Canonical Analysis of Double Null Relativistic Hamiltonian Dynamics

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ABSTRACT

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CANONICAL ANALYSIS OF DOUBLE NULL RELATIVISTIC HAMILTONIAN DYNAMICS

BY PAUL LAMBERT

In this thesis we consider a canonical analysis of double null General Relativity. We start with an introduction to Lagrangian and Hamiltonian dynamics, which introduces many of the techniques that are used throughout this thesis. Then in chapter 2 we introduce the canonical quantisation process, and perform the analysis on electromagnetism and General Relativity to help clarify the steps involved.

In chapter 3 we introduce the double null formulation of General Relativity. From this understanding we calculate the canonical analysis of this description. The complexity of the resulting constraints provides the motivation to introduce Ashtekar variables, the topic of chapter 4. This chapter also introduces $SO(3)$ variables. The new variables are then used in the canonical analysis of the double null description of General Relativity. Two different methods are considered and while the resulting constraint algebra remains the same, the two methods have different advantages.

Contents

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Finally I would like to thank God; my Lord and friend.

Notation

$\mathbf{A}, \mathbf{B}, \mathbf{C}, ...$	$SO(3)$ indices$(1, 2, 3)$
$\alpha, \beta, ...$	coordinate indices $(0, ..., 3)$
$a, b, c, ...$	time coordinate indices $(0, 1)$
$i, j, k, ...$	space coordinate indices (1,..,3 for 3+1 or 2, 3 for 2+2)
$\boldsymbol{\alpha}, \boldsymbol{\beta}, ...$	frame indices $(0, ..., 3)$
$\mathbf{a}, \mathbf{b}, \mathbf{c}, ...$	time frame indices $(0, 1)$
$\mathbf{i}, \mathbf{j}, \mathbf{k}, ...$	space frame indices (1,..,3 for 3+1 or 2, 3 for 2+2)

∂ or ,	partial derivative
$\Box$	space-time covariant derivative
∇ or ;	covariant derivative on leaves of foliation
D	exterior covariant derivative
$\pounds$	space-time Lie derivative
$\mathcal{L}$	space Lie derivative

$\tilde{N}$	density of weight one
$\underset{\sim}{N}$	density of weight minus one
$\bar{N}_{\alpha\beta}$	trace free part of $N_{\alpha\beta}$

$g^{\alpha\beta}$	space-time metric
$\gamma^{\mu\nu}$	induced metric

${}^4A_\alpha{}^{\alpha\beta}$	four connection
${}^4R_{\alpha\beta}{}^{\alpha\beta}$	four curvature
$A_\alpha{}^{\alpha\beta}$	connection in spatial surface
$R_{\alpha\beta}{}^{\alpha\beta}$	curvature in spatial surface
$\mathcal{A}_\alpha{}^{\alpha\beta}$	self-dual connection
$\mathcal{R}_{\alpha\beta}{}^{\alpha\beta}$	self-dual curvature

$\epsilon_{\mathbf{0123}} = -i$

$\epsilon_{\alpha\beta\gamma\delta} = 4!\, \delta^{\mathbf{0}}_{[\alpha} \delta^{\mathbf{1}}_{\beta} \delta^{\mathbf{2}}_{\gamma} \delta^{\mathbf{3}}_{\delta]}$

$*$	Hodge-dual
$\star$	star operator
$\{f, g\}$	Poisson brackets
$[f, g]$	commutator bracket
L	Lagrangian
$\mathcal{L}$	Lagrangian density
H	Hamiltonian
$\mathcal{H}$	Hamiltonian density

Chapter 1

Introduction

The concept of gravity was first described by Newton through his study of planetary motion. Newton understood that all objects on earth are subject to a force that pulls them towards the earth's centre. An opposing force must be applied to lift an object off the earth's surface, but if this opposing force is removed then the object falls back to the ground. To explain this, Newton described a force that attracted massive objects towards each other. With this new force, called gravity, and the recently developed mathematical techniques of calculus, Newton was able to understand planetary orbits.

One main problem with Newton's work was that it failed to describe how this force arises. This was overcome when Einstein first wrote about General Relativity. Before Einstein had accomplished this famous work he had already established himself in the scientific community with his special theory of Relativity, which explains how light can propagate through a vacuum, such as space, while all other waves required a medium in which to travel. This theory explained the results of a recent experiment by Michealson and Morely. It also resulted in breaking many strongly held beliefs concerning the nature of the universe held by scientists at that time.

Einstein was not content with this work but wanted to incorporate gravity into his theory of relativity. This work took him many years to complete, but resulted in a new understanding of gravity. In his work Einstein proposed that gravity was the result of mass curving 'space-time'. This meant that the force of gravity was a consequence of the geometry of a four dimensional manifold called space-time.

This new understanding of a force was unique; no other force had been so beautifully described by the curvature of geometry. Even to the present day forces other than gravity have been described in terms of an exchange of particles rather than from a geometrical basis. This common particle exchange approach enabled theories describing the forces to be combined, but gravity stood apart; distinctly unique. Unlike all the other physical theories, gravity is described by geometry and it has therefore proved difficult to combine it with these other forces. Einstein spent many years trying to bridge this divide, but to no avail.

To start with the desire for a complete theory that described both General Relativity and the theory of particle interaction (called quantum field theory) was one of aesthetics. Now science is trying to understand regions of space that require a description of large masses on a small scale. To do this a clear understanding of how the two theories of General Relativity and quantum field theory interact with each other. One of the first approaches was called canonical quantisation. The work in this thesis starts to apply this technique to a particular description of gravity: double null. The method of canonical quantisation, as well as other quantisation methods, is based on the fact that both General Relativity and quantum field theory can be described in a Hamiltonian form. Therefore we now spend some time introducing Lagrangian and Hamiltonian techniques that are used throughout this thesis.

1.1 Lagrangian and Hamiltonian dynamics in finite dimensions

An instantaneous phase space of a system is described in terms of a set of N generalised coordinates q^k and their velocities $\dot{q}^k$ (where k runs from $1, 2, .., N$). This gives the configuration of the system. The initial state of the system is described by a point in the phase space, and over an interval of time the state of the system will evolve, resulting in a curve.

Let T denote the kinetic energy and V the potential energy. Then we can define the Lagrangian $L = T - V$. In this introduction we will consider only Lagrangians without an

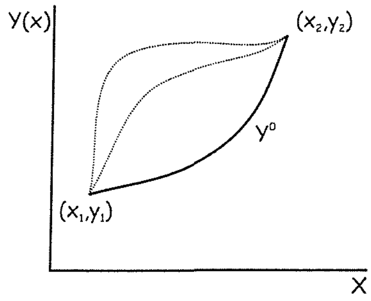

Figure 1.1: *Varying paths of a one dimensional system.*

explicit time dependance. Therefore the Lagrangian is a function of only the generalised coordinates q^k and their time derivatives $\dot{q}^k$.

Now that we have introduced the Lagrangian we will consider Hamilton's Principle which states that the motion of a system from time t_1 to time t_2 is such that the line integral

$$I = \int_{t_1}^{t_2} L(q^k, \dot{q}^k)\,\mathrm{d}t,$$

is an extremum. I is called the action. This allows us to say that the variation of the line integral I is zero:

$$\delta I = \delta \int_{t_1}^{t_2} L(q^k, \dot{q}^k)\,\mathrm{d}t = 0. \tag{1.1}$$

We now show that a necessary and sufficient condition for the above is that

$$\frac{\partial L}{\partial q^k} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}^k}\right), \tag{1.2}$$

which are just the Lagrange equations (sometimes called equations of motion).

We start by labeling all the possible paths, $q_k(x)$, the system could take with a parameter, α, and a variation $\eta_k(t)$. We take the path $\alpha = 0$ to be the extremum path (see figure 1.1).
Therefore:

$$q_k(t, \alpha) = q_k^0(t) + \alpha\eta_k(t), \tag{1.3}$$

where $q_k^0(t)$ is the solution at the extremum. For fixed variations of $\eta_k(t)$ we consider the action I as a function of a parameter α and let

$$I(\alpha) = I(q_k^0(t) + \alpha\eta_k(t)). \tag{1.4}$$

First we consider:

$$\begin{aligned}
\frac{\mathrm{d}I}{\mathrm{d}\alpha} &= \int_{t_1}^{t_2} \frac{\partial L}{\partial q_k}\frac{\partial q_k}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_k}\frac{\partial \dot{q}_k}{\partial \alpha}\,\mathrm{d}t \\
&= \int_{t_1}^{t_2} \frac{\partial L}{\partial q_k}\frac{\partial q_k}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_k}\frac{\partial^2 q_k}{\partial t\partial \alpha}\,\mathrm{d}t \\
&= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_k} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_k}\right)\frac{\partial q_k}{\partial \alpha}\,\mathrm{d}t.
\end{aligned}$$

In the last step we used an integration by parts on the second term and the fact that the variation vanishes at t_1 and t_2. We now introduce the variation in terms of the parameter α.

$$\delta I = \left.\frac{\mathrm{d}I}{\mathrm{d}\alpha}\right|_{\alpha=0}, \qquad \delta q_k = \left.\frac{\partial q_k(t,\alpha)}{\partial \alpha}\right|_{\alpha=0} = \eta_k(t). \tag{1.5}$$

Then, using the above:

$$\delta I = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_k} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_k}\right)\delta q_k\,\mathrm{d}t. \tag{1.6}$$

Since the variation δq_k is arbitrary, $\delta I = 0$ if and only if

$$\frac{\partial L}{\partial q_k} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_k} = 0. \tag{1.7}$$

Hence, we have shown that through a variational approach the Lagrange equations follow from the Hamilton's principle. It is possible to generalise this approach to allow for higher derivatives, or several parameters x^α. Such generalisations can be found in Goldstein (1969).

1.1.1 Hamiltonian description for finite dimensional systems

The system of equations given by the Lagrange equations (1.2) are second order. The phase space for the Lagrangian description uses q^k and $\dot{q}^k$ with time as a parameter. We now introduce an alternative form, in which the velocities used above are replaced by generalised momenta, p_k, defined by:

$$p_k := \frac{\partial L(q^k, \dot{q}^k)}{\partial \dot{q}^k}. \tag{1.8}$$

This change of basis is performed through a Legendre transformation which also defines the Hamiltonian:

$$H(q^k, p_k) = \dot{q}^k p_k - L(q^k, \dot{q}_k). \tag{1.9}$$

From this definition we get the system of equations

$$\dot{q}^k = \frac{\partial H}{\partial p_k} \tag{1.10a}$$

$$\dot{p}_k = -\frac{\partial H}{\partial q^k}. \tag{1.10b}$$

The equations (1.10a, 1.10b) are known as the canonical equations of Hamilton, and they replace the Lagrange equations. We can see from these equations that we have a system of first order equations replacing the second order Lagrange equations. In solving a dynamical system, the first step is to obtain the Lagrangian, and then to perform the Legendre transformation to obtain a Hamiltonian. Finally we obtain and solve the canonical equations. The Legendre transformation is possible only if the dynamical system is not constrained. We consider constrained systems when we discuss the Dirac-Bergman algorithm in the following chapter.

1.1.2 Poisson brackets

If $F(q^k, p_k)$ and $G(q^k, p_k)$ are two arbitrary functions, then their Poisson bracket is defined by:

$$\{F, G\} = \frac{\partial F}{\partial q^k}\frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q^k}. \tag{1.11}$$

From this definition we can easily show the fundamental Poisson brackets satisfy:

$$\{q^k, q^l\} = 0 \tag{1.12a}$$

$$\{p_k, p_l\} = 0 \tag{1.12b}$$

$$\{q^k, p_l\} = \delta^k_l, \tag{1.12c}$$

and they are independent of the canonical coordinates chosen. It is possible to give the canonical equations in terms of the Poisson brackets as:

$$\dot{q}^k = \{q^k, H\} \tag{1.13}$$

$$\dot{p}^k = \{p_k, H\}. \tag{1.14}$$

More generally if $f = f\big(q(t), p(t)\big)$ then $\dot{f} = \{f, H\}$ so that functions whose Poisson bracket with the Hamiltonian is zero must be constants of the motion.

1.2 Lagrangian and Hamiltonian dynamics for field theories

In the section above we considered a finite dimensional system. Many physical systems are described by field theory, in which the Lagrangian depends upon position as well as time. Such a system can be thought of as an infinite dimensional system. To obtain equations of motion for such systems Hamilton's Principle must be generalised to:

$$\delta I = \delta \int_{t_1}^{t_2} \int \mathcal{L}\, \mathrm{d}^3x\, \mathrm{d}t = 0, \tag{1.15}$$

where $\mathcal{L}(q^\lambda, \dot{q}^\lambda, q^\lambda{}_{,i}, x^i)$ is called the Lagrangian density and $q^\lambda = q^\lambda(t, x^i)$ are the fields in which the index λ labels the different fields in the system. The Lagrangian density is a scalar density of weight one. The relationship between the Lagrangian and the Lagrangian density is given by:

$$L = \int \mathcal{L}\, \mathrm{d}^3x. \tag{1.16}$$

In the finite dimensional case we were able to show that Hamilton's Principle was satisfied if and only if the Lagrange equations (1.2) were satisfied. In the infinite dimensional case we can also show that Hamilton's principle (1.15) is satisfied when the generalised Lagrange equations,

$$\frac{\partial \mathcal{L}}{\partial q^\lambda} - \partial_i \left(\frac{\partial \mathcal{L}}{\partial q^\lambda{}_{,i}} \right) - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}^\lambda} \right) = 0, \tag{1.17}$$

are satisfied. The proof which shows this is just a generalisation of the finite dimensional case shown earlier (see Goldstein 1969).

The functional derivative of a second order Lagrangian density is given by, $\mathcal{L}$:

$$\frac{\delta \mathcal{L}}{\delta q^\lambda} \equiv \frac{\partial \mathcal{L}}{\partial q^\lambda} - \partial_i \left(\frac{\partial \mathcal{L}}{\partial q^\lambda{}_{,i}} \right). \tag{1.18}$$

Then using this notation we may state the Lagrange equations as

$$\frac{\delta \mathcal{L}}{\delta q^\lambda} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}^\lambda} \right) = 0, \tag{1.19}$$

which are similar in appearance to (1.2).

1.2.1 Hamiltonian description for continuous systems

We now require a Hamiltonian description for a continuous system. We start with the definition of the momenta:

$$\tilde{p}_\lambda = \frac{\delta \mathcal{L}}{\delta \dot{q}^\lambda}. \tag{1.20}$$

We note that unlike the finite dimensional system, here the momenta are densities. Therefore the Hamiltonian we define using these momenta is also a density of weight one. Using the definition of a Hamiltonian density, $\mathcal{H} = \tilde{p}_\lambda q^\lambda - \mathcal{L}$, we obtain:

$$H = \int \mathcal{H}(q^\lambda, q^\lambda{}_{,i}, \tilde{p}_\lambda, \tilde{p}_{\lambda\,,i}, x^i)\, \mathrm{d}^3 x. \tag{1.21}$$

Note that in some cases the canonical variables can themselves be densities and therefore the momenta may not simply have a weight of one. We require particular care when calculating Lie derivatives or partial derivatives of densitised variables.

The canonical equations are expressed in the same way as the finite dimensional case, except we use functional derivatives rather than partial derivatives:

$$\dot{q}^{\lambda} = \frac{\delta H}{\delta \tilde{p}_{\lambda}} \tag{1.22a}$$

$$\dot{\tilde{p}}_{\lambda} = -\frac{\delta H}{\delta q^{\lambda}} \tag{1.22b}$$

1.2.2 Poisson brackets in infinite dimensions

Now that we are working with infinite dimensions we are required to redefine the Poisson bracket to be

$$\{F(x), G(y)\}_{x^0=y^0} = \int \left(\frac{\delta F(x)}{\delta q^{\lambda}(z)} \frac{\delta G(y)}{\delta \tilde{p}_{\lambda}(z)} - \frac{\delta F(x)}{\delta \tilde{p}_{\lambda}(z)} \frac{\delta G(y)}{\delta q^{\lambda}(z)} \right) \mathrm{d}^3 z. \tag{1.23}$$

We can see in the above that in moving from a discrete definition to a continuous one we have replaced the partial derivatives with functional derivatives. We also changed the sum in the discrete Poisson bracket to an integral, but a sum on the discrete field label arises. The other addition to (1.23) is the condition $x^0 = y^0$, which states that the Poisson bracket is evaluated for 'equal time'. In the future we will not state this condition explicitly. Using (1.23) the canonical equations are expressed as:

$$\dot{q}^{\lambda} = \{q^{\lambda}, H\} \tag{1.24a}$$

$$\dot{\tilde{p}}_{\lambda} = \{\tilde{p}_{\lambda}, H\}, \tag{1.24b}$$

and the fundamental Poisson brackets are:

$$\{q^{\lambda}(x), q^{\kappa}(y)\} = 0 \tag{1.25a}$$

$$\{\tilde{p}_{\lambda}(x), \tilde{p}_{\kappa}(y)\} = 0 \tag{1.25b}$$

$$\{q^{\lambda}(x), \tilde{p}_{\kappa}(y)\} = \delta^{\lambda}_{\kappa} \delta(x, \tilde{y}). \tag{1.25c}$$

This introduces the Dirac delta function $\delta(x, \tilde{y})$, which is a bidensity (see DeWitt 1967).

1.2.3 Dealing with the delta function

In all our future work we will be using the infinite dimensional definition of the Poisson bracket, and therefore we are often required to integrate the Dirac delta function. The Dirac delta function has the following property:

$$\int \tilde{f}(z)\delta(x,z)\,\mathrm{d}^3z = \tilde{f}(x), \tag{1.26}$$

where $\tilde{f}$ is a density of weight $+1$. A densitised delta function arises when we calculate a functional derivative of a density with a density, for example:

$$\frac{\delta\tilde{p}_\lambda(x)}{\delta\tilde{p}_\kappa(y)} = \delta^\lambda_\kappa\tilde{\delta}(x,y). \tag{1.27}$$

With a densitised delta function we get the property:

$$\int g(z)\tilde{\delta}(x,z)\,\mathrm{d}^3z = g(x), \tag{1.28}$$

where g is a function.

The bidensity that arose in the fundamental Poisson brackets given above has no weight on the the first argument, x, and weight one on its second, y. This results in the properties:

$$\int g(x)\delta(x,\tilde{y})\,\mathrm{d}^3x = g(y) \tag{1.29a}$$

$$\int \tilde{f}(y)\delta(x,\tilde{y})\,\mathrm{d}^3y = \tilde{f}(x). \tag{1.29b}$$

1.3 Canonical Quantisation

In Dirac (1964), Dirac showed how it is possible to move from a constrained classical theory described by Hamiltonian dynamics to a corresponding quantum theory. The basic idea involves replacing the Poisson bracket of the canonical variables with the commutator of the corresponding quantum operators according to:

$$[f, g] \to \frac{i}{\hbar}\{\hat{f}, \hat{g}\}. \tag{1.30}$$

For many theories this transition is too simplistic because in general, classical observables do not necessarily have unique quantum observables associated to them, or the theory contains constraints. Dirac proposed an 'algorithm' (see Dirac 1964) which can be used to carry out this procedure for theories containing constraints. This algorithm is now called the Dirac-Bergman algorithm and the form of quantisation is called Canonical Quantisation; we will outline only the details here as a more in depth discussion is given in chapter 2.

The fundamentals of this approach are to express the classical theory in terms of a Hamiltonian, before obtaining the first class constraints. First class constraints are constraints that generate infinitesimal transformations, ie. they change the canonical variables without changing the physical state. The remaining constraints are called second class. Once the first class constraints have all been obtained an algebra is generated by calculating the Poisson bracket relations between them. We then move to the quantum theory by replacing the Poisson brackets with commutator relations as given above. The second class constraints are eliminated by the definition of new variables.

Despite issues relating to some finer points of the algorithm, in particular the construction of quantum observables, canonical analysis has had notable success in quantising electromagnetism to obtain the theory quantum electrodynamics. Other field theories have also been quantised, such as Yang-Mills theory, but General Relativity has proved to be too difficult.

At about the same time canonical quantisation was developed, an alternative method was being established. This method involved solving the Hamiltonian through path-integral methods. In a similar manner to the canonical quantisation method it was successful for Electromagnetism, and some other 'simpler' theories, but it has not so far been used to quantise General Relativity, although it has had some success with string theory.

1.4 General Relativity

In this section we derive the Einstein field equations from a Lagrangian density. Although this was not the approach that Einstein originally took when he first derived his field equations, it is the most direct approach and the one that will be most useful for this thesis.

Any field theory can be described by a Lagrangian density, but before we state the Lagrangian for General Relativity we first introduce a metric $g_{\alpha\beta}$. A metric is a symmetric covariant tensor of rank 2, which can be used to measure the infinitesimal intervals $\mathrm{d}s^2 = g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta$ on a manifold, M. The covariant metric $g_{\alpha\beta}$ has an inverse $g^{\alpha\beta}$, and together they can be used to raise and lower tensorial indices. The metric can also be used to define a connection on the manifold, called the metric connection:

$$\Gamma^\alpha_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\delta}\left(g_{\delta\gamma,\beta} + g_{\delta\beta,\gamma} - g_{\beta\gamma,\delta}\right). \tag{1.31}$$

The connection is used to define a covariant derivative:

$$\Box_\gamma X^{\alpha\dots}_{\beta\dots} := \partial_\gamma X^{\alpha\dots}_{\beta\dots} + \Gamma^\alpha_{\delta\gamma} X^{\delta\dots}_{\beta\dots} - \Gamma^\delta_{\beta\gamma} X^{\alpha\dots}_{\delta\dots} \tag{1.32}$$

Note that we have used $\Box$ to denote the space-time covariant derivative in order to reserve the usual notation, ∇, for later use as the covariant derivative induced on a 3-surface (see Isenberg & Nester 1979).

As well as the covariant derivative the metric connection is also used to define the curvature or Riemann tensor:

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha_{\beta\delta,\gamma} - \Gamma^\alpha_{\beta\gamma,\delta} + \Gamma^\epsilon_{\beta\delta}\Gamma^\alpha_{\epsilon\gamma} - \Gamma^\epsilon_{\beta\gamma}\Gamma^\alpha_{\epsilon\delta}. \tag{1.33}$$

If we use the metric to contract two Riemann indices then we obtain the Ricci tensor:

$$R_{\alpha\beta} = g^{\gamma\delta} R_{\delta\alpha\gamma\beta}, \tag{1.34}$$

while a further contraction defines the Ricci scaler:

$$R = g^{\alpha\beta} R_{\alpha\beta}. \tag{1.35}$$

We now define the Einstein tensor:

$$G_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R, \tag{1.36}$$

which is symmetric, and satisfies the contracted Bianchi identities:

$$\Box_\beta G_\alpha^{\ \beta} \equiv 0. \tag{1.37}$$

Now that we have defined the tensors above we can introduce the Einstein-Hilbert Lagrangian, from which we can obtain the Einstein equations.

$$I = \int_\Omega R(-g)^{1/2}\,\mathrm{d}\Omega. \tag{1.38}$$

g is the determinant of the metric $g_{\alpha\beta}$, and R is the Ricci scaler defined above (1.35). Before we can apply Hamilton's principle to the action above, we first require some identities:

$$\delta g^{\alpha\delta} = -g^{\alpha\beta} g^{\gamma\delta} \delta g_{\beta\gamma}, \tag{1.39a}$$

$$\delta(-g)^{1/2} = \tfrac{1}{2}(-g)^{1/2} g^{\alpha\beta} \delta g_{\alpha\beta}. \tag{1.39b}$$

We now regard the action as a function of $g_{\alpha\beta}$ and its first two derivatives and vary the action (1.38) to obtain:

$$\delta I = \int_\Omega \left[\delta(-g)^{1/2} g^{\alpha\beta} + (-g)^{1/2} \delta g^{\alpha\beta}\right] R_{\alpha\beta} + (-g)^{1/2} g^{\alpha\beta} \delta R_{\alpha\beta}\,\mathrm{d}\Omega. \tag{1.40}$$

The last term vanishes because after using the Palatini identity:

$$\delta R^\alpha_{\ \beta\gamma\delta} = \Box_\gamma(\delta\Gamma^\alpha_{\beta\delta}) - \Box_\delta(\delta\Gamma^\alpha_{\ \beta\gamma}), \tag{1.41}$$

the integral can be converted to a surface integral, which vanishes because variations at the boundary of Ω are assumed to vanish. Therefore (1.40) reduces to:

$$\begin{aligned}\delta I &= \int_\Omega R_{\alpha\beta}\left[\delta(-g)^{1/2}g^{\alpha\beta} + (-g)^{1/2}\delta g^{\alpha\beta}\right]\,\mathrm{d}\Omega\\ &= -\int_\Omega \left[R^{\gamma\delta} - \tfrac{1}{2}\gamma^{\gamma\delta}R\right]\delta g_{\gamma\delta}(-g)^{1/2}\,\mathrm{d}\Omega. \end{aligned} \tag{1.42}$$

Now applying Hamilton's principle, we obtain

$$\delta I = 0 \quad \Longrightarrow \quad R^{\gamma\delta} - \tfrac{1}{2}\gamma^{\gamma\delta}R = 0,$$

and hence

$$G^{\gamma\delta} = 0. \tag{1.43}$$

We have therefore shown that the Einstein equations can be obtained from the variation of an action $I[g_{\alpha\beta}, g_{\alpha\beta,\gamma}, g_{\alpha\beta,\gamma\delta}]$.

There are various other methods for obtaining the Einstein equations from the Einstein-Hilbert action. One such method introduced by Palatini considers the connection variables to be independent of the metric, therefore the action $I[g_{\alpha\beta}, \Gamma^{\alpha}_{\beta\gamma}]$ becomes first order. This method is covered in more detail in chapter 4. An alternative is to replace the metric variables with a frame $\theta^{\boldsymbol{\alpha}}_{\alpha}$ and the metric connection with the Ricci rotation coefficients $A^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}\gamma}$ to obtain an action of the form $I[\theta^{\boldsymbol{\alpha}}_{\alpha}, A^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}\gamma}]$. The connection is now the connection between frames, $A^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}\gamma}$, not the metric connection previously introduced.

1.4.1 $3+1$ decomposition

In order to transform the Lagrangian description above into a Hamiltonian one, we require an evolution direction. This requires a decomposition of both the manifold and the tensor fields into a 3+1 form. The details of this calculation can be found in the following chapter and so we will not cover them here, but rather give some history and context for the work.

One of the main benefits of writing General Relativity in a Hamiltonian form is that it is then suitable for Canonical quantisation. One of the earliest attempts at this was undertaken by Arnowitt *et al.* (1960), and is known as the ADM approach. In their work they derived a Hamiltonian for General Relativity, and obtained the constraints that arise due to General Relativity being a constrained dynamical system. They were also able to give a geometrical understanding to the constraints. It was expected that from this work it would be possible to complete the canonical quantisation process for General Relativity. Unfortunately this was not the case. There were a number of reasons for this, but one significant problem was the non-polynomial nature of the constraints.

1.4.2 Ashtekar variables

In order to overcome the complexity of the constraints Ashtekar (1991) introduced a new set of variables that result in polynomial constraints which are of a simular form to those in Yang-Mills theory. The ADM approach had used the Einstein-Hilbert action, and taken the metric as the canonical variables. Therefore the action contains first and second derivatives of the canonical variables. This leads to complicated constraints which are second order partial derivatives of the metric. The Ashtekar approach uses objects constructed from the connection and frame as canonical variables, but also extends them by complexifying them (this allows the variables to take complex values). The remarkable aspect of this approach is that it is possible to split the action into two parts, both of which result in the Einstein equations independently. Therefore we need to consider only part of the action; this is described in more detail in chapter 4. The result of Ashtekar's work is that it is possible to obtain constraints that are polynomial in the canonical variables. Another benefit of working with this framework is that it allows for topologies in which the metric is degenerate.

Although Ashtekar's work was a big step forward in simplifying the constraints, the scalar constraint still caused difficulty. In the years that followed many attempts were made to overcome this obstacle. One such attempt was introduced by Jacobson & Smolin (1988) where they used Wilson loops to obtain a large class of solutions. Despite the effort made, this loop representation has not succeeded in the full quantisation of General Relativity. We will not discuss loop quantisation in any detail because it is beyond the scope of this

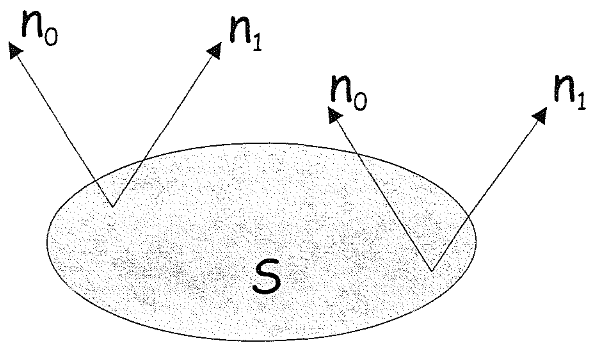

Figure 1.2: Double null space-time illustrating null directions.

thesis. At the same time that loop quantisation progressed alternative approaches, see below, tried to overcome these difficulties using canonical quantisation.

1.4.3 Alternative approaches to Canonical quantisation

Following from the work of Torre (1986), see chapter 3, it was realised that by making the evolution direction null, the scalar constraint became second class and therefore did not require explicit quantisation. With this understanding Goldberg *et al.* (1992) started the canonical quantisation approach with a $3 + 1$ null approach (see Appendix B). In their work they were able to obtain a Hamiltonian description of General Relativity from which they started the canonical quantisation process. Just as in the ADM approach, they obtained a system of constraints, although not all the constraints were first class. Unfortunately setting the evolution direction to be null introduced an additional problem: the evolution direction becomes tangential to the three surface. Therefore evolving the three surface required the construction of a normal to the three surface. This results in extra freedom that complicates the resulting first class constraints by introducing null rotations.

In a $2+2$ double null formulation the problem of a tangential evolution direction does not arise. Both null directions are normal to the two surface (see figure 1.2). Therefore it was expected that this approach would overcome many of the obstacles of earlier attempts, and this is the motivation for the work presented in this thesis.

Chapter 2

Canonical Analysis

2.1 Introduction

In the 1940's Dirac realised that by performing some analysis on a field theory represented by a Hamiltonian he could understand the theory's underlying structure. With this understanding Dirac also outlined how one might be able to quantise this field theory. It was thought that if only this could be applied to General Relativity then we would have a description of quantum gravity. After Dirac introduced this method of quantisation there was a lot of interest in applying the method to different field theories. Electromagnetism was successfully quantised, along with the Yang-Mills theory. Unfortunately, despite the progress made with these simpler theories General Relativity proved to be much more complicated.

Although 60 years on some progress has been made, many serious obstacles remain. Despite this, the method is still useful in its own right to aid understanding of classical field theories, and there is still hope that these obstacles may one day be overcome.

In the following sections we shall outline the steps required to complete the quantisation of a field theory. We shall first outline the canonical analysis of an action. We then look at some examples to clarify the method, before finally continuing with the remainder of the process.

2.2 Dirac-Bergmann algorithm / Canonical Analysis

In this section we outline the steps of the canonical analysis based on the Dirac-Bergmann algorithm.

We will assume that the theory we wish to quantise can be represented by an action in the following way:

$$I := \int_M \mathcal{L}\,\mathrm{d}^4x, \qquad \mathcal{L} = \mathcal{L}(q^\lambda, q^\lambda{}_{,\mu}, x^i) \tag{2.1}$$

where $\mathcal{L}$ is a Lagrangian density. We also assume that our field theory is defined on a manifold M which can be written $M = \Sigma \times \mathbb{R}$, where Σ are space-like hypersurfaces of constant time and form a foliation of M. We also assume a metric, g, is given on M along with a connection and 4-d covariant derivative, which we will denote by a $\Box$ to distinguish it from the 3-d covariant derivative ∇.

2.2.1 Step 1: $3+1$ decomposition

The first stage of the analysis is the full decomposition of the action into a space plus time description. Both the fields and the derivatives acting on them need to be decomposed.

We introduce a frame $\{\theta^0, \theta^{\mathrm{i}}\}$ and its respective dual $\{e_0, e_{\mathrm{i}}\}$ which have been adapted so that e_{i} forms a tangent basis to Σ. Using this frame we define projections onto Σ and its normal with:

$$P^0 := e_0^\alpha \theta_\beta^0 \quad P_\beta^\alpha := e_{\mathrm{i}}^\alpha \theta_\beta^{\mathrm{i}}. \tag{2.2}$$

Therefore any tensor can be split into its spatial and normal part. For example, the metric is split $g = -\theta^0 \otimes \theta^0 + \eta_{\mathrm{ij}} \theta^{\mathrm{i}} \otimes \theta^{\mathrm{j}}$.

Next we need to decompose the derivatives. We can define new covariant and Lie derivatives of functions on Σ using the projection operators:

$$\nabla_{e_{\mathrm{i}}}(Z) := P(\Box_{e_{\mathrm{i}}} Z) \quad \mathcal{L}_{e_{\mathrm{i}}}(Z) = P(\pounds_{e_{\mathrm{i}}} Z), \tag{2.3}$$

where $P(Z)$ denotes a product of projection operators sufficient to project all components of Z onto the hypersurface.

These define derivatives which only lie on the hypersurface. To consider derivatives off the hypersurface we need to extend the definition of $\mathcal{L}$ given above. For any spatial tensor Z, and space-time vector field X, we define:

$$\mathcal{L}_X(Z) = P(\pounds_X Z). \tag{2.4}$$

We now have defined space-covariant derivatives on and off the hypersurface. Therefore we can decompose the space-time covariant derivative into derivatives defined on the hypersurface. To do this we shall use the results of Isenberg & Nester (1979), and give the full decomposition of $\square_\alpha V^\beta$ and $\square_\alpha V_\beta$ as,

$$\begin{aligned}
&\square_{e_0} V^{\mathbf{0}} = \mathcal{L}_{e_0} V^{\mathbf{0}} + a_{\mathbf{j}} V^{\mathbf{j}} && \square_{e_0} V_{\mathbf{0}} = \mathcal{L}_{e_0} V_{\mathbf{0}} - a^{\mathbf{j}} V_{\mathbf{j}} \\
&\square_{e_{\mathbf{m}}} V^{\mathbf{0}} = \nabla_{\mathbf{m}} V^{\mathbf{0}} - K_{\mathbf{mj}} V^{\mathbf{j}} && \square_{e_{\mathbf{m}}} V_{\mathbf{0}} = \nabla_{\mathbf{m}} V_{\mathbf{0}} + K^{\mathbf{j}}{}_{\mathbf{m}} V_{\mathbf{j}} \\
&\square_{e_0} V^{\mathbf{s}} = \mathcal{L}_{e_0} V^{\mathbf{s}} + a^{\mathbf{s}} V^{\mathbf{0}} - K^{\mathbf{s}}{}_{\mathbf{j}} V^{\mathbf{j}} && \square_{e_0} V_{\mathbf{s}} = \mathcal{L}_{e_0} V_{\mathbf{s}} - a_{\mathbf{s}} V_{\mathbf{0}} + K^{\mathbf{j}}{}_{\mathbf{s}} V_{\mathbf{j}} \\
&\square_{e_{\mathbf{m}}} V^{\mathbf{s}} = \nabla_{\mathbf{m}} V^{\mathbf{s}} - K_{\mathbf{m}}{}^{\mathbf{s}} V^{\mathbf{0}} && \square_{e_{\mathbf{m}}} V_{\mathbf{s}} = \nabla_{\mathbf{m}} V_{\mathbf{s}} + K_{\mathbf{sm}} V_{\mathbf{0}}.
\end{aligned} \tag{2.5}$$

Above and subsequently we use bold indices to indicate frame components. The notation K_{ij} represents the extrinsic curvature and a^s denotes the acceleration in the e_0 direction.

Before we proceed to describe the other steps in the analysis, we must first define a derivative along the foliation. So far all derivatives have been projected onto the hypersurface. We shall also require a derivative which will evolve the hypersurface. This is performed by $\mathcal{L}_{\partial/\partial t}$, where $\partial/\partial t$ is a vector field mapped to 1 by the form $\mathrm{d}t$. This derivative allows us to evolve a tensor along the foliation. Therefore we shall denote it by

$$\dot{Z} := \mathcal{L}_{\partial/\partial t} Z. \tag{2.6}$$

The vector field $\partial/\partial t$ can be expressed in terms of the adapted frame, see figure 2.1. Hence,

$$\frac{\partial}{\partial t} = N e_{\mathbf{0}} + N^{\mathbf{i}} e_{\mathbf{i}}.$$

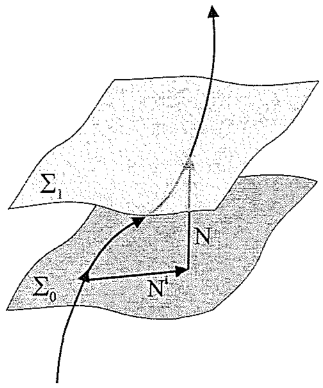

Figure 2.1: *Lapse and shift vectors in a* $3+1$ *foliation with one spacial dimension removed.*

The scalars N, $N^{\mathbf{i}}$ are often referred to as the lapse and shift respectively. We can therefore write:

$$\mathcal{L}_{\partial/\partial t} = N\mathcal{L}_{e_0} + \mathcal{L}_{\vec{N}}. \tag{2.7}$$

Only the first term governs the evolution of a tensor field; the second term generates translations within the hypersurface.

2.2.2 Step 2: Define conjugate momenta

We now use the decomposed action to define the conjugate momenta for each of the canonical variables, q^λ. The corresponding momentum is defined by

$$p_\lambda := \frac{\partial \mathcal{L}}{\partial \dot{q}^\lambda}. \tag{2.8}$$

The canonical variables, q^λ, and their momenta, p_λ, describe the phase space, Γ, of the theory. We define the Poisson bracket for a field theory (see 1.23):

$$\{F(p_\lambda, q^\lambda), G(p_\lambda, q^\lambda)\} = \int_z \left(\frac{\delta F}{\delta q^\lambda}\frac{\delta G}{\delta p_\lambda} - \frac{\delta F}{\delta p_\lambda}\frac{\delta G}{\delta q^\lambda} \right) \mathrm{d}^3 z, \tag{2.9}$$

where we define

$$\frac{\delta F}{\delta q^\lambda} = \frac{\partial F}{\partial q^\lambda} - \partial_i \frac{\partial F}{\partial(\partial_i\, q^\lambda)}. \tag{2.10}$$

2.2.3 Step 3: Primary constraints

Primary constraints are equations that constrain the phase space to generate a reduced phase space ($\bar{\Gamma}$). If the rank of the Hessian, $W_{\lambda\zeta} = \frac{\partial p_\zeta}{\partial \dot{q}^\lambda}$, is maximal, ie. has the same number of independent momenta as field variables, then we would be able to solve the system for all momenta and so easily transform from the Lagrangian to the Hamiltonian description. In many physical systems, however, this is not the case. Constraints often arise due to restrictions in the system, or due to the existence of conserved quantities. In electromagnetism such constraints arise due to the gauge freedoms which exist within the theory. In many constrained dynamical systems some momenta vanish, therefore primary constraints are in the form $p_\lambda = 0$. In this case the variables are called cyclic, and we shall discuss a 'shortcut' method of dealing with them later in the chapter.

The use of Poison brackets in field theory requires all the phase space variables to be independent. Hence all the primary constraints must be arise after the Poisson brackets have been calculated. Therefore, they are included into the Lagrangian with the use of Lagrange multipliers, which we will denote by u_m. The primary constraints are then obtained through variations of the Lagrangian density with respect to the multipliers. Imposing the constraints before the Poisson bracket is calculated would result in too few equations and hence a loss of information.

2.2.4 Step 4: Deriving the Hamiltonian

Now that we have defined all the momenta and found the constraints we can transform the decomposed Lagrangian into a Hamiltonian. This is accomplished using the definition

$$\mathcal{H} = p_\lambda \dot{q}^\lambda - \mathcal{L}. \tag{2.11}$$

The first stage is to substitute the momenta into the Lagrangian density using their definitions given by (2.8). It might be possible, at this stage to write the Lagrangian density in the form $\mathcal{L} = \dot{q}^\lambda p_\lambda - F(q^\lambda, p_\lambda)$. In this case one can simply interpret the density $F(q^\lambda, p_\lambda)$ as the Hamiltonian. Otherwise we would use (2.11) to obtain the Hamiltonian density replacing the $\dot{q}$ by p. Following the terminology of Dirac (1964) the Hamiltonian without any primary constraints is called the 'base Hamiltonian' and is denoted by H_0, whereas the Hamiltonian containing primary constraints is called the 'primary Hamiltonian' and is denoted H_p.

2.2.5 Step 5: Constraint analysis.

This is the main algorithmic part of the analysis. In order for the field theory to be consistent we must ensure that the primary constraints hold for every time step. This is achieved by using the equation

$$\dot{Z} = \{Z, H\}, \tag{2.12}$$

where Z is a function of the q's and p's and their spatial derivatives. We must therefore ensure that, when calculated, the Poisson brackets of all the primary constraints with the primary Hamiltonian are zero on the reduced phase space. When this occurs we shall denote it as being weakly zero, $\phi \approx 0$, in order to distinguish it from being zero on the full phase space. Thus, for every primary constraint, ϕ_m,

$$\{\phi_m, \mathcal{H}_p\} \approx 0. \tag{2.13}$$

Unless the Poisson brackets vanish identically, additional constraints will arise. These constraints, although only weak, will reduce the number of independent variables contained within the Hamiltonian, ie. the dimension of $\bar{\Gamma}$. If the resulting constraints include multiplier terms they will be called multiplier equations, while the remaining constraints are called secondary constraints. We also require these secondary constraints, denoted by $\chi_k \approx 0$, to be conserved. Therefore we calculate

$$\{\chi_k, H_p\} \approx 0. \tag{2.14}$$

If these Poisson brackets continue to present additional constraints then we get additional secondary constraints and we repeat as before until all constraints are weakly conserved or define multipliers. After this we have $2N - M - K$ independent variables (where M and K are the number of primary and secondary constraints respectively) and we have restricted the multipliers u_m with the multiplier equations.

2.2.6 Step 6: Evolution equations

Next we determine the evolution equations for the canonical variables. These are given by

$$\dot{q} = \{q, H\} \qquad \dot{p} = \{p, H\}. \tag{2.15}$$

Once these have been calculated we have a well posed initial value procedure for calculating the field equations of the action. This is outlined below:

a) Specify initial data on a chosen hypersurface for the canonical variables, which satisfies all of the constraints.

b) Choose initial data for the unknown multipliers.

c) Evolve the canonical variables using (2.15). If any of the geometric fields are not defined by the canonical variables or the constraints, then they must be specified independently on each hypersurface.

d) Evolve the multipliers by a chosen method.

Once the canonical variables are defined through all time, we can reconstruct the $3+1$ decomposition to reconstruct the full space-time variables.

2.2.7 A shortcut

Let us consider a shortcut that can be used in the canonical analysis we have been discussing. This shortcut makes use of the primary constraints that arise due to the cyclic variables mentioned previously. Instead of introducing the cyclic variables as primary constraints and therefore including them in the whole constraint analysis, we can consider them merely as Lagrange multipliers. No information has been lost by using this method. We must, however ensure that the constraint arising from $\delta\mathcal{H}/\delta Z$ is preserved by the evolution. The implies that

$$\left\{\frac{\delta\mathcal{H}}{\delta Z}, H_p\right\} \approx 0 \tag{2.16}$$

where Z is our cyclic variable. We must also ensure the resulting constraint is preserved for all time. If Z remains undefined by the end of the analysis then we may specify it in any way we desire. This is illustrated by the treatment of the lapse and shift in the example of vacuum General Relativity.

2.3 Examples of Canonical Analysis

Although this is not the end of the algorithm we shall break from the theory at this point with some examples illustrating the steps already outlined above. We shall first consider the simple case of electromagnetism in a curved background and then move on to investigate the more complicated case of General Relativity.

2.3.1 Maxwell's theory

In this section we shall apply the canonical analysis to the theory of electromagnetism on a fixed curved space time background. We will work in the frame description because future work is also based on the frame approach, although using the coordinate approach would result in the same equations. Note that in this example we are using the Minkowski frame metric with respect to an orthonormal frame so that $\eta^{\alpha\beta} = (-1, +1, +1, +1)$. However in chapter 5 onwards we will work exclusively in a null basis.

Step 1: We start by writing the usual Lagrangian density for electromagnetism:

$$\mathcal{L} = -\tfrac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \sqrt{-g} \tag{2.17}$$

where

$$F_{\alpha\beta} = \Box_\alpha A_\beta - \Box_\alpha A_\beta . \tag{2.18}$$

When expressed in terms of the frame, this results in

$$\mathcal{L} = -\tfrac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \sqrt{-g} \tag{2.19}$$

where

$$F_{\alpha\beta} = \Box_\alpha A_\beta - \Box_\beta A_\alpha. \tag{2.20}$$

Note the use of bold indices to indicate components with respect to a frame.

Alternatively we may write this as:

$$\mathcal{L} = \int -\tfrac{1}{4}(\Box_\alpha A_\beta - \Box_\beta A_\alpha)(\Box_\gamma A_\delta - \Box_\delta A_\gamma)\eta^{\alpha\gamma}\eta^{\beta\delta}\sqrt{-g}\,\mathrm{d}x^4. \tag{2.21}$$

We now choose a $3+1$ foliation of the space-time and decompose (2.21) into this form. Note that $\eta^{\mathbf{00}} = -1$.

$$\begin{aligned}
\mathcal{L} = -\tfrac{1}{4}\textstyle\int \eta^{\mathbf{00}}\eta^{\beta\delta}&(\Box_\mathbf{0} A_\beta - \Box_\beta A_\mathbf{0})(\Box_\mathbf{0} A_\delta - \Box_\delta A_\mathbf{0})\\
&+ \eta^{\mathbf{ik}}\eta^{\beta\delta}(\Box_\mathbf{i} A_\beta - \Box_\beta A_\mathbf{i})(\Box_\mathbf{k} A_\delta - \Box_\delta A_\mathbf{k})\sqrt{-g}\,\mathrm{d}x^4\\
= \int \tfrac{1}{2}\eta^{\mathbf{ij}}&(\Box_\mathbf{0} A_\mathbf{i} - \Box_\mathbf{i} A_\mathbf{0})(\Box_\mathbf{0} A_\mathbf{j} - \Box_\mathbf{j} A_\mathbf{0})\\
&- \tfrac{1}{4}\eta^{\mathbf{ik}}\eta^{\mathbf{jl}}(\Box_\mathbf{i} A_\mathbf{j} - \Box_\mathbf{j} A_\mathbf{i})(\Box_\mathbf{k} A_\mathbf{l} - \Box_\mathbf{l} A_\mathbf{k})\sqrt{-g}\,\mathrm{d}x^4.
\end{aligned}$$

The 4-dimensional covariant derivative acting on a vector can be decomposed into spatial derivatives and projected Lie derivatives along the 'time' direction. This is achieved using the identities (2.5), thereby obtaining a fully decomposed Lagrangian.

$$\begin{aligned}
= \int \tfrac{1}{2}\eta^{\mathbf{ij}}&(\mathcal{L}_{e_0} A_\mathbf{i} - a_\mathbf{i} A_\mathbf{0} + K^\mathbf{k}{}_\mathbf{i} A_\mathbf{k} - \nabla_\mathbf{i} A_\mathbf{0} - K^\mathbf{k}{}_\mathbf{i} A_\mathbf{k})\\
&(\mathcal{L}_{e_0} A_\mathbf{j} - a_\mathbf{j} A_\mathbf{0} + K^\mathbf{k}{}_\mathbf{j} A_\mathbf{k} - \nabla_\mathbf{j} A_\mathbf{0} - K^\mathbf{k}{}_\mathbf{j} A_\mathbf{k})\\
&- \tfrac{1}{4}\eta^{\mathbf{ik}}\eta^{\mathbf{jl}}(\nabla_\mathbf{i} A_\mathbf{j} + K_\mathbf{ij} A_\mathbf{0} - \nabla_\mathbf{j} A_\mathbf{i} - K_\mathbf{ij} A_\mathbf{0})\\
&(\nabla_\mathbf{k} A_\mathbf{l} + K_\mathbf{kl} A_\mathbf{0} - \nabla_\mathbf{l} A_\mathbf{k} - K_\mathbf{kl} A_\mathbf{0})\sqrt{-g}\,\mathrm{d}x^4\\
= \int \tfrac{1}{2}\eta^{\mathbf{ij}}&(\mathcal{L}_{e_0} A_\mathbf{i} - a_\mathbf{i} A_\mathbf{0} - \nabla_\mathbf{i} A_\mathbf{0})(\mathcal{L}_{e_0} A_\mathbf{j} - a_\mathbf{j} A_\mathbf{0} - \nabla_\mathbf{j} A_\mathbf{0})\\
&- \tfrac{1}{4}\eta^{\mathbf{ik}}\eta^{\mathbf{jl}}(\nabla_\mathbf{i} A_\mathbf{j} - \nabla_\mathbf{j} A_\mathbf{i})(\nabla_\mathbf{k} A_\mathbf{l} - \nabla_\mathbf{l} A_\mathbf{k})\sqrt{-g}\,\mathrm{d}x^4.
\end{aligned} \tag{2.22}$$

Step 2: We now define the conjugate momenta:

$$\tilde{\pi}^{\alpha} := \frac{\partial \mathcal{L}}{\partial \dot{A}_{\alpha}} \tag{2.23}$$

where N is the lapse.

$$\tilde{\pi}^{0} = 0 \tag{2.24}$$

$$\begin{aligned} \tilde{\pi}^{\mathrm{p}} &= \frac{1}{N}\eta^{\mathrm{pi}}[\mathcal{L}_{e_0}A_{\mathrm{i}} - a_{\mathrm{i}}A_0 - \nabla_{\mathrm{i}}A_0]\sqrt{-g} \\ &= -F^{0\mathrm{p}}\gamma^{1/2}. \end{aligned} \tag{2.25}$$

Since $\tilde{\pi}^{\alpha}$ is conjugate to A_{α} we have the canonical relations

$$\begin{aligned} \{A_0(x), \tilde{\pi}^0(y)\} &= \delta(x, \tilde{y}) \\ \{A_{\mathrm{i}}(x), \tilde{\pi}^{\mathrm{j}}(y)\} &= \delta^{\mathrm{j}}_{\mathrm{i}}\delta(x, \tilde{y}). \end{aligned}$$

Step 3: We see from (2.24) that the definition of the momenta has resulted in a primary constraint $\tilde{\pi}^0 \approx 0$. This also reveals A_0 to be a cyclic variable, although in the interest of clarity we will not use the shortcut method for this variable.

Step 4: We now introduce the momenta into the Lagrangian (2.22).

$$\mathcal{L} = \int \frac{N^2}{2\sqrt{-g}}\tilde{\pi}^{\mathrm{i}}\tilde{\pi}_{\mathrm{i}} - \tfrac{1}{4}\eta^{\mathrm{ik}}\eta^{\mathrm{jl}}(\nabla_{\mathrm{i}}A_{\mathrm{j}} - \nabla_{\mathrm{j}}A_{\mathrm{i}})(\nabla_{\mathrm{k}}A_{\mathrm{l}} - \nabla_{\mathrm{l}}A_{\mathrm{k}})\sqrt{-g}\,\mathrm{d}x^4.$$

From this we are able to use $H = p^{\alpha}\dot{q}_{\alpha} - \mathcal{L}$ to derive the Hamiltonian.

$$\mathcal{H}_0 = \int \tilde{\pi}_{\mathrm{i}}\dot{A}_{\mathrm{i}} - \frac{N}{2\gamma^{1/2}}\tilde{\pi}^{\mathrm{i}}\tilde{\pi}_{\mathrm{i}} + \tfrac{1}{4}\eta^{\mathrm{ik}}\eta^{\mathrm{jl}}(\nabla_{\mathrm{i}}A_{\mathrm{j}} - \nabla_{\mathrm{j}}A_{\mathrm{i}})(\nabla_{\mathrm{k}}A_{\mathrm{l}} - \nabla_{\mathrm{l}}A_{\mathrm{k}})N\gamma^{1/2}\,\mathrm{d}^3x. \tag{2.26}$$

We now look to replace the time derivative of A_{i} with another expression that uses only the variables, their spatial derivatives and introduces shift variables. We start with the definition of $\dot{A}_i$.

$$\dot{A}_{\mathrm{i}} = N\,\mathcal{L}_{e_0}A_{\mathrm{i}} + \mathcal{L}_{\vec{N}}A_{\mathrm{i}} \tag{2.27}$$

where we can re-write (2.25) to obtain

$$\mathcal{L}_{e_0} A_\mathrm{i} = \frac{\tilde{\pi}_\mathrm{i} N}{\sqrt{-g}} + a_\mathrm{i} A_\mathbf{0} + \nabla_\mathrm{i} A_\mathbf{0}. \tag{2.28}$$

The lapse and shift are required to relate two points on different equal time surfaces, but it is important to note at this point that since we are working on a fixed background they are not variables of the field theory. This is different to the ADM form of General Relativity in which the lapse and shift variables are part of the field theory. Despite this difference, in both cases we remain free to choose the lapse and the shift.

After substituting this expression into (2.26) we find

$$\begin{aligned} H_0 = \int \frac{N}{2\gamma^{1/2}} \tilde{\pi}^\mathrm{i} \tilde{\pi}_\mathrm{i} + N \tilde{\pi}^\mathrm{i} (a_\mathrm{i} A_\mathbf{0} + \nabla_\mathrm{i} A_\mathbf{0}) + \tilde{\pi}^\mathrm{i} \, \mathcal{L}_{\vec{N}} A_\mathrm{i} \\ + \tfrac{1}{4} \eta^\mathrm{ik} \eta^\mathrm{jl} (\nabla_\mathrm{i} A_\mathrm{j} - \nabla_\mathrm{j} A_\mathrm{i})(\nabla_\mathbf{k} A_\mathrm{l} - \nabla_\mathrm{l} A_\mathrm{k}) N \gamma^{1/2} \, \mathrm{d}^3 x. \end{aligned} \tag{2.29}$$

Therefore we obtain the primary Hamiltonain

$$H_p = H_0 + \int N u^0 \tilde{\pi}^\mathbf{0} \, \mathrm{d}^3 x. \tag{2.30}$$

Step 5: We now use the definition of the Poisson bracket (2.9) to propagate the primary constraint, $\pi^\mathbf{0} = 0$.

$$\{\tilde{\pi}^\mathbf{0}, H_p\} = \int \frac{\delta \tilde{\pi}^\mathbf{0}(x)}{\delta A_\alpha(z)} \frac{\delta H_p(y)}{\delta \tilde{\pi}^\alpha(z)} - \frac{\delta \tilde{\pi}^\mathbf{0}(x)}{\delta \tilde{\pi}^\alpha(z)} \frac{\delta H_p(y)}{\delta A_\alpha(z)} \, \mathrm{d}^3 z,$$

which leads to:

$$\begin{aligned} \frac{\delta \tilde{\pi}^\mathbf{0}(x)}{\delta \tilde{\pi}^\mathbf{0}(z)} &= \tilde{\delta}^3(x, z) \\ \frac{\delta H_p(y)}{\delta A_\mathbf{0}(z)} &= \int N \tilde{\pi}^\mathrm{i}(y) a_\mathrm{i} \delta^3(y, z) \, \mathrm{d}^3 y - \nabla_\mathrm{i} \int \tilde{\pi}^\mathrm{i}(y) N \delta^3(y, z) \, \mathrm{d}^3 y \\ &= N \tilde{\pi}^\mathrm{i}(z) a_\mathrm{i} - \nabla_\mathrm{i}(\tilde{\pi}^\mathrm{i}(z) N) \\ &= -N \nabla_\mathrm{i} [\tilde{\pi}^\mathrm{i}(z)] \end{aligned}$$

$$\{\tilde{\pi}^0, H_p\} = \int N\nabla_{\mathrm{i}}[\tilde{\pi}^{\mathrm{i}}(z)]\delta^3(x,z)\,\mathrm{d}^3z$$
$$= N\nabla_{\mathrm{i}}[\tilde{\pi}^{\mathrm{i}}(x)]. \tag{2.31}$$

We therefore obtain a secondary constraint, $\psi \equiv \nabla_{\mathrm{i}}[\tilde{\pi}^{\mathrm{i}}(x)] \approx 0$. From this constraint we obtain one of the Maxwell equations, $\nabla_{\alpha}E^{\alpha} = 0$, where $\tilde{\pi}^{\alpha}$ is identified as the electric field E^{α}.

We now propagate ψ to ensure that the constraint holds for all time. We use the following formula

$$\tilde{\pi}^{\mathrm{i}}\ \mathcal{L}_{\vec{N}}A_{\mathrm{i}} = \tilde{\pi}^{\mathrm{i}}(N^{\mathrm{j}}\nabla_{\mathrm{j}}A_{\mathrm{i}} + A_{\mathrm{j}}\nabla_{\mathrm{i}}N^{\mathrm{j}})$$

to express the Lie derivative in terms of the canonical variables.

$$\begin{aligned}
\{\nabla_{\mathrm{p}}\tilde{\pi}^{\mathrm{p}}, \mathcal{H}\} &= \int_z \nabla_{\mathrm{p}}\left(\tilde{\delta}(x,z)\right)\frac{\delta H}{\delta A^{\mathrm{p}}(z)}\,\mathrm{d}^3z \\
&= \nabla_{\mathrm{p}}\left(\frac{\delta H}{\delta A^{\mathrm{p}}(x)}\right) \\
&= \nabla_{\mathrm{p}}\Big\{-\tilde{\pi}^{\mathrm{i}}\nabla_{\mathrm{i}}(N^{\mathrm{p}}) - \nabla_{\mathrm{j}}\left[\tilde{\pi}^{\mathrm{p}}N^{\mathrm{j}}\right] \\
&\quad + \tfrac{1}{4}\eta^{\mathrm{ik}}\eta^{\mathrm{jl}}\big[-\nabla_{\mathrm{i}}(F_{\mathrm{kl}}N\gamma^{1/2})\delta^{\mathrm{p}}_{\mathrm{j}} + \nabla_{\mathrm{j}}(F_{\mathrm{kl}}N\gamma^{1/2})\delta^{\mathrm{p}}_{\mathrm{i}} - \nabla_{\mathrm{k}}(F_{\mathrm{ij}}N\gamma^{1/2})\delta^{\mathrm{p}}_{\mathrm{l}} \\
&\quad + \nabla_{\mathrm{l}}(F_{\mathrm{ij}}N\gamma^{1/2})\delta^{\mathrm{p}}_{\mathrm{k}}\big]\Big\} \\
&= \nabla_{\mathrm{p}}\left\{-\tilde{\pi}^{\mathrm{i}}\nabla_{\mathrm{i}}(N^{\mathrm{p}}) + \nabla_{\mathrm{i}}(\tilde{\pi}^{\mathrm{p}}N^{\mathrm{i}}) + \nabla_{\mathrm{i}}(F^{\mathrm{ip}}N\gamma^{1/2})\right\} \\
&= -\nabla_{\mathrm{i}}(\tilde{\pi}^{\mathrm{j}})\nabla_{\mathrm{j}}(N^{\mathrm{i}}) - \tilde{\pi}^{\mathrm{j}}\nabla_{\mathrm{i}}\nabla_{\mathrm{j}}(N^{\mathrm{i}}) + \nabla_{\mathrm{i}}(\tilde{\pi}^{\mathrm{i}})\nabla_{\mathrm{j}}(N^{\mathrm{j}}) + \tilde{\pi}^{\mathrm{i}}\nabla_{\mathrm{i}}\nabla_{\mathrm{j}}(N^{\mathrm{j}}) \\
&\quad + \nabla_{\mathrm{i}}(N^{\mathrm{j}})\nabla_{\mathrm{j}}(\tilde{\pi}^{\mathrm{i}}) + N^{\mathrm{j}}\nabla_{\mathrm{i}}\nabla_{\mathrm{j}}(\tilde{\pi}^{\mathrm{i}}) + \nabla_{\mathrm{i}}\nabla_{\mathrm{j}}(F^{\mathrm{ij}}N\gamma^{1/2})
\end{aligned} \tag{2.32}$$

We now use the formula, $(\nabla_i \nabla_j - \nabla_j \nabla_i)\tilde{\pi}^i = R^i{}_{kij}\,\tilde{\pi}^k$ and the constraint, $\psi \approx 0$ to obtain,

$$
\begin{aligned}
&= -R^i{}_{kij}\tilde{\pi}^j N^k - \tilde{\pi}^j \nabla_j \nabla_i (N^i) + \tilde{\pi}^i \nabla_i \nabla_j (N^j) + R^i{}_{kij} N^j \tilde{\pi}^k \\
&\quad + N^j \nabla_j \nabla_i (\tilde{\pi}^i) + \tfrac{1}{2}(\nabla_i \nabla_j - \nabla_j \nabla_i)(F^{ji} N \gamma^{1/2}) \\
&= \tfrac{1}{2}(R^i{}_{kij} F^{kj} N \gamma^{1/2} + R^j{}_{kij} F^{ik} N \gamma^{1/2}) \\
&= \tfrac{1}{2}(R_{kj} F^{kj} N \gamma^{1/2} + R_{kj} F^{kj} N \gamma^{1/2}) \\
&= R_{kj} F^{kj} N \gamma^{1/2} \\
&= 0.
\end{aligned}
$$

In the last step we the fact that F^{ij} is anti symmetric while R_{ij} is symmetric.

Step 6: We next calculate the evolution equations for the canonical variables. We shall first consider $\dot{A}_p$:

$$
\begin{aligned}
\dot{A}_p &= \frac{\delta H}{\delta \tilde{\pi}^p(z)} \\
&= \int \left[\frac{N}{\gamma^{1/2}} \tilde{\pi}_p(x) + N a_p A_0(x) + N \nabla_p A_0(x) + \mathcal{L}_{\vec{N}} A_p(x) \right] \tilde{\delta}^3(x,z)\, \mathrm{d}^3 x \\
&= \frac{N}{\gamma^{1/2}} \tilde{\pi}_p + N a_p A_0 + N \nabla_p A_0 + \mathcal{L}_{\vec{N}} A_p. \qquad (2.33)
\end{aligned}
$$

We find that this is identical to (2.28)

We now calculate the evolution equation for the conjugate momenta. Note that we use the notation, $\tilde{\pi}^p = \gamma^{1/2}\pi^p$.

$$
\begin{aligned}
\dot{\tilde{\pi}}^p &= \{\tilde{\pi}^p(x), \mathcal{H}\} = -\frac{\delta \mathcal{H}}{\delta A_p(x)} \\
&= -\tilde{\pi}^i \nabla_i (N^p) - \nabla_j \left[\tilde{\pi}^p N^j\right] \\
&+ \tfrac{1}{4}\eta^{ik}\eta^{jl}\big[- \nabla_i (F_{kl} N \gamma^{1/2})\delta^p_j + \nabla_j (F_{kl} N \gamma^{1/2})\delta^p_i - \nabla_k (F_{ij} N \gamma^{1/2})\delta^p_l + \nabla_l (F_{ij} N \gamma^{1/2})\delta^p_k\big] \\
&= -\tilde{\pi}^i \nabla_i (N^p) + \nabla_i (\tilde{\pi}^p N^i) + \nabla_i (F^{ip} N \sqrt{h}) \\
&= \left[-\pi^i \nabla_i (N^p) + \pi^p \nabla_i N^i + N^i \nabla_i (\pi^p) + N \nabla_i (F^{ip}) + F^{ip} \nabla_i (N)\right] \gamma^{1/2}. \qquad (2.34)
\end{aligned}
$$

In order to relate this to Maxwell's field equations we have to re-write the left hand side:

$$\begin{aligned}
\dot{\tilde{\pi}}^{\mathrm{p}} &= \mathcal{L}_{\frac{\partial}{\partial t}}(\pi^{\mathrm{p}}\gamma^{1/2}) = \gamma^{1/2}\mathcal{L}_{\frac{\partial}{\partial t}}\pi^{\mathrm{p}} + \pi^{\mathrm{p}}\mathcal{L}_{\frac{\partial}{\partial t}}\gamma^{1/2} \\
&= \gamma^{1/2}\dot{\pi}^{\mathrm{p}} + \pi^{p}\left[N\mathcal{L}_{e_0}\gamma^{1/2} + \mathcal{L}_{\vec{N}}\gamma^{1/2}\right] \\
&= \gamma^{1/2}\dot{\pi}^{\mathrm{p}} + \pi^{\mathrm{p}}\tfrac{1}{2}\gamma^{1/2}N\gamma^{ij}\mathcal{L}_{e_0}\gamma_{ij} + \tfrac{1}{2}\pi^{\mathrm{p}}\gamma^{1/2}\gamma^{ij}\mathcal{L}_{\vec{N}}\gamma_{ij} \\
&= \gamma^{1/2}\left[\dot{\pi}^{\mathrm{p}} - \pi^{\mathrm{p}}NK + \pi^{\mathrm{p}}\nabla_{\mathrm{i}}N^{\mathrm{i}}\right].
\end{aligned} \tag{2.35}$$

We now substitute this into (2.34) to obtain,

$$\begin{aligned}
\dot{\pi}^{\mathrm{i}} &= \mathcal{L}_{\vec{N}}\pi^{\mathrm{i}} + \pi^{\mathrm{i}}NK + N\nabla_{\mathrm{j}}(F^{\mathrm{ji}}) + F^{\mathrm{ip}}\nabla_{\mathrm{i}}(N) \\
\Rightarrow \frac{1}{N}\left\{\mathcal{L}_{\frac{\partial}{\partial t}}\pi^{\mathrm{i}} - \mathcal{L}_{\vec{N}}\pi^{\mathrm{i}}\right\} &+ \nabla_{\mathrm{j}}(F^{\mathrm{ij}}) - \pi^{\mathrm{i}}K - \frac{1}{N}F^{\mathrm{ji}}\nabla_{\mathrm{j}}(N) = 0 \\
\mathcal{L}_{e_0}\pi^{\mathrm{i}} - F^{\mathrm{i0}}K + \nabla_{\mathrm{j}}(F^{\mathrm{ij}}) + F^{\mathrm{ij}}a_{\mathrm{j}} &= 0 \\
\mathcal{L}_{e_0}F^{\mathrm{i0}} - K^{\mathrm{i}}{}_{\mathrm{j}}F^{\mathrm{j0}} + F^{\mathrm{ij}}a_{\mathrm{j}} + \nabla_{\mathrm{j}}F^{\mathrm{ij}} - F^{\mathrm{0j}}K^{\mathrm{i}}{}_{\mathrm{j}} - F^{\mathrm{i0}}K &= 0 \\
\Box_0 F^{\mathrm{i0}} + \Box_{\mathrm{j}}F^{\mathrm{ij}} &= 0 \\
\Box_\alpha F^{\mathrm{i}\alpha} &= 0.
\end{aligned} \tag{2.36}$$

This gives the remaining Maxwell's field equations, $\Box_\alpha F^{\mathrm{i}\alpha} = 0$. We may also express Maxwell's equation in the more familiar way, $\nabla \times B - \dot{E} = 0$.

Now that we have finished the constraint analysis and we have a well posed initial value problem. If A_{i} and π^{i} are given on an initial surface satisfying the secondary constraints (2.31), then they will be propagated by the evolution equations (2.33) and (2.34). The lapse and shift may be freely chosen, along with the cyclic variable A_0. The constraints ensure that π^0 remains zero for all time. This concludes the canonical analysis for covariant electromagnetic theory.

2.3.2 Vacuum General Relativity

In this section we shall illustrate the canonical treatment of the Einstein-Hilbert action. (These calculations are based on the approach in Isenberg & Nester (1979)). From this action we will be able to derive the constraints and the Hamiltonian equations of motion.

Before we can start in earnest we first state various identities which are used in this section. The first set of identities arise from the assumption that we are working with a torsion free theory. From this assumption we obtain the following equations,

$$K_{ab} = K_{ba} \tag{2.37a}$$

$$a_b = e_b(lnN) = \frac{1}{N}\nabla_b(N) \tag{2.37b}$$

$$K_{ab} = -\tfrac{1}{2}\mathcal{L}_{e_0}\gamma_{ab}. \tag{2.37c}$$

The decomposition of the space-time curvature is greatly simplified by these 'no torsion' conditions. The resulting equations are know as the Gauss-Codazzi equations which are given below:

$${}^4R^d{}_{0a0} = \gamma^{db}\,\mathcal{L}_{e_0}K_{ba} + K^{db}K_{ba} + \frac{1}{N}\nabla_a\nabla^d(N) \tag{2.38a}$$

$${}^4R^0{}_{cab} = \nabla_b K_{ac} - \nabla_a K_{bc} \tag{2.38b}$$

$${}^4R^d{}_{cab} = R^d{}_{cab} + K^d{}_a K_{bc} - K^d{}_b K_{ac}. \tag{2.38c}$$

Using these equations we are able to decompose the Einstein-Hilbert action,

$$\begin{aligned} I &= \int {}^4R\sqrt{-g}\,\mathrm{d}^4x \\ &= \int N\gamma^{1/2} g^{\alpha\beta\,4}R^\delta{}_{\alpha\delta\beta}\,\mathrm{d}^4x, \end{aligned}$$

into a $3+1$ form where $g^{\alpha\beta} = -e_0e_0 + \gamma^{ab}e_ae_b$.

Hence,

$$I = \int N\gamma^{1/2}[g^{00}R^\delta{}_{0\delta0} + g^{ab}R^0{}_{a0b} + g^{ab}R^c{}_{acb}]\,\mathrm{d}^4x.$$

We also note that $g^{ab}R^{0}{}_{a0b} = -R^{a}{}_{0a0}$. Therefore

$$\begin{aligned} I &= \int N\gamma^{1/2}\{-2(\gamma^{ab}\,\mathcal{L}_{e_0}K_{ba} + K^{ad}K_{da} + \frac{1}{N}\nabla_a\nabla^a(N)) \\ &\qquad + \gamma^{ab}(R^{c}{}_{acb} + K^{c}_{\ c}K_{ab} - K^{c}_{\ b}K_{ca})\}\,\mathrm{d}^4x \\ &= \int N\gamma^{1/2}\left(R + K^{c}_{\ c}K^{a}_{\ a} - 3K^{c}_{\ b}K^{b}_{\ c} - \frac{2}{N}\nabla^2(N) - 2\gamma^{ab}\,\mathcal{L}_{e_0}K_{ab}\right)\mathrm{d}^4x. \end{aligned} \tag{2.39}$$

In this current form the last term contains a second time derivative of the metric, because $K_{ab} = -1/2\mathcal{L}_{e_0}\gamma_{ab}$. This derivative can be moved using an integration by parts, leaving an additional boundary term. We first rewrite (2.39),

$$\begin{aligned} I = \int \mathrm{d}t \int N\gamma^{1/2}\left(R + K^{c}_{\ c}K^{a}_{\ a} - 3K^{c}_{\ b}K^{b}_{\ c} - 2\frac{1}{N}\nabla^2(N)\right) \\ - 2\gamma^{ab}\gamma^{1/2}\mathcal{L}_{\frac{\partial}{\partial t}}K_{ab} + 2\gamma^{ab}\gamma^{1/2}\mathcal{L}_{\vec{N}}K_{ab}\,\mathrm{d}^4x, \end{aligned} \tag{2.40}$$

then focus on the last two terms. We integrate the penultimate term by parts, which results in the last two terms becoming,

$$\begin{aligned} &-2\int \gamma^{ab}K_{ab}\gamma^{1/2}\,\mathrm{d}x^3\Big|_{t_i}^{t_f} + \int \mathrm{d}t\int 2K_{ab}\mathcal{L}_{\frac{\partial}{\partial t}}(\gamma^{ab}\gamma^{1/2}) + 2\gamma^{ab}\gamma^{1/2}\mathcal{L}_{\vec{N}}K_{ab}\,\mathrm{d}^4x \\ &= -2\int \gamma^{1/2}K\,\mathrm{d}x^3\Big|_{t_i}^{t_f} + \int \mathrm{d}t\int \Big[2K_{ab}\gamma^{1/2}\left(N\mathcal{L}_{e_0}\gamma^{ab} + \mathcal{L}_{\vec{N}}\gamma^{ab}\right) \\ &\quad + 2K_{ab}\gamma^{ab}\left(N\mathcal{L}_{e_0}\gamma^{1/2} + \mathcal{L}_{\vec{N}}\gamma^{1/2}\right) + 2\gamma^{ab}\gamma^{1/2}\mathcal{L}_{\vec{N}}K_{ab}\Big]\,\mathrm{d}^4x. \end{aligned}$$

Collecting the $\mathcal{L}_{\vec{N}}$ terms and using $2K^{ab} = \mathcal{L}_{e_0}\gamma^{ab}$ we obtain,

$$\begin{aligned} &= -2\int \gamma^{1/2}K\,\mathrm{d}x^3\Big|_{t_i}^{t_f} + \int \mathrm{d}t\int 4K_{ab}K^{ab}N\gamma^{1/2} + KN\gamma^{1/2}\gamma^{ab}\mathcal{L}_{e_0}\gamma_{ab} + 2\mathcal{L}_{\vec{N}}(k\gamma^{1/2})\,\mathrm{d}^4x \\ &= -2\int \gamma^{1/2}K\,\mathrm{d}x^3\Big|_{t_i}^{t_f} + \int \mathrm{d}t\int 4K_{ab}K^{ab}N\gamma^{1/2} - 2KN\gamma^{1/2}\gamma^{ab}K_{ab} \\ &\qquad\qquad + 2\gamma^{1/2}N^a\nabla_aK + k\gamma^{1/2}\gamma^{ab}\mathcal{L}_{\vec{N}}\gamma_{ab}\,\mathrm{d}^4x \\ &= -2\int \gamma^{1/2}K\,\mathrm{d}x^3\Big|_{t_i}^{t_f} + \int \mathrm{d}t\int N\gamma^{1/2}\left(4K^{a}_{\ b}K^{b}_{\ a} - 2K^{m}_{\ m}K^{n}_{\ n}\right) \\ &\qquad\qquad + 2\gamma^{1/2}\left(N^a\nabla_aK + K\nabla_aN^a\right)\,\mathrm{d}^4x \\ &= -2\int \gamma^{1/2}K\,\mathrm{d}x^3\Big|_{t_i}^{t_f} + \int \mathrm{d}t\int N\gamma^{1/2}\left(4K^{a}_{\ b}K^{b}_{\ a} - 2K^{m}_{\ m}K^{n}_{\ n}\right) + 2\gamma^{1/2}\nabla_a(K^{m}_{\ m}N^a)\,\mathrm{d}^4x. \end{aligned}$$

Substituting this back into (2.40) we obtain,

$$\begin{aligned} I = &\int \mathrm{d}t \int N\gamma^{1/2} \left(R + K^a_b K^a_b - K^m_m K^n_n\right) \mathrm{d}^3x \\ &- 2\int \gamma^{1/2} K \,\mathrm{d}^3x \Big|_{t_i}^{t_f} \\ &- 2\int \mathrm{d}t \int \nabla_a \left(\nabla^a (N\gamma^{1/2}) - \gamma^{1/2} K^m_m N^a\right) \mathrm{d}^3x. \end{aligned} \tag{2.41}$$

The last two lines are boundary terms.

We now express (2.41) in terms of the metric and its derivatives. We will make use of the equation

$$K_{ab} = -\tfrac{1}{2}\mathcal{L}_{e_0}\gamma_{ab} = -\tfrac{1}{2N}\left(\mathcal{L}_{\frac{\partial}{\partial t}}\gamma_{ab} - \mathcal{L}_{\vec{N}}\gamma_{ab}\right). \tag{2.42}$$

When this is substituted into (2.41) we obtain

$$I = \int \mathrm{d}t \int \mathrm{d}^3x \left(NR + \frac{1}{4N}\left(\dot{\gamma}_{ab} - \mathcal{L}_{\vec{N}}\gamma_{ab}\right)\left(\dot{\gamma}_{cd} - \mathcal{L}_{\vec{N}}\gamma_{cd}\right)\left(\gamma^{ac}\gamma^{db} - \gamma^{ab}\gamma^{cd}\right)\right) + \text{boundary terms.} \tag{2.43}$$

We now have a Lagrangian which is a function of the metric, its derivatives, the lapse and the shift. Note that the lapse, and shift variables are cyclic, and therefore using the shortcut method we define only the variables conjugate to the metric,

$$\begin{aligned} \tilde{\pi}^{pq} = \frac{\partial \mathcal{L}}{\partial \dot{\gamma}_{pq}} &= \frac{\gamma^{1/2}}{4N}\left(\gamma^{ac}\gamma^{db} - \gamma^{ab}\gamma^{cd}\right)\left[\left(\dot{\gamma}_{cd} - \mathcal{L}_{\vec{N}}\gamma_{cd}\right)\delta^p_a\delta^q_b + \left(\dot{\gamma}_{ab} - \mathcal{L}_{\vec{N}}\gamma_{ab}\right)\delta^p_c\delta^q_d\right] \\ &= \frac{\gamma^{1/2}}{2N}\left(\dot{\gamma}_{cd} - \mathcal{L}_{\vec{N}}\gamma_{cd}\right)\left(\gamma^{pc}\gamma^{qd} - \gamma^{qp}\gamma^{cd}\right). \end{aligned}$$

This can be re-expressed in terms of K_{ab} using (2.42):

$$\begin{aligned} \tilde{\pi}^{ab} &= -\gamma^{1/2} K_{cd}\left(\gamma^{ac}\gamma^{bd} - \gamma^{ba}\gamma^{cd}\right) \\ &= \gamma^{1/2}\left(\gamma^{ab}K - K^{ab}\right). \end{aligned} \tag{2.44}$$

We can rearrange this to provide a definition of $\dot{\gamma}_{ab}$ as a function of γ_{ab} and $\tilde{\pi}^{ab}$. In the following calculation we use the result arising from the trace of (2.44), ie.$\tilde{\pi}^m_{\ m} = 2K\gamma^{1/2}$

$$\begin{aligned}
K_{ab} &= \gamma_{ab}K - \gamma^{-1/2}\tilde{\pi}_{ab} \\
&= \tfrac{1}{2}\gamma^{-1/2}\gamma_{ab}\tilde{\pi}^n_{\ n} - \gamma^{-1/2}\tilde{\pi}_{ab} \\
\Rightarrow \qquad -\frac{1}{2N}\left(\dot{\gamma}_{ab} - \mathcal{L}_{\vec{N}}\gamma_{ab}\right) &= \tfrac{1}{2}\gamma_{ab}\tilde{\pi}^m_{\ m}\gamma^{-1/2} - \gamma^{-1/2}\tilde{\pi}_{ab} \\
\Rightarrow \qquad \dot{\gamma}_{ab} &= N\gamma^{-1/2}\left(2\tilde{\pi}_{ab} - \gamma_{ab}\tilde{\pi}^m_{\ m}\right) + \mathcal{L}_{\vec{N}}\gamma_{ab}.
\end{aligned} \tag{2.45}$$

We now make the transition from the Lagrangian to the Hamiltonian description. We define the base Hamiltonian density by $\mathcal{H} = p^\lambda q_\lambda - \mathcal{L}$. We will also use (2.45) to replace the time derivative of the metric. The boundary terms remain the same.

$$\begin{aligned}
H &= \int \left[\tilde{\pi}^{ab}\dot{\gamma}_{ab} - \left(R + K^a_{\ b}K^b_{\ a} - K^m_{\ m}K^n_{\ n}\right)N\gamma^{1/2}\right] \mathrm{d}^3x \\
&= \int \Big[\tilde{\pi}^{ab}N\gamma^{-1/2}\left(2\tilde{\pi}_{ab} - \gamma_{ab}\tilde{\pi}^m_{\ m}\right) + N\gamma^{-1/2}K^m_{\ m}K^n_{\ n} - N\gamma^{1/2}K^a_{\ b}K^b_{\ a} - N\gamma^{1/2}R \\
&\qquad + \tilde{\pi}^{ab}\mathcal{L}_{\vec{N}}\gamma_{ab}\Big] \mathrm{d}^3x \\
&= \int \Big[\tilde{\pi}^{ab}N\gamma^{-1/2}\left(2\tilde{\pi}_{ab} - \gamma_{ab}\tilde{\pi}^m_{\ m}\right) + N\gamma^{1/2}\left(-3KK + 2\delta^a_b KK^b_{\ a} - K^a_{\ b}K^b_{\ a} + 2KK\right) \\
&\qquad - N\gamma^{1/2}R + \tilde{\pi}^{ab}\mathcal{L}_{\vec{N}}\gamma_{ab}\Big] \mathrm{d}^3x \\
&= \int \Big[\tilde{\pi}^{ab}N\gamma^{-1/2}\left(2\tilde{\pi}_{ab} - \gamma_{ab}\tilde{\pi}^m_{\ m}\right) + N\gamma^{1/2}\left[-\left(\delta^a_b K - K^a_{\ b}\right)\left(\delta^b_a K - K^b_{\ a}\right) + 2KK\right] \\
&\qquad - N\gamma^{1/2}R + \tilde{\pi}^{ab}\mathcal{L}_{\vec{N}}\gamma_{ab}\Big] \mathrm{d}^3x \\
&= \int \Big[\tilde{\pi}^{ab}N\gamma^{-1/2}\left(2\tilde{\pi}_{ab} - \gamma_{ab}\tilde{\pi}^m_{\ m}\right) - N\gamma^{-1/2}\tilde{\pi}^a_{\ b}\tilde{\pi}^b_{\ a} + \tfrac{1}{2}N\gamma^{-1/2}\tilde{\pi}^a_{\ a}\tilde{\pi}^m_{\ m} - N\gamma^{1/2}R \\
&\qquad + \tilde{\pi}^{ab}\mathcal{L}_{\vec{N}}\gamma_{ab}\Big] \mathrm{d}^3x \\
&= \int N\left[\gamma^{-1/2}\left(\tilde{\pi}^a_{\ b}\tilde{\pi}^b_{\ a} - \tfrac{1}{2}\tilde{\pi}^a_{\ a}\tilde{\pi}^m_{\ m}\right) - \gamma^{1/2}R\right] \mathrm{d}^3x + \int \tilde{\pi}^{ab}\mathcal{L}_{\vec{N}}\gamma_{ab}\,\mathrm{d}^3x \\
&\qquad + \text{boundary terms}
\end{aligned} \tag{2.46}$$

Although we described this as the base Hamiltonian, it is also the primary Hamiltonian because there are no primary constraints (remember that the lapse and shift are cyclic and we are using the shortcut method). If we make use of writing the Lie derivative in terms of the covariant derivative and then integrate by parts, as shown:

$$\int \tilde{\pi}^{ab}\mathcal{L}_{\vec{N}}\gamma_{ab}\,\mathrm{d}^3x = \int -\left(N_a\nabla_b\tilde{\pi}^{ab} + N_b\nabla_a\tilde{\pi}^{ba}\right)\,\mathrm{d}^3x + \text{boundary terms}$$
$$= \int -2N^a\nabla_b\tilde{\pi}^b{}_a\,\mathrm{d}^3x,$$

we are able to re-write the expression $\tilde{\pi}^{ab}\mathcal{L}_{\vec{N}}\gamma_{ab}$ to show that the Hamiltonian can be expressed in the form

$$H = \int \left(N\left[\gamma^{-1/2}\left(\tilde{\pi}^a{}_b\tilde{\pi}^b{}_a - \tfrac{1}{2}\tilde{\pi}^a{}_a\tilde{\pi}^m{}_m\right) + \gamma^{1/2}R\right] - N^a\left[2\nabla_m\tilde{\pi}^m{}_a\right]\right)\mathrm{d}^3x. \tag{2.47}$$

Hence we can rewrite (2.47) in the form

$$H = \int \left(N\mathcal{H}' + N^a\mathcal{H}'_a\right)\mathrm{d}^3x, \tag{2.48}$$

where

$$\mathcal{H}' = \gamma^{-1/2}\left(\tilde{\pi}^a{}_b\tilde{\pi}^b{}_a - \tfrac{1}{2}\tilde{\pi}^a{}_a\tilde{\pi}^m{}_m\right) - \gamma^{1/2}R \tag{2.49a}$$
$$\mathcal{H}'_a = -2\nabla_m\tilde{\pi}^m{}_a. \tag{2.49b}$$

Variation with respect to the lapse and shift results in four secondary constraints, which are (2.49a) and (2.49b). Further propagation of these does not lead to any additional constraints. They are automatically preserved in time due to the kinematics of the decomposition.

We now look to calculate the evolution of the canonical variables. We start by rewriting (2.46) in terms of the canonical variables:

$$H = \int \left(N\gamma^{1/2}\left[-\tfrac{1}{2}\left(\tilde{\pi}^{mn}\gamma_{mn}\right)^2 + \tilde{\pi}^{ma}\tilde{\pi}^{nb}\gamma_{an}\gamma_{bm}\right] - N\gamma^{1/2}\gamma^{cd}R_{cd} + \tilde{\pi}^{mn}\mathcal{L}_{\vec{N}}\gamma_{mn}\right)\mathrm{d}^3x, \tag{2.50}$$

then calculate the evolution equation for the 3-metric, γ_{ab}.

$$\dot{\gamma}_{ab} = \{\gamma_{ab}(x), H(y)\} = \int \frac{\delta\gamma_{ab}(x)}{\delta\gamma_{cd}(z)}\frac{\delta H(y)}{\delta\tilde{\pi}^{cd}(z)}\,\mathrm{d}^3z$$

$$
\begin{aligned}
\frac{\delta H}{\delta \tilde{\pi}^{cd}(z)} &= \int \left(-N\gamma^{-1/2}\gamma_{cd}\tilde{\pi}^{mn}\gamma_{mn} + 2N\gamma^{-1/2}\tilde{\pi}_{cd} + \mathcal{L}_{\vec{N}}\gamma_{cd}\right)\tilde{\delta}^3(y,z)\,\mathrm{d}^3z \\
&= 2N\gamma^{-1/2}\left(\tilde{\pi}_{ab}(z) - \tfrac{1}{2}\tilde{\pi}^m{}_m(z)\gamma_{ab}\right) + \mathcal{L}_{\vec{N}}\gamma_{ab} \\
\Rightarrow \qquad \dot{\gamma}_{ab} &= \int \delta^3(x,z)\left(2N\gamma^{-1/2}\left(\tilde{\pi}_{ab}(z) - \tfrac{1}{2}\tilde{\pi}^m{}_m(z)\gamma_{ab}\right) + \mathcal{L}_{\vec{N}}\gamma_{ab}\right)\mathrm{d}^3z \\
&= 2N\gamma^{-1/2}\left(\tilde{\pi}_{ab}(x) - \tfrac{1}{2}\tilde{\pi}^m{}_m(x)\gamma_{ab}\right) + \mathcal{L}_{\vec{N}}\gamma_{ab}. \qquad (2.51)
\end{aligned}
$$

Before we calculate the other evolution equation we will derive some useful identities.

a)

$$
\frac{\partial \gamma}{\partial \gamma_{ab}} = \gamma\gamma^{ab} \qquad \Rightarrow \qquad \frac{\partial \gamma^{1/2}}{\partial \gamma_{ab}} = \tfrac{1}{2}\gamma^{-1/2}\gamma\gamma^{ab} = \tfrac{1}{2}\gamma^{1/2}\gamma^{ab} \qquad (2.52)
$$

$$
\& \qquad \frac{\partial \gamma^{-1/2}}{\partial \gamma_{ab}} = -\tfrac{1}{2}\gamma^{-1/2}\gamma^{ab} \qquad (2.53)
$$

b)

$$
\begin{aligned}
\gamma^{ab}\gamma_{bc} &= \delta^a_c \\
\Rightarrow \qquad \left(\delta\gamma^{ab}\right)\gamma_{bc} &= -\gamma^{ab}\left(\delta\gamma_{bc}\right) \\
\Rightarrow \qquad \left(\delta\gamma^{ab}\right)\gamma_{bc}\gamma^{ce} &= -\gamma^{ab}\gamma^{ce}\left(\delta\gamma_{bc}\right) \\
\Rightarrow \qquad \delta\gamma^{ae} &= -\gamma^{ab}\gamma^{ce}\delta\gamma_{bc} \qquad (2.54)
\end{aligned}
$$

c) We wish to find an expression for $\delta I/\delta\gamma_{ab}$ where

$$
\begin{aligned}
\delta I = \int N\gamma^{1/2}\gamma^{cd}\delta R_{cd} &= \int N\gamma^{1/2}\gamma^{cd}\left(\nabla_a(\delta\Gamma^a{}_{cd}) - \nabla_d(\delta\Gamma^a{}_{ca})\right) \\
&= \int \nabla_a(N\gamma^{1/2}\gamma^{cd}\delta\Gamma^a{}_{cd}) - \nabla_d(N\gamma^{1/2}\gamma^{cd}\delta\Gamma^a{}_{ca}) \\
&\quad - \nabla_a(N\gamma^{1/2}\gamma^{cd})\delta\Gamma^a{}_{cd} + \nabla_d(N\gamma^{1/2}\gamma^{cd})\delta\Gamma^a{}_{ca}.
\end{aligned}
$$

$\delta\Gamma^a{}_{bc}$ is a tensorial term. It therefore remains the same in all coordinates. If we assume geodesic coordinates then, $\delta\Gamma^a{}_{bc} \overset{*}{=} \frac{1}{2}\gamma^{ab}[\delta(\gamma_{cb,d}) + \delta(\gamma_{db,c}) - \delta(\gamma_{cd,b})]$ because $\gamma_{ab,c} = 0$. We separate the integral and consider the penultimate term first:

$$\int -\nabla_a(N\gamma^{1/2}\gamma^{cd})\delta\Gamma^a{}_{cd} \overset{*}{=} \int -\nabla_a(N\gamma^{1/2}\gamma^{cd})\tfrac{1}{2}\gamma^{ab}[\delta(\gamma_{cb,d}) + \delta(\gamma_{db,c}) - \delta(\gamma_{cd,b})] \tag{2.55a}$$

$$\begin{aligned} &\overset{*}{=} \int \tfrac{1}{2}\gamma^{ab}\nabla_d\nabla_a(N\gamma^{1/2}\gamma^{cd})\delta\gamma_{cb} + \tfrac{1}{2}\gamma^{ab}\nabla_c\nabla_a(N\gamma^{1/2}\gamma^{cd})\delta\gamma_{db} \\ &\qquad - \tfrac{1}{2}\gamma^{ab}\nabla_b\nabla_a(N\gamma^{1/2}\gamma^{cd})\delta\gamma_{cd}\,\mathrm{d}^3x + \text{boundary terms} \end{aligned} \tag{2.55b}$$

$$= \int \left[\nabla^a\nabla^b(N\gamma^{1/2}) - \tfrac{1}{2}\nabla^2(N\gamma^{1/2}\gamma^{ab})\right]\delta\gamma_{ab}\,\mathrm{d}^3x.$$

We have been able to convert the partial derivative in (2.55a) to covariant derivatives in (2.55b), because the term being differentiated, after the integration by parts, is a density. The final result is tensorial so that this equation is then true for all coordinates.

The final term becomes,

$$\begin{aligned} \int \nabla_d(N\gamma^{1/2}\gamma^{cd})\delta\Gamma^a{}_{ca}\,\mathrm{d}^3x &= \int \nabla_d(N\gamma^{1/2}\gamma^{cd})\tfrac{1}{2}\gamma^{ab}[\delta(\gamma_{cb,a}) + \delta(\gamma_{ab,c}) - \delta(\gamma_{ca,b})]\,\mathrm{d}^3x \\ &= \int -\tfrac{1}{2}\nabla_a\nabla_d(N\gamma^{1/2}\gamma^{cd}\gamma^{ab})\delta\gamma_{cb} - \tfrac{1}{2}\gamma^{ab}\nabla_c\nabla_d(N\gamma^{1/2}\gamma^{cd})\delta\gamma_{ab} \\ &\qquad + \tfrac{1}{2}\gamma^{ab}\nabla_b\nabla_d(N\gamma^{1/2}\gamma^{cd})\delta\gamma_{ca}\,\mathrm{d}^3x + \text{boundary terms} \\ &= \int -\tfrac{1}{2}\gamma^{ab}\nabla^2(N\gamma^{1/2})\delta\gamma_{ab}\,\mathrm{d}^3x + \text{boundary terms.} \end{aligned}$$

Putting these terms back into the initial expression we find that

$$\delta I = \int \left[\nabla^a\nabla^b(N\gamma^{1/2}) - \gamma^{ab}\nabla^2(N\gamma^{1/2})\right]\delta\gamma_{ab}\,\mathrm{d}^3x + \text{ boundary terms.} \tag{2.56}$$

The final identity we need gives an expression for the Lie derivative of a density in terms of covariant derivatives.

$$\begin{aligned} \mathcal{L}_{\vec{N}}\tilde{\pi}^{cd} &= \gamma^{1/2}\mathcal{L}_{\vec{N}}\pi^{cd} + \pi^{cd}\mathcal{L}_{\vec{N}}\gamma^{1/2} \\ &= \gamma^{1/2}\mathcal{L}_{\vec{N}}\pi^{cd} + \tfrac{1}{2}\pi^{cd}\gamma^{1/2}\gamma^{ab}\mathcal{L}_{\vec{N}}\gamma_{ab} \\ &= \gamma^{1/2}N^a\nabla_a(\pi^{cd}) - \tilde{\pi}^{cn}\nabla_n(N^d) - \tilde{\pi}^{nd}\nabla_n(N^c) + \tfrac{1}{2}\tilde{\pi}^{cd}\gamma^{ab}(\nabla_a(N_b) - \nabla_b(N_a)) \\ &= N^a\nabla_a(\tilde{\pi}^{cd}) + \tilde{\pi}^{cd}\nabla_a(N^a) - \tilde{\pi}^{cn}\nabla_n(N^d) - \tilde{\pi}^{nd}\nabla_n(N^c). \end{aligned} \tag{2.57}$$

We now start calculating the evolution equations for $\tilde{\pi}^{ab}$. Using (2.50) and ignoring the boundary terms we obtain

$$\dot{\tilde{\pi}}^{cd} = \{\tilde{\pi}^{cd}, \mathcal{H}\} = -\int_z \delta^3(x,z)\frac{\delta\mathcal{H}}{\delta\gamma_{cd}(z)}\,\mathrm{d}^3z \tag{2.58}$$

$$\begin{aligned} &= \tfrac{1}{2}\gamma^{-1/2}\gamma^{cd}N\left[-\tfrac{1}{2}\left(\tilde{\pi}^{mn}\gamma_{mn}\right)^2 + \tilde{\pi}^{ma}\tilde{\pi}^{nb}\gamma_{an}\gamma_{bm}\right] + N\gamma^{-1/2}\tilde{\pi}^{cd}\tilde{\pi}^{m}_{\ m} \\ &\quad - N\gamma^{-1/2}\left[\tilde{\pi}^{mc}\tilde{\pi}^{db}\gamma_{bm} + \tilde{\pi}^{da}\tilde{\pi}^{nc}\gamma_{an}\right] + \tfrac{1}{2}N\gamma^{1/2}\gamma^{cd}R - N\gamma^{1/2}R^{cd} \\ &\quad + \gamma^{1/2}\nabla^c\nabla^d(N) - \gamma^{1/2}\gamma^{cd}\nabla^2(N) + \nabla_a(N^a\tilde{\pi}^{cd}) - \tilde{\pi}^{cn}\nabla_n(N^d) \\ &\quad - \tilde{\pi}^{md}\nabla_m(N^c), \end{aligned} \tag{2.59}$$

where we have used (2.52) (2.54) and (2.56). So applying (2.57) we can write

$$\begin{aligned} \dot{\tilde{\pi}}^{cd} &= -N\gamma^{1/2}(R^{cd} - \tfrac{1}{2}\gamma^{cd}R) + N\gamma^{-1/2}(\tilde{\pi}^{cd}\tilde{\pi}^{m}_{\ m} - 2\tilde{\pi}^{c}_{\ m}\tilde{\pi}^{dm}) \\ &\quad + \tfrac{1}{2}\gamma^{-1/2}\gamma^{cd}N(\tilde{\pi}^{m}_{\ n}\tilde{\pi}^{n}_{\ m} - \tfrac{1}{2}\tilde{\pi}^{m}_{\ m}\tilde{\pi}^{n}_{\ n}) + \gamma^{1/2}\nabla^c\nabla^d(N) - \gamma^{1/2}\gamma^{cd}\nabla^2(N) + \mathcal{L}_{\vec{N}}\tilde{\pi}^{cd}. \end{aligned} \tag{2.60}$$

We have now completed the first stage of the Dirac-Bergmann algorithm, using the short cut method. This gives us a well-defined initial value system because we choose initial data for γ_{ab} and $\tilde{\pi}^{ab}$, ensuring it satisfies (2.49a) and (2.49b). The lapse and shift must be chosen throughout the space-time. We then use (2.51) and (2.60) to evolve the canonical variables through the space-time.

The four equations, (2.49a), (2.49b), (2.51) and (2.60) can be expressed more simply if we replace $\tilde{\pi}_{ab}$ with K_{ab} using (2.44). Before we start we will derive some more identities which will be used in the simplification.

a) Taking the trace of (2.44) we find

$$\tilde{\pi}^{m}_{\ m} = 2K\gamma^{1/2} \equiv 2\tilde{K}.$$

Note the use of $\tilde{K}$ to represent $K\gamma^{1/2}$.

b) Also from (2.44), $\tilde{\pi}^{ab} = \gamma^{ab}\tilde{K} - \tilde{K}^{ab}$, we obtain

$$
\begin{aligned}
\gamma^{-1/2}\tilde{\pi}^{ab} &= \gamma^{ab}K - K^{ab} \\
&= \tfrac{1}{2}\gamma^{ab}\tilde{\pi}^{m}_{\ m} - K^{ab} \\
K^{ab} &= \gamma^{-1/2}\left(\tfrac{1}{2}\gamma^{ab}\tilde{\pi}^{m}_{\ m} - \tilde{\pi}^{ab}\right) \\
\tilde{K}^{ab} &= \tfrac{1}{2}\gamma^{ab}\tilde{\pi}^{m}_{\ m} - \tilde{\pi}^{ab}.
\end{aligned} \tag{2.61}
$$

c) We now use (2.44) to replace $\tilde{\pi}_{ij}$ with $\tilde{K}^{ij}$ in the following expression:

$$
\begin{aligned}
N\gamma^{1/2}(\tilde{\pi}^{c}_{\ e}\tilde{\pi}^{m}_{\ m} - 2\tilde{\pi}^{c}_{\ m}\tilde{\pi}^{m}_{\ e}) &= N\gamma^{-1/2}\left[(\delta^{c}_{e}\tilde{K} - \tilde{K}^{c}_{\ e})(2\tilde{K}) - 2(\delta^{c}_{m}\tilde{K} - \tilde{K}^{c}_{\ m})(\delta^{m}_{e}\tilde{K} - \tilde{K}^{m}_{\ e})\right] \\
&= N\gamma^{-1/2}\left[2\delta^{c}_{e}\tilde{K}\tilde{K} - 2\tilde{K}\tilde{K}^{c}_{\ e} - 2\delta^{c}_{e}\tilde{K}\tilde{K} + 4\tilde{K}\tilde{K}^{c}_{\ m}\delta^{m}_{e} - 2\tilde{K}^{c}_{\ m}\tilde{K}^{m}_{\ e}\right] \\
&= -2N\gamma^{-1/2}\left[\tilde{K}^{c}_{\ n}\tilde{K}^{n}_{\ e} - \tilde{K}\tilde{K}^{c}_{\ e}\right].
\end{aligned} \tag{2.62}
$$

We start with simplifying (2.49a):

$$
\begin{aligned}
\gamma^{-1/2}\mathcal{H}' &= \gamma^{-1}\left[-\tfrac{1}{2}(\tilde{\pi}^{m}_{\ m})^2 + \tilde{\pi}^{m}_{\ n}\tilde{\pi}^{n}_{\ m}\right] - R \\
&= -2KK + (\delta^{m}_{n}K - K^{m}_{\ n})(\delta^{n}_{m}K - K^{n}_{\ m}) - R \\
&= -2KK + 3KK - \delta^{m}_{n}K^{n}_{\ m}K - K^{m}_{\ n}\delta^{n}_{m}K + K^{m}_{\ n}K^{n}_{\ m} - R \\
&= K^{m}_{\ n}K^{n}_{\ m} - KK - R.
\end{aligned} \tag{2.63}
$$

Equation (2.49b) becomes

$$
\begin{aligned}
\gamma^{-1/2}\mathcal{H}'_a &= -2\gamma^{-1/2}\nabla_b(\tilde{\pi}^{b}_{\ a}) \\
&= -2\nabla_b(\delta^{b}_{a}K - K^{b}_{\ a}) \\
&= 2(\nabla_b K^{b}_{\ a} - \nabla_a K),
\end{aligned} \tag{2.64}
$$

while (2.51) simply becomes

$$
\begin{aligned}
\dot{\gamma}_{ab} &= 2N\gamma^{-1/2}(\tilde{\pi}_{ab} - \tfrac{1}{2}\gamma_{ab}\tilde{\pi}^{m}_{\ m}) + \mathcal{L}_{\vec{N}}\gamma_{ab} \\
&= -2NK_{ab} + \mathcal{L}_{\vec{N}}\gamma_{ab}.
\end{aligned} \tag{2.65}
$$

The final equation, (2.60), takes a bit more work. To start with we show that

$$\tilde{\pi}^a{}_b = \delta^a_b \tilde{K} - \tilde{K}^a{}_b \Rightarrow \dot{\tilde{\pi}}^a{}_b = \delta^a_b \dot{\tilde{K}} - \dot{\tilde{K}}^a{}_b,$$

then

$$\dot{\tilde{\pi}}^c{}_e = \dot{\tilde{\pi}}^{cd}\gamma_{de} + \tilde{\pi}^{cd}\dot{\gamma}_{de} \Rightarrow \dot{\tilde{\pi}}^{cd}\gamma_{de} = \delta^c_e \dot{\tilde{K}} - \dot{\tilde{K}}^a{}_b - \tilde{\pi}^{cd}\dot{\gamma}_{de}. \tag{2.66}$$

We now multiply (2.60) by γ_{de}.

$$\begin{aligned}(\dot{\tilde{\pi}}^{cd})\gamma_{de} = &-N\gamma^{1/2}(R^c{}_e - \tfrac{1}{2}\delta^c_e R) + N\gamma^{-1/2}(\tilde{\pi}^c{}_e\tilde{\pi}^m{}_m - 2\tilde{\pi}^c{}_m\tilde{\pi}^m{}_e)\\ &+ \tfrac{1}{2}\gamma^{-1/2}\delta^c_e N(\tilde{\pi}^m{}_n\tilde{\pi}^n{}_m - \tfrac{1}{2}\tilde{\pi}^m{}_m\tilde{\pi}^n{}_n) + \gamma^{1/2}\nabla^c\nabla_e(N) - \gamma^{1/2}\delta^c_e\nabla^2(N) + \gamma_{de}\mathcal{L}_{\vec{N}}\tilde{\pi}^{cd}.\end{aligned} \tag{2.67}$$

Now using the identity derived earlier in (2.66) and (2.62), we show that

$$\begin{aligned}\delta^c_e\dot{\tilde{K}} - \dot{\tilde{K}}^a{}_b = &-N\gamma^{1/2}(R^c{}_e - \tfrac{1}{2}\delta^c_e R) + 2N\gamma^{-1/2}\left[\tilde{K}^c{}_n\tilde{K}^n{}_e - \tilde{K}\tilde{K}^c{}_e\right]\\ &- \tfrac{1}{2}N\gamma^{-1/2}\delta^c_e(\tilde{K}^c{}_n\tilde{K}^n{}_e - \tilde{K}\tilde{K}^c{}_e) + \gamma^{1/2}\nabla^c\nabla_e(N) - \gamma^{1/2}\delta^c_e\nabla^2(N)\\ &+ \gamma_{de}\mathcal{L}_{\vec{N}}\tilde{\pi}^{cd} + \tilde{\pi}^{cd}\dot{\gamma}_{de}.\end{aligned} \tag{2.68}$$

We shall now spend some time simplifying the last two terms of the expression above.

$$\begin{aligned}\gamma_{de}\mathcal{L}_{\vec{N}}\tilde{\pi}^{cd} + \tilde{\pi}^{cd}\dot{\gamma}_{de} &= \tilde{\pi}^{cd}N\mathcal{L}_{e_0}\gamma_{de} + \tilde{\pi}^{cd}\mathcal{L}_{\vec{N}}\gamma_{de} + \gamma_{de}\mathcal{L}_{\vec{N}}\tilde{\pi}^{cd}\\ &= -2\tilde{\pi}^{cd}K_{de}N + \mathcal{L}_{\vec{N}}\tilde{\pi}^c{}_e\\ &= -2(\gamma^{cd}\tilde{K} - \tilde{K}^{cd})\tilde{K}_{de}N\gamma^{-1/2} + \mathcal{L}_{\vec{N}}\tilde{\pi}^c{}_e\\ &= 2(\tilde{K}^c{}_d\tilde{K}^d{}_e - \tilde{K}\tilde{K}^c{}_e)\gamma^{-1/2}N + \mathcal{L}_{\vec{N}}(\delta^c_e\tilde{K} - \tilde{K}^c{}_e).\end{aligned}$$

We now substitute this back into (2.68),

$$\begin{aligned}\delta^c_e\dot{\tilde{K}} - \dot{\tilde{K}}^c{}_e = &-N\gamma^{1/2}(R^c{}_e - \tfrac{1}{2}\delta^c_e R) - \tfrac{1}{2}N\gamma^{-1/2}\delta^c_e(\tilde{K}^m{}_n\tilde{K}^n{}_m - \tilde{K}\tilde{K}^c{}_e)\\ &+ \gamma^{1/2}\nabla^c\nabla_e(N) - \gamma^{1/2}\delta^c_e\nabla^2(N) + \mathcal{L}_{\vec{N}}(\delta^c_e\tilde{K} - \tilde{K}^c{}_e).\end{aligned} \tag{2.69}$$

We then take the trace of this equation to obtain an expression for $\dot{\tilde{K}}$.

$$-2\dot{\tilde{K}} = -\tfrac{1}{2}N\gamma^{1/2}R - N\gamma^{-1/2}\tfrac{3}{2}\left(\tilde{K}^a{}_b\tilde{K}^b{}_a - \tilde{K}\tilde{K}\right) + 2\gamma^{1/2}\nabla^2(N) - \mathcal{L}_{\vec{N}}(2\tilde{K}).$$

We can substitute this expression into (2.69)

$$\begin{aligned}
\dot{\tilde{K}}^c{}_e &= N\gamma^{1/2}(R^c{}_e - \tfrac{1}{2}\delta^c_e R) - \tfrac{1}{2}N\gamma^{-1/2}(\delta^c_e\tilde{K}^m{}_n\tilde{K}^n{}_m - \tilde{K}\tilde{K}^c{}_e) - \gamma^{1/2}\nabla^c\nabla_e(N) \\
&\quad + \gamma^{1/2}\delta^c_e\nabla^2(N) - \mathcal{L}_{\vec{N}}(\delta^c_e\tilde{K} - \tilde{K}^c{}_e) + \delta^c_e\big[-\tfrac{1}{4}N\gamma^{1/2}R + N\gamma^{-1/2}\,\tfrac{3}{4}\left(\tilde{K}^a{}_b\tilde{K}^b{}_a - \tilde{K}\tilde{K}\right) \\
&\quad - \gamma^{1/2}\nabla^2(N) + \mathcal{L}_{\vec{N}}(\tilde{K})\big] \\
&= N\gamma^{1/2}R^c{}_e - \gamma^{1/2}\nabla^c\nabla_e(N) + \mathcal{L}_{\vec{N}}(\tilde{K}^c{}_e) + \tfrac{1}{4}\delta^c_e\left[N\gamma^{-1/2}(\tilde{K}^a{}_b\tilde{K}^b{}_a - \tilde{K}\tilde{K}) - N\gamma^{1/2}R\right] \\
&= N\gamma^{1/2}R^c{}_e - \gamma^{1/2}\nabla^c\nabla_e(N) + \mathcal{L}_{\vec{N}}(\tilde{K}^c{}_e). \qquad (2.70)
\end{aligned}$$

In the last step we made use of the constraint (2.49a). We now focus on the left hand side of (2.70).

$$\begin{aligned}
\dot{\tilde{K}}^c{}_e &= \dot{K}^c{}_e\gamma^{1/2} + K^c{}_e\dot{\gamma}^{1/2} \\
&= \dot{K}^c{}_e\gamma^{1/2} + K^c{}_e N\mathcal{L}_{e_0}\gamma^{1/2} + K^c{}_e\mathcal{L}_{\vec{N}}\gamma^{1/2} \\
&= \dot{K}^c{}_e\gamma^{1/2} + \tfrac{1}{2}K^c{}_e N\gamma^{1/2}\gamma^{ab}\mathcal{L}_{e_0}\gamma_{ab} + K^c{}_e\mathcal{L}_{\vec{N}}\gamma^{1/2} \\
&= \dot{K}^c{}_e\gamma^{1/2} - N\gamma^{1/2}KK^c{}_e + K^c{}_e\mathcal{L}_{\vec{N}}\gamma^{1/2}.
\end{aligned}$$

By substituting this into (2.70) we finally get the expression

$$\dot{K}^c{}_e = N(R^c{}_e + KK^c{}_e) - \nabla^c\nabla_e(N) + \mathcal{L}_{\vec{N}}K^c{}_e. \qquad (2.71)$$

We have now finished simplifying all the equations. In summary, the simplified constraint equations and evolution equations are

$$\gamma^{-1/2}\mathcal{H}' = K^a_{\ b}K^b_{\ a} - KK - R \tag{2.72a}$$

$$\gamma^{-1/2}\mathcal{H}'_a = 2(\nabla_b K^b_{\ a} - \nabla_a K) \tag{2.72b}$$

$$\dot{\gamma}_{ab} = -2NK_{ab} + \mathcal{L}_{\vec{N}}\gamma_{ab} \tag{2.72c}$$

$$\dot{K}^a_{\ b} = N(R^a_{\ b} + KK^a_{\ b}) - \nabla^a\nabla_b(N) + \mathcal{L}_{\vec{N}}K^a_{\ b}. \tag{2.72d}$$

This is the same as (4.18) in Isenberg & Nester (1979).

2.4 Remaining steps of constraint analysis

Having shown examples to clarify the method of canonical analysis, we will now proceed with the remaining steps of the Dirac-Bergmann algorithm. We first consider grouping the constraints into first and second class. We will then give a brief overview of the steps which, when possible, lead one to a quantisation of the original field theory.

2.4.1 First and Second class constraints

A first class constraint is a linear combination of the primary and secondary constraints whose Poisson bracket with every other constraint is at least weakly zero. If a Poisson bracket is not weakly zero then the constraint is said to be second class. Therefore we are able to rewrite the primary and secondary constraints as linear independent first and second class constraints. There are therefore many different expressions for the first and second class constraints depending on which linear combinations are taken. We may think of these as being different yet equivalent descriptions of the same physical system. Although every description is equally valid, we will find that some descriptions will be more advantageous in revealing the underlining gauge transformation generated by the first class constraints. It is these first class constraints that generate infinitesimal gauge transformations on the reduced phase space. The algebra generated by the Poisson brackets of the first class constraints is closed.

It is now also possible to ascertain the number of degrees of freedom. Without any constraints the number of degrees of freedom is the same as the number of variables. In a theory which has constraints, the number of degrees of freedom will decrease by one for every independent constraint and gauge freedom. The number of constraints can easily be found by totalling the number of constraints obtained through the canonical analysis. The presence of gauge freedoms can be found in the indeterminacy of the evolution equations, ie. the evolution equation is determined by an unknown multiplier, N. There may exist more gauge conditions than undefined multipliers, and therefore to obtain the number of degrees of freedom we would have to find the number of gauge constraints of the theory. Fortunately, in all meaningful cases, the number of gauge constraints is the same as the number of first class constraints. Therefore the number of degrees of freedom can be calculated by,

$$\tfrac{1}{2}(N - 2F - S), \tag{2.73}$$

where N is the number of variables (including the momenta), F is the number of first class constraints and S is then number of second class constraints. In general this is as far as one can go to determine the degrees of freedom. In many cases the theory is too complicated to isolate the degrees of freedom. A common approach would be to introduce a gauge condition which breaks the gauge freedom of the theory. Therefore the first class constraints become second class. Since all constraints are second class you now have one system of equations that may be solved explicitly for all variables, except the variables that contain the degrees of freedom. This then, leaves us with an understanding of the true degrees of freedom contained in our theory.

When the first class algebra is calculated it is common to smear the constraints with test functions. This avoids added complication from integrating products of delta functions. This can be seen in the following example.

As an example we shall consider the constraints that were obtained from the constraint analysis of vacuum General Relativity. There are four constraints given by (2.49a) and (2.49b); they are all first class. To show this we first calculate the Poisson bracket of γ_{ij} and $\tilde{\pi}_{ij}$ with $\mathcal{H}_i'$ smeared by with a test function (see below). Then we calculate the

Poisson brackets of the constraints with each other. Therefore:

$$
\begin{aligned}
\{\gamma_{ij}(x), \int_y f^k \mathcal{H}'_k \, \mathrm{d}^3y\} &= \{\gamma_{ij}(x), \int 2\nabla_q(f^k)\tilde{\pi}^{pq}\gamma_{pk}\, \mathrm{d}^3y\} \\
&= \int_z \delta(x,z) \left[\int_y \left(f^k_{;j}\gamma_{ik} + f^k_{;i}\gamma_{jk} \right) \tilde{\delta}(y,z)\, \mathrm{d}^3y \right] \mathrm{d}^3z \\
&= \gamma_{ik} f^k_{;j} + \gamma_{jk} f^k_{;i} \\
&= \mathcal{L}_f \gamma_{ij} \qquad (2.74) \\
\{\tilde{\pi}^{ij}(x), \int_y f^k \mathcal{H}'_k \, \mathrm{d}y^3\} &= \{\tilde{\pi}^{ij}(x), \int_y f^k \left[-2(\gamma_{kp}\tilde{\pi}^{pq})_{,q} + \tilde{\pi}^{mn}\gamma_{mn,k}) \right] \mathrm{d}^3y\} \\
&= \int_z -\tilde{\delta}(x,z) \left[\int_y \left(f^i_{,q}\tilde{\pi}^{jq} + f^j_{,q}\tilde{\pi}^{iq} - (\tilde{\pi}^{ij}f^k)_{,k} \right) \delta(y,z)\, \mathrm{d}^3y \right] \\
&= -f^i_{,q}\tilde{\pi}^{jq} - f^j_{,q}\tilde{\pi}^{iq} + (\tilde{\pi}^{ij}f^k)_{,k} \\
&= \mathcal{L}_f \tilde{\pi}^{ij}. \qquad (2.75)
\end{aligned}
$$

In the first stage of (2.75) we have expressed the constraint $\mathcal{H}'_k$ in terms of partial derivatives instead of the covariant derivative. Using these two equations we are able to deduce that the Poisson bracket of $\mathcal{H}'(\gamma_{ij}, \tilde{\pi}^{ij})$ with $\mathcal{H}'_k$ is

$$
\left\{ \int g\mathcal{H}' \, \mathrm{d}^3x, \int f^k \mathcal{H}'_k \, \mathrm{d}^3y \right\} = -\int_x \mathcal{H}' \mathcal{L}_f g \, \mathrm{d}^3x \approx 0. \qquad (2.76)
$$

We can clearly see from (2.74) and (2.75) that the constraint $\mathcal{H}'_i$ generates diffeomorphisms in the hypersurface. This reveals the invariance of the action to infinitesimal coordinate transformations $x^i \to x'^i$, which are given by $x'^i = x^i + \delta x^i$, where $\delta x^i = f^i(x)$

The Poisson bracket $\{\int f^i \mathcal{H}'_i \, \mathrm{d}^3x, \int g^j \mathcal{H}'_j \, \mathrm{d}^3y\}$ can also be calculated using the same method as above. Therefore using the identities (2.74) and (2.75) we obtain:

$$
\left\{ \int f^i \mathcal{H}'_i \, \mathrm{d}^x, \int g^j \mathcal{H}'_j \, \mathrm{d}y \right\} = -\int H'_k \mathcal{L}_g f^k \, \mathrm{d}^3x. \qquad (2.77)
$$

The final Poisson bracket, $\{\int f\mathcal{H}' \, \mathrm{d}^3x, \int g\mathcal{H}' \, \mathrm{d}^3y\}$ can also be calculated. We see from (2.49a) that most of the terms will commute because they contain no derivatives, or free indices. The only term that will not commute is the term that contains the curvature R.

Therefore, before we calculate the Poisson bracket we will first vary the curvature:

$$\begin{aligned}\delta(\gamma^{1/2}fR) &= (\tfrac{1}{2}\gamma^{1/2}\gamma^{pq})Rf\delta\gamma_{pq} + \gamma^{1/2}f\delta(\gamma^{pq})R_{pq} + \gamma^{1/2}f\gamma^{pq}\delta R_{pq} \\ &= -\gamma^{1/2}f(R^{pq} - \tfrac{1}{2}\gamma^{pq}R)\delta\gamma_{pq} - \gamma^{1/2}f\gamma^{pq}\left[\delta\Gamma_{lq}{}^{l}{}_{;p} - \delta\Gamma_{pq}{}^{l}{}_{;l}\right]. \end{aligned} \tag{2.78}$$

We separate this and consider the penultimate term first:

$$\begin{aligned}(\gamma^{1/2}f\gamma^{pq})_{;p}\delta\Gamma_{lq}{}^{l} &= (\gamma^{1/2}f\gamma^{pq})_{;p}\left[\tfrac{1}{2}\gamma^{lk}(\delta\gamma_{lk,q} + \delta\gamma_{qk,l} - \delta\gamma_{lq,k})\right] \\ &= (\gamma^{1/2}f\gamma^{pq}\gamma^{lk})_{;p}\left[\tfrac{1}{2}\gamma_{lk,q}\right]\end{aligned}$$

and then the final term

$$\begin{aligned}(\gamma^{1/2}f\gamma^{pq})_{;l}\delta\Gamma_{pq}{}^{l} &= (\gamma^{1/2}f\gamma^{pq})_{;l}\left[\tfrac{1}{2}\gamma^{lk}(\delta\gamma_{pk,q} + \delta\gamma_{qk,p} - \delta\gamma_{pq,k})\right] \\ &= (\gamma^{1/2}f\gamma^{pq}\gamma^{lk})_{;l}\left[\gamma_{pk,q} - \tfrac{1}{2}\gamma_{pq,k}\right].\end{aligned}$$

So combining these with (2.78) results in:

$$\delta(\gamma^{1/2}fR) = -\gamma^{1/2}f(R^{pq} - \tfrac{1}{2}\gamma^{pq}R)\delta\gamma_{pq} - \left[\gamma^{1/2}f(\gamma^{pk}\gamma^{lq} - \gamma^{pq}\gamma^{lk})\right]_{;lk}\delta\gamma_{pq} \tag{2.79}$$

Again we can see from the above that it is only the last term which will contribute to terms arising from the Poisson bracket. We now calculate this Poisson bracket:

$$\begin{aligned}
\left\{\int f\mathcal{H}'\,\mathrm{d}^3x, \int g\mathcal{H}'\,\mathrm{d}^3y\right\} = \int_z &\left[\int_x -\left(\gamma^{1/2}f(\gamma^{pk}\gamma^{lq}-\gamma^{pq}\gamma^{lk})\right)_{;lk}\delta(x,z)\,\mathrm{d}^3x\right] \\
&\left[\int_y 2g\gamma^{-1/2}(\tilde{\pi}_{pq}-\tfrac{1}{2}\gamma_{pq}\tilde{\pi})\tilde{\delta}(y,z)\,\mathrm{d}^3y\right] \\
&-\left[\int_x 2f\gamma^{-1/2}(\tilde{\pi}_{pq}-\tfrac{1}{2}\gamma_{pq}\tilde{\pi})\tilde{\delta}(x,z)\,\mathrm{d}^3x\right] \\
&-\left[\int_y \left(\gamma^{1/2}g(\gamma^{pk}\gamma^{lq}-\gamma^{pq}\gamma^{lk})\right)_{;lk}\delta(y,z)\,\mathrm{d}^3y\right]\mathrm{d}^3z \\
= \int_z &-2f_{;kl}g\tilde{\pi}^{kl}+f_{;lk}g\tilde{\pi}\gamma^{lk}+2f_{;lk}g\tilde{\pi}-3\gamma^{lk}f_{;lk}g\tilde{\pi}\gamma^{lk} \\
&+2g_{;kl}f\tilde{\pi}^{kl}-g_{;lk}f\tilde{\pi}\gamma^{lk}-2g_{;lk}f\tilde{\pi}+3\gamma^{lk}g_{;lk}f\tilde{\pi}\gamma^{lk}\,\mathrm{d}^3z \\
= \int_z &+2\tilde{\pi}^{kl}{}_{;k}\left[gf_{;l}-fg_{;l}\right]\mathrm{d}^3z \\
= \int_z &\mathcal{H}^i\left[fg_{;i}-gf_{;i}\right]\mathrm{d}^3z. \qquad (2.80)
\end{aligned}$$

Combining these results gives us the first class constraint algebra:

$$\left\{\int f^i\mathcal{H}'_i\,\mathrm{d}^x, \int g^j\mathcal{H}'_j\,\mathrm{d}^3y\right\} = \int_x H'_k\mathcal{L}_f g^k\,\mathrm{d}^3x \qquad (2.81a)$$

$$\left\{\int f^k\mathcal{H}'_k\,\mathrm{d}^3x, \int g\mathcal{H}'\,\mathrm{d}^3y\right\} = \int_x \mathcal{H}'\mathcal{L}_f g\,\mathrm{d}^3x \qquad (2.81b)$$

$$\left\{\int g\mathcal{H}'\,\mathrm{d}^3x, \int f\mathcal{H}'\,\mathrm{d}^3y\right\} = \int \mathcal{H}'_i\gamma^{ij}\left[fg_{,j}-gf_{,j}\right]. \qquad (2.81c)$$

We now consider how many degrees of freedom we have in our field theory. We know that all four constraints are first class, and that there are no secondary constraints. The total number of canonical variables arising from the symmetric variables $\gamma_{ij}, \tilde{\pi}^{ij}$ is twelve. Therefore applying these numbers to the formula outline above (2.73) results in the number of degrees of freedom $\frac{1}{2}\big(12-(2\times 4)\big) = 2$. As expected this is consistent with General Relativity.

We finally consider the geometric content of the gauge transformations generated by the other constraint $\mathcal{H}'$. To simplify things we introduce a gauge condition in which

$N = 1, N^i = 0$, then the Hamiltonian (2.47) reduces to:

$$H = \int \mathcal{H}' \, \mathrm{d}^3 x. \tag{2.82}$$

From this we would expect the constraint $\mathcal{H}'$ to generate the dynamics. If we calculate the Poisson bracket below,

$$\begin{aligned}\{\gamma_{ij}(x), \int_y f\mathcal{H}' \, \mathrm{d}y^3\} &= \int_z \delta^3(x,z)\Big[\int_y f\gamma^{-1/2}\big(\tilde{\pi}^b{}_d\gamma_{bj}\gamma_{di} + \tilde{\pi}^a{}_c\gamma_{ic}\gamma_{aj} \\ &\qquad - \tfrac{1}{2}\tilde{\pi}^b{}_d\gamma_{ij}\gamma_{bd} - \tfrac{1}{2}\tilde{\pi}^a{}_c\gamma_{ij}\gamma_{ac}\big)\tilde{\delta}^3(y,z)\,\mathrm{d}^3 y\Big] \\ &= \gamma^{-1/2} f\,(2\tilde{\pi}_{ij} - \tilde{\pi}\ \gamma_{ij})\,. \end{aligned} \tag{2.83}$$

We can see from (2.51) that this is just $\dot{\gamma}_{ij}$. Therefore $\mathcal{H}'$ is responsible for the dynamics, which leads to it being called the Hamiltonian constraint, while the constraint $\mathcal{H}'_i$ is called the momentum constraint.

2.4.2 Reduction of second class constraints

In order to promote the Poisson bracket relations to operator relations, the constraints must be gauge invariant. Therefore, if our system contains second class constraints the transition is not straightforward. There are methods to deal with such systems. The first we shall look at is the method of using modified brackets, as introduced by Dirac. These brackets are Poisson brackets which have been adapted by the addition of terms which cancel those given by the secondary constraints. Hence, if we define a matrix of Poisson brackets among the second class constraints as

$$\triangle_{ij}(x-y) = \{\chi_i(x), \chi_j(y)\},$$

then we define the Dirac bracket as

$$\{A(x), B(y)\}_D := \{A(x), B(y)\} - \int_\Sigma \mathrm{d}w^3\,\mathrm{d}z^3\{A(x), \chi_i(w)\}(\Delta^{-1})^{ij}(w-z)\{\chi_j(x), B(y)\},$$

which will be zero with all constraints. Although this seems very straight forward, problems arise in practice due to the inversion of the matrix $\triangle_{ij}$. We are assured of its existence by the linear independence of the second class constraints, but actually calculating it can be very cumbersome in physical situations.

An alternative method to constructing the Dirac bracket is the introduction of new variables, called starred variables. These variables are then invariant under Poisson bracket relations with all constraints. They are defined by

$$^{*}A(x) := A(x) - \int_{\Sigma} udw^3\,\mathrm{d}z^3\{A(x), \chi_i(w)\}(\triangle^{-1})^{ij}(w-z)\chi_j(x). \tag{2.84}$$

Therefore, the Poisson bracket of the starred variables is the same as the Dirac bracket with the unstarred variables. At first glance the new variables seem as complicated to define as the Dirac bracket. However, in practice this is not the case. If we define the new variables as a linear integral combination of the second class constraints,

$$^{*}A(x) := A(x) + \int_{\Sigma} \mathrm{d}^3y\Lambda^i(x-y)\chi_i(y), \tag{2.85}$$

where Λ^i is an appropriate distribution to be determined, then suitable combinations can be determined using the requirement that the new variables have a vanishing Poisson bracket with the secondary constraints. This avoids the calculation of $(\triangle^{-1})^{ij}$. Examples using starred variables can be found in Soteriou (1992).

By using either method, Dirac brackets or starred variables, we are able to eliminate the second class constraints, thereby leaving only first class constraints. This then enables us to generate an algebra from the gauge invariant variables, which in turn allows us to proceed with the canonical quantisation.

2.4.3 Quantum theory: the final steps

In this section we shall give a brief outline of the steps involved in transforming the Dirac algebra, $\mathcal{S}$, into a coherent quantum theory. This area is beyond the scope of this report, but we shall outline it for completeness. For more detailed analysis the author recommends reading Ashtekar (1991).

Step 1:

First, each element ($\mathcal{F}$) of $\mathcal{S}$ must be unambiguously promoted to a quantum operator $\hat{\mathcal{F}}$. The algebra of $\mathcal{S}$ must not be so large so that the process is hindered by factor order problems, and yet large enough to keep the quantum operators unambiguous.

Step 2:

Construct an algebra generated by these operators by imposing commutation relations: $[\hat{\mathcal{F}}, \hat{\mathcal{G}}] = i\hbar\{F, G\}_D$. We will denote the algebra $\mathcal{A}$.

Step 3:

Define a $\star$ relation on the algebra. This requires $\hat{G} = (\hat{F})^\star$ when two classical variables are complex conjugates of each other, and that the $\star$ relation satisfies the following relationships

$$(\hat{F} + \lambda\hat{G})^\star = \hat{F}^\star + (\bar{\lambda})\hat{G}^\star$$
$$(\hat{F}\hat{G})^\star = \hat{G}^\star\hat{F}^\star$$
$$(\hat{F}^\star)^\star = F.$$

We denote the resulting algebra $\mathcal{A}^\star$

Step 4:

Find a representation of the algebra $\mathcal{A}$ by using operators on a complex vector space V.

Step 5:

Obtain quantum analogs of the classical constraints. Find the linear subspace V_{phy} of V which is annihilated by all quantum constraints. This is the space of physical quantum states.

Step 6:

Introduce an inner-product on V_{phy}, such that the $\star$ relations become the Hermitian relations on the Hilbert space.

Step 7:

Interpret the adjoint operators and devise a method to compute their spectra and eigenvectors. We then need to discover the transformation generated by the Hamiltonian which can be interpreted as the "time evolution".

If all steps have been completed then we have we a quantum representation of the original action.

2.5 Summary

To summarise, we have discussed the Dirac-Bergmann algorithm, which enables us to decompose a field theory from the action, and have provided some examples to help clarify aspects this process.

Although some progress has been made in General Relativity, a full decomposition has still to be completed. Initial progress slowed considerably due to the complexity of the constraints that were obtained. Due to their non polynomial nature the Hamiltonian constraint does not give a differential operator on quantisation but a pseudo-differential operator, and therefore we do not know how to use operator ordering in the quantum space. Therefore further progress along the route of canonical quantisation of General Relativity stalled.

Chapter 3

Canonical analysis in a $2+2$ foliation

The previous chapter explored the application of the Dirac-Bergmann algorithm to a $3+1$ decomposition of the space time. Although this form of decomposition is the most common, it is by no means the only type. d'Inverno & Smallwood (1980) introduced an alternative approach to space-time decomposition. In their work they introduce a $2+2$ decomposition, in which the space-time is decomposed into a timelike 2-surface and a spacelike 2-surface. In the analysis of the initial value problem it was found that the true degrees of freedom reside in the conformal metric induced on the spatial two surface; therefore the variables that generate the degrees of freedom should be easily isolated. It was because of this attractive property that a canonical analysis of the 2+2 decomposition was thought to be of value. In order to give a Hamiltonian description the timelike 2-surface is decomposed into an evolution direction and non evolutionary direction. This was first attempted by Torre (1986). In this work Torre performed the canonical analysis of a $2+2$ description of space-time. In his work, however he did not set two directions to be null; he kept the evolution direction time-like instead of null. Therefore his approach was not fully $2+2$ but rather a $2+1+1$ approach, see figure 3.1. In this chapter we will extend the work in Torre's paper so that the evolution direction is also null. This is a more natural approach to a $2+2$ foliation (see Hayward 1993)

We shall start by describing the general $2+2$ metric decomposition, which provides an introduction to the $2+2$ formulation. We will not consider many other details such as the Einstein equations, but rather restrict ourselves to details required for this chapter

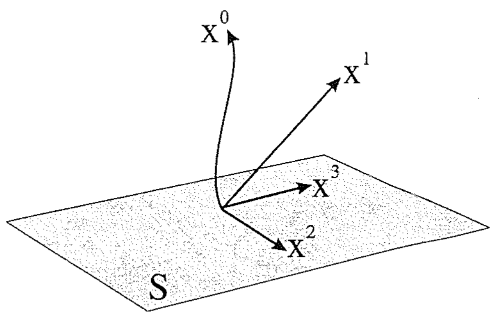

Figure 3.1: *Figure showing Torre's foliation with x^0 as the evolution direction and x^1 the null direction. x^i spans the spacial surface S.*

and those later in this thesis.

We will then perform the Dirac-Bergmann algorithm on a modified Lagrangian so that the evolution direction is null on the reduced phase-space. This results in the derivation of a first class algebra, which is given a geometrical interpretation in the $2+2$ geometry.

We conclude this chapter with a discussion about the complicated nature of the constraints and how this might benefit from the introduction of the Ashtekar approach. We will also compare our first class algebra with the algebra obtained by Torre. This will provide us with some understanding of the first class algebra to expect when we consider a $2+2$ Ashtekar approach.

3.1 $2+2$ metric decomposition

A foliation of co-dimension two can be described by two closed one forms $n^{\mathbf{0}}$ and $n^{\mathbf{1}}$, therefore locally $n^{\mathrm{a}} = d\phi^{\mathrm{a}}$. In our work we consider the surfaces $\phi^{\mathbf{0}} = $ const to be equal time surfaces. These two forms generate hypersurfaces defined by:

$$\Sigma_{\mathbf{a}} : \phi^{\mathrm{a}}(x^{\alpha}) = \text{ const.}$$

These two families of three-dimensional hypersurfaces intersect, thereby generating a family two-dimensional surface $\{S\}$, (figure 3.2).

$$\{S\} = \{\Sigma_0\} \cap \{\Sigma_1\}$$

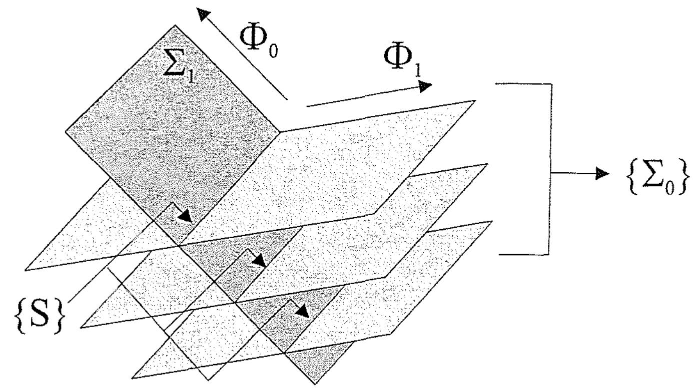

Figure 3.2: *Intersection of two hypersurfaces to form $\{S\}$, where one dimension has been removed.*

Let $n_{\mathbf{a}}$ be the dyad for the vector basis which is dual to $n^{\mathbf{a}}$. Therefore:

$$n^{\mathbf{a}}_{\alpha} n^{\alpha}_{\mathbf{b}} = \delta^{\mathbf{a}}_{\mathbf{b}}$$

The one-forms $n^{\mathbf{a}}$ form a basis for a family of spaces orthogonal to $\{S\}$, which we will label $\{T\}$. The vector basis that spans $\{T\}$ is given by $n_{\mathbf{a}}$. These basis vectors do not necessarily commute, which by Frobenius' theorem implies the spaces $\{T\}$ need not be surface forming. We define the surfaces $\{S\}$ to be space-like, while the distributions, $\{T\}$, will be considered to be time-like.

We can use $n_{\mathbf{a}}$ to define a 2×2 matrix,

$$N_{\mathbf{ab}} = g_{\alpha\beta} n^{\alpha}_{\mathbf{a}} n^{\beta}_{\mathbf{b}}, \tag{3.1}$$

which then implies that

$$\begin{aligned} n^{\alpha}_{\mathbf{a}} &= g^{\alpha\beta} N_{\mathbf{ab}} n^{\mathbf{b}}_{\beta} \\ n^{\mathbf{a}}_{\alpha} &= g_{\alpha\beta} N^{\mathbf{ab}} n^{\beta}_{\mathbf{b}}. \end{aligned}$$

In order to represent information in the $2 + 2$ foliation, we project it onto $\{S\}$ and $\{T\}$

using the respective projectors

$$S^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} - n^{\alpha}_{\mathbf{a}} n^{\mathbf{a}}_{\beta} \tag{3.2}$$

$$T^{\alpha}_{\beta} = n^{\alpha}_{\mathbf{a}} n^{\mathbf{a}}_{\beta} \tag{3.3}$$

By using these projectors we can decompose any space-time tensor into tensors defined on S and T. Tensors defined on T can be reduced to scalars by contracting them with the dyad basis vectors or 1-forms.

Projecting the metric $g_{\alpha\beta}$ gives metrics induced on $\{S\}$ and $\{T\}$.

$$\begin{aligned} \gamma_{\alpha\beta} &= S^{\gamma}_{\alpha} S^{\delta}_{\beta} g_{\gamma\delta} \\ &= S^{\gamma}_{\alpha} S_{\beta\gamma} \\ &= S_{\alpha\beta} \end{aligned} \tag{3.4}$$

$$\begin{aligned} h_{\alpha\beta} &= T^{\gamma}_{\alpha} T^{\delta}_{\beta} g_{\gamma\delta} \\ &= T^{\gamma}_{\alpha} T_{\gamma\beta} \\ &= T_{\alpha\beta}. \end{aligned} \tag{3.5}$$

It is worth noting at this point that

$$\gamma_{\mathbf{ab}} = \gamma_{\alpha\beta} n^{\alpha}_{\mathbf{a}} n^{\beta}_{\mathbf{b}} = 0 \tag{3.6}$$

$$h_{\mathbf{ab}} = h_{\alpha\beta} n^{\alpha}_{\mathbf{a}} n^{\beta}_{\mathbf{b}} = N_{\mathbf{ab}}. \tag{3.7}$$

Therefore, we can understand $N_{\mathbf{00}}$ and $N_{\mathbf{11}}$ as the lapse of $\{S\}$ in $\{\Sigma_0\}$ and $\{\Sigma_1\}$ respectively.

Let us define two vectors $e_{\mathbf{a}}$ which connect the family of two surfaces of $\{S\}$, see figure 3.3. These are known as the rigging vectors. We define them by

$$e_{\mathbf{a}} = n_{\mathbf{a}} + b_{\mathbf{a}} \tag{3.8}$$

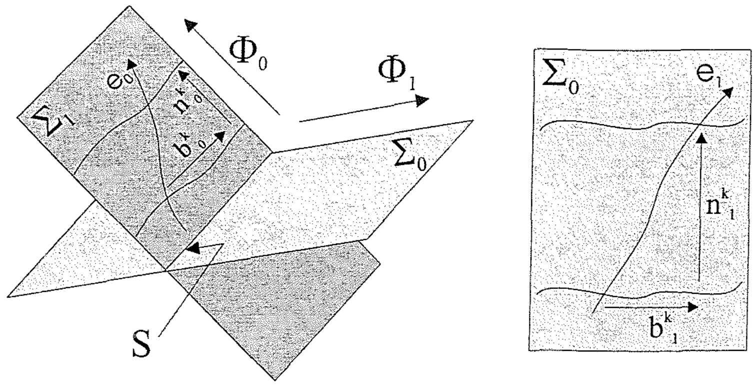

Figure 3.3: *Lapse and shift vectors in the two null directions where one spacial dimension has been removed.*

where $b_{\mathbf{a}}$ is considered to be the shift vector, and

$$n^{\mathbf{a}}_{\alpha} b^{\alpha}_{\mathbf{b}} = 0. \tag{3.9}$$

Although, as already stated, $n_{\mathbf{a}}$ do not necessarily commute, it is always possible to choose $b_{\mathbf{a}}$ in such a way that the rigging vectors $e_{\mathbf{a}}$ commute. A consequence of this is that each $e_{\mathbf{a}}$ is tangent to a congruence of curves in $\{\Sigma_{\mathbf{a}}\}$, which by construction, are parametrised by $\phi^{\mathbf{a}}(x^{\alpha})$. Therefore by choosing coordinates so that $\phi^{\mathbf{0}}(x^{\alpha}) = x^0, \phi^{\mathbf{1}}(x^{\alpha}) = x^1$, and x^2, x^3 are constant, then

$$e_{\mathbf{a}} = \frac{\partial}{\partial x^a}.$$

From (3.8) we can write

$$n_{\mathbf{0}} = e_{\mathbf{0}} - b_{\mathbf{0}} = (1, 0, b^i_0)$$

$$n_{\mathbf{1}} = e_{\mathbf{1}} - b_{\mathbf{1}} = (0, 1, b^i_1).$$

This then allows us to calculate the metric components.

$$\begin{aligned}
g_{ab} &= g(e_{\mathbf{a}}, e_{\mathbf{b}}) \\
&= g(n_{\mathbf{a}} + b_{\mathbf{a}}, n_{\mathbf{b}} + b_{\mathbf{b}}) \\
&= g(n_{\mathbf{a}}, n_{\mathbf{b}}) + g(b_{\mathbf{a}} b_{\mathbf{b}}) \\
&= N_{ab} + \gamma_{ij} b^i_a b^j_b \\
g_{ai} &= g(n_{\mathbf{a}} + b_{\mathbf{a}}, e_{\mathbf{i}}) \\
&= \gamma_{ij} b^j_a \\
g_{ij} &= \gamma_{ij}.
\end{aligned}$$

Note at this point that $N_{\mathbf{ab}} = N_{ab}$ by the choice of $n_{\mathbf{a}}$.

Therefore, we obtain the metric

$$g_{\alpha\beta} = \begin{pmatrix} N_{ab} + \gamma_{ij} b^i{}_a b^j{}_b & \gamma_{ij} b^j{}_a \\ \gamma_{ij} b^j{}_a & \gamma_{ij} \end{pmatrix} \tag{3.10}$$

and with inverse

$$g^{\alpha\beta} = \begin{pmatrix} N^{ab} & -N^{ab} b^i{}_b \\ -N^{ab} b^i{}_b & \gamma^{ij} + N^{ab} b^i{}_a b^j{}_b \end{pmatrix}. \tag{3.11}$$

In a similar manner to the metric the vacuum Einstein equations

$$G^{\alpha\beta} = 0,$$

also decompose into three groups:

$$\begin{aligned}
G^{\mathbf{ab}} &= n^{\mathbf{a}}_{\alpha} n^{\mathbf{b}}_{\beta} G^{\alpha\beta} = 0 \\
G^{\mathbf{ai}} &= 0 \\
G^{\mathbf{ij}} &= 0.
\end{aligned}$$

The advantage of this $2+2$ formulation is that after the analysis of the field equations we find that the two gravitational degrees of freedom can be chosen to lie within the conformal 2-structure $\bar{\gamma}_{ij}$ where

$$\begin{aligned}\gamma_{ij} =& \gamma\bar{\gamma}_{ij} \\ \gamma =& |\gamma_{ij}|.\end{aligned} \tag{3.12}$$

In most calculations we will consider a frame that has been adapted to the foliation. This means that we restrict the frame e_α in such a way that e_i are tangent to $\{S\}$. This will greatly simplify the calculations. We can see from (3.11) that to obtain a double null foliation, ($g^{00} = g^{11} = 0$), we require both $N^{00} = 0$ and $N^{11} = 0$. These are the two constraints that we impose on the Lagrangian and this is the topic of the next section.

3.2 Double null canonical analysis

We start our work from the $2+2$ Lagrangian that Torre derived in his calculations. As we have already mentioned Torre imposed only one null condition and therefore only the x^1 direction was null. Although this was achieved by the introduction of a Lagrange multiplier, Torre combined all the terms that multiplied the N^{11} variable into the multiplier. This effectively set N^{11} and its spatial derivatives to zero and therefore in order to obtain all the Einstein equations he had to make a particular choice for the multiplier that introduces the null condition. After the canonical analysis Torre was able to isolate the First class constraints, and calculate the algebra associated with them.

In this section we are going to extend this work of Torre to ensure that two directions are null. This will then give us a true double null $2+2$ first class algebra. As we are starting from Torre's Lagrangian the two null conditions are going to be treated differently. The condition that Torre had set from the very start we will keep just as he had introduced. The second null condition $N^{00} = 0$ will also be introduced via a Lagrangian multiplier, but it will remain freely specifiable until the constraint is introduced after the canonical analysis. This is just the same as the $\tilde{\pi}^0$ constraint in electromagnetism, previously considered. Although we have said that this work gives us a 'true' $2+2$ first class algebra, we still choose x^0 to be the 'time' direction. This breaks the symmetry between x^0 and x^1. This will be discussed in the final chapter.

We will follow the work of Torre as closely as possible, although the index notation follows the convention used throughout this thesis. We will write N_{ab} as

$$N_0 := N_{00}, N := N_{01}, N_1 := N_{11}.$$

We start our work from the Lagrangian derived by Torre, $\mathcal{L}(N_{ab}, b_a{}^i, \gamma^{ij})$:

$$\begin{aligned}\mathcal{L} = \tfrac{1}{2}N^{-3}\gamma^{1/2}N_{1,0}(2N_0\mathcal{L}_{n_1}N - N\mathcal{L}_{n_1}N_0) + N^{-1}\left[\gamma^{1/2}_{,0}(2\mathcal{L}_{n_1}N - \mathcal{L}_{n_0}N_1) - \gamma^{1/2}N_{,k}b_1{}^k{}_{,0}\right] \\ + \tfrac{1}{2}\gamma^{1/2}N^{-1}\Big[\gamma_{ij}(\mathcal{L}_{n_0}b_1{}^i - b_0{}^i{}_{,1}) + (\gamma_{mn}\gamma_{kl} - \gamma_{mk}\gamma_{nl})\big(N\mathcal{L}_{n_0}\gamma^{mn}\mathcal{L}_{n_1}\gamma^{kl} - \tfrac{1}{2}N_0\mathcal{L}_{n_1}\gamma^{mn}\mathcal{L}_{n_1}\gamma^{kl}\big)\Big] \\ + N\gamma^{1/2}({}^3R + \tfrac{1}{2}N^{-2}\nabla_k N\nabla^k N) - N^{-1}\left[\gamma^{1/2}\nabla_k b_0{}^k\mathcal{L}_{n_1}N + \mathcal{L}_{n_1}\gamma^{1/2}(b_0{}^k N_{,k} + \mathcal{L}_{n_1}N_0)\right] \\ + \mu_1 N_1 + \mu_0(N_0)^2. \quad (3.13)\end{aligned}$$

In the above ${}^3R$ is the curvature in the spacial two surface. Following Torre we have already set N_1 to be null, although its time derivatives remain undefined at this point. Following Goldberg *et al.* (1992) we have introduced the extra null condition, $N_0 = 0$, as a squared term because we are imposing a condition on a cyclic variable. This simplifies the constraint analysis by avoiding the occurrence of a multiplier equation at the next stage, which is a trick that we will find useful later in our calculations.

We now define the conjugate momenta below:

$$\tilde{P}^0 := \frac{\delta\mathcal{L}}{\delta\dot{N}_0} \quad = 0 \tag{3.14a}$$

$$\tilde{P} := \frac{\delta\mathcal{L}}{\delta\dot{N}} \quad = 0 \tag{3.14b}$$

$$\begin{aligned}\tilde{P}^1 := \frac{\delta\mathcal{L}}{\delta\dot{N}_1} \quad &= -N^{-1}\mathcal{L}_{n_0}\gamma^{1/2} + \tfrac{1}{2}N^{-3}\gamma^{1/2}(2N_0\mathcal{L}_{n_1}N - N\mathcal{L}_{n_1}N_0)\\ &= -N^{-1}\mathcal{L}_{n_0}\gamma^{1/2} - \tfrac{1}{2}\gamma^{1/2}\mathcal{L}_{n_1}(N^{-2}N_0)\end{aligned} \tag{3.14c}$$

$$\begin{aligned}\tilde{P}^1{}_k := \frac{\delta\mathcal{L}}{\delta\dot{b}_1{}^k} \quad &= -N^{-1}\gamma^{1/2}N_{,k} + \tfrac{1}{2}N^{-1}\left[\gamma_{kj}(\mathcal{L}_{n_0}b_1{}^j - b_0{}^j{}_{,1}) + \gamma_{ik}(\mathcal{L}_{n_0}b_1{}^i - b_0{}^i{}_{,1})\right]\\ &= N^{-1}\gamma^{1/2}\gamma_{kj}(\mathcal{L}_{n_0}b_1{}^j - b_0{}^j{}_{,1}) - N^{-1}\gamma^{1/2}N_{,k}\end{aligned} \tag{3.14d}$$

$$\tilde{P}^0{}_k := \frac{\delta\mathcal{L}}{\delta\dot{b}_0{}^k} \quad = 0 \tag{3.14e}$$

$$\tilde{\Pi}_{kl} := \frac{\delta\mathcal{L}}{\delta\dot{\gamma}^{kl}} \quad = \tfrac{1}{2}\gamma^{1/2}N^{-1}(\gamma_{mn}\gamma_{kl} - \gamma_{km}\gamma_{ln})N\mathcal{L}_{n_1}\gamma^{mn} + \tfrac{1}{2}\gamma^{1/2}\gamma_{kl}N^{-1}(\mathcal{L}_{n_0}N_1 - 2\mathcal{L}_{n_1}N) \tag{3.14f}$$

These variables and their respective momenta satisfy the equal-time Poisson brackets:

$$\begin{aligned}\{N_0(x), \tilde{P}^0(y)\} &= \delta(x, \tilde{y})\\ \{N(x), \tilde{P}(y)\} &= \delta(x, \tilde{y})\\ \{N_1(x), \tilde{P}^1(y)\} &= \delta(x, \tilde{y})\\ \{b_A^i(x), \tilde{P}^B_j(y)\} &= \delta^B_A\delta^i_j\delta(x, \tilde{y})\\ \{\gamma^{ij}(x), \tilde{\Pi}_{kl}(y)\} &= \delta^i_{(k}\delta^j_{l)}\delta(x, \tilde{y})\end{aligned}$$

Hence, from these definitions of the momenta we are able to show that the primary constraints are:

$$N_1 \approx 0, \qquad N_0 \approx 0, \qquad \tilde{P}^0 \approx 0, \qquad \tilde{P} \approx 0, \qquad \tilde{P}^0{}_k \approx 0. \tag{3.15}$$

An additional constraint arises from (3.14f). At first glance it is clear that this equation is not a primary constraint, due to the time derivative in the penultimate term. But as we shall see, this derivative occurs only in the trace of $\tilde{\Pi}_{ij}$. Therefore the trace free part of (3.14f) is in fact also a primary constraint. Let $\tilde{\Pi}$ be the trace of $\tilde{\Pi}_{ij}$:

$$\begin{aligned}\tilde{\Pi} := \tilde{\Pi}_{ij}\gamma^{ij} &= N^{-1}\gamma^{1/2}(\mathcal{L}_{n_0}N_1 - 2\mathcal{L}_{n_1}N) + \tfrac{1}{2}\gamma^{1/2}(2\gamma_{kl} - \gamma_{kl})\mathcal{L}_{n_1}\gamma^{kl} \\ &= N^{-1}\gamma^{1/2}(\mathcal{L}_{n_0}N_1 - 2\mathcal{L}_{n_1}N) + \tfrac{1}{2}\gamma^{1/2}\gamma_{kl}\mathcal{L}_{n_1}\gamma^{kl}.\end{aligned}$$

Let $\bar{\Pi}_{ij}$ denote the trace free part of $\tilde{\Pi}_{ij}$ defined by

$$\begin{aligned}\bar{\Pi}_{ij} &= \tilde{\Pi}_{ij} - \tfrac{1}{2}\gamma_{ij}\tilde{\Pi} \\ &= \tfrac{1}{2}\gamma^{1/2}(\gamma_{ij}\gamma_{kl} - \gamma_{ik}\gamma_{jl})\mathcal{L}_{n_1}\gamma^{kl} - \tfrac{1}{2}\gamma^{1/2}\gamma_{ij}N^{-1}(\mathcal{L}_{n_0}N_1 - 2\mathcal{L}_{n_1}N) \\ &\qquad - \tfrac{1}{2}\gamma_{ij}\left[N^{-1}\gamma^{1/2}(\mathcal{L}_{n_0}N_1 - 2\mathcal{L}_{n_1}N) + \tfrac{1}{2}\gamma^{1/2}\gamma_{kl}\mathcal{L}_{n_1}\gamma^{kl}\right] \\ &= \tfrac{1}{2}\gamma^{1/2}(\gamma_{ij}\gamma_{kl} - \gamma_{ik}\gamma_{jl} - \tfrac{1}{2}\gamma_{ij}\gamma_{kl})\mathcal{L}_{n_1}\gamma^{kl} \\ &= \tfrac{1}{2}\gamma^{1/2}(\tfrac{1}{2}\gamma_{ij}\gamma_{kl} - \gamma_{ik}\gamma_{jl})\mathcal{L}_{n_1}\gamma^{kl} \\ &= -\tfrac{1}{2}\gamma_{ik}\gamma_{jl}\gamma^{kl}\mathcal{L}_{n_1}\gamma^{1/2} - \tfrac{1}{2}\gamma^{1/2}\gamma_{ik}\gamma_{jl}\mathcal{L}_{n_1}\gamma^{kl} \\ &= -\tfrac{1}{2}\gamma_{ik}\gamma_{jl}\mathcal{L}_{n_1}(\gamma^{1/2}\gamma^{kl}) \\ \Rightarrow X_{ij} : &= \bar{\Pi}_{ij} + \tfrac{1}{2}\gamma_{ik}\gamma_{jl}\mathcal{L}_{n_1}(\gamma^{1/2}\gamma^{kl}) = 0 \end{aligned} \tag{3.16}$$

Adding (3.16) to the other primary constraints (3.15) we see that from a total phase space of twenty field variables we have eight primary constraints. After the Legendre transformation we derive the Hamiltonian density:

$$\begin{aligned}\mathcal{H} = N_0N^{-1}\Phi_0 + N\Phi + b_0{}^k\Phi_k + \chi\big[\tfrac{1}{2}N_0N^{-1}(\gamma^{-1/2}\mathcal{L}_{n_1}\gamma^{1/2} - N^{-1}\mathcal{L}_{n_1}N) + N\gamma^{-1/2}\tilde{P}^1 \\ + \tfrac{1}{2}\mathcal{L}_{n_1}(N_0N^{-1})\big] + \mu_1N_1 + \tilde{\mu}_0(N_0)^2 + \lambda_0\tilde{P}^0 + \lambda\tilde{P} + \lambda^i\tilde{P}^0_{\ i} + \lambda^{ij}X_{ij}\end{aligned} \tag{3.17}$$

We define

$$\begin{aligned}\Phi_0 &:= \tfrac{1}{2}\tilde{\Pi}_{ij}\mathcal{L}_{n_1}\gamma^{ij} - \mathcal{L}_{n_1}\mathcal{L}_{n_1}\gamma^{1/2} \\ \Phi &:= \tfrac{1}{2}\gamma^{-1/2}\tilde{P}^1_{\ m}\tilde{P}^{1m} - 2N^{-1}\mathcal{L}_{n_1}\Big(N[\tilde{P}^1 + \tfrac{1}{2}\mathcal{L}_{n_1}(\gamma^{1/2}N^{-2}N_0)]\Big) - \nabla_m\tilde{P}^{1m} - \gamma^{1/2}\ {}^3R \\ \Phi_k &:= -\mathcal{L}_{n_1}P^1_{\ k} + 2\nabla^m\Pi_{km} \\ \chi &:= \Pi + 2\gamma^{1/2}N^{-1}\mathcal{L}_{n_1}N + \mathcal{L}_{n_1}\gamma^{1/2}.\end{aligned}$$

We now use the Dirac-Bergmann algorithm to determine the complete set of constraints. This is achieved by propagating the variables in the usual way, $\dot{Z} = \{Z, H\}$

$$\dot{N}_0 = \lambda_0 \tag{3.18a}$$

$$\dot{N} = \lambda \tag{3.18b}$$

$$\dot{N}_1 = N\gamma^{1/2}\chi \tag{3.18c}$$

$$\dot{b}_0{}^i = \lambda^i \tag{3.18d}$$

$$\dot{b}_1{}^i = N\gamma^{-1/2}P^{1i} + \nabla^i N + \mathcal{L}_{n_1} b_0{}^i \tag{3.18e}$$

We shall split the equation of motion for the two metric into its trace and trace free parts. This is because the conjugate momenta to γ^{ij} is Π_{ij}, which has already been split to obtain an additional constraint. Therefore, the trace of the propagation of γ^{ij} results in an equation of motion, whereas the trace-free part results in a multiplier equation: this is shown below.

$$\begin{aligned}
\gamma^{ij}{}_{,0} &= \tfrac{1}{2}N_0 N^{-1}\mathcal{L}_{n_1}\gamma^{ij} - 2\nabla^j b_0{}^i + \lambda^{ij} - \tfrac{1}{2}\lambda^{pq}\gamma_{pq}\gamma^{ij} \\
&\quad + \gamma^{ij}\left[\tfrac{1}{2}N_0 N^{-1}(\gamma^{-1/2}\mathcal{L}_{n_1}\gamma^{1/2} - N^{-1}\mathcal{L}_{n_1}N) + N\gamma^{-1/2}P^1 + \tfrac{1}{2}\mathcal{L}_{n_1}(N_0 N^{-1})\right] \\
\gamma_{ij}\gamma^{ij}{}_{,0} &= -2\nabla_i b_0{}^i + \tfrac{1}{2}N_0 N^{-1}\gamma_{ij}\mathcal{L}_{n_1}\gamma^{ij} \\
&\quad + 2\left[\tfrac{1}{2}N_0 N^{-1}(\gamma^{-1/2}\mathcal{L}_{n_1}\gamma^{1/2} - N^{-1}\mathcal{L}_{n_1}N) + N\gamma^{-1/2}P^1 + \tfrac{1}{2}\mathcal{L}_{n_1}(N_0 N^{-1})\right] \\
&= -2\nabla_i b_0{}^i + 2N\gamma^{-1/2}P^1 + \tfrac{1}{2}N_0 N^{-1}\gamma_{ij}\mathcal{L}_{n_1}\gamma^{ij} + N_0 N^{-1}\gamma^{-1/2}\mathcal{L}_{n_1}\gamma^{1/2} \\
&\quad - N_0 N^{-2}\mathcal{L}_{n_1}N + \mathcal{L}_{n_1}(N_0 N^{-1}) \\
&= -2\nabla_i b_0{}^i + 2N\gamma^{-1/2}P^1 + N\mathcal{L}_{n_1}(N^{-2}N_0).
\end{aligned} \tag{3.19}$$

We define the trace free part of $\dot{\gamma}^{ij}$ by

$$\begin{aligned}
\bar{\dot{\gamma}}^{ij} :&= \gamma^{ij}{}_{,0} - \tfrac{1}{2}\gamma^{ij}\gamma_{lm}\gamma^{lm}{}_{,0} \\
&= \tfrac{1}{2}N_0N^{-1}\mathcal{L}_{n_1}\gamma^{ij} - 2\nabla^j b_0{}^i + \bar{\lambda}^{ij} - \gamma^{ij}\nabla_m b^m_0 - N\gamma^{-1/2}\gamma^{ij}P - \tfrac{1}{2}\gamma^{ij}N\mathcal{L}_{n_1}(N^{-2}N_0) \\
&\quad + \gamma^{ij}\left[\tfrac{1}{2}N_0N^{-1}(\gamma^{-1/2}\mathcal{L}_{n_1}\gamma^{1/2} - N^{-1}\mathcal{L}_{n_1}N) + N\gamma^{-1/2}P^1 + \tfrac{1}{2}\mathcal{L}_{n_1}(N_0N^{-1})\right] \\
&= \bar{\lambda}^{ij} - 2\nabla^j b_0{}^i - \gamma^{ij}\nabla_m b_0{}^m + \tfrac{1}{2}N_0N^{-1}(\mathcal{L}_{n_1}\gamma^{ij} + \gamma^{ij}\gamma^{-1/2}\mathcal{L}_{n_1}\gamma^{1/2}) \\
&= \tfrac{1}{2}N_0N^{-1}(\mathcal{L}_{n_1}\gamma^{ij} - \tfrac{1}{2}\gamma^{ij}\gamma_{lm}\mathcal{L}_{n_1}\gamma^{lm}) + \mathcal{L}_{b_0}\bar{\gamma}^{ij} + \bar{\lambda}^{ij} \\
&= \tfrac{1}{2}N_0N^{-1}\mathcal{L}_{n_1}\bar{\gamma}^{ij} + \mathcal{L}_{b_0}\bar{\gamma}^{ij} + \bar{\lambda}^{ij}.
\end{aligned} \tag{3.20}$$

We can now see that (3.20) defines the multipliers $\bar{\lambda}^{ij}$ while (3.19) is an equation of motion.

We now propagate the primary constraints. We start by ensuring the slicing condition $N_1 = 0$ is true for all time. Therefore $N_{1,0} = 0$, which implies by (3.18) that

$$\chi \approx 0 \tag{3.21}$$

On the other hand the propagation of N_0 defines simply the multiplier λ_0 (see 3.18), and not an additional constraint.

We now go on to propagate the remaining primary constraints:

$$\tilde{P}^0{}_{,0} = N^{-1}\Phi_0 + \tfrac{1}{2}\chi\left[N^{-1}(\gamma^{-1/2}\mathcal{L}_{n_1}\gamma^{1/2} - N^{-1}\mathcal{L}_{n_1}N) - \mathcal{L}_{n_1}(N^{-1})\right] + 2\mu_0N^0 \tag{3.22a}$$

$$\approx N^{-1}\Phi_0 \tag{3.22b}$$

$$\tilde{P}^0{}_{k,0} = \Phi_k \tag{3.22c}$$

$$
\begin{aligned}
\tilde{P}_{,0} = & -N_0 N^{-2}\phi_0 + \tfrac{1}{2}\gamma^{1/2}\tilde{P}^1{}_m \tilde{P}^{1m} - \nabla_m \tilde{P}^{1m} - \gamma^{1/2 s}R \\
& - 2\gamma^{1/2}N^{-2}\mathcal{L}_{n_1}N\left[\tfrac{1}{2}N_0 N^{-1}(\gamma^{-1/2}\mathcal{L}_{n_1}\gamma^{1/2} - N^{-1}\mathcal{L}_{n_1}N) + N\gamma^{-1/2}\tilde{P}^1 + \tfrac{1}{2}\mathcal{L}_{n_1}(N_0 N^{-1})\right] \\
& - 2\mathcal{L}_{n_1}\left(\gamma^{1/2}N^{-1}\left[\tfrac{1}{2}N_0 N^{-1}(\gamma^{-1/2}\mathcal{L}_{n_1}\gamma^{1/2} - N^{-1}\mathcal{L}_{n_1}N) + N\gamma^{-1/2}\tilde{P}^1 + \tfrac{1}{2}\mathcal{L}_{n_1}(N_0 N^{-1})\right]\right) \\
& - \tfrac{1}{2}\chi[N_0 N^{-2}(\gamma^{-1/2}\mathcal{L}_{n_1}(\gamma^{1/2})] + \chi N_0 N^{-3}\mathcal{L}_{n_1}(N) + \tfrac{1}{2}\mathcal{L}_{n_1}(\chi N_0 N^{-2}) \\
& + \chi\gamma^{-1/2}\tilde{P}^1 + \tfrac{1}{2}N_0 N^{-2}\mathcal{L}_{n_1}(\chi) \\
= & -N_0 N^{-2}\phi_0 + \tfrac{1}{2}\gamma^{1/2}\tilde{P}^1{}_m \tilde{P}^{1m} - \nabla_m \tilde{P}^{1m} - \gamma^{1/2 s}R \\
& - 2N^{-1}\mathcal{L}_{n_1}\Big(\gamma^{1/2}\left[\tfrac{1}{2}N_0 N^{-1}(\gamma^{-1/2}\mathcal{L}_{n_1}\gamma^{1/2} - N^{-1}\mathcal{L}_{n_1}N) + N\gamma^{-1/2}\tilde{P}^1 + \tfrac{1}{2}\mathcal{L}_{n_1}(N_0 N^{-1})\right]\Big) \\
& + \chi\left[-\tfrac{1}{2}N_0 N^{-2}\left(\gamma^{-1/2}\mathcal{L}_{n_1}(\gamma^{1/2}) - N^{-1}\mathcal{L}_{n_1}(N)\right) + \gamma^{-1/2}\tilde{P}^1 - \tfrac{1}{2}\mathcal{L}_{n_1}(N_0 N^{-2})\right] \\
= & -N_0 N^{-2}\phi_0 + \tfrac{1}{2}\gamma^{1/2}\tilde{P}^1{}_m \tilde{P}^{1m} - \nabla_m \tilde{P}^{1m} - \gamma^{1/2 s}R - 2N^{-1}\mathcal{L}_{n_1}(N\tilde{P}^1) \\
& - 2N^{-1}\mathcal{L}_{n_1}\left(\tfrac{1}{2}\mathcal{L}_{n_1}(N_0 N^{-1}\gamma^{1/2}) - \tfrac{1}{2}\mathcal{L}_{n_1}(N)\gamma^{1/2}N_0 N^{-2}\right) \\
& + \chi\left[-\tfrac{1}{2}N_0 N^{-2}\left(\gamma^{-1/2}\mathcal{L}_{n_1}(\gamma^{1/2}) - N^{-1}\mathcal{L}_{n_1}(N)\right) + \gamma^{-1/2}\tilde{P}^1 - \tfrac{1}{2}\mathcal{L}_{n_1}(N_0 N^{-2})\right] \\
= & -N_0 N^{-2}\phi_0 + \tfrac{1}{2}\gamma^{1/2}\tilde{P}^1{}_m \tilde{P}^{1m} - \nabla_m \tilde{P}^{1m} - \gamma^{1/2 s}R \\
& - 2N^{-1}\mathcal{L}_{n_1}\Big(N[\tilde{P}^1 + \tfrac{1}{2}\mathcal{L}_{n_1}(\gamma^{1/2}N^{-2}N_0)]\Big) \\
& + \chi\left[-\tfrac{1}{2}N_0 N^{-2}\left(\gamma^{-1/2}\mathcal{L}_{n_1}(\gamma^{1/2}) - N^{-1}\mathcal{L}_{n_1}(N)\right) + \gamma^{-1/2}\tilde{P}^1 - \tfrac{1}{2}\mathcal{L}_{n_1}(N_0 N^{-2})\right] \\
= & \Phi - N_0 N^{-2}\phi_0 \\
& + \chi\left[-\tfrac{1}{2}N_0 N^{-2}\left(\gamma^{-1/2}\mathcal{L}_{n_1}(\gamma^{1/2}) - N^{-1}\mathcal{L}_{n_1}(N)\right) + \gamma^{-1/2}\tilde{P}^1 - \tfrac{1}{2}\mathcal{L}_{n_1}(N_0 N^{-2})\right]
\end{aligned} \tag{3.22d}
$$

$$
\approx \phi \tag{3.22e}
$$

When we propagate these four constraints we will get extra equations arising from only the weak form of the constraint. To help clarify this, we will propagate the constraint $\phi + \chi F(q^\lambda, p_\lambda)$.

$$
\begin{aligned}
\left\{\phi + \chi F(q^\lambda, p_\lambda), H\right\} &= \{\phi, H\} + \{\chi, H\}\, F(qq^\lambda, p_\lambda) + \chi\left\{F(q^\lambda, p_\lambda), H\right\} \\
&\approx \{\phi, H\}
\end{aligned}
$$

We can see that the last term will be zero because it is weakly zero by the constraint χ, while the middle term is zero as we have already ensured χ is time independent. Therefore when we propagate secondary equations (3.22) we need to propagate only their weak form. These are automatically preserved because they define the Einstein components $G^{0\mu}$, which in turn are preserved by the Bianchi identities. This implies that preserving the constraints (3.15) in time results in five secondary constraints.

$$\chi \approx 0 \qquad \Phi_0 \approx 0 \qquad \Phi \approx 0 \qquad \Phi_k \approx 0. \tag{3.23}$$

Propagating (3.16) leads to a multiplier equation. We have now ensured that all the primary constraints are conserved for all time. The aim of this work is to obtain a constraint algebra. We therefore move straight on to obtaining the first class constraints. It is possible to obtain all the Einstein equations from the constraints and the equations of motion, using Torre's original work.

3.2.1 First class algebra

The constraints so far obtained are not necessarily first class. Some will require adapting by the addition of linear combinations of the other constraints. Geometric expectations will dictate the constraints that need adapting as well as the linear combinations required. For example, we would expect that two of the first class constraints would generate diffeomorphisms along the spatial surface. Therefore we adapt the constraint ψ_k to ensure transformations of variables generated by the constraint are in line with our geometric expectations and then check that they are first class.

Therefore we adapt three secondary constraints in the following way:

$$\begin{aligned}
\psi_k :=& \phi_k + \tilde{P}^1 \nabla_k N_1 + \tilde{P} \nabla_k N + \tilde{P}^0 \nabla_k N_0 \\
=& 2\nabla^m \tilde{\Pi}_{km} - \mathcal{L}_{n_1} \tilde{P}^1_k + \tilde{P}^1 \nabla_k N_1 + \tilde{P} \nabla_k N + \tilde{P}^0 \nabla_k N_0 \\
\psi_0 :=& \phi_0 \\
=& \tfrac{1}{2} \tilde{\Pi}_{ij} \mathcal{L}_{n_1}(\gamma^{ij}) - 2\mathcal{L}_{n_1} \mathcal{L}_{n_1}(\gamma^{1/2}).
\end{aligned}$$

To ensure the constraints are first class we must calculate the Poisson brackets between each of the constraints. We will start with ψ_k, but before we calculate its Poisson brackets with other constraints we will calculate its Poisson bracket with the canonical variables.

$$\begin{aligned}
\{\gamma^{mn}, \int_y 2f^k\nabla_l(\tilde{\Pi}_{pk}\gamma^{pl})\} &= \int_z \delta(x,z)\int_y \gamma^{ml}\nabla_l(f^n) + \gamma^{nl}\nabla_l(f^m)\,\mathrm{d}^3y \\
&= \nabla^m(f^n) + \nabla^n(f^m) \equiv \mathcal{L}_f\gamma^{mn} &&(3.24a)\\
\{b_1^k, \int_y -f^m\mathcal{L}_{n_1}P_m^1\} &= \int_z \delta(x,z)\int_y \mathcal{L}_{n_1}f^k\tilde{\delta}(y,z)\,\mathrm{d}y\,\mathrm{d}z \\
&= \mathcal{L}_f b_1^k &&(3.24b)\\
\{N_0, \int_y f^k\tilde{P}^0\nabla_k N_0\,\mathrm{d}^3y\} &= \int_z \delta(x,z)\int_y \left(f^k\nabla_k N_0\tilde{\delta}(y,z)\,\mathrm{d}^y\right)\,\mathrm{d}^3z \\
&= f^k\nabla_k N_0 \quad = \quad \mathcal{L}_f N_0 &&(3.24c)\\
\{N, \int_y f^k\tilde{P}P\nabla_k N\,\mathrm{d}^3y\} &= \int_z \delta(x,z)\int_y \left(f^k\nabla_k(N)\tilde{\delta}(y,z)\right) \\
&= \mathcal{L}_f N &&(3.24d)\\
\{N_1, \int_y f^kP^1\nabla_k(N)\,\mathrm{d}y\} &= \int_z \delta(x,z)\int_y f^k\nabla_k(N)\tilde{\delta}(y,z)\,\mathrm{d}y\,\mathrm{d}z \\
&= \mathcal{L}_f N_1 &&(3.24e)
\end{aligned}$$

$$\{\tilde{\Pi}_{mn}, \int_y 2f^k\nabla_l(\tilde{\Pi}_{pk}\gamma^{pl})\,\mathrm{d}^3y\} = \mathcal{L}_f\tilde{\Pi}_{mn} \qquad (3.24f)$$

$$\begin{aligned}
\{\tilde{P}_k^1, \int_y -f^k\mathcal{L}_{n_1}\tilde{P}_k^1\,\mathrm{d}^3y\} &= \{\tilde{P}_k^1, \int_y f^k\mathcal{L}_{b_1}\tilde{P}_k^1\,\mathrm{d}^3y\} \\
f^k\mathcal{L}_{b_1}\tilde{P}_k^1 &= f^k\gamma^{1/2}\mathcal{L}_{b_1}P_k^1 + f^kP_k^1\mathcal{L}_{b_1}\gamma^{1/2} \\
&= f^k\tilde{P}_l^1 b^l_{1,k} + f^k(b_1^l\tilde{P}_k^1)_{,l} \\
\{\tilde{P}_k^1, \int_y f^k\Phi_k\,\mathrm{d}^3y\} &= \{\tilde{P}_p^1, \int_y -f^k_{,l}\tilde{P}_k^1 b_1^l - (f^k\tilde{P}_l^1)_{,k}b_1^l\,\mathrm{d}^3y\} \\
&= \int_z -\tilde{\delta}(x,z)\int_y \left[-f^k_{,p}\tilde{P}_k^1 - (f^k\tilde{P}_p^1)_{,k}\right]\delta(y,z)\,\mathrm{d}^3y\,\mathrm{d}^3z \\
&= f^k_{,p}\tilde{P}_k^1 - (f^k\tilde{P}_p^1)_{,k} \equiv \mathcal{L}_f\tilde{P}_p^1 &&(3.24g)
\end{aligned}$$

$$\{\tilde{P}^0, \int_y f^k \tilde{P}^0 \nabla_k(N_0)\,\mathrm{d}^3y\} = \mathcal{L}_f \tilde{P}^0 \tag{3.24h}$$

$$\{\tilde{P}, \int_y f^k \tilde{P} \nabla_k(N)\,\mathrm{d}^3y\} = \mathcal{L}_f \tilde{P} \tag{3.24i}$$

$$\begin{aligned}\{\tilde{P}^1, \int_y f^k \tilde{P}^1 \nabla_k(N_1)\,\mathrm{d}y\} &= \int_z -\tilde{\delta}(x,z) \int_y -\nabla_k(f^k \tilde{P}^1)\delta(y,z)\,\mathrm{d}y\,\mathrm{d}z \\ &= \nabla_k(f^k \tilde{P}^1) \equiv \mathcal{L}_f \tilde{P}^1 \end{aligned} \tag{3.24j}$$

Using equations (3.24) we can easily calculate the Poisson brackets of ψ_k:

$$\begin{aligned}\left\{\int_x f^k \psi_k\,\mathrm{d}^3x, \int_y g^l \psi_l\,\mathrm{d}^3y\right\} = \Big\{ \int_x f^k [&2\nabla_m(\tilde{\Pi}_{pk})\gamma^{pm} - \mathcal{L}_{n_1}\tilde{P}^1_k + \tilde{P}^1 \nabla_k N_1 + \tilde{P}\nabla_k N \\ &+ \tilde{P}^0 \nabla_k N_0]\,\mathrm{d}^3x, \int_y g^l \psi_l\,\mathrm{d}^3y \Big\}.\end{aligned}$$

We shall break this down, term by term. The first term is given below:

$$\begin{aligned}&\int_z \int_x 2\nabla_q(\tilde{\Pi}_{pk}) f^k \delta(x,z)\,\mathrm{d}^3x \int_y \mathcal{L}_g \gamma^{pq}\tilde{\delta}(y,z)\,\mathrm{d}^3y\,\mathrm{d}^3z \\ &\quad + \int_z \int_x 2\nabla_m(f^q \gamma^{pm})\tilde{\delta}(x,z)\,\mathrm{d}^3x \int_y \mathcal{L}_g \tilde{\Pi}_{pq}\delta(y,z)\,\mathrm{d}^3y\,\mathrm{d}^3z \\ &= -\int 2\left[\gamma^{pq}\nabla_q(\tilde{\Pi}_{pk})\mathcal{L}_g f^k + f^k \gamma^{pq}\mathcal{L}_g \nabla_q(\tilde{\Pi}_{pq}) - \nabla_m(f^q\gamma^{pm})\mathcal{L}_g(\tilde{\Pi}_{pz})\right]\,\mathrm{d}^3z \\ &= -\int 2\left[\gamma^{pq}\nabla_q(\tilde{\Pi}_{pk})\mathcal{L}_g f^k + f^k \gamma^{pq}\partial_q \mathcal{L}_g(\tilde{\Pi}_{pq}) - f^q\gamma^{pm}\partial_m \mathcal{L}_g(\tilde{\Pi}_{pq})\right]\,\mathrm{d}^3z \\ &= \int 2\nabla^p(\tilde{\Pi}_{pk})\mathcal{L}_f g^k\,\mathrm{d}^3z.\end{aligned} \tag{3.25}$$

The second term that arises is:

$$
\begin{aligned}
&\left\{\int_x f^k[\tilde{P}^1_l b^l_{1,k} + (\tilde{P}^1_k b^l_1)_{,l}]\,\mathrm{d}^3x, \int_y g^l\psi_l\,\mathrm{d}y\right\} \\
&= \int_z\int_x -\left[f^k_{,p}\tilde{P}^1_k + (f^k\tilde{P}^1_p)_{,k}\right]\delta(x,z)\,\mathrm{d}^3x\,[\mathcal{L}_{n_1}g^p] \\
&\quad - \int_x (-f^p_{,l}b^l_1 + f^k b^p_{1,k})\,\delta(x,z)\,\mathrm{d}^3x\left[-\mathcal{L}_g\tilde{P}^1_p\right]\mathrm{d}^3z \\
&= \int_z \left[f^k_{,p}\tilde{P}^1_k + (f^k\tilde{P}^1_p)_{,k}\right][b^l_1 g^p_{,l} - g^l b^p_{1,l}] - [f^p_{,l}b^l_1 - f^k b^p_{1,k}]\left[g^l_{,p}\tilde{P}^1_l + (g^l\tilde{P}^1_p)_{,l}\right]\mathrm{d}^3z \\
&= \int_z -(P^1_k b^l_1)_{,l}\,[g^p f^k_{,p} - f^p g^k_{,p}] - (b^p_{1,k}\tilde{P}^1_p)\,[g^p f^k_{,p} - f^p g^k_{,p}]\,\mathrm{d}^3z \\
&= \int_z -\mathcal{L}_{n_1}P^1_k\mathcal{L}_f g^k\,\mathrm{d}^3z.
\end{aligned} \tag{3.26}
$$

The third term becomes:

$$
\begin{aligned}
&\int_z\int_x -\nabla_k(\tilde{P}^1 f^k)\delta(x,z)\,\mathrm{d}^3x\int_y g^l\nabla_l(N_1)\delta(y,z)\,\mathrm{d}^3y \\
&\quad - \int_x f^k\nabla_k(N_1)\delta(x,z)\,\mathrm{d}^3x\int_y -\nabla_l(\tilde{P}^1 g^l)\delta(y,z)\,\mathrm{d}^3y\,\mathrm{d}^3z \\
&= \int_z -\nabla_k(\tilde{P}^1 f^k)g^l\nabla_l(N^1) + \nabla_l(\tilde{P}^1 g^l)f^k\nabla_k(N_1)\,\mathrm{d}^3z \\
&= \int_z f^k\tilde{P}^1\nabla_k(g^l)\nabla_l(N_1) - \tilde{P}^1 g^l\nabla_l(f^k)\nabla_k(N_1) + \tilde{P}^1 f^k g^l\nabla_{[k}\nabla_{l]}(N_0)\,\mathrm{d}^3z \\
&= \int_z \tilde{P}^1\nabla_l(N_1)\mathcal{L}_f g^l\,\mathrm{d}^3z.
\end{aligned} \tag{3.27}
$$

The remaining terms are the same as the above. Therefore, combining all the terms gives the final result:

$$
\left\{\int_x f^k\psi_k\,\mathrm{d}^3x, \int_y g^l\psi_l\,\mathrm{d}^3y\right\} = \int_z \psi_k\mathcal{L}_f g^k\,\mathrm{d}^3z \tag{3.28}
$$

For the remaining Poisson brackets the calculations are similar and we shall state the outcome. We now present a summary of the Poisson brackets for ψ_k.

$$
\begin{aligned}
\left\{\int_x f N_1 \,\mathrm{d}^3x, \int_y g^j \psi_j \,\mathrm{d}^3y\right\} &= \int_z \mathcal{L}_g N_1 \,\mathrm{d}^3z \approx 0 \\
\left\{\int_x f^k \tilde{P}^0_k \,\mathrm{d}^3x, \int_y g^j \psi_j \,\mathrm{d}^3y\right\} &= 0 \\
\left\{\int_x f \tilde{P} \,\mathrm{d}^3x, \int_y g^j \psi_j \,\mathrm{d}^3y\right\} &= \int_z f \mathcal{L}_g \tilde{P} \,\mathrm{d}^3z \approx 0 \\
\left\{\int_x f \tilde{P}_0 \,\mathrm{d}^3x, \int_y g^j \psi_j \,\mathrm{d}^3y\right\} &= \int_z f \mathcal{L}_g \tilde{P}_0 \,\mathrm{d}^3z \approx 0 \\
\left\{\int_x f^{ij} X_{ij} \,\mathrm{d}^3x, \int_y g^k \psi_k \,\mathrm{d}^3y\right\} &= \int_z f^{ij} \mathcal{L}_g X_{ij} \,\mathrm{d}^3z \approx 0 \\
\left\{\int_x f \psi_0 \,\mathrm{d}^3x, \int_y g^j \psi_j \,\mathrm{d}^3y\right\} &= \int_z f \mathcal{L}_g \psi_0 \,\mathrm{d}^3z \approx 0 \\
\left\{\int_x f \chi \,\mathrm{d}^3x, \int_y g^j \psi_j \,\mathrm{d}^3y\right\} &= \int_z f \mathcal{L}_g \chi \,\mathrm{d}^3z \approx 0 \\
\left\{\int_x f N_0 \,\mathrm{d}^3x, \int_y g^j \psi_j \,\mathrm{d}^3y\right\} &= \int_z f \mathcal{L}_g N_0 \,\mathrm{d}^3z \approx 0 \\
\left\{\int_x f \phi \,\mathrm{d}^3x, \int_y g^j \psi_j \,\mathrm{d}^3y\right\} &= \int_z f \mathcal{L}_g \phi \,\mathrm{d}^3z \approx 0 \\
\left\{\int_x f^k \psi_k \,\mathrm{d}^3x, \int_y g^j \psi_j \,\mathrm{d}^3y\right\} &= \int_z \psi_k \mathcal{L}_f g^k \,\mathrm{d}^3z \approx 0
\end{aligned}
$$

We can see clearly from these equations that ψ_k is a first class constraint. It is straightforward to show that $\tilde{P}^0_k$ is also first class. We can see from the Poisson brackets of ψ_0 with the other constraints that ψ_0 is first class.

$$
\begin{aligned}
\left\{\int_x f N_1 \,\mathrm{d}^3x, \int_y g \psi_0 \,\mathrm{d}^3y\right\} &= 0 \\
\left\{\int_x f^k \tilde{P}^0_k \,\mathrm{d}^3x, \int_y g \psi_0 \,\mathrm{d}^3y\right\} &= 0 \\
\left\{\int_x f \tilde{P} \,\mathrm{d}^3x, \int_y g \psi_0 \,\mathrm{d}^3y\right\} &= 0 \\
\left\{\int_x f \tilde{P}_0 \,\mathrm{d}^3x, \int_y g \psi_0 \,\mathrm{d}^3y\right\} &= 0 \\
\left\{\int_x f^{ij} X_{ij} \,\mathrm{d}^3x, \int_y g \psi_0 \,\mathrm{d}^3y\right\} &= \int_z X_{ij}\, g \mathcal{L}_{n_1}(f^{ij}) \,\mathrm{d}^3z \approx 0
\end{aligned}
$$

$$\left\{\int_x f\psi_0 \,\mathrm{d}^3x, \int_y g\psi_0 \,\mathrm{d}^3y\right\} \approx \int_z \psi_o \left(f\mathcal{L}_{n_1}g - g\mathcal{L}_{n_1}f\right) \mathrm{d}^3z \approx 0$$
$$\left\{\int_x f\chi \,\mathrm{d}^3x, \int_y g\psi_0 \,\mathrm{d}^3y\right\} = \int_z \chi g\mathcal{L}_{n_1}f \,\mathrm{d}^3z \approx 0$$
$$\left\{\int_x fN_0 \,\mathrm{d}^3x, \int_y g\psi_0 \,\mathrm{d}^3y\right\} = 0$$
$$\left\{\int_x f\phi \,\mathrm{d}^3x, \int_y g\psi_0 \,\mathrm{d}^3y\right\} = \int_z f\mathcal{L}_g\phi \,\mathrm{d}^3z \approx 0$$
$$\left\{\int_x f^k\psi_k \,\mathrm{d}^3x, \int_y g\psi_0 \,\mathrm{d}^3y\right\} = -\int_z g\mathcal{L}_f\psi_0 \,\mathrm{d}^3z \approx 0.$$

Therefore, we have five first class constraints, leaving the remaining eight constraints to be second class. These are

$$\phi \approx 0, \qquad N_0 \approx 0, \qquad N_1 \approx 0, \qquad X_{ij} \approx 0, \qquad \tilde{P} \approx 0, \qquad \chi \approx 0, \qquad \tilde{P}^0 \approx 0 \tag{3.30}$$

The first class algebra is given by

$$\left\{\int_x f^k\tilde{P}^0_k \,\mathrm{d}^3x, \int_y g^j\tilde{P}^0_j \,\mathrm{d}^3y\right\} = 0 \tag{3.31a}$$
$$\left\{\int_x f^k\tilde{P}^0_k \,\mathrm{d}^3x, \int_y g^j\psi_j \,\mathrm{d}^3y\right\} = 0 \tag{3.31b}$$
$$\left\{\int_x f^k\tilde{P}^0_k \,\mathrm{d}^3x, \int_y g\psi_0 \,\mathrm{d}^3y\right\} = 0 \tag{3.31c}$$
$$\left\{\int_x f^k\psi_k \,\mathrm{d}^3x, \int_y g^j\psi_j \,\mathrm{d}^3y\right\} = \int_z \psi_k\mathcal{L}_f g^k \,\mathrm{d}^3z \approx 0 \tag{3.31d}$$
$$\left\{\int_x f\psi_0 \,\mathrm{d}^3x, \int_y g^j\psi_j \,\mathrm{d}^3y\right\} = \int_z f\mathcal{L}_g\psi_0 \,\mathrm{d}^3z \approx 0 \tag{3.31e}$$
$$\left\{\int_x f\psi_0 \,\mathrm{d}^3x, \int_y g\psi_0 \,\mathrm{d}^3y\right\} = \int_z \psi_0(f\mathcal{L}_{n_1}g - g\mathcal{L}_{n_1}f) \,\mathrm{d}^3z \approx 0. \tag{3.31f}$$

Now that we have obtained the first and second class constraints we may find the dimension of the reduced phase-space. In this double null description we have a phase space of twenty variables. We obtain five first class and eight second class constraints. Putting these results into the standard formula (2.73) we obtain $\frac{1}{2}(20-2(5)-8) = 1$. This results in one true degree of freedom, which is what we would expect from a null foliation of the space-time as Penrose states:

> *There is the curious feature of these null data that apparently it is sufficient to have one-half as much information per point as in the corresponding Cauchy problem.* (Penrose 1980)

Geometrically the two constraints $\tilde{P}^0_k$ represent the gauge freedom of the shift variables b^k_0. We use (3.24) to provide the infinitesimal transformations of the constraint ψ_k:

$$
\begin{aligned}
\delta\gamma_{mn} &= \left\{\gamma_{mn}, \int_x f^k\psi_k \, \mathrm{d}^3x\right\} = \mathcal{L}_f\gamma_{mn} \\
\delta b^k_0 &= \left\{b^k_0, \int_x f^k\psi_k \, \mathrm{d}^3x\right\} = 0 \\
\delta b^k_1 &= \left\{b^k_1, \int_x f^k\psi_k \, \mathrm{d}^3x\right\} = \mathcal{L}_f b^k_1 \\
\delta N_0 &= \left\{N_0, \int_x f^k\psi_k \, \mathrm{d}^3x\right\} = \mathcal{L}_f N_0 \approx 0 \\
\delta N &= \left\{N, \int_x f^k\psi_k \, \mathrm{d}^3x\right\} = \mathcal{L}_f N \\
\delta N_1 &= \left\{N_1, \int_x f^k\psi_k \, \mathrm{d}^3x\right\} \mathcal{L}_f N_1 \approx 0.
\end{aligned}
$$

This shows that this constraint generates diffeomorphism's along the spatial two surfaces. We also note from the above that the variables b^k_0 and b^k_1 generate different transformations. This can be understood when we realise that b^k_0 lie in the Σ_1 plane and are therefore shifts to the evolution direction, while b^k_1 lie in the Σ_0 plane and are therefore part of the decomposed three metric.

The final first class constraint, ψ_0, generates Lie derivatives along the n^1 direction. This can be seen when we consider the transformations generated by ψ_0 on the canonical variables:

$$\delta\gamma_{mn} = \left\{\gamma_{mn}, \int_x f\psi_0 \, \mathrm{d}^3x\right\} = f\mathcal{L}_{n_1}\gamma_{mn} \tag{3.32a}$$

$$\delta b_a^k = \left\{b_a^k, \int_x f\psi_0 \, \mathrm{d}^3x\right\} \approx 0 \tag{3.32b}$$

$$\delta N_0 = \left\{N_0, \int_x f\psi_0 \, \mathrm{d}^3x\right\} = 0 \tag{3.32c}$$

$$\delta N = \left\{N, \int_x f\psi_0 \, \mathrm{d}^3x\right\} = \mathcal{L}_{n_1}(Nf) \tag{3.32d}$$

$$\delta N_1 = \left\{N_1, \int_x f\psi_0 \, \mathrm{d}^3x\right\} \approx 0 \tag{3.32e}$$

As we have already explained b_1^k is analogous to part of the three metric in the ADM description. Therefore we might expect that δb_1^k would result in a Lie derivative term instead of the trivial solution that is obtained. To understand why this occurs we introduce an adapted coordinate basis on the Σ_0 hypersurface, (see fig 3.2), which we will denote:

$$\{e_{\mathbf{1}}^\alpha, e_{\mathbf{2}}^\alpha, e_{\mathbf{3}}^\alpha\} = \{\mathrm{e}_\mu^\alpha\} \quad (\mu, \nu = 1, 2, 3) \tag{3.33}$$

We introduce the induced metric on the hypersurface, $(h_{\mu\nu})$, which has the following components in the above basis:

$$h_{11} = \gamma_{ij} b_1^i b_1^j$$
$$h_{1i} = b_{1i}$$
$$h_{ij} = \gamma_{ij}.$$

This can be seen from (3.10).

When we consider the Lie derivative of a vector v^α acting on the induced metric

$$\mathcal{L}_v h_{\mu\nu} = v^\alpha \partial_\alpha h_{\mu\nu} + h_{\alpha\nu}\partial_\mu v^\alpha + h_{\mu\alpha}\partial_\nu v^\alpha, \tag{3.34}$$

and apply it to h_{1i} with $v^\alpha = fn_1^\alpha$ we get the identity:

$$
\begin{aligned}
\mathcal{L}_{fn_1} h_{1i} &= fn_1^\alpha \partial_\alpha h_{1i} + h_{\alpha i}\partial_1(fn_1^\alpha) + h_{1\alpha}\partial_i(fn_1^\alpha) \\
&= f b_1^j \mathcal{L}_{n_1} h_{ij} \\
&= \mathcal{L}_{fn_1} b_{1i} - f h_{ij}\mathcal{L}_{n_1} b_1^j \\
\Rightarrow \mathcal{L}_{n_1} b_1^i &= 0.
\end{aligned}
$$

Therefore, the Lie derivative term we expected from δb_1^i in (3.32), is actually equal to zero.

3.3 Discussion

In this chapter we extended the work of Torre so that the double null foliation was considered. As we have just described, the resulting first class constraints are associated with the diffeomorphism freedoms in the Σ_0 hypersurface. This result was also found in ADM except that in this chapter, due to the 2+2 foliation, the diffeomorphism constraints are split into two on the spacial two-surface and one in the null n_1 direction. In the ADM analysis one first class constraint, the Hamiltonian constraint, is dynamical. This is due to the presence of the derivative on the right hand side of (2.81c) depending upon the form of the momentum constraint. This meant that the first class algebra did not form a Lie algebra. In the double null analysis all of the first class constraints are kinematic, and therefore we might expect that the first class constraint algebra did form a Lie algebra; unfortunately this is not the case. If we look at (3.31) we see that the last term has a $\mathcal{L}_{n_1}$ term. $\mathcal{L}_{n_1} = \mathcal{L}_{e_1} - \mathcal{L}_{b_1}$, and therefore the first class algebra contains the gauge dependent variables b_1^k. The first class algebra could be made a Lie algebra if a gauge condition was introduced that set these variables to zero; Torre (1986) has a discussion on suitable gauges.

In Torre's work three of the first class constraints were obtained due to the gauge freedom of the lapse and shift variables. In the double null description only two of these gauge freedoms remain. We used the gauge choice of the lapse to specify that the evolution direction is null after the Poisson bracket calculations.

Now that we have the canonical analysis for a double null description of General Relativity we could continue to see how far along the Dirac-Bergman algorithm we can go. Unfortunately due to the complexity of the constraints and the first class algebra not being a Lie algebra we would find further progress towards quantising this field theory just as difficult as ADM; we therefore do not pursue this analysis any further. Instead we will introduce a change of variables that will overcome the current obstacles by providing polynomial first class constraints as well as a Lie algebra for the first class algebra. These new variables were introduced by Ashtekar and are the subject of the next chapter.

Chapter 4

Ashtekar variables

Although the work of Arnowitt *et al.* (1960) gave a canonical analysis of the $3+1$ decomposition and obtained a first class algebra for the constraints, the remaining steps of canonical quantisation proved to be difficult.

Some years later Ashtekar (1987) proposed a change of variables which overcame some of the problems which had hindered earlier work in canonical quantisation and brought new life to this area of research. It had been known for some time that working with the connection and the curvature simplifies the canonical analysis, but also results in additional second class constraints. When these are solved for kinematic variables, and then substituted into the first class constraints, we arrive at the same system of first class equations as ADM. Although the Ashtekar approach uses the connection and frame as variables they are complexified and all the constraints are first class. As we shall find out, the complexification of the connection also allows us to split the action into anti-self-dual and self-dual parts, but only one part is required.

The constraints that result from the canonical analysis are all first class, polynomial and have geometrical interpretation. This last point is important because while other variables have resulted in polynomial first class constraints they lacked a geometrical understanding which becomes important in the latter stages of the quantisation process.

Another benefit of the Ashtekar approach is that the resulting Hamiltonian has a similar structure to Yang-Mills' theory. Unlike General Relativity this field theory has been quantised, and it is thought that techniques used in that process can be adapted for use with General Relativity.

In this chapter we shall introduce Ashtekar variables, before introducing the variables that we shall use in the latter chapters. In this chapter we shall follow the approach of Giulini (1994). We consider the approach given by Ashtekar (1991) in Appendix A. In the first section we shall show how the Einstein equations are obtained through a variation of the Einstein Hilbert action with respect to the connection and frame. This was first achieved by Palatini and it is his approach which is most common. We choose not to use this form, but instead use the that of by Giulini (1994). We do this so that in the following section we may simply adapt the approach to show that only the self-dual part of the complexified action is required.

The final section introduces the variables that we shall be using for the remainder of the thesis. A local isomorphism is exploited to adapt the complex self-dual variables into $SO(3)$ variables. We then give the structure equations and Bianchi identities in this new basis. The Einstein equations are then given using this notation and we close the chapter with a self-dual action written using the $SO(3)$ basis and curvature.

4.1 Variation of the connection and frame

We will start by introducing some new notation and identities. We start with a $SO(1,3)$ valued connection which is denoted by the 1-form A^{α}_{β} and the curvature denoted by a 2-form Ω^{α}_{β}. We introduce the space-time exterior covariant derivative D given by

$$D\lambda := \mathrm{d}\lambda + [A, \lambda], \tag{4.1}$$

where λ is an $SO(1,3)$ valued 1-form. The following two derivatives

$$\begin{aligned} D^{A}\lambda :&= \mathrm{d}\lambda + [A, \lambda] \\ D^{A'}\lambda :&= \mathrm{d}\lambda + [A', \lambda] \end{aligned}$$

are defined using two different connections (A and A').These are related in the following way

$$D^{A'}\lambda = D^{A}\lambda + [A' - A, \lambda]. \tag{4.2}$$

Applying (4.1) to the tetrad θ^{α} results in the first Cartan structure equation:

$$D\theta^{\alpha} = d\theta^{\alpha} + A^{\alpha}_{\beta} \wedge \theta^{\beta} = T^{\alpha} \tag{4.3}$$

where T denotes the torsion which we will take to be zero.

We express the curvature 2-form in terms of the connection by:

$$\Omega^{\alpha}_{\beta} = \mathrm{d}A^{\alpha}_{\beta} + \tfrac{1}{2}[A, A]^{\alpha}_{\beta} \tag{4.4}$$

which results in

$$D\Omega^{\alpha}_{\beta} = \mathrm{d}\Omega^{\alpha}_{\beta} + [A, \Omega]^{\alpha}_{\beta} = 0. \tag{4.5}$$

The Hodge dual of a n form in 4 dimensions is given by:

$$* \left(\theta^{\alpha_1} \wedge ... \wedge \theta^{\alpha_n}\right) := \frac{1}{(4-n)!}\epsilon^{\alpha_1 ... \alpha_n}{}_{\alpha_{n+1} ... \alpha_4}\theta^{\alpha_{n+1}} \wedge ... \wedge \theta^{4}. \tag{4.6}$$

where we take $\epsilon_{0123} = 1, \quad \epsilon^{0123} = -1$. We also make use of the inner product of two 2-forms:

$$\lambda = \frac{1}{2}\lambda_{\alpha\beta}\theta^{\alpha} \wedge \theta^{\beta} \qquad\qquad \sigma = \frac{1}{2}\sigma_{\alpha\beta}\theta^{\alpha} \wedge \theta^{\beta}$$

given by

$$g(\lambda, \sigma) := \lambda_{\alpha\beta}g^{\alpha\gamma}g^{\beta\delta}\sigma_{\gamma\delta} = \lambda_{\alpha\beta}\sigma^{\alpha\beta}.$$

We now have the useful result that

$$\lambda \wedge *\sigma = g(\lambda, \sigma)\epsilon. \tag{4.7}$$

Now that we have introduced some useful identities and definitions we can start calculating the Einstein equations through a variation of the tetrad and connection. We start by writing the Einstein-Hilbert action in terms of the curvature two-form:

$$\begin{aligned} I := & \int \Omega_{\alpha\beta} \wedge *(\theta^\alpha \wedge \theta^\beta) && (4.8) \\ = & \frac{1}{2}\int R_{\alpha\beta\gamma\delta}(\theta^\gamma \wedge \theta^\delta) \wedge *(\theta^\alpha \wedge \theta^\beta), \\ = & \frac{1}{2}\int R_{\alpha\beta\gamma\delta}\, g(\theta^\gamma \wedge \theta^\delta, \theta^\alpha \wedge \theta^\beta)\epsilon \\ = & \frac{1}{2}\int R_{\alpha\beta\gamma\delta}(\eta^{\alpha\gamma}\eta^{\beta\delta} - \eta^{\alpha\delta}\eta^{\beta\gamma})\epsilon \\ = & \int R\epsilon && (4.9) \end{aligned}$$

Therefore we are going to use the action $I[\theta^\alpha, A^\alpha_\beta]$ given by (4.8) to replace the standard Einstein-Hilbert action. We now consider the tetrad θ^α and the connection A^α_β to be independent. Due to this we have increased the number of variables, and therefore we would expect a greater number of equations resulting from variations of the connection. Palatini discovered that these extra equations showed the connection is the metric connection. To obtain this we first use (4.4) to calculate the variation of the curvature with respect to the connection:

$$\delta\Omega_{\alpha\beta} = \mathrm{d}(\delta A_{\alpha\beta}) + [A, \delta A]_{\alpha\beta} = D(\delta A_{\alpha\beta}). \tag{4.10}$$

Therefore when we calculate the variation of the action with respect to the connection we obtain:

$$\begin{aligned} \delta I = & \int \delta A_{\alpha\beta} \wedge D * (\theta^\alpha \wedge \theta^\beta) \\ = & \frac{1}{2}\int \delta A_{\alpha\beta} \wedge \epsilon^{\alpha\beta}{}_{\gamma\beta} D(\theta^\gamma \wedge \theta^\delta). && (4.11) \end{aligned}$$

From the requirement that

$$\frac{\delta I}{\delta A_{\alpha\beta}} = 0 \tag{4.12}$$

we obtain the result

$$D(\theta^\alpha \wedge \theta^\beta) = 0. \tag{4.13}$$

We write the connection $A = \Gamma + \Lambda$ where Γ is the Levi-Civita (or metric) connection, then show that $\Lambda = 0$. In order to do this we first express the result (4.13) using (4.2) to obtain

$$D^A(\theta^\alpha \wedge \theta^\beta) = D^\Gamma(\theta^\alpha \wedge \theta^\beta) + \Lambda^\alpha_\gamma \wedge \theta^\gamma \wedge \theta^\beta + \Lambda^\beta_\gamma \wedge \theta^\alpha \wedge \theta^\gamma \tag{4.14}$$

$$= (\Lambda^\alpha_{[\gamma\delta]}\delta^\beta_\epsilon + \Lambda^\beta_{[\gamma\epsilon]}\delta^\alpha_\delta)\theta^\gamma \wedge \theta^\delta \wedge \theta^\epsilon. \tag{4.15}$$

We have used the vanishing torsion of Γ to remove the term $D^\Gamma(\theta^\alpha \wedge \theta^\beta)$. The remaining term vanishes if the cyclic sum in $\gamma - \delta - \epsilon$ of the coefficient does. When the above is contracted on β and ϵ we get the expression

$$\Lambda^\alpha_{[\gamma\delta]} + \Lambda^\beta_{[\gamma\beta]}\delta^\alpha_\delta - \Lambda^\beta_{[\delta\beta]}\delta^\alpha_\gamma = 0. \tag{4.16}$$

which when contracted on α and δ results in

$$\Lambda^\alpha_{[\gamma\alpha]} = 0. \tag{4.17}$$

This, when substituted into (4.16) gives

$$\Lambda^\alpha_{[\gamma\delta]} = 0. \tag{4.18}$$

Thus Λ is symmetric in its bottom two indices. If we define the covariant tensor $\Lambda_{\alpha\beta\gamma} := \eta_{\beta\delta}\Lambda^\delta_{\alpha\gamma}$, then we have a covariant tensor that is symmetric on its first and third indices, while also being anti-symmetric in the second and third indices. A tensor with these properties must be zero, as can be seen from interchanging the indices. Therefore we have finally shown that because Λ is zero, $A = \Gamma$ and hence the connection A is the Levi-Civita connection.

We now consider the variation of the action with respect to the tetrad θ^α.

$$
\begin{aligned}
I &= \int \tfrac{1}{2}\Omega_{\alpha\beta}\epsilon^{\alpha\beta}{}_{\gamma\delta} \wedge \theta^\gamma \wedge \theta^\delta \\
\Rightarrow \delta I &= \int \epsilon^{\alpha\beta}{}_{\gamma\delta}\Omega_{\alpha\beta} \wedge \theta^\gamma \wedge \delta\theta^\delta \\
&= \int \tfrac{1}{2}(\epsilon^{\alpha\beta}{}_{\gamma\delta}R_{\alpha\beta\nu\mu}\theta^\nu \wedge \theta^\mu \wedge \theta^\gamma) \wedge \delta\theta^\delta \\
&= \int \tfrac{1}{2}(\epsilon^{\alpha\beta\gamma\delta}R^{\alpha\beta}{}_{\nu\mu}\epsilon^{\nu\mu\gamma\epsilon}\eta_{\epsilon\lambda} * \theta^\lambda) \wedge \delta\theta^\delta \\
&= -\int \tfrac{1}{2}(3!\delta^\nu_{[\alpha}\delta^\mu_\beta\delta^\epsilon_{\delta]}R^{\alpha\beta}{}_{\nu\mu}\epsilon^{\nu\mu\gamma\epsilon}\eta_{\epsilon\lambda} * \theta^\lambda) \wedge \delta\theta^\delta \\
&= 2\int (R_{\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta}R) * \theta^\alpha \wedge \delta\theta^\beta.
\end{aligned} \tag{4.19}
$$

So finally Hamiltons principle states that

$$
\begin{aligned}
\frac{\delta I}{\delta\theta^\beta} &= 0 \\
\Rightarrow R_{\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta}R &= 0 \\
\Rightarrow G_{\alpha\beta} &= 0.
\end{aligned}
$$

Therefore by considering the connection and the tetrad to be independent variables we are able to derive the Einstein equations as well as showing the connection is the metric connection. It was stated at the beginning of this chapter that only the self-dual part was necessary to obtain these equations. In the following section we shall prove that this is indeed the case.

4.2 Complex General Relativity and Self-dual variation

If we take a two form, $(\theta^\alpha \wedge \theta^\beta)$, then using (4.6) we can calculate its Hodge dual

$$
*(\theta^\alpha \wedge \theta^\beta) = \tfrac{1}{2}\epsilon^{\alpha\beta}{}_{\gamma\delta}\theta^\gamma \wedge \theta^\delta \tag{4.20}
$$

and its double Hodge dual

$$
\begin{aligned}
**(\theta^\alpha \wedge \theta^\beta) &= \tfrac{1}{2}\epsilon^{\alpha\beta}{}_{\gamma\delta} * (\theta^\gamma \wedge \theta^\delta) \\
&= \tfrac{1}{4}\epsilon^{\alpha\beta}{}_{\gamma\delta}\epsilon^{\gamma\delta}{}_{\varsigma\eta}\theta^\varsigma \wedge \theta^\eta \\
&= \tfrac{1}{4}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\gamma\delta\varsigma\eta}\theta^\varsigma \wedge \theta^\eta \\
&= -\tfrac{1}{2}\delta^\alpha_{[\varsigma}\delta^\beta_{\eta]}\theta^\varsigma \wedge \theta^\eta \\
&= -\theta^\alpha \wedge \theta^\beta.
\end{aligned} \tag{4.21}
$$

Therefore the eigenvalues of $*$ are $\pm i$:

$$
\begin{aligned}
*\lambda &= i\lambda \quad && \text{self-dual} \\
*\lambda &= -i\lambda && \text{anti self-dual}
\end{aligned} \tag{4.22}
$$

We now extend the framework outlined in the previous section to include complex frames and connections. Due to this extension we are no longer calculating General Relativity, but a generalisation of the Einstein equations because the metric can be complex, although this complexification does not change the nature of the manifold which remains real. Therefore in order to obtain General Relativity and not a complexified General Relativity we must add extra conditions, called reality conditions, to obtain a real metric. It is worth noting that from this point we will always be considering a complexified General Relativity unless otherwise stated. We leave any further discussion on the reality conditions until chapter 7.

The structure group for the connection and tetrad is now $SO(1,3)_{\mathbb{C}}$, which has Lie algebra $so(1,3)_{\mathbb{C}}$. If we represent $so(1,3)_{\mathbb{C}}$ using complex bivectors $\sigma_{\alpha\beta}$ we may introduce the operator $\star$, which gives the dual in the Lie algebra by

$$
\star\,\sigma_{\alpha\beta} := \tfrac{1}{2}\epsilon_{\alpha\beta}{}^{\gamma\delta}\sigma_{\gamma\delta}. \tag{4.23}
$$

This also squares to minus the identity so that we may define self-dual and anti self-dual elements of the Lie algebra according to:

$$\begin{aligned} \star\sigma &= i\sigma \quad \text{self-dual} \\ \star\sigma &= -i\sigma \quad \text{anti-self-dual} \end{aligned} . \tag{4.24}$$

This allows us to split the Lie algebra, into self-dual and anti self-dual parts,

$$so(1,3)_{\mathbb{C}} = so(1,3)^{(+)}_{\mathbb{C}} \oplus so(1,3)^{(-)}_{\mathbb{C}}. \tag{4.25}$$

We also use this operator to split forms which take values in the Lie algebra into their self-dual and anti self-dual parts by introducing two projectors $P^{\pm} = \frac{1}{2}(1 \mp i\star)$. Therefore we decompose the connection and curvature as follows:

$$\begin{aligned} A &= P^{(+)}A + P^{(-)}A = {}^{(+)}A + {}^{(-)}A \\ \Omega &= P^{(+)}\Omega + P^{(-)}\Omega = {}^{(+)}\Omega + {}^{(-)}\Omega = \Omega({}^{(+)}A) + \Omega({}^{(-)}A). \end{aligned}$$

We now decompose the action by splitting the curvature into the self-dual and anti self-dual parts.

$$I = I^{(+)} + I^{(-)} = \int {}^{(+)}\Omega_{\alpha\beta} \wedge *(\theta^\alpha \wedge \theta^\beta) + \int {}^{(-)}\Omega_{\alpha\beta} \wedge *(\theta^\alpha \wedge \theta^\beta), \tag{4.26}$$

which using $\Omega_{\alpha\beta} \wedge *(\theta^\alpha \wedge \theta^\beta) = \star\Omega_{\alpha\beta} \wedge (\theta^\alpha \wedge \theta^\beta)$ and (4.24) can be expressed

$$= i \int {}^{(+)}\Omega_{\alpha\beta} \wedge (\theta^\alpha \wedge \theta^\beta) - i \int {}^{(-)}\Omega_{\alpha\beta} \wedge (\theta^\alpha \wedge \theta^\beta). \tag{4.27}$$

We will now consider the two parts of the action independently. The self-dual part is a function of the self-dual connection and the frame, whereas the anti-self-dual part is a function of the anti-self-dual connection and the frame. In both cases the connection and the frame are considered to be independent.

To calculate the variation of the self-dual action, $I^{(+)}$, with respect to the the tetrad and self-dual curvature, we will use a method analogous to the one used in the previous section. The anti self-dual case follows in exactly the same way and therefore we do not include it here. We start by calculating the variation of the connection:

$$\delta I^{(+)} = i \int \delta^{(+)} A_{\alpha\beta} \wedge {}^{(+)}D(\theta^\alpha \wedge \theta^\beta) \tag{4.28}$$

Thus we get the result that

$$^{(+)}D(\theta^\alpha \wedge \theta^\beta) = 0. \tag{4.29}$$

If we set ${}^{(+)}A = {}^{(+)}\Gamma + {}^{(+)}\Lambda$ and follow the argument given above, we show that ${}^{(+)}\Lambda = 0$ and therefore

$$^{(+)}A = {}^{(+)}\Gamma = \text{self-dual part of the Levi-Civita (metric) connection.}$$

The surprising result is that when we calculate the variation of the self-dual action with the tetrad, we obtain the full Einstein equations, as is shown below.

$$\frac{\delta I^{(+)}}{\delta\theta^\beta} = 2i^{(+)}\Omega_{\alpha\beta} \wedge \theta^\alpha \tag{4.30}$$

$$= i\left(\Omega_{\alpha\beta} - i\tfrac{1}{2}\epsilon^{\gamma\delta}{}_{\alpha\beta}\Omega_{\gamma\delta}\right) \wedge \theta^\alpha. \tag{4.31}$$

Then, using the fact that $\Omega_{\alpha\beta} \wedge \theta^\alpha = R_{\alpha\beta\gamma\delta} \wedge \theta^\gamma \wedge \theta^\delta \wedge \theta^\alpha = 0$, which is the result of the Bianchi identities, we get the same expression as we did using the standard variational principle, (4.19)

$$\frac{\delta I^{(+)}}{\delta\theta^\beta} = \tfrac{1}{2}\epsilon^{\gamma\delta}{}_{\alpha\beta}\Omega_{\gamma\delta} \wedge \theta^\alpha. \tag{4.32}$$

Using the same line of argument as before results in the equations

$$\Rightarrow R_{\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta}R = 0. \tag{4.33}$$

Although we will call these the Einstein equations it is important not to forget that these are the complexified generalisation of the Einstein equations.

In this section we have shown that by complexifying the variables we need to consider only the self-dual part of the action to obtain the Einstein equations. This result is the basis of the self-dual variables which we will introduce in the following section.

4.3 $SO(3)_{\mathbb{C}}$ variables

In the previous section we used a complex connection and curvature, which both have the Lie algebra $so(1,3)_{\mathbb{C}}$. This was split into self-dual and anti self-dual parts, denoted $so(1,3)^{(+)}_{\mathbb{C}} \oplus so(1,3)^{(-)}_{\mathbb{C}}$. We then demonstrated that only the self-dual part is required to obtain the full Einstein equations. In the following chapters we will exploit a local isomorphism that exists between $so(1,3)^{(+)}_{\mathbb{C}}$ and $so(3)_{\mathbb{C}}$. This isomorphism enables us to replace the self-dual connection and curvature used above with a connection and curvature that have a $SO(3)_{\mathbb{C}}$ basis of complex self-dual two forms, $S^{\mathbf{A}}$ (bold upper Latin indices range from $1,2,3$). The basis of self dual two forms is given below:

$$S^{\mathbf{1}} = \tfrac{1}{2}(\theta^{\mathbf{1}} \wedge \theta^{\mathbf{0}} + \theta^{\mathbf{3}} \wedge \theta^{\mathbf{2}}) \tag{4.34a}$$

$$S^{\mathbf{2}} = \theta^{\mathbf{1}} \wedge \theta^{\mathbf{2}} \tag{4.34b}$$

$$S^{\mathbf{3}} = \theta^{\mathbf{3}} \wedge \theta^{\mathbf{0}}. \tag{4.34c}$$

When we state that these two forms are self-dual then we require $*S^{\mathbf{A}} = iS^{\mathbf{A}}$ where $*$ is the Hodge-dual (4.6). An example is given below:

$$\begin{aligned}
*S^{\mathbf{2}} &= *(\theta^{\mathbf{1}} \wedge \theta^{\mathbf{2}}) \\
&= \tfrac{1}{2}\epsilon^{\mathbf{12}}{}_{\alpha\beta}(\theta^{\alpha} \wedge \theta^{\beta}) \\
&= \tfrac{1}{2}\eta^{\mathbf{1}\gamma}\eta^{\mathbf{2}\delta}\epsilon_{\gamma\delta\alpha\beta}(\theta^{\alpha} \wedge \theta^{\beta}) \\
&= -\tfrac{1}{2}\epsilon_{\mathbf{0}\,\mathbf{3}\,\alpha\beta}(\theta^{\alpha} \wedge \theta^{\beta}) \\
&= i\theta^{\mathbf{1}} \wedge \theta^{\mathbf{2}} \\
&= iS^{\mathbf{2}}
\end{aligned}$$

where we have used

$$\eta^{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \epsilon_{\mathbf{0\,1\,2\,3}} = -i.$$

If we define the metric $g^{\mathbf{AB}} := g(S^{\mathbf{A}}, S^{\mathbf{B}})$, and use the identity $g(P,Q) = *(P \wedge Q)$ then we can show that

$$\begin{aligned} g^{\mathbf{11}} = g(S^{\mathbf{1}}, S^{\mathbf{1}}) &= *(S^{\mathbf{1}} \wedge S^{\mathbf{1}}) \\ &= *\tfrac{1}{2}(\theta^{\mathbf{0}} \wedge \theta^{\mathbf{1}} \wedge \theta^{\mathbf{2}} \wedge \theta^{\mathbf{3}}) \\ &= \tfrac{1}{2}. \end{aligned}$$

Therefore, by calculating all of the components we obtain the $SO(3)_{\mathbb{C}}$ metric

$$g^{\mathbf{AB}} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \tag{4.35}$$

and the inverse

$$g_{\mathbf{AB}} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{4.36}$$

The metric and its inverse are used to raise and lower the self-dual triad indices.

The connection and curvature used in the earlier section have SO(3) representations:

$$\Gamma^{\mathbf{1}} = \tfrac{1}{2}(\mathcal{A}^{\mathbf{01}} + \mathcal{A}^{\mathbf{23}}) \tag{4.37a}$$

$$\Gamma^{\mathbf{2}} = \mathcal{A}^{\mathbf{21}} \tag{4.37b}$$

$$\Gamma^{\mathbf{3}} = \mathcal{A}^{\mathbf{03}} \tag{4.37c}$$

and

$$R^1 = \tfrac{1}{2}(\mathcal{R}^{01} + \mathcal{R}^{23}) \tag{4.38a}$$

$$R^2 = \mathcal{R}^{21} \tag{4.38b}$$

$$R^3 = \mathcal{R}^{03}. \tag{4.38c}$$

These forms satisfy the first and second Cartan equations

$$dS^{\mathbf{A}} + 2\eta^{\mathbf{A}}{}_{\mathbf{BC}}\Gamma^{\mathbf{B}} \wedge S^{\mathbf{C}} = 0 \tag{4.39a}$$

and

$$d\Gamma^{\mathbf{A}} + \eta^{\mathbf{A}}{}_{\mathbf{BC}}\Gamma^{\mathbf{B}} \wedge \Gamma^{\mathbf{C}} = R^{\mathbf{A}}, \tag{4.39b}$$

where $\eta_{\mathbf{ABC}} = \eta_{[\mathbf{ABC}]}$ and $\eta_{\mathbf{123}} = 1$.

Exterior differentiation of the above equations results in the first and second Bianchi identities

$$\eta_{\mathbf{ABC}}R^{\mathbf{B}} \wedge S^{\mathbf{C}} = 0 \tag{4.40a}$$

$$dR^{\mathbf{A}} + 2\eta^{\mathbf{A}}{}_{\mathbf{BC}}\Gamma^{\mathbf{B}} \wedge R^{\mathbf{C}} = 0. \tag{4.40b}$$

The relationship between the tetrad components of the Einstein tensor $G^{\alpha}{}_{\beta}$, and the components of the self-dual curvature 2-forms $R^{\mathbf{A}}{}_{\alpha\beta}$ is given by

$$\tfrac{1}{3!}G_{\alpha}{}^{\beta}\epsilon_{\beta\gamma\delta\sigma}\theta^{\gamma} \wedge \theta^{\delta} \wedge \theta^{\sigma} = 2i(e_{\alpha} \lrcorner S^{\mathbf{A}}) \wedge R_{\mathbf{A}}. \tag{4.41}$$

The following steps lead to an equation more useful in determining the Einstein components in terms of the 2-form curvature. Contracting (4.41) with θ^λ gives

$$
\begin{aligned}
&\tfrac{1}{3!}G_\alpha^{\ \beta}\epsilon_{\beta\gamma\delta\sigma}\theta^\gamma \wedge \theta^\delta \wedge \theta^\sigma \wedge \theta^\lambda = 2i(e_\alpha \lrcorner S^{\mathbf{A}}) \wedge R_{\mathbf{A}} \wedge \theta^\lambda \\
&\Rightarrow G_\alpha^{\ \beta}\epsilon_{\beta\gamma\delta\sigma}\theta^\beta \wedge \theta^\gamma \wedge \theta^\delta \wedge \theta^\sigma = 2i(e_\alpha \lrcorner S^{\mathbf{A}}) \wedge \theta^\lambda \wedge R_{\mathbf{A}} \\
&\qquad \Rightarrow G^\beta{}_\alpha \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 = -2(e_\alpha \lrcorner S^{\mathbf{A}}) \wedge \theta^\beta \wedge R^{\mathbf{B}} g_{\mathbf{AB}},
\end{aligned}
$$

It is now a simple matter to express the Einstein components explicitly in terms of the curvature. We will use the first component as an example to illustrate the method. In this example we calculate $e_0 \lrcorner S^A$. Using

$$e_\alpha \lrcorner \theta^\beta = \delta_\alpha^\beta$$

as well as

$$e_\alpha \lrcorner \theta^\beta \wedge \theta^\alpha = e_\alpha \lrcorner(-\theta^\alpha \wedge \theta^\beta) = -\theta^\beta$$

we can show that

$$
\begin{aligned}
e_0 \lrcorner S^1 &= e_0 \lrcorner \tfrac{1}{2}(\theta^1 \wedge \theta^0 + \theta^3 \wedge \theta^2) \\
&= \tfrac{1}{2}(-\theta^1).
\end{aligned}
$$

Using these calculations we obtain

$$
\begin{aligned}
G^{\mathbf{0}}{}_{\mathbf{0}}\theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 &= -2(e_0 \lrcorner S^{\mathbf{A}}) \wedge \theta^0 \wedge R^{\mathbf{B}} g_{\mathbf{AB}} \\
&= -2(-\tfrac{2}{2}\theta^1 \wedge \theta^0 \wedge R^1 + \theta^3 \wedge \theta^0 \wedge R^2) \\
&= -2(R^1_{23} - R^2_{12})\theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3.
\end{aligned}
$$

This process results in the relationships

$$
\begin{array}{ll}
G^{0}{}_{0} = -2(R^{1}_{23} - R^{2}_{12}) & G^{0}{}_{1} = 2R^{3}_{13} \\
G^{0}{}_{2} = -2(R^{3}_{32} + R^{1}_{12}) & G^{0}{}_{3} = 2R^{1}_{31} \\
G^{1}{}_{0} = 2R^{2}_{20} & G^{1}{}_{1} = 2(R^{1}_{32} + R^{3}_{30}) \\
G^{1}{}_{2} = 2R^{1}_{02} & G^{1}{}_{3} = 2(R^{2}_{23} + R^{1}_{03}) \\
G^{2}{}_{0} = 2(R^{1}_{03} + R^{2}_{01}) & G^{2}{}_{1} = 2R^{1}_{13} \\
G^{2}{}_{2} = -2(R^{1}_{01} + R^{3}_{03}) & G^{2}{}_{3} = -2R^{2}_{13} \\
G^{3}{}_{0} = -2R^{1}_{02} & G^{3}{}_{1} = -2(R^{1}_{12} + R^{3}_{10}) \\
G^{3}{}_{2} = 2R^{3}_{02} & G^{3}{}_{3} = -2(R^{1}_{01} + R^{2}_{21}).
\end{array}
\tag{4.42}
$$

This process defines sixteen Einstein components, but we are aware that the symmetric Einstein tensor should have only ten independent components. In order to reveal which of the ten Einstein components given above are independent, we write the first Bianchi identities (4.40a) explicitly:

$$\eta_{\mathbf{ABC}} R^{\mathbf{B}} \wedge S^{\mathbf{C}} = 0$$

$$\Longrightarrow R^{2} \wedge S^{3} - R^{3} \wedge S^{2} = 0$$

$$\Rightarrow R^{2}_{12}(\theta^{1} \wedge \theta^{2} \wedge \theta^{3} \wedge \theta^{0}) + R^{3}_{03}(\theta^{0} \wedge \theta^{3} \wedge \theta^{1} \wedge \theta^{1}) = 0$$

$$\Rightarrow R^{2}_{12} + R^{3}_{03} = 0 \tag{4.43}$$

$$\Longrightarrow R^{3} \wedge S^{1} - R^{1} \wedge S^{3} = 0$$

$$\Rightarrow R^{3}_{01} + R^{3}_{23} - 2R^{1}_{12} = 0 \tag{4.44}$$

$$\Longrightarrow R^{1} \wedge S^{2} - R^{2} \wedge S^{1} = 0$$

$$\Rightarrow R^{2}_{01} + R^{2}_{23} + 2R^{1}_{03} = 0. \tag{4.45}$$

We can immediately see from (4.42) that $G^{0}{}_{3} = -G^{2}{}_{1}$ and $G^{3}{}_{0} = -G^{1}{}_{2}$. When the three identities above are used in (4.42) we reveal the remaining four identities on the Einstein tensor.

$$
\begin{array}{ll}
G^{0}{}_{0} = G^{1}{}_{1} & G^{2}{}_{2} = G^{3}{}_{3} \\
G^{0}{}_{2} = -G^{3}{}_{1} & G^{1}{}_{3} = -G^{2}{}_{0}.
\end{array}
$$

These identities are not unexpected because they may also be obtained using the property that the Einstein tensor is symmetric.
Therefore from the original twelve Einstein components $G^{\alpha}{}_{\beta}$, we have ten independent components, as expected. These are given below:

$$\begin{array}{ll} G^{0}{}_{0} = -2(R^{1}_{23} - R^{2}_{12}) & G^{0}{}_{1} = 2R^{3}_{13} \\ G^{0}{}_{2} = -2(R^{3}_{32} + R^{1}_{12}) & G^{0}{}_{3} = 2R^{1}_{31} \\ G^{1}{}_{0} = 2R^{2}_{20} & G^{1}{}_{2} = 2R^{1}_{02} \\ G^{1}{}_{3} = 2(R^{2}_{23} + R^{1}_{03}) & G^{2}{}_{3} = -2R^{2}_{13} \\ G^{3}{}_{2} = 2R^{3}_{02} & G^{3}{}_{3} = -2(R^{1}_{01} + R^{2}_{21}). \end{array} \tag{4.46}$$

We now redefine the self-dual part of the action (4.27) in terms of the $SO(3)$ basis and curvature. This action was introduced by Samuel, Jacobson and Smolin (see Samuel 1987; Jacobson & Smolin 1989).

$$I = \int R^{\mathbf{A}} \wedge S^{\mathbf{B}} g_{\mathbf{AB}}. \tag{4.47}$$

Although in this section we obtained the Einstein components from (4.41), it is also possible to obtain them from a variation of the action given above.

4.4 Summary

In this chapter we have introduced the Ashtekar approach to canonical analysis. We have shown that if the variable space is extended to allow complex variables then only the self-dual action is required to obtain all the Einstein equations.

Then in the last section we adapted the connection and the curvature so that they are defined by a $SO(3)$ basis, which leads to a new action using these variables. We are now in the position to apply these new variables to the $2+2$ foliation that was introduced earlier.

Chapter 5

Canonical Analysis of $2+2$ Hamiltonian using $SO(3)$ variables

5.1 Introduction

In this chapter we shall use the $SO(3)$ variables outlined in the penultimate section of chapter 3 and apply them to a double null $2+2$ formulation, given in chapter 3. We do this to simplify the constraint algebra we found earlier, by using the Ashtekar approach. This work extends the Lagrangian description of d'Inverno & Vickers (1995) by obtaining the Hamiltonian description and calculating the constraint algebra.

In the first section we introduce a general basis of 1-forms that are suitable for working in a $2+2$ formulation. We then calculate the conditions required to ensure a double null foliation; these are called the slicing conditions. In this section we also introduce the densitised $SO(3)$ basis. These variables can be expressed in terms of the basis of 1-forms, and therefore we have a choice of variables. We will also decompose the connection into a $2+2$ form and introduce the covariant derivative that acts on the SO(3) variables.

After this ground work we will be in a position to calculate the Lagrangian from the action given at the end of the last chapter; this is the topic of section two. In the third section we transform the Lagrangian description to the Hamiltonian description and perform the canonical analysis. From the constraints obtained in this analysis we are able to derive the Einstein equations. This then leads us, in section six, to calculate the first class constraints, and from these the first class algebra. A geometrical interpretation of the first class constraints is then given before we conclude this section with some closing remarks.

Just as in the previous double null analysis we are required to choose which direction we shall take as our evolution direction. In this and later chapters we shall take the evolution direction to be the x^0 direction. As we have stated earlier this choice breaks the symmetry of the double null description.

5.2 $2+2$ tetrad, connection and curvature

We start this work by expressing the general basis of 1-forms that was introduced in d'Inverno & Vickers (1995).

$$\theta^{\mathbf{a}} = \mu^{\mathbf{a}}{}_b dx^b + \alpha^{\mathbf{a}}{}_i (dx^i + s^i{}_b dx^b) \tag{5.1a}$$

$$\theta^{\mathbf{i}} = \nu^{\mathbf{i}}{}_j (dx^j + s^j{}_a dx^a) \tag{5.1b}$$

$\mu^{\mathbf{a}}{}_b$ and $s^i{}_a$ are the lapse and shift. We can see that the four 2×2 matrix variables contain 16 degrees of freedom. These comprise 10 metric and 6 Lorentz freedoms. The dual basis is given by

$$e_{\mathbf{a}} = u_{\mathbf{a}}{}^b \left(\frac{\partial}{\partial x^b} - s^i{}_b \frac{\partial}{\partial x^i}\right) \tag{5.2a}$$

$$e_{\mathbf{i}} = v_{\mathbf{i}}{}^j \frac{\partial}{\partial x^j} + \alpha^{\mathbf{a}}{}_j v^j{}_{\mathbf{i}} \left(u_{\mathbf{a}}{}^b s^j{}_b \frac{\partial}{\partial x^j} - u_{\mathbf{a}}{}^b \frac{\partial}{\partial x^b} \right) \tag{5.2b}$$

where the 2×2 matrices $u_{\mathbf{a}}{}^b$ and $v_{\mathbf{i}}{}^j$ are defined to be inverses of $\mu^{\mathbf{a}}{}_b$ and $\nu^{\mathbf{i}}{}_j$ respectively, so that

$$u_{\mathbf{a}}{}^b \mu^{\mathbf{a}}{}_c = \delta^b_c, \qquad u_{\mathbf{a}}{}^c \mu^{\mathbf{b}}{}_c = \delta^{\mathbf{b}}_{\mathbf{a}}$$

$$v_{\mathbf{i}}{}^j \nu^{\mathbf{i}}{}_k = \delta^j_k, \qquad v_{\mathbf{i}}{}^k \nu^{\mathbf{j}}{}_k = \delta^{\mathbf{j}}_{\mathbf{i}}.$$

At this point we shall greatly simplify all future calculations by working in a adapted frame. This means that we work with a frame where $e_{\mathbf{i}}$ are tangent to $\{S\}$. Our basis (5.1) requires that the alphas vanish, ie. $\alpha^{\alpha}_{i} = 0$. This reduction in the degrees of freedom decreases the number of Lorentz transformations by four to a two parameter subgroup of spin and boost transformations. Unlike the similar calculations in the work by Goldberg *et al.* (1992), choosing an adapted frame does not automatically result in x^1 becoming null, therefore additional constraints are required to ensure x^0 and x^1 are null; these constraints are called the slicing constraints.

The double null slicing condition requires that:

$$\begin{aligned} g^{00} &= g^{\alpha\beta}\theta^{0}_{\alpha}\theta^{0}_{\beta} = 2\mu^{0}{}_{0}\mu^{0}{}_{1} = 0 \\ g^{11} &= g^{\alpha\beta}\theta^{1}_{\alpha}\theta^{1}_{\beta} = 2\mu^{1}{}_{0}\mu^{1}{}_{1} = 0. \end{aligned}$$

The volume form is given by

$$\begin{aligned} V &= -i\theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \\ &= -i\mu\nu dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \end{aligned}$$

which implies that μ, ν are non-zero. Therefore to satisfy the null conditions as well as the condition that $\mu = \mu^{0}{}_{0}\mu^{1}{}_{1} - \mu^{0}{}_{1}\mu^{1}{}_{1}$ is non zero, we require that either

$$\mu^{0}{}_{0} = \mu^{1}{}_{1} = 0 \tag{5.3}$$

or

$$\mu^{0}{}_{1} = \mu^{1}{}_{0} = 0 \tag{5.4}$$

be satisfied. Although in future we shall require (5.4), there has been no loss of generality because a change to the other condition is equivalent to interchanging the coordinates x^0 and x^1.

In the Ashtekar approach we consider in Appendix A, the Lagrangian was a function of the hypersurface connection variables ($\mathcal{A}_{\mu}{}^{\mathrm{ij}}$), the lapse ($\underset{\sim}{N}$), the shift ($N^{\mu}$) and the densitised frame projected onto the hypersurface ($\tilde{E}^{\mu}_{\mathrm{i}}$). These frame variables define the metric on the hypersurface. Therefore in our $2+2$ approach we shall also use connection variables, the lapse $(\mu^{\mathbf{a}}{}_b)$ and the shift $(s^i{}_a)$. It is worth pointing out here that only $s^i{}_0$ are the traditional shifts to the evolution direction. The remaining shift terms $s^i{}_1$ would form part of the three metric. We will see this difference expressed through the roles the constraint equations play, which we will comment on later. At present the induced 2-metric is defined by the variables $\nu^{\mathbf{i}}{}_j$, hence the conformal factor of the two metric is given by ν, the determinant of the variables $\nu^{\mathbf{i}}{}_j$. In order to follow the Ashtekar approach we need to introduce densitised $SO(3)$ variables to replace the variables $\nu^i{}_j$. This is achieved by first expressing the $SO(3)$ basis (4.34) in terms of the 1-form adapted basis variables given at the start of this section.

$$
\begin{aligned}
S^1 &= \tfrac{1}{2}(\theta^{\mathbf{1}} \wedge \theta^{\mathbf{0}} + \theta^{\mathbf{3}} \wedge \theta^{\mathbf{2}}) \\
&= \tfrac{1}{2}[(\mu^{\mathbf{1}}{}_a\mu^{\mathbf{0}}{}_b + \nu^{\mathbf{3}}{}_i s^i{}_a \nu^{\mathbf{2}}{}_j s^j{}_b)dx^a \wedge dx^b - (\nu^{\mathbf{2}}{}_j s^j{}_a \nu^{\mathbf{3}}{}_i - \nu^{\mathbf{3}}{}_j s^j{}_a \nu^{\mathbf{2}}{}_i)dx^a \wedge dx^i \\
S^2 &= \theta^{\mathbf{1}} \wedge \theta^{\mathbf{2}} \\
&= (\mu^{\mathbf{1}}{}_a \nu^{\mathbf{2}}{}_i s^i{}_b)dx^a \wedge dx^b + (\mu^{\mathbf{1}}{}_a \nu^{\mathbf{2}}{}_i)dx^a \wedge dx^i + (\nu^{\mathbf{3}}{}_i \nu^{\mathbf{2}}{}_j)dx^i \wedge dx^j \\
S^3 &= \theta^{\mathbf{3}} \wedge \theta^{\mathbf{0}} \\
&= \nu^{\mathbf{3}}{}_j(dx^j + s^j{}_a dx^a) \wedge (\mu^{\mathbf{0}}{}_b dx^b) \\
&= (\nu^{\mathbf{3}}{}_i s^i{}_a \mu^{\mathbf{0}}{}_b)dx^a \wedge dx^b - (\nu^{\mathbf{3}}{}_i \mu^{\mathbf{0}}{}_a)dx^a \wedge dx^i.
\end{aligned}
\tag{5.5}
$$

Then we introduce the new variables, which are densitised versions of the $S^{\mathbf{A}}$ basis variables.

$$\tilde{\Sigma}_{\mathbf{A}}^{\alpha\beta} = \tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta} S^{\mathbf{B}}_{\gamma\delta} g_{\mathbf{AB}}, \tag{5.6}$$

where $g_{\mathbf{AB}}$ is defined by (4.36). We express the Sigma variables in terms of the tetrad variables using (5.5), to obtain the system of equations:

$$(\tilde{\Sigma}_{\mathbf{1}}^{01}, \tilde{\Sigma}_{\mathbf{2}}^{01}, \tilde{\Sigma}_{\mathbf{3}}^{01}) = (-\nu, 0, 0) \tag{5.7a}$$

$$(\tilde{\Sigma}_{\mathbf{1}}^{23}, \tilde{\Sigma}_{\mathbf{2}}^{23}, \tilde{\Sigma}_{\mathbf{3}}^{23}) = (-\mu - \bar{s}, \mu^{\mathbf{0}}{}_0 s^k{}_1 \nu^{\mathbf{3}}{}_k - \mu^{\mathbf{0}}{}_1 \nu^{\mathbf{3}}{}_k s^k{}_0, \mu^{\mathbf{1}}{}_1 \nu^{\mathbf{2}}{}_k s^k{}_0 - \mu^{\mathbf{1}}{}_0 \nu^{\mathbf{2}}{}_k s^k{}_1) \tag{5.7b}$$

$$(\tilde{\Sigma}_{\mathbf{1}}^{ai}, \tilde{\Sigma}_{\mathbf{2}}^{ai}, \tilde{\Sigma}_{\mathbf{3}}^{ai}) = -\epsilon^{ab}\epsilon^{ij}(\nu^{\mathbf{3}}{}_k s^k{}_b \nu^{\mathbf{2}}{}_j - \nu^{\mathbf{2}}{}_k s^k{}_b \nu^{\mathbf{3}}{}_j, \mu^{\mathbf{0}}{}_b \nu^{\mathbf{3}}{}_j, -\mu^{\mathbf{1}}{}_b \nu^{\mathbf{2}}{}_j). \tag{5.7c}$$

We may consider (5.7c) as a system of equations that allow us to determine $\nu^{\mathbf{i}}{}_j$ in terms of the twelve $\tilde{\Sigma}_{\mathbf{A}}^{ai}$. This indicates that there are eight constraints in this over determined system of equations. Therefore, in order to use these densitised variables to replace the variables $\nu^{\mathbf{i}}{}_{j}$, we require further constraints to be introduced. We may obtain four of these from (5.7c). First,

$$\begin{aligned}\tilde{\Sigma}_{\mathbf{2}}{}^{ai} &= -\epsilon^{ab}\epsilon^{ij}\mu^{\mathbf{0}}{}_{b}\nu^{\mathbf{3}}{}_{j} \\ \Rightarrow \tilde{\Sigma}_{\mathbf{2}}{}^{0i} &= -\epsilon^{ij}\mu^{\mathbf{0}}{}_{1}\nu^{\mathbf{3}}{}_{j}, \qquad \tilde{\Sigma}_{\mathbf{2}}{}^{1i} = \epsilon^{ij}\mu^{\mathbf{0}}{}_{0}\nu^{\mathbf{3}}{}_{j}\end{aligned}$$

and by multiplying the former by $\mu^{\mathbf{0}}{}_{0}$ and the latter by $\mu^{\mathbf{0}}{}_{1}$, we obtain

$$\begin{aligned}\mu^{\mathbf{0}}{}_{0}\tilde{\Sigma}_{\mathbf{2}}{}^{0i} + \mu^{\mathbf{0}}{}_{1}\tilde{\Sigma}_{\mathbf{2}}{}^{1i} &= 0 \\ \Rightarrow C^{i} \equiv \mu^{\mathbf{0}}{}_{a}\tilde{\Sigma}_{\mathbf{2}}{}^{ai} &= 0.\end{aligned} \tag{5.8}$$

Using this method with $\tilde{\Sigma}_{3}{}^{ai}$ we obtain two further constraints

$$\tilde{C}^{i} \equiv \mu^{\mathbf{1}}{}_{a}\tilde{\Sigma}_{\mathbf{3}}{}^{ai} = 0. \tag{5.9}$$

The final four constraints are obtained through manipulating the definitions of $\tilde{\Sigma}_{1}{}^{ai}$.

$$\begin{aligned}\tilde{\Sigma}_{\mathbf{1}}{}^{ai} &= -\epsilon^{ab}\epsilon^{ij}(s^{k}{}_{b}\nu^{\mathbf{3}}{}_{k}\nu^{\mathbf{2}}{}_{j} - s^{k}{}_{b}\nu^{\mathbf{2}}{}_{k}\nu^{\mathbf{3}}{}_{j}) \\ &= \epsilon^{ab}s^{i}{}_{b}\nu \\ \Rightarrow C^{i}{}_{a} &\equiv s^{i}{}_{a}\tilde{\Sigma}_{\mathbf{1}}{}^{01} - \epsilon_{ab}\tilde{\Sigma}_{\mathbf{1}}{}^{bi} = 0\end{aligned} \tag{5.10}$$

The expressions for $\nu^{\mathbf{i}}{}_{j}$ in terms of the Sigmas are given by

$$\begin{aligned}\nu^{\mathbf{2}}{}_{j} &= \mu^{-1}\mu^{\mathbf{0}}{}_{a}\tilde{\Sigma}_{\mathbf{3}}{}^{ai}\epsilon_{ij} \\ \nu^{\mathbf{3}}{}_{j} &= \mu^{-1}\mu^{\mathbf{1}}{}_{a}\tilde{\Sigma}_{\mathbf{2}}{}^{ai}\epsilon_{ij}\end{aligned}$$

which, when combined, gives us

$$\mu\nu = \tilde{\Sigma}_{\mathbf{2}}{}^{ai}\tilde{\Sigma}_{\mathbf{3}}{}^{bj}\epsilon_{ab}\epsilon_{ij}.$$

In 2 + 2 formalism we wish to include the conformal factor of the induced metric, ν, as a variable of the Lagrangian. We therefore require an additional constraint, which we construct from the equation above.

$$\hat{C} \equiv \tilde{\Sigma}_{\mathbf{2}}{}^{ai}\tilde{\Sigma}_{\mathbf{3}}{}^{bj}\epsilon_{ab}\epsilon_{ij} - \mu\nu = 0. \tag{5.11}$$

Note that in the following calculations we shall use (5.7a) to replace ν with the variable $\tilde{\Sigma}_{\mathbf{1}}{}^{01}$.

Before we can derive the 2 + 2 Lagrangian we decompose the $SO(3)$ connection and curvature into a 2 + 2 form. We have already introduced the $SO(3)$ connection 1-form $\Gamma^{\mathbf{A}}$, which decomposes into the 2 + 2 form given below:

$$\Gamma^{\mathbf{A}} = \Gamma^{\mathbf{A}}_{\ \mu} dx^{\mu} = A^{\mathbf{A}}_{\ i} dx^{i} + B^{\mathbf{A}}_{\ a} dx^{a}. \tag{5.12}$$

The curvature 2-forms $R^{\mathbf{A}}$ are defined by

$$\begin{aligned}
R^{\mathbf{A}} &= \mathrm{d}\Gamma^{\mathbf{A}} + \eta^{\mathbf{A}}_{\ \mathbf{BC}}\Gamma^{\mathbf{B}} \wedge \Gamma^{\mathbf{C}} \\
&= A^{\mathbf{A}}_{\ i,\alpha}\, \mathrm{d}x^{\alpha} \wedge \mathrm{d}x^{i} + B^{\mathbf{A}}_{\ a,\alpha}\, \mathrm{d}x^{\alpha} \wedge \mathrm{d}x^{a} + \eta^{\mathbf{A}}_{\ \mathbf{BC}}\Gamma^{\mathbf{B}}_{\ \alpha}\Gamma^{\mathbf{C}}_{\ \beta}\, \mathrm{d}x^{\alpha} \wedge \mathrm{d}x^{\beta} \\
\Rightarrow R^{\mathbf{A}}_{\ ab} &= -B^{\mathbf{A}}_{\ a,b} + B^{\mathbf{A}}_{\ b,a} + 2\eta^{\mathbf{A}}_{\ \mathbf{BC}}B^{\mathbf{B}}_{\ a}B^{\mathbf{C}}_{\ b} && (5.13\text{a}) \\
R^{\mathbf{A}}_{\ ai} &= -B^{\mathbf{A}}_{\ a,i} + A^{\mathbf{A}}_{\ i,a} + 2\eta^{\mathbf{A}}_{\ \mathbf{BC}}B^{\mathbf{B}}_{\ a}A^{\mathbf{C}}_{\ i} && (5.13\text{b}) \\
R^{\mathbf{A}}_{\ ij} &= -A^{\mathbf{A}}_{\ i,j} + A^{\mathbf{A}}_{\ j,i} + 2\eta^{\mathbf{A}}_{\ \mathbf{BC}}A^{\mathbf{B}}_{\ i}A^{\mathbf{C}}_{\ j}. && (5.13\text{c})
\end{aligned}$$

We will find it useful to introduce two differential operators which are the restrictions of the four-dimensional self-dual SO(3) covariant derivative to the spaces S and T. Their actions on $SO(3)$ valued functions $f^{\mathbf{A}}$ and $f_{\mathbf{A}}$ are given by

$$\begin{aligned}
D_a f^{\mathbf{A}} &= f^{\mathbf{A}}_{\ ,a} + 2\eta^{\mathbf{A}}_{\ \mathbf{BC}}B^{\mathbf{B}}_{\ a}f^{\mathbf{C}} \qquad & D_i f^{\mathbf{A}} &= f^{\mathbf{A}}_{\ ,i} + 2\eta^{\mathbf{A}}_{\ \mathbf{BC}}A^{\mathbf{B}}_{\ i}f^{\mathbf{C}} && (5.14\text{a}) \\
D_a f_{\mathbf{A}} &= f_{\mathbf{A},a} - 2\eta^{\mathbf{C}}_{\ \mathbf{BA}}B^{\mathbf{B}}_{\ a}f_{\mathbf{C}} \qquad & D_i f_{\mathbf{A}} &= f_{\mathbf{A},i} - 2\eta^{\mathbf{C}}_{\ \mathbf{BA}}A^{\mathbf{B}}_{\ i}f_{\mathbf{C}}. && (5.14\text{b})
\end{aligned}$$

We may use these differential operators to express the curvature 2-forms:

$$R^{\mathbf{A}}{}_{ab} = B^{\mathbf{A}}_{b,a} - D_b B^{\mathbf{A}}_{a} \tag{5.15a}$$

$$R^{\mathbf{A}}{}_{ai} = A^{\mathbf{A}}_{i,a} - D_i B^{\mathbf{A}}_{a} \tag{5.15b}$$

$$R^{\mathbf{A}}{}_{ij} = A^{\mathbf{A}}_{j,i} - D_j A^{\mathbf{A}}_{i}. \tag{5.15c}$$

5.3 Derivation of the Lagrangian

We are now in a position to calculate the Lagrangian. We start from the action introduced in the last chapter (4.47):

$$I = \int R^{\mathbf{A}} \wedge S^{\mathbf{B}} g_{\mathbf{AB}}. \tag{5.16}$$

From this action we are able to give the Lagrangian in terms of our variables as:

$$L = \int \left(R^{\mathbf{A}}{}_{01} \tilde{\Sigma}_{\mathbf{A}}{}^{01} + R^{\mathbf{A}}{}_{23} \tilde{\Sigma}_{\mathbf{A}}{}^{23} + R^{\mathbf{A}}{}_{ai} \tilde{\Sigma}_{\mathbf{A}}{}^{ai} \right) \mathrm{d}^4 x.$$

The curvature terms $R^{\mathbf{A}}{}_{01}$ and $R^{\mathbf{A}}{}_{0i}$ contain time derivatives of the connection. Therefore we write these curvature terms explicitly so all time derivatives can easily be seen from the Lagrangian. We then replace $\tilde{\Sigma}_{\mathbf{A}}{}^{23}$ with the frame variables, using (5.7) to obtain:

$$\begin{aligned}
\tilde{\Sigma}_{\mathbf{1}}{}^{23} &= -\mu - s_0^k \left(s_1^l \nu^{\mathbf{3}}{}_l \nu^{\mathbf{2}}{}_k - s_1^l \nu^{\mathbf{2}}{}_l \nu^{\mathbf{3}}{}_k \right) \\
&\Rightarrow R^{\mathbf{1}}{}_{23} \tilde{\Sigma}_{\mathbf{1}}{}^{23} = -R^{\mathbf{1}}{}_{23} \mu - s_0^i \tilde{\Sigma}_{\mathbf{1}}{}^{0j} R^{\mathbf{1}}{}_{23} \\
\tilde{\Sigma}_{\mathbf{2}}{}^{23} &= -s_0^k \nu^{\mathbf{3}}{}_k \mu^{\mathbf{0}}{}_1 + s_1^k \nu^{\mathbf{3}}{}_k \mu^{\mathbf{0}}{}_1 \\
&\Rightarrow R^{\mathbf{2}}{}_{23} \tilde{\Sigma}_{\mathbf{2}}{}^{23} = -s_0^i \tilde{\Sigma}_{\mathbf{2}}{}^{0j} R^{\mathbf{2}}{}_{ij} - s_1^i \tilde{\Sigma}_{\mathbf{2}}{}^{1j} R^{\mathbf{2}}{}_{ij} \\
\tilde{\Sigma}_{\mathbf{3}}{}^{23} &= s_0^k \nu^{\mathbf{2}}{}_k \mu^{\mathbf{1}}{}_1 - s_1^k \nu^{\mathbf{2}}{}_k \mu^{\mathbf{1}}{}_0 \\
&\Rightarrow R^{\mathbf{3}}{}_{23} \tilde{\Sigma}_{\mathbf{3}}{}^{23} = -s_0^i \tilde{\Sigma}_{\mathbf{3}}{}^{0j} R^{\mathbf{3}}{}_{ij} - s_1^i \tilde{\Sigma}_{\mathbf{3}}{}^{1j} R^{\mathbf{3}}{}_{ij}.
\end{aligned}$$

Using these identities we may write the Lagrangian density given below:

$$\begin{aligned}
\mathcal{L} = \tilde{\Sigma}_{\mathbf{A}}{}^{0i} A^{\mathbf{A}}_{i,0} + \tilde{\Sigma}_{\mathbf{A}}{}^{01} B^{\mathbf{A}}_{1,0} + B^{\mathbf{A}}_0 D_1 \tilde{\Sigma}_{\mathbf{A}}{}^{01} + B^{\mathbf{A}}_0 D_i \tilde{\Sigma}_{\mathbf{A}}{}^{0i} - \mu R^{\mathbf{1}}{}_{23} - s_0^i R^{\mathbf{A}}{}_{ij} \tilde{\Sigma}_{\mathbf{A}}{}^{0j} \\
- s_1^i (R^{\mathbf{2}}{}_{ij} \tilde{\Sigma}_{\mathbf{2}}{}^{1j} + R^{\mathbf{3}}{}_{ij} \tilde{\Sigma}_{\mathbf{3}}{}^{1j}) + R^{\mathbf{A}}{}_{1i} \tilde{\Sigma}_{\mathbf{A}}{}^{1i}.
\end{aligned} \tag{5.17}$$

Then we introduce the primary constraints (5.8, 5.9, 5.10, 5.11) with the use of Lagrangian multipliers to obtain the primary Lagrangian

$$\begin{aligned} L = \int \Big(& \tilde{\Sigma}_{\mathbf{A}}{}^{0i} A^{\mathbf{A}}_{i,0} + \tilde{\Sigma}_{\mathbf{A}}{}^{01} B^{\mathbf{A}}_{1,0} + B^{\mathbf{A}}_{0} D_1 \tilde{\Sigma}_{\mathbf{A}}{}^{01} + B^{\mathbf{A}}_{0} D_i \tilde{\Sigma}_{\mathbf{A}}{}^{0i} - \mu R^{\mathbf{1}}{}_{23} - s^i_0 R^{\mathbf{A}}{}_{ij} \tilde{\Sigma}_{\mathbf{A}}{}^{0j} \\ & - s^i_1 (R^{\mathbf{2}}{}_{ij} \tilde{\Sigma}_{\mathbf{2}}{}^{1j} + R^{\mathbf{3}}{}_{ij} \tilde{\Sigma}_{\mathbf{3}}{}^{1j}) + R^{\mathbf{A}}{}_{1i} \tilde{\Sigma}_{\mathbf{A}}{}^{1i} + \lambda_i C^i + \tilde{\lambda}_i \tilde{C}^i + \hat{\lambda} \hat{C} + \lambda^a_i C^i_a \\ & + \rho (\mu^0{}_1)^2 + \tilde{\rho} (\mu^1{}_0)^2 + \tau^2 \Sigma_{\mathbf{2}}{}^{01} + \tau^3 \Sigma_{\mathbf{3}}{}^{01} \Big) \, \mathrm{d}^4 x, \end{aligned} \tag{5.18}$$

where we have denoted:

$$C^i_a = s^i_a \tilde{\Sigma}_{\mathbf{1}}{}^{01} - \epsilon_{ab} \tilde{\Sigma}_{\mathbf{1}}{}^{bi} \tag{5.19a}$$

$$C^i = \mu^{\mathbf{0}}{}_a \tilde{\Sigma}_{\mathbf{2}}{}^{ai} \tag{5.19b}$$

$$\tilde{C}^i = \mu^{\mathbf{1}}{}_a \tilde{\Sigma}_{\mathbf{3}}{}^{ai} \tag{5.19c}$$

$$\hat{C} = \tilde{\Sigma}_{\mathbf{2}}{}^{ai} \tilde{\Sigma}_{\mathbf{3}}{}^{bj} \epsilon_{ab} \epsilon_{ij} + \mu \tilde{\Sigma}_{\mathbf{1}}{}^{01}. \tag{5.19d}$$

It is worth noting that if one imposes the double null slicing condition the constraints C^i and $\tilde{C}^i$ become

$$C^i = \tilde{\Sigma}_{\mathbf{2}}{}^{0i} \qquad\qquad \tilde{C}^i = \tilde{\Sigma}_{\mathbf{3}}{}^{1i}. \tag{5.20}$$

At this point we may stay in the Lagrangian description and calculate the Einstein equations and the structure equations through variation of the Lagrangian with respect to the different variables and multipliers (see d'Inverno & Vickers 1995). We do not pursue this here as our focus is on performing the canonical analysis on the Hamiltonian description, and through this we also obtain the Einstein equations.

5.4 Hamiltonian description

The Lagrangian density is of the form $\mathcal{L} = p^\lambda \dot{q}_\lambda - \mathcal{H}$, and therefore we can see directly that the canonical variables are $A^{\mathbf{A}}_i$ and $B^{\mathbf{A}}_1$, and have the respective momenta $\tilde{\Sigma}_{\mathbf{A}}{}^{0i}$ and $\tilde{\Sigma}_{\mathbf{A}}{}^{01}$. We can simply read off the Hamiltonian to give:

$$H = \int \mu R^{\mathbf{1}}{}_{23} + s^i_0 R^{\mathbf{A}}{}_{ij} \tilde{\Sigma}_{\mathbf{A}}{}^{0j} + s^i_1 (R^{\mathbf{2}}{}_{ij} \tilde{\Sigma}_{\mathbf{2}}{}^{1j} + R^{\mathbf{3}}{}_{ij} \tilde{\Sigma}_{\mathbf{3}}{}^{1j}) - R^{\mathbf{A}}{}_{1i} \tilde{\Sigma}_{\mathbf{A}}{}^{1i} - B^{\mathbf{A}}_0 (D_1 \Sigma_{\mathbf{A}}{}^{\mathbf{01}} + D_i \Sigma_{\mathbf{A}}{}^{0i})$$
$$+\lambda_i C^i + \tilde{\lambda}_i \tilde{C}^i + \hat{\lambda} \hat{C} + \lambda^a_i C^i_a + \rho(\mu^0{}_1)^2 + \tilde{\rho}(\mu^1{}_0)^2 + \tau^{\mathbf{2}} \tilde{\Sigma}_{\mathbf{2}}{}^{01} + \tau^{\mathbf{3}} \tilde{\Sigma}_{\mathbf{3}}{}^{01} \tag{5.21}$$

where the canonical Poisson brackets are:

$$\left\{ A^{\mathbf{A}}_i(x), \tilde{\Sigma}_{\mathbf{B}}{}^{0j}(y) \right\} = \delta^{\mathbf{A}}_{\mathbf{B}} \delta^j_i \delta(x, \tilde{y}) \tag{5.22a}$$
$$\left\{ B^{\mathbf{A}}_1(x), \tilde{\Sigma}_{\mathbf{B}}{}^{01}(y) \right\} = \delta^{\mathbf{A}}_{\mathbf{B}} \delta(x, \tilde{y}). \tag{5.22b}$$

In this approach we will use the shortcut method discussed in the second chapter. In the shortcut method we treat cyclic variables as multipliers. Therefore, in the Hamiltonian above, we consider the variables $\mu^{\mathrm{a}}{}_b, s^i{}_a, \tilde{\Sigma}_{\mathbf{A}}{}^{1i}, B^{\mathbf{A}}_0$ to be multipliers. It is worth noting at this point that some of the constraints that were introduced into the Lagrangian are not constraints on the canonical variables, but constraints on the cyclic variables. Therefore they reduce to multiplier equations, for example $\tilde{C}^i = \mu^{\mathbf{1}}{}_a \tilde{\Sigma}_{\mathbf{3}}{}^{ai}$. As a result of the original thirteen constraints only four:

$$C^i = 0, \qquad \tilde{\Sigma}_{\mathbf{2}}{}^{01} = 0, \qquad \tilde{\Sigma}_{\mathbf{3}}{}^{01} = 0, \tag{5.23}$$

are actually primary constraints.

We now start the constraint analysis algorithm by varying the Hamiltonian with respect to the multipliers to obtain the primary constraints. Variation with respect to the cyclic variables leads to the equations:

$$\frac{\delta H}{\delta \mu^{\mathbf{0}}{}_{0}} = -\mu^{\mathbf{1}}{}_{1} R^{\mathbf{1}}{}_{23} - \mu^{\mathbf{1}}{}_{1} \tilde{\Sigma}_{\mathbf{1}}{}^{01} \hat{\lambda} - \lambda_i \tilde{\Sigma}_{\mathbf{2}}{}^{0i} \tag{5.24a}$$

$$\frac{\delta H}{\delta \mu^{\mathbf{1}}{}_{1}} = -\mu^{\mathbf{0}}{}_{0} R^{\mathbf{1}}{}_{23} - \mu^{\mathbf{0}}{}_{0} \tilde{\Sigma}_{\mathbf{1}}{}^{01} \hat{\lambda} - \tilde{\lambda}_i \tilde{\Sigma}_{\mathbf{3}}{}^{1i} \tag{5.24b}$$

$$\frac{\delta H}{\delta \mu^{\mathbf{0}}{}_{1}} = \mu^{\mathbf{1}}{}_{0} R^{\mathbf{1}}{}_{23} + \mu^{\mathbf{1}}{}_{0} \tilde{\Sigma}_{\mathbf{1}}{}^{01} \hat{\lambda} - \lambda_i \tilde{\Sigma}_{\mathbf{2}}{}^{1i} \tag{5.24c}$$

$$\frac{\delta H}{\delta \mu^{\mathbf{1}}{}_{0}} = \mu^{\mathbf{0}}{}_{1} R^{\mathbf{1}}{}_{23} + \mu^{\mathbf{0}}{}_{1} \tilde{\Sigma}_{\mathbf{1}}{}^{01} \hat{\lambda} - \tilde{\lambda}_i \tilde{\Sigma}_{\mathbf{3}}{}^{0i} \tag{5.24d}$$

$$\frac{\delta H}{\delta s^{i}{}_{0}} = R^{\mathbf{A}}{}_{ij} \tilde{\Sigma}_{\mathbf{A}}{}^{0j} + \lambda_i^0 \tilde{\Sigma}_{\mathbf{1}}{}^{01} \tag{5.25a}$$

$$\frac{\delta H}{\delta s^{i}{}_{1}} = R^{\mathbf{2}}{}_{ij} \tilde{\Sigma}_{\mathbf{2}}{}^{1j} + R^{\mathbf{3}}{}_{ij} \tilde{\Sigma}_{\mathbf{3}}{}^{1j} + \lambda_i^1 \tilde{\Sigma}_{\mathbf{1}}{}^{01} \tag{5.25b}$$

$$\frac{\delta H}{\delta \tilde{\Sigma}_{\mathbf{1}}{}^{1p}} = R^{\mathbf{1}}{}_{1p} + \lambda_p^0 \tag{5.26a}$$

$$\frac{\delta H}{\delta \tilde{\Sigma}_{\mathbf{2}}{}^{1p}} = R^{\mathbf{2}}{}_{1p} - R^{\mathbf{2}}{}_{jp} s_1^j - \lambda_p \mu^{\mathbf{0}}{}_{1} + \hat{\lambda} \tilde{\Sigma}_{\mathbf{3}}{}^{0j} \epsilon_{pj} \tag{5.26b}$$

$$\frac{\delta H}{\delta \tilde{\Sigma}_{\mathbf{3}}{}^{1p}} = R^{\mathbf{3}}{}_{1p} - R^{\mathbf{3}}{}_{jp} s_1^j - \tilde{\lambda}_p \mu^{\mathbf{1}}{}_{1} + \hat{\lambda} \tilde{\Sigma}_{\mathbf{2}}{}^{0j} \epsilon_{pj} \tag{5.26c}$$

$$\frac{\delta H}{\delta B_0^{\mathbf{A}}} = D_1 \tilde{\Sigma}_{\mathbf{A}}{}^{01} + D_i \tilde{\Sigma}_{\mathbf{A}}{}^{0i}. \tag{5.27}$$

We now propagate the primary constraints (5.23) using $\dot{Z} = \{Z, H\}$:

$$\dot{C}^i = \mu^{\mathbf{0}}{}_{0} \dot{\tilde{\Sigma}}_{\mathbf{2}}{}^{0i} \tag{5.28}$$

$$\dot{\tilde{\Sigma}}_{\mathbf{2}}{}^{01} = \tilde{\Sigma}_{\mathbf{2}}{}^{1i}{}_{,i} + A_i^{\mathbf{3}}\tilde{\Sigma}_{\mathbf{1}}{}^{1i} + 2A_i^{\mathbf{1}}\tilde{\Sigma}_{\mathbf{2}}{}^{1i} + B_0^{\mathbf{3}}\tilde{\Sigma}_{\mathbf{1}}{}^{01} + B_0^{\mathbf{1}}\tilde{\Sigma}_{\mathbf{2}}{}^{01} \tag{5.29a}$$

$$\dot{\tilde{\Sigma}}_{\mathbf{3}}{}^{01} = \tilde{\Sigma}_{\mathbf{3}}{}^{1i}{}_{,i} - A_i^{\mathbf{2}}\tilde{\Sigma}_{\mathbf{1}}{}^{1i} - 2A_i^{\mathbf{1}}\tilde{\Sigma}_{\mathbf{3}}{}^{1i} - B_0^{\mathbf{2}}\tilde{\Sigma}_{\mathbf{1}}{}^{01} - B_0^{\mathbf{1}}\tilde{\Sigma}_{\mathbf{3}}{}^{01}. \tag{5.29b}$$

We must now check which of the above equations are secondary equations and which define multipliers. We first see that (5.25b) defines the multipliers $\lambda_p^1 = -(R^{\mathbf{2}}{}_{pj}\tilde{\Sigma}_{\mathbf{2}}{}^{1j} + R^{\mathbf{3}}_{pj}\tilde{\Sigma}_{\mathbf{3}}{}^{1j})/\tilde{\Sigma}_{\mathbf{1}}{}^{01}$. Equation (5.24a) defines $\hat{\lambda}$, $\hat{\lambda} \approx -R^{\mathbf{1}}{}_{23}/\tilde{\Sigma}_{\mathbf{1}}{}^{01}$. If this is substituted into (5.24b) then it becomes weakly zero. Also, after substituting $\hat{\lambda}$ into equation (5.24c), the multiplier equation $\lambda_i\tilde{\Sigma}_{\mathbf{2}}{}^{1i} \approx 0$ is obtained. We use (5.26a) to define the multipliers $\lambda_p^0 = -R^{\mathbf{1}}{}_{1p}$, and (5.26c) to define $\mu^1{}_1\lambda_p = R^{\mathbf{3}}{}_{1p} - R^{\mathbf{3}}{}_{ip}s_1^i + R^{\mathbf{1}}{}_{ip}\tilde{\Sigma}_{\mathbf{2}}{}^{0i}/\tilde{\Sigma}_{\mathbf{1}}{}^{01}$. Equations (5.28) define the cyclic variables $\tilde{\Sigma}_{\mathbf{2}}{}^{1i}$, while the final equations (5.29a) and (5.29b) define $B_0^{\mathbf{2}}$ and $B_0^{\mathbf{3}}$. This leaves us with eight secondary constraints (5.24d, 5.27, 5.26b, 5.27), which can be written:

$$\frac{\delta H}{\delta \mu_0^1} \approx \tilde{\Sigma}_{\mathbf{3}}{}^{0p}\left(R^{\mathbf{3}}{}_{1p}\tilde{\Sigma}_{\mathbf{1}}{}^{01} + R^{\mathbf{3}}{}_{ip}\tilde{\Sigma}_{\mathbf{1}}{}^{0i} + R^{\mathbf{1}}{}_{ip}\tilde{\Sigma}_{\mathbf{2}}{}^{0i}\right) \tag{5.30a}$$

$$\frac{\delta H}{\delta \tilde{\Sigma}_{\mathbf{2}}{}^{1p}} \approx R^{\mathbf{2}}{}_{1p}\tilde{\Sigma}_{\mathbf{1}}{}^{01} + R^{\mathbf{2}}{}_{ip}\tilde{\Sigma}_{\mathbf{1}}{}^{0i} + R^{\mathbf{1}}{}_{ip}\tilde{\Sigma}_{\mathbf{3}}{}^{0i} \tag{5.30b}$$

$$\frac{\delta H}{\delta s_0^p} \approx -R^{\mathbf{A}}{}_{pj}\tilde{\Sigma}_{\mathbf{A}}{}^{0j} + R^{\mathbf{A}}{}_{1p}\Sigma_A{}^{01} \tag{5.30c}$$

$$\frac{\delta H}{\delta B_0^{\mathbf{A}}} \approx D_1\tilde{\Sigma}_{\mathbf{A}}{}^{01} + D_i\tilde{\Sigma}_{\mathbf{A}}{}^{0i}. \tag{5.30d}$$

Therefore at this point we have a phase space of 18 variables, with 4 primary constraints and 8 secondary constraints. We now propagate the secondary constraints to check for any tertiary constraints. We will show in the next section that the first five equations given by (5.30) define the Einstein constraint equations and are therefore automatically preserved by the Bianchi identities. When we calculate $\ddot{B}_0^{\mathbf{1}}$ we find that it is identically zero on the reduced phase space, whereas $\ddot{B}_0^{\mathbf{2}}$ and $\ddot{B}_0^{\mathbf{3}}$ define the multipliers $\tau^{\mathbf{2}}$ and $\tau^{\mathbf{3}}$.

Now that we have completed the constraint analysis we shall give the equations of motion:

$$\dot{A}_p^{\mathbf{1}} = D_p B_0^{\mathbf{1}} + R^{\mathbf{1}}{}_{ip} s_0^i - R^{\mathbf{2}}{}_{pj} \tilde{\Sigma}_{\mathbf{2}}{}^{1j} \left(\tilde{\Sigma}_{\mathbf{1}}{}^{01}\right)^{-1} \tag{5.31a}$$

$$\dot{A}_p^{\mathbf{2}} = D_p B_0^{\mathbf{2}} + R^{\mathbf{2}}{}_{ip} s_0^i + \mu^0{}_0 \lambda_p - R^{\mathbf{1}}{}_{pj} \tilde{\Sigma}_{\mathbf{3}}{}^{1j} \left(\tilde{\Sigma}_{\mathbf{1}}{}^{01}\right)^{-1} \tag{5.31b}$$

$$\dot{A}_p^{\mathbf{3}} = D_p B_0^{\mathbf{3}} + R^{\mathbf{3}}{}_{ip} s_0^i - R^{\mathbf{1}}{}_{pj} \tilde{\Sigma}_{\mathbf{2}}{}^{1j} \left(\tilde{\Sigma}_{\mathbf{1}}{}^{01}\right)^{-1} \tag{5.31c}$$

$$\dot{B}_1^{\mathbf{1}} = D_1 B_0^{\mathbf{1}} + \mu \hat{\lambda} + \lambda_i^a s_a^i \tag{5.32a}$$

$$\dot{B}_1^{\mathbf{2}} = D_1 B_0^{\mathbf{2}} + \tau^{\mathbf{2}} \tag{5.32b}$$

$$\dot{B}_1^{\mathbf{3}} = D_1 B_0^{\mathbf{3}} + \tau^{\mathbf{3}} \tag{5.32c}$$

$$\dot{\tilde{\Sigma}}_{\mathbf{1}}{}^{0i} = 2D_j \left(\tilde{\Sigma}_{\mathbf{1}}{}^{a[i} s_a^{j]}\right) - D_1(\tilde{\Sigma}_{\mathbf{1}}{}^{1i}) + \epsilon^{ij}(\mu - s\tilde{\Sigma}_{\mathbf{1}}{}^{01})_{,j} + 2\eta^{\mathbf{C}}{}_{\mathbf{B1}} B_0^{\mathbf{B}} \tilde{\Sigma}_{\mathbf{C}}^{0i} \tag{5.33a}$$

$$\dot{\tilde{\Sigma}}_{\mathbf{2}}{}^{0i} = 2D_j \left(\tilde{\Sigma}_{\mathbf{2}}{}^{a[i} s_a^{j]}\right) - D_1(\tilde{\Sigma}_{\mathbf{2}}{}^{1i}) + \epsilon^{ij} A_j^{\mathbf{3}}(\mu - s\tilde{\Sigma}_{\mathbf{1}}{}^{01}) + 2\eta^{\mathbf{C}}{}_{\mathbf{B2}} B_0^{\mathbf{B}} \tilde{\Sigma}_{\mathbf{C}}^{0i} \tag{5.33b}$$

$$\dot{\tilde{\Sigma}}_{\mathbf{3}}{}^{0i} = 2D_j \left(\tilde{\Sigma}_{\mathbf{3}}{}^{a[i} s_a^{j]}\right) - D_1(\tilde{\Sigma}_{\mathbf{3}}{}^{1i}) + \epsilon^{ij} A_j^{\mathbf{2}}(\mu - s\tilde{\Sigma}_{\mathbf{1}}{}^{01}) + 2\eta^{\mathbf{C}}{}_{\mathbf{B3}} B_0^{\mathbf{B}} \tilde{\Sigma}_{\mathbf{C}}^{0i} \tag{5.33c}$$

$$\dot{\tilde{\Sigma}}_{\mathbf{A}}{}^{01} = D_i \tilde{\Sigma}_{\mathbf{A}}{}^{1i} + 2\eta^{\mathbf{C}}{}_{\mathbf{BA}} B_0^{\mathbf{B}} \tilde{\Sigma}_{\mathbf{C}}{}^{01}. \tag{5.34}$$

5.5 Einstein equations

We now show that the equations which we have obtained so far contain the ten Einstein equations (4.46). In order to do this we first represent the Einstein equations in terms of the variables used in the Hamiltonian description. As an example of the method an explicit calculation is given below:

$$
\begin{aligned}
G^{\mathbf{1}}{}_{\mathbf{3}} &= 2(R^{\mathbf{2}}{}_{\mathbf{23}} + R^{\mathbf{1}}{}_{\mathbf{03}}) \\
&= 2\left(e^{\alpha}_{\mathbf{2}} e^{\beta}_{\mathbf{3}} R^{\mathbf{2}}{}_{\alpha\beta} + e^{\alpha}_{\mathbf{0}} e^{\beta}_{\mathbf{3}} R^{\mathbf{1}}{}_{\alpha\beta}\right) \\
&= 2\left(v^i_{\mathbf{2}} v^j_{\mathbf{3}} R^{\mathbf{2}}{}_{ij} + u^a_{\mathbf{0}} v^j_{\mathbf{3}} R^{\mathbf{1}}{}_{aj} - u^a_{\mathbf{0}} s^i_a v^j_{\mathbf{3}} R^{\mathbf{1}}{}_{ij}\right) \\
&\approx 2\left(v^i_{\mathbf{2}} v^j_{\mathbf{3}} R^{\mathbf{2}}{}_{ij} - (R^{\mathbf{1}}{}_{0j} \tilde{\Sigma}_{\mathbf{3}}{}^{0j} - R^{\mathbf{1}}{}_{ij} s^i_0 \tilde{\Sigma}_{\mathbf{3}}{}^{0j}) uv\right) \\
\tilde{\Sigma}_{\mathbf{1}}{}^{01} G^{\mathbf{1}}{}_{\mathbf{3}} &\approx 2\left(R^{\mathbf{2}}{}_{ij} \tilde{\Sigma}_{\mathbf{2}}{}^{1i} - R^{\mathbf{1}}{}_{0j} \tilde{\Sigma}_{\mathbf{1}}{}^{01} + R^{\mathbf{1}}{}_{ij} \tilde{\Sigma}_{\mathbf{1}}{}^{1i}\right) \tilde{\Sigma}_{\mathbf{3}}{}^{0j} uv. \qquad (5.35)
\end{aligned}
$$

After performing similar calculations we obtain the system of equations given below. First the constraint equations:

$$\tilde{\Sigma}_{\mathbf{1}}{}^{01} G^{\mathbf{0}}{}_{\mathbf{0}} \approx 2uv\left(R^{\mathbf{2}}{}_{1j} \tilde{\Sigma}_{\mathbf{1}}{}^{01} + R^{\mathbf{2}}{}_{ij} \tilde{\Sigma}_{\mathbf{1}}{}^{0i} + R^{\mathbf{1}}{}_{ij} \tilde{\Sigma}_{\mathbf{3}}{}^{0i}\right) \tilde{\Sigma}_{\mathbf{2}}{}^{1j} \qquad (5.36a)$$

$$\tilde{\Sigma}_{\mathbf{1}}{}^{01} G^{\mathbf{0}}{}_{\mathbf{1}} \approx -2(u^1{}_1)^2 v\left(R^{\mathbf{3}}{}_{1j} \tilde{\Sigma}_{\mathbf{1}}{}^{01} + R^{\mathbf{3}}{}_{ij} \tilde{\Sigma}_{\mathbf{1}}{}^{0i}\right) \tilde{\Sigma}_{\mathbf{3}}{}^{0j} \qquad (5.36b)$$

$$\tilde{\Sigma}_{\mathbf{1}}{}^{01} G^{\mathbf{0}}{}_{\mathbf{2}} \approx -2uv\left(R^{\mathbf{1}}{}_{1j} \tilde{\Sigma}_{\mathbf{1}}{}^{01} + R^{\mathbf{3}}{}_{ij} \tilde{\Sigma}_{\mathbf{3}}{}^{0i} + R^{\mathbf{1}}{}_{ij} \tilde{\Sigma}_{\mathbf{1}}{}^{0i}\right) \tilde{\Sigma}_{\mathbf{2}}{}^{1j} \qquad (5.36c)$$

$$\tilde{\Sigma}_{\mathbf{1}}{}^{01} G^{\mathbf{0}}{}_{\mathbf{3}} \approx -2(u^1{}_1)^2 v\left(R^{\mathbf{1}}{}_{1j} \tilde{\Sigma}_{\mathbf{1}}{}^{01} + R^{\mathbf{1}}{}_{ij} \tilde{\Sigma}_{\mathbf{1}}{}^{0i}\right) \tilde{\Sigma}_{\mathbf{3}}{}^{0j} \qquad (5.36d)$$

$$\tilde{\Sigma}_{\mathbf{1}}{}^{01} G^{\mathbf{2}}{}_{\mathbf{3}} \approx -2(u^1{}_1)^2 v\left(R^{\mathbf{2}}{}_{1j} \tilde{\Sigma}_{\mathbf{1}}{}^{01} + R^{\mathbf{2}}{}_{ij} \tilde{\Sigma}_{\mathbf{1}}{}^{0i}\right) \tilde{\Sigma}_{\mathbf{3}}{}^{0j} \qquad (5.36e)$$

$$\qquad (5.36f)$$

and then the evolution equations:

$$\tilde{\Sigma}_{\mathbf{1}}{}^{01} G^{\mathbf{1}}{}_{\mathbf{0}} \approx -2(u^0{}_0)^2 v\left(R^{\mathbf{2}}{}_{0j} \tilde{\Sigma}_{\mathbf{1}}{}^{01} - R^{\mathbf{2}}{}_{ij} \tilde{\Sigma}_{\mathbf{1}}{}^{1i}\right) \tilde{\Sigma}_{\mathbf{2}}{}^{1j} \qquad (5.36g)$$

$$\tilde{\Sigma}_{\mathbf{1}}{}^{01} G^{\mathbf{1}}{}_{\mathbf{2}} \approx -2(u^0{}_0)^2 v\left(R^{\mathbf{1}}{}_{0j} \tilde{\Sigma}_{\mathbf{1}}{}^{01} - R^{\mathbf{1}}{}_{ij} \tilde{\Sigma}_{\mathbf{1}}{}^{1i}\right) \tilde{\Sigma}_{\mathbf{2}}{}^{1j} \qquad (5.36h)$$

$$\tilde{\Sigma}_{\mathbf{1}}{}^{01} G^{\mathbf{1}}{}_{\mathbf{3}} \approx -2uv\left(R^{\mathbf{1}}{}_{0j} \tilde{\Sigma}_{\mathbf{1}}{}^{01} - R^{\mathbf{1}}{}_{ij} \tilde{\Sigma}_{\mathbf{1}}{}^{1i} - R^{\mathbf{2}}{}_{ij} \tilde{\Sigma}_{\mathbf{2}}{}^{1i}\right) \tilde{\Sigma}_{\mathbf{3}}{}^{0j} \qquad (5.36i)$$

$$\tilde{\Sigma}_{\mathbf{1}}{}^{01} G^{\mathbf{3}}{}_{\mathbf{2}} \approx -2uv\left(R^{\mathbf{3}}{}_{0j} \tilde{\Sigma}_{\mathbf{1}}{}^{01} - R^{\mathbf{3}}{}_{ij} \tilde{\Sigma}_{\mathbf{1}}{}^{1i}\right) \tilde{\Sigma}_{\mathbf{2}}{}^{1j} \qquad (5.36j)$$

$$
\begin{aligned}
\tilde{\Sigma}_{\mathbf{1}}{}^{01} G^{\mathbf{3}}{}_{\mathbf{3}} \approx 2uv\Big[&\left(R^{\mathbf{1}}{}_{0i} \tilde{\Sigma}_{\mathbf{1}}{}^{01} + R^{\mathbf{1}}{}_{ij} \tilde{\Sigma}_{\mathbf{1}}{}^{1j} + R^{\mathbf{2}}{}_{ij} \tilde{\Sigma}_{\mathbf{2}}{}^{1j}\right) \tilde{\Sigma}_{\mathbf{1}}{}^{0i} \\
&+ \left(R^{\mathbf{2}}{}_{1i} \tilde{\Sigma}_{\mathbf{2}}{}^{1i} - R^{\mathbf{1}}{}_{1i} \tilde{\Sigma}_{\mathbf{1}}{}^{1i} + R^{\mathbf{1}}{}_{01} \tilde{\Sigma}_{\mathbf{1}}{}^{01}\right) \tilde{\Sigma}_{\mathbf{1}}{}^{01}\Big]. \qquad (5.36k)
\end{aligned}
$$

We can now see straight away that the first five equations are determined by the secondary constraints in the following manner:

$$\tilde{\Sigma}_{\mathbf{1}}{}^{01}G^{\mathbf{0}}{}_{\mathbf{0}} = 2uv\tilde{\Sigma}_{\mathbf{2}}{}^{1i}\frac{\delta H}{\delta\tilde{\Sigma}_{\mathbf{2}}{}^{1i}} \tag{5.37a}$$

$$\tilde{\Sigma}_{\mathbf{1}}{}^{01}G^{\mathbf{0}}{}_{\mathbf{1}} = -2(u^{1}{}_{1})^{2}v\frac{\delta H}{\delta\mu_{0}^{1}} \tag{5.37b}$$

$$\tilde{\Sigma}_{\mathbf{1}}{}^{01}G^{\mathbf{0}}{}_{\mathbf{2}} = -2uv\tilde{\Sigma}_{\mathbf{2}}{}^{1i}\frac{\delta H}{\delta s_{0}^{i}} \tag{5.37c}$$

$$\tilde{\Sigma}_{\mathbf{1}}{}^{01}G^{\mathbf{0}}{}_{\mathbf{3}} = 2(u^{1}{}_{1})^{2}v\tilde{\Sigma}_{\mathbf{3}}{}^{0i}\frac{\delta H}{\delta s_{0}^{i}} \tag{5.37d}$$

$$\tilde{\Sigma}_{\mathbf{1}}{}^{01}G^{\mathbf{2}}{}_{\mathbf{3}} = 2(u^{1}{}_{1})^{2}v\tilde{\Sigma}_{\mathbf{3}}{}^{0i}\frac{\delta H}{\delta\tilde{\Sigma}_{\mathbf{2}}{}^{1i}}. \tag{5.37e}$$

In order to show that the equations of motion (5.31) and (5.32a) express the remaining Einstein equations, we rewrite them. (5.31a) gives

$$-\dot{A}_{p}^{\mathbf{1}} + D_{p}B_{0}^{\mathbf{1}} + R^{\mathbf{1}}{}_{ip}s_{0}^{i} - R^{\mathbf{2}}{}_{pj}\tilde{\Sigma}_{\mathbf{2}}{}^{1j}\left(\tilde{\Sigma}_{\mathbf{1}}{}^{01}\right)^{-1} = 0.$$

Using the definition of $R^{\mathbf{1}}{}_{0i}$ and the constraints C_{0}^{i} we obtain the equation,

$$\Rightarrow -R^{\mathbf{1}}{}_{0p}\tilde{\Sigma}_{\mathbf{1}}{}^{01} + R^{\mathbf{1}}{}_{ip}\tilde{\Sigma}_{\mathbf{1}}{}^{1i} - R^{\mathbf{2}}{}_{pj}\tilde{\Sigma}_{\mathbf{2}}{}^{1j} \approx 0. \tag{5.38a}$$

In a similar way we rewrite the remaining equations (5.31b, 5.31c and 5.32a) to obtain:

$$\left(-R^{\mathbf{2}}{}_{0p} + R^{\mathbf{2}}{}_{ip}\tilde{\Sigma}_{\mathbf{1}}{}^{1i}\right)\tilde{\Sigma}_{\mathbf{2}}{}^{1p} \approx 0 \tag{5.38b}$$

$$-R^{\mathbf{3}}{}_{0p}\tilde{\Sigma}_{\mathbf{1}}{}^{01} + R^{\mathbf{3}}{}_{ip}\tilde{\Sigma}_{\mathbf{1}}{}^{1i} - R^{\mathbf{1}}{}_{pi}\tilde{\Sigma}_{\mathbf{2}}{}^{1i} \approx 0 \tag{5.38c}$$

$$R^{\mathbf{1}}{}_{01}\tilde{\Sigma}_{\mathbf{1}}{}^{01} - R^{\mathbf{1}}{}_{1i}\tilde{\Sigma}_{\mathbf{1}}{}^{1i} + R^{\mathbf{2}}{}_{1i}\tilde{\Sigma}_{\mathbf{2}}{}^{1i} \approx 0. \tag{5.38d}$$

The last equation takes a bit more working, and uses the constraint (5.11). Equations (5.38a) define $G^{\mathbf{1}}{}_{\mathbf{2}} \approx 0$ and $G^{\mathbf{1}}{}_{\mathbf{3}} \approx 0$. Equation (5.38b) defines $G^{\mathbf{1}}{}_{\mathbf{0}} \approx 0$, and (5.38c) defines $G^{\mathbf{3}}{}_{\mathbf{2}} \approx 0$. The final constraint component $G^{\mathbf{3}}{}_{\mathbf{3}} \approx 0$ is defined by (5.38a) and (5.38d). We have now shown that the constraint equations and equations of evolution imply the Einstein equations.

5.5.1 Structure equations

In the self-dual approach considered in section 5.2, we obtained not only the Einstein equations, but also some structure equations. These were derived through the variation of the connection variables and resulted in the equation ${}^{(+)}\Box(\theta^{\alpha} \wedge \theta^{\beta}) = 0$. When we changed the variable basis to the $SO(3)_{\mathbb{C}}$ basis (4.34) we obtained the structure equations, $dS^{\mathbf{A}} + 2\eta^{\mathbf{A}}{}_{\mathbf{BC}}\Gamma^{\mathbf{B}} \wedge S^{\mathbf{C}} = 0$ (4.39a). When this is expressed in terms of the Sigma variables, we obtain the equations $D_{\alpha}\tilde{\Sigma}_{\mathbf{A}}{}^{\gamma\alpha} = 0$. Therefore we should expect to derive these equations as well as the Einstein equations through variations of the Hamiltonian.

We would normally expect the structure equations to come from the equations of motion, but this is not completely true in this case. The equations of motion (5.33) and (5.34) can be written in the forms $-D_{\alpha}\tilde{\Sigma}_{\mathbf{A}}{}^{\alpha i} = 0$ and $D_{\alpha}\tilde{\Sigma}_{\mathbf{A}}{}^{1\alpha} = 0$ respectively. The remaining structure equations are not found in the equations of motion but in the constraint equations; this is a result of using the shortcut method. The constraint equations (5.27) can be written as $D_{\alpha}\tilde{\Sigma}_{\mathbf{A}}{}^{0\alpha} = 0$. Combining these constraint equations with the equations of motion (5.34), we obtain $D_{\alpha}\tilde{\Sigma}_{\mathbf{A}}{}^{\gamma\alpha}$. Hence we have shown that the structure equations are also defined by the equations of motion, along with some constraints obtained from the Hamiltonian.

When expressed in terms of the basis of 1-forms using equations (5.6) and (4.34), these structure equations $D_{\alpha}\tilde{\Sigma}_{\mathbf{A}}{}^{\gamma\alpha}$ are equivalent to equation (4.29). This shows that the connection induced on the basis of 1-forms is just the self-dual part of the Levi-Civita connection.

5.6 First Class constraints

We calculate the first class constraints in just the same way as the double null analysis in chapter 3. Guided by a geometric understanding we take linear combinations of the four primary and eight secondary constraints. We start by considering the secondary constraints that arise from the variation of the multipliers s_0^p, which we adapt with the addition of the constraint (5.27), multiplied with the canonical variables $A_p^{\mathbf{A}}$. This is expressed below:

$$\psi_p := R^{\mathbf{A}}{}_{ip}\tilde{\Sigma}_{\mathbf{A}}{}^{0i} + R^{\mathbf{A}}{}_{1p}\tilde{\Sigma}_{\mathbf{A}}{}^{01} + A^{\mathbf{A}}_p\left(D_1\tilde{\Sigma}_{\mathbf{A}}{}^{01} + D_i\tilde{\Sigma}_{\mathbf{A}}{}^{0i}\right) = 0 \tag{5.39a}$$

$$= B^{\mathbf{A}}_{1,p}\tilde{\Sigma}_{\mathbf{A}}{}^{01} + A^{\mathbf{A}}_{i,p}\tilde{\Sigma}_{\mathbf{A}}{}^{0i} - (A^{\mathbf{A}}_p\tilde{\Sigma}_{\mathbf{A}}{}^{01})_{,1} - (A^{\mathbf{A}}_p\tilde{\Sigma}_{\mathbf{A}}{}^{0j})_{,j} = 0. \tag{5.39b}$$

Another first class constraint can be obtained from adapting the constraint (5.30a). We first rewrite this constraint using (5.30b) then use (5.30c) as show below:

$$\begin{aligned}
\tilde{\Sigma}_{\mathbf{3}}{}^{0p}\left(R^{\mathbf{3}}{}_{1p}\tilde{\Sigma}_{\mathbf{1}}{}^{01} + R^{\mathbf{3}}{}_{ip}\tilde{\Sigma}_{\mathbf{1}}{}^{0i} + R^{\mathbf{1}}{}_{ip}\tilde{\Sigma}_{\mathbf{2}}{}^{0i}\right) &= 0\\
\Rightarrow \tilde{\Sigma}_{\mathbf{1}}{}^{01}\left(R^{\mathbf{3}}{}_{1p}\tilde{\Sigma}_{\mathbf{3}}{}^{0p} + R^{\mathbf{2}}{}_{1p}\tilde{\Sigma}_{\mathbf{2}}{}^{0p}\right) + \tilde{\Sigma}_{\mathbf{1}}{}^{0i}\left(R^{\mathbf{3}}{}_{ip}\tilde{\Sigma}_{\mathbf{3}}{}^{0p} + R^{\mathbf{2}}{}_{ip}\tilde{\Sigma}_{\mathbf{2}}{}^{0p}\right) &= 0\\
\Rightarrow \tilde{\Sigma}_{\mathbf{1}}{}^{01}R^{\mathbf{A}}{}_{1i}\tilde{\Sigma}_{\mathbf{A}}{}^{0i} - R^{\mathbf{1}}{}_{ij}\tilde{\Sigma}_{\mathbf{1}}{}^{0i}\tilde{\Sigma}_{\mathbf{1}}{}^{0j} &= 0\\
\Rightarrow R^{\mathbf{A}}{}_{1i}\tilde{\Sigma}_{\mathbf{A}}{}^{0i} &= 0.
\end{aligned} \tag{5.40}$$

Then, to obtain the first class constraint we adapt it in a similar manner to the previous constraint, ψ_p.

$$\psi_1 := R^{\mathbf{A}}{}_{i1}\tilde{\Sigma}_{\mathbf{A}}{}^{0i} + B^{\mathbf{A}}_1\left(D_1\tilde{\Sigma}_{\mathbf{A}}{}^{01} + D_i\tilde{\Sigma}_{\mathbf{A}}{}^{0i}\right) = 0 \tag{5.41a}$$

$$= B^{\mathbf{A}}_{1,1}\tilde{\Sigma}_{\mathbf{A}}{}^{01} + A^{\mathbf{A}}_{i,1}\tilde{\Sigma}_{\mathbf{A}}{}^{0i} - (B^{\mathbf{A}}_1\tilde{\Sigma}_{\mathbf{A}}{}^{01})_{,1} - (B^{\mathbf{A}}_1\tilde{\Sigma}_{\mathbf{A}}{}^{0j})_{,j} = 0. \tag{5.41b}$$

Before we continue to calculate the final first class constraint, it is worth noting that if we combine (5.41) with (5.39) we get

$$\psi_B := R^{\mathbf{A}}{}_{BC}\tilde{\Sigma}_{\mathbf{A}}{}^{0C} - A^{\mathbf{A}}_B D_C\tilde{\Sigma}_{\mathbf{A}}{}^{0C} = 0 \tag{5.42a}$$

$$= A^{\mathbf{A}}_{C,B}\tilde{\Sigma}_{\mathbf{A}}{}^{0C} - (A^{\mathbf{A}}_B\tilde{\Sigma}_{\mathbf{A}}{}^{0C})_{,C} = 0, \tag{5.42b}$$

where the unbold indices A, B, C are coordinate indices range over $1, 2, 3$, and we have also introduced $A^{\mathbf{A}}_B := (B^{\mathbf{A}}_1, A^{\mathbf{A}}_i)$. This shows that we could replace constraints ψ_1 and ψ_p with ψ_B, which would act just like the constraint (A.10b) in the Ashtekar case. Instead, in line with our foliation of the spacetime, we keep the constraints separate.

The final first class constraint is the equation obtained from $\dot{B}_0^1$:

$$\mathcal{G}_1 := D_1\tilde{\Sigma}_1{}^{01} + D_i\tilde{\Sigma}_1{}^{0i} = 0. \tag{5.43}$$

After this analysis we have four first class constraints, which implies that the remaining eight constraints are second class. We can now calculate the number of degrees of freedom that our theory contains using the standard formula (2.73):

$$\tfrac{1}{2}\,(18 - 2(4) - 8) = 1. \tag{5.44}$$

This is the number of degrees of freedom that we would expect in the null setting (see Penrose 1980). To see the $2+2$ geometric structure we choose smearing functions such that f, g are on the hypersurface Σ_0 (see figure 3.2), f^1, g^1 only varies in the x^1 direction so that $f^1{}_{,i} = g^1{}_{,i} = 0$ and f^i, g^i only varies in the x^i direction so that $f^i{}_{,1} = g^i{}_{,1} = 0$. Then we find:

$$\left\{\int_x f\,\mathcal{G}_1\,\mathrm{d}^3x, \int_y g\,\mathcal{G}_1\,\mathrm{d}^3y\right\} = 0 \tag{5.45a}$$

$$\left\{\int_x f\,\mathcal{G}_1\,\mathrm{d}^3x, \int_y g^1\,\psi_1\,\mathrm{d}^3y\right\} = \int_z f\mathcal{L}_g\mathcal{G}_1\,\mathrm{d}^3z \tag{5.45b}$$

$$\left\{\int_x f\,\mathcal{G}_1\,\mathrm{d}^3x, \int_y g^i\psi_i\,\mathrm{d}^3y\right\} = \int_z f\mathcal{L}_g\mathcal{G}_1\,\mathrm{d}^3z \tag{5.45c}$$

$$\left\{\int_x f^1\psi_1\,\mathrm{d}^3x, \int_y g^1\,\psi_1\,\mathrm{d}^3y\right\} = \int_z \psi_1\mathcal{L}_f g^1\,\mathrm{d}^3z \tag{5.45d}$$

$$\left\{\int_x f^1\psi_1\,\mathrm{d}^3x, \int_y g^p\,\psi_p\,\mathrm{d}^3y\right\} = \int \psi_1\mathcal{L}_g f^1\,\mathrm{d}^3z \tag{5.45e}$$

$$\left\{\int_x f^p\psi_p\,\mathrm{d}^3x, \int_y g^q\,\psi_q\,\mathrm{d}^3y\right\} = \int \psi_q\mathcal{L}_f g^q\,\mathrm{d}^3z. \tag{5.45f}$$

These may be combined to obtain the algebra on Σ_0, which has virtually the same form as (B.14) of Appendix B (eg. $\psi_1 g^1 + \psi_i\, g^i = \psi_C\, g^C$).

We now wish to ascertain the geometric interpretation of the first class constraints. In order to do this we calculate the infinitesimal transformations of the canonical variables generated by the constraints. First we consider ψ_i:

$$\delta A^{\mathbf{A}}_{\ i} = \left\{A^{\mathbf{A}}_i, \psi_i(g^i)\right\} = \mathcal{L}_g A^{\mathbf{A}}_{\ i} \tag{5.46a}$$

$$\delta B^{\mathbf{A}}_{\ 1} = \left\{B^{\mathbf{A}}_1, \psi_i(g^i)\right\} = \mathcal{L}_g B^{\mathbf{A}}_{\ 1}. \tag{5.46b}$$

Then we consider ψ_1:

$$\delta A^{\mathbf{A}}_{\ i} = \left\{A^{\mathbf{A}}_i, \psi_1(g^1)\right\} = \mathcal{L}_g A^{\mathbf{A}}_{\ i} \tag{5.47a}$$

$$\delta B^{\mathbf{A}}_{\ 1} = \left\{B^{\mathbf{A}}_0, \psi_1(g^1)\right\} = \mathcal{L}_g B^{\mathbf{A}}_{\ 1}. \tag{5.47b}$$

We can now see from (5.46) that the constraint ψ_i generates the diffeomorphisms within the spatial two surface, while from (5.47) we see that ψ_1 generates the diffeomorphism along the x^1 direction. These three constraints were not unexpected because they also appear in the double null analysis in chapter 3. The remaining constraints do not have an analogous constraint in the work in chapter 3. To understand the transformations generated by the remaining constraints we first have to understand the effect of spin and boost transformations, and this is the subject of the following section.

5.7 Spin and boost transformations

The Poisson bracket with the remaining first class constraint (5.43) generates the self-dual spin and boost transformations. This can be seen when we look at the infinitesimal transformations that these constraints generate. First we will consider how the spin and boost transformations effect the different variables. Then we compare these to the infinitesimal transformations generated by the first class constraint (5.43).

To understand the effect of complex spin and boost transformations on tetrad variables we first consider their effect on the 1-forms:

$$\begin{aligned} \theta^0 &\longrightarrow \rho r \theta^0 \\ \theta^1 &\longrightarrow \rho^{-1} r^{-1} \theta^1 \\ \theta^2 &\longrightarrow \rho r^{-1} \theta^2 \\ \theta^3 &\longrightarrow \rho^{-1} r \theta^3 \end{aligned} \tag{5.48}$$

Since we are working in a complexified space we do not require $r = \bar{\rho}$, where $\bar{\rho}$ is the complex conjugate of ρ. Using 5.48 along with the basis (5.1) leads to the following transformations of the tetrad variables:

$$\begin{aligned} &\mu^{\mathbf{0}}{}_a \longrightarrow \rho r \mu^{\mathbf{0}}{}_a \\ &\mu^{\mathbf{1}}{}_a \longrightarrow \rho^{-1} r^{-1} \mu^{\mathbf{1}}{}_a \\ &\nu^{\mathbf{2}}{}_i \longrightarrow \rho r^{-1} \nu^{\mathbf{2}}{}_i, \quad s^2{}_a \longrightarrow s^2{}_a \\ &\nu^{\mathbf{3}}{}_i \longrightarrow \rho^{-1} r \nu^{\mathbf{3}}{}_i, \quad s^3{}_a \longrightarrow s^3{}_a. \end{aligned} \tag{5.49}$$

Using the system of equations (5.7c), along with (5.49), we can calculate how the Sigma variables transform:

$$\begin{aligned} &\tilde{\Sigma}_{\mathbf{1}}{}^{01} \longrightarrow \tilde{\Sigma}_{\mathbf{1}}{}^{01} && \tilde{\Sigma}_{\mathbf{2}}{}^{02} \longrightarrow r^2 \tilde{\Sigma}_{\mathbf{2}}{}^{02} \\ &\tilde{\Sigma}_{\mathbf{1}}{}^{02} \longrightarrow \tilde{\Sigma}_{\mathbf{1}}{}^{02} && \tilde{\Sigma}_{\mathbf{2}}{}^{03} \longrightarrow r^2 \tilde{\Sigma}_{\mathbf{2}}{}^{03} \\ &\tilde{\Sigma}_{\mathbf{1}}{}^{03} \longrightarrow \tilde{\Sigma}_{\mathbf{1}}{}^{03} && \tilde{\Sigma}_{\mathbf{2}}{}^{12} \longrightarrow r^2 \tilde{\Sigma}_{\mathbf{2}}{}^{12} \\ &\tilde{\Sigma}_{\mathbf{1}}{}^{23} \longrightarrow \tilde{\Sigma}_{\mathbf{1}}{}^{23} && \tilde{\Sigma}_{\mathbf{2}}{}^{13} \longrightarrow r^2 \tilde{\Sigma}_{\mathbf{2}}{}^{13} \\ &\tilde{\Sigma}_{\mathbf{2}}{}^{23} \longrightarrow r^2 \tilde{\Sigma}_{\mathbf{2}}{}^{23} && \tilde{\Sigma}_{\mathbf{3}}{}^{02} \longrightarrow r^{-2} \tilde{\Sigma}_{\mathbf{3}}{}^{02} \\ &\tilde{\Sigma}_{\mathbf{3}}{}^{23} \longrightarrow r^{-2} \tilde{\Sigma}_{\mathbf{3}}{}^{23} && \tilde{\Sigma}_{\mathbf{3}}{}^{03} \longrightarrow r^{-2} \tilde{\Sigma}_{\mathbf{3}}{}^{03} \\ &\tilde{\Sigma}_{\mathbf{1}}{}^{12} \longrightarrow \tilde{\Sigma}_{\mathbf{1}}{}^{12} && \tilde{\Sigma}_{\mathbf{3}}{}^{12} \longrightarrow r^{-2} \tilde{\Sigma}_{\mathbf{3}}{}^{12} \\ &\tilde{\Sigma}_{\mathbf{1}}{}^{13} \longrightarrow \tilde{\Sigma}_{\mathbf{1}}{}^{13} && \tilde{\Sigma}_{\mathbf{3}}{}^{13} \longrightarrow r^{-2} \tilde{\Sigma}_{\mathbf{3}}{}^{13}. \end{aligned} \tag{5.50}$$

Note that r is the self-dual and ρ the anti self-dual spin and boost freedom. This can be seen in the result that the Sigma variables, which are self-dual, are transformed by only r and never ρ.

In order to compare these with the transformations generated by the constraints, we will consider only infinitesimal transformations, meaning:

$$\begin{aligned} \rho &\to (1+\delta\rho), & \rho^{-1} &\to (1-\delta\rho) \\ r &\to (1+\delta r), & r^{-1} &\to (1-\delta r) \\ r^2 &\to (1+2\delta r), & r^{-2} &\to (1-2\delta r). \end{aligned}$$

Therefore infinitesimal transformations of the variables result in:

$$\begin{aligned} \mu^{\mathbf{0}}{}_a + \delta\mu^{\mathbf{0}}{}_a &\to (1+\delta\rho)(1+\delta r)\mu^0{}_a \\ &= \mu^0{}_a + (\delta\rho + \delta r)\mu^0{}_a \\ \Rightarrow \delta\mu^{\mathbf{0}}{}_a &= (\delta\rho+\delta r)\mu^0{}_a. \end{aligned}$$

We summarise the results for all the variables below:

$$\delta\mu^{\mathbf{0}}{}_a = (\delta\rho+\delta r)\mu^{\mathbf{0}}{}_a \qquad \delta\mu^{\mathbf{1}}{}_a = -(\delta\rho+\delta r)\mu^{\mathbf{1}}{}_a \tag{5.51a}$$

$$\delta\nu^{\mathbf{2}}{}_i = (\delta\rho-\delta r)\nu^{\mathbf{2}}{}_i \qquad \delta\nu^{\mathbf{3}}{}_i = -(\delta\rho-\delta r)\nu^{\mathbf{3}}{}_i \tag{5.51b}$$

$$\delta\tilde{\Sigma}_{\mathbf{2}}{}^{23} = 2\delta r\tilde{\Sigma}_{\mathbf{2}}{}^{23} \qquad \delta\tilde{\Sigma}_{\mathbf{3}}{}^{23} = -2\delta r\tilde{\Sigma}_{\mathbf{3}}{}^{23} \tag{5.51c}$$

$$\delta\tilde{\Sigma}_{\mathbf{2}}{}^{02} = 2\delta r\tilde{\Sigma}_{\mathbf{2}}{}^{02} \qquad \delta\tilde{\Sigma}_{\mathbf{2}}{}^{03} = 2\delta r\tilde{\Sigma}_{\mathbf{2}}{}^{03} \tag{5.51d}$$

$$\delta\tilde{\Sigma}_{\mathbf{2}}{}^{12} = 2\delta r\tilde{\Sigma}_{\mathbf{2}}{}^{12} \qquad \delta\tilde{\Sigma}_{\mathbf{2}}{}^{13} = 2\delta r\tilde{\Sigma}_{\mathbf{2}}{}^{13} \tag{5.51e}$$

$$\delta\tilde{\Sigma}_{\mathbf{3}}{}^{02} = -2\delta r\tilde{\Sigma}_{\mathbf{3}}{}^{02} \qquad \delta\tilde{\Sigma}_{\mathbf{3}}{}^{03} = -2\delta r\tilde{\Sigma}_{\mathbf{3}}{}^{03} \tag{5.51f}$$

$$\delta\tilde{\Sigma}_{\mathbf{3}}{}^{12} = -2\delta r\tilde{\Sigma}_{\mathbf{3}}{}^{12} \qquad \delta\tilde{\Sigma}_{\mathbf{3}}{}^{13} = -2\delta r\tilde{\Sigma}_{\mathbf{3}}{}^{13}. \tag{5.51g}$$

s^i_a do not change under infinitesimal spin and boost transformations.

We can use (5.48) to show how the $SO(3)$ triad (4.34) transforms under spin and boost transformations. We represent this in the matrix form:

$$S^{\mathbf{A}} \to (\Lambda^{-1})^{\mathbf{A}}{}_{\mathbf{B}} S^{\mathbf{B}} \quad \text{where} \quad (\Lambda)^{\mathbf{A}}{}_{\mathbf{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^{-2} \end{bmatrix}$$

Under a gauge transformation the connection transforms according to:

$$\Gamma^{\mathbf{A}} \longrightarrow \eta_{\mathbf{B}}{}^{\mathbf{AC}}(\Lambda^{-1})^{\mathbf{B}}{}_{\mathbf{D}}\, \mathrm{d}(\Lambda)^{\mathbf{D}}{}_{\mathbf{C}} + \eta_{\mathbf{B}}{}^{\mathbf{AC}}\eta^{\mathbf{F}}{}_{\mathbf{DE}}(\Lambda^{-1})^{\mathbf{B}}{}_{\mathbf{F}}(\Lambda)^{\mathbf{D}}{}_{\mathbf{C}}\Gamma^{\mathbf{E}}. \tag{5.52}$$

Using this we find the infinitesimal transformations of the connection variables, $A_i^{\mathbf{A}}$ and $B_a^{\mathbf{A}}$, are given by:

$$\begin{aligned}
\delta\Gamma^{\mathbf{1}} \to (\delta r)_{,\alpha}\,\mathrm{d}x^{\alpha} \quad &\Rightarrow \quad \begin{aligned}\delta A_i^{\mathbf{1}} &\to \delta r_{,i}\\ \delta B_a^{\mathbf{1}} &\to \delta r_{,a}\end{aligned}\\
\delta\Gamma^{\mathbf{2}} \to -2\delta r\Gamma^{\mathbf{2}} \quad &\Rightarrow \quad \begin{aligned}\delta A_i^{\mathbf{2}} &\to -2A_i^{\mathbf{2}}\delta r\\ \delta B_a^{\mathbf{2}} &\to -2B_a^{\mathbf{2}}\delta r\end{aligned}\\
\delta\Gamma^{\mathbf{3}} \to -2\delta r\Gamma^{\mathbf{3}} \quad &\Rightarrow \quad \begin{aligned}\delta A_i^{\mathbf{3}} &\to 2A_i^{\mathbf{3}}\delta r\\ \delta B_a^{\mathbf{3}} &\to 2B_a^{\mathbf{3}}\delta r.\end{aligned}
\end{aligned} \tag{5.53}$$

Now that we have calculated the effect of spin and boost transformations on the variables, we will now calculate the infinitesimal transformations generated by the first class constraint $\mathcal{G}_1$, which is given by (5.43):

$$\delta A_i^{\mathbf{A}} = \{A_i^{\mathbf{A}}, \mathcal{G}_1(g)\} = -g_{,i}\delta_{\mathbf{1}}^{\mathbf{A}} - 2gA_i^{\mathbf{2}}\delta_{\mathbf{2}}^{\mathbf{A}} + 2gA_i^{\mathbf{3}}\delta_{\mathbf{3}}^{\mathbf{A}} \tag{5.54a}$$

$$\delta B_1^{\mathbf{A}} = \{B_1^{\mathbf{A}}, \mathcal{G}_1(g)\} = -g_{,1}\delta_{\mathbf{1}}^{\mathbf{A}} - 2gB_1^{\mathbf{2}}\delta_{\mathbf{2}}^{\mathbf{A}} + 2gB_1^{\mathbf{3}}\delta_{\mathbf{3}}^{\mathbf{A}}. \tag{5.54b}$$

When these equations are compared with (5.53), we can see that $\mathcal{G}_1$ generates the self-dual spin and boost transformations.

5.8 Conclusion

In this chapter we have applied canonical analysis using an adapted Ashtekar approach, to a double null description of General Relativity. We started from a $SO(3)$ action, and from this we obtained a Lagrangian density. From this we calculated the Hamiltonian, on which we performed the canonical analysis. This lead us to obtaining four primary constraints and eight linearly independent secondary constraints. By taking particular linear combinations of these twelve constraints, we revealed four first class constraints. Two of these constraints, ψ_p, generate the diffeomorphisms within the spatial hypersurface (S);

while one constraint, ψ_1, generates the diffeomorphism in the x^1 direction. The final first class constraint, as shown above, generated the self-dual spin and boost transformations. These transformations were not present in the original double null analysis (see chapter 3) but arise here due to the extra freedom that is introduced from working with a frame.

In the formulation of General Relativity used in this chapter some of the variables, $\mu_{\mathfrak{b}}^{\mathfrak{a}}$, contain an anti self-dual part. We would therefore expect to obtain another first class constraint that generates the anti self-dual null rotations. This did not arise in our calculation because we used the shortcut method, which meant that the variables that contained anti self-dual parts were multipliers not variables. Hence no additional first class constraint was obtained. If we were to perform the canonical analysis without the shortcut method we find an additional first class constraint arises which generates the anti self-dual null rotations.

While this work has been successful in obtaining a first class algebra, it does not appear completely satisfactory to work with a mixture of tetrad variables and 2-form variables. In the next chapter we consider an alternative approach in which we work entirely with the two-form variables, $\tilde{\Sigma}_A{}^{\alpha\beta}$.

Chapter 6

$2+2$ field equations using Connection variables

6.1 Introduction

In the last chapter we used both tetrad variables and $SO(3)$ variables. This enabled us to retain an analogy with the double null formalism that was used in chapter 3 through the identification of the lapse and shift variables. In this section we will use only the $SO(3)$ variables. This results in a simpler constraint analysis.

In the first section we use the action (4.47) to obtain a Lagrangian description expressed in terms of the connection variables and the densitised $SO(3)$ basis variables, $\tilde{\Sigma}_{\mathbf{A}}{}^{\alpha\beta}$, that were introduced in the last section. However these variables are not independent and some constraints exist in the Sigma variables. These are found and introduced into the Lagrangian using Lagrange multipliers. This then leads us, in the following section, to a Hamiltonian description.

In the next section we show that all the Einstein equations are defined by the constraints and equations of motion. After this we continue to calculate the first class constraints and obtain the first class algebra. At the end of this section we discuss the geometrical interpretation of the first class constraints.

6.2 2+2 Connection Variables

We start with the action

$$I = \int R^{\mathbf{A}} \wedge S^{\mathbf{B}} g_{\mathbf{AB}},$$

which leads to the Lagrangian density:

$$\begin{aligned}\mathcal{L} &= R^{\mathbf{A}}{}_{\alpha\beta}\tilde{\Sigma}_{\mathbf{A}}{}^{\alpha\beta} \\ &= \tilde{\Sigma}_{\mathbf{A}}{}^{0i}A^{\mathbf{A}}_{i,0} + \tilde{\Sigma}_{\mathbf{A}}{}^{01}B^{\mathbf{A}}_{1,0} + B^{\mathbf{A}}_{0}D_1\tilde{\Sigma}_{\mathbf{A}}{}^{01} + B^{\mathbf{A}}_{0}D_i\tilde{\Sigma}_{\mathbf{A}}{}^{0i} + R^{\mathbf{A}}{}_{1i}\tilde{\Sigma}_{\mathbf{A}}{}^{1i} + R^{\mathbf{A}}{}_{23}\tilde{\Sigma}_{\mathbf{A}}{}^{23}. \end{aligned} \tag{6.1}$$

In the previous chapter we worked with the twenty three variables $\tilde{\Sigma}_{\mathbf{A}}^{01}, \tilde{\Sigma}_{\mathbf{A}}^{ai}, \mu^{\mathbf{a}}{}_b, s^i{}_a$ in addition to the connection terms ($A^{\mathbf{A}}_i$ and $B^{\mathbf{A}}_a$). Within these twenty three variables there are thirteen constraints which left ten degrees of freedom: two spin and boost freedoms and eight degrees of freedom for the double null metric (this includes 10 for the standard metric with the two slicing conditions). In this chapter we will be making use of all eighteen Sigma variables $\tilde{\Sigma}_{\mathbf{A}}^{\alpha\beta}$ which, as shown below, have to satisfy nine constraints. This means that we have only nine degrees of freedom; one less than chapter 5. In order to understand this loss we consider the spin and boost transformations (see 5.49 and 5.50). The sigma variables are self-dual, unlike the variables $\mu^{\mathbf{a}}{}_b$ and therefore we do not have the anti self-dual spin and boost freedom present in the earlier formalism. Therefore the work in this chapter will contain one less degree of freedom than previous work.

We now obtain the nine constraints. The first constraints (6.2) are found by expressing the results $S^{\mathbf{1}} \wedge S^{\mathbf{2}} = 0$ and $S^{\mathbf{1}} \wedge S^{\mathbf{3}} = 0$ in terms of $\tilde{\Sigma}_{\mathbf{A}}{}^{\alpha\beta}$.

$$\epsilon_{\alpha\beta\gamma\delta}\tilde{\Sigma}_{\mathbf{1}}{}^{\alpha\beta}\tilde{\Sigma}_{\mathbf{2}}{}^{\gamma\delta} = 0 \tag{6.2a}$$

$$\epsilon_{\alpha\beta\gamma\delta}\tilde{\Sigma}_{\mathbf{1}}{}^{\alpha\beta}\tilde{\Sigma}_{\mathbf{3}}{}^{\gamma\delta} = 0. \tag{6.2b}$$

Then, by rearranging the system of equations (5.7) we obtain the four constraints:

$$\epsilon_{ij}\tilde{\Sigma}_{\mathbf{2}}{}^{0i}\tilde{\Sigma}_{\mathbf{2}}{}^{1j} = 0 \tag{6.3a}$$

$$\epsilon_{ij}\tilde{\Sigma}_{\mathbf{3}}{}^{0i}\tilde{\Sigma}_{\mathbf{3}}{}^{1j} = 0 \tag{6.3b}$$

$$\tilde{\Sigma}_{\mathbf{2}}{}^{01} = 0 \tag{6.3c}$$

$$\tilde{\Sigma}_{\mathbf{3}}{}^{01} = 0. \tag{6.3d}$$

The remaining two constraints are found by manipulating equations (5.7c) to find $\mu_0^{\mathbf{1}}\mu_0^{\mathbf{0}}$ and $\mu_1^{\mathbf{0}}\mu_1^{\mathbf{1}}$ in terms of the Sigma variables. The slicing conditions (5.4) then give

$$\epsilon_{ij}\tilde{\Sigma}_{\mathbf{2}}{}^{1i}\tilde{\Sigma}_{\mathbf{3}}{}^{1j} = 0 \tag{6.4a}$$

$$\epsilon_{ij}\tilde{\Sigma}_{\mathbf{2}}{}^{0i}\tilde{\Sigma}_{\mathbf{3}}{}^{0j} = 0. \tag{6.4b}$$

It is worth noting at this point that the constraints (6.2a), (6.3a), (6.3b) and (6.4) can be combined with the requirement that the volume form be non-vanishing

$$\tilde{\Sigma}_{\mathbf{2}}{}^{ai}\tilde{\Sigma}_{\mathbf{3}}{}^{bj}\epsilon_{ab}\epsilon_{ij} \neq 0,$$

where they reduce to the requirement that either

$$\tilde{\Sigma}_{\mathbf{2}}{}^{02} = \tilde{\Sigma}_{\mathbf{2}}{}^{03} = \tilde{\Sigma}_{\mathbf{3}}{}^{12} = \tilde{\Sigma}_{\mathbf{3}}{}^{13} = 0, \tag{6.5a}$$

or

$$\tilde{\Sigma}_{\mathbf{3}}{}^{02} = \tilde{\Sigma}_{\mathbf{3}}{}^{03} = \tilde{\Sigma}_{\mathbf{2}}{}^{13} = \tilde{\Sigma}_{\mathbf{2}}{}^{12} = 0. \tag{6.5b}$$

The two conditions are interchanged on relabelling and there is no loss of generality in choosing the former condition, which turns out to agree with the choice of slicing condition used in the previous chapter. With this choice the following simplified constraint equations are obtained.

$$\begin{aligned}
C_1 &\equiv \epsilon_{abcd}\tilde{\Sigma}_{\mathbf{1}}{}^{ab}\tilde{\Sigma}_{\mathbf{2}}{}^{cd} = 0 & C_2 &\equiv \epsilon_{abcd}\tilde{\Sigma}_{\mathbf{1}}{}^{ab}\tilde{\Sigma}_{\mathbf{3}}{}^{cd} = 0 \\
C_3 &\equiv \tilde{\Sigma}_{\mathbf{2}}{}^{01} = 0 & C_4 &\equiv \tilde{\Sigma}_{\mathbf{3}}{}^{01} = 0 \\
C_5 &\equiv \tilde{\Sigma}_{\mathbf{2}}{}^{02} = 0 & C_6 &\equiv \tilde{\Sigma}_{\mathbf{2}}{}^{03} = 0 \\
C_7 &\equiv (\tilde{\Sigma}_{\mathbf{3}}{}^{12})^2 = 0 & C_8 &\equiv (\tilde{\Sigma}_{\mathbf{3}}{}^{13})^2 = 0.
\end{aligned} \tag{6.6}$$

Note that as in chapter 3 we have squared the final two constraints because the variables are cyclic.

Our final constraint comes from expressing the constraint $\hat{C}$ (see 5.11) in terms of the Sigmas. This results in the constraint

$$C_9 \equiv \epsilon_{ab}\tilde{\Sigma}_{\mathbf{1}}{}^{a2}\tilde{\Sigma}_{\mathbf{1}}{}^{b3} + \epsilon_{ab}\epsilon_{ij}\tilde{\Sigma}_{\mathbf{2}}{}^{ai}\tilde{\Sigma}_{\mathbf{3}}{}^{bj} - \tilde{\Sigma}_{\mathbf{1}}{}^{23}\tilde{\Sigma}_{\mathbf{1}}{}^{01}. \tag{6.7}$$

If we fix the anti-self dual gauge freedom in the choice of $\mu^{\mathbf{a}}{}_{b}$ the map from the (constrained) frame and $SO(3)$ variables used in chapter 5 to the space of Sigmas satisfying the above nine constraints is invertible at the linearised level. By the inverse function theorem we can consider the two descriptions of double null general relativity to be equivalent. We therefore consider our Lagrangian to be given by (6.1) and use Lagrange multipliers to introduce the primary constraints.

$$\mathcal{L} = \tilde{\Sigma}_{\mathbf{A}}{}^{0i}A^{\mathbf{A}}_{i,0} + \tilde{\Sigma}_{\mathbf{A}}{}^{01}B^{\mathbf{A}}_{1,0} + B^{\mathbf{A}}_{0}\left(D_1\tilde{\Sigma}_{\mathbf{A}}{}^{01} + D_i\tilde{\Sigma}_{\mathbf{A}}{}^{0i}\right) + R^{\mathbf{A}}{}_{1i}\tilde{\Sigma}_{\mathbf{A}}{}^{1i} + R^{\mathbf{A}}{}_{23}\tilde{\Sigma}_{\mathbf{A}}{}^{23} - \lambda^{\alpha}C_{\alpha}. \tag{6.8}$$

In the equation above α sums from $1, ..., 9$. The Lagrangian is now in an appropriate form to transfer easily to the Hamiltonian description.

Before moving to the Hamiltonian description we could calculate the Einstein equations and structure equations. However, these calculations would be very similar to the Hamiltonian equations and therefore, to save repetition, we leave them and move straight onto the Hamiltonian description.

6.3 Hamiltonian description

In the usual manner we can 'read off' from (6.8) that the momenta to the canonical variables $A_i^{\mathbf{A}}$ and $B_1^{\mathbf{A}}$ are $\tilde{\Sigma}_{\mathbf{A}}{}^{0i}$ and $\tilde{\Sigma}_{\mathbf{A}}{}^{01}$. We can also see that the Hamiltonian density is given by

$$\mathcal{H} = -B_0^{\mathbf{A}}\left(D_1\tilde{\Sigma}_{\mathbf{A}}{}^{01} + D_i\tilde{\Sigma}_{\mathbf{A}}{}^{0i}\right) - R^{\mathbf{A}}{}_{23}\tilde{\Sigma}_{\mathbf{A}}{}^{23} - R^{\mathbf{A}}{}_{1i}\tilde{\Sigma}_{\mathbf{A}}{}^{1i} + \lambda^\alpha C_\alpha + \xi_{\mathbf{A}}{}^{23}P^{\mathbf{A}}{}_{23} + \xi_{\mathbf{A}}{}^{1i}P^{\mathbf{A}}{}_{1i} + \xi^{\mathbf{A}}P_{\mathbf{A}}, \tag{6.9}$$

where we have introduced the momenta $P^{\mathbf{A}}{}_{23}$, $P^{\mathbf{A}}{}_{1i}$, $\tilde{P}_{\mathbf{A}}$ for the cyclic variables $\tilde{\Sigma}_{\mathbf{A}}^{23}$, $\tilde{\Sigma}_{\mathbf{A}}{}^{1i}$, $B_0^{\mathbf{A}}$. This results in additional primary constraints which have been introduced into the Hamiltonian using the Lagrangian multipliers $\xi_{\mathbf{A}}{}^{23}, \xi_{\mathbf{A}}{}^{1i}, \xi_{\mathbf{A}}$.

The canonical Poisson brackets are given by:

$$\left\{A_i^{\mathbf{A}}(x), \tilde{\Sigma}_{\mathbf{B}}{}^{0j}(y)\right\} = \delta^{\mathbf{A}}_{\mathbf{B}}\delta_i^j\,\delta(x,\tilde{y}) \tag{6.10a}$$

$$\left\{B_1^{\mathbf{A}}(x), \tilde{\Sigma}_{\mathbf{B}}{}^{01}(y)\right\} = \delta^{\mathbf{A}}_{\mathbf{B}}\delta(x,\tilde{y}) \tag{6.10b}$$

$$\left\{B_0^{\mathbf{A}}(x), \tilde{P}_{\mathbf{B}}(y)\right\} = \delta^{\mathbf{A}}_{\mathbf{B}}\delta(x,\tilde{y}) \tag{6.10c}$$

$$\left\{\tilde{\Sigma}_{\mathbf{A}}^{23}(x), P^{\mathbf{B}}_{23}(y)\right\} = \delta^{\mathbf{A}}_{\mathbf{B}}\delta(\tilde{x},y) \tag{6.10d}$$

$$\left\{\tilde{\Sigma}_{\mathbf{A}}{}^{1i}(x), P^{\mathbf{B}}_{1j}(y)\right\} = \delta^{\mathbf{A}}_{\mathbf{B}}\delta_j^i\,\delta(\tilde{x},y). \tag{6.10e}$$

We have a total of twenty one primary constraints introduced into the Hamiltonian. Following the Dirac-Bergamnn algorithm we propagate the primary constraints:

$$\dot{\tilde{P}}_{\mathbf{A}} = D_1\tilde{\Sigma}_{\mathbf{A}}{}^{01} + D_i\tilde{\Sigma}_{\mathbf{A}}{}^{0i} \tag{6.11a}$$

$$\dot{P}^{\mathbf{A}}{}_{23} = R^{\mathbf{A}}{}_{23} - \delta_1^{\mathbf{A}}(\tilde{\Sigma}_{\mathbf{2}}{}^{02}\lambda^1 + \tilde{\Sigma}_{\mathbf{3}}{}^{01}\lambda^2 - \tilde{\Sigma}_{\mathbf{1}}{}^{01}\lambda^9) - \delta_2^{\mathbf{A}}\tilde{\Sigma}_{\mathbf{1}}{}^{01}\lambda^1 - \delta_3^{\mathbf{A}}\tilde{\Sigma}_{\mathbf{1}}{}^{01}\lambda^2 \tag{6.11b}$$

$$\begin{aligned}\dot{P}^{\mathbf{A}}{}_{12} = R^{\mathbf{A}}{}_{12} &- \delta_{\mathbf{1}}^{\mathbf{A}}(\tilde{\Sigma}_{\mathbf{2}}{}^{03}\lambda^1 + \tilde{\Sigma}_{\mathbf{3}}{}^{03}\lambda^2 - \tilde{\Sigma}_{\mathbf{1}}{}^{03}\lambda^9) - \delta_{\mathbf{2}}^{\mathbf{A}}(\tilde{\Sigma}_{\mathbf{1}}{}^{03}\lambda^1 - \tilde{\Sigma}_{\mathbf{3}}{}^{03}\lambda^9)\\ &- \delta_{\mathbf{3}}^{\mathbf{A}}(\tilde{\Sigma}_{\mathbf{1}}{}^{03}\lambda^2 - \tilde{\Sigma}_{\mathbf{2}}{}^{03}\lambda^9 + 2\tilde{\Sigma}_{\mathbf{3}}{}^{12}\lambda^7)\end{aligned} \tag{6.11c}$$

$$\begin{aligned}\dot{P}^{\mathbf{A}}{}_{13} = R^{\mathbf{A}}{}_{13} &+ \delta_{\mathbf{1}}^{\mathbf{A}}(\tilde{\Sigma}_{\mathbf{2}}{}^{02}\lambda^1 + \tilde{\Sigma}_{\mathbf{3}}{}^{02}\lambda^2 - \tilde{\Sigma}_{\mathbf{1}}{}^{02}\lambda^9) + \delta_{\mathbf{2}}^{\mathbf{A}}(\tilde{\Sigma}_{\mathbf{1}}{}^{02}\lambda^1 - \tilde{\Sigma}_{\mathbf{3}}{}^{02}\lambda^9)\\ &+ \delta_{\mathbf{3}}^{\mathbf{A}}(\tilde{\Sigma}_{\mathbf{1}}{}^{02}\lambda^2 - \tilde{\Sigma}_{\mathbf{2}}{}^{02}\lambda^9 + 2\tilde{\Sigma}_{\mathbf{3}}{}^{13}\lambda^8)\end{aligned} \tag{6.11d}$$

$$\dot{C}_3 = \left(D_i\tilde{\Sigma}_{\mathbf{A}}{}^{1i} + 2\eta^c{}_{\mathbf{AB}}B_0^{\mathbf{B}}\tilde{\Sigma}_{\mathbf{C}}{}^{01}\right)\delta_{\mathbf{2}}^{\mathbf{A}} \tag{6.12a}$$

$$\dot{C}_4 = \left(D_i\tilde{\Sigma}_{\mathbf{A}}{}^{1i} + 2\eta^c{}_{\mathbf{AB}}B_0^{\mathbf{B}}\tilde{\Sigma}_{\mathbf{C}}{}^{01}\right)\delta_{3}^{\mathbf{A}} \tag{6.12b}$$

$$\dot{C}_{5,6} \equiv \dot{\tilde{\Sigma}}_{\mathbf{2}}{}^{0i} = -2\eta^{\mathbf{C}}{}_{\mathbf{2A}}B_0^{\mathbf{B}}\tilde{\Sigma}_{\mathbf{C}}^{0i} - \tilde{\Sigma}_{\mathbf{2}}{}^{1i}{}_{,1} + 2\eta^{\mathbf{A}}{}_{\mathbf{B2}}B_1^{\mathbf{B}}\Sigma_{\mathbf{A}}{}^{1i} + D_j\tilde{\Sigma}_{\mathbf{2}}{}^{ij} \tag{6.12c}$$

$$\dot{C}_{7,8} \equiv \dot{\Sigma}_{\mathbf{3}}{}^{1i} = \xi_{\mathbf{3}}{}^{1i} \tag{6.12d}$$

The equations $\dot{C}_1$, $\dot{C}_2$ and $\dot{C}_9$ define the multipliers $\xi_{\mathbf{2}}{}^{23}$, $\xi_{\mathbf{3}}{}^{23}$ and $\xi_{\mathbf{1}}{}^{23}$ respectively. $\dot{C}_7$ and $\dot{C}_8$ define the multipliers $\xi_{\mathbf{3}}{}^{1i}$. Equation (6.11b) define the multipliers $\lambda^1\tilde{\Sigma}_{\mathbf{1}}{}^{01} = R^{\mathbf{2}}{}_{23}$, $\lambda^2\tilde{\Sigma}_{\mathbf{1}}{}^{01} = R^{\mathbf{3}}{}_{23}$ and $\lambda^9\tilde{\Sigma}_{\mathbf{1}}{}^{01} \approx -R^{\mathbf{1}}{}_{23}$. This leaves thirteen secondary constraints given by:

$$\dot{P}^{\mathbf{1}}{}_{1i} \approx R^{\mathbf{1}}{}_{1i}\tilde{\Sigma}_{\mathbf{1}}{}^{01} - R^{\mathbf{1}}{}_{ij}\tilde{\Sigma}_{\mathbf{1}}{}^{0j} - R^{\mathbf{2}}{}_{ij}\tilde{\Sigma}_{\mathbf{2}}{}^{0j} - R^{\mathbf{3}}{}_{ij}\tilde{\Sigma}_{\mathbf{3}}{}^{0j} \approx 0 \tag{6.13a}$$

$$\dot{P}^{\mathbf{2}}{}_{1i} \approx R^{\mathbf{2}}{}_{1i}\tilde{\Sigma}_{\mathbf{1}}{}^{01} - R^{\mathbf{2}}{}_{ij}\tilde{\Sigma}_{\mathbf{1}}{}^{0j} - R^{\mathbf{1}}{}_{ij}\tilde{\Sigma}_{\mathbf{3}}{}^{0j} = 0 \tag{6.13b}$$

$$\dot{P}^{\mathbf{3}}{}_{1i} \approx R^{\mathbf{3}}{}_{1i}\tilde{\Sigma}_{\mathbf{1}}{}^{01} - R^{\mathbf{3}}{}_{ij}\tilde{\Sigma}_{\mathbf{1}}{}^{0j} - R^{\mathbf{1}}{}_{ij}\tilde{\Sigma}_{\mathbf{2}}{}^{0j} = 0 \tag{6.13c}$$

$$\dot{\tilde{P}}_{\mathbf{A}} = D_1\tilde{\Sigma}_{\mathbf{A}}{}^{01} + D_i\tilde{\Sigma}_{\mathbf{A}}^{0i} = 0 \tag{6.13d}$$

$$\dot{C}_3 = \left(D_i\tilde{\Sigma}_{\mathbf{A}}{}^{1i} + 2\eta^{\mathbf{C}}{}_{\mathbf{AB}}B_0^{\mathbf{B}}\tilde{\Sigma}_{\mathbf{C}}{}^{01}\right)\delta_{\mathbf{2}}^{\mathbf{A}} = 0 \tag{6.13e}$$

$$\dot{C}_4 = \left(D_i\tilde{\Sigma}_{\mathbf{A}}{}^{1i} + 2\eta^{\mathbf{C}}{}_{\mathbf{AB}}B_0^{\mathbf{B}}\tilde{\Sigma}_{\mathbf{C}}{}^{01}\right)\delta_{\mathbf{3}}^{\mathbf{A}} = 0 \tag{6.13f}$$

$$\dot{\tilde{\Sigma}}_{\mathbf{2}}{}^{0i} = -2\eta^{\mathbf{C}}{}_{\mathbf{2A}}B_0^{\mathbf{B}}\tilde{\Sigma}_{\mathbf{C}}^{0i} - \Sigma_{\mathbf{2}}{}^{1i}{}_{,1} + 2\eta^{\mathbf{A}}{}_{\mathbf{B2}}B_1^{\mathbf{B}}\Sigma_{\mathbf{A}}{}^{1i} + D_j\tilde{\Sigma}_{\mathbf{2}}{}^{ij} = 0. \tag{6.13g}$$

The Dirac-Bergmann algorithm then requires us to propagate these secondary constraints to ensure that no additional constraints arise. Before this is done we split the two secondary constraints (6.13c) in the following way:

$$\tilde{\Sigma}_{\mathbf{3}}{}^{0i}\dot{P}^{\mathbf{3}}{}_{1i} = \tilde{\Sigma}_{\mathbf{3}}{}^{0i}(R^{\mathbf{3}}{}_{1i}\tilde{\Sigma}_{\mathbf{1}}{}^{01} - R^{\mathbf{3}}{}_{ij}\tilde{\Sigma}_{\mathbf{1}}{}^{0j} - R^{\mathbf{1}}{}_{ij}\tilde{\Sigma}_{\mathbf{2}}{}^{0j}) \tag{6.14a}$$

$$\tilde{\Sigma}_{\mathbf{1}}{}^{1i}\dot{P}^{\mathbf{3}}{}_{1i} = \tilde{\Sigma}_{\mathbf{1}}{}^{1i}(R^{\mathbf{3}}{}_{1i}\tilde{\Sigma}_{\mathbf{1}}{}^{01} - R^{\mathbf{3}}{}_{ij}\tilde{\Sigma}_{\mathbf{1}}{}^{0j} - R^{\mathbf{1}}{}_{ij}\tilde{\Sigma}_{\mathbf{2}}{}^{0j}). \tag{6.14b}$$

As we shall show the five equations (6.13a, 6.13b and 6.14a) define the five Einstein constraint equations and are therefore conserved by the Bianchi identity. Propagation of (6.13d) is identically zero for $A = 0$, and defines the multipliers λ^3 and λ^4 for the remaining values of A. Equations (6.13e) and (6.13f) define the multipliers $\xi^{\mathbf{3}}$ and $\xi^{\mathbf{2}}$ when propagated. Propagation of (6.13g) define the multipliers ${\xi_{\mathbf{2}}}^{1i}$, and finally propagation of (6.14b) leads to a multiplier equation containing the multipliers ${\xi_{\mathbf{1}}}^{1i}$. Therefore no additional secondary constraints arise.

We now calculate the equations of motion.

$$\begin{aligned}\dot{A}_i^{\mathbf{A}} = D_i B_0^{\mathbf{A}} - \delta_{\mathbf{1}}^{\mathbf{A}} \epsilon_{ij} ({\tilde{\Sigma}_{\mathbf{2}}}^{0j} \lambda^1 + {\tilde{\Sigma}_{\mathbf{3}}}^{0j} \lambda^2 - {\tilde{\Sigma}_{\mathbf{1}}}^{1j} \lambda^9) - \delta_{\mathbf{2}}^{\mathbf{A}} \epsilon_{ij} ({\tilde{\Sigma}_{\mathbf{1}}}^{1j} \lambda^1 - {\tilde{\Sigma}_{\mathbf{3}}}^{1j} \lambda^9) \\ -\delta_{\mathbf{3}}^{\mathbf{A}} \epsilon_{ij} ({\tilde{\Sigma}_{\mathbf{1}}}^{1j} \lambda^2 - {\tilde{\Sigma}_{\mathbf{2}}}^{1j} \lambda^9)\end{aligned} \tag{6.15}$$

which results in:

$$\dot{A}_i^{\mathbf{1}} = D_i B_0^{\mathbf{1}} - \left({R^{\mathbf{2}}}_{ij} {\tilde{\Sigma}_{\mathbf{2}}}^{0j} + {R^{\mathbf{3}}}_{ij} {\tilde{\Sigma}_{\mathbf{3}}}^{0j} + {R^{\mathbf{1}}}_{ij} {\tilde{\Sigma}_{\mathbf{1}}}^{1j} \right) / {\tilde{\Sigma}_{\mathbf{1}}}^{01} \tag{6.16a}$$

$$\dot{A}_i^{\mathbf{2}} = D_i B_0^{\mathbf{2}} - ({R^{\mathbf{2}}}_{ij} {\tilde{\Sigma}_{\mathbf{1}}}^{1j} + {R^{\mathbf{1}}}_{ij} {\tilde{\Sigma}_{\mathbf{3}}}^{1j}) / {\tilde{\Sigma}_{\mathbf{1}}}^{01} \tag{6.16b}$$

$$\dot{A}_i^{\mathbf{3}} = D_i B_0^{\mathbf{3}} - ({R^{\mathbf{3}}}_{ij} {\tilde{\Sigma}_{\mathbf{1}}}^{1j} + {R^{\mathbf{1}}}_{ij} {\tilde{\Sigma}_{\mathbf{2}}}^{1j}) / {\tilde{\Sigma}_{\mathbf{1}}}^{01}, \tag{6.16c}$$

while the remaining equations of motion are:

$$\dot{\tilde{\Sigma}}{}_{\mathbf{A}}^{01} = 2\eta^{\mathbf{C}}{}_{\mathbf{BA}} B_0^{\mathbf{B}} \tilde{\Sigma}_{\mathbf{C}}^{01} + D_i \tilde{\Sigma}_{\mathbf{A}}^{1i} \tag{6.16d}$$

$$\dot{\tilde{\Sigma}}{}_{\mathbf{A}}^{0i} = 2\eta^{\mathbf{C}}{}_{\mathbf{BA}} B_0^{\mathbf{B}} \tilde{\Sigma}_{\mathbf{C}}^{0i} - D_1 \tilde{\Sigma}_{\mathbf{A}}^{1i} - D_j \tilde{\Sigma}_{\mathbf{A}}^{ji} \tag{6.16e}$$

$$\begin{aligned}\dot{B}_1^{\mathbf{A}} = D_1 B_0^{\mathbf{A}} + \delta_{\mathbf{1}}^{\mathbf{A}} ({\tilde{\Sigma}_{\mathbf{2}}}^{23} \lambda^1 + {\tilde{\Sigma}_{\mathbf{3}}}^{23} \lambda^2 - {\tilde{\Sigma}_{\mathbf{1}}}^{23} \lambda^9) + \delta_{\mathbf{2}}^{\mathbf{A}} ({\tilde{\Sigma}_{\mathbf{1}}}^{ij} \lambda^1 + \lambda^3) \\ + \delta_{\mathbf{3}}^{\mathbf{A}} ({\tilde{\Sigma}_{\mathbf{1}}}^{ij} \lambda^2 + \lambda^4).\end{aligned} \tag{6.16f}$$

6.4 Einstein Equations

Since we have now finished the canonical analysis and have shown that no additional constraints remain, we derive the Einstein equations. We use five of the secondary equations

(6.13a, 6.13b and 6.14a), with (5.36) to define the Einstein constraint equations.

$$\dot{P}^{\mathbf{2}}{}_{1j}\tilde{\Sigma}_{\mathbf{2}}{}^{1j} = \left(R^{\mathbf{2}}{}_{1j}\tilde{\Sigma}_{\mathbf{1}}{}^{01} + R^{\mathbf{2}}{}_{ij}\tilde{\Sigma}_{\mathbf{1}}{}^{0i} + R^{\mathbf{1}}{}_{ij}\tilde{\Sigma}_{\mathbf{3}}{}^{0i}\right)\tilde{\Sigma}_{\mathbf{2}}{}^{1j} \approx 0 \Longleftrightarrow G^{\mathbf{0}}{}_{\mathbf{0}} \approx 0 \tag{6.17a}$$

$$\dot{P}^{\mathbf{3}}{}_{1j}\tilde{\Sigma}_{\mathbf{3}}{}^{0j} = \left(R^{\mathbf{3}}{}_{1j}\tilde{\Sigma}_{\mathbf{1}}{}^{01} + R^{\mathbf{3}}{}_{ij}\tilde{\Sigma}_{\mathbf{1}}{}^{0i} + R^{\mathbf{1}}{}_{ij}\tilde{\Sigma}_{\mathbf{2}}{}^{0i}\right)\tilde{\Sigma}_{\mathbf{3}}{}^{0j} \approx 0 \Longleftrightarrow G^{\mathbf{0}}{}_{\mathbf{1}} \approx 0 \tag{6.17b}$$

$$\dot{P}^{\mathbf{1}}{}_{1j}\tilde{\Sigma}_{\mathbf{2}}{}^{1j} = \left(R^{\mathbf{1}}{}_{1j}\tilde{\Sigma}_{\mathbf{1}}{}^{01} + R^{\mathbf{1}}{}_{ij}\tilde{\Sigma}_{\mathbf{1}}{}^{0i} + R^{\mathbf{3}}{}_{ij}\tilde{\Sigma}_{\mathbf{3}}{}^{0i}\right)\tilde{\Sigma}_{\mathbf{2}}{}^{1j} \approx 0 \Longleftrightarrow G^{\mathbf{0}}{}_{\mathbf{2}} \approx 0 \tag{6.17c}$$

$$\dot{P}^{\mathbf{1}}{}_{1j}\tilde{\Sigma}_{\mathbf{3}}{}^{0j} \approx \left(R^{\mathbf{1}}{}_{1j}\tilde{\Sigma}_{\mathbf{1}}{}^{01} + R^{\mathbf{1}}{}_{ij}\tilde{\Sigma}_{\mathbf{1}}{}^{0i} + R^{\mathbf{3}}{}_{ij}\tilde{\Sigma}_{\mathbf{3}}{}^{0i}\right)\tilde{\Sigma}_{\mathbf{3}}{}^{0j} \approx 0 \Longleftrightarrow G^{\mathbf{0}}{}_{\mathbf{3}} \approx 0 \tag{6.17d}$$

$$\dot{P}^{\mathbf{2}}{}_{1j}\tilde{\Sigma}_{\mathbf{3}}{}^{0j} \approx \left(R^{\mathbf{2}}{}_{1j}\tilde{\Sigma}_{\mathbf{1}}{}^{01} + R^{\mathbf{2}}{}_{ij}\tilde{\Sigma}_{\mathbf{1}}{}^{0i} + R^{\mathbf{1}}{}_{ij}\tilde{\Sigma}_{\mathbf{3}}{}^{0i}\right)\tilde{\Sigma}_{\mathbf{3}}{}^{0j} \approx 0 \Longleftrightarrow G^{\mathbf{2}}{}_{\mathbf{3}} \approx 0. \tag{6.17e}$$

The remaining Einstein equations are obtained from the equations of motion. Therefore we can show using (6.16) and (5.36) that

$$\dot{A}^{\mathbf{2}}_{i}\tilde{\Sigma}_{\mathbf{2}}{}^{1i} \approx -\left(R^{\mathbf{2}}{}_{0i}\tilde{\Sigma}_{\mathbf{1}}{}^{01} + R^{\mathbf{1}}{}_{ij}\tilde{\Sigma}_{\mathbf{3}}{}^{1j} + R^{\mathbf{2}}{}_{ij}\tilde{\Sigma}_{\mathbf{1}}{}^{1j}\right)\tilde{\Sigma}_{\mathbf{2}}{}^{1i} \approx 0 \Longleftrightarrow G^{\mathbf{1}}{}_{\mathbf{0}} \approx 0 \tag{6.18a}$$

$$\dot{A}^{\mathbf{1}}_{i}\tilde{\Sigma}_{\mathbf{2}}{}^{1i} \approx -\left(R^{\mathbf{1}}{}_{0i}\tilde{\Sigma}_{\mathbf{1}}{}^{01} + R^{\mathbf{1}}{}_{ij}\tilde{\Sigma}_{\mathbf{1}}{}^{1j} + R^{\mathbf{2}}{}_{ij}\tilde{\Sigma}_{\mathbf{2}}{}^{1j}\right)\tilde{\Sigma}_{\mathbf{2}}{}^{1i} \approx 0 \Longleftrightarrow G^{\mathbf{1}}{}_{\mathbf{2}} \approx 0 \tag{6.18b}$$

$$\dot{A}^{\mathbf{1}}_{i}\tilde{\Sigma}_{\mathbf{3}}{}^{0i} \approx -\left(R^{\mathbf{1}}{}_{0i}\tilde{\Sigma}_{\mathbf{1}}{}^{01} + R^{\mathbf{1}}{}_{ij}\tilde{\Sigma}_{\mathbf{1}}{}^{1j} + R^{\mathbf{2}}{}_{ij}\tilde{\Sigma}_{\mathbf{2}}{}^{1j}\right)\tilde{\Sigma}_{\mathbf{3}}{}^{0i} \approx 0 \Longleftrightarrow G^{\mathbf{1}}{}_{\mathbf{3}} \approx 0 \tag{6.18c}$$

$$\dot{A}^{\mathbf{3}}_{i}\tilde{\Sigma}_{\mathbf{2}}{}^{1i} \approx -\left(R^{\mathbf{3}}{}_{0i}\tilde{\Sigma}_{\mathbf{1}}{}^{01} + R^{\mathbf{1}}{}_{ij}\tilde{\Sigma}_{\mathbf{2}}{}^{1j} + R^{\mathbf{3}}{}_{ij}\tilde{\Sigma}_{\mathbf{1}}{}^{0j}\right)\tilde{\Sigma}_{\mathbf{2}}{}^{1i} \approx 0 \Longleftrightarrow G^{\mathbf{3}}{}_{\mathbf{2}} \approx 0 \tag{6.18d}$$

$$\begin{aligned}\dot{A}^{\mathbf{1}}_{i}\tilde{\Sigma}_{\mathbf{1}}{}^{0i} + \dot{B}^{\mathbf{1}}_{1}\tilde{\Sigma}_{\mathbf{1}}{}^{01} &\approx -\left(R^{\mathbf{1}}{}_{0i}\tilde{\Sigma}_{\mathbf{1}}{}^{01} + R^{\mathbf{1}}{}_{ij}\tilde{\Sigma}_{\mathbf{1}}{}^{1j} + R^{\mathbf{2}}{}_{ij}\tilde{\Sigma}_{\mathbf{2}}{}^{1j}\right)\tilde{\Sigma}_{\mathbf{1}}{}^{0i}\\ &-\left(R^{\mathbf{1}}{}_{01}\tilde{\Sigma}_{\mathbf{1}}{}^{01} - R^{\mathbf{1}}{}_{23}\tilde{\Sigma}_{\mathbf{1}}{}^{23} - R^{\mathbf{2}}{}_{23}\tilde{\Sigma}_{\mathbf{2}}{}^{23} - R^{\mathbf{3}}{}_{23}\tilde{\Sigma}_{\mathbf{3}}{}^{23}\right)\tilde{\Sigma}_{\mathbf{1}}{}^{01} \approx 0 \Longleftrightarrow G^{\mathbf{3}}{}_{\mathbf{3}} \approx 0.\end{aligned} \tag{6.18e}$$

We have shown that we can derive all the Einstein equations from the constraint and evolution equations. The structure equations are obtained in exactly the same way as in the previous chapter. The final stage is to ascertain which constraints are first class and calculate their algebra.

6.5 First class constraints

We now move to the next stage of the Dirac-Bergmann algorithm and calculate the first class equations, followed by the first class constraint algebra. We can see that some of the secondary constraints (6.13) are the same as (5.30) and therefore we adapt them in the same manor to obtain four first class constraints. Due to not using the shortcut method in this chapter an additional two first class constraints are obtained. These are the primary constraints $P^{\mathbf{1}}{}_{23} = 0$ and $\tilde{P}_1 = 0$.

We now derive the remaining four first class constraints. We start by adding the Gauss constraint (6.13d), as well as the constraints $\tilde{\Sigma}_{\mathbf{2}}{}^{01} \approx 0$ and $\tilde{\Sigma}_{\mathbf{3}}{}^{01} \approx 0$ to (6.13a):

$$-R^{\mathbf{1}}{}_{1j}\tilde{\Sigma}_{\mathbf{1}}{}^{01} - R^{\mathbf{2}}{}_{ij}\tilde{\Sigma}_{\mathbf{2}}{}^{0i} - R^{\mathbf{3}}{}_{ij}\tilde{\Sigma}_{\mathbf{3}}{}^{0i} - R^{\mathbf{1}}{}_{ij}\tilde{\Sigma}_{\mathbf{1}}{}^{0i} - R^{\mathbf{2}}{}_{1j}\tilde{\Sigma}_{\mathbf{2}}{}^{01} - R^{\mathbf{3}}{}_{1j}\tilde{\Sigma}_{\mathbf{3}}{}^{01}$$
$$-A_j^{\mathbf{A}}\left(D_1\tilde{\Sigma}_{\mathbf{A}}{}^{01} + D_i\tilde{\Sigma}_{\mathbf{A}}{}^{0i}\right) \approx 0$$
$$\Rightarrow R^{\mathbf{A}}{}_{j1}\tilde{\Sigma}_{\mathbf{A}}{}^{01} + R^{\mathbf{A}}{}_{ji}\tilde{\Sigma}_{\mathbf{A}}{}^{0i} - A_j^{\mathbf{A}}\left(D_1\tilde{\Sigma}_{\mathbf{A}}{}^{01} + D_i\tilde{\Sigma}_{\mathbf{A}}{}^{0i}\right) \approx 0 \qquad (6.19a)$$
$$\Rightarrow \tilde{\Sigma}_{\mathbf{A}}{}^{01}(-A^{\mathbf{A}}_{j,1} + D_jB_1^{\mathbf{A}}) - \tilde{\Sigma}_{\mathbf{A}}{}^{0i}(D_iA_j^{\mathbf{A}} - A^{\mathbf{A}}_{i,j}) - A_j^{\mathbf{A}}D_i\tilde{\Sigma}_{\mathbf{A}}{}^{0i} - A_j^{\mathbf{A}}D_1\tilde{\Sigma}_{\mathbf{A}}{}^{01} \approx 0$$
$$\Rightarrow B^{\mathbf{A}}_{1,j}\tilde{\Sigma}_{\mathbf{A}}{}^{01} + A^{\mathbf{A}}_{i,j}\tilde{\Sigma}_{\mathbf{A}}{}^{0i} - \left(A_j^{\mathbf{A}}\tilde{\Sigma}_{\mathbf{A}}{}^{0i}\right)_{,i} - \left(A_j^{\mathbf{A}}\tilde{\Sigma}_{\mathbf{A}}{}^{01}\right)_{,1} \approx 0. \qquad (6.19b)$$

Note that this is just the same as the constraint (5.39).

Another first class constraint can be obtained by combining (6.13b) and (6.14a).

$$\tilde{\Sigma}_{\mathbf{2}}{}^{0i}\dot{P}^{\mathbf{2}}{}_{1i} + \tilde{\Sigma}_{\mathbf{3}}{}^{0i}\dot{P}^{\mathbf{3}}{}_{1i} \approx 0$$
$$\Rightarrow \tilde{\Sigma}_{\mathbf{1}}{}^{01}\left(R^{\mathbf{2}}{}_{1i}\tilde{\Sigma}_{\mathbf{2}}{}^{0i} + R^{\mathbf{3}}{}_{1i}\tilde{\Sigma}_{\mathbf{3}}{}^{0i}\right) + \tilde{\Sigma}_{\mathbf{1}}{}^{0j}\left(R^{\mathbf{2}}{}_{ji}\tilde{\Sigma}_{\mathbf{2}}{}^{0i} + R^{\mathbf{3}}{}_{ji}\tilde{\Sigma}_{\mathbf{3}}{}^{0i}\right) \approx 0,$$

using (6.13a) results in:

$$\Rightarrow \tilde{\Sigma}_{\mathbf{1}}{}^{01}\left(R^{\mathbf{A}}{}_{1i}\tilde{\Sigma}_{\mathbf{A}}{}^{0i}\right) \approx 0$$
$$\Rightarrow \left(R^{\mathbf{A}}{}_{1i}\tilde{\Sigma}_{\mathbf{A}}{}^{0i}\right) \approx 0. \qquad (6.20a)$$

Then in a similar manner to the previous constraint we add the Gauss constraint multiplied by $-B_1^{\mathbf{A}}$

$$\Rightarrow \left(R^{\mathbf{A}}{}_{1i}\tilde{\Sigma}_{\mathbf{A}}{}^{0i}\right) - B_1^{\mathbf{A}}\left(D_1\tilde{\Sigma}_{\mathbf{A}}{}^{01} + D_i\tilde{\Sigma}_{\mathbf{A}}{}^{0i}\right) \approx 0 \qquad (6.20b)$$
$$\Rightarrow A^{\mathbf{A}}_{i,1}\tilde{\Sigma}_{\mathbf{A}}{}^{0i} - \tilde{\Sigma}_{\mathbf{A}}{}^{0i}D_iB_1^{\mathbf{A}} - B_1^{\mathbf{A}}D_1\tilde{\Sigma}_{\mathbf{A}}{}^{01} - B_1^{\mathbf{A}}D_i\tilde{\Sigma}_{\mathbf{A}}{}^{0i} \approx 0$$
$$\Rightarrow B^{\mathbf{A}}_{1,1}\tilde{\Sigma}_{\mathbf{A}}{}^{01} + A^{\mathbf{A}}_{i,1}\tilde{\Sigma}_{\mathbf{A}}{}^{0i} - (B_1^{\mathbf{A}}\tilde{\Sigma}_{\mathbf{A}}{}^{01})_{,1} - (B_1^{\mathbf{A}}\tilde{\Sigma}_{\mathbf{A}}{}^{0i})_{,i} \approx 0, \qquad (6.20c)$$

which is the same as (5.41).

Just as in the previous chapter the final first class constraint arises from the propagation of the momenta $\tilde{P}_{\mathbf{1}}$, (6.11a), which results in the Gauss constraint:

$$D_1\tilde{\Sigma}_{\mathbf{1}}{}^{01} + D_i\tilde{\Sigma}_{\mathbf{1}}{}^{0i} = 0. \tag{6.21}$$

We are required to include some additional terms because we are not using the shortcut method. These terms are required to ensure the cyclic variables transform correctly.

$$D_1\tilde{\Sigma}_{\mathbf{1}}{}^{01} + D_i\tilde{\Sigma}_{\mathbf{1}}{}^{0i} + 2\eta^{\mathbf{C}}{}_{\mathbf{B1}}P^{\mathbf{B}}{}_{1i}\tilde{\Sigma}_{\mathbf{C}}{}^{1i} + 2\eta^{\mathbf{C}}{}_{\mathbf{B1}}P^{\mathbf{B}}{}_{23}\tilde{\Sigma}_{\mathbf{C}}{}^{23} - 2\eta^{\mathbf{C}}{}_{\mathbf{B1}}B^{\mathbf{B}}_0\tilde{P}_{\mathbf{C}} \approx 0. \tag{6.22}$$

Extra terms are also required in the three constraints ψ_1, ψ_p to ensure the correct transformation of the cyclic variables. These are shown in the summary of the six first class constraints given below:

$$P^{\mathbf{1}}{}_{23} = 0 \tag{6.23a}$$

$$\tilde{P}_{\mathbf{1}} = 0 \tag{6.23b}$$

$$\mathcal{G}_{\mathbf{1}} := D_1\tilde{\Sigma}_{\mathbf{1}}{}^{01} + D_i\tilde{\Sigma}_{\mathbf{1}}{}^{0i} + 2\eta^{\mathbf{C}}{}_{\mathbf{B1}}P^{\mathbf{B}}{}_{1i}\tilde{\Sigma}_{\mathbf{C}}{}^{1i} + 2\eta^{\mathbf{C}}{}_{\mathbf{B1}}P^{\mathbf{B}}{}_{23}\tilde{\Sigma}_{\mathbf{C}}{}^{23} - 2\eta^{\mathbf{C}}{}_{\mathbf{B1}}B^{\mathbf{B}}_0\tilde{P}_{\mathbf{C}} \approx 0 \tag{6.23c}$$

$$\begin{aligned}\psi_1 := {} & B^{\mathbf{A}}_{1,1}\tilde{\Sigma}_{\mathbf{A}}{}^{01} + A^{\mathbf{A}}_{i,1}\tilde{\Sigma}_{\mathbf{A}}{}^{0i} - (B^{\mathbf{A}}_1\tilde{\Sigma}_{\mathbf{A}}{}^{01})_{,1} - (B^{\mathbf{A}}_1\tilde{\Sigma}_{\mathbf{A}}{}^{0i})_{,i} \\ & + B^{\mathbf{A}}_{0,1}\tilde{P}_{\mathbf{A}} - \tilde{\Sigma}_{\mathbf{A}}{}^{23}P^{\mathbf{A}}{}_{23,1} - \tilde{\Sigma}_{\mathbf{A}}{}^{1i}P^{\mathbf{A}}{}_{1i,1} + (\tilde{\Sigma}_{\mathbf{A}}{}^{1i}P^{\mathbf{A}}{}_{1i})_{,1} \approx 0\end{aligned} \tag{6.23d}$$

$$\begin{aligned}\psi_p := {} & B^{\mathbf{A}}_{1,j}\tilde{\Sigma}_{\mathbf{A}}{}^{01} + A^{\mathbf{A}}_{i,j}\tilde{\Sigma}_{\mathbf{A}}{}^{0i} - (A^{\mathbf{A}}_j\tilde{\Sigma}_{\mathbf{A}}{}^{0i})_{,i} - (A^{\mathbf{A}}_j\tilde{\Sigma}_{\mathbf{A}}{}^{01})_{,1} \\ & + B^{\mathbf{A}}_{o,j}\tilde{P}_{\mathbf{A}} - \tilde{\Sigma}_{\mathbf{A}}{}^{23}P^{\mathbf{A}}{}_{23,j} - \tilde{\Sigma}_{\mathbf{A}}{}^{1i}P^{\mathbf{A}}{}_{1i,j} + (\tilde{\Sigma}_{\mathbf{A}}{}^{1i}P^{\mathbf{A}}{}_{1j})_{,i} \approx 0.\end{aligned} \tag{6.23e}$$

The remaining twenty eight constraints, shown below, are second class.

$$\begin{array}{llll} P^{\mathbf{A}}{}_{1i} = 0 & P^{\mathbf{2}}{}_{23} = 0 & P^{\mathbf{3}}{}_{23} = 0 & \\ \tilde{P}_{\mathbf{2}} = 0 & \tilde{P}_{\mathbf{3}} = 0 & C_\alpha = 0 & \alpha = 1, ..., 9 \end{array}$$

$$\left(D_1\tilde{\Sigma}_{\mathbf{A}}{}^{01} + D_i\tilde{\Sigma}_{\mathbf{A}}{}^{0i}\right)\delta_{\mathbf{2}}^{\mathbf{A}} = 0 \qquad \left(D_1\tilde{\Sigma}_{\mathbf{A}}{}^{01} + D_i\tilde{\Sigma}_{\mathbf{A}}{}^{0i}\right)\delta_{\mathbf{3}}^{\mathbf{A}} = 0$$
$$\left(D_i\tilde{\Sigma}_{\mathbf{A}}{}^{1i} + 2\eta^{\mathbf{c}}{}_{\mathbf{AB}}B_0^{\mathbf{B}}\tilde{\Sigma}_{\mathbf{C}}{}^{01}\right)\delta_{\mathbf{2}}^{\mathbf{A}} = 0 \qquad \left(D_i\tilde{\Sigma}_{A}{}^{1i} + 2\eta^{\mathbf{c}}{}_{\mathbf{AB}}B_0^{\mathbf{B}}\tilde{\Sigma}_{\mathbf{C}}{}^{01}\right)\delta_{\mathbf{3}}^{\mathbf{A}} = 0$$
$$\left(R^{\mathbf{A}}{}_{1j}\tilde{\Sigma}_{\mathbf{1}}{}^{01} + R^{\mathbf{A}}{}_{ij}\tilde{\Sigma}_{\mathbf{1}}{}^{0i}\right)\delta_{\mathbf{2}}^{\mathbf{A}} = 0 \qquad \left(R^{\mathbf{A}}{}_{1j}\tilde{\Sigma}_{\mathbf{1}}{}^{01} + R^{\mathbf{A}}{}_{ij}\tilde{\Sigma}_{\mathbf{1}}{}^{0i}\right)\delta_{\mathbf{3}}^{\mathbf{A}} = 0$$
$$D_1\tilde{\Sigma}_{\mathbf{2}}{}^{i1} + D_j\tilde{\Sigma}_{\mathbf{2}}{}^{ij} - 2\eta^{\mathbf{C}}{}_{\mathbf{2A}}B_0^{\mathbf{B}}\tilde{\Sigma}_{\mathbf{C}}{}^{0i} = 0 \qquad \tilde{\Sigma}_{\mathbf{1}}{}^{1i}(R^{\mathbf{3}}{}_{1i}\tilde{\Sigma}_{\mathbf{1}}{}^{01} - R^{\mathbf{3}}{}_{ij}\tilde{\Sigma}_{\mathbf{1}}{}^{0j} - R^{\mathbf{1}}{}_{ij}\tilde{\Sigma}_{\mathbf{2}}{}^{0j}) = 0.$$

We can see that many of our first class constraints are the same as those calculated in the previous chapter, but in this chapter we obtain an additional two constraints (6.23a) and (6.23b). This is due to not using the shortcut method.

We now check the number of degrees of freedom using the standard formula (2.73) which gives $\frac{1}{2}(42 - 2(6) - 28) = 1$ degree of freedom. This is just what is to be expected from a null formulation of general relativity, and is the same as calculated in earlier chapters.

Now that we are confident that we have obtained all the first class constraints we calculate the first class algebra. Below we show only those term that are not strongly zero:

$$\left\{\int_x \tilde{f}P^{\mathbf{1}}{}_{23}\,\mathrm{d}^3x, \int_y g^i\psi_i\,\mathrm{d}^3y\right\} = \int_z \tilde{f}\mathcal{L}_g P^{\mathbf{1}}{}_{23}\,\mathrm{d}^3z$$
$$\left\{\int_x \tilde{f}P^{\mathbf{1}}{}_{23}\,\mathrm{d}^3x, \int_y g^1\,\psi_1\,\mathrm{d}^3y\right\} = \int_z \tilde{f}\mathcal{L}_g P^{\mathbf{1}}{}_{23}\,\mathrm{d}^3z$$
$$\left\{\int_x f\tilde{P}_{\mathbf{1}}\,\mathrm{d}^3x, \int_y g^i\psi_i\,\mathrm{d}^3y\right\} = \int_z f\mathcal{L}_g\tilde{P}_{\mathbf{1}}\,\mathrm{d}^3z$$
$$\left\{\int_x f\tilde{P}_{\mathbf{1}}\,\mathrm{d}^3x, \int_y g^1\,\psi_1\,\mathrm{d}^3y\right\} = \int_z f\mathcal{L}_g\tilde{P}_{\mathbf{1}}\,\mathrm{d}^3z$$
$$\left\{\int_x f\,\mathcal{G}_{\mathbf{1}}\,\mathrm{d}^3x, \int_y g\,\psi_1\,\mathrm{d}^3y\right\} = \int_z f\mathcal{L}_g\mathcal{G}_{\mathbf{1}}\,\mathrm{d}^3z$$
$$\left\{\int_x f\,\mathcal{G}_{\mathbf{1}}\,\mathrm{d}^3x, \int_y g^i\psi_i\,\mathrm{d}^3y\right\} = \int_z f\mathcal{L}_g\mathcal{G}_{\mathbf{1}}\,\mathrm{d}^3z$$
$$\left\{\int_x f^1\psi_1\,\mathrm{d}^3x, \int_y g^1\,\psi_1\,\mathrm{d}^3y\right\} = \int_z f^1\mathcal{L}_g\psi_1\,\mathrm{d}^3z$$
$$\left\{\int_x f^1\psi_1\,\mathrm{d}^3x, \int_y g^i\,\psi_i\,\mathrm{d}^3y\right\} = \int_z f^1\mathcal{L}_g\psi_i\,\mathrm{d}^3z$$
$$\left\{\int_x f^i\psi_i\,\mathrm{d}^3x, \int_y g^j\,\psi_j\,\mathrm{d}^3y\right\} = \int_z f^i\mathcal{L}_g\psi_i\,\mathrm{d}^3z.$$

Now that we have calculated the first class algebra, we will give the geometrical interpretation of the first class constraints. We use the same method as before and calculate the infinitesimal transformations the constraints generate. We do not consider the constraints $\tilde{P}_1$ and $P^1{}_{23}$ as they just indicate the gauge freedom to choose the variables B^1_0 and $\tilde{\Sigma}_1{}^{23}$. Therefore we start with the constraint ψ_1:

$$\delta A^{\mathbf{A}}_i = \left\{A^{\mathbf{A}}_i, \psi_1(g^1)\right\} = \mathcal{L}_g A^{\mathbf{A}}_i \tag{6.24a}$$
$$\delta B^{\mathbf{A}}_1 = \left\{B^{\mathbf{A}}_0, \psi_1(g^1)\right\} = \mathcal{L}_g B^{\mathbf{A}}_1 \tag{6.24b}$$
$$\delta B^{\mathbf{A}}_0 = \left\{B^{\mathbf{A}}_0, \psi_1(g^1)\right\} = \mathcal{L}_g B^{\mathbf{A}}_0 \tag{6.24c}$$
$$\delta \tilde{\Sigma}_{\mathbf{A}}{}^{23} = \left\{\tilde{\Sigma}_{\mathbf{A}}{}^{23}, \psi_1(g^1)\right\} = \mathcal{L}_g \tilde{\Sigma}_{\mathbf{A}}{}^{23} \tag{6.24d}$$
$$\delta \tilde{\Sigma}_{\mathbf{A}}{}^{1i} = \left\{\tilde{\Sigma}_{\mathbf{A}}{}^{1i}, \psi_1(g^1)\right\} = \mathcal{L}_g \tilde{\Sigma}_{\mathbf{A}}{}^{1i}. \tag{6.24e}$$

We can see from the above that this constraint generates the diffeomorphism in the x^1 direction as before. Then we look at the constraint ψ_i:

$$\delta A^{\mathbf{A}}_i = \left\{A^{\mathbf{A}}_i, \psi_i(g^i)\right\} = \mathcal{L}_g A^{\mathbf{A}}_i \tag{6.25a}$$
$$\delta B^{\mathbf{A}}_1 = \left\{B^{\mathbf{A}}_0, \psi_i(g^i)\right\} = \mathcal{L}_g B^{\mathbf{A}}_1 \tag{6.25b}$$
$$\delta B^{\mathbf{A}}_0 = \left\{B^{\mathbf{A}}_0, \psi_i(g^i)\right\} = \mathcal{L}_g B^{\mathbf{A}}_0 \tag{6.25c}$$
$$\delta \tilde{\Sigma}_{\mathbf{A}}{}^{23} = \left\{\tilde{\Sigma}_{\mathbf{A}}{}^{23}, \psi_i(g^i)\right\} = \mathcal{L}_g \tilde{\Sigma}_{\mathbf{A}}{}^{23} \tag{6.25d}$$
$$\delta \tilde{\Sigma}_{\mathbf{A}}{}^{1i} = \left\{\tilde{\Sigma}_{\mathbf{A}}{}^{1i}, \psi_i(g^i)\right\} = \mathcal{L}_g \tilde{\Sigma}_{\mathbf{A}}{}^{1i}. \tag{6.25e}$$

We see, just as before, that ψ_i generates the diffeomorphisms in the two surface.

In an analogous manner to before, the constraint $\mathcal{G}_1$ generates the self-dual spin and boost transformations. This is seen by comparing the infinitesimal transformations given below with those of (5.51) and (5.53):

$$\delta A_i^{\mathbf{A}} = \left\{A_i^{\mathbf{A}}, \mathcal{G}_1(g)\right\} = -g_{,i}\delta_{\mathbf{1}}^{\mathbf{A}} - 2gA_i^{\mathbf{2}}\delta_{\mathbf{2}}^{\mathbf{A}} + 2gA_i^{\mathbf{3}}\delta_{\mathbf{3}}^{\mathbf{A}} \tag{6.26a}$$

$$\delta B_1^{\mathbf{A}} = \left\{B_1^{\mathbf{A}}, \mathcal{G}_1(g)\right\} = -g_{,1}\delta_{\mathbf{1}}^{\mathbf{A}} - 2gB_1^{\mathbf{2}}\delta_{\mathbf{2}}^{\mathbf{A}} + 2gB_1^{\mathbf{3}}\delta_{\mathbf{3}}^{\mathbf{A}} \tag{6.26b}$$

$$\delta B_0^{\mathbf{A}} = \left\{B_0^{\mathbf{A}}, \mathcal{G}_1(g)\right\} = -2gB_0^{\mathbf{2}}\delta_{\mathbf{2}}^{\mathbf{A}} + 2gB_0^{\mathbf{3}}\delta_{\mathbf{3}}^{\mathbf{A}} \tag{6.26c}$$

$$\delta \tilde{\Sigma}_{\mathbf{A}}{}^{23} = \left\{\tilde{\Sigma}_{\mathbf{A}}{}^{23}, \mathcal{G}_1(g)\right\} = -2g\tilde{\Sigma}_{\mathbf{2}}{}^{23}\delta_{\mathbf{2}}^{\mathbf{A}} + 2g\tilde{\Sigma}_{\mathbf{3}}{}^{23}\delta_{\mathbf{3}}^{\mathbf{A}} \tag{6.26d}$$

$$\delta \tilde{\Sigma}_{\mathbf{A}}{}^{1i} = \left\{\tilde{\Sigma}_{\mathbf{A}}{}^{1i}, \mathcal{G}_1(g)\right\} = -2g\tilde{\Sigma}_{\mathbf{2}}{}^{1i}\delta_{\mathbf{2}}^{\mathbf{A}} + 2g\tilde{\Sigma}_{\mathbf{3}}{}^{1i}\delta_{\mathbf{3}}^{\mathbf{A}}. \tag{6.26e}$$

This concludes our analysis of the first class constraints.

6.6 Summary

In this chapter we have derived a double null Hamiltonian using only the connection and the adapted $SO(3)$ triad. This has resulted in a simplified system of equations. By applying the shortcut method, further simplification could have been obtained. A problem with the shortcut method arises when considering the constraint (6.14b), because we would have multiplied a constraint with a multiplier, and then called it a multiplier equation. It was for this reason that the shortcut method was not used.

The work of this chapter resulted in a first class algebra for General Relativity using only self-dual variables. All the first class constraints could be given a geometrical interpretation. It can be seen from these interpretations how they relate to the Ashtekar approach, or that of Goldberg *et al.* (1992) described in the Appendix. We see in these approaches that the constraints (6.23d) and (6.23e) are the double null representation of the momentum constraints (A.10b) and (B.13a). The constraint (6.23c) is equivalent to the constraints (A.10c) and (B.13b). The greater number of constraints in (A.10c) or (B.13b) relates to the greater freedom these descriptions contain. The Hamiltonian constraint (A.10a) is split within the second class constraints.

Chapter 7

Discussion

The area of quantum gravity is still one of the key areas of research in General Relativity. Despite the many efforts by eminent scientists, a full satisfactory description has not been obtained. This reveals the complexity of the issues that surround quantising gravity. Some of these issues were revealed when the canonical method of quantisation was applied to the ADM description of General Relativity (see Arnowitt *et al.* 1960). This work showed that the Hamiltonian constraint was ill-defined at the quantum level, which meant the remaining steps of the quantisation process could not be completed.

Although attempts to quantise gravity based on the ADM approach failed, the failure was not due to an inherent difficulty with the canonical quantisation method but rather to the particular structure of the constraints in the ADM formalism. Therefore Torre (1986) adapted the description of General Relativity to a $2 + 2$ description, but he set only the non evolution direction to be null, and not using the full double null approach. His work revealed the fact that the Hamiltonian constraint becomes second class in null descriptions. Therefore issues that occurred with the ADM description could be circumvented using alternative methods that remove the second class constraints (for example replacing the Poisson brackets with Dirac brackets). As we have discussed while this approach overcomes the previous difficulties, the complexity of the constraints means that the canonical quantisation can not be concluded.

These issues regarding the complexity of the constraints led Ashtekar to devise alternative variables which simplified the constraints. In his work he used variables that were complex and self-dual. Using such variables he was able to obtain constraints that were polynomial. Unfortunately in the Ashtekar approach the Hamiltonian constraint was first class, and as in the ADM case this constraint was not well defined at the quantum level; hence the canonical process was never completed.

Goldberg *et al.* (1992) used a version of Ashtekar variables and applied it to a null description of General Relativity, which built on Torre's work that indicated the Hamiltonian constraint would become second class. This managed to overcome some difficulties found in previous work, but unfortunately in their work they choose a $3+1$ description and set the hypersurface to be null. This resulted in complicated first class constraints which preserve the slicing.

Therefore in this thesis we set out to apply the first stage of the canonical quantisation process (the canonical analysis) to a double null description of General Relativity using self-dual complex variables. The advantage of using a double null description of General Relativity is that the null directions are both normal to the two surface, and therefore the only gauge freedoms that remain are the spin and boost transformations. This should overcome the more complicated first class constraints that occur in Goldberg *et al.* (1992). Another advantage of using the double null method is that the Hamiltonian constraint is not a first class constraint. The advantage of using the self-dual variables is that they result in polynomial constraints. Therefore using the double null method overcomes some of the obstacles of earlier approaches.

In chapter 3 we approached this work first by extending the work of Torre to allow for two null directions. The motivation for using a double null approach is that a spacelike 2-surface naturally singles out two null directions so that the situation is geometrically simpler than for a null hypersurface where one has the gauge freedom arising form the lack of a canonical normal direction. However despite this simplification the non-polynomial nature of the constraints makes it very difficult to make progress with the later stages of the Dirac-Bergman algorithm. Despite this, the geometric analysis of the constraints in this situation provides us with valuable information when we come to analyse the constraints in the self-dual double null formulations used in chapters 5 and 6.

In chapter 4 we introduce a description of General Relativity in terms of self-dual 2-forms that is closely related to the use of Ashtekar variables. The use of these variables considerably simplifies the constraint analysis because it results in polynomial constraints. However, as outlined above problems relating to the nature of the Hamiltonian constraint and also the fact that the constraint algebra does not form a Lie algebra, remain with this approach.

In chapter 5 we went on to use the self-dual variables in a double null setting. Following Goldberg *et al.* (1992) we used a mixture of tetrad variables and densitised 2-forms. This enabled us to compare our work with their analysis and also had the advantage of the tetrad variables being similar to those used in chapter 3. This enabled us to use the geometric insight gained from chapter 3 to make an intelligent guess at which combinations of primary and secondary constraints would result in first class constraints with a clear geometrical interpretation. Although in theory it is possible to apply the Dirac-Bergman constraint analysis in a purely algorithmic fashion, in practice it is just too complicated to do this without some geometrical insight. The outcome of the work in chapter 5 was the construction of a polynomial first class constraint algebra that also formed a Lie algebra. In theory this should make the next step of the quantisation process easier.

In chapter 5 (see 5.42) we related our first class constraints to those obtained using the standard $3+1$ Ashtekar and also to those obtained by Goldberg *et al* using a 3+1 null slicing. In particular, by combining the diffeomorphisms in the 2-surface with those of the null generators in the hypersurface one obtains the three hypersurface diffeomorphism constraints found by Goldberg *et al.* (1992). In order to continue with the canonical quantisation process all constraints must be first class. A common method to accomplish this would be to use starred variables (an example can be found in Soteriou 1992). An alternative method would be to replace the Poisson bracket with the Dirac bracket. However the mixture of tetrad and self-dual 2-form variables mean that the relationship between the variables is rather complicated, and rather than pursue the next step of the process we attempt to move onto a description entirely in terms of the self-dual 2-forms.

In chapter 6 we worked solely with the self-dual densitised 2-form variables. These variables are not independent but have to satisfy a number of constraints if they are to represent 2-forms which are derived from a null frame. In general, these conditions are very complicated, but when combined with the double null condition they simplify considerably. This results in the same first class algebra that was obtained in chapter 5, yet with a simpler constraint structure which is particularly evident with the second class constraints. It was only because of the work in chapters 3 and 5 that it was possible to have the geometric insight required to make the appropriate combination of primary and secondary constraints that form the first class constraints.

The result of the work in chapter 6 was the derivation of a double null first class constraint algebra which also formed a Lie algebra. The constraints were polynomial with the Hamiltonian constraint becoming second class. Therefore some of the difficulties that occur in the earlier formulations of General Relativity do not arise. There were four constraints containing geometrical meaning: two of the constraints generated infinitesimal transformations in the spatial two surface $\{S\}$; one constraint gave the infinitesimal transformations in the x^1 direction; and the final constraint generated the self-dual spin and boost gauge freedoms. The ease with which we obtained the geometrical understanding of the first class constraints was a result of using a double null approach.

The work in chapter 6 contains not only the first class constraints, but also some second class constraints; just as in chapter 5. Therefore in a similar manner we would first have to consider the second class constraints before pursuing any further a quantum description of gravity. This may be achieved by the use of starred variables or Dirac brackets already outlined. Once only a first class algebra remains, we may progress towards a quantum description using the steps outlined in section 2.4.3. This would involve promoting the first class constraints to unambiguous quantum operators, from which an algebra could then be constructed by replacing the Dirac or Poisson brackets with commutation relations. Once this has been achieved, further steps are required before a complete and coherent quantum description can be obtained.

The final issue that needs discussing are the 'reality constraints'. From chapter 4 onwards we worked with a complexified version of General Relativity. However from a physical point of view we wanted to be able to regain the real version of the theory. In terms of the of 1-forms θ^α this is accomplished by a requirement that they correspond to a complex null basis formed from a real orthogonal basis. This requires that θ^α satisfies the equations:

$$\bar{\theta}^0 = \theta^0 \tag{7.1a}$$

$$\bar{\theta}^1 = \theta^0 \tag{7.1b}$$

$$\bar{\theta}^2 = \theta^3 \tag{7.1c}$$

It can easily be shown that this implies that the self-dual 2-forms $S^{\mathbf{A}}$ satisfy the six complex constraints:

$$S^{\mathbf{A}} \wedge \bar{S}^{\mathbf{B}} = 0. \tag{7.2}$$

Conversely if these conditions are satisfied one can find a basis θ^α such that the $S^{\mathbf{A}}$ are given by (4.34) and satisfy (7.1). So in terms of the $\tilde{\Sigma}_A{}^{\alpha\beta}$ variables the reality conditions are given by:

$$\epsilon_{\alpha\beta\gamma\delta}\tilde{\Sigma}_A{}^{\alpha\beta}\bar{\tilde{\Sigma}}_{\mathbf{B}}{}^{\gamma\delta} = 0. \tag{7.3}$$

Although the work in this thesis has used a double null 2+2 description of general relativity to describe the geometry at the Hamiltonian level we have broken the symmetry by singling out one of the two null directions as an evolution direction. We have already commented on the result of combining three of the first class constraints to give constraints that generate the diffeomorphism freedom of the hypersurfaces $\{\Sigma_0\}$. This reveals the $3+1$ nature of the canonical analysis we used. Even though we were using a double null description of gravity, we were required to choose an evolution direction, x^0. Therefore our description is really propagating the hypersurface Σ_0, just as earlier methods had done. This choice also breaks the symmetry between the two null direction x^0 and x^1. Therefore a natural extension to this work would be to consider a description in which both x^0 and x^1 are considered as evolution directions. This would maintain the symmetry and thus reveal the double null structure more clearly. Some work in this area has already been accomplished by Hayward (1993) in which he considered a Hamiltonian with two evolution directions. Before we build on this work we would have to be able to define Poisson brackets that are defined with two evolution directions. Work has been done in this area by Matteucci (2003), but it remains incomplete. At present it appears that three different Poisson brackets would be required: one for the surfaces $\{T\}$; one for the surfaces $\{S\}$ and the third would be needed to cross the two 2-surfaces (see Figure 3.2).

We finally concluded that although a quantum description of gravity remains a distant goal we have been able to overcome some of the obstacles of earlier methods, and provided a good base for future work in this area.

Appendix A

Introduction of Ashtekar variables

In chapter 4 we introduced self-dual variables using the approach of Giulini (1994). Then we used a local isomorphism to change the complex self-dual variables into the $SO(3)$ variables. This was not the original approach of Ashtekar (1991). In his work he performed a $3+1$ decomposition on the complex self-dual connection and curvature. Using a local isomorphism he was able to replace the pull back self-dual connection and curvature which occur in the Lagrangian, with a self-dual connection and curvature defined on the three surface. In this chapter we are going to outline this approach and show how these variables simplify the constraint equations.

We will use only the self-dual part of the action (4.27) because this is all that is required to obtain all the Einstein equations. $e^{\alpha}_{\boldsymbol{\alpha}}$ is the tetrad, and the self-dual curvature defined on the space-time is given by ${}^4\mathcal{R}_{\alpha\beta}{}^{\boldsymbol{\alpha\beta}}$. This curvature is just the same as the self-dual curvature ${}^{(+)}\Omega_{\boldsymbol{\alpha\beta}}$ (see 4.4), however, we drop the ${}^{(+)}$ because we will use only self-dual curvature. The ${}^{(+)}$ is replaced with ^{4} to help distinguish it from the self-dual curvature defined on the hypersurface, $\mathcal{R}_{\alpha\beta}{}^{\mathrm{ij}}$, that will be introduced later in this section. The action in this notation is given below:

$$I = \int (-g)^{1/2} e^{\alpha}_{\boldsymbol{\alpha}} e^{\beta}_{\boldsymbol{\beta}}\, {}^4\mathcal{R}_{\alpha\beta}{}^{\boldsymbol{\alpha\beta}}. \tag{A.1}$$

The self-dual curvature is defined by a self-dual connection, ${}^4\mathcal{A}_{\alpha}^{\boldsymbol{\alpha\beta}}$.

We now decompose the frame into the $3+1$ foliation, in which we consider a vector field t^α, whose integral curves intercept the hypersurfaces once and are transverse at its interception. This vector field can be decomposed as in section 2.2 so that $t^\alpha = Nn^\alpha + N^\alpha$. We define projection operators $q^\alpha_\beta := \delta^\alpha_\beta + n_\beta n^\alpha$ and n_α, where n^α is the unit normal to the hypersurfaces. Using these projections we split the frame into its normal and tangential parts,

$$e^\alpha_{\boldsymbol{\alpha}} = q^\alpha_\beta e^\beta_{\boldsymbol{\alpha}} - n^\alpha n_\beta e^\beta_{\boldsymbol{\alpha}} \tag{A.2}$$

$$:= E^\alpha_{\boldsymbol{\alpha}} - n^\alpha n_{\boldsymbol{\alpha}}. \tag{A.3}$$

Substituting the above into the action (A.1) we obtain

$$I = \int (N\gamma^{1/2})\big(E^\alpha_{\boldsymbol{\alpha}} E^\beta_{\boldsymbol{\beta}} {}^4\mathcal{R}_{\alpha\beta}{}^{\boldsymbol{\alpha\beta}} - 2E^\alpha_{\boldsymbol{\beta}} n^\beta n_{\boldsymbol{\alpha}} {}^4\mathcal{R}_{\alpha\beta}{}^{\boldsymbol{\alpha\beta}}\big). \tag{A.4}$$

We now introduce the projected frame density $\tilde{E}^\alpha_{\boldsymbol{\alpha}} := \gamma^{1/2} E^\alpha_{\boldsymbol{\alpha}}$, and use the self-dual identity of the self-dual curvature, ${}^4\mathcal{R}_{\alpha\beta}{}^{\boldsymbol{\alpha\beta}} + \frac{1}{2} i \epsilon^{\boldsymbol{\alpha\beta}}{}_{\boldsymbol{\gamma\delta}} {}^4\mathcal{R}_{\alpha\beta}{}^{\boldsymbol{\gamma\delta}} = 0$, to obtain:

$$I = \int \underset{\sim}{N} \tilde{E}^\alpha_{\boldsymbol{\alpha}} \tilde{E}^\beta_{\boldsymbol{\beta}} {}^4\mathcal{R}_{\alpha\beta}{}^{\boldsymbol{\alpha\beta}} + iNn^\alpha \tilde{E}^\beta_{\boldsymbol{\beta}} n_{\boldsymbol{\alpha}} \epsilon^{\boldsymbol{\alpha\beta}}{}_{\boldsymbol{\gamma\delta}} {}^4\mathcal{R}_{\alpha\beta}{}^{\boldsymbol{\gamma\delta}} \tag{A.5}$$

$$= \int \underset{\sim}{N} \tilde{E}^\alpha_{\boldsymbol{\alpha}} \tilde{E}^\beta_{\boldsymbol{\beta}} {}^4\mathcal{R}_{\alpha\beta}{}^{\boldsymbol{\alpha\beta}} - i\tilde{E}^\beta_{\boldsymbol{\beta}} (t^\alpha - N^\alpha) \epsilon^{\boldsymbol{\beta}}{}_{\boldsymbol{\gamma\delta}} {}^4\mathcal{R}_{\alpha\beta}{}^{\boldsymbol{\gamma\delta}} \tag{A.6}$$

where we have used $t^\alpha = Nn^\alpha + N^\alpha$ given above, and defined $\epsilon^{\boldsymbol{\beta}}{}_{\boldsymbol{\gamma\delta}} = \epsilon^{\boldsymbol{\alpha\beta\gamma\delta}} n_{\boldsymbol{\alpha}}$. By expressing $t^\alpha\, {}^4\mathcal{R}_{\alpha\beta}{}^{\boldsymbol{\alpha\beta}}$ in terms of the self-dual connection, ${}^4\!\mathcal{A}_\alpha^{\boldsymbol{\alpha\beta}}$, we obtain:

$$\begin{aligned} t^\alpha\, {}^4\mathcal{R}_{\alpha\beta}{}^{\boldsymbol{\alpha\beta}} &= t^\alpha ({}^4\!\mathcal{A}^{\boldsymbol{\alpha\beta}}_{\beta\ ,\alpha} - {}^4\!\mathcal{A}^{\boldsymbol{\alpha\beta}}_{\alpha\ ,\beta} + [{}^4\!\mathcal{A}_\alpha, {}^4\!\mathcal{A}_\beta]^{\boldsymbol{\alpha\beta}}) \\ &= \pounds_t {}^4\!\mathcal{A}^{\boldsymbol{\alpha\beta}}_\beta - (t^\alpha {}^4\!\mathcal{A}^{\boldsymbol{\alpha\beta}}_\alpha)_{,\beta} + t^\alpha [{}^4\!\mathcal{A}_\alpha, {}^4\!\mathcal{A}_\beta]^{\boldsymbol{\alpha\beta}} \\ &= \pounds_t {}^4\!\mathcal{A}^{\boldsymbol{\alpha\beta}}_\beta - {}^4D_\beta (t^\alpha {}^4\!\mathcal{A}^{\boldsymbol{\alpha\beta}}_\alpha). \end{aligned}$$

By substituting this into (A.6) we obtain:

$$I = \int \underset{\sim}{N} \tilde{E}^\alpha_{\boldsymbol{\alpha}} \tilde{E}^\beta_{\boldsymbol{\beta}} {}^4\mathcal{R}_{\alpha\beta}{}^{\boldsymbol{\alpha\beta}} - i\tilde{E}^\beta_{\boldsymbol{\beta}} N^\alpha \epsilon^{\boldsymbol{\beta}}{}_{\boldsymbol{\gamma\delta}} {}^4\mathcal{R}_{\alpha\beta}{}^{\boldsymbol{\gamma\delta}} + i\tilde{E}^\beta_{\boldsymbol{\beta}} \epsilon^{\boldsymbol{\beta}}{}_{\boldsymbol{\gamma\delta}} \left[\pounds_t {}^4\!\mathcal{A}_\beta{}^{\boldsymbol{\gamma\delta}} - {}^4D_\beta ({}^4\!\mathcal{A}_\alpha{}^{\boldsymbol{\gamma\delta}} t^\alpha)\right]. \tag{A.7}$$

We can see that in the action above the self-dual curvature is always projected into the hypersurface by the projected frame density. For example $\tilde{E}^{\alpha}_{\alpha}{}^{4}\mathcal{R}_{\alpha\beta}{}^{\alpha\beta} = \tilde{E}^{\alpha}_{\mathrm{i}}{}^{4}\mathcal{R}_{\alpha\beta}{}^{\mathrm{i}\beta}$. We can therefore replace the curvature with its pull-back, $\mathcal{R}_{\mu\nu}{}^{\mathrm{ij}}$, which remarkably is defined by the pull-back of the connection. To explain this we shall consider the Lie algebra of the connection. The algebra of a complex self-dual connection, ${}^{4}\!\mathcal{A}_{\alpha}{}^{\gamma\delta}$, is $sl(2,\mathbb{C})$, which is isomorphic to $so(3)_{\mathbb{C}}$. This is the same as the Lie algebra of the connection defined on the hypersurface, denoted by $\mathcal{A}_{\alpha}{}^{\mathrm{ij}}$. This also enables us to replace the space-time exterior covariant derivative, ${}^{4}D_{\alpha}$, with the derivative defined on the hypersurface, D_{α}. We can now express the Lagrangian using these pull-back variables. We will use a dot to denote the Lie derivative with respect to t^{α} and define $\mathcal{A}_{0}{}^{\mathrm{ij}} := t^{\alpha}\mathcal{A}_{\alpha}{}^{\mathrm{ij}}$. The Lagrangian is then given by.

$$L = \int -i\tilde{E}^{\mu}_{\mathrm{i}}\epsilon^{\mathrm{i}}{}_{\mathrm{ji}}\dot{\mathcal{A}}_{\mu}{}^{\mathrm{ji}} + i\tilde{E}^{\nu}_{\mathrm{i}}\epsilon^{\mathrm{i}}{}_{\mathrm{jk}}N^{\mu}\mathcal{R}_{\mu\nu}{}^{\mathrm{jk}} - i(\mathcal{A}_{0}{}^{\mathrm{ij}})D_{\mu}(\tilde{E}^{\mu}_{\mathrm{k}}\epsilon^{\mathrm{k}}{}_{\mathrm{ij}}) + \underset{\sim}{N}\tilde{E}^{\mu}_{\mathrm{i}}\tilde{E}^{\nu}_{\mathrm{j}}\mathcal{R}_{\mu\nu}{}^{\mathrm{ij}}\,\mathrm{d}^{3}x \quad \text{(A.8)}$$

In the current form, $\mathcal{L} = p\dot{q} - \mathcal{H}$, and so we are able to 'read off' the canonical variables as well as the Hamiltonian. Therefore we find the configuration variables are the connections, $\mathcal{A}_{\mu}{}^{\mathrm{ij}}$, whose corresponding canonical momenta are the self-dual parts of $-i\tilde{E}^{\mu}_{\mathrm{i}}\epsilon^{\mathrm{i}}{}_{\mathrm{jk}} \equiv \tilde{E}^{\mu}_{[\mathrm{i}}n_{\mathrm{j}]} - \frac{i}{2}\tilde{E}^{\mu}_{\mathrm{k}}\epsilon^{\mathrm{k}}{}_{\mathrm{ij}} =: \tilde{\Pi}^{\mu}_{\mathrm{ij}}$. We are using the shortcut method and therefore we treat the variables $N^{\mu}, \underset{\sim}{N}, \mathcal{A}_{0}{}^{\mathrm{ij}}$ as multipliers and do not introduce additional momenta.

We now express the Hamiltonian using the canonical variables:

$$H = \int N^{\mu}\tilde{\Pi}^{\nu}_{\mathrm{ij}}\mathcal{R}_{\mu\nu}{}^{\mathrm{ij}} - (\mathcal{A}_{0}{}^{\mathrm{ij}})D_{\mu}(\tilde{\Pi}^{\mu}_{\mathrm{ij}}) - \underset{\sim}{N}\tilde{\Pi}^{\mu}{}_{\mathrm{i}}{}^{\mathrm{j}}\tilde{\Pi}^{\nu}{}_{\mathrm{j}}{}^{\mathrm{k}}\mathcal{R}_{\mu\nu\mathrm{k}}{}^{\mathrm{i}}\,\mathrm{d}^{3}x. \quad \text{(A.9)}$$

Due to the shortcut method there are no primary constraints, but variation of the multipliers result in the secondary constraints:

$$\tilde{\Pi}^{\mu}{}_{\mathrm{i}}{}^{\mathrm{j}}\tilde{\Pi}^{\nu}{}_{\mathrm{j}}{}^{\mathrm{k}}\mathcal{R}_{\mu\nu\mathrm{k}}{}^{\mathrm{i}} = 0 \quad \text{(A.10a)}$$

$$\tilde{\Pi}^{\nu}_{\mathrm{ij}}\mathcal{R}_{\mu\nu}{}^{\mathrm{ij}} = 0 \quad \text{(A.10b)}$$

$$D_{\mu}(\tilde{\Pi}^{\mu}_{\mathrm{ij}}) = 0. \quad \text{(A.10c)}$$

We can show that these seven constraints are all first class. The first constraint is called the Hamiltonian constraint, while the second constraint (A.10b) is called the momentum constraint. These are similar to the earlier constraints found in chapter 1. The final constraint (A.10c) is called the Gauss constraint. In the ADM Hamiltonian the Hamiltonian and momentum constraints generate diffeomorphisms in the normal and tangential directions. In Ashtekar's approach the momentum constraints have to be adapted by including the Gauss constraint in order to generate the diffeomorphisms. The resulting constraint is given below:

$$\tilde{\Pi}^{\nu}_{\mathrm{ij}} \mathcal{R}_{\mu\nu}{}^{\mathrm{ij}} - \mathcal{A}^{\mathrm{ij}}_{\nu} D_{\mu}(\tilde{\Pi}^{\mu}_{\mathrm{ij}}) = 0. \tag{A.11}$$

Note that this is similar to the adaption made to the first class constraints in chapters 5 and 6.

The Gauss constraints generate rotations of the frame indices. When we perform the standard counting to show the degrees of freedom, we find that there are $(18-2(7))/2 = 2$ degrees of freedom.

As stated in the beginning of this section the Ashtekar variables that we are using are complex. Therefore the current solution is for complexified General Relativity, which allows complex metrics. To guarantee a real metric we need to impose some extra constraints on the equations. These constraints are not considered to be primary or secondary constraints of the theory, rather they 'filter' the solution space so that we consider only solutions that generate real metrics. In order to obtain a real metric the expression $\tilde{\Pi}^{\mu}{}_{\mathrm{ij}}\tilde{\Pi}^{\nu\,\mathrm{ij}}$ must be real. This however, is not enough because we also have to ensure that the metric remains real for all time, which implies the time derivative of the metric must be real. This is achieved by calculating the Poisson bracket of the metric with the Hamiltonian. Using the Hamiltonian (A.9) we get the time derivative of the metric to be $\tilde{\Pi}^{(\mu}{}_{\mathrm{ij}} D_{\gamma} \left[\tilde{\Pi}^{|\gamma|}, \tilde{\Pi}^{\nu)}\right]^{\mathrm{ij}}$. We must ensure that this is real. We next may check that this condition is preserved for all time. If it is then providing these conditions are satisfied initially then they are also satisfied for all time, which then implies that using the canonical method above results in General Relativity.

Appendix B

Null $3+1$ canonical analysis

In this appendix we outline the work by Goldberg, Robinson and Soteriou (see Goldberg *et al.* 1992; Goldberg & Soteriou 1995). In their work they used a $3+1$ foliation of spacetime and made the hypersurfaces null through the use of a Lagrange multiplier. This work uses a similar approach to the one found in chapter 5. We note that because we are working in a $3+1$ form in this appendix the index $i, j, k..$ and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ sum between $1, 2, 3$.

We first introduce the null basis of one forms and the corresponding tetrad basis:

$$\theta^{\mathbf{0}} = N\,\mathrm{d}t + \alpha_i(N^i\,\mathrm{d}t + \mathrm{d}x^i) \tag{B.1a}$$

$$\theta^{\mathbf{i}} = \nu^{\mathbf{i}}{}_j(N^j\,\mathrm{d}t + \mathrm{d}x^j). \tag{B.1b}$$

$$e_{\mathbf{0}} = \frac{1}{N}\left(\frac{\partial}{\partial t} - N^i\frac{\partial}{\partial x^i}\right), \tag{B.1c}$$

$$e_{\mathbf{i}} = v_{\mathbf{i}}{}^j\frac{\partial}{\partial x^j} + \frac{\alpha_{\mathbf{i}}}{N}\left(N^j\frac{\partial}{\partial x^j} - \frac{\partial}{\partial t}\right), \tag{B.1d}$$

where $v_{\mathbf{a}}{}^i\nu^{\mathbf{b}}{}_i = \delta^{\mathbf{b}}{}_{\mathbf{a}}$. We see in the above that the the frame is adapted when $\alpha_i = 0$. The null condition is given by $\alpha_1 + \alpha_2\alpha_3 = 0$. α_2 and α_3 can be set to zero using a gauge freedom, which leaves the null condition as $\alpha_1 = 0$. From this point we will not use the index on the alpha variables because there is only one of them.

We split the connection $\Gamma^{\mathbf{A}}$ (4.37) into the $3+1$ form:

$$\Gamma^{\mathbf{A}} = A_i^{\mathbf{A}}\,\mathrm{d}x^i + B^{\mathbf{A}}\,\mathrm{d}t, \tag{B.2}$$

which substituting into (4.39b) results in:

$$R^{\mathbf{A}}{}_{ij} = 2A^{\mathbf{A}}_{[i,j]} - 2\eta^{\mathbf{A}}{}_{\mathbf{BC}}A^{\mathbf{B}}{}_{i}A^{\mathbf{C}}{}_{j}, \tag{B.3a}$$

$$R^{\mathbf{A}}{}_{0i} = D_i B^{\mathbf{A}} - A^{\mathbf{A}}{}_{i,0}. \tag{B.3b}$$

Where the derivative D_i is defined by:

$$D_i f^{\mathbf{A}} := f^{\mathbf{A}}_{,i} + 2\eta^{\mathbf{A}}_{\mathbf{BC}}A^{\mathbf{A}}_{i}f^{\mathbf{C}}. \tag{B.4}$$

Then using these definitions, the $SO(3)$ basis given by (4.34) and the action (4.47) we obtain the Lagrangian density:

$$\mathcal{L} = A^{\mathbf{A}}{}_{i,0}\tilde{\Sigma}_{\mathbf{A}}{}^{i} + B^{\mathbf{A}}D_i\tilde{\Sigma}_{\mathbf{A}}{}^{i} + R^{\mathbf{A}}{}_{ij}N^i\tilde{\Sigma}_{\mathbf{A}}{}^{j} - \underset{\sim}{N}\tilde{v}^i(R^{1}{}_{ij}\tilde{\Sigma}_{\mathbf{3}}{}^{i} + R^{2}{}_{ij}\tilde{\Sigma}_{\mathbf{1}}{}^{i})$$
$$+ \mu_i(\tilde{\Sigma}_{\mathbf{2}}{}^{i} + \alpha\tilde{v}^i) + \rho(\alpha)^2. \tag{B.5}$$

where:

$$\tilde{v}^i := vv^i_{\mathbf{2}}, \tag{B.6}$$

$$\tilde{\Sigma}_{\mathbf{1}}{}^{i} := -vv^i_{\mathbf{1}} \quad \tilde{\Sigma}_{\mathbf{2}}{}^{i} := -\alpha vv^i_{\mathbf{2}} \quad \tilde{\Sigma}_{\mathbf{3}}{}^{i} := -vv^i_{\mathbf{3}}. \tag{B.7}$$

Note in the above Lagrangian we have introduced the null condition $\alpha = 0$, though the use a the multiplier ρ. The equations (B.7) are used to replace the variables $v^i_{\mathbf{1}}$ and $v^i_{\mathbf{3}}$ with $\tilde{\Sigma}_{\mathbf{1}}{}^{i}$ and $\tilde{\Sigma}_{\mathbf{3}}{}^{i}$ in the Lagrangian. The variables $\tilde{\Sigma}_{\mathbf{2}}{}^{i}$ are zero on the null space, and hence they can not replace the variables $\tilde{v}^i$. Therefore both sets of variables ($\tilde{\Sigma}_{\mathbf{2}}{}^{i}$ and $\tilde{v}^i$) are used in the Lagrangian which results in an additional three constraints $\tilde{\Sigma}_{\mathbf{2}}{}^{i} + \alpha\tilde{v}^i$. This Lagrangian is in the form $\mathcal{L} = \dot{q}^\lambda p_\lambda - \mathcal{H}(q^\lambda p_\lambda)$ and we can see straight away that the canonical variables are $A^{\mathbf{A}}_i$ with conjugate momenta $\tilde{\Sigma}_{\mathbf{A}}{}^{i}$. All the remaining variables are cyclic and therefore they will be treated as multipliers via the shortcut method. This results in the Hamiltonian density:

$$\mathcal{H} = \underset{\sim}{N}\mathcal{H}_0 + N^i\mathcal{H}_i - B^{\mathbf{A}}\mathcal{G}_{\mathbf{A}} - \mu_i C^i - \rho(\alpha^2), \tag{B.8}$$

where we have defined:

$$\mathcal{H}_0 \equiv v^i({R^{\mathbf{1}}}_{ij}\tilde{\Sigma}_{\mathbf{3}}{}^j + {R^{\mathbf{2}}}_{ij}\tilde{\Sigma}_{\mathbf{1}}{}^j) \tag{B.9a}$$

$$\mathcal{H}_i \equiv -{R^{\mathbf{A}}}_{ij}\tilde{\Sigma}_{\mathbf{A}}{}^j \tag{B.9b}$$

$$\mathcal{G}_{\mathbf{A}} \equiv D_i(\tilde{\Sigma}_{\mathbf{A}}{}^i) \tag{B.9c}$$

$$C^i \equiv \tilde{\Sigma}_{\mathbf{2}}{}^i + \alpha v^i. \tag{B.9d}$$

The above are constraints that arise from variation with respect to $\underset{\sim}{N}, N^i, B^{\mathbf{A}}$ and μ_i respectively. We get additional constraint equations from varying with respect to ρ, α and $\tilde{v}^i$:

$$\alpha = 0, \tag{B.9e}$$

$$\tilde{v}^i\mu_i = 0, \tag{B.9f}$$

$$\phi_i := R^{\mathbf{1}}_{ij}\tilde{\Sigma}_{\mathbf{3}}{}^j + {R^{\mathbf{2}}}_{ij}\tilde{\Sigma}_{\mathbf{1}}{}^j = 0. \tag{B.9g}$$

Note that $\tilde{v}^i\phi_i = \mathcal{H}_0$ and $\tilde{\Sigma}_{\mathbf{1}}{}^i\phi_i = \tilde{\Sigma}_{\mathbf{3}}{}^i\mathcal{H}_i$. Therefore (B.9g) contains only one independent equation; this will be labeled $\phi_i\tilde{\Sigma}_{\mathbf{3}}{}^i$.

We now need to propagate the constraint equations to ensure they remain true for all time. Propagation of the constraint $C^i = 0$ gives:

$$\chi^i := 2{\delta^{\mathbf{B}}}_2 D_j(\underset{\sim}{N}\tilde{v}^{[i}\tilde{\Sigma}_{\mathbf{B}}{}^{j]}{Q^{\mathbf{B}}}_{\mathbf{A}}) - 2{A^{\mathbf{3}}}_j\underset{\sim}{N}^{[i}\tilde{\Sigma}_{\mathbf{1}}{}^{j]} - B^{\mathbf{3}}\tilde{\Sigma}_{\mathbf{1}}{}^i = 0 \tag{B.10}$$

$$Q^{\mathbf{B}}_{\mathbf{A}} := \delta^{\mathbf{B}}_{\mathbf{3}}\delta^{\mathbf{1}}_{\mathbf{A}} + \delta^{\mathbf{B}}_{\mathbf{1}}\delta^{\mathbf{2}}_{\mathbf{A}}.$$

Propagation of the constraint $\mathcal{G}_{\mathbf{3}} = 0$ results in a multiplier equation which constrains the multiplier μ_i:

$$\mu_i\tilde{\Sigma}_{\mathbf{1}}{}^i = {R^{\mathbf{1}}}_{i0}\tilde{\Sigma}_{\mathbf{3}}{}^i - {R^{\mathbf{1}}}_{ij}\tilde{\Sigma}_{\mathbf{3}}{}^i N^j. \tag{B.11}$$

The constraints (B.9e) (B.9f) and (B.10) are multiplier equations and therefore do not require propagation. Propagation of remaining constraints $(\mathcal{H}_0, \mathcal{H}_i, \mathcal{G}_\mathbf{1}, \mathcal{G}_\mathbf{2}, \phi_i \tilde{\Sigma}_\mathbf{3}{}^i)$ does not result in any additional equations.

The equations of motion are then given by:

$$A^\mathbf{1}{}_{i,0} = \{A^\mathbf{1}{}_i, H_p\}$$

$$= \delta^\mathbf{1}{}_\mathbf{A} D_i B^\mathbf{A} + N^j R^\mathbf{1}{}_{ij} - \mathrm{N} v^j R^\mathbf{2}{}_{ij} \tag{B.12a}$$

$$A^\mathbf{2}{}_{i,0} = \delta^\mathbf{2}{}_\mathbf{A} D_i B^\mathbf{A} + N^j R^\mathbf{2}{}_{ij} - \mu_i \tag{B.12b}$$

$$A^\mathbf{3}{}_{i,0} = \delta^\mathbf{3}{}_\mathbf{A} D_i B^\mathbf{A} + N^j R^\mathbf{3}{}_{ij} - \mathrm{N} v^j R^\mathbf{1}{}_{ij} \tag{B.12c}$$

$$\tilde{\Sigma}_\mathbf{1}{}^i{}_{,0} = 2\delta^\mathbf{A}_\mathbf{1} D_j(\mathrm{N} v^{[i} \tilde{\Sigma}_\mathbf{B}{}^{j]} Q^\mathbf{B}{}_\mathbf{A}) - 2\delta^\mathbf{A}_\mathbf{1} D_j(n^{[i} \tilde{\Sigma}_\mathbf{A}{}^{j]}) - 2B^\mathbf{3} \tilde{\Sigma}_\mathbf{3}{}^i \tag{B.12d}$$

$$\tilde{\Sigma}_\mathbf{3}{}^i{}_{,0} = 2\delta^\mathbf{A}_\mathbf{3} D_j(\mathrm{N} v^{[i} \tilde{\Sigma}_\mathbf{B}{}^{j]} Q^\mathbf{B}{}_\mathbf{A}) - 2\delta^\mathbf{A}_\mathbf{3} D_j(n^{[i} \tilde{\Sigma}_\mathbf{A}{}^{j]}) - 2B^\mathbf{1} \tilde{\Sigma}_\mathbf{3}{}^i + B^\mathbf{2} \tilde{\Sigma}_\mathbf{1}{}^i \tag{B.12e}$$

After this analysis we have eleven constraints and six multiplier conditions. Five of the constraints, $\mathcal{H}_i, \mathcal{G}_\mathbf{1}$ and $\mathcal{G}_\mathbf{2}$, after being adapted, are first class. The remaining constraints, $\mathcal{G}_\mathbf{3}, \mathcal{H}_0$ and $\phi_i \tilde{\Sigma}_\mathbf{3}{}^i$ are second class. To calculate the first class algebra the constraints are smeared with test functions:

$$H(Y^i) := \int Y^i H_i \, \mathrm{d}^3 x \tag{B.13a}$$

$$G(M^1, M^2) := \int (M^1 \mathcal{G}_1 + M^2 \mathcal{G}_2) \, \mathrm{d}^3 x. \tag{B.13b}$$

The algebra is then given by:

$$\{H(Y^i), H(Z^j)\} = H(\mathcal{L}_Y Z^j) \tag{B.14a}$$

$$\{H(Y^i), G(M^1, M^2)\} = G(\mathcal{L}_Y M^1, \mathcal{L}_Y M^2) \tag{B.14b}$$

$$\{G(M^1, M^2), G(K^1, K^2)\} = G\left(0, 2(M^2 K^1 - M^1 K^2)\right). \tag{B.14c}$$

The constraint $H(Y^i)$ generates the diffeomorphisms within the three surface, which results in the infinitesimal transformations:

$$\delta \underset{\sim}{N} = \mathcal{L}_Y \underset{\sim}{N} \tag{B.15a}$$

$$\delta N^i = \mathcal{L}_Y N^i \tag{B.15b}$$

$$\delta \tilde{\Sigma}_{\mathbf{A}}{}^{i} = \mathcal{L}_Y \tilde{\Sigma}_{\mathbf{A}}{}^{i} \tag{B.15c}$$

$$\delta v^i = \mathcal{L}_Y v^i \tag{B.15d}$$

$$\delta A^{\mathbf{A}}_{i} = \mathcal{L}_Y A^{\mathbf{A}}_{i} \tag{B.15e}$$

$$\delta B^{\mathbf{3}} = \mathcal{L}_Y B^{\mathbf{3}}. \tag{B.15f}$$

The other first class constraints, $G(M^1, M^2)$, generate the self-dual null rotations:

$$\delta \underset{\sim}{N} = 2M^2 \underset{\sim}{N} \tag{B.16a}$$

$$\delta N^i = M^2 \underset{\sim}{N} v^i \tag{B.16b}$$

$$\delta \tilde{\Sigma}_{\mathbf{A}}{}^{i} = -\delta^{\mathbf{3}}_{\mathbf{A}} (2M^1 \tilde{\Sigma}_{\mathbf{3}}{}^{i} + M^2 \tilde{\Sigma}_{\mathbf{1}}{}^{i}) \tag{B.16c}$$

$$\delta v^i = 0 \tag{B.16d}$$

$$\delta A^{\mathbf{A}}_{i} = \delta^{\mathbf{B}}_{\mathbf{1}} (-M^1{}_{,i} + M^2 A^{\mathbf{3}}{}_{i}) - 2\delta^{\mathbf{B}}_{\mathbf{2}} D_i M^2 + 2\delta^{\mathbf{B}}_{\mathbf{3}} M^1 A^{\mathbf{3}}{}_{i} \tag{B.16e}$$

$$\delta B^{\mathbf{3}} = 2M^1 B^{\mathbf{3}}. \tag{B.16f}$$

This concludes the null $3+1$ canonical analysis.

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