D-branes, KK-theory and duality on noncommutative spaces

J. Brodzki[(a)], **V. Mathai**[(b)], **J. Rosenberg**[(c)] **and R.J. Szabo**[(d)]

[(a)] School of Mathematics, University of Southampton
Southampton SO17 1BJ, UK
[(b)] Department of Pure Mathematics, University of Adelaide
Adelaide 5005, Australia
[(c)] Department of Mathematics, University of Maryland
College Park, MD 20742, USA
[(d)] Department of Mathematics and
Maxwell Institute for Mathematical Sciences
Heriot-Watt University, Riccarton, Edinburgh EH14 4AS, UK

E-mail: j.brodzki@soton.ac.uk , mathai.varghese@adelaide.edu.au , jmr@math.umd.edu
R.J.Szabo@ma.hw.ac.uk

Abstract. We present a new categorical classification framework for D-brane charges on noncommutative manifolds using methods of bivariant K-theory. We describe several applications including an explicit formula for D-brane charge in cyclic homology, a refinement of open string T-duality, and a general criterion for cancellation of global worldsheet anomalies.

1. Introduction

This article centres in part around the following physical question: *What is a D-brane?* More precisely, given a closed string background X, what are the possible states of D-branes in X? At the level of the worldsheet field theory of open strings in X, this is a problem of finding the consistent boundary conditions in the underlying boundary conformal field theory. As we will discuss, when this worldsheet perspective is combined with the target space classification of D-branes in terms of Fredholm modules over a suitable C^*-algebra, a powerful categorical description of D-branes and their charges emerges. This is particularly useful for those boundary states which have no geometric description. In certain instances these "non-geometric" backgrounds can be interpreted as noncommutative manifolds, *i.e.*, as separable noncommutative C^*-algebras. The formalism that we review in the following was developed in detail in refs. [1, 2], and it allows for the construction of general charge vectors for D-branes on these noncommutative spaces.

This point of view becomes particularly fruitful for considerations involving compactifications with H-flux. Consider, for example, a principal torus bundle $X \to M$ with constant H-flux. Applying a T-duality transformation along the fibre gives a space which does not always admit a global riemannian description. Instead, one can double the dualized directions and use elements of the T-duality group as transition functions between local patches. This is called a "T-fold" [3]. In some examples, one can show [4] that the *open* string metric on a T-fold is precisely the metric on an associated continuous field of stabilized noncommutative tori fibred over M which corresponds to a certain crossed product C^*-algebra [5, 6]. Thus the open string version of a T-fold can be generally regarded as a globally defined, noncommutative C^*-algebra. Performing additional T-duality transformations along the base leads in some instances to nonassociative tori in the fibre directions [7]–[9]. This example, wherein the action of T-duality is realized by taking a certain crossed product algebra, motivates an axiomatic definition of topological open string T-duality. This generalizes and refines the more common examples of T-duality between noncommutative spaces in terms of Morita equivalence [10] to a special type of "KK-equivalence", which defines a T-duality action that is of order two up to Morita equivalence.

From a purely mathematical perspective of noncommutative geometry, the framework needed to achieve the physical constructions above develops more tools for dealing with noncommutative spaces in general. These include the appropriate noncommutative versions of Poincaré duality and orientation, topological invariants of noncommutative spaces such as the Todd genus, and a noncommutative version of the Grothendieck-Riemann-Roch theorem which is directly linked to the general formula for D-brane charge. All of this is defined and developed in the purely algebraic framework of separable C^*-algebras.

2. D-branes and K-theory

It is well-known that D-brane charges and Ramond-Ramond fields in Type II superstring theory without H-flux are classified topologically by the complex K-theory of spacetime X [11]–[17]. We will begin by briefly reviewing some salient features of this classification that we will generalize later on to more generic noncommutative settings.

2.1. A simple observation

Let X be a compact spin^c manifold. Poincaré duality in cohomology states that the natural bilinear pairing

$$(x,y)_{\rm H} = \big\langle x \smile y \,,\, [X]\big\rangle \tag{2.1}$$

between cohomology classes x, y of X in complementary degree is non-degenerate. If α, β are de Rham representatives of x, y, then this pairing is just $(x,y)_{\rm H} = \int_X \alpha\wedge\beta$. On the other hand, in K-theory the natural bilinear pairing between complex vector bundles $E, F \to X$ is given by the index of the twisted Dirac operator

$$(E,F)_{\rm K} = {\rm index}\big(\not{D}_{E\otimes F}\big) \tag{2.2}$$

associated to the spin^c structure on X. The Chern character gives a natural, $\mathbb{Z}_2$-graded ring isomorphism

$${\rm ch} \,:\, {\rm K}(X)\otimes\mathbb{Q} \xrightarrow{\approx} {\rm H}(X,\mathbb{Q}) \tag{2.3}$$

but it doesn't preserve these bilinear forms. However, by the Atiyah-Singer index theorem one has

$${\rm index}\big(\not{D}_{E\otimes F}\big) = \big\langle {\rm ch}(E\otimes F) \smile {\rm Todd}(X)\,,\,[X]\big\rangle \ , \tag{2.4}$$

so we get an isometry by replacing the isomorphism (2.3) with the "twisted" Chern character

$${\rm ch} \ \longrightarrow\ \sqrt{{\rm Todd}(X)} \smile {\rm ch} \ . \tag{2.5}$$

Here ${\rm Todd}(X) \in {\rm H}(X,\mathbb{Q})$ is the invertible Todd characteristic class of the tangent bundle of X, which can be expressed in terms of the Pontrjagin classes of X along with a degree two characteristic class $c_1 \in {\rm H}^2(X,\mathbb{Z})$ whose reduction modulo 2 is the second Stiefel-Whitney class $w_2(X)$. This almost trivial observation plays a crucial role in what follows.

2.2. D-brane charges

A natural geometric description of a D-brane in X is provided by a topological K-cycle (W, E, f) in X [18]–[22], where $f : W \hookrightarrow X$ is a closed, embedded spin^c submanifold of X (the brane worldvolume), and $E \to W$ is the Chan-Paton gauge bundle equipped with a hermitean connection and regarded as an element of the topological K-theory group $\mathrm{K}^0(W)$. The collection of K-cycles forms an additive category under disjoint union. The quotient of this category of D-branes by Baum-Douglas "gauge" equivalence [23] is isomorphic to the K-homology of X, defined as the group of stable homotopy classes of *Fredholm modules* over the commutative C^*-algebra $\mathcal{A} = C(X)$ of continuous functions on X. The isomorphism is generated by associating to a K-cycle (W, E, f) the (unbounded) Fredholm module $(\mathcal{H}, \rho, D\!\!\!/_E^{(W)})$, where $\mathcal{H} = L^2(W, S \otimes E)$ with $S \to W$ the spinor bundle over the D-brane worldvolume, the $*$-representation $\rho(\phi) = m_{\phi\circ f}$ of $\phi \in \mathcal{A}$ on the separable Hilbert space $\mathcal{H}$ is given by pointwise multiplication by the function $\phi \circ f$, and $D\!\!\!/_E^{(W)}$ is the Dirac operator associated to the spin^c structure on W.

It follows that D-branes naturally provide K-homology classes on X. They are dual to K-theory classes $f_!(E) \in \mathrm{K}^d(X)$, where $f_!$ is the K-theoretic Gysin pushforward map and d is the codimension of W in X. The *Ramond-Ramond charge* of a D-brane supported on W with Chan-Paton bundle $E \in \mathrm{K}^0(W)$ is the element of $\mathrm{H}(X, \mathbb{Q})$ given by

$$Q(W, E) = \mathrm{ch}\big(f_!(E)\big) \smile \sqrt{\mathrm{Todd}(X)} \ . \tag{2.6}$$

This is known as the *Minasian-Moore formula* [11]. One of our goals in the following will be to generalize this construction to generic noncommutative settings.

3. D-branes and bivariant K-theory

We will now propose a powerful categorical classification of D-branes which extends the descriptions provided by K-theory and K-homology in a unified manner. Our proposal is motivated by the structure of the open string algebras and bimodules that arise in the underlying worldsheet boundary conformal field theory, which enable us to treat the collection of allowed D-brane boundary conditions as a certain category.

3.1. D-brane categories

Open string fields define relative maps $(\Sigma, \partial\Sigma) \to (X, W)$ from an oriented Riemann surface Σ with boundary $\partial\Sigma$. Not all maps are allowed. They are constrained by the requirements of worldsheet conformal and modular invariance (such as the Cardy conditions), as well as by cancellation of global worldsheet anomalies. These constraints are viewed as equations of motion in the underlying boundary conformal field theory. For example, in Type II superstring theory in the absence of H-flux, this is just our previous requirement that the worldvolume W be a spin^c manifold. Classically, this means that a D-brane may be regarded as a suitable boundary condition in the boundary conformal field theory. It is not presently known what is meant generally by a "quantum D-brane" in the underlying quantum boundary conformal field theory. In the following we will propose an algebraic characterization of quantum D-branes in the context of separable C^*-algebras.

The crucial observation is that the concatenation of open string vertex operators defines algebras and bimodules. We take $\Sigma = \mathbb{R} \times I$, where $\mathbb{R}$ parametrizes the time evolution and the interval $I = [0, 1]$ parametrizes the space coordinates of the open strings in X. Let us label the allowed D-brane boundary conditions by $a, b, \dots$. An a-b open string has a boundary conditions at its $t = 0$ end and b boundary conditions at its $t = 1$ end. The set of a-a open strings forms a noncommutative algebra $\mathcal{D}_a$ of open string fields as the vertex operator algebra of observables in the boundary conformal field theory. The opposite algebra $\mathcal{D}_a^{\rm o}$, *i.e.*, the algebra with the same underlying vector space as $\mathcal{D}_a$ but with the product reversed, is obtained by reversing the orientations of the a-a open strings. The set of a-b boundary conditions, on the other hand, forms a $\mathcal{D}_a$-$\mathcal{D}_b$ bimodule $\mathcal{E}_{ab}$. The dual bimodule $\mathcal{E}_{ab}^\vee = \mathcal{E}_{ba}$ is obtained by reversing the orientations of the a-b open strings. Note that $\mathcal{E}_{aa} = \mathcal{D}_a$ is the trivial $\mathcal{D}_a$-bimodule obtained by letting $\mathcal{D}_a$ act on itself by multiplication from the left and from the right.

We would now like to define an additive category whose objects are the D-brane algebras $\mathcal{D}_a$, and the morphisms between any two objects $\mathcal{D}_a, \mathcal{D}_b$ is precisely the open string bimodule $\mathcal{E}_{ab}$. This means that for any three boundary conditions a, b, c there should be a $\mathbb{C}$-bilinear map

$$\mathcal{E}_{ab} \times \mathcal{E}_{bc} \longrightarrow \mathcal{E}_{ac} \tag{3.1}$$

which defines the associative composition law in this category. A natural guess for this map is the canonical open string vertex which combines an a-b open string field with a b-c open string field into an a-c open string field. However, the operator product expansion on the underlying open string vertex operator algebras is not generally associative, and in general does not even lead to a well-defined map (3.1) [2]. We therefore need some other way to define and compose the morphisms between the objects $\mathcal{D}_a$ of our category.

3.2. *Seiberg-Witten limit*

An example of a situation in which the assignment (3.1) is well-defined was worked out by Seiberg and Witten [10] (see also ref. [17]) in the case of open string boundary conditions of maximal support on an n-torus $X = \mathbb{T}^n$ with a constant B-field. The Seiberg-Witten limit of the boundary conformal field theory in X amounts to simultaneously sending both the string tension T and the B-field to infinity whilst keeping their ratio B/T a finite constant. One also needs to scale the closed string metric g to 0. This low-energy limit keeps only zero modes of the string fields. Quantization of the point particle at the endpoint of an a-a open string gives a Hilbert space $\mathcal{H}_a$ which is a module for the noncommutative C^*-algebra of a noncommutative torus $\mathcal{D}_a$. The algebra $\mathcal{D}_a \otimes \mathcal{D}_b$ acts irreducibly on the Hilbert space $\mathcal{E}_{ab} = \mathcal{H}_a \otimes \mathcal{H}_b^\vee$, and the map (3.1) in this case is given by

$$V_{ac}(t') = \lim_{t \to t'} V_{ab}(t) \cdot V_{bc}(t') \ . \tag{3.2}$$

Here $V_{ab}(t)$, $t \in I$ are the open string vertex operators for the boundary conditions labelled by a, b, and the product in eq. (3.2) is the operator product expansion taken in the Seiberg-Witten limit. The map (3.1) is now well-defined as the conformal dimensions of all vertex operators, being proportional to g/T, vanish in the limit [2]. In addition, the operator product expansion (3.2) is associative in the limit, and hence the map (3.1) extends to a map

$$\mathcal{E}_{ab} \otimes_{\mathcal{D}_b} \mathcal{E}_{bc} \longrightarrow \mathcal{E}_{ac} \ . \tag{3.3}$$

Because the noncommutative algebras $\mathcal{D}_a$ contain the complete set of observables for boundary conditions of maximal support, they act irreducibly on the quantum mechanical Hilbert spaces and there are natural identifications

$$\mathcal{D}_a \cong \mathcal{E}_{ab} \otimes_{\mathcal{D}_b} \mathcal{E}_{ba} \qquad \text{and} \qquad \mathcal{D}_b \cong \mathcal{E}_{ba} \otimes_{\mathcal{D}_a} \mathcal{E}_{ab} \ . \tag{3.4}$$

These relations mean that the open string bimodule $\mathcal{E}_{ab}$ is a *Morita equivalence bimodule*, expressing a *T-duality* between the noncommutative tori $\mathcal{D}_a$ and $\mathcal{D}_b$ [10, 17].

3.3. *KK-theory*

Motivated by the situation described by the Seiberg-Witten limit of boundary conformal field theory, we will assume that there is a suitable extension or "deformation" of the open string bimodule $\mathcal{E}_{ab}$ to a *Kasparov bimodule* $(\mathcal{E}_{ab}, F_{ab})$. In the Seiberg-Witten limit described in Section 3.2 above, *i.e.*, when $\mathcal{E}_{ab}$ is a Morita equivalence bimodule, this is a "trivial" bimodule $(\mathcal{E}_{ab}, 0)$. The Kasparov bimodules $(\mathcal{E}_{ab}, F_{ab})$ generalize Fredholm modules and their stable homotopy classes define the $\mathbb{Z}_2$-graded *bivariant K-theory* or *KK-theory* group $\mathrm{KK}_\bullet(\mathcal{D}_a, \mathcal{D}_b)$. Elements of this group may be regarded as "generalized" morphisms $\mathcal{D}_a \to \mathcal{D}_b$ between separable C^*-algebras. More precisely, there is an additive category whose objects are separable C^*-algebras $\mathcal{A}$ and whose morphisms between two objects $\mathcal{A}, \mathcal{B}$ are exactly the elements of $\mathrm{KK}_\bullet(\mathcal{A}, \mathcal{B})$. In particular, if $\phi : \mathcal{A} \to \mathcal{B}$ is $*$-homomorphism of C^*-algebras, then there is a canonically defined element $[\phi] \in \mathrm{KK}(\mathcal{A}, \mathcal{B})$ which is represented by the "Morita-type" bimodule $(\mathcal{B}, \phi, 0)$. This categorical point of view enables one to uniquely characterize the bivariant K-theory groups by the properties of homotopy invariance, stability under compact perturbation, and split exactness on the category of separable C^*-algebras and $*$-homomorphisms [24].

The KK-category is the one we shall take as our toy model for a "category of D-branes". It is not an abelian category, but it admits the structure of a triangulated category [25] and hence has properties similar to the more commonly used categories of topological D-branes [26, 27]. We assume that, at least under certain circumstances, it is the appropriate category for dealing with quantum D-branes for energies outside the classical regime of the infinite tension limit, along the lines suggested in ref. [17]. Further properties of this category are discussed below. The groups $\mathrm{KK}_\bullet(\mathcal{A}, \mathcal{B})$ unify the K-theory and K-homology of C^*-algebras, which arise as special cases. When $\mathcal{A} = \mathbb{C}$ the group $\mathrm{KK}_\bullet(\mathbb{C}, \mathcal{B}) = \mathrm{K}_\bullet(\mathcal{B})$ is the K-theory of $\mathcal{B}$. On the other hand, when $\mathcal{B} = \mathbb{C}$ the group $\mathrm{KK}_\bullet(\mathcal{A}, \mathbb{C}) = \mathrm{K}^\bullet(\mathcal{A})$ is the K-homology of $\mathcal{A}$, as in this case a Kasparov bimodule is the same thing as a Fredholm module over the algebra $\mathcal{A}$.

3.4. Intersection product

Although the groups $\mathrm{KK}_\bullet(\mathcal{A}, \mathcal{B})$ naturally incorporate both the K-theory and K-homology classifications of D-branes, the bivariant version of K-theory is much more powerful than K-theory or K-homology alone. This is due to the existence of the bilinear, associative *intersection product*

$$\otimes_{\mathcal{B}} : \mathrm{KK}_i(\mathcal{A}, \mathcal{B}) \times \mathrm{KK}_j(\mathcal{B}, \mathcal{C}) \longrightarrow \mathrm{KK}_{i+j}(\mathcal{A}, \mathcal{C}) . \tag{3.5}$$

The definition of this product is notoriously difficult. In Section 5 we will see how to describe it explicitly on the category of smooth manifolds. The product (3.5) is compatible with composition of morphisms. If $\phi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{B} \to \mathcal{C}$ are $*$-homomorphisms of separable C^*-algebras, then

$$[\phi] \otimes_{\mathcal{B}} [\psi] = [\psi \circ \phi] . \tag{3.6}$$

The intersection product makes the group $\mathrm{KK}_0(\mathcal{A}, \mathcal{A})$ into a ring with unit $1_{\mathcal{A}} = [\mathrm{id}_{\mathcal{A}}]$.

Any fixed element $\alpha \in \mathrm{KK}_d(\mathcal{A}, \mathcal{B})$ determines homomorphisms in K-theory and K-homology by left and right multiplication

$$\otimes_{\mathcal{A}} \alpha : \mathrm{K}_j(\mathcal{A}) \longrightarrow \mathrm{K}_{j+d}(\mathcal{B}) \qquad \text{and} \qquad \alpha \otimes_{\mathcal{B}} : \mathrm{K}^j(\mathcal{B}) \longrightarrow \mathrm{K}^{j+d}(\mathcal{A}) . \tag{3.7}$$

If α is *invertible*, *i.e.*, if there exists an element $\beta \in \mathrm{KK}_{-d}(\mathcal{B}, \mathcal{A})$ such that $\alpha \otimes_{\mathcal{B}} \beta = 1_{\mathcal{A}}$ and $\beta \otimes_{\mathcal{A}} \alpha = 1_{\mathcal{B}}$, then the maps (3.7) induce isomorphisms

$$\mathrm{K}_j(\mathcal{A}) \cong \mathrm{K}_{j+d}(\mathcal{B}) \qquad \text{and} \qquad \mathrm{K}^j(\mathcal{B}) \cong \mathrm{K}^{j+d}(\mathcal{A}) . \tag{3.8}$$

In this case the algebras $\mathcal{A}$ and $\mathcal{B}$ are said to be *KK-equivalent*. For example, by eq. (3.4) it follows that a Morita equivalence implies a KK-equivalence with invertible class $\alpha = [(\mathcal{E}_{ab}, 0)]$. The converse, however, is not generally true.

The Kasparov intersection product defines the associative composition law in the KK-category. It also yields the additional structure of a tensor category with multiplication bifunctor given by the spatial tensor product on objects, the external Kasparov product on morphisms, and with identity object the one-dimensional C^*-algebra $\mathbb{C}$. More precisely, this data defines a "relaxed" or "weak" monoidal category whereby associativity of the tensor product only holds up to natural isomorphism. A diagrammatic calculus in this tensor category was introduced in ref. [1] and developed more extensively in ref. [2].

4. Duality and worldsheet anomaly cancellation

We will now apply KK-theory to formulate the notion of noncommutative Poincaré duality, introduced originally by Connes [28]. This can be applied to formulate target space consistency conditions on noncommutative D-branes represented by generic separable C^*-algebras, and hence it selects the consistent sets of D-branes from the KK-category. Moreover, it implies that the K-theory and K-homology classifications of D-branes are equivalent.

4.1. Poincaré duality

Let $\mathcal{A}$ be a separable C^*-algebra and $\mathcal{A}^{\rm o}$ its opposite algebra. We say that $\mathcal{A}$ is a *Poincaré duality (PD) algebra* if there exists a *fundamental class* $\Delta \in {\rm KK}_d(\mathcal{A}\otimes\mathcal{A}^{\rm o},\mathbb{C}) = {\rm K}^d(\mathcal{A}\otimes\mathcal{A}^{\rm o})$ in K-homology with an inverse class $\Delta^\vee \in {\rm KK}_{-d}(\mathbb{C},\mathcal{A}\otimes\mathcal{A}^{\rm o}) = {\rm K}_{-d}(\mathcal{A}\otimes\mathcal{A}^{\rm o})$ in K-theory such that

$$\begin{aligned} \Delta^\vee \otimes_{\mathcal{A}^{\rm o}} \Delta &= 1_{\mathcal{A}} \in {\rm KK}_0(\mathcal{A},\mathcal{A}) \ , \\ \Delta^\vee \otimes_{\mathcal{A}} \Delta &= (-1)^d \, 1_{\mathcal{A}^{\rm o}} \in {\rm KK}_0(\mathcal{A}^{\rm o},\mathcal{A}^{\rm o}) \ . \end{aligned} \tag{4.1}$$

The opposite algebra is used in this definition to describe $\mathcal{A}$-bimodules as $(\mathcal{A}\otimes\mathcal{A}^{\rm o})$-modules, and the sign in eq. (4.1) depends on the orientation of the Bott element. This data determines inverse isomorphisms

$${\rm K}_i(\mathcal{A}) \xrightarrow{\otimes_{\mathcal{A}}\Delta} {\rm K}^{i+d}(\mathcal{A}^{\rm o}) = {\rm K}^{i+d}(\mathcal{A}) \qquad \text{and} \qquad {\rm K}^i(\mathcal{A}) = {\rm K}^i(\mathcal{A}^{\rm o}) \xrightarrow{\Delta^\vee\otimes_{\mathcal{A}^{\rm o}}} {\rm K}_{i-d}(\mathcal{A}) \tag{4.2}$$

between the K-theory and K-homology of the algebra $\mathcal{A}$. More generally, by replacing $\mathcal{A}^{\rm o}$ everywhere in the above by another separable C^*-algebra $\mathcal{B}$ gives the notion of *PD pairs* $(\mathcal{A},\mathcal{B})$. The moduli space of fundamental classes of a given algebra $\mathcal{A}$ is isomorphic to the group of invertible elements in the unital ring ${\rm KK}_0(\mathcal{A},\mathcal{A})$ [1]. This space is in general larger than the set of all K-orientations or "spin^c structures" discussed below.

A simple example of a PD pair $(\mathcal{A},\mathcal{B})$ is provided by taking $\mathcal{A}=C_0(X)$ to be the algebra of continuous functions vanishing at infinity on a complete oriented manifold X, and either $\mathcal{B}=C_0(T^*X)$ or $\mathcal{B}=C_0(X,{\rm Cliff}(T^*X))$ where ${\rm Cliff}(T^*X)$ is the Clifford algebra bundle of the cotangent bundle over X. The fundamental class Δ in this case is given by the Dirac operator constructed on ${\rm Cliff}(T^*X)$. When X is spin^c, $\mathcal{A}$ is itself a PD algebra with fundamental class Δ the spin^c Dirac operator $\not{D}$ induced on the diagonal of $X\times X$, and $\Delta^\vee$ is the Bott element. The two-dimensional noncommutative tori $\mathbb{T}^2_\theta$ are examples of noncommutative PD algebras [1, 28].

4.2. K-orientation

Suppose that $f:\mathcal{A}\to\mathcal{B}$ is a $*$-homomorphism of separable C^*-algebras in a suitable category. Then a *K-orientation* for f is a functorial way of associating an element $f! \in {\rm KK}_d(\mathcal{B},\mathcal{A})$. This determines a *Gysin* or "*wrong way*" *homomorphism* by right multiplication

$$f_! = \otimes_{\mathcal{B}} f! \, : \, {\rm K}_\bullet(\mathcal{B}) \longrightarrow {\rm K}_{\bullet+d}(\mathcal{A}) \ . \tag{4.3}$$

If $\mathcal{A}$ and $\mathcal{B}$ are both PD algebras, then any morphism $f:\mathcal{A}\to\mathcal{B}$ is K-oriented with K-orientation

$$f! = (-1)^{d_{\mathcal{A}}} \, \Delta^\vee_{\mathcal{A}} \otimes_{\mathcal{A}^{\rm o}} [f^{\rm o}] \otimes_{\mathcal{B}^{\rm o}} \Delta_{\mathcal{B}} \tag{4.4}$$

and $d = d_{\mathcal{A}} - d_{\mathcal{B}}$. We have used the fact that the involution $\mathcal{A} \to \mathcal{A}^{\rm o}$ on the stable homotopy category of C^*-algebras passes to the KK-category, and $[f^{\rm o}]$ is the KK-class of the $*$-homomorphism $f^{\rm o} : \mathcal{A}^{\rm o} \to \mathcal{B}^{\rm o}$ defined by $f^{\rm o}(x^{\rm o}) = f(x)^{\rm o}$ for $x \in \mathcal{A}$. The functoriality of the construction (4.4), *i.e.*, that $g! \otimes_{\mathcal{B}} f! = (g \circ f)!$ for any other morphism $g : \mathcal{B} \to \mathcal{C}$ of PD algebras, follows by associativity of the Kasparov intersection product.

For example, let $f : W \hookrightarrow X$ be a smooth proper embedding of codimension d between smooth compact manifolds such that the normal bundle $\nu = f^*(TX)/TW$ over W in X is spin^c. Then a K-orientation for f is determined by the element

$$f! = i^W! \otimes_{C_0(\nu)} j! \tag{4.5}$$

of $\mathrm{KK}_d(C(W), C(X))$, where $i^W!$ is the invertible element of the KK-theory group $\mathrm{KK}_d(C(W), C_0(\nu))$ determined by the Thom isomorphism of the zero section embedding $i^W : W \hookrightarrow \nu$, and $j!$ is the element of $\mathrm{KK}_0(C_0(\nu), C(X))$ induced by the extension by zero. When X is spin^c, the spin^c condition on ν is equivalent to a spin^c structure on W and is just the Freed-Witten anomaly cancellation condition for a D-brane supported on W in Type II superstring theory on X without H-flux [15]. Thus any D-brane (W, E, f) in X determines a canonically defined KK-theory element $f! \in \mathrm{KK}(C(W), C(X))$. Our notion of K-orientation may be regarded in this way as a generalization of the Freed-Witten condition to more general (noncommutative) spacetime geometries and D-branes. For example, the construction of the K-orientation (4.5) can be extended to arbitrary smooth proper maps $f : W \to X$ between smooth manifolds for which the bundle $TW \oplus f^*(TX)$ over W is spin^c [2]. Again when X itself is spin^c, so is W and this corresponds to a D-brane wrapping a generally non-representable cycle in X associated to a generic Baum-Douglas K-cycle [21].

5. Open string T-duality

We will now apply our considerations thus far to give a very general, axiomatic description of T-duality in open string theory which refines the usual notions of topological T-duality. The formulation may be motivated by the correspondence picture for KK-theory, introduced originally for the KK-theory of manifolds by Baum, Connes and Skandalis [29], which provides an explicit description of the intersection product and of the KK-category itself.

5.1. Correspondences

Let X, Y be smooth manifolds. A *correspondence* is given by a diagram

$$\begin{array}{ccccc} & & (Z,E) & & \\ & {}^{f}\swarrow & & \searrow^{g} & \\ X & & & & Y \end{array} \tag{5.1}$$

where Z is a smooth manifold, E is a complex vector bundle over Z, $f : Z \to X$ is smooth and proper, and $g : Z \to Y$ is K-oriented. Any correspondence naturally defines a class $g_!(f^*(-) \otimes E)$ in the bivariant K-theory group $\mathrm{KK}(X, Y) := \mathrm{KK}(C_0(X), C_0(Y))$. This gives a geometrical realization of the analytic index for families of elliptic operators on X parametrized by Y. The collection of all correspondences for X, Y forms an additive category under disjoint union. The quotient of this category by the suitable notions of cobordism, direct sum and vector bundle modification is isomorphic to the KK-theory group $\mathrm{KK}(X, Y)$ [2]. When $Y = \mathrm{pt}$, this definition reduces to the Baum-Douglas K-homology of X. When $X = \mathrm{pt}$, it mimicks the Atiyah-Bott-Shapiro construction of D-brane charge as an element of the K-theory of spacetime Y [12, 14, 21].

A major advantage of the correspondence picture is that the intersection product on KK-theory, which is difficult to define in the analytic setting, is particularly simple in this geometric setting. It is the bilinear associative map

$$\otimes_M \ : \ \mathrm{KK}(X,M)\times\mathrm{KK}(M,Y) \ \longrightarrow \ \mathrm{KK}(X,Y) \tag{5.2}$$

which is defined by sending two correspondences

$$\begin{array}{ccccccc} & (Z_1,E_1) & & & (Z_2,E_2) & & \\ {}^{f}\swarrow & & \searrow^{g_M} & {}^{f_M}\swarrow & & \searrow^{g} & \\ X & & & M & & & Y \end{array} \tag{5.3}$$

to the correspondence

$$[Z,E]=[Z_1,E_1]\otimes_M[Z_2,E_2] \tag{5.4}$$

with $Z=Z_1\times_M Z_2$ and $E=E_1\boxtimes E_2$. This definition requires a transversality condition on the two maps f_M and g_M in order to ensure that the fibred product Z is a smooth manifold.

5.2. Fourier-Mukai transform

While the correspondence picture is mathematically useful because it gives a somewhat more precise meaning to the interpretation of KK-theory classes as “generalized morphisms”, its main physical appeal is that it strongly resembles a smooth analog of the Fourier-Mukai transform and is thus intimately related to T-duality [30]. We will now make this observation precise. Let M be a smooth manifold, and let $\mathbb{T}^n\cong\mathbb{R}^n/\mathbb{Z}^n$ be an n-dimensional torus. Let $\widehat{\mathbb{T}}^n$ be the dual n-torus, which is canonically isomorphic to the Picard group $\mathrm{Pic}^0(\mathbb{T}^n)$ of flat line bundles over $\mathbb{T}^n$. The Poincaré line bundle is the unique line bundle $\mathcal{P}_0$ over the product $\mathbb{T}^n\times\widehat{\mathbb{T}}^n$ such that for any point $\widehat{t}\in\widehat{\mathbb{T}}^n$ the restriction $(\mathcal{P}_0)_{\widehat{t}}$ to $\mathbb{T}^n\times\{\,\widehat{t}\,\}$ represents the element of $\mathrm{Pic}^0(\mathbb{T}^n)$ corresponding to $\widehat{t}$, and such that the restriction bundle $\mathcal{P}_0|_{\{0\}\times\widehat{\mathbb{T}}^n}$ is trivial.

Consider the diagram

$$\begin{array}{ccc} & \big(M\times\mathbb{T}^n\times\widehat{\mathbb{T}}^n\,,\,\mathcal{P}\big) & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ M\times\mathbb{T}^n & & M\times\widehat{\mathbb{T}}^n \end{array} \tag{5.5}$$

where p_1,p_2 are the natural projections and $\mathcal{P}$ is the pullback of the Poincaré line bundle to $M\times\mathbb{T}^n\times\widehat{\mathbb{T}}^n$ by projection. The (smooth) Fourier-Mukai transform is then the isomorphism of K-theory

$$T_! \ : \ \mathrm{K}^\bullet\big(M\times\mathbb{T}^n\big) \ \xrightarrow{\approx} \ \mathrm{K}^{\bullet+n}\big(M\times\widehat{\mathbb{T}}^n\big) \tag{5.6}$$

given by

$$T_!(-)=(p_2)_!\big(p_1^*(-)\otimes\mathcal{P}\big) \ . \tag{5.7}$$

It follows that *topological T-duality* is a correspondence, which may be described somewhat more explicitly as follows (see ref. [1] for more details).

By Rieffel’s imprimitivity theorem, there is a Morita equivalence

$$C_0(M\times\mathbb{T}^n)\rtimes\mathbb{R}^n \ \sim \ C_0(M)\otimes C^*(\mathbb{R}^n)\cong C_0\big(M\times\widehat{\mathbb{T}}^n\big) \tag{5.8}$$

where the locally compact abelian group $\mathbb{R}^n$ acts trivially on M and by translations on $\mathbb{R}^n/\mathbb{Z}^n$. By the Connes-Thom isomorphism, this then defines a *KK-equivalence*

$$\alpha \ \in \ \mathrm{KK}_n\big(M\times\mathbb{T}^n\,,\,M\times\widehat{\mathbb{T}}^n\big) \ . \tag{5.9}$$

The invertible element α may be interpreted analytically as the families Dirac operator, and its inverse is the Bott element. It follows that topological T-duality may be interpreted algebraically as taking a crossed product with the natural action of $\mathbb{R}^n$ on the C^*-algebra $C_0(M \times \mathbb{T}^n)$. By Takai duality, there is a Morita equivalence

$$\big(C_0(M \times \mathbb{T}^n) \rtimes \mathbb{R}^n\big) \rtimes \mathbb{R}^n \sim C_0(M \times \mathbb{T}^n) \tag{5.10}$$

and hence the duality is of order two up to Morita equivalence.

These constructions all generalize [1, 2] to the examples, discussed in Section 1, of D-branes in a spacetime X which is a principal torus bundle $X \to M$ in a constant H-flux background. In this case D-branes are classified by the twisted K-theory of X, defined as the K-theory of the continuous trace algebra $\mathcal{A} = CT(X,H)$. This is a noncommutative C^*-algebra with spectrum equal to X and Dixmier-Douady invariant equal to $H \in \mathrm{H}^3(X,\mathbb{Z})$. See ref. [2] for other noncommutative examples of correspondences.

5.3. Axiomatic T-duality

The interpretations of T-duality in the examples considered in Section 5.2 above lead to the following general, algebraic characterization of topological open string T-duality. Consider a suitable category of separable C^*-algebras, possibly equipped with some extra structure (such as the $\mathbb{R}^n$-actions used in Section 5.2 above), whose objects $\mathcal{A}$ are called *T-dualizable algebras* and satisfy the following properties:

(i) There exists a covariant functor $\mathcal{A} \mapsto T(\mathcal{A})$ sending the algebra $\mathcal{A}$ to the *T-dual* of $\mathcal{A}$;

(ii) There exists a functorial map $\mathcal{A} \mapsto \alpha_{\mathcal{A}} \in \mathrm{KK}(\mathcal{A}, T(\mathcal{A}))$ such that the class $\alpha_{\mathcal{A}}$ is a KK-equivalence; and

(iii) The algebras $\mathcal{A}$ and $T(T(\mathcal{A}))$ are Morita equivalent, and the class $\alpha_{\mathcal{A}} \otimes_{T(\mathcal{A})} \alpha_{T(\mathcal{A})}$ is the associated KK-equivalence.

This realization of T-duality as a particular functorial kind of involutive KK-equivalence gives a refinement of the more commonly used notion of topological T-duality at the level of K-theory alone [2]. This can be seen by noticing that, for C^*-algebras $\mathcal{A}, \mathcal{B}$ which are KK-equivalent to commutative C^*-algebras, there is a universal coefficient theorem presenting the abelian group $\mathrm{KK}_\bullet(\mathcal{A}, \mathcal{B})$ as an extension of $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_\bullet(\mathcal{A}), \mathrm{K}_\bullet(\mathcal{B}))$ by $\mathrm{Ext}_{\mathbb{Z}}(\mathrm{K}_{\bullet+1}(\mathcal{A}), \mathrm{K}_\bullet(\mathcal{B}))$ [31]. The characterization described here has the advantage that, unlike the non-geometric examples such as T-folds, our T-dual spacetimes are *globally* defined, at the cost of possibly being noncommutative.

6. D-brane charges on noncommutative spaces

We will finally give the noncommutative versions of the constructions of D-brane charges presented in Section 2. For this, we will first have to deal with some topics of independent mathematical interest, which are concerned with the problem of constructing certain topological invariants for noncommutative spaces. In particular, we will describe a noncommutative version of the Riemann-Roch theorem.

6.1. Local cyclic cohomology

To proceed further, we need to find an appropriate cohomological analog of Kasparov's KK-functor. In particular, we need a bivariant cohomology theory which is defined on a similar class of algebras as KK-theory, which possesses similar algebraic and topological properties, and which provides an appropriate receptacle for a suitably defined Chern character. The best suited theory for our purposes is Puschnigg's theory [32] which can be defined on large classes of topological and bornological algebras, as well as for separable C^*-algebras. We will now briefly summarize the main ingredients of this cyclic cohomology theory.

Let $\mathcal{A}$ be a unital algebra, and let $T\mathcal{A} = \bigoplus_{n\geq 0} \mathcal{A}^{\otimes n}$ be the quasi-free tensor algebra of $\mathcal{A}$. On $T\mathcal{A}$, define the algebra of *noncommutative differential forms* in degree n by

$$\Omega^n(\mathcal{A}) = \mathcal{A}^{\otimes(n+1)} \oplus \mathcal{A}^{\otimes n} \ . \tag{6.1}$$

On the graded vector space $\Omega^\bullet(\mathcal{A})$ there is a differential d of degree +1 defined on the splitting (6.1) by

$$\mathrm{d} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{6.2}$$

and one has an isomorphism

$$\Omega^n(\mathcal{A}) \cong \mathrm{Span}_{\mathbb{C}}\big\{a_0 \ \mathrm{d}a_1 \cdots \mathrm{d}a_n \ \big| \ a_0, a_1, \dots, a_n \in \mathcal{A}\big\} \ . \tag{6.3}$$

We are interested in a suitable completion of the algebra $\Omega^\bullet(\mathcal{A})$ which can be described as a certain deformation $X(T\mathcal{A})$ of the tensor algebra. This is the $\mathbb{Z}_2$-graded *X-complex* defined by

$$X(T\mathcal{A}) \ : \ \Omega^0(T\mathcal{A}) = T\mathcal{A} \ \underset{\mathrm{b}}{\overset{\natural\circ\mathrm{d}}{\rightleftarrows}} \ \Omega^1(T\mathcal{A})_\natural = \Omega^1(T\mathcal{A}) \,\big/\, \big[\Omega^1(T\mathcal{A}) \, , \, \Omega^1(T\mathcal{A})\big] \ , \tag{6.4}$$

where b is the nilpotent operator defined by $\omega_0 \ \mathrm{d}\omega_1 \mapsto [\omega_0, \omega_1]$ for $\omega_0, \omega_1 \in T\mathcal{A}$ and $\natural$ denotes the quotient map $\Omega^1(T\mathcal{A}) \to \Omega^1(T\mathcal{A})_\natural$.

With some additional structure [1], one can then define the $\mathbb{Z}_2$-graded *bivariant local cyclic cohomology*

$$\mathrm{HL}_\bullet(\mathcal{A}, \mathcal{B}) = \mathrm{H}_\bullet\big(\mathrm{Hom}_{\mathbb{C}}(\,\widehat{X}(T\mathcal{A}), \widehat{X}(T\mathcal{B})\,)\,,\,\partial\big) \tag{6.5}$$

where ∂ is a differential determined by the X-complex (6.4) which makes $\mathrm{Hom}_{\mathbb{C}}(\widehat{X}(T\mathcal{A}), \widehat{X}(T\mathcal{B}))$ into a $\mathbb{Z}_2$-graded complex of bounded maps, and

$$\widehat{X}(T\mathcal{A}) \ : \ \prod_{n\geq 0} \Omega^{2n}(\mathcal{A}) \ \rightleftarrows \ \prod_{n\geq 0} \Omega^{2n+1}(\mathcal{A}) \tag{6.6}$$

is the Puschnigg completion of $X(T\mathcal{A})$. This bivariant cyclic cohomology theory is the closest one in structure to Kasparov's KK-theory. In particular, most of our previous definitions and constructions in bivariant K-theory have obvious analogs in local bivariant cyclic cohomology. The key property of this theory is the existence of a "good" $\mathbb{Z}_2$-graded bivariant Chern character map

$$\mathrm{ch} \ : \ \mathrm{KK}(\mathcal{A}, \mathcal{B}) \ \longrightarrow \ \mathrm{HL}(\mathcal{A}, \mathcal{B}) \tag{6.7}$$

for any two separable C^*-algebras $\mathcal{A}$ and $\mathcal{B}$. The homomorphism (6.7) is functorial and multiplicative.

As an explicit example, let X be a compact oriented manifold of dimension d. Then the inclusion

$$C^\infty(X) \ \hookrightarrow \ C(X) \tag{6.8}$$

of the Fréchet algebra of smooth functions on X determines an invertible element of $\mathrm{HL}(C^\infty(X), C(X))$ [1], and hence an HL-equivalence. Thus $\mathrm{HL}(C(X)) \cong \mathrm{HL}(C^\infty(X))$, for both homology and cohomology. On the other hand, $\mathrm{HL}(C^\infty(X))$ is isomorphic to the standard periodic cyclic homology $\mathrm{HP}(C^\infty(X))$ [32]. Moreover, the action of the boundary map b in eq. (6.4) is trivial in this case and the Puschnigg complex (6.6) reduces to the periodic complexified de Rham complex $(\Omega^\bullet(X), \mathrm{d})$, where d is the usual de Rham exterior derivative on

X. Connes' version of the Hochschild-Kostant-Rosenberg theorem gives a quasi-isomorphism $\mu : \Omega^n(C^\infty(X)) \to \Omega^n(X)$ which is implemented by sending a noncommutative n-form to a differential n-form,

$$\mu\big(f^0 \ \mathrm{d}f^1 \cdots \mathrm{d}f^n\big) = \tfrac{1}{n!} \ f^0 \ \mathrm{d}f^1 \wedge \cdots \wedge \mathrm{d}f^n \ , \quad f^i \in C^\infty(X) \ . \tag{6.9}$$

It follows that the periodic cyclic homology of the algebra $C^\infty(X)$ is canonically isomorphic to the periodic de Rham cohomology of X. Putting everything together we thus have the $\mathbb{Z}_2$-graded isomorphism

$$\mathrm{HL}_\bullet\big(C(X)\big) \cong \mathrm{H}^\bullet_{\mathrm{dR}}(X) \ . \tag{6.10}$$

Moreover, the image of the class $[\varphi]$ of the cyclic d-cocycle

$$\varphi\big(f^0 \, , \, f^1 \, , \, \dots \, , \, f^d\big) = \frac{1}{d!} \int_X f^0 \ \mathrm{d}f^1 \wedge \cdots \wedge \mathrm{d}f^d \tag{6.11}$$

under the homomorphism $\mathrm{HP}^\bullet(C^\infty(X)) \cong \mathrm{HL}^\bullet(C(X)) \to \mathrm{HL}^\bullet(C(X) \otimes C(X))$ induced by the product map is the orientation fundamental class $\Xi \in \mathrm{HL}^d(C(X) \otimes C(X))$ of X in cyclic cohomology. Higher degree homology classes of X are obtained by associating in this way a cyclic k-cocycle with any closed k-current C on X.

6.2. *Todd classes*

Let $\mathcal{A}$ be a PD algebra with fundamental K-homology class $\Delta \in \mathrm{K}^d(\mathcal{A} \otimes \mathcal{A}^{\mathrm{o}})$ and fundamental cyclic cohomology class $\Xi \in \mathrm{HL}^d(\mathcal{A} \otimes \mathcal{A}^{\mathrm{o}})$. Then the *Todd class* of $\mathcal{A}$ is the element of the unital ring $\mathrm{HL}_0(\mathcal{A}, \mathcal{A})$ given by

$$\mathrm{Todd}(\mathcal{A}) := \Xi^\vee \otimes_{\mathcal{A}^{\mathrm{o}}} \mathrm{ch}(\Delta) \ . \tag{6.12}$$

The Todd class depends "covariantly" on the choices of fundamental classes [1], and it is invertible with inverse class given by

$$\mathrm{Todd}(\mathcal{A})^{-1} = (-1)^d \ \mathrm{ch}\big(\Delta^\vee\big) \otimes_{\mathcal{A}^{\mathrm{o}}} \Xi \ . \tag{6.13}$$

This definition can be motivated by the following example. Let $\mathcal{A} = C(X)$ where X is a compact complex manifold. Then $\mathcal{A}$ is a PD algebra with KK-theory fundamental class Δ given by the Dolbeault operator ∂ on $X \times X$, and with HL-theory fundamental class Ξ induced by the orientation cycle $[X]$ determining Poincaré duality in rational homology of X. As before, there is an isomorphism $\mathrm{HL}(\mathcal{A}) \cong \mathrm{HP}(C^\infty(X))$, and so by the universal coefficient theorem we may identify $\mathrm{HL}(\mathcal{A}, \mathcal{A}) \cong \mathrm{End}(\mathrm{H}(X, \mathbb{Q}))$. Then $\mathrm{Todd}(\mathcal{A}) = \smile \ \mathrm{Todd}(X)$ is cup product with the usual Todd characteristic class $\mathrm{Todd}(X) \in \mathrm{H}(X, \mathbb{Q})$.

6.3. *Grothendieck-Riemann-Roch theorem*

For any K-oriented morphism $f : \mathcal{A} \to \mathcal{B}$ of separable C^*-algebras, we can compare the bivariant cyclic cohomology class $\mathrm{ch}(f!)$ with the HL-theory orientation class $f*$ in $\mathrm{HL}_d(\mathcal{B}, \mathcal{A})$. If $\mathcal{A}$ and $\mathcal{B}$ are both PD algebras, then $d = d_{\mathcal{A}} - d_{\mathcal{B}}$ and one has the noncommutative *Grothendieck-Riemann-Roch formula*

$$\mathrm{ch}(f!) = (-1)^{d_{\mathcal{B}}} \ \mathrm{Todd}(\mathcal{B}) \otimes_{\mathcal{B}} (f*) \otimes_{\mathcal{A}} \mathrm{Todd}(\mathcal{A})^{-1} \ . \tag{6.14}$$

This may be proven by writing out both sides of eq. (6.14) using the various definitions and multiplicativity properties of the bivariant Chern character, and then simplifying using associativity properties of the intersection product [1, 2]. In terms of the associated Gysin maps, the formula (6.14) yields a commutative diagram

$$\begin{array}{ccc} \mathrm{K}_\bullet(\mathcal{B}) & \xrightarrow{f_!} & \mathrm{K}_{\bullet+d}(\mathcal{A}) \\ {\scriptstyle \mathrm{ch}\otimes_{\mathcal{B}}\mathrm{Todd}(\mathcal{B})}\downarrow & & \downarrow{\scriptstyle \mathrm{ch}\otimes_{\mathcal{A}}\mathrm{Todd}(\mathcal{A})} \\ \mathrm{HL}_\bullet(\mathcal{B}) & \xrightarrow[f_*]{} & \mathrm{HL}_{\bullet+d}(\mathcal{A}) \ . \end{array} \tag{6.15}$$

See ref. [2] for some applications of this theorem.

6.4. Isometric pairing formula

Let $\mathcal{A}$ be a PD algebra. A fundamental K-homology class Δ for $\mathcal{A}$ is called *symmetric* if $\sigma(\Delta)^{\rm o} = \Delta$ in $\mathrm{K}^d(\mathcal{A}\otimes\mathcal{A}^{\rm o})$, where σ is the map on K-homology induced by the flip involution $\mathcal{A}\otimes\mathcal{A}^{\rm o}\to\mathcal{A}^{\rm o}\otimes\mathcal{A}$ defined by $x\otimes y^{\rm o}\mapsto y^{\rm o}\otimes x$ for $x,y\in\mathcal{A}$. If $\mathcal{A}$ satisfies the universal coefficient theorem for local bivariant cyclic cohomology, then $\mathrm{HL}_\bullet(\mathcal{A},\mathcal{A})\cong\mathrm{End}(\mathrm{HL}_\bullet(\mathcal{A}))$. If $\mathrm{HL}_\bullet(\mathcal{A})$ is a finite-dimensional vector space, then the bivariant Chern character (6.7) induces an isomorphism

$$\mathrm{ch}\ :\ \mathrm{KK}(\mathcal{A},\mathcal{B})\otimes_{\mathbb{Z}}\mathbb{C}\cong\mathrm{Hom}_{\mathbb{C}}\big(\mathrm{K}_\bullet(\mathcal{A})\otimes_{\mathbb{Z}}\mathbb{C}\,,\,\mathrm{K}_\bullet(\mathcal{B})\otimes_{\mathbb{Z}}\mathbb{C}\big)\ \xrightarrow{\approx}\ \mathrm{HL}(\mathcal{A},\mathcal{B}) \tag{6.16}$$

for any separable C^*-algebra $\mathcal{B}$ which is KK-equivalent to a commutative C^*-algebra. If in addition $\mathcal{A}$ has symmetric (even-dimensional) cyclic and K-theory fundamental classes Ξ and Δ, then the *modified Chern character*

$$\mathrm{ch}\otimes_{\mathcal{A}}\sqrt{\mathrm{Todd}(\mathcal{A})}\ :\ \mathrm{K}_\bullet(\mathcal{A})\ \longrightarrow\ \mathrm{HL}_\bullet(\mathcal{A}) \tag{6.17}$$

is an isometry with respect to the inner products

$$(\alpha,\beta)_{\mathrm{K}} = \big(\alpha\otimes\beta^{\rm o}\big)\otimes_{\mathcal{A}\otimes\mathcal{A}^{\rm o}}\Delta \tag{6.18}$$

on the K-theory of $\mathcal{A}$ and

$$(x,y)_{\mathrm{HL}} = \big(x\otimes y^{\rm o}\big)\otimes_{\mathcal{A}\otimes\mathcal{A}^{\rm o}}\Xi \tag{6.19}$$

on the local cyclic homology of $\mathcal{A}$.

The proof of this result can be found in refs. [1, 2]. Let us comment on some of the ingredients that go into this formula. The assumptions made on the local cyclic homology of $\mathcal{A}$ enable one to identify the Todd class $\mathrm{Todd}(\mathcal{A})$ as an element of $GL(\mathrm{HL}_0(\mathcal{A}))\cong GL(n,\mathbb{C})$, where $n:=\dim_{\mathbb{C}}(\mathrm{HL}_0(\mathcal{A}))$. One can then define its square root using the Jordan canonical form, and then consider $\sqrt{\mathrm{Todd}(\mathcal{A})}$ as an element of the ring $\mathrm{HL}_0(\mathcal{A},\mathcal{A})$ again by using the universal coefficient theorem. This square root is not unique, but we fix a choice. In some instances the Todd class may be self-adjoint and positive with respect to an inner product on the vector space $\mathrm{HL}_0(\mathcal{A})$, which may help to fix a canonical choice. The symmetry hypothesis made on the fundamental classes of $\mathcal{A}$ ensure that the pairings (6.18) and (6.19) define symmetric bilinear forms. Then, by using the definition of the intersection product, one can show that the pairing (6.18) generalizes the index pairing (2.2) on topological K-theory. Indeed, in the commutative case, this result is essentially the KK-theory version of the Atiyah-Singer index theorem.

Using the modified Chern character (6.17), we finally arrive at a noncommutative version of the Minasian-Moore formula (2.6) valued in $\mathrm{HL}_\bullet(\mathcal{A})$ and given by

$$Q(\mathcal{D},\xi)=\mathrm{ch}\big(f_!(\xi)\big)\otimes_{\mathcal{A}}\sqrt{\mathrm{Todd}(\mathcal{A})}\ , \tag{6.20}$$

for noncommutative D-branes $\mathcal{A},\mathcal{D}$ with K-oriented morphism $f:\mathcal{A}\to\mathcal{D}$ and Chan-Paton bundle $\xi\in\mathrm{K}_\bullet(\mathcal{D})$. By using the noncommutative Grothendieck-Riemann-Roch formula (6.14), one can formulate criteria under which the charge vector (6.20) can be expressed as the pullback f_* of a "Wess-Zumino class" in $\mathrm{HL}_\bullet(\mathcal{D})$ and hence formulate conditions under which the noncommutative D-brane charge is invariant under T-duality transformations. See ref. [2] for an explicit description of this and for applications of the noncommutative charge formula (6.20).

Acknowledgments

R.J.S. would like to thank the organizers of the Alessandria and Orsay meetings for the invitation and hospitality in a stimulating atmosphere. He also thanks P. Bieliavsky, P. Turner and S. Willerton for helpful comments. V.M. was supported by the Australian Research Council. J.R. was partially supported by US NSF grant DMS-0504212. R.J.S. was supported in part by the EU-RTN Network Grant MRTN-CT-2004-005104.

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