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Metric and analytic properties of $\mathbb{R}$-trees

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ABSTRACT

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In this thesis we consider metric and analytic properties of non-discrete metric spaces. The particular example we work with is that of $\mathbb{R}$-trees. We construct large scale Lipschitz maps which we use to prove that the Hilbert space compression of $\mathbb{R}$-trees is equal to one. We provide an overview of the current variations of property A and move on to develop some new definitions. We then discuss which classes of $\mathbb{R}$-trees have property A, also known as a weak form of amenability. Finally we review results linking property A, uniform embeddability and exactness.

Contents

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Chapter 1

Introduction

Groups appear in the study of geometry of spaces as a way to encode symmetries of the space. In recent years it emerged that geometry can be used to study properties of groups. Several metric properties of groups have been defined, for example, property A, which is a weaker version of amenability, has been introduced by Yu in his work on the Novikov Conjecture.

If a metric space has property A then it is embeddable in Hilbert space. This means that the geometry of the metric space resembles, up to a controlled amount of distortion, the geometry of a subset of a Hilbert space. This is a fairly strong geometric condition. For discrete groups property A is equivalent to exactness, which is an analytic condition. This provides an example of how geometry and analysis of groups interact in a nontrivial way.

If a group G is amenable, then there is a kind of averaging operation on G which is invariant under translation by elements of the group. More formally, we say that a locally compact group G is amenable if and only if there is an invariant mean (i.e. a state) on $L^\infty(G)$. For discrete groups this simplifies to the condition that G is amenable if and only if it has a finitely additive left invariant probability measure. All finite groups and all abelian groups are amenable, whereas any group containing a free subgroup on two generators is not amenable. In the case of discrete groups, amenability implies both property A and exactness.

Property A, amenability, exactness and embeddability in Hilbert space are all interesting analytic and geometric properties of groups. A key conjecture in this area is the Baum-Connes conjecture which states that for every countable group G, the analytic assembly map $\mu_i^G : K_i^G(\underline{E}G) \to K_i((C_r^*(G))$ (for $i = 0, 1$) is an isomorphism [1]. The left hand side of the conjecture is the geometric (or topological) side, where we consider the equivariant K-homology of the classifying space $\underline{E}G$, whereas the right hand side uses the topological K-theory for C^*-algebras. It is known that, among others, every amenable group satisfies the Baum-Connes conjecture. On the other hand, Yu has demonstrated that every group with property A is uniformly embeddable in a Hilbert space which implies that such a group satisfies the Novikov conjecture.

Property A was first defined by Yu in [22] for discrete groups. He also defined it for discrete and then non-discrete metric spaces, but the non-discrete property A wasn't developed much initially. Since then many mathematicians, including Higson and Roe [9] and Tu [18], have studied property A for discrete groups and metric spaces. The work by Tu introduced many equivalent ways to define property A.

In this thesis we endeavour to extend these methods to non-discrete metric spaces. While in the discrete case there is a large number of equivalent descriptions of property A, it is not immediately clear which of those will be the most useful in the non-discrete case. To gain insight into this question we study $\mathbb{R}$-trees. An $\mathbb{R}$-tree is not, in general, locally compact which leads to a number of technical issues to be resolved. On the other hand $\mathbb{R}$-trees provide a nice example because they are equipped with a unique Lebesgue measure constructed by Valette in [4]. We find that one of the many definitions given by Tu, which we shall call property A_{L^1}, is very useful in this context.

Our work in the non-discrete case is modelled on a number of previous results which we briefly describe here. Dranishnikov and Januszkiewicz in [7] proved that trees satisfy property A. They also proved that in the case of spaces with bounded geometry property A passes to subspaces and finite products of property A spaces. The main application of these results was to prove that every Coxeter group is Higson-Roe amenable [7, Theorem A], i.e. that every Coxeter group admits a topologically amenable action on a compact space.

The non-discrete case has not been developed so well. Tu remarks that while many of his conditions introduced in [18] can be stated in the non-discrete case, they will not in general be equivalent to Yu's definition. I have put together much of this information, along with some new results. Most of the work done in this thesis, along with the work by Brodzki, Niblo and Wright, assumes a slightly weaker version of property A given by Tu.

In chapter 3 we describe an invariant called the Hilbert space compression. This was introduced by Guentner and Kaminker in [8]. It assigns a number to a metric space, or a group, which indicates how much it needs to be twisted, or compressed, to embed into a Hilbert space. It was proved in [8], and details are given later in this thesis, that the value of the Hilbert space compression lies between 0 and 1. If the Hilbert space compression is non-zero, then the metric space is uniformly embeddable in a Hilbert space by means of a large scale Lipschitz map. Even more can be said in the case of discrete groups: if the Hilbert space compression of a group Γ is greater than $\frac{1}{2}$ then Γ is exact. In [8, Proposition 4.2] Guentner and Kaminker prove that the free group on two generators has Hilbert space compression equal to one, and hence this group is exact. Later in this thesis we use a similar method to prove that $\mathbb{R}$-trees have Hilbert space compression equal to one.

Dadarlat and Guentner have expanded Gromov's work on uniform embeddability in a Hilbert space. They also detail how uniform embeddability (and hence Hilbert space compression) links to exactness and property A. Specifically, in [5] they defined exactness for a metric space, and one of the main theorems in the paper shows that for any metric space exactness implies uniform embeddability. Also, if the metric space is discrete we have that property A implies exactness, and finally if it is discrete and has bounded geometry, then exactness is equivalent to property A.

In [12], Ozawa defined a property for discrete groups which we shall refer to as property O. A group Γ has property O if for any finite subset $E \subset \Gamma$ and any $\epsilon > 0$, there are a finite subset $F \subset \Gamma$ and a positive kernel $u : \Gamma \times \Gamma \to \mathbb{C}$ such that (i) $u(s,t) \neq 0$ only if $st^{-1} \in F$, and (ii) $|1 - u(s,t)| < \epsilon$ if $st^{-1} \in E$. The main result in Ozawa's paper [12, Theorem 3] proves that for a discrete group G, property O is equivalent to the reduced C^*-algebra of G being exact, which in turn is equivalent to the uniform Roe algebra of G being nuclear.

Work linking property A, property O and the nuclearity of the Roe algebra for discrete groups has been done by Campbell [3] and Brodzki, Niblo and Wright [2]. These results can be briefly described as follows.

In a paper by Campbell [3] kernels satisfying property O were constructed for both amenable and free groups via their Cayley graphs, thus showing that these groups have property O and hence are exact and have property A. The construction of these kernels depends on the geometric properties of the group, whereas the equivalent property A depends on the metric.

The main idea in [2] by Brodzki, Niblo and Wright, is to define an invariant on discrete bounded geometry metric spaces which in some sense measures how much like a group it is. This is done by assigning a number to the space depending on how close we can get to embedding the space into a group. We know that if X is a discrete bounded geometry space then if X has property A this implies that that $C_u^*(X)$ is nuclear which in turn implies that $C_u^*(X)$ is exact. Brodzki, Niblo and Wright prove that if the space X is locally uniformly embeddable into a countable discrete group, then property A, nuclearity of the uniform Roe algebra and exactness of the uniform Roe algebra are all equivalent. Note here that X is locally uniformly embeddable in Y means that $\text{fin}(X)$ (all finite subsets of X) is uniformly embeddable in $\text{fin}(Y)$. They also prove that if a certain example of a disjoint union of graphs with bounded valencies is embeddable in a discrete group, then every bounded geometry space is locally uniformly embeddable in this same group.

Overview of the thesis

The work in this thesis is organised in the following way. Background material is given in chapter 2. The first section gives definitions and many basic examples from coarse geometry. We also prove a few lemmas linking coarse equivalence, uniform embeddings and quasi-isometries. Next we give a very brief description of coarse metric spaces and provide an example of the bounded coarse structure which is used later on in this thesis. After that there is a section on measure theory, again giving many basic definitions. The final section of the background chapter is on C*-algebras. We start this section with the definition of an inner product, progress through norms to the definition of a C*-algebra. Finally we define states on a C*-algebra and give evaluation maps as an example.

Chapter 3 starts off with the definition of an $\mathbb{R}$-tree and we provide some examples of $\mathbb{R}$-trees. In section 3.2 we then give a detailed description of the Hilbert space compression of a metric space and provide an overview of the paper [8] by Guentner and Kaminker. The main result in this section is Proposition 3.13 which states that if the Hilbert space compression of a metric space X is nonzero, then X is uniformly embeddable in Hilbert space. This then leads on to the final section of this chapter, section 3.3 in which we construct a family of large-scale Lipschitz maps on an $\mathbb{R}$-tree X. We then use these in Theorem 3.19 to prove that the Hilbert space compression of X is equal to 1.

We start chapter 4 with the definition of property A for a discrete metric space, as given by Yu in [22]. Following this are details of work on property A by Higson-Roe, Tu, Dadarlat-Guentner, Brodzki-Niblo-Wright and myself. The main result of this chapter is given in Theorem 4.8 where we state and prove ten equivalent definitions of property A for discrete bounded geometry metric spaces.

Following on from the previous chapter, chapter 5 is on property A for non-discrete metric spaces. First we give details of how Theorem 4.8 extends to the non-discrete case. Then we provide the definition of property A_b based on states of $C_0(X)$ and show how it links to the previous definitions of property A for both the non-discrete, and the discrete bounded geometry case. Finally we provide another two definitions of property A, by Roe. Once again giving details on how they relate to the previous definitions.

We discuss the permanence properties of property A in chapter 6. Section 6.1 is devoted to discrete metric spaces whereas section 6.2 is for non-discrete metric spaces. Both sections are then split down into three subsections, each one for a different permanence property. First we prove that if two metric spaces X and Y are coarsely equivalent, and if X has property A then so does Y. The second property we consider is products of property A spaces. We finish with a statement that property A passes to subspaces.

In chapter 7 we go back to the example of $\mathbb{R}$-trees. In this chapter we use property A_{L^1}, from [18], and we prove that $\mathbb{R}$-trees have property A.

Chapter 8 then looks at other properties of metric spaces, for example exactness and uniform embeddability. Much of the first part of this chapter is based on a paper by Dadarlat and Guentner [5]. We prove theorems and proofs linking property A to exactness and uniform embeddability. In the final section we turn to a paper by Nowak, [11] and give details of an interesting metric space which does not have property A, but is uniformly embeddable in Hilbert space.

Chapter 2

Background

2.1 Coarse Geometry

In this section we provide the reader with some background on coarse geometry of metric spaces. The general idea of coarse geometry is to ignore details of the metric space at small distances and to concentrate instead on what happens on the large scale. So it's like looking at the metric space from further and further away and then only worrying about what can still be seen. A simple example is that of the integers and the real line. If one places dots on the number line at each integer point, and then stands far enough away from it, it will look like the whole number line is covered, i.e. the integers are in fact equivalent (in coarse geometry) to the whole real line.

As with the introduction of anything new, we have to start with definitions. The following definitions and examples should help the reader to grasp the basics of coarse geometry of metric spaces.

Firstly recall that a map $f : X \to Y$ is a Lipschitz map if there exists $C > 0$ such that $d_Y(f(x), f(y)) \leq Cd_X(x, y)$. We then define a large-scale Lipschitz map as follows.

Definition 2.1. [8, Definition 2.2]
A map $f : X \to Y$ is **large-scale Lipschitz** if there exist $C > 0$ and $D \geq 0$ such that $d_Y(f(x), f(y)) \leq Cd_X(x, y) + D$.

We can see that it makes sense for this to be called a large-scale Lipschitz map, because as $d_X(x, y) \to \infty$ (assuming X is unbounded) the extra constant D becomes less important. So in some sense, from a distance large-scale Lipschitz maps look just like Lipschitz maps.

Example 2.2. Some examples of large-scale Lipschitz maps are

- Lipschitz maps - these are also large-scale Lipschitz maps where the constant $D = 0$.
- Isometries - these are homeomorphisms which preserve distance, and therefore are large-scale Lipschitz maps, where $C = 1$ and $D = 0$.
- Quasi-isometries - these are functions of the form $f : X \to Y$ such that $\exists C > 0, D \geq 0$ such that $\frac{1}{C}d_X(x,y) - D \leq d_Y(f(x), f(y)) \leq Cd_X(x,y) + D$. It can be seen that the righthand side of this inequality is the definition of large-scale Lipschitz maps.

Definition 2.3. [8, page 1]
A **uniform embedding** of a metric space X into Y is a function $f : X \to Y$ for which there exist non-decreasing functions $\rho_\pm : [0,\infty) \to \mathbb{R}$ such that $\lim_{r\to\infty} \rho_\pm(r) = \infty$ (i.e. $\rho_\pm$ are proper) and such that for all x, $y \in X$

$$\rho_-(d_X(x,y)) \leq d_Y(f(x), f(y)) \leq \rho_+(d_X(x,y))$$

A metric space X is said to be **uniformly embeddable in Hilbert space** (or just uniformly embeddable) iff there exists a uniform embedding from X to a Hilbert space $\mathcal{H}$.

Example 2.4. Consider the integers $\mathbb{Z}$, and the reals $\mathbb{R}$, with the usual distance function. Then there is a map $f : \mathbb{R} \to \mathbb{Z}$ defined by $x \mapsto \lfloor x \rfloor$, i.e. x maps to the maximum integer not greater than itself. Then it can easily be seen that

$$d(x,y) - 1 \leq d(\lfloor x \rfloor, \lfloor y \rfloor) \leq d(x,y) + 1$$

where $d(x,y) \pm 1 : [0,\infty) \to \mathbb{R}$ and $\lim_{d(x,y)\to\infty} d(x,y) \pm 1 = \infty$. Therefore f is a uniform embedding from $\mathbb{R}$ to $\mathbb{Z}$.

Now let $g : \mathbb{Z} \to \mathbb{R}$ be the identity map i.e. $x \mapsto x$. Then

$$d(x,y) \leq d(g(x), g(y)) \leq d(x,y)$$

where $d(x,y) : [0,\infty) \to \mathbb{R}$ and $\lim_{d(x,y)\to\infty} d(x,y) = \infty$. Therefore g is a uniform embedding from $\mathbb{Z}$ to $\mathbb{R}$.

Definition 2.5. [14, page 6]
Let X and Y be metric spaces, and let $f : X \to Y$ be a map, then

(1) f is metrically **proper** if the inverse image under f of each bounded subset of Y is a bounded subset of X.

(2) f is uniformly **bornologous** if for every $R > 0$ there is $S > 0$ such that $d(x, x') < R \Rightarrow d(f(x), f(x')) < S$.

(3) f is **coarse** if it is proper and bornologous, i.e. it satisfies both (1) and (2).

Two maps $f : X \to Y$ and $f' : X \to Y$ are **close** if $d(f(x), f'(x))$ is bounded uniformly in X.

Two metric spaces X and Y are **coarsely equivalent** if there exist coarse maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are close to the identity maps on Y and on X respectively.

Example 2.6. $\mathbb{Z}$ and $\mathbb{R}$ are coarsely equivalent. Let f and g be the maps defined above; $f(x) = \lfloor x \rfloor$ and $g(x) = x$. They are both proper and bornologous and therefore are coarse maps. Now $f \circ g = f = g \circ f$, so we need only check that f is close to the identity. $d(f(x), x) \leq 1$ and is uniformly bounded in $\mathbb{R}$ as required, and so $\mathbb{Z}$ and $\mathbb{R}$ are coarsely equivalent.

Lemma 2.7. *If $f : X \to Y$ is a uniform embedding, then X and $f(X) \subset Y$ are coarsely equivalent.*

Proof. Let $f : X \to Y$ be a uniform embedding. Then there exist functions $\rho_\pm : [0, \infty) \to \mathbb{R}$ such that $\lim_{r\to\infty} \rho_\pm(r) = \infty$ (i.e. $\rho_\pm$ are proper maps) and $\forall x, y \in X$,

$$\rho_-(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho_+(d_X(x, y))$$

We need to show that $f : X \to f(X)$ is a coarse map and we need to define a map $g : f(X) \to X$ which is coarse and which is close to the identity map when composed with f both ways round. Firstly we shall consider the map f.

Given $S > 0$ such that $d(f(x), f(y)) \leq S$, we have by assumption that $\rho_-(d(x, y)) \leq S$. i.e. there exists $R > 0$ such that $\rho_-(d(x, y)) \leq S \Rightarrow d(x, y) < R$, and therefore f is a proper map.

Given $R > 0$, then for any $x, y \in X$ such that $d(x,y) < R$, we have that $d(f(x), f(y)) \leq \rho_+(R)$ by assumption. But $\rho_+(R) \in \mathbb{R}$, so set $\rho_+(R) = S$, and we have that f is bornologous. Combining this with the above result we have that f is a coarse map.

We define $g : f(X) \to X$ as follows. Every $u \in f(X)$ has at least one pre-image in X. By the axiom of choice we can choose one such pre-image, for example x and then we define $g(u) = x = f^{-1}(u)$.

We now check that g is a coarse map. Firstly, given $R_1 > 0$ such that $d(g(u), g(v)) \leq R_1$, then let $x = g(u)$ and $y = g(v)$, so $u = f(x)$ and $v = f(y)$. Then we have that given $R_1 > 0$ such that $d(x,y) \leq R_1$, then by the uniform embedding f, $d(f(x), f(y)) \leq \rho_+(R_1) = S_1$. And so g is proper.

Similarly, let $f(x) = u$ and $f(y) = v$, then by assumption $d(x, g(u)) < K$ and $d(y, g(y)) < K$. So given $S_1 > 0$, then for any $u, v \in f(X)$ such that $d(u,v) = d(f(x), f(y)) < S_1$, then by f being a uniform embedding we have that $\rho_-(d(x,y)) \leq S_1 \Rightarrow \exists R_1 > 0$ so that $d(x,y) < R_1$. Then by the triangle inequality and the definition of g we have

$$\begin{aligned} d(g(u), g(v)) &\leq d(g(u), x) + d(x,y) + d(g(v), y) \\ &\leq d(x,y) + 2K \\ &< R_1 + 2K \\ &= R_2 \end{aligned}$$

and so g is bornologous and therefore is a coarse map.

Finally we need to check that $g \circ f$ and $f \circ g$ compose to approximate the identity maps on X and $f(X)$ respectively. Consider $g \circ f(x)$. We chose g above to be such that f is the inverse of g, and g is 'approximately' the inverse of f. By this we mean that there exists $K > 0$ such that if $f(x) = f(x')$ then $d(x, x') \leq K$, and therefore $d(g \circ f(x), x) \leq K$. This means that $g \circ f(x)$ is within distance K of x for all $x \in X$, i.e. $g \circ f$ is close to the identity. Now $f \circ g(v) = v$ so $f \circ g = id_{f(X)}$ as required.

Therefore X and $f(X)$ are coarsely equivalent. □

A subset Z of Y is **coarsely dense** if there is some $R > 0$ such that every point Y lies within R of a point in Z. Our earlier illustration of the real line and the integers is a good example here. The set $\mathbb{Z}$ of integers is a subset of the set of real numbers $\mathbb{R}$ and there exists some $R > 0$, specifically $R = \frac{1}{2}$ such that every point in $\mathbb{R}$ lies within distance $\frac{1}{2}$ of an integer. In fact by the above lemma and the previous example, showing that there is a uniform embedding $f : \mathbb{R} \to \mathbb{Z}$ we have the stronger condition that $\mathbb{R}$ and $\mathbb{Z}$ are coarsely equivalent.

Example 2.8. An example of coarse equivalence is based on a map which is a quasi-isometry. Let $f : X \to Y$ be a quasi-isometry, so there exist constants $C > 0$ and $D \geq 0$ such that for all $x, x' \in X$,

$$\frac{1}{C} d_X(x, x') - D \leq d_y(f(x), f(x')) \leq C d_X(x, x') + D$$

Now f is a uniform embedding as $\rho_- = \frac{1}{C} d_X(x, x') - D$ and $\rho_+ = C d_X(x, x') + D$. Therefore by Lemma 2.7, X and $f(X) \subseteq Y$ are coarsely equivalent.

Definition 2.9. Two metric spaces X and Y are **quasi-isometric** if there is a quasi-isometry $f : X \to Y$ and if every point in Y is distance at most D from some point of $f(x)$.

Lemma 2.10. *Let X and Y be metric spaces. Then X and Y are quasi-isometric iff there exist large-scale Lipschitz maps $f : X \to Y$ and $g : Y \to X$ such that $d(g \circ f(x), x)$ and $d(f \circ g(y), y)$ are bounded.*

Proof. First we assume that X and Y are quasi-isometric, so there is a map $f : X \to Y$ with constants $C > 0$, $D \geq 0$ such that

$$\frac{1}{C} d(x, x') - D \leq d(f(x), f(x')) \leq C d(x, x') + D$$

and $f(X)$ is D-dense in Y. This gives us the map $f : X \to Y$ which is large-scale Lipschitz and we just need to define $g : Y \to X$. We define g as follows:

- if $y \in Y$ is equal to some $f(x)$, then we define $g(y) = x$. If there is more than one inverse image of f, we use the axiom of choice to pick one. Now since f is a quasi-isometry the inverse images of f are at most distance CD apart.

- if $y \in Y$ is not equal to some $f(x)$, then it is within distance D of some $f(x)$ by assumption. Hence we define $g(y)$ to be an inverse image of this $f(x)$, again using the axiom of choice if necessary.

This means that $g \circ f(x)$ and $f \circ g(y)$ are 'close' to the identity maps. Specifically, $d(g \circ f(x), x) \leq CD$ and $d(f \circ g(y), y) \leq D$, so these distances are bounded as required.

Finally we need to check that $g : Y \to X$ is a large-scale Lipschitz map. We do this in two stages:

- let $y, y' \in f(X)$, then let $x = g(y)$ and $x' = g(y')$, so $y = f(x)$ and $y' = f(x')$. Then from the condition on f we have

$$\frac{1}{C} d(x, x') - D \leq d(f(x), f(x'))$$

and substituting in we get

$$\frac{1}{C} d(g(y), g(y')) - D \leq d(y, y')$$

Therefore $d(g(y), g(y')) \leq Cd(y, y') + CD$.

- if $y, y' \in Y \backslash f(X)$ then there exists $y_0, y'_0 \in f(X)$ such that $d(y, y_0) \leq D$ and $d(y', y'_0) \leq D$ and such that $g(y) = g(y_0)$ and $g(y') = g(y'_0)$. Then $d(y_0, y'_0) \leq d(y, y') + 2D$ and so

$$d(g(y), g(y')) \leq Cd(y_0, y'_0) + CD = Cd(y, y') + 3CD$$

Conversely, we assume that there exist constants $C, C' > 0$, $D, D' \geq 0$ and $S, S' > 0$ such that for all $x \in X$ and $y \in Y$

$$d(f(x), f(x')) \leq Cd(x, x') + D \text{ and } d(g(y), g(y')) \leq C'd(y, y') + D'$$

and

$$d(g \circ f(x), x) \leq S \text{ and } d(f \circ g(y), y) \leq S'$$

Then we have an upper estimate for the distance between two points in X given by

$$\begin{aligned} d(x,x') &\leq d(g\circ f(x),x) + d(g\circ f(x), g\circ f(x')) + d(g\circ f(x'),x') \\ &\leq S + C'd(f(x),f(x')) + D' + S \\ &= (2S+D') + C'd(f(x),f(x')) \end{aligned}$$

So, combining this with the large-scale Lipschitz assumption, we have

$$\frac{1}{C'}d(x,x') - \frac{2S+D'}{C'} \leq d(f(x),f(x')) \leq Cd(x,x') + D$$

Now if we let $A = \max\{C, C'\}$ and $B = \max\{D, \frac{2S+D'}{C'}\}$, then

$$\frac{1}{A}d(x,x') - B \leq d(f(x),f(x')) \leq Ad(x,x') + B$$

And we have that f is a quasi-isometric map from X to Y. Now we need to show that $f(X)$ is B-dense in Y. i.e. given $y \in Y$ there exists some $f(x) \in Y$ such that $d(f(x),y) \leq B$.

Given $y \in Y$ we know that $d(f\circ g(y),y) \leq S'$, so let $x = g(y)$, and then we have $d(f(x),y) \leq S'$. Then if we let $B' = max\{B, S'\}$ we have:

$$\frac{1}{A}d(x,x') - B' \leq d(f(x),f(x')) \leq Ad(x,x') + B'$$

and $f(X)$ is B'-dense in Y, as required. □

Lemma 2.11. *If two metric spaces X and Y are quasi-isometric then X and Y are coarsely equivalent.*

Proof. The condition that $f\circ g$ and $g \circ f$ are close to the identity maps on Y and X in the definition of coarse equivalence can be re-stated as $d(g\circ f(x),x)$ and $d(f\circ g(y),y)$ are bounded. Therefore from Lemma 2.10 the result follows as large-scale Lipschitz maps are examples of coarse maps. So if f and g are large-scale Lipschitz maps with the required properties, then more specifically they are coarse maps. □

2.2 Coarse Metric Spaces

Definition 2.12. [14, Definition 2.3] A **coarse structure** on a set X is a collection $\mathcal{E}$ of subsets $X \times X$, called the **controlled sets** for the coarse structure, which contains the diagonal and is closed under the formation of subsets, inverses, products and finite unions. A set equipped with a coarse structure is called a **coarse space**. A function $f(x)$ is said to be **supported within a controlled neighbourhood of x**, if the set $\{(x, y) : y \in \text{supp} f(x)\}$ is a controlled set.

Example 2.13. An example of a coarse structure on a metric space X is when the controlled sets of X are those contained within a finite width of the diagonal. This is called the **bounded coarse structure**. Formally, we let (X, d) be a metric space and $\mathcal{E}$ be the collection of subsets $E \subseteq X \times X$ for which $\sup\{d(x, x') : (x, x') \in E\}$ is finite.

2.3 General Measure Theory

This section gives some background on measures on metric spaces. We use this later on when considering $\mathbb{R}$-trees and measured walls spaces, as well as property A.

Given a metric space X, a **measure** m on X is a function which maps the σ-ring generated by open sets of X to non-negative real values (this is sometimes called a positive measure). It is a function such that the measure of the union of mutually disjoint subsets is equal to the sum of their measures, i.e. if $E_1, E_2, \dots E_n \subset X$ with $E_i \cap E_j = \phi$ for $i \neq j$ then $m(\bigcup_n E_n) = \sum_n m(E_n)$. Throughout this section we shall assume that m is a measure on a metric space X.

One can also define complex measures which are measures whose range is the complex numbers, not non-negative reals. In this case there exists a positive measure (which from now on we shall refer to just as a measure) which is defined as follows.

Definition 2.14. Let m be a complex measure. Then there exists a positive measure associated to m which is denoted by $|m|$ and is called the **total variation** of m. We define this as $|m|(X) = \sup \sum_i |m(X_i)|$ where the supremum is taken over all partitions $\bigcup X_i$ of X into measurable subsets X_i. If the measure m is a positive measure, then $m = |m|$.

One use of the total variation is when taking the difference of two measures. Let μ and ν be two measures. We note that the difference $\mu - \nu$ is not necessarily positive. Therefore we take the modulus of the difference to give a positive value, $|\mu - \nu|$. This is the total variation.

Next we introduce the idea of taking the norm of a measure.

Definition 2.15. If m is a complex measure on X, then we define the **norm** of m to be

$$\|m\|_1 = |m|(X) = \int_X d|m|$$

There are several different kinds of measures which can be defined by adding extra conditions to the definition of a measure. There are a few which are going to be useful later on and so we define them in a moment, but first we require a couple more definitions.

Definition 2.16. A **σ-algebra** is a collection of subsets of X which contains X and $\emptyset$, and is closed under complements and countable unions. The **Borel space** of X is the pair (X, b) where B is the σ-algebra generated by the open sets. A **Borel subset** is any element of the Borel space.

Example 2.17. The power set of X, which is the set of all subsets of X is a sigma algebra.

Definition 2.18. A measure m on X is called a **bounded measure** if the measure of the whole space is finite, i.e. if $m(X) < \infty$.

A measure m on X is **regular** if for every Borel subset A of X:

$$\begin{aligned} m(A) &= \sup\{m(C) : C \subseteq A,\ C \,\text{is closed}\} \\ &= \inf\{m(U) : A \subseteq U,\ U \,\text{is open}\} \end{aligned}$$

A measure m on a space X is a **probability measure** if it is a positive measure and $m(X) = 1$. We shall denote the set of all probability measures on X by $Prob(X)$.

Definition 2.19. [13, Theorem 2.1, Definition 2.1]
If X is a separable metric space (i.e. it has a countable dense subspace) and m is a probability measure on X, then there exists a unique closed set C_m, called the **support of m**, satisfying

(i) $m(C_m) = 1$, and

(ii) if D is any closed set such that $m(D) = 1$, then $C_m \subseteq D$.

Another way to think of the support of a measure m is that it is the complement of the largest open set on which the measure is zero.

Lemma 2.20. *[13, Theorem 2.1]*
C_m is the set of all points $x \in X$ having the property that for each open set U containing x, $m(U) > 0$.

Proof. If $x \in X \setminus C_m$ and U_x is an open set containing x then $m(U_x) = 0$ as otherwise $x \in C_m$. Conversely, if $x \in C_m$ and U_x is an open set containing x then $m(U_x) > 0$ as otherwise $C_m \setminus U_x \subset C_m$ such that $m(C_m \setminus U_x) = 1$ which contradicts the fact that C_m is the support of the measure m. □

Example 2.21. Let X be a finite metric space, and define the probability measure m of a subset to be equal to the number of elements in that subset divided by the size of X, i.e. if $A \subseteq X$ then $m(A) = \frac{|A|}{|X|}$. In this case the support of m is the whole of X.

Now, if there are appropriate conditions on a map $f : X \longrightarrow Y$ between metric spaces, and there is a measure on X, then one can induce a measure on the metric space Y which we define as follows.

Theorem 2.22. *A map $f : X \longrightarrow Y$ is* **measurable** *if the inverse image of a Borel set in Y is measurable in X. Furthermore if m is a regular measure on a Borel space X, then every mapping $f : X \longrightarrow Y$ which is measurable induces a measure $f_*(m)$ on Y such that*

$$f_*(m)(B) = m(f^{-1}(B))$$

The measure $f_*(m)$ is called the **image measure**.

It is useful to note how the supports of a measure and its image measure relate to each other. The following Lemma gives the details of this.

Lemma 2.23. *Let $f : X \to Y$ and let m be a regular probability measure on X, so that $f_*(m)$ is its image measure on Y. Then the support of the measure $f_*(m)$ is equal to the closure of $f(\text{support of } m)$.*

Proof. For ease of notation we shall write C_X for the support of the measure m on X, and C_Y for the support of the measure $f_*(m)$ on Y.

From the definition of the support of a measure $f_*(m)$ we have that C_Y is a closed set such that

(i) $f_*(m)(C_Y) = 1$, and

(ii) if $f_*(m)(D) = 1$, then $C_Y \subseteq D$.

And we have similar conditions for the measure m.

From condition (i) and the definition of the image measure, we have that:

$$1 = f_*(m)(C_Y) = m(f^{-1}(C_Y))$$

Therefore $m(f^{-1}(C_Y)) = 1$. But $m(C_X) = 1$ by the definition of support. By this same definition we have that $C_X \subseteq f^{-1}(C_Y)$, i.e.

$$\overline{f(C_X)} \subseteq C_Y$$

By the definition of the image measure and support of a measure we have

$$f_*(m)(f(C_X)) = m(f^{-1}f(C_X)) = m(C_X) = 1$$

so $f_*(m)(f(C_X)) = 1$. Then, by the definition of supports, we have

$$C_Y \subseteq \overline{f(C_X)}$$

So combining the two results here we have that

$$C_Y = \overline{f(C_X)}$$

□

Lemma 2.24. *Given a probability measure m, the image measure $f_*(m)$ is also a probability measure.*

Proof. By definition $f_*(m)(B) = m(f^{-1}(B))$. As m is a probability measure, $m(f^{-1}(B))$ takes values between 0 and 1 and therefore $f_*(m)(B)$ also takes values in this range. In addition $f_*(m)(Y) = m(f^{-1}(Y)) = m(X) = 1$ as m is a probability measure. Therefore $f_*(m)$ is a probability measure as required. □

2.4 C*-algebras

We start with some basic definitions of inner product and norms which lead to the definition of algebras, states and positivity.

Definition 2.25. Let V be a real or complex vector space. An **inner product** on V is a mapping $\langle .\,,.\rangle : V \times V \to \mathbb{R}$ which satisfies

1. $\langle x, y\rangle = \overline{\langle y, x\rangle}$;
2. $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$;
3. $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$; and
4. $\langle x, x\rangle \geq 0$ with equality only when $x = 0$,

for all $x, y, z \in V$ and for all $\lambda \in \mathbb{C}$.

An **inner product space** is a pair $(V, \langle .\,,.\rangle)$ where V is a complex vector space and $\langle .\,,.\rangle$ is an inner product on V. The **norm** of a vector in an inner product space is defined to be $\langle x, x\rangle^{\frac{1}{2}}$, and is denoted by $\|x\|$.

Example 2.26.

1. Let X be a metric space and $l^2(X)$ be the space of functions on X which are square summable and have countable support, i.e. if $f \in l^2(X)$ then $\sum_{x\in X} |f(x)|^2 < \infty$. Then we define the inner product for all $f, g \in l^2(X)$ as
$$\langle f, g\rangle = \sum_{x\in X} f(x) \cdot \overline{g(x)}$$

2. Let $f, g \in C[0,1]$ the set of continuous complex valued functions on the interval $[0,1]$ with point wise addition and scalar multiplication. Then we define the inner product on these functions as follows

$$\langle f, g \rangle = \int_0^1 f(t) \cdot \overline{g(t)} dt$$

We can define the norm directly by:

Definition 2.27. Let V be a complex vector space. A **norm** on V is a mapping $\|\cdot\| : V \longrightarrow \mathbb{R}$ which satisfies

N1. $\|x\| > 0$ if $x \neq 0$;

N2. $\|\lambda x\| = |\lambda|\|x\|$ for all scalars $\lambda \in \mathbb{C}$ and vectors $x \in V$; and

N3. $\|x + y\| \leq \|x\| + \|y\|$ for all vectors $x, y \in V$.

Example 2.28.

1. For $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ define the norm of x by $\|x\| = (|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2)^{\frac{1}{2}}$. This is know as the euclidean length of a vector x.

2. Let X be a metric space, and $l^\infty(X)$ be complex valued bounded functions on X with pointwise addition and scalar multiplication. Then define the norm as follows, let $f \in l^\infty(X)$ then

$$\|f\|_\infty = \sup_{x \in X} |f(x)|$$

A **normed space** is a pair $(V, \|\cdot\|)$ where V is a real or complex vector space and $\|\cdot\|$ is a norm on V. A **Banach space** is a normed space which is a complete metric space (i.e. every Cauchy sequence converges) with respect to the metric induced by its norm. A **Hilbert space** is an inner product space which is a complete metric space with respect to the metric induced by its inner product.

Definition 2.29. An **algebra** is a vector space A over a field F with a multiplication defined on it which turns it into a ring such that $f(ab) = (fa)b = a(fb)$ for all $a, b \in A$ and $f \in F$. A **Banach algebra** is an algebra A over a field F that has a norm relative to which A is a Banach space and such that $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$.

Definition 2.30. An **involution** * (called the **adjoint**) is a map of A into A such that for all $a, b \in A$ and $\lambda \in \mathbf{C}$,

$$\begin{array}{rclcrcl} (a+b)^* & = & a^* + b^* & \quad & (\lambda a)^* & = & \overline{\lambda} a^* \\ (a^*)^* & = & a & & (ab)^* & = & b^* a^* \end{array}$$

Definition 2.31. A **C*-algebra** is a Banach algebra A with an involution and the additional norm condition $||a^*a|| = ||a||^2$ for all $a \in A$.

Example 2.32. One example of a C*-algebra is the set of bounded operators on a Hilbert space. Let E and F be Hilbert spaces, then an operator $T : E \longrightarrow F$ is bounded if there exists $M \geq 0$ such that $||Tx|| \leq M||x||$, for all $x \in E$. The adjoint is defined to be the unique operator $T^* : F \longrightarrow E$ such that $\langle Tx, y\rangle_F = \langle x, T^*y\rangle_E$ for $x \in E$ and $y \in F$. This then satisfies the norm condition $||T|| = ||T^*||$.

Example 2.33. We define by $C_0(X)$ the algebra of bounded continuous functions which tend to zero outside compact sets. $C_0(X)$ is a C*-algebra where we define the involution $*$ as the complex conjugate and we use the supremum norm. Let $f, g \in C_0(X)$, then $(f \cdot g)(x) = f(x) \cdot g(x)$, $f^*(x) = \overline{f(x)}$ and $||f|| = \sup_{x \in X} |f(x)|$. We show that the additional norm condition for a C*-algebra is satisfied below:

$$\begin{aligned} ||f^*f|| &= \sup_{x \in X} |f^*(x) \cdot f(x)| \\ &= \sup_{x \in X} |\overline{f(x)} \cdot f(x)| \\ &= \sup_{x \in X} |f(x)|^2 \\ &= ||f||^2 \end{aligned}$$

The Gelfand-Naimark theorem states that every commutative C*-algebra A is *-isomorphic to the algebra $C_0(X)$.

Definition 2.34. Let A be a C*-algebra, then the **spectrum** of $a \in A$, $spec(a)$ is the set of $\lambda \in \mathbb{C}$ such that $\lambda I - a$ is not invertible.

[6, Proposition 1.6.1] A hermitian element $a \in A$ (i.e. satisfies $a = a^*$) is said to be a **positive** element iff the following equivalent conditions hold:

(i) its spectrum is contained in the real half line, i.e. $spec(a) \in [0, \infty)$;

(ii) a is of the form gg^* for some $g \in A$;

(iii) a is of the form h^2 for some hermitian $h \in A$.

Lemma 2.35. *Consider the C*-algebra, $C_0(X)$, and let $f \in C_0(X)$. Then $f(x) \geq 0$ iff f is a positive element of $C_0(X)$.*

Proof. Firstly, let $f(x) \geq 0$ for all $x \in X$, and define $h(x) = \sqrt{f(x)}$ for all $x \in X$. Then h is real valued (and so hermitian), bounded, continuous, and vanishes where f does and so $h \in C_0(X)$ and $f = h^2$, and so f is positive.

Similarly, let f be a positive element and so $f = gg^*$ for some $g \in C_0(X)$. Then

$$f(x) = g(x) \cdot \overline{g(x)} = |g(x)|^2 \geq 0$$

and therefore $f(x) \geq 0$ for all $x \in X$ as required. □

Definition 2.36. A **functional** on a C*-algebra A is a linear map $\phi : A \to \mathbb{C}$. A functional $\phi(f)$ is said to be **positive** iff given $f(x) \geq 0$ for all $x \in X$ then $\phi(f) \geq 0$. A **state** on A is a positive linear functional on A with unit norm.

Example 2.37. An example of states on $C_0(X)$ are evaluation maps, ev_x. For every $x \in X$ the evaluation map ev_x is defined by $ev_x(f) = f(x)$. So it evaluates functions of $C_0(X)$ at the point $x \in X$. Note that these maps are in fact algebra homomorphisms as

$$ev_x(f) \cdot ev_x(g) = f(x) \cdot g(x) = (f \cdot g)(x) = ev_x(f \cdot g)$$

We now prove that they are states on $C_0(X)$. It can easily be seen that they are positive functionals, as if $f(x) \geq 0$ for all $x \in X$, then $ev_x(f) = f(x) \geq 0$. Now we need to prove that they have unit norm. Using the supremum norm we get:

$$\begin{aligned} \|ev_x\| &= \sup_{\|f\| \leq 1} |ev_x(f)| \\ &= \sup_{\|f\| \leq 1} |f(x)| \end{aligned}$$

But by the definition of the supremum norm $\|f\|_\infty \leq 1$ iff $|f(x)| \leq 1$ for all $x \in X$, and so

$$\|ev_x\| \leq 1$$

for all $x \in X$. Note that there exists a function $g \in C_0(X)$ such that $g(x) = 1$ and $g(y) \leq 1$ for all $y \neq x$ then $ev_x(g) = 1$ and $\|g\| \leq 1$. So

$$\|ev_x\| \geq |g(x)| = 1$$

Combining the two results we get $\|ev_x\| = 1$ as required.

Chapter 3

$\mathbb{R}$-trees and Hilbert Space Compression

Guentner and Kaminker, in [8], recently developed an invariant which can be assigned to any metric space; this is the Hilbert space compression. We give details of this later, but it basically gives us a measure of how much we need to compress or twist a space, using a large-scale Lipschitz map, to embed it into Hilbert space. We shall see later that the Hilbert space compression takes values in the interval $[0, 1]$. Guentner and Kaminker also show that if the Hilbert space compression of a finitely generated discrete group is strictly greater than $\frac{1}{2}$, then the group is exact. We consider this theory for non-discrete metric spaces, specifically the example of $\mathbb{R}$-trees. The main result of this chapter is the following theorem.

Theorem (3.19). *The Hilbert space compression of an $\mathbb{R}$-tree X is equal to 1.*

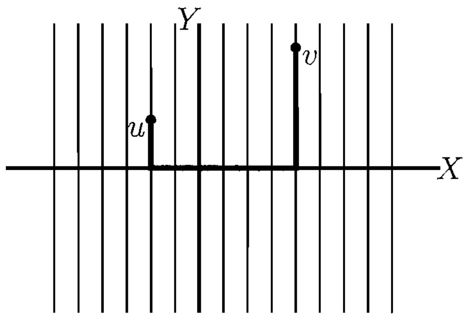
3.1 $\mathbb{R}$-trees

Definition 3.1. Let X be a metric space. Then an **arc** in X is a subset of X homeomorphic to a compact interval of $\mathbb{R}$ and a **segment** is a subset which is isometric to an interval of $\mathbb{R}$. A **real tree** (or $\mathbb{R}$-tree) is a metric space X, with a distance function d, such that any two distinct points $x, y \in X$ belong to a unique arc $[x, y]$, which is a closed segment.

Example 3.2. Consider the metric space $\mathbb{R}^2$ with the distance function defined as below:

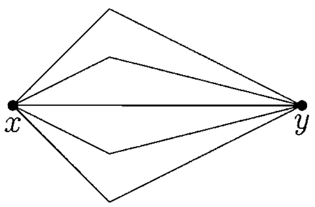
Let $u = (x_1, y_1)$ and $v = (x_2, y_2)$, then

$$d(u, v) = \begin{cases} |y_1 - y_2| & \text{if } x_1 = x_2 \\ |y_1| + |y_2| + |x_1 - x_2| & \text{otherwise} \end{cases}$$

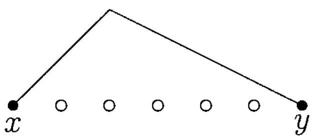


Example 3.3. Here are some examples of metric spaces which are not $\mathbb{R}$-trees.

1) $\mathbb{R}^2$ with the euclidean metric, as two points $x, y \in X$ do not belong to a unique arc, as shown in the diagram below:

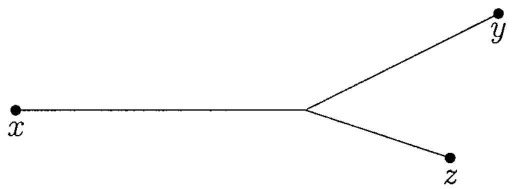


2) The following curve considered as part of $\mathbb{R}^2$ with the euclidean metric is not an $\mathbb{R}$-tree as the unique arc joining x and y is not a segment in $\mathbb{R}^2$. We have illustrated the relevant segment by a dotted line in the diagram.



One important feature of an $\mathbb{R}$-tree is that it has a tripod structure. Given three points x, y and z in an $\mathbb{R}$-tree, there are unique geodesics $[x, y]$, $[y, z]$

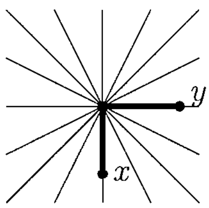
and $[x, z]$ which have a common point of intersection (called the centre). The endpoints x, y and z are called the vertices of the tripod. So given three points in an $\mathbb{R}$-tree, they uniquely define the vertices of a tripod (we shall say they form a tripod). For example the points x, y and z form a tripod as shown below.



$\mathbb{R}$-trees are equipped with a measure, the Lebesgue measure, which was constructed by Valette in [19]. First consider the example of $\mathbb{R}^2$ equipped with the French railway metric:

$$d(x, y) = \begin{cases} 0 & x = y \\ d_2(x, (0,0)) + d_2((0,0), y) & \text{if } x \neq y \end{cases}$$

where $d_2(x, y) = (|x_1 - y_1|^2 + |x_2 - y_2|^2)^{\frac{1}{2}}$ is the euclidean metric. This looks like a point (the origin) with an infinite number of 'spokes', and to travel between any two points on different spokes one must go via the origin.



Note that any open neighbourhood of the origin contains infinitely many 'spokes' and so is not relatively compact. This causes problems when trying to define the Borel measure. We instead use the σ-algebra $\mathcal{B}$ generated by segments in X, and define the measure on X as follows:

Proposition 3.4. *[19, Proposition 3]*
Let X be an $\mathbb{R}$-tree. For all $x, y \in X$ we denote by $[x, y]$ the segment joining these two points. There exists a unique measure μ on X, supported on $\mathcal{B}$ such that for all $x, y \in X$, $\mu[x, y] = d(x, y)$. This measure is called the Lebesgue measure on X.

3.2 Hilbert Space Compression

In this section we give an overview of [8], a paper by Guentner and Kaminker where Hilbert space compression is defined.

First recall that a large-scale Lipschitz map is a map $f : X \to Y$ such that there exist constants $C > 0$ and $D \geq 0$ such that

$$d_Y(f(x), f(y)) \leq Cd_X(x, y) + D$$

We have already provided some examples of large-scale Lipschitz maps but here we provide another example which is relevant to the Hilbert space compression.

Example 3.5. Consider $\mathbb{F}_2$ the free group on two generators, and let $X = (V, E)$ be the Cayley graph of $\mathbb{F}_2$ where V is the set of vertices and E the set of edges. Let $\mathcal{H} = l^2(E)$, then define $f : \mathbb{F}_2 \to \mathcal{H}$ by

$$f(s) = \delta_{e_1(s)} + \cdots + \delta_{e_k(s)}$$

where δ_e is the Dirac function of the edge e and $e_1(s), \cdots, e_k(s)$ are the edges on the unique path in the Cayley graph from $s \in \mathbb{F}_2$ to the identity $1 \in \mathbb{F}_2$. So $k = d(s, 1)$, and therefore $\|f(s)\| = \sqrt{d(s, 1)}$ and

$$\|f(s) - f(t)\| = \sqrt{d(s, t)} \leq d(s, t) + 1$$

Therefore f is a large-scale Lipschitz map. Computing the compression is more difficult. This is proved in [8, Proposition 4.2] by weighting the edges and creating a family of large-scale Lipschitz maps. We use a similar approach later when we calculate the Hilbert space compression of $\mathbb{R}$-trees.

Definition 3.6. [8] We denote the set of large-scale Lipschitz maps from X to Y by $Lip^{ls}(X, Y)$. The **compression**, ρ_f, of $f \in Lip^{ls}(X, Y)$ is defined by

$$\rho_f(r) = \inf_{d_X(x,y) \geq r} \{d_Y(f(x), f(y))\}$$

From the definition we can see that the compression, ρ_f of f provides a lower bound for $d_Y(f(x), f(y))$ when $d_X(x, y) \geq r$, whereas the large-scale Lipschitz condition gives an upper bound.

Example 3.7. Below are a couple of examples of large-scale Lipschitz maps and estimates of their compression.

- If f is an isometry, then $C = 1$, $D = 0$ and the compression is estimated by

$$\rho_f(r) = \inf_{d_X(x,y)\geq r} \{d_Y(f(x), f(y))\} = \inf_{d_X(x,y)\geq r} \{d_X(x,y)\} \geq r$$

  Note that in an unbounded geodesic space we get $\rho_f(r) = r$.

- If f is a quasi-isometry, we have that $\frac{1}{C} d_X(x,y) - D \leq d_Y(f(x), f(y))$, then the compression can be estimated as follows

$$\rho_f(r) = \inf_{d_X(x,y)\geq r} d_Y(f(x), f(y)) \geq \inf_{d_X(x,y)\geq r} \frac{1}{C} d_X(x,y) - D \geq \frac{r}{C} - D$$

- The compression for the map $f : \mathbb{F}_2 \to \mathcal{H}$ described above can be estimated as follows:

  We have that $\|f(s) - f(t)\| = \sqrt{d(s,t)}$ and f is a large-scale Lipschitz map. So using this and the definition of compression we have:

$$\begin{aligned} \rho_f(r) &= \inf_{d_{\mathbb{F}_2}(s,t)\geq r} \|f(s) - f(t)\| \\ &= \inf_{d_{\mathbb{F}_2}(s,t)\geq r} \sqrt{d(s,t)} \end{aligned}$$

  And so as each edge has length equal to one we have the following estimate for the compression: $\sqrt{r} \leq \rho_f(r) \leq \sqrt{r+1}$.

Definition 3.8. [8] Let X be an unbounded metric space. Then we define the **Hilbert space compression** of X as follows:

(i) The **asymptotic compression**, R_f, of a large-scale Lipschitz map $f \in Lip^{ls}(X,Y)$ is

$$R_f = \liminf_{r\to\infty} \frac{\log \rho_f^*(r)}{\log r}$$

where $\rho_f^*(r) = \max\{\rho_f(r), 1\}$.

(ii) The **compression** of X in Y is $R(X,Y) = \sup\{R_f : f \in Lip^{ls}(X,Y)\}$.

(iii) If Y is a separable Hilbert space, then the **Hilbert space compression** of X is $R(X) = R(X, \mathcal{H})$, where $\mathcal{H}$ is any separable Hilbert space. (i.e. the Hilbert space compression does not depend on the Hilbert space. We prove that this is true later in Theorem 3.12.)

First consider the definition of the asymptotic compression. We have introduced a new function $\rho_f^*(r)$. The definition would still be valid mathematically if we let $\rho_f = \rho_f^*$, but it would result in R_f being able to take negative values, which doesn't make sense when we are talking about compression. So to eliminate this, we have set ρ_f^* to be at least equal to 1, which means that R_f is non-negative.

As the notation $\liminf_{r\to\infty}$ may be unfamiliar we give an explanation here. First define a set L by

$$L = \left\{ a | a = \lim_{r_n \to \infty} \frac{\log \rho_f^*(r_n)}{\log r_n} \text{ for some sequence } r_n \right\}$$

and then define $R_f = \inf L$. Note that if $\frac{\log \rho_f^*(r)}{\log r}$ has a limit as $r \to \infty$, then the above definition simplifies as $\liminf = \lim$.

Lemma 3.9. *[8, Proposition 2.3] The compression, $R(X,Y)$ takes values in the range 0 to 1. As does the asymptotic compression, R_f.*

Proof. Firstly assume that there do not exist any large-scale Lipschitz maps $f : X \to Y$. But then the compression $\rho_f(r)$, and consequently the Hilbert space compression $R(X,Y)$, are not defined. So if $R(X,Y)$ exists, as in this Lemma, we know that there exists a map $f : X \to Y$ with $C > 0$ and $D \geq 0$ such that $d_Y(f(x), f(y)) \leq C d_X(x,y) + D$. Furthermore

$$\rho_f(r) = \inf_{d_X(x,y) \geq r} d_Y(f(x), f(y))$$

and

$$R_f = \liminf_{r \to \infty} \frac{\log \rho_f^*(r)}{\log r}$$

where $\rho_f^*(r) = \max\{\rho_f(r), 1\}$.

Now we consider the lower bound for $R(X, Y)$. This occurs when R_f is as small as possible. R_f takes its minimum value when $\rho_f^*(r)$ is at its minimum. So let $\rho_f^*(r) = 1$, and then, $\log(\rho_f^*(r)) = 0$, which in turn leads to $R_f = 0$. But, $\rho_f^*(r) = 1$ only if $\rho_f(r) \leq 1$. Now if $\rho_f(r) \leq 1$ for all $f \in Lip^{ls}(XY)$ then $R_f = 0$ for all $f \in Lip^{ls}(X, Y)$ and therefore $R(X, Y) = 0$ giving the lower bound.

Next we consider the upper bound for $R(X, Y)$ and follow the proof of Proposition 2.3 in [8]. As X is unbounded there exist sequences x_n and $y_n \in X$ such that $r_n = d_X(x_n, y_n) \to \infty$. From the definitions above, we then have

$$\begin{aligned} \rho_f(r_n) &= \inf_{d_X(x,y) \geq r_n} d_Y(f(x), f(y)) \\ &\leq \inf_{d_X(x,y) \geq r_n} C d_X(x, y) + D \\ &= C r_n + D \end{aligned}$$

The upper bound will occur when $\rho_f(r_n) = C r_n + D > 1$. If this is the case, then we get

$$R_f = \liminf_{r \to \infty} \frac{\log \rho_f(r)}{\log r} \leq \liminf_{n \to \infty} \frac{\log(C r_n + D)}{\log r} = 1$$

We also have that $R(X, Y) = 1$. So we have proved $0 \leq R(X, Y) \leq 1$ and $0 \leq R_f \leq 1$, as required. □

Lemma 3.10. *[8, Proposition 2.10]*
Let $\phi : X_1 \to X_2$ be a quasi-isometry. Then $R_\phi = 1$.

Proof. By Lemma 3.9 we have that $R_\phi \leq 1$. Now, as ϕ is a quasi-isometry, there exist constants $C > 0$ and $D \geq 0$ such that for all $x, x' \in X_1$

$$\frac{1}{C} d_{X_1}(x, x') - D \leq d_{X_2}(\phi(x), \phi(x')) \leq C d_{X_1}(x, x') + D$$

Now,

$$\begin{aligned}\rho_\phi(r) &= \inf_{d_{X_1}(x,x')\geq r} d_{X_2}(\phi(x),\phi(x')) \\ &\geq \inf_{d_{X_1}(x,x')\geq r} C^{-1}d_{X_1}(x,x') - D \\ &\geq C^{-1}r - D\end{aligned}$$

Therefore

$$\begin{aligned}R_\phi &= \liminf_{r\to\infty} \frac{\log \rho_\phi^*(r)}{\log r} \\ &\geq \liminf_{r\to\infty} \frac{\log(C^{-1}r - D)}{\log r} = 1\end{aligned}$$

We have shown that $R_\phi \leq 1$ and that $R_\phi \geq 1$, and so we conclude that $R_\phi = 1$. □

Lemma 3.11. *[8, Proposition 2.11]*
Let $f \in Lip^{ls}(X,Y)$ and $g \in Lip^{ls}(Z,X)$. Then $f \circ g \in Lip^{ls}(Z,Y)$ and $R_{fog} \geq R_f R_g$.

Proof. If f and g are large-scale Lipschitz maps, then we have the following two equations:

$$d_Y(f(x),f(x')) \leq C_1 d_X(x,x') + D_1$$

$$d_X(g(z),g(z')) \leq C_2 d_Z(z,z') + D_2$$

Combining these we get that

$$\begin{aligned}d_Y(f(g(z)),f(g(z'))) &\leq C_1 d_X(g(z),g(z')) + D_1 \\ &\leq C_1(C_2 d_Z(z,z') + D_2) + D_1 \\ &= C_1C_2 d_Z(z,z') + C_1D_2 + D_1\end{aligned}$$

And therefore $f \circ g$ is also a large-scale Lipschitz map.

Now we need to prove that $\rho_{fog}(r) \geq \rho_f(\rho_g(r))$. We start by considering the right hand side of this equation:

$$\begin{aligned}\rho_f(\rho_g(r)) &= \rho_f\left(\inf_{d_Z(z,z')\geq r} d_X(g(z), g(z'))\right) \\ &= \inf d(f(x), f(x'))\end{aligned}$$

where the infimum is taken over all $x, x' \in X$ which satisfy $d_X(x, x') \geq \inf_{d_Z(z,z')\geq r}\{d_X(g(z), g(z'))\}$. By changing this condition to $d_X(g(z), g(z')) \geq \inf_{d_Z(z,z')\geq r}\{d_X(g(z), g(z'))\}$ we are considering only a subset of the previous set and hence the infimum does not decrease. So we have

$$\rho_f(\rho_g(r)) \leq \inf d(f(g(z)), f(g(z')))$$

where the infimum is taken over all $g(z), g(z') \in X$ which satisfy

$$d_X(g(z), g(z')) \geq \inf_{d_Z(z,z')\geq r}\{d_X(g(z), g(z'))\}$$

Now just consider the condition on the infimum. This is trivial for any $z, z' \in Z$ such that $d_Z(z, z') \geq r$. So we get

$$\begin{aligned}\rho_f(\rho_g(r)) &\leq \inf_{d_Z(z,z')\geq r} d(f(g(z)), f(g(z'))) \\ &= \rho_{fog}(r)\end{aligned}$$

as required. Now if the increasing function $\rho_g(r)$ is bounded, by b say, then

$$\begin{aligned}R_g &= \liminf_{r\to\infty} \frac{\log \rho_g^*(r)}{\log r} \\ &\leq \liminf_{r\to\infty} \frac{\log b}{\log r} \\ &= 0\end{aligned}$$

and hence $R_{fog} \geq 0$ which we know from the definition.

So let us assume that $\lim_{r\to\infty} \rho_g(r) = \infty$. Combining this with the inequality

$\rho_{f\circ g}(r) \geq \rho_f(\rho_g(r))$, we get:

$$\begin{aligned} R_{f\circ g} &= \liminf_{r\to\infty} \frac{\log \rho^*_{f\circ g}(r)}{\log r} \\ &\geq \liminf_{r\to\infty} \left(\frac{\log \rho^*_f(\rho_g(r))}{\log \rho_g(r)}\right)\left(\frac{\log \rho_g(r)}{\log r}\right) \\ &= R_f R_g \end{aligned}$$

□

This means that the asymptotic compression of the composition of two large-scale Lipschitz maps is greater than or equal to the multiplication of the asymptotic compression of each of the individual large-scale Lipschitz maps. Combining Lemma 3.10 and Lemma 3.11, we get that if f is a quasi-isometry, then the asymptotic compression of the composition of the maps is greater than or equal to the asymptotic compression of g.

Theorem 3.12. *Let X be a metric space with unbounded metric and H_1 and H_2 be separable Hilbert spaces. Then $R(X, H_1) = R(X, H_2)$. (i.e. the Hilbert space compression does not depend on the choice of Hilbert space.)*

Proof. H_1 is a separable Hilbert space, and so is isometric to $l^2(\mathbb{N})$ [17, Theorem 3.21-B], i.e. there is a one-to-one correspondence between elements of H_1 and elements of $l^2(\mathbb{N})$ which preserves distance. As the same is true for H_2, one can see that H_1 and H_2 are in fact isometric. Hence there exist isometries $q_1 : H_1 \longrightarrow H_2$ and $q_2 : H_2 \longrightarrow H_1$ such that $R_{q_1} = R_{q_2} = 1$. Now, using the Lemma 3.10 and Lemma 3.11 we have that

$$\begin{aligned} R(X, H_2) &= \sup\{R_g : g \in Lip^{ls}(X, H_2)\} \\ &\geq \sup\{R_f : f \in Lip^{ls}(X, H_1)\} R_{q_1} \\ &= R(X, H_1) \end{aligned}$$

and

$$\begin{aligned}R(X,H_1) &= \sup\{R_f : f \in Lip^{ls}(X,H_1)\} \\ &\geq \sup\{R_g : g \in Lip^{ls}(X,H_2)\}R_{q_2} \\ &= R(X,H_2)\end{aligned}$$

Combining these we get

$$R(X,H_2) = R(X,H_1)$$

and hence the Hilbert space compression of X does not depend on the choice of Hilbert space. □

Given a large-scale Lipschitz map $f : X \to Y$ it can easily be seen that if we define the function $\rho_+(d_X(x,y))$ (from the definition of uniform embeddings) to be $Cd_X(x,y) + D$, then from the definition of large-scale Lipschitz this will be an upper bound for $d_Y(f(x), f(y))$ and it will be proper, ie. $\lim_{r\to\infty} \rho_+(r) = +\infty$. Now, the compression function ρ_f is a lower bound for $d_Y(f(x), f(y))$ and so, if it is proper then X is uniformly embeddable in Y.

In general it is not easy to show that a map is proper. So we turn to a proposition by Guentner and Kaminker which relates uniform embeddability of a space to its Hilbert space compression.

Proposition 3.13. *[8, Proposition 3.1]*
Let X be a metric space. If the Hilbert space compression of X is nonzero, then X is uniformly embeddable in Hilbert space.

Proof. Given a metric space X with $R(X) > 0$, there exists a function $f \in Lip^{ls}(X, \mathcal{H})$ such that $R_f > 0$, formally, there exists $\epsilon > 0$ such that $R_f > \epsilon$.

$$\liminf_{r\to\infty} \frac{\log \rho_f^*(r)}{\log r} > \epsilon$$

where $\rho_f^*(r) = \max\{\rho_f(r), 1\}$. If $\rho_f^*(r) = 1$ then $R_f = 0$, therefore we must have $\rho_f^*(r) = \rho_f(r) > 1$, and the condition becomes that there exists $\epsilon > 0$ such that

$$\liminf_{r\to\infty} \frac{\log \rho_f(r)}{\log r} > \epsilon$$

i.e. for large enough r,

$$\begin{aligned} \frac{\log \rho_f(r)}{\log r} &> \frac{\epsilon}{2} \\ \log \rho_f(r) &> \frac{\epsilon}{2}\log(r) \\ \rho_f(r) &> r^{\frac{\epsilon}{2}} \end{aligned}$$

Therefore $\lim_{r\to\infty} \rho_f(r) = \lim_{r\to\infty} r^{\frac{\epsilon}{2}} \to \infty$ as required. □

Thus to show that X is uniformly embeddable in Hilbert space one needs to find a large-scale Lipschitz map with nonzero compression. This will show that $R(X) > 0$, and then by Proposition 3.13 we will have proven that X embeds uniformly in Hilbert space.

3.3 Hilbert Space Compression of $\mathbb{R}$-trees

In this section and throughout this thesis on our work on $\mathbb{R}$-trees we note that the Hilbert spaces we are mapping into are not necessarily separable. Therefore we can not assume that the Hilbert space compression is unaltered by changing the Hilbert space. From here on in we shall consider only the Hilbert space $l^2(X)$, for any $\mathbb{R}$-tree.

Definition 3.14. Let X be an $\mathbb{R}$-tree. Fix a base point $u \in X$, let $\epsilon > 0$, and define a family of maps on X by:

$$\phi_v^\epsilon(x) = \begin{cases} d(v,x)^\epsilon & \text{if } x \in [v,u) \\ 0 & \text{otherwise} \end{cases}$$

where $v, x \in X$.

Proposition 3.15. *Let X be an $\mathbb{R}$-tree. Then for every $v \in X$, ϕ_v^ϵ is an l^2 function.*

Proof. First we fix a base point $u \in X$ and we note that given a point $v \in X$ there is a unique arc joining v to u.

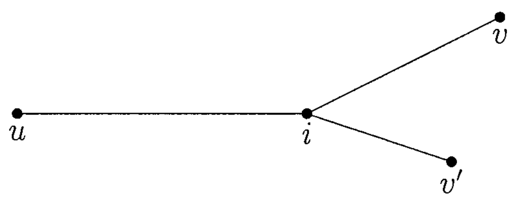
Recall that $\mathbb{R}$-trees are equipped with Lebesgue measure (see [4]) where the interval $[v, u)$ is isometric to the half-open interval on the real line $[0, d(v, u))$. Using this we show that the map $\phi_v^\epsilon(x)$ is square integrable.

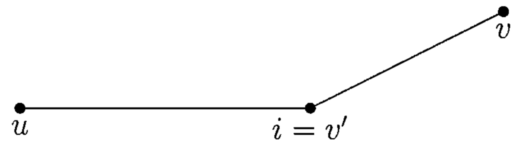
$$\begin{aligned}
\|\phi_v^\epsilon\|^2 &= \int_X |\phi_v^\epsilon(x)|^2 d\mu \qquad \text{where } \mu \text{ is the Lebesgue measure on } X \\
&= \int_{(v,u)} |\phi_v^\epsilon(x)|^2 d\mu \qquad \text{as } \operatorname{supp} \phi_v^\epsilon(x) = [v, u) \\
&= \int_0^{d(v,u)} |t^\epsilon|^2 dt \qquad \text{where } dt \text{ is the Lebesgue measure on } \mathbb{R} \\
&= \int_0^{d(v,u)} t^{2\epsilon} dt \\
&= \left[\frac{t^{2\epsilon+1}}{2\epsilon+1} \right]_0^{d(v,u)} \\
&= \frac{[d(v,u)]^{2\epsilon+1}}{2\epsilon+1} \\
&< \infty \text{ as long as } d(v,u) < \infty
\end{aligned}$$

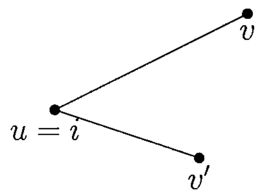
Therefore ϕ_v^ϵ is an l^2 function on X for all $\epsilon > 0$ and all $v \in X$. □

Proposition 3.16. *For every $0 < \epsilon < \frac{1}{2}$, the map $F^\epsilon : X \to l^2(X)$ defined by $v \mapsto \phi_v^\epsilon$ is a large-scale Lipschitz map.*

Proof. We shall denote by d the distance in X and distance in $l^2(X)$ is the norm $\|\cdot\|$. We shall show directly that $v \mapsto \phi_v^\epsilon$ is a large-scale Lipschitz map. First we need to recall some structure of an $\mathbb{R}$-tree. Given any three points $u, v, v' \in X$ there is a unique arc joining u to v, and another unique arc joining u to v'. These two arcs will intersect at a unique point i, and the points u, v, v' give rise to a tripod of one of the following forms:







We need to calculate an upper bound for the norm $\|\phi_v^\epsilon - \phi_{v'}^\epsilon\|$. To do this we first note that the support of this expression is contained in a tripod of the above form (i.e. it is zero elsewhere). Thus to calculate an estimate, we can split the norm into 3 parts,

(i) for $x \in [v, i]$;

(ii) for $x \in [v', i]$;

(iii) for $x \in [i, u]$.

We consider each case in turn and then sum the results.

Without loss of generality, we shall let $d(v, u) \geq d(v', u)$. Also, it is without loss of generality that we shall assume the tripod is set up as in the top diagram above. The other possibilities are when $i = v'$, but in this situation we would just have zero contribution to the norm from case (ii), giving us a smaller upper bound for the norm. Alternatively we could have that $i = u$, but again this would just result in case (iii) giving a zero contribution to the norm. And thus our estimate will be suitable.

Case (i) $\boldsymbol{x} \in [\boldsymbol{v}, \boldsymbol{i}]$

$$\begin{aligned}
\|\phi_v^\epsilon - \phi_{v'}^\epsilon\|^2 &= \int_0^{d(v,i)} t^{2\epsilon} dt \\
&= \left[\frac{t^{2\epsilon+1}}{2\epsilon+1}\right]_0^{d(v,i)} \\
&= \frac{[d(v,i)]^{2\epsilon+1}}{2\epsilon+1} - 0 \\
&= \frac{[d(v,i)]^{2\epsilon+1}}{2\epsilon+1}
\end{aligned}$$

Therefore taking square roots of both sides we get:

$$\|\phi_v^\epsilon - \phi_{v'}^\epsilon\| = \frac{[d(v,i)]^{\epsilon+\frac{1}{2}}}{(2\epsilon+1)^{\frac{1}{2}}}$$

Case (ii) $\boldsymbol{x} \in [\boldsymbol{v}', \boldsymbol{i}]$

This is basically the same as case (i). Note that in some cases this will be equal to zero (specifically when v' lies on the path from u to v, i.e. when v' coincides with i).

$$\|\phi_v^\epsilon - \phi_{v'}^\epsilon\|^2 = \frac{[d(v',i)]^{2\epsilon+1}}{2\epsilon+1}$$

And so

$$\|\phi_v^\epsilon - \phi_{v'}^\epsilon\| = \frac{[d(v',i)]^{\epsilon+\frac{1}{2}}}{(2\epsilon+1)^{\frac{1}{2}}}$$

Case (iii) $\boldsymbol{x} \in [\boldsymbol{i}, \boldsymbol{u}]$

In the following equations we use the notation $(s^\epsilon)' = \frac{d}{ds}(s^\epsilon)$.

We note that $[d(v,i)+t]^\epsilon - [d(v',i)+t]^\epsilon = \int_{d(v',i)+t}^{d(v,i)+t} (s^\epsilon)' ds$ and so we have

$\|\phi_v^\epsilon - \phi_{v'}^\epsilon\|^2 = \int_0^{d(i,u)} \left(\int_{d(v',i)+t}^{d(v,i)+t} (s^\epsilon)' ds \right)^2 dt$

We use the Cauchy-Schwartz inequality to rewrite this expansion as follows:

In our case the norm is defined by integration, and the inner product is the integral squared, and so we get:

$$\begin{aligned}
\left(\int_{d(v',i)+t}^{d(v,i)+t} (s^\epsilon)' ds \right)^2 &\leq \left(\int_{d(v',i)+t}^{d(v,i)+t} ds \right) \left(\int_{d(v',i)+t}^{d(v,i)+t} ((s^\epsilon)')^2 ds \right) \\
&= [d(v,i) - d(v',i)] \int_{d(v',i)+t}^{d(v,i)+t} ((s^\epsilon)')^2 ds \\
&= [d(v,i) - d(v',i)] \int_{d(v',i)+t}^{d(v,i)+t} (\epsilon s^{\epsilon-1})^2 ds
\end{aligned}$$

So substituting this into the above estimate for the norm we get:

$$\int_0^{d(i,u)} \left(\int_{d(v',i)+t}^{d(v,i)+t} (s^\epsilon)' ds \right)^2 dt$$

$$\leq [d(v,i) - d(v',i)] \int_0^{d(i,u)} \int_{d(v',i)+t}^{d(v,i)+t} \epsilon^2 s^{2\epsilon-2} ds dt$$

Now we calculate the first integral, which gives us:

$$[d(v,i) - d(v',i)] \int_0^{d(i,u)} \int_{d(v',i)+t}^{d(v,i)+t} \epsilon^2 s^{2\epsilon-2} ds dt$$

$$\begin{aligned}
&= [d(v,i) - d(v',i)] \int_0^{d(i,u)} \left[\frac{\epsilon^2 s^{2\epsilon-1}}{2\epsilon - 1} \right]_{d(v',i)+t}^{d(v,i)+t} dt \\
&= [d(v,i) - d(v',i)] \int_0^{d(i,u)} \frac{\epsilon^2}{2\epsilon - 1} \left([d(v,i)+t]^{2\epsilon-1} - [d(v',i)+t]^{2\epsilon-1} \right) dt
\end{aligned}$$

Now, calculating the second integral, gives us:

$$[d(v,i)-d(v',i)]\int_0^{d(i,u)} \frac{\epsilon^2}{2\epsilon-1}\left([d(v,i)+t]^{2\epsilon-1}-[d(v',i)+t]^{2\epsilon-1}\right)dt$$

$$\begin{aligned}
&= [d(v,i)-d(v',i)]\frac{\epsilon^2}{2\epsilon-1}\left[\frac{[d(v,i)+t]^{2\epsilon}}{2\epsilon}-\frac{[d(v',i)+t]^{2\epsilon}}{2\epsilon}\right]_0^{d(i,u)} \\
&= [d(v,i)-d(v',i)]\frac{\epsilon}{2(2\epsilon-1)}\left[[d(v,i)+d(i,u)]^{2\epsilon}-[d(v',i)+d(i,u)]^{2\epsilon}\right. \\
&\quad \left.-[d(v,i)]^{2\epsilon}+[d(v',i)]^{2\epsilon}\right]
\end{aligned}$$

Recall that i lies on the arc from v to u and so $d(v,i)+d(i,u)=d(v,u)$. And similarly i lies on the arc from v' to u. So using this and some rearranging, the estimate becomes:

$$\begin{aligned}
&[d(v,i)-d(v',i)]\frac{\epsilon}{2(2\epsilon-1)}\left[[d(v,i)+d(i,u)]^{2\epsilon}-[d(v',i)+d(i,u)]^{2\epsilon}\right. \\
&\quad \left.-[d(v,i)]^{2\epsilon}+[d(v',i)]^{2\epsilon}\right] \\
&= [d(v,i)-d(v',i)]\frac{\epsilon}{2(2\epsilon-1)}\left[[d(v,u)]^{2\epsilon}-[d(v',u)]^{2\epsilon}\right. \\
&\quad \left.-[d(v,i)]^{2\epsilon}+[d(v',i)]^{2\epsilon}\right] \\
&= [d(v,i)-d(v',i)]\frac{\epsilon}{2(1-2\epsilon)}\left[-[d(v,u)]^{2\epsilon}+[d(v',u)]^{2\epsilon}\right. \\
&\quad \left.+[d(v,i)]^{2\epsilon}-[d(v',i)]^{2\epsilon}\right]
\end{aligned}$$

Recall that $d(v,u)\geq d(v',u)$, $0<\epsilon<\frac{1}{2}$ and that we have the tripod structure in the $\mathbb{R}$-tree. Therefore we can deduce that $d(v,i)-d(v',i)\leq d(v,v')$, also $[d(v,i)]^{2\epsilon}-[d(v',i)]^{2\epsilon}\leq[d(v,i)]^{2\epsilon}\leq[d(v,v')]^{2\epsilon}$, and $[d(v',u)]^{2\epsilon}-[d(v,u)]^{2\epsilon}\leq 0$.

So the above formula for the norm in case (iii) reduces to:

$$
\begin{aligned}
\|\phi_v^\epsilon - \phi_{v'}^\epsilon\|^2 &\leq \frac{\epsilon d(v,v')}{2(1-2\epsilon)}\left[0 + [d(v,v')]^{2\epsilon}\right] \\
&= \frac{\epsilon[d(v,v')]^{2\epsilon+1}}{2(1-2\epsilon)}
\end{aligned}
$$

Then taking square roots of both sides we get:

$$
\|\phi_v^\epsilon - \phi_{v'}^\epsilon\| \leq \frac{\epsilon^{\frac{1}{2}}[d(v,v')]^{\epsilon+\frac{1}{2}}}{(2-4\epsilon)^{\frac{1}{2}}}
$$

Now combining the estimates for the norm over the three sections, we get:

$$
\|\phi_v^\epsilon - \phi_{v'}^\epsilon\| \leq \frac{[d(v,i)]^{\epsilon+\frac{1}{2}}}{(2\epsilon+1)^{\frac{1}{2}}} + \frac{[d(v',i)]^{\epsilon+\frac{1}{2}}}{(2\epsilon+1)^{\frac{1}{2}}} + \frac{\epsilon^{\frac{1}{2}}[d(v,v')]^{\epsilon+\frac{1}{2}}}{(2-4\epsilon)^{\frac{1}{2}}}
$$

But $d(v,i) \leq d(v,v')$ and $d(v',i) \leq d(v,v')$ and so

$$
\begin{aligned}
\|\phi_v^\epsilon - \phi_{v'}^\epsilon\| &\leq \frac{[d(v,i)]^{\epsilon+\frac{1}{2}}}{(2\epsilon+1)^{\frac{1}{2}}} + \frac{[d(v',i)]^{\epsilon+\frac{1}{2}}}{(2\epsilon+1)^{\frac{1}{2}}} + \frac{\epsilon^{\frac{1}{2}}[d(v,v')]^{\epsilon+\frac{1}{2}}}{(2-4\epsilon)^{\frac{1}{2}}} \\
&\leq \frac{[d(v,v')]^{\epsilon+\frac{1}{2}}}{(2\epsilon+1)^{\frac{1}{2}}} + \frac{[d(v,v')]^{\epsilon+\frac{1}{2}}}{(2\epsilon+1)^{\frac{1}{2}}} + \frac{\epsilon^{\frac{1}{2}}[d(v,v')]^{\epsilon+\frac{1}{2}}}{(2-4\epsilon)^{\frac{1}{2}}} \\
&= [d(v,v')]^{\epsilon+\frac{1}{2}}\left(\frac{2}{(2\epsilon+1)^{\frac{1}{2}}} + \frac{\epsilon^{\frac{1}{2}}}{(2-4\epsilon)^{\frac{1}{2}}}\right)
\end{aligned}
$$

In order to prove a large-scale Lipschitz type estimate for the norm on the left we need to consider two cases.

Recall that $0 < \epsilon < \frac{1}{2}$, so $\frac{1}{2} < \epsilon + \frac{1}{2} < 1$, therefore:

If $d(v,v') \geq 1$, then

$$
\|\phi_v^\epsilon - \phi_{v'}^\epsilon\| \leq d(v,v')\left(\frac{2}{(2\epsilon+1)^{\frac{1}{2}}} + \frac{\epsilon^{\frac{1}{2}}}{(2-4\epsilon)^{\frac{1}{2}}}\right)
$$

and we put

$$C = \frac{2}{(2\epsilon+1)^{\frac{1}{2}}} + \frac{\epsilon^{\frac{1}{2}}}{(2-4\epsilon)^{\frac{1}{2}}}$$

Now if $d(v,v') < 1$, then $[d(v,v')]^{\epsilon+\frac{1}{2}} < 1$, therefore

$$\|\phi_v^\epsilon - \phi_{v'}^\epsilon\| \leq \left(\frac{2}{(2\epsilon+1)^{\frac{1}{2}}} + \frac{\epsilon^{\frac{1}{2}}}{(2-4\epsilon)^{\frac{1}{2}}} \right)$$

and we let

$$D = \frac{2}{(2\epsilon+1)^{\frac{1}{2}}} + \frac{\epsilon^{\frac{1}{2}}}{(2-4\epsilon)^{\frac{1}{2}}}$$

Therefore combining the above two cases, we have that for all values of $d(v,v')$, the norm can be approximated by:

$$\|\phi_v^\epsilon - \phi_{v'}^\epsilon\| \leq Cd(v,v') + D$$

where

$$C = D = \frac{2}{(2\epsilon+1)^{\frac{1}{2}}} + \frac{\epsilon^{\frac{1}{2}}}{(2-4\epsilon)^{\frac{1}{2}}}$$

depends only on the choice of ϵ, and so for each $0 < \epsilon < \frac{1}{2}$ the map $F^\epsilon : X \to l^2(X)$ is a large-scale Lipschitz map as required. □

Now, before we can calculate the compression of the maps F^ϵ we need to calculate a lower bound for $\|\phi_v^\epsilon - \phi_{v'}^\epsilon\|$ when $d(v,v') \geq r$. The compression will then be greater than or equal to this lower bound.

Proposition 3.17. *Let ϕ_v^ϵ be the family of maps defined in Definition 3.14. Then, given $0 < \epsilon < \frac{1}{2}$, there exists a constant $C_\epsilon > 0$ such that*

$$\|\phi_v^\epsilon - \phi_{v'}^\epsilon\|^2 \geq C_\epsilon r^{1+2\epsilon}$$

for all $v, v' \in X$ such that $d(v,v') \geq r$.

Proof. Let $d(v,v') \geq r$ and assume, as above, without loss of generality, that $d(v,u) \geq d(v',u)$, then $d(v,i) \geq \frac{r}{2}$. We know that the norm over the tripod u, i, v, v' is equal to the sum of the norm of each section of the tripod, and therefore is greater than or equal to the norm over the section from v to i. Therefore

$$\begin{aligned}
\|\phi_v^\epsilon - \phi_{v'}^\epsilon\|^2 &\geq \int_0^{\frac{r}{2}} t^{2\epsilon} dt \\
&= \left[\frac{t^{2\epsilon+1}}{2\epsilon+1}\right]_0^{\frac{r}{2}} \\
&= \frac{\left(\frac{r}{2}\right)^{2\epsilon+1}}{2\epsilon+1} - 0 \\
&= \frac{r^{2\epsilon+1}}{(2^{2\epsilon+1})(2\epsilon+1)} \\
&= C_\epsilon r^{2\epsilon+1}
\end{aligned}$$

where

$$C_\epsilon = \frac{1}{(2^{2\epsilon+1})(2\epsilon+1)}$$

Therefore

$$\|\phi_v^\epsilon - \phi_{v'}^\epsilon\| \geq C_\epsilon^{\frac{1}{2}} r^{\epsilon+\frac{1}{2}}$$

□

Lemma 3.18. *The asymptotic compression of the map F^ϵ is greater than or equal to $\frac{1}{2}+\epsilon$ when $0 < \epsilon < \frac{1}{2}$.*

Proof. Recall that by definition

$$\rho_{F^\epsilon}(r) = \inf_{d(v,v') \geq r} \|\phi_v^\epsilon - \phi_{v'}^\epsilon\|$$

and so we have

$$\rho_{F^\epsilon}(r) \geq C_\epsilon^{\frac{1}{2}} r^{\epsilon+\frac{1}{2}}$$

for all r.

From this we can calculate the asymptotic compression:

$$\begin{aligned} R_{F^\epsilon} &= \liminf_{r\to\infty} \frac{\log \rho_{F^\epsilon}(r)}{\log(r)} \\ &\geq \liminf_{r\to\infty} \frac{\log C_\epsilon^{\frac{1}{2}} r^{\epsilon+\frac{1}{2}}}{\log r} \\ &= \epsilon + \frac{1}{2} \end{aligned}$$

□

Theorem 3.19. *Let X be an $\mathbb{R}$-tree. The Hilbert space compression of X is equal to 1.*

Proof. Given $0 < \epsilon < \frac{1}{2}$ we have constructed a family of maps F^ϵ which are large-scale Lipschitz and have asymptotic compression greater than or equal to $\frac{1}{2}$. Therefore these maps show that we can embed an $\mathbb{R}$-tree X into the Hilbert space $l^2(X)$.

Hence if we let ϵ tend to $\frac{1}{2}$, we have that R_ϕ tends to 1. Hence the Hilbert space compression of X is 1. □

Theorem 3.20. *Any $\mathbb{R}$-tree X is uniformly embeddable in Hilbert space.*

Proof. X has Hilbert space compression 1, and by Proposition 3.13 we know that if the Hilbert space compression of a metric space X is non-zero then X is uniformly embeddable in Hilbert space. □

Chapter 4

Property A for Discrete Metric Spaces

Guoliang Yu, in [22], defined property A for discrete spaces as an analogue of the Følner condition for amenable groups. He then defined property A for non-discrete spaces in terms of discrete subspaces. In this chapter we give Yu's original definition along with various other equivalent definitions which have been developed, some which hold only for discrete bounded geometry metric spaces and some for more general metric spaces. We also give theorems and proofs relating to the permanence properties of property A in the discrete and general cases.

4.1 Property A as defined by Yu

The original definition of property A was by Yu in [22] and is stated for discrete metric spaces as follows.

Definition 4.1. [22, Definition 2.1]
A discrete metric space X is said to have **property A** if for any $R > 0$, $\epsilon > 0$, there exists a family of finite subsets $\{A_x\}_{x \in X}$ of $X \times \mathbb{N}$ such that

(1) $(x, 1) \in A_x$ for all $x \in X$;

(2) $\frac{|A_x \triangle A_{x'}|}{|A_x \cap A_{x'}|} < \epsilon$ for all $x, x' \in X$ satisfying $d(x, x') \leq R$ (where for each finite set A, $|A|$ is the number of elements in A); and

(3) $\exists S > 0$ such that if $(y, m) \in A_x$, $(z, n) \in A_x$ for some $x \in X$, then $d(y, z) \leq S$.

Yu then defined property A for non-discrete metric spaces in terms of his definition for discrete metric spaces. We call metric spaces without restrictions, like discreteness or bounded geometry, **general** metric spaces.

Definition 4.2. [22, page 206]
A general metric space X is said to have **property A** if there exists a discrete subspace U of X such that

(1) there exists $c > 0$ for which $d(x, U) \leq c$ for all $x \in X$; and

(2) U has property A.

4.2 Higson and Roe's work

In [9] Yu's definition of property A was restated as follows:

Definition 4.3. [9, Definition 3.2]
Let $\text{fin}(Z \times \mathbb{N})$ be the set of all finite non-empty subsets of $Z \times \mathbb{N}$. The discrete metric space Z has **property A** if there are maps $A_n : Z \longrightarrow \text{fin}(Z \times \mathbb{N})$, where $n = 1, 2, \cdots$ such that:

1) for each n there is some $R > 0$ for which

$A_n(z) \subset \{(z', j) \in Z \times \mathbb{N} | d(z, z') < R\}$, for every $z \in Z$; and

2) for every $K > 0$, $\lim\limits_{n \to \infty} \sup\limits_{d(z,w)<K} \dfrac{|A_n(z) \triangle A_n(w)|}{|A_n(z) \cap A_n(w)|} = 0$

It can be seen that this definition is the same as that of Yu's, by letting the subset A_x from Yu's definition be the image of the point $x \in X$ under the map A_n for some specific n. Note that we have a sequence of maps in Definition 4.3 which we use for condition (2) where we show that the limit as $n \longrightarrow \infty$ is zero, whereas in Definition 4.1 we show that this fraction is not greater than ϵ. But we can chose ϵ to be as close to zero as we like, so even though the definitions look different they are in fact equivalent.

Remark 4.4. Whether or not a discrete metric space has property A depends only on the coarse equivalence class of the metric on Z [9]. Later on in this thesis, in Theorem 6.1 we prove that property A is a coarse invariant.

Higson and Roe then go on to reformulate the definition for discrete bounded geometry metric spaces. First we need to recall a couple of definitions.

Definition 4.5. A countable discrete metric space Z has **bounded geometry** if for every $C > 0$ there is an absolute bound on the number of elements in any ball within Z of radius C.

We will denote by $Prob(Z)$ the set of **Borel probability measures** on Z, i.e. the set of functions $b : Z \to [0,1]$ such that $\sum_{z \in Z} b(z) = 1$.

The following lemma is stated without proof in [9]. I provide a proof below.

Lemma 4.6. *$Prob(Z)$ is a subset of $l^1(Z)$.*

Proof. $l^1(Z)$ is the set of functions $f : Z \to \mathbb{R}$ such that $\sum_{z \in Z} |f(z)| < \infty$.

Now if $g \in Prob(Z)$, then $g : Z \to [0,1] \subset \mathbb{R}$, and

$$\sum_{z \in Z} |g(z)| = \sum_{x \in Z} g(z) = 1 < \infty$$

Thus $g \in Prob(Z) \Rightarrow g \in l^1(Z)$. Therefore $Prob(Z) \subseteq l^1(Z)$ as required. □

Proposition 4.7. *[9, Lemma 3.5]*
If Z is a discrete metric space of bounded geometry then Z has property A if and only if there is a sequence of maps $a^n : Z \to Prob(Z)$ such that

(1) for every n there is some $R > 0$ such that for every $z \in Z$, $supp(a^n_z) \subset \{z' \in Z | d(z, z') < R\}$; and

(2) for every $K > 0$, $\lim_{n \to \infty} \sup_{d(z,w) < K} \|a^n_z - a^n_w\|_1 = 0$

Proof. This was proved in [9] on page 5, but here we explain it in greater detail.

First we shall assume that we have maps $a^n \in Prob(Z) \subset l^1(Z)$ satisfying conditions (1) and (2) in Proposition 4.7, and we show therefore that Z has property A. By (1) we have that for each n there exists R so that a^n_z is supported within a ball of radius R. Z also has bounded geometry and so the number of elements of Z in the support of a^n_z is uniformly bounded. Therefore the number of non-zero values of the function a^n_z will be uniformly bounded. Let us assume for the moment that the values of the function a^n_z are rational. Then for each n there is a natural number M (take for instance M to be the product of all the denominators of the non-zero values of a_z^n when written in reduced form) such that if $z \in Z$, then the function a_z^n assumes only values in the range $\frac{0}{M}, \frac{1}{M}, \frac{2}{M}, \cdots \frac{M}{M}$, and as $a_z^n \in Prob(Z)$, $\sum_{z' \in Z} a_z^n(z') = 1$.

We now need to consider the case when at least one value of a_z^n is irrational. In this case we can follow a similar argument, but instead of the function assuming values in the range $\frac{0}{M}, \frac{1}{M}, \frac{2}{M}, \cdots \frac{M}{M}$, it will assume values within ϵ of those values in the range. By increasing M we can make ϵ as small as we like. The rest of the proof relies on taking limits of supremums and the ϵ will become irrelevant. Therefore we shall continue the proof for the case when the values of a_z^n are rational.

We define $A_n(z) \subset Z \times \mathbb{N}$ by the relation

$$(z', j) \in A_n(z) \Leftrightarrow \frac{j}{M} \leq a_z^n(z')$$

We notice that if $d(z, z') > R$ then $z' \notin \text{supp}(a_z^n)$ therefore $a_z^n(z') = 0$, and so there is no element in $A_n(z)$ of the form (z', j). Therefore all the elements of $A_n(z)$ are of the form (z', j) where $d(z, z') < R$. So the $A_n(z)$ satisfy part (1) of Definition 4.3. We also note that to each $a_z^n(z')$ we assign a number $\frac{j_{z'}}{M}$ such that there are $j_{z'}$ elements in $A_n(z)$ of the form $(z', \cdot)$. Therefore

$$\begin{aligned}
|A_n(z)| &= |\{(z', \cdot) : d(z, z') < R\}| \\
&= \sum_{z' \in X} |\{(z', \cdot) \in A_n(z)\}| \\
&= \sum_{z' \in X} j_{z'} \\
&= M \sum_{z' \in X} \frac{j_{z'}}{M} \\
&= M \sum_{z' \in Z} a_z^n(z') \\
&= M \qquad \text{as } a_z^n \text{ is a probability measure}
\end{aligned}$$

Therefore $|A_n(z)| = M$ for every $z \in Z$.

Recall that $a_z^n(z')$ is of the form $\frac{j}{M}$, where $0 \leq j \leq M$. So when we consider the size of the symmetric difference $|A_n(z)\triangle A_n(w)|$ and the norm $\|a_z^n - a_w^n\|$ we need to add in a factor of M, giving us:

$$|A_n(z)\triangle A_n(w)| = M\|a_z^n - a_w^n\|_1 = |A_n(z)|\|a_z^n - a_w^n\|_1$$

And so by (2) of Proposition 4.7, for every $K > 0$,

$$\lim_{n\to\infty} \sup_{d(z,w)<K} \|a_z^n - a_w^n\|_1 = \lim_{n\to\infty} \sup_{d(z,w)<K} \frac{|A_n(z)\triangle A_n(w)|}{|A_n(z)|} = 0$$

And therefore by [9, lemma 3.4],

$$\lim_{n\to\infty} \sup_{d(z,w)<K} \frac{|A_n(z)\triangle A_n(w)|}{|A_n(z)\cap A_n(w)|} = 0$$

so Z has property A.

Conversely, suppose that Z has property A. We define $a^n : Z \to Prob(Z)$ as follows:

$$a_z^n(z')|A_n(z)| = |\{j|(z',j) \in A_n(z)\}|$$

We note that $\|a_z^n \cdot |A_n(z)|\|_1 = \sum_{t\in Z} |\{j|(t,j) \in A_n(z)\}| = |A_n(z)|$, and so $\|a_z^n\| = 1$ and therefore $a_z^n \in Prob(Z)$.

The support condition is satisfied as $a_z^n(z') = \dfrac{|\{j|(z',j) \in A_n(z)\}|}{|A_n(z)|} = 0$ if $d(z,z') > R$ by definition of $A_n(z)$. Therefore $a_z^n(z') = 0$ for $d(z,z') > R$, so the support of a_z^n is contained within a ball of radius R.

Now we prove condition (2). From the definition of a_z^n we have,

$$\begin{aligned}&\|a_z^n \cdot |A_n(z)| - a_w^n \cdot |A_n(w)|\|_1 \\ &\quad = \sum_{t\in Z} \big|\, |\{j|(t,j) \in A_n(z)\}| - |\{j'|(t,j') \in A_n(w)\}|\,\big|\end{aligned} \tag{A}$$

This is not greater than the sum

$$\sum_{(t,j)\in Z\times\mathbb{N}} \left|\, |\{j|(t,j)\in A_n(z)\}| - |\{j'|(t,j')\in A_n(w)\}|\,\right| \qquad (B)$$

as everything which contributes to equation (A) will also contribute to equation (B), whereas there will be some elements (t,j) which cancel out in equation (A) but are added together in equation (B). This can be seen more easily if we split equation (B) into 4 sums.

$$\begin{aligned}
&\sum_{(t,j)\in Z\times\mathbb{N}} \left|\, |\{j|(t,j)\in A_n(z)\}| - |\{j'|(t,j')\in A_n(w)\}|\,\right| \\
&= \sum_{(t,j)\in A_n(z)\setminus A_n(w)} \left|\, |\{j|(t,j)\in A_n(z)\}| - |\{j'|(t,j')\in A_n(w)\}|\,\right| \\
&+ \sum_{(t,j)\in A_n(z)\cap A_n(w)} \left|\, |\{j|(t,j)\in A_n(z)\}| - |\{j'|(t,j')\in A_n(w)\}|\,\right| \\
&+ \sum_{(t,j)\in A_n(w)\setminus A_n(z)} \left|\, |\{j|(t,j)\in A_n(z)\}| - |\{j'|(t,j')\in A_n(w)\}|\,\right| \\
&+ \sum_{(t,j)\notin A_n(z)\cup A_n(w)} \left|\, |\{j|(t,j)\in A_n(z)\}| - |\{j'|(t,j')\in A_n(w)\}|\,\right|
\end{aligned}$$

The last summation will be equal to zero, as both parts of it will be zero. The second summation will also be zero, as there will be an equal number of js as j's, so they will cancel out. So we only have to consider the first and third summation, i.e. where $(t,j)\in A_n(z)\setminus A_n(w)$ or $(t,j)\in A_n(w)\setminus A_n(z)$. It can be seen that each of these summations will be equal to the size of the set we are summing over. Therefore the sum of these 4 summations is equal to the symmetric difference of $A_n(z)$ and $A_n(w)$. So we can write,

$$\|a_z^n\cdot|A_n(z)| - a_w^n\cdot|A_n(w)|\|_1 \le |A_n(z)\triangle A_n(w)|$$

and so dividing through by $|A_n(z)|$ and taking supremums and limits we get

$$\lim_{n\to\infty}\sup_{d(z,w)<K}\left\|a_z^n - a_w^n\frac{|A_n(w)|}{|A_n(z)|}\right\|_1 \le \lim_{n\to\infty}\sup_{d(z,w)<K}\frac{|A_n(z)\triangle A_n(w)|}{|A_n(z)|} = 0$$

But by [9, lemma 3.4]

$$\lim_{n\to\infty} \sup_{d(z,w)<K} \frac{|A_n(z) \triangle A_n(w)|}{|A_n(z)|} = 0 \Rightarrow \lim_{n\to\infty} \sup_{d(z,w)<K} \frac{|A_n(w)|}{|A_n(z)|} = 1$$

and so

$$\lim_{n\to\infty} \sup_{d(z,w)<K} \|a_z^n - a_w^n\|_1 = 0$$

as required. □

4.3 Equivalent definitions of property A for discrete bounded geometry metric spaces

In his paper [18] Tu lists many definitions of property A for discrete bounded geometry metric spaces, which he proves are all equivalent. Also, in a recent preprint by Brodzki, Niblo and Wright some more equivalent definitions are introduced, and finally some other conditions were suggested by Brodzki, Niblo and Wright and are proved by myself. As X is a discrete metric space the measure we use is the counting measure. Below we provide a list of various characterizations of property A.

In the following Theorem, statements (2), (3), (4), (8) and (9) come from Tu's paper [18, Proposition 3.2]; the statement (7) is due to Dadarlat and Guentner [5, Proposition 2.5], while statement (10) is taken from the paper by Brodzki, Niblo and Wright [2, Theorem 6]. In statements (5) and (6) we modify and prove a similar Proposition from [2, Theorem 6] from the l^2 context to l^1.

Theorem 4.8. *Let X be a discrete metric space with bounded geometry. The following are equivalent:*

(1) X has property A,

(2) $\forall R > 0$, $\forall \epsilon > 0$, $\exists S > 0$, $\exists (\xi_x)_{x\in X}$, $\xi_x \in l^1(X)$, $\mathrm{supp}(\xi_x) \subset B(x,S)$, $\|\xi_x\|_{l^1(X)} = 1$, and $\|\xi_x - \xi_y\|_{l^1(X)} \leq \epsilon$ whenever $d(x,y) \leq R$.

(3) $\forall R > 0$, $\forall \epsilon > 0$, $\exists S > 0$, $\exists (\chi_x)_{x\in X}$, $\chi_x \in l^1(X)$, $\mathrm{supp}(\chi_x) \subset B(x,S)$, $\frac{\|\chi_x - \chi_y\|_{l^1(X)}}{\|\chi_x\|_{l^1(X)}} \leq \epsilon$ whenever $d(x,y) \leq R$.

(4) $\forall R > 0$, $\forall \epsilon > 0$, $\exists S > 0$, $\exists (\eta_x)_{x \in X}$, $\eta_x \in l^2(X)$, $\text{supp}(\eta_x) \subset B(x, S)$, $\|\eta_x\|_{l^2(X)} = 1$, *and* $\|\eta_x - \eta_y\|_{l^2(X)} \leq \epsilon$ *whenever* $d(x, y) \leq R$.

(5) $\forall R > 0$, $\forall \epsilon > 0$, $\forall \delta > 0$, $\exists S > 0$, $\exists (\alpha_x)_{x \in X}$, $\alpha_x \in l^1(X)$, $\|\alpha_x\|_{l^1(X)} = 1$, $\|\alpha_x - \alpha_y\|_{l^1(X)} \leq \epsilon$ *whenever* $d(x, y) \leq R$, *and the restriction of* α_x *to* $B(x, S)$ *has norm at least* $1 - \delta$.

(6) $\exists \delta < 1$ *such that* $\forall R > 0$, $\forall \epsilon > 0$, $\exists S > 0$, $\exists (\beta_x)_{x \in X}$, $\beta_x \in l^1(X)$, $\|\beta_x\|_{l^1(X)} = 1$, $\|\beta_x - \beta_y\|_{l^1(X)} \leq \epsilon$ *whenever* $d(x, y) \leq R$, *and the restriction of* β_x *to* $B(x, S)$ *has norm at least* $1 - \delta$, *and the restriction of* β_x *to the set* $B(x, R + S) \setminus B(x, S)$ *has norm at most* ϵ.

(7) $\forall R > 0$, $\forall \epsilon > 0$, $\exists S > 0$, $\gamma_x \in \mathcal{H}$, $\|\gamma_x\|_{l^2(X)} = 1$, $\|\gamma_x - \gamma_y\|_{l^2(X)} \leq \epsilon$ *whenever* $d(x, y) \leq R$, *and* $\langle \gamma_x, \gamma_y \rangle = 0$ *whenever* $d(x, y) \geq S$.

(8) $\forall R > 0$, $\forall \epsilon > 0$, $\exists S > 0$, $\exists (\zeta_x)_{x \in X}$, $\zeta_x \in l^2(X \times \mathbb{N})$, $\text{supp}(\zeta_x) \subset B(x, S) \times \mathbb{N}$, $\|\zeta_x\|_{l^2(X \times \mathbb{N})} = 1$, *and* $\|\zeta_x - \zeta_y\|_{l^2(X \times \mathbb{N})} \leq \epsilon$ *whenever* $d(x, y) \leq R$.

(9) $\forall R > 0$, $\forall \epsilon > 0$, $\exists S > 0$, $\exists \phi : X \times X \to \mathbb{R}$ *of positive type such that* $\text{supp}(\phi) \subset \Delta_S$ *and* $|1 - \phi(x, y)| \leq \epsilon$ *whenever* $d(x, y) \leq R$.

(10) $\forall R > 0$, $\forall \epsilon > 0$, $\exists \psi : X \times X \to \mathbb{C}$ *of positive type such that* $|1 - \psi(x, y)| \leq \epsilon$ *whenever* $d(x, y) \leq R$ *and convolution of an* l^2 *function with* ψ *defines a bounded operator in the uniform Roe algebra* $C_u^*(X)$.

Proof. (1) $\Leftrightarrow$ (2) First we note that $\||\xi_x| - |\xi_y|\|_{l^1(X))} \leq \|\xi_x - \xi_y\|_{l^1(X)}$ and therefore we can assume that ξ_x is non-negative. Therefore

$$1 = \|\xi_x\|_1 = \sum_{x \in X} |\xi_x| = \sum_{x \in X} \xi_x$$

So for all $x \in X$, $0 \leq \xi_x \leq 1$ and ξ_x sums to 1, therefore $\xi_x \in Prob(X)$. Similarly, any function $a_x^n \in Prob(X)$ is an l^1 function, as $Prob(X) \subset l^1(X)$ [9].

We now turn to Proposition 4.7, and we can show that (2) is equivalent to this definition. We take $\xi_x = a_x^n$ and then the support conditions are equivalent and ξ_x has unit norm. The other condition follows by taking $\epsilon = \frac{1}{n}$.

(2) $\Rightarrow$ (3) This follows immediately, by letting $\chi_x = \xi_x$.

(3) $\Rightarrow$ (2) Let χ_x be as in (3) and let $\xi_x = \frac{\chi_x}{\|\chi_x\|_{l^1(X)}}$. Then the support condition follows immediately, as ξ_x is non-zero only if χ_x is non-zero. Furthermore, by using the triangle inequality, $\|\chi_x\|_1 = \|\chi_x - \chi_y + \chi_y\|_1 \leq \|\chi_x - \chi_y\|_1 + \|\chi_y\|_1$ and by rearranging to get $|\|\chi_y\|_1 - \|\chi_x\|_1| \leq \|\chi_x - \chi_y\|_1$, we have that

$$\begin{aligned}
\|\xi_x - \xi_y\|_1 &= \left\| \frac{\chi_x}{\|\chi_x\|_1} - \frac{\chi_y}{\|\chi_y\|_1} \right\|_1 \\
&= \left\| \frac{\chi_x - \chi_y}{\|\chi_x\|_1} + \chi_y \left(\frac{1}{\|\chi_x\|_1} - \frac{1}{\|\chi_y\|_1} \right) \right\|_1 \\
&\leq \frac{\|\chi_x - \chi_y\|_1}{\|\chi_x\|_1} + \|\chi_y\|_1 \left| \frac{1}{\|\chi_x\|_1} - \frac{1}{\|\chi_y\|_1} \right| \\
&= \frac{\|\chi_x - \chi_y\|_1}{\|\chi_x\|_1} + \left| \frac{\|\chi_y\|_1}{\|\chi_x\|_1} - 1 \right| \\
&= \frac{\|\chi_x - \chi_y\|_1}{\|\chi_x\|_1} + \frac{|\|\chi_y\|_1 - \|\chi_x\|_1|}{\|\chi_x\|_1} \\
&= \frac{\|\chi_x - \chi_y\|_1 + |\|\chi_y\|_1 - \|\chi_x\|_1|}{\|\chi_x\|_1} \\
&\leq \frac{2\|\chi_x - \chi_y\|_1}{\|\chi_x\|_1} \\
&\leq 2\epsilon \ \text{ whenever } d(x, y) \leq R
\end{aligned}$$

(2) $\Rightarrow$ (4) Let ξ_x be as in (2) and then define $\eta_x = |\xi_x|^{\frac{1}{2}}$. Then η_x has unit norm, the support condition follows and we have

$$
\begin{aligned}
\|\eta_x - \eta_y\|^2_{l^2(X)} &= \sum_{z\in X} |\eta_x(z) - \eta_y(z)|^2 \\
&\leq \sum_{z\in X} (\eta_x + \eta_y)|\eta_x(z) - \eta_y(z)| \\
&= \sum_{z\in X} |\eta_x(z)^2 - \eta_y(z)^2| \\
&= \sum_{z\in X} |\, |\xi_x(z)| - |\xi_y(z)|\, | \\
&= \||\xi_x| - |\xi_y|\|_{l^1(X)} \\
&\leq \|\xi_x - \xi_y\|_{l^1(X)} \\
&\leq \epsilon \ \text{ whenever } d(x,y) \leq R
\end{aligned}
$$

(4) $\Rightarrow$ (2) Let η_x be as in (4). We can suppose that $\eta_x \geq 0$, and then let $\xi_x = \eta_x^2$. As before ξ_x has unit norm and the support condition follows immediately. Recall that the Cauchy-Schwarz inequality states $|\langle x, y\rangle| \leq \|x\|_{l^2(X)}\|y\|_{l^2(X)}$, we use this below.

$$
\begin{aligned}
\|\xi_x - \xi_y\|_{l^1(X)} &= \sum_{z\in X} |\eta_x(z)^2 - \eta_y(z)^2| \\
&= \sum_{z\in X} |\eta_x(z) - \eta_y(z)|(\eta_x(z) + \eta_y(z)) \\
&= |\langle |\eta_x(z) - \eta_y(z)|, \eta_x(z) + \eta_y(z)\rangle| \\
&\leq \|\eta_x - \eta_y\|_{l^2(X)} \|\eta_x + \eta_y\|_{l^2(X)} \\
&\leq \|\eta_x - \eta_y\|_{l^2(X)} \left(\|\eta_x\|_{l^2(X)} + \|\eta_y\|_{l^2(X)}\right) \\
&= 2\|\eta_x - \eta_y\|_{l^2(X)} \\
&\leq 2\epsilon \ \text{ whenever } d(x,y) \leq R
\end{aligned}
$$

(2) $\Rightarrow$ (5) Let ξ_x be as in (2), and define $\alpha_x = \xi_x$. Then α_x has unit norm and $\|\alpha_x - \alpha_y\|_{l^1(X)} \leq \epsilon$ whenever $d(x,y) \leq R$ so we just need to check the support condition. But $supp(\alpha_x) = supp(\xi_x) \subset B(x,S)$ therefore $\|\alpha_x\|$ restricted to $B(x,S)$ is equal to the norm over the whole space, which is 1. Therefore for all $\delta > 0$, $\|\alpha_x\|$ restricted to $B(x,S)$ is at least $1-\delta$, as required.

(5) $\Rightarrow$ (6) Let α_x be as in (5), and define $\beta_x = \alpha_x$. Let $\delta < 1$, and given R and ϵ, define $\delta' = \min\{\delta, \epsilon\}$. We will denote the norm of β_x restricted to $B(x,S)$ by $\|\beta_x\|_{B(x,S)}$, then from (5) $\|\beta_x\|_{B(x,S)} \geq 1-\delta'$, and so

$$\begin{aligned} \|\beta_x\|_{B(x,R+S)\setminus B(x,S)} &= \|\beta_x\|_{B(x,R+S)} - \|\beta_x\|_{B(x,S)} \\ &\leq 1-(1-\delta') \\ &= \delta' \leq \epsilon \end{aligned}$$

So the norm of β_x restricted to $B(x, R+S)\setminus B(x,S)$ is at most ϵ as required.

(6) $\Rightarrow$ (2) Let β_x be as in (6) and fix δ. Then given R and ϵ, we define

$$\xi'_x(z) = \begin{cases} \beta_x(z) & \text{if } z \in B(x, R+S) \\ 0 & \text{otherwise} \end{cases}$$

For ease of notation we shall let $C = B(x,R+S) \setminus B(y,R+S)$, $D = B(y,R+S) \setminus B(x,R+S)$ and $E = B(x,R+S) \cap B(y,R+S)$. So that $\|\xi'_x\|_C$ denotes the norm of ξ'_x restricted to the set $B(x,R+S) \setminus B(y,R+S)$. Also, we let $F = B(x,R+S) \setminus B(x,S)$ and $G = B(y,R+S) \setminus B(y,S)$.

Notice that for $d(x,y) \leq R$, $B(y,S) \subset B(x,R+S)$ and $B(x,S) \subset B(y,R+S)$. So $D \subset G$ and $C \subset F$. So for $d(x,y) \leq R$ we have:

$$\begin{aligned} \|\xi'_x - \xi'_y\|_{l^1(X)} &= \sum_{z\in X} |\xi'_x(z) - \xi'_y(z)| \\ &= \sum_{z\in C} |\xi'_x(z)| + \sum_{z\in D} |\xi'_y(z)| + \sum_{z\in E} |\xi'_x(z) - \xi'_y(z)| \\ &= \|\xi'_x\|_C + \|\xi'_y\|_D + \|\xi'_x - \xi'_y\|_E \\ &\leq \|\xi'_x\|_F + \|\xi'_y\|_G + \|\xi'_x - \xi'_y\|_E \\ &\leq \epsilon + \epsilon + \epsilon \\ &= 3\epsilon \end{aligned}$$

Therefore, for $d(x,y) \leq R$,

$$\|\xi'_x - \xi'_y\|_{l^1(X)} \leq 3\epsilon$$

We then define $\xi_x = \frac{\xi'_x}{\|\xi'_x\|}$, and therefore $\|\xi_x\| = \left\|\frac{\xi'_x}{\|\xi'_x\|}\right\| = 1$ as required. Also, $supp(\xi_x) = supp(\xi'_x) \subset B(x, R+S)$ as required. Finally, by a similar argument to (3) $\Rightarrow$ (2), we have that for $d(x,y) \leq R$

$$\begin{aligned} \|\xi_x - \xi_y\|_1 &= \left\| \frac{\xi'_x}{\|\xi'_x\|} - \frac{\xi'_y}{\|\xi'_y\|} \right\| \\ &\leq \frac{2\|\xi'_x - \xi'_y\|}{\|\xi'_x\|} \\ &\leq \frac{2(3\epsilon)}{1-\delta} \\ &\leq \frac{6\epsilon}{1-\delta} \end{aligned}$$

But δ is fixed, so we choose ϵ to make this arbitrarily small.

(4) $\Rightarrow$ (8) Let $\zeta_x(z) = (\eta_x(z), 1) \in l^2(X \times \mathbb{N})$ and the conditions follow.

(8) $\Rightarrow$ (4) Let ζ_x be as in (8) and let $\eta_x(z) = \|\zeta_x(z, \cdot)\|_{l^2(\mathbb{N})}$. Then the support condition follows as $\eta_x(z) = 0$ only if $\|\zeta(z, \cdot)\|_{l^2(\mathbb{N})} = 0$. Additionally,

$$\begin{aligned} \|\eta_x - \eta_y\|^2_{l^2(X)} &= \sum_{z \in X} \left| \|\zeta_x(z,\cdot)\|_{l^2(\mathbb{N})} - \|\zeta_y(z,\cdot)\|_{l^2(\mathbb{N})} \right|^2 \\ &= \sum_{z \in X} \|\zeta_x(z,\cdot) - \zeta_y(z,\cdot)\|^2_{l^2(\mathbb{N})} \\ &= \|\zeta_x - \zeta_y\|^2_{l^2(X \times \mathbb{N})} \\ &\leq \epsilon \text{ whenever } d(x,y) \leq R \end{aligned}$$

(4) $\Rightarrow$ (9) Let η_x be as in (4) and let $\phi(x,y) = \langle \eta_x, \eta_y \rangle$. Then $\phi(x,y)$ is of positive type (see [18, page 118]),

$$supp(\phi) = \{(x,y) \in X \times X : \phi(x,y) = \langle \eta_x, \eta_y \rangle \neq 0\}$$

But by assumption $supp(\eta_x) \subset B(x,S)$ and $supp(\eta_y) \subset B(y,S)$ and therefore η_x and η_y are both zero when $d(x,y) > 2S$, i.e

$$supp(\phi) \subset \{(x,y) \in X \times X : d(x,y) \leq 2S\}$$

And if $d(x,y) \leq R$, then

$$\begin{aligned}
\|\eta_x - \eta_y\|^2_{l^2(X)} &= \langle \eta_x, \eta_x \rangle - \langle \eta_x, \eta_y \rangle - \langle \eta_y, \eta_x \rangle + \langle \eta_y, \eta_y \rangle \\
&= 2 - 2\langle \eta_x, \eta_y \rangle \\
&= 2(1 - \phi(x,y))
\end{aligned}$$

Therefore

$$\begin{aligned}
1 - \phi(x,y) &= \frac{1}{2}\|\eta_x - \eta_y\|^2_{l^2(X)} \\
&\leq \frac{1}{2}\epsilon^2
\end{aligned}$$

(9) $\Rightarrow$ (4) Let ϕ be as in (9) and suppose that $\epsilon \leq \frac{1}{2}$. Let

$$(T_\phi \eta)(x) = \sum_{y \in X} \phi(x,y)\eta(y)$$

Using the definition of inner products, the definition of T_ϕ, and by simple rearranging, for all $\xi, \eta \in l^2(X)$, we have

$$\begin{aligned}
|\langle \xi, T_\phi \eta \rangle| &= \left| \sum_{x \in X} \xi(x)(T_\phi \eta)(x) \right| \\
&= \left| \sum_{x \in X} \xi(x) \left(\sum_{y \in X} \phi(x,y)\eta(y) \right) \right| \\
&= \left| \sum_{x,y \in X} \phi(x,y)\xi(x)\eta(y) \right| \\
&\leq \sum_{x,y \in X} |\phi(x,y)| \cdot |\xi(x)| \cdot |\eta(y)|
\end{aligned}$$

Next we rewrite ϕ as $(\phi^{\frac{1}{2}})^2$ and then use the Cauchy-Schwarz inequailty $|\langle x, y \rangle| \leq \|x\|_2 \|y\|_2$.

$$
\begin{aligned}
&= \sum_{x,y\in X} |\phi(x,y)|^{\frac{1}{2}} \cdot |\xi(x)| \cdot |\phi(x,y)|^{\frac{1}{2}}.|\eta(y)| \\
&\leq \left(\sum_{x,y\in X} \left(|\phi(x,y)|^{\frac{1}{2}}|\xi(x)|\right)^2\right)^{\frac{1}{2}} \left(\sum_{x,y\in X} \left(|\phi(x,y)|^{\frac{1}{2}}|\eta(y)|\right)^2\right)^{\frac{1}{2}} \quad \text{by C.S} \\
&= \left(\sum_{x,y\in X} |\phi(x,y)| \cdot |\xi(x)|^2\right)^{\frac{1}{2}} \left(\sum_{x,y\in X} |\phi(x,y)| \cdot |\eta(y)|^2\right)^{\frac{1}{2}}
\end{aligned}
$$

But ϕ is symmetric so we can change $\phi(x,y)$ to $\phi(y,x)$ in the second summation, and then we relabel throughout that sum, swapping x with y, so that the notation is consistent in both sums.

$$
\begin{aligned}
&= \left(\sum_{x,y\in X} |\phi(x,y)| \cdot |\xi(x)|^2\right)^{\frac{1}{2}} \left(\sum_{x,y\in X} |\phi(x,y)| \cdot |\eta(x)|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{x\in X} \left(|\xi(x)|^2 \sum_{y\in X} |\phi(x,y)|\right)\right)^{\frac{1}{2}} \left(\sum_{x\in X} \left(|\eta(x)|^2 \sum_{y\in X} |\phi(x,y)|\right)\right)^{\frac{1}{2}}
\end{aligned}
$$

We can then split the double sums into single sums, by taking the supremum over x of the sum over y, and we rearrange.

$$
\begin{aligned}
&\leq \left(\sum_{x\in X} |\xi(x)|^2\right)^{\frac{1}{2}} \left(\sup_x \sum_{y\in X} |\phi(x,y)|\right)^{\frac{1}{2}} \left(\sum_{x\in X} |\eta(x)|^2\right)^{\frac{1}{2}} \left(\sup_x \sum_{y\in X} |\phi(x,y)|\right)^{\frac{1}{2}} \\
&\leq \left(\sup_{x\in X} \sum_{y\in X} |\phi(x,y)|\right) \|\xi\|_{l^2(X)} \|\eta\|_{l^2(X)}
\end{aligned}
$$

Since ϕ is of positive type,

$$
\sup_{x,y} |\phi(x,y)| \leq \sup_{x\in X} \phi(x,x) \leq 1+\epsilon \leq 2
$$

Recall that a space is of bounded geometry if the number of elements in a ball of a given radius is uniformly bounded, i.e. for each R there exists a constant $N(R) > 0$ such that $\forall x \in X$, $|B(x,R)| \leq N(R)$. Therefore combining the above calculations we have that for all $\xi, \eta \in l^2(X)$,

$$
\begin{aligned}
|\langle \xi, T_\phi \eta \rangle| &\leq \left(\sup_{x \in X} \sum_{y \in X} |\phi(x,y)| \right) \|\xi\|_{l^2(X)} \|\eta\|_{l^2(X)} \\
&\leq 2N(S) \|\xi\|_{l^2(X)} \|\eta\|_{l^2(X)}
\end{aligned}
$$

Therefore if we let $\xi = T_\phi \eta$, then we get

$$
|\langle T_\phi \eta, T_\phi \eta \rangle| \leq 2N(S) \|T_\phi \eta\|_{l^2(X)} \|\eta\|_{l^2(X)}
$$

i.e.

$$
\|T_\phi \eta\| \leq 2N(S) \|\eta\|_{l^2(X)}
$$

And so we conclude that T_ϕ is a bounded operator on $l^2(X)$.

Next we turn to Young's book on Hilbert spaces [21]. The adjoint T_ϕ^* of T_ϕ satisfies $\langle \xi, T_\phi \eta \rangle = \langle T_\phi^* \xi, \eta \rangle$. We calculated $\langle \xi, T_\phi \eta \rangle$ above, so here we calculate $\langle T_\phi \xi, \eta \rangle$ and show that they are equal, and therefore $T_\phi = T_\phi^*$.

$$
\begin{aligned}
\langle T_\phi \xi, \eta \rangle &= \sum_{x \in X} (T_\phi \xi)(x) \eta(x) \\
&= \sum_{x \in X} \left(\sum_{y \in X} \phi(x,y) \xi(y) \right) \eta(x) \\
&= \sum_{x,y \in X} \phi(x,y) \xi(y) \eta(x)
\end{aligned}
$$

Next we swap x and y throughout, and finally replace $\phi(x,y)$ with $\phi(y,x)$ as they are equal by assumption. So

$$
\begin{aligned}
\langle T_\phi \xi, \eta \rangle &= \sum_{x,y \in X} \phi(y,x) \xi(x) \eta(y) \\
&= \sum_{x,y \in X} \phi(x,y) \xi(x) \eta(y) \\
&= \langle \xi, T_\phi \eta \rangle
\end{aligned}
$$

And so $T_\phi = T_\phi^*$. Therefore T_ϕ is a Hermitian operator, see [21, definition 7.17], so we can use [21, theorem 7.18] which states: if A is a Hermitian operator on a Hilbert space H, then $\|A\| = \sup_{\|x\|=1} |\langle Ax, x\rangle|$. And we get:

$$\begin{aligned}
\|T_\phi\| &= \sup_{\|\alpha\|=1} |\langle T_\phi \alpha, \alpha\rangle| \\
&\leq \sup_{\|\alpha\|=1} 2N(S)\|\alpha\|^2 \\
&= 2N(S)
\end{aligned}$$

Also, note that T_ϕ is a positive operator since

$$\langle \eta, T_\phi \eta\rangle = \sum_{x,y\in X} \phi(x,y)\eta(x)\eta(y) \geq 0$$

The first equality is true by the definition of inner products, the second inequality is true by definition as ϕ is a positive definite kernel.

Let p be a polynomial such that $0 \leq p(t)$ and $|p(t)^2 - t| \leq \epsilon$ on $[0, 2N(S)]$. Let $\phi_1 = p(\phi)$, where $p(\phi)$ is obtained using the convolution product of ϕ with itself. Recall that the definition of the convolution product of two functions ϕ and ψ is:

$$(\phi * \psi)(x,y) = \sum_{z\in X} \phi(x,z)\psi(z,y)$$

Let $(e_x)_{x\in X}$ be the canonical basis of $l^2(X)$. Let

$$\eta'_x = \phi_1(x,\cdot), \qquad \eta_x = \frac{\eta'_x}{\|\eta'_x\|_{l^2(X)}}$$

Then we have

$$\begin{aligned}
\langle \eta'_x, \eta'_y\rangle &= \sum_{z\in X} \phi_1(x,z)\phi_1(y,z) && \text{by defn of inner product} \\
&= \sum_{z\in X} \phi_1(x,z)\phi_1(z,y) && \text{as } \phi \text{ and therefore } \phi_1 \text{ are symmetric} \\
&= (\phi_1 * \phi_1)(x,y) && \text{by defn of convolution product} \\
&= (p^2(\phi))(x,y) && \text{by defn of } p
\end{aligned}$$

Aside:
$T_\phi e_y(x) = \sum_{z\in X} \phi(x,z)e_y(x) = \phi(x,y)$ as all other terms are zero. Therefore $\langle e_x, T_\phi e_y\rangle = \sum_{z\in X} e_x(z)T_\phi e_y(z) = T_\phi e_y(x) = \phi(x,y)$. This, and a similar for calculation $\langle e_x, p^2(T_\phi)e_y\rangle$ gives

$$\langle e_x, T_\phi e_y\rangle = \phi(x,y), \quad \langle e_x, p^2(T_\phi)e_y\rangle = p^2(\phi(x,y))$$

And combining these gives

$$\langle e_x, (p^2(T_\phi) - T_\phi)e_y\rangle = (p^2(\phi) - \phi)(x,y)$$

which we use in the following calculation.

$$\begin{aligned}
\left|\langle \eta'_x, \eta'_y\rangle - \phi(x,y)\right| &= \left|(p^2(\phi) - \phi)(x,y)\right| \\
&= \left|\langle e_x, (p^2(T_\phi) - T_\phi)e_y\rangle\right| \\
&\leq \|e_x\|_{l^2(X)}\|p^2(T_\phi) - T_\phi\|_{l^2(X)}\|e_y\|_{l^2(X)} \\
&= \|p^2(T_\phi) - T_\phi\|_{l^2(X)} \\
&\leq \epsilon
\end{aligned}$$

as $\|e_x\| = \|e_y\| = 1$, $\|T_\phi\| \leq 2N(S)$ and $|p(t)^2 - t| \leq \epsilon$ on $[0, 2N(S)]$.

Therefore $\left|\langle \eta'_x, \eta'_y\rangle - 1\right| \leq \left|\langle \eta'_x, \eta'_y\rangle - \phi(x,y)\right| + |1 - \phi(x,y)| \leq 2\epsilon$ whenever $d(x,y) \leq R$. Thus $\langle \eta'_x, \eta'_x\rangle - 1 \leq 2\epsilon$, so $\langle \eta'_x, \eta'_x\rangle \leq 1 + 2\epsilon$ for all $x \in X$ and $\langle \eta'_x, \eta'_y\rangle \geq 1 - 2\epsilon$ for all $x, y \in X$ such that $d(x,y) \leq R$. We use these in the calculation below:

$$\begin{aligned}
1 - \langle \eta_x, \eta_y\rangle &= 1 - \left\langle \frac{\eta'_x}{\|\eta'_x\|_{l^2(X)}}, \frac{\eta'_y}{\|\eta'_y\|_{l^2(X)}} \right\rangle \\
&= 1 - \frac{\langle \eta'_x, \eta'_y\rangle}{\langle \eta'_x, \eta'_x\rangle^{\frac{1}{2}}\langle \eta'_y, \eta'_y\rangle^{\frac{1}{2}}} \\
&\leq 1 - \frac{1 - 2\epsilon}{1 + 2\epsilon} \\
&= \frac{4\epsilon}{1 + 2\epsilon} \\
&\leq 4\epsilon \quad \text{when}\, d(x,y) \leq R
\end{aligned}$$

From our proof of (4) $\Rightarrow$ (9), we have that $\|\eta_x - \eta_y\|^2_{l^2(X)} = 2 - 2\langle \eta_x, \eta_y \rangle$ and therefore $\|\eta_x - \eta_y\|_{l^2(X)} = \sqrt{2 - 2\langle \eta_x, \eta_y \rangle} \leq \sqrt{2.4\epsilon} = \sqrt{8\epsilon}$.

Finally, we consider the support condition. Recall that $supp(\phi) \subset \{(x, y) \in X \times X : d(x, y) \leq S\}$ and that $(\phi * \phi)(x, y) = \sum_{z \in X} \phi(x, z)\phi(z, y)$ therefore this is non zero only when $d(x, z) \leq S$ and $d(z, y) \leq S$, i.e. when $d(x, y) \leq 2S$. It can therefore easily be seen that $(\phi * \cdots * \phi)(x, y)$ is non zero only when $d(x, y) \leq nS$, if ϕ is convoluted with itself n times. Therefore if p is of degree n, we can write $p(\phi) = a_0 + a_1\phi + \cdots + a_n(\phi * \cdots * \phi)$ and therefore $supp(\eta_x) \subset B(x, nS)$.

(7) $\Rightarrow$ (9)
Let $\gamma_x \in \mathcal{H}$ be as in (7), and define $\phi(x, y) = Re\langle \gamma_x, \gamma_y \rangle$. Then $\phi(x, y)$ is of positive type by definition (see [18, page 118]). Also $\phi(x, y) = Re\langle \gamma_x, \gamma_y \rangle = 0$ when $d(x, y) \geq S$ i.e. $supp(\phi) \subset \{(x, y) : d(x, y) \leq S\} = \delta_S$, so the support condition holds.

Then as in (4) $\Rightarrow$ (9),

$$\begin{aligned} \|\gamma_x - \gamma_y\|_2^2 &= 2(1 - \phi(x, y)) \\ 1 - \phi(x, y) &\leq \frac{1}{2}\epsilon^2 && \text{whenever } d(x, y) \leq R \\ |1 - \phi(x, y)| &\leq \frac{1}{2}\epsilon^2 && \text{whenever } d(x, y) \leq R \end{aligned}$$

(4) $\Rightarrow$ (7)
Let η_x be as in (4), and define $\gamma_x = \eta_x$. Then $\gamma_x \in l^2(X)$ which is a Hilbert space, $\|\gamma_x\|_2 = 1$, and $\|\gamma_x - \gamma_y\|_2 \leq \epsilon$ whenever $d(x, y) \leq R$.

In addition $supp(\eta_x) = supp(\gamma_x) \subset B(x, S)$, i.e. $\gamma_x(z) = 0$ whenever $d(x, z) \geq S$. Therefore

$$\langle \gamma_x, \gamma_y \rangle = \sum_{z \in X} \gamma_x(z) \cdot \gamma_y(z)$$

If $\gamma_x(z)$ or $\gamma_y(z)$ equals 0, then $\langle \gamma_x, \gamma_y \rangle = 0$, i.e. if $d(x, z)$ or $d(y, z) \geq S$ then $\langle \gamma_x, \gamma_y \rangle = 0$, i.e. if $d(x, y) \geq 2S$ then $\langle \gamma_x, \gamma_y \rangle = 0$, as required.

(9) $\Rightarrow$ (10)
Let ϕ be as in (10) and define $\psi : X \times X \to \mathbb{C}$ by $\psi = \phi$. Then $\|1 - \psi(x,y)\| \leq \epsilon$ whenever $d(x,y) \leq R$. And we have $supp(\psi) = supp(\phi) \subset \{(x,y) : d(x,y) \leq S\}$, i.e. $\psi = 0$ whenever $d(x,y) \geq S$, i.e. ψ is a finite propagation kernel as defined in [2, definition 4].

Convolution of an l^2 function with ψ defines an operator $Op(\psi)$ by

$$(Op(\psi)\xi)(x) = \sum_{y \in X} \psi(x,y)\xi(y)$$

Using the fact that X has bounded geometry along with the same calculations for T_ϕ in (9) $\Rightarrow$ (4) we see that Op is a bounded operator. Specifically if we let $N(S)$ be the maximum number of points in any ball of radius S, then $\|Op(\psi)\xi\| \leq 2N(S)\|\xi\|$. Then as ψ has finite propagation, so does $Op(\psi)$, and therefore $Op(\psi) \in C^*_u(X)$, as required.

(10) $\Rightarrow$ (4)
Let ψ and $Op(\psi)$ be as in (10). Then as in the proof of (9) $\Rightarrow$ (4) as ψ is of positive type, $Op(\psi)$ is a positive operator, i.e. $\langle \xi, Op(\psi)\xi \rangle \geq 0$ for all $\xi \in l^2(X)$. Then by [21, theorem 12.4] there exists a unique positive square root $Op(\psi)^{\frac{1}{2}}$ of $Op(\psi)$, i.e. there exists a unique positive operator, $Op(v)$ such that $(Op(v))^2 = Op(\psi)$. Now, as $Op(v)$ is a positive operator it is bounded. Furthermore $(Op(v))^2 = Op(u)$ and $Op(u)$ has finite propagation, so $Op(v)$ does too, and therefore $Op(v) \in C^*_u(X)$.

We now truncate $Op(v)$ so it satisfies the support condition, and call this new operator $Op(w)$. Note that all the kernels are symmetric. In addition, $Op(w)$ satisfies the following approximation inequality, which we shall use later.

$$\|Op(v) - Op(w)\|_2 < \min\left(\epsilon, \frac{\epsilon}{2(\|Op(v)\|_2 + \epsilon)}\right)$$

From this it follows that $\|Op(w)\|_2 \leq \|Op(v)\|_2 + \epsilon$ so we have

$$\begin{aligned}\|Op(v)^2 - Op(w)^2\| &\leq \|Op(w)\|.\|Op(v) - Op(w)\| + \|Op(v) - Op(w)\|.\|Op(w)\| \\ &= \|Op(v) - Op(w)\| \left(\|Op(v)\| + \|Op(w)\|\right) \\ &< \epsilon\end{aligned}$$

Now, let $\eta'_x \in l^2(X)$ be the vector with entries $\eta'_x(z) = w(z,x)$. Now, from the definition of inner products, we can see that

$$\langle \eta'_x, \eta'_y \rangle = \sum_{z\in X} \overline{w(z,x)}w(z,y) = \sum_{z\in X} w(z,x)w(z,y)$$

i.e. the kernel $\langle \eta'_x, \eta'_y \rangle$ consists of the matrix entries of the operator $Op(w)^2$. Since $Op(w)^2$ differs from $Op(v)^2$ by at most ϵ it follows that for each $z \in X$ $Op(w)^2(z)$ differs from $Op(v)^2(z)$ by at most ϵ and so the kernel $\langle \eta'_x, \eta'_y \rangle$ differs from $u(x,y)$ entrywise by at most ϵ. Hence $\| \langle \eta'_x, \eta'_y \rangle \| \leq 3\epsilon$ whenever $d(x,y) \leq R$, and therefore by the calculation in (7) $\Rightarrow$ (9) $\|\eta'_x\| \leq \sqrt{6\epsilon}$ whenever $d(x,y) \leq R$.

Finally, $\|\eta'_x\|^2 = \langle \eta'_x, \eta'_y \rangle$ which we can approximate by the following:

$$d(x,y) \leq R \Rightarrow |u(x,y) - 1| \leq \epsilon \Rightarrow 1 - \epsilon < u(x,y)$$

and

$$|\langle \eta'_x, \eta'_y \rangle - u(x,y)| \leq \epsilon \Rightarrow (x,y) < \epsilon + \langle \eta'_x, \eta'_y \rangle$$

And therefore, $1 - \epsilon \leq u(x,y) \leq \epsilon + \langle \eta'_x, \eta'_y \rangle$. And so

$$\|\eta'_x\|^2 = \langle \eta'_x, \eta'_y \rangle \geq 1 - 2\epsilon$$

We define $\eta_x = \frac{\eta'_x}{\|\eta'_x\|_2}$ and then by the calculations in (6) $\Rightarrow$ (4) we see that for all $d(x,y) \leq R$,

$$\|\eta_x - \eta_y\|_2 \leq 2\sqrt{\frac{6\epsilon}{1-2\epsilon}}$$

□

Chapter 5

Property A for non-discrete metric spaces

In Theorem 4.8 we have given many equivalent definitions of property A for discrete bounded geometry spaces. We now remove the restriction that the metric space X be discrete, and we also allow it to have unbounded geometry. X is now what we refer to as a **general** metric space.

For general metric spaces one defines property A in terms of conditions on families of vectors in $l^1(X)$ or $l^2(X)$. Here $l^1(X)$ and $l^2(X)$ are defined with respect to the counting measure. However, this seems too restrictive for more complicated spaces like $\mathbb{R}$-trees. So in the case when a metric space is equipped with a measure μ, one can state definitions similar to some of those in Theorem 4.8, replacing $l^1(X)$ and $l^2(X)$ with $L^1(X)$ and $L^2(X)$ respectively, defined with respect to the measure μ. In this way we obtain equivalent definitions of property A that depend on the measure μ, but are not in general equivalent to Yu's original definition. However a definition of this type would be satisfactory if it still implied embeddability in Hilbert space and exactness.

In the following theorem, we assume that X is equipped with the counting measure. The numbering system may seem a little unusual, but I have used numbers to tie-in with the the earlier result in Theorem 4.8. As before, statements (2), (3), (4) and (8) come from Tu's paper [18, Proposition 3.2]; the statement (7) is due to Dadarlat and Guentner [5, Proposition 2.5]; and in statements (5) and (6) we modify and prove a similar Proposition from [2, Theorem 6] from the l^2 context to l^1.

Theorem 5.1. *Let X be a general (non-discrete and unbounded geometry) metric space. Then X has property A $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8), where conditions (2), (3), $\cdots$ (8) are as in Theorem 4.8.*

Proof. The equivalences of conditions (2), (3), (4), (5), (6), (7) and (8) have all been proved in Theorem 4.8, and extend immediately to the non-discrete case. Conditions (9) and (10) have been omitted here as they rely on X having bounded geometry and therefore in this case may no longer be equivalent. □

The definition from the above theorem that we find most useful is the L^1-norm version, condition (2), which we shall denote property A_{L^1}. We state it explicitly below.

Definition 5.2. [18, Proposition 3.2] A general metric space X, has property A_{L^1} iff $\forall R > 0$, $\forall \epsilon > 0$, $\exists S > 0$, $\exists (\xi_x)_{x \in X}$, with $\xi_x \in L^1(X)$, such that

(1) $supp(\xi_x) \subset B(x, S)$;

(2) $\|\xi_x\|_{L^1(X)} = 1$; and

(3) $\|\xi_x - \xi_y\|_{L^1(X)} \leq \epsilon$ whenever $d(x, y) \leq R$.

5.1 Property A and States on $C_0(X)$

Throughout the rest of this chapter $C_0^b(X)$ will denote the C*-algebra of bounded, continuous functions which tend to zero outside *bounded* subsets of X, i.e. $f \in C_0^b(X)$ iff for every $\epsilon > 0$ there exists a bounded subset $B \subset X$ such that $\sup_{x \notin B} |f(x)| < \epsilon$. This is different to the usual form of $C_0(X)$, which we shall denote by $C_0^c(X)$, by using bounded instead of compact sets.

One might ask why we want this new definition. There are several reasons. When working with general metric spaces it is convenient to state support conditions of functions using balls, perform estimates using balls, and construct functions which are supported in balls. In a space which is not locally compact (for example, an $\mathbb{R}$-tree), these balls are not compact, but are bounded and therefore we can use this new definition $C_0^b(X)$.

Definition 5.3. We define $Prob(X)$ to be the set of states of the C*-algebra $C_0^b(X)$, i.e. $Prob(X)$ is the set of all positive linear functionals on $C_0^b(X)$ of norm 1. We recall that $\phi(f)$ is positive if given $f(x) \geq 0 \, \forall x \in X$ then $\phi(f) \geq 0$.

Given a vector space E and its dual E' then the weak topology on E is defined by, $x_n \overset{w}{\rightarrow} x$ (x_n tends to x weakly) iff for all $f \in E'$, $f(x_n) \rightarrow f(x)$. We can then induce a topology on E', called the weak-* topology, by $f_n \overset{w*}{\rightarrow} f$ iff for all $x \in E$, $f_n(x) \rightarrow f(x)$.

Definition 5.4. We say that a general metric space has property A_b if there exists a sequence of weak-* continuous maps $a_n : X \rightarrow Prob(X)$ such that

(1) for each n there is an R such that, for each x, the measure $a_n(x)$ is supported within $B(x, R)$; and

(2) for each $S > 0$, as $n \rightarrow \infty$, $\sup_{d(x,y)<S} \|a_n(x) - a_n(y)\| \rightarrow 0$.

This appears to be the most general version of property A that we have so far, so the obvious question to ask, is how does this relate to the other definitions of property A that we have. Note that it is equivalent to property A_{L^1} when X is discrete and has bounded geometry. We prove this later in the chapter in Theorem 5.9. As for it's relation to property A_{L^1} for a general metric space, this is answered in the following theorem.

Theorem 5.5. *A general metric space with a measure, (X, μ) which satisfies property A_{L^1}, also satisfies property A_b.*

Proof. We have a general metric space X, and $\forall R > 0$, $\forall \epsilon > 0$, $\exists S > 0$, $\exists (\xi_x)_{x \in X}$, with $\xi_x \in L^1(X)$, such that

(1) $supp(\xi_x) \subset B(x, S)$;

(2) $\|\xi_x\|_{L^1(X)} = 1$; and

(3) $\|\xi_x - \xi_y\|_{L^1(X)} \leq \epsilon$ whenever $d(x, y) \leq R$.

We note that for a given ϵ we have a family of functions, say $(\xi_x^{1/\epsilon})_{x \in X}$ which satisfy the above conditions. We replace ϵ with $\frac{1}{n}$ and then for each $n > 0$ we have a family $(\xi_x^n)_{x \in X}$ such that $supp(\xi_x^n) \subset B(x, S)$; $\|\xi_x^n\|_{L^1(X)} = 1$; and

$\|\xi^n_x - \xi^n_y\|_{L^1(X)} \leq \frac{1}{n}$ whenever $d(x,y) \leq R$. We use this at the end of the proof. Additionally, as we saw in the proof of Proposition 4.8 (1) $\Leftrightarrow$ (2), we can assume that ξ^n_x is non-negative, which we will use later in the proof.

Now, for a given $n > 0$, ξ^n_x is an L^1 function on X so ξ^n_x determines a state $\phi^n_x(f)$ on $C^b_0(X)$ with norm 1 defined by:

$$\phi^n_x(f) = \langle f, \phi^n_x \rangle = \int_{z \in X} \xi^n_x(z) f(z) dz$$

$\phi^n_x(f)$ is in fact a state on $C^b_0(X)$, and we prove this now.

By definition $\xi^n_x(z)$ is non-zero only on a finite set, $z \in B(x,S)$, therefore $\xi^n_x(z)f(z)$ is non-zero only when $z \in B(x,S)$. Therefore

$$\phi^n_x(f) = \int_{B(x,S)} \xi^n_x(z) f(z) dz$$

where f is a bounded, continuous function which tends to zero outside bounded sets.

We need to check that $\phi^n_x(f)$ has norm 1, and is a positive linear functional. Then we know that $\phi^n_x(f)$ is a state on $C^b_0(X)$ with norm 1, i.e. $\phi^n_x(f) \in Prob(X)$ as required. To do this we define a function $f' \in C^b_0(X)$ and use this to get a lower bound of 1 for the norm of ϕ^n_x, and then we prove that the upper bound is also 1, and therefore $\|\phi^n_x\| = 1$.

We define

$$f'(z) = \begin{cases} 0 & \text{for } z \notin B(x,S) \\ 1 - \frac{d(z,x)}{S} & \text{for } z \in B(x,S) \end{cases}$$

Now it can be seen that f' is bounded as $f'(z) \in [0,1]$, is continuous inside and outside of $B(x,S)$, and tends to zero outside bounded sets. So all that is left to show is that f' is continuous on the boundary of $B(x,S)$. This can easily be seen by considering a point on the boundary, say u. This is such that $d(u,x) = S$, so $1 - \frac{d(u,x)}{S} = 1 - \frac{S}{S} = 0$ which agrees with the value for $f'(z)$ if $z \notin B(x,S)$. Note that it is important that we are using $C^b_0(X)$ here rather than the usual $C_0(X)$, as this function f' would not belong to $C_0(X)$ because the ball is not necessarily compact in general metric spaces (for example in $\mathbb{R}$-trees).

Now, $\|f'(z)\| = \sup_{z\in X}|f'(z)| = |f'(x)| = 1-0 = 1$, therefore, we can use f' as an example of a function f with norm not greater than 1 in the following equations:

$$\begin{aligned}
\|\phi_x^n\| &= \sup_{\|f\|\leq 1} |\langle f, \phi_x^n\rangle| \\
&\geq |\langle f', \phi_x^n\rangle| \\
&= \phi_x^n(f') \\
&= \int_{z\in X} \xi_x^n(z) f'(z) dz \\
&= \int_{z\in B(x,S)} \xi_x^n(z)\cdot 1 dz \\
&= \int_{z\in X} \xi_x^n(z) dz \\
&= \|\xi_x^n(z)\| = 1
\end{aligned}$$

In the above equations we use the fact that $supp(\xi_x^n) \subset B(x,S)$, and therefore $\int_{z\in B(x,S)} \xi_x^n(z)\cdot 1dz = \int_{z\in X}\xi_x^n(z)dz$. So $\|\phi_x^n\| \geq 1$.

Next we use the Cauchy-Schwartz inequality to gain an upper bound for the norm of ϕ_x^n.

$$\begin{aligned}
\|\phi_x^n\| &= \sup_{\|f\|\leq 1} |\langle f, \phi_x^n\rangle| \\
&= \sup_{\|f\|\leq 1} |\phi_x^n(f)| \\
&= \sup_{\|f\|\leq 1} \left|\int_{z\in X} \xi_x^n(z) f(z) dz\right| \\
&= \sup_{\|f\|\leq 1} |\langle f, \xi_x^n\rangle| \\
&\leq \sup_{\|f\|\leq 1} \|f\|\cdot\|\xi_x^n\| \\
&\leq \|\xi_x^n\| = 1
\end{aligned}$$

So $\|\phi_x^n\| \leq 1$ and $\|\phi_x^n\| \geq 1$, therefore $\|\phi_x^n\| = 1$, as required.

We also need to show that $\phi_x^n(f)$ is a positive linear functional, i.e. if $f(z) \geq 0$, $\forall z \in X$ then $\phi_x^n(f) \geq 0$. But $0 \leq \xi_x^n(z) \leq 1$, $\forall z \in X$ by definition. So if $f(z) \geq 0$, $\forall z \in X$, then

$$\xi_x^n(z)f(z) \geq 0 \, \forall z \in X \Longrightarrow \phi_x^n(f) = \int_{z \in X} \xi_x^n(z)f(z)dz \geq 0$$

as required. Therefore $\phi_x^n(f)$ is a state on $C_0^b(X)$, and so by the definition of $Prob(X)$ we have $\phi_x^n(f) \in Prob(X)$, as required.

Now we need to check the other conditions of property A_b, the support condition and the norm estimate for the difference of two such states. Firstly, we check that the support condition is satisfied.

$\phi_x^n(f) = \int_{z \in X} \xi_x^n(z)f(z)dz$, but $supp(\xi_x^n) \subset B(x, S)$, and therefore $\xi_x^n(z) = 0$ whenever $d(x, z) \geq S$.

We define the support of a state on $C_0^b(X)$ by

$$supp(\phi_x^n) = \bigcap_{\substack{U \subset X \\ U \text{ closed}}} \left\{ \exists f \in C_0^b(X), supp(f) \subset U, \text{ and } \phi_x^n(f) \neq 0 \right\}$$

Now, as before we define $f'(z) \in C_0^b(X)$ to be

$$f'(z) = \begin{cases} 0 & \text{for } z \notin B(x, S) \\ 1 - \frac{d(z,x)}{S} & \text{for } z \in B(x, S) \end{cases}$$

Now, $B(x, S)$ is a closed subset of X and with f' defined as above, we have $supp(f') \subset B(x, S)$, and

$$\phi_x^n(f') = \int_{z \in X} \xi_x^n(z)f'(z)dz \neq 0$$

as $supp(f') \cap supp(\xi_x^n) \neq \emptyset$.

The support of ϕ_x^n is the intersection of all such subsets. Therefore, we know that $supp(\phi_x^n) \subset B(x, S)$. And so the support condition is satisfied.

Finally, by the Cauchy-Schwartz inequality

$$\begin{aligned}
\|\phi_x^n(f) - \phi_y^n(f)\| &= \sup_{\|f\|\leq 1} |\langle f, \phi_x^n - \phi_y^n \rangle| \\
&= \sup_{\|f\|\leq 1} \left| \int_{z\in X} (\xi_x^n - \xi_y^n)(z) f(z) dz \right| \\
&\leq \sup_{\|f\|\leq 1} \|f\| \cdot \|\xi_x^n - \xi_y^n\| \\
&\leq \|\xi_x^n - \xi_y^n\| \\
&\leq \frac{1}{n}
\end{aligned}$$

whenever $d(x,y) \leq R$. So, $\sup_{d(x,y)<R} \|\xi_x^n - \xi_y^n\| \to 0$ as $n \to \infty$. Therefore

$$\sup_{d(x,y)<R} \|\phi_x^n - \phi_y^n\| \to 0$$

as $n \to \infty$.

And so X satisfies property A_b. □

Corollary 5.6. *An $\mathbb{R}$-tree X which satisfies property A_{L^1} also satisfies property A_b.*

Proof. This follows immediately from Theorem 5.5 above as an $\mathbb{R}$-tree is an example of a general metric space with a measure. So if it satisfies property A_{L^1} then it also satisfies property A_b. The discussion of $\mathbb{R}$-trees satisfying property A_{L^1} is given in chapter 7 and specifically in Theorem 7.2. □

If we are given a tree T, and regard each of its edges as intervals of length 1, and if we define the distance between two points as the length of the arc joining them, then we have an $\mathbb{R}$-tree, i.e. the geometric realization of a tree is an $\mathbb{R}$-tree.

Example 5.7. Consider the geometric realization of the Cayley graph of $\mathbb{F}_\infty$. This is an unbounded $\mathbb{R}$-tree with a ray, and hence by Theorem 7.1 and Corollary 5.6 it has the property A_{L^1}, and hence property A_b.

5.2 Property A_b for discrete bounded geometry metric spaces

Here we make the connection between property A_b and the usual property A for discrete bounded geometry metric spaces. We prove that under these conditions they are in fact equivalent. To do this we first have to consider $C_0^b(X)$ for discrete bounded geometry metric spaces. In the following Lemma we prove that the dual of $C_0^b(X)$ is equivalent to $l^1(X)$.

Lemma 5.8. *Let X be a discrete bounded geometry space. Then the dual of $C_0^b(X)$, denoted $(C_0^b(X))'$ is isomorphic to $l^1(X)$.*

Proof. This is an extension of a classical result in analysis, see for example [20, Example 1, page 114] where Yosida proves this duality for sequences.

Let $\phi \in (C_0^b(X))'$. We want to show that there exists $g_\phi = \{\lambda_x\} \in l^1(X)$ such that for all $f \in C_0^b(X)$ we have

$$\langle f, \phi \rangle = \sum_{x \in X} \lambda(x) f(x)$$

and $||\phi|| = ||g_\phi||$. Conversely, any $g = \{\lambda_x\} \in l^1(X)$ defines $\phi_g \in (C_0^b(X))'$ such that for any $f \in C_0^b(X)$,

$$\langle f, \phi_g \rangle = \sum_{x \in X} \lambda(x) f(x)$$

and $||\phi_g|| = ||g||$.

$\{\delta_x\}$ are unit vectors, and in $C_0^b(X)$ we use the supremum norm, $||f|| = \sup_x |f(x)|$. For any $f \in C_0^b(X)$ and $\phi \in (C_0^b(X))'$, we fix a base point $y \in X$ and then for all $\epsilon > 0$ there exists $K > 0$ such that

$$\left\| \sum_{x \in B(y,K)} f(x)\delta_x - f \right\| = ||f(x)||_{x \notin B(y,K)} = \sup_{x \notin B(y,K)} |f(x)| < \epsilon$$

as $f(x)$ tends to zero outside of bounded sets, and $B(y, K)$ is a bounded set. And so, by the definition of strong limit (s-lim) we have

$$\lim_{k\to\infty}\left\|\sum_{x\in B(y,K)} f(x)\delta_x - f\right\| = 0 \Longleftrightarrow \operatorname*{s-lim}_{k\to\infty}\sum_{x\in B(y,K)} f(x)\delta_x = f$$

Now,

$$\begin{aligned}
\phi(f) &= \langle f,\phi\rangle \\
&= \lim_{k\to\infty}\left\langle \sum_{x\in B(y,K)} f(x)\delta_x,\phi\right\rangle \\
&= \lim_{k\to\infty}\phi\left(\sum_{x\in B(y,K)} f(x)\delta_x\right) \\
&= \lim_{k\to\infty}\sum_{x\in B(y,K)} f(x)\phi(\delta_x) \\
&= \lim_{k\to\infty}\sum_{x\in B(y,K)} f(x)\lambda_x
\end{aligned}$$

where we define $\lambda_x = \phi(\delta_x) \in \mathbb{C}$. So, given $\phi \in (C_0^b(X))'$ we define

$$f(x)_{B(y,K)} = \begin{cases} \frac{|\lambda_x|}{\lambda_x} & \text{for } \lambda_x \neq 0 \text{ and } x \in B(y,K) \\ 0 & \text{otherwise} \end{cases}$$

Then $|f(x)_{B(y,K)}|$ is equal to 1 or 0, and so $f(x)_{B(y,K)} \leq 1$. Therefore

$$\begin{aligned}
\|\phi\| &= \sup_{\|f\|\leq 1} |\langle f,\phi\rangle| \\
&\geq |\langle f_{B(y,K)},\phi\rangle| \\
&= \phi(f_{B(y,K)}) \\
&= \sum_{x\in B(y,K)} f(x)_{B(y,K)}\lambda_x \\
&= \sum_{x\in B(y,K)} |\lambda_x|
\end{aligned}$$

But $\|\lambda_x\|_1 = \sum_{x\in X}|\lambda_x|$, so by letting $K\to\infty$, we have $\|\phi\| \geq \|\lambda_x\|_1 = \|g_\phi\|$, and $\|\phi\| < \infty$ as ϕ is a linear functional, and hence $\|g_\phi\| < \infty$, so $g_\phi \in l^1(X)$

as required.

Conversely, if $g = \{\lambda_x\} \in l^1(X)$, then for all $f \in C_0^b(X)$

$$\langle f, g \rangle = \left| \sum_{x \in X} f(x)\lambda_x \right| \leq \|f\| \cdot \|g\|$$

by the Cauchy-Schwarz inequality. So g defines $\phi_g \in (C_0^b(X))'$ with

$$\phi_g(f) = \langle f, \phi_g \rangle = \sum_{x \in X} f(x)\lambda_x$$

And,

$$\begin{aligned} \|\phi_g\| &= \sup_{\|f\| \leq 1} |\langle f, \phi_g \rangle| \\ &= \sup_{\|f\| \leq 1} |\langle f, g \rangle| \\ &\leq \sup_{\|f\| \leq 1} \|f\| \cdot \|g\| \\ &\leq \|g\| \end{aligned}$$

Therefore $\|\phi_g\| = \|g\|$ as required.

In fact, by the above two arguments it can be seen that by applying both processes in succession we get back to where we started:

$$\phi \to g_\phi \to \phi_{g_\phi} = \phi \to g_\phi$$

So given a state on $C_0^b(X)$ there is an associated l^1 function which is equivalent and vice versa. □

Theorem 5.9. *Property A_b is equivalent to the l^1-norm version if X is a discrete bounded geometry metric space.*

Proof. Given a state on $C_0^b(X)$ it is equivalent (by Lemma 5.8) to an l^1-function, and vice versa. This theorem then follows as the conditions for property A_b and the l^1-norm version are equivalent. □

The above Theorem establishes the equivalence of property A_b to the usual property A for discrete bounded geometry metric spaces.

5.3 Roe's property A

On the other hand we show that for not necessarily discrete bounded geometry proper metric spaces, property A_b is equivalent to the following version of property A proposed by Roe in a recent paper [16]. We shall call this property A_R. It is worth noting here that Roe uses $Prob(X)$ to be the set of states on the C*-algebra $C_0^c(X)$. Also, $C_0^c(X) = C_0^b(X)$ for bounded geometry proper spaces. This is proved below in Theorem 5.11.

Definition 5.10. Let X be a bounded geometry, proper metric space, then X has property A_R if there exist a sequence of weak-* continuous maps $f_n : X \longrightarrow Prob(X)$ such that

(i) for each n there is an r such that for each x, the measure $f_n(x)$ is supported within $B(x, r)$, and

(ii) for each $S > 0$, as $n \longrightarrow \infty$, $\sup_{d(x,y)<S} \|f_n(x) - f_n(y)\| \longrightarrow 0$

Theorem 5.11. *Let X be a bounded geometry, proper metric space. Then X has property A_b if and only if X satisfies property A_R.*

Proof. We prove that the two algebras $C_0^b(X)$ and $C_0^c(X)$ are in fact equal when X is a bounded geometry proper metric space, and therefore the two Properties A_b and A_R are equivalent.

Recall that X is a proper metric space if and only if closed bounded subsets are compact. In the following calculations we shall assume that B is a bounded set and K is a compact set. Therefore, using standard notation we have that $\overline{B}$ (the closure of B) is a closed and bounded set and B^c is the complement of B in X, i.e. $X \setminus B$.

$$f \in C_0^b(X) \Longleftrightarrow \forall \epsilon\, \exists B \text{ such that } \sup_{x \in X \setminus B} |f(x)| < \epsilon$$

Now, $\overline{B}$ is closed and bounded, and therefore compact, and $B \subseteq \overline{B}$ therefore $\overline{B}^C \subseteq B^C$ and so

$$\sup_{x \notin \overline{B}} |f(x)| \leq \sup_{x \notin B} |f(x)| < \epsilon \Rightarrow f \in C_0^c(X)$$

Conversely,

$$f \in C_0^c(X) \iff \forall \epsilon \, \exists K \text{ such that } \sup_{x \in X \setminus K} |f(x)| < \epsilon$$

But K is a compact set, which is therefore closed and bounded, so

$$f \in C_0^c(X) \Rightarrow f \in C_0^b(X)$$

So for a proper metric space X we have

$$f \in C_0^b(X) \iff f \in C_0^c(X)$$

i.e.

$$C_0^b(X) = C_0^c(X)$$

So property A_b and property A_R are equivalent. □

There is another definition of property A due to Roe which was stated without proof in a lecture [15]. Below we assume that X is equipped with the bounded coarse structure, and prove that this version of property A is equivalent to property A_{L^1}, and therefore is equivalent to all the other conditions given in Theorem 5.1.

Theorem 5.12. *[15]*
Let X be a general metric space equipped with the counting measure. Then X has property A_{L^1} iff there exist maps $m^n : X \to Prob(X)$ such that for each n, for all x, m_x^n is supported within a controlled neighbourhood of x and as $n \to \infty$, $\|m_x^n - m_y^n\| \to 0$ uniformly on controlled subsets of $X \times X$.

Proof. $\Rightarrow$
Let ξ_x be as in Theorem 4.8 condition (2) and define $m_x^n = \xi_x$, where $n = \frac{1}{\epsilon}$. Then as in the proof of Theorem 4.8 (1) $\Leftrightarrow$ (2), we can assume that ξ_x is non-negative, and therefore is a probability measure. Also, $\text{supp}(m_x^n) = \text{supp}(\xi_x) \subset B(x, S)$, which is a controlled set as for any $y \in B(x, S)$, $d(x, y) \leq S$. And finally,

$$\begin{aligned}
\|\xi_x - \xi_y\|_{L^1(X)} &\leq \epsilon && \text{when}\, d(x,y) \leq R \\
\|m_x^n - m_y^n\|_{L^1(X)} &\leq \frac{1}{n} && \text{when}\, d(x,y) \leq R \\
\|m_x^n - m_y^n\|_{L^1(X)} &\to 0 && \text{as}\, n \to \infty\, \text{when}\, d(x,y) \leq R
\end{aligned}$$

Therefore as $n \to \infty$, $\|m_x^n - m_y^n\|_{L^1(X)} \to 0$ uniformly when (x,y) belongs to a controlled set, as required.

$\Leftarrow$

Let $m^n : X \to Prob(X)$ be as in the theorem, and define $\xi_x^n = m^n(x)$. Then for any given n, we have that $\|\xi_x^n\| = \|m_x^n\| = 1$ as m_x^n is a probability function. In addition, m_x^n is supported on a controlled neighbourhood of x. With the bounded coarse structure on X, this is the same as $\exists S > 0$ such that $\{(x,y) : y \in \text{supp}(m_x^n)\}$ with $d(x,y) < S$. Therefore $\text{supp}(\xi_x^n) = \text{supp}(m_x^n) \subset B(x,S)$.

Finally, as $n \to \infty$, $\|m_x^n - m_y^n\| \to 0$ uniformly on controlled subsets of $X \times X$. But in the bounded coarse structure controlled sets are just sets of finite width, therefore $\forall R > 0$, as $n \to \infty$, $\|m_x^n - m_y^n\| \to 0$ uniformly when $d(x,y) \leq R$. i.e. $\forall R > 0$, $\forall \epsilon > 0$, $\exists N > 0$ such that $\|m_x^n - m_y^n\| \leq \epsilon\ \forall n \geq N$ and when $d(x,y) \leq R$.

So $\forall R > 0$, $\forall \epsilon > 0$, $\exists N > 0$ such that $\|\xi_x^n - \xi_y^n\| \leq \epsilon\ \forall n \geq N$ and when $d(x,y) \leq R$. $\square$

Chapter 6

Permanence properties of property A

6.1 Discrete metric spaces

We discuss the permanence properties of property A, i.e. that it is a coarse invariant, that it passes to subspaces and to product spaces. Much of the work done by Yu and others was firstly done for discrete metric spaces, and then for non-discrete ones. Here we shall consider the permanence properties in the same order (doing the discrete case first), but some of the proofs for the discrete case will be left till later as they follow directly as corollaries of the non-discrete case.

6.1.1 Coarse invariance

In [22] it was stated without proof that "property A is invariant under quasi-isometry". Below I prove a more general statement, that property A is a coarse invariant.

First, recall that two metric spaces X and Y are **quasi-isometric** if there exist large-scale Lipschitz maps $f : X \to Y$ and $g : Y \to X$ such that $d(g \circ f(x), x)$ and $d(f \circ g(y), y)$ are bounded. More generally, two metric spaces X and Y are **coarsely equivalent** if there exist coarse maps $f : X \to Y$ and $g : Y \to X$, such that $d(g \circ f(x), x)$ and $d(f \circ g(y), y)$ are bounded.

Theorem 6.1. *Given two discrete metric spaces X and Y which are coarsely equivalent. If X has property A then so does Y.*

Proof. We assume that there are maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ exhibiting a coarse equivalence. Also there are finite subsets $\{A_x\}_{x \in X}$ of $X \times \mathbb{N}$ which satisfy the conditions of property A as stated in Definition 4.1.

We define subsets of $Y \times \mathbb{N}$ by

$$\{C_y\} = (f \times I)(\{A_{f^{-1}(y)}\})$$

Firstly we check that $(y, 1) \in C_y$ for all $y \in Y$.

We know that $(x, 1) \in A_x$ for all $x \in X$, and therefore

$$\begin{aligned} (f^{-1}(y), 1) \in A_{f^{-1}(y)} &\Rightarrow (f \circ f^{-1}(y), 1) \in C_y \\ &\Rightarrow (y, 1) \in C_y \end{aligned}$$

So condition 1 of property A is satisfied.

We notice that

$$\left|(f \times I)(A_{f^{-1}(y)}) \cap (f \times I)(A_{f^{-1}(y')})\right| \geq \left|A_{f^{-1}(y)} \cap A_{f^{-1}(y')}\right|$$

because all the elements which contribute to the right hand side also contribute to the left hand side, but the left hand side may also contain extra elements, for example there may be elements in $A_{f^{-1}(y)}$ and $A_{f^{-1}(y')}$ which are not the same, but whose images under $(f \times I)$ are the same.

By similar reasoning

$$\begin{aligned} &\left|(f \times I)(A_{f^{-1}(y)}) - (f \times I)(A_{f^{-1}(y')})\right| + \left|(f \times I)(A_{f^{-1}(y')}) - (f \times I)(A_{f^{-1}(y)})\right| \\ &\leq \left|A_{f^{-1}(y)} - A_{f^{-1}(y')}\right| + \left|A_{f^{-1}(y')} - A_{f^{-1}(y)}\right| \end{aligned}$$

And therefore

$$
\begin{aligned}
&\frac{|C_y - C_{y'}| + |C_{y'} - C_y|}{|C_y \cap C_{y'}|} \\
&\quad = \frac{\left|(f \times I)(A_{f^{-1}(y)}) - (f \times I)(A_{f^{-1}(y')})\right| + \left|(f \times I)(A_{f^{-1}(y')}) - (f \times I)(A_{f^{-1}(y)})\right|}{\left|(f \times I)(A_{f^{-1}(y)}) \cap (f \times I)(A_{f^{-1}(y')})\right|} \\
&\quad \leq \frac{\left|A_{f^{-1}(y)} - A_{f^{-1}(y')}\right| + \left|A_{f^{-1}(y')} - A_{f^{-1}(y)}\right|}{\left|A_{f^{-1}(y)} \cap A_{f^{-1}(y')}\right|} \\
&\quad < \epsilon
\end{aligned}
$$

for all $f^{-1}(y), f^{-1}(y') \in X$ such that $d(f^{-1}(y), f^{-1}(y')) \leq r$.

But f is a coarse map, and so $\forall r$, $\exists R$ such that

$$
\begin{aligned}
d(f^{-1}(y), f^{-1}(y')) \leq r &\Rightarrow d(f \circ f^{-1}(y), f \circ f^{-1}(y')) \leq R \\
&\Rightarrow d(y, y') \leq R
\end{aligned}
$$

i.e. when $d(y, y') \leq R$,

$$
\frac{|C_y - C_{y'}| + |C_{y'} - C_y|}{|C_y \cap C_{y'}|} < \epsilon
$$

And finally, let $(y, m) \in C_y$ and $(y', n) \in C_y$. Then $(f^{-1}(y), m) \in A_{f^{-1}(y)}$ and $(f^{-1}(y'), n) \in A_{f^{-1}(y)}$ which implies that there exists $s > 0$ such that $d(f^{-1}(y), f^{-1}(y')) \leq s$.

But f is a coarse map, and so $\forall s$, $\exists S$ such that

$$
\begin{aligned}
d(f^{-1}(y), f^{-1}(y')) \leq s &\Rightarrow d(f \circ f^{-1}(y), f \circ f^{-1}(y')) \leq S \\
&\Rightarrow d(y, y') \leq S
\end{aligned}
$$

i.e. $\exists S > 0$ such that if $(y, m) \in C_y$ and $(y', n) \in C_y$ then $d(y, y') \leq S$. □

Corollary 6.2. *Let X and Y be two quasi-isometric discrete metric spaces. Then if X has property A so does Y.*

Proof. This follows from Theorem 6.1 as large-scale Lipschitz maps are examples of coarse maps, and so quasi-isometric spaces are examples of coarsely equivalent spaces. □

Example 6.3. Theorem 6.1 establishes property A as an invariant of the coarse structure. For example, consider a finitely generated discrete group with a set of generators S, and another set of generators S'. Let X be the metric space given by the word length metric l on the set of generators S, and X' be the metric space given by the word length metric l' on the set of generators S'. We can show that X and X' are quasi-isometric.

Each generator $s_i \in S$ can be written in terms of the generators in S' (and their inverses). Then for each s_i there exists k_i such that $l(s_i) = 1$ and $l'(s_i) = k_i$. As there are finitely many generators in S, we define $K = max\{k_i\}$. Let g be an element of the group, then $l'(g) \leq K \cdot l(g)$, for all $g \in G$. Similarly, there is a constant C, so that $l(g) \leq C \cdot l'(g)$ for all $g \in G$. Let $f : X \to X'$ be the function which maps elements of G written in terms of the generators in S to the same element written in terms of the generators in S', and vice versa for $g : X' \to X$. Then, $f \circ g = id_{X'}$ and $g \circ f = id_X$, and we have shown that f and g are large-scale Lipshitz maps and therefore X and X' are quasi-isometric.

Therefore, by Corollary 6.2, we can see that property A on metric spaces of discrete groups does not depend on the choice of generators. However the choice of the length function may be significant.

Example 6.4. Consider the free group on infinitely many generators, and consider the word length metric on $\mathbb{F}_\infty$. This does not have bounded geometry, as in the ball around the origin of radius 1 there are infinitely many elements, which makes it more difficult to decide if it has property A or not. So we turn to a lemma by Tu [18, lemma 2.1] which states that for any countable discrete group G there is a length function on G so that the resulting metric space has bounded geometry. So, we define a metric space on $\mathbb{F}_\infty$ with an appropriate length function (which exists by Tu's lemma) so that we have bounded geometry. Then by [5, proposition 2.10], which we state and prove later on, this metric space of $\mathbb{F}_\infty$ with the appropriate length function has property A.

If we consider $\mathbb{F}_\infty$ with the word length metric, then by [10, corollary 5.3] we do know that the group is exact, but we can not say from this whether or not it has property A. So, as is shown in this example calculations are simplified by choosing the right length function. It is worth noting that for all countable discrete groups, by Tu's lemma [18, lemma 2.1], we can find a length function so that the metric space has bounded geometry and then we can use one of our many equivalent definitions to check if it has property A.

6.1.2 Products of property A spaces

In [7, Proposition 2] Dranishnikov and Januszkiewicz stated a Proposition regarding the product of discrete, bounded geometry property A spaces. In the next chapter we extend this to the non-discrete case and therefore the following Proposition follows from Proposition 6.10.

Proposition 6.5. *Let $Z = Z_1 \times Z_2$ be the product of two discrete bounded geometry metric spaces with the l_1 metric. Assume that Z_1 and Z_2 have property A. Then Z has property A.*

Proof. In Proposition 6.10 we assume that Z_1 and Z_2 are non-discrete, unbounded geometry metric spaces and prove that if they both have property A then so does their product. Proposition 6.5 is a specific case of Proposition 6.10, so the proof follows. □

6.1.3 Subspaces of property A spaces

In [7, Proposition 3] Dranishnikov and Januszkiewicz proved that property A passes to subspaces of discrete bounded geometry metric spaces. In Theorem 6.11 we provide a different proof for property A passing to subspaces of non-discrete metric spaces. The discrete case follows from the non-discrete case, Theorem 6.11, which will be proved in the next chapter.

Proposition 6.6. *Let $Y \subset X$ be discrete metric spaces with bornologous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g = id_Y$. Then if X has property A_b so does Y.*

Proof. This is a specific case of Theorem 6.11. Here we have the extra condition that Z is discrete. The proof follows from that of Theorem 6.11. □

6.2 Non-discrete metric spaces

6.2.1 Coarse invariance

In this section we consider property A as defined by Yu, see Definition 4.2.

Theorem 6.7. *Let X and Y be two coarsely equivalent general metric spaces. Then if X has property A, so does Y.*

Proof. To prove this we first recall that earlier we proved this for the discrete case in Theorem 6.1. So here we shall use the definition of property A by Yu, Definition 4.2 and consider coarsely dense discrete subspaces to which we can then apply Theorem 6.1.

We assume that X and Y are coarsely equivalent, so we have coarse maps $f : X \to Y$ and $g : Y \to X$ such that $d(g \circ f(x), x)$ and $d(f \circ g(y), y)$ are bounded. As f and g are coarse, we also have that $\forall r_1$, $\exists R_1$ such that

$$d(x, x') \leq r_1 \Rightarrow d(f(x), f(x')) \leq R_1$$

and $\forall r_2$, $\exists R_2$ such that

$$d(y, y') \leq r_2 \Rightarrow d(g(y), g(y') \leq R_2$$

By assumption X has property A, and therefore there exists a discrete subspace U of X such that there exists $c_1 > 0$ for which $d(x, U) \leq c_1$ for all $x \in X$, and U has property A.

There exists a coarsely dense discrete subspace V, of Y, and so we define a new map $f' : U \to V$ which is close to the map f for elements of U by $f'(u) = f(u)$ if $f(x) \in V$, and as V is coarsely dense in Y, there exists $S_1 > 0$ such that $d_V(f(u), f'(u)) < S_1$ for all $u \in U$.

Similarly we define a map $g' : V \to U$ which is close to g, so there exists $S_2 > 0$ such that $d_U(g(v), g'(v)) < S_2$ for all $v \in V$

So we have two coarsely equivalent non-discrete metric spaces X and Y, and discrete subsets U and V respectively. We want to show that U and V are coarsely equivalent, as we can then use Theorem 6.1. We do this next.

f and g are coarse maps. Let $d(u, u') \leq r_1$. Then

$$\begin{aligned} d_V(f'(u), f'(u')) &\leq d_Y(f'(u), f(u)) + d_Y(f(u), f(u')) + d_Y(f(u'), f'(u')) \\ &\leq S_1 + R_1 + S_1 \\ &= R_1 + 2S_1 \end{aligned}$$

Similarly, for $d(v, v') \leq r_2$

$$d(g'(v), g'(v')) \leq R_2 + 2S_2$$

Therefore f' and g' are coarse maps as required.

We know that $d(g \circ f(x), x)$ and $d(f \circ g(y), y)$ are bounded and we need to check that $d(g' \circ f'(u), u)$ and $d(f' \circ g'(v), v)$ are bounded.

For all $v \in V$, $d(g'(v), g(v)) \leq S_2$ by the definition of g', and similarly for all $u \in U$, $d(f'(u), f(u)) \leq S_1$ by the definition of f'. So as g is a coarse map and $f'(u)$ and $f(u) \in Y$, there exists $C_2 > 0$ such that $d(g \circ f'(u), g \circ f(u)) \leq C_2$.

Now, using the triangle inequality, the above approximations, and noting that $f'(u) \in V$, and $f'(u)$ and $f(u) \in Y$, we have that

$$\begin{aligned} d(g' \circ f'(u), g \circ f(u)) &\leq d(g' \circ f'(u), g \circ f'(u)) + d(g \circ f'(u), g \circ f(u)) \\ &\leq S_2 + C_2 \end{aligned}$$

Using the triangle inequality again, we have

$$\begin{aligned} d(g' \circ f'(u), u) &\leq d(g' \circ f'(u), g \circ f(u)) + d(g \circ f(u), u) \\ &\leq S_2 + C_2 + d(g \circ f(u), u) \end{aligned}$$

But by assumption $d(g \circ f(u), u)$ is bounded, and therefore $d(g' \circ f'(u), u)$ is bounded. Similarly, we can show that as $d(f \circ g(v), v)$ is bounded, $d(f' \circ g'(v), v)$ is bounded too.

And so f' and g' are coarse maps between U and V with appropriate conditions so that U and V are coarsely equivalent. Therefore we have the following set up:

$$\begin{array}{ccll} X & \stackrel{c.e.}{\sim} & Y & \text{non} - \text{discrete} \\ \cup & & \cup & \\ U & \stackrel{c.e.}{\sim} & V & \text{discrete} \end{array}$$

Therefore, if X has property A, then there exists a coarsely dense discrete subspace U with property A. But if U has property A, then V has property A by Theorem 6.1. If V has property A, then Y also has property A (by Yu's definition). And so property A is a coarse invariant for non-discrete metric spaces. □

Using Theorem 6.7 we can now prove that bounded metric spaces have property A.

Lemma 6.8. *Any bounded metric space Y is coarsely equivalent to a point $z \in Y$.*

Proof. If Y is bounded then there exists a constant, C say, so that given any two points $y, y' \in Y$ we have $d(y, y') \leq C$. Let $Z = \{z\}$ be the metric space containing just one point z of Y. Then define $f : Y \rightarrow Z$ by $y \mapsto z$ for all $y \in Y$, and $g : Z \rightarrow Y$ by $z \mapsto z$, i.e. g is the identity map on Z.

These maps are both proper and bornologous, and hence are coarse maps. Also $f \circ g(z) = z$ and $g \circ f(y) = z$, with $d(y, z) \leq C$, so $g \circ f$ is 'close' to the identity map on Y. Hence Y and Z are coarsely equivalent metric spaces, as required. □

Theorem 6.9. *Any bounded metric space, Y, satisfies property A.*

Proof. From Lemma 6.8 we know that any bounded metric space is coarsely equivalent to a point. It can easily be seen that a point (viewed as a metric space) satisfies the three conditions of Yu's property A.

We also know that property A is a coarse invariant, by Thereom 6.7, and so any bounded metric space does have property A. □

6.2.2 Products of property A spaces

In [7, Proposition 2] Dranishnikov and Januszkiewicz stated a proposition regarding the product of discrete, bounded geometry property A spaces. We extend this below to the non-discrete case. These spaces are assumed to have a measure such that if $Z = Z_1 \times Z_2$ then $\mu_Z = \mu_{Z_1} \times \mu_{Z_2}$.

Theorem 6.10. *Let $Z = Z_1 \times Z_2$ be the product of two metric spaces with the L_1 metric. Assume that Z_1 and Z_2 have property A_{L^1}. Then Z has property A_{L^1}.*

Proof. Let $\beta_{z_1} : Z_1 \to L^1(Z_1)$ and $\gamma_{z_2} : Z_2 \to L^1(Z_2)$ be families of functions from the definition of property A for Z_1 and Z_2 respectively. Then define a family of functions $\alpha_z : Z \to L^1(Z)$ by

$$\alpha_{(z_1,z_2)} = \beta_{z_1} \cdot \gamma_{z_2}$$

Firstly, we shall check that $\|a_{(z_1,z_2)}\|_{L^1(Z)} = 1$.

$$\begin{aligned}
\|a_{(z_1,z_2)}\|_{L^1(Z)} &= \int_Z |\beta_{z_1}(s) \cdot \gamma_{z_2}(t)| d\mu_Z \\
&= \int_{Z_1} \int_{Z_2} |\beta_{z_1}(s)| \cdot |\gamma_{z_2}(t)| d\mu_{Z_1} d\mu_{Z_2} \\
&= \int_{Z_1} |\beta_{z_1}(s)| \left(\int_{Z_2} |\gamma_{z_2}(t)| d\mu_{Z_1} \right) d\mu_{Z_2} \\
&= \int_{Z_1} |\beta_{z_1}(s)| \cdot \|\gamma_{z_2}\|_{L^1(Z_2)} d\mu_{Z_1} \\
&= \|\gamma_{z_2}\|_{L^1(Z_2)} \int_{Z_1} |\beta_{z_1}(s)| d\mu_{Z_1} \\
&= \|\gamma_{z_2}\|_{L^1(Z_2)} \|\beta_{z_1}\|_{L^1(Z_1)} \\
&= 1 \qquad\qquad \text{by definition}
\end{aligned}$$

Now we move on to consider the support condition. By definition there exist $R_1 > 0$ and $R_2 > 0$ such that $supp(\beta_{z_1}) \subset B(z_1, R_1)$ and $supp(\gamma_{z_2}) \subset B(z_2, R_2)$, i.e.

if $d(z_1, z_1') \geq R_1$ then $\beta_{z_1}(z_1') = 0$ and if $d(z_2, z_2') \geq R_2$ then $\gamma_{z_2}(z_2') = 0$

So if either $d(z_1, z_1') \geq R_1$, $d(z_2, z_2') \geq R_2$ or both, then $\alpha_{(z_1,z_2)}(z_1', z_2') = 0$.

Let $R = R_1 + R_2$ and let $d((z_1, z_2), (z_1', z_2')) \geq R$ then $\alpha_{(z_1,z_2)}(z_1', z_2') = 0$, i.e.

$$supp(\alpha_z) \subset B(z, R)$$

Finally we turn to the convergence condition.

$$\begin{aligned}
&\|\alpha_z - \alpha_w\|_{L^1(Z)} \\
&\quad = \int_Z |\beta_{z_1}(s)\gamma_{z_2}(t) - \beta_{w_1}(s)\gamma_{w_2}(t)|\, du \\
&\quad = \int_Z |\beta_{z_1}(s)\gamma_{z_2}(t) - \beta_{w_1}(s)\gamma_{z_2}(t) + \beta_{w_1}(s)\gamma_{z_2}(t) - \beta_{w_1}(s)\gamma_{w_2}(t)|\, du
\end{aligned}$$

In the equations above we have added and subtracted an extra term so that we can split up and factorise the integral, as below. We then split the integrals over Z into integrals over Z_1 and Z_2 and use the fact that the norms of γ_{z_2} and β_{w_1} are equal to 1.

$$\begin{aligned}
&\leq \int_Z |\beta_{z_1}(s) - \beta_{w_1}(s)| \cdot |\gamma_{z_2}(t)|\, du + \int_Z |\gamma_{z_2}(t) - \gamma_{w_2}(t)| \cdot |\beta_{w_1}(s)|\, du \\
&= \|\gamma_{z_2}\|_{L^1(Z_2)} \int_{Z_1} |\beta_{z_1}(s) - b_{w_1}(s)|\, ds + \|\beta_{w_1}\|_{L^1(Z_1)} \int_{Z_2} |\gamma_{z_2}(t) - \gamma_{w_2}(t)|\, dt \\
&= \|\beta_{z_1} - \beta_{w_1}\|_{L^1(Z_1)} + \|\gamma_{z_2} - \gamma_{w_2}\|_{L^1(Z_2)} \\
&\leq 2\epsilon
\end{aligned}$$

whenever $d(z_1, w_1) \leq K_1$ and $d(z_2, w_2) \leq K_2$. Therefore

$$\|\alpha_v - \alpha_w\|_{L^1(Z)} \leq 2\epsilon$$

whenever $d((z_1, z_2), (w_1, w_2)) \leq K = \min\{K_1, K_2\}$. □

6.2.3 Subspaces of property A spaces

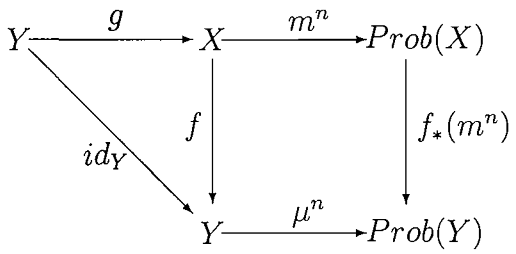
Theorem 6.11. *Let $Y \subset X$ be general, coarse metric spaces with bornologous maps $g : Y \to X$ and $f : X \to Y$ such that f is closed and $f \circ g = id_Y$ (the identity map on Y). If X has property A_b then Y also has property A_b.*

Proof. First we note that every state on $C_0^b(X)$ defines a probability measure and vice versa. We shall from now on refer only to probability measures.

We shall first construct a family μ_y^n of probability measures on Y. As X has property A_b we know that for every n there is a map m^n from the space X to the set of probability measures on X, defined by $m^n : X \to Prob(X)$, $x \mapsto m_x^n$. For each n one can then define a map μ^n from Y to the space of probability measures on Y, i.e. $\mu^n : Y \to Prob(Y)$, by using image measures in the following way. If $y \in Y$, we define

$$\mu_y^n = f_*(m_{g(y)}^n)$$

By Lemma 2.24 we know that μ_y^n is probability measure. So we have relationships between the spaces as shown in the diagram below which commutes for every n.



Our main task is to show that the family $\{\mu_y^n\}$ satisfies property A_b. We begin with the support condition.

$$supp(\mu_y^n) = supp(f_*(m_{g(y)}^n))$$

But from Lemma 2.23 we have that

$$\text{supp}(f_*(m_{g(y)}^n)) = \overline{f(supp(m_{g(y)}^n))}$$

Now, f is a closed map, and so it maps closed subsets to subsets, and by definition $\text{supp}(m_{g(y)}^n)$ is closed and so $f(\text{supp}(m_{g(y)}^n))$ is also closed. Also, X has property A_b and therefore there exists $R > 0$ such that

$$\begin{aligned} supp(\mu_y^n) &= supp(f_*(m_{g(y)}^n)) \\ &= f(supp(m_{g(y)}^n)) \\ &\subset f(B(g(y), R) \end{aligned}$$

Also, f is bornologous, and so for every K_1 there exists $C_1 > 0$ such that $d(x,x') < K_1 \Rightarrow d(f(x), f(x')) < C_1$. So we can write the above as, for every R there exists $S > 0$ such that $f(B(x,R)) \subset B(f(x), S)$, i.e.

$$supp(\mu_y^n) \subset B(f \circ g(y), S) = B(y, S)$$

Thus μ_y^n satisfies condition (1) of property A_b.

Next we prove that the family $\{\mu_y^n\}$ satisfies the control condition of property A_b. We need to show that for $T > 0$, as $n \to \infty$, $\sup_{d(y,v)<T} \|\mu_y^n - \mu_v^n\| \to 0$.

So, for $d(y,v) < T$ we have:

$$\begin{aligned} \|\mu_y^n - \mu_v^n\|_{L^1(Y)} &= \|f_*(m_{g(y)}^n) - f_*(m_{g(v)}^n)\|_{L^1(Y)} \\ &= |f_*(m_{g(y)}^n) - f_*(m_{g(v)}^n)|(Y) \\ &= |m_{g(y)}^n - m_{g(v)}^n|(f^{-1}(Y)) \\ &= |m_{g(y)}^n - m_{g(v)}^n|(X) \\ &= \|m_{g(y)}^n - m_{g(v)}^n\|_{L^1(X)} \end{aligned}$$

But g is bornologous, so for every K_2 there exists $C_2 > 0$ such that $d(y,y') < K_2 \Rightarrow d(g(y), g(y')) < C_2$. So, for each K_2 there exists C_2 such that as $n \to \infty$

$$\sup_{d(y,v)<K_2} \|\mu_y^n - \mu_v^n\| \leq \sup_{d(g(y),g(v))<C_2} \|m_{g(y)}^n - m_{g(v)}^n\| \to 0$$

So, for each K_2

$$\sup_{d(y,v)<K_2} \|\mu_y^n - \mu_v^n\| \to 0$$

as $n \to \infty$, as required. And thus Y has property A_b. □

Remark 6.12. We note that if Y is a closed subspace of a property A_b space X, and if f is a bornologous retraction (a continuous map of a space onto a subspace which leaves each point of the subspace fixed) from X to Y, then Y has property A_b.

If $Y \subset X$ and both spaces are discrete then the maps f and g are always closed. So for the discrete case, we only need the restrictions that f and g are bornologous and that $f \circ g = id_Y$.

Corollary 6.13. *Let Y be a coarsely dense discrete subspace of a general metric space X. Then if X has property A_b, so does Y.*

Proof. This follows from Theorem 6.11 where we let g be the identity map, and f be the retraction of X to Y. As Y is coarsely dense in X these maps are both bornologous and satisfy $f \circ g = id_Y$. So if X has property A_b so does any coarsely dense discrete subspace Y. □

Chapter 7

Property A and $\mathbb{R}$-trees

Recall that Yu's definition of property A for non-discrete spaces, Definition 4.2, states that X has property A if there exists a discrete subspace Z of X such that there exists $c > 0$ for which $d(x, Z) \leq c$ for all $x \in X$; and Z has property A.

Theorem 7.1. *Any unbounded $\mathbb{R}$-tree X which admits a ray satisfies property A_{L^1}.*

Proof. We follow closely the proof by Dranishnikov and Januszkiewicz in [7] of a similar result for simplicial trees.

Let $\mathcal{R} = [0, \infty)$ be the positive half-line. We fix a basepoint $u \in X$ and a ray γ_u in X which is defined to be an isometry $\gamma_u : [0, \infty) \to X$, where $\gamma_u(0) = u$. Then given a point $x \in X$ we note that there exists a unique point $v \in \gamma_u$ such that for all points $p \in \gamma_u$ the arc from x to p contains v. We then define γ_x to be the unique ray which consists of the interval $[x, v]$ and then follows γ_u starting at v.

Now, we denote by $[x, \gamma_x(n)]$ the segment along the ray γ_x from x to a point which is distance n from x, which we call $\gamma_x(n)$, where $n \in \mathbb{R}$.

For each x, we define a sequence of functions (ξ_x^n) by:

$$\xi_x^n(z) = \begin{cases} \frac{1}{n} & \text{if } z \in [x, \gamma_x(n)] \\ 0 & \text{otherwise} \end{cases}$$

As we shall see in a moment, this definition will allow us to control the norm $\|\xi_x^n - \xi_y^n\|$. In particular, we shall be able to choose an n such that this norm is bounded above by ϵ as required by Definition 5.2.

For all $n \in \mathcal{R}$ and $x \in X$, the function ξ_x^n is supported on the interval $[x, \gamma_x(n)]$ and its value is constant and equal to $\frac{1}{n}$ along this interval. The interval $[x, \gamma_x(n)]$ is isometric to the interval $[0, n] \subset \mathbb{R}$ and so using Valette's construction of the Lebesgue measure μ on X we can write:

$$\begin{aligned} \|\xi_x^n\|_{L^1(X)} &= \int_X |\xi_x^n(y)| d\mu \\ &= \int_0^n \frac{1}{n} dt \\ &= 1 \end{aligned}$$

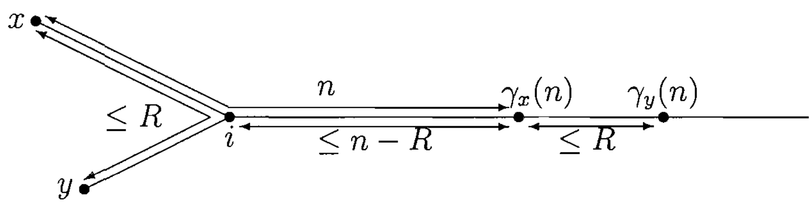
where dt is the Lebesgue measure on $\mathbb{R}$. Hence for all x, n, ξ_x^n is a unit vector in $L^1(X)$, and so condition (2) of property A is satisfied.

Furthermore, note that $\text{supp}(\xi_x^n) = [x, \gamma_x(n)] \subset B(x, n)$, so letting $S \geq n$, we prove condition (1).

Finally we consider two points in the $\mathbb{R}$-tree which lie within distance R of each other, i.e. $d(x, y) \leq R$ for $x, y \in X$. Then the intervals $[x, \gamma_x(n)]$ and $[y, \gamma_y(n)]$ overlap along an interval of length greater than or equal to $n - R$. Assume that i is the point on both γ_x and γ_y where the two rays meet. Without loss of generality let $d(x, i) \geq d(y, i)$. Then

$$[x, \gamma_x(n)] \cap [y, \gamma_y(n)] = [i, \gamma_x(n)]$$

which has length greater than or equal to $n - R$, as shown in the picture.



We can estimate the norm $\|\xi_x^n - \xi_y^n\|$ as follows.

Given that $\mathrm{supp}(\xi_x^n) = [x, \gamma_x(n)]$, $\mathrm{supp}(\xi_y^n) = [y, \gamma_y(n)]$ and $\xi_x^n = \xi_y^n$ on $[x, \gamma_x(n)] \cap [y, \gamma_y(n)] = [i, \gamma_x(n)]$. We have:

$$\begin{aligned}\|\xi_x^n - \xi_y^n\|_{L^1(X)} &= \int_X |\xi_x^n - \xi_y^n| d\mu \\ &= \int_{\mathrm{supp}(\xi_x^n)\triangle \mathrm{supp}(\xi_y^n)} |\xi_x^n - \xi_y^n| d\mu\end{aligned}$$

We note that

$$\mathrm{supp}(\xi_x^n)\triangle\mathrm{supp}(\xi_y^n) = [x, i] \cup [y, i] \cup [\gamma_x(n), \gamma_y(n)]$$

and the total length of intervals on the right is at most $2R$. Therefore

$$\begin{aligned}\|\xi_x^n - \xi_y^n\|_{L^1(X)} &\leq \int_0^{2R} \frac{1}{n} dt \\ &= \frac{2R}{n}\end{aligned}$$

For a given ϵ we choose n so that $\dfrac{2R}{n} < \epsilon$ to complete the proof. □

Theorem 7.2. *Any $\mathbb{R}$-tree X which is either bounded, or admits a ray satisfies property A_{L^1} and hence property A_b.*

Proof. In Theorem 6.9 we have proved the case for bounded $\mathbb{R}$-trees (as property A according to Yu implies property A_{L^1} by Theorem 5.1). Furthermore in Theorem 7.1 we have proved it for unbounded $\mathbb{R}$-trees which admit a ray. By Corollary 5.6 it follows that these $\mathbb{R}$-trees also satisfy property A_b. □

Corollary 7.3. *Products of $\mathbb{R}$-trees which are bounded or admit a ray have property A_{L^1} and hence property A_b.*

Proof. This follows directly from Theorem 6.10 that states that products of property A_{L^1} spaces also have property A_{L^1} along with Corollary 5.6. □

Chapter 8

Further Properties of Metric Spaces

8.1 Embeddability in Hilbert Space

We turn to a recent paper by Dadarlat and Guentner [5] which proves that discrete, bounded geometry metric spaces with property A are exact and therefore are embeddable in Hilbert space. Note that in [5] the definitions of property A which were used are the ones in Proposition 4.8 parts (2) and (7), which use l^1 functions with support in a finite ball, and Hilbert space valued functions where the support condition is written in terms of the inner product, respectively.

Proposition 8.1. *[5, Proposition 2.1]*
Let X be a metric space, then X is **uniformly embeddable** *if and only if for every $R > 0$ and for every $\epsilon > 0$ there exists a Hilbert space valued map $\xi : X \to \mathcal{H}$, $(\xi_x)_{x \in X}$, such that $\|\xi_x\| = 1$ for all $x \in X$ and such that*

(i) $\sup\{\|\xi_x - \xi_{x'}\| : d(x,x') \leq R,\ x, x' \in X\} \leq \epsilon$, *and*

(ii) $\lim_{S\to\infty} \sup\{|\langle \xi_x, \xi_{x'}\rangle| : d(x,x') \geq S,\ x, x' \in X\} = 0$.

Or equivalently,

(iii) $\sup\{|1 - \langle \xi_x, \xi_{x'}\rangle| : d(x,x') \leq R,\ x, x' \in X\} \leq \epsilon$, *and*

(iv) $\lim_{S\to\infty} \inf\{\|\xi_x - \xi_{x'}\| : d(x,x') \geq S,\ x, x' \in X\} = \sqrt{2}$.

Proof. We follow closely the proof provided in [5] by Dadarlat and Guentner, giving additional details where required.

Firstly, we show the equivalence of conditions (i) and (ii) to (iii) and (iv) respectively.

$$\begin{aligned}\|\xi_x - \xi_{x'}\|^2 &= \langle \xi_x - \xi_{x'}, \xi_x - \xi_{x'} \rangle \\ &= \langle \xi_x, \xi_x \rangle - 2\langle \xi_x, \xi_{x'} \rangle + \langle \xi_{x'}, \xi_{x'} \rangle \\ &= 2 - 2\langle \xi_x, \xi_{x'} \rangle\end{aligned}$$

Therefore conditions (i) and (iii) are equivalent, as

$$\|\xi_x - \xi_{x'}\| \leq \epsilon \text{ iff } 1 - \langle \xi_x, \xi_{x'} \rangle \leq \frac{\epsilon^2}{2}$$

Now, from condition (ii), we have that for all $x, x' \in X$ such that $d(x, x') \geq S$, $\lim_{S\to\infty} \sup |\langle \xi_x, \xi_{x'} \rangle| = 0$, i.e. for all such $x, x' \in X$, $\langle \xi_x, \xi_{x'} \rangle \leq 0$. But $\|\xi_x - \xi_{x'}\|^2 = 2 - 2\langle \xi_x, \xi_{x'} \rangle \geq 2$. And so $\|\xi_x - \xi_{x'}\| \geq \sqrt{2}$. Therefore $(ii) \Rightarrow (iv)$, and vice versa.

Now let us assume that X is uniformly embeddable in a Hilbert space $\mathcal{H}$. Then there exists a function $F : X \to \mathcal{H}$ which is a uniform embedding. Therefore there exist non-decreasing functions $\rho_\pm : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{t\to\infty} \rho_\pm(t) = \infty$ and for all $x, x' \in X$

$$\rho_-(d_X(x, x')) \leq d_{\mathcal{H}}(F(x), F(x')) \leq \rho_+(d_X(x, x'))$$

We start by constructing a Hilbert space $Exp(\mathcal{H})$ (which was first introduced by Foch) by direct summation and tensor product multiplication of the Hilbert space $\mathcal{H}$ with itself. Let

$$Exp(\mathcal{H}) = \mathbb{R} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus (\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) \oplus \cdots$$

and define a map $Exp : \mathcal{H} \to Exp(\mathcal{H})$ by

$$Exp(\zeta) = 1 \oplus \zeta \oplus \left(\frac{1}{\sqrt{2!}}\zeta \otimes \zeta\right) \oplus \left(\frac{1}{\sqrt{3!}}\zeta \otimes \zeta \otimes \zeta\right) \oplus \cdots$$

Next we calculate the inner product of a couple of these functions, but first it is worth noting that each of the spaces will be orthogonal to the other ones in the direct sum, and hence any cross-multiplication terms in the inner product will be zero. From background definitions on tensor products we know that $\langle a \otimes b, c \otimes d \rangle := \langle a, c \rangle \langle b, d \rangle$. So for all $\zeta, \zeta' \in \mathcal{H}$

$$\begin{aligned}
&\langle Exp(\zeta), Exp(\zeta') \rangle \\
&= \left\langle 1 \oplus \zeta \oplus \left(\frac{1}{\sqrt{2!}} \zeta \otimes \zeta \right) \oplus \cdots, 1 \oplus \zeta' \oplus \left(\frac{1}{\sqrt{2!}} \zeta' \otimes \zeta' \right) \oplus \cdots \right\rangle \\
&= \langle 1, 1 \rangle + \langle \zeta, \zeta' \rangle + \frac{1}{2!} \langle \zeta \otimes \zeta, \zeta' \otimes \zeta' \rangle + \cdots \\
&= \langle 1, 1 \rangle + \langle \zeta, \zeta' \rangle + \frac{1}{2!} \langle \zeta, \zeta' \rangle^2 + \cdots \\
&= e^{\langle \zeta, \zeta' \rangle} \qquad (*)
\end{aligned}$$

Now for all $t > 0$ we define a function $\xi : X \to Exp(\mathcal{H})$ by

$$\xi_x = e^{-t\|F(x)\|^2} Exp(\sqrt{2t} F(x))$$

Then for all $x, x' \in X$

$$\begin{aligned}
\langle \xi_x, \xi'_x \rangle &= \left\langle e^{-t\|F(x)\|^2} Exp(\sqrt{2t} F(x)), e^{-t\|F(x')\|^2} Exp(\sqrt{2t} F(x')) \right\rangle \\
&= e^{-t\|F(x)\|^2} e^{-t\|F(x')\|^2} \left\langle Exp(\sqrt{2t} F(x)) Exp(\sqrt{2t} F(x')) \right\rangle \\
&= e^{-t\|F(x)\|^2} e^{-t\|F(x')\|^2} e^{2t\langle F(x), F(x') \rangle} \qquad \text{by } (*) \\
&= e^{-t\|F(x)\|^2} e^{-t\|F(x')\|^2} e^{t(\|F(x)\|^2 + \|F(x')\|^2 - \|F(x) - F(x')\|^2)} \\
&= e^{-t\|F(x) - F(x')\|^2}
\end{aligned}$$

And so $\|\xi_x\|^2 = \langle \xi_x, \xi_x \rangle = e^0 = 1$ for all $x \in X$.

Furthermore F is a uniform embedding so

$$\rho_-(d_X(x, x'))^2 \leq \|F(x), F(x')\|^2 \leq \rho_+(d_X(x, x'))^2$$

and so

$$e^{-t(\rho_+(d(x,x')))^2} \leq \langle \xi'_x \xi_{x'} \rangle = e^{-t\|F(x) - F(x')\|^2} \leq e^{-t(\rho_-(d(x,x')))^2}$$

Now if we let $t = \epsilon(1+\rho_+(R)^2)^{-1}$, then we get

$$e^{-\frac{\epsilon\rho_+(d(x,x')))^2}{1+\rho_+(R)^2}} \leq \langle \xi'_x, \xi_{x'} \rangle \leq e^{-\frac{\epsilon\rho_-(d(x,x')))^2}{1+\rho_+(R)^2}}$$

So if $d(x,x') \leq R$ we have

$$\left| 1 - e^{-\frac{\epsilon\rho_+(R)^2}{1+\rho_+(R)^2}} \right| \geq |1 - \langle \xi'_x, \xi_{x'} \rangle|$$

We can make ϵ as small as we like in the above equation, and therefore the left hand side is as close to zero as we like, say δ. And then $|1 - \langle \xi'_x \xi_{x'} \rangle| \leq \delta$ and condition (iii) in the proposition is satisfied.

Now let $d(x,x') \geq S$, then $\rho_-(d(x,x'))^2 \geq \rho_-(S)^2$ and so

$$\langle \xi'_x, \xi_{x'} \rangle \leq e^{-\frac{\epsilon\rho_-(d(x,x'))^2}{1+\rho_+(R)^2}} \leq e^{-\frac{\epsilon\rho_-(S)^2}{1+\rho_+(R)^2}}$$

But $\lim_{S\to\infty} e^{-\frac{\epsilon\rho_-(S)^2}{1+\rho_+(R)^2}} = 0$ and so $\lim_{S\to\infty}\langle \xi'_x, \xi_{x'} \rangle = 0$ which gives us condition (ii) of Proposition 8.1.

Conversely, we assume we have the conditions in Proposition 8.1 and prove that we have a uniform embedding. We start by rewriting the conditions of a uniform embedding, as below.

There exists a sequence of maps, $\eta_n : X \to \mathcal{H}_n$, from X into Hilbert spaces and a sequence of numbers $S_0 = 0 < S_1 < S_2 < \cdots$ increasing to infinity such that for every $n \geq 1$ and every $x, x' \in X$

(a) $\|\eta_n(x)\| = 1$;

(b) $\|\eta_n(x) - \eta_n(x')\| \leq \frac{1}{n}$ whenever $d(x,x') \leq \sqrt{n}$; and

(c) $\|\eta_n(x) - \eta_n(x')\| \geq 1$ whenever $d(x,x') \geq S_n$.

Notice that condition (b) is just condition (i) with $R = \sqrt{n}$ and condition (c) is condition (iv) with $S_n = S$ and taking a smaller lower bound 1 instead of $\sqrt{2}$.

We now choose a base point $x_0 \in X$, and define $F : X \to \bigoplus_{n=1}^{\infty} \mathcal{H}_n$ by

$$F(x) = \frac{1}{2}\left(\eta_1(x) - \eta_1(x_0) \oplus \eta_2(x) - \eta_2(x_0) \oplus \eta_3(x) - \eta_3(x_0) \oplus \cdots\right)$$

Firstly, we show that F is a well defined function. We use the fact that if $d(x, x_0) \leq \sqrt{n}$ then $\|\eta_n(x) - \eta_n(x_0)\| \leq \frac{1}{n}$.

$$\begin{aligned}
\|F(x)\|^2 &= \frac{1}{4}\sum_{n=1}^{\infty} \|\eta_n(x) - \eta_n(x_0)\|^2 \\
&= \frac{1}{4}\left(\sum_{n=1}^{\lfloor d(x,x_0)^2 \rfloor} \|\eta_n(x) - \eta_n(x_0)\|^2 + \sum_{n=\lfloor d(x,x_0)^2\rfloor+1}^{\infty} \|\eta_n(x) - \eta_n(x_0)\|^2\right) \\
&\leq \frac{1}{4}\left(\sum_{n=1}^{\lfloor d(x,x_0)^2 \rfloor} \|\eta_n(x) - \eta_n(x_0)\|^2 + \sum_{n=\lfloor d(x,x_0)^2\rfloor+1}^{\infty} \frac{1}{n^2}\right)
\end{aligned}$$

Now the first sum is finite as there are finitely many terms, and the second sum converges, therefore F is well defined as $\|F(x)\| < \infty$.

Next we calculate an upper estimate for the norm using condition (b). If $d(x, x') \leq \sqrt{n}$ then $\|\eta_n(x) - \eta_n(x')\| \leq \frac{1}{n}$, as follows:

$$\begin{aligned}
\|F(x) - F(x')\|^2 &= \frac{1}{4}\|\eta_1(x) - \eta_1(x')\|^2 + \frac{1}{4}\|\eta_2(x) - \eta_2(x')\|^2 + \cdots \\
&= \frac{1}{4}\sum_{n=1}^{\infty} \|\eta_n(x) - \eta_n(x')\|^2 \\
&\leq \frac{1}{4}\left(\sum_{n=1}^{\lfloor d(x,x')^2 \rfloor} \|\eta_n(x) - \eta_n(x')\|^2 + \sum_{n=\lfloor d(x,x')^2\rfloor+1}^{\infty} \frac{1}{n^2}\right)
\end{aligned}$$

Now $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. For simplicity, we shall use 2 as the upper bound of this sum, giving

$$\begin{aligned}
\|F(x) - F(x')\|^2 &\leq \frac{1}{4} \sum_{n=1}^{\lfloor d(x,x')^2 \rfloor} \|\eta_n(x) - \eta_n(x')\|^2 + \frac{1}{2} \\
&\leq \frac{1}{4} \sum_{n=1}^{\lfloor d(x,x')^2 \rfloor} \|\eta_n(x)\|^2 + \|\eta_n(x')\|^2 + \frac{1}{2} \\
&\leq \frac{1}{4} \sum_{n=1}^{\lfloor d(x,x')^2 \rfloor} 2 + \frac{1}{2} \\
&\leq \frac{1}{2} \lfloor d(x,x')^2 \rfloor + \frac{1}{2}
\end{aligned}$$

So for all $x, x' \in X$

$$\|F(x) - F(x')\| \leq \frac{1}{\sqrt{2}}(d(x,x') + 1)$$

This function is an upper bound for the norm and obviously tends to infinity as $d(x,x') \to \infty$, and so this is our ρ_+ function for the uniform embedding.

We now use condition (c) to calculate a lower bound for the norm. We also assume that $S_{k-1} \leq d(x,x') < S_k$, so that for all $n \leq k-1$ we have $\|\eta_n(x) - \eta_n(x')\| \geq 1$.

$$\begin{aligned}
\|F(x) - F(x')\|^2 &= \frac{1}{4} \sum_{n=1}^{\infty} \|\eta_n(x) - \eta_n(x')\|^2 \\
&\geq \frac{1}{4} \left(\sum_{n=1}^{k-1} 1 + \sum_{n=k}^{\infty} \|\eta_n(x) - \eta_n(x')\|^2 \right) \\
&\geq \frac{1}{4} \sum_{n=1}^{k-1} 1 \\
&= \frac{1}{4}(k-1)
\end{aligned}$$

Therefore for all $x, x' \in X$ we have

$$\|F(x) - F(x')\| \geq \frac{\sqrt{n-1}}{2}\chi_{[S_{n-1},S_n)}$$

As $S_k \to \infty$ when $k \to \infty$ we can see that the lower bound function given above tends to infinity as $d(x, x') \to \infty$. Therefore F is a uniform embedding. □

As before we turn to $\mathbb{R}$-trees as our example of a general metric space. We have already shown, in Theorem 7.1 that any unbounded $\mathbb{R}$-tree which admits a ray satisfies property A. We now use this, and the fact that uniform embeddability passes to subspaces, along with the above Theorem to prove that all $\mathbb{R}$-trees are uniformly embeddable in Hilbert space. Note that this re-proves our earlier result of Theorem 3.20 where we proved uniform embeddability by calculating the Hilbert space compression of $\mathbb{R}$-trees.

Theorem 8.2. *$\mathbb{R}$-trees are uniformly embeddable in Hilbert space.*

Proof. We show this by first considering unbounded $\mathbb{R}$-trees which admit a ray, and using the functions which we used to prove that they have property A in Theorem 7.1.

Let X be an $\mathbb{R}$-tree with a ray, and define functions

$$\xi_x^n(z) = \begin{cases} \frac{1}{n} & \text{if } z \in [x, \gamma_x(n)] \\ 0 & \text{otherwise} \end{cases}$$

Then by Theorem 7.1 $\|\xi_x\|_{L^1(X)} = 1$, and $\|\xi_x - \xi_y\|_{L^1(X)} \leq \epsilon$ whenever $d(x, y) \leq R$, which gives us condition (i) in Proposition 8.1 above.

We also have that $\exists S > 0$ such that $\text{supp}(\xi_x) \subset B(x, S)$ and $\text{supp}(\xi_x') \subset B(x', S)$. Therefore if $d(x, x') > 2S$ then

$$\text{supp}(\xi_x) \subset B(x, S) \cap \text{supp}(\xi_x') \subset B(x', S) = \emptyset$$

and so $\|\xi_x - \xi_{x'}\|^2 = \|\xi_x\|^2 + \|\xi_{x'}\|^2 = 2$ and hence condition (iv) is also satisfied.

Now, by [5, Corollary 4.5] the subspace of a uniformly embeddable metric space is also uniformly embeddable.

So, given an $\mathbb{R}$-tree Y without a ray, we attach a ray to it which gives us a new $\mathbb{R}$-tree, say X. Using the above proof, X is uniformly embeddable and therefore as $Y \subset X$, so is Y.

Hence all $\mathbb{R}$-trees are uniformly embeddable. $\square$

8.2 Exact metric spaces

Let X be a set. A **partition of unity** on X is a family of maps $(\phi_i)_{i\in I}$, with $\phi_i : X \to [0,1]$, and such that $\sum_{i\in I} \phi_i(x) = 1$ for all $x \in X$. We say that $(\phi_i)_{i\in I}$ is **subordinated to a cover** $\mathcal{U} = (U_i)_{i\in I}$ of X if each ϕ_i vanishes outside U_i [5].

Definition 8.3. [5, Definition 2.7]
A metric space X is **exact** if $\forall R > 0$, $\epsilon > 0$ there exists a partition of unity $(\phi_i)_{i\in I}$ on X subordinated to a cover $\mathcal{U} = (U_i)_{i\in I}$ and such that

i) $\forall x, y \in X$ with $d(x,y) \leq R$, $\sum_{i\in I} |\phi_i(x) - \phi_i(y)| \leq \epsilon$, and

ii) the cover $\mathcal{U} = (U_i)_{i\in I}$ is uniformly bounded, i.e. $\sup_{i\in I} diam(U_i) < \infty$.

We now turn to a result of Dadarlat and Guentner which links property A, exactness and uniform embeddability of metric spaces. We give their proof, but go through it in more detail.

Proposition 8.4. *[5, Proposition 2.10]*
Let X be a metric space.

(a) If X is discrete and has property A, then X is exact.

(b) If X is discrete and has bounded geometry then X is exact if and only if it has property A.

(c) If X is exact then X is uniformly embeddable in Hilbert space.

Proof.
(a)
Let X be a discrete metric space with property A (as defined by Tu), so we have functions $\xi : X \to l^1(X)$ (which we assume to be non-negative) such

that $\forall R > 0$, $\forall \epsilon > 0$, $\exists S > 0$ such that $\|\xi_x\|_1 = 1$, $\|\xi_x - \xi_y\|_1 \leq \epsilon$ whenever $d(x,y) \leq R$, and $supp(\xi_x) \subset B(x,S)$.

We define a family of functions $(\alpha_z)_{z\in X}$ by $\alpha_z(x) = \xi_x(z)$. First we note that α_z is non-negative and $\sum_{z\in X} \alpha_z(x) = \sum_{x\in X} \xi_x(z) = \|\xi_x\| = 1$, so it is a partition of unity, as required.

If $U_z = \{x \in X : \alpha_z(x) > 0\}$ then $U_z = \{x \in X : \xi_x(z) > 0\}$, but there exists $S > 0$ such that for all $x \in X$, $supp(\xi_x) \subset B(x,S)$. Therefore if $d(x,z) > S$ then $\xi_x(z) = 0 = \alpha_z(x)$, and so $U_z \subset B(z,S)$ and the cover $\mathcal{U} = (U_z)_{z\in X}$ is uniformly bounded as required.

Now, $\sum_{z\in X} |\alpha_z(x) - \alpha_z(y)| = \sum_{z\in X} |\xi_x(z) - \xi_y(z)| = \|\xi_x(z) - \xi_y(z)\| \leq \epsilon$ if $d(x,y) \leq R$.

All the conditions from the definition of exactness have been satisfied, and so X is exact.

(b)
$(\Rightarrow)$
We assume that our discrete bounded geometry metric space X is exact, and so there is a partition of unity $(\alpha_i)_{i\in I}$ on X which is subordinated to a cover $\mathcal{U} = (U_i)_{i\in I}$ such that if $d(x,y) \leq R$ then $\sum_{i\in I} |\alpha_i(x) - \alpha_i(y)| \leq \epsilon$ and the cover is uniformly bounded.

We define $\beta : X \to l^2(I)$, a Hilbert space, by $\beta_x(i) = \alpha_i(x)^{\frac{1}{2}}$. Then

$$\|\beta_x\|^2_{l^2(I)} = \sum_{i\in I} \beta_x(i)^2 = \sum_{i\in I} (\alpha_i(x)^{\frac{1}{2}})^2 = \sum_{i\in I} \alpha_i(x) = 1$$

If $d(x,y) \leq R$ then, using the inequality $|a^{\frac{1}{2}} - b^{\frac{1}{2}}|^2 \leq |a - b|$, we have

$$\|\beta_x - \beta_y\|^2_{l^2(I)} = \sum_{i\in I} |\alpha_i(x)^{\frac{1}{2}} - \alpha_i(y)^{\frac{1}{2}}|^2 \leq \sum_{i\in I} |\alpha_i(x) - \alpha_i(y)| \leq \epsilon$$

Finally, if $d(x,y) > \sup_{i\in I} diam(U_i)$ then either $\alpha_i(x)$ or $\alpha_i(y)$ or both are zero and so

$$\langle \beta_x, \beta_y \rangle = \sum_{i \in I} \beta_x(i) \cdot \beta_y(i) = \sum_{i \in I} \alpha_i(x)^{\frac{1}{2}} \cdot \alpha_i(y)^{\frac{1}{2}} = 0$$

i.e. there exists $S = \sup_{i \in I} diam(U_i)$ such that $d(x, y) > S \Rightarrow \langle \beta_x, \beta_y \rangle = 0$

The functions $\beta : X \longrightarrow l^2(I)$ satisfy the conditions of property A in Proposition 4.8 part (7), as required.

(⇐)
We assume that our discrete bounded geometry metric space X has property A, so there is a Hilbert space valued function $\beta : X \longrightarrow l^2(I)$ such that $\forall R > 0$, $\forall \epsilon > 0$, $\exists S > 0$ so that $\|\beta_x\|_{l^2(I)} = 1$, $\|\beta_x - \beta_y\|_{l^2(I)} \leq \epsilon$ whenever $d(x, y) \leq R$, and $\langle \beta_x, \beta_y \rangle = 0$ whenever $d(x, y) \geq S$.

We define $\alpha : X \longrightarrow l^2(I)$, a Hilbert space, by $\alpha_x(i) = \beta_i(x)^2$. Then as above $\sum_{i \in I} \alpha_i(x) = 1$ and if $d(x, y) \geq S$, $\sup_{i \in I} diam(U_i) < \infty$.

Finally, if $d(x, y) \leq R$ then we have

$$\sum_{i \in I} |\alpha_i(x) - \alpha_i(y)| = \sum_{i \in I} |\beta_x(i)^2 - \beta_y(i)^2| = \|\beta_x - \beta_y\|^2 \leq \sqrt{\epsilon}$$

So the functions $\alpha : X \longrightarrow l^2(I)$ satisfy the conditions of the definition of exactness as required.

(c)
The proof of this is very similar to that of part (b) (⇒), but here we do not assume that the metric space is discrete. As before we assume X is exact, and therefore there is a partition of unity $(\alpha_i)_{i \in I}$ on X which is subordinated to a cover $\mathcal{U} = (U_i)_{i \in I}$ such that if $d(x, y) \leq R$ then $\sum_{i \in I} |\alpha_i(x) - \alpha_i(y)| \leq \epsilon$ and the cover is uniformly bounded.

We define $\beta : X \longrightarrow l^2(I)$, by $\beta_x(i) = \alpha_i(x)^{\frac{1}{2}}$. And therefore $\|\beta_x\|^2_{l^2(I)} = 1$, $\|\beta_x - \beta_y\|^2_{l^2(I)} \leq \epsilon$ whenever $d(x, y) \leq R$, and $\langle \beta_x, \beta_y \rangle = 0$ whenever $d(x, y) > \sup_{i \in I} diam(U_i)$. So,

$$\lim_{S \to \infty} \sup\{|\langle \beta_x, \beta_y \rangle| : d(x, y) \geq S\} = 0$$

Therefore the functions $\beta : X \to l^2(I)$ satisfy the conditions of the definition of uniform embeddability into Hilbert space, as required. $\square$

Example 8.5. Let X be a discrete metric space which has finitely many elements, and fix some basepoint $x_0 \in X$. Then let $\xi_x = \delta_{x_0}$ for all $x \in X$. These functions satisfy the conditions of property A as

(i) $||\xi_x|| = ||\delta_{x_0}|| = 1$;

(ii) $||\xi_x - \xi_{x'}|| = \sum_{z\in X} |\xi_x(z) - \xi_{x'}(z)| = \sum_{z\in X} |\delta_{x_0}(z) - \delta_{x_0}(z)| = 0$; and

(iii) $supp(\xi_x) = d(x, x_0) \leq S = \max_{x,x'\in X} d(x, x') < \infty$.

Then, using the proof of Proposition 8.4 we define a partition of unity $\phi_z(x)$ as follows

$$\phi_z(x) = \xi_x(z) = \begin{cases} 1 & \text{if } z = x_0 \\ 0 & \text{otherwise} \end{cases}$$

where the cover is the sets

$$U_z = \begin{cases} X & \text{if } z = x_0 \\ \emptyset & \text{otherwise} \end{cases}$$

Therefore $\sum_{z\in X} |\phi_z(x) - \phi_z(y)| = 0$ for all $x, y \in X$ and $\sup_{z\in X} diam(U_z) = diam(x) < \infty$. And hence X is exact.

Example 8.6. An example of a discrete metric space with property A is a tree. This is a corollary of Theorem 7.2 as trees are a discrete example of $\mathbb{R}$-trees. Therefore, by Proposition 8.4 and its proof, we see that trees considered as metric spaces are exact.

Specifically, let X be a tree with a ray. Using the notation defined in Theorem 7.1 we fix a basepoint $u \in X$ and a ray γ_u in X where $\gamma_u(0) = u$. For each x we define a sequence of functions (ξ_x^n) by:

$$\xi_x^n(z) = \begin{cases} \frac{1}{n} & \text{if } z \in [x, \gamma_x(n)] \\ 0 & \text{otherwise} \end{cases}$$

Then we define a family of functions $\alpha_z(x) = \xi_x(z)$, i.e.

$$\alpha_z^n(x) = \begin{cases} \frac{1}{n} & \text{if } z \in [x, \gamma_x(n)] \\ 0 & \text{otherwise} \end{cases}$$

For each n these functions form a partition of unity on X and satisfy the conditions for X to be exact.

In Yu's original paper about property A, he states and proves the following theorem regarding embeddability of property A spaces into Hilbert space.

Theorem 8.7. *[22, Theorem 2.2]*
If a discrete metric space Γ has property A, then Γ admits a uniform embedding into Hilbert space.

We note that this follows from Proposition 8.4 parts (a) and (c). Alternatively one can turn to [22] for Yu's original proof. Here we follow Yu's proof, but give a couple of lemmas first and will then give the proof of this theorem.

Recall from Definition 4.1 that for a discrete metric space Γ with property A there exists a family of finite subsets $\{A_\gamma\}_{\gamma\in\Gamma}$ of $\Gamma \times \mathbb{N}$ which satisfy the three conditions of property A.

Lemma 8.8. *Let $\gamma, \gamma' \in \Gamma$. Then if $d(\gamma, \gamma') > 2R$ the intersection $A_\gamma \cap A_{\gamma'}$ is empty.*

Proof. From condition (2) of the definition we have that $\exists R > 0$ such that if $(x, m) \in A_\gamma$, $(y, n) \in A_\gamma$, then $d(x, y) \leq R$.

So if $d(\gamma, \gamma') > R$, then $(\gamma, m) \notin A_{\gamma'}$ and $(\gamma', n) \notin A_\gamma$. But this is not a strong enough condition to say anything about the intersection $A_\gamma \cap A_{\gamma'}$. So instead let $d(\gamma, \gamma') > 2R$. Then for all $z \in \Gamma$ either $d(\gamma, z) > R$ or $d(z, \gamma') > R$, or both. And hence $(z, n) \in A_{\gamma'}$ or $(z, n) \in A_\gamma$ or neither, and so the intersection $A_\gamma \cap A_{\gamma'}$ is empty. □

Proof of Theorem 8.7. Let H be the Hilbert space defined as

$$H = \bigoplus_{k=1}^{\infty} l^2(\Gamma \times \mathbb{N})$$

Fix a basepoint $\gamma_0 \in \Gamma$ and define a function f from Γ to H which is a uniform embedding and which depends on the characteristic function of the property A sets of Γ as follows:

$$f(\gamma) = \sum_{k=1}^{\infty} \left(\frac{\chi_{A_\gamma^{(k)}}}{(|A_\gamma^{(k)}|)^{\frac{1}{2}}} - \frac{\chi_{A_{\gamma_0}^{(k)}}}{(|A_{\gamma_0}^{(k)}|)^{\frac{1}{2}}} \right)$$

We return to this proof in a moment, after the following Lemma. □

Lemma 8.9. *A discrete metric space X has Yu's property A iff for any $r > 0$ and $\epsilon > 0$ there exist a family of finite subsets $\{A_\gamma^{(k)}\}_{\gamma \in \Gamma}$ in $\Gamma \times \mathbb{N}$ such that*

(1) $(\gamma, 1) \in A_\gamma^{(k)}$ *for all* $\gamma \in \Gamma$;

(2) $\exists R_k > 0$ *such that if* $(x, m) \in A_\gamma^{(k)}$, $(y, n) \in A_\gamma^{(k)}$ *for some* k *and* $x, y \in X$, *then* $d(x, y) \leq R_k$; *and*

(3) $\left\| \frac{\chi_{A_\gamma^{(k)}}}{|A_\gamma^{(k)}|^{\frac{1}{2}}} - \frac{\chi_{A_{\gamma'}^{(k)}}}{|A_{\gamma'}^{(k)}|^{\frac{1}{2}}} \right\|_2 < \frac{1}{2^k}$.

Proof. From Yu's definition of property A, Definition 4.1, we have conditions (1) and (2) as above, and then condition (3) was

$\frac{|A_\gamma - A_{\gamma'}| + |A_{\gamma'} - A_\gamma|}{|A_\gamma \cap A_{\gamma'}|} < \epsilon$ for all γ and $\gamma' \in \Gamma$ satisfying $d(\gamma, \gamma') \leq r$.

We now show that these two conditions are in fact equivalent. Firstly, for ease of notation, let $A = A_\gamma^{(k)}$ and $B = A_{\gamma'}^{(k)}$. Then,

$$
\begin{aligned}
&\left\| \frac{\chi_A}{|A|^{\frac{1}{2}}} - \frac{\chi_B}{|B|^{\frac{1}{2}}} \right\|_2 \\
&= \left\| \frac{|B|^{\frac{1}{2}}\chi_A - |A|^{\frac{1}{2}}\chi_B}{|A|^{\frac{1}{2}}|B|^{\frac{1}{2}}} \right\|_2 \\
&= \frac{\left(|A \setminus B||B| + |A \cap B|||B|^{\frac{1}{2}} - |A|^{\frac{1}{2}}|^2 + |B \setminus A||A|\right)^{\frac{1}{2}}}{|A|^{\frac{1}{2}}|B|^{\frac{1}{2}}} \\
&= \frac{\left(|A \setminus B||B| + |A \cap B||B| - 2|A \cap B||B|^{\frac{1}{2}}|A|^{\frac{1}{2}} + |A \cap B||A| + |B \setminus A||A|\right)^{\frac{1}{2}}}{|A|^{\frac{1}{2}}|B|^{\frac{1}{2}}} \\
&= \frac{\sqrt{2}\left(|A||B| - |A \cap B||B|^{\frac{1}{2}}|A|^{\frac{1}{2}}\right)^{\frac{1}{2}}}{|A|^{\frac{1}{2}}|B|^{\frac{1}{2}}} \\
&= \sqrt{2}\left(1 - \frac{|A \cap B|}{|A|^{\frac{1}{2}}|B|^{\frac{1}{2}}}\right)^{\frac{1}{2}} \\
&\leq \sqrt{2}\left(1 - \frac{|A \cap B|}{|A|^{\frac{1}{2}}|B|^{\frac{1}{2}}}\right)
\end{aligned}
$$

Now, $|A \cup B| \geq |A|$ and $|A \cup B| \geq |B|$, hence taking by taking square roots of both equations, and then multiplying, we have that $|A \cup B| \geq |A|^{\frac{1}{2}}|B|^{\frac{1}{2}}$. We use this substitution in the above equations to get

$$
\begin{aligned}
\sqrt{2}\left(1 - \frac{|A \cap B|}{|A|^{\frac{1}{2}}|B|^{\frac{1}{2}}}\right) &\leq \sqrt{2}\left(\frac{|A \cup B|}{|A|^{\frac{1}{2}}|B|^{\frac{1}{2}}} - \frac{|A \cap B|}{|A|^{\frac{1}{2}}|B|^{\frac{1}{2}}}\right) \\
&\leq \sqrt{2}\left(\frac{|A \cup B| - |A \cap B|}{|A|^{\frac{1}{2}}|B|^{\frac{1}{2}}}\right) \\
&\leq \sqrt{2}\left(\frac{|A \setminus B| + |B \setminus A|}{|A|^{\frac{1}{2}}|B|^{\frac{1}{2}}}\right)
\end{aligned}
$$

But, we also have that $|A| \geq |A \cap B|$ and $|B| \geq |A \cap B|$, so as before, by taking square roots of both equations and multiplying we have $|A|^{\frac{1}{2}}|B|^{\frac{1}{2}} \geq |A \cap B|$. We substitute this in to the above equations and get

$$\sqrt{2}\left(\frac{|A\setminus B|+|B\setminus A|}{|A|^{\frac{1}{2}}|B|^{\frac{1}{2}}}\right)\leq\sqrt{2}\left(\frac{|A\setminus B|+|B\setminus A|}{|A\cap B|}\right)$$

Hence we have that

$$\left\|\frac{\chi_A}{|A|^{\frac{1}{2}}}-\frac{\chi_B}{|B|^{\frac{1}{2}}}\right\|_2\leq\sqrt{2}\left(\frac{|A\setminus B|+|B\setminus A|}{|A\cap B|}\right)$$

But, by assumption the right hand side is less than $\sqrt{2}\epsilon$ whenever $d(\gamma,\gamma')\leq r$.

Now, from the definition of property A, for any $r>0$ and $\epsilon>0$ there exist a family of sets with the appropriate properties, we can in fact choose $r=k$ and $\epsilon=\frac{1}{2^{k+\frac{1}{2}}}$, then we get the inequality in Yu's proof:

$$\left\|\frac{\chi_A}{|A|^{\frac{1}{2}}}-\frac{\chi_B}{|B|^{\frac{1}{2}}}\right\|_2<\frac{1}{2^k} \qquad (*)$$

for all γ and $\gamma'\in\Gamma$ which satisfy $d(\gamma,\gamma')\leq k$. So given two points close enough to one another, we can see that the above norm is small. We shall use this later in proving that we have a uniform embedding from Γ into a Hilbert space. □

Proof of Theorem 8.7 continued.
We return now to the proof of the theorem.

The upper bound estimate for the norm which is given in (*) works for all $k>d(\gamma,\gamma_0)=D$. We use this now to prove that f is a well defined function.

$$
\begin{aligned}
\|f(\gamma)\|^2_{l^2(\Gamma\times\mathbb{N})} &= \sum_{k=1}^{\infty}\left\|\frac{\chi_{A^{(k)}_{\gamma}}}{(|A^{(k)}_{\gamma}|)^{\frac{1}{2}}}-\frac{\chi_{A^{(k)}_{\gamma_0}}}{(|A^{(k)}_{\gamma_0}|)^{\frac{1}{2}}}\right\|^2\\
&< \sum_{k=1}^{\lfloor D\rfloor}\left\|\frac{\chi_{A^{(k)}_{\gamma}}}{(|A^{(k)}_{\gamma}|)^{\frac{1}{2}}}-\frac{\chi_{A^{(k)}_{\gamma_0}}}{(|A^{(k)}_{\gamma_0}|)^{\frac{1}{2}}}\right\|^2+\sum_{k=\lfloor D\rfloor+1}^{\infty}\left(\frac{1}{2^k}\right)^2\\
&= \sum_{k=1}^{\lfloor D\rfloor}\left\|\frac{\chi_{A^{(k)}_{\gamma}}}{(|A^{(k)}_{\gamma}|)^{\frac{1}{2}}}-\frac{\chi_{A^{(k)}_{\gamma_0}}}{(|A^{(k)}_{\gamma_0}|)^{\frac{1}{2}}}\right\|^2+\sum_{k=\lfloor D\rfloor+1}^{\infty}\frac{1}{4^k}\\
&< \sum_{k=1}^{\lfloor D\rfloor}\left\|\frac{\chi_{A^{(k)}_{\gamma}}}{(|A^{(k)}_{\gamma}|)^{\frac{1}{2}}}-\frac{\chi_{A^{(k)}_{\gamma_0}}}{(|A^{(k)}_{\gamma_0}|)^{\frac{1}{2}}}\right\|^2+1
\end{aligned}
$$

Now, the sum we are left with is finite, and so $\|f(\gamma)\|_2 < \infty$, and f is well defined.

It is just left to prove that this function is indeed a uniform embedding. Firstly we shall consider the upper bound.

Let $d(\gamma,\gamma') = d$, then by the calculation above, but substituting γ' for γ_0 we get

$$
\begin{aligned}
\|f(\gamma)-f(\gamma')\|_{l^2(\Gamma\times\mathbb{N})} &= \left\|\sum_{k=1}^{\infty}\frac{\chi_{A^{(k)}_{\gamma}}}{(|A^{(k)}_{\gamma}|)^{\frac{1}{2}}}-\frac{\chi_{A^{(k)}_{\gamma'}}}{(|A^{(k)}_{\gamma'}|)^{\frac{1}{2}}}\right\|\\
&< \sum_{k=1}^{\lfloor d\rfloor}\left\|\frac{\chi_{A^{(k)}_{\gamma}}}{(|A^{(k)}_{\gamma}|)^{\frac{1}{2}}}-\frac{\chi_{A^{(k)}_{\gamma_0}}}{(|A^{(k)}_{\gamma_0}|)^{\frac{1}{2}}}\right\|+1\\
&\leq \sum_{k=1}^{\lfloor d\rfloor}\left(\left\|\frac{\chi_{A^{(k)}_{\gamma}}}{(|A^{(k)}_{\gamma}|)^{\frac{1}{2}}}\right\|+\left\|\frac{\chi_{A^{(k)}_{\gamma_0}}}{(|A^{(k)}_{\gamma_0}|)^{\frac{1}{2}}}\right\|\right)+1\\
&\leq \sum_{k=1}^{\lfloor d\rfloor}(1+1)+1\\
&= 2\lfloor d\rfloor+1\\
&\leq 2d(\gamma,\gamma')+1
\end{aligned}
$$

So for all $\gamma,\gamma'\in\Gamma$

$$\|f(\gamma) - f(\gamma')\|_{l^2(\Gamma\times\mathbb{N})} < 2d(\gamma,\gamma') + 1$$

and we have the function $\rho_+(d(\gamma,\gamma')) = 2d(\gamma,\gamma') + 1$ for the upper bound of the uniform embedding.

Now we turn to the lower bound. The norm $\|f(\gamma) - f(\gamma')\|_{l^2(\Gamma\times\mathbb{N})}$ is obviously bounded below by zero, but for a uniform embedding we need a lower function $\rho_-(d(\gamma,\gamma'))$ which tends to infinity.

$$\left\| \frac{\chi_{A_\gamma^{(k)}}}{|A_\gamma^{(k)}|^{\frac{1}{2}}} - \frac{\chi_{A_{\gamma'}^{(k)}}}{|A_{\gamma'}^{(k)}|^{\frac{1}{2}}} \right\|^2 = 2 - \frac{2|A_\gamma^{(k)} \cap A_{\gamma'}^{(k)}|}{|A_\gamma^{(k)}|^{\frac{1}{2}}|A_{\gamma'}^{(k)}|^{\frac{1}{2}}}$$

Given k, let $d(\gamma,\gamma') > 2R_k$ so that $A_\gamma^{(k)} \cap A_{\gamma'}^{(k)} = \emptyset$ and

$$\left\| \frac{\chi_{A_\gamma^{(k)}}}{|A_\gamma^{(k)}|^{\frac{1}{2}}} - \frac{\chi_{A_{\gamma'}^{(k)}}}{|A_{\gamma'}^{(k)}|^{\frac{1}{2}}} \right\|^2 = 2$$

We shall use this in a moment but first we define an increasing sequence S_k by

$$S_k = \max\{2S_{k-1}, 2R_1, 2R_2, \cdots 2R_k\}$$

Let $d(\gamma,\gamma') \geq S_n$ then $d(\gamma,\gamma') \geq R_k$ for each $k = 1, 2, \cdots n$, and so $\left\| \frac{\chi_{A_\gamma^{(k)}}}{|A_\gamma^{(k)}|^{\frac{1}{2}}} - \frac{\chi_{A_{\gamma'}^{(k)}}}{|A_{\gamma'}^{(k)}|^{\frac{1}{2}}} \right\|^2 = 2$ for each $k = 1, 2, \cdots n$. This then gives the following estimate for the norm

$$\begin{aligned} \|f(\gamma), f(\gamma')\|^2 &= \sum_{k=1}^{\infty} \left\| \frac{\chi_{A_\gamma^{(k)}}}{|A_\gamma^{(k)}|^{\frac{1}{2}}} - \frac{\chi_{A_{\gamma'}^{(k)}}}{|A_{\gamma'}^{(k)}|^{\frac{1}{2}}} \right\|^2 \\ &> \sum_{k=1}^{n} 2 \\ &= 2n \end{aligned}$$

Therefore $\|f(\gamma), f(\gamma')\| > \sqrt{2n}$ if $S_n \leq d(\gamma,\gamma') < S_{n+1}$. So a lower estimate, $\rho_-(d(\gamma,\gamma'))$, for this norm is given by

$$\|f(\gamma), f(\gamma')\| > \sqrt{2n}\chi_{[S_n,S_{n+1})}$$

This is a suitable lower bound function for the uniform embedding because it tends to infinity as $d(\gamma, \gamma') \to \infty$ (as the S_k's form an increasing sequence). Therefore f is a uniform embedding. □

8.3 Nowak's example of metric spaces without property A

The reverse implication of Theorem 8.7 is not true, i.e. given a metric space X which is uniformly embeddable in Hilbert space it is not necessarily true that X has property A. An example of a metric space of this kind is given in [11]. Before we give Nowak's theorem and proof, we need some background definitions from his paper [11].

Definition 8.10. We denote by $Prob(X)$ the space of positive probability measures on X, i.e. $Prob(X) = \{f \in l_1(X) : \|f\|_1 = 1 \,\text{and}\, f \geq 0\}$. Also, if $\gamma \in \Gamma$, where Γ is a group and $f \in Prob(X)$ then by $\gamma \cdot f$ we denote the translation of f by the element γ, i.e. $(\gamma \cdot f)(g) = f(\gamma^{-1}g)$ for all $g \in \gamma$.

Proposition 8.11. *[11, Proposition 2.2]*
Let X be a discrete metric space with property A, then X satisfies the following conditions; for every $R > 0$ and $\epsilon > 0$ there exists a map $\xi : X \to Prob(X)$ and $S \in \mathbb{R}$ such that

(i) $\|\xi(x) - \xi(y)\|_1 \leq \epsilon$ whenever $d(x, y) \leq R$; and

(ii) $supp\xi(x) \subseteq B(x, S)$ for every $x \in X$.

Proof. This Proposition was stated without proof in [11]. We note that this is equivalent to condition (2) of Theorem 4.8, except that we have the extra condition that $\xi(x)$ must be positive. This was dealt with in the proof of (1) $\Leftrightarrow$ (2) of Theorem 4.8. □

Definition 8.12. [11, Definition 3.1] Let X be a discrete metric space with property A, so it satisfies the conditions in the above Proposition. Then we denote $S_X(\xi, R, \epsilon) = \inf S$ where the infimum is taken over all $S > 0$ satisfying condition (ii) above.

We define $diam_X^A(R,\epsilon) = \inf_\xi S_x(\xi, R, \epsilon)$ where the infimum is taken over all maps $\xi : X \to Prob(X)$ which satisfy condition (i) with the given R and ϵ for all $x, y \in X$ such that $d(x,y) \leq R$.

Finally, if Γ is a finitely generated group then for a given $R > 0$ and $\epsilon > 0$ we denote by $diam_\Gamma^{\mathcal{F}}(R,\epsilon)$ the smallest S such that there exists a function $f \in Prob(X)$ such that $suppf \subseteq B(e,S)$ and $\|f - \gamma \cdot f\|_1 \leq \epsilon$ for all $\gamma \in \Gamma$ which satisfies $d(\gamma, e) \leq R$.

Proposition 8.13. *[11, Proposition 3.2]*
If a discrete metric space X has property A then $diam_X^A(R,\epsilon)$ is finite for every $R > 0$ and $\epsilon > 0$.

Proof. This can be seen easily from the previous Proposition and definition. We shall give a proof by contradiction. Assume that for X there exists some R and ϵ such that $diam_X^A(R,\epsilon)$ is infinite. From the definition we have that $diam_X^A(R,\epsilon) = \inf_\xi S_x(\xi,R,\epsilon) = \infty$, and therefore $S_x(\xi,R,\epsilon) = \infty$ for all maps $\xi : X \to Prob(X)$ satisfying condition (i).

This then in turn implies that $S_X(\xi,R,\epsilon) = \inf S = \infty$, and hence the maps $\xi(x)$ are supported on balls of infinite radius. But the definition of property A states that the support is finite and hence we have our contradiction. □

We state (without proof) two theorems from Nowak's paper regarding finitely generated amenable groups. These are required later in proving that our metric space doesn't have property A.

Theorem 8.14. *[11, Theorem 3.3]*
Let Γ be a finitely generated amenable group and fix $R \geq 1$ and $\epsilon > 0$. Then $diam_\Gamma^A(R,\epsilon) = diam_\Gamma^{\mathcal{F}}(R,\epsilon)$.

Theorem 8.15. *[11, Theorem 4.3]*
Let Γ be a finitely generated amenable group. Then for any $0 < \epsilon < 2$, $\liminf_{n\to\infty} diam_{\Gamma^n}^{\mathcal{F}}(1,\epsilon) = \infty$.

The spaces which do not have property A are defined as the disjoint union of bounded, locally finite metric spaces with property A which have diameters growing to infinity. We now need to define the metric on such a space, and can then give the theorem.

Definition 8.16. [11]
Given a sequence $\{(X_n, d_n)\}_{n=1}^{\infty}$ we will make the disjoint union $\chi = \coprod X_n$ into a metric space by giving it a metric d_χ such that

(1) d_χ restricted to X_n is d_n;

(2) $d_\chi(X_n, X_{n+1}) \geq n + 1$; and

(3) if $n \leq m$ we have $d_\chi(X_n, X_m) = \sum_{k=n}^{m-1} d_\chi(X_k, X_{k+1})$

Theorem 8.17. *[11, Theorem 5.1]*
Let Γ be a finite group. The (locally finite) metric space $\chi_\Gamma = \coprod_{n=1}^{\infty} \Gamma^n$ does not have property A, but it is uniformly embeddable in Hilbert space.

Note that in the paper [11], Nowak uses the phrase coarse embedding for what we are calling uniform embedding into Hilbert space.

Proof. We prove the first part by contradiction. Let Γ be defined as in the theorem and let $\chi_\Gamma = \coprod_{n=1}^{\infty} \Gamma^n$ have property A.

So by Proposition 8.13 we have that $diam^{\mathcal{A}}_{\chi_\Gamma}(R, \epsilon)$ is finite for every $R > 0$ and $\epsilon > 0$. So specifically,

$$diam^{\mathcal{A}}_{\chi_\Gamma}(1, \epsilon) < \infty$$

for every $0 < \epsilon < 2$ (a restriction we need later to use Theorem 8.15).

For large enough n, $B_{\chi_\Gamma}(x, S) = B_{\Gamma^n}(x, S)$ for all $x \in \Gamma^n \subset \chi_\Gamma$ which implies that the restriction of the maps $\xi(x)$ to each Γ^n also satisfies $diam^{\mathcal{A}}_{\Gamma^n}(1, \epsilon) < \infty$, and hence for every $n \in \mathbb{N}$ we have $diam^{\mathcal{A}}_{\Gamma^n}(1, \epsilon) < \infty$, i.e.

$$\sup_{n \in \mathbb{N}} diam^{\mathcal{A}}_{\Gamma^n}(1, \epsilon) < \infty$$

But, by Theorem 8.14

$$diam^{\mathcal{F}}_{\Gamma^n}(1, \epsilon) = diam^{\mathcal{A}}_{\Gamma^n}(1, \epsilon)$$

and by Theorem 8.15

$$diam^{\mathcal{F}}_{\Gamma^n}(1,\epsilon) \to \infty$$

as $n \to \infty$, which gives us a contradiction. Therefore χ_Γ does not have property A.

Now consider Γ as a finite metric space. There then exists a map $f : \Gamma \to l^1(\Gamma)$ which is biLipschitz. This can easily be seen, as Γ is finite, so we consider all possible pairs $(x, y) \in \Gamma \times \Gamma$ and calculate the constants L_i such that $\frac{1}{L_i} d(x, y) \leq d(f(x), f(y)) \leq L_i(x, y)$, and then let $L = \sup\{L_i\}$. Thus for all $x, y \in \Gamma$

$$\frac{1}{L} d(x,y) \leq d(f(x), f(y)) \leq L(x,y)$$

Then we define the product map

$$f^n = f \times f \times \cdots \times f : \Gamma^n \to \sum_{i=1}^{n} l^1(\Gamma)$$

This is also biLipschitz, with the same constant, as we can see in the calculations below:

$$\begin{aligned} & d_{\sum_{i=1}^n l^1(\Gamma)} \left(f(x_1, \cdots x_n), f(y_1, \cdots y_n)\right) \\ &\quad = d_{l^1(\Gamma)} \left(f(x_1), f(y_1)\right) + \cdots d_{l^1(\Gamma)} \left(f(x_n), f(y_n)\right) \\ &\quad \leq L d_\Gamma(x_1, y_1) + \cdots L_\Gamma d(x_n, y_n) \\ &\quad = L \left(d_\Gamma(x_1, y_1) + \cdots d_\Gamma(x_n, y_n)\right) \\ &\quad = L d_{\Gamma^n} \left((x_1, \cdots x_n), (y_1, \cdots y_n)\right) \end{aligned}$$

and similarly for the lower bound. Therefore for all $n \in \mathbb{N}$ there is a uniform embedding from Γ^n into $\sum_{i=1}^n l^1(\Gamma)$ with the same constant L.

Now, $l^1(\Gamma)$ is the space of infinite sequences $\{x_m\}$ such that $\sum_{m=1}^\infty |x_m| < \infty$, so we can define a map $g^n : \sum_{i=1}^n l^1(\Gamma) \to l^1(\Gamma)$ as follows:

First let $\{a_{1m}\} \oplus \{a_{2m}\} \oplus \cdots \{a_{nm}\}$ be an element of $\sum_{i=1}^n l^1(\Gamma)$, so that $\sum_{m=1}^\infty |a_{im}| < \infty$ for all $i = 1, 2, \cdots n$. Then we define

$$g^n(\{a_{1m}\} \oplus \{a_{2m}\} \oplus \cdots \{a_{nm}\}) = \{a_{11}, a_{21}, \cdots a_{n1}, a_{12}, \cdots a_{n2}, a_{13} \cdots\}$$

And we know that $\sum_{i=1}^{n} \sum_{m=1}^{\infty} |a_{im}| < \infty$, so

$$|a_{11}| + |a_{21}| + \cdots |a_{n1}| + |a_{12}| + \cdots |a_{n2}| + |a_{13}| + \cdots < \infty$$

and $g^n(\{a_{1m}\} \oplus \{a_{2m}\} \oplus \cdots \{a_{nm}\}) \in l^1(\Gamma)$.

In fact g^n is a bijective isometry from $\sum_{i=1}^{n} l^1(\Gamma)$ to $l^1(\Gamma)$ and so we have uniform embeddings for each n from Γ^n into $l^1(\Gamma)$, each with the same constant. We can then define a new function $F : \chi_\Gamma \to l^1(\Gamma)$ such that the restriction of F to each Γ^n is $g^n \circ f^n$. This new function F is also a uniform embedding with constant L. Hence, χ_Γ is uniformly embeddable in $l^1(\Gamma)$.

But from [11] we know that the properties of uniform embeddability into l^p for $1 \leq p \leq 2$ are all equivalent. And hence χ_Γ embeds uniformly into the Hilbert space $l^2(\Gamma)$ as required. □

A simple example of such a metric space, is by letting Γ be the group with 2 elements. Then Γ^n is the n-dimensional cube and we define χ_Γ to be the disjoint union of all such n-dimensional cubes.

An interesting open question is whether this metric space χ_Γ is exact or not. If it is exact, then we know that exactness does not imply property A. If on the other hand it is not exact, then we shall know that uniform embeddability in a Hilbert space does not imply exactness (as [11] proves also that χ_Γ is uniformly embeddable). One way to approach this problem is:

Consider the metric spaces given by Γ^n for each $n \in \mathbb{N}$ separately, they are discrete and finite. Therefore by Example 8.5 they have property A and are exact, and there are a family of partitions of unity (indexed by n) given by

$$\phi_z^n(x) = \begin{cases} 1 & \text{if } z = x_0^n \\ 0 & \text{otherwise} \end{cases}$$

where the cover is the sets

$$U_z^n = \begin{cases} \Gamma^n & \text{if } z = x_0^n \\ \emptyset & \text{otherwise} \end{cases}$$

where x_0^n is the basepoint in Γ^n.

Then in χ_Γ the basepoint α_0 is the union of the basepoints in each Γ^n and for $\alpha, \beta \in \chi_\Gamma$ we define functions $\psi_\alpha(\beta)$ as

$$\psi_\alpha(\beta) = \begin{cases} 1 & \text{if } \alpha = \alpha_0 \\ 0 & \text{otherwise} \end{cases}$$

Then given $\beta, \beta' \in \chi_\Gamma$ we have $\sum_{\alpha \in \chi_\Gamma} |\psi_\alpha(\beta) - \psi_\alpha(\beta')| = 0$. But $\sup_{\alpha \in \chi_\Gamma} diam(U_\alpha) = \infty$.

ψ is therefore not a suitable partition of unity to show that χ_Γ is exact as we don't have enough control over the growth rate of the covering sets U_α. Unfortunately it does not prove that χ_Γ is not exact either.

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