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Embedding of some finite geometries into Riemann Surfaces

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ABSTRACT

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In this thesis we provide examples of a new approach to the field of finite geometries, namely by considering the conformal embedding of a finite geometry into a Riemann surface. A finite geometry is a particular arrangement of a finite number of points and lines satisfying some well known axioms. We will cover the first two examples of the family of Hadamard designs, which are the Fano plane and the 3-biplane.

Riemann surfaces and dessins are introduced and explained in chapter one. We explore their common relationship to cocompact Fuchsian groups and display some results regarding the calculation of their automorphisms groups. We also describe the three most common geometric representations of a dessin: those by Cori, James and Walsh.

Chapter two is divided into two different parts. In the first one we cover the family of finite groups $PSL(2,p)$ where p is a prime number, particularly for the cases where $p \in \{5,7,11\}$. In the second part of the chapter we introduce the family of Hecke groups H^q and their special congruence subgroups, with special regards to the cases where $q = 3$ and $q = 5$.

In chapter three we cover finite geometries and their properties. Projective planes and biplanes are studied in different sections paying special attention to the Fano plane as our chosen representative for the projective planes and the 3-biplane.

Finally in chapter four we make extensive use of all the preliminar material by finding and describing several conformal embeddings of the Fano plane and the 3-biplane into Riemann surfaces, especially into those Riemann surfaces with automorphism group isomorphic to $PSL(2,p)$ and that can be uniformized by a special congruence subgroup of H^q.

To Concha and Naira

“Y miro
y al mirar,
miro al Sur siempre.
Al Sur
del Sur mismo,
al Sur de todas las cosas”

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Contents

Preface

A map, or more generally a dessin, is a particular arrangement of faces, vertices and edges that have a very rich and widely explored relationship with Riemann surfaces and Fuchsian triangle groups. See for example [**JS1**], [**Sch**] and [**Wal**]. The theory that originally covered conformal embeddings of maps into Riemann Surfaces, has been progressively expanded to embrace an ever increasing class of subjects, such as anticonformal embeddings of hypermaps into Klein surfaces and their relationship with N.E.C. triangle groups (see [**BS**] or [**IS**]). In chapter one we introduce Riemann surfaces, Fuchsian groups, dessins and some of their properties and relationships, since they constitute the foundation stone of the examples displayed in chapter four.

It was first observed by Walsh [**Wal**] in 1975 that it is possible to embed the incidence relation of the Fano plane as a map into a torus. In 1986 [**Si4**] Singerman described another embedding of the Fano plane as a regular dessin into a Riemann surface. A regular dessin is one possesing the highest possible symmetry in a surface, and Singerman's embedding proved particularly interesting, since it linked together three well known and widely studied objects: the Fano plane, Klein's quartic and their automorphism groups which are isomorphic to $PSL(2,7)$. Furthermore it is possible to describe the bits of that embedding by means of the cusp set of the action of $\Gamma(7)$ on the upper half plane, where $\Gamma(7)$ stands for the principal congruence subgroup of level seven of the modular group.

All objects highlighted in Singerman's embedding are exceptional on their own rights. Klein's quartic, among other interesting properties, is the Hurwitz surface with the smallest genus (a Hurwitz surface is one that attains Hurwitz's upper bound for the number of automorphisms of a surface, which is $84(g-1)$ for $g \geq 2$) and therefore it is a highly symmetric surface. The Fano plane is the smallest finite projective plane, and arises as the projective space over the field $\mathbb{Z}_2$, while $PSL(2,7)$ enjoys a special status within the $PSL(2,p)$ family that is only shared by $PSL(2,5)$ and $PSL(2,11)$ as it was proven in a result due to Evariste Galois.

What we have done is to fully explore what we considered to be the parallel case to the Fano plane embedding described by D. Singerman, finding some intriguing and beautiful relationships on our way. Since Singerman's embedding fully covers $PSL(2,7)$, we have been guided by the idea of exploring the remaining case in Galois' result, which is $PSL(2,11)$, finding objects that relate to each other in a similar way to those involved with $PSL(2,7)$. As a starting point for our work, I have to acknowledge a very inspirational paper by Konstant: *'The graph of the truncated Icosahedron and the last letter of Galois'* (see [**Ko**]).

Starting then with $PSL(2,11)$ we chose the 3-biplane as the best candidate for a structure to embed for several reasons. The first one is because it is a finite geometry with automorphism group isomorphic to $PSL(2,11)$, which is a necessary condition and that is fairly similar to the Fano plane. Both of them, the Fano plane and the 3-biplane are the smallest examples of Hadamard designs, and the symmetric arrangement of the 55 flags structure of the 3-biplane into 11 blocks of 5 flags closely resembles the 21 flags, 7 blocks of 3 flags of the Fano plane. On the other hand, there is a clear relationship between the flags set of the 3-biplane and the vertices of a truncated icosahedron (which is basically the relationship between $PSL(2,11)$ and its subgroups isomorphic to $A_5 = PSL(2,5)$), and that relationship mirrors that of the Fano plane and the truncated cube (which is again that between $PSL(2,7)$ and its S_4 subgroups).

In spite of finding examples of 3-biplane conformal embeddings as a regular dessin, we thought it more appropriate to drop some hypothesis in the embedding in order to find examples with the rich structure underlying Singerman's embedding. Our main example is therefore no longer a dessin, since the components of the complement of the embedded graph in the surface are not simply connected. This is highlighted by the fact that we do not use a triangle group, but a Fuchsian group with three periods and signature $(1, +, [5, 5, 11])$.

When we explore this example we realized that H^5 emerges in a natural way to play a role similar to that of H^3 in the Fano plane case, further enhancing the singular nature of these two embeddings, since H^3 and H^5 stand in a class of their own within H^q as the only two Hecke groups with a cusp set equal to $\mathbb{Q}(\lambda_q) \cup \{\infty\}$. As a last idea, we use the action of $H^5(4 - \sqrt{5})$ to describe the flags of the embedding, and the vertex set of the truncated icosahedron that relates to it.

Chapter One

Riemann Surfaces and Dessins

In the first half of this chapter we introduce the concept of Riemann surface and provide some preliminary definitions and results. For a broader introduction to Riemann surfaces see either [**Mir**] or [**JS2**]. The notion of compact Riemann surface will be explored following two different but closely related approaches:

· as 2-manifolds with an analytic structure,

· as the quotient by the action of a cocompact Fuchsian group.

Issues related to Fuchsian and NEC groups have also been discussed within the first part of the chapter.

In the second half we have examined the general properties of dessins and their conformal embeddings into Riemann surfaces.

A more general approach to the subject by substituting Riemann surfaces with Klein surfaces has been outlined, and references to the topic of dessins embedding in Klein surfaces are provided.

1.1. Riemann surfaces

Roughly, a *Riemann surface* is a space which, locally, looks like an open set in the complex plane. To make the concept of "looks like" a bit more mathematical, we need to define a *chart* on a topological space X.

A *chart* on a two dimensional topological space X is a homeomorphism

$$\phi_i : U_i \longrightarrow V_i$$

where $U_i \subset X$ is an open set in X, and $V_i \subset \mathbb{C}$ is an open set in the complex plane. The open subset U_i is called the *domain* of the chart ϕ_i. We say that the chart ϕ_i

is *centred* at $p \in U_i$ if $\phi_i(p) = 0$, and it is easy to see that all a chart does is to give local complex coordinates in its domain.

As the charts are local homeomorphisms, they can be inverted, and we will call

$$\phi_{ij} := \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i((U_i \cap U_j)$$

the *transition function* between ϕ_i and ϕ_j. Two charts are *analytically compatible* if the transition function between both of them is analytic or if the intersection of their domains is empty. A set of charts $\{\phi_i : U_i \to V_i\}$ on X is called an *analytic atlas* (also called a *complex atlas*) on X if $\{U_i\}$ is an open cover of X and all the charts in the set are compatible. We will call the charts in an analytic atlas *analytically compatible charts* (or *complex charts*).

It is easy to define an equivalence relation within the atlases of X, two atlases A and B are *equivalent* if every chart of one is compatible with every chart of the other, and an equivalence class of analytic atlases is called a *complex structure* (or *analytic structure*) on the surface X.

Definition 1.1.1. A second countable, connected, Hausdorff topological space with a complex structure is called a *Riemann surface*.

Although the previous definition allows non-compact surfaces, throughout this work we will usually mean a "compact Riemann surface" every time we refer to a Riemann surface.

We can induce local orientations around each point of a Riemann surface by "pull-back" of the orientation of the complex plane via a local chart. These local orientations are well defined, independent of the choice of chart in the atlas and they induce a global orientation on the Riemann surface so that the concept of "clockwise" and "anticlockwise" rotation around a point in a Riemann surface is well defined.

Given two Riemann surfaces X and Y with complex atlases Φ and Ψ, we can define analytic mappings between X and Y in terms of their charts.

Definition 1.1.2. Let X and Y be Riemann surfaces with complex atlases $\Phi = \{\phi_i : U_i \to V_i\}$ and $\Psi = \{\psi_j : U_j \to V_j\}$ respectively. We say that the map $f : X \to Y$ is *analytic* at $p \in X$ if there exist analytic charts $\phi_i : U_i \to V_i$ with $p \in U_i$ and $\psi_j : U_j \to V_j$ with $f(p) \in U_j$ such that $\psi_j \circ f \circ \phi_i^{-1}$ is an analytic function at $\phi_i(p)$.

A map will be analytic in X if and only if it is analytic at every point of X. By definition, if a map is analytic at p, it is analytic in a neighbourhood of p. Using similar principles, we can define meromorphic and analytic functions on Riemann surfaces.

It is easy to see that the complex plane $\mathbb{C}$ admits a complex structure and so is a Riemann surface (although not a compact one). By means of the stereographic projection, the one point compactification of $\mathbb{C}$, $\Sigma = \mathbb{C} \cup \{\infty\}$ also forms a Riemann surface which is called the *Riemann Sphere*. Analytic and meromorphic functions on a Riemann surface X (in the traditional sense of complex analysis) can be thought of as being analytic mappings from X to $\mathbb{C}$ and Σ respectively in the sense of Riemann surfaces.

If a Riemann surface locally looks like an open set in the complex plane, there is a more general object, that we will cover for the sake of completeness called a *Klein surface*, for a thorough introduction to Klein surfaces see [**AG**]. A Klein surface has the property of being locally like an open set in the closed upper half-plane,

$$\mathbb{C}^+ = \{a + bi \ : \ a, b \in \mathbb{R} \text{ and } b \geq 0\}.$$

with the subspace topology that $\mathbb{C}^+$ inherits from the usual topology on $\mathbb{C}$.

Since there are two different classes of open sets in the upper half plane: those that intersect its boundary, and those that are contained in the interior of the upper half plane, we need to define transition functions that cope with this characteristic.

We will say that a complex function f is *antianalytic* (or *anticonformal*) if $\partial f / \partial z = 0$, and f will be called *dianalytic* if it is either analytic or antianalytic in each connected component of its domain. It is easy to see that a function that is both analytic and antianalytic in the same connected component of the domain, is constant on that component. We can generalize the concept of analytic chart to that of *dianalytic chart* on X: a homeomorphism $\phi_i : U_i \to V_i$, where U_i is an open subset of X and V_i is an open subset of either $\mathbb{C}$ or $\mathbb{C}^+$. If ϕ_i is centred at $p \in X$ and V_i is an open set of $\mathbb{C}^+$ but not of $\mathbb{C}$, we will say that p is in the *boundary* of X (we use ∂X to refer to the boundary of X). Two charts will be *dianalytically compatible* if their transition functions are dianalytic, and a set of charts $\{\phi_i : U_i \to V_i\}$ on X is called a *dianalytic atlas* on X if $\{U_i\}$ is an open cover of X and all the transition functions are dianalytic.

Two dianalytic atlases A and B will be *dianalytically equivalent* if $A \cup B$ is a dianalytic atlas (that is, if every chart of one is dianalytically compatible with every chart of the other), and an equivalence class of dianalytic atlases on X is a *dianalytic structure* on X.

Definition 1.1.3. A 2-manifold X with a dianalytic structure A on X will be called a *Klein surface.*

According to the previous definition, any Riemann surface is in fact a Klein surface, since any analytic atlas is dianalytic, and any open set of $\mathbb{C}$ is homeomorphic to an open set in $\mathbb{C}^+$. To avoid confusion, we will reserve the term "Klein surface" to its traditional meaning, that is, any surface X with a dianalytic structure that is not orientable or has at least one boundary component, and we will use "Riemann surface" to refer to Klein surfaces that are orientable and have no boundary. In general, Klein surfaces will be assumed to be compact. As we have already stated we will mainly study symmetric Riemann surfaces throughout this work.

Let $f : X \to Y$ be an analytic map defined at p which is not constant, then (see [**Mir**]) there is a unique integer $m \geq 1$ such that there are local analytic charts ϕ_i and ψ_j centred at p and $f(p)$ respectively, such that $\psi_j \circ f \circ \phi_i^{-1}(z) = z^m$. We will call m the *multiplicity* of f at p. The multiplicity of a map at one point is independent of the choice of charts, and by taking an element $y \in Y$ we can define the *degree* of f at y as the sum of the multiplicities of f at the points of X mapping to y:

$$d_y(f) = \sum_{p \in f^{-1}(y)} mult_p(f)$$

The degree of f at y is constant, independent of y, and so we will just call it the degree of f.

Given an analytic function $f : X \to Y$ of degree n, there is a finite set of points $C(f) \subset Y$ so that $|f^{-1}(y)| = n$ for any $y \in Y - C(f)$ and $1 \leq |f^{-1}(y)| < n$ for every $y \in C(f)$, f is then an *n-sheeted covering*, *branched* if $C(f) \neq \emptyset$ and *unbranched* otherwise. The elements of $C(f)$ are called the *critical values* or *branch points* of f, the elements $x \in X$ with multiplicity greater than one are called the *critical points* or *ramification points* of f, and those with multiplicity one are called *regular points.* Both, the set of critical values and of critical points are finite.

Definition 1.1.4. Two Riemann surfaces X and Y are *conformally equivalent* or *isomorphic* if there exist an analytic bijection $f : X \to Y$.

Proposition 1.1.5. An analytic map between compact Riemann surfaces is an isomorphism if and only if it has degree one.

Proof See Corollary 4.10 in [**Mir**].

An analytic bijection $f : X \to X$ is called an orientation preserving automorphism of X, and the set $Aut^+(X)$ of all orientation preserving automorphisms of X forms a group under composition. Every Riemann surface is orientable, and we will define two different classes of automorphisms, those that preserve orientation (*conformal automorphisms*), and those that reverse orientation (or *anti-conformal automorphisms*). The latter being of capital importance when dealing with symmetric Riemann surfaces.

Definition 1.1.6. A Riemann surface R is called *symmetric* if we can define an anticonformal involution $\sigma : R \to R$.

For the cases when R is symmetric, we call $Aut(R)$ (or $\overline{Aut(R)}$) the group of dianalytic automorphisms of R, it is clear that

$$[Aut(R) : Aut^+(R)] = 2$$

if R is symmetric and it is 1 otherwise.

1.2. Fuchsian groups and NEC groups

In this section we will introduce the main features of discrete groups of plane isometries, regarding as a plane $\mathcal{P}$ one of the following spaces:

- The hyperbolic plane $\mathcal{U}$ with automorphism group

$$Aut(\mathcal{U}) = PGL(2, \mathbb{R}) \qquad Aut^+(\mathcal{U}) = PSL(2, \mathbb{R})$$

- The sphere Σ, with automorphism group

$$Aut(\Sigma) = \overline{PGL(2, \mathbb{C})} \qquad Aut^+(\Sigma) = PGL(2, \mathbb{C})$$

- The Euclidean plane $\mathbb{C}$ with automorphism group

$$Aut(\mathbb{C}) = \overline{Aff(1, \mathbb{C})} \qquad Aut^+(\mathbb{C}) = Aff(1, \mathbb{C}).$$

Where the bar above the groups means that we extend their natural action using complex conjugation.

In this work we will make the standard abuse of notation and use matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{to refer to maps} \quad z \longrightarrow \frac{az+b}{cz+d}$$

and for a field K we will call

$$GL(2,K) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in K, \quad ad-bc \not\equiv 0 \pmod{K} \right\}$$

$$PGL(2,K) = \frac{GL(2,K)}{(\lambda I)} \text{ where } \lambda \in K - \{0\}$$

$$SL(2,K) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in K, \quad ad-bc \equiv 1 \pmod{K} \right\}$$

$$PSL(2,K) = \frac{SL(2,K)}{(\pm I)}$$

The reason for giving a special consideration to these three Riemann surfaces, will become clear after the following well known theorems:

The Uniformisation Theorem 1.2.1. *Every simply connected Riemann surface is conformally equivalent to the hyperbolic plane $\mathcal{U}$, the sphere Σ or the Euclidean plane $\mathbb{C}$.*

The above theorem is due to F. Klein, H. Poincaré and P. Koebe, for a proof using modern notation see [**Bea2**]. A proof of the theorem below is easily obtained from results in [**JS2**] §4.19 and §5.7.

Theorem 1.2.2. Let R be a compact Riemann surface, then there is a discrete subgroup G of $Aut^+(\mathcal{P}_R)$ (where $\mathcal{P}_R$ is either $\mathcal{U}$, Σ or $\mathbb{C}$), acting on $\mathcal{P}_R$ without fixed points, such that R is isomorphic to the quotient space $\mathcal{P}_R/G$.

Given a group G of plane isometries, we will say that it is *non orientable* if it contains any orientation reversing isometries (if it contains reflections or glide reflections), otherwise G is an *orientable group.* The set of all the orientation preserving isometries of a given non orientable group G is itself a subgroup of G of index two, and we will usually refer to it as G^+. A discrete subgroup of $PSL(2,\mathbb{R})$ is called a *Fuchsian group*, and a discrete subgroup of $Aut^+(\mathcal{P})$ with a compact quotient space (a cocompact group) is a *crystallographic group.* Crystallographic groups may have torsion.

We can classify the elements of $PSL(2, \mathbb{R})$ (and therefore the elements of any Fuchsian group) by the number of fixed points in $\mathcal{U}$, this classification can be done in terms of the trace of the matrices.

We will say that an element $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{R})$ is

· *elliptic* if the action of M fixes two conjugate points in $\mathbb{C}$, one of them is in $\mathcal{U}$. We have that $tr(M) = |a + d| < 2$.

· *parabolic* if the action of M fixes only one point on $\mathbb{R} \cup \{\infty\}$, $|a + d| = 2$.

· *hyperbolic* if M fixes two points in $\mathbb{R} \cup \{\infty\}$, we get that $|a + d| > 2$.

It is worth thinking of elliptic elements as finite order rotations about a point in the open upper-half plane, while parabolic elements can be seen as infinite order rotations about a point in $\mathbb{R} \cup \{\infty\}$ and hyperbolic elements as translations in the closed upper-half plane.

Any fixed point by a parabolic element of a Fuchsian group G is called a *cusp* (or *parabolic point*), and the set of all cusps by elements of G is called the cusp set of G. The number of orbits of the action of G in its cusp set is called the *parabolic class number* of G.

A point $a \in \mathbb{R} \cup \{\infty\}$ is a *limit point* of a Fuchsian group Γ if there is a $x \in \mathcal{U}$ and a sequence $\mathcal{G} = g_1,\ g_2, \ldots,\ g_n, \ldots$ with $g_i \in \Gamma$ such that the sequence

$$\mathcal{G}(x) = g_1(x),\ g_2(x), \ldots,\ g_n(x), \ldots$$

converges to a. The set of all limits points of Γ is $L(\Gamma)$, the *limit set* of Γ. Every point a fixed by a parabolic or hyperbolic element of Γ is a limit point of Γ. If we call $O(\Gamma) = \mathbb{R} - L(\Gamma)$ the *ordinary set* of Γ, we will say that the Fuchsian group Γ is of the *first kind* if $O(\Gamma)$ is empty, otherwise we will say that Γ is of the *second kind*. If Γ is of the second kind, then $L(\Gamma)$ is nowhere dense in $\mathbb{R}$. We will only deal with first kind Fuchsian groups.

By the uniformisation theorem, we know that for each Riemann surface R, there is a set of torsion free crystallographic groups $\mathcal{G}_R = \{G_i\}$ that uniformize R, these groups are called *surface groups* of R, and it can be proved that $\mathcal{G}_R$ is a class of conjugate groups in $Aut^+(\mathcal{P})$ so that R is isomorphic to R' if and only if $\mathcal{G}_R = \mathcal{G}_{R'}$.

A *fundamental region* for a crystallographic group G is a closed, connected subset F of $\mathcal{P}$ such that:

· for every point $x \in \mathcal{P}$ there is a point $y \in F$ such that $x \in Gy$.

· No two points in the interior of F are in the same G orbit.

· F is locally finite, that is, any compact set $C \subset \mathcal{P}$ intersects only finitely many images of F.

It is easy to see that if R is uniformized by G, and F is a fundamental region for G, then (see for example [**JS2**] Th. 5.9.6.):

$$\frac{\mathcal{P}_R}{G} \cong \frac{F}{G}.$$

A *fundamental polygon* is a special type of fundamental region whose boundary is a union of geodesic segments that we call the *sides* of the polygon. Given any fundamental polygon P and s a side of P then there is precisely one side of P that we call $\overline{s}$ and one element $g \in G$ so that $g(s) = \overline{s}$ (it is possible that $s = \overline{s}$) and $g(P)$ is adjacent to P along s. We say that s and $\overline{s}$ are *paired* by g.

Fundamental polygons are very useful to find presentations for subgroups of Fuchsian groups by means of a technique known as Reidemeister-Schreier's method. It can be proved [**Bea**] that given P a fundamental polygon for the Fuchsian group Γ, the side pairings elements of P generate Γ, so we can associate to the fundamental polygon P a set of generators for Γ

$$\Phi = \{\beta_1, \beta_2, \ldots, \beta_n\}$$

consisting of side pairings elements of P.

Suppose that we have a subgroup $\Delta < \Gamma$ and we call C_R a whole set of right coset representatives for Δ in Γ so that

$$\Gamma = \bigsqcup_{c \in C_R} \Delta c,$$

($\bigsqcup$ denotes disjoint union), we say that C_R is a *(right) Schreier transversal* for Δ in Γ if whenever $c_1 c_2 \ldots c_n$ is a word in Φ with $c_1 c_2 \ldots c_n \in C_R$ then we have that $c_1 c_2 \ldots c_{n-1} \in C_R$.

Once we have a Schreier transversal for Δ in Γ, we can describe a set of generators for Δ. For any element $g \in \Gamma$ there is a unique $r \in C_R$ such that $\Delta r = \Delta g$ and we call $\overline{g} = r$. Then the set

$$B = \left\{ r\beta(\overline{r\beta})^{-1} \mid r \in R_C,\ \beta \in \Phi \right\}$$

generates Δ and is called a set of *Schreier generators* for Δ in Γ.

With a set of Schreier generators and transversals, we can find a presentation for Δ in Γ. We assume that Γ has presentation $< X \mid R >$ and we call

$$\tilde{R} = \{crc^{-1} \; | c \in C_R, \;\; r \in R\}.$$

If we write the elements of $\tilde{R}$ in terms of those in B and call this new set $\tilde{R}_B$, a presentation for Δ is given by $< B \mid \tilde{R}_B >$.

Example 1.2.3. We will calculate a set of Schreier generators for the Fuchsian group $\Delta = (1; [3,3,7])$ inside $\Gamma = [3,4,7]$ where Γ has presentation

$$< A, B, C \mid A^3 = B^4 = C^7 = ABC = 1 > .$$

We have that $\Phi = \{A, C\}$ is a set of generators for Γ while

$$C_R = \{1, A, C, C^2, \ldots, C^6\}$$

constitutes a Schreier transversal for Δ in Γ, and we get the following diagram (see Theorem 1.2.5. and Example 1.2.6.)

$$\begin{array}{ccccccccc}
 & \Delta C^5 & \Delta A & \Delta C & \Delta C^3 & \Delta C^6 & \Delta C^4 & \Delta & \Delta C^2 \\
*A & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
 & \Delta A & \Delta & \Delta C & \Delta C^2 & \Delta C^3 & \Delta C^4 & \Delta C^5 & \Delta C^6 \\
*C & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 & \Delta A & \Delta C & \Delta C^2 & \Delta C^3 & \Delta C^4 & \Delta C^5 & \Delta C^6 & \Delta
\end{array}$$

for the action of A and C on the right cosets of Δ.

We can now easily calculate the set of Schreier generators, if we delete obvious repetitions, like elements inverses, we get:

$$B = \{CAC^{-1}, C^2AC^{-3}, C^3AC^{-6}, C^4AC^{-4}, C^5A, C^6AC^{-2}, ACA^{-1}\}$$

and we have that Δ has presentation $< B \mid R >$, where R are the relations of Γ. In order to get a standard presentation for Δ we might have to consider changing the generators in B by some of its conjugates in Δ, that has been done in the last section of chapter four, where this particular pair of groups have been considered.

The following result shows how to construct a fundamental polygon for $\Delta < \Gamma$.

Theorem 1.2.4. Let $\Delta < \Gamma$ be a pair of finitely generated Fuchsian groups such that Γ has fundamental polygon P and C_R is a right Schreier transversal for Δ in Γ. Then a fundamental polygon for Δ is

$$P' = \bigcup_{r \in R_C} rP$$

and the sides of P' are paired by the Schreier generators corresponding to R_C.

Let now $R = \mathcal{P}/G$ for some $G \in \mathcal{G}_R$, then if we call $N_+(G)$ the normaliser of G in $Aut^+(\mathcal{P})$, we have that (see Theorem 5.9.4. in **[JS2]**):

$$Aut^+(R) \cong \frac{N_+(G)}{G},$$

and furthermore, any group of orientation preserving automorphisms of R is isomorphic to N/G, for some group $N \leq Aut^+(\mathcal{P})$ so that $G \triangleleft N$.

As we have mentioned before, the group of automorphisms of $\mathcal{P}$ contains orientation reversing elements as well, so we may consider what happens when we allow orientation reversing elements in the previous theory. Although there is no generic name for a discrete subgroup of $Aut(\mathcal{P})$ with orientation reversing elements, we will be mainly concerned with crystallographic groups, and in this case we use NEC as an acronym for Non Euclidean Crystallographic, referring to cocompact discrete subgroups of $Aut(\mathcal{P})$.

We will follow with groups terminology the same rules already stated for surfaces, and therefore we will call crystallographic groups those NEC groups that contain no orientation reversing element, reserving the name *proper NEC group* (or just NEC group) for those which do. If G is an NEC group, its *canonical Fuchsian subgroup* is G^+. Since the set of Riemann surfaces arising from the action of Euclidean and spherical groups is very limited in scope (for instance, no surface of genus $g \geq 2$ is formed in such a way), we may sometimes use the word "Fuchsian" in a general way to refer to a group that contains no orientation reversing elements, although it may not be a hyperbolic group.

A Riemann surface R admits an antianalytic involution if the normaliser of $G \in \mathcal{G}_R$ in $Aut(\mathcal{P})$ contains orientation reversing elements. If this is the case, we have that:

$$Aut(R) \cong \frac{N(G)}{G}.$$

It is obvious that $N_+(G) = N^+(G)$ (where we define $N^+(G) := N(G)^+$), and that $N^+(G) = N(G)$ if and only if R is not symmetric.

A special class of crystallographic groups are triangle groups. A *(proper NEC) triangle group* is defined as the group of isometries generated by the reflections on the side of a triangle with angles $\frac{\pi}{l}$, $\frac{\pi}{m}$ and $\frac{\pi}{n}$ and is represented by $\Gamma(l, m, n)$, which we will usually shorten to (l, m, n) if no confussion arises. We will say that a triangle group is *spherical*, *Euclidean* or *hyperbolic* if $\frac{\pi}{l} + \frac{\pi}{m} + \frac{\pi}{n}$ is greater than, equal to or smaller than π.

The canonical Fuchsian group of a triangle group, is also called a triangle group, and in that case the group is defined as the group generated by rotations of order n, m and l at the vertices of a triangle of angles $\frac{\pi}{l}$, $\frac{\pi}{m}$ and $\frac{\pi}{n}$. In general it will be easy to know from the context if we are talking about orientation preserving or reversing groups, and we will use $\Gamma[l, m, n]$ to denote a Fuchsian triangle group and $\Gamma(l, m, n)$ to denote a proper NEC triangle group.

It is clear that:

$$\Gamma^+(l, m, n) = \Gamma[l, m, n].$$

The importance of triangle groups will become apparent in the next section, due to its relationship to the Belyĭ functions (see **[JS3]** and **[Sch]**) and the fundamental groups of maps and hypermaps. We can define a triangle group using only algebraic terms as the group with the following presentation:

$$\Gamma(l, m, n) =< x, y, z \mid x^2 = y^2 = z^2 = (xy)^l = (yz)^m = (zx)^n = 1 >$$

$$\Gamma[l, m, n] =< a, b, c \mid a^l = b^m = c^n = abc = 1 >$$

Macbeath and Wilkie proved that for each crystallographic (or NEC) group G with compact quotient space there is a signature and a "marked polygon" from which a canonical presentation for G may be derived, a *marked polygon* being a fundamental polygon for G together with the identifications on the boundary of

F produced by the action of G. See **[Wil]**, **[Mac]**; the explanation here has been taken from **[Wat]**. The presentation for G is given by:

$$< x_i, c_{jk}, e_j, a_p, b_q \mid x_i^{m_i} = c_{jk}^2 = (c_{jk-1}c_{jk})^{n_{jk}} = c_{j0}e_jc_{jt_j}e_j^{-1} = AD = 1 >$$

Where $A = x_1...x_re_1...e_s$, $i = 1...r$, $j = 1...s$, $k = 0...t_j$, $p = 1...g$, $q = 1...h$ with $r \geq 0$, $j \geq 0$, $t_j \geq 0$, $g \geq 0$, $h \in \{0, g\}$, $n_{jk} \geq 2$, $m_i \geq 2$ and:

- $D = a_1b_1a_1^{-1}b_1^{-1}...a_gb_ga_g^{-1}b_g^{-1}$ if $h = g$.
- $D = a_1^2...a_g^2$ if $h \neq g$.

The numbers m_i are called the *proper periods* of the group, while the n_{jk}'s are the *link periods*, and will correspond to the branch points lying inside the quotient space or in its boundary respectively. The quotient space $\frac{\mathcal{P}}{G}$ has genus g and is orientable if $h = g$ and non orientable otherwise. The number of boundary components of the quotient space is s. If $s = 0$ and there are no proper periods, G is a crystallographic surface group, while if $s \neq 0$ and there are no proper or link periods, then G is an NEC surface group.

Periods m_i (or n_{jk}) can be infinity, in which case the relations of the form $x_i^\infty = 1$ (or $(c_{jk-1}c_{jk})^\infty = 1$) are omitted from the presentation. Any finite order element of G is either conjugate to x_i^f or to $(c_{jk-1}c_{jk})^f$ (for a certain i, j, k, f), therefore if G is a Fuchsian group, all its finite order elements are conjugate to its elliptic generators.

To every presentation (and therefore to every group) we can associate a signature which will be

$$(g; +; [m_1, ..., m_r]; \{(n_{11}, ..., n_{1t_1}), ..., (n_{s1}, ..., n_{st_s})\})$$

in the orientable case, and

$$(g; -; [m_1, ..., m_r]; \{(n_{11}, ..., n_{1t_1}), ..., (n_{s1}, ..., n_{st_s})\})$$

in the non-orientable. When $r = 3$, $s = 0$ and $g = 0$ we have the Fuchsian triangle group $\Gamma[m_1, m_2, m_3]$, while if $r = 0$, $g = 0$, $s = 1$ and $t_1 = 3$, we get the NEC triangle group that we have called $\Gamma(n_{11}, n_{12}, n_{13})$.

Singerman [**Si3**] determined the hyperbolic area of a fundamental region of a NEC group G. In the orientable case it is:

$$\mu(G) = 2\pi\left(2g - 2 + s + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{j=1}^{s}\sum_{k=1}^{t_j}\left(1 - \frac{1}{n_{jk}}\right)\right)$$

while in the non-orientable case is:

$$\mu(G) = 2\pi\left(g - 2 + s + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{j=1}^{s}\sum_{k=1}^{t_j}\left(1 - \frac{1}{n_{jk}}\right)\right)$$

The particular case of the first formula, that covers the case where G does not have any boundary components ($s = 0$) and therefore no link periods is usually known as *Riemann-Hurwitz formula.* It follows from Theorem 1.2.4. that if G_1 is a subgroup of G of index k, then

$$\mu(G_1) = k\mu(G).$$

We will end the present section with a Theorem by David Singerman [**Si1**] concerning subgroups of Fuchsian groups with finite index. Although the original theorem is more general, we will only use it for cocompact Fuchsian groups. This theorem allows us to work out the signature of the subgroup, given the signature of the big group and the actions of its generators on the cosets of the subgroup. A generalization of it for NEC groups was done by Hoare [**Ho**].

Singerman Theorem 1.2.5. Let G be a Fuchsian group with signature

$$(g; [m_1, \ \dots \ , m_r])$$

then G contains a subgroup F of finite index N and signature

$$(h; [n_{11}, n_{12}, \ \dots \ , n_{1p_1}, \ \dots \ , n_{r_1}, n_{r2}, \ \dots \ , n_{rp_r}])$$

if and only if:

- There is a permutation group $H < S^N$ transitive on N points and an epimorphism $\Theta : G \to H$ so that the permutations $\Theta(x_i)$ has cycles of length m_i and precisely p_i other cycles whose lengths are $m_i/n_{i1}, \ \dots \ , m_i/n_{ip_i}$,
- $\frac{\mu(F)}{\mu(G)} = N.$

Example 1.2.6. Let G be the group $[3, 4, 7]$ with presentation as in Example 1.2.3. and consider the map $\Theta : G \longrightarrow S^8$ given by:

$$A \longmapsto (1,7,2)(4,5,8)(3)(6)$$
$$B \longmapsto (1,2,5,8)(3,7,6,4)$$
$$C \longmapsto (1)(2,3,4,5,6,7,8)$$

If we call $\Theta(G) = H$ it is immediate that we are mapping G onto a transitive subgroup of S^8. $\Theta(A)$ has two cycles of length 3 that produce no other periods in F and two cycles of length 1 that produce two 3 periods in F. $\Theta(B)$ produce no period of F since all its cycles have length 4, on the other hand $\Theta(C)$ induces a 7 period in F since it has got one cycle of length 1. We have seen that F has signature $(g; [3, 3, 7])$ and the condition $\mu(F) = 8\mu(G)$ implies that $g = 1$.

1.3. Maps

In this section we want to explore the relationship between maps and hypermaps. We will define maps and hypermaps over Riemann surfaces and explain the geometric and algebraic approach to these objects. A current term, due to Grothendieck, to refer to both maps and hypermaps is "dessin d'enfants" or just *dessin*. For an introduction to maps theory see [**JS1**], for a discussion of the wide scope of dessin theory see [**Sch**].

Intuitively, a map is a decomposition of a surface into polygonal two-cells or faces. That leaves a structure on the surface composed by the interior of the polygons (faces), edges and vertices that constitute the map, for an easy example of these we can think of the platonic solids as maps on the surface of the sphere, picture *Fig. 1.1. a)* below depicts the map on the sphere induced by a tetrahedron.

The set of vertices and edges of the map constitute a graph which is embedded into the surface. Although we will allow a very general class of graph, we will require the graph to be connected and that to every vertex there is only a finite number of incident edges, this number is the *valency* of the vertex. Our graph can contain edges with one or two vertices, the edges with two vertices correspond to our everyday perception of an edge, as for the edges with only one vertex, they can either be a *loop* or a *free edge*, the difference being that a loop is homeomorphic to a circle, while a free edge is homeomorphic to a segment. Every face would be surrounded by a finite number of edges, and this number is the *valency* of the face, faces should be simply-connected. An *allowed graph* is a graph satisfying the previous conditions. For a picture of a generic map on a sphere see *Fig. 1.1. b)*

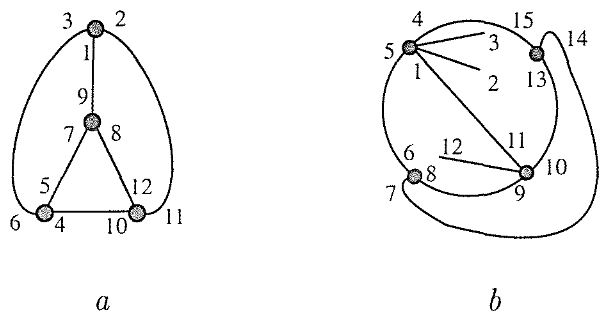


Fig. 1.1. Two examples of maps on the sphere

A *hypergraph* is a generalization of this idea where we allow any finite number of vertices to lie on an edge. We will meet some hypergraphs when we describe finite geometries in chapter three. Our only restriction on the structure of the hypergraph is that it is connected, and that to every vertex there is only a finite number of incident edges. We can define a *(hyper) graph isomorphism* as a bijection f between the vertex sets of two (hyper)graphs that induces a bijection between the edges set so that the incidence structure is preserved, that is, if

$$P \in E \Longleftrightarrow f(P) \in f(E)$$

where P is any vertex contained in the edge E.

We will start with the topological approach to a map, then describe the algebraic structure of it, and use algebraic notions to generalize the concept of map to that of a hypermap. Nevertheless, after the topological and the algebraic approaches have been explained, we will need to provide maps with a richer structure, namely the analytic structure of their underlying Riemann surface.

Definition 1.3.1. A *map* $\mathcal{M}$ on a compact surface X is an embedding of an allowed graph $\mathcal{G}$ into X such that the components of $X - \mathcal{G}$ are simply-connected. We call the components of $X - \mathcal{G}$ the *faces* of $\mathcal{M}$.

Definition 1.3.2. A *topological map* $Top(\mathcal{M})$ is a triple $(\mathcal{G}, \mathcal{V}, S)$ where $\mathcal{G}$ is an allowed graph, $\mathcal{V}$ its set of vertices and S is the underlying surface.

Definition 1.3.3. Let e be an edge of a map $\mathcal{M}$ with endpoints e_1 and e_2 (that are identified in the case of loops), and let m be a point in e which is not e_1 nor e_2. We can now divide e into two segments m_1 and m_2. A *dart* of a map $\mathcal{M}$ is a pair consisting of a vertex and an incident segment. If there are no free edges then every edge has two darts.

In *Fig. 1.2.* we can see the three different kinds of edges on a map with their corresponding darts.

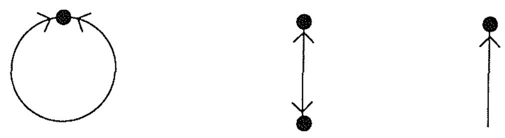


Fig. 1.2. Loop, standard edge and free edge

We can associate an algebraic structure to the topological construction described above. We will first consider the map $\mathcal{M}$ with its set of darts that we will call Ω^+, the superscript $^+$ means that we are working with maps on orientable structures, since we are only interested in Riemann surfaces, we will not cover the definition of maps for Klein surfaces, for a survey of that topic see **[BS]** and **[IS]**. We define two permutations of Ω^+ as follows:

· r_0 consists of cycles formed by going round each vertex in an anticlockwise direction.

· r_1 is the product of the transpositions that interchange the two darts on an edge or loop, and fix the dart on a free edge.

The product $r_2 = r_1 r_0^{-1}$ consist of cycles which define anticlockwise rotations of the darts about each face of $\mathcal{M}$. If we let $G_+ =< r_0, r_1 >$ be the group generated by r_0 and r_1, G_+ is a subgroup of S^{Ω^+},the group of permutations on the elements of Ω^+, that is transitive because the graph underlying $\mathcal{M}$ is connected. G_+ is the *monodromy group* of $\mathcal{M}$, we could have called it the *oriented* monodromy group, but as we have already stated, we will only consider maps on orientable surfaces. We define the *algebraic map* of $\mathcal{M}$ to be:

$$(Alg\ \mathcal{M})^+ = (G_+, \Omega^+, r_0, r_1).$$

Example 1.3.4. The tetrahedron in *Fig. 1.1.a)* has $\Omega^+ = \{n \mid n = 1...12\}$ as its set of darts, and the permutations are:

$r_0 \to (1,2,3)(4,5,6)(7,8,9)(10,11,12)$

$r_1 \to (1,9)(2,11)(3,6)(4,10)(5,7)(8,12)$

$r_2 \to (1,8,11)(2,10,6)(3,5,9)(4,12,7)$

Example 1.3.5. The map in *Fig. 1.1.b)* has $\Omega^+ = \{n \mid n = 1...15\}$ as its set of darts. The permutations are:

$r_0 \to (1,2,3,4,5)(6,7,8)(9,10,11,12)(13,14,15)$

$r_1 \to (1,11)(2)(3)(4,15)(5,6)(7,14)(8,9)(10,13)(12)$

$r_2 \to (1,10,15,3,2)(4,14,6)(5,8,12,11)(7,13,9)$

Definition 1.3.6. We define an *algebraic map* as a quadruple:

$$\mathcal{A}^+ := (G_+, \Omega^+, r_0, r_1)$$

where Ω^+ is a finite set, G_+ is a transitive subgroup of the group of permutations S^{Ω^+} generated by r_0 and r_1, and where r_1 is a product of disjoint transpositions. We call $r_2 = r_1 r_0^{-1}$. To any oriented algebraic map $\mathcal{A}^+$, there is a topological map $\mathcal{M}$ so that $Alg^+(\mathcal{M}) = \mathcal{A}^+$.

If r_0 and r_2 have orders m and n respectively, we say that $\mathcal{M}$ (resp. $\mathcal{A}^+$) has *type* (m,n), that is, the *l.c.m.* of the valencies of vertices and faces are m and n respectively.

Given the triangle group $\Gamma[m,2,n]$ with a presentation of the form:

$$\Gamma[m,2,n] =< a, b \mid a^m = b^2 = (ba^{-1})^n = 1 >$$

and a map $\mathcal{M}$ of type (m,n) with $Alg^+(\mathcal{M}) = (G_+, \Omega^+, r_0, r_1)$, we can define an epimorphism $\theta : \Gamma[m,2,n] \longrightarrow G_+$ sending a to r_0 and b to r_1.

Let $G_{+\alpha} = \{g \in G_+ | \alpha g = \alpha\}$ be the stabiliser of $\alpha \in \Omega^+$, then $M = \theta^{-1}(G_{+\alpha})$ is called a *fundamental group* of $\mathcal{M}$ in $\Gamma[m,2,n]$. A different choice of dart α will yield a group conjugated to M in $\Gamma[m,2,n]$, hence to every map $\mathcal{M}$ of type (m,n), we can associate a conjugacy class of fundamental groups in $\Gamma[m,2,n]$ (see [**JS1**]).

Next, we identify Ω^+ with the set of right M-cosets in $\Gamma[m,2,n]$ by the bijection $Mh \to \alpha(h\theta)$. Taking $M^* = Ker(\theta)$, the core of M (the intersection of all the conjugates of M by elements of Γ), we can identify G_+ with the quotient group

$\Gamma[m,2,n]/M^*$ by $M^*g \to g\theta$ so that the action of a (respectively b) corresponds to the permutation r_0 (respec. r_1) in G_+. In this way we see that any finite index subgroup $M \leq [m,2,n]$ is a fundamental group of some oriented map.

It is possible to generalize the concept of dart, so that we can define maps on Klein surfaces, this generalization relates in a natural way to the inclusion of crystallographic triangle groups into NEC triangle groups, see [**BS**].

Another important property of a fundamental group M of a map $\mathcal{M}$ is that the surface S in which the map is embedded is uniformized (not uniquely) by M in the sense that:

$$S \cong \frac{\mathcal{U}}{M}.$$

To every topological map corresponds an algebraic map, and from any algebraic map we can recover a topological map. Unfortunately, this relationship does not make both categories equivalent, since there are many ways of altering a topological map via an homeomorphism of the underlying surface without a counterpart in the side of algebraic maps. We want to have a geometrical definition of map that allows us to translate results from the algebraic category to the geometrical one, and in order to do so, we need to define maps on surfaces in such a way that the isomorphisms between them correspond to isomorphisms between their underlying surfaces. These maps when considered on Riemann surfaces were originally called *Riemann maps* by [**JS**], we will call them *analytic maps*, since their isomorphisms correspond to mappings that preserve the analytic structure of their underlying surfaces. The first step to define them is to define universal maps and quotients of maps.

The NEC triangle group $\Gamma(2,m,n)$ acts naturally on $\mathcal{P}$ and has a triangle T of angles $a = \frac{\pi}{2}$, $b = \frac{\pi}{m}$ and $c = \frac{\pi}{n}$ as its fundamental polygon. If we draw a dart on the side ab, and let Γ act on ab with that dart, we get a map on $\mathcal{P}$ with all the edges of the same length and such that all vertices have order m and all faces have order n, this is the *universal topological map* $\hat{\mathcal{M}}$ *of type (m,n)*.

Let $\mathcal{M} = (\mathcal{G}, \mathcal{V}, S)$ be a topological map of type (m,n), then there is an algebraic map $\mathcal{A} = Alg(\mathcal{M})$ associated with it, from where we can calculate a fundamental group M of $\mathcal{M}$. We have seen that $M \leq (2,m,n)$ and therefore M acts on the universal map $\hat{\mathcal{M}}$ of type (m,n) sending vertices to vertices, edges to edges and face centres to face centres. The quotient of this action is a map $\check{\mathcal{M}}$ of type (m,n) which is naturally embedded in the surface S by the covering $\pi : \mathcal{P} \to \frac{\mathcal{P}}{M}$ and which is topologically equivalent to $\mathcal{M}$. $\check{\mathcal{M}}$ is an *analytic* map of $\mathcal{M}$.

Any analytic map, is in particular a topological map. We have proved that to any algebraic map $\mathcal{A}$ there is at least one topological map $\mathcal{M}$ so that $\mathcal{A} = Alg(\mathcal{M})$.

Definition 1.3.7. Two maps $\mathcal{M}$ and $\mathcal{M}'$ of type (m,n) embedded in the surfaces S and S' are called *analytically equivalent* if there is an analytic isomorphism

$$f : S \longrightarrow S'$$

between the surfaces that takes the set of vertices, edges and faces of $\mathcal{M}$ to the set of vertices, edges and faces of $\mathcal{M}'$ respectively.

If we consider the definition of an algebraic map, there is no reason why we should restrict ourselves to quadruples where r_1 has order two, and not consider groups where this condition is dropped. In fact, this property corresponds to the fact that the edges have only two vertices, so by allowing a higher order in r_1, we will study maps whose edges contain more than two vertices, and whose fundamental group is a finite index subgroup of the triangle group $[l,n,m]$. Those objects are what we call *oriented hypermaps.*

1.4. Hypermaps

Definition 1.4.1. An *oriented algebraic hypermap* $\mathcal{A}$ consists of a finite set of objects Ω^+ that we will call the *bits* (or *brins*) of $\mathcal{A}$, together with two permutations $r_0, r_1 \in S^{\Omega^+}$ and a group $G_+ =< r_0, r_1 >$ such that G_+ is transitive in Ω^+.

(As before we will drop the word "orientable" since we will only consider Riemann surfaces). We call G_+ the *monodromy group* of the hypermap $\mathcal{A}$. The cycles r_0 and r_1 correspond to *hypervertices* and *hyperedges* in precisely the same way as the cycles generating the monodromy group of a map corresponds to vertices and edges, while the permutation $r_2 = (r_0 r_1)^{-1}$ corresponds to *hyperfaces.* The hypermap is represented by the quadruple $(G_+, \Omega^+, r_0, r_1)$, and if r_i has order l_i we say that $\mathcal{A}$ has *type* (l_0, l_1, l_2). The *degree* of $\mathcal{A}$ is equal to $|\Omega^+|$.

We extend the notion of fundamental group to hypermaps by considering the natural epimorphism $\theta : \Gamma[l_0, l_1, l_2] \to G_+$ in the same way as we did with maps. We therefore can define a *fundamental group* of $\mathcal{A}$ to be $M = \theta^{-1}(G_{+\alpha})$. We will use the word *algebraic dessin* to refer to either an oriented algebraic map or hypermap indistinctly.

Although from the algebraic point of view there is little difference between a map and a hypermap, it is not intuitively immediate how to view a map on a surface when we allow each edge to meet as many vertices as necessary, several authors have suggested different ways of doing it, we will first give an abstract definition of topological hypermap on Riemann surfaces and then describe the representations by *Cori*, *James* and *Walsh*.

Definition 1.4.2. A *topological hypermap* $\mathcal{H}$ on a Riemann surface S is a triple (S, E, V) where E and V are closed subsets of S such that:

a) $\Omega^+ := E \cap V$ is a non-empty finite set. Ω^+ is called the set of *bits* (or *brins*).

b) $E \cup V$ is connected.

c) The components of E and V are homeomorphic to closed discs. The components of E are called *hyperedges* while the component of V are the *hypervertices*.

d) The components of $S - (E \cup V)$ are homeomorphic to open discs and are called *hyperfaces*.

We will call *topological dessin* an object which is either a topological map or hypermap on a Riemann or Klein surface.

For an explanation of dessins on Klein surfaces see [**IS**].

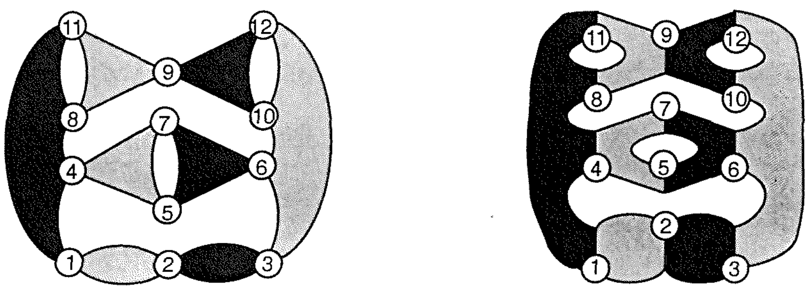
The Cori representation

In the Cori representation (see [**Co**]) hypervertices and hyperedges are represented by closed polygons called 0-faces and 1-faces respectively (we will colour 0-faces with a light grey and 1-faces with a dark grey), each polygon having as many sides as the length of the cycle it represents, while hyperfaces are represented by open polygons called 2-faces each one having twice as many sides as the order of the hyperface they represent (2-faces are painted white). 0-faces are mutually disjoint, as are 1-faces, and they intersect each other only in their vertices, which represent the bits, each bit shared by precisely one hypervertex and one hyperedge. As the hypermap is embedded in an orientable surface R, we can define an anticlockwise permutation of the bits around each 0-face and each 1-face consistent with the orientation of R, these permutations corresponds to r_0 and r_1 respectively. If we call V the set of all 0-faces, and E the set of all 1-faces, then $V \cup E$ is a closed set in R, and the components of $R - (V \cup E)$ are the 2-faces, which are open polygons that induce the permutation $r_2 = (r_0 r_1^{-1})$. As 2-faces have an even number of bits, r_2 permute the bits acting as the square of the clockwise permutation around each hyperface.

The Cori representation is very easy to draw and shows some symmetries of the hypermap in a clear way, but has the drawback of differentiating between hypervertices, hyperedges and hyperfaces in a way that seems unnatural if we regard the algebraic definition of the hypermap.

The James representation

The James representation (see [**Ja**]) displays the trinity among hypervertices, hyperedges and hyperfaces in a more natural way. This time 0-faces, 1-faces and 2-faces will all be polygons with twice as many sides as the orders of the elements they represent. The intersection of any two i-faces is always empty, and 0-faces and 1-faces intersect along one side. To place the bits, we need to consider the three-valent map on R defined by the sides of these polygons, in such a map every vertex has order three and each edge separates faces of different labels (there are no two i-faces with a common edge). Going anticlockwise around each vertex, we get a permutation of the faces that is either $(0, 1, 2)$ or $(0, 2, 1)$, if the permutation is $(0, 1, 2)$, the vertex is called a bit. Once the bits are placed, the permutations r_0, r_1 and r_2 follow in the same way as with the Cori representation.



Cori James

Fig. 1.3. Cori and James representations of the hypermap $\mathcal{H}_e$.

To go from the Cori representation to the James representation, we push together the hypervertices and hyperedges with a common bit, thereby changing the bit in the Cori representation (one point) to an edge in the James representation one of whose vertices is a bit, and doubling the number of sides of every 0-face and 1-face.

The Walsh representation

In order to introduce this representation first presented in [**Wal**], we need to define a *bipartite map*, that is, a map in which the vertices have one of two colours, and such that all edges connect vertices of different colours. Starting now at the Cori representation, we substitute every 0-face by a 0-vertex (where the number of the vertex refers to its colour) and every 1-face by a 1-vertex, and we join two vertices together if and only if their respective 0-face and 1-face intersect in a bit, therefore substituting each bit shared by the 0-face and 1-face for an edge joining the two new vertices. We have now a bipartite map on the surface, with the same number of edges as the number of bits of the original hypermap, the vertices in the bipartite map stand for the hypervertices and hyperedges, while the faces of the bipartite map represents the hyperfaces of the original hypermap. The permutation r_0 (resp. r_1) corresponds to anticlockwise rotation around each 0-vertex (resp. 1-vertex) while r_2 corresponds to the clockwise permutation of the edges around a face.

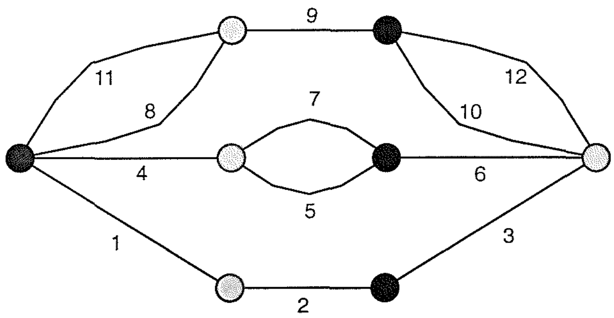


Fig. 1.4. Walsh representation of $\mathcal{H}_e$.

Example 1.4.3. The hypermap $\mathcal{H}_e$ has $\Omega^+ = \{n \mid n = 1...12\}$ as its set of bits, and the permutations are:

$r_0 \to (1,2)(3,12,10,6)(4,5,7)(8,9,11)$

$r_1 \to (2,3)(5,6,7)(9,10,12)(1,4,8,11)$

$r_2 \to (1,8,11)(2,10,6)(3,5,9)(4,12,7)$

We have already defined the cycles r_0, r_1 and $r_2 = r_1 r_0^{-1}$ for the hypermap $\mathcal{H}$ when we explained the Cori model. If we let $G_+ :=< r_0, r_1 >$, G_+ is a transitive subgroup of S^{Ω^+}, and we define the *oriented algebraic hypermap* of $\mathcal{H}$ to be:

$$Alg^{\ +}(\mathcal{H}) = (G_+, \Omega^+, r_0, r_1).$$

We say that the hypermap has *type* (l, m, n) if r_0, r_1 and r_2 have orders l, m and n respectively.

It is clear that if any element of the set $\{l, m, n\}$ is 2, then the quadruple will be an algebraic map up to duality. We will use the name *algebraic dessin* to refer to either algebraic maps or algebraic hypermaps.

We can embed $\mathcal{H}$ into the surface:

$$S \cong \frac{\mathcal{U}}{M}$$

which is a Riemann surface since M is a subgroup of a crystallographic triangle group.

To define an analytic structure on hypermaps, we follow a similar procedure to the one explained in the case of maps. We first define the *universal hypermap* of type (l, m, n) by using a triangle with angles π/l (labelled 0), π/m (labelled 1) and π/n (labelled 2). By the action of $\Gamma(l, m, n)$ on that triangle, we get a map over $\mathcal{P}$ with three different types of vertices, if we delete the vertices of type 2 and the edges meeting these vertices, and paint with white the vertices of type 0 and with black those of type 1, we get a Walsh representation of the *universal hypermap* of type (l, m, n) which we shall call $\breve{\mathcal{H}}(l, m, n)$ or simply $\breve{\mathcal{H}}$ when no confusion arises.

If $\mathcal{H} = (S, E, V)$ is a topological hypermap of type (l, m, n), we can calculate its algebraic hypermap and its fundamental group $H \leq \Gamma(l, m, n)$. Letting H act on $\breve{\mathcal{H}}$, we get a quotient hypermap $\breve{\mathcal{H}}/H$ embedded into $\mathcal{P}/H$, and we call it the dianalytic hypermap corresponding to $\mathcal{H}$. If H contains no orientation reversing elements, by using the Fuchsian triangle group $\Gamma[l, m, n]$ for the *orientable universal hypermap* we get an analytic hypermap corresponding to $\mathcal{H}$.

Definition 1.4.4. Two hypermaps $\mathcal{H}$ and $\mathcal{H}'$ of type (l, m, n) embedded in the surfaces S and S' are *analytically equivalent* if there is an analytic isomorphism between the surfaces that sends the set of hypervertices, hyperedges and hyperfaces of $\mathcal{H}$ to their corresponding sets in $\mathcal{H}'$.

1.5. Group of Automorphisms of a dessin

Since we have defined maps and hypermaps as particular cases of dessins, I will define morphisms in the category of dessins, and the natural restrictions of these morphisms can be translated in an easy way to either maps or hypermaps. We will use the name vertices, edges and faces of a dessin to refer to either vertices, edges and faces of a map, or hypervertices, hyperedges and hyperfaces of a hypermap. Whenever no distinction is needed, I will use "bits" to refer to darts and bits.

Results in this section are well known in dessin theory, and most of them have been reworded from [**JS1**]. When they are more general than the original result, the proof still follows along the same lines.

Definition 1.5.1. We will say that ϕ is a *topological morphism* between the dessins $\mathcal{D}$ and $\mathcal{D}'$ with topological structures given by (S, E, V) and (S', E', V'), if $\phi : S \to S'$ is a topological covering between the underlying surfaces (possibly ramified, all branch points having finite order), preserving orientation and so that $\phi(E) = E'$ and $\phi(V) = V'$.

We say that $\mathcal{D}$ *covers* $\mathcal{D}'$ if there is a morphism from $\mathcal{D}$ to $\mathcal{D}'$.

Theorem 1.5.2. Let $\mathcal{D}$ and $\mathcal{D}'$ be dessins of type (l, m, n), then $\mathcal{D}$ covers $\mathcal{D}'$ if and only if there are groups M and M' which are representatives of the conjugacy classes of fundamental groups of $\mathcal{D}$ and $\mathcal{D}'$ respectively, such that $M \leq M'$.

Definition 1.5.3. Given two dessins $\mathcal{D}$ and $\mathcal{D}'$ embedded respectively in S and S', we will say that $\phi : \mathcal{D} \to \mathcal{D}'$ is an *analytic morphism* between the dessins if ϕ is a topological morphism between the dessins and an analytic morphism between the underlying surfaces.

This definition of morphism between dessins is more restrictive than the one originally suggested in [**JS1**], since we impose the condition that the morphism preserves the analytic structure of the surface. It corresponds to the "Riemann morphism" between maps described in that paper.

Definition 1.5.4. We will say that the pair of functions $\psi = (f, g)$ is an *algebraic morphism* between the dessins $\mathcal{D}$ and $\mathcal{D}'$ with algebraic structures given by (G, Ω, r_0, r_1) and by $(G', \Omega', r'_0, r'_1)$ respectively, if $f : \Omega \to \Omega'$ is an onto function and $g : G \to G'$ is a group epimorphism such that $g(r_0) = r'_0$, $g(r_1) = r'_1$ and the following diagram commutes:

$$\begin{array}{ccc} \Omega \times G & \longrightarrow & \Omega \\ f \downarrow \quad \downarrow g & & \downarrow f \\ \Omega' \times G' & \longrightarrow & \Omega' \end{array}$$

that is, $f(ar) = f(a)g(r)$, for any $r \in G$ acting on any $a \in \Omega$.

We will say that $\psi = (f, g)$ as in Definition 1.5.4. is an *algebraic isomorphism* between the dessins $\mathcal{D}$ and $\mathcal{D}'$ with algebraic structure as above if $f : \Omega \to \Omega'$ is a bijective map and $g : G \to G'$ is a group isomorphism.

Two topological dessins $\mathcal{D}$ and $\mathcal{D}'$ are *isomorphic* if there exists a topological morphism ϕ between them, such that ϕ is a homeomorphism between the underlying surfaces, therefore the group of topological automorphisms of a dessin is too ample for our purposes, on the other hand, the group of analytic automorphisms of a dessin is finite, and in fact is a subgroup of the group of automorphisms of the underlying surface. The analytic definition of isomorphism introduces some rigidity in the structure of topological dessin that is needed if we want to define a one to one correspondence between algebraic and geometric structures.

Definition 1.5.5. Two dessins $\mathcal{D}$ and $\mathcal{D}'$ are *analytically equivalent* or just *equivalent* if there is an analytic isomorphism between them.

Proposition 1.5.6. Any dessin $\mathcal{D}$ which is equivalent to an analytic dessin $\mathcal{D}'$ is an analytic dessin itself.

Corollary 1.5.7. The following results are equivalent for dessins embedded in Riemann surfaces:

· The analytic dessins $\mathcal{D}$ and $\mathcal{D}'$ are isomorphic.

· The algebraic dessins $Alg(\mathcal{D})$ and $Alg(\mathcal{D}')$ are isomorphic.

· The oriented algebraic dessins $Alg^+(\mathcal{D})$ and $Alg^+(\mathcal{D}')$ are isomorphic.

We can now define the class $\overline{\mathcal{D}}$ of all the analytic dessins isomorphic to $\mathcal{D}$ and the class $\overline{\mathcal{A}}$ of all the algebraic dessins isomorphic to $\mathcal{A}$.

Theorem 1.5.8. For any class $\overline{\mathcal{A}}$ there is only one class $\overline{\mathcal{D}}$ such that, given $\mathcal{D} \in \overline{\mathcal{D}}$, $Alg(\mathcal{D}) \in \bar{\mathcal{A}}$. And for any class $\overline{\mathcal{D}}$ there is only one class $\overline{\mathcal{A}}$ so that the analytic dessins associated with the algebraic dessins in $\overline{\mathcal{A}}$ are all contained in $\overline{\mathcal{D}}$.

Corollary 1.5.9. Analytic dessins with morphisms and algebraic dessins with morphisms are two equivalent categories.

Therefore, we can define the functors An and Alg^+ that induce mutually inverse bijections between the isomorphism classes of oriented algebraic and analytic dessins of finite type.

Theorem 1.5.10. Let D_1 and D_2 be subgroups of $\Gamma[l_0, l_1, l_2]$ of finite index so that they are fundamental groups of some dessins, then, they give rise to isomorphic analytic dessins if and only if they are conjugate in $\Gamma[l_0, l_1, l_2]$.

Theorem 1.5.11. If $\mathcal{D}$ has fundamental group $D \leq \Gamma(l_0, l_1, l_2)$, then the full group of automorphisms of $\mathcal{D}$ is:

$$Aut(\mathcal{D}) \cong \frac{N_{(l_0,l_1,l_2)}(D)}{D}$$

where $N_{(l_0,l_1,l_2)}(D)$ is the normalizer of D in $\Gamma(l_0, l_1, l_2)$. Now we can assume that the dessin can be embedded in a Riemann surface, in which case, the group of orientation preserving automorphisms of $\mathcal{D}$ is:

$$Aut^+(\mathcal{D}) \cong \frac{N_{[l_0,l_1,l_2]}(D)}{D}.$$

We will say that a dessin is *regular* if its group of automorphisms acts transitively on the set of bits. Regular dessins corresponds to dessins with the greatest degree of symmetry. If we relax the transitivity condition, and request only that all the vertices have the same valency, all the edges have the same valency, and all the faces have the same valency, we get *uniform* dessins. Every regular dessin is uniform, but the converse is false.

Chapter Two

$PSL(2,p)$ and Hecke groups

This chapter is divided into two main parts. The first one (2.A.) studies the structure of the family of finite groups $PSL(2,p)$ where $p \in \mathbb{Z}$ is prime, focusing in particular in the cases where $p \in \{5,7,11\}$. These cases are covered in three separate sections.

In the second part (2.B.) we will introduce H^n (for $n \geq 3$), the family of Hecke groups of the first kind. Two separate sections have been used to describe the main properties of H^3, which is the modular group, and H^5.

While the Hecke groups H^3 and H^5 are important for our work because they provide us with a tool for the arithmetic characterization of the two main combinatorial structures embeddings described in chapter four, (Singerman's embedding of the Fano plane, and the 3-biplane embedding into $R_{g=70}$), we are also interested in $PSL(2,7)$ and $PSL(2,11)$, because they are isomorphic to the automorphism groups of the Fano plane and the 3-biplane respectively.

2.A. The structure of $PSL(2,p)$

Let us call $SL(2,p) := SL_2(\mathbb{Z}_p)$ where p is a prime and $p > 2$, i.e. the set of all 2×2 matrices with coefficients in the field $\mathbb{Z}_p$ such that the determinant is equal to 1. $PSL(2,p)$ is $SL(2,p)$ modulo its center, that is, modulo $(I, -I)$ where I is the identity matrix.

Proposition 2.A.1. $PSL(2,p)$ where $p \geq 3$ prime, is a non-commutative finite group of order

$$\frac{p(p-1)(p+1)}{2}.$$

Proof That $PSL(2,p)$ where $p \geq 3$ prime is non-commutative is easily seen by considering for example matrices

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

in $PSL(2,p)$ and checking that $ab \neq ba$.

To calculate its order we will count all matrices $\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ in $SL(2,p)$.

- $xy \neq 0$: There are $p-1$ choices for x and $p-1$ choices for y, we can then choose t in p ways and for each choice of x, y, t there is only one possible choice for z. That yields $p(p-1)^2$ elements of this form.
- $xy = 0$: If we assume $x = 0$ then there are p choices for t and $p-1$ choices for y. Once x, y, t are fixed there is only one possible value for z. If $y = 0$ the same reasoning applies and so there are $2p(p-1)$ matrices of this form.

We have proved that $|SL(2,p)| = p(p-1)(p+1)$ and the order of $PSL(2,p)$ follows naturally from the fact that $|PSL(2,p)| = [SL(2,p) : (\pm I)]$. $\square$

It is well known that every group $PSL(2,p)$ acts transitively on a set of $p+1$ elements, and we will prove it geometrically in the following lines. In order to see this action geometrically, we call $\mathbb{P}_p$ the projective line defined over the field $\mathbb{Z}_p$. Every element in $\mathbb{P}_p = \{0, 1, \ldots, p-1, \infty\}$ can be represented by a pair of homogeneous coordinates $[r:s]$(where $r, s \in \mathbb{Z}_p$), so that $[r:s]$ stands for:

$$\begin{cases} \infty & \text{if } s = 0 \\ \frac{r}{s} \pmod p & \text{if } s \neq 0 \end{cases}$$

where $\frac{r}{s} = rs^{-1}$.

As we have already mentioned in chapter 1, we will be using the standard abuse of notation whereby elements of $PSL(2,p)$ are thought of as matrices, and we will associate to every element $M \in PSL(2,p)$ a bijection of $\mathbb{P}_p$ defined in the usual way:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,p) \qquad \longleftrightarrow \qquad f_M(z) := \frac{az+b}{cz+d}$$

where as usual $f_M(\infty) = ac^{-1}$ if $c \not\equiv 0 \pmod p$ and $f_M(\infty) = \infty$ if $c \equiv 0 \pmod p$.

It is easy to see that the image of z by f_M does not depend on the choice of homogeneous coordinates for z. As f_M is a bijection in the finite set $\mathbb{P}_p$, we can treat it as a finite permutation and find its representation as a product of disjoint cycles. That this action is transitive is easy to see since

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in PSL(2,p)$$

and the action of a on the points of $\mathbb{P}_p$ is a cyclic permutation of the finite points (i.e. those which admit coordinates $[r:1]$, $r \in \mathbb{Z}_p$) while the action of b swaps ∞ and 0.

Proposition 2.A.2. *$PSL(2,p)$ acts doubly transitively on $\mathbb{P}_p$.*

Proof If we consider the ordered pair $(a,b) \in \mathbb{P}_p$ with $a \not\equiv b \pmod p$ and $\infty \notin \{a,b\}$, we see that the matrix

$$M_{(a,b)} = \begin{pmatrix} \frac{b}{b-a} & a \\ \frac{1}{b-a} & 1 \end{pmatrix} \in PSL(2,p)$$

takes the ordered pair $(0,\infty)$ to (a,b). If $\infty \in \{a,b\}$, we need to consider two cases separately:

· If $b = \infty$ we take $\frac{b}{b-a} = 1$ and $\frac{1}{b-a} = 0$, and then $M_{(a,b)}$ sends $(0,\infty)$ to (a,b).

· If $a = \infty$ then we can take

$$M_{(a,b)} = \begin{pmatrix} b & -1 \\ 1 & 0 \end{pmatrix}$$

and the action is doubly transitive.

However, the action of $PSL(2,p)$ on $\mathbb{P}_p$ is not triply transitive in general, as we can see with $\{0,1,\infty\}$ and $\{0,1,3\}$ for $p = 7$.

Proposition 2.A.3. Given $p, q \in \mathbb{Z}$ prime numbers so that $p \equiv 1 \pmod q$, there is only one non commutative group G of order $|G| = pq$ that we shall call G_{pq}. The structure of G_{pq} is given by

$$C_p \rtimes C_q \ = < f, e \mid f^p = e^q = efe^{-1}f^{-u} = 1 >$$

where $u \in \mathbb{Z}_p^*$ and $ord_{\mathbb{Z}_p^*}(u) = q$. Therefore the structure of the semidirect product is independent of the choice of u.

Proof As p is prime there is a subgroup C_p of order p in G, and by Sylow's theorem the number N of these subgroups satisfy

$$N \equiv 1 \pmod p \qquad \text{and} \qquad N \mid q.$$

Therefore $N = 1$ and using Sylow's theorem again $C_p \triangleleft G$ and the structure of G is given by:

$$< f, e \mid f^p = e^q = efe^{-1}f^{-u} = 1 >$$

where $u \in \mathbb{Z}_p$.

· Assume that $ord_{\mathbb{Z}_p^*}(u) \neq q$, then either $u = 1$ or $u^q \not\equiv 1 \pmod p$. If $u = 1$ it is easy to see that G is commutative and therefore of no interest to us. On the other hand, if $u^q \not\equiv 1 \pmod p$, using $efe^{-1}f^{-u} = 1$ we get $e^s fe^{-s} = f^{u^s}$ and by making $s = q$ we get $f = f^{u^q}$. By elementary group theory $(u^q - 1) \equiv 0 \pmod p$, which is a contradiction.

· Once we have proved that $ord_{\mathbb{Z}_p^*}(u) = q$, all that remains is to prove that for different u, the corresponding groups are isomorphic. To do so we choose a new generator $e_s = e^s$ (where $s \not\equiv 0 \pmod q$) so that we can rewrite the last identity in the presentation of G as $e_s f e_s^{-1} f^{-u^s} = 1$. As q is prime $ord_{\mathbb{Z}_p^*}(u^s) = q$ and u^s runs through all order q elements in $\mathbb{Z}_p^*$ therefore making it possible to derive any other presentation from the one in proposition 2.A.3.

Note If q is not prime, the result is not true, as we can see with the following non-commutative and non isomorphic groups of order 20:

$$G =< h, f \mid h^4 = f^5 = hfh^{-1}f = 1 >$$

$$G' =< h, f \mid h^4 = f^5 = hfh^{-1}f^3 = 1 >$$

In general, given $p \equiv 1 \pmod n$, the number of non-isomorphic groups of the form $C_p \rtimes C_n$ is $d(n)$, that is, the number of divisors of n.

Corollary 2.A.4. Let $p \in \mathbb{Z}$ be a prime $p \geq 3$ and q another prime so that $p \equiv 1 \pmod{q}$. There are only two groups of order pq:

· The cyclic group that we shall call C_{pq}.

· The semidirect product $C_p \rtimes C_q = G_{pq}$ and is presented as in the previous proposition. This group has one normal subgroup of order p that we shall call $C_p =< f >$, and p subgroups of order q, that are conjugate by the action of the elements of C_p.

Proof Let us call H any group so that $|H| = pq$ where p and q are as in the hypothesis. If H is non commutative, we are in the hypothesis of the previous proposition and we have finished. That $C_p \lhd H$ is unique and normal is true by Sylow's theorem. If the order q subgroups were not conjugate by f, then one of them should be normal in H (since f^s for a certain s will belong to its normalizer) which contradicts Sylow's theorem.

If H is commutative it must be the direct product of a C_p and a C_q, and therefore, we have $H \cong C_{pq}$.

Proposition 2.A.5. Let $p \in \mathbb{Z}$ be a prime $p \geq 3$ and $n = \frac{p-1}{2}$. If we consider the action of $PSL(2,p)$ on $\mathbb{P}_p$, where $x \in \mathbb{P}_p$. We get that $Stab(x) \cong C_p \rtimes C_n$ with presentation:

$$< e, f \mid f^p = e^n = efe^{-1}f^{-u} = 1 >$$

where $u \in \mathbb{Z}_p^*$ and $ord_{Z_p^*}(u) = n$.

Proof Since the action of $PSL(2,p)$ on $\mathbb{P}_p$ is transitive, we can choose $x = \infty$ without loss of generality, and we see that

$$|Stab(\infty)| = \frac{|PSL(2,p)|}{|\mathbb{P}_p|} = \frac{p(p-1)}{2}.$$

In fact, $Stab(\infty)$ consists of any element of the form $\begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix}$ where $a, b \in \mathbb{Z}_p$ and $a \neq 0$. If we call

$$f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad e = \begin{pmatrix} a & 1 \\ 0 & \frac{1}{a} \end{pmatrix}$$

where $ord_{\mathbb{Z}_p^*}(a) = 2n = p-1$, we have that $ord(f) = p$ and $ord(e) = n = \frac{p-1}{2}$, therefore e and f generate $Stab(\infty)$. We can see that $efe^{-1} = f^{a^2}$ and then the result is proved.

Lemma 2.A.6. Let p be a prime $p \geq 3$, $m \in \mathbb{Z}$ a positive integer so that $p \equiv 1 \pmod m$. The following two subgroups of $GL(2,p)$ have order pm:

$$\mathcal{G}_{pm} = \left\{ \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} \mid a, b \in \mathbb{Z}_p,\ a^m \equiv 1 \pmod p \right\}$$

$$\mathcal{H}_{pm} = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{Z}_p,\ a^m \equiv 1 \pmod p \right\}$$

$\mathcal{G}_{pm}$ has presentation

$$< h, f \mid h^m = f^p = hfh^{-1}f^{-s^{2r}} = 1 >$$

while a presentation for $\mathcal{H}_{pm}$ is given by

$$< h, f \mid h^m = f^p = hfh^{-1}f^{-s^{r}} = 1 >$$

where $s \in \mathbb{Z}_p^*$ and $ord_{\mathbb{Z}_p^*}(s) = m$ and $(r, m) = 1$.

· If m is odd we have that $\mathcal{G}_{pm} \cong \mathcal{H}_{pm}$.

· If m is even the two groups are non isomorphic.

Proof An easy process of counting elements shows that the order of both groups is pm, and using Sylow's theorems we get that the group structure is $C_p \rtimes C_m$, furthermore the groups are non commutative.

Once we have identified the structure of both groups, all we need to do is choose suitable generators f and h of order p and m respectively and study the action of h on f. A choice for $\mathcal{G}_{pm}$ is $f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $h = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}$ where $ord_{Z_p^*}(a) = m$. It is immediate to check that $hfh^{-1}f^{-a} = 1$ and the other possibilities arise when one substitute h with h^s where $(s, m) = 1$.

A choice of generators for $\mathcal{H}_{pm}$ is f as before and $h = \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}$, bearing this in mind, the rest of the proof is as above.

If m is odd we can define the isomorphism $\phi : \mathcal{G}_{pm} \longrightarrow \mathcal{H}_{pm}$ by

$$\phi \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} = \begin{pmatrix} a^2 & ba \\ 0 & 1 \end{pmatrix}.$$

By working with the presentations, it is easy to see that no isomorphism between the groups can be defined in the case where m is even.

Proposition 2.A.7. Let $p, q \in \mathbb{Z}$ be primes, $p > 5$ and $q > 2$ so that $p \equiv 1 \pmod{q}$, and let G_{pq} be as in Corollary 2.A.4. There are only two groups H up to isomorphism so that $G_{pq} < H$ and $[H : G_{pq}] = 2$. The two possibilities for H are either :

$$H \cong \mathcal{G}_{2pq} \cong G_{pq} \times C_2 \qquad \text{or} \qquad H \cong \mathcal{H}_{2pq}.$$

Proof As $[H : G_{pq}] = 2$, we have that $G_{pq} \triangleleft H$, so in order to describe H we need to find all possible order two automorphisms $\sigma : G_{pq} \longrightarrow G_{pq}$. Taking G_{pq} presented as in Proposition 2.A.3., and using the information from Corollary 2.A.4. we get that σ must satisfy:

$$\sigma(f) = f^a \qquad \sigma(e) = f^s e^g f^{-s}$$

where $a, s \in \mathbb{Z}_p$ and $b \in \mathbb{Z}_q$.

· As $\sigma^2(f) = f$ we have that $a \equiv \pm 1 \pmod{p}$.

· As σ has to preserve the relations of G_{pq}, from

$$1 = \sigma(efe^{-1}f^{-u}) \qquad \text{we get} \qquad e^b f^a e^{-b} = f^{ua}$$

but in particular we know that $e^b f^a e^{-b} = f^{au^b}$, and since $a \not\equiv 0 \pmod{p}$, we have $u^{b-1} \equiv 1 \pmod{p}$ and so $b \equiv 1 \pmod{q}$.

· If $a \equiv 1 \pmod{p}$, from $\sigma^2(e) = e$ we obtain $s = 0$ since otherwise G_{pq} should be commutative.

We have proved that either σ is equal to the identity $\sigma_1 : G_{pq} \longrightarrow G_{pq}$ or is a representative of the following family of order two automorphisms:

$$\sigma_2(f) = f^{-1} \qquad \sigma_2(e) = f^s e f^{-s}$$

where $s \in \mathbb{Z}_p$.

Since there is at least one element of order two in $H - G_{pq}$, we can call it g and assume that it induces the automorphism of G_{pq}, i.e. $\sigma_i(x) = gxg^{-1}$ for $i \in \{1, 2\}$ and $x \in G_{pq}$.

In the case of σ_1 it is easy to see that g commutes with the generators of G_{pq} and therefore we obtain $H \cong G_{pq} \times C_2$ and from there

$$H \cong \mathcal{G}_{2pq}.$$

For the action of σ_2, we need to prove that the same group H is generated regardless of the choice of $s \in \mathbb{Z}_p$. We can see that the equations relating the three generators of H are:

$$1)\ efe^{-1}f^{-u} = 1 \quad 2)\ gfgf = 1 \quad 3)\ gegf^{-s}e^{-1}f^{s} = 1.$$

We can choose to change the generator e for $E = f^n e f^{-n}$, where $n \in \mathbb{Z}_p$. If we do so we get the following expressions

$$1)\ EfE^{-1}f^{-u} = 1 \quad 2)\ gfgf = 1 \quad 3)\ gEgE^{-1}f^{(2n-s)(1-u)} = 1.$$

As $u \neq 1$ and $\mathbb{Z}_p$ is a field, if we let $n = \frac{s}{2}$ we can easily show that any choice of s yields the same group structure as $s = 0$. $\square$

Given a group G, we will say that the subgroups $F, H < G$ are *complementary* if given any element $g \in G$ we can express it in a unique way as a product $g = fh$ where $f \in F$ and $h \in H$. If that is the case, we will write $G = F \cdot H$. It should be noticed that two subgroups $F, H < G$ are complementary if and only if $G = FH$ and $F \cap H = \{1\}$.

A very interesting point about $PSL(2,p)$ goes back to Galois, who proved the following result:

Theorem 2.A.8. For $p > 11$, there is no subgroup of $PSL(2,p)$ which is complementary to any of its p-subgroups.

That means that for $p > 11$ we cannot get $PSL(2,p) \cong F \cdot \mathbb{Z}_p$, which implies that there is no transitive action on p points for $PSL(2,p)$ if $p > 11$. This can be seen in the following way: as p is a factor of $|PSL(2,p)|$ and p^2 is not (see Proposition 2.A.1.), if there is a transitive action of $PSL(2,p)$ on p points $\{x_1, \ldots, x_p\}$, no order p element of $PSL(2,p)$ can fix any $\{x_i\}$ (otherwise p^2 will divide $|PSL(2,p)|$), which means that given $g \in PSL(2,p)$ of order p, we have that $PSL(2,p) = Stab(x_i) \cdot < g >$ for any $i \in \{1, \ldots, p\}$.

For the cases where $p \leq 11$ and $PSL(2,p)$ simple, that is, for $p = 5, 7, 11$, we know that the following holds:

$$PSL(2,5) = A_4 \cdot C_5 \qquad PSL(2,7) = S_4 \cdot C_7 \qquad PSL(2,11) = A_5 \cdot C_{11}$$

and therefore $PSL(2,p)$ acts transitively on p points. Through this work we will try to explore this special feature geometrically, especially for the cases where $p = 7$ and $p = 11$.

2.1. The structure of $PSL(2,5)$

We will not go into too much length for the case $p = 5$, but for the sake of completeness, we will explain two ways of seeing its geometrical action on five points. $PSL(2,5)$ is isomorphic to A_5, the alternating group on five elements that is isomorphic to the group of symmetries of an icosahedron.

Lemma 2.1.1. A_5 is a simple group and it contains:

· No proper subgroup of order higher than 12.

· 6 conjugate subgroups of order 5.

· 10 conjugate subgroups of order 3.

· 5 conjugate subgroups of order 12, each one isomorphic to A_4.

This is not a full description of all the subgroups of A_5 and it merely describes those subgroups order that are interesting for our work. For each conjugacy class, we can find five different embeddings of A_4 into A_5, and its algebraic action on five points is readily seen as its action on these embeddings by conjugation. Our geometrical examples will mirror this action.

· Given an icosahedron, we can see its face centres as the vertices of five tetrahedra inscribed in it. The symmetry group of the icosahedron permutes the inscribed tetrahedra transitively.

· For another view of the same phenomenon, we can consider the set of edges of the icosahedron and form pairs of antipodal edges. Thus we obtain fifteen pairs of edges so that each pair defines a rectangle inscribed in the icosahedron. The fifteen rectangles form triples of mutually orthogonal elements, and the action of $PSL(2,5)$ will permute transitively the five triples.

It is easy to see how these two examples relate to the algebraic action explained above, since the symmetry group of both the tetrahedron and a triple of mutually orthogonal rectangles is isomorphic to A_4.

2.2. The structure of $PSL(2,7)$

We will represent $PSL(2,7)$ as the group generated by the following matrices

$$a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad c = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$$

where $a^2 = b^3 = c^7 = abc = I$. This group has order 168 and appears as the group of automorphisms of two important classical geometric objects: the Fano plane, which will be covered together with other examples of finite geometries in the next chapter, and Klein's quartic, that will be described in chapter four.

Lemma 2.2.1. [Kl1] $PSL(2,7)$ is a simple group of order 168 that contains:

· No proper subgroup of order higher than 24.

· $\binom{8}{2} = 28$ subgroups of order 3 in one conjugacy class.

· 21 subgroups isomorphic to C_4 in one conjugacy class.

· 8 subgroups of order 7 in one conjugacy class.

· 8 subgroups of order 21 in one conjugacy class.

· $7 \times 2 = 14$ subgroups with 24 elements in two conjugacy classes. All of them are isomorphic to S_4.

Lemma 2.2.2. If we consider the action of $PSL(2,7)$ on $\mathbb{P}_7$ we see that:

1· There is a one-one correspondence between the C_3 subgroups of $PSL(2,7)$ and the two point subsets $\{a,b\}$ of $\mathbb{P}_7$ given by the fixed point set of the action of each C_3 subgroup on $\mathbb{P}_7$.

If a C_3 subgroup corresponds to $\{a,b\}$, we will refer to it as $C_3^{a,b}$

2· There is a one-one correspondence between the C_7 subgroups of $PSL(2,7)$ and the points of $\mathbb{P}_7$ given by the fixed point of the action of each C_7 subgroup on $\mathbb{P}_7$.

If a C_7 subgroup corresponds to $a \in \mathbb{P}_7$, we will refer to it as C_7^a

3· Each subgroup of order 21 is isomorphic to $C_7 \rtimes C_3$. There is a one-one correspondence between the subgroups of order 21 and the stabilisers of elements in $\mathbb{P}_7$.

If a group of order 21 is the stabiliser of $a \in \mathbb{P}_7$ we will refer to it as G_{21}^a.

Proof

1· We can take any order three element and calculate its cycle structure on $\mathbb{P}_7$, for example:

$$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \in PSL(2,7) \quad \text{produces} \quad (0\ 6\ \infty)\ (1\ 3\ 5)\ (2)\ (4)$$

and we see that it fixes $\{2\}$ and $\{4\}$. Given that all order three elements are conjugate, they all have the same cycle structure and therefore every C_3 fixes two points in $\mathbb{P}_7$. Since $PSL(2,7)$ acts transitively on pairs of points of $\mathbb{P}_7$, the proof is finished.

2· For the case of C_7 we can do the same, if we start with $z \to z+1$, it only fixes ∞. Since the group action is transitive on $\mathbb{P}_7$, the rest follows.

3· From Proposition 2.A.5. we know that for any $a \in \mathbb{P}_7$

$$Stab(a) \cong C_7 \rtimes C_3.$$

Since there are only eight subgroups of order 21, each one is the stabiliser of a point. $\square$

SUBGROUPS OF $PSL(2,7)$ OF ORDER 24
IN TWO CONJUGACY CLASSES

Class P	Class L
$\{\{0,1\}\{2,4\}\{3,6\}\{5,\infty\}\}$	$\{\{0,1\}\{2,5\}\{3,\infty\}\{4,6\}\}$
$\{\{1,2\}\{3,5\}\{4,0\}\{6,\infty\}\}$	$\{\{1,2\}\{3,6\}\{4,\infty\}\{5,0\}\}$
$\{\{2,3\}\{4,6\}\{5,1\}\{0,\infty\}\}$	$\{\{2,3\}\{4,0\}\{5,\infty\}\{6,1\}\}$
$\{\{3,4\}\{5,0\}\{6,2\}\{1,\infty\}\}$	$\{\{3,4\}\{5,1\}\{6,\infty\}\{0,2\}\}$
$\{\{4,5\}\{6,1\}\{0,3\}\{2,\infty\}\}$	$\{\{4,5\}\{6,2\}\{0,\infty\}\{1,3\}\}$
$\{\{5,6\}\{0,2\}\{1,4\}\{3,\infty\}\}$	$\{\{5,6\}\{0,3\}\{1,\infty\}\{2,4\}\}$
$\{\{6,0\}\{1,3\}\{2,5\}\{4,\infty\}\}$	$\{\{6,0\}\{1,4\}\{2,\infty\}\{3,5\}\}$

We will consider the geometric actions of $PSL(2,7)$ on seven points in chapter four. Before we do that, we need to highlight the fact that the symmetric group on four elements S_4 contains four subgroups of order three. We have already seen that we can characterize all C_3 in $PSL(2,7)$ by their pair of fixed points (i.e. refer to $C_3^{a,b}$ as $\{a,b\}$), and we will label all subgroups of order 24 in $PSL(2,7)$ using this characterization, that is, we will refer to each of them by the four pairs of points that characterize its order three subgroups. That information is displayed in the previous table and will be used in the next two chapters, the 14 embeddings of S_4 have been separated into two conjugacy classes that we shall call class P and class L. (see Example 2.3.3. in the following section).

2.3. The structure of PSL(2,11)

We can represent the group $PSL(2,11)$ in many different ways that are related to triangle Fuchsian groups. We can see it as the group generated by the matrices (or any other conjugation of this triple):

$$a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 8 \end{pmatrix}, \quad \text{and} \quad c = \begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix}$$

where $a^2 = b^5 = c^{11} = abc = I$. This representation is useful when we study epimorphisms from Fuchsian groups with three periods of type $[2,5,11]$. On the other hand we can see it as the group generated by the matrices:

$$a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad c = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

where $a^2 = b^3 = c^{11} = abc = I$, and we will use these matrices for groups of type $[2,3,11]$. There are other ways of presenting this group that we will cover in chapter four.

$PSL(2,11)$ has order 660 and beside being the biggest group of the family of $PSL(2,p)$ that acts transitively on p points, it is the automorphism group of the 3-biplane, as we shall prove in chapter three.

Lemma 2.3.1. [Di] *$PSL(2,11)$ is a simple group of order 660 and contains:*

· *No proper subgroup of order higher than 60.*

· *$\binom{12}{2} = 66$ subgroups of order five in one conjugacy class.*

· *12 subgroups of order eleven in one conjugacy class.*

- 12 subgroups of order 55 in one conjugacy class.
- $2 \times 11 = 22$ subgroups of order 60 in two conjugacy classes. All of them are isomorphic to A_5.

This list is not complete, there are some other subgroups of $PSL(2, 11)$ that are not mentioned here, however, it is complete for every order that is mentioned in the lemma.

Lemma 2.3.2. If we consider the action of $PSL(2, 11)$ on $\mathbb{P}_{11}$ we see that:

- There is a one-one correspondence between the C_5 subgroups of $PSL(2, 11)$ and the two point subsets $\{a, b\}$ of $\mathbb{P}_{11}$ given by the fixed point set of the action of each C_5 subgroup on $\mathbb{P}_{11}$.

  If a C_5 subgroup corresponds to $\{a, b\}$, we will refer to it as $C_5^{a,b}$
- There is a one-one correspondence between the C_{11} subgroups of $PSL(2, 11)$ and the points of $\mathbb{P}_{11}$ given by the fixed point of the action of each C_{11} subgroup on $\mathbb{P}_{11}$.

  If a C_{11} subgroup corresponds to $a \in \mathbb{P}_{11}$, we will refer to it as C_{11}^a
- Each subgroup of order 55 is isomorphic to $C_{11} \rtimes C_5$. There is a one-one correspondence between the subgroups of order 55 and the stabilisers of elements in $\mathbb{P}_7$.

  If a group of order 55 is the stabiliser of $a \in \mathbb{P}_7$ we will refer to it as G_{55}^a.

The proof for the above lemma is similar to the proof for the corresponding one in the case $p = 7$ (Lemma 2.2.2.) so we will omit it here.

The geometric actions of $PSL(2, 11)$ on eleven points will be covered in chapter 4, and as we have done before we need to explain a way of classifying the embeddings of A_5 into $PSL(2, 11)$. We have seen that there are six possible embeddings of C_5 into A_5, and that each $C_5 < PSL(2, 11)$ can be labelled with its pair of fixed points on $\mathbb{P}_{11}$. Following the same idea as in $PSL(2, 7)$, we will characterize every $A_5 < PSL(2, 11)$ by a sextuple of pairs of points as it is shown in the table below.

SUBGROUPS OF $PSL(2,11)$ OF ORDER 60 IN TWO CONJUGACY CLASSES	
Class P	Class L
$\{\{0,1\}\{4,6\}\{7,10\}\{5,9\}\{3,8\}\{2,\infty\}\}$	$\{\{0,1\}\{6,8\}\{2,5\}\{3,7\}\{4,9\}\{10,\infty\}\}$
$\{\{1,2\}\{5,7\}\{8,0\}\{6,10\}\{4,9\}\{3,\infty\}\}$	$\{\{1,2\}\{7,9\}\{3,6\}\{4,8\}\{5,10\}\{0,\infty\}\}$
$\{\{2,3\}\{6,8\}\{9,1\}\{7,0\}\{5,10\}\{4,\infty\}\}$	$\{\{2,3\}\{8,10\}\{4,7\}\{5,9\}\{6,0\}\{1,\infty\}\}$
$\{\{3,4\}\{7,9\}\{10,2\}\{8,1\}\{6,0\}\{5,\infty\}\}$	$\{\{3,4\}\{9,0\}\{5,8\}\{6,10\}\{7,1\}\{2,\infty\}\}$
$\{\{4,5\}\{8,10\}\{0,3\}\{9,2\}\{7,1\}\{6,\infty\}\}$	$\{\{4,5\}\{10,1\}\{6,9\}\{7,0\}\{8,2\}\{3,\infty\}\}$
$\{\{5,6\}\{9,0\}\{1,4\}\{10,3\}\{8,2\}\{7,\infty\}\}$	$\{\{5,6\}\{0,2\}\{7,10\}\{8,1\}\{9,3\}\{4,\infty\}\}$
$\{\{6,7\}\{10,1\}\{2,5\}\{0,4\}\{9,3\}\{8,\infty\}\}$	$\{\{6,7\}\{1,3\}\{8,0\}\{9,2\}\{10,4\}\{5,\infty\}\}$
$\{\{7,8\}\{0,2\}\{3,6\}\{1,5\}\{10,4\}\{9,\infty\}\}$	$\{\{7,8\}\{2,4\}\{9,1\}\{10,3\}\{0,5\}\{6,\infty\}\}$
$\{\{8,9\}\{1,3\}\{4,7\}\{2,6\}\{0,5\}\{10,\infty\}\}$	$\{\{8,9\}\{3,5\}\{10,2\}\{0,4\}\{1,6\}\{7,\infty\}\}$
$\{\{9,10\}\{2,4\}\{5,8\}\{3,7\}\{1,6\}\{0,\infty\}\}$	$\{\{9,10\}\{4,6\}\{0,3\}\{1,5\}\{2,7\}\{8,\infty\}\}$
$\{\{10,0\}\{3,5\}\{6,9\}\{4,8\}\{2,7\}\{1,\infty\}\}$	$\{\{10,0\}\{5,7\}\{1,4\}\{2,6\}\{3,8\}\{9,\infty\}\}$

Example 2.3.3. Let us consider the subgroup G of $PSL(2,11)$ generated by the matrices:

$$a = \begin{pmatrix} 9 & 0 \\ 6 & 5 \end{pmatrix} \qquad b = \begin{pmatrix} 9 & 10 \\ 0 & 5 \end{pmatrix}.$$

If we consider the action of a on $\mathbb{P}_{11}$ we see that it corresponds to the permutation

$$(1,\infty,7,10,9)\ (2,3,5,6,4)\ (0)\ (8)$$

on the other hand, b corresponds to

$$(1,6,4,7,8)\ (2,10,9,5,0)\ (3)\ (\infty).$$

We can label $< a >= C_5^{0,8} = \{0,8\}$ and $< b >= C_5^{3,\infty} = \{3,\infty\}$ and see that $G \cong A_5$. Studying the action of G on the set of pairs of points of $\mathbb{P}_{11}$, we see that the pairs fixed by a and b belong to the same orbit, which is the sextuple of pairs that characterizes G:

$$\Big\{\ \{1,2\}\ \{5,7\}\ \{8,0\}\ \{6,10\}\ \{4,9\}\ \{3,\infty\}\ \Big\}.$$

And therefore those are the six subgroups isomorphic to C_5 contained in G.

2.B. The Hecke group H^n

In general, a Hecke group H_λ is a discrete subgroup of $PSL(2,\mathbb{R})$ that is generated by the following two transformations:

$$X : z \to -\frac{1}{z} \quad \text{and} \quad Z : z \to z + \lambda$$

In a result due to Hecke **[He]**, any group generated by X and Z as before is discrete if and only if $\lambda \geq 2$ or $\lambda = \lambda_n = 2\cos\frac{\pi}{n}$, where $n \in \mathbb{N}$ and $n \geq 3$. The Hecke groups where $\lambda > 2$, are Fuchsian groups of the second kind, with a limit set that is nowhere dense in $\mathbb{R}$, and they are of no interest to us. The case $\lambda = 2$ gives a Fuchsian group of the first kind but with a non compact quotient, as for the case $\lambda = \lambda_n$ they are Fuchsian groups of the first kind, and we will refer to them as H^n. Every time we mention "Hecke group" in this work, we shall mean Hecke group of the first kind. In particular, we see that

$$H^n \leq PSL(2, \mathbb{Z}[\lambda_n])$$

and for $n \geq 4$ the inclusion is strict.

In this work we will deal with H^3, which is the modular group $\Gamma = PSL(2,\mathbb{Z})$, and the group H^5 where λ_5 is the golden ratio.

$$\lambda_5 = \frac{1+\sqrt{5}}{2}$$

When considered in terms of their presentation, the groups H^n are triangle groups with signature $[2, n, \infty]$, where X is a representative for the elliptic elements of order 2, Z is a parabolic element and

$$Y = XZ^{-1} \ : z \longrightarrow \frac{-1}{z - \lambda_n}$$

is a representative for the elliptic elements of order n, thus there is one conjugate class of subgroups of order n.

We can find a fundamental domain for H^n, which will be a hyperbolic triangle in $\mathcal{U}$ with one or two of its vertices in $\partial\mathcal{U} = \widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. A standard choice for a fundamental region of H^n is the hyperbolic triangle with vertices 0, $e^{\frac{\pi i}{n}}$ and ∞; another choice is the hyperbolic triangle with vertices $-e^{\frac{-\pi i}{n}}$, $e^{\frac{\pi i}{n}}$ and ∞.

In general, the description of the elements of H^n cannot be explicitly done, however it is possible to describe them for the cases $n = 3$, 4 and 6. We will cover $n = 3$ in a later section while the other two cases are very similar to one another:

$$\begin{cases} n = 4 & \text{we have} \quad \lambda_4 = \sqrt{2} \\ n = 6 & \text{we have} \quad \lambda_6 = \sqrt{3}. \end{cases}$$

We will now prove that a matrix M is an element of H^{n_k} where $n_k = 2(k+1)$ and $m_k = (k+1)$, $(k = 1, 2)$ if M can be written in one of the following ways:

$$M_o = \begin{pmatrix} a\sqrt{m_k} & b \\ c & d\sqrt{m_k} \end{pmatrix} \qquad \text{or} \qquad M_e = \begin{pmatrix} a & b\sqrt{m_k} \\ c\sqrt{m_k} & d \end{pmatrix}$$

where $det(M) = 1$ and a, b, c, $d \in \mathbb{Z}$.

It is easy to see that any matrix in H^{n_k} must have this form. Any matrix in H^{n_k} is represented by a word in X and Z, and any word in X and Z with only one X would be of the form

$$W^1_{(r,s)} = Z^r X Z^s = \begin{pmatrix} r\sqrt{m_k} & rsm_k - 1 \\ 1 & s\sqrt{m_k} \end{pmatrix}$$

for a certain $r, s \in \mathbb{Z}$. We will prove that all words with an odd number of X in them have the form of the matrix M_o while the words with an even number of X have the form of M_e. We will proceed by induction and first assume that the result is true for any word W_l with l or less X in it.

Take a word W_{l+1} with $l+1$ appearances of X, we can express it as $W_{l+1} = W^1_{(r,s)} W_l$ for certain $r, s \in \mathbb{Z}$ and for a certain word W_l with l appearances of X. By hypothesis, if l is odd then

$$W_l = \begin{pmatrix} a\sqrt{m_k} & b \\ c & d\sqrt{m_k} \end{pmatrix}$$

where a, b, c, $d \in \mathbb{Z}$ and an easy calculation shows that $W^1_{(r,s)} W_l$ corresponds to a matrix of the form M_e. If l is even we choose W_l from the set of matrices of M_e and proceed in the same way.

As for the set of cusps, it is not known what the cusp set is for a general n, although we know that the cusp set of $PSL(2, \mathbb{Z}[\lambda_n])$ is $\mathbb{Q}(\lambda_n) \cup \{\infty\}$ (which we represent by $\widehat{\mathbb{Q}}(\lambda_n)$) and thus that of H^n must be a subset of $\widehat{\mathbb{Q}}(\lambda_n)$. One of the best results for this problem is due to Leutbecher [**Leu2**] who proved that for the values $n \in \{3,4,5,6,8,10,12\}$ the cusp set of H^n is:

$$\lambda_n \mathbb{Q}(\lambda_n^2) \cup \{\infty\}.$$

Therefore in the case $n = 3$, the cusp set is $\widehat{\mathbb{Q}}$ while for the case $n = 5$ its cusp set is $\widehat{\mathbb{Q}}(\sqrt{5})$, and in both cases the cusp set is the whole field completed with infinity. It turns out that these are the only two cases where the equality $Cusps(H^n) = \widehat{\mathbb{Q}}(\lambda_n)$ holds, since Wolfart [**Wol**] proved that it is false for any other case except perhaps $n = 9$ and Seybold proved later that $n = 9$ is not possible either [**Ro**].

The field $\mathbb{Q}(\lambda_n)$ is a number field, that is, it is a finite extension of $\mathbb{Q}$, and its degree is $\frac{\phi(2n)}{2}$ where ϕ is the Euler function. In fact it is the maximum real subfield of the number field $\mathbb{Q}(\epsilon_{2n})$, where ϵ_n is the nth root of unity, and we can easily calculate its degree since $[\mathbb{Q}(\epsilon_{2n}) : \mathbb{Q}(\lambda_n)] = 2$. The ring of algebraic integers of the field $\mathbb{Q}(\lambda_n)$ is $\mathbb{Z}[\lambda_n]$ (see [**Was**]), that means that if $x \in \mathbb{Q}(\lambda_n)$ is a solution for

$$x^m + a_{m-1}x^{m-1} + \cdots + a_0 = 0$$

where $a_i \in \mathbb{Z}$, then $x \in \mathbb{Z}[\lambda_n]$. And furthermore, given $y \in \mathbb{Z}[\lambda_n]$ and $d = \frac{\phi(2n)}{2}$, then y have the form

$$y = b_{d-1}\lambda_n^{d-1} + b_{d-2}\lambda_n^{d-2} + \cdots + b_0$$

where $b_i \in \mathbb{Z}$.

If we are to study the structure of the ideals of $\mathbb{Z}[\lambda_n]$, it suffices to study its prime ideals, since $\mathbb{Z}[\lambda_n]$ is the integral ring of a number field, it is a Unique Factorization Domain (UFD). We can go even further, since for $n < 68$ every $\mathbb{Z}[\lambda_n]$ is a PID (see [**Was**]) and every ideal in it is principal. Since we will only deal with $n = 3$ and $n = 5$ that result covers our two chosen cases.

Given P a prime ideal of $\mathbb{Z}[\lambda_n]$ (where $n < 68$) we will say that $Norm(P) = m$ if $m \in \mathbb{Z}$ is the smallest positive rational integer so that $P \mid m$. There is a general definition of the $Norm$ of an ideal where the class number is different from 1, but we will not need it here.

Our last concern is going to be the group of units of the rings $\mathbb{Z}[\lambda_n]$, but in this case we will only cover the cases we are going to use. For $n = 3, 5$ the integers rings are $\mathbb{Z}$ and $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$ respectively, while the group of units in the rings is $\{1, \ -1\}$ for the case $n = 3$, and is the cyclic group generated by λ_5 for the case $n = 5$.

A standard feature of any Hecke group is that its cusp set can be represented by finite continued fractions.

Notation We will use the notation $\lfloor \frac{a}{b} \rfloor$ to indicate a term inside a continued fractions expansion, and we will display the terms in a continued fractions expansion ordered, between brackets and separated by commas to make clear the non-commutative nature of this representation, so that

$$a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \cdots}}}}$$

will be represented by

$$\left(\lfloor a_0 \rfloor, \left\lfloor \frac{b_0}{a_1} \right\rfloor, \left\lfloor \frac{b_1}{a_2} \right\rfloor, \left\lfloor \frac{b_2}{a_3} \right\rfloor, \left\lfloor \frac{b_3}{a_4} \right\rfloor, \ldots \right).$$

Theorem 2.B.1. Given H^n a Hecke group, $\mathcal{C}$ is a cusp of H^n if and only if it can be expressed as a finite continued fraction of λ_n of the form:

$$\left(\lfloor a_0\lambda_n \rfloor, \left\lfloor \frac{-1}{a_1\lambda_n} \right\rfloor, \left\lfloor \frac{-1}{a_2\lambda_n} \right\rfloor, \ldots, \left\lfloor \frac{-1}{a_m\lambda_n} \right\rfloor\right)$$

where $a_i \in \mathbb{Z}$.

Proof H^n is generated by X and Z, so any element in H^n is a finite word in these two letters. As any parabolic element of H^n is conjugate to a power of XZX (that fixes 0), any cusp of H^n is the image of 0 by an element of H^n. Let us assume that the cusp $\mathcal{C}$ is the image of 0 by the word W defined by

$$W = Z^{a_0}XZ^{a_2}XZ^{a_3}\cdots Z^{a_{(m-1)}}XZ^{a_m}$$

where $a_i \in \mathbb{Z}$.

It is easy to see that

$$Z^a(z) = z + a\lambda_n, \ \ XZ^a(z) = \frac{-1}{z + a\lambda_n} \text{ and } Z^bXZ^a(z) = \frac{-1}{z + a\lambda_n} + b\lambda_n$$

so that when we consider the representation of W, we get

$$W(z) = \left(\lfloor a_0 \lambda_n \rfloor , \left\lfloor \frac{-1}{a_1 \lambda_n} \right\rfloor , \left\lfloor \frac{-1}{a_2 \lambda_n} \right\rfloor , \ldots , \left\lfloor \frac{-1}{a_m \lambda_n + z} \right\rfloor \right)$$

and we see that $W(0)$ is as in the hypothesis. To prove the converse it is enough to reverse the argument so that the proof is finished. $\square$

Since H^n is a free product of its generators, we can impose conditions on $W \in H^n$ to make its expression in terms of X and Z unique, and therefore we can as well consider the continued fraction representation for every cusp unique [**Ro**].

Example 2.B.2. We will show how the previous theorem works. Let

$$\frac{89 + 107\sqrt{5}}{38}$$

be a cusp of H^5, an easy calculation shows that it can be expressed as

$$\left(\lfloor 5\lambda_5 \rfloor , \left\lfloor \frac{-1}{-\lambda_5} \right\rfloor , \left\lfloor \frac{-1}{3\lambda_5} \right\rfloor \right)$$

and it is therefore the image of

$$Z^5 X Z^{-1} X Z^3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Another interesting area for the study of Hecke groups are their subgroups, in particular the special congruence subgroups. Given the group $PSL(2, \mathbb{Z}[\lambda_n])$ and I an ideal in $\mathbb{Z}[\lambda_n]$, we can define the *special congruence subgroups* of $PSL(2, \mathbb{Z}[\lambda_n])$ for I as:

$$PSL(2, \mathbb{Z}[\lambda_n])(I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}[\lambda_n]) \mid a \equiv d \equiv 1, \ b \equiv c \equiv 0 \pmod{I} \right\}$$

$$PSL(2, \mathbb{Z}[\lambda_n])_1(I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}[\lambda_n]) \mid a \equiv d \equiv 1, \ c \equiv 0 \pmod{I} \right\}$$

$$PSL(2, \mathbb{Z}[\lambda_n])_0(I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}[\lambda_n]) \mid c \equiv 0 \pmod{I} \right\}$$

We call $PSL(2,\mathbb{Z}[\lambda_n])(I)$ the *principal congruence subgroup* of $PSL(2,\mathbb{Z}[\lambda_n])$ for I. We can extend these definitions to H^n and define the special congruence subgroups of H^n as:

$$H^n_*(I) = PSL(2,\mathbb{Z}[\lambda_n])_*(I) \cap H^n$$

where the subscript $_*$ can be 0, 1 or omitted.

We will proceed to describe some relations satisfied by the special congruence subgroups of $PSL(2,\mathbb{Z}[\lambda_n])$. That the principal congruence subgroup satisfy

$$PSL(2,\mathbb{Z}[\lambda_n])(I) \triangleleft PSL(2,\mathbb{Z}[\lambda_n])_1(I)$$

is easy to see if we take the group epimorphism

$$\psi : PSL(2,\mathbb{Z}[\lambda_n])_1(I) \longrightarrow \frac{\mathbb{Z}[\lambda_n]}{I} \quad \text{defined by} \quad \psi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = [b]$$

where $[b]$ denotes the class of b inside $\frac{\mathbb{Z}[\lambda_n]}{I}$, and calculate $Ker(\psi)$.

The previous result is actually a particular case of a more general one since $PSL(2,\mathbb{Z}[\lambda_n])(I)$ is actually normal inside the group $PSL(2,\mathbb{Z}[\lambda_n])$ for which we need to consider

$$\Psi : PSL(2,\mathbb{Z}[\lambda_n]) \longrightarrow PSL\left(2,\frac{\mathbb{Z}[\lambda_n]}{I}\right) \quad \text{given by} \quad \Psi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} [a] & [b] \\ [c] & [d] \end{pmatrix}$$

where $[x]$ is the class of x in $\frac{\mathbb{Z}[\lambda_n]}{I}$, it is clear that $PSL(2,\mathbb{Z}[\lambda_n])(I) = Ker(\Psi)$. We have as well that

$$PSL(2,\mathbb{Z}[\lambda_n])_1(I) \triangleleft PSL(2,\mathbb{Z}[\lambda_n])_0(I).$$

This we can see if we consider the epimorphism

$$\chi : PSL(2,\mathbb{Z}[\lambda_n])_0(I) \longrightarrow \frac{U\big(\mathbb{Z}[\lambda_n]/I\big)}{\{\pm 1\}} \quad \text{given by} \quad \chi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = [a]$$

where $[a]$ is the class of a in $\frac{U(\mathbb{Z}[\lambda_n]/I)}{\{\pm 1\}}$, and $U\big(\mathbb{Z}[\lambda_n]/I\big)$ is the set of units of $\mathbb{Z}[\lambda_n]/I$. We can check that $Ker(\chi) = PSL(2,\mathbb{Z}[\lambda_n])_1(I)$. It follows in a natural way that the same happens to the special congruence subgroups of H^n.

$$H^n(I) \triangleleft H^n_1(I) \triangleleft H^n_0(I) < H^n \quad \text{and} \quad H^n(I) \triangleleft H^n$$

Example 2.B.3. The above inclusions are not necessarily strict, if we take H^5 and the ideal in $\mathbb{Z}[\lambda_5]$ generated by (2), we can see that

$$H^5(2) \triangleleft H_1^5(2) = H_0^5(2) < H^5$$

As for the indices of the special congruence subgroups of H^n for a given n, in general they are not known. The indices of the subgroups of $PSL(2, \mathbb{Z}[\lambda_n])$ (see [**Hur**]) provide an upper bound, but there are examples where these bounds are not attained.

$$[PSL(2, \mathbb{Z}[\lambda_n]) : PSL(2, \mathbb{Z}[\lambda_n])_0(I)] = Norm(I) \prod_{P|I} \left(1 + \frac{1}{Norm(P)}\right)$$

$$[PSL(2, \mathbb{Z}[\lambda_n]) : PSL(2, \mathbb{Z}[\lambda_n])(I)] = Norm(I)^3 \prod_{P|I} \left(1 - \frac{1}{Norm(P)^2}\right)$$

$$[PSL(2, \mathbb{Z}[\lambda_n])_1(I) : PSL(2, \mathbb{Z}[\lambda_n])(I)] = Norm(I)$$

where $Norm(I)$ stands for the norm of the ideal I, and P runs through the distinct prime divisors of I.

There are many cases where the indices of the special congruence subgroups of H^n, attain the upper bound provided by the indices of the special congruence subgroups of $PSL(2, \mathbb{Z}[\lambda_n])$, as it is shown in the following theorem by Frye [**Fr**]

Theorem 2.B.4. Let

$$I = \prod_{i=1}^{k} P_i^{e_i}$$

be the prime factorization of an ideal I of $\mathbb{Z}[\lambda_n]$ with $(6n, I) = 1$. If $n = 3$ (that is, for the modular group) let also be $(5, I) = 1$. Let $I^* = I \cap \mathbb{Q}(\lambda_n^2)$ and let $Norm(I)$ be the $Norm$ of I in $\mathbb{Q}(\lambda_n^2)$. Then

$$[H^n : H^n(I)] = 2^s Norm(I^*)^3 \prod_{P|I^*} \left(1 - \frac{1}{Norm(P)^2}\right)$$

with

$$s = \begin{cases} 0 & \text{if there is } i \neq j \text{ such that } P_i \cap \mathbb{Q}(\lambda_n) = P_j \cap \mathbb{Q}(\lambda_n) \\ 0 & \text{if there is } i \text{ such that } \frac{H^n}{H^n(P_i)} \cong PGL(2, \frac{\mathbb{Z}[\lambda_n]}{P_i}) \\ -1 & \text{otherwise} \end{cases}$$

and P running through the prime divisors of I^*.

2.4. The modular group H^3

For the case where $n = 3$, $\lambda_3 = 1$ and we get the most studied of all Hecke groups, namely the modular group Γ

$$\Gamma = PSL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{R}) \mid a,b,c,d \in \mathbb{Z} \right\}$$

With presentation $\Gamma =< X, Y_3 \mid X^2 = Y_3^3 = 1 >$ and signature $[2,3,\infty]$.

For a picture of a modular region (a fundamental region of the modular group) see *Fig. 2.1.* below.

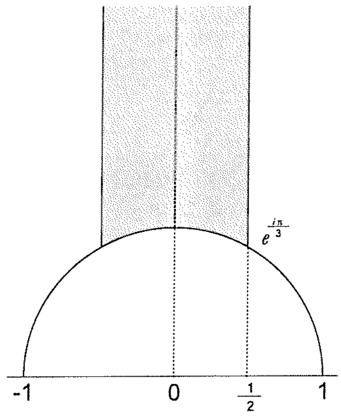


Fig. 2.1. A modular region

Proposition 2.4.1. The orbit of ∞ under the action of Γ is $\widehat{\mathbb{Q}}$.

Proof Let $\frac{a}{c} \in \mathbb{Q}$ be an irreducible fraction, as $(a,c) = 1$ we can find $d, b \in \mathbb{Z}$ so that $ad - bc = 1$, and therefore

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z})$$

if we call $T: z \longrightarrow \frac{az+b}{cz+d}$ it is clear that $T(\infty) = \frac{a}{c}$.

Corollary 2.4.2. The cusp set of Γ is $\widehat{\mathbb{Q}}$ and its action on its cusp set is transitive.

Proof That ∞ is a cusp of Γ is straightforward since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$, is parabolic and fixes ∞. $\square$

The indices of the special congruence subgroups of H^3 can be easily calculated as a particular case of the formulae provided at the end of section 2.B. (for a direct proof see [**Schoe**]):

$$[\Gamma : \Gamma_0(N)] = N \prod_{p|N} \left(1 + \frac{1}{p}\right)$$

$$[\Gamma : \Gamma_1(N)] = \frac{N^2}{2} \prod_{p|N} \left(1 - \frac{1}{p^2}\right)$$

$$[\Gamma : \Gamma(N)] = N\ [\Gamma : \Gamma_1(N)]$$

for any $N > 2$ and the product running along the positive prime divisors of N.

We will take for granted the first equality, which is widely known, and use it to prove the other two. In order to do so we will use the functions ψ and χ defined in section 2.B. in the context of the special congruence subgroups of $PSL(2, \mathbb{Z}[\lambda_n])$. When $n = 3$, $\Gamma = PSL(2, \mathbb{Z}[\lambda_3])$ so we can use χ to prove that

$$\frac{\Gamma_0(N)}{\Gamma_1(N)} \cong \frac{U(\mathbb{Z}_N)}{\{\pm 1\}} \quad \text{and therefore} \quad [\Gamma_0(N) : \Gamma_1(N)] = \frac{\varphi(N)}{2}$$

where φ is the Euler function. On the other hand, if we use ψ we can prove that

$$\frac{\Gamma_1(N)}{\Gamma(N)} \cong \mathbb{Z}_N \quad \text{and therefore} \quad [\Gamma_1(N) : \Gamma(N)] = N.$$

For any integer N of $\mathbb{Z}$ we will say that the congruence subgroups of Γ for N are of *level* N. We know that $\Gamma(N)$ has no elliptic elements if $N \geq 2$ so that if we fill in the cusps of $X = \frac{\mathcal{U}}{\Gamma(N)}$ we get in X the structure of a compact Riemann surface.

Another interesting result, which follows immediately from the discussion of special congruence subgroups of $PSL(2, \mathbb{Z}[\lambda_n])$ in section 2.B. is:

Theorem 2.4.3. If $p \in \mathbb{Z}$ is prime, we get that:

$$\frac{\Gamma}{\Gamma(p)} \cong PSL(2, p).$$

We can describe the action of the principal congruence group of level N on the cusp set of Γ. In fact for any N **[Schoe]**:

Theorem 2.4.4. Given two cusps of Γ, $\frac{p}{q}$ and $\frac{p'}{q'}$ if they are expressed as irreducible fractions in $\mathbb{Q}$, then the equivalence condition under the action of $\Gamma(N)$ is given by:

$$\frac{p}{q} \sim \frac{p'}{q'} \Leftrightarrow \begin{cases} p \equiv \pm p' \pmod N \\ \qquad \text{and} \\ q \equiv \pm q' \pmod N \end{cases}$$

Corollary 2.4.5. The number of inequivalent cusps under the action of $\Gamma(N)$ for $N \in \mathbb{Z}$ is:

$$\frac{1}{2} N^2 \prod_{p|N} \left(1 - \frac{1}{p^2}\right)$$

if $N \geq 3$ and 3 if $N = 2$.

Proof The proof follows from the above formulae as

$$Stab(\infty) \cong \mathbb{Z}_N \quad \text{in} \quad \frac{\Gamma}{\Gamma(N)}.$$

2.5. The Hecke group H^5

The next interesting Hecke group is $n = 5$ and that for a number of different reasons. It is together with $n \in \{3, 4, 6\}$ the only other case where $\mathbb{Q}(\lambda_n)$ is a quadratic extension of $\mathbb{Q}$, and together with $n = 3$ the only case where the cusp set is the completed field $\widehat{\mathbb{Q}}(\lambda_n)$. Unfortunately we do not have any explicit description of the matrices of H^5, although we can by-pass this problem by the use of continued fractions.

Theorem 2.5.1. **[Leut]** Given $\alpha \in \mathbb{Q}(\sqrt{5})$, it admits a finite continued fractions representation of the form:

$$\alpha = \left(\lfloor a_0 \lambda_5 \rfloor , \left\lfloor \frac{-1}{a_1 \lambda_5} \right\rfloor , \left\lfloor \frac{-1}{a_2 \lambda_5} \right\rfloor , \ \dots \ , \left\lfloor \frac{-1}{a_n \lambda_5} \right\rfloor \right)$$

where $a_i \in \mathbb{Z}$.

Corollary 2.5.2. The cusp set of H^5 is $\widehat{\mathbb{Q}}(\lambda_5)$.

Proof Immediate from the previous theorem and Theorem 2.B.1. □

Once we have described the cusp set of H^5 it is natural to try to answer similar questions to those solved for $n = 3$, like finding the index of the special congruence subgroups of H^5 or the action of the principal congruence subgroup for the ideal I, $H^5(I)$ over the cusp set of H^5. The following explicit calculation of the index $[H^5 : H_0^5(I)]$ can be found in **[CLLT]**.

Theorem 2.5.3. Given I a non-zero prime ideal of $\mathbb{Z}[\lambda_5]$, then:

$$[H^5 : H_0^5(I)] = \begin{cases} 5 & \text{if } I = (2) \\ 6 & \text{if } I = (2+\lambda_5) \\ p^2+1 & \text{if } I = (p) \text{ where } p \equiv \pm 2 \pmod 5,\, p \neq 2 \\ p+1 & \text{in any other case.} \end{cases}$$

where p is the positive rational prime contained in I.

We will prove a theorem for the action of $H^5(I)$ on the cusp set of H^5 that is similar to the one explained for $n = 3$. Before doing so we need to define the canonical form of a cusp of H^5.

Let $\frac{r}{s} \in \mathbb{Q}(\lambda_5)$ be a cusp of H^5, we can find a unique expression of $\frac{r}{s}$ as a continued fraction of λ_5 of the form:

$$\frac{n}{m} = \left(\lfloor a_0\lambda_5 \rfloor, \left\lfloor \frac{-1}{a_1\lambda_5} \right\rfloor, \left\lfloor \frac{-1}{a_2\lambda_5} \right\rfloor, \ldots, \left\lfloor \frac{-1}{a_n\lambda_5} \right\rfloor \right)$$

where $a_i \in \mathbb{Z}$. We will define a reduction process in order to construct the canonical form of $\frac{r}{s}$. Given a finite continued fraction of the form

$$\left(\left\lfloor \frac{b_0}{a_0} \right\rfloor, \ldots, \left\lfloor \frac{b_{n-2}}{a_{n-2}} \right\rfloor, \left\lfloor \frac{b_{n-1}}{a_{n-1}} \right\rfloor, \left\lfloor \frac{b_n}{a_n} \right\rfloor \right)$$

it is clear that the following expression represents the same number and is one step smaller:

$$\left(\left\lfloor \frac{b_0}{a_0} \right\rfloor, \ldots, \left\lfloor \frac{b_{n-2}}{a_{n-2}} \right\rfloor, \left\lfloor \frac{b_{n-1}a_n}{a_{n-1}a_n + b_n} \right\rfloor \right),$$

where for technical reasons we will not allow any form of fraction simplification within the last term of the expression.

If we apply this process to the continued fractions expansion of $\frac{n}{m}$ in terms of λ_5 as defined in Theorem 2.5.1., what we get is a fraction (that should not be reduced) whose numerator P and denominator Q are an expression in a_i and λ_5.

We say that $\begin{pmatrix} P \\ Q \end{pmatrix}$ is the *canonical form* of the fraction $\frac{n}{m}$ under λ_5. It is clear that there is a $l \in \mathbb{Z}$ such that $P = \lambda_5^l n$ and $Q = \lambda_5^l m$ so that $\frac{P}{Q} = \frac{n}{m}$. The main advantage of using the canonical form of a fraction is that it allows us to find a matrix $M \in H^5$ that takes it to either 0 or ∞. **[Ro]**

Theorem 2.5.4. Given

$$\begin{pmatrix} P \\ Q \end{pmatrix} \text{ where } \frac{P}{Q} \in \mathbb{Q}(\sqrt{5})$$

a fraction in the canonical form, then there is a matrix $M \in H^5$ so that

$$M \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} P \\ Q \end{pmatrix}.$$

Proof Taking X and Z in their matrix form,

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & \lambda_5 \\ 0 & 1 \end{pmatrix}$$

and

$$\frac{P}{Q} = \left(\lfloor a_0 \lambda_5 \rfloor, \left\lfloor \frac{-1}{a_1 \lambda_5} \right\rfloor, \left\lfloor \frac{-1}{a_2 \lambda_5} \right\rfloor, \ldots, \left\lfloor \frac{-1}{a_n \lambda_5} \right\rfloor \right)$$

it is easy to see that

$$M = Z^{a_0} X Z^{a_2} X Z^{a_3} \cdots Z^{a_{(n-1)}} X Z^{a_n}$$

satisfies all the above.

Example 2.5.5. Using the calculations from the Example 2.B.2., we see that

$$\begin{pmatrix} 38\lambda_5 + 15 \\ 3\lambda_5 + 4 \end{pmatrix} = \begin{pmatrix} 34 + 19\sqrt{5} \\ \frac{11+3\sqrt{5}}{2} \end{pmatrix}$$

is the canonical form for

$$\frac{89 + 107\sqrt{5}}{38} = \left(\lfloor 5\lambda_5 \rfloor, \left\lfloor \frac{-1}{-\lambda_5} \right\rfloor, \left\lfloor \frac{-1}{3\lambda_5} \right\rfloor \right)$$

because there is a matrix $M \in H^5$ with the form:

$$\begin{pmatrix} * & 34 + 19\sqrt{5} \\ * & \frac{11+3\sqrt{5}}{2} \end{pmatrix}.$$

Proposition 2.5.6. Given two cusps of H^5, $\frac{P}{Q}$ and $\frac{P'}{Q'}$ expressed in the canonical form. They are equivalent under the action of $H^5(I)$ for I an ideal of $\mathbb{Z}[\lambda_5]$ if the following holds:

$$\frac{P}{Q} \sim \frac{P'}{Q'} \Leftrightarrow \begin{cases} P \equiv \pm P' \mod(I) \\ \quad\text{and} \\ Q \equiv \pm Q' \mod(I) \end{cases}$$

Proof Let $\frac{P}{Q}$ and $\frac{P'}{Q'}$ be two cusps in the canonical form. Then there are $a, b, c, d \in \mathbb{Z}[\lambda_5]$ so that:

$$L_1 = \begin{pmatrix} a & P \\ b & Q \end{pmatrix}, \; L_2 = \begin{pmatrix} c & P' \\ d & Q' \end{pmatrix} \in H^5 \text{ and}$$

$$L_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} P \\ Q \end{pmatrix} \quad \text{and} \quad L_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} P' \\ Q' \end{pmatrix}.$$

Any transformation $V \in H^5$ such that $V \begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} P' \\ Q' \end{pmatrix}$ has the form

$$V = L_2 X Z^k X L_1^{-1}$$

for some $k \in \mathbb{Z}$. The condition for $V \in H^5(I)$, that is:

$$L_2 X Z^k X L_1^{-1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod (I)$$

forces $L_2^{-1} L_1 \equiv X Z^k X \mod (I)$, and that implies:

$$\begin{aligned} &1) \quad Pd - Qc \equiv \pm 1 \mod (I) \\ &2) \quad ad - cb \equiv k\lambda_5 \mod (I) \\ &3) \quad PQ' - P'Q \equiv 0 \mod (I) \\ &4) \quad Q'a - P'b \equiv \pm 1 \mod (I) \end{aligned}$$

As we know that in particular $V \in PSL(2, \mathbb{Z}[\lambda_5])$ we can ignore 2). From 3) $PQ' \equiv P'Q$ so if we multiply 1) by Q' and use that $dP' - cQ' = 1$ we get that $Q \equiv \pm Q' \mod (I)$. Using the same method and multiplying by P' we get $P \equiv \pm P' \mod (I)$, since the solutions satisfy 4) as well, the proof is finished.

Corollary 2.5.7. Given I a non-zero prime ideal of $\mathbb{Z}[\lambda_5]$, the number of inequivalent cusps for $H^5(I)$ is

$$\frac{(p^2-1)}{2}$$

where p is the rational prime in I.

The following theorem provides the group structure of the quotient of H^5 by a principal congruence subgroup of it **[Can]**.

Theorem 2.5.8. Given I a non-zero prime ideal of $\mathbb{Z}[\lambda_5]$ and p the integer prime inside I, we get that:

$$\frac{H^5}{H^5(I)} \cong \begin{cases} PSL(2,p) & \text{if } p \equiv \pm 1 \pmod{10} \\ PSL(2,p^2) & \text{if } p \equiv \pm 3 \pmod{10} \\ D_5 & \text{if } p = 2 \\ A_5 & \text{if } p = 3, 5. \end{cases}$$

In $\mathbb{Z}[\lambda_5]$ the ideal (11) is not prime as $(11) = (4+\sqrt{5})(4-\sqrt{5})$, so in order to study $PSL(2,11)$ we need choose one of the factors as I, and we will take $I = (4-\sqrt{5})$ since $4-\sqrt{5}$ and $4+\sqrt{5}$ are conjugate in $\mathbb{Z}[\lambda_5]$. The quotient groups that arise for I are:

$$\frac{H^5}{H^5(4-\sqrt{5})} \cong PSL(2,11) \quad \text{and} \quad \frac{H^5_0(4-\sqrt{5})}{H^5(4-\sqrt{5})} \cong C_{11} \rtimes C_5$$

Chapter Three

Finite Geometries

The aim of this chapter is to introduce incidence structures, that will lead to (finite) geometries and designs. Among them we will pay special regard to projective planes and biplanes, particularly to the structure of the Fano plane and that of the first three biplanes, paying special attention to the biplane of order three or 3-biplane.

Among other connections that will be explored in the next chapter, the Fano plane and the 3-biplane are the first two examples of Hadamard designs. Most of the information displayed here can be found in [**HP**] and [**Po**], as well as proofs for most of the results in this chapter, unless otherwise stated.

3.1. Introduction

An *incidence structure* or simply a *structure* $\mathcal{S}$ is a pair of non-empty sets $\mathcal{P}$ and $\mathcal{L}$, which we shall call *points* set $\mathcal{P}$ and *lines* set $\mathcal{L}$ with an *incidence relation* $I_{\mathcal{S}} \subset \mathcal{P} \times \mathcal{L}$ consisting of a non-empty set of pairs (P, l) where P is a point and l a line. We will assume that the structures have a finite number of points and lines, that is, that they are *finite* structures.

If $(P, l) \in I_{\mathcal{S}}$, we say that P belongs to l, that P and l are *incident* or that (P, l) is an *incidence pair*. The usual name for lines in structures is *blocks*, but we prefer lines since we will rapidly move into geometries.

We can represent the incidence relation $I_{\mathcal{S}}$ of a finite structure $\mathcal{S}$ with m points and n lines by a $m \times n$ matrix $M_{\mathcal{S}}$ that we call an *incidence matrix* of $\mathcal{S}$.

To construct an incidence matrix of $\mathcal{S}$ we need to index the points and lines sets $\mathcal{P} = \{P_i\}_{i=1}^m$, $\mathcal{L} = \{l_j\}_{j=1}^n$ and define $M_{\mathcal{S}}$ as the matrix with coefficients (M_{ij}) where $i = 1, \ldots, m$, $j = 1, \ldots, n$ and:

$$M_{ij} = \begin{cases} 1 & \text{if } (P_i, l_j) \in I_{\mathcal{S}} \\ 0 & \text{otherwise.} \end{cases}$$

$M_{\mathcal{S}}$ is not uniquely determined, but it contains all the relevant information about the structure $\mathcal{S}$. In fact, given $M_{\mathcal{S}}$ and $M'_{\mathcal{S}}$ two incidence matrices for the same structure $\mathcal{S}$, there are two permutation matrices (i.e. matrices so that in every row and column all the coefficients are 0 except for one that is 1) R and S so that:

$$RM_{\mathcal{S}}S = M'_{\mathcal{S}}.$$

We will say that a finite structure $\mathcal{S}$ is *connected* if for any two points P and Q there is a sequence of incidence pairs of the form

$$\left[(P, l),\ (P_1, l),\ (P_1, l_1),\ (P_2, l_1),\ (P_2, l_2),\ \ldots\ ,\ (Q, l')\right]$$

that connects P and Q.

With the previous definition of structure there are some pathological cases that we want to avoid:

- Two lines (resp. points) are *repeated* if they are incident with the same set of points (resp. lines), if we remove all repetitions (i.e. we get rid of all repeated elements of one kind but one), we say that we have *reduced* the structure. It is clear that reducing the structure is equivalent to eliminating all repeated rows and columns in the incidence matrix.
- We say that an element (point or line) is *isolated* when it is in none or just one pair of the incidence relation, and we say it is *full* when it is incident with all the elements of the other type. We *standardize* a structure if we remove all isolated and full elements. On the other hand, we standardize an incidence matrix when we remove all columns or rows that have either all coefficients equal to zero, all coefficients but one equal to zero, or no coefficients at all equal to zero.

A structure is *totally reduced* if it is reduced and standardized, we will call *geometry* a connected totally reduced structure. This definition of geometry is more restrictive than the usual one (that allows full and repeated elements) but we are not interested in the anomalous cases. Although in the process of reducing or standarizing a given structure $\mathcal{S}$ we might convert normal elements into repeated, isolated or full elements and therefore will have to remove these elements as well, the process is essentially well defined. Since the structures we are considering are finite, their total reduction is either the empty set or a geometry that we will call $\mathcal{G}_{\mathcal{S}}$.

In a reduced structure we can consider any line l as a subset of $\mathcal{P}$, so that P is incident with l if $P \in l$. We will say that a structure is *uniform* if all the lines contain the same number of points and *regular* if all the points are incident with the same number of lines, we will say that a uniform structure with k points in every line is *trivial* if every set of k points is incident with at least one line. A *finite geometry* is one with a finite number of points.

We define the *flags* (also called *bits* because they play a similar role to that of bits in dessins) of a finite geometry as an incidence pair, so that every pair (P, l) in the incidence relation corresponds to a flag.

Given two structures $\mathcal{S}$ and $\mathcal{T}$, an *isomorphism* between them is a bijective map from the point set of $\mathcal{S}$ to the point set of $\mathcal{T}$ that induces a bijective map between the line sets so that the incidence relations are preserved. An *automorphism* is an isomorphism of a structure into itself. These definitions extend in a straightforward way to finite geometries.

Since an incidence matrix contains all relevant information about a structure, we can characterize isomorphic structures in terms of their incidence matrices: two structures $\mathcal{S}$ and $\mathcal{T}$ are isomorphic if and only if there are incidence matrices $M_{\mathcal{S}}$ of $\mathcal{S}$ and $M_{\mathcal{T}}$ of $\mathcal{T}$, and two permutation matrices R, S such that:

$$RM_{\mathcal{S}}S = M_{\mathcal{T}}.$$

From the definition of geometry and the dual nature of repeated, full and isolated elements, it is clear that points and lines play an interchangeable role, and if we swap points for lines in a geometry $\mathcal{G}$ we get its *dual* geometry, $\mathcal{G}^T$. The use of the superscript ^{T} describe the fact that the incidence matrix of $\mathcal{G}^T$ is the transpose matrix of $M_{\mathcal{G}}$.

Proposition 3.1.1. Given a geometry $\mathcal{G}$:

· $\mathcal{G}$ is uniform if and only if $\mathcal{G}^T$ is regular.

· $(\mathcal{G}^T)^T = \mathcal{G}$.

We can consider as well the set of complements of the lines of a geometry $\mathcal{G}$ as a new set of lines, if $\mathcal{P}$ with this new set of lines is a geometry, we will call the resulting incidence structure the *complement geometry* of $\mathcal{G}$.

Example 3.1.2. Let $\mathcal{S}$ be the geometry with points $\mathcal{P} = \{P_1, P_2, P_3, P_4\}$ lines $\mathcal{L} = \{l_1, l_2, l_3, l_4\}$, and incidence relation given by:

$$l_1 = \{P_1, P_2, P_3\} \quad l_2 = \{P_1, P_3, P_4\} \quad l_3 = \{P_1, P_2, P_4\} \quad l_4 = \{P_2, P_3, P_4\}$$

We will call $\mathcal{S}'$ its complement structure, whose incidence relation is trivially given by:

$$l_1' = \{P_4\} \quad l_2' = \{P_2\} \quad l_3' = \{P_3\} \quad l_4' = \{P_1\}.$$

If we totally reduce $\mathcal{S}'$, we end up with an empty incidence structure (since every point belongs to a single line) therefore making it impossible to get the complement geometry of $\mathcal{S}$.

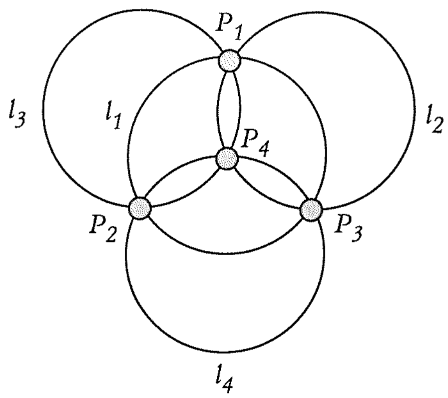


Fig. 3.1. Picture of the geometry of example 3.1.2.

Note We could easily avoid the problem showed in Example 3.1.2. by changing the definition of "full element" to include any element that is incident with all or all but one of the elements of the other kind. The complement of a geometry defined in that way will be already a totally reduced structure, and therefore a geometry. We have not done so because with that definition we would have excluded some structures commonly regarded as geometries, as $\mathcal{S}$ in the example above.

Proposition 3.1.3. If $\mathcal{G}$ is a geometry and $\mathcal{C}$ its complement structure:

· $\mathcal{G}$ is uniform if and only if $\mathcal{C}$ is uniform.

· $\mathcal{G}$ is regular if and only if $\mathcal{C}$ is regular.

Proposition 3.1.4. Given $\mathcal{G}$ a geometry, consider $\mathcal{G}^T$ its dual geometry and $\mathcal{C}$ its complement structure, we have:

$$Aut(\mathcal{G}) \cong Aut(\mathcal{G}^T) \cong Aut(\mathcal{C}).$$

Proof $\Phi \in Aut(\mathcal{G})$ is defined by a pair of bijections (ϕ, ϕ') where

$$\phi : \mathcal{P} \longrightarrow \mathcal{P} \quad \text{and} \quad \phi' : \mathcal{L} \longrightarrow \mathcal{L}$$

so that $(P, l) \in I_{\mathcal{G}} \Longleftrightarrow \big(\phi(P), \phi'(l)\big) \in I_{\mathcal{G}}$. It is now easy to see that

$$\Phi \in Aut(\mathcal{G}) \Longleftrightarrow \Phi' = (\phi', \phi) \in Aut(\mathcal{G}^T).$$

For the proof of the second part, we will consider $\mathcal{L}$ as a subset of $Power(\mathcal{P})$, the set of subsets of $\mathcal{P}$ and define $\mathcal{L}_{\mathcal{C}}$ as the set

$$\mathcal{L}_{\mathcal{C}} = \{l \mid l = \mathcal{P} - l',\ l' \in \mathcal{L}\}.$$

For a given element $\Phi \in Aut(\mathcal{G})$, as the definition of $\phi' : \mathcal{L} \longrightarrow \mathcal{L}$ is consistent with the pointwise extension of $\phi : \mathcal{P} \longrightarrow \mathcal{P}$, we can define $\psi : \mathcal{L}_{\mathcal{C}} \longrightarrow \mathcal{L}_{\mathcal{C}}$ by $\psi(l) = \mathcal{P} - \phi(\mathcal{P} - l)$ and in that way we have proved that

$$\Phi = (\phi, \phi') \in Aut(\mathcal{G}) \Longleftrightarrow \Psi = (\phi, \psi) \in Aut(\mathcal{C}). \qquad \square$$

A uniform geometry is called a *design*. If there are exactly λ lines through every t points of a design, we say that we have a $t - (v, k, \lambda)$ design, where v is the number of points of the design, and k the number of points in each line. In particular any $t - (v, k, \lambda)$ design is regular. We will say that a design is *square* if its incidence matrix is a square matrix, that is if there are as many lines as points in the design, a square $2 - (v, k, \lambda)$ design is called a *symmetric* design.

Theorem 3.1.5. In general, for a $t-(v,k,\lambda)$ design, if we call b the number of lines in the design and r the number of lines through a point, the two following equalities hold:

- $b = \lambda \frac{v(v-1)\cdots(v-t+1)}{k(k-1)\cdots(k-t+1)}$
- if $t > 0$ then $bk = vr$.

Proof The proof is an easy counting exercise. Since every t-uple of points defines λ lines and from a given line we can get $\binom{k}{t}$ t-uples of points, the number of lines is

$$b = \frac{\lambda\binom{v}{t}}{\binom{k}{t}}$$

The design is regular because by a similar argument the number of lines through any given point is

$$\frac{\lambda\binom{v-1}{t-1}}{\binom{k-1}{t-1}}$$

and the rest follows.

Proposition 3.1.6. In a symmetric design with parameters (v,k,λ), we have

$$v = \frac{k(k-1)+\lambda}{\lambda}.$$

Proof For the proof we will fix a point $P \in \mathcal{P}$ and consider all the flags of the form (Q,l) where $Q \neq P$ and $P \in l$ and we will count them in two different ways.

As there are r lines through P and each one contains k points, we have $r(k-1)$ flags of that sort. On the other hand, there are $v-1$ choices for Q and for every choice there are λ different possibilities for l yielding a total of $\lambda(v-1)$ flags. Since the design is symmetric $v = b$ and therefore $r = k$. $\square$

We could also have proved Proposition 3.1.6. as a Corollary to Theorem 3.1.5. where $b = v$ and $t = 2$.

We will only deal with designs such that $t = 2$, that is, there is a fixed number of lines λ through every two points of the design. We will define the *order* n for such a design as $n = r - \lambda$ where r is the number of lines through a point. There is a general definition of order for a $t-(v,k,\lambda)$ design (where $t \geq 2$), but it is of no interest for us here.

As designs are in particular geometries, the relations between a general design, its dual and its complement structure, follow easily from similar results about structures and geometries. A more difficult question is that of the relation among a $t-(v,k,\lambda)$ design, its dual and its complement. Here we will only tackle that question for square $t=2$ designs.

Proposition 3.1.7. Given a symmetric design $\mathcal{D}$ with parameters (v,k,λ), its dual $\mathcal{D}^T$ is a symmetric design with the same parameters.

Proof That $\mathcal{D}^T$ is a design is easy to see since as $\mathcal{D}$ is regular (becase it is symmetric), $\mathcal{D}^T$ is a uniform geometry.

As $\mathcal{D}$ is symmetric, if we call b the number of lines and r the number of lines through a point as in Theorem 3.1.5. following that theorem we get that $b=v$ and $r=k$. We still have to prove that every two lines of $\mathcal{D}$ intersect in precisely λ points, and we will use the incidence matrix $M_{\mathcal{D}}$ for this part.

If we call I_v the identity matrix of order v and J_v a v by v matrix with all its entries 1, we get that

$$M_{\mathcal{D}} \cdot M_{\mathcal{D}}^T = rI_v + \lambda(J_v - I_v)$$

where the element rI_v on the right means that there are r lines trough each point and the element $\lambda(J_v - I_v)$ comes from the fact that any two points determine λ lines. It is easy to prove that

$$|M_{\mathcal{D}} M_{\mathcal{D}}^T| = (r-\lambda)^{v-1}\big(r + (n-1)\lambda\big)$$

and since $r > \lambda > 0$, $M_{\mathcal{D}}$ is an invertible matrix.

From there we get:

$$M_{\mathcal{D}} \cdot \left(M_{\mathcal{D}}^T \cdot M_{\mathcal{D}}\right) = \left(M_{\mathcal{D}} \cdot M_{\mathcal{D}}^T\right) \cdot M_{\mathcal{D}} = rM_{\mathcal{D}} + \lambda\left(J_v \cdot M_{\mathcal{D}} - M_{\mathcal{D}}\right)$$

but

$$J_v \cdot M_{\mathcal{D}} = kJ_v = rJ_v = M_{\mathcal{D}} \cdot J_v$$

and since $M_{\mathcal{D}}$ is non singular, we get that

$$M_{\mathcal{D}}^T \cdot M_{\mathcal{D}} = rI_v + \lambda(J_v - I_v)$$

and therefore any two lines of $\mathcal{D}$ have precisely λ points in common.

Theorem 3.1.8. Given a non trivial symmetric design $\mathcal{D}$ with parameters (v,k,λ), its complement geometry is a symmetric design with the same order as $\mathcal{D}$ and parameters $(v, v-k, v-2k+\lambda)$.

Theorem 3.1.9. For a non trivial symmetric design of order n and parameters (v, k, λ), we have that

$$4n - 1 \leq v \leq n^2 + n + 1.$$

Proof By Proposition 3.1.6. we have that:

$$v = \frac{k(k-1) + \lambda}{\lambda}.$$

In this kind of design $n = k - \lambda$ ($v = b$ by symmetry, hence $k = r$ by Theorem 3.1.5.) which yields:

$$v = \frac{n(n-1)}{\lambda} + \lambda + 2n.$$

If we consider the above expression as a function $v(\lambda)$ where $1 \leq \lambda \leq n(n-1)$ (if $\lambda > n(n-1)$ then $v(\lambda)$ would be a rational number, which is impossible), we see that it has a concave graph with a minimum at

$$\lambda_0 = \sqrt{n(n-1)}$$

and two maxima at

$$\lambda_1 = 1 \text{ and } \lambda_2 = n(n-1).$$

The two natural numbers closest to λ_0 are $(n-1)$ and n, and we have that

$$v(n) = v(n-1) = 4n - 1$$

so $v(\lambda) \geq 4n - 1$. On the other hand, for the two maxima values we get that

$$v(1) = v(n^2 - n) = n^2 + n + 1$$

and therefore $v(\lambda) \leq n^2 + n + 1$.

The two extreme cases of this theorem actually occur:

- For $v = 4n - 1$ we get that either the design or its complement have parameters $2 - (4\lambda + 3, 2\lambda + 1, \lambda)$. Any symmetric design with that set of parameters and for any λ is called a *Hadamard 2-design* or simply a *$\mathcal{H}$-design.*
- For $v = n^2 + n + 1$ we have that either the design or its complement is a projective plane, as we shall see in the next section.

There is a limited amount of information regarding the existence of designs for a given set of parameters (t, v, k, λ) or about the number of non equivalent desings for one choice of parameters. The main result about non existence of square $2-(v, k, \lambda)$ designs is (for a proof see [**HP**]):

Theorem 3.1.10. (Bruck-Ryser-Chowla) If $v,\ k,\ \lambda \in \mathbb{Z}$ satisfy

$$(v-1)\lambda = k(k-1)$$

then for the existence of a symmetric (v, k, λ) design it is necessary that:

- If v is even then $k - \lambda$ is a square.
- If v is odd, then $z^2 = (k-\lambda)x^2 + (-1)^{\frac{v-1}{2}}\lambda y^2$ has a non-trivial solution in integers x, y and z.

The following corollary is very important when dealing with projective planes, since it restricts the possible orders for them.

Corollary 3.1.11. If a symmetric design with $\lambda = 1$ and order n exists, and

$$n \equiv 1 \pmod 4 \quad \text{or}$$
$$n \equiv 2 \pmod 4$$

then n can be expressed as a sum $a^2 + b^2$ where $a, b \in \mathbb{Z}$.

Proof Since the design is symmetric and $\lambda = 1$, by Proposition 3.1.6. we have that

$$v = k(k-1) + 1 = (n+1)n + 1.$$

Assume now $n \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$, we then have $v \equiv 3 \pmod 4$ and therefore $(-1)^{\frac{v-1}{2}} = -1$ so that from the second statement in the previous theorem, we have $z^2 = nx^2 - y^2$ and then

$$n = \frac{z^2}{x^2} + \frac{y^2}{x^2}$$

where $x, y, z, n \in \mathbb{Z}$. It is well known that an integer is the sum of two square rationals if and only if it is the sum of two square integers. $\square$

We can obtain similar results about the existence of $\mathcal{H}$-designs of a given order by using Hadamard matrices. A *Hadamard matrix* of order n is a square n by n matrix H whose entries are either $+1$ or -1 and such that $HH^T = nI$. We can define a generalized version of the permutation matrix that we have used when dealing with incidence matrices, a *generalized permutation matrix* is a square matrix whose entries are ± 1 or 0 and such that in each row and column all entries but one are zero.

Theorem 3.1.12. If H is a Hadamard matrix of order n and R, S are generalized permutation matrices of order n, then $H' = RHS$ is a Hadamard matrix of order n. We will say that H and H' are *Hadamard equivalent.*

It is clear that a class of Hadamard equivalent matrices of order n is stable under the action of the group of all row and column permutations and sign changes. In fact, for every class of Hadamard equivalent matrices of order n we can choose a matrix in the class such that all entries in the first row and column are 1 (which is obviously not unique for a given class). We will call such a matrix a *normalized Hadamard matrix.*

The following result impose a restriction on the order of possible Hadamard matrices:

Theorem 3.1.13. If H is a Hadamard matrix of order n then n must satisfy one of the following conditions:

$$\begin{cases} n = 1 \\ n = 2 \\ n \equiv 0 \pmod 4. \end{cases}$$

As one would suspect, there is a strong relation between Hadamard matrices and $\mathcal{H}$-designs:

Theorem 3.1.14. Let H be a normalized Hadamard matrix of order $n > 4$. If we delete the first row and column of H and substitute all -1 entries by 0, we obtain a matrix $M_{\mathcal{S}}$, which is an incidence matrix for a square design $\mathcal{S}$ with parameters

$$2 - \left(n - 1, \frac{n-2}{2}, \frac{n-4}{4}\right)$$

that is, a $\mathcal{H}$-design. Conversely the incidence matrix of any Hadamard 2-design for a given λ becomes a normalized Hadamard matrix of order $4(\lambda + 1)$ by the reverse procedure.

Example 3.1.15. H is a normalized Hadamard matrix of order 8, if we apply the method described above to it, we get M_{Fano} which is an incidence matrix for a square design with parameters $2-(7,3,1)$ that is in fact the Fano plane and that will be described in the next section.

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 \end{pmatrix} \qquad M_{Fano} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

As we will see, the result in this example is in fact true for any normalized Hadamard matrix of order 8. By Theorem 3.1.14. any such a matrix will yield an incidence matrix for a symmetric $2-(7,3,1)$ design, whose complement is a symmetric $2-(7,4,2)$ (see Theorem 3.1.8.). As there is only one symmetric $2-(7,4,2)$ design (the 2-biplane, see the proof of Proposition 3.3.5.), there is only one symmetric $2-(7,3,1)$ design, the Fano Plane.

3.2. Projective planes

A *projective plane* is a geometry that satisfies the following three axioms:

· Two distinct points are contained in a unique line.

· Two distinct lines intersect in a unique point.

· There exist four points of which no three are incident with the same line.

Theorem 3.2.1. A finite projective plane is the same as a symmetric design $(v,k,1)$ with $k>2$.

Proof Given $\mathcal{D}$ a symmetric $(v,k,1)$ design, since $\lambda=1$ it is true that any two distinct points are contained in a unique line, and by Proposition 3.1.7. the second projective plane axiom is also true. As $k>2$, by Proposition 3.1.6. we have that

$$v = k(k-1)+1 > 3(k-1) > k.$$

As $v>k$ we can choose three points P_1, P_2, P_3 not in the same line, and call $l_{\{i,j\}}$ the line incident with P_i and P_j. It is easy to see that

$$|l_{\{1,2\}} \cup l_{\{2,3\}} \cup l_{\{1,3\}}| = 3(k-1)$$

and so we can choose a fourth point P_4 of the design so that the third axiom is also satisfied.

To prove the reverse, we need to show that a projective plane is uniform and square. We can take a line l and a point $P \notin l$. By the first and second axiom, there are as many lines incident with P as there are points in l and we shall call this number k. Let us take l' any line incident with P, by the third axiom we can choose $Q \notin l'$, P', Q' so that no three of them are incident with the same line. If we call l'' the line defined by P' and Q', l'' has k points, and therefore there are k lines incident with Q, which means that there are k points in l' and the geometry is uniform. By Theorem 3.1.5. the geometry is also square, and by the first axiom $t = 2$ and $\lambda = 1$. $\square$

We define the *order* of a finite projective plane as $k - 1$, that is the number n, such that there are $n + 1$ points in each line, and $n + 1$ lines through each point.

Corollary 3.2.2. For any symmetric (v, k, λ) design with $\lambda = 1$, we have that $k = n + 1$ and $v = n^2 + n + 1$ where n is the order of the design.

There is an easy way to construct finite projective planes for which we shall need the following well known result about finite fields:

Theorem 3.2.3. If K is a finite field then $|K| = p^r$ where p is prime and $r \in \mathbb{Z}$, $r \geq 1$. And for any p^r as before there is a unique finite field of order p^r that we shall denote F_{p^r}.

If we take K to be a finite field, and V to be a 3 dimensional vector space over K, we can define $P(K)$ to be the structure whose points are the 1-dimensional subspaces of V, and whose lines are the 2-dimensional subspaces of V, while the incidence structure is given by the subspace inclusion in V.

Proposition 3.2.4. $P(K)$ is a projective plane of type $2-(n^2+n+1, n+1, 1)$ where $n = |K|$.

Proof To define a 1-dim subspace of V we need to consider any non zero vector of V, and the fact that any 1-dim subspace contains $n - 1$ such vectors. Following that we see that $P(K)$ contains $\frac{n^3-1}{n-1} = n^2 + n + 1$ points.

Given a 2-dim subspace, it contains $n^2 - 1$ non zero vectors, and reasoning as before we get that it contains $\frac{n^2-1}{n-1} = n + 1$ 1-dim subspaces, that is, each line in $P(K)$ contains $n + 1$ points. Obviously any two different 1-dim subspaces generate a 2-dim subspace. The incidence structure for the projective plane follows easily from the properties of a vector space.

Corollary 3.2.5. There is at least one projective plane of order n for any $n = p^r$ where p is prime and $r \in \mathbb{Z}$, $r \geq 1$.

The two main results about the existence and non-existence of projective planes are Corollary 3.1.11. and the corollary above. Little is known in general for any other values of n and the smallest value for which we do not know whether there is a projective plane with such order is $n = 10$. There are many projective planes that are not constructed in the way mentioned above, for more information on the topic see **[HP]**.

Another important issue is the calculation of the automorphism group of a projective plane. Although we do not know a general way of calculating automorphism groups for an arbitrary plane, we do know it for $P(K)$ when $|K| = p^r$ and p prime. What follows is the Fundamental Theorem of Projective Geometry, that can be found in many standard books on projective geometry, we will only use what applies to finite projective planes.

Theorem 3.2.6. Fundamental Theorem of Projective Geometry
Let K be a finite field of order $|K| = p^r$ and p prime, then

$$Aut\,(P(K)) \cong PGL(3, K) \rtimes Aut(K)$$

Corollary 3.2.7. Let K be a finite field of order p where p is a prime, then

$$Aut\,(P(K)) \cong PGL(3, p)$$

Proof From Theorem 3.2.6. and the fact that if $K \cong \mathbb{Z}_p$ as a field, then $Aut(K) \cong \{1\}$.

The smallest possible non-trivial finite projective plane is the *Fano Plane* composed by seven points and seven lines, so that there are three points in any line, and there are three lines going through any given point, that is, it is a $2-(7,3,1)$ design. It corresponds to $P(\mathbb{Z}_2)$. The Fano plane is a *homogeneous geometry*, which means that any two different points of it are indistinguishable, i.e. there is an automorphism of the geometry carrying one point to the other.

The Fano plane is the only projective plane of order 2, and it is easy to see that it has 21 bits. We will see that its full group of automorphisms has order 168 and it is isomorphic to $PSL(2,7)$. For the most traditional picture of the Fano plane with its flags labelled, see *Fig. 3.2.*

Beside being related to $\mathbb{Z}_2$, the Fano plane is closely related to the finite field $\mathbb{Z}_7$. In fact, if we consider the set of quadratic residues mod 7, that is, the set $\{1,2,4\}$ and let the transformation $z \longrightarrow z+1 \pmod 7$ act on the set of triples of $\mathbb{Z}_7$, the orbit of $\{1,2,4\}$ by this action is:

$$\{1,2,4\} \ \{2,3,5\} \ \{3,4,6\} \ \{4,5,0\} \ \{5,6,1\} \ \{6,0,2\} \ \{0,1,3\}$$

and if we call each of these sets a line, and consider every element $n \in \mathbb{Z}_7$ as a point of a structure, we can see that what we get is the incidence relation of the Fano plane.

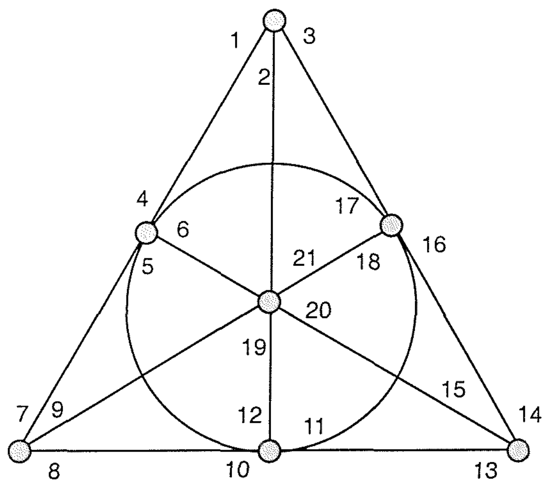


Fig. 3.2. Fano plane

Since the Fano plane represents a highly symmetric hypergraph, it is only natural to try to find ways of embedding it as a conformal structure in a highly symmetric way in a surface. The optimum way of doing this will be to find an embedding that keeps both its symmetries and its combinatorial properties, while corresponding to a smooth embedding into a surface.

If we try to embed it as a regular hypermap inside a Riemann surface, it is knows that it can be done in two different ways, as a $(3,3,3)$ dessin in the triangular torus [**Wa**] or as a $(3,3,7)$ dessin inside Klein's quartic [**Si4**]. Other embeddings of finite geometries in Riemann surfaces as dessins are also discussed in [**Si4**]. Embeddings of finite geometries as regular dessins into Riemann surfaces are closely related to the existence of big subgroups inside the automorphism group of the geometry.

Corollary 3.2.8. The automorphism group of the Fano plane is isomorphic to:

$$PSL(2,7)$$

which is the simple group of order 168.

Proof Immediate from Corollary 3.2.7. and the fact that (for a proof see for example **[Di]**)

$$PGL(3,2) \cong PSL(2,7).$$

3.3. Biplanes

A generalization of finite projective planes arises naturally when we allow two lines to intersect in more than a point (respectively when we allow two points to define more than one line), if λ is the number of points that belong to two different lines, we will call these generalization λ-*plane* (*bi-* and *tri-* for the cases $\lambda = 2,\ 3$). So a *biplane* is a symmetric geometry that satisfies the following two axioms:

· Two distinct points are contained in exactly two distinct lines.

· Two distinct lines intersect in exactly two distinct points.

Finite biplanes have been much less studied than standard projective planes, and we only know 17 different examples of them **[Po]**. We will associate with every biplane its *order*, which is the number n such that every line of it has $n+2$ points. As far as we know nobody has tried to find smooth embeddings of biplanes in Riemann surfaces before. The 17 known biplanes have orders in the set $\{1, 2, 3, 4, 7, 9, 11\}$. By Theorem 3.1.10. there are no biplanes of orders $\{5, 6, 8\}$. The first example of a biplane is a $2-(4,3,2)$ design, which is a biplane of order 1. It is immediate to see that there is only one possible incidence structure for such a biplane where every line is determined by the only point not belonging to it. The biplane of order 1 has $vk = 12$ flags and can be embedded into the sphere, since we can see it as a regular map of type (3,2,3) i.e. a tetrahedron. For two different pictures of the biplane of order 1 see *Fig. 3.3.* We have already seen that its complement is not a geometry in Example 3.1.2.

The second example of a biplane is a $2-(7,4,2)$ design which corresponds to biplanes of order 2. We will later see that there is only one possible incidence structure for biplanes of order 2 and 3 (see Theorem 3.3.3. and the proof of Proposition 3.3.5.), although that is not the case for biplanes of higher orders, as there are 3 biplanes of order 4 [**Hus**], and 4 biplanes of order 7 [**MS**].

The biplane of order 2 is the complement of the Fano plane, it has 7 points and 7 lines, but there are four points in each line making a total of 28 bits. Its group of automorphisms is again $PSL(2,7)$ (see Proposition 3.1.4. or Proposition 3.3.5.) and it is a $2-(7,4,2)$ design. For a picture of this biplane see *Fig. 3.4.* where the thick circle is highlighted because it also meets the central vertex.

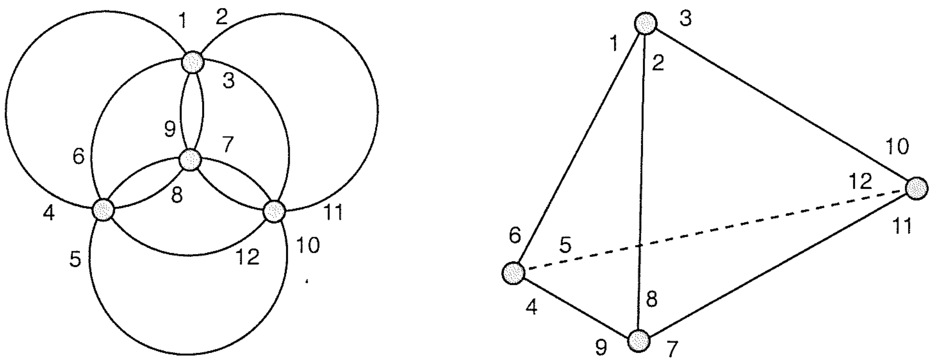


Fig. 3.3. 1-biplane on the plane and embedded as a dessin in the sphere

Since there is no subgroup of $PSL(2,7)$ of index 6, it is not possible to embed this structure as a regular hypermap or hypergraph into a Riemann surface with $PSL(2,7)$ as its automorphisms group. Nevertheless, we will show an alternative representation of its incidence structure into Klein's Riemann surface in chapter four.

The biplane we are most interested in is the biplane of order 3, which has 11 points and 11 lines, and five points in each line making a total of 55 bits. Its full group of automorphisms has order 660 and is isomorphic to $PSL(2,11)$. Since it is the only biplane of order three, we will refer to it as the 3-biplane. It is a $2-(11,5,2)$ design.

The 3-biplane is related to the quadratic residue classes of $\mathbb{Z}_{11}$ in precisely the same way as the Fano plane with $\mathbb{Z}_7$. That is, if we take the set of quadratic residues mod 11, $A=\{1,3,4,5,9\}$ and construct eleven sets of five elements by considering the action of $\phi : z \longmapsto z+1 \pmod{11}$, we can regard the elements $n \in \mathbb{Z}_{11}$ as points of the 3-biplane where the lines are represented by $\phi^r(A)$ where $r \in \mathbb{Z}_{11}$.

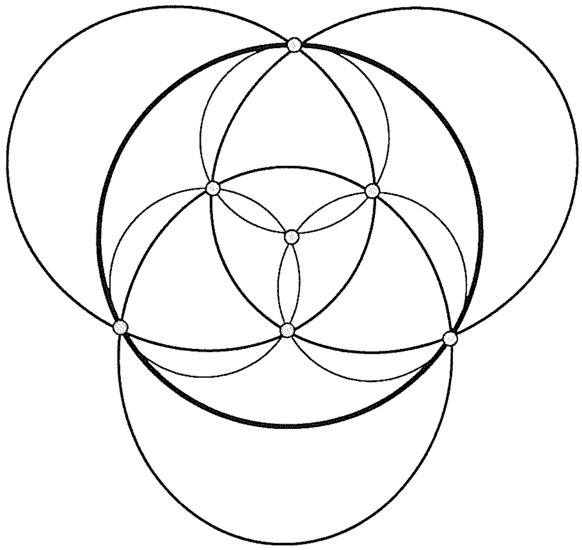


Fig. 3.4. 2-biplane

To calculate automorphism groups of biplanes we need to introduce Hussain graphs [**Hus**]. We will see their importance when dealing with the automorphism group of the 3-biplane. A *Hussain graph* is a graph associated to any line l in a biplane B and any point $Q \notin l$. The vertices of the graph are those of the line l, and two vertices $P, P' \in l$ are joined by an edge of the Hussain graph if and only if there is a line of the biplane through P, P' and Q.

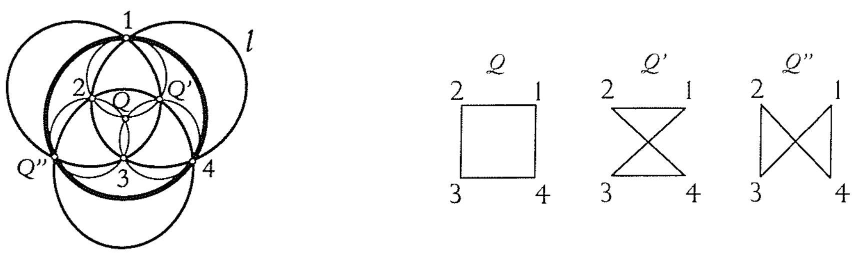


Fig. 3.5. 2-biplane and the l-Hussain set

We will use the notation l-Hussain set (or simply l-H) to refer to the set of Hussain graphs associated with the line l and $[Q, l]$-Hussain graph (or $[Q, l]$-H) to refer to a particular graph in the l-Hussain set. The importance of Hussain graphs is that a line $l \in B$ and the whole set of l-Hussain graphs completely determine the biplane B, for a proof see Theorem 3.34 in [**HP**].

We will say that the Hussain graphs $[Q,l]$-H and $[Q',l']$-H are isomorphic, if they are isomorphic as graphs. Two l-Hussain sets l-H and l'-H are isomorphic if there is a bijection from the point set of l-H to the point set of l'-H that induces isomorphisms between their graphs.

Proposition 3.3.1. Given a biplane B and a line $l \in B$ any two l-Hussain graphs share exactly two edges and these two edges do not share a vertex. Furthermore, every Hussain graph is a divalent graph.

Proof Let us choose two points $Q', Q \notin l$, since B is a biplane, there are two lines through Q and Q' that we will call l_1 and l_2. Four new points appear when we consider the intersections $l_1 \cap l = \{P_1, P_2\}$, $l_2 \cap l = \{P_3, P_4\}$, where $P_i \neq P_j$ if $i \neq j$, since otherwise l_1 and l_2 will have more than two points in common.

It is immediate to see now that $\overline{P_1P_2}$ and $\overline{P_3P_4}$ are two shared edges by $[Q,l]$-H and $[Q',l]$-H and that they do not share a vertex. If there were a third common edge, that will mean that there is a third line through Q and Q' different from l_1 and l_2, in contradiction to the fact that B is a biplane.

Given P any vertex in $[Q,l]$-H, there are only two lines going through P and Q and each one defines an edge of P, therefore any vertex has only two edges and the graph is divalent.

Corollary 3.3.2. A $[Q,l]$-Hussain graph is a disjoint union of polygons whose vertices are the points of l.

Proof Immediate since the graph is divalent.

Theorem 3.3.3. Two biplanes B and B' are isomorphic if and only if for any line $l \in B$ there is a line $l' \in B'$ such that the l-Hussain set is isomorphic to the l'-Hussain set.

Proof Take $l \in B$ to be a line, and $\phi : \mathcal{P} \longrightarrow \mathcal{P}'$ the bijection between the point sets induced by the isomorphism.

We will call $l' = \phi(l)$ and assume Q a point of B and $Q \notin l$. Let us call $Q' = \phi(Q)$ and choose $P_1, P_2 \in l$ so that the edge $\overline{P_1P_2}$ is in $[Q,l]$-H. That means that there is a third line l_1 in B so that $\{P_1, P_2, Q\} \in l_1$, which in turns mean that $\{\phi(P_1), \phi(P_2), \phi(Q)\}$ are points of the set $\phi(l_1)$, that is a line in B', since ϕ is an isomorphism. Therefore the edge $\overline{\phi(P_1)\phi(P_2)}$ is in $[Q', l']$-H and the graphs are isomorphic.

Let us assume now that there are two sets l-H and l'-H that are isomorphic and that $\psi : l\text{-H} \longrightarrow l'\text{-H}$ is the bijection between the point sets of l and l' that defines the isomorphism, so we have already defined the images of any point in l. Take $Q \notin l$ a point of B, ψ takes $[Q, l]$-H to a graph in l'-H that we call $[Q', l']$-H, so that we can define $\psi(Q) = Q'$ and therefore we have extended the definition of ψ to a bijection of the whole point set of B. All we need to prove is that it preserves the incidence structure.

That ψ takes lines to lines is clear, since given a line l_1 in B different from l, we know that $l_1 \cap l = \{P_1, P_2\}$ and taking $Q \in l_1$ and $Q \notin l$, as $[Q, l]$-H goes to $[Q', l']$-H, there must be a line l'_1 through $\psi(P_1)$ and $\psi(P_2)$ different from l', and so we call $\psi(l_1) = l'_1$. We have then a bijection of the point set that preserves the lines and the incidence relation.

Corollary 3.3.4. Given B and B' two biplanes, and lines $l \in B$ and $l' \in B'$, there is a one-to-one correspondence between the set of isomorphisms $\phi : B \longrightarrow B'$ such that $\phi(l) = l'$ and the set of isomorphisms of Hussain sets taking l-H to l'-H.

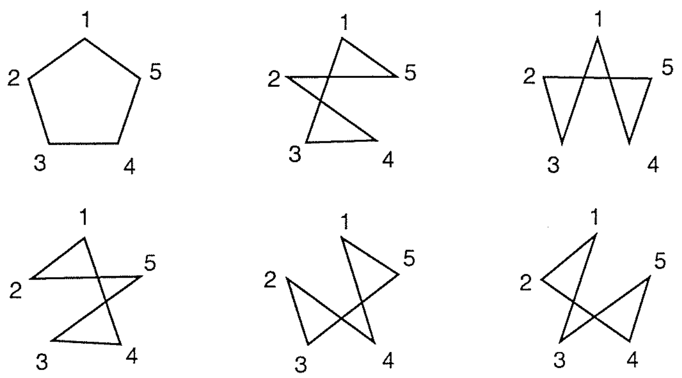


Fig. 3.6. Set of l-Hussain graphs for the 3-biplane

Proposition 3.3.5. The automorphism group of the biplanes of order 1, 2 and 3 are isomorphic respectively to S_4, $PSL(2,7)$ and $PSL(2,11)$.

Proof Let us call B_1, B_2 and B_3 the biplanes of order 1, 2 and 3. As B_1 has only four points, $Aut(B_1) \leq S_4$. If we consider its model as a tetrahedron, we see that the orientation preserving and orientation reversing symmetries of the tetrahedron are an automorphism of B_1, therefore $Aut(B_1) \cong S_4$.

For B_2 we use the fact that B_2 is the complement of the Fano plane, and Proposition 3.1.4. In that way $Aut(B_2) \cong Aut(Fano) \cong PSL(2,7)$. We could as well have calculated $Aut(B_2)$ using its set of l-H graphs, if we had done so we would have seen that ther is only one way of drawing an l-H set (see *Fig. 3.5.*) for a 2 biplane, and therefore by Proposition 3.3.3. there is only one biplane of order 2.

Finally for B_3 we need to consider its l-Hussain graphs. A divalent graph on five points must be a pentagon, and there are only five ways of drawing a divalent graph on five vertices so that it shares exactly two non consecutive edges with the pentagon. The six elements of any l-Hussain graph of B_3 can be seen in *Fig. 3.6.*

Since there is only one way of representing an l-H set, all the l-Hussain graphs are isomorphic, (and hence, by Proposition 3.3.3. there is only one possible biplane of order three) and there are automorphisms of B_3 taking any line to any other line, i.e. the automorphism group is transitive in the line set, and therefore in the set of l-H sets. Furthermore, we will see that the group of automorphism of an l-H set is isomorphic to A_5:

It must be a subgroup of S_5 but it does not contain any transpositions or 4-cycles, since the action of both transpositions and 4-cycles always fix two consecutive edges of the pentagon, therefore it is a subgroup of A_5. If we consider the action of a 3-cycle we see that all of them are automorphisms of the set of l-Hussain graphs, so the stabiliser of any l-H set is in fact A_5. As there are 11 lines in B_3 and the action is transitive, $|Aut(B_3)| = 660$.

We will now show that $Aut(B_3)$ is simple, and since there is only one simple group of order 660, that will prove that $Aut(B_3) \cong PSL(2,11)$.

Take l a line of B_3, as the automorphism group of the l-H set is isomorphic to the alternating group A_5, by Corollary 3.3.4. we get that

$$Stab(l) \cong A_5.$$

Furthermore, since $Aut(B_3)$ is transitive on the set of lines, all the stabilizers are conjugate and $Stab(l) \ntriangleleft Aut(B_3)$ for any $l \in \mathcal{L}_{B_3}$.

Let us assume that $G \cong C_{11}$ and $G \triangleleft Aut(B_3)$. For this to happen there must be only one 11-Sylow subgroup in $Aut(B_3)$. Since any two lines l, l' share two points, using Hussain graphs it is easy to see that

$$Stab(l) \cap Stab(l') \cong S_3,$$

on the other hand, given l and l' two lines, there are precisely three other lines l''_i, $i \in \{1, 2, 3\}$ such that

$$|Stab(l) \cap Stab(l') \cap Stab(l''_i)| = 2.$$

As any element of $Aut(B_3)$ of order different from 11 belongs to a line stabilizer, we can count all elements of order different to 11 by using the previous facts, and we get

$$1 + \underbrace{11 \cdot 59}_{\text{elements of } A_5} - \underbrace{5 \cdot \binom{11}{2}}_{\text{elements of } S_3} + \underbrace{3 \cdot \binom{11}{2}}_{\text{intersections of 3 Stab}} = 540$$

so there are 120 elements of order 11, that yields 12 subgroups isomorphic to C_{11} and that shows that $G \ntriangleleft Aut(B_3)$.

Let us take now $1 \neq G \triangleleft Aut(B_3)$ and $x \in G$ of order different from 11. There is a line $l \in \mathcal{L}_{B_3}$ such that $x \in Stab(l)$, and therefore $1 \neq G \cap Stab(l) \triangleleft Stab(l)$. As $Stab(l)$ is simple and it is not normal in $Aut(B_3)$, we have that $G = Aut(B_3)$. □

There are a number of reasons that justify the study of the biplane of order three: the biplane provides a geometrical model for the action of $PSL(2, 11)$ in the same way as the icosahedron and the Fano plane are models for $PSL(2, 5)$ and $PSL(2, 7)$ respectively, therefore throwing some light into the structure of the group. As a result of this similarity it is possible to find embeddings for the truncated icosahedron in a Riemann surface (using the same idea that allows us to embed a truncated cube in Klein's Riemann surface). Finally, both the combinatorial structure of the 3-biplane and that of the vertices of the truncated icosahedron correspond to the combinatorial structure of the cusps of certain congruence subgroups of the Hecke group H^5. Some of the ideas developed in this work were found in [**Ko**].

Chapter Four

Embedding of finite Geometries

In this chapter we will explore several embeddings of the Fano plane and the 3-biplane inside a Riemann surface, together with other geometric structures that are related to them, such as the truncated cube and the truncated icosahedron. The Fano plane can be embedded as a regular hypermap in the torus and in Klein's quartic. These are the only embeddings of the Fano plane as a regular hypermap in a Riemann surface.

We will also discuss two different kind of embeddings of the 3-biplane:

- as a regular hypermap inside a Riemann surface, there are two possibilities, inside a surface of genus $g = 12$ and a surface of genus $g = 15$.
- as a bipartite graph, there are three possibilities, inside a surface with genus $g = 70$, a surface with genus $g = 125$ and a surface with genus $g = 180$ We will only cover the first two cases in this work.

Each of the two embeddings of the 3-biplane as a graph that we are going to study here generalizes different aspects of the Fano plane embeddings mentioned above. Using them we will explore some interesting relationships among the groups $PSL(2, p)$ where $p \in \{5,\ 7,\ 11\}$.

4.A. The Fano plane as a dessin

Since we know that the Fano plane has 21 bits, in order to embed it as a regular hypermap, we need to find an epimorphism from a triangle group $\Delta = [l, m, n]$ into a group of order 21 such that its kernel is torsion free. The torsion free kernel condition together with the fact that the Fano plane has no automorphism of order 21 forces $\{l,\ m,\ n\} \subset \{3,\ 7\}$.

On the other hand, we have already seen in chapter two that there are only two non-isomorphic groups of order 21, one of them is the cyclic group C_{21}, and the other one is the semidirect product $C_7 \rtimes C_3$ which we call G_{21} and that is defined by the presentation:

$$< e,\ f \mid e^3 = f^7 = efe^{-1}f^{-u} = 1 >$$

where we will assume $u = 2$ without loss of generality.

It is clear that there is no surface kernel epimorphism from the triangle group $\Delta = [l, m, n]$ (where $\{l,\ m,\ n\} \in \{3,\ 7\}$) into C_{21} since there should be an element in Δ of order 21, which contradicts the assumptions, so we only need to consider G_{21} as the image of the epimorphism. Since $C_7 \triangleleft G_{21}$, there is only one subgroup of order 7 in G_{21}, and thus there are only two possible triangle groups for the epimorphism, either $[3, 3, 3]$ or $[3, 3, 7]$. We will see that both of them produce an embedding of the Fano plane.

4.1. The Fano plane in the torus

This embedding was found by Walsh in 1975 [**Wa**] and corresponds to a regular hypermap on a torus. It is related to the toroidal embedding of the complete graph K_7. We start by getting a presentation for $[3, 3, 3]$ given by:

$$< a,\ b,\ c \mid a^3 = b^3 = c^3 = abc = 1 >$$

where we can define the epimorphism $\phi : \ [3, 3, 3] \longrightarrow G_{21}$ by (G_{21} presented as above):

$$\phi(a) = e \qquad \phi(b) = fef^{-1} \qquad \phi(c) = fe^{-2}f^5$$

and using Riemann-Hurwitz formula it is easy to see that $Ker(\phi)$ has signature $(1; \underline{\quad})$. It is therefore a torsion free group and corresponds to a torus that we shall call T. For a picture of a Walsh representation of the embedding see *Fig. 4.1.* where the torus is marked with the thin continuous line.

It is easy to calculate the modulus of this torus, which is $\tau = \frac{1+\sqrt{3}i}{2}$. The automorphism group $Aut(\mathcal{M})$ of any regular map $\mathcal{M}$ on the torus is finite and described in [**CM**], while the automorphism group $Aut(T)$ of any torus T is continuous and hence infinite. If we calculate $Aut(T)$ for this torus, it turns out that $Aut(T)$ is isomorphic to the groups quotient $\frac{H}{\Lambda}$, where H is the subgroup of $Aut(\mathbb{C})$ generated by the transformations:

$$z \longmapsto z + \frac{5+\sqrt{3}i}{2}\lambda, \qquad z \longmapsto \frac{-1}{z}, \qquad z \longmapsto z + b\mu, \qquad z \longmapsto (z-b)e^{\frac{2\pi i}{3}} + b$$

whith $\lambda, \mu \in \mathbb{R}$ and $b = \frac{3+5\sqrt{3}i}{2}$, and Λ is the torus lattice generated by the translations

$$z \longmapsto z + \frac{5+\sqrt{3}i}{2} \quad \text{and} \quad z \longmapsto z + \frac{3+5\sqrt{3}i}{2}.$$

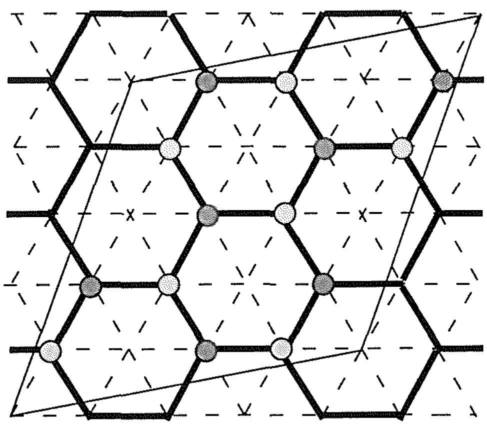


Fig. 4.1. Walsh embedding of the Fano plane

4.2. The Fano plane in Klein's quartic

This embedding was found by D. Singerman [**Si4**] in 1986. It corresponds to an embedding of the Fano plane as a regular hypermap of type $(3,3,7)$ into a Riemann surface with automorphism group isomorphic to the automorphism group of the Fano Plane. It is therefore an embedding into a highly symmetric surface.

As before we will assume that the presentation of $[3,3,7]$ and that of G_{21} are respectively:

$$< a,\ b,\ c \mid a^3 = b^3 = c^7 = abc = 1 > \qquad < e,\ f \mid e^3 = f^7 = efe^{-1}f^{-2} = 1 >$$

Our starting point is an epimorphism $\phi\ : [3,3,7] \longrightarrow G_{21}$ defined by

$$\phi(a) = e \qquad \phi(b) = fe^2f^{-1} \qquad \phi(c) = f^{-1}$$

and by Riemann-Hurwitz formula we see that the signature of $Ker(\phi)$ is $(3;\underline{\quad})$, we will call $K_{g=3}$ the Riemann surface uniformized by $Ker(\phi)$.

$[3,3,7]$ is not a maximal triangle group, since it can be embedded into $[2,3,7]$, which is maximal [**Si2**], with index 8. If we consider a presentation for $[2,3,7]$ of the form

$$< A, B, C \mid A^2 = B^3 = C^7 = ABC = 1 >$$

one of the possible set of equations (there are eight ways of doing the embedding) for the embedding $i : [3,3,7] \longrightarrow [2,3,7]$ is :

$$i(a) = C^2AC^4 \qquad i(b) = C^4AC^2 \qquad i(c) = ACA$$

where we can check that $i(abc) = C^2ACAC^2ACA = CBCB = 1$.

In order to calculate the automorphism group of $K_{g=3}$ we need to find the normalizer of $Ker(\phi)$ inside $PSL(2,\mathbb{R})$. We will show that $Ker(\phi) \vartriangleleft [2,3,7]$, and since $[2,3,7]$ is a maximal triangle group (see [**Si2**]), we will have proved that it is the normalizer of $Ker(\phi)$. That fact will prove that $K_{g=3}$ is Klein's quartic.

$$\begin{array}{ccccc}
[2,3,7] & \xrightarrow{\quad\Phi\quad} & PSL(2,7) & \xrightarrow{\quad\Psi\quad} & Aut(K_{g=3}) \\
i\uparrow 8 & & \uparrow 8 & & \uparrow 8 \\
[3,3,7] & \xrightarrow{\quad\phi\quad} & G_{21}^{\infty} & \longrightarrow & Aut(\mathcal{H}_{[3,3,7]}) \\
\uparrow 3 & & \uparrow 3 & & \uparrow 3 \\
[7,7,7] & \longrightarrow & C_7^{\infty} & \longrightarrow & S \\
\uparrow 7 & & \uparrow 7 & & \uparrow 7 \\
Ker(\phi) & \longrightarrow & \{1\} & \longrightarrow & \{1\}
\end{array}$$

We define the group epimorphism $\Phi : [2,3,7] \longrightarrow PSL(2,7)$ by:

$$\Phi(A) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \Phi(B) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \qquad \Phi(C) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

and we choose a representation of G_{21} as $G_{21}^{\infty} < PSL(2,7)$ where

$$e = \begin{pmatrix} 3 & 1 \\ 0 & 5 \end{pmatrix} \qquad f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

with the above presentations we see that $\phi(x) = \Phi(i(x))$ for any $x \in [3,3,7]$. As the index of $Ker(\phi)$ inside $[2,3,7]$ is 168 we know that $Ker(\phi) = Ker(\Phi)$ and therefore $Ker(\phi) \triangleleft [2,3,7]$. We have then proved that

$$Aut(K_{g=3}) \cong PSL(2,7)$$

and that characterizes $K_{g=3}$ as Klein's quartic, which we can express in complex projective coordinates as [**Kl1**]:

$$x^3y + y^3z + z^3x = 0.$$

The eight different embeddings of $[3,3,7]$ into $[2,3,7]$ correspond to the eight different embeddings of a G_{21} into $PSL(2,7)$, each one corresponding to $Stab(x)$ for a given $x \in \mathbb{P}_7$. Since we have already chosen G_{21}^{∞}, we get the stack of groups shown in the previous page, where $Aut(K_{g=3})$ has been explicitly calculated by Klein in [**Kl1**].

In the diagram from the previous page $Aut(\mathcal{H}_{[3,3,7]})$ refers to the automorphism group of the dessin, while S refers to the stabilizer of the face centres of the dessin, the horizontal lines on the left are group epimorphisms, those on the right are group isomorphisms, and vertical arrows designate group inclusions.

We will only show the embedding that corresponds to G_{21}^{∞} in our notation because as we have said, the others are conjugate to it by automorphisms of the surface.

In order to display the hypermap on the surface, we need to consider 21 special points on $K_{g=3}$, and for simplicity we will refer to $\Psi(H) < Aut(K_{g=3})$ as H whenever there is no confusion possible. The action of each subgroup of order seven of $Aut(K_{g=3})$ fixes three points on the surface ([**Kl1**] or direct calculation), and there are not two subgroups that fix the same point, so if we consider all the points fixed by each of these subgroups, we get 24 distinguished points, which are actually the Weierstrass points of Klein's quartic.

As every triple of points corresponds to the action of a C_7 group, and every C_7 in $PSL(2,7)$ can be associated to a point in $\mathbb{P}_7$, we can see that the action of G_{21}^{∞} on the Weierstrass points splits them into two stable sets, one of them with three points in it (those fixed by C_7^{∞}), the other with 21. We can see the first set as the face centres of our dessin on the surface, while the second set corresponds to the set of bits of the dessin. For a Cori representation of the embedding see *Fig. 4.2.* where the identification of the edges of the polygon follows the rule

$$2n+1 \sim 2n+6 \pmod{14} \qquad \text{where} \qquad n \in \{0,1,...,6\}.$$

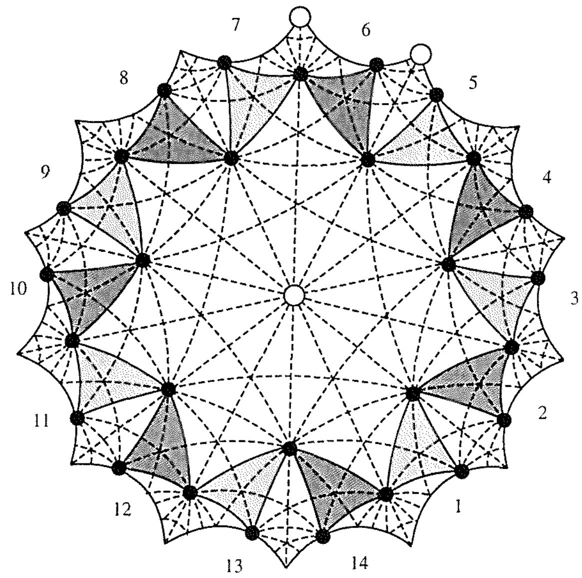


Fig. 4.2. Singerman's embedding of the Fano plane

The lighter coloured hyperbolic triangles in *Fig. 4.2.* represent hypervertices while the darker ones are hyperedges of the dessin, the black points show the bits and the three face centres of the dessin are highlighted with a light coloured circle.

4.3. On the Fano plane and Klein's quartic

The embedding of the Fano plane into $K_{g=3}$ is also interesting for its relationship with the cusp set of the principal congruence subgroup of the modular group of level seven, its relationship with the embedding of the truncated cube in $K_{g=3}$ and the possibility of providing a geometrical model for the 2-biplane.

We have already mentioned in chapter two that the cusp set of the modular group Γ is $\widehat{\mathbb{Q}}$, that $\Gamma(N)$ is torsion free for every $N \geq 2$ and that $\frac{\Gamma}{\Gamma(p)} \cong PSL(2,p)$ where $p \in \mathbb{Z}$ is prime. We can actually use a stack of groups similar to the one displayed in previous sections to calculate the cusp set of the action of the principal congruence group of level seven, $\Gamma(7)$ over $\mathcal{U}$ with the structure of the Fano plane.

It is clear that there is an epimorphism from Γ into $[2, 3, 7]$, since a presentation for Γ is given by

$$< X, Y, Z \mid X^2 = Y^3 = XYZ = 1 > .$$

We can easily define it by projecting X onto A, Y onto B and Z onto C (where $[2, 3, 7]$ is presented as in section 4.2.), we will call such epimorphism σ. If we consider the special congruence subgroups of level seven, we get the following stack:

$$\begin{array}{ccc}
\Gamma & \xrightarrow{\quad\sigma\quad} & [2,3,7] \\
\uparrow 8 & & \uparrow 8 \\
\Gamma_0(7) & \xrightarrow{\qquad\quad} & [3,3,7] \\
\uparrow 3 & & \uparrow 3 \\
\Gamma_1(7) & \xrightarrow{\qquad\quad} & [7,7,7] \\
\uparrow 7 & & \uparrow 7 \\
\Gamma(7) & \xrightarrow{\qquad\quad} & KS_{g=3}
\end{array}$$

Where $\frac{\mathcal{U}}{\Gamma(7)}$ is a Riemann surface of genus $g = 3$ with 24 punctures.

We will follow Klein in [**Kl1**] to obtain a picture of a fundamental region of $\Gamma(7)$ in the upper-half plane with a triangulation by triangles of type $[2, 3, \infty]$. To do so we use the transformation $z \longmapsto z + 1$ to generate six copies of *Fig. 4.3.* (as described in [**Kl1**]) and paste them together along consecutive vertical sides. That leaves us with a hyperbolic polygon with vertices:

$$\frac{\pm 3 + 7n}{7}, \; \frac{\pm 1 + 3n}{3}, \; \frac{\pm 2 + 7n}{7}, \; \frac{-1 + 2n}{2}, \; n$$

where $n \in \{0, \; ..., \; 6\}$.

As we know, the punctures on the surface of $\frac{\mathcal{U}}{\Gamma(7)}$ correspond to points fixed by parabolic elements, and all parabolic elements in Γ project by σ onto elements of order seven in $[2, 3, 7]$, therefore the 24 punctures will project onto the 24 Weierstrass points of $K_{g=3}$. Reversing the reasoning, as the projection is one to one, the cusp set of $\frac{\mathcal{U}}{\Gamma(7)}$ inherits the Fano plane structure defined using the Weierstrass points of $K_{g=3}$. This embedding, although being very closely related to the one in $K_{g=3}$ is more interesting, since we can use arithmetic to describe the Fano plane structure in the set of punctures of $\Gamma(7)$.

We will now obtain the set of cusps of $\frac{\mathcal{U}}{\Gamma(7)}$ out of the set of vertices of a fundamental region of $\Gamma(7)$ described above.

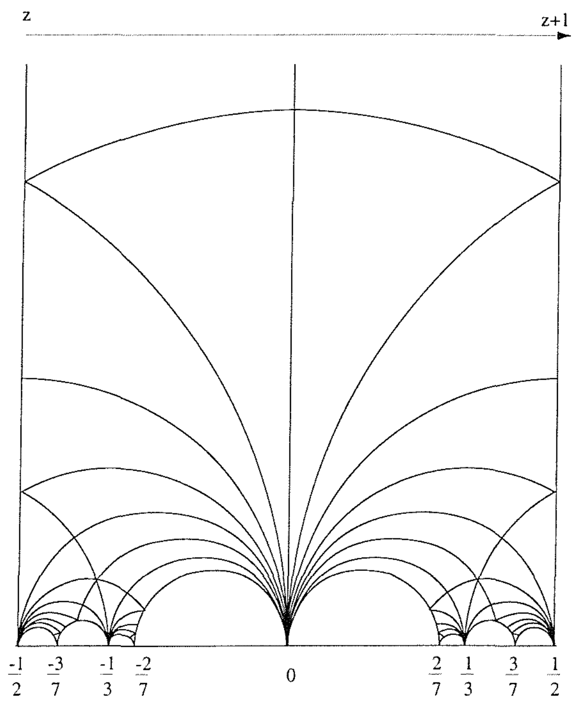


Fig. 4.3. Pattern of the triangulation of $\Gamma(7)$ *by* $[2, 3, \infty]$.

By Theorem 2.4.4. two irreducible fractions $\frac{a}{b}$ and $\frac{a'}{b'}$ represent the same cusp under the action of $\Gamma(7)$ if and only if $a \equiv \pm a' \pmod 7$ and $b \equiv \pm b' \pmod 7$, and we therefore see that either a cusp is equal to ∞, $\frac{2}{7}$, $\frac{3}{7}$ or it belongs to one of the following families:

$$n, \ \frac{1+3n}{3}, \ \frac{1+2n}{2}$$

where $n \in \{0, \ldots, 6\}$. Since the first three cusps are fixed by the action of $z \longmapsto z+1$ and thus correspond to the face centres of the Fano plane embedding associated to G_{21}^{∞}, the other 21 cusps constitute the bits of that same embedding.

We will prove that for this embedding two cusps $\frac{a}{b}$ and $\frac{a'}{b'}$ in their irreducible form share an edge if and only if

$$ab' - a'b \equiv \pm 1 \pmod 7.$$

We assume that the condition holds for a couple of cusps $\frac{a}{b}$ and $\frac{a'}{b'}$ sharing an edge. If we let the order seven element act on the shared edge, we get new edges defined by the couples

$$\frac{a+nb}{b}, \quad \frac{a'+nb'}{b'}$$

where $1 \leq n \leq 6$ and all of them trivially satisfy the condition.

On the other hand, any order three element

$$A = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in G_{21}^{\infty}$$

satisfies $tr(A) \equiv \pm 1 \pmod 7$, and if we assume $\frac{a}{b}$ and $\frac{a'}{b'}$ to be two cusps as before, it is trivial to see that

$$\frac{xa+yb}{zb}, \quad \frac{xa'+yb'}{zb'}$$

also satisfy the condition.

It suffices to show that there is at least one edge that satisfies the condition: let us take the bit represented by 0, an edge of the embedding can only join it to the following points

$$\pm\frac{1}{2}, \ \pm\frac{1}{3}, \ \pm 1, \ \pm 2.$$

The last four points must be discarded since any edge connecting them to 0 will imply that two hypervertices (or hyperedges) share a bit, which is impossible. On the other hand, there are precisely four edges incident to any one bit, and in this case the four edges are represented by the first four points. It is now trivial to check that any of them satisfies the condition above.

4.4. Embeddings related to $PSL(2,7)$

In this section we want to study other geometric structures that can be described in terms of the embedding covered in sections 4.2. and 4.3. As we have already seen that Klein's quartic is equivalent to $\frac{\mathcal{U}}{\Gamma(7)}$ without considering its punctures, we will only consider the description of the embeddings into Klein's quartic, their extension to the cusp set of $\frac{\mathcal{U}}{\Gamma(7)}$ should follow easily.

The other "big" subgroup of $PSL(2,7)$ has 24 elements and is isomorphic to S_4, we will show its relationship to the 24 Weierstrass points of Klein's quartic. In the previous section we were forced to split the set of Weierstrass points into two subsets, mainly because we were considering the action of a group of order 21, in this section we will study the action of S_4 over the set of Weierstrass points and in order to do that we require the following lemma.

Lemma 4.4.1. There are fourteen groups of order 24 (isomorphic to S_4) inside $PSL(2,7)$. The action of any of these groups, when considered as subgroups of $Aut(K_{g=3})$, on the Weierstrass points of $K_{g=3}$ is transitive.

Proof The first part follows from **[Di]**, was proved in **[Kl1]** and has been calculated in section 2.2. For the second part, we only need to consider that each Wierstrass point in $K_{g=3}$ is the fixed point of a unique $C_7 < PSL(2,7)$ and that any proper subgroup of $PSL(2,7)$ whose order is a multiple of 7 is isomorphic to either C_7 or G_{21} **[Kl1]**. Thus the only elements in S_4 that could fix a Wierstrass point are those of order three.

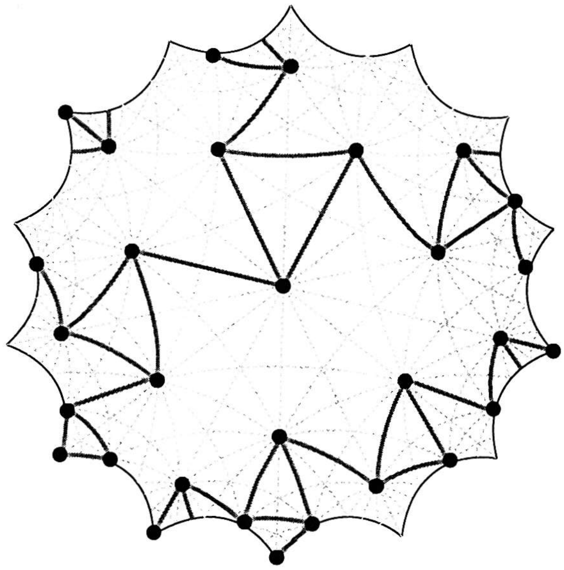


Fig. 4.4. Embedding of the truncated cube in $K_{g=3}$

On the other hand, any element of order three is associated to two C_7 groups and therefore it fixes their corresponding sets of Wierstrass points (although it does not fix the actual points). Since any element of order three fixes just two points on $K_{g=3}$, it is immediate that it cannot fix a Wierstrass point, therefore the stabilizer of any point is trivial (when considering the action of an S_4) and the action of S_4 on the set of Weierstrass points is transitive. □

Every embedding of an S_4 into $PSL(2,7)$ is related to an embedding of a truncated cube into $K_{g=3}$ in which the Weierstrass points of $K_{g=3}$ are the vertices of the truncated cube. We can choose one of the groups S_4 displayed in the table in chapter two, each one containing four subgroups isomorphic to C_3. The action of any $C_3 < Aut(K_{g=3})$ on $K_{g=3}$ fixes two points on the surface, therefore to every embedding of an S_4 into $PSL(2,7)$ we can attach four pairs of points on the surface of Klein's quartic, each of these pairs will define a diagonal of the (truncated) cube.

Each point fixed by an element of order three is the centre of an equilateral triangle of area $\frac{\pi}{7}$ whose vertices are three Weierstrass points (since they are fixed by elements of order 7). In that way we get a triangle corresponding to the truncation of one vertex of the cube for every point fixed by an element of order three in the S_4 we are considering (i.e. two triangles for each C_3). Repeating the same operation with every subgroup C_3 in S_4 we get the eight triangular faces of the truncated cube. The remaining edges of the truncated cube are easy to find, since every two cycle of S_4 will fix two other points on the surface, these points correspond to edges centres of the cube. For a picture of a truncated cube inside $K_{g=3}$ see *Fig. 4.4*.

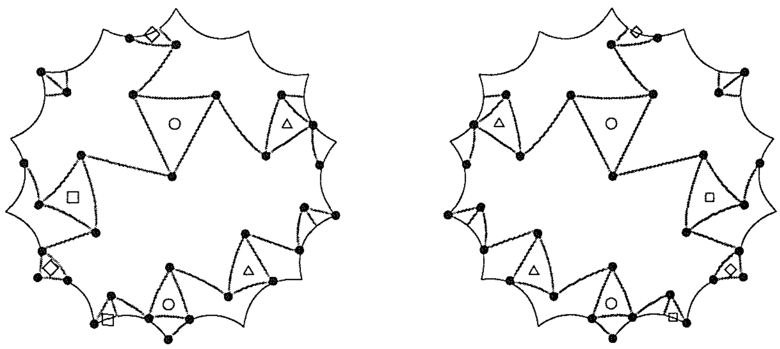


Fig. 4.5. Representatives of the two classes with their diagonals marked

There are fourteen such embeddings into two conjugacy classes, but if we allow orientation reversing automorphism of $K_{g=3}$ they will form a unique conjugacy class. To visualize the remaining six embeddings in *Fig. 4.4.* class it suffices to rotate the truncated cube around the center point of the hyperbolic polygon shown in *Fig. 4.4.* To get a representative of the second class, we have to reflect the truncated cube along the vertical diagonal of the polygon. To finish with this section we intend to provide a model for the geometric action of $PSL(2,7)$ on seven points alongside with a model for the 2-biplane inside $K_{g=3}$.

The algebraic action of $PSL(2,7)$ on seven points is easy to describe as the action of $PSL(2,7)$ on its seven subgroups isomorphic to S_4 within one conjugacy class. Considering that every S_4 in $Aut(K_{g=3})$ is connected to an embedding of the truncated cube, we can see the geometric action of $PSL(2,7)$ on seven points as the permutation of the seven truncated cubes in a conjugacy class.

For a diagram of the 14 truncated cubes inside $K_{g=3}$ we only need to rotate the class representatives displayed in *Fig. 4.5.* around the central vertex of the hyperbolic polygon.

We can now consider the diagonals of each truncated cube in a conjugacy class as a bit of a combinatorial structure and thus we get 28 bits in seven sets of four. We can call each of these sets *lines*, so that each line is in fact a truncated cube within a conjugacy class. Furthermore, we can choose seven new sets of four diagonals, that we shall call *points*, so that every point shares only a bit with four lines, and every line shares just one bit with four points. In doing so what we get is a model for the 2-biplane inside $K_{g=3}$. We have seen that each diagonal can be associated to a $C_3 < PSL(2,7)$ and that the set of "lines" is identified with the set of subgroups S_4 in one conjugacy class. Using a similar reason, the set of "points" is identified with the set of subgroups S_4 in the other conjugacy class, as we can see in the table in chapter two, where P stands for points and L for lines, and the coordinates of the C_3 label the incidence structure of the bits of the 2-biplane.

4.B. The 3-biplane as a dessin

The 3-biplane has 55 bits, so to embed it as a regular hypermap our first concern is the structure of the groups of order 55. We have already seen in chapter two that there are only two groups of order 55: the cyclic group C_{55} and the semidirect product $C_{11} \rtimes C_5$ that we call G_{55} and has a presentation:

$$< e, f \mid e^5 = f^{11} = efe^{-1}f^{-u} = 1 >$$

where $u \in \{3, 4, 5, 9\}$. We shall use $u = 4$ unless otherwise stated.

Using the same ideas explained in the study of the embeddings of the Fano plane, we can discard C_{55} (there is no subgroup of $PSL(2, 11)$ isomorphic to C_{55}) and state that there are only two possible embeddings for the 3-biplane as a regular hypermap, these embeddings arise when we consider the triangle groups $[5, 5, 5]$ and $[5, 5, 11]$.

4.5. Embedding the 3-biplane inside $R_{g=12}$

This embedding corresponds to a hypermap of type $[5, 5, 5]$ inside a Riemann surface of genus $g = 12$. It arises when one consider the group $[5, 5, 5]$ with presentation

$$< a, b, c \mid a^5 = b^5 = c^5 = abc = 1 >$$

and the epimorphism $\phi : [5, 5, 5] \longrightarrow G_{55}$ defined by:

$$\phi(a) = e \qquad \phi(b) = fe \qquad \phi(c) = e^{-1}f^{-1}e^{-1}$$

that induces the following stack of groups.

$$\begin{array}{ccc}
[5,5,5] & \xrightarrow{\quad\phi\quad} & G_{55} \\
\uparrow 5 & & \uparrow 5 \\
(2;_) & \longrightarrow & C_{11} \\
\uparrow 11 & & \uparrow 11 \\
Ker(\phi) & \longrightarrow & \{1\}
\end{array}$$

$[5, 5, 5]$ is not a maximal triangle group because it satisfies **[Si2]**

$$[5, 5, 5] \overset{3}{<} [3, 3, 5] \overset{2}{\triangleleft} [2, 3, 10] \quad \text{or} \quad [5, 5, 5] \overset{2}{\triangleleft} [2, 5, 10]$$

and there are no other inclusions for $[5,5,5]$. As it is not maximal, we need to do some extra calculations to find $Aut(R_{g=12})$.

If we take $[2,5,10]$ with presentation

$$< A, B, C \mid A^2 = B^5 = C^{10} = ABC = 1 >$$

we can define the inclusion $i : [5,5,5] \longrightarrow [2,5,10]$ by

$$i(a) = ABA, \quad i(b) = B \quad i(c) = C^2.$$

Let us consider a group isomorphic to $\mathcal{H}_{110}$ (defined as in Lemma 2.A.6.) with presentation:

$$< e, f, g \mid e^5 = f^{11} = g^2 = efe^{-1}f^{-4} = gfgf = gege^{-1} = 1 >$$

and the map $\Phi : [2,5,10] \longrightarrow H_{110}$ given by:

$$\Phi(A) = f^7 g \quad \Phi(B) = fe \quad \Phi(C) = ge^{-1}f^5$$

which we can prove to be an epimorphism since $\Phi(AC) = e^{-1}$. We can see that $Ker(\Phi)$ is torsion free in the usual way. If we calculate the restriction of Φ to $[5,5,5]$, we see that $\Phi(i(x)) = \phi(x)$ for any $x \in [5,5,5]$ and using indexes calculations as we have done before, $Ker(\phi) \lhd [2,5,10]$. As $[2,5,10]$ is a maximal triangle group, we have proved that

$$Aut(R_{g=12}) \cong H_{110}.$$

Since $[2,5,10]$ is a triangle group, we can consider the dessin induced by the inclusion $Ker(\phi) \lhd [2,5,10]$ which is a dessin with 110 bits that derives from the 3-biplane embedding by a standard procedure that we call *Walsh double.* We can obtain a geometric representation of a Walsh double of a hypermap by taking its Walsh representation and painting all vertices black, the bipartite map we get is a Walsh double of the original hypermap.

4.6. Embedding the 3-biplane inside $R_{g=15}$

This embedding appears when one consider the triangle group $[5,5,11]$ with presentation

$$< a, b, c \mid a^5 = b^5 = c^{11} = abc = 1 >$$

and the epimorphism $\phi : [5,5,11] \longrightarrow G_{55}$ given by:

$$\phi(a) = e \quad \phi(b) = e^{-1}f^{-1} \quad \phi(c) = f.$$

Using Riemann-Hurwitz formula, we see that the underlying Riemann surface has genus $g = 15$ and we get the following group diagram:

$$\begin{array}{ccc}
[5,5,11] & \xrightarrow{\quad\phi\quad} & G_{55} \\
\uparrow 5 & & \uparrow 5 \\
(0;[11^5]) & \xrightarrow{\qquad} & C_{11} \\
\uparrow 11 & & \uparrow 11 \\
Ker(\phi) & \xrightarrow{\qquad} & \{1\}
\end{array}$$

As in the previous case $[5,5,11]$ is not a maximal triangle group [**Si2**], and is not contained in any other triangle group except:

$$[5,5,11] \overset{2}{\lhd} [2,5,22].$$

We will prove nevertheless that $|Aut(R_{g=15})| = 55$.

Using the same ideas as before, we suppose there is a torsion free kernel epimorphism $\Phi : [2,5,22] \longrightarrow H$, where H is a group such that $[H : G_{55}] = 2$. Since H must have an element of order 22, $H \neq \mathcal{H}_{110}$. On the other hand, if $H = G_{110} \cong G_{55} \times C_2$, $\Phi(AB)$ must have order ten, where A and B are respectively elements of order two and five in $[2,5,22]$, and therefore it is impossible to define such an epimorphism, so we have proved that

$$Aut(R_{g=15}) \cong G_{55}.$$

4.C. The 3-biplane as a conformal graph

The two examples mentioned above are the only possible embeddings of a 3-biplane as a regular dessin in a Riemann surface, as any other epimorphism from a triangle group $[l, m, n]$ to G_{55} will produce torsion. Unfortunately none of them is into a Riemann surface with automorphism group isomorphic to $PSL(2, 11)$.

Since we are trying to find embeddings of the 3-biplane that mirror as closely as possible the characteristics of Singerman's embedding of the Fano plane, there are two important questions that we need to solve:

· Is there any other way of embedding the 3-biplane in a "rigid way" into a Riemann surface S such that $Aut(S) \cong PSL(2, 11)$? That is, is it possible to embed the bipartite graph representing the 3-biplane incidence structure conformally into a Riemann surface S such that $Aut(S) \cong PSL(2, 11)$ and so that the stabilizer of the embedding is isomorphic to G_{55}?

· Among the solutions to the previous question, can any of the surfaces S be uniformized by a principal congruence group G of a Hecke group H^q in such a way that the bits of the embedding can be thought of as cusps of G?

The answer to both questions is afirmative, there are at least two surfaces S that solve the first question, and one of them is a positive answer to the second.

We need to relax some of the conditions for the embedding, since we know that there is no solution for it among dessins. We chose to relax the conditions relating to the "faces" of the embedding (as seen in a Walsh representation), therefore our embedding will not be a dessin, but only a bipartite graph, since the connected components of the complement of the graph (faces) will no longer be simply connected. This relaxation implies algebraically that we will consider groups Δ' with three periods and signature $(g, [l', m', n'])$ where $g \neq 0$, as candidates for the epimorphism onto G_{55}. Unfortunately these groups are not rigid in the sense of Fuchsian groups, and so we will place a restriction on them by studying only those that are subgroups of triangle groups $[l, m, n]$ that projects epimorphically onto $PSL(2, 11)$.

Let us start then with a triangle group $\Delta = [l, m, n]$ that projects epimorphically via Φ onto $PSL(2, 11)$ so that its kernel has index 660 and is torsion free. The torsion free kernel condition limits the choice of periods for the triangle group to the set $\{2, 3, 4, 5, 6, 11\}$. We need to impose a further condition: that there is a subgroup $\Delta' < \Delta$ with $[\Delta : \Delta'] = 12$ and signature

$$\Delta' =< g; [l_1, l_2, l_3] >,$$

that is, three finite periods ($l_i \neq 0$) such that the restriction $\Phi_{\Delta'}$ is an epimorphism onto G_{55}. This implies that Δ' has one of the following signatures:

$$\Delta' = (g; [5, 5, 11]) \qquad \text{or} \qquad \Delta' = (g; [5, 5, 5])$$

which implies that there is at least a period of order multiple of 5 in Δ, using the condition that limits the choice of periods in Δ, we can assume that $\Delta = [5, m, n]$. Since the index of Δ' in Δ is 12, we have that $12\mu(\Delta) = \mu(\Delta')$, that is:

$$\frac{10g+2}{5} = 12\left(1 - \frac{1}{5} - \frac{1}{m} - \frac{1}{n}\right) \qquad \text{or} \qquad \frac{110g+28}{55} = 12\left(1 - \frac{1}{5} - \frac{1}{m} - \frac{1}{n}\right)$$

where $m, n \in \{2,\ 3,\ 4,\ 5,\ 6,\ 11\}$ and $g \in \mathbb{N}$. Out of the set of hyperbolic triangle groups that could be candidates for Δ, we can see that only $[2, 5, 11]$, $[3, 5, 11]$ and $[5, 6, 11]$ satisfy all the conditions, and therefore are the only ones that might allow an embedding of the kind described above.

In the first case ($[2, 5, 11]$), which seems to be the most interesting, the embedding will be into a Riemann surface of genus $g = 70$. In that case we will be able to extend the 3-biplane structure to the cusp set of a congruence subgroup of the Hecke group H^5.

In the second case the underlying surface has genus $g = 125$, while in the last case Δ' has signature $(3; [5, 5, 11])$ and the underlying surface has genus $g = 180$. In all three cases the bipartite graph has type $(5, 5, 11)$ meaning that it is invariant by rotations of order eleven around its "face" centres, or rotations of order five around its vertices. We will only cover the first two cases as the third one does not show any interesting properties not covered by the other two cases.

4.7. Embedding the 3-biplane inside $R_{g=70}$

In this section we will explain the embedding of the 3-biplane that most closely resembles that of the Fano plane in Klein's quartic. We start with the triangle group $[2, 5, 11]$ with the following presentation:

$$< A, B, C \mid A^2 = B^5 = C^{11} = ABC = 1 >$$

and we define a map $\Phi :\ [2, 5, 11] \longrightarrow PSL(2, 11)$ by:

$$\Phi(A) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \Phi(B) = \begin{pmatrix} 0 & 1 \\ -1 & 8 \end{pmatrix},\ \Phi(C) = \begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix}.$$

It is clear that $Ker(\Phi)$ is torsion free, since every finite order element in $[2,5,11]$ is projected onto an element of $PSL(2,11)$ of the same order, thus $Ker(\Phi)$ uniformizes a Riemann surface of genus $g = 70$ (Riemann-Hurwitz formula) that we call $R_{g=70}$. By its construction and the fact that $[2,5,11]$ is a maximal triangle group, we have proved that

$$Aut(R_{g=70}) \cong PSL(2,11).$$

We know that $F = (1;[5,5,11]) < [2,5,11]$ and if we take a presentation for F given by:

$$< a,b,c,x,y \mid a^5 = b^5 = c^{11} = xyx^{-1}y^{-1}abc = 1 >$$

we see that $i : F \longrightarrow [2,5,11]$ can be defined by:

$$i(a) = C^{-7}AC^2AC^{-1}AC^7 \qquad i(b) = C^{-8}AC^9$$
$$i(c) = AC^{-1}A \qquad i(x) = C^{-1}AC^6 \qquad i(y) = C^{-7}AC^4.$$

If we consider now the restriction $\phi : (1,[5,5,11]) \longrightarrow PSL(2,11)$ defined as $\phi(x) = \Phi(i(x))$ for any $x \in (1,[5,5,11])$, we see that $Img(\phi) = G_{55}^{\infty}$ and so we can construct the following stack of groups where horizontal lines denote group epimorphisms and vertical arrows symbolize group inclusions.

$$\begin{array}{ccc}
[2,5,11] & \xrightarrow{\quad\Phi\quad} & PSL(2,11) \\
\uparrow 12 & & \uparrow 12 \\
(1;[5,5,11]) & \xrightarrow{\quad\phi\quad} & G_{55}^{\infty} \\
\uparrow 55 & & \uparrow 55 \\
Ker(\phi) & \xrightarrow{\qquad} & \{1\}
\end{array}$$

To find the structure of the bipartite graph on the surface we need to consider one set of 22 points that will correspond to the vertices of the bipartite graph, and another one of 55 points that correspond to edge centres (bits).

We will call H the image of the embedding of the group G_{55}^{∞} into the automorphism group of $R_{g=70}$. In chapter two we have seen that there are twelve different subgroups of order 11 in $PSL(2,11)$. If we consider them as subgroups of the automorphism group of $R_{g=70}$, we can see that each subgroup C_{11} fixes five points on the surface, and no two subgroups fix the same point, so we have 60 distinguished points on the surface. If we remove from this set the five points fixed by the action of H (i.e. the five points fixed by C_{11}^{∞}), we obtain a collection of 55 points that are transitively permuted by the action of H and that corresponds to the bits of the embedded graph.

The remaining 22 points arise naturally when we consider the action of the order five subgroups of H. There are 11 such subgroups, each one fixes two points on the surface, and therefore we have found the 22 points we needed. The choice of colour is arbitrary, we need to fix one of the C_5, for example $C_5^{\infty,0}$, and paint one of its fix points white, the other black. If we let C_{11}^{∞} acts on this pair of coloured points, we will get the colouring for the remaining.

Since there are 12 different embeddings of G_{55} in $PSL(2,11)$, there are 12 ways of embedding the bipartite graph (not considering swapping the colours of the vertices), but all of them are conjugate by an automorphism of the surface.

4.8. Embedding the 3-biplane inside $R_{g=125}$

This embedding arises when one consider the triangle group $[3,5,11]$ with presentation:

$$< A, B, C \mid A^3 = B^5 = C^{11} = ABC = 1 >$$

and the group epimorphism $\Phi : \ [3,5,11] \longrightarrow PSL(2,11)$ defined by:

$$\Phi(A) = \begin{pmatrix} 1 & 9 \\ 7 & 9 \end{pmatrix}, \quad \Phi(B) = \begin{pmatrix} 7 & 10 \\ 1 & 0 \end{pmatrix}, \quad \Phi(C) = \begin{pmatrix} 4 & 1 \\ 8 & 5 \end{pmatrix}.$$

It is easy to check that Φ is an epimorphism with a torsion free kernel, and that $Ker(\Phi)$ uniformizes a Riemann surface of genus $g = 125$ that we call $R_{g=125}$. We can prove that $F = (2, [5,5,11])$ with presentation

$$< x, y, z, t, a, b, c \mid a^5 = b^5 = c^{11} = xyx^{-1}y^{-1}ztz^{-1}t^{-1}abc = 1 >$$

is a subgroup of $[3,5,11]$ and using Schreier's method find the following equations for the inclusion $i : F \longrightarrow [3,5,11]$:

$$i(x) = C^4A^2C^{-1}A^2C^4 \qquad i(y) = C^8AC^5 \qquad i(z) = C^8A^2C^9 \qquad i(t) = C^3AC^2$$

$$i(a) = C^3AC^{-1}A^2C^{-2}(C^{-1}A^2)C^2ACA^2C^{-3}$$

$$i(b) = C^3AC^{-1}A^2C^{-2}(A^2C^{-1})C^2ACA^2C^{-3} \qquad i(c) = C^4A^2CAC^{-4}$$

With all the group maps defined as before, we can restrict Φ to F in which case we obtain a group epimorphism onto G_{55}^{∞} and the following stack of groups:

$$\begin{array}{ccc}
[3,5,11] & \xrightarrow{\quad\phi\quad} & PSL(2,11) \\
\uparrow 12 & & \uparrow 12 \\
(2;[5,5,11]) & \xrightarrow{\qquad} & G_{55}^{\infty} \\
\uparrow 11 & & \uparrow 11 \\
Ker(\phi) & \xrightarrow{\qquad} & \{1\}
\end{array}$$

So what we get is another conformal embedding of the 3-biplane as a $(5,5,11)$ bipartite graph inside $R_{g=125}$.

4.9. Other considerations for the 3-biplane and $R_{g=70}$

Among all the possibilities displayed above, the embedding of the 3-biplane into $R_{g=70}$ as a bipartite graph seems the most interesting one because we can find a fair amount of relations between it and Singerman's embedding of the Fano plane. Among them we will give a 3-biplane structure to the cusp set of $H^5(4-\sqrt{5})$ and show a geometric action of $PSL(2,11)$ on eleven objects on the surface $R_{g=70}$.

In chapter two we have introduced the Hecke group H^5 and we saw that its signature as a Fuchsian group is $[2,5,\infty]$. Following the ideas explained in Singerman's embedding, we can easily define an epimorphism $\Psi : H^5 \longrightarrow [2,5,11]$ given by (presentations for both groups as used before):

$$\Psi(X) = A \qquad \Psi(Y) = B \qquad \Psi(Z) = C$$

that will allow us to extend the 3-biplane structure from $R_{g=70}$ to some cusp set of congruence subgroups of H^5. If we pull-back the subgroups of $[2,5,11]$ by means of Ψ we get the group diagram displayed in next page.

In this case the role of the ideal (7) in the description of Klein's surface is played either by $(4+\sqrt{5})$ or $(4-\sqrt{5})$, the reason for this is that (11), which would be the natural choice, is not longer a prime ideal in $\mathbb{Z}[\lambda_5]$ since it can be factorized as the product of the previously mentioned ideals. It is trivial to see that Ψ extends easily to an epimorphism onto $PSL(2,11)$ as the following diagram shows.

$$
\begin{array}{ccccc}
H^5 & \xrightarrow{\quad\Psi\quad} & [2,5,11] & \xrightarrow{\quad\Phi\quad} & PSL(2,11) \\
\uparrow 12 & & \uparrow 12 & & \uparrow 12 \\
H^5_0(4-\sqrt{5}) & \longrightarrow & (1;5,5,11) & \longrightarrow & G^{\infty}_{55} \\
\uparrow 5 & & \uparrow 5 & & \uparrow 5 \\
H^5_1(4-\sqrt{5}) & \longrightarrow & (5;11^5) & \longrightarrow & C^{\infty}_{11} \\
\uparrow 11 & & \uparrow 11 & & \uparrow 11 \\
H^5(4-\sqrt{5}) & \longrightarrow & S_{g=70} & \longrightarrow & \{1\}
\end{array}
$$

So what we get is a Riemann surface of genus 70 with 60 punctures, defined by $\frac{\mathcal{U}}{H^5(4-\sqrt{5})}$. Since the parabolic elements of $[2,5,\infty]$ projects onto elements of order 11 in $[2,5,11]$, the puncture set of this surface projects onto the points of $R_{g=70}$ that are fixed by elements of order 11, and reversing the projection, the cusp set of $H^5(4-\sqrt{5})$ inherits the structure of the 3-biplane.

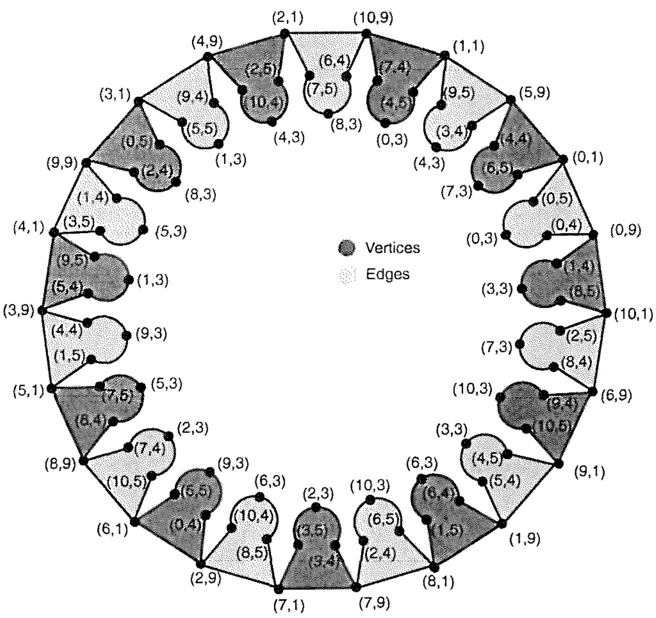


Fig. 4.6. Representation of the 3-biplane in the cusps of $\frac{\mathcal{U}}{H^5(4-\sqrt{5})}$

We have seen in Proposition 2.5.6. that the cusp set of $H^5(4-\sqrt{5})$ in the upper half-plane can be described in terms of fractions of $\mathbb{Q}(\sqrt{5})$ in their canonical form. We will now label the cusps so that we can obtain a picture of the 3-biplane embedding associated to the action of G^{∞}_{55} in the cusp set.

We will represent a fraction in its canonical form $\frac{P}{Q} \in \mathbb{Q}(\sqrt{5})$ by the coordinates (a, b) where $a, b \in \{0,\ 1,\ \ldots,\ 10\}$, if and only if:

$$\begin{cases} P \equiv \pm a \mod(4-\sqrt{5}) \\ \qquad \text{and} \\ Q \equiv \pm b \mod(4-\sqrt{5}) \end{cases}$$

Using that notation we get the incidence diagram in *Fig. 4.6.* where darker pawns represent vertices, lighter pawns represent edges and two points with the same coordinates are identified.

4.10. Embeddings related to *PSL*(2,11)

In this section we will describe other geometrical structures that can be described in terms of embeddings into $R_{g=70}$ and we will see a geometrical action of $PSL(2,11)$ on eleven objects. As we have seen in the previous section, we can consider the embeddings as lying on $R_{g=70}$ or as displayed in the cusp set of $H^5(I)$ where

$$I = (4+\sqrt{5}) \qquad \text{or} \qquad I = (4-\sqrt{5}).$$

The biggest proper subgroup of $PSL(2,11)$ has order 60 and as we have seen, it is isomorphic to $PSL(2,5)$. We have already mentioned that there are 22 of them into two conjugacy classes. They can be seen as the symmetry group of the icosahedron, in fact, if we take the 60 points fixed by elements of order 11 on $R_{g=70}$, we will see that they correspond to the vertices of a truncated icosahedron (a football).

Let us consider any of the possible embeddings of A_5 into $PSL(2,11)$, there are six subgroups of order 5 inside it, and each C_5 fixes two points on the surface of $R_{g=70}$ when we see them as subgroups of its automorphism group. Each pair of fixed points by a C_5 can be seen as a diagonal of the (truncated) icosahedron embedded on the surface. Every point in that pair is the centre of an hyperbolic equilateral pentagon, whose vertices are points fixed by elements of order 11 and in that manner we obtain the 12 pentagonal faces of the truncated icosahedron on the surface $R_{g=70}$. For the hexagonal faces we can proceed in any of two ways: we can study the action of the subgroups of order 3 on the set of C_5, or consider the action of the subgroups of order two on the surface, each subgroup of order two fixes two points on the surface that correspond to edge centres of hexagonal edges of the truncated icosahedron. With some work and a fair amount of calculation it is possible to obtain a picture for the truncated icosahedron inside $R_{g=70}$, although given the big genus of the surface and the low order of the highest order automorphism of the surface, it shows as much as it conceals and so we will not display it here.

There are twenty two embeddings of a truncated icosahedron inside $R_{g=70}$, distributed into two conjugacy classes, although orientation reversing automorphisms of the surface will take one class onto the other. We will finish this section by showing a model for *a geometric action of* $PSL(2,11)$ *on eleven points* that is related to a model for the 3-triplane inside $R_{g=70}$ (The definition of a triplane follows naturally as a generalization of the one for a biplane).

The geometric action of $PSL(2,11)$ on eleven objects is related to the combinatorial objects that are stabilized by $A_5 < PSL(2,11)$ and that can be embedded into $R_{g=70}$, so the truncated icosahedron is the obvious choice for it. If we consider the eleven truncated icosahedra inside a conjugacy class, each one is linked to one embedding of A_5 inside a conjugacy class, and so the algebraic action of $PSL(2,11)$ on the A_5 of one class is analogous to its geometric action on the embedding of the truncated icosahedron corresponding to that A_5.

Following the same ideas displayed in the sections regarding the Fano plane, we can consider the diagonals of each embedding of a truncated icosahedron as bits of a combinatorial structure, and we can label them using the labelling for the subgroup isomorphic to C_5 in $PSL(2,11)$ whose pull-back into $Aut(R_{g=70})$ fixes that diagonal. If we do so, we can take the icosahedrons in one conjugacy class as "lines" of an incidence structure (*Class L* in the table in chapter 2) and the icosahedrons in the other conjugacy class as "points" (*Class P*). If we check the resulting incidence structure, we see that we obtained a model for the 3-triplane, that is, the complement geometry of the 3-biplane inside $R_{g=70}$.

4.11. Another example of a graph embedding

In this section we will describe an embedding of the Fano plane into a surface of genus $g = 24$ that we will call $R_{g=24}$. We will embed the Fano plane as a conformal graph in the same way as we have done with the 3-biplane. This is the only possible rigid embedding of the Fano plane as a bipartite graph with non simply connected faces into a surface with automorphism group isomorphic to $PSL(2,7)$.

We will consider the triangle group $[3,4,7]$ with presentation:

$$< A, B, C \mid A^3 = B^4 = C^7 = ABC = 1 >$$

and the epimorphism $\Phi : [3,4,7] \longrightarrow PSL(2,7)$ given by:

$$\Phi(A) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \quad \Phi(B) = \begin{pmatrix} -1 & 5 \\ -1 & 4 \end{pmatrix} \quad \Phi(C) = \begin{pmatrix} 6 & 3 \\ 0 & 6 \end{pmatrix}.$$

It is easy to check that Φ is an epimorphism and that the group $Ker(\Phi)$ is torsion free and therefore it uniformizes a Riemann surface that has genus $g = 24$.

We will see now that the Fuchsian group $(1;[3,3,7])$ with presentation:

$$< a, b, c, x, y \mid a^3 = b^3 = c^7 = xyx^{-1}y^{-1}abc = 1 >$$

is an index 8 subgroup of $[3,4,7]$, with the inclusion given by:

$$i(x) = C^6AC^{-2} \qquad i(y) = C^3A^2C^{-2}$$

$$i(a) = C^3A^2C^{-1}ACAC^{-3} = 1 \qquad i(b) = C^4AC^{-4} \qquad i(c) = C^5A^2CAC^{-5}$$

If we restrict Φ to ϕ in the usual way $\phi(z) = \Phi(i(z))$ for any $z \in (1;[3,3,7])$ we get the following stack of groups and an embedding of the Fano plane as a bipartite graph inside $R_{g=24}$.

$$\begin{array}{ccc}
[3,4,7] & \xrightarrow{\quad\Phi\quad} & PSL(2,7) \\
i\uparrow 8 & & \uparrow 8 \\
(1;[3,3,7]) & \xrightarrow{\quad\phi\quad} & G_{21}^{\infty} \\
\uparrow 3 & & \uparrow 3 \\
(3;[7,7,7]) & \xrightarrow{\qquad} & C_7^{\infty} \\
\uparrow 7 & & \uparrow 7 \\
Ker(\phi) & \xrightarrow{\qquad} & \{1\}
\end{array}$$

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