limsup_{t \to \infty} \ |x(t)| \tag{1.13}$$

can be used to analyse the ultimate behaviour of the closed loop.

Having a bounded cost functional is a primary goal of any control design. Note that any stable system should have a finite cost. Particularly, we are interested in an asymptotically stable system for which (1.13) should converge to zero. Most adaptive control schemes satisfy some asymptotic performance criteria, e.g. their tracking error converges to zero. However this is not true for robust adaptive controllers. For example, we will see that the goal of dead-zone based robust adaptive controllers is merely convergence to some neighbourhood of origin rather than asymptotic stability. Another example is the projection based scheme for which the desired asymptotic performance is not guaranteed.

Transient performance is often more important. Particularly in practical applications, the question is most likely to be how fast a controller achieves its goal. Analytical quantification and systematic improvement of transient performance are open problems in adaptive control [35]. The ultimate objective of transient performance is to determine the cost value a-priori by setting the initial conditions and closed loop parameters.

For traditional adaptive linear control, even singular performance bounds such as $\int x(t)^2 dt$ are hard to quantify a-priori, and there is no systematic way to improve them. In fact, it was shown in [82] that poor initial parameter estimates may result in unacceptable transient behaviour. The design of adaptive controllers with improved transient performance is a current research topic. Among the others, Fu [15, 14] modified the single-input single-output MRAC scheme by a variable structure design (VSD), resulting in faster response, hence improving the transient performance. However, VSD suffers from r.h.s. discontinuity – complicating its analysis as leading to chattering problem. Narendra and Boskovic [55] attempted to combine the advantages of direct, indirect and VSD methods to improve the transient performance of robust adaptive control. The direct and indirect component guaranteed stability, while the variable structure component improved transient response. Other efforts include the use of fixed compensators [72, 4] to depress the effect of estimation error, and imposing a passive identifier with 'higher order tuners' [50, 57, 77]. However, in all of these controllers, only parametric uncertainties were considered and robustness was not discussed.

The adaptive backstepping design proposed by Krstic *et al.* in [36, 37] guaranteed good bound on transient performance in the ideal case. The dependence of these bounds on design parameters and initial conditions presented a systematic way for transient performance improvement [38, 35]. Recently a certainty equivalence adaptive controller has been proposed for linear adaptive control by Zhang and Ioannou [83] by combining backstepping designs with a normalised adaptive law. The resulting adaptive controller guarantees stability and transient performance without introducing higher order nonlinear terms or extra parameter estimates.

The transient cost functionals used in all the above research are singular, i.e. they depend only on state $x(\cdot)$ or output $y(\cdot)$, ignoring the control input $u(\cdot)$. The necessity of taking control input $u(\cdot)$ into account is obvious; it represents the cost which has to be paid to achieve the control objectives and it is an important factor in any application which cannot be unlimited both practically and financially. The financial constraint on input is well-known and does not need any example to clarify! Some examples of practical constraints include saturation ($\|u(\cdot)\|_{\mathcal{L}^\infty}$) or saturation rate($\|\dot{u}(\cdot)\|_{\mathcal{L}^\infty}$). Since all actuators (the physical device that produces the control signal) have some level of saturation, this affects the 'actual' transient performance [13]. Another example of a practical constraint is the fuel consumption ($\int u(t)^2 dt$) of a typical mechanical controller. However, despite their importance, none of these effects would be indicated directly in singular transient performance. So, it is particularly important to consider non-singular costs in the performance index.

The first work taking non-singular transient performance for adaptive control was by French *et al.* [11, 12, 10]. They presented a constructive bound on a-priori determined non-singular transient performance. They used the following non-singular quadratic cost functional:

$$J(x(\cdot), u(\cdot)) = \int_T x(t)^T Q x(t) + k u(t)^2 dt. \tag{1.14}$$

where $T = t \geq 0 \mid x(t) \notin \Omega_0$ for some residual set Ω_0 containing the origin. $x^T Q x$ represents a penalty on the deviation of the state x from the origin with Q the weighting matrix, and the term ku^2 is a weighted cost of control.

In this thesis as in [67, 68] we will consider bounds on the non-singular transient performance cost which penalises the $\mathcal{L}^\infty$ norm of state, control effort and control derivative, i.e.

$$J(x(\cdot), u(\cdot)) = \|x(\cdot)\|_{\mathcal{L}^\infty} + \|u(\cdot)\|_{\mathcal{L}^\infty} + \|\dot{u}(\cdot)\|_{\mathcal{L}^\infty}. \tag{1.15}$$

1.4 Motivation and Contributions of the Dissertation

As noted above, the discontinuous switching activity of the dead-zone and projection modifications leads to some potential problems such as possible loss of uniqueness of solutions and chattering. Motivated by the relatively old idea of hysteresis switching, we developed an alternative for dead-zone modification. We will prove the uniqueness and boundedness of solutions, and robustness of the system with respect to bounded disturbance. We also compare hysteresis dead-zone with conventional dead-zone and address its advantages. For example, in this method sliding motions are avoided and the chattering effect can be mitigated.

The main body of this dissertation is motivated by the fact that each of the designs mentioned in section 1.1.3 have different advantages and drawbacks. For example, dead-zone modifications require a-priori knowledge of the disturbance level, and only achieve convergence of the output to some pre-specified neighbourhood of the origin (whilst keeping all signals bounded). In particular if the disturbance vanishes, then the dead-zone controller does not typically achieve convergence to zero, the convergence remains to the pre-specified neighbourhood of the origin. On the other hand, projection modifications generally achieve boundedness of all signals, and furthermore have the desirable property that if no disturbances are present the output converges to zero. However, an arbitrarily small $\mathcal{L}^\infty$ disturbance can completely destroy any convergence of the state.

This illustrates that in the case of asymptotic performance, there are some known characterisations of *good* and *bad* behaviour. However, there are many situations in which we cannot definitively state whether a projection or dead-zone controller is superior even when only considering asymptotic performance. This motivates the need for a comparative theory to choose the *best* alternative, which is the main subject of this thesis.

There are two main challenges in developing any comparative results: firstly we must identify a problem domain in which both dead-zone and projection designs can be meaningfully compared, secondly we must identify a suitable criteria of comparison. The transient performance cost functional is a good comparison criteria as it is meaningful (as noted in 1.3) and analytical results can be derived.

We will develop a set of results which allow analytical comparisons to be made between two robust adaptive designs. In particular, we will compare dead-zone and projection based adaptive controllers with respect to transient performance. The transient performance measure will be nonsingular i.e. penalise both state and control effort. Two rigorous results are presented demonstrating situations in which the dead-zone controller is superior to the projection controller and vice versa.

These results will be extended to minimum phase linear systems with relative degree one and also nonlinear systems in the form of integrator chain. We also show that the results are applicable for hysteresis dead-zone controllers.

1.5 Summary of Contents

This dissertation is divided into six chapters among which chapters 3,4, and 5 contain the new results. These chapters follow almost the same structure as they seek the same goals for different systems. Every chapter starts with some well-known concepts required for the proof of the main results.

In chapter 3, we start with the well-known concept of adaptive control. The necessity of taking disturbance into account is illustrated with a simple example. Then the idea of robust adaptive control is explained in detail. Next, we explain the dead-zone and projection modifications, prove their robustness and discuss their advantages and drawbacks. Next, the idea of hysteresis dead-zone and how it can deal with the problems arising from the r.h.s discontinuity of the differential equations will be illustrated briefly. We defer the formal definition and the complete proof of stability until chapter 5.

In the rest of chapter 3 we also build a frame-work for transient performance comparison between the dead-zone and projection methods. Theorem **I/Ia** show the situation under which a dead-zone/hysteresis dead-zone based controller is superior to that of a projection based controller with respect to transient cost. Theorems **II/IIa** are the complementary result. In order to generate a systematic procedure, we will first prove these theorems for our simple example. This enables us to avoid any restrictive assumptions whilst expressing the idea simply.

The result of chapter 3 will be extended to minimum phase linear systems with relative degree one in chapter 4. First, the well-known concept of non-identifier-based high-gain adaptive control is introduced. Based on this, the dead-zone and projection based controllers for such systems will be analysed. Finally, the results of Theorems **I**, **II** will be established for minimum phase linear systems with relative degree one.

In chapter 5, we aim to extend the main results to that of nonlinear systems in the form of integrator chain. Following the same structure as chapters 3 and 4, we will define the system and explain the adaptive design very briefly. Next we examine the properties of dead-zone and projection based controllers for such systems. The formal definition and stability analysis of hysteresis dead-zone proposed in chapter 3 will be given next. Comparing this method to conventional dead-zone, we will address the advantages of such controllers.

In order to generalise Theorems **I**, **Ia**, **II**, and **IIa** to nonlinear systems, we consider the case of scalar nonlinear systems and integrator chain separately. The reason being the technicalities involved in nonlinear systems and the need to make some assumptions a-priori. We will see that these assumptions become more restrictive when considering integrator chain.

We conclude by indicating the directions for future work.

Chapter 2

Preliminaries

2.1 Introduction

In this chapter, we briefly review some necessary elements of mathematical analysis which are used in this dissertation. First, we give some definitions and basic facts from elementary analysis. Second, the concepts of the existence and uniqueness of the solutions in a system of differential equations with right hand side discontinuity will be discussed. Such systems will occur widely in this thesis. Finally we briefly review some necessary stability theorems. The presentation of the material on this chapter closely follows [27, 34, 35] for section 2.2 and [30, 8] for sections 2.3-2.4.

2.2 Fundamental Preliminaries

2.2.1 Normed Space

We start with definitions of normed spaces and several useful technical lemmas:

Definition 2.1. A vector space $\mathcal{X}$ over the field $\mathbb{R}$ is a set of vectors together with two operations – addition and scalar multiplication by real numbers, such that the following properties hold:

- For any vector $x, y, z \in \mathcal{X}$, the sum $x + y$ is defined, $x + y \in \mathcal{X}$, $x + y = y + x$,

$(x + y) + z = x + (y + z)$, there exists a zero vector $0 \in \mathcal{X}$ such that for all $x \in \mathcal{X}$, $x + 0 = x$, and for any $x \in \mathcal{X}$ there exists a unique element denoted by $-x$ such that $x + (-x) = 0$.

- For any numbers α, $\beta \in \mathbb{R}$, scalar multiplication αx is defined, $\alpha x \in \mathcal{X}$, $1 \cdot x = x$, $(\alpha\beta)x = \alpha(\beta x) = \beta(\alpha x)$, $(\alpha + \beta)x = \alpha x + \beta x$, and $\alpha(x + y) = \alpha x + \alpha y$.

Associating a *length* to each object in a vector space, we have a normed vector space:

Definition 2.2. A vector space $\mathcal{X}$ is a normed vector space if to each vector $x \in \mathcal{X}$, there is a real-valued norm $\|x\|_{\mathcal{X}}$ which satisfies:

- $\|x\|_{\mathcal{X}} \geq 0$ for all $x \in \mathcal{X}$, with $\|x\|_{\mathcal{X}} = 0$ if and only if $x = 0$,
- $\|\alpha x\|_{\mathcal{X}} = |\alpha| \; \|x\|_{\mathcal{X}}$, for all $\alpha \in \mathbb{R}$ and $x \in \mathcal{X}$,
- $\|x + y\|_{\mathcal{X}} \leq \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{X}}$ for all $x, y \in \mathcal{X}$.

We define 'signal' as a time domain real valued function i.e. $f : \mathbb{R}^+ \to \mathbb{R}^n$. All inputs, states, outputs in control systems are signals which are typically belong to a 'space of signals', which is often taken as a normed space $\mathcal{L}^p$ of functions $\mathbb{R}^+ \to \mathbb{R}^n$.

The $\mathcal{L}^p$ norm of a signal $x : \mathbb{R}^+ \to \mathbb{R}^n$ is defined by

$$\|x(\cdot)\|_{\mathcal{L}^p} := \left(\int_0^\infty |x(\tau)|^p \, d\tau \right)^{1/p}, \qquad 1 \leq p < \infty, \tag{2.1}$$

and

$$\|x(\cdot)\|_{\mathcal{L}^\infty} := \sup_{t \geq 0} |x(t)|, \qquad p = \infty. \tag{2.2}$$

where the right hand side norms is the Euclidean norm on $\mathbb{R}^n$ where the integral (2.1) is defined in the sense of Lebesgue[1]. If $\|x(\cdot)\|_{\mathcal{L}^p} < \infty$, we say that $x(\cdot) \in \mathcal{L}^p$ [2]. The following lemmas show some useful properties of $\mathcal{L}^p$ spaces.

[1]Throughout this dissertation, all integrals are defined in the sense of Lebesgue.

[2]Or more precisely, $[x(\cdot)] \in \mathcal{L}^p$ where $[\cdot]$ denotes the equivalence class of signals which differ only on a set of measure zero. In common with standard practice we will not distinguish further between $x(\cdot)$ and $[x(\cdot)]$.

Lemma 2.1. **(Hölder's Inequality)** If $p > 1$ and $1/p+1/q = 1$ then $f(\cdot) \in \mathcal{L}^p$, $g(\cdot) \in \mathcal{L}^q$ implies that $f(\cdot)g(\cdot) \in \mathcal{L}^1$ and

$$\|f(\cdot)\, g(\cdot)\|_{\mathcal{L}^1} \leq \|f(\cdot)\|_{\mathcal{L}^p}\, \|g(\cdot)\|_{\mathcal{L}^q}. \tag{2.3}$$

If $p = 2$, then $q = 2$ and lemma 2.1 yields the *Cauchy–Schwartz* inequality:

$$\|f(\cdot)\, g(\cdot)\|_{\mathcal{L}^1} \leq \|f(\cdot)\|_{\mathcal{L}^2}\, \|g(\cdot)\|_{\mathcal{L}^2}. \tag{2.4}$$

Lemma 2.2. **(Minkowski's Inequality)** For $p \geq 1$, $f(\cdot), g(\cdot) \in \mathcal{L}^p$ implies that $f(\cdot) + g(\cdot) \in \mathcal{L}^p$ and

$$\|f + g\|_{\mathcal{L}^p} \leq \|f\|_{\mathcal{L}^p} + \|g\|_{\mathcal{L}^p} \tag{2.5}$$

In particular Lemma 2.2 is the triangle inequality required to show that $\mathcal{L}^p$ is a normed vector space, for $p \geq 1$.

2.2.2 Properties of Functions

Let us start with some basic definitions. Since most of the functions we consider have a real-valued time domain, we restrict our definitions to the domain $\mathbb{R}^+$.

Definition 2.3. A function $f : \mathbb{R}^+ \to \mathbb{R}^n$ is continuous on $\mathbb{R}^+$, if and only if $\forall t_0 \in \mathbb{R}^+$, given $\varepsilon > 0$ there is $\delta(\varepsilon, t_0) > 0$ such that $|t - t_0| < \delta(\varepsilon, t_0)$ implies that $|f(t) - f(t_0)| < \varepsilon$. A function $f(t)$ is uniformly continuous if δ is only dependant on ε, i.e. $\forall\varepsilon > 0 \;\; \exists\delta(\varepsilon) > 0 \;\; s.t. \;\; |t - t_0| < \delta(\varepsilon) \Rightarrow |f(t) - f(t_0| < \varepsilon$. $f(t)$ is said to be piecewise continuous if $f(t)$ is continuous on any finite interval $[t_0, t_1] \subset \mathbb{R}^+$ except for a finite number of points.

Definition 2.4. A function $f : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$, if and only if for all $\varepsilon > 0$ there is $\delta > 0$ such that for any finite collection of sub intervals (α_i, β_i) of $[a, b]$ for which $\sum_{i=1}^n |\alpha_i - \beta_i| < \delta$, we have $\sum_{i=a}^n |f(\alpha_i) - f(\beta_i)| < \varepsilon$.

Definition 2.5. A function $f(\cdot)$ is bounded with respect to $\mathcal{L}^p$, if and only if there is a positive number c such that $\|f(\cdot)\|_p \leq c$. By the expression '$f(\cdot)$ is bounded', we mean $f(\cdot)$ is bounded with respect to $\mathcal{L}^\infty$.

Remark. Note the following facts which are frequently used in the analysis of adaptive schemes.

- if $f(\cdot)$ is absolutely continuous on domain [a,b], then $f(\cdot)$ is differentiable almost everywhere (a.e.)[3] in $[a,b]$.
- $\lim_{t\to\infty} \dot{f}(t) = 0$ does not imply that $f(t)$ has a limit as $t \to \infty$.
- $\lim_{t\to\infty} f(t) = c$ for some constant $c \in \mathbb{R}$ does not imply that $\dot{f}(t) \to 0$ as $t \to \infty$.
- A function $f(\cdot) \in \mathcal{L}^1$ may not be bounded and, conversely, a bounded function need not belong to $\mathcal{L}^1$. However, if $f(\cdot) \in \mathcal{L}^1 \cap \mathcal{L}^\infty$, then $f(\cdot) \in \mathcal{L}^p$ for $1 \leq p < \infty$.
- $f(\cdot) \in \mathcal{L}^p$, $1 \leq p < \infty$ does not imply that $f(t) \to 0$ as $t \to \infty$. This is not even true if $f(t)$ is bounded, but see Lemmas 2.3 and 2.5 below.

Lemma 2.3. **(Barbalat's Lemma)** If $f(t)$ is a uniformly continuous function, such that $\lim_{t\to\infty} \int_0^t f(\tau)d\tau$ exists and is finite, then

$$f(t) \to 0 \quad \text{as} \quad t \to \infty$$

Proof. See Lemma 3.2.6 in [27]. □

Note that if $\dot{f}(\cdot) \in \mathcal{L}^\infty$ then $f(t)$ is uniformly continuous on $\mathbb{R}^+$, hence an easy sufficient condition for the uniform continuity of $f(t)$ is the boundedness of $\dot{f}(\cdot)$.

Corollary 2.1. If $\dot{f}(\cdot) \in \mathcal{L}^\infty$, and $f(\cdot) \in \mathcal{L}^p$ for some $p \in [1,\infty)$, then $f(t) \to 0$ as $t \to \infty$.

Lemma 2.4. Consider nonnegative scalar functions $f : \mathbb{R}^+ \to \mathbb{R}^+$, $g : \mathbb{R}^+ \to \mathbb{R}^+$. If $f(t) \leq g(t)$ $\forall t \geq 0$ and $g(\cdot) \in \mathcal{L}^p$, then $f(\cdot) \in \mathcal{L}^p$ for all $p \geq 1$.

Lemma 2.5. Suppose $f : \mathbb{R}^+ \to \mathbb{R}^n$ is any absolutely continuous function. Suppose $f(\cdot) \in \mathcal{L}^p(0,\infty)$ for some $p \in [1,\infty)$ and $\dot{f}(\cdot) \in \mathcal{L}^q(0,\infty)$ for some $q \in [1,\infty]$. Then
(i) $f(\cdot) \in \mathcal{L}^j(0,\infty)$ for all $j \in [p,\infty)$
(ii) $\lim_{t\to\infty} f(t) = 0$.

Proof. See Lemma 2.1.7 in [22]. □

[3]A property of Ω is said to hold **almost everywhere** if the set of points in Ω where this property fails had measure zero.

2.2.3 Sets and Set-valued Functions

Let M be a subset of $\mathbb{R}^n$. A point p is said to be a *limit point* of M if there is a divergent sequence $\{b_k\}$ of distinct members of M such that for all $\epsilon > 0$ there exists $K > 0$ such that $\|b_k - p\| \le \epsilon$ for all $k > K$. A set M is said to be *closed* if it contains all its limit points, *compact* if it is closed and bounded in $\mathbb{R}^n$, and *convex* if for any two of its points a and b, all the points of a segment joining a and b also belong to M, i.e. given any $a, b \in M$, we have $\gamma\, a + (1-\gamma)\, b \in M$ for all $0 \le \gamma \le 1$. Given a set M in a vector space L, there exist convex (closed) sets which contain M. Among them, the smallest convex (closed) set is called the *convex hull* (*convex closure*) and is denoted by $conv\, M$ ($\overline{conv}\, M$). Such a set always exist, and is the intersection of all convex (convex closed) sets containing M.

If for each point p of a set $D \subset \mathbb{R}^m$ there corresponds a non-empty closed set $F(p) \subset \mathbb{R}^n$, then F is a *set-valued function.* Let $F(M)$ denote the image of the set-valued function on a set M, i.e. $F(M) = \bigcup_{p \in M} F(p)$. The norm of a set-valued function on a set M is defined as follows:

$$\|F(M)\| = \sup_{p \in M} \sup_{x \in F(p)} |x|. \tag{2.6}$$

A set-valued function F is bounded on a set M, if $\|F(M)\| < \infty$.

Throughout this thesis we define $m(M)$ the measure of a set M in the sense of Lebesgue. Specifically we note that the Lebesgue measure of a countable set is zero and $m([a, b]) = b - a$

2.3 Existence and Uniqueness

We will consider systems which can be represented by a state equation:

$$\Sigma: \ \dot{x} = f(x, t), \qquad x(t_0) = x_0. \tag{2.7}$$

where $x \in \mathbb{R}^n$. We expect that starting from a given initial state, the system Σ will evolve and its state will be defined in the (at least immediate) future time $t > t_0$. That is the local existence of a solution for system Σ. Moreover, with deterministic system, we expect that repeating the same experiment in the same conditions yields to the same result i.e. the uniqueness of the solution. This is the question of existence and uniqueness which is addressed in this section.

2.3.1 Piecewise Continuous Differential Equations

Let us denote the set of continuous and k time differentiable real valued functions on domain Ω by $\mathcal{C}(\Omega), \mathcal{C}^k(\Omega)$ respectively. We start with continuous differential equations:

Definition 2.6. Let $J_T = \{t \mid t_0 \leq t < T\}$ for some $T > t_0$. A continuous function $x : J_T \longrightarrow \mathbb{R}^n$ is said to be a solution of (2.7) over an interval J_T if $x(\cdot) \in \mathcal{C}^1(\mathbb{R}^+)$, $x(t_0) = x_0$ and $\dot{x} = f(x,t)$ for all $t \in J_T$.

For the case that the right hand side of (2.7) is piecewise continuous in t, there may be a finite set of points of time for which the solution $x(t)$ is not differentiable. We rewrite the differential equation (2.7) in its equivalent Carathéodory (integral) form:

$$x(t) = x_0 + \int_{t_0}^{t} f(x(\tau),\tau)\, d\tau \tag{2.8}$$

Definition 2.7. A function $x : J_T \longrightarrow \mathbb{R}^n$ is a Carathéodory solution (C-solution) of (2.7) if $x(t_0) = x_0$, $x(\cdot)$ is absolutely continuous on each compact subinterval of J_T and $\dot{x} = f(x,t)$ is satisfied a.e. on J_T.

The existence and uniqueness of the solution of a piecewise continuous differential equation can be investigated using the locally Lipschitz condition.

Lemma 2.6. A function $f : \mathbb{R}^n \times [t_0, t_1] \longrightarrow \mathbb{R}^n$ is said to be *locally Lipschitz* on a domain (open and connected set) $D \subset \mathbb{R}^n$ if for any compact subset $D_0 \subset D$ there exists a Lipschitz constant L such that

$$\|f(x,t) - f(y,t)\| \leq L\|x - y\| \qquad \forall x, y \in D_0, \quad \forall t \in [t_0, t_1] \tag{2.9}$$

The Lipschitz property is by default assumed to be uniform in t. A definition for *semi-globally Lipschitz* function follows by requiring inequality (2.9) to hold uniformly (with the same constant L) for $x, y \in D$. If (2.9) holds uniformly for all $x, y \in \mathbb{R}^n$ then $f(x,t)$ is *globally Lipschitz* in x.

The following lemmas demonstrate the relation between Lipschitz property and continuous differentiability:

Lemma 2.7. Let $f(x,t)$ be continuous on $D \times [t_0, t_1]$ for some domain $D \in \mathbb{R}^n$. If the Jacobian matrix $[\partial f/\partial x]$ exists and is continuous on $D \times [t_0, t_1]$, then $f(x,t)$ is locally Lipschitz in x on $D \times [t_0, t_1]$.

Lemma 2.8. In addition to above, if $[\partial f/\partial x]$ is uniformly bounded on $D \times [t_0, t_1]$, then $f(x,t)$ is semi-globally Lipschitz in x. $f(x,t)$ is globally Lipschitz in x if this holds on $\mathbb{R}^n \times [t_0, t_1]$.

Theorem 2.1. Let $f(x,t)$ be piecewise continuous in t and locally Lipschitz in x on a domain $D \subset \mathbb{R}^n$. Then there exists some $\delta > 0$ such that the state equation (2.7) has a unique C-solution over $J_\delta := [t_0, t_0 + \delta] \subset J_T$.

Proof. See theorem 2.2 in [30]. □

Theorem 2.1 is a local theorem since it guarantees existence and uniqueness only over an interval J_δ, where δ may be very small. It is possible to extend J_δ to a maximal interval of existence J_m by repeating Theorem 2.1 as follows: Start with $x_0 = x(t_0)$ and apply the theorem to establish a unique solution over $J_\delta = [t_0, t_0 + \delta]$ where δ is depend on x_0. Now denote $t_1 = t_0 + \delta$ and take $x_1 = x(t_1)$ as new initial state. If all conditions of the theorem satisfy at $(x(t_1), t_1)$ then there exist $\delta_2 > 0$ depend on x_1 such that the unique solution exists over $[t_1, t_1 + \delta_2]$. Concatenating the time pieces yields to a unique solution over $J_{\delta+\delta_2} = [t_0, t_0 + \delta + \delta_2]$. The procedure can be repeated until the conditions of Theorem 2.1 cease to hold.

Corollary 2.2. There is a maximum interval $J_m := [t_0, t_0 + m)$ where the unique solution starting at (x_0, t_0) exists[4].

Still it is possible that $J_m \subset J_T$. The solution can be extended further only if we have additional knowledge of its behaviour. The following theorem is a case in point:

[4] See [46] for the proof.

Theorem 2.2. Let $f(x,t)$ be piecewise continuous in t and locally Lipschitz in x on domain $D \subset \mathbb{R}^n$. Let W be a compact subset of D and suppose the initial point $(x_0, t_0) \in W \times [t_0, t_1]$. Suppose it is known that every solution of (2.7) lies entirely in W. Then, there is a unique solution that is defined over $J_\infty := [t_0, \infty)$.

Proof. By Theorem 2.1 there exist a unique solution over J_δ. Since the solution never leaves W, the condition of Theorem 2.1 holds for ever. So the solution can be extended indefinitely by repeated application of the theorem. □

The above results imply that if $f(x,t)$ is locally Lipschitz in x, then either a unique C-solution of (2.7) exists globally, i.e. in the interval J_∞, or there exists a finite time T_f such that as $t \to T_f$, the trajectory $x(\cdot)$ leaves any compact set. If the latter occurs, then the solution is said to have a *finite escape time.*

Seen from the control systems point of view, the existence and uniqueness of a solution resulting in the following definitions:

Definition 2.8. The system Σ given by (2.7) is said to be *well-defined at* $x_0 = x(t_0)$ if there exists a solution of (2.7) on $[t_0, \infty)$ in the sense of Carathéodory for the initial state $x_0 \in \mathbb{R}^n$. If the solution is unique, Σ is said to be *well-posed at* x_0. The system Σ is *well-posed* if it is well-posed at every initial state $x_0 \in \mathbb{R}^n$.

2.3.2 Discontinuous Differential Equations

As we mentioned in section 1.1, switching schemes are common methods in robust adaptive control theory. This results in a set of differential equations with discontinuous right hand sides for which, the definition, existence and uniqueness a of solution does not follow from the classical theory of differential equations.

The definition of a solution, given in 2.7, as an absolutely continuous function satisfying the equation almost everywhere is not always applicable for the discontinuous case. This is due to the inadequacy of the C-solution to properly define the trajectory on the discontinuity surface. Here is an example:

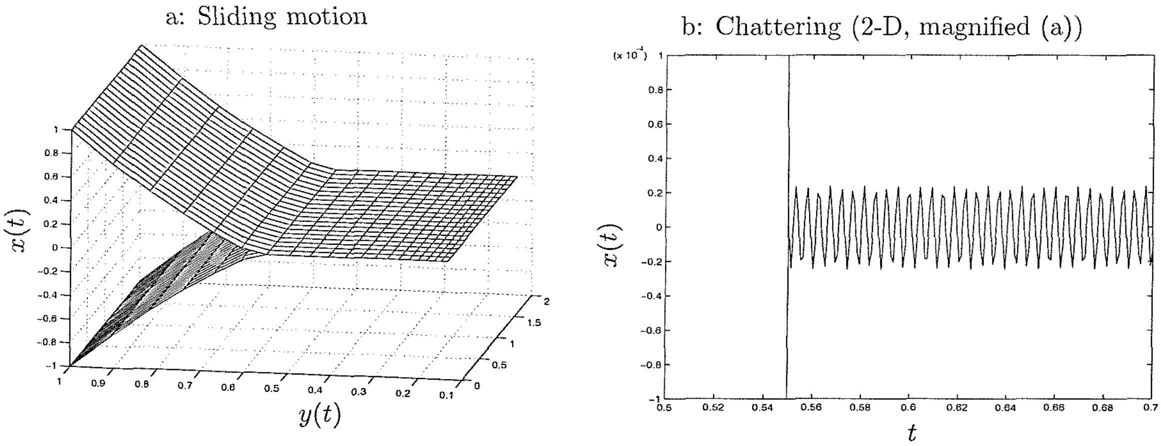

FIGURE 2.1: Sliding motion and chattering

Example 2.1. Consider the following differential equation system:

$$\dot{x} = -2x - \operatorname{sgn}(x) \qquad x(0) = x_0, \tag{2.10}$$

$$\dot{y} = -y \qquad y(0) = y_0. \tag{2.11}$$

Note that the equation (2.11) has a unique C-solution $y(t) = y_0\, e^{-t}$, and we only add (2.11) to the system for clarity of presentation of Fig. 2.1. For $x \geq 0$, where sgn(x) = 1, the solution of (2.10) is given by $x(t) = -1/2 + c_1\, e^{-2t}$, while for $x < 0$ we have that sgn(x) = −1 and the solution is $x(t) = 1/2 + c_2\, e^{-2t}$. Fig. 2.1-a demonstrates the results for a specific initial value. As t increases, each solution reaches the surface $x = 0$. The direction vector $\dot{x} = -2x - \text{sgn(x)}$ is negative if x is positive, i.e. $\dot{x}_+ < 0$, and positive if x is negative i.e. $\dot{x}_- > 0$ which means that when we are on the surface of discontinuity $x = 0$, the vector fields prevent the solution from leaving this surface either upward or downward. So any meaningful solution to this equation has to *slide* on this surface. On the other hand, the function $x(t) = 0$, as a solution, does not satisfy the differential equation (2.10) since $x(t) = 0$ implies that $\dot{x} = 0$ while the right hand side of (2.10) has the value $-\text{sgn(x)} = -1 \neq 0$ for $x(t) = 0$; Hence the C-solution does not exist.

Remark on Example 2.1 Theoretically, the sliding motion causes infinite rate switching. However, in practice, there would be some small delay in the switching operation. This delay causes the trajectory of $x(\cdot)$ to go back and forth between the two regions with a high frequency. This behaviour is known as *chattering* (Fig. 2.1-b).

There are several definitions of generalised solutions for the differential equation with discontinuous right hand side [40]. One possibility would be to interpret solutions in terms of differential inclusions. In particular, we consider solutions in the Filippov sense.

Let a physical system be described outside the δ-neighbourhood of a set of discontinuity M by the differential equation (2.7) where f is assumed to be defined on the domain $G := \mathbb{R}^n \times \mathbb{R}^+$ consists a finite number of sub-domains in each of which f is continuous up to the boundary[5] and a set M of measure zero consists of boundary points of these sub-domains. The set of points of discontinuity M often consists of a finite number of hypersurfaces.

Definition 2.9. A function $x : J_T \to \mathbb{R}^n$ is a Filippov solution (F-solution) of (2.7) if $x(0) = x_0$, $x(t)$ is absolutely continuous on each compact subinterval of J_T and $x(t)$ satisfies the differential inclusion:

$$\dot{x} \in F(x,t) = \bigcap_{\delta>0} \bigcap_{m(M)=0} \overline{conv}\, f(B_\delta(x,t) - M\,, t) \tag{2.12}$$

almost everywhere on J_T. The set-valued function $F(x,t)$ is the smallest closed convex set containing all limit values of $f(y,t)$ as $y \to x$, $t = const.$ Approaching x, y spans almost the whole neighbourhood(except for the set of measure zero) of the point x.

The negligence of sets M of measure zero, is the crucial point in definition (2.12) since it allows the solution to ignore possible misbehaviour of f on these sets [61]. The ball $B_\delta(x,t)$ of radius δ is introduced in order to provide space for this concept to work and is ultimately annihilated by taking $\delta \to 0$ via the first intersection in (2.12).

The solution of a typical discontinuous right hand side differential equation can behave in different ways, namely *regular* or *sliding* motion, as we illustrate in following:

Consider the case where M consists of a smooth surface $S = \{(x,t) \in M \mid \varphi(x,t) = 0\}$. The surface S partitions the G space into two domains G^- and G^+ (Fig. 2.2). Let $f^-(x,t)$ and $f^+(x,t)$ be the limit values of function $f(y,t)$ as y approaching x from G^-

[5]A function is continuous in the domain up to the boundary if, when its argument approaches the boundary, the function tends to finite limit, possibly to different limits for different boundary points.

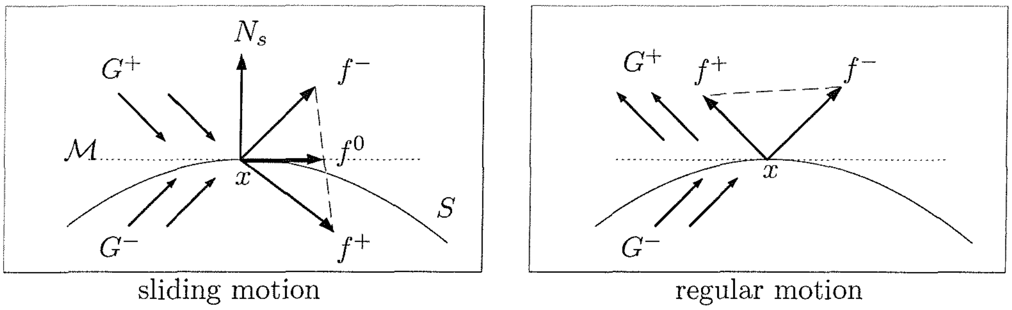

FIGURE 2.2: Filippov's solutions

and G^+ respectively

$$f^-(x,t) = \lim_{(y,t)\in G^-,\, y\to x} f(y,t) \qquad f^+(x,t) = \lim_{(y,t)\in G^+,\, y\to x} f(y,t). \tag{2.13}$$

Let $\mathcal{M}$ be the plane tangential to S and $N_s(x)$ be the normal to the surface S at the point x directed toward G^+. We denote the projections of the vectors $f^-(x,t)$ and $f^+(x,t)$ onto $N_s(x)$ by $f_N^-(x,t)$ and $f_N^+(x,t)$ respectively:

$$f_N^- = \frac{\nabla\varphi \cdot f^-(x,t)}{|\nabla\varphi|}, \qquad f_N^+ = \frac{\nabla\varphi \cdot f^+(x,t)}{|\nabla\varphi|} \tag{2.14}$$

The set $F(x,t)$ thus can be defined by (2.12) as a linear segment joining the end points of the vectors $f^-(x,t)$ and $f^+(x,t)$. For $t_0 < t^* < t_1$, if this segment lies on one side of $\mathcal{M}$ at $x^* = x(t^*)$, then both $f^-(x^*,t^*)$ and $f^+(x^*,t^*)$ point in the same region i.e. $f_N^- f_N^+ > 0$, and therefore the solution approach S on one side and leave S on the other side. In this case, which is referred to as *regular motion*, the F-solution and the C-solution are equivalent.

If this segment intersects the plane $\mathcal{M}$, that is, $f_N^- \geq 0$, $f_N^+ \leq 0$ and $(f_N^- - f_N^+) > 0$, then both $f^-(x^*,t^*)$ and $f^+(x^*,t^*)$ are directed towards the surface S forcing the trajectory to remain on S. This behaviour is referred to as *sliding motion*. In this case the C-solution does not exist. The intersection point of the segment $F(x,t)$ and tangent plane $\mathcal{M}$ is the endpoint of the vector $f^o(x^*,t^*)$ which determines the velocity of motion along the surface S in the x space:

$$\dot{x} = f^0(x,t) = \alpha f^+(x,t) + (1-\alpha) f^-(x,t), \qquad 0 \leq \alpha \leq 1, \tag{2.15}$$

which means that the function $x(t)$ satisfying (2.15) is assumed to be a solution of equation (2.7) in the sense of definition 2.9. α is calculated such that the trajectory remains on the surface S.

The existence and continuation of the solution requires (Theorems 4,5 of [8]) that $f^+(x,t), f^-(x,t)$ be locally Lipschitz in x away from S (i.e. in G^- and G^+). For uniqueness of the solutions (see Lemma 2 and Theorem 2 in [8]) the case when trajectories point away from S along both f^+ and f^- (i.e. $f_N^- < 0$, $f_N^+ > 0$) needs to be disallowed.

It can be shown [8, 69] that a system with a step discontinuity at S has a unique F-solutions if at each point $(x^*, t^*) \in S$ at least one of the inequalities

$$f_N^+(x^*, t^*) < 0, \quad or \quad f_N^-(x^*, t^*) > 0 \tag{2.16}$$

is satisfied. Therefore, if $f^-(x,t)$ and $f^+(x,t)$ are locally Lipschitz in x in the regions G^- and G^+ respectively, the uniqueness of the F-solution guarantees if (2.16) hold.

2.4 Lyapunov Stability of Dynamic Systems

In this section we briefly review some well known theorems regarding to the Lyapunov stability which are used in this thesis. The proof of theorems are well known and can be found in almost all text books in nonlinear control (see e.g. [30]). The systems that we are consider in this section have autonomous dynamics represented in the following form:

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0, \tag{2.17}$$

where $f : U \to \mathbb{R}^n$ is a vector field of class C^1 on $U \subset \mathbb{R}^n$ which is taken to be locally Lipschitz. Let us start with reviewing some key concepts:

Definition 2.10. The origin $\bar{x} = 0$ is called an *equilibrium point* of (2.17) if $f(0) = 0$.

The equilibrium point at the origin is said to be stable if given x_0 close to origin, the trajectory $x(t)$ remains in a neighbourhood of origin thereafter. More precisely:

Definition 2.11. The equilibrium point $\bar{x} = 0$ of (2.17) is said to be *stable* if given $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ so that

$$\|x_0\| < \delta(\epsilon) \Longrightarrow \|x(t)\| < \epsilon, \qquad \forall t \geq 0. \tag{2.18}$$

$\bar{x} = 0$ is asymptotically stable if (i) $\bar{x} = 0$ is stable and (ii) $\bar{x} = 0$ is attractive i.e. δ can be chosen such that

$$\|x_0\| < \delta \Longrightarrow \lim_{t \to \infty} |x(t)| = 0. \tag{2.19}$$

Definition 2.12. A function $\gamma : [0, r) \to \mathbb{R}^+$ belongs to class $\mathcal{K}$, if it is continuous, strictly increasing, and $\gamma(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $r = \infty$ and $\gamma(s) \to \infty$ as $s \to \infty$.

Definition 2.13. A continuous function $V : U \to \mathbb{R}^+$ is called a *positive semidefinite function*, if there exists a continuous function $\gamma : U \to \mathbb{R}^+$, $\gamma(\cdot) \geq 0$ on U, such that

$$V(0) = 0, \quad V(x) \geq \gamma(\|x\|), \quad \forall x \in \mathbb{R}^n. \tag{2.20}$$

$V(\cdot)$ is called *positive definite* if $\gamma(\cdot) \in \mathcal{K}$, and is *positive definite and radially unbounded* if $\gamma(\cdot) \in \mathcal{K}_\infty$. $V(\cdot)$ is *negative semidefinite* or *negative definite* if $-V(\cdot)$ is positive semidefinite or positive definite, respectively.

Theorem 2.3. **(Lyapunov's Theorem)** Consider the dynamical system (2.17) and let $\bar{x} = 0 \in U$ be an equilibrium point. Suppose there exists a continuously differentiable positive definite function $V(x) : U \to \mathbb{R}$ and denote

$$\dot{V}(x(t)) = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x} f(x(t)), \tag{2.21}$$

the time derivative of $V(\cdot)$ along the trajectories of (2.17). Then the equilibrium point at the origin is stable if $\dot{V}(x)$ is negative semidefinite and it is asymptotically stable if $\dot{V}(x)$ is negative definite.

In the case of a negative semidefinite $\dot{V}$, more can be said. This is given by LaSalle's Invariance Theorem:

Definition 2.14. Let $x(t)$ be a solution of (2.17). By an *invariant set* of (2.17) we mean a set M such that if the trajectory belongs to M at some time instant t_1, it remains in M for all future and past time, i.e. $\forall t \in \mathbb{R}$

$$x(t_1) \in M \Longrightarrow x(t) \in M. \tag{2.22}$$

If (2.22) is true only for all future time $t \geq t_1$, we say M is *positively invariant set.*

Theorem 2.4. **(LaSalle's Theorem)** Let Ω be positively invariant set of (2.17). Consider $V : \Omega \to \mathbb{R}$ of class $\mathcal{C}^1$ such that $\dot{V}(x) \leq 0$, $\forall x \in \Omega$. Let $E = \{x \in \Omega \mid \dot{V}(x) = 0\}$, and let M be the largest invariant set in E. Then every solution $x(t)$ starting in Ω approaches M as $t \to \infty$.

The following corollary addresses a situation in which asymptotic stability can be guaranteed:

Corollary 2.3. Consider the dynamical system (2.17) and let $\bar{x} = 0$ be an equilibrium point. Suppose there exists a positive definite function $V : D \to \mathbb{R}$ of class $\mathcal{C}^1$ such that $\dot{V}(x)$ is negative semidefinite in a neighbourhood D of origin. Let $E = \{x \in D \mid \dot{V}(x) = 0\}$, and suppose that no solution other than $x(t) \equiv 0$ can stay forever in E. Then $\bar{x} = 0$ is asymptotically stable.

The above results are true *globally* if we let domains $U, D, \Omega = \mathbb{R}^n$ and replace V by a continuously differentiable, positive definite, radially unbounded function.

Finally the following lemma is useful in bounding V from $\dot{V}$:

Lemma 2.9. Suppose $f(\cdot) \in \mathcal{C}(\mathbb{R}^+)$ and $V(\cdot) \in \mathcal{C}^1(\mathbb{R}^+)$ and α, β be finite positive constants. Then

$$\dot{V}(t) \leq -\alpha V(t) + \beta f(t)^2, \qquad \forall t \geq t_0 \geq 0,$$

implies that

$$V(t) \leq V(t_0)\, e^{-\alpha(t-t_0)} + \beta \int_{t_0}^{t} e^{-\alpha(t-\tau)}\, f(\tau)^2 d\tau, \qquad \forall t \geq t_0 \geq 0.$$

Moreover, if $f(\cdot) \in \mathcal{L}^2$, then $V(\cdot) \in \mathcal{L}^1$ and

$$\|V(\cdot)\|_{\mathcal{L}^1} \leq \frac{1}{\alpha}\left(V(0) + \beta \|f(\cdot)\|^2_{\mathcal{L}^2}\right)$$

Proof. See Appendix B in [35]. □

Chapter 3

Adaptive Control: Robustness and Performance

3.1 Introduction

The first step in control design is building a *sufficiently accurate* mathematical model of the plant. The term 'sufficiently accurate' has been used since, apart from very simple systems, having an *exact* mathematical model of a physical plant is almost impossible and even so, the implementation is practically inappropriate [27]. So by modeling, we adequately capture essential features of the plant for analysis and control synthesis. This abstraction introduces the *system uncertainty* phenomenon which is the difference between the nominal mathematical model and the actual physical plant. System uncertainty generally includes other concepts such as the imperfect knowledge of the system dynamics, and unknown or uncertain system parameters.

As we mentioned in section 1.1, system uncertainties, by their nature, cannot be modelled in general, and therefore can lead to a wide range of problems, from performance reduction to eventually loss of stability. Hence, any effective control system should be, up to some degree, *robust* against uncertainties. *Robust control* is the specific area in control theory that address the control of these uncertainties. When the uncertainties appear only in system parameters, they can be addressed by the area of *adaptive* *control.* In addition to system uncertainty, an applicable adaptive controller must be capable of dealing with *environmental uncertainties* such as noise and other (bounded) disturbances. That is the subject of our interest: *robust adaptive control.*

The contents of this chapter have been arranged into two parts. In the first part, we will briefly review the idea of adaptive control by a simple example. Then we will illustrate the necessity of considering robustness in adaptive control schemes by showing how fragile the design is when bounded disturbances are present. Amongst the proposed modifications to achieve robustness, we will examine two well known methods, namely *dead-zone* [66] and *projection* [33] based controllers. Stating the properties of dead-zone and projection modifications, we will notice that these two mechanisms, like other switching schemes, suffer from the difficulty of coping with differential equations with discontinuous right hand sides. Our contribution to this part is offering an alternative solution, so-called *hysteresis dead-zone*, to overcome this problem. The hysteresis algorithm enables us to replace the discontinuity of the adaptive law in state x by a piecewise continuity in time, whose solution can be interpreted classically by section 2.3.1. The same approach can be used for projection controllers though there is a well known simpler method referred to as 'smooth projection' [62].

Alongside the hysteresis algorithm, our goal is to elucidate the advantages and drawbacks of dead-zone and projection methods. We will observe that both methods require appropriate a-priori knowledge of unknown factors, i.e. a-priori knowledge is required of either the:

- Maximum disturbance level for dead-zone modification, $d_{\max}$.
- Maximum parametric uncertainty level for projection modification, $\theta_{\max}$.

Therefore, the accomplishment of each method directly depends on the quality of our knowledge of the uncertainty. For example, consider a perturbed physical system with poor a-priori information about an unknown parameter θ, and suppose we want to use a projection based controller. We will show in section 3.4.3 that in order to construct such controller, we need to define a convex set Π based on $\theta_{\max}$ (the a-priori known upper bound for $|\theta|$). We also need to make sure that $\theta \in \Pi$. Therefore, if the a-priori $\theta_{\max}$ leads us to a wrong decision about the size of Π in such a way that $\theta \notin \Pi$, then practically we prevent the parameter estimator $\hat{\theta}$ from reaching its desired value θ. The same sort of conclusion can be made with dead-zone modification when, based on poor (or unreliable) information $d_{\max}$, the size of the dead-zone has been chosen wrongly. So, reliable a-priori information is essential. Note that the two types of controller require different information.

On the other hand, with regards to asymptotic performance, the behaviours of these two controllers are different. Dead-zone controllers only achieve convergence of the state/output/error to some pre-specified neighbourhood of the origin (whilst keeping all signals bounded). In particular if the disturbance vanishes, then the dead-zone controller does not typically achieve convergence to zero: the convergence guarantee remains only to the pre-specified neighbourhood of the origin [54]. Projection based controllers generally achieve only boundedness of all signals, but have the desirable property that if no disturbances are present, then the state/output/error converges to zero, however, an arbitrarily small $\mathcal{L}^\infty$ disturbance can completely destroy any convergence of the output[27].

This illustrates that in the case of asymptotic performance, there are some circumstances under which it can be clearly stated as to which adaptive controller is superior. However, there are many situations in which we cannot definitively state whether a projection or dead-zone controller is superior even when only considering asymptotic performance.

In the second part of this chapter, we will establish a general frame-work for a comparative theory to choose the *best* alternative with respect to a cost functional. We will define a *transient performance cost functional* ($\mathcal{P}$) as our criterion of comparison. Moreover, in contrast with the most results in adaptive control which are confined to singular performances, the transient performance measure will be nonsingular, i.e. penalise both the state (x) and the input (u) of the plant. We will identify circumstances in which a dead-zone based adaptive controller is superior to the projection based adaptive controller with respect to $\mathcal{P}$, and *vice versa.*

A simple scalar system has been used to develop the results and illustrate the trade-offs between the designs in the simplest manner. Finally, a structural procedure is proposed to develop the idea for the more complex systems described in later chapters.

3.2 System Description and Basic Design

Consider the following SISO system

$$\Sigma(x_0, \theta, d(\cdot)): \quad \begin{aligned} \dot{x}(t) &= f(x(t), \theta, u(t), d(t)), \qquad x(0) = x_0 \\ y(t) &= h(x(t)), \end{aligned} \tag{3.1}$$

where $x(\cdot) \in \mathbb{R}^n$ is the state, $u(\cdot), y(\cdot) \in \mathbb{R}$ are the control and output respectively, $d(\cdot)$ belongs to a class of bounded disturbance signals $\mathcal{D}$, and θ represents the parametric uncertainty of the system. We assume f, h are continuous and u, d are piecewise continuous.

Equation (3.1) represents a set of systems for different choices of θ. As explained in section 1.1.2, the objective of an adaptive control design is to provide a *single* controller that can be applied to any member of family (3.1). That is, defining a controller

$$\Xi: \quad u(t) = j(\hat{\theta}(t), z(t)) \tag{3.2}$$

$$\dot{\hat{\theta}}(t) = g(\hat{\theta}(t), z(t)), \qquad \hat{\theta}(0) \in \mathbb{R}, \tag{3.3}$$

consisting of a feedback law $j(\hat{\theta}, z)$ which, in turn, depends on a *tuning* parameter $\hat{\theta}(\cdot)$ generated by *adaptive law* (3.3). We require for all choices of θ, $d(\cdot) \in \mathcal{D}$ that this controller fulfills some or all of the following objectives:

G1. Existence (and uniqueness) of the solution of the closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi)$.

G2. Boundedness of closed loop signals $x(\cdot), \hat{\theta}(\cdot), u(\cdot)$.

G3. Asymptotic stability of the system state/output.

The function $z(\cdot)$ is defined based on the control strategy. If the state vector x is measurable, we let $z(t) := x(t)$. For the output feedback systems $z(t) := y(t)$.

In order to achieve G1–G3, we need to specify functions f and h. From now on we assume f belongs to some known class of linear/nonlinear systems for which there exists an adaptive controller in the form of (3.2)–(3.3), such that the disturbance free ($\mathcal{D} = \{0\}$)

closed loop system satisfies G1–G3. Particularly, we are interested in minimum phase linear systems with relative degree one (chapter 4), and nonlinear systems in the form of integrator chain (chapter 5).

In the following we explain the idea of robust adaptive control by using a simple example.

3.3 A Simple Example

Let us define the system (3.1) in the following form:

$$\begin{aligned} \Sigma(x_0, \theta, d(\cdot)) : \quad \dot{x}(t) &= \theta x(t) + u(t) + d(t), \qquad x(0) = x_0 \\ y(t) &= x(t). \end{aligned} \tag{3.4}$$

Suppose $y(\cdot) \in \mathbb{R}$ is available for measurement and $\theta \in \mathbb{R}$ is an unknown constant. Let $\mathcal{D} = \{0\}$ and suppose the design objective is to asymptotically stabilise the system i.e. to find a controller so that $x(t) \to 0$ as $t \to \infty$.

We start designing the controller Ξ in (3.2) by choosing a feedback law

$$u(t) = -ax(t) - \hat{\theta}(t)x(t), \tag{3.5}$$

for some constant $a > 0$. This yields to the following differential equation:

$$\dot{x}(t) = -ax(t) + \tilde{\theta}(t)x(t), \qquad x(0) = x_0, \tag{3.6}$$

where $\tilde{\theta}(t) := \theta - \hat{\theta}(t)$. Defining the Lyapunov function

$$V(x(t), \tilde{\theta}(t)) = \frac{1}{2}x(t)^2 + \frac{1}{2}\tilde{\theta}(t)^2, \tag{3.7}$$

we observe that

$$\dot{V}(x(t), \tilde{\theta}(t)) = -ax(t)^2 + \tilde{\theta}(t)\left(x(t)^2 - \dot{\hat{\theta}}(t)\right), \tag{3.8}$$

which can be made negative semi-definite by choosing the *adaptive law*

$$\dot{\hat{\theta}}(t) = x(t)^2, \qquad \hat{\theta}(0) \in \mathbb{R}. \tag{3.9}$$

It follows by LaSalle's theorem 2.4 that $x(\cdot), u(\cdot) \in \mathcal{L}^\infty$, and $x(t) \to 0$ as $t \to \infty$.

So, we have given an *adaptive controller* Ξ which is a *time varying* feedback containing a parameter estimator which is updated on-line such that the closed loop system asymptotic stability is preserved:

$$\begin{aligned} \Xi: \quad & u(t) = -ax(t) - \hat{\theta}(t)x(t) \\ & \dot{\hat{\theta}}(t) = x(t)^2, \qquad \hat{\theta}(0) = 0. \end{aligned} \tag{3.10}$$

3.3.1 Parameter Drift

It is well known that even a small $\mathcal{L}^\infty$ disturbance may cause a drift of the parameter estimates $\hat{\theta}(\cdot)$, (see eg. [6]). The following example illustrates this phenomenon.

Example 3.1. Consider the closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi)$ defined by (3.4), (3.10) respectively where $x_0 = 2, a = 1$ and $d(t) = 1$ for all t. The graph of the system trajectory $x(t)$ and parameter estimator $\hat{\theta}(t)$ are shown in Fig. 3.1.

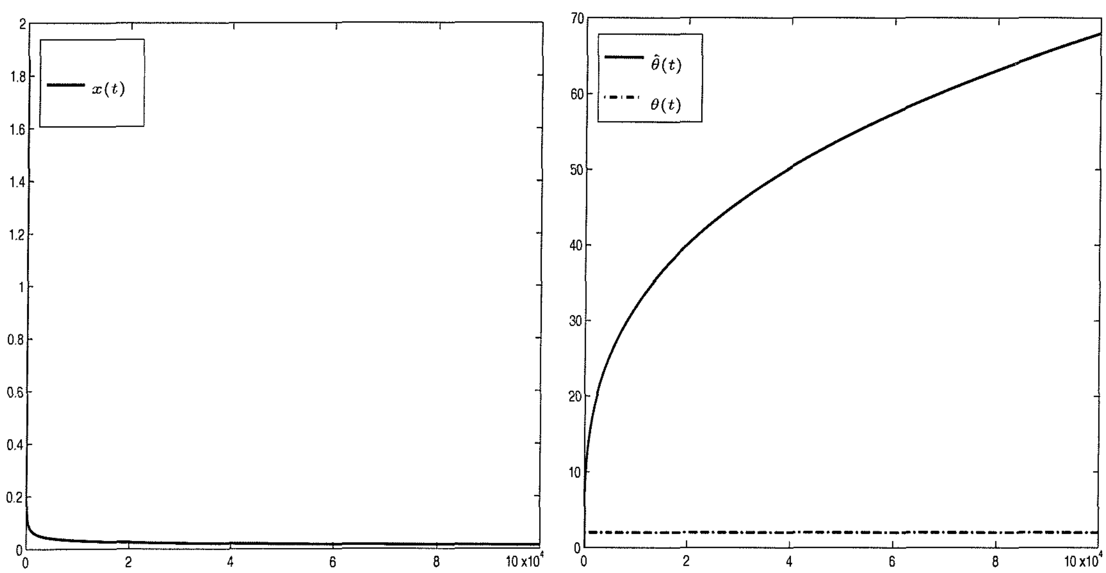

FIGURE 3.1: Parameter drift

As it has been shown in Fig. 3.1 the simulation illustrates $\hat{\theta}(t) \to \infty$ as $t \to \infty$. This can be proved analytically (see proposition 3.1 later in section 3.5.3). An interesting point in Figure 3.1 is that, despite the drift in $\hat{\theta}(t)$, the asymptotic stability of system trajectory $x(\cdot)$ has been achieved, i.e. $\lim_{t\to\infty} x(t) = 0$ (see the proofs of theorems 4.1

and 5.1 in later chapters). That is, the singular transient performance (1.12) or the singular asymptotic performance (1.13) are bounded.

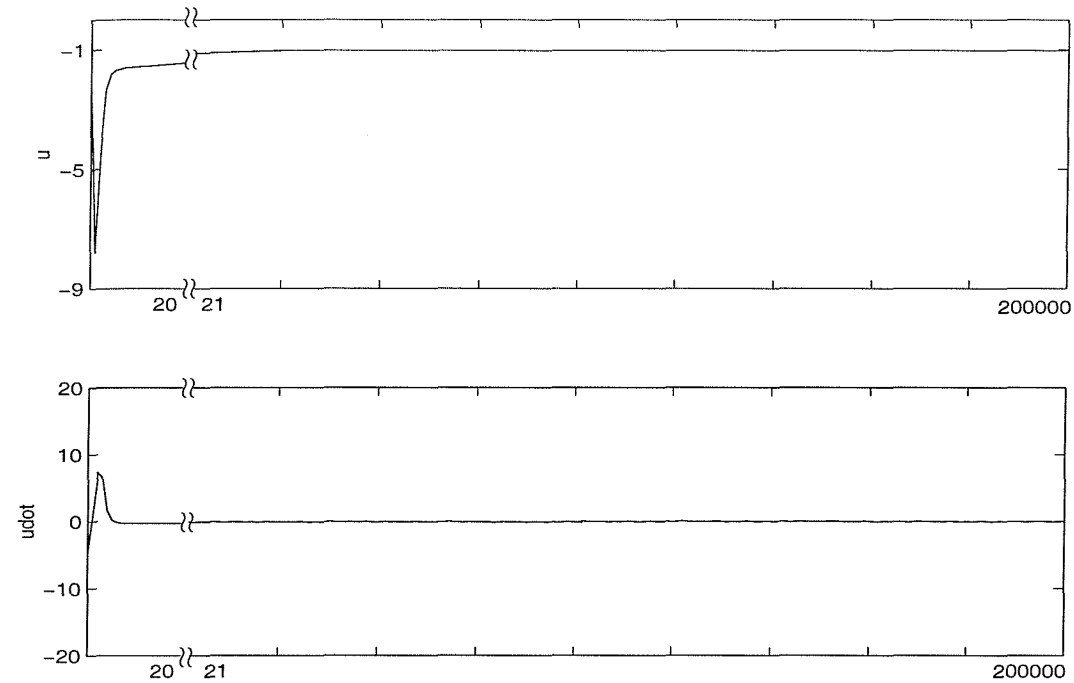

FIGURE 3.2: $u(t)$ and $\dot{u}(t)$ for example 3.1 (constant disturbance)

One of our main goals in this thesis is to explicitly pinpoint the problem caused by parameter drift. We show that these problems will be highlighted, if we take non-singular transient performance into account. By non-singular, we mean considering the transient performance of both state $x(\cdot)$ and control $u(\cdot)$. The result is not straightforward since, surprisingly, even the control $u(\cdot)$ behaves well under some circumstances. For instance, it has been shown in Fig. 3.2 that for the above example $\|u(\cdot)\|_{\mathcal{L}^\infty} < 6$ and $u(t) \to 1$ as $t \to \infty$. In fact, one can prove that if $\lim_{t\to\infty} u(t)$ exists for such systems, then $\lim_{t\to\infty} u(t) = d$ (see Lemma A.1 in appendix). From this, for this example one may reach to the following conclusion: $\hat{\theta}(\cdot)$ is only a *virtual* parameter which has no correspondence in real physical system and therefore it is possible that it can drift to a very large value without any 'bad' effect on 'system performance'.

However, taking $\dot{u}(t)$ into account, we can show that even a small step change on disturbance $d(\cdot)$, can cause some problem (Fig. 3.3). This motivates the introduction of the control derivative into cost functional.

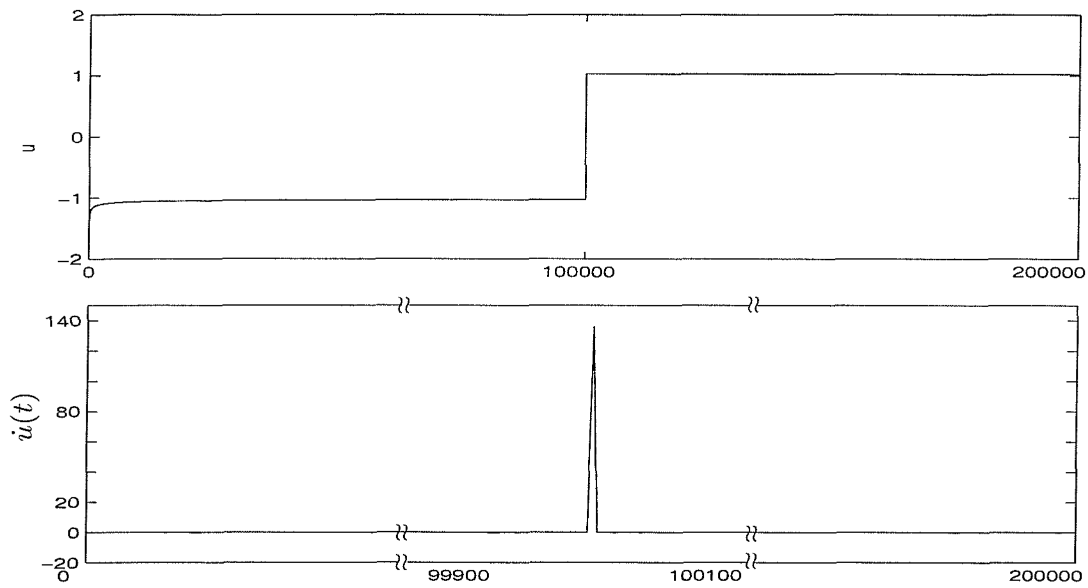

FIGURE 3.3: $u(t)$ and $\dot{u}(t)$ for example 3.1 (step disturbance)

Once we have shown the effect of bounded disturbances on transient performance of the robust adaptive control systems, the need to consider non-singular transient performance will be clarified.

3.4 Modified Algorithms

As we have mentioned in section 1.1.3, the adaptation law is often modified to avoid parameter drift. However, it is important to note that the modification algorithms are a *trade-off* between robustness and the ideal properties G1-G3. In the following sections, we briefly describe two popular methods, i.e. the dead-zone and projection modifications.

3.4.1 Dead-zone Modification

The idea of the dead-zone modification [6, 60, 66] is to divide the state/output space into two mutually exclusive regions by a smooth switching surface, and modify the adaptive law in (3.3) so that the adaptive mechanism is 'switched off' when measurable signal $z(\cdot)$ (state $x(\cdot)$ or output $y(\cdot)$) lies inside a region Ω_0 where the disturbance has a destabilising effect on the dynamics. A-priori knowledge of the size of the disturbance is typically used to define the size of the dead-zone. Let $d_{\max}$ be the a-priori known of the upper bound of the disturbance level, i.e. $d_{\max} \geq \|d(\cdot)\|_{\mathcal{L}^\infty}$ for all $d(\cdot) \in \mathcal{D}$. For scalar systems (3.4), the dead-zone region $\Omega_0(d_{\max})$ can be simply defined by $\Omega_0(d_{\max}) = [-\eta_0, \eta_0]$, where $\eta_0 = \varrho(d_{\max})$ and $\varrho : \mathbb{R}^+ \to \mathbb{R}^+$. For higher order systems with measurable state vector x, the dead-zone region $\Omega_0(d_{\max})$ is defined as follows:

$$\Omega_0(d_{\max}) = \{\, x \mid x^T P x \leq \varrho(d_{\max})^2 \,\}, \tag{3.11}$$

where P is an appropriate symmetric positive definite matrix.

We modify the adaptive law (3.3) by

$$\dot{\hat{\theta}}(t) = D_{\Omega_o(d_{\max})}(z)\, g(\hat{\theta}(t), z(t)), \qquad \hat{\theta}(0) = 0, \tag{3.12}$$

where $D_\Phi(z) := 0$ if $z \in \Phi$ and $D_\Phi(z) := 1$, elsewhere. Consequently the dead-zone controller $\Xi_D(d_{\max})$ is defined by

$$\begin{aligned} \Xi_D(d_{\max}) :\quad & u(t) = j(\hat{\theta}(t), z(t)) \\ & \dot{\hat{\theta}}(t) = D_{\Omega_0(d_{\max})}(z)\, g(\hat{\theta}(t), z(t)), \quad \hat{\theta}(0) = 0, \quad \eta_0 = \varrho(d_{\max}), \end{aligned} \tag{3.13}$$

We will study the robustness of the closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi_D(d_{\max}))$ for linear and certain nonlinear systems and for various choices of $j(\cdot,\cdot), g(\cdot,\cdot)$ in the following two chapters. We will prove that:

D1) The closed loop solution exist (but it is not necessary unique).

D2) All closed loop signals $x(\cdot), \hat{\theta}(\cdot), u(\cdot)$ are bounded.

D3) The state/output $z(t)$ converges to Ω_0 as $t \to \infty$.

The following theorem analyses the stability of the closed loop in the form of example 3.3, for which the dead-zone controller is defined as follows:

$$\begin{aligned} \Xi_D(d_{\max}) :\quad & u(t) = -ax(t) - \hat{\theta}(t)x(t) \\ & \dot{\hat{\theta}}(t) = D_{\Omega_0(d_{\max})}(x)\, x(t)^2, \quad \hat{\theta}(0) = 0, \quad \eta_0 := \varrho(d_{\max}) = \frac{d_{\max}}{a}. \end{aligned} \tag{3.14}$$

We will delay the complete proof to chapter 5 in which the stability of more complex systems will be examined.

Theorem 3.1. Consider the closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi_D(d_{\max}))$ defined by (3.4), (3.14), where $d(\cdot) \in \mathcal{L}^\infty$. Suppose $d_{\max}$ is such that $d_{\max} \geq \|d(\cdot)\|_{\mathcal{L}^\infty}$. Define continuous function

$$V_0(x_0, \theta, d_{\max}) := \frac{1}{2}\max(x_0^2, \eta_0^2) + \frac{1}{2}\theta^2. \tag{3.15}$$

Then for any $x_0 \in \mathbb{R}$,

D1. The solution $(x(\cdot), \hat{\theta}(\cdot)) : \mathbb{R}^+ \to \mathbb{R}^2$ exists.

D2. $x(\cdot), u(\cdot), \hat{\theta}(\cdot)$ are uniformly bounded as continuous functions of $V_0(x_0, \theta, d_{\max})$.

D3. $x(t) \to \Omega_0$ as $t \to \infty$.

Proof. This is a simple application of Theorem 5.2. To bypass the technical difficulties arising from discontinuous r.h.s. differential equations and avoid repetition, we defer proving D1,D3 until then (see Theorem 5.2). An outline proof of D2 (which is directly required later in this chapter) is as follows: Define the Lyapunov function

$$V(x(t), \tilde{\theta}(t)) = \frac{1}{2}x(t)^2 + \frac{1}{2}\tilde{\theta}(t)^2. \tag{3.16}$$

Considering different situations of x_0 inside, outside, or on the boundary of the dead-zone $\Omega_0(d_{\max})$, eventually yield to $V(x(t), \tilde{\theta}(t)) \leq V_0(x_0, \theta, d_{\max})$ for all $t \geq 0$ (see Theorem 5.2). From this and (3.16),(3.14) one can easily bound $x(\cdot), \hat{\theta}(\cdot), u(\cdot)$ uniformly as a continuous function of $V_0 := V_0(x_0, \theta, d_{\max})$:

$$|x(t)| \leq \sqrt{2V_0}\,, \quad |\hat{\theta}(t)| \leq 2\sqrt{2V_0}\,, \quad |u(t)| \leq a\sqrt{2V_0} + 4V_0. \tag{3.17}$$

□

Figure 3.4 illustrates the result for the example 3.1.

Remark 3.1. The right choice of the size of dead-zone $\Omega_0(d_{\max}) = [-\eta_0, \eta_0]$ is the crucial point in this design. We have assumed that the disturbance $d(\cdot)$ belongs to a class of bounded disturbances $\mathcal{D}$. However, the actual disturbance level is not known. In fact the upper bound $d_{\max}$ is assumed to be the only available a-priori information about

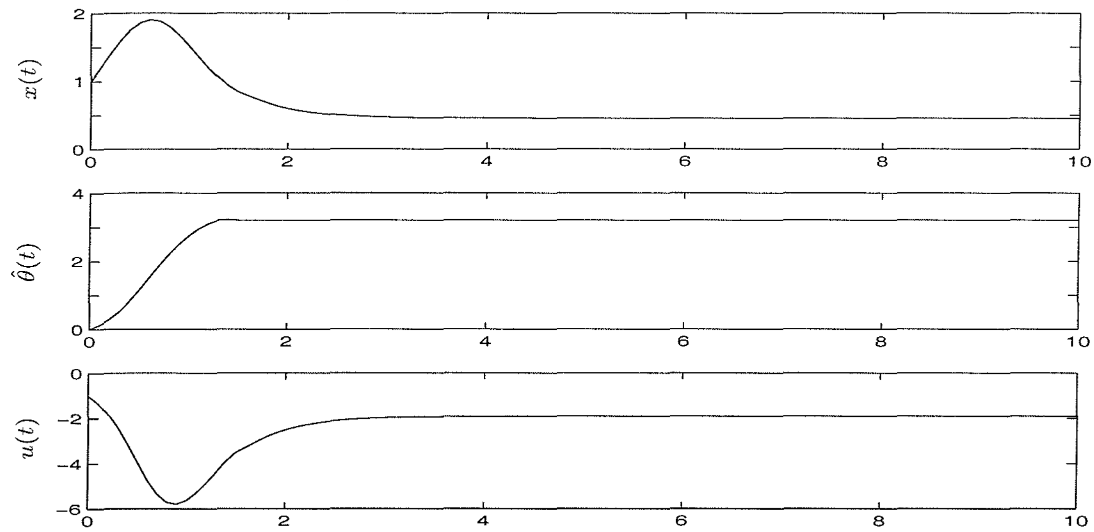

FIGURE 3.4: Dead-zone modification for example 3.3

the disturbance $d(\cdot)$. Therefore, it seems to be logical if we rely on this knowledge and define η_0 as a function of $d_{\max}$ (see the discussion in section 3.5.5). As we have shown in Theorem 3.1, a suitable choice of $\eta_0 = d_{\max}/a$ assures the boundedness of signals and converges $x(t)$ to Ω_0. However, by this choice, the control design is directly dependent on the reliability of a-priori $d_{\max}$. Observe that $|\Omega_0| \to \infty$ as $d_{\max} \to \infty$ i.e. $\dot{\hat{\theta}}(t) = 0$ for all $t \in [0, \omega)$, $\omega \to \infty$. In other words, the most conservative information yields to the least adaptation ($\dot{\hat{\theta}}(t) = 0$). We will discuss this matter extensively in section 3.5 by showing that how conservative information potentially degrades the transient performance of the closed loop (Theorem **II**).

Remark 3.2. Due to the switching nature of the dead-zone modification, the differential equations governing the closed loop have discontinuous r.h.s. for which, the classical definition of a solution is not valid. We, therefore, consider solutions in the Filippov sense. However, as we mentioned in chapter 2.3.2, there is a possibility of loss of uniqueness as well as sliding motion, or practical chattering. To overcome these problems we introduce so-called 'hysteresis dead-zone'.

3.4.2 Hysteresis Dead-zone

Although the sliding motion is essential in sliding mode robust control, it is an undesirable phenomenon from the adaptive control point of view. In practice, sliding motions cause chattering and also may lead to a theoretical loss of uniqueness of solutions. To overcome this problem one should avoid the circumstances which cause sliding motion, i.e. the conditions we explained in section 2.3.2. One method is to modify the differential equations in such a way that the right hand side discontinuity is replaced by piecewise continuity in time. This can be implemented by adding a 'hysteresis' effect to the switching mechanism [45, 51, 79]. Motivated by this relatively old idea, we developed an alternative for dead-zone modification. Hysteresis dead-zone controllers, firstly introduced by Brogliato and Neto [3], have some important analytical as well as practical advantages over conventional dead-zone based controllers. We leave the formal definitions and stability analysis to chapter 5. In this section we simply illustrate the idea.

Consider the dead-zone region Ω_0 in (3.11). Let us define another region Ω_1:

$$\Omega_1 = \{ x \mid |x| \leq \eta_1 \}, \tag{3.18}$$

where $\eta_1 = (1 + \beta)\eta_0$ for some small $\beta > 0$. We also define 'switching' time sequences $\{t_i\}$, 'storing' sequence $\{t_i^s\}$, and 'restoring' time sequence $\{t_i^r\}$. We replace operator D_{Ω_0} by its hysteresis version $H_{\Omega_0,\,\Omega_1}$, and define hysteresis dead-zone controller on the interval $t_i^r \leq t < t_{i+1}^r$ for $i \geq 0$ as follows:

$$\begin{aligned}\Xi_H(d_{\max}) : u(t) &= j(\hat{\theta}(t), z(t)) \\ \dot{\hat{\theta}}(t) &= H_{\Omega_0,\Omega_1}(z)\; g(\hat{\theta}(t), z(t)),\ \hat{\theta}(0) = 0,\ \eta_0 = \varrho(d_{\max}),\ \eta_1 = (1+\beta)\eta_0, \\ \hat{\theta}(t_i^r) &= \hat{\theta}(t_i^s) \qquad\qquad t_i^r \leq t < t_{i+1}^r,\ \ i \geq 0\end{aligned} \tag{3.19}$$

where H_{Ω_0,Ω_1} represent the action described in Fig. 3.5. Similar to dead-zone, $d_{\max}$ represents the *a-priori knowledge* of the upper bound of the disturbance level. The proper definition of t_i, t_i^s, t_i^r and H_{Ω_0,Ω_1} will be given in section 5.3.2.

The operating procedure has been shown in Fig. 3.5. Let us start outside Ω_1. When the trajectory hits the boundary of Ω_1 at time t_i^s, the value of the parameter estimator is *stored*, but the adaptation still continues until the trajectory reaches Ω_0 at time t_i. In this stage, the adaptation is switched off and remains off until the trajectory passes the

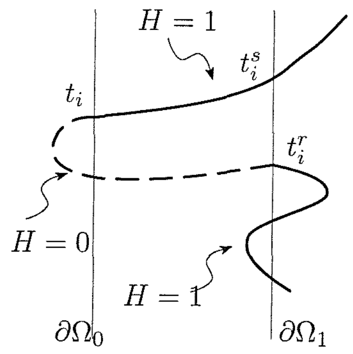

FIGURE 3.5: Hysteresis dead-zone

dead-zone Ω_0 and hits the boundary Ω_1 again at time t_i^r. At this point, the saved value of parameter estimator is *restored* and the adaptation turns on.

This delay in switching is the essential property of the hysteresis and results in eliminating the possibility of sliding motion. In fact the system trajectory either has a regular motion or *oscillate* back and forth across the switching surface at a finite frequency, which is controllable by some parameters such that η_0, η_1, etc. This ensures piecewise continuous right hand side differential equations for which the sufficient condition for the uniqueness of the C-solutions always hold, and the solution is well-defined.

The following theorem shows the properties of the closed loop system of example 3.3 and the controller:

$$\begin{aligned} \Xi_H(d_{\max}) : u(t) &= -ax(t) - \hat{\theta}(t)x(t) \\ \dot{\hat{\theta}}(t) &= H_{\Omega_0,\Omega_1}(x)\, x(t)^2, \quad \hat{\theta}(0) = 0, \ \eta_0 = \frac{d_{\max}}{a}, \ \eta_1 = (1+\beta)\frac{d_{\max}}{a}, \ \beta > 0 \\ \hat{\theta}(t_i^r) &= \hat{\theta}(t_i^s) \qquad t_i^r \le t < t_{i+1}^r, \ \ i \ge 0. \end{aligned} \tag{3.20}$$

Theorem 3.2. Consider the closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi_H(d_{\max}))$ defined by (3.4), (3.20), where $d(\cdot) \in \mathcal{L}^\infty$. Assume that $d_{\max}$ is such that $d_{\max} \ge \|d(\cdot)\|_{\mathcal{L}^\infty}$. Define the continuous function

$$\acute{V}_0(x_0, \theta, d_{\max}) := \frac{1}{2}x_0^2 + \frac{1}{2}\theta^2 + (\beta^2 + 2\beta)\eta_0^2. \tag{3.21}$$

Then for any $x_0 \in \mathbb{R}$,

H1. The solution $(x(\cdot), \hat{\theta}(\cdot)) : \mathbb{R}^+ \to \mathbb{R}^2$ exists and is unique.

H2. $x(\cdot), u(\cdot), \hat{\theta}(\cdot)$ are uniformly bounded as continuous functions of $\acute{V}_0(x_0, \theta, d_{\max})$.

H3. $x(t) \to \Omega_1$ as $t \to \infty$.

Proof. This result is a special case of Theorem 5.3 in section 5.3.2, and we defer the proof of H1, H3 until then. In order to proof H2, it can be shown (see Theorem 5.3) that

$$V(x(t), \tilde{\theta}(t)) := \frac{1}{2}x(t)^2 + \frac{1}{2}\tilde{\theta}(t)^2 \leq V(x_0, 0) + \eta_1^2 - \eta_0^2 := \acute{V}_0(x_0, \theta, \eta_0). \tag{3.22}$$

Therefore, similar to (3.17) each closed loop signal $x(\cdot), \hat{\theta}(\cdot), u(\cdot)$ can be made uniformly bounded as a continuous function of $\acute{V}_0 := \acute{V}_0(x_0, \theta, d_{\max})$:

$$|x(t)| \leq \sqrt{2\acute{V}_0}\,, \quad |\hat{\theta}(t)| \leq 2\sqrt{2\acute{V}_0}\,, \quad |u(t)| \leq a\sqrt{2\acute{V}_0} + 4\acute{V}_0. \tag{3.23}$$

□

Remark 3.3. In addition to the above properties, the hysteresis dead-zone has some other analytical and practical advantages. The system can be analysed using simpler regular piecewise continuous differential equations, and the simulation is faster as the usual integrator methods can be used rather than stiff ODE solvers. We will address these matters in more detail in chapter 5.

3.4.3 Parameter Projection

Projection is an alternative method to eliminate parameter drift by keeping the parameter estimates within some a priori defined bounds [6, 33]. Let $\theta_{\max}$ be the *a-priori* *knowledge* of the parametric uncertainty level, which is defined as the upper bound of $|\theta|$. Define the convex set

$$\Pi(\theta_{\max}) = \{\hat{\theta}(t) \in \mathbb{R}^m \mid P_{\theta_{\max}}(\hat{\theta}(t)) \leq 0\}, \tag{3.24}$$

where $P_{\theta_{\max}}(\cdot) : \mathbb{R}^m \to \mathbb{R}$ is considered as a smooth function that represent the surface $S : P_{\theta_{\max}}(\hat{\theta}(t)) = 0$, for all t. As an example

$$P_{\theta_{\max}}(\hat{\theta}(t)) := \hat{\theta}(t)^T \hat{\theta}(t) - \theta_{\max}^2, \tag{3.25}$$

Denote $\Pi^o(\theta_{\max})$, $\partial\Pi(\theta_{\max})$, the interior and the boundary of $\Pi(\theta_{\max})$ respectively and observe that $\nabla_{\hat{\theta}} P_{\theta_{\max}}$ represents an outward normal vector at $\hat{\theta} \in \partial\Pi(\theta_{\max})$. Consider the unmodified adaptive law (3.3)

$$\dot{\hat{\theta}}(t) = g(\hat{\theta}(t), z(t)), \qquad \hat{\theta}(0) \in \mathbb{R}. \tag{3.26}$$

The idea behind this method is to project $g(\cdot,\cdot)$ on the hyperplane tangent to $\partial\Pi(\theta_{\max})$ at $\hat{\theta}(t)$ when $\hat{\theta}(t)$ is on the boundary $\partial\Pi(\theta_{\max})$ and $g(\cdot,\cdot)$ pointing outward (Fig. 3.6).

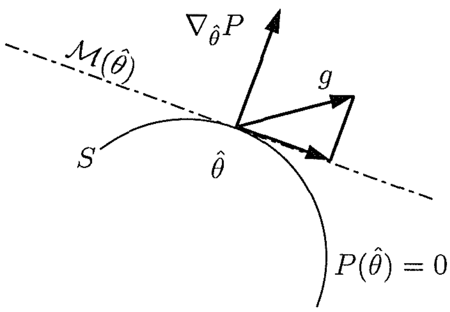

FIGURE 3.6: Projection operator

For compactness we drop the functions' arguments from our notation. In order to evaluate the $\text{Proj}_{\Pi(\theta_{\max})}(g, \hat{\theta})$, we observe that for some $\alpha \in \mathbb{R}$,

$$\nabla_{\hat{\theta}} P^T \ \text{Proj}_{\Pi(\theta_{\max})}(g, \hat{\theta}) = 0, \tag{3.27}$$

$$g = \alpha \nabla_{\hat{\theta}} P + \text{Proj}_{\Pi(\theta_{\max})}(g, \hat{\theta}). \tag{3.28}$$

Substituting (3.27) in the multiplication of (3.28) by $\nabla_{\hat{\theta}} P^T$, we have

$$\alpha = \frac{\nabla_{\hat{\theta}} P^T \, g}{\nabla_{\hat{\theta}} P^T \nabla_{\hat{\theta}} P}, \tag{3.29}$$

therefore the operator Proj $_{\Pi(\theta_{\max})} : \mathbb{R}^m \times \Pi \to \mathbb{R}^m, \;\; (g, \hat{\theta}) \mapsto$ Proj $_{\Pi(\theta_{\max})}(g, \hat{\theta})$ is defined as follows

$$\text{Proj}_{\,\Pi(\theta_{\max})}(g, \hat{\theta}) = \begin{cases} g, & \hat{\theta} \in \Pi^o \;\text{ or }\; \nabla_{\hat{\theta}} P^T g \leq 0 \\ \left(I - \dfrac{\nabla_{\hat{\theta}} P \nabla_{\hat{\theta}} P^T}{\nabla_{\hat{\theta}} P^T \nabla_{\hat{\theta}} P} \right) g & \hat{\theta} \in \partial\Pi \;\text{ and }\; \nabla_{\hat{\theta}} P^T g > 0. \end{cases} \tag{3.30}$$

The modified adaptive law is taken to be $\dot{\hat{\theta}} = \text{Proj}\,(g, \hat{\theta}), \;\; \hat{\theta}(0) = 0$. Consequently the projection controller $\Xi_P(\theta_{\max})$ is defined by

$$\begin{aligned} \Xi_P(\theta_{\max}) : \;\; & u(t) = j(\hat{\theta}(t), z(t)) \\ & \dot{\hat{\theta}}(t) = \text{Proj}_{\,\Pi(\theta_{\max})}(\, g, \hat{\theta}\,), \qquad \hat{\theta}(0) = 0. \end{aligned} \tag{3.31}$$

For the scalar systems where $\theta \in \mathbb{R}$, a simplified version of parameter projection can be obtained by defining $\Pi(\theta_{\max}) := [-\theta_{\max}\,, \, \theta_{\max}]$. Then, (3.30) is reduced to a simple switching mechanism:

$$\text{Proj}_{\,\Pi(\theta_{\max})}(g, \hat{\theta}) = \begin{cases} g, & |\hat{\theta}| < \theta_{\max} \;\text{ or }\; \hat{\theta}\, g \leq 0 \\ 0, & |\hat{\theta}| = \theta_{\max} \;\text{ and }\; \hat{\theta}\, g > 0. \end{cases} \tag{3.32}$$

The robustness of the closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi_P(\theta_{\max}))$ for linear and certain non-linear systems will be discussed in the next two chapters. It will be proven that in the presence of bounded disturbances, (i) the solution $(x(\cdot), \hat{\theta}(\cdot))$ exists, and (ii) all closed loop signals $x(\cdot), \hat{\theta}(\cdot), u(\cdot)$ are bounded. Note that the projection method does not guarantee asymptotic stability.

For scalar system as in section (3.3), the projection controller is

$$\begin{aligned} \Xi_P(\theta_{\max}) : \;\; & u(t) = -ax(t) - \hat{\theta}(t)x(t) \\ & \dot{\hat{\theta}}(t) = \text{Proj}_{\,\Pi(\theta_{\max})}\left(x(t)^2\right), \qquad \hat{\theta}(0) = 0. \end{aligned} \tag{3.33}$$

The monotonicity of $\hat{\theta}(t)$ indicates that starting from $\hat{\theta}(0) = 0$, $\hat{\theta}$ increases until it hits the boundary $\theta_{\max}$ and remains constant thereafter. That is $\Pi(\theta_{\max}) = [0, \theta_{\max}]$ and

$$\begin{aligned} &\dot{\hat{\theta}}(t) = x(t)^2, \quad \hat{\delta}(0) = 0, \qquad &&\forall t \in [0, T_m], \\ &\hat{\theta}(t) = \theta_{\max}, &&\forall t \in [T_m, \infty), \end{aligned} \tag{3.34}$$

where $T_m = \inf\{t \geq 0 \mid \hat{\theta}(t) = \theta_{\max}\}$. Observe that (3.34) easily satisfies (3.32) and (3.33). The following theorem examines the properties (i)–(ii) for example 3.3:

Theorem 3.3. Consider the closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi_P(\theta_{\max}))$ defined by (3.4),(3.33). Assume $\theta_{\max}$ is such that $|\theta| \leq \theta_{\max}$. Then, for any $x_0 \in \mathbb{R}$:

P1. The solution $(x(\cdot), \hat{\theta}(\cdot)) : \mathbb{R}^+ \to \mathbb{R}^2$ exists.

P2. $x(\cdot), u(\cdot), \hat{\theta}(\cdot)$ are uniformly bounded as a continuous function of $x_0, \|d\|, \theta_{\max}$.

Proof. Again to avoid repetition we defer the proof of P1 to later chapters (see Theorem 5.4). The proof of P2 is as follows: Defining the Lyapunov function

$$V(x(t), \tilde{\theta}(t)) = \frac{1}{2}x(t)^2 + \frac{1}{2}\tilde{\theta}(t)^2, \tag{3.35}$$

we observe that

$$\dot{V}(x(t), \tilde{\theta}(t)) = -ax(t)^2 + x(t)d(t) + \tilde{\theta}(t)x(t)^2 - \tilde{\theta}(t)\text{Proj}_{\,\Pi(\theta_{\max})}(x(t)^2). \tag{3.36}$$

It can be easily shown from (3.34) and (3.32) that

$$-\tilde{\theta}(t)\text{Proj}_{\,\Pi(\theta_{\max})}(x(t)^2) \leq -\tilde{\theta}(t)x(t)^2. \tag{3.37}$$

Therefore

$$\begin{aligned} \dot{V}(x(t), \tilde{\theta}(t)) &\leq -ax(t)^2 + x(t)d(t) = a\left(-\frac{x(t)^2}{2} - \frac{1}{2}\left(x(t) - \frac{d(t)}{a}\right)^2 + \frac{d(t)^2}{2a^2}\right) \\ &\leq a\left(-\frac{x(t)^2}{2} + \frac{d(t)^2}{2a^2}\right) \leq \frac{-a}{2}|x(t)|^2 + \frac{1}{2a}\|d(t)\|^2. \end{aligned} \tag{3.38}$$

Adding and subtracting $kV(x(t),\tilde{\theta}(t))$ for some $k>0$ yields

$$\begin{aligned}\dot{V}(x(t),\tilde{\theta}(t)) &\leq -kV(x(t),\tilde{\theta}(t)) - \frac{-a}{2}|x(t)|^2 + \frac{1}{2a}\|d(t)\|^2 + k\left(\frac{1}{2}x(t)^2+\frac{1}{2}\tilde{\theta}(t)^2\right)\\ &\leq -kV(x(t),\tilde{\theta}(t)) - (\frac{a}{2}-k)|x(t)|^2 + \frac{k}{2}\tilde{\theta}(t)^2 + \frac{1}{2a}\|d(t)\|^2 \qquad (3.39)\end{aligned}$$

Choosing $k<a/2$, the second term in (3.39) is negative. Note that by (3.34) $\hat{\theta}(t)\leq\theta_{\max}$. We also have, by assumption, that $|\theta|\leq\theta_{\max}$. Define

$$V^*(\theta_{\max},\|d\|) := \frac{1}{2}\theta_{\max}^2 + \frac{\|d\|^2}{2ka}, \tag{3.40}$$

and observe that

$$\dot{V}(x(t),\tilde{\theta}(t)) \leq -k(V(x(t),\tilde{\theta}(t)) - V^*) \tag{3.41}$$

which implies that, $\dot{V}(x(t),\tilde{\theta}(t))\leq 0$ for all $V\geq V^*$. It follows that

$$V(x(t),\tilde{\theta}(t)) \leq V_0'(x_0,\|d\|,\theta_{\max}) := \max\{V(x_0,0),V^*(\theta_{\max},\|d\|)\}, \quad \forall t\geq 0. \tag{3.42}$$

The uniform boundedness of $x(\cdot),u(\cdot)$ as a continuous function of $V_0'(x_0,\|d\|,\theta_{\max})$ follows from (3.35),(3.34), (3.42) and (3.33):

$$|\hat{\theta}(t)|\leq\theta_{\max}, \quad |x(t)|\leq\sqrt{2V_0'}, \quad |u(t)|\leq(a+\theta_{\max})\sqrt{2V_0'}. \tag{3.43}$$

□

Figure 3.7 illustrates the result for the example 3.3.

Remark 3.4. Projection modification relies upon the a-priori information $\theta_{\max}$. In fact $|\theta|\leq\theta_{\max}$ is a necessary condition in Theorem 3.3. This can be clarified by a simple system $\dot{x}=k^*x+u$, $u=-\hat{\theta}x$. If $\Pi(\theta_{\max})=[-\theta_{\max},\theta_{\max}]$ such that $k^*-\theta_{\max}>0$, then the best that can happen to the closed loop is $\dot{x}=(k^*-\theta_{\max})x$ resulting in $x(t)\to\infty$.

The reliability of $\theta_{\max}$ is vital in defining $\Pi(\theta_{\max})$. We will show in proof of Theorem **I** of section 3.5 (Proposition 3.3) that for sufficiently conservative $\theta_{\max}$, the behaviour of a projection modification tends to that of the unmodified design, hence parameter drift. Theorem **I** demonstrates the effect of such a choice on transient performance of

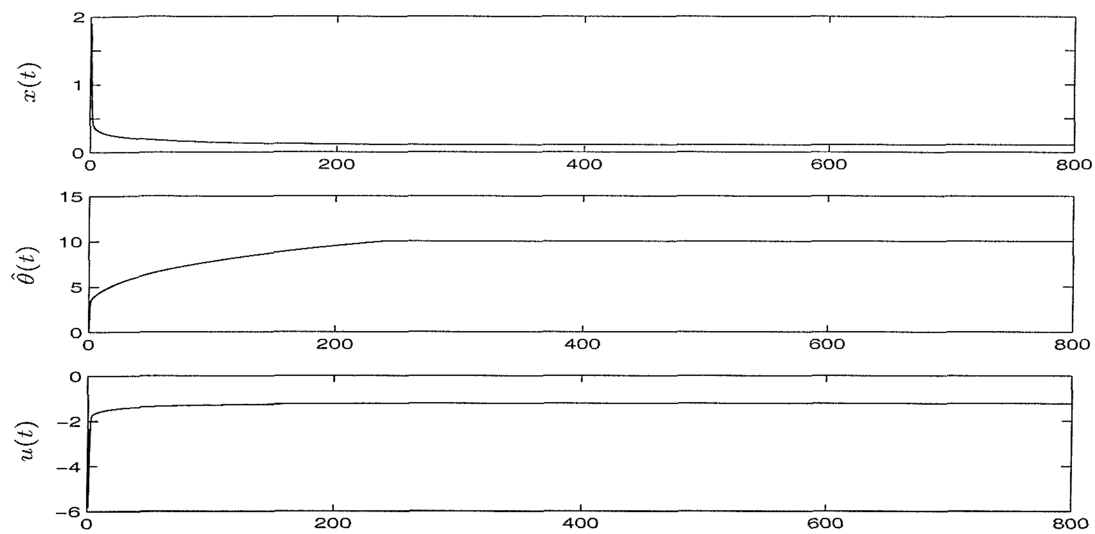

FIGURE 3.7: Projection modification for example 3.3

the closed loop system.

Remark 3.5. Projection has also some switching activity on the boundary of H leading to r.h.s. discontinuity of differential equations. However, one can avoid the discontinuous switching by modifying (3.30) to so-called smooth or 'soft-projection' [62] for which the classical theory of differential equations is valid.

3.5 Performance Comparison

As we mentioned in section 3.1, choosing the 'best' alternative for robust adaptive control is only possible if we have some *rich* a-priori information for one method and *poor* a-priori knowledge for the others. By 'rich' information, we mean enough reliable, and relevant knowledge. For example, consider a system for which we have very good reliable a-priori knowledge about the behaviour of the disturbances $d(\cdot)$, and little unreliable information about the uncertain system parameter θ. In such case, we have no doubt that dead-zone controller is the best alternative in robust adaptive design. On the other hand, intuitively, the 'projection' controller is the best alternative[1] if we have rich information about the system parameter θ and poor knowledge about disturbance $d(\cdot)$.

However, there is some grey margin between these two cases where one cannot easily

[1]Note that in this thesis we only compare dead-zone and projection based controllers. So, in all statements we simply ignore all other modification methods.

decide which method is more efficient. This motivates the idea of generating a decision algorithm by building an analytical theory for comparing these modifications.

The foundation of any comparative theory is built on some measurable meaningful criteria of comparison. All methods to be compared must rely on these criteria. Furthermore, the comparative theory must give answers to the so-called 'black and white cases' without any ambiguity. Dead-zone and projection modification deal with disturbances from different points of view. So, as a criterion of comparison, one should choose a feasible common ground which illuminates the strength and weakness of each method. The remarks 3.1 and 3.4 show that the 'a-priori information' upon which the robust adaptive controllers are designed ($d_{\max}$ for dead-zone and $\theta_{\max}$ for projection) would be a good choice as we have shown above that the answer to the 'black and white' cases are unambiguous.

We are interested in the worst case scenario and we choose a non-singular transient performance cost functional since it is meaningful for either methods and analytical results can be derived. Note that we are not concerned with the comparison of asymptotic performance, as it has been studied previously, see eg. [54] and the references therein.

In this section, we build a general framework for the rest of dissertation. First, a transient performance cost functional $\mathcal{P}$ will be defined. Then, two theorems will be introduced to identify circumstances in which a dead-zone based adaptive controller is superior to the projection based adaptive controller with respect to $\mathcal{P}$, and *vice versa*. We will develop the results by considering a simple scalar system and finally a structural procedure will be proposed to extend the idea to the more complex systems considered in later chapters.

3.5.1 Cost Functional

Consider a generic class of SISO system-controller interconnection (Σ, Ξ) with initial condition x_0, unknown parameter θ, bounded disturbance $d(\cdot)$, and control input $u(\cdot)$. We will compare the performance of the controllers with respect to the following worst case non-singular transient cost functional:

$$\mathcal{P}\left(\Sigma\left(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)\right), \Xi\right) = \sup_{x_0 \in \mathcal{X}_0(\gamma)} \sup_{\theta \in \Delta(\delta)} \sup_{d \in \mathcal{D}(\epsilon)} \left(\|x(\cdot)\|_{\mathcal{L}^\infty} + \|u(\cdot)\|_{\mathcal{L}^\infty} + \|\dot{u}(\cdot)\|_{\mathcal{L}^\infty}\right) \tag{3.44}$$

where

$$\begin{aligned} \mathcal{D}(\epsilon) &:= \{d(\cdot) \mid \|d(\cdot)\|_{\mathcal{L}^\infty} \leq \epsilon\}, \\ \Delta(\delta) &:= \{\theta \mid |\theta| \leq \delta\}, \\ \mathcal{X}_0(\gamma) &:= \{x_0 \mid \|x_0\| \leq \gamma\} \end{aligned} \tag{3.45}$$

for some $\epsilon, \delta \geq 0$ and $\gamma > 0$.

It is important to note that all the forthcoming theorems have established the comparative results based on the a-priori information $d_{\max}$ and $\theta_{\max}$. So, we will consider the (natural) assumption that any change in $d_{\max}$ affects the size of dead-zone $\Omega_0(d_{\max})$, and also $\Pi(\theta_{\max})$ is not independent of $\theta_{\max}$.

In the following, we will establish the main results for the scalar system described in section 3.3. This will help us to build a structural framework for developing the results for more complex systems in the later chapters.

3.5.2 Main Results

Theorem I. *Consider the system $\Sigma(x_0, \theta, d(\cdot))$ defined by* (3.4) *and the dead-zone and projection controllers $\Xi_D(d_{\max})$ and $\Xi_P(\theta_{\max})$ defined by* (3.14), (3.33) *respectively. Consider the transient performance cost functional* (3.44). *Then for all $d_{\max} \geq \epsilon$, there exists $\theta^*_{\max} \geq \delta$ such that for all $\theta_{\max} \geq \theta^*_{\max}$,*

$$\mathcal{P}(\Sigma\left(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)\right), \Xi_P(\theta_{\max})) > \mathcal{P}(\Sigma\left(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)\right), \Xi_D(d_{\max})). \tag{3.46}$$

This theorem can be interpreted as stating that if the a-priori knowledge of the parametric uncertainty level $\theta_{\max}$ is sufficiently conservative ($\theta_{\max} \geq \theta^*_{\max}$), then the dead-zone based design will out-perform the projection based design.

Similar result can be obtain with respect to hysteresis dead-zone as follows:

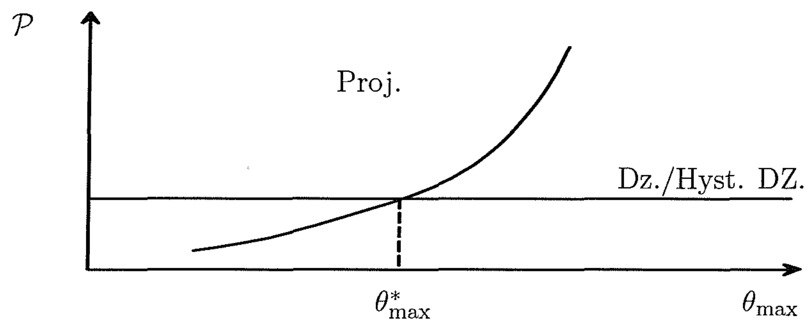

FIGURE 3.8: Statement of Theorems **I**, **Ia**

Theorem Ia. *Consider the system* $\Sigma(x_0, \theta, d(\cdot))$ *defined by* (3.4) *and the hysteresis dead-zone and projection controllers* $\Xi_H(d_{\max})$ *and* $\Xi_P(\theta_{\max})$ *defined by* (3.20), (3.33) *respectively. Consider the transient performance cost functional* (3.44). *Then for all* $d_{\max} \geq \epsilon$, *there exists* $\theta^*_{\max} \geq \delta$ *such that for all* $\theta_{\max} \geq \theta^*_{\max}$,

$$\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_P(\theta_{\max})) > \mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_H(d_{\max})). \quad (3.47)$$

In fact, as it has been shown in Fig. 3.8, we will prove the stronger results that the ratio between the two costs can be made arbitrarily large (Fig. 3.8). That is, for dead-zone controller:

$$\frac{\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_P(\theta_{\max}))}{\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_D(d_{\max}))} \to \infty \text{ as } \theta_{\max} \to \infty, \quad \forall d_{\max} \geq \epsilon. \quad (3.48)$$

Alternatively, for hysteresis dead-zone controller we have

$$\frac{\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_P(\theta_{\max}))}{\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_H(d_{\max}))} \to \infty \text{ as } \theta_{\max} \to \infty, \quad \forall d_{\max} \geq \epsilon. \quad (3.49)$$

The complementary results are as follows:

Theorem II. *Consider the system* $\Sigma(x_0, \theta, d(\cdot))$ *defined by* (3.4) *and the dead-zone and projection controllers* $\Xi_D(d_{\max})$ *and* $\Xi_P(\theta_{\max})$ *defined by* (3.14), (3.33) *respectively. Consider the transient performance cost functional* (3.44). *Then* $\exists \delta > 0$ *such that*

$\forall\theta_{\max} \geq \delta \;\; \exists d^*_{\max} \geq \epsilon$ *such that* $\forall d_{\max} \geq d^*_{\max}$,

$$\mathcal{P}(\,\Sigma\,(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_D(d_{\max})\,) \;>\; \mathcal{P}(\,\Sigma\,(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_P(\theta_{\max})\,). \tag{3.50}$$

Theorem IIa. *Consider the system* $\Sigma(x_0, \theta, d(\cdot))$ *defined by* (3.4) *and the hysteresis dead-zone and projection controllers* $\Xi_H(d_{\max})$ *and* $\Xi_P(\theta_{\max})$ *defined by* (3.20), (3.33) *respectively. Consider the transient performance cost functional* (3.44). *Then* $\exists\delta > 0$ *such that* $\forall\theta_{\max} \geq \delta \;\; \exists d^*_{\max} \geq \epsilon$ *such that* $\forall d_{\max} \geq d^*_{\max}$,

$$\mathcal{P}(\,\Sigma\,(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_H(d_{\max})\,) \;>\; \mathcal{P}(\,\Sigma\,(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_P(\theta_{\max})\,). \tag{3.51}$$

These theorems can be interpreted as stating that above a certain uncertainty level δ, if the a-priori knowledge $d_{\max}$ of the disturbance level is sufficiently conservative ($d_{\max} \geq d^*_{\max}$), then the projection design will out-perform the dead-zone/hysteresis dead-zone design (Fig. 3.9).

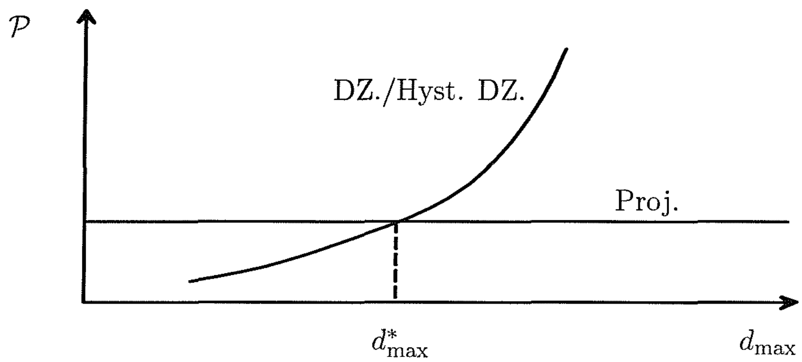

FIGURE 3.9: Statement of Theorems **II**, **IIa**

As we have shown in Fig. 3.9, we will prove that the ratio between the two costs can be made arbitrarily large i.e. for Theorem **II**:

$$\frac{\mathcal{P}(\,\Sigma\,(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_D(d_{\max})\,)}{\mathcal{P}(\,\Sigma\,(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_P(\theta_{\max})\,)} \rightarrow \infty \;\text{ as } d_{\max} \rightarrow \infty, \quad \forall\theta_{\max} \geq \delta, \tag{3.52}$$

and for Theorem **IIa**

$$\frac{\mathcal{P}(\,\Sigma\,(\mathcal{X}_0(\gamma),\Delta(\delta),\mathcal{D}(\epsilon)),\Xi_H(d_{\max})\,)}{\mathcal{P}(\,\Sigma\,(\mathcal{X}_0(\gamma),\Delta(\delta),\mathcal{D}(\epsilon)),\Xi_P(\theta_{\max})\,)} \to \infty \quad \text{as } d_{\max} \to \infty, \quad \forall \theta_{\max} \geq \delta. \tag{3.53}$$

3.5.3 Proof of Theorems I, Ia

In order to prove theorem **I**, **Ia**, we use the following procedure:

1. Firstly, we will prove that Ξ, the unmodified controller (3.10) causes parameter drift whilst the state trajectory converges. That is, for any $x_0 \in \mathcal{X}_0(\gamma)$, $\theta \in \Delta(\delta)$ and for $d \equiv \epsilon \in \mathcal{D}(\epsilon)$, we have

$$|\hat{\theta}(t)| \to \infty, \quad |x(t)| \to 0 \qquad \text{as } t \to \infty. \tag{3.54}$$

2. From this we prove that

$$\mathcal{P}(\Sigma\,(\mathcal{X}_0(\gamma),\Delta(\delta),\mathcal{D}(\epsilon)),\Xi) = \infty. \tag{3.55}$$

3. The projection modification design, $\Xi_P(\theta_{\max})$ is shown to have the property that

$$\mathcal{P}(\Sigma\,(\mathcal{X}_0(\gamma),\Delta(\delta),\mathcal{D}(\epsilon)),\Xi_P(\theta_{\max})) \to \infty, \quad \text{as } \theta_{\max} \to \infty. \tag{3.56}$$

4. Finally the proof will be completed by showing that for dead-zone controller

$$\mathcal{P}(\Sigma\,(\mathcal{X}_0(\gamma),\Delta(\delta),\mathcal{D}(\epsilon)),\Xi_D(d_{\max})) < \infty, \qquad \forall d_{\max} \geq \epsilon. \tag{3.57}$$

   Alternatively, for hysteresis dead-zone controller:

$$\mathcal{P}(\Sigma\,(\mathcal{X}_0(\gamma),\Delta(\delta),\mathcal{D}(\epsilon)),\Xi_H(d_{\max})) < \infty, \qquad \forall d_{\max} \geq \epsilon, \tag{3.58}$$

Let us follow the procedure by establishing some propositions:

Proposition 3.1. Consider the closed loop system $(\Sigma(x_0,\theta,d(\cdot)),\Xi)$ defined by (3.4), (3.10). Let $d(\cdot) = \epsilon$ for some $\epsilon > 0$. Then

$$|x(t)| \to 0, \ |\hat{\theta}(t)| \to \infty \qquad \text{as } t \to \infty. \tag{3.59}$$

Proof. The proof closely follows the examples 8,9 in [16] and [17]: Defining $\zeta : \mathbb{R} \to \mathbb{R}$, $\hat{\theta}(t) \mapsto \theta - \hat{\theta}(t) - a$, the closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi)$ can be rewritten in the following form:

$$\begin{aligned} \dot{x}(t) &= \epsilon + \zeta(\hat{\theta}(t))\, x(t), \qquad x(0) = x_0, && (3.60) \\ \dot{\hat{\theta}}(t) &= x(t)^2. && (3.61) \end{aligned}$$

(1) $\hat{\theta}(t) \to \infty$ as $t \to \infty$. Seeking a contradiction. Assume there exists $M_1 > 0$ such that $\hat{\theta}(t) < M_1$ for all t. Then by (3.61) $x(\cdot) \in \mathcal{L}^2[0, \infty)$. Since $\zeta(\hat{\theta})$ is continuous, there exists $M_2 > 0$ such that $\zeta(\hat{\theta}) < M_2$ for all t. Therefore by (3.60)

$$\int_0^t (\dot{x}(\tau) - \epsilon)^2 d\tau \leq M_2^2 M_1, \quad \forall t. \tag{3.62}$$

Hence

$$M_2^2 M_1 \geq \int_0^t \dot{x}(\tau)^2 d\tau - 2\epsilon \int_0^t \dot{x}(\tau) d\tau \; + \epsilon^2 t \geq -2\epsilon(x(t) - x_0) + \epsilon^2 t. \tag{3.63}$$

It follows that

$$x(t) \geq -\frac{M_2^2 M_1}{2\epsilon} + x_0 + \frac{\epsilon t}{2}, \quad \forall t. \tag{3.64}$$

It contradicts the assumption $x(t) \in \mathcal{L}^2[0, \infty)$; Hence $\hat{\theta}(t) \to \infty$ as $t \to \infty$.

(2) $x(t) \to 0$ as $t \to \infty$. Denote $\alpha(t) = -\zeta(\hat{\theta}(t))$ and observe that $\alpha(\cdot)$ is continuous monotonically non-decreasing and $\alpha(t) \to \infty$ as $t \to \infty$. Define

$$t' = \int_0^t \alpha(\tau) d\tau =: \kappa(t). \tag{3.65}$$

$\kappa(t)$ is monotonically increasing on $[t_0, \infty)$ for some $t_0 > 0$ for which $\alpha(t_0) > 0$. Changing the argument t to t' and noting $\hat{x}(t') := x(\kappa^{-1}(t'))$, and $dt' = \alpha(t)dt$, equation (3.60) becomes

$$\frac{d}{dt'}\hat{x}(t') = \frac{\epsilon}{\alpha(\kappa^{-1}(t'))} - \hat{x}(t') := \mu(t') - \hat{x}(t'). \tag{3.66}$$

Observe that $\mu(t') \to 0$ monotonically as $t' \to \infty$. Let $t'_0 = \kappa(t_0)$ and define $\delta_1 = |\hat{x}(t'_0)|$ and $\delta_2 = |\mu(t'_0)|$. Then for $t' \geq t'_0$

$$|\hat{x}(t')| = \left| e^{-t'+t'_0}\hat{x}(t'_0) + \int_{t'_0}^{t'} e^{-t'+\tau}\mu(\tau)d\tau \right| \leq e^{-t'+t'_0}\delta_1 + \delta_2. \tag{3.67}$$

Now we claim that $\hat{x}(t') \to 0$ as $t' \to \infty$, i.e. given any $\beta > 0$ there exists t'_1 such that $|\hat{x}(t')| \leq \beta$ for all $t' \geq t'_1$. To see this, choose t'_0 so that $\delta_2 \leq \beta/2$ and choose $t'_1 = t'_0 + \ln(2\delta_1/\beta)$ if $2\delta_1/\beta > 1$, otherwise, choose $t'_1 = t'_0$. This ensures that for any $t' \geq t'_1$

$$|\hat{x}(t')| \leq e^{-t'+t'_0}\delta_1 + \delta_2 \leq e^{-t'_1+t'_0}\delta_1 + \delta_2 \leq \beta. \tag{3.68}$$

It follows that $|x(t)| \leq \beta$ for all $t \geq \kappa^{-1}(t'_1)$ which completes the proof.

□

Remark 3.6. The above proof is limited to this particular example and cannot be straightforwardly generalised. We will provide an alternative more general proof (see propositions 4.2, 5.1, and 5.10) in later chapters.

Proposition 3.2. The closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi)$ defined by (3.4), (3.10) has the following property:

$$\mathcal{P}(\Sigma\,(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi) = \infty. \tag{3.69}$$

Proof. For ease of the notation let us denote $\limsup_{t\to\infty}$ by $\overline{\lim}$. We choose $d(\cdot) = \epsilon \neq 0$ and let $x_0 \in \mathcal{X}_0(\gamma)$, $\theta \in \Delta(\delta)$.

Suppose for contradiction $\mathcal{P}(\Sigma(x_0, \theta, d(\cdot)), \Xi) < \infty$. Consider $\dot{x}(t)$. There are two cases either 1. $\overline{\lim}\,|\dot{x}(t)| = \infty$ or 2. $\overline{\lim}\,|\dot{x}(t)| < \infty$:

1. Suppose $\overline{\lim}\,|\dot{x}(t)| = \infty$, i.e. $\overline{\lim}\,|-ax(t) + (\theta - \hat{\theta}(t))x(t) + \epsilon| = \infty$. Since $x(t) \to 0$

by Proposition 3.1, therefore $\overline{\lim}\, |\hat{\theta}(t)x(t)| = \infty$. It follows that

$$\|u(\cdot)\|_{\mathcal{L}^\infty} \geq \|\hat{\theta}(\cdot)x(\cdot)\|_{\mathcal{L}^\infty} = \infty, \tag{3.70}$$

i.e $\mathcal{P} = \infty$, which is a contradiction.

2. Suppose $\overline{\lim}\, |\dot{x}(t)| < \infty$. Considering $\overline{\lim}\, \dot{u}(t)$, by applying (3.59), we observe that

$$\overline{\lim} \left(\dot{u}(t) - \left(a + \hat{\theta}(t)\right) \left(\hat{\theta}(t)x(t) - \epsilon\right)\right) \to 0 \quad \text{as} \quad t \to \infty. \tag{3.71}$$

Now there are two possibilities: either i) $\hat{\theta}(t)x(t) \not\to \epsilon$ (including the possibility that $\lim_{t\to\infty} \hat{\theta}(t)x(t)$ does not exist), or ii) $\lim_{t\to\infty} \hat{\theta}(t)x(t) = \epsilon$

(a) Suppose $\lim_{t\to\infty} \hat{\theta}(t)x(t)$ does not exist or $\hat{\theta}(t)x(t) \not\to \epsilon$. It follows by (3.59) that $\|\dot{u}(\cdot)\|_{\mathcal{L}^\infty} = \infty$; hence contradiction.

2.b. Suppose $\lim_{t\to\infty} \hat{\theta}(t)x(t) = \epsilon$. By (3.59) we have that

$$\forall \hat{\theta}^* > 0 \;\; \exists T > 0 \;\; s.t. \;\; \forall t > T \quad \hat{\theta}(t) > \hat{\theta}^*. \tag{3.72}$$

Now we choose $d_2(\cdot)$ as follows

$$d_2(t) = \begin{cases} \epsilon & t \leq T \\ -\epsilon & t > T \end{cases} \tag{3.73}$$

Note that $d_2(t) = d(t)$ for all $t \leq T$. With this choice, by continuity and causality, we have that

$$\lim_{t\to T+} x(t) = x(T), \quad \lim_{t\to T+} \hat{\theta}(t) = \hat{\theta}(T) \tag{3.74}$$

where $\lim_{t\to T+}$ denote $\lim_{t\to T, t>T}$. It follows that

$$\left(\lim_{t\to T+} \dot{u}(t)\right) - \dot{u}(T) = 2(a + \hat{\theta}(T))\epsilon \geq 2(a + \hat{\theta}^*)\epsilon. \tag{3.75}$$

By choosing a suitable $\hat{\theta}^*$, it follows that $\hat{\theta}(T)$ can be made arbitrarily large and hence the difference (3.75) is arbitrarily large. Then either $\dot{u}(T)$ is large or $\lim_{t\to T^+} \dot{u}(t)$ is large, therefore $\|\dot{u}(\cdot)\|_{\mathcal{L}^\infty}$ can be made arbitrarily large hence contradiction.

Therefore at least one component of (3.44) diverges, i.e. $\mathcal{P}(\Sigma(x_0, \theta, d(\cdot)), \Xi) = \infty$; hence

$$\mathcal{P}(\Sigma\left(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)\right), \Xi) \geq \mathcal{P}(\Sigma(x_0, \theta, d(\cdot)), \Xi) = \infty. \tag{3.76}$$

□

Proposition 3.3. The closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi_P(\theta_{\max}))$ defined by (3.4), (3.31) has the following property:

$$\mathcal{P}(\Sigma\left(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)\right), \Xi_P(\theta_{\max})) \to \infty, \quad \text{as } \theta_{\max} \to \infty. \tag{3.77}$$

Proof. It is convenient to define

$$\mathcal{P}_{[0,T]}\left(\Sigma(x_0, \theta, d(\cdot)), \Xi\right) = \left(\|x(\cdot)\|_{\mathcal{L}^\infty[0,T]} + \|u(\cdot)\|_{\mathcal{L}^\infty[0,T]} + \|\dot{u}(\cdot)\|_{\mathcal{L}^\infty[0,T]}\right) \tag{3.78}$$

Now let $M > 0$. By Proposition 3.2 there exists $x_0 \in \mathcal{X}_0$, $d(\cdot) \in \mathcal{D}(\epsilon)$, $\theta \in \Delta(\delta)$ so that

$$\mathcal{P}_{[0,\infty)}(\Sigma(x_0, \theta, d(\cdot)), \Xi) \geq 2M. \tag{3.79}$$

It follows that $\exists T > 0$ s.t. $\mathcal{P}_{[0,T]}(\Sigma(x_0, \theta, d(\cdot)), \Xi) \geq M$. Since $\theta_{\max}$ diverges, by choosing

$$\theta_{\max} = 2\hat{\theta}(T), \tag{3.80}$$

we have that $\theta_{\max} > \hat{\theta}(T)$, i.e. the unmodified and the projection designs are identical on $[0, T]$, i.e

$$\mathcal{P}_{[0,T]}(\Sigma(x_0, \theta, d(\cdot)), \Xi_P(\theta_{\max})) = \mathcal{P}_{[0,T]}(\Sigma(x_0, \theta, d(\cdot)), \Xi) \geq M. \tag{3.81}$$

Therefore

$$\mathcal{P}(\Sigma\left(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)\right), \Xi_P(\theta_{\max})) \geq \mathcal{P}_{[0,T]}(\Sigma(x_0, \theta, d(\cdot)), \Xi_P(\theta_{\max})) \geq M. \tag{3.82}$$

Since this holds for all $M > 0$, this completes the proof. □

Proposition 3.4. The closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi_D(d_{\max}))$ defined by (3.4), (3.13) has the following property:

$$\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_D(d_{\max})) < \infty, \quad \forall d_{\max} \geq \epsilon. \tag{3.83}$$

Proof. Let $x_0 \in \mathcal{X}_0(\gamma)$, $\theta \in \Delta(\delta)$ and $d \in \mathcal{D}(\epsilon)$. The uniform boundedness of signals $x(\cdot), \hat{\theta}(\cdot), u(\cdot)$ as a continuous function of $V_0(x_0, \theta, d_{\max})$ follow directly by property D2 of Theorem 3.1. Therefore by (3.17) and (3.6)

$$|\dot{x}(t)| \leq 6V_0 + a\sqrt{2V_0} + \epsilon. \tag{3.84}$$

It follows

$$|\dot{u}(t)| \leq (a + |\hat{\theta}(t)|)|\dot{x}(t)| + |x(t)|^3 \leq (14V_0 + a^2 + 2\epsilon)\sqrt{2V_0} + a(10V_0 + \epsilon), \tag{3.85}$$

i.e. $\dot{u}(\cdot)$ is uniformly bounded in terms of a continuous function of $V_0(x_0, \theta, d_{\max})$. It follows that

$$\mathcal{P}(\Sigma(x_0, \theta, d(\cdot)), \Xi_D(d_{\max})) \leq M\left(V_0(x_0, \theta, d_{\max})\right), \tag{3.86}$$

for some continuous $M(V_0(x_0, \theta, d_{\max})) < \infty$. Taking the supremum over system parameters x_0, θ, d implies that for all $d_{\max} \geq \epsilon$,

$$\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_D(d_{\max})) \leq \sup_{x_0 \in \mathcal{X}_0(\gamma)} \sup_{\theta \in \Delta(\delta)} \sup_{d \in \mathcal{D}(\epsilon)} M\left(V_0(x_0, \theta, d_{\max})\right) < \infty. \tag{3.87}$$

□

Proposition 3.5. The closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi_H(d_{\max}))$ defined by (3.4), (3.19) has the following property:

$$\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_H(d_{\max})) < \infty, \quad \forall d_{\max} \geq \epsilon. \tag{3.88}$$

Proof. Let $x_0 \in \mathcal{X}_0(\gamma)$, $\theta \in \Delta(\delta)$ and $d \in \mathcal{D}(\epsilon)$. The uniform boundedness of signals $x(\cdot), \hat{\theta}(\cdot), u(\cdot)$ as a continuous function of $\acute{V}_0(x_0, \theta, d_{\max})$ defined by (3.21) follow by inequalities (3.23). The claim of proposition follows by replacing $V_0(x_0, \theta, d_{\max})$ by $\acute{V}_0(x_0, \theta, d_{\max})$ in the proof of Proposition 3.4. □

Following the procedure 3.5.3, the above propositions suffice to establish the Theorem **I**, **Ia** as follows:

*Proof of Theorem **I**.*

This is a simple consequence of Proposition 3.3 and Proposition 3.4. □

*Proof of Theorem **Ia**.*

The proof follows by Proposition 3.3 and Proposition 3.5. □

3.5.4 Proof of Theorem II, IIa

Proposition 3.6. Consider the closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi_P(\theta_{\max}))$ defined by (3.4), (3.33). Consider the transient performance cost functional defined in (3.44). Then

$$\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_P(\theta_{\max})) < \infty, \qquad \forall \theta_{\max} \geq \delta. \tag{3.89}$$

Proof. Let $x_0 \in \mathcal{X}_0(\gamma)$, $\theta \in \Delta(\delta)$ and $d \in \mathcal{D}(\epsilon)$. By Theorem 3.3, $x(\cdot), \hat{\theta}(\cdot), u(\cdot)$ are uniformly bounded as a continuous function of $V_0'(x_0, \|d\|, \theta_{\max})$. Therefore by (3.43) and (3.6),

$$|\dot{x}(t)| \leq (a + 2\theta_{\max})\sqrt{2V_0'} + \epsilon. \tag{3.90}$$

It follows that

$$|\dot{u}(t)| \leq \left((a + 2\theta_{\max})^2 + 2V_0'\right)\sqrt{2V_0'} + (a + \theta_{\max})\epsilon, \tag{3.91}$$

i.e. $\dot{u}(\cdot)$ is uniformly bounded as a continuous function of $V_0'(x_0, \|d\|, \theta_{\max})$. It follows that

$$\mathcal{P}(\Sigma(x_0, \theta, d(\cdot)), \Xi_H(d_{\max})) \leq M\left(V_0(x_0, \|d\|, \theta_{\max})\right), \tag{3.92}$$

for some continuous $M(V_0(x_0, \|d\|, \theta_{\max})) < \infty$. Taking the supremum over system parameters x_0, θ, d implies that for all $\theta_{\max} \geq \delta$,

$$\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_H(d_{\max})) \leq \sup_{x_0 \in \mathcal{X}_0(\gamma)} \sup_{\theta \in \Delta(\delta)} \sup_{d \in \mathcal{D}(\epsilon)} M\left(V_0(x_0, \|d\|, \theta_{\max})\right) < \infty. \tag{3.93}$$

□

Proposition 3.7. Consider the closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi_D(d_{\max}))$ defined by (3.4), (3.14). Then $\exists \delta > 0$ such that

$$\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_D(d_{\max})) \to \infty, \quad \text{as} \quad d_{\max} \to \infty. \tag{3.94}$$

Proof. Note that by choice of $\eta_0 := \varrho(d_{\max}) = d_{\max}/a$ we have that $|\Omega_0| \to \infty$ as $d_{\max} \to \infty$. Let $x_0 \in \mathcal{X}_0(\gamma)$, $\theta \in \Delta(\delta)$ and $d \in \mathcal{D}(\epsilon)$. Suppose $x_0 \in \Omega_0$. We define τ as follows:

$$\tau = \begin{cases} \infty & \text{if } x(t) \in \Omega_0 \quad \forall t \geq 0 \\ \inf\{t \geq 0 \mid x(t) \in \partial\Omega_0\}, & \text{otherwise} \end{cases} \tag{3.95}$$

and observe that $\dot{\hat{\theta}}(t) = 0$ for all $t \in [0, \tau)$ i.e. $\hat{\theta} = 0$ since $\hat{\theta}(0) = 0$. Therefore

$$\dot{x}(t) = (\theta - a)x(t) + d(t) \qquad \forall t \in [0, \tau) \tag{3.96}$$

By (3.45), $\exists \delta > 0$ for which $\theta \geq a$. Hence if $d = \epsilon$, then $|\dot{x}(t)| > \epsilon$ for all $t \in [0, \tau)$. Now we claim that $\tau < \infty$. To prove this, suppose for contradiction that $\tau = \infty$. It follows that $|\dot{x}(t)| > \epsilon$ for all $t > 0$, i.e. $|x(t)| \to \infty$ as $t \to \infty$ i.e. $x(t)$ hits the boundary $\partial\Omega_0$ in finite time. Hence contradiction, therefore $\tau < \infty$. It follows that

$$\|x(\cdot)\|_{\mathcal{L}^\infty} \geq |x(\tau)| = |\partial\Omega_0|. \tag{3.97}$$

If $x_0 \notin \Omega_0$ therefore $\|x(\cdot)\|_{\mathcal{L}^\infty} \geq |x_0| \geq |\partial\Omega_0|$. The proof is completed by taking $d_{\max} \to \infty$ i.e. $|\Omega_0| \to \infty$. It follows that

$$\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_D(d_{\max})) \geq \mathcal{P}(\Sigma(x_0, \theta, d(\cdot)), \Xi_D(d_{\max})) = \infty. \tag{3.98}$$

□

Proposition 3.8. Consider the closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi_H(d_{\max}))$ defined by (3.4), (3.20). Then $\exists \delta > 0$ such that

$$\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_H(d_{\max})) \to \infty, \quad \text{as} \quad d_{\max} \to \infty. \tag{3.99}$$

Proof. See the proof of Proposition 3.7. □

*Proof of Theorem **II**.*

This is a simple consequence of Proposition 3.7 and Proposition 3.6. □

*Proof of Theorem **IIa**.*

From Proposition 3.8 and Proposition 3.6 the claim of theorem follows. □

The proof of above theorems are heavily based on the very natural assumption that the size of dead-zone is a divergent function of a-priori information on disturbance level. In particular, $\eta_0 := d_{\max}/a$ implies that $\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_D(d_{\max})) \to \infty$ as $d_{\max} \to \infty$. In the following section we show that the other choices of η_0 also yield the same result.

3.5.5 Choices of Dead-zone

Suppose $d(\cdot) = \epsilon$ for some $\epsilon > 0$ and let $\Omega_0 = [-\eta_0, \eta_0]$, where $\eta_0 = \varrho(d_{\max})$. The choice of $\varrho(d_{\max})$ encounters the following possible cases[2]:

(*i*) $\eta_0 := \varrho(d_{\max}) \to \infty$ as $d_{\max} \to \infty$. Therefore $\mathcal{P} = \infty$ by Proposition 3.7.

(*ii*) $\eta_0 := \varrho(d_{\max}) \to 0$ as $d_{\max} \to \infty$. By shrinking the dead-zone, we have a sequence of modified controllers $\Xi_D(d_{\max})$ tending to unmodified controller Ξ. It follows that as $d_{\max} \to \infty$, the performance of the sequence of modified closed loops $\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_D(d_{\max}))$ tends to the performance of that of unmodified closed loop $\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi)$ for which by Proposition 3.2, $\mathcal{P} = \infty$, therefore

$$\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_D(d_{\max})) \to \infty, \quad \text{as } d_{\max} \to \infty. \tag{3.100}$$

(*iii*) $\eta_0 := \varrho(d_{\max}) \leq c$ as $d_{\max} \to \infty$. Recall the closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi_D(d_{\max}))$ defined by (3.4), (3.14). We have shown in Theorem 3.1 that the choice of $\eta_0^* = d_{\max}/a$ suggested by Lyapunov theory suffices to establish D1-D3. However, it is

[2]Other cases such as oscillatory but bounded $\varrho(\cdot)$ can be handled suitably by considering monotonic subsequences.

well known that Lyapunov method provides only a sufficient condition for stability and in fact there are systems for which $x(t) \rightarrow \Omega_0 = [-c, c]$ where $c < \eta_0^*$. Note that even if $x(t) \not\rightarrow \Omega_0$, it remains bounded since by proof of Theorem (4.2), $|x(t)| > \eta_0^*$ yields a negative semi-definite Lyapunov function. Let us Rearrange (3.4) to

$$\dot{x}(t) = -\hat{\theta}(t)x(t) + (\theta - a)x(t) + d(t). \tag{3.101}$$

Suppose for contradiction $x(t) \not\rightarrow \Omega_0$. Then there must exists a positive divergent sequence $\{t_k\}_{k\geq 1}$ such that $x(t_k)^2 > c^2$. There are two possible cases: either (*1*) $\overline{\lim}\, |\dot{x}(t)| = \infty$ or (*2*) $\overline{\lim}\, |\dot{x}(t)| \leq M < \infty$.

1. Suppose $\overline{\lim}\, |\dot{x}(t)| = \infty$ i.e. $\overline{\lim}\, | - \hat{\theta}(t)x(t) + (\theta - a)x(t) + d(t)| = \infty$. It follows that $\hat{\theta}(t) \rightarrow \infty$ as $t \rightarrow \infty$ since $x(\cdot), d(\cdot)$ are bounded. it follows that there exist $t^* > 0$ such that for all $t > t^*$ the negative term $-\hat{\theta}(t)$ dominates the other (possibly positive) terms in (3.101) resulting in $x(t) \rightarrow 0$ as $t \rightarrow \infty$, hence contradiction.

2. If $\overline{\lim}\, |\dot{x}(t)| \leq M < \infty$, then $x(t)$ is uniformly continuous, i.e. for $\epsilon = c/2$

$$\exists \omega > 0 \; s.t. \; \forall \tau \in [0, \omega], \; \forall t > 0, \qquad |x(t) - x(t + \tau)| < \frac{c}{2}. \tag{3.102}$$

   Therefore $|x(t_k) - x(t_k + \tau)| < c/2$ and since $|x(t_k)| \geq c$, we have that $|x(t_k + \tau)| > c/2$ i.e. $|x(t)| \geq c/2$ for all $t \in [t_k, t_k + \omega]$. With no loss of generality, we may assume $t_{k+1} - t_k \geq \omega$. It follows that

$$\hat{\theta}(t_k + \omega) = \int_0^{t_k+\omega} \dot{\hat{\theta}}(\tau)d\tau = \int_0^{t_k+\omega} x^2(\tau)d\tau \geq \frac{c^2}{4}k\omega. \tag{3.103}$$

   Therefore $\hat{\theta}(t_k + \omega) \rightarrow \infty$ as $k \rightarrow \infty$, i.e. $\hat{\theta}(t) \rightarrow \infty$ as $t \rightarrow \infty$, hence contradiction.

It follows that by choice of $\eta_0 = c$, property D2 of Theorem 3.1 holds also for system (3.4) for all $t > t^*$. Hence $\mathcal{P} < \infty$.

However in the following, we will illustrate that this is not true if the controllers generalised for tracking problems. Consider system (3.4) and define $e(t) := x(t) - x_{ref}(t)$ where $x_{ref}(\cdot)$ is a reference signal. The objective is for $x(\cdot)$ to approximately track the reference signal $x_{ref}(\cdot)$, i.e. $e(t) \to \Omega_0$ as $t \to \infty$. Let us define the following tracking controller:

$$\begin{aligned} u(t) &= -ae(t) - \hat{\theta}(t)x(t) + \dot{x}_{ref}(t), && (3.104)\\ \dot{\hat{\theta}}(t) &= D_{\Omega_0}(e)\, x(t)e(t), \qquad \hat{\theta}(0) = 0, && (3.105) \end{aligned}$$

Observe that given $x_{ref} = 0$, the tracking controller is identical to the dead-zone controller (3.14), i.e. stabilisation can be considered as a special case of tracking where $x_{ref} = 0$. In the presence of disturbances, a routine calculation yields to

$$\dot{e}(t) = -ae(t) + (\theta - \hat{\theta}(t))x(t) + d(t). \tag{3.106}$$

The choice of $\eta_0 := d_{\max}/a$ is suggested by Lyapunov analysis and implies $e(t) \to \Omega_0$ as $t \to \infty$. However, inspired by the above explanation one may choose $\eta_0 := c$. The following example illustrates the closed loop response to such a choice.

Example 3.2. Consider the closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi_D(d_{\max}))$ defined by (3.101), (3.14), where

$$a = 1, \quad \theta = 2, \quad d(\cdot) = 100, \quad c = 10, \quad x_{ref} = 10\sin(t). \tag{3.107}$$

The behaviour of the closed loop signals have been shown in Fig. 3.10.

As it has been shown in Fig. 3.10, the parameter estimator $\hat{\theta}(\cdot)$ drifts. Comparing this situation with that of unmodified controller Fig. 3.1, and Fig. 3.3, one can easily build a similar setup as Proposition 3.2 to achieve $\mathcal{P} = \infty$. Therefore, this provides a motivation for the choice of dead-zone $\varrho(d_{\max}) = d_{\max}/a$.

Another alternative method for tracking is λ–tracking [24] whose aim is 'practical tracking' i.e. $e(t) \to \lambda$ as $t \to \infty$, where $\lambda > 0$ represents 'prescribed accuracy'. The λ–tracking controller is given by:

$$\begin{aligned} u(t) &= -\hat{\theta}(t)e(t), && (3.108)\\ \dot{\hat{\theta}}(t) &= D'_{\Omega_0}(e)\, |e(t)|, && (3.109) \end{aligned}$$

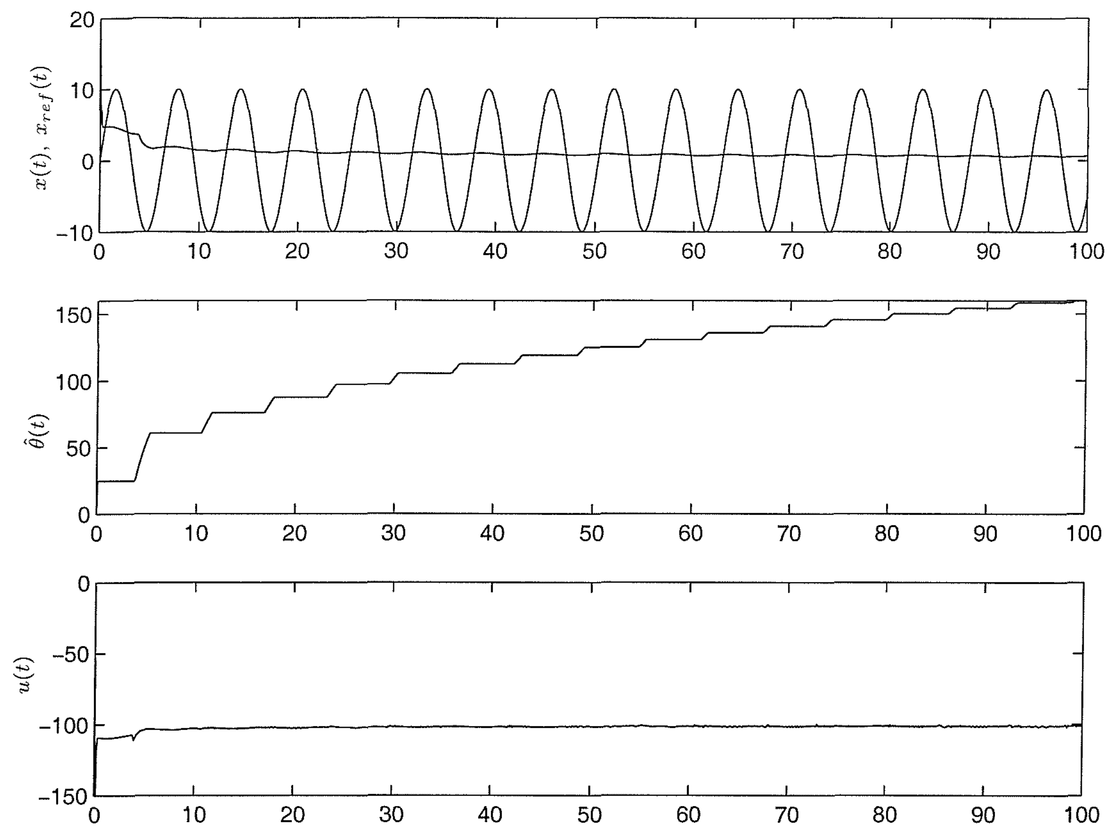

FIGURE 3.10: Tracking for example 3.2

where 'smooth' dead-zone function $D'_{\Omega_0}(e) = 0$ if $e \in \Omega_0$, $D_{\Omega_0}(e) = |e| - \lambda$ elsewhere. A routing substitution yields the following error equation:

$$\dot{e}(t) = (\theta - \hat{\theta}(t))e(t) + d(t) + \theta x_{ref}(t) - \dot{x}_{ref}(t), \tag{3.110}$$

Loosely speaking, the difference between these two methods is that the normal tracking drives the rendered error $e(t)$ to Ω_0 by canceling the unwanted terms while λ–tracking achieves its goal by dominating the unwanted terms. (see control input $u(\cdot)$ in (3.104) and (3.108)). As it has been shown in Fig. 3.11, λ–tracking achieves better closed loop response on this particular example.

However, there are some situations in which the λ–tracking controller results in an arbitrarily large cost. Let x_{ref} be constant and $d(\cdot) = 0$. Note that even if no disturbances are present, we still need to define the dead-zone. Since $e(t) \to [0, \lambda]$ as $t \to \infty$, then there exists $t^* > 0$ such that $e(t^*) < 2\lambda$ and $\frac{1}{2}\frac{d}{dt}e(t^*)^2 = \dot{e}(t^*)e(t^*) \leq 0$. It follows by (3.110) that, $(\hat{\theta}(t^*) - \theta) \geq \theta x_{ref}(t^*)/(2\lambda)$. From this,

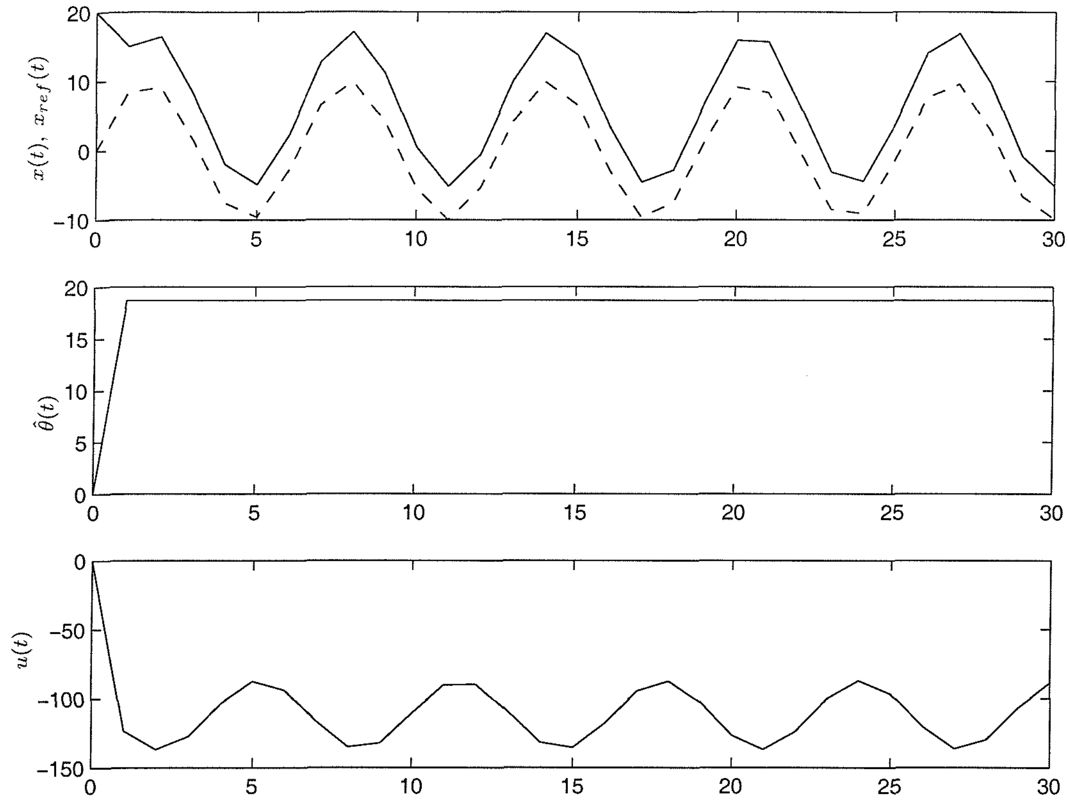

FIGURE 3.11: λ–tracking for example 3.2

one can easily show that the parameter estimator $\hat{\theta}(\cdot)$ in this design is of order $|x_{ref}(\cdot)|/\lambda$. So, $\hat{\theta}(t)$ can be made arbitrarily large by increasing accuracy ($\lambda \to 0$). Once a large parameter $\hat{\theta}$ has been obtained, a similar approach as Proposition 3.2 can be used to show that $\mathcal{P}$ can be made arbitrarily large. Consider $d'(t) := \dot{x}_{ref} - \theta x_{ref}$ as a bounded disturbance, and choose $x_{ref}(t) \in \mathcal{C}^1$ a.e. so that $d'(T)$ has a step change of sign similar to (3.73), where T is the time at which $\hat{\theta}(T)$ is sufficiently large. Such reference signals are quite common in applications, e.g. x_{ref} can be considered as the output (current/voltage) of an RC/LC circuit with step input. Renaming 'e' in (3.108)–(3.110) into 'x' and following the procedure explained in proof of Proposition 3.2 implies that $\mathcal{P}$ can be made arbitrarily large. Note that if $d = 0$, then the adaptive law (3.105) in dead-zone tracking controller is identical to that of standard tracking controller resulting in convergence of error to zero and boundedness of all closed loop signals, i.e. $\lim_{\eta_0 \to 0} \mathcal{P} < \infty$.

Finally, as we have mentioned in different point of this thesis, we emphasis that the basis of all comparative results established by theorems **I**, **Ia**, **II**, and **IIa** is founded on the quality of the corresponding a-priori information.

3.6 Summary and Discussion

This chapter introduced the main concepts of this dissertation by exploring the motivations behind the ideas on a very simple scalar system.

Firstly, we briefly reviewed some known concepts in the robust adaptive control literature including dead-zone and projection based controllers. Then we introduced the idea of hysteresis dead-zone and listed its advantages compared to the conventional dead-zone. Deferring the technicalities until chapter 5, we showed that the hysteresis dead-zone solves the major problems of the conventional dead-zone in the sense that the solution of the closed loop system is unique, the sliding motions are avoided and chattering effect can be mitigated.

Secondly, we have established two rigorous results demonstrating situations in which we can compare projection and dead-zone based adaptive controllers with respect to a worst case non-singular transient cost functional. We have shown that

- The dead-zone/hysteresis dead-zone controller outperforms the projection controller when the a-priori information on uncertain system parameter θ is sufficiently conservative.
- The projection controller outperforms the dead-zone/hsysteresis dead-zone controller when the a-priori information on disturbance level is sufficiently conservative.

Our results are based on the a-priori information $d_{\max}$ and $\theta_{\max}$. We showed that the other choices of controllers independent of a-priori information such as λ–tracking controllers can be driven to the same conclusions.

The subject of next chapters is to generalise the results of this chapter to minimum phase linear systems with relative degree one, and nonlinear systems in the form of integrator chain.

Chapter 4

Robustness and Performance Comparison: Linear Systems

4.1 Introduction

As discussed in the previous chapters, adaptive control is suitable for physical systems whose mathematical model contains an uncertain parameter θ. A common feature of adaptive designs is the construction of a time varying parameter $\hat{\theta}(t)$ whose value is controlled by an adaptive law. So, any adaptive controller consists of two parts: a feedback law and an adaptive law. There are two approaches in designing an adaptive controller namely: identifier-based and non-identifier-based.

The role of the adaptive law in an 'identifier-based' or 'indirect' adaptive controller is to attempt to 'identify' or 'estimate' the parameter θ. Once we have enough information such that $\hat{\theta}(t)$, which is referred to as the 'parameter estimator', reaches its desire value θ, the system can be regulated by means of the feedback law, which is often constructed via certainty equivalence.

In contrast to the above method, the objective of a 'non-identifier-based' or 'direct' adaptive control is simply to control the unknown plant. In this method no plant parameter estimation takes place. Instead the adaptation strategy uses certain information about the plant to find suitable methods of system regulation. In other words, the adaptive law has no interest in learning or estimating θ, but merely attempts to seek out a stabilising value of $\hat{\theta}$. This idea was initially proposed in 1980's by Morse [49] and Nussbaum [56]. Their idea has been developed since 1985 into two frameworks; that of Mårtensson [43], and that of Willems and Byrnes [80]. These form the foundation of the theory of 'universal adaptive control'. Additional contributions have been made by Ilchmann [22], Hicks [20] and Townley [73, 74] in the area of linear systems, Ryan [65], and Mårtensson [44] in nonlinear systems, and Logemann [41] in distributed parameter systems.

With regards to extending the results of chapter 3 to finite dimensional linear systems, we explain the concept of high-gain non-identifier-based adaptive control design first. Then we show that the dead-zone and projection modifications are appropriate solutions to the problem of robustness when linear systems are perturbed by bounded disturbances. Finally, the results of theorems **I** and **II** in the previous chapter will be extended to relative degree one, minimum phase linear systems.

4.2 System Description

Consider a class of SISO linear time invariant plant described by

$$y = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}(u+d) \tag{4.1}$$

where $a_i, b_j \in \mathbb{R}$ $\;0 \le i \le n-1,\; 0 \le j \le m$ are unknown constants and $d(\cdot)$ belongs to a class of bounded disturbances $\mathcal{D} \subset \mathcal{L}^\infty[0,\infty)$. We assume that only output $y(\cdot)$ is available for measurement. A minimal state space realisation of the plant in canonical observer form can be obtained as follows:

$$\begin{aligned} \Sigma(x_0, \theta, d(\cdot)): \quad & \dot{x}(t) = Ax(t) + B(u(t) + d(t)) \qquad x(0) = x_0, \\ & y(t) = Cx(t), \end{aligned} \tag{4.2}$$

in which $x(\cdot), B, C^T \in \mathbb{R}^n$, $A \in \mathbb{R}^{(n\times n)}$, and

$$A = \begin{bmatrix} -a_{n-1} & 1 & 0 & \dots & 0 \\ -a_{n-2} & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \dots & 1 \\ -a_0 & 0 & 0 & \dots & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0(\rho-1) \\ b_m \\ \vdots \\ b_1 \\ b_0 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}, \quad (4.3)$$

where $\rho = n - m$ is the relative degree of the system. We emphasise that by non-identifier-based control, we are not estimating the unknown parameter θ. However, for the sake of compatibility with the general system notation used in other parts of this thesis, we let

$$\theta = (a_0, \dots, a_{n-1}, b_0, \dots, b_m). \quad (4.4)$$

We aim to extend the result of the previous chapter to the linear system (4.2) of relative degree one. So, for the rest of this chapter we assume the following hold:

C1. The plant is minimum phase i.e. $b(s) = b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0$ is Hurwitz.

C2. The plant order n is known, $\rho = 1$, and $b_m = b_{n-1} > 0$.

4.2.1 High-Gain Control Design

In this section, we introduce the well known concept of 'high-gain controllers'. The presentation of the material on this section closely follows [22].

Lemma 4.1. Consider the system $\Sigma(x_0, \theta, d(\cdot))$ defined by (4.2), where C1,C2 hold. Suppose $d(t) = 0$. Define the controller:

$$\Xi : \; u(t) = -k(t)\, y(t), \quad (4.5)$$

where $k : \mathbb{R}^+ \to \mathbb{R}$ is a piecewise continuous function. Suppose there exists $t^* \in \mathbb{R}^+$, and $k^* \in \mathbb{R}$ such that $k(t) \geq k^*$ for all $t > t^*$. Then for k^* sufficiently large, the closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi)$ is exponentially stable.

Proof. Substituting (4.5) into (4.2) yields

$$\dot{x}(t) = (A - k(t)BC)\, x(t). \tag{4.6}$$

Following the High-Gain Lemma in [22], let us define coordinate transformation matrices S, S^{-1} as follows:

$$S := \left[B(CB)^{-1}, T\right], \quad S^{-1} = \left[C^T, N^T\right]^T, \quad N = \left[(b_{m-1}/b_m \ldots b_0/b_m)^T;\, I_{(m-1)}\right]. \tag{4.7}$$

where $T \in \mathbb{R}^{n\times(n-1)}$ denotes a basis matrix of *ker* C. Observe that S, S^{-1} depend continuously on θ. Transfering the coordinate of (4.6) by (4.7), we have

$$S^{-1}\, BC\, S = \begin{bmatrix} b_m & 0 \\ 0 & 0 \end{bmatrix}, \qquad S^{-1}AS = \begin{bmatrix} \bar{a}_1 & \bar{A}_2 \\ \bar{A}_3 & \bar{A}_4 \end{bmatrix}, \tag{4.8}$$

where $\bar{a}_1 \in \mathbb{R}$, $\bar{A}_2^T, \bar{A}_3 \in \mathbb{R}^{n-1}$ and $\bar{A}_4 \in \mathbb{R}^{(n-1)\times(n-1)}$. Let us denote

$$\bar{x}(t) := (y(t), z(t)^T)^T = S^{-1}x(t), \tag{4.9}$$

and substitute (4.9) in (4.6). By (4.8) we have that

$$\begin{bmatrix} \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} \bar{a}_1 - b_m k(t) & \bar{A}_2 \\ \bar{A}_3 & \bar{A}_4 \end{bmatrix} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}. \tag{4.10}$$

Since $b_m > 0$ and $k(t) > k^* \;\; \forall t > t^*$, for sufficient large k^*, we have that

$$\bar{a}_1 - b_m k(t) < 0 \qquad \forall t > t^*. \tag{4.11}$$

Now we show that $\bar{A}_4$ is also stable. To this end, we apply Schur's formula to (4.10) and observe that $\forall \lambda \neq \bar{a}_1 - b_m k(t)$

$$|\lambda I_n - A + k(t)BC| = (\lambda - \bar{a}_1 + b_m k(t))\, |\lambda I_{n-1} - \bar{A}_4 - \xi_\lambda(k)|, \tag{4.12}$$

where

$$\xi_\lambda : \mathbb{R} \to \mathbb{R}^{m\times m}, \quad k \mapsto \bar{A}_3(\lambda - \bar{a}_1 + b_m k)^{-1}\bar{A}_2. \tag{4.13}$$

Note that

$$\lim_{k\to\infty} \|\xi_\lambda(k)\| = 0, \quad \forall \lambda \in \mathbb{R}, \tag{4.14}$$

i.e. the term $\lambda I_{n-1} - \bar{A}_4 - \xi_\lambda(k)$ in (4.12) can be interpreted as a combination of a fixed part $\lambda I_{n-1} - \bar{A}_4$ and a vanishing part $\xi_\lambda(k)$. Therefore as $k \to \infty$, the coefficients of the polynomial in λ of $|\lambda I_{n-1} - \bar{A}_4 - \xi_\lambda(k)|$ converge to the coefficients of polynomial of $|\lambda I_{n-1} - \bar{A}_4|$. It follows by (4.12) that for all $k(t) \geq k^*$ for sufficiently large k^*, the eigenvalues of $A - k(t)BC$ are approaching a set including the eigenvalues of $\bar{A}_4$.

$$\sigma(A - k(t)BC) \to \{\bar{a}_1 - b_m k(t)\} \cup \sigma(\bar{A}_4). \tag{4.15}$$

The classical theory of LTI control (e.g. a root locus argument) indicates that there exists $\epsilon, \mu^* > 0$ such that if $\mu \geq \mu^*$, then $Re\,\sigma(A - \mu BC) < -\epsilon$. Hence, there exists $t^* > 0$ such that $k(t) \geq k^* = \mu^*$ for all $t \geq t^*$, so the time varying spectrum of $A - k(t)BC$ satisfies

$$Re\,\sigma(A - k(t)BC) < -\epsilon. \tag{4.16}$$

Hence by (4.15), $\bar{A}_4$ is stable[1].

Therefore there exists a symmetric positive definite matrix R such that

$$R\bar{A}_4 + \bar{A}_4^T R = -I_{n-1}. \tag{4.17}$$

Now we define the Lyapunov function

$$V(y(t), z(t)) = \frac{1}{2}y(t)^2 + z(t)^T R z(t), \quad V_0 := V(y(0), z(0)). \tag{4.18}$$

[1]Note that showing that a time-varying linear system with time varying spectrum with eigenvalues whose real parts are less than $-\epsilon$ does not necessary imply exponential stability

The time derivative of $V(y(t), z(t))$ is

$$\begin{aligned}\dot{V}(y(t), z(t)) &= y(t)\left((\bar{a}_1 - b_m k(t))y(t) + \bar{A}_2 z(t)\right) + y(t)\bar{A}_3^T R z(t) + z(t)^T R \bar{A}_3 y(t) \\ &\quad + z(t)^T(\bar{A}_4^T R + R\bar{A}_4)z(t) \\ &= -(b_m k(t) - \bar{a}_1)y(t)^2 + y(t)\left(\bar{A}_2 + 2\bar{A}_3^T R\right) z(t) - \|z(t)\|^2 \\ &\leq -(b_m k(t) - M)y(t)^2 - \frac{1}{2}\|z(t)\|^2,\end{aligned} \tag{4.19}$$

where $M := |\bar{a}_1| + \left(\|\bar{A}_2\| + 2\|R\|\,\|\bar{A}_3\|\right)^2/2$. We choose k^* such that $b_m k^* - M > 1/2$. Since by assumption there exists t^* such that $k(t) > k^*\ \forall t \geq t^*$, we observe that

$$\dot{V}(y(t), z(t)) \leq -\alpha V(y(t), z(t)) \qquad \forall t \geq t^*, \tag{4.20}$$

where $\alpha = \min\{1, 1/\overline{\lambda}(R)\}$. It follows that $V(y(t), z(t)) \leq V_0 e^{-\alpha t}$ for all $t \geq t^*$, therefore by (4.18) and (4.9) that $x(t)$ is exponentially stable. Thus completing the proof. □

Later in this chapter we frequently use the same Lyapunov function (4.18). A compact notation of the above calculations are given in the following corollary:

Corollary 4.1. Denote

$$D(k) = A - kBC, \tag{4.21}$$

for some $k > 0$, and let $\bar{D} = S^{-1}DS$. Then

$$\bar{D}(k^*)^T P + P\bar{D}(k^*) \leq -Q \tag{4.22}$$

where the symmetric positive definite matrices P and Q are

$$P = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & R \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2}I_{n-1} \end{bmatrix}, \tag{4.23}$$

and R is defined by (4.17).

Lemma (4.1) can be used to establish a 'non-identifier-based adaptive control' design by defining $\hat{\delta} : \mathbb{R}^+ \to \mathbb{R}$ as the tuning function of the controller

$$\Xi : \quad u(t) = -\hat{\delta}(t)\, y(t), \qquad \dot{\hat{\delta}}(t) = y(t)^2 \qquad \hat{\delta}(0) = 0. \tag{4.24}$$

The stability of the closed loop system where $\mathcal{D} = \{0\}$ can be achieved by the following theorem:

Theorem 4.1. Consider the closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi)$ defined by (4.2) and (4.24), where C1,C2 hold. If $d(t) = 0$ then the closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi)$ is asymptotically stable.

Proof. Replacing $k(t)$ with $\hat{\delta}(t)$ in (4.10), we have that

$$\dot{y}(t) = (\bar{a}_1 - b_m\hat{\delta}(t))y(t) + \bar{A}_2 z(t) \tag{4.25}$$

$$\dot{z}(t) = \bar{A}_4 z(t) + \bar{A}_3 y(t) \tag{4.26}$$

First, we show that $\hat{\delta}(\cdot) \in \mathcal{L}^\infty$. Suppose for contradiction $\hat{\delta}(t)$ diverges, then by Lemma 4.1, $y(t)$ exponentially decays, so $y(\cdot) \in \mathcal{L}^2$, hence

$$\hat{\delta}(t) = \int_0^t y(s)^2 ds \leq \int_0^\infty y(s)^2 ds = \|y(\cdot)\|_{\mathcal{L}^2}^2 < \infty, \tag{4.27}$$

hence contradiction. Therefore $\hat{\delta} \in \mathcal{L}^\infty$. Now since $\hat{\delta}(t)$ is monotonic, either (i) there exists $t^* \in \mathbb{R}^+, k^* \in \mathbb{R}$ such that $\hat{\delta}(t) \geq k^*$ for all $t \geq t^*$ or (ii) $\hat{\delta}(t) < k^*$ for all $t \geq 0$. The first case satisfies the conditions of Lemma 4.1 and the claim of lemma follows.

If $\hat{\delta}(t) < k^*$ for all $t \geq 0$, then by the second equation in (4.24)

$$\int_0^\infty y(s)^2 ds < k^*, \tag{4.28}$$

i.e $y(\cdot) \in \mathcal{L}^2$. Since $\bar{A}_4$ is stable, (4.26) can be considered as an $\mathcal{L}^2$ input $y(t)$ to a stable system $\dot{z}(t) = \bar{A}_4 z(t)$, which follows that $z(\cdot) \in \mathcal{L}^2$. Since $\hat{\delta}(\cdot)$ is bounded, it follows by (4.25) that $\dot{y}(\cdot) \in \mathcal{L}^2$ and by (4.26) that $\dot{z}(\cdot) \in \mathcal{L}^2$. Hence by Lemma 2.5, $y(t) \to 0$, $z(t) \to 0$. Thus completing the proof. □

If $d(t) \neq 0$, we need to modify the adaptive law as we discuss in the next section.

4.3 Modified Controllers

4.3.1 Dead-zone Based Controllers

Based on the definition of the dead-zone modification described in section 3.4.1 and theorem 4.1, the corresponding parameters $j(\cdot)$, $g(\cdot)$, Ω_0 for system (4.2) are as follows:

$$\Omega_0(d_{\max}) := [-\eta_0, \eta_0], \quad \eta_0 := \varrho(d_{\max}), \quad j := -\hat{\delta}(t)y(t), \quad g := y(t)^2. \tag{4.29}$$

Therefore similar to (3.13), the dead-zone controller for linear system (4.2) is

$$\begin{aligned} \Xi_D(d_{\max}): \quad & u(t) = -\hat{\delta}(t)y(t)) \\ & \dot{\hat{\delta}}(t) = D_{\Omega_0(d_{\max})}(y)\, y(t)^2, \quad \hat{\delta}(0) = 0, \quad \eta_0 = d_{\max} \end{aligned} \tag{4.30}$$

where

$$D_{\Omega_0(d_{\max})}(y) = \begin{cases} 0, & y \in \Omega_0(d_{\max}) \\ 1, & y \notin \Omega_0(d_{\max}) \end{cases} \tag{4.31}$$

In order to examine properties D1–D3 explained in section 3.4.1, we note that since definition (4.31) introduces a r.h.s. discontinuity in differential equation (4.30), we must define the solution $(x(\cdot), \hat{\delta}(\cdot))$ in a meaningful sense e.g. Filippov, and also, we need to consider the possibility of loosing the uniqueness of the solution. However, in this section we take the advantages of so-called 'smooth dead-zone' defined by

$$D'_{\Omega_0(d_{\max})}(y) = \begin{cases} 0, & y \in \Omega_0(d_{\max}) \\ |y| - \eta_0, & y \notin \Omega_0(d_{\max}) \end{cases} \tag{4.32}$$

We slightly modify the controller (4.30) into:

$$\begin{aligned} \Xi_{\acute{D}}(d_{\max}): \quad & u(t) = -\hat{\delta}(t)y(t)) \\ & \dot{\hat{\delta}}(t) = D'_{\Omega_0(d_{\max})}(y)\, |y(t)|, \quad \hat{\delta}(0) = 0, \quad \eta_0 = d_{\max}, \end{aligned} \tag{4.33}$$

Then the existence and uniqueness of the solution of the closed loop follows directly from the classical theory of differential equations. The following theorem establishes the properties of such controllers:

Theorem 4.2. Consider the closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi_{\hat{D}}(d_{\max}))$ defined by (4.2), (4.33), where C1,C2 hold and $d(\cdot)$ is bounded. Assume that $d_{\max}$ is such that $\|d(\cdot)\|_{\mathcal{L}^\infty} \leq d_{\max}$. Then for any $x_0 \in \mathbb{R}^n$, the following properties hold:

D1. There exist a unique solution $(x(\cdot), \hat{\delta}(\cdot)) : \mathbb{R}^+ \to \mathbb{R}^{(n+1)}$.

D2. The closed loop signals $x(\cdot), \hat{\delta}(\cdot), u(\cdot)$ are uniformly bounded as a continuous function of $x_0, \theta, d_{\max}$.[2]

D3. $y(t) \to \Omega_0$ as $t \to \infty$.

Before we give the proof, we first present a preliminary Lemma:

Lemma 4.2. Suppose M is an stable matrix. Let the positive definite matrix G be the solution of the Lyapunov equation

$$GM + M^T G = -I. \tag{4.34}$$

Then there exist $k, k_1 > 0$ such that

$$\|e^{Mt}\| \leq k_1 e^{-kt}, \quad k := 1/\overline{\lambda}(G), \quad k_1 := \sqrt{\frac{\overline{\lambda}(G)}{\underline{\lambda}(G)}}. \tag{4.35}$$

Proof. Define the Lyapunov function $V(t) := V(s(t))$:

$$V(s(t)) = s(t)^T G s(t), \tag{4.36}$$

where $s(t) = s_0 e^{Mt}$ is the solution of the differential equation

$$\dot{s}(t) = Ms(t), \qquad s_0 = s(0) \in \mathbb{R}^n. \tag{4.37}$$

[2]The function has domain $\mathbb{R}^n \times \mathcal{S} \times [0, \infty)$, where $\mathcal{S} := \{\theta \mid \Sigma(x_0, \theta, d(\cdot)) \text{ satisfies C1,C2}\}$.

Observe that by (4.37),(4.34),

$$\dot{V}(t) = s(t)^T(GM + M^TG)s(t) = -s(t)^Ts(t). \tag{4.38}$$

The general inequality

$$\underline{\lambda}(G)\|s(t)\|^2 \leq s(t)^TGs(t) \leq \overline{\lambda}(G)\|s(t)\|^2, \tag{4.39}$$

together with (4.36), (4.38) implies that

$$\dot{V}(t) \leq -kV(t), \quad k := 1/\overline{\lambda}(G), \tag{4.40}$$

which has the solution

$$V(t) \leq V_0e^{-kt}. \tag{4.41}$$

From this and (4.36),(4.34) one can obtain

$$\|s(t)\| \leq k_1\|s_0\|e^{-kt}, \quad k_1 := \sqrt{\frac{\overline{\lambda}(G)}{\underline{\lambda}(G)}}. \tag{4.42}$$

Substituting $s(t) = s_0e^{Mt}$ into (4.42) yields

$$\frac{\|s_0e^{Mt}\|}{\|s_0\|} \leq k_1e^{-kt}. \tag{4.43}$$

Finally by taking the supremum over s_0 of both sides, by definition of the induced norm of a matrix M, we have

$$\|e^{Mt}\| = \sup_{\|s_0\|\neq 0} \frac{\|s_0e^{Mt}\|}{\|s_0\|} \leq k_1e^{-kt}. \tag{4.44}$$

□

We now give the proof of Theorem 4.2. Despite their different objectives, this proof is derived from the technique used in the proof of 'λ–tracking' based on [22] but with a significant extension to obtain property D2.

proof of Theorem 4.2. The continuity of D'_{Ω_0} implies by classical result of differential equations that there exists a unique solution $(x(\cdot), \hat{\delta}(\cdot))$ over its maximal interval of existence $[0, \omega)$ for some $\omega > 0$. Using the transformation defined by (4.7), we obtain

$$\begin{aligned} \dot{y}(t) &= \left(\bar{a}_1 - b_m\hat{\delta}(t)\right) y(t) + \bar{A}_2 z(t) + b_m d(t), & y(0) &= Cx_0, & (4.45)\\ \dot{z}(t) &= \bar{A}_3 y(t) + \bar{A}_4 z(t), & z(0) &= Nx_0, & (4.46)\\ \dot{\hat{\delta}}(t) &= D'_{\Omega_0}(y)\,|y(t)|, & \hat{\delta}(0) &= 0 & (4.47) \end{aligned}$$

Define the $\mathcal{C}^1$ function

$$V(y(t)) := \begin{cases} 0, & y(t) \in \Omega_0(d_{\max}), \\ \frac{1}{2}(|y(t)| - \eta_0)^2, & y(t) \notin \Omega_0(d_{\max}), \end{cases} \tag{4.48}$$

and observe that

$$\dot{V}(y(t)) = \zeta(t)\dot{y}(t) \qquad \forall t \in [0, \omega), \tag{4.49}$$

where the continuous function $\zeta : \mathbb{R} \to \mathbb{R}$ is

$$\zeta(t) := \begin{cases} 0, & y(t) \in \Omega_0(d_{\max}). \\ (|y(t)| - \eta_0)\dfrac{y(t)}{|y(t)|}, & y(t) \notin \Omega_0(d_{\max}). \end{cases} \tag{4.50}$$

Note that by (4.33)

$$\dot{\hat{\delta}}(t) = \zeta(t)y(t) = |\zeta(t)|\,|y(t)|, \tag{4.51}$$

and by the continuity of (4.50), $|y(t)|\,|\zeta(t)| \geq \eta_0\,|\zeta(t)|$, or

$$|\zeta(t)| \leq \eta_0^{-1}|\zeta(t)|\,|y(t)| = \eta_0^{-1}\dot{\hat{\delta}}(t). \tag{4.52}$$

Substituting (4.45) in (4.49), we have for all $t \in [0, \omega)$,

$$\begin{aligned} \dot{V}(y(t)) &= (\bar{a}_1 - b_m\hat{\delta}(t))y(t)\zeta(t) + \bar{A}_2 z(t)\zeta(t) + b_m d(t)\zeta(t) \\ &\leq (\bar{a}_1 - b_m\hat{\delta}(t))\dot{\hat{\delta}}(t) + b_m\|d(\cdot)\|_{\mathcal{L}^\infty}\,|\zeta(t)| + \|\bar{A}_2\|\,\|z(t)\|\,|\zeta(t)| \\ &\leq (M_1 - b_m\hat{\delta}(t))\dot{\hat{\delta}}(t) + \|\bar{A}_2\|\,\|z(t)\|\,|\zeta(t)|, \end{aligned} \tag{4.53}$$

where M_1 denotes the continuous function $M_1(\theta, d_{\max}) := |\bar{a}_1| + \eta_0^{-1} b_m \|d(\cdot)\|_{\mathcal{L}^\infty}$. Note that the continuous dependency of M_1 on θ follows from definition (4.4) and transformation (4.8) which also depends continuously on θ. M_1 is also a continuous function of $d_{\max}$ by definition (4.33).

Now we derive a relation between the second part of (4.53) and $\hat{\delta}(t)$. Rewrite equation (4.46) as

$$\dot{z}(t) = \bar{A}_4 z(t) + \bar{A}_3 \zeta(t) + h(t), \tag{4.54}$$

where

$$h(t) := \bar{A}_3(y(t) - \zeta(t)). \tag{4.55}$$

Note that since

$$y - \zeta = \begin{cases} y, & |y| < \eta_0 \\ \eta_0 \dfrac{y}{|y|}, & |y| \geq \eta_0 \end{cases} \tag{4.56}$$

we have that $|h(t)| \leq \eta_0 \|\bar{A}_3\|$. Since $\bar{A}_4$ is exponentially stable, by Lemma 4.2,

$$M_2 e^{-\mu t} \geq \|e^{\bar{A}_4 t}\|, \quad M_2 := M_2(\theta) = \sqrt{\frac{\overline{\lambda}(R)}{\underline{\lambda}(R)}}, \quad \mu := \mu(\theta) = 1/\overline{\lambda}(R). \tag{4.57}$$

M_2 and μ depend continuously on θ since $\bar{A}_4$ depends continuously on θ, hence R and its eigenvalues are continuously dependent on θ. Therefore by equation (4.54), we have

$$\begin{aligned} \|z(t)\| &\leq M_2 e^{-\mu t}\|z_0\| + M_2 \int_0^t e^{-\mu(t-s)}(\|\bar{A}_3\|\,|\zeta(s)| + |h(s)|)\,ds \\ &\leq M_2 e^{-\mu t}\|z_0\| + M_2\|\bar{A}_3\| \int_0^t e^{-\mu(t-s)}|\zeta(s)|ds + M_2\|\bar{A}_3\|\eta_0\mu^{-1}(1 - e^{-\mu t}) \\ &\leq M_3\left(1 + \int_0^t e^{-\mu(t-s)}|\zeta(s)|\,ds\right), \end{aligned} \tag{4.58}$$

where $M_3 := M_3(x_0, \theta, d_{\max}) := M_2\left[\|z_0\| + \|\bar{A}_3\|\eta_0\mu^{-1}\right]$. The dependency of continuous function M_3 to $\|x_0\|, \theta$ follows from the continuous dependency of S^{-1}, μ and M_2 on θ. From inequality (4.58) it follows that

$$\int_0^t \|z(s)\|\,|\zeta(s)|\,ds \leq M_3 \int_0^t |\zeta(s)| + M_3 \int_0^t |\zeta(s)| \int_0^s e^{-\mu(s-\tau)}|\zeta(\tau)|\,d\tau\,ds. \tag{4.59}$$

Using the Cauchy-Schwartz inequality (2.4), we have that

$$\int_0^t |\zeta(s)| \int_0^s e^{-\mu(s-\tau)}|\zeta(\tau)|\, d\tau\, ds \leq \|\zeta(\cdot)\|_{\mathcal{L}^2(0,\,t)} \cdot \left\| \int_0^{\bullet} e^{-\mu(\bullet-\tau)}|\zeta(\tau)| d\tau \right\|_{\mathcal{L}^2(0,\,t)}. \tag{4.60}$$

An application of the following inequality [76]

$$\left\| \int_0^{\bullet} e^{-\mu(\bullet-\tau)}|\zeta(\tau)| d\tau \right\|_{\mathcal{L}^2(0,\,t)} \leq \|e^{-\mu \bullet}\|_{\mathcal{L}^1(0,\,t)} \|\zeta(\cdot)\|_{\mathcal{L}^2(0,\,t)} \leq \mu^{-1} \|\zeta(\cdot)\|_{\mathcal{L}^2(0,\,t)}, \tag{4.61}$$

and the fact that $|\zeta(t)| \leq \eta_0^{-1} \dot{\hat{\delta}}(t)$ by (4.52), yields by (4.59), (4.61),

$$\begin{aligned}\int_0^t \|z(s)\|\, |\zeta(s)| ds &\leq M_4 \int_0^t |\zeta(s)| + \zeta(s)^2\, ds \\ &\leq M_4 \int_0^t (1+\eta_0^{-1})|\zeta(s)|\, |y(s)|\, ds \\ &\leq M_5 \hat{\delta}(t),\end{aligned} \tag{4.62}$$

where $M_4 := M_4(x_0, \theta, d_{\max}) := M_3(1+\mu^{-1})$, $M_5 : M_5(x_0, \theta, d_{\max}) := M_4(1+\eta_0^{-1})$. Now, we can calculate $V(\cdot)$ in terms of $\hat{\delta}(\cdot)$ by integrating (4.53) over $[0, t]$

$$\begin{aligned}V(y(t)) &\leq V(y(0)) + \int_0^t \left(M_1 - b_m \hat{\delta}(s)\right) \dot{\hat{\delta}}(s) ds \; + \|\bar{A}_2\| M_5 \hat{\delta}(t) \\ &\leq V_0 + M_6 \hat{\delta}(t) - \frac{b_m}{2} \hat{\delta}(t)^2,\end{aligned} \tag{4.63}$$

where $V_0 := V(y(0))$, and $M_6 := M_6(x_0, \theta, d_{\max}) := M_1 + \|\bar{A}_2\| M_5$. The positive definiteness of $V(y(t))$ implies that

$$-\frac{b_m}{2}\hat{\delta}(t)^2 + M_6 \hat{\delta}(t) + V_0 \geq 0. \tag{4.64}$$

Solving the quadratic inequality (4.64) along $\hat{\delta}(t)$, we have that

$$\frac{M_6 - M_7}{b_m} < \hat{\delta}(t) < \frac{M_6 + M_7}{b_m}, \tag{4.65}$$

where $M_7 := M_7(x_0, \theta, d_{\max}) := \sqrt{M_6^2 + 2 b_m V_0} > M_6$. We discard the negative lower bound of (4.65) due to the fact that by (4.47), $\hat{\delta}(0) = 0$ and $\hat{\delta}(t)$ is non decreasing. Therefore

$$\hat{\delta}(t) \leq \frac{M_6 + M_7}{b_m}. \tag{4.66}$$

The continuous dependency on $x_0, \theta, d_{\max}$ of M_4–M_7 is a consequence of the continuous dependency of M_3, μ and the coordinate transformation (4.8). We now define $V^* := V^*(x_0, \theta, d_{\max}) = V_0 + M_6(M_6 + M_7)/b_m$ which is continuously dependent on $x_0, \theta, d_{\max}$. Then

$$V(y(t)) < V^*(x_0, \theta, d_{\max}) \qquad \forall t \in [0, \omega], \tag{4.67}$$

The uniform boundedness of $y(\cdot)$ in terms of V^* on $[0, \omega)$ follows from (4.48), (4.67). Therefore by (4.50), $\zeta(\cdot)$ is uniformly bounded in terms of a continuous function of $V^*(x_0, \theta, d_{\max})$ on $[0, \omega)$. Hence by (4.58), $z(\cdot)$ is uniformly bounded as a continuous function of $V^*(x_0, \theta, d_{\max})$ on $[0, \omega)$. It follows by (4.9) that $x(\cdot) \in \mathcal{L}^\infty(0, \omega)$ uniformly as a continuous function of $V^*(x_0, \theta, d_{\max})$ on $[0, \omega)$. The continuity of the closed loop equations (4.45)–(4.47) and the boundedness of the solution $(x(\cdot), \hat{\delta}(\cdot))$ implies that $\omega = \infty$. Finally the uniform boundedness of $u(\cdot)$ as a continuous function of $V^*(x_0, \theta, d_{\max})$ follows from (4.33). Thus establishing D1–D2.

In order to prove D3, we observe that by (4.51),(4.52) and the boundedness of $z(\cdot)$

$$\|\bar{A}_2\| \, \|z(t)\| \, |\zeta(t)| \leq \eta_0^{-1} \|\bar{A}_2\| \, \|z(t)\| \, \dot{\hat{\delta}}(t) := M_8 \dot{\hat{\delta}}(t). \tag{4.68}$$

Substituting (4.68) in (4.53) and defining $M_9 := \max_{t \geq 0}\{M_1 - b_m \hat{\delta}(t) + M_8\}$, we have that

$$\dot{V}(y(t)) \leq (M_1 - b_m \hat{\delta}(t) + M_8)\dot{\hat{\delta}}(t) \leq -\dot{\hat{\delta}}(t) + (M_9 + 1)\dot{\hat{\delta}}(t). \tag{4.69}$$

Now, we define $W(y(t), \hat{\delta}(t)) := V(y(t)) - (M_9 + 1)\hat{\delta}(t)$ and observe that

$$\dot{W}(y(t), \hat{\delta}(t)) \leq -\dot{\hat{\delta}}(t)) := -D'_{\Omega_0}(y) \, |y(t)| \leq 0. \tag{4.70}$$

Therefore by LaSalle's theorem 2.4, the solution $(x(t), \hat{\delta}(t))$ approaches the largest invariant set in $\{(x, \hat{\delta}) \in \mathbb{R}^{(n+1)} \mid |y(t)| \leq \eta_0\}$ as $t \to \infty$, hence proving D3, and completing the proof. □

4.3.2 Projection Based Controllers

For a SISO linear output feedback system, the convex set $\Pi(\delta_{\max})$ in (3.24) can be simply defined by $\Pi(\delta_{\max}) := [0, \delta_{\max}]$ where $\delta_{\max}$ is a strict upper bound for δ_θ and

$$\delta_\theta = \inf\{\, \delta \geq 0 \mid A - \tilde{\delta}BC \text{ is Hurwitz } \forall \tilde{\delta} \geq \delta\}. \tag{4.71}$$

Let T_m be the first time instance that $\hat{\delta}$ hits the boundary $\delta_{\max}$:

$$T_m = \inf\{t \geq 0 \mid \hat{\delta}(t) = \delta_{\max}\}. \tag{4.72}$$

Then the projection controller is simply defined as follows:

$$\begin{aligned} \Xi_P(\delta_{\max}): \quad & u(t) = -\hat{\delta}(t)y(t) \\ & \dot{\hat{\delta}}(t) = y(t)^2, \quad \hat{\delta}(0) = 0, \qquad \forall t \in [0, T_m], \\ & \hat{\delta}(t) = \delta_{\max}, \qquad\qquad\quad\; \forall t \in [T_m, \infty). \end{aligned} \tag{4.73}$$

We denote the respective closed loop system by $(\Sigma(x_0, \theta, d(\cdot)), \Xi_P(\delta_{\max}))$. The stability of the closed loop is examined in the following theorem.

Theorem 4.3. Consider the closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi_P(\delta_{\max}))$ defined by (4.2), (4.73), where C1,C2 hold and $d(\cdot) \in \mathcal{L}^\infty$. Assume that $\delta_{\max}$ is such that $\delta_\theta < \delta_{\max}$. Then for any $x_0 \in \mathbb{R}^n$, the following properties hold:

P1. The solution $(x(\cdot), \hat{\delta}(\cdot)) : \mathbb{R}^+ \to \mathbb{R}^{(n+1)}$ exist.

P2. all closed loop signals $x(\cdot), \hat{\delta}(\cdot), u(\cdot)$ are uniformly bounded as a continuous function of $x_0, \theta, \|d\|, \delta_{\max}$.[3]

Proof. Since the right hand side of the differential equations (4.2) and (4.73) are locally Lipschitz, an absolutely continuous local solution exists. Let $(x(\cdot), \hat{\delta}(\cdot))$ denote a solutions of $(\Sigma(x_0, \theta, d(\cdot)), \Xi_P(\delta_{\max}))$ on a maximum interval of existence $[0, \omega)$ for some $\omega \in [0, \infty)$.

[3]The function has domain $\mathbb{R}^n \times \mathcal{S} \times [0, \infty)$, where $\mathcal{S} := \{\theta \mid \Sigma(x_0, \theta, d(\cdot)) \text{ satisfies C1,C2}\}$.

By definition of projection modification (4.73), $\hat{\delta}(t) \leq \delta_{\max}$ for all $t \geq 0$. Choose k^* defined in Lemma 4.1 such that $k^* = (\delta_{\max} + \delta_\theta)/2$ and observe that $k^* \in \Pi(\delta_{\max})$. The monotonicity of $\hat{\delta}(t)$ implies that either (i) $\hat{\delta}(t) < k^*$ for all $t \geq 0$, or (ii) there exists $t^* \geq 0$ such that $\hat{\delta}(t) > k^*$ for all $t > t^*$.

(i) If $\hat{\delta}(t) < k^* < \delta_{\max}$ for all $t \in [0, \omega)$ then by (4.73) $\dot{\hat{\delta}}(t) = y(t)^2$ for all $t \in [0, \omega)$. It follows by the same argument as Theorem 4.1 that $y(\cdot) \in \mathcal{L}^2[0, \omega)$ and by (4.46), $z(\cdot)$, $\dot{z}(\cdot) \in \mathcal{L}^2[0, \omega)$. An explicit bound on $\|z\|_{\mathcal{L}^2}$ can be obtained as follows:

$$\|z\|_{\mathcal{L}^2[0,\omega)} \leq \|z\|_{\mathcal{L}^2[0,\infty)} \leq \|z_0 e^{\bar{A}_4 t}\| + \|(sI - \bar{A}_4)^{-1}\bar{A}_3\|_{H_\infty} \|y'\|_{\mathcal{L}^2[0,\infty)}, \tag{4.74}$$

where

$$y'(t) = \begin{cases} y(t), & 0 \leq t < \omega \\ 0, & \omega \leq t < \infty \end{cases} \tag{4.75}$$

Note that since $y(\cdot) \in \mathcal{L}^2[0, \omega)$ and $\bar{A}_4$ is exponentially stable, by Lemma 4.2, there exists M_0, ν such that

$$M_0 e^{-\nu t} \geq \|e^{\bar{A}_4 t}\| \quad M_0 := M_0(\theta) = \sqrt{\frac{\overline{\lambda}(R)}{\underline{\lambda}(R)}}, \quad \nu : \nu(\theta) = 1/\overline{\lambda}(R), \tag{4.76}$$

Therefore

$$\begin{aligned} \|z\|_{\mathcal{L}^2[0,\omega)} &\leq M_0 e^{-\nu t}\|z_0\| + \|(sI - \bar{A}_4)^{-1}\bar{A}_3\|_{H_\infty} \|y\|_{\mathcal{L}^2[0,\omega)} \\ &\leq M_0 e^{-\nu t}\|z_0\| + \|(sI - \bar{A}_4)^{-1}\bar{A}_3\|_{H_\infty} \sqrt{k^*}. \end{aligned} \tag{4.77}$$

A uniform bound on $z(\cdot)$ is also required to show the boundedness of $x(\cdot)$. Applying (4.76) to the solution of (4.46), we have that and

$$\begin{aligned} \|z(t)\| &\leq M_0 e^{-\nu t}\|z_0\| + M_0 \int_0^t e^{-\nu(t-s)} \|\bar{A}_3\| \, |y(s)| \, ds \\ &\leq M_1 \left(1 + \int_0^t e^{-\nu(t-s)} |y(s)| \, ds\right), \end{aligned} \tag{4.78}$$

where $M_1 := M_1(\|x_0\|, \theta) := M_0(\|z_0\| + \|\bar{A}_3\|)$. It follows that $z(\cdot)$ is uniformly bounded as a continuous function of $\|x_0\|, \theta, \delta_{\max}$. It remains to show $y(\cdot) \in$

$\mathcal{L}^\infty[0,\omega)$. To this end, consider (4.45) and define a Lyapunov function:

$$V_1(y(t)) := \frac{1}{2}y(t)^2, \qquad \forall t \in [0,\omega). \tag{4.79}$$

We let $V_1(t)$ denote $V_1(y(t))$. The time derivative of (4.79) is

$$\begin{aligned}\dot{V}_1(t) &= (\bar{a}_1 - b_m\hat{\delta}(t))y(t)^2 + \bar{A}_2 z(t)y(t) + b_m d(t)y(t) \\ &= \frac{-1}{2}y(t)^2 + (\bar{a}_1+1)y(t)^2 - b_m\hat{\delta}(t)y(t)^2 + \bar{A}_2 z(t)y(t) - \frac{1}{2}y(t)^2 + b_m d(t)y(t) \\ &\le -V_1(t) + (\bar{a}_1+1)y(t)^2 + \bar{A}_2 z(t)y(t) + \frac{b_m^2}{2}\|d(\cdot)\|^2_{\mathcal{L}^\infty} \end{aligned} \tag{4.80}$$

where (4.80) follows from the Young's inequality and noting that $b_m\hat{\delta}(t) > 0$ for all $t \ge 0$. In order to obtain $V_1(\omega)$, Let us define the continuous function $M_2(\theta, \|d\|) := \frac{b_m^2}{2}\|d(\cdot)\|^2_{\mathcal{L}^\infty}$ and denote

$$F_1 := \{\, t \in [0,\omega) \ \mid V_1(t) > M_2(\theta, \|d\|)\,\}. \tag{4.81}$$

Note that $m(F_1) < \omega$. F_1 can be written as the union of all maximal disjointed connected intervals $G_a = [t_a^-, t_a^+]$, i.e. $F_1 = \bigcup_{a\in A} G_a$. Resorting G_a's on an increasing sequence

$$A_n := \{a \in A \ \mid \ m(G_a) \ge 1/n\}, \tag{4.82}$$

one can rewrite $F_1 := \bigcup_{n\ge 1}\bigcup_{a\in A_n} G_a$. Not that, as $m(F_1) < \infty$, the cardinality of each A_n is finite. Let us define the function $v_n : \mathbb{R} \to \mathbb{R}$ as

$$v_n(t) = \mathcal{X}_{\bigcup\limits_{a\in A_n} G_a}\dot{V}_1(t), \tag{4.83}$$

where the characteristic function $\mathcal{X}_\sigma(t) = 1$ if $t \in \sigma$, $\mathcal{X}_\sigma(t) = 0$ elsewhere. Observe that $v_n \leq v_{n+1}$ almost everywhere for each n, and also observe that

$$\begin{aligned}
\int v_n &= \sum_{a \in A_n} V_1(t_a^+) - V_1(t_a^-) = \int_{t_a^-}^{t_a^+} \dot{V}_1(s)ds \\
&\leq (\bar{a}_1 + 1) \int_{t_a^-}^{t_a^+} y(s)^2 ds + \int_{t_a^-}^{t_a^+} \bar{A}_2 z(s) y(s) ds \\
&\leq \bar{a}_1 \|y(\cdot)\|^2_{\mathcal{L}^2[0,\omega)} + \|\bar{A}_2\| \, \|y(\cdot)\|_{\mathcal{L}^2[0,\omega)} \, \|z(\cdot)\|_{\mathcal{L}^2[0,\omega)} \\
&\leq \bar{a}_1 \|y(\cdot)\|^2_{\mathcal{L}^2[0,\omega)} + \|\bar{A}_2\| \, \|y(\cdot)\|_{\mathcal{L}^2[0,\omega)} \Big(M_0 e^{-\nu t} \|z_0\| + \sqrt{k^*} \|(sI - \bar{A}_4)^{-1} \bar{A}_3\|_{H_\infty} \Big) \\
&\leq \delta_{\max} \left(\bar{a}_1 + \|\bar{A}_2\| \left(M_0 e^{-\nu t} \|z_0\| + \sqrt{k^*} \, \|(sI - \bar{A}_4)^{-1} \bar{A}_3\|_{H_\infty} \right) \right) \\
&= M_3(x_0, \theta, \delta_{\max}).
\end{aligned} \tag{4.84}$$

The continuous dependency of M_3 on θ follows by the continuity of the transformation (4.8), and the continuity of $\|(sI - \bar{A}_4)^{-1} \bar{A}_3\|_{H_\infty}$ on the entries of the matrices $\bar{A}_3, \bar{A}_4$ [85].

Inequality (4.84) implies that $\sup_n \int v_n < \infty$. Therefore by the Monotone Convergence Theorem [78]

$$\int_{F_1} \dot{V}_1(t)dt = \lim_{n \to \infty} \int_{\bigcup\limits_{a \in A_n} G_a} \dot{V}_1(t)dt \leq M_3(x_0, \theta, \delta_{\max}). \tag{4.85}$$

Now we can calculate $y(\omega)$ as follows:

$$\begin{aligned}
\frac{1}{2} y(\omega)^2 := V_1(\omega) &= V_1(0) + \int_{F_1} \dot{V}_1(\tau)d\tau + \int_{[0,\omega)\setminus F_1} \dot{V}_1(\tau)d\tau \\
&\leq y(0)^2 + M_3(x_0, \theta, \delta_{\max}) + M_2(\theta, \|d\|).
\end{aligned} \tag{4.86}$$

That is $y(\cdot)$ is uniformly bounded on $[0, \omega)$ in terms of a continuous function of $x_0, \theta, \|d\|, \delta_{\max}$. It follows by (4.78), (4.9) that $x(\cdot)$ is uniformly bounded on $[0, \omega)$ as a continuous function of $x_0, \theta, \|d\|, \delta_{\max}$.

(*ii*) Suppose there exists $t^* \geq 0$ such that $\hat{\delta}(t) > k^*$ for all $t > t^*$. Define the Lyapunov function

$$V_2(\bar{x}(t)) = \bar{x}(t)^T P \bar{x}(t), \qquad \forall t \in [0, \omega) \tag{4.87}$$

where $\bar{x}(t), P$ are defined by (4.9) and (4.23) respectively. Denoting

$$\bar{b} := (P + P^T)\bar{B}, \tag{4.88}$$

the time derivative of $V(t) := V(\bar{x}(t))$ is

$$\begin{aligned}
\dot{V}_2(t) =&\bar{x}(t)^T P\dot{\bar{x}}(t) + \dot{\bar{x}}(t)^T P\bar{x}(t),\\
=&\bar{x}(t)^T P\left(\bar{D}(\hat{\delta}(t))\bar{x}(t) + \bar{B}d(t)\right) + \left(\bar{D}(\hat{\delta})\bar{x} + \bar{B}d(t)\right)^T P\bar{x}(t),\\
=&\bar{x}(t)^T\left(P\bar{D}(\hat{\delta}(t)) + \bar{D}^T(\hat{\delta}(t))P\right)\bar{x}(t) + \bar{x}(t)^T P\bar{B}d(t) + d(t)\bar{B}^T P\bar{x}(t)\\
=&\bar{x}(t)^T\left(P\bar{D}(\hat{\delta}(t)) + \bar{D}^T(\hat{\delta}(t))P\right)\bar{x}(t) + \bar{x}(t)^T\,\bar{b}\,d(t).
\end{aligned} \tag{4.89}$$

Let

$$\mu(\theta, \|d\|) := \frac{\sqrt{\underline{\lambda}(P)}}{\underline{\lambda}(Q)}\,|\bar{b}|\,\|d(\cdot)\|_{\mathcal{L}^\infty}, \tag{4.90}$$

then μ is continuously depend on θ by definition (4.88) and the continuity of the transformation (4.8). Applying Corollary 4.1, we observe that

$$\dot{V}_2(t) \;\leq\; -\bar{x}(t)^T Q\bar{x}(t) + \bar{x}(t)^T\bar{b}d(t), \tag{4.91}$$

$$\leq\; -\left(\underline{\lambda}(Q) - \frac{\sqrt{\underline{\lambda}(P)}}{\sqrt{V_2(t)}}\,|\bar{b}|\,\|d(\cdot)\|_{\mathcal{L}^\infty}\right)\|\bar{x}(t)\|^2, \tag{4.92}$$

$$\leq\; -\underline{\lambda}(Q)\left(1 - \frac{\mu(\theta, \|d\|)}{\sqrt{V_2(t)}}\right)\|\bar{x}(t)\|^2, \tag{4.93}$$

where (4.92) follows from the general inequality

$$\underline{\lambda}(\Gamma)\|x(t)\|^2 \leq x(t)^T\Gamma x(t) \leq \overline{\lambda}(\Gamma)\|x(t)\|^2, \tag{4.94}$$

where Γ is a symmetric positive definite matrix. Inequality (4.93) for all $t > t^*$ implies that $\dot{V}_2(\bar{x}(t)) < 0$ for all $V_2(t) > \mu(\theta, \|d\|)^2$, i.e.

$$V(\bar{x}(t)) \leq V'(\bar{x}^*, \theta, \|d\|) := \max\{\,V_2(\bar{x}^*)\,,\,\mu(\theta, \|d\|)^2\,\} \quad \forall t > t^*, \tag{4.95}$$

where $\bar{x}^* := \bar{x}(t^*)$. Therefore, by (4.87), $\bar{x}(\cdot)$ is uniformly bounded on $[t^*, \infty)$ in terms of $V'(\bar{x}^*, \theta, \|d\|)$. A uniform bound on $\bar{x}(\cdot)$ on $[0, t^*)$ and a bound on the end point $\bar{x}(t^*)$ as a continuous function of $x_0, \theta, \|d\|, \delta_{\max}$ follows directly by the argument of part (i) where $\hat{\delta}(t) < k^* < \delta_{\max}$ for all $t \in [0, t^*)$. Hence by the continuity of the transformation (4.8), $x(\cdot)$ is uniformly bounded on $[0, \omega)$ as a continuous function of $x_0, \theta, \|d\|, \delta_{\max}$.

The boundedness of $x(t)$ over $[0, \omega)$ implies that $x(\cdot)$ cannot have a finite escape time. $\hat{\delta}(\cdot)$ is also known, by definition (4.73), that never leave the set $\Pi(\delta_{\max})$. Hence by Corollary 2.2, $\omega = \infty$ i.e. the solution $(x(\cdot), \hat{\delta}(\cdot))$ exists for all $t \in [0, \infty)$ and is uniformly bounded as a continuous function of $x_0, \theta, \|d\|, \delta_{\max}$. Finally, the uniform boundedness of $u(\cdot)$ as a continuous function of $x_0, \theta, \|d\|, \delta_{\max}$ is followed by (4.73).

□

4.4 Performance Comparison

We will investigate theorems **I** and **II** of the last chapter for the linear system described in (4.2). Define

$$\Delta(\delta) = \{\theta \mid A - \delta BC \text{ is Hurwitz and C1, C2 hold}\}, \quad \delta \geq 0, \tag{4.96}$$

where θ is given by (4.4). Let Λ be any compact subset of $\Delta(\delta)$. Define $\mathcal{X}_0(\gamma), \mathcal{D}(\epsilon)$ as in (3.45) by

$$\begin{aligned} \mathcal{D}(\epsilon) &:= \{d(\cdot) \mid \|d(\cdot)\|_{\mathcal{L}^\infty} \leq \epsilon\}, \quad && \epsilon \geq 0, \\ \mathcal{X}_0(\gamma) &:= \{x_0 \mid \|x_0\| \leq \gamma\}, && \gamma > 0. \end{aligned} \tag{4.97}$$

The cost functional is now taken to be

$$\mathcal{P}\left(\Sigma\left(\mathcal{X}_0(\gamma), \Lambda, \mathcal{D}(\epsilon)\right), \Xi\right) = \sup_{x_0 \in \mathcal{X}_0(\gamma)} \sup_{\theta \in \Lambda} \sup_{d \in \mathcal{D}(\epsilon)} \left(\|x(\cdot)\|_{\mathcal{L}^\infty} + \|u(\cdot)\|_{\mathcal{L}^\infty} + \|\dot{u}(\cdot)\|_{\mathcal{L}^\infty}\right). \tag{4.98}$$

There are elements on the boundary of $\Delta(\delta)$ which do not satisfy C1,C2 and for which the closed loop is not stable, hence generating an infinite cost. Therefore the second supremum cannot be taken over $\Delta(\delta)$. This is reflected in the bounds obtained in theorems 4.2 and 4.3 which depend on:

1. $1/b_m$: $b_m = 0$ corresponds to violating the relative degree one assumption C2 (see inequality (4.66)).

2. $\sigma(R)$: $\underline{\lambda}(R) = 0$ corresponds to R not being positive definite, ie. $\bar{A}_4$ not being Hurwitz, ie. the system not being minimum phase C1 (see (4.57) and the dependency of M_3, M_4 on μ^{-1}).

3. $\|(sI - \bar{A}_4)^{-1}\bar{A}_3\|_{H_\infty}$: $(\|(sI - \bar{A}_4)^{-1}\bar{A}_3\|_{H_\infty} = \infty$ corresponds to $\bar{A}_4$ not being Hurwitz, which again corresponds to violating the minimum phase behaviour C1 (see relations (4.77)).

4.4.1 Theorem I

Theorem I. *Consider the system* $\Sigma(x_0, \theta, d(\cdot))$ *and the controllers* $\Xi_{\hat{D}}(d_{\max})$ *and* $\Xi_P(\delta_{\max})$ *defined by* (4.2), (4.33) *and* (4.73) *respectively, where C1,C2 hold. Let* $\Lambda \subset \Delta(\delta)$ *be compact. Consider the transient performance cost functional* (4.98). *Then* $\forall d_{\max} \geq \epsilon$, $\exists \delta^*_{\max} \geq \delta$ *such that* $\forall \delta_{\max} \geq \delta^*_{\max}$,

$$\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Lambda, \mathcal{D}(\epsilon)), \Xi_P(\delta_{\max})) > \mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Lambda, \mathcal{D}(\epsilon)), \Xi_{\hat{D}}(d_{\max})). \tag{4.99}$$

In order to follow the procedure used in section 3.5.3, we give an alternative proof for Proposition 3.1 as follows:

Lemma 4.3. Consider the system $\dot{z} = f(z)$ where f is continuous. Then $\lim_{t\to\infty} z(t) = z^*$ implies that z^* is an equilibrium point.

Proof. z^* is an equilibrium if and only if $f(z^*) = 0$. So for a contradiction, suppose without loss of generality that $f(z^*) > 0$. Since f is continuous, there exist a set $\mathcal{V}$ of neighbourhood zero such that $f(x) > \epsilon = |f(z^*)/2|$, $\forall x \in \mathcal{V}$. Since $z(t) \to z^*$ as $t \to \infty$, it follows that there exists a time instant $T = \sup_{t>0}\{z(t) \notin \mathcal{V}\}$. Therefore

$$z(\infty) - \sup_{z\in\mathcal{V}} |z| = \int_T^\infty \dot{z} d\tau > \int_T^\infty \epsilon \, d\tau = \infty,$$

hence contradiction.

□

Proposition 4.1. Consider the closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi)$ defined by (4.2), (4.24), where C1,C2 hold and $d(t) = \epsilon$ for some $\epsilon \neq 0$. Then

$$\|x(t)\| \to 0 \text{ as } t \to \infty \Longleftrightarrow \hat{\delta}(t) \to \infty \text{ as } t \to \infty. \tag{4.100}$$

Proof. $\to$) Suppose for contradiction $\hat{\delta}(t) \not\to \infty$. Then $\hat{\delta}(t) \to \hat{\delta}^* < \infty$, since $\hat{\delta}(t)$ is monotonic by (4.24). Therefore $(x(t), \hat{\delta}(t)) = (0, \hat{\delta}^*)$ is an equilibrium point of closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi)$ by Lemma 4.3. Hence $(0, \hat{\delta}^*)$ must be a solution of the following equations:

$$\begin{aligned}
x_2(t) - a_{n-1}x_1(t) + b_m(\epsilon - \hat{\delta}(t)x_1(t)) &= 0, \\
x_3(t) - a_{n-2}x_1(t) + b_{m-1}(\epsilon - \hat{\delta}(t)x_1(t)) &= 0, \\
&\vdots \\
-a_0x_1(t) + b_0(\epsilon - \hat{\delta}(t)x_1(t)) &= 0, \\
x_1(t)^2 &= 0.
\end{aligned} \tag{4.101}$$

But $b_0 \neq 0$ since the system is minimum phase. We also have $\epsilon \neq 0$. Therefore $(x(t), \hat{\delta}(t)) = (0, \hat{\delta}^*)$ cannot be a solution of (4.101), hence contradiction.

$\leftarrow$) Define the Lyapunov function

$$V(\bar{x}(t)) = \bar{x}(t)^T P \bar{x}(t), \tag{4.102}$$

where $\bar{x}(t), P$ are defined by (4.9) and (4.23) respectively. Denote $\bar{B} = S^{-1}B$ and

$\bar{b} = (P + P^T)\bar{B}$. Define

$$\varphi(t) := \bar{x}(t)^T \left(P\bar{D}(\hat{\delta}(t) - k^*) + \bar{D}(\hat{\delta}(t) - k^*)^T P \right) x(t). \tag{4.103}$$

The time derivative of $V(\bar{x}(t))$ is:

$$\begin{aligned}
\dot{V}(\bar{x}(t)) &= \bar{x}(t)^T \left(P\bar{D}(k^*) + \bar{D}^T(k^*)P \right) \bar{x}(t) + \bar{x}(t)^T P\bar{B}\epsilon + \epsilon\bar{B}^T P\bar{x}(t) + \varphi(t) \\
&\leq -\bar{x}(t)^T Q\bar{x}(t) + \bar{x}(t)^T \bar{b}\epsilon + \varphi(t), \qquad (4.104) \\
&\leq -\underline{\lambda}(Q)\|\bar{x}(t)\|^2 + \|\bar{x}(t)\|\,|\bar{b}|\,|\epsilon| + \varphi(t), \\
&\leq -\left(\frac{\underline{\lambda}(Q)\|\bar{x}(t)\|^2}{2} - \varphi(t) \right) + \frac{|\bar{b}|^2\,|\epsilon|^2}{2\underline{\lambda}(Q)}. \qquad (4.105)
\end{aligned}$$

where equation (4.104) follows from (4.22), and inequality (4.105) follows from Young's inequality. Therefore $V(\cdot)$ is decreasing if

$$\frac{\underline{\lambda}(Q)\|\bar{x}(t)\|^2}{2} - \varphi(t) \geq \frac{|\bar{b}|^2\,|\epsilon|^2}{2\underline{\lambda}(Q)}. \tag{4.106}$$

Now, we claim the convergence of $\bar{x}(\cdot)$: if $\|\bar{x}(t)\| \not\to 0$ as $t \to \infty$ then either 1. $\liminf_{t\to\infty} \|\bar{x}(t)\| > 0$ or 2. $\liminf_{t\to\infty} \|\bar{x}(t)\| = 0$:

1. Suppose $\liminf_{t\to\infty} \|\bar{x}(t)\| > 0$. Then there exists $\epsilon' > 0$ s.t. $\|\bar{x}(t)\| > \epsilon'$ $\forall t$. It follows by (4.103) that $\varphi(t) \to -\infty$ as $\hat{\delta}(t) \to \infty$, hence by (4.104), $\dot{V} \to -\infty$ as $t \to \infty$, i.e.

$$\forall M > 0 \;\; \exists T > 0 \;\; s.t. \;\; \forall t \geq T \;\; \dot{V}(\bar{x}(t)) \leq -M, \tag{4.107}$$

   which implies that $V(\bar{x}(t)) \to -\infty$ as $t \to \infty$. This contradicts the positive definiteness of $V(\cdot)$.

2. If $\liminf_{t\to\infty} \|\bar{x}(t)\| = 0$, then there exists $\epsilon' > 0$, and a positive divergent sequence $\{t_k\}_{k\geq 1}$ such that $\dot{V}(\bar{x}(t_k)) > 0$ and $\|\bar{x}(t_k)\| > \epsilon'$. Since by (4.103), $\varphi(t_k) \to -\infty$ as $k \to \infty$, it follows that (4.106) holds at time t_k, hence contradiction.

Therefore $\|\bar{x}(t)\| \to 0$ as $t \to \infty$; hence $x(t) \to 0$ by (4.9).

□

Proposition 4.2. Consider the closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi)$ defined by (4.2), (4.24), where C1,C2 hold and $d(t) = \epsilon$ for some $\epsilon \neq 0$. If $x(t)$ is uniformly continuous, then as $t \to \infty$

$$\|x(t)\| \to 0, \quad \hat{\delta}(t) \to \infty. \tag{4.108}$$

Proof. Firstly we show that $y(t) \to 0$ as $t \to \infty$. From this we will prove that $\hat{\delta}(t) \to \infty$ and finally by Proposition 4.1, we conclude that $\|x(t)\| \to 0$ as $t \to \infty$. Suppose for contradiction $y(t) \not\to 0$. Then there must exists a positive divergent sequence $\{t_k\}_{k\geq 1}$ for which $y(t_k) \geq M$ for some $M > 0$ i.e.

$$\exists M > 0 \qquad \exists \{t_k\}_{k\geq 1},\ t_k \to \infty \qquad s.t. \qquad y(t_k) \geq M. \tag{4.109}$$

$y(t)$ is uniformly continuous since, by assumption, $x(t)$ is uniformly continuous i.e. for $\epsilon = M/2$

$$\exists \omega > 0 \ s.t. \ \forall \tau \in [0, \omega],\ \forall t > 0, \qquad |y(t) - y(t+\tau)| < \frac{M}{2}. \tag{4.110}$$

Therefore

$$|y(t_k) - y(t_k + \tau)| < \frac{M}{2}, \tag{4.111}$$

and since $y(t_k) \geq M$, we have that $y(t_k + \tau) > M/2$ i.e.

$$y(t) \geq \frac{M}{2}, \qquad \forall t \in [t_k, t_k + \omega]. \tag{4.112}$$

Hence,

$$\int_{t_k}^{t_k+\omega} y^2(\tau) d\tau \geq \frac{M^2}{4} \omega. \tag{4.113}$$

With no loss of generality, we may assume $t_{k+1} - t_k \geq \omega$. It follows by (4.24) and (4.113)

$$\hat{\delta}(t_k + \omega) = \int_0^{t_k+\omega} \dot{\hat{\delta}}(\tau) d\tau = \int_0^{t_k+\omega} y^2(\tau) d\tau \geq \frac{M^2}{4} k\omega, \tag{4.114}$$

so $\hat{\delta}(t_k + \omega) \to \infty$ as $k \to \infty$. It follows by Proposition 4.1 that $\|x(t)\| \to 0$ as $t \to \infty$, therefore $y(t) \to 0$ by (4.2), hence contradiction.

Now we have $y(t) = x_1(t) \to 0$ and we claim $\hat{\delta}(t) \to \infty$. Suppose for contradiction $\hat{\delta}(t) \not\to \infty$. Then $\hat{\delta}(t) \to \hat{\delta}^* < \infty$, since $\hat{\delta}(t)$ is monotonic by (4.24). Substitute this into (4.2), we have

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) - (a_{n-1} + \hat{\delta}^* b_m)x_1(t) + b_m\epsilon, \qquad (4.115)\\
\dot{x}_2(t) &= x_3(t) - (a_{n-2} + \hat{\delta}^* b_{m-1})x_1(t) + b_{m-1}\epsilon,\\
&\vdots\\
\dot{x}_{n-1}(t) &= x_n(t) - (a_1 + \hat{\delta}^* b_1)x_1(t) + b_1\epsilon, \qquad (4.116)\\
\dot{x}_n(t) &= -(a_0 + \hat{\delta}^* b_0)x_1(t) + b_0\epsilon, \qquad (4.117)
\end{aligned}$$

where by minimum phase property of system, $b_i\epsilon \neq 0,\ i \in [0, m]$. As $x_1(t) \to 0$, equation (4.117) implies that $x_n(t) \to \infty$, since $x(\cdot)$ is uniformly continuous. It follows by (4.116) that $x_{n-1}(t) \to \infty$, and cascading the argument yields to $x_1(t) \to \infty$ as $t \to \infty$, hence contradiction. Therefore $\hat{\delta}(t) \to \infty$. From this and Proposition 4.1, the claim of the proposition follows.

□

Proposition 4.3. Consider the closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi)$ defined by (4.2), (4.24) where C1,C2 hold. Let $\Lambda \subset \Delta(\delta)$ be compact. Consider the transient performance cost functional (4.98). Then

$$\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Lambda, \mathcal{D}(\epsilon)), \Xi) = \infty. \qquad (4.118)$$

The proof is slightly more complicated than that of Proposition 3.2. Fig. 4.1 shows the procedure.

Proof. Let $x_0 \in \mathcal{X}_0(\gamma)$, $\theta \in \Lambda$, and choose $d(t) = \epsilon \neq 0$. Denote $\limsup_{t\to\infty}$ by $\overline{\lim}$. Suppose for contradiction $\mathcal{P}(\Sigma(x_0, \theta, d(\cdot)), \Xi) < \infty$. Consider $\dot{x}(t)$. There are two cases either 1. $\overline{\lim}\,\|\dot{x}(t)\| = \infty$ or 2. $\overline{\lim}\,\|\dot{x}(t)\| < \infty$:

1. Suppose $\overline{\lim}\,\|\dot{x}(t)\| = \infty$, i.e. $\overline{\lim}\,\|Ax(t) + Bu(t) + B\epsilon\| = \infty$. Therefore either

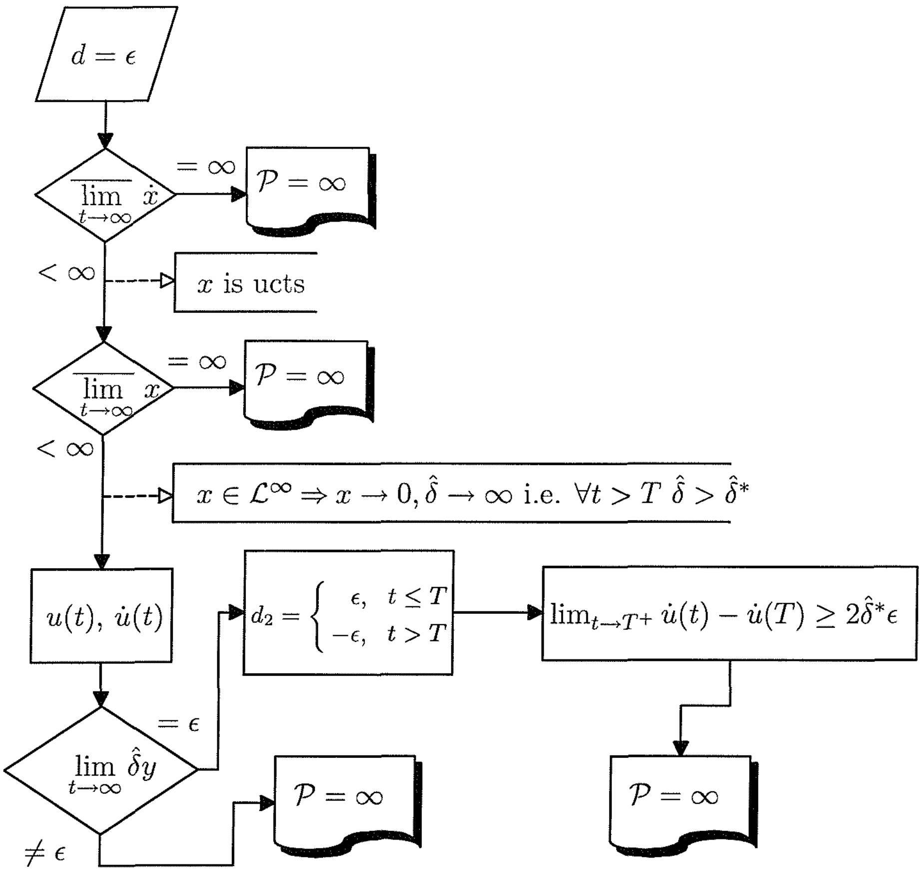

FIGURE 4.1: Proof of Proposition 4.3.

(a) $\overline{\lim}\,\|x(t)\| = \infty$, which implies that $\|x(\cdot)\|_{\mathcal{L}^\infty} = \infty$, hence contradiction, or

(b) $\overline{\lim}\,\|x(t)\| < \infty$, therefore $\overline{\lim}\, u(t) = \infty$ i.e. $\|u(\cdot)\|_{\mathcal{L}^\infty} = \infty$. Hence contradiction.

2. Suppose $\overline{\lim}\,\|\dot{x}(t)\| < \infty$ i.e. $x(\cdot)$ is uniformly continuous. Therefore by Proposition 4.2

$$\|x(t)\| \to 0, \quad \hat{\delta}(t) \to \infty \quad \text{as } t \to \infty. \tag{4.119}$$

Considering $\overline{\lim}\,\dot{u}(t)$, we observe that

$$\overline{\lim}\ \dot{u}(t) = \overline{\lim}\ \left[-y(t)^3 - \hat{\delta}(t)\left(CAx(t) - CB\left(\hat{\delta}(t)y(t) - \epsilon\right)\right)\right]. \tag{4.120}$$

Note that $CB \neq 0$ since the relative degree $\rho = 1$. Now there are two possible cases, either a) $\hat{\delta}(t)y(t) \not\to \epsilon$ (including the possibility that $\lim_{t\to\infty} \hat{\delta}(t)y(t)$ does not exist), or b) $\lim_{t\to\infty} \hat{\delta}(t)y(t) = \epsilon$

(a) Suppose $\lim_{t\to\infty} \hat{\delta}(t)y(t)$ does not exist or $\hat{\delta}(t)y(t) \not\to \epsilon$ as $t \to \infty$. It follows by (4.119) that $\|\dot{u}(\cdot)\|_{\mathcal{L}^\infty} = \infty$; hence contradiction.

(b) Suppose $\lim_{t\to\infty} \hat{\delta}(t)y(t) = \epsilon$. By (4.119) we have that

$$\forall \hat{\delta}^* > 0 \ \ \exists T > 0 \ \ s.t. \ \ \hat{\delta}(t) > \hat{\delta}^* \quad \forall t > T. \tag{4.121}$$

Now we choose $d_2(\cdot)$ as follows

$$d_2(t) = \begin{cases} \epsilon & t \leq T \\ -\epsilon & t > T \end{cases} \tag{4.122}$$

Note that $d_2(t) = d(t)$ for all $t \leq T$. With this choice, by continuity and causality, we have that

$$\lim_{t\to T^+} x(t) = x(T), \quad \lim_{t\to T^+} \hat{\delta}(t) = \hat{\delta}(T) \tag{4.123}$$

where $\lim_{t\to T^+}$ denote $\lim_{t\to T, t>T}$. It follows that

$$\left(\lim_{t\to T^+} \dot{u}(t)\right) - \dot{u}(T) = 2\hat{\delta}(T)CB\epsilon \geq 2\hat{\delta}^* b_m \epsilon. \tag{4.124}$$

By choosing a suitable $\hat{\delta}^*$, it follows that $\hat{\delta}(T)$ can be made arbitrarily large and hence the difference (4.124) is arbitrarily large. Then either $\dot{u}(T)$ is large or $\lim_{t\to T^+} \dot{u}(t)$ is large, therefore $\|\dot{u}(\cdot)\|_{\mathcal{L}^\infty}$ can be made arbitrarily large. Hence contradiction.

Therefore at least one component of (4.98) diverges, hence

$$\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Lambda, \mathcal{D}(\epsilon)), \Xi) \geq \mathcal{P}(\Sigma(x_0, \theta, d(\cdot)), \Xi) = \infty. \tag{4.125}$$

□

Proposition 4.4. Consider the closed $(\Sigma(x_0, \theta, d(\cdot)), \Xi_P(\delta_{\max}))$ defined by (4.2), (4.73) where C1,C2 hold. Let $\Lambda \subset \Delta(\delta)$ be compact. Consider the transient performance cost functional (4.98). Then

$$\mathcal{P}(\Sigma\left(\mathcal{X}_0(\gamma), \Lambda, \mathcal{D}(\epsilon)\right), \Xi_P(\delta_{\max})) \to \infty \quad \text{as} \quad \delta_{\max} \to \infty. \tag{4.126}$$

Proof. The proof is simply followed by replacing $\theta_{\max}$ by $\delta_{\max}$ in the proof of Proposition 3.3. □

Proposition 4.5. Consider the closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi_{\hat{D}}(d_{\max}))$ defined by (4.2), (4.33) where C1,C2 hold. Let $\Lambda \subset \Delta(\delta)$ be compact. Consider the transient performance cost functional (4.98). Then

$$\mathcal{P}(\Sigma\left(\mathcal{X}_0(\gamma), \Lambda, \mathcal{D}(\epsilon)\right), \Xi_{\hat{D}}(d_{\max})) < \infty, \qquad \forall d_{\max} > \epsilon. \tag{4.127}$$

Proof. Let $x_0 \in \mathcal{X}_0(\gamma)$, $\theta \in \Lambda$ and $d \in \mathcal{D}(\epsilon)$. A direct application of Property P2 of Theorem 4.2 guarantees the uniformly boundedness of $x(\cdot), \hat{\delta}(\cdot), u(\cdot)$ as a continuous function of $V^*(x_0, \theta, d_{\max})$. It follows that

$$\dot{u}(t) = -D'_{\Omega_0}|y(t)|y(t)^2 - \hat{\delta}(t)C\left(\left(A - \hat{\delta}(t)BC\right)x(t) + Bd(t)\right), \tag{4.128}$$

is uniformly bounded in terms of a continuous function of $V^*(x_0, \theta, d_{\max})$. Therefore

$$\mathcal{P}(\Sigma(x_0, \theta, d(\cdot)), \Xi_{\hat{D}}(d_{\max})) \leq M(V^*(x_0, \theta, d_{\max})), \tag{4.129}$$

for some continuous $M(V^*(x_0, \theta, d_{\max}) < \infty$. It follows by the same argument of Proposition 3.4 that for all $d_{\max} > \epsilon$,

$$\mathcal{P}(\Sigma\left(\mathcal{X}_0(\gamma), \Lambda, \mathcal{D}(\epsilon)\right), \Xi_{\hat{D}}(d_{\max})) \leq \sup_{x_0 \in \mathcal{X}_0(\gamma)} \sup_{\theta \in \Lambda} \sup_{d \in \mathcal{D}(\epsilon)} M(V^*(x_0, \theta, d_{\max})) < \infty. \tag{4.130}$$

□

The above propositions suffice to prove Theorem **I**:

*Proof of Theorem **I**.*

This is a simple consequence of Proposition 4.4 and Proposition 4.5. □

4.4.2 Theorem II

Theorem II. *Consider the system* $\Sigma(x_0, \theta, d(\cdot))$ *and the controllers* $\Xi_{\hat{D}}(d_{\max})$ *and* $\Xi_P(\delta_{\max})$ *defined by* (4.2), (4.33) *and* (4.73) *respectively where C1,C2 hold. Let* $\Lambda \subset \Delta(\delta)$ *be compact. Consider the transient performance cost functional* (4.98). *Then* $\exists\delta > 0$ *such that* $\forall\delta_{\max} \geq \delta$ $\exists d^*_{\max} \geq \epsilon$ *such that* $\forall d_{\max} \geq d^*_{\max}$,

$$\mathcal{P}(\Sigma\left(\mathcal{X}_0(\gamma), \Lambda, \mathcal{D}(\epsilon)\right), \Xi_{\hat{D}}(d_{\max})) > \mathcal{P}(\Sigma\left(\mathcal{X}_0(\gamma), \Lambda, \mathcal{D}(\epsilon)\right), \Xi_P(\delta_{\max})). \quad (4.131)$$

Proposition 4.6. Consider the closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi_{\hat{D}}(d_{\max}))$ defined by (4.2), (4.33) where C1,C2 hold. Let $\Lambda \subset \Delta(\delta)$ be compact. Consider the transient performance cost functional (4.98). Then $\exists\delta > 0$ such that

$$\mathcal{P}(\Sigma\left(\mathcal{X}_0(\gamma), \Lambda, \mathcal{D}(\epsilon)\right), \Xi_{\hat{D}}(d_{\max})) \to \infty, \quad \text{as } d_{\max} \to \infty. \quad (4.132)$$

Proof. Note that by choice (4.33), $\|\Omega_0\| \to \infty$ as $d_{\max} \to \infty$ (c.f. to the discussion in section 3.5.5). Similarly to the proof of Proposition 3.7, we start with $y_0 \in \Omega_0$ (i.e. $\gamma < \eta_0$) and define

$$\tau = \begin{cases} \infty & \text{if } y(t) \in \Omega_0 \quad \forall t \geq 0 \\ \inf\{t \geq 0 \mid y(t) \in \partial\Omega_0\}, & \text{otherwise.} \end{cases} \quad (4.133)$$

Note that by dead-zone definition (4.33), $\dot{\hat{\delta}}(t) = 0$ for all $t \in [0, \tau)$ i.e. $\hat{\delta}(t) = 0$ for all $t \in [0, \tau)$ since $\hat{\delta}(0) = 0$. Therefore

$$\dot{x}(t) = Ax(t) + Bd(t) \qquad \forall t \in [0, \tau). \quad (4.134)$$

Now by (4.96) there exists $\theta \in \Lambda$ for which A is not stable. Hence if $d(t) = \epsilon$, then $\|\dot{x}(t)\| > \epsilon$ for all $t \in [0, \tau)$ i.e. the output $y(t) := x_1(t)$ hits the boundary $\partial\Omega_0$ in finite time, hence $\tau < \infty$. It follows that

$$\|x(\cdot)\|_{\mathcal{L}^\infty} \geq |y(\tau)| = |\partial\Omega_0|. \quad (4.135)$$

If $y_0 = x_1(0) \notin \Omega_0$ then we are naturally outside the dead-zone i.e. again $\|x(\cdot)\|_{\mathcal{L}^\infty} \geq |y_0| \geq |\partial\Omega_0|$. The proof is completed by taking $d_{\max} \to \infty$ i.e. $|\Omega_0| \to \infty$. Hence

$\mathcal{P}(\Sigma(x_0, \theta, d(\cdot)), \Xi_D(d_{\max})) = \infty$, therefore $\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Lambda, \mathcal{D}(\epsilon)), \Xi_D(d_{\max})) = \infty$. □

Proposition 4.7. Suppose C1,C2 hold. Let $\Lambda \subset \Delta(\delta)$ be compact. Then the closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi_P(\delta_{\max}))$ defined by (4.2), (4.73) has the property

$$\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Lambda, \mathcal{D}(\epsilon)), \Xi_P(\delta_{\max})) < \infty, \qquad \forall \delta_{\max} \geq \delta. \tag{4.136}$$

Proof. Let $x_0 \in \mathcal{X}_0(\gamma)$, $\theta \in \Lambda$, $d \in \mathcal{D}(\epsilon)$. A direct application of Theorem 4.3 guarantees the uniform boundedness of signals $x(\cdot)$, $\hat{\delta}(\cdot)$, $u(\cdot)$ of the closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi_P(\delta_{\max}))$ as a continuous function of $x_0, \theta, \|d\|, \delta_{\max}$. It follows that

$$\dot{u}(t) = -y(t)^3 - \hat{\delta}(t) C\left(Ax(t) + B(u(t) + d(t))\right) \tag{4.137}$$

is uniformly bounded in terms of a continuous function of $x_0, \theta, \|d\|, \delta_{\max}$, hence

$$\mathcal{P}(\Sigma(x_0, \theta, d(\cdot)), \Xi_P(\delta_{\max})) \leq M(x_0, \theta, \|d\|, \delta_{\max}) < \infty, \tag{4.138}$$

where M is continuous. The claim of proposition follows by taking the supremum over system parameters x_0, θ, d as in (4.97) on both sides of (4.138), and observing that by uniform boundedness, the right hand side remains bounded for all $\delta_{\max} \geq \delta$. □

Proof of Theorem ***II***.

This is a simple consequence of Proposition 4.6 and Proposition 4.7.

□

4.5 Summary and Discussion

We extended the results of the last chapter to minimum phase linear systems with relative degree one. We introduce the concept of high-gain adaptive control first. Then, we defined dead-zone and projection modified controllers when bounded disturbances present. In the stability proof for dead-zone modification, we used the so-called smooth dead-zone and benefited from the proof of λ–tracking theorem. Finally we extended the result of Theorem **I** and **II** for minimum phase, relative degree one linear systems by following the procedure explained in chapter 3.

Propositions 4.3–4.7 are proved by extensions of the analogous proofs in chapter 3, however Proposition 4.2 was proved by a new technique[4].

In the next chapter, we extend the results of Theorems **I**, **Ia**, **II**, and **IIa** to nonlinear systems in the form of integrator chain.

[4]An alternative proof of Proposition 4.2 due to A. Ilchmann [21] is given in Appendix A.

Chapter 5

Robustness and Performance Comparison: Nonlinear Systems

5.1 Introduction

Producing a mathematical model of a physical system often requires dealing with different types of physical laws. Nonlinearity is an intrinsic part of almost all physical systems. Therefore, the resulting mathematical model is most likely nonlinear. No systematic mathematical tools yet exist to help find necessary and sufficient conditions to guarantee the stability and performance of nonlinear systems in general form [31]. Therefore we usually consider some classes of nonlinear plants for which the stability and performance problem is solvable. The class of nonlinear system we are considering in this chapter is known as an 'integrator chain' which encompass a large class of physical systems.

An intuitive approach to achieve stability in nonlinear systems is to apply a feedback control which *exactly cancels* the nonlinear terms appearing in the input-output map. In theory, this renders the closed loop system linear, but in many practical situations the uncertainty of the system prevents the nonlinear term being cancelled 'exactly', so some nonlinearities remain. The necessity of considering these uncertainties in the control design motivates the idea of *nonlinear adaptive control.* Like their linear cousins, the resulting nonlinear adaptive controllers are susceptible to phenomena such as parameter drift when small disturbances are present. The idea of modifying the adaptive law described in chapter 3 can be extended to nonlinear systems to assure robustness in the presence of bounded disturbances.

We will extend the topics of previous chapters to integrator chain nonlinear systems. First, we introduce the nonlinear adaptive control scheme. Then, we develop the dead-zone and the projection based controllers to achieve robustness when bounded disturbances are present. The concept of hysteresis dead-zone, introduced in section 3.4.2, will be mathematically synthesised in section 5.3.2. Finally the results of theorems **I**, **Ia** and **II**, **IIa** will be extended to integrator chain in section 5.4.

5.2 System Description and Adaptive Design

By an 'integrator chain', we mean the following SISO nonlinear system:

$$\begin{aligned}\Sigma(x_0,\theta,d(\cdot)):\ \dot{x}_i(t) &= x_{i+1}(t) \qquad 1 \le i \le n-1\\ \dot{x}_n(t) &= \theta^T\phi(x(t)) + u(t) + d(t) \qquad (5.1)\\ y(t) &= x_1(t),\end{aligned}$$

where the state vector $x(\cdot) \in \mathbb{R}^n$ is available for measurement, $\theta \in \mathbb{R}^m$ is unknown constant parameter and $\phi : \mathbb{R}^n \to \mathbb{R}^m$ is a known smooth function.

Suppose the control objective is to asymptotically stabilise the system, that is, to find a control law that guarantees the boundedness of all closed loop signals and forces the system trajectory to converge to zero asymptotically. Define the feedback law

$$u(t) := -a^T x(t) - \hat{\theta}(t)^T\phi(x(t)), \qquad (5.2)$$

where $a = [a_1, \ldots, a_n]^T$ is chosen such that the matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_1 & -a_2 & -a_3 & \ldots & -a_n \end{bmatrix} \tag{5.3}$$

is Hurwitz. Let P, Q be symmetric positive definite matrices satisfying the Lyapunov equation:

$$A^T P + PA = -Q, \tag{5.4}$$

and define the weighting vector $b := (P + P^T)B$ where $B := (0, \ldots, 0, 1)^T$.

The signal $\hat{\theta} : \mathbb{R}^+ \to \mathbb{R}^m$ in (5.2) represents the adaptive estimator of θ and is updated by the online adaptive law:

$$\dot{\hat{\theta}}(t) = \alpha\, x(t)^T\, b\, \phi(x(t)), \qquad \hat{\theta}(0) = 0, \tag{5.5}$$

where $\alpha > 0$ is the adaptation gain. Thus, the controller Ξ consists the control law and the adaptive law as follows:

$$\begin{aligned} \Xi: \ u(t) &= -a^T x(t) - \hat{\theta}(t)^T \phi(x(t)) \\ \dot{\hat{\theta}}(t) &= \alpha\, x(t)^T\, b\, \phi(x(t)) \qquad \hat{\theta}(0) = 0. \end{aligned} \tag{5.6}$$

We denote the respective closed loop systems by $(\Sigma(x_0, \theta, d(\cdot)), \Xi)$ defined by (5.1), (5.6). We frequently use the following compact combination of (5.1),(5.2):

$$\dot{x}(t) = Ax(t) + B(\,(\theta - \hat{\theta}(t))^T \phi(x(t)) + d(t)\,), \qquad x(0) = x_0. \tag{5.7}$$

The following well known theorem [35] establishes the stability of the closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi)$ where $d(\cdot) \in \mathcal{D} = \{0\}$.

Theorem 5.1. Consider the closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi)$ defined by (5.1) and (5.6), where $d(\cdot) = 0$. Then (i) the closed loop is well-posed, (ii) all closed loop signals $x(\cdot), u(\cdot), \hat{\theta}(\cdot)$

are bounded and *(iii)* $x(t) \to 0$ as $t \to \infty$.

Proof. Since the right hand side of the closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi)$ is continuous and locally Lipschitz, the existence and uniqueness of the solution $(x(\cdot), \hat{\theta}(\cdot))$ follows by theorem 2.1 once the boundedness of solution has been shown. Let $\tilde{\theta}(t) = \theta - \hat{\theta}(t)$ and define the Lyapunov function

$$V(x(t), \tilde{\theta}(t)) = x(t)^T P x(t) + \frac{1}{2\alpha}\, \tilde{\theta}(t)^T \tilde{\theta}(t). \tag{5.8}$$

The time derivative of $V(x(t), \tilde{\theta}(t))$ is

$$\begin{aligned} \dot{V}(x(t), \tilde{\theta}(t)) &= x(t)^T P \dot{x}(t) + \dot{x}(t)^T P x(t) - \frac{1}{2\alpha} \tilde{\theta}(t)^T \dot{\hat{\theta}}(t) - \frac{1}{2\alpha} \dot{\hat{\theta}}(t)^T \tilde{\theta}(t), \\ &= x(t)^T P \left(A x(t) + B \tilde{\theta}(t)^T \phi(x(t)) \right) \\ &\quad + \left(A x(t) + B \tilde{\theta}(t)^T \phi(x(t)) \right)^T P x(t) - \frac{1}{\alpha} \tilde{\theta}(t)^T \dot{\hat{\theta}}(t), & (5.9) \\ &= -x(t)^T Q x(t) + x(t)^T b\, \tilde{\theta}(t)^T \phi(x(t)) \\ &\quad - \frac{1}{\alpha} \tilde{\theta}(t)^T \left\{ \alpha x(t)^T b\, \phi(x(t)) \right\}, & (5.10) \\ &= -x(t)^T Q x(t) & (5.11) \\ &\leq -\underline{\lambda}(Q) \|x(t)\|^2. & (5.12) \end{aligned}$$

Inequality (5.12) implies that $V(t) = V(x(t), \hat{\theta}(t)) \in \mathcal{L}^\infty$, therefore $x(\cdot), \hat{\theta}(\cdot) \in \mathcal{L}^\infty$. The boundedness of $u(\cdot)$ then follows from (5.6). Finally, the asymptotic stability of the state $x(t)$ is followed by LaSalle theorem 2.4 and the fact that by (5.12), $V(t)$ is negative semidefinite. □

Similar to previous chapters, we use the dead-zone and projection modification of the adaptive law 5.5 to achieve robustness when $d(\cdot) \neq 0$. We are interested in the following features:

(i) The existence and uniqueness of the closed loop solution $(\, x(\cdot), \hat{\theta}(\cdot)\,)$,

(ii) The boundedness of closed loop signals $x(\cdot), \hat{\theta}(\cdot), u(\cdot)$,

(iii) The system *practical* asymptotic stability i.e. convergence of $x(t)$ to a small neighbourhood of origin.

5.3 Algorithm Modification

5.3.1 Dead-zone Modification

Based on the description of the dead-zone modification described in chapter 3.4.1 and Theorem 5.1, we define the dead-zone region $\Omega_0(d_{\max})$:

$$\Omega_0(d_{\max}) = \{\, x \mid x^T P x \leq \eta_0^2 \,\}, \tag{5.13}$$

where

$$\eta_0 := \varrho(d_{\max}) = \frac{\sqrt{\overline{\lambda}(P)}}{\underline{\lambda}(Q)}\, |b|\, d_{\max}. \tag{5.14}$$

Therefore, the dead-zone controller is defined as follows:

$$\begin{aligned} \Xi_D(d_{\max}): \quad & u(t) = -a^T x(t) - \hat{\theta}(t)^T \phi(x(t)) \\ & \dot{\hat{\theta}}(t) = D_{\Omega_0(d_{\max})}(x)\, \alpha\, x(t)^T\, b\, \phi(x(t)), \qquad \hat{\theta}(0) = 0, \end{aligned} \tag{5.15}$$

The respective closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi_D(d_{\max}))$ is given by equations (5.1), (5.15). The following theorem examines the properties (i)–(iii) for the closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi_D(d_{\max}))$:

Theorem 5.2. Consider the closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi_D(d_{\max}))$ defined by (5.1), (5.15), where $d(\cdot) \in \mathcal{L}^\infty$ and η_0 defined by (5.14). Define

$$V_0(x_0, \theta, d_{\max}) = \max(x_0^T P x_0, \eta_0^2) + \frac{1}{2\alpha}\theta^T \theta. \tag{5.16}$$

Assume that $d_{\max}$ is such that $\|d(\cdot)\|_{\mathcal{L}^\infty} \leq d_{\max}$. Then for any $x_0 \in \mathbb{R}^n$, the following properties hold:

D1. The solution $(\, x(\cdot), \hat{\theta}(\cdot)\,) : \mathbb{R}^+ \to \mathbb{R}^{n+m}$ exists.

D2. All closed loop signals $x(\cdot), u(\cdot), \hat{\theta}(\cdot)$ are uniformly bounded as a continuous function of $V_0(x_0, \theta, d_{\max})$.

D3. $x(t) \to \Omega_0$ as $t \to \infty$.

Proof. Due to the r.h.s. discontinuity of dead-zone controller (5.15), the solution of the closed loop is considered in Filippov's sense. The complete proof has been established in [11]. A sketch of the proof is as follows: Let

$$T_1 = \{t \geq 0 \mid x(t) \in \mathbb{R}^n \setminus \Omega_0\} \tag{5.17}$$

and define the Lyapunov function

$$V(x(t), \tilde{\theta}(t)) = x(t)^T P x(t) + \frac{1}{2\alpha}\tilde{\theta}(t)^T\tilde{\theta}(t). \tag{5.18}$$

It has been shown that if

(i) $\dot{V}(x(t), \tilde{\theta}(t)) < 0$ for all $t \in T_1$,

(ii) the left hand derivative $D_{-}x(t)^T P\, x(t) = 0$ and $D_{-}V(x(t), \tilde{\theta}(t)) \leq 0$ for all $t \geq 0$ such that $x(t) \in \partial\Omega_0$, then the solution $(\, x(\cdot), \hat{\theta}(\cdot)\,)$ exists for all $t \geq 0$ and $V(x(t), \tilde{\theta}(t)) \leq V_0(x_0, \theta, d_{\max})$ for all $t \geq 0$.

In order to verify condition (i), observe that for all $t \in T_1$,

$$\overline{\lambda}(P)\|x(t)\|^2 \geq x(t)^T P x(t) > \eta_0^2. \tag{5.19}$$

Substituting condition (5.14) in (5.19) and noting that $\|d\| \leq d_{\max}$, we have

$$\overline{\lambda}(P)\|x(t)\|^2 > \frac{\overline{\lambda}(P)|b|^2\,\|d(t)\|^2}{\underline{\lambda}^2(Q)}, \quad \forall t \in T_1, \tag{5.20}$$

therefore

$$\underline{\lambda}(Q) > \frac{|b|}{\|x(t)\|}\,\|d(t)\|, \qquad \forall t \in T_1. \tag{5.21}$$

A similar calculation as (5.9)-(5.10) shows that for all $t \in T_1$:

$$\dot{V}(x(t), \tilde{\theta}(t)) = -x(t)^T Q x(t) + x(t)^T\, b\, d(t) \leq -\left(\underline{\lambda}(Q) - \frac{|b|}{\|x(t)\|}\|d(t)\|\right)\|x(t)\|^2. \tag{5.22}$$

Hence by (5.21), $\dot{V}(x(t), \tilde{\theta}(t)) < 0$ on T_1.

Condition (ii) is established as follows: On the boundary $\partial\Omega_0$, $\;D_{-}x(t)^T P\, x(t) = 0$

since $x(t)^T P x(t) = \eta_0^2$. Therefore by (5.10)

$$0 = -x(t)^T Q x(t) + x(t)^T b \left[\tilde{\theta}(t)^T \phi(x(t)) + d(t)\right], \tag{5.23}$$

or

$$x(t)^T b \tilde{\theta}(t)^T \phi(x(t)) = x(t)^T Q x(t) - x(t)^T b d(t). \tag{5.24}$$

In order to calculate $D\_V(x(t), \tilde{\theta}(t))$, we note that by the definition of the solution at the point of discontinuity [8], for some $\lambda = \lambda(t) \in [0, 1]$, we have $\dot{\hat{\theta}}(t) = \lambda \alpha x(t)^T b \phi(x(t))$. Hence by (5.24) and (5.22)

$$D\_V(x(t), \tilde{\theta}(t)) = -\lambda x(t)^T b \tilde{\theta}(t) \phi(x(t)) = \lambda \left(-x(t)^T Q x(t) + x(t)^T b d(t)\right) \leq 0, \tag{5.25}$$

thus satisfying condition (*ii*). It follows that the solution exists for all $t \geq 0$; hence D1.

A uniform bound over signals $x(\cdot)$, $\hat{\theta}(\cdot)$ can be obtained as a continuous function of $V_0 := V_0(x_0, \theta, d_{\max})$ since $V(x(t), \tilde{\theta}(t)) \leq V_0$ for all $t \geq 0$.

$$\|x(t)\| \leq \sqrt{V_0/\underline{\lambda}(P)}\,, \qquad \|\hat{\theta}(t)\| \leq 2\sqrt{2\alpha V_0}. \tag{5.26}$$

Finally the uniform boundedness of $u(\cdot)$ in terms of a continuous function of $V_0(x_0, \theta, d_{\max})$ follows from (5.6) and the continuity of $\phi(\cdot)$.

The proof of D3 is similar to H3 of the next section. To avoid repetition and to represent a complete proof for hysteresis dead-zone, we omit the proof of D3 here and refer the reader to the proof of H3 in next section.

□

5.3.2 Hysteresis Dead-zone

In this section, we establish a mathematical description of the hysteresis dead-zone mentioned in section 3.4.2. Given the stability analysis, we address the advantages of this method compared to conventional dead-zone.

5.3.2.1 Definition

The idea behind hysteresis dead-zone is to define a switching function such that the 'switching on' and 'switching off' operations of the adaptive law do not both occur simultaneously on the switching surface. Let us denote $\mathcal{Z}$, the history of the system trajectory on a specific time:

$$\mathcal{Z} := \{(x_{[0,t_1)}, t_2) \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^n) \times \mathbb{R}^+ \mid t_1 = t_2\}. \quad (5.27)$$

Definition 5.1. The hysteresis dead-zone function $H : \mathcal{Z} \to \{0, 1\}$ is defined by:

$$H_{\Omega_0(d_{\max}),\Omega_1}\left(x_{[0,t)}, t\right) := \begin{cases} 1, & \text{if } x(t) \in \mathbb{R}^n \setminus \Omega_1 \\ 0, & \text{if } x(t) \in \Omega_0(d_{\max}) \\ S(t), & \text{if } x(t) \in \Omega_1 \setminus \Omega_0(d_{\max}) \end{cases} \quad (5.28)$$

where

$$\Omega_0(d_{\max}) = \{x \mid x^T P x \leq \eta_0^2\} \subsetneq \Omega_1, \quad (5.29)$$

$$\Omega_1 = \{x \mid x^T P x \leq \eta_1^2\} \subset \mathbb{R}^n, \quad (5.30)$$

and

$$S(t) = \begin{cases} 1 & \text{if } t = 0 \text{ and } x_0 \in \Omega_1 \setminus \Omega_0(d_{\max}), \\ \lim\limits_{\tau \to t^-} H_{\Omega_0(d_{\max}),\Omega_1}\left(x_{[0,\tau)}, \tau\right) & \text{elsewhere,} \end{cases} \quad (5.31)$$

where by $\lim_{\tau \to t^-}$ we mean $\lim_{\tau \to t, \tau < t}$. Note that $\eta_0 = \varrho(d_{\max})$ is chosen based on the a-priori knowledge $d_{\max}$, and η_1 is defined such that $\eta_1 > \eta_0$. We let $\eta_1 = (1 + \beta)\eta_0$ for some small $\beta > 0$ since we will see that the system trajectory asymptotically converges to Ω_1.

For the compactness of notation we denote $H_{\Omega_0(d_{\max}),\Omega_1}\left(x_{[0,t)}, t\right)$ by $H(t)$. We also define the time instances where the system trajectory is on the boundary of Ω_1 and also time periods where we switch off the adaptation:

$$T_{\partial\Omega_1} = \{ \, t \geq 0 \mid x(t) \in \partial\Omega_1 \, \} \tag{5.32}$$

$$T_{\Omega_0} = \{ \, t \geq 0 \mid H(t^-) = 1 \, , \; H(t) = 0 \, \} \tag{5.33}$$

Finally, using the above sets, we define a 'switching' sequence $\{t_i\}_{i \geq 0}$, a 'storing sequence' $\{t_i^s\}_{i \geq 0}$ and a 'restoring' sequence $\{t_i^r\}_{i \geq 0}$ (see Fig. 5.1) as follows:

$$\begin{aligned} t_i &= \inf_{t \geq t_{i-1} \, , \, t \in T_{\Omega_0}} t \qquad i \geq 1, \qquad t_0 = 0, \\ t_i^s &= \sup_{t < t_i \, , \, t \in T_{\partial\Omega_1}} t \qquad i \geq 1, \qquad t_0^s = 0, \\ t_i^r &= \inf_{t > t_i \, , \, t \in T_{\partial\Omega_1}} t \qquad i \geq 1, \qquad t_0^r = 0. \end{aligned} \tag{5.34}$$

Now we define the modified adaptive controller $\Xi_H(d_{\max})$ on the interval $t_i^r \leq t < t_{i+1}^r$

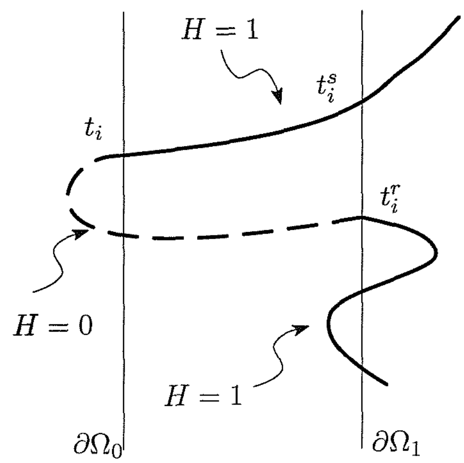

FIGURE 5.1: Hysteresis dead-zone

for $i \geq 0, \; \beta > 0$:

$$\begin{aligned} \Xi_H(d_{\max}) : u(t) &= -a^T x(t) - \hat{\theta}(t)^T \phi(x(t)) \\ \dot{\hat{\theta}}(t) &= \alpha \, x(t)^T b H(t) \phi(x(t)), \quad \hat{\theta}(0) = 0, \; \eta_0 = \varrho(d_{\max}), \; \eta_1 = (1+\beta)\eta_0, \\ \hat{\theta}(t_i^r) &= \hat{\theta}(t_i^s) \qquad\qquad t_i^r \leq t < t_{i+1}^r, \;\; i \geq 0 \end{aligned} \tag{5.35}$$

Consequently, we denote the respective closed loop system by $(\Sigma(x_0, \theta, d(\cdot)), \Xi_H(d_{\max}))$.

5.3.2.2 Stability Analysis

We show the properties of the closed loop system by the following theorem:

Theorem 5.3. Consider the closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi_H(d_{\max}))$ defined by (5.1), (5.35) where $d(\cdot) \in \mathcal{L}^\infty$. Assume that $d_{\max}$ is such that $\|d(\cdot)\|_{\mathcal{L}^\infty} \leq d_{\max}$. Suppose

$$\eta_0 : \varrho(d_{\max}) = \frac{\sqrt{\overline{\lambda}(P)}}{\underline{\lambda}(Q)} |b| \, d_{\max}. \tag{5.36}$$

Define

$$V_0' := V_0'(x_0, \theta, d_{\max}) = x_0^T P x_0 + \frac{1}{2\alpha}\theta^T\theta + (\eta_1^2 - \eta_0^2). \tag{5.37}$$

Then for any $x_0 \in \mathbb{R}^n$ the followings hold:

H1. The solution $(x(\cdot), \hat{\theta}(\cdot)) : \mathbb{R}^+ \to \mathbb{R}^{n+m}$ exists and is unique.

H2. All closed loop signals $x(\cdot), u(\cdot), \hat{\theta}(\cdot)$ are uniformly bounded as a continuous function of $V_0'(x_0, \theta, d_{\max})$.

H3. $x(t) \to \Omega_1$ as $t \to \infty$.

Proof. We prove H1 by induction. Suppose $x_0 \in \mathbb{R}^n \setminus \Omega_0(d_{\max})$. Observe that this includes $x_0 \in \mathbb{R}^n \setminus \Omega_1$ in which $H(0) = 1$ by (5.28) and the case where $x_0 \in \Omega_1 \setminus \Omega_0$ in which again $H(0) = 1$ by definition (5.31). Since the r.h.s. of the closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi_D(d_{\max}))$ is locally Lipschitz, by corollary 2.2, a unique solution $(x(\cdot), \hat{\theta}(\cdot))$ exists on its maximal interval of existence $[0, \omega)$ for some $\omega > 0$. Define a Lyapunov function

$$V(x(t), \tilde{\theta}(t)) = x(t)^T P x(t) + \frac{1}{2\alpha}\tilde{\theta}(t)^T\tilde{\theta}(t). \tag{5.38}$$

and by an abuse of notation we denote $V(x(t), \hat{\theta}(t))$ by $V(t)$. We take:

$$J_i = \begin{cases} \text{a)} \;\; V(t) \leq V_0', & t_i < t < t_i^r \\ \text{b)} \;\; V(t) \leq V(0) & t_i^s \leq t \leq t_i \;\text{ or }\; t_i^r \leq t \leq t_{i+1}^s \end{cases} \tag{5.39}$$

to be the inductive hypothesis. Consider the induction step $J_i \Rightarrow J_{i+1}$. Let us split the case into several time intervals:

1. Suppose $t^s_{i+1} \leq t \leq t_{i+1}$. Calculating $\dot{V}(t)$ in the same manner as (5.9)-(5.10) yields

$$\begin{aligned}\dot{V}(t) &= -x(t)^T Qx(t) + x(t)^T b\,\tilde{\theta}(t)^T \phi(x(t)) + x(t)^T\, b\, d(t) \\ &\quad - \frac{1}{\alpha}\tilde{\theta}(t)^T \left\{\alpha x(t)^T\, b\, H(t)\, \phi(x(t))\right\} \\ &= -x(t)^T Qx(t) + x(t)^T\, b\, d(t).\end{aligned} \tag{5.40}$$

Note that by (5.35), the adaptation is 'On' in this interval, i.e. $H(t) = 1$. It follows that

$$\dot{V}(t) \leq -\left(\underline{\lambda}(Q) - \frac{|b|}{\|x(t)\|}\|d(t)\|\right)\|x(t)\|^2. \tag{5.41}$$

Since $x(t) \in \mathbb{R}^n \backslash \Omega_0$, by (5.29) and the same inequality as (5.19)–(5.21), we observe that by (5.36), $\dot{V}(t) \leq 0$ on $t^s_{i+1} \leq t \leq t_{i+1}$, i.e.

$$V(t_{i+1}) \leq V(t) \leq V(t^s_{i+1}) \leq V(0), \tag{5.42}$$

where the last inequality follows from b) of statement J_i.

2. Suppose $t_{i+1} \leq t \leq t^r_{i+1}$. The adaptation is 'Off' ($H(t) = 0$) and we have

$$V(t) \leq V(t_{i+1}) + \eta_1^2 - \eta_0^2. \tag{5.43}$$

But $V(t_i) \leq V(0)$ by (5.42), so

$$V(t) \leq V_0' \qquad t_{i+1} \leq t \leq t^r_{i+1} \tag{5.44}$$

3. Suppose $t^r_{i+1} \leq t \leq t^s_{i+2}$. Note that by (5.35)

$$x(t^r_{i+1})^T Px(t^r_{i+1}) = \eta_1^2 = x(t^s_{i+1})^T Px(t^s_{i+1}) \tag{5.45}$$

as $x(t^r_{i+1})\,,\, x(t^s_{i+1}) \in \partial\Omega_0$ and $\hat{\theta}(t^r_{i+1}) = \hat{\theta}(t^s_{i+1})$. It follows that

$$V(t^r_{i+1}) = V(t^s_{i+1}). \tag{5.46}$$

In same manner as step 1, we can show that $\dot{V}(t) \leq 0$ for $t^r_{i+1} \leq t \leq t^s_{i+2}$. Thus

$$V(t^s_{i+2}) \leq V(t) \leq V(t^r_{i+1}) = V(t^s_{i+1}) \leq V(0), \tag{5.47}$$

where again the last inequality follows from b) of statement J_i.

The result of the above steps shows that by assuming J_i is true, we have proved J_{i+1}:

$$J_{i+1} = \begin{cases} \text{a)} \quad V(t) \leq V_0', & t_{i+1} < t < t^r_{i+1} \\ \text{b)} \quad V(t) \leq V(0) & t^s_{i+1} \leq t \leq t_{i+1} \text{ or } t^r_{i+1} \leq t \leq t^s_{i+2} \end{cases} \tag{5.48}$$

is true. The case J_0 is also true, since by (5.34), $t^s_0 = t^r_0 = t_0 = 0$ and for $0 \leq t \leq t^s_1$ we have $H(t) = 1$ and by the same argument as in step 1, $\dot{V}(t) \leq 0$ i.e. $V(t) \leq V(0)$, therefore J_0 is true. Hence, by induction, (5.39) is true for all $i \geq 0$. Therefore by (5.38), $x(t), \hat{\theta}(t)$ are bounded for all $t \leq t_i$.

In order to prove well-posedness, i.e. $\omega = \infty$, we consider the two cases $|T_{\Omega_0}| < \infty$, and $|T_{\Omega_0}| = \infty$, i.e. the cases of whether T_{Ω_0} has a maximal element or not:

(a). Suppose t_I is the largest element in T_{Ω_0} i.e the switching is stopped for all $t > t_I$. The boundedness of $x(\cdot), \hat{\theta}(\cdot)$ on $[0, t_I]$ follows from the induction. For the interval $[t_I, \infty)$ either $x(t)$ remains in Ω_1 for all $t > t_I$ or there exists $t > t_I$ such that $x(t) \in \partial\Omega_1$:

  (*i*). If $x(t) \in \Omega_1$ for all $t > t_I$ then $\hat{\theta}(t)$ remains constant by (5.35) since $H(t) = 0$ for all $t \in [t_I, \infty)$. It follows that $x(\cdot), \hat{\theta}(\cdot)$ remain bounded.

  (*ii*). If there exists $t^r_I > t_I$ such that $x(t^r_I) \in \partial\Omega_1$, then $H(t) = 1$ for all $t \in [t^r_I, \infty)$. It follows by the same argument as step 1 that $\dot{V}(t) \leq 0$ for all $t \geq t^r_I$; hence the closed loop signals $x(\cdot), \hat{\theta}(\cdot)$ remain bounded.

(b). Suppose T_{Ω_0} does not have a maximal element. For well posedness, it suffices to show that $\lim_{i\to\infty} t_i = +\infty$. To this end, let $\tilde{t} = \lim_{i\to\infty} t_i$. Observe that $\dot{x}(\cdot)$ is bounded on $[0, \tilde{t})$ since all signals appearing on the r.h.s. of equation (5.7) are bounded on $[0, \tilde{t})$. It follows that

$$t_i \geq m\left(\{t \mid x(t) \in \Omega_1 \setminus \Omega_0\}\right) \cap [0, t_i] \geq \frac{i(\eta_1^2 - \eta_0^2)}{\sup\limits_{t \in [\,0, \tilde{t}\,)} \nu(t)} \longrightarrow \infty \quad as \quad i \longrightarrow \infty, \tag{5.49}$$

where $\nu(t) = \frac{d}{dt} x(t)^T P x(t)$. So, boundedness is global.

Looking at the system in each interval $[t_i, t_i^r]$, $[t_i^r, t_{i+1}]$, and $[t_I, \infty)$, we have two continuous differential equations, either of which has a unique solution by theorem 2.1, since there is no finite escape time. Therefore the solution is unique; hence H1.

If $x_0 \in \Omega_0$ then either $x(\cdot)$ remains in Ω_0 forever, in which the solution exists, is unique and bounded and the conclusions hold. Otherwise a similar argument as step 2 guarantees the boundedness of $V(t)$ until the system trajectory hits the boundary of $\partial\Omega_1$. Let t^* be the first time instant in which $x(t^*) \in \partial\Omega_1$ (i.e. $x(t^*) \in \mathbb{R}^n \setminus \Omega_0$) and define $x_0^* = x(t^*)$. Then the system $\Sigma(x_0, \theta, d(\cdot))$ can be considered as $\Sigma(x_0, \theta, d(\cdot))$ for all $0 \leq t < t^*$ and $\Sigma(x_0^*, \theta, d(\cdot))$ for all $t \geq t^*$ either of which holds H1. Note that due to regular motion on the boundary Ω_1, the closed loop is well behaved on transition at $t = t^*$. Hence H1 hold regardless of initial conditions.

The property H2 directly follows form the fact that $V(t) \leq V_0'(x_0, \theta, d_{\max})$ for all $t \geq 0$. It follows by (5.38) that $x(\cdot)$, $\hat{\theta}(\cdot)$ are uniformly bounded as a continuous function of $V_0'(x_0, \theta, d_{\max})$. Therefore by (5.1), (5.35), $u(\cdot)$ is uniformly bounded in terms of a continuous function of $V_0'(x_0, \theta, d_{\max})$.

In order to prove property H3, let us define the function

$$\xi : \mathbb{R}^+ \to \mathbb{R}, \qquad t \mapsto \left(x(t)^T P x(t) - \eta_1^2\right) \mathcal{X}_{T_1}(t), \tag{5.50}$$

where $T_1 = \{t \geq 0 \mid x(t) \in \mathbb{R}^n \setminus \Omega_1\}$ and the characteristic function $\mathcal{X}_\omega(t) = 1$ if $t \in \omega$, $\mathcal{X}_\omega(t) = 0$ elsewhere. Then

$$\begin{aligned} \int_0^\infty |\xi(t)|\, dt &= \int_{T_1} (x(t)^T P x(t) - \eta_1^2)\, dt \\ &\leq \int_{T_1} x(t)^T P x(t)\, dt \leq \overline{\lambda}(P) \int_{T_1} \|x(t)\|^2 dt \end{aligned} \tag{5.51}$$

$$\leq \frac{\overline{\lambda}(P)}{\underline{\lambda}(Q) - w} \int_{T_1} -\dot{V}\, dt \tag{5.52}$$

where $w = \frac{|b|}{\|x(t)\|}\|d(t)\|$. Note that by (5.36), for all $t \in T_1$,

$$\max(w) \leq \frac{|b|\,\|d(\cdot)\|_{\mathcal{L}^\infty}}{(1+\beta)\eta_0} \leq \frac{\underline{\lambda}(Q)}{(1+\beta)\sqrt{\overline{\lambda}(P)}} < \underline{\lambda}(Q), \tag{5.53}$$

since $\beta > 0$. It follows that the coefficient of the integral in (5.52) is bounded. In order to calculate the integral term of (5.52), we use (5.46) to construct a telescopic sum:

$$\begin{aligned}\int_{T_1} -\dot{V}(t)dt &= \sum_{i=1}^{I} \int_{t_{i-1}^r}^{t_i^s} -\dot{V}(t)dt = \sum_{i=1}^{I} V(t_{i-1}^r) - V(t_i^s) = \sum_{i=1}^{I} V(t_{i-1}^s) - V(t_i^s) \\ &= V(t_0^s) - V(t_I^s) \\ &\leq V(t_0^s) = V(0)\end{aligned} \tag{5.54}$$

Applying (5.54) to (5.52) yields

$$\int_0^\infty |\xi(t)|\, dt < \infty. \tag{5.55}$$

We also observe that $\dot{\xi}(\cdot) \in \mathcal{L}^\infty$, since $x(\cdot), \dot{x}(\cdot) \in \mathcal{L}^\infty$. Therefore, using the Barbalat's lemma (2.3), $\xi(t) \to 0$ as $t \to \infty$; hence property H3. □

5.3.2.3 Hysteresis Dead-zone vs Dead-zone

A comparison between the conventional dead-zone and hysteresis dead-zone can be summarised as follows:

(*i.*) Like the normal dead-zone, the hysteresis dead-zone also defines two exclusive regions (inside and outside Ω_1) in the state space, but, with hysteresis, switching from one mode to another does not occur on the switching surface, but rather after the system trajectory has traveled a further distance of $\|\Omega_1\|_p - \|\Omega_0\|_p$. This means there is no point that can be forced by two vector fields f^+ and f^- simultaneously in time. In fact the right hand side discontinuity of $D(x)$ in dead-zone is replaced by a piecewise continuity $H(t,x)$ in t, which behaves well as we explained in section

2.3.1. Hence, the sliding motion (or chattering with some high uncontrollable frequency) does not happen. Inequality (5.48-b) shows that the speed of traveling between two switching modes ('on' and 'off') is not 'infinitely fast', in other words, $t_i - t_i^s \geq t'$ for some $t' > 0$, $\forall i \geq 0$, hence T_{Ω_0} is countable. So, the trajectory 'oscillate' back and forth across the switching surface with a finite frequency which is controllable by parameters such as η_0, η_1, etc.

(*ii.*) In contrast with the normal dead-zone, the hysteresis dead-zone is well-posed. This is a consequence of eliminating the counteracts simultaneity vector fields f^+, f^- on switching surface, which guarantees a 'regular switching motion'; hence the uniqueness of the solution.

(*iii.*) Since the *fast* dynamics due to chattering does not exist in control systems defined by hysteresis dead-zone, the dynamic equations are not stiff. Therefore using the normal integration method, one can simulate such a system in a much shorter time compared to normal dead-zone.

(*iv.*) The stability analysis is considerably simplified. The reason is that we do not need to deal with different possible situations at the point of discontinuity. The sufficient condition for the uniqueness of the C-solutions always holds, and the solution is well-defined.

5.3.3 Projection Modification

Following the definition described in section 3.4.3, we modify the adaptive law (5.1) as follows: Similar to (3.24), we define

$$\Pi(\theta_{\max}) = \{\hat{\theta}(t) \in \mathbb{R}^m \mid P_{\theta_{\max}}(\hat{\theta}(t)) \leq 0\}. \tag{5.56}$$

Recall the modified adaptive law

$$\dot{\hat{\theta}}(t) = \text{Proj}_{\,\Pi(\theta_{\max})}(g, \hat{\theta}). \tag{5.57}$$

where the argument of the projection operator is the unmodified adaptive law, and

$$\text{Proj}_{\,\Pi(\theta_{\max})}(g,\hat{\theta}) := \begin{cases} g, & \hat{\theta} \in \Pi^{o}(\theta_{\max}) \ \text{ or } \ \nabla_{\hat{\theta}} P^{T} g \leq 0 \\ \left(I - \dfrac{\nabla_{\hat{\theta}} P \nabla_{\hat{\theta}} P^{T}}{\nabla_{\hat{\theta}} P^{T} \nabla_{\hat{\theta}} P}\right) g & \hat{\theta} \in \partial\Pi(\theta_{\max}) \ \text{ and } \ \nabla_{\hat{\theta}} P^{T} g > 0. \end{cases} \tag{5.58}$$

Accommodating (5.57) by the adaptive law in (5.6), we have

$$\begin{aligned} \Xi_P(\theta_{\max}):\ & u(t) = -a^{T}x(t) - \hat{\theta}(t)^{T}\phi(x(t)) \\ & \dot{\hat{\theta}}(t) = \text{Proj}_{\,\Pi(\theta_{\max})}\left(\alpha\, x(t)^{T}\, b\, \phi(x(t))\right), \qquad \hat{\theta}(0) = 0. \end{aligned} \tag{5.59}$$

Consequently, we denote the respective closed loop system by $(\Sigma(x_0,\theta,d(\cdot)), \Xi_P(\theta_{\max}))$.

Lemma 5.1. The following consequences of the projection operator are important. For the compactness of notations, we denote $\text{Proj}\,(\cdot,\cdot) := \text{Proj}_{\,\Pi(\theta_{\max})}(\cdot,\cdot)$.

L1: Let Γ be a symmetric matrix, then $\text{Proj}\,(\Gamma\,(g,\hat{\theta})) = \Gamma\ \text{Proj}\,(g,\hat{\theta})$.

L2: $\text{Proj}\,(g,\hat{\theta})^{T}\text{Proj}\,(g,\hat{\theta}) \leq g^{T}g, \qquad \forall\hat{\theta} \in \Pi(\theta_{\max})$.

L3: Assume $g, \hat{\theta} \in \mathcal{C}^{1}$ Then, on its domain of definition, $\hat{\theta}(t)$ remains in $\Pi(\theta_{\max})$.

L4: $-(\theta-\hat{\theta})^{T}\,\text{Proj}\,(g,\hat{\theta}) \leq -(\theta-\hat{\theta})g, \qquad \forall\hat{\theta},\theta \in \Pi(\theta_{\max})$.

Proof. 1. When $\hat{\theta}$ is inside $\Pi(\theta_{\max})$ or g pointing inward, we have $\text{Proj}\,(g,\hat{\theta}) = g$ and L1 trivially holds. Otherwise by (5.58) we have

$$\text{Proj}\,(\Gamma\,(g,\hat{\theta})) = \left(I - \frac{\nabla_{\hat{\theta}} P \nabla_{\hat{\theta}} P^{T}}{\nabla_{\hat{\theta}} P^{T} \nabla_{\hat{\theta}} P}\right) \Gamma\, g. \tag{5.60}$$

Since $\nabla_{\hat{\theta}} P \nabla_{\hat{\theta}} P^{T}$ is symmetric, we have that $\nabla_{\hat{\theta}} P \nabla_{\hat{\theta}} P^{T}\,\Gamma = \Gamma\ \nabla_{\hat{\theta}} P \nabla_{\hat{\theta}} P^{T}$. Therefore $\text{Proj}\,(\Gamma\,(g,\hat{\theta})) = \Gamma\ \text{Proj}\,(g,\hat{\theta})$.

2. For the part that Proj $(g, \hat{\theta}) = g$, L2 trivially holds by equality. Otherwise by (5.58) we have

$$\begin{aligned}\text{Proj}\,(g,\hat{\theta})^T\,\text{Proj}\,(g,\hat{\theta}) &= \left(g^T - g^T\frac{\nabla_{\hat{\theta}}P\nabla_{\hat{\theta}}P^T}{\nabla_{\hat{\theta}}P^T\nabla_{\hat{\theta}}P}\right)\left(g - \frac{\nabla_{\hat{\theta}}P\nabla_{\hat{\theta}}P^T}{\nabla_{\hat{\theta}}P^T\nabla_{\hat{\theta}}P}g\right)\\ &= g^Tg - 2\frac{g^T\nabla_{\hat{\theta}}P\nabla_{\hat{\theta}}P^Tg}{\nabla_{\hat{\theta}}P^T\nabla_{\hat{\theta}}P} + \frac{g^T\nabla_{\hat{\theta}}P\nabla_{\hat{\theta}}P^T\nabla_{\hat{\theta}}P\nabla_{\hat{\theta}}P^Tg}{\nabla_{\hat{\theta}}P^T\nabla_{\hat{\theta}}P\nabla_{\hat{\theta}}P^T\nabla_{\hat{\theta}}P}\\ &= g^Tg - \frac{(\nabla_{\hat{\theta}}P^Tg)^2}{\nabla_{\hat{\theta}}P^T\nabla_{\hat{\theta}}P}\\ &\le g^Tg.\end{aligned} \tag{5.61}$$

3. Since Proj $(g, \hat{\theta})$ is one of the vectors in the tangent hyperplane, and by orthogonality of the normal vector $\nabla_{\hat{\theta}}P$ with any vector in the tangent hyperplane, we observe,

$$\nabla_{\hat{\theta}}P\,\text{Proj}\,(g,\hat{\theta}) = \begin{cases}\nabla_{\hat{\theta}}P^Tg, & \hat{\theta}\in\Pi^o(\theta_{\max}) \text{ or } \nabla_{\hat{\theta}}P^Tg\le 0\\ 0, & \hat{\theta}\in\partial\Pi(\theta_{\max}) \text{ and } \nabla_{\hat{\theta}}P^Tg > 0\end{cases} \tag{5.62}$$

which means that the vector Proj $(g, \hat{\theta})$ either points inside $\Pi(\theta_{\max})$ or is tangential to the hyperplane of $\partial\Pi(\theta_{\max})$ at $\hat{\theta}$. Therefore $\hat{\theta}(t)$ remains in $\Pi(\theta_{\max})$ as far as the solution exists and $\hat{\theta}(0) \in \Pi(\theta_{\max})$.

4. When Proj $(g, \hat{\theta}) = g$, L4 holds with equality. On the boundary $\partial\Pi(\theta_{\max})$ we have $(\theta - \hat{\theta})^T\nabla_{\hat{\theta}}P \le 0$ since $\theta \in \Pi(\theta_{\max})$ and $\Pi(\theta_{\max})$ is convex. Also observe that by definition, $\nabla_{\hat{\theta}}P^Tg > 0$ on the boundary $\partial\Pi(\theta_{\max})$. Hence

$$\begin{aligned}-(\theta-\hat{\theta})^T\,\text{Proj}\,(g,\hat{\theta}) &= -(\theta-\hat{\theta})^Tg + \frac{(\theta-\hat{\theta})^T\nabla_{\hat{\theta}}P\;\nabla_{\hat{\theta}}P^Tg}{\nabla_{\hat{\theta}}P^T\nabla_{\hat{\theta}}P}\\ &\le -(\theta-\hat{\theta})^Tg.\end{aligned} \tag{5.63}$$

□

The following theorem show the robustness of the closed loop system.

Theorem 5.4. Consider the closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi_P(\theta_{\max}))$ defined by (5.1), (5.59) where $d(\cdot) \in \mathcal{L}^\infty$. Suppose $\theta_{\max}$ is such that $|\theta| \leq \theta_{\max}$. Then for any $x_0 \in \mathbb{R}^n$, the following properties hold:

P1. The solution $(x(\cdot), \hat{\delta}(\cdot)) : \mathbb{R}^+ \to \mathbb{R}^{(n+1)}$ exist.

P2. All closed loop signals $x(\cdot), \hat{\theta}(\cdot), u(\cdot)$ are uniformly bounded as a continuous function of x_0, $\|d\|, \theta_{\max}$.

Proof. In order to prove the existence of the solution, let us map the situation to that of section 2.3.2. We observe that the boundary $\partial\Pi(\theta_{\max})$ defines the switching surface S. The convexity of $\Pi(\theta_{\max})$ enables us to define G^- by $\Pi(\theta_{\max})$, the normal vector N by $\nabla_{\hat{\theta}} P$, and the vector field f^- by g (see Fig. 2.2, and Fig. 3.6). Note that $f^+ = 0$. Now since the right hand side of differential equations (5.1) and (5.59) are locally Lipschitz on G^-, an absolutely continuous local solution exists. Suppose $(x(\cdot), \hat{\theta}(\cdot))$ is one of the solutions of $(\Sigma(x_0, \theta, d(\cdot)), \Xi_P(\theta_{\max}))$ on its maximum interval of existence $[0, \omega)$ for some $\omega \in [0, \infty)$. We define the same Lyapunov function

$$V(x(t), \tilde{\theta}(t)) = x(t)^T P x(t) + \frac{1}{2\alpha}\, \tilde{\theta}(t)^T \tilde{\theta}(t) \qquad \forall t \in [0, \omega) \tag{5.64}$$

and by the same argument as (5.9)-(5.10), the time derivative of $V(x(t), \tilde{\theta}(t))$ is

$$\begin{aligned} \dot{V}(x(t), \tilde{\theta}(t)) = &-x(t)^T Q x(t) + x(t)^T b\, \tilde{\theta}(t)^T \phi(x(t)) + x(t)^T\, b\, d(t) \\ &- \frac{1}{\alpha}\, \tilde{\theta}(t)^T \text{Proj}\ (\alpha x(t)^T\, b\, \phi(x(t))). \end{aligned} \tag{5.65}$$

Observe that by properties L1 and L4 in Lemma 5.1:

$$\begin{aligned} -\frac{1}{\alpha}\, \tilde{\theta}(t)^T \text{Proj}\ (\alpha x(t)^T\, b\, \phi(x(t))) &= -\tilde{\theta}(t)^T \text{Proj}\ (x(t)^T\, b\, \phi(x(t))) \\ &\leq -x(t)^T\, b\, \tilde{\theta}(t)^T \phi(x(t)). \end{aligned} \tag{5.66}$$

Therefore for all $t \in [0, \omega)$,

$$\dot{V}(x(t), \tilde{\theta}(t)) \leq -x(t)^T Q x(t) + x(t)^T\, b\, d(t) \leq -\underline{\lambda}(Q) \|x(t)\|^2 + x(t)^T b d(t). \tag{5.67}$$

Let b_i denote the elements of vector b and apply the completion of squares to (5.67)

$$
\begin{aligned}
-\underline{\lambda}(Q)\|x(t)\|^2 + x(t)^T bd(t) &= \sum_{i=1}^{n} \underline{\lambda}(Q) \left(-x_i(t)^2 + x_i(t) \frac{b_i d(t)}{\underline{\lambda}(Q)} \right) \\
&= \sum_{i=1}^{n} \underline{\lambda}(Q) \left(-\frac{x_i(t)^2}{2} - \frac{1}{2} \left(x_i(t) - \frac{b_i d(t)}{\underline{\lambda}(Q)} \right)^2 + \frac{(b_i\, d(t))^2}{2\, \underline{\lambda}^2(Q)} \right) \\
&\leq \sum_{i=1}^{n} \underline{\lambda}(Q) \left(-\frac{x_i(t)^2}{2} + \frac{(b_i\, d(t))^2}{2\, \underline{\lambda}^2(Q)} \right) \\
&\leq \frac{-\underline{\lambda}(Q)}{2} \|x(t)\|^2 + \gamma \|d(t)\|^2,
\end{aligned} \tag{5.68}
$$

where $\gamma = (|b|^2)/(2\underline{\lambda}(Q))$. Therefore

$$
\dot{V}(x(t), \tilde{\theta}(t)) \leq \frac{-\underline{\lambda}(Q)}{2} \|x(t)\|^2 + \gamma \|d(t)\|^2, \quad \forall t \in [0, \omega). \tag{5.69}
$$

Adding and subtracting the term $kV(x(t), \tilde{\theta}(t))$ for some $k > 0$, we have

$$
\begin{aligned}
\dot{V}(x(t), \tilde{\theta}(t)) \;\; &\leq \;\; -kV(x(t), \tilde{\theta}(t)) - \frac{\underline{\lambda}(Q)}{2} \|x(t)\|^2 + \gamma \|d(t)\|^2 \\
&\quad + k \left(x(t)^T P x(t) + \frac{1}{2\alpha} \tilde{\theta}(t)^T \tilde{\theta}(t) \right) \qquad (5.70) \\
&\leq \;\; -kV(x(t), \tilde{\theta}(t)) - \left(\frac{\underline{\lambda}(Q)}{2} - k\overline{\lambda}(P) \right) \|x(t)\|^2 \\
&\quad + \frac{k}{2\alpha} \tilde{\theta}(t)^T \tilde{\theta}(t) + \gamma \|d(t)\|^2. \qquad (5.71)
\end{aligned}
$$

Choosing $k < \underline{\lambda}(Q)/2\overline{\lambda}(P)$, the second term in (5.71) is negative. The boundedness of $\hat{\theta}(\cdot)$ directly follows from property L3 in Lemma 5.1, therefore

$$
\tilde{\theta}(t)^T \tilde{\theta}(t) \leq (\theta - \hat{\theta}_m)^T (\theta - \hat{\theta}_m), \tag{5.72}
$$

where

$$
\hat{\theta}_m = \sup_{t \in T} \hat{\theta}(t), \qquad T = \{t \geq 0 \;\mid\; \hat{\theta}(t) \in \partial\Pi(\theta_{\max})\}. \tag{5.73}
$$

Defining

$$
V^* := V^*(\|d\|, \theta_{\max}) := \frac{1}{2\alpha} (\theta - \hat{\theta}_m)^T (\theta - \hat{\theta}_m) + \frac{\gamma}{k} \|d\|^2, \tag{5.74}
$$

implies that by (5.71),

$$\dot{V}(x(t),\tilde{\theta}(t)) \leq -k\left(V(x(t),\tilde{\theta}(t)) - V^*\right), \quad \forall t \in [0,\omega), \tag{5.75}$$

which implies that $\dot{V}(x(t),\tilde{\theta}(t)) \leq 0$ for all $V \geq V^*$. It follows that for all $t \in [0,\omega)$,

$$V(x(t),\tilde{\theta}(t)) \leq V_0'(x_0,\|d\|,\theta_{\max}) := \max\{\, V(x_0,0)\,,\, V^*(\|d\|,\theta_{\max})\,\}. \tag{5.76}$$

The uniform boundedness of $V(\cdot,\cdot)$ on $[0,\omega)$ in terms of V_0' implies that $x(\cdot),\hat{\theta}(\cdot) \in \mathcal{L}^\infty(0,\,\omega)$ uniformly as a continuous function of V_0'. Therefore equation (5.7), the boundedness of $d(\cdot)$ and continuity of $\phi(\cdot)$, imply that $x(t)$ cannot have a finite escape time and thus by Corollary 2.2, $\omega = \infty$, i.e. the solution $(\,x(\cdot),\hat{\theta}(\cdot)\,)$ exists for all $t \in [0,\infty)$ and in uniformly bounded as a continuous function of $V_0'(x_0,\|d\|,\theta_{\max})$. Finally the uniform boundedness of $u(\cdot)$ in terms of a continuous function of $V_0'(x_0,\|d\|,\theta_{\max})$ follows from the uniform boundedness of $x(\cdot),\hat{\theta}(\cdot)$ and (5.59). Thus completing the proof. □

Remark 5.1. The uniqueness of the solution can potentially be lost due to the discontinuity of the projection operator. Recall that if at any point on the switching surface the trajectories point away from S along both f^+ and f^- (i.e. $f_N^- < 0,\ f_N^+ > 0$), then the uniqueness of solution ceases to hold. In our case we have

$$f_N^+ = 0\ , \qquad f_N^- = \frac{\nabla_{\hat{\theta}} P^T g}{\|\nabla_{\hat{\theta}} P\|} \leq 0. \tag{5.77}$$

Suppose the solution is on the boundary $\partial\Pi(\theta_{\max})$. f_N^- shows that the solution either remains on the tangential hyperplane $\mathcal{M}$, or returns inside $\Pi^o(\theta_{\max})$, while $f_N^+ = 0$ shows that the solution remains on $\mathcal{M}$. Therefore every solution reaching the boundary $\partial\Pi$ has two possible paths and therefore uniqueness cannot be guaranteed.

5.4 Performance Comparison

In order to generalise theorems **I**, **Ia**, **II**, and **IIa** to nonlinear systems of form (5.1), some assumptions have to be made to establish parameter drift. We believe that starting with a simple scalar nonlinear system helps to gain a better understanding of the assumptions.

5.4.1 Scalar Case

Consider the following class of SISO scalar nonlinear system

$$\Sigma(x_0, \theta, d(\cdot)): \quad \dot{x}(t) = \theta\phi(x(t)) + u(t) + d(t), \qquad x(0) = x_0, \tag{5.78}$$

where $\phi(\cdot)$ is a known smooth real valued function which is assumed to satisfy some or all of the following conditions at various points in this section:

$$\begin{cases} \text{a)} \;\; x = 0 \Longleftrightarrow \phi(x) = 0, \\ \text{b)} \;\; \left.\dfrac{\partial\phi(x)}{\partial x}\right|_{x=0} > 0, \\ \text{c)} \;\; \inf\limits_{x\in\mathbb{R}} \left|\dfrac{\phi(x)}{x}\right| \geq \beta > 0. \end{cases} \tag{5.79}$$

Note that the equivalent results subject to a sign change in (5.79)-b also hold, i.e. where $\frac{\partial\phi(x)}{\partial x}|_{x=0} < 0$. We will use assumptions a) and b) to establish parameter drift in Theorems **I**, **Ia**, and assumption c) in Theorem **II**, **IIa**.

We will consider the unmodified controller:

$$\begin{aligned} \Xi: \;\; u(t) &= -ax(t) - \hat{\theta}(t)\phi(x(t)) \\ \dot{\hat{\theta}}(t) &= \alpha x(t)\phi(x(t)), \qquad \hat{\theta}(0) = 0, \end{aligned} \tag{5.80}$$

the dead-zone controller:

$$\begin{aligned} \Xi_D(d_{\max}): \;\; u(t) &= -ax(t) - \hat{\theta}(t)\phi(x(t)) \\ \dot{\hat{\theta}}(t) &= \alpha D_{\Omega_0(d_{\max})}(x)\, x(t)\phi(x(t)), \quad \hat{\theta}(0) = 0, \quad \eta_0 = \frac{d_{\max}}{a}, \end{aligned} \tag{5.81}$$

the hysteresis dead-zone controller on the interval $t_i^r \le t < t_{i+1}^r$ for $i \ge 0,\ \beta > 0$:

$$\begin{aligned}\Xi_H(d_{\max}):\ & u(t) = -ax(t) - \hat{\theta}(t)\phi(x(t)) \\ & \dot{\hat{\theta}}(t) = \alpha H(t)\, x(t)\phi(x(t)), \quad \hat{\theta}(0) = 0, \eta_0 = \frac{d_{\max}}{a},\ \eta_1 = (1+\beta)\frac{d_{\max}}{a}, \\ & \hat{\theta}(t_i^r) = \hat{\theta}(t_i^s) \qquad t_i^r \le t < t_{i+1}^r,\ \ i \ge 0.\end{aligned} \tag{5.82}$$

and the projection controller:

$$\begin{aligned}\Xi_P(\theta_{\max}):\ & u(t) = -ax(t) - \hat{\theta}(t)\phi(x(t)) \\ & \dot{\hat{\theta}}(t) = \alpha \text{Proj}_{\ \Pi(\theta_{\max})} \left(x(t)\phi(x(t))\right), \qquad \hat{\theta}(0) = 0,\end{aligned} \tag{5.83}$$

where $d_{\max}$, $\theta_{\max}$ are the a-priori knowledge of disturbance level and parameter uncertainty level respectively.

The transient performance cost functional (3.44) is defined by

$$\mathcal{P}\left(\Sigma\left(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)\right), \Xi\right) = \sup_{x_0 \in \mathcal{X}_0(\gamma)} \ \sup_{\theta \in \Delta(\delta)} \ \sup_{d \in \mathcal{D}(\epsilon)} \left(\|x(\cdot)\|_{\mathcal{L}^\infty} + \|u(\cdot)\|_{\mathcal{L}^\infty} + \|\dot{u}(\cdot)\|_{\mathcal{L}^\infty}\right) \tag{5.84}$$

where $\gamma \le \eta_0$, and $\Delta(\delta), \mathcal{D}(\epsilon)$ are defined in (3.45)

5.4.1.1 Theorem I, Ia

Following the procedure explained in section 3.5.3, we give the following propositions:

Proposition 5.1. Suppose $\phi(\cdot)$ satisfies conditions 5.79-a, 5.79-b. Consider the closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi)$ defined by (5.78), (5.80), where $d(\cdot) = \epsilon$ for some $\epsilon \ne 0$. Then as $t \to \infty$ the followings hold:

i. $x(t) \to 0 \Longleftrightarrow \hat{\theta}(t) \to \infty$.

ii. If $x(t)$ is bounded and uniformly continuous then $x(t) \to 0,\ \hat{\theta}(t) \to \infty$.

Proof. The immediate consequence of 5.79-a, 5.79-b implies that $\hat{\theta}(\cdot)$ is monotonically increasing. The proof is in the same manner as Propositions 4.1, 4.2: Suppose $x(t) \to 0$. Seeking a contradiction leads to the existence of an equilibrium point $(0, \hat{\theta}^*)$ for some $\hat{\theta}^*$.

It contradicts the fact that, given $\epsilon \neq 0$ and assumption 5.79-a, the closed loop has no equilibrium point. The proof of the sufficient part of (i) follows by defining Lyapunov function $V(x(t)) = x(t)^2/2$ and observing that

$$\dot{V}(x(t)) = -ax(t)^2 + \epsilon x(t) + \varphi(t), \tag{5.85}$$

where in the interest of brevity we have denoted

$$\varphi(t) := \varphi(x, \phi, \hat{\theta}) = (\theta - \hat{\theta}(t))x(t)\phi(x(t)) \tag{5.86}$$

It follows that $V(x(t))$ is decreasing if

$$ax(t)^2/2 - \varphi(t) \geq \epsilon^2/2a, \tag{5.87}$$

which is similar to inequality (4.106). The rest of the proof of part (i) follows from a similar argument as in Proposition 4.1.

We prove part (ii) by contradiction: Suppose $x(t) \not\to 0$ as $t \to \infty$. Replacing $y(t)$ by $x(t)$ in Proposition 4.2, we observe that the relations (4.109)–(4.112) hold for $x(t)$, i.e.

$$x(t) \geq \frac{M}{2}, \qquad \forall t \in [\,t_k, t_k + \delta\,]. \tag{5.88}$$

Now by (5.79), the boundedness of $x(\cdot)$, and the continuity of $\phi(\cdot)$, we have that $\phi(x(t)) \geq \beta > 0$ for some β, i.e.

$$\exists N > 0 \quad s.t. \quad \alpha\, x(t)\phi(x(t)) \geq N, \qquad \forall t \in [\,t_k, t_k + \delta\,]. \tag{5.89}$$

It follows that

$$\int_{t_k}^{t_k+\delta} \alpha\, x(\tau)\phi(x(\tau))\, d\tau \geq N\delta. \tag{5.90}$$

The rest of the proof is as the same as Proposition 4.2. □

Proposition 5.2. Suppose $\phi(\cdot)$ satisfies conditions 5.79-a and 5.79-b. Consider the closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi)$ defined by equations (5.78), (5.80) and the transient performance cost functional $\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi)$. Then

$$\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi) = \infty. \tag{5.91}$$

Proof. The spirit of the proof is similar to Propositions 3.2 and 4.3. Let $x_0 \in \mathcal{X}_0(\gamma)$, $\theta \in \Delta(\delta)$, and choose $d(t) = \epsilon \neq 0$. Suppose for contradiction $\mathcal{P}(\Sigma(x_0, \theta, d(\cdot)), \Xi) < \infty$. Consider $\dot{x}(t)$. There are two cases: either 1. $\overline{\lim}\,\dot{x}(t) = \infty$ or 2. $\overline{\lim}\,\dot{x}(t) < \infty$. The proof of the first case is similar to before. Suppose $\overline{\lim}\,\dot{x}(t) < \infty$ i.e. $x(t)$ is uniformly continuous. Again, there are two possibilities: either a) $\overline{\lim}\,x(t) = \infty$, or b) $\overline{\lim}\,x(t) < \infty$. The former yields to $\|x(\cdot)\|_{\mathcal{L}^\infty} = \infty$, hence contradiction. The latter follows the boundedness of $x(\cdot)$, therefore by Proposition 5.1

$$x(t) \to 0, \quad \hat{\theta}(t) \to \infty \quad \text{as} \quad t \to \infty. \tag{5.92}$$

Applying (5.92) to $\overline{\lim}\,\dot{u}(t)$, we observe that

$$\overline{\lim}\,\dot{u}(t) = \overline{\lim}\left[-\left(\hat{\theta}(t)\phi(x(t)) - \epsilon\right)\left(a + \hat{\theta}(t)\frac{\partial \phi(x)}{\partial x}\right)\right]. \tag{5.93}$$

Now there are two possibilities: either i) $\hat{\theta}(t)\phi(x(t)) \not\to \epsilon$ (including the possibility that $\lim \hat{\theta}(t)\phi(x(t))$ does not exist), or ii) $\lim \hat{\theta}(t)\phi(x(t)) = \epsilon$:

1. Suppose $\lim_{t\to\infty} \hat{\theta}(t)\phi(x(t))$ does not exist or $\hat{\theta}(t)\phi(x(t)) \not\to \epsilon$. Since

$$a + \hat{\theta}(t)\frac{\partial \phi(x)}{\partial x} \to \infty \quad \text{as} \quad x(t) \to 0, \;\; \hat{\theta}(t) \to \infty, \tag{5.94}$$

   it follows by (5.79-b) that $\|\dot{u}(\cdot)\|_{\mathcal{L}^\infty} = \infty$; hence contradiction.

2. Suppose $\lim_{t\to\infty} \hat{\theta}(t)\phi(x(t)) = \epsilon$. By the smoothness of $\phi(\cdot)$ and the same argument as Propositions 3.2 and 4.3, there exists $T > 0$ such that $\hat{\theta}(T) > \hat{\theta}^*$. So by choosing $d(\cdot)$ similar to (3.73) we observe that,

$$\left(\lim_{t\to T^+} \dot{u}(t)\right) - \dot{u}(T) = 2\epsilon(a + \hat{\theta}(T)\frac{\partial \phi(x(T))}{\partial x}) \geq 2\hat{\theta}^*\frac{\partial \phi(x(T))}{\partial x}\epsilon. \tag{5.95}$$

   Considering assumption 5.79-b, choosing a suitable $\hat{\theta}^*$, can make the difference

(5.95) arbitrarily large. Then either $\dot{u}(T)$ is large or $\lim_{t\to T^+} \dot{u}(t)$ is large, therefore $\|\dot{u}(\cdot)\|_{\mathcal{L}^\infty}$ can be made arbitrarily large hence contradiction.

The proof is completed since $\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi) \geq \mathcal{P}(\Sigma(x_0, \theta, d(\cdot)), \Xi) = \infty$.

□

Proposition 5.3. Suppose $\phi(\cdot)$ satisfies conditions 5.79-a, 5.79-b. Consider the closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi_P(\theta_{\max}))$ defined by equations (5.78),(5.83). Then

$$\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_P(\theta_{\max})) \to \infty \quad \text{as} \quad \theta_{\max} \to \infty. \tag{5.96}$$

Proof. See the proof of Proposition 3.3 and Remark 3.4. □

Proposition 5.4. The closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi_D(d_{\max}))$ defined by (5.78), (5.81) has the property

$$\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_D(d_{\max})) < \infty, \qquad \forall d_{\max} > \epsilon. \tag{5.97}$$

Proof. Let $x_0 \in \mathcal{X}_0(\gamma)$, $\theta \in \Delta(\delta)$ and $d \in \mathcal{D}(\epsilon)$. The uniform boundedness of signals $x(\cdot), \hat{\theta}(\cdot), u(\cdot)$ as a continuous function of $V_0(x_0, \theta, d_{\max})$ defined by (5.16) follow from Theorem 5.2. A similar calculation as Proposition 3.4 on (5.78) implies the uniform boundedness of $\dot{x}(\cdot)$ in terms of a continuous function of $V_0(x_0, \theta, d_{\max})$. So

$$\dot{u}(t) = -a\dot{x}(t) - \hat{\theta}(t)\frac{\partial \phi(x)}{\partial x}\dot{x}(t) - \alpha\, D_{\Omega_0} x(t)\phi(x(t))^2, \tag{5.98}$$

is uniformly bounded in terms of a continuous function of $V_0(x_0, \theta, d_{\max})$. That is

$$\mathcal{P}(\Sigma(x_0, \theta, d(\cdot)), \Xi_D(d_{\max})) \leq M(V_0(x_0, \theta, d_{\max})), \tag{5.99}$$

for some $M(V_0(x_0, \theta, d_{\max})) < \infty$. The proof is completed by taking the supremum over system arguments x_0, θ, d:

$$\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_D(d_{\max})) < \infty, \qquad \forall d_{\max} > \epsilon. \tag{5.100}$$

□

Proposition 5.5. The closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi_H(d_{\max}))$ defined by (5.78), (5.82) has the property

$$\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_H(d_{\max})) < \infty, \qquad \forall d_{\max} > \epsilon. \tag{5.101}$$

Proof. Replace $V_0(x_0, \theta, d_{\max})$ with $V_0'(x_0, \theta, d_{\max})$ defined by (5.37) and follow the proof of Proposition 5.4. □

Theorem I. *Suppose $\phi(\cdot)$ satisfies conditions 5.79-a, 5.79-b. Consider the system $\Sigma(x_0, \theta, d(\cdot))$ and the controllers $\Xi_D(d_{\max})$ and $\Xi_P(\theta_{\max})$ defined by* (5.78), (5.81) *and* (5.83) *respectively. Consider the transient performance cost functional* (5.84). *Then $\forall d_{\max} \geq \epsilon$, $\exists \theta^*_{\max} \geq \delta$ such that $\forall \theta_{\max} \geq \theta^*_{\max}$,*

$$\mathcal{P}(\,\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_P(\theta_{\max})\,) > \mathcal{P}(\,\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_D(d_{\max})\,). \tag{5.102}$$

Proof. The proof is followed simply from propositions 5.3 and 5.4. □

Theorem Ia. *Suppose $\phi(\cdot)$ satisfies conditions 5.79-a, 5.79-b. Consider the system $\Sigma(x_0, \theta, d(\cdot))$ and the controllers $\Xi_H(d_{\max})$ and $\Xi_P(\theta_{\max})$ defined by* (5.78), (5.82) *and* (5.83) *respectively. Consider the transient performance cost functional* (5.84). *Then $\forall d_{\max} \geq \epsilon$, $\exists \theta^*_{\max} \geq \delta$ such that $\forall \theta_{\max} \geq \theta^*_{\max}$,*

$$\mathcal{P}(\,\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_P(\theta_{\max})\,) > \mathcal{P}(\,\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_H(d_{\max})\,). \tag{5.103}$$

Proof. The proof is a simple consequence of Propositions 5.3 and Proposition 5.5. □

5.4.1.2 Theorem II, IIa

Again, we refer the reader to the discussion on the possible choices of dead-zone in section 3.5.5.

Proposition 5.6. Suppose $\phi(\cdot)$ satisfies condition 5.79-c. Consider the closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi_D(d_{\max}))$ defined by (5.78), (5.81). Then $\exists \delta > 0$ such that

$$\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_D(d_{\max})) \to \infty, \quad \text{as } d_{\max} \to \infty. \tag{5.104}$$

Proof. Let $x_0 > 0$ and replace equation (3.96) by

$$\dot{x}(t) = -ax(t) + \theta\phi(x(t)) + d(t) \qquad \forall t \in [0, \tau). \tag{5.105}$$

A direct consequence of 5.79-c is

$$\sup_{x\in\mathbb{R}} \left| \frac{x}{\phi(x)} \right| \le M < \infty. \tag{5.106}$$

Therefore there exists $\delta = 2aM$ such that if $\theta = \delta$, then $\forall x(t) > 0$,

$$-ax + \theta\phi(x) = \left(-a\frac{x}{\phi(x)} + \theta \right) \phi(x) = \left(-a\frac{|x|}{|\phi(x)|} + \theta \right) |\phi(x)| \ge aM|\phi(x)| > 0, \tag{5.107}$$

The rest of the proof is followed by Proposition 3.7. □

Proposition 5.7. Suppose $\phi(\cdot)$ satisfies condition 5.79-c. Consider the closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi_H(d_{\max}))$ defined by (5.78), (5.82). Then $\exists\delta > 0$ such that

$$\mathcal{P}(\Sigma\left(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)\right), \Xi_H(d_{\max})) \to \infty, \quad \text{as } d_{\max} \to \infty. \tag{5.108}$$

Proof. See proof of Proposition 5.6. □

Proposition 5.8. The closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi_P(\theta_{\max}))$ defined by (5.78), (5.83) has the property

$$\mathcal{P}(\Sigma\left(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)\right), \Xi_P(\theta_{\max})) < \infty, \qquad \forall\theta_{\max} \ge \delta. \tag{5.109}$$

Proof. See the proof of Proposition 3.6. □

Theorem II. *Suppose $\phi(\cdot)$ satisfies condition 5.79-c. Consider the system $\Sigma(x_0, \theta, d(\cdot))$ and the controllers $\Xi_D(d_{\max})$ and $\Xi_P(\theta_{\max})$ defined by* (5.78), (5.81) *and* (5.83) *respectively. Consider the cost functional* (5.84). *Then $\exists\delta > 0$ such that $\forall\theta_{\max} \ge \delta$ $\exists d^*_{\max} \ge \epsilon$ such that $\forall d_{\max} \ge d^*_{\max}$,*

$$\mathcal{P}(\Sigma\left(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)\right), \Xi_D(d_{\max})) > \mathcal{P}(\Sigma\left(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)\right), \Xi_P(\theta_{\max})). \tag{5.110}$$

Proof. This is a simple consequence of Proposition 5.6 and Proposition 5.8. □

Theorem IIa. *Suppose $\phi(\cdot)$ satisfies condition 5.79-c. Consider the system $\Sigma(x_0, \theta, d(\cdot))$ and the controllers $\Xi_H(d_{\max})$ and $\Xi_P(\theta_{\max})$ defined by* (5.78), (5.82) *and* (5.83) *respectively. Consider the cost functional* (5.84). *Then $\exists\delta > 0$ such that $\forall\theta_{\max} \geq \delta$ $\exists d^*_{\max} \geq \epsilon$ such that $\forall d_{\max} \geq d^*_{\max}$,*

$$\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_H(d_{\max})) > \mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_P(\theta_{\max})). \qquad (5.111)$$

Proof. This is a simple consequence of Proposition 5.7 and Proposition 5.8. □

5.4.2 Chain of Integrators

Suppose $\theta \in \mathbb{R}$ in system (5.1). It follows by (5.7) that

$$\dot{x}(t) = Ax(t) + (\,B(\theta - \hat{\theta}(t))\phi(x(t)) + d(t)\,), \qquad x(0) = x_0. \qquad (5.112)$$

Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be such that some or all of the following conditions satisfied at various points in this section

$$\begin{cases} \text{a)} \;\; x = 0 \Longleftrightarrow \phi(x) = 0, \\ \text{b)} \;\; x^T b\phi(x) \geq 0, \\ \text{c)} \;\; \left.\dfrac{\partial\phi(x)}{\partial x_n}\right|_{x=0} > 0, \\ \text{d)} \;\; \inf\limits_{x\in\mathbb{R}^n} \dfrac{|\phi(x)|}{\|x\|} \geq \beta > 0. \end{cases} \qquad (5.113)$$

5.4.2.1 Theorem I, Ia

Proposition 5.9. Suppose $\theta \in \mathbb{R}$ and $\phi(\cdot)$ satisfies conditions 5.113-a, 5.113-b. Consider the closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi)$ defined by (5.1), (5.6), where $d(\cdot) = \epsilon$ for some $\epsilon \neq 0$.

Then

$$\|x(t)\| \to 0 \quad \text{as} \quad t \to \infty \Longleftrightarrow \hat{\theta}(t) \to \infty \quad \text{as} \quad t \to \infty \tag{5.114}$$

Proof.

$\rightarrow$) Seeking a contradiction. If $\hat{\theta}(t) \not\to \infty$ then by 5.113-a, 5.113-b, $\hat{\theta}(t) \to \hat{\theta}^*$, i.e. by Lemma 4.3 $(0, \hat{\theta}^*)$ is an equilibrium point for some $\hat{\theta}^*$. This contradicts the fact that the given $\epsilon \neq 0$ and assumption 5.79-a, the closed loop system

$$\begin{aligned} -a^T x + (\theta - \hat{\theta})\phi(x) + \epsilon &= 0 \\ \alpha\, x^T\, b\, \phi(x) &= 0 \end{aligned} \tag{5.115}$$

has no equilibrium point.

$\leftarrow$) Defining the Lyapunov function

$$V(x(t)) = x(t)^T P x(t), \tag{5.116}$$

yields

$$\dot{V}(t) = -x(t)^T Q x(t) + x(t)^T b((\theta - \hat{\theta}(t)) + \epsilon) \tag{5.117}$$

$$\leq -\underline{\lambda}(Q)\|x(t)\|^2 + \|x(t)\|\,|b|\,|\epsilon| + \varphi(t) \tag{5.118}$$

where

$$\varphi(t) = x(t)^T b\,(\theta - \hat{\theta}(t))\phi(x(t)). \tag{5.119}$$

Note that condition 5.113-b implies that as $\hat{\theta}(t) \to \infty$ we have $\varphi(t) \to -\infty$ for all $x(t) \neq 0$. The claim of the proof then follows using the same argument as in Proposition 4.1, replacing $\bar{x}(\cdot)$ with $x(\cdot)$ and $\hat{\delta}(\cdot)$ by $\hat{\theta}(\cdot)$.

□

Proposition 5.10. Suppose $\theta \in \mathbb{R}$ and $\phi(\cdot)$ satisfies conditions 5.113-a, 5.113-b. Consider the closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi)$ defined by (5.1), (5.6), where $d(t) = \epsilon$ for some $\epsilon \neq 0$.

If $x(\cdot)$ is bounded and uniformly continuous then as $t \to \infty$

$$\|x(t)\| \to 0, \quad \hat{\theta}(t) \to \infty. \tag{5.120}$$

Proof. The proof is divided into two parts:

(i) Firstly we show that $x(t)^T b \to 0$ as $t \to \infty$.

(ii) Then we prove that $\hat{\theta}(t) \to \infty$ which implies by Proposition 5.9 that $x(t) \to 0$.

(1). Replacing $y(t)$ by $x(t)^T b$ in proof of Proposition (4.2) and following (4.109)–(4.112) yields

$$x(t)^T b \geq \frac{M}{2}, \qquad \forall t \in [\, t_k, t_k + \omega\,]. \tag{5.121}$$

The boundedness of $x(\cdot)$ implies that

$$\|x(\cdot)\|_{\mathcal{L}^\infty} \leq K < \infty. \tag{5.122}$$

By (5.113-b), the fact that $x(t)^T b \neq 0 \Rightarrow x(t) \neq 0$ and the negation of (5.113-a) we see that

$$\phi(x(t)) \geq \beta, \qquad \forall t \in [\, t_k, t_k + \omega\,], \tag{5.123}$$

where

$$0 < \beta = \inf \left\{ \phi(x(t)) \;\mid\; \|x(t)\| \leq K \;\text{ and }\; x(t)^T b \geq \frac{M}{2} \right\}. \tag{5.124}$$

It follows that

$$\exists N > 0 \quad s.t. \quad \alpha x(t)^T b \phi(x(t)) \geq N, \qquad \forall t \in [\, t_k, t_k + \omega\,]. \tag{5.125}$$

So

$$\int_{t_k}^{t_k+\omega} \hat{\theta}(\tau) d\tau \geq \omega N. \tag{5.126}$$

With no loss of generality, we may assume $t_{k+1} - t_k \geq \omega$. It follows that

$$\hat{\theta}(t_k + \omega) = \int_0^{t_k+\omega} \dot{\hat{\theta}}(\tau) d\tau \geq k\omega N, \tag{5.127}$$

so $\hat{\theta}(t_k+\omega) \to \infty$ as $k \to \infty$, hence $\hat{\theta}(t) \to \infty$ as $t \to \infty$. Therefore by Proposition 5.9 $\|x(t)\| \to 0$, i.e. $x(t)^T b \to 0$, hence contradiction.

(*2*). Suppose for contradiction $\hat{\theta} \not\to \infty$. Then $\hat{\theta}(t) \to \hat{\theta}^* < \infty$, since $\hat{\theta}(t)$ is monotonically increasing by 5.113-a, 5.113-b. Note that $x(t)^T b \to 0 \Rightarrow \varphi(t) \to 0$. Adding and subtracting $kV(x(t)$ for $k \leq \underline{\lambda}(Q)/\overline{\lambda}(P)$ to (5.118), we have that

$$V(x(t)) \leq V_0 e^{-kt} + e^{-kt} \int_0^t e^{k\tau} \varphi(\tau) d\tau. \tag{5.128}$$

Consider the integral term in (5.128). There are two possibilities: either it is uniformly bounded, or not:

(*a*). Suppose the integral term in (5.128) is uniformly bounded, then

$$\exists M > 0 \;\; s.t. \;\; \left| \int_0^t e^{k\tau} \varphi(\tau) d\tau \right| < M, \quad \forall t \geq 0. \tag{5.129}$$

Therefore as $t \to \infty$, the term e^{-kt} dominate the integral and therefore, $\lim_{t\to\infty} V(x(t)) = 0$, i.e. $x(t) \to 0$ by (5.116); hence contradiction by Proposition 5.9.

(*b*). Suppose the integral term in (5.128) is not uniformly bounded, then the integrand $e^{kt}\varphi(t)$ is either bounded or unbounded:

(*i*). Suppose for all $t \geq 0$, we have $|e^{kt}\varphi(t)| < N$ for some $N > 0$. Then

$$\lim_{t\to\infty} V(x(t)) \leq \lim_{t\to\infty} (V_0 e^{-kt} + e^{-kt} N t) = 0, \tag{5.130}$$

hence by positive definiteness of $V(\cdot)$, we have $\lim_{t\to\infty} V(x(t)) = 0$, hence contradiction.

(*ii*). The last case is where both integrand $e^{kt}\varphi(t)$ and the integral are unbounded. In this case, as $t \to \infty$, (5.128) is of indeterminant form. Applying L'Hôpital's rule, we observe that

$$\lim_{t\to\infty} e^{-kt} \int_0^t e^{k\tau} \varphi(\tau) d\tau = \lim_{t\to\infty} \frac{\int_0^t e^{k\tau}\varphi(\tau) d\tau}{e^{kt}} = \lim_{t\to\infty} \frac{\varphi(t)}{k}. \tag{5.131}$$

So

$$\lim_{t\to\infty} V(t) \leq \lim_{t\to\infty} \frac{\varphi(t)}{k} \to 0, \tag{5.132}$$

hence contradiction.

Therefore $\hat{\theta} \to \infty$ and by Proposition 5.9, $x \to 0$. Thus completing the proof.

□

Proposition 5.11. Suppose $\theta \in \mathbb{R}$ and $\phi(\cdot)$ satisfies conditions 5.113–a,b,c and consider the closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi)$ defined by (5.1),(5.6). Consider the transient performance cost functional (5.84). Then

$$\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi) = \infty. \tag{5.133}$$

Proof. Let $x_0 \in \mathcal{X}_0(\gamma)$, $\theta \in \Delta(\delta)$, and choose $d(t) = \epsilon \neq 0$. Suppose for contradiction $\mathcal{P}(\Sigma(x_0, \theta, d(\cdot)), \Xi) < \infty$. Replace the scalar $x(t)$ by $\|x(t)\|$ and repeat the argument in Proposition 5.2 until (5.92). Therefore either $\mathcal{P} = \infty$ or $x(t)$ is bounded and uniformly continuous. The first case is a contradiction. The second case implies by Proposition (5.10) that

$$\|x(t)\| \to 0, \quad \hat{\theta}(t) \to \infty. \tag{5.134}$$

It follows that $x(t)^T b \to 0$ and by 5.113-a that $\phi(x) \to 0$. Applying these limits to $\overline{\lim}\, \dot{u}(t)$, we have that

$$\begin{aligned}\overline{\lim}\, \dot{u}(t) &= \overline{\lim} \left(-a_n \dot{x}_n(t) - \hat{\theta}(t) \frac{\partial \phi(x)}{\partial x_n} \dot{x}_n(t) - x(t)^T b \phi(x(t))^2 \right) \\ &\quad - \overline{\lim} \sum_{i=1}^{n-1} \left(a_i - \hat{\theta}(t) \frac{\partial \phi(x)}{\partial x_i} \right) x_{i+1}(t) \\ &= \overline{\lim}\, (\hat{\theta}(t)\phi(x(t)) - \epsilon) a_n + \hat{\theta}(t) \left(\frac{\partial \phi(x)}{\partial x_n} (\hat{\theta}(t)\phi(x(t)) - \epsilon) + \sum_{i=1}^{n-1} \frac{\partial \phi(x)}{\partial x_i} x_{i+1}(t) \right),\end{aligned} \tag{5.135}$$

but by (5.134),

$$\lim_{t\to\infty} \sum_{i=1}^{n-1} \frac{\partial \phi}{\partial x_i} x_{i+1} = 0. \tag{5.136}$$

Therefore

$$\overline{\lim}\, \dot{u}(t) = \overline{\lim} \left(\hat{\theta}(t)\phi(x(t)) - \epsilon \right) \left(a_n + \hat{\theta}(t) \frac{\partial \phi(x)}{\partial x_n} \right), \tag{5.137}$$

which is similar to (5.93). The rest of the proof is similar to Proposition 5.2 by considering condition 5.113–c and replacing $\partial\phi(x)/\partial x$ by $\partial\phi(x)/\partial x_n$. □

Proposition 5.12. Suppose $\theta \in \mathbb{R}$ and $\phi(\cdot)$ satisfies conditions 5.113–a,b,c. Consider the closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi_P(\theta_{\max}))$ defined by equations (5.78),(5.83). Then

$$\mathcal{P}(\Sigma\left(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)\right), \Xi_P(\theta_{\max})) \to \infty \quad \text{as} \quad \theta_{\max} \to \infty. \tag{5.138}$$

Proof. See the proof of Proposition 3.3 and Remark 3.4. □

Proposition 5.13. Suppose $\theta \in \mathbb{R}$. The closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi_D(d_{\max}))$ defined by (5.1), (5.15) has the property

$$\mathcal{P}(\Sigma\left(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)\right), \Xi_D(d_{\max})) < \infty, \qquad \forall d_{\max} > \epsilon. \tag{5.139}$$

Proof. Let $x_0 \in \mathcal{X}_0(\gamma)$, $\theta \in \Delta(\delta)$ and $d \in \mathcal{D}(\epsilon)$. The uniform boundedness of closed loop signals $x(\cdot), \hat{\theta}(\cdot), u(\cdot)$ as a continuous function of $V_0(x_0, \theta, d_{\max})$ defined by (5.16) follows directly from Theorem 5.2. Therefore by (5.7), $\dot{x}(\cdot)$ is uniformly bounded in terms of a continuous function of $V_0(x_0, \theta, d_{\max})$. Hence

$$\dot{u}(t) = -a_n \dot{x}_n(t) - \hat{\theta}(t)\frac{\partial\phi(x)}{\partial x_n}\dot{x}_n(t) - \sum_{i=1}^{n-1}\left(a_i - \hat{\theta}(t)\frac{\partial\phi(x)}{\partial x_i}\right)x_{i+1}(t) - x(t)^T b\phi(x(t))^2, \tag{5.140}$$

is uniformly bounded as a continuous function of $V_0(x_0, \theta, d_{\max})$. Therefore

$$\mathcal{P}(\Sigma(x_0, \theta, d(\cdot)), \Xi_D(d_{\max})) \le M(V_0(x_0, \theta, d_{\max})), \tag{5.141}$$

for some $M(V_0(x_0, \theta, d_{\max})) < \infty$; hence $\mathcal{P}(\Sigma\left(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)\right), \Xi_D(d_{\max})) < \infty$. □

Proposition 5.14. Suppose $\theta \in \mathbb{R}$. The closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi_H(d_{\max}))$ defined by (5.1), (5.35) has the property

$$\mathcal{P}(\Sigma\left(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)\right), \Xi_H(d_{\max})) < \infty, \qquad \forall d_{\max} > \epsilon. \tag{5.142}$$

Proof. Replace $V_0(x_0, \theta, d_{\max})$ with $V_0'(x_0, \theta, d_{\max})$ defined in (5.37) and follow the proof of Proposition 5.13. □

*Proof of Theorem **I**.*

The proof is a consequence of Proposition 5.12 and Proposition 5.13. □

*Proof of Theorem **Ia**.*

This is a simple consequence of Proposition 5.12 and Proposition 5.14. □

5.4.2.2 Theorem II, IIa

Proposition 5.15. Suppose $\theta \in \mathbb{R}$ and $\phi(\cdot)$ satisfies condition 5.113-d. Consider the closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi_D(d_{\max}))$ defined by (5.1), (5.15). Then $\exists \delta > 0$ such that

$$\mathcal{P}(\Sigma\,(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_D(d_{\max})) \to \infty, \quad \text{as } d_{\max} \to \infty. \tag{5.143}$$

Proof. Condition 5.113-d implies that

$$|a^T x| \leq \|a\|\|x\| \leq \frac{M\|a\|}{|\theta|}|\theta\phi(x)|. \tag{5.144}$$

Choosing $d(\cdot) = \epsilon > 0$, and $\delta = M\|a\|$. Replace equation (5.105) by

$$\dot{x}_n(t) = -a^T x(t) + \theta\phi(x(t)) + d(t) \qquad \forall t \in [0, \tau), \tag{5.145}$$

and observe that if $\theta = \delta$, then $\dot{x}_n(t) > 0$, $\forall x(t) > 0$. It follows by cascading the result alongside the chain of integrators (5.112) that $\|x(t)\| \to \infty$ as $t \to \infty$. The rest of the proof is the same as Proposition 3.7. □

Proposition 5.16. Suppose $\theta \in \mathbb{R}$ and $\phi(\cdot)$ satisfies condition 5.113-d. Consider the closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi_H(d_{\max}))$ defined by (5.1), (5.35). Then $\exists \delta > 0$ such that

$$\mathcal{P}(\Sigma\,(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_H(d_{\max})) \to \infty, \quad \text{as } d_{\max} \to \infty. \tag{5.146}$$

Proof. See the proof of Proposition 5.15. □

Proposition 5.17. Suppose $\theta \in \mathbb{R}$. The closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi_P(\theta_{\max}))$ defined by (5.1), (5.59) has the property

$$\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_P(\theta_{\max})) < \infty, \qquad \forall \theta_{\max} \geq \delta. \tag{5.147}$$

Proof. Property P2 of Theorem 5.4 guarantees the uniform boundedness of $x(\cdot)$, $\hat{\theta}(\cdot)$ and $u(\cdot)$ in terms of $V_0(x_0, \|d\|, \theta_{\max})$ defined in (5.76) for all $x_0 \in \mathcal{X}_0(\gamma)$, $\theta \in \Delta(\delta)$ and $d \in \mathcal{D}(\epsilon)$. It follows by (5.112) that $\dot{x}_n(\cdot)$ is uniformly bounded in terms of a continuous function of $V_0(x_0, \|d\|, \theta_{\max})$. The continuity of $\phi(\cdot)$ together with (5.135) implies that $\dot{u}(\cdot)$ is uniformly bounded as a continuous function of $V_0(x_0, \|d\|, \theta_{\max})$. It follows that

$$\mathcal{P}(\Sigma(x_0, \theta, d(\cdot)), \Xi_P(\theta_{\max})) \leq M(V_0(x_0, \|d\|, \theta_{\max})), \tag{5.148}$$

where continuous function $M(V_0(x_0, \|d\|, \theta_{\max})) < \infty$. Taking the supremum over system arguments x_0, θ, d implies that for all $\theta_{\max} \geq \delta$,

$$\mathcal{P}(\Sigma(\mathcal{X}_0(\gamma), \Delta(\delta), \mathcal{D}(\epsilon)), \Xi_P(\theta_{\max})) < \infty. \tag{5.149}$$

□

*Proof of Theorem **II**.*

The proof is a consequence of Proposition 5.15 and Proposition 5.17. □

*Proof of Theorem **IIa**.*

This is a simple consequence of Proposition 5.16 and Proposition 5.17. □

5.5 Summary and Discussion

In this chapter we investigated robust adaptive control designs for the integrator chain class of nonlinear systems. We showed how a typical adaptive design stabilises such systems when no disturbances are present. The Dead-zone and Projection modification have been defined to assure robustness of adaptive systems in the presence of bounded disturbances.

Next, we defined and analysed the hysteresis dead-zone modification for such nonlinear systems. Comparing this method and conventional dead-zone, we discussed the advantages of hysteresis dead-zone e.g. uniqueness of solution, controllable finite frequency chattering, and efficiency of the algorithm with respect to complexity and simulation. Application of hysteresis switching schemes are still an interesting line of research in the adaptive control literature, see e.g. [32, 52, 58] and references therein.

Generalising Theorems **I**, **Ia**, **II**, and **IIa** to the chain of integrators was discussed. Some assumptions have been made to ensure that parameter drift occurs. We started with a simple first order nonlinear system and showed that the assumption become more restrictive when considering higher order systems.

Chapter 6

Concluding Remarks

In this final chapter, we conclude by summarising the contributions of this dissertation and indicating some directions for future work.

6.1 Contribution of This Dissertation

6.1.1 Primary Contributions

By considering a non-singular performance cost functional for a variety of systems (scalar systems, minimum phase linear systems with relative degree one and nonlinear systems in the form of integrator chain), we have established two rigorous results comparing the performance of the dead-zone and the projection based robust adaptive control systems:

- The dead-zone based controller outperforms the projection based controller when the a-priori information on the uncertainty level is sufficiently conservative.
- The projection controller outperforms the dead-zone controller when the a-priori information on the disturbance level is sufficiently conservative.

These are the first analytical performance comparison results in robust adaptive control literature which take control effort into account The methodology is based on the a-priori information available for each design and the novelty of analysis is to employ non-singular cost to formulate the problem.

This case study has shown that a quantitative cost based approach can be utilised to assess relative benefits of different robust adaptive controllers.

6.1.2 Secondary Contribution

Motivated by a relatively old idea, an alternative for dead-zone modification has been developed. Hysteresis dead-zone controllers have some important analytical as well as practical advantages over to conventional dead-zone based controllers. For example, the solution of the closed loop is unique, sliding motion is mitigated, and the efficiency of the algorithm has been improved with respect to complexity and simulation. The comparative analysis was also applied to hysteresis dead-zone controllers.

6.2 Recommendations for Future Work

There are a number of directions in which the results can be fruitfully generalised, for example:

- Relaxing the assumptions 5.113-b appeared in the proof of integrator chain. Currently the proof relies on the assumption $x^T b\phi(x) \geq 0$ which restricts the choices of $\phi(x)$. So, further generalisations of the current approach requires finding less restrictive assumptions to ensure parameter drift.

- Generalisation of the results to the minimum phase linear system with relative degree n. One possible method proposed by French [9] is motivated as follows: Denote Σ_n the relative degree n plant satisfying $C1, C2$ of section 4.2 (except $\rho = 1$), and note that there exists a controller Ξ_1 for which the closed loop (Σ_1, Ξ_1) is well posed. Let us explain the idea for $n = 2$. Define $z(s) := b'_n s$ and observe that $z(s)\Sigma_2 = \Sigma_1$, i.e. $(z(s)\Sigma_2, \Xi_1)$ is well posed. The equivalency of $(z(s)\Sigma_2, \Xi_1)$ and $(\Sigma_2, z(s)\Xi_1)$ implies the well posedness of the latter. However, $z(s)\Xi_1$ is not proper, so we add a filter term to obtain the proper controller $M/(s+M)\, z(s)\Xi_1$. Now, we need to show that the filter term is small in some meaningful sense. If $\delta(z(s)\Sigma_2\,,\, M/(s+M)\, z(s)\Sigma_2) \to 0$ as $M \to \infty$, where $\delta(\cdot,\cdot)$ denotes the gap metric between two plants [81, 7], then $(\Sigma_2, z(s)\Xi_1)$ is well posed. A generalisation of Theorems **I**, **II** may be possible by similar reasonings.

- Generalisation of the result to strict feedback systems, via backstepping controllers.
- Establishing whether the same results can be given for the alternative costs, for example, $\mathcal{P} = \|x(\cdot)\|_{\mathcal{L}^\infty} + \|u(\cdot)\|_{\mathcal{L}^\infty}$.
- Whilst we have given our result in an $\mathcal{L}^\infty$ setting, we expect that many meaningful generalisations are possible with different signal norms. In particular, generalisation to LQ performance of control and state transient performance remain the subject of future work.
- The techniques developed in this thesis can be extended to comparison of many other modified algorithms e.g. σ-modification, relative dead-zone, etc. In fact, similar comparisons could be made for all manner of controller pairs, see e.g. [10].

Appendix A

Lemma A.1. Consider the closed loop $(\Sigma(x_0, \theta, d(\cdot)), \Xi)$ defined by (3.4),(3.10), where $d(\cdot) = \epsilon,$. Suppose there exists $L \in \mathbb{R}$ such that $\lim_{t\to\infty} \hat{\theta}(t)\, x(t) = L$. Then $L = \epsilon$.

Proof. Let g and g' be defined as follows:

$$g = \hat{\theta}x = \frac{x}{1/\hat{\theta}}, \qquad g' = \frac{\frac{d}{dt}x}{\frac{d}{dt}(1/\hat{\theta})}. \tag{A.1}$$

A relationship between g, g' can be obtain as follows:

$$g' = \frac{-\hat{\theta}^2(-ax + \theta x + \epsilon)}{x^2} + \frac{\hat{\theta}^2}{x^2}g, \tag{A.2}$$

therefore

$$g = \frac{-ax + \theta x + \epsilon}{1 - \frac{x^2}{\hat{\theta}^2}\frac{g'}{g}}. \tag{A.3}$$

Now since there exist L s.t. $\lim_{t\to\infty} g = L$ then by considering the Taylor series of g, g' (these exist since the r.h.s. of the equations (3.4),(3.10) are analytic, hence the solutions are analytic.) we have $\lim_{t\to\infty} g' = \lim_{t\to\infty} g = L$, i.e $\lim_{t\to\infty} g'/g = 1$. It follows that

$$\lim_{t\to\infty} g = \lim_{t\to\infty}\left(\frac{-ax + \theta x + \epsilon}{1 - \frac{x^2}{\hat{\theta}^2}\frac{g'}{g}}\right) = \frac{\lim_{t\to\infty}(-ax + \theta x + \epsilon)}{\lim_{t\to\infty}(1 - \frac{x^2}{\hat{\theta}^2}\frac{g'}{g})} = \frac{\epsilon}{1 - \lim_{t\to\infty}\frac{x^2}{\hat{\theta}^2}}. \tag{A.4}$$

Note that by Proposition 3.1, $\lim_{t\to\infty}\left(\frac{x^2}{\hat{\theta}^2}\right) = 0$. Hence

$$L = \lim_{t\to\infty} \hat{\theta}x = \lim_{t\to\infty} g = \epsilon. \tag{A.5}$$

□

Proposition A.1. Suppose $\rho = 1$ and C1,C2 hold. Consider the closed loop system $(\Sigma(x_0, \theta, d(\cdot)), \Xi)$ defined by (4.2), (4.24), where $d(t) = \epsilon$ for some $\epsilon \neq 0$. Then as $t \to \infty$

$$\|x(t)\| \to 0, \quad \hat{\delta}(t) \to \infty. \tag{A.6}$$

Proof (A. Ilchmann [21]). Consider (4.2) in the equivalent form

$$\begin{aligned} \dot{y}(t) &= \left(\bar{a}_1 - CB\hat{\delta}(t)\right) y(t) + \bar{A}_2 z(t) + CB\epsilon \\ \dot{z}(t) &= \bar{A}_3 y(t) + \bar{A}_4 z(t), \\ \dot{\hat{\delta}}(t) &= y(t)^2. \end{aligned} \tag{A.7}$$

(*i*) $\lim_{t\to\infty} \hat{\theta}(t) = \infty$.

Seeking a contradiction, assume $\hat{\theta}(t) \to \hat{\theta}^* \in \mathbb{R}$ as $t \to \infty$. Then (A.7) yields $y \in \mathcal{L}^2(0, \infty)$ and invoking asymptotic stability of $\bar{A}_4$ and (A.7) again gives $z \in \mathcal{L}^2(0, \infty)$. We may rewrite the first equation in (A.7) as

$$\dot{y}(t) = -y(t) + \psi(t) + CB\epsilon, \tag{A.8}$$

where

$$\psi(t) := [1 + \bar{a}_1 - CB\hat{\theta}(t)]y(t) + \bar{A}_2 z(t). \tag{A.9}$$

Since $\psi \in \mathcal{L}^2(0, \infty)$, by an application of Lemma 4.1 in [23][1], we conclude

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \left\{ e^{-t} y(0) + \int_0^t e^{-(t-s)} [\psi(t) + CB\epsilon] ds \right\} = CB\epsilon, \tag{A.10}$$

which contradicts $y \in \mathcal{L}^2(0, \infty)$.

(*ii*) $\lim_{t\to\infty} x(t) = 0$

This follows from Proposition 4.1.

□

[1]The convolution of an exponential with an $\mathcal{L}^2$-function gives a function which leads to 0.

Bibliography

[1] J. A. Aseltine, A. R. Mancini, and C.W. Sarture. A survey of adaptive control systems. *IRE Transactions on Automatic control*, 3:102–108, 1958.

[2] K. J. Åström and B. Wittenmark. *Adaptive Control.* Addison–Wesley, 1989.

[3] B. Brogliato and A. T. Neto. Practical stabilization of a class of nonlinear systems with partially known uncertainties. *Automatica*, 31(1):145–150, 1995.

[4] A. Datta. Transient performance improvement in continuous time model reference adaptive control: An $\mathcal{L}_1$ formulation. *In Proc. American Control Conference,San Francisco*, pages 294–295, 1993.

[5] R. R. Drenick and R. A. Shahbender. Adaptive servomechanisms. *AIEE Transactions*, 79:286–292, 1957.

[6] B. Egardt. *Stability of adaptive controllers.* Lecture notes in Control and Information Sciences, Springer–Verlag, New York, 1979.

[7] A. K. El-Sakkary. The gap metric: Robustness of stabilization of feedback systems. *IEEE Trans. on Automatic Control*, 30:240–247, 1985.

[8] A. F. Filippov. *Differential Equations with Discontinues Righthand Sides.* Kluwer Academic Publishers, 1988.

[9] M. French. Adaptive control and robustness in the gap metric. *preprint.*

[10] M. French. An analytical comparison between the nonsingular quadratic performance of robust and adaptive backstepping designs. *IEEE Trans. on Automatic Control*, 47(4):670–675, 2002.

[11] M. French, E. Rogers, and Cs. Szepesvári. Uncertainty, performance, and model dependency in approximate adaptive nonlinear control. *IEEE Trans. on Automatic Control*, 45(2):353–358, 2000.

[12] M. French, Cs. Szepesvári, and E. Rogers. LQ performance bounds for adaptive output feedback controllers for functionally uncertain nonlinear systems. *Automatica*, 38:683–693, 2002.

[13] B. Friedland. *Control system Design: An Introduction to State-Space Methods.* McGraw–Hill, 1986.

[14] L. C. Fu. A new approach to robust model reference adaptive control for a class of plants. *International Journal of Control*, 53:1359–1375, 1991.

[15] L. C. Fu. A new robust MRAC using variable structure design for relative degree two plants. *Automatica*, 28:911–925, 1992.

[16] T. T. Georgiou and M. C. Smith. Robustness analysis of nonlinear feedback systems: An input–output approach. *IEEE Trans. on Automatic Control*, 42(9):1200–1221, 1997.

[17] T. T. Georgiou and M. C. Smith. Remarks on 'robustness analysis of nonlinear feedback systems: An input–output approach'. *IEEE Trans. on Automatic Control*, 46(1):171–172, 2001.

[18] G. C. Goodwin and D. Q. Mayne. A parameter estimation perspective of continuous time model reference adaptive control. *Automatica*, 23, 1987.

[19] L. P. Grayson. Design via Lyapunov's second method. *In Proc. 4th Joint Conference on Automatic control*, pages 589–595, 1963.

[20] A. C. Hicks and S. Townley. On exact solutions of differential equations arising in universal adaptive stabilization. *Systems and Control Letters*, 20:117–129, 1993.

[21] A. Ilchmann. Private communication.

[22] A. Ilchmann. *Non–Identifier–Based High–Gain Adaptive Control.* Lecture notes in Control and Information Sciences, Springer–Verlag, London, 1993.

[23] A. Ilchmann and I.M.Y. Mareels. *On stability radii of slowley time–varying systems*, chapter in *Advances in mathematical system theory*, F. Colonius, U. Helmke, D. Pratzel–Wolters, and F. Wirth, Eds, pages 55–75. Verlag, Boston, 2000.

[24] A. Ilchmann and E. P. Ryan. Universal λ-tracking for nonlinearly perturbed systems in the presence of noise. *Automatica*, 30:337–346, 1994.

[25] P. A. Ioannou and P. V. Kokotovic. *Adaptive Systems with Reduced Models.* Springer-Verlag, New York, 1983.

[26] P. A. Ioannou and J. Sun. Theory and design of robust direct and indirect adaptive control schemes. *International Journal of Control*, 47:775–813, 1988.

[27] P. A. Ioannou and J. Sun. *Robust Adaptive Control.* Prentice Hall, 1996.

[28] A. Isidori. *Nonlinear Control Systems.* Spring–Verlag, Heidelberg, 1989.

[29] G. W. Johnson. Synthesis of control systems with stability constraints via the direct method of Lyapunov. *IEEE Trans. on Automatic Control*, 9:270–273, 1964.

[30] H. K. Khalil. *Nonlinear Systems.* Macmillan, 1992.

[31] P. Kokotovic and M. Arcak. Constructive nonlinear control: A historical perspective. *Automatica*, 37:637–662, 2001.

[32] E. B. Kosmatopoulos and P. A. Ioannou. A switching adaptive controller for feedback linearizable systems. *IEEE Trans. on Automatic Control*, 44:742–750, 1999.

[33] G. Kreisselmeier and K. S. Narendra. Stable model reference adaptive control in the presence of bounded disturbances. *IEEE Trans. on Automatic Control*, 27(6):1169–1175, 1982.

[34] E. Kreyszig. *Introductory Functional Analysis with Applications.* John Wiley & Sons, 1978.

[35] M. Kristic, I. Kanellakopoulos, and P. Kokotovic. *Nonlinear and Adaptive Control Design.* John Wiley & Sons, New York, 1995.

[36] M. Krstic, I. Kanellakopoulos, and P. V. Kokotovic. Nonlinear design of adaptive controllers for linear systems. *IEEE Trans. on Automatic Control*, 39:738–752, 1994.

[37] M. Krstic, I. Kanellakopoulos, and P. V. Kokotovic. Passivity and parametric robustness of a new class of adaptive systems. *Automatica*, 30:1703–1716, 1994.

[38] M. Krstic, P. V. Kokotovic, and I. Kanellakopoulos. Transient performance improvement with a new class of adaptive controllers. *Systems and Control Letters*, 21:451–461, 1993.

[39] I. D. Landau. *Adaptive Control: The Model Reference Approach.* Dekker, New York, 1979.

[40] Y. S. Ledyaev and E. Sontag. A remark on robust stabilization of general asymptotically controllable systems. *Proc. Conf. on Information Sciences and Systems (CISS 97)*, pages 246–251, 1997.

[41] H. Logemann and S. Townley. Adaptive stabilization without identification for distributed parameter systems: An overview. *IMA Journal Mathematic Control and Information*, 20:175–206, 1997.

[42] R. Lozano, J. Collado, and S. Mondie. Model reference robust adaptive control without a–priori knowledge of the high frequency gain. *IEEE Trans. on Automatic Control*, 35:71–78, 1990.

[43] B. Mårtensson. The order of any stabilizing regulator is sufficient a priori information for adaptive stabilization. *Systems and Control Letters*, 6:87–91, 1985.

[44] B. Mårtensson. Remarks on adaptive stabilization of first order nonlinear systems. *Systems and Control Letters*, 14:1–7, 1990.

[45] R. H. Middleton, G. C. Goodwin, D. J. Hil, and D. Q. Mayne. Design issues in adaptive control. *IEEE Trans. on Automatic Control*, 33(1):50–58, 1988.

[46] R. K. Miller and A. N. Michel. *Ordinary Differential Equations.* Academic Press,, New York, 1982.

[47] R. V. Monopoli. Synthesis techniques employing the direct method. *IEEE Trans. on Automatic Control*, 10:369–370, 1965.

[48] R. V. Monopoli. Discussion on two theorem on the second method. *IEEE Trans. on Automatic Control*, 11:140–141, 1966.

[49] A. S. Morse. *An Adaptive Control for Globally Stabilizing Linear Systems with Unknown High-frequency Gain.* Springer–Verlag, Berlin, 1984.

[50] A. S. Morse. High–order parameter tuners for the adaptive control of nonlinear systems. *In Proc. US–Italy Joint Seminar Systems, Models Feedback Theorem Applications, Italy*, 1992.

[51] A. S. Morse, D. Q. Mayne, and G. C. Goodwin. Application of hysteresis switching in parameter adaptive control. *IEEE Trans. on Automatic Control*, 37:1343–1354, 1992.

[52] S. Morse. Supervisory control of families of linear set–point controllers –part 1: Exact matching. *IEEE Trans. on Automatic Control*, 41:1413–1431, 1996.

[53] K. S. Narendra and A. M. Annaswamy. Robust adaptive control in the presence of bounded disturbance. *IEEE Trans. on Automatic Control*, 31:306–216, 1986.

[54] K. S. Narendra and A. M. Annaswamy. *Stable Adaptive Systems.* Prentice–Hall, 1989.

[55] K. S. Narendra and J. D. Boskovic. A combined direct, indirect, and variable structure method for robust adaptive control. *IEEE Trans. on Automatic Control*, 1992.

[56] R. D. Nussbaum. Some remarks on a conjecture in parameter adaptive control. *Systems and Control Letters*, 3:243–246, 1983.

[57] R. Ortega. On Morse's new adaptive controller: Parameter convergence and transient performance. *IEEE Trans. on Automatic Control*, 38:1191–1202, 1993.

[58] F. M. pait and S. Morse. A cyclic switching strategy for parameter adaptive control. *IEEE Trans. on Automatic Control*, 39:1172–1183, 1994.

[59] P. C. Parks. Lyapunov redesign of model reference adaptive systems. *IEEE Trans. on Automatic Control*, 11:362–365, 1966.

[60] B. B. Peterson and K. S. Narendra. Bounded error adaptive control. *IEEE Trans. on Automatic Control*, 27(6):1161–1168, 1982.

[61] M. M. Polycarpou and P. A. Ioannou. On the existence and uniqueness of solutions in adaptive control systems. *IEEE Trans. on Automatic Control*, 38(3):474–479, 1993.

[62] J. B. Pomet and L. Praly. Adaptive nonlinear regulation: Estimation from the Lyapunov equation. *IEEE Trans. on Automatic Control*, 37(6):729–740, 1992.

[63] L. Praly. Robustness of indirect adaptive control based on pole placement design. *IFAC workshop on adaptive control, CA*, 1983.

[64] C. Rohrs, L. Valavani, M. Athans, and G. Stein. Robustness of continuoustime adaptive control algorithms in the presence of unmodeled dynamics. *IEEE Trans. on Automatic Control*, 30:881–889, 1985.

[65] E. P. Ryan. A universal adaptive stabilizer for class of nonlinear systems. *Systems and Control Letters*, 16:209–218, 1991.

[66] C. Samson. Stability analysis of adaptively controlled systems subject to bounded disturbances. *Automatica*, 19(1):81–86, 1983.

[67] A. Sanei and M. French. A non–singular performance comparison between two robust adaptive control designs. *In Proc. 40th IEEE Conference on Decision and Control*, pages 2043–2048, 2001.

[68] A. Sanei and M. French. A comparison between robust adaptive controllers w.r.t. a non–singular transient cost. *In Proc. 15th International Symposium on Mathematical Theory of Networks and System(MTNS)*, CD-ROM, 2002.

[69] S. Sastry. *Nonlinear Systems: Analysis, Stability and Control.* Springer–Verlag, New York, 1999.

[70] S. Sastry and M. Bodson. *Adaptive Control: Stability, Convergence, and Robustness.* Prentice–Hall, 1989.

[71] P. R. Stromer. Adaptive or self–optimizing control systems – a bibliography. *IRE Transactions on Automatic control*, 4:65–68, 1959.

[72] J. Sun. A modified model reference adaptive control scheme for improved transient performance. *IEEE Trans. on Automatic Control*, 38:1255–1259, 1993.

[73] S. Townley. Generic properties of universal adaptive stabilization schemes. *In Proc. 31st IEEE Conference on Decision and Control*, pages 3630–3631, 1992.

[74] S. Townley. Topological aspects of universal adaptive stabilization. *SIAM Journal Control and Optimization*, 34:1044–1070, 1996.

[75] V. I. Utkin. Variable structure systems with sliding modes. *IEEE Trans. on Automatic and Control*, 22:212–222, 1977.

[76] M. Vidyasagar. *Nonlinear Systems Analysis.* Prentice–Hall, 1993.

[77] L. Wang and D. R. Mudgett. Improvement of transient response in adaptive control using modified high order tuning. *In Proc. American Control Conference, San Francisco,*, pages 282–286, 1993.

[78] A. J. Weir. *Lebesgue Integration and Measure.* Cambridge University Press, 1991.

[79] S. R. Weller and G. C. Goodwin. Hysteresis switching adaptive control of linear multivariable systems. *IEEE Trans. on Automatic Control*, 39:1360–1375, 1994.

[80] J. C. Willwms and C. I. Byrnes. Global adaptive stabilization in the absence of information on the sign of the high–frequency gain. *In Lecture Notes in control and information science, Springer–Verlag, Berlin*, 62:49–57, 1984.

[81] G. Zames and A. K. El-Sakkary. Unstable systems and feedback: The gap metric. *In Proc. Allerton conference*, pages 380–385, 1980.

[82] Z. Zang and R. R. Bitmead. Transient bounds for adaptive control systems. *In Proc. 30th IEEE Conference on Decision and Control*, pages 171–175, 1990.

[83] Y. Zhang and P. A. Ioannou. A new linear adaptive controller: Design, analysis and performance. *IEEE Trans. on Automatic Control*, 45:883–897, 2000.

[84] F. Zhao and V. I. Utkin. Adaptive simulation and control of variable–structure control system in sliding regimes. Technical Report OSU–CISRC–3/95–TR10, The Ohio State University, 1995.

[85] K. Zhou, J. C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice–Hall, 1996.