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BOUNDARIES OF RELATIVELY HYPERBOLIC GROUPS

by

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ABSTRACT

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In this thesis we will investigate two issues on relatively hyperbolic groups which will be treated independently in two parts.

In the first part of this thesis we characterise relatively hyperbolic groups as geometrically finite convergence groups. More precisely, we show the following. Suppose M is a non-empty perfect compact metrisable space, and suppose that a group, Γ, acts as a convergence group on M such that M consists only conical limit points and bounded parabolic points. Suppose also that the stabiliser of each bounded parabolic point is finitely generated. Then Γ is relatively hyperbolic, and M is equivariantly homeomorphic to the boundary of Γ. We also give another aspect of this characterisation by showing under the above assumptions that Γ acts also as a cusp uniform group on the space of triples of M.

In the second part, we describe a condition on the minimal compactifications of maximal parabolic subgroups of a relatively hyperbolic group, Γ. We prove that the dynamical system of Γ on its boundary is finitely presented if we assume that this conditions is satisfied by maximal parabolic subgroups of Γ. We also give examples of groups where this condition is satisfied.

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PREFACE

In this thesis we investigate two problems relating relatively hyperbolic groups, their boundaries and the dynamics arising from the action of relatively hyperbolic groups on their boundaries. These two problems are treated separately in two different parts. Neither of the parts refers directly to the other and each can be read independently. Since both problems are fairly distinct we give one introduction for each part rather than a global introduction, and we repeat in each part some key definitions, like the definition of a relatively hyperbolic group and its boundary.

I visited for the period 10 March - 18 June 2001 the Institut de Recheche Avancée de Mathématique at the University of Strasbourg. The essential work of the second part of this thesis was carried out in the form of discussions between myself and the coauthor, F. Dahmani during this visit. The backbone of the work, as well as the ingredients and their applications in the proof, were formulated at these discussions. Later independently each author wrote his part of the work deriving from this foundation. Sections 2, 4, 5, 6 represent primarily my part of the work, while Sections 1, 3 and 7 were written by the coauthor. I give in Sections 4, 5 and 6 very detailed proof of the main result which enables algorithmic computations and which represent my own approach to the proof. Later the coauthor subsequently produced a condensed version of the proof in the form of a joint preprint.

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Part 1

A TOPOLOGICAL CHARACTERISATION OF RELATIVELY HYPERBOLIC GROUPS

by Aslı Yaman

0. INTRODUCTION

The main objective of this paper is to describe which dynamical, topological properties characterise relatively hyperbolic groups in terms of their boundaries. More precisely we prove the following theorem:

Theorem 0.1. *Suppose that M is a non-empty, perfect and metrisable compactum, and Γ is a convergence group acting on M such that M consists only of conical limit points and bounded parabolic points. Suppose also that the quotient of the set of bounded parabolic points by Γ is finite and the stabiliser of each bounded parabolic point is finitely generated. Then Γ is hyperbolic relative to the set of its maximal parabolic subgroups and M is equivariantly homeomorphic to the boundary of Γ.*

Theorem 0.1 has been shown by Bowditch in [Bo4] in the absence of bounded parabolic points. Therefore to simplify the presentation we will only explicitly give the proof in the presence of bounded parabolic points. However we mention that this proof can be adapted with more technicality to the general case. Henceforth, we assume that M contains parabolic points.

The hypothesis that the stabiliser of each bounded parabolic point is finitely generated does not play any role in the proof of the theorem, but it is there merely to satisfy hypothesis (4) of Definition 2 of relative hyperbolicity as given below. Moreover the hypothesis that the quotient of the set of bounded parabolic points by Γ is finite can also be omitted since we can see by Theorem 1B of [Tu2] that if Γ is a convergence group acting on M such that the action is geometrically finite then there are only finitely many Γ-orbits of bounded parabolic points. On the other hand we note that, if we assume that the quotient of the set of bounded parabolic points by Γ is finite, Theorem 0.1 gives as a consequence the result of Tukia, as we shall explain later after giving the definition of relative hyperbolicity.

Before we give an outline of this paper, we will give the definitions necessary to understand Theorem 0.1. Given a group, Γ, acting on a locally compact topological space we say that the action is *properly discontinuous* if each compact subset of the space meets only finitely many translates of itself under Γ. Let Γ be a group acting by homeomorphisms on a perfect metrisable compactum, M. We denote by $\Theta_3(M)$ the set of ordered distinct triples of M. We say that Γ is a *convergence group* if the induced action on $\Theta_3(M)$ is properly discontinuous. Suppose, now, that Γ is a convergence group. A *parabolic point*, x, is a fixed point of a parabolic subgroup of Γ (see Section 5). We say that a parabolic point, x, is a *bounded parabolic point* if $(M\backslash\{x\})/\operatorname{Stab}_\Gamma(x)$ is compact. A *conical limit point* $x \in M$ is a point such that there is a sequence $\{g_i\}_i \subseteq \Gamma$ and two distinct points a, b in M such that $g_i(x)$ converges to a and $g_i(y)$ converges to b for all $y \in M\backslash\{x\}$. We say that Γ is *geometrically finite* if every point of M is either conical limit point or bounded parabolic point.

The study of convergence groups was introduced by Gehring and Martin [GeM] in order to describe the dynamical properties of a Kleinian group acting on a unit sphere $\mathbf{S}^n$ of $\mathbf{R}^{n+1}$. (Here, a Kleinian group is a subgroup of Isom($\mathbf{H}^n$) acting properly discontinuously on $\mathbf{H}^n$ with boundary $\partial\mathbf{H}^n$.) The notion of convergence group was, later on, generalised to compact Hausdorff spaces by several people, such as Tukia, Freden and Bowditch [Tu1, F, Bo5]. One can see that the notion of conical limit point has an important role in theory of Kleinian groups. They were originally introduced by Hedlund [He] and used by several authors, for example [BeaM, Tu2 ,Bo5, Bo4, Su, N]. They are also known as "points of approximation" [BeaM] or " radial limit points". One can also see that the definition of geometrical finiteness for convergence group is a natural generalisation of the definition of Beardon-Maskit given in the case of Kleinian groups acting on the unit sphere of $\mathbf{R}^3$ in [BeaM]. In fact, it can also be understood as a generalisation of the classical formulation of geometrical finiteness in the case of 3-dimensional Kleinian groups, which are Kleinian groups acting on $\mathbf{H}^3$. To see this we will briefly outline different definitions of geometrical finiteness for a 3-dimensional Kleinian group, Γ.

The most intuitive and original definition of geometrical finiteness for a 3- dimensional Kleinian group, Γ, demands that Γ should possess a finite sided convex fundamental polyhedron. Ahlfors and Greenberg were among the first people using this definition to study the theory of 3-dimensional Kleinian groups. Ahlfors introduced the notion to prove that if Γ is geometrically finite then the measure of limit set, Λ, of Γ is either full or zero [Ah]. At the same time it has been shown by Greenberg in [Gre] that there are 3-dimensional Kleinian groups which are finitely generated but not geometrically finite. To show this Greenberg used the examples of finitely generated groups due originally to Bers, [Ber], and proved that they are not geometrically finite.

Marden's geometrical finiteness definition involves the quotient of the discontinuity domain, $\Omega(\Gamma)$, under Γ. Thus Γ is geometrically finite if $(\mathbf{H}^3 \cup \Omega(\Gamma))/\Gamma$ has finitely many topological ends and if each topological end can be identified with an end of $(\mathbf{H}^3 \cup \Omega(G))/G$ where G is a maximal parabolic subgroup of Γ, [Mar].

Thurston gives a decomposition of a hyperbolic orbifold into the "thin part" and the "thick part". We denote by $\text{hull}(\Lambda(\Gamma))$ the convex hull of the limit set $\Lambda(\Gamma)$, which is the minimal convex subset of $\mathbf{H}^3$ whose closure contains $\Lambda(\Gamma)$. From its definition, $\text{hull}\,\Lambda(\Gamma)$ is Γ-invariant and we can consider the convex core of $N = \mathbf{H}^3/\Gamma$, $\text{core}(N) = \text{hull}\,\Lambda(\Gamma)/\Gamma$. Now Γ is geometrically finite if the thick part of $\text{core}\,N$ is compact, or, alternatively, if the r-neighbourhood of $\text{core}(N)$ has finite volume for every $r > 0$, [Th].

The equivalence of these definitions for 3-dimensional Kleinian groups can be found, for example, in [Th, MatT]. The generalisation of these definitions was given, and their equivalence was proved, by Bowditch for Kleinian groups in all dimensions [Bo2] and for groups acting on a complete simply connected Riemanian manifold of pinched negative curvature [Bo3]. In dimension two (unlike dimension three), geometrically finite Kleinian groups are precisely the finitely generated Kleinian groups, (which are also called Fuchsian groups). This result, which emphasises the importance of geometrically finite groups, seems to be proved for different degrees of generality by Fenchel, Nielsen , Heins and Greenberg, and later on, several other alternative proofs were given (see for example [Bea]).

The notion of a relatively hyperbolic group was introduced by Gromov in [Gro2] and has been elaborated on in various papers, for example [Farb, Sz, Bo7]. In this work, we refer especially to the paper, [Bo7], of Bowditch, where relative hyperbolicity is formulated by giving two equivalent definitions. We will consider the second one of these definitions to characterise Relatively Hyperbolic groups in the proofs given in both parts of this work.

Let Γ be a group and $\mathcal{G}$ be a collection of subgroups.

Definition 1. We say that Γ *hyperbolic relative to* $\mathcal{G}$, if Γ admits a properly discontinuous isometric action on a path-metric space, Σ, with the following properties:
(1) Σ is proper (i.e. complete and locally compact) and hyperbolic,
(2) every point of the boundary of Σ is either a conical limit point or a bounded parabolic point,
(3) the elements of $\mathcal{G}$ are precisely the maximal parabolic subgroups of Γ,
(4) every element of $\mathcal{G}$ is finitely generated.

We refer the elements of the set $\mathcal{G}$ as *peripheral subgroups.*

It follows from this definition that relative hyperbolicity can be understood as a generalisation of Gromov hyperbolic groups and of geometrically finite Kleinian groups. In fact, in former case, there are no peripheral subgroups and we obtain exactly the definition of a hyperbolic group. In the latter case the space, Σ, in question can be taken as $\mathrm{hull}(\Lambda(\Gamma))$, and so every point of the boundary of Σ is either a conical limit point or a bounded parabolic point. Thus in this case the peripheral subgroups are the maximal parabolic subgroups.

Definition 2. We say that Γ is *hyperbolic relative to* $\mathcal{G}$, if Γ admits an action on a connected graph, $\mathcal{K}$, with the following properties:
(1) $\mathcal{K}$ is hyperbolic and each edge of $\mathcal{K}$ is contained in only finitely many circuits of length n for any given integer n,
(2) there are finitely many Γ-orbits of edges, and each edge stabiliser is finite,
(3) the elements of $\mathcal{G}$ are precisely the vertex stabilisers of infinite valence of $\mathcal{K}$,
(4) every element of $\mathcal{G}$ is finitely generated.

The first definition is a modified formulation of the original definition introduced by Gromov. It gives a dynamical characterisation of relatively hyperbolic groups in terms of a group action on a hyperbolic space. The content of Theorem 0.1 is that one can reconstruct a space, Σ, given only an action of a relatively hyperbolic group Γ on its boundary. (Note that the space Σ is not necessarily quasi-isometric to any other one given by Definition 1.) The second definition characterises relative hyperbolicity in terms of a group action on a "hyperbolic Γ-set" (for the definition see Section 7 of Part 1). This latter notion was introduced by Bowditch in order to prove the equivalence of Definition 1 and Definition 2 and to analyse further the theory of relatively hyperbolic groups.

Tukia has shown in [Tu2] the following theorem (Theorem 1B of [Tu2]):

Theorem 0.2. Given a convergence group, Γ, acting on a compact metric space, M, M consists of only conical limit points and bounded parabolic points if and only if $\Theta_3(M)/\Gamma$ is the union of a compact set and a finite number of Γ-quotients of cusp neighbourhoods of bounded parabolic points.

In other words the action is geometrically finite if and only if Γ is "cusp uniform" in the terminology of Tukia introduced in [Tu2]. Cusp uniformity for Γ means that $\Theta_3(M)/\Gamma$ is the union of a compact set and a finite number of Γ-quotients of cusp neighbourhoods of bounded parabolic points. Thus this result, together with Theorem 0.1, gives a condition for relative hyperbolicity. The proof of the direction "only if" of Theorem 0.2, namely if M consists of only conical limit points and bounded parabolic points then $\Theta_3(M)/\Gamma$ is the union of a compact set and finite number of Γ-quotients of cusp neighbourhoods of bounded parabolic points, is more complicated than the other direction. However as we already mentioned, in the case where the quotient of the set of bounded parabolic points by Γ is finite, Theorem 0.1 gives another proof of this direction (see Section 9, Proposition 9.1).

The main idea of the proof of Theorem 0.1 is the following. As we already mentioned Theorem 0.1 is proved in the absence of parabolic points in M by Bowditch ([Bo4]). In this work he construct a "system of annuli" on M which gives rise to a hyperbolic path quasimetric ρ, on the set of distinct triples, $\Theta_3(M)$, of M and proves that the orbit $\Gamma\theta$ of any point θ of $\Theta_3(M)$ is quasidense in $(\Theta_3(M), \rho)$. From this he shows that the Cayley Graph of Γ is hyperbolic. Moreover he shows that points of M, namely conical limit points, can naturally be identified with points of $\partial\Theta_3(M)$ and hence with points of $\partial\Gamma$ where ∂Q denotes the Gromov boundary of a hyperbolic space Q. To prove Theorem 0.1 we mimic some of these ideas and we construct a "system of annuli" on M which gives rise to a hyperbolic path quasimetric ρ, on the set of distinct triples of M union the set of bounded parabolic points. We see that, in our case, differently from the case of [Bo4] we push parabolic points to a bounded distance of the points of $\Theta_3 M$. To achieve this, we generalise, for a geometrically finite actions, the annulus system given in [Bo4]. Moreover we note that the restriction of the quasimetric thus obtained, to the set of bounded parabolic points is also path hyperbolic. Using this hyperbolic path quasimetric on the set of bounded parabolic points and the geometrically finite action, we construct a graph satisfying all properties required by Definition 2 of relative hyperbolicity.

The structure of this paper, in outline, is as follows. In Section 1, we introduce some basic notation and terminology and define the notion of "hyperbolic crossratio". In Section 2, we show which conditions for an annulus system defined on M give a hyperbolic crossratio on M and ensure a "dichotomy" between the points of M, in other words give rise to a partition of M into the set of "conical points" and its complement. In Section 3, we discuss how a finite subset of M can be approximated by trees in the case where we have an appropriate annulus system. In Section 4, we show how a suitable annulus system gives rise to a hyperbolic path quasimetric on the set of distinct triples union of the non conical points of M. In Section 5, we introduce convergence actions and relate the annulus systems of previous sections to convergence groups acting geometrically finitely on M, and we obtain the set of bounded parabolic points as a hyperbolic path quasimetric space. In Section 6, we give a construction of an annulus system, which has all the properties required to enable us to obtain the previous results. In Section 7, we construct a graph, $\mathcal{K}$, satisfying all the properties demanded by definition 2 of relative hyperbolicity. We prove Theorem 7.1, which tell us that Γ is a relatively hyperbolic group. In section 8 we describe the boundary, $\partial\Gamma$, of a relatively hyperbolic group, Γ, and prove that in our situation $\partial\Gamma$ is homeomorphic to M (Theorem 8.2). In Section 9, we give a correspondence between $\Theta_3(M) \cup M$ and the space, $\Sigma \cup \partial\Sigma$ (defined from Definition 1 of relative hyperbolicity), and using this we prove one direction of Theorem 0.2.

1. CROSSRATIOS

The aim of this section is to describe the notion of a "hyperbolic crossratio". In fact, the notion of a crossratio, which is a map defined symmetrically on 4-tuples of a set, has been introduced and used by several authors in different contexts [O,P]. For example it is known that the boundary of a hyperbolic group admits a crossratio, where the crossratio of four points x, y, z, w of the boundary could be interpreted as the distance, up to an additive constant, between two bi-infinite geodesics, one connecting x and y and the other z and w.

For this work we will use specifically the definition of hyperbolic crossratio given by Bowditch in his paper [Bo4]. But before giving this definition we need to introduce some notation and conventions, which could appear rather formal to the reader but which will be used throughout the rest of this work.

For $p, q, k \in \mathbf{R}$ we will write $p \simeq_k q$, $p \preceq_k q$ and $p \ll_k q$ to mean respectively

$|p-q| \leq k$, $p \leq q+k$ and $p \leq q-k$. We notice that if $p \simeq_k q$ and $q \simeq_k r$ then $p \simeq_{2k} r$. This is true also for $\preceq_k$ and $\ll_k$. Thus, since it is possible to find the final constant after a finite number of manipulations, we will frequently drop the subscripts indicating the additive constant, and we will simply use the notation $\simeq, \preceq, \ll$.

Let M be a set. Let $\Theta_n(M)$ be the set of distinct n-tuples (ordered), in other words M^n minus the large diagonal. Consider a map $(..|..) : \Theta_4(M) \to [0,\infty)$ written $[(x,y,z,w) \mapsto (xy|zw)]$ satisfying the symmetry condition $(xy|zw) = (yx|zw) = (zw|xy)$. Such a map will be called a *crossratio* on M.

Definition. We will say $(..|..)$ on M is *k-hyperbolic* if it satisfies:

(B1) Given any subset $\{x,y,z,w\} \subseteq M$ of four distinct elements we can find a permutation of x,y,z,w so that

$$(xz|yw) \simeq_k 0 \text{ and } (xw|yz) \simeq_k 0.$$

(B2) For all distinct $x,y,z,w,u \in M$ we can find a permutation of x,y,z,w,u so that

$$(xz|yu) \simeq_k (xy|wu),$$
$$(xu|zw) \simeq_k (yu|zw),$$
$$(xy|zw) \simeq_k (xy|zu) + (xu|zw),$$

and $(ab|cd) \simeq_k 0$ for all other possibilities where $a,b,c,d \in \{x,y,z,w,u\}$ are distinct.

We will say that $(..|..)$ is *hyperbolic* if it is k-hyperbolic for some k.

In the case where M is a tree the properties B1 and B2 can be illustrated respectively by Figure 1.1.1 and Figure 1.1.2.

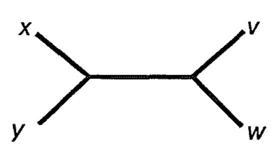

Figure 1.1.1

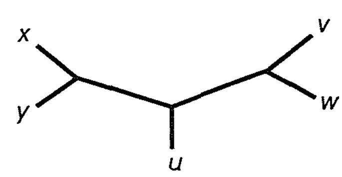

Figure 1.1.2

Notation. Given $x,y,z,w,u \in M$ and $k \in \mathbf{R}$ we write

$(xy:zw)_k \Leftrightarrow (xz|yw) \simeq_k 0$ and $(xw|yz) \simeq_k 0$,

$(xy:u:zw)_k \Leftrightarrow (xu:zw)_k, (xy:zw)_k, (uy:zw)_k, (xy:uw)_k$ and $(xy:zu)_k$,

$(x.y.z.w)_k \Leftrightarrow (xy|zw) \simeq_k 0, (xz|yw) \simeq_k 0$ and $(xw|yz) \simeq_k 0$.

Normally the additive constant k will be taken to be equal to the constant of hyperbolicity of the crossratio. Thus we will frequently omit the subscript indicating k and we will use for $x, y, z, w, u \in M$ the notation $(xy : zw)$, $(xy : u : zw)$ and $(x.y.z.w)$. Otherwise the constant k will be indicated in the notation.

Given $x, y, z, w, u \in M$ it is easy to see that $(x.y.z.w)_k$ is satisfied if and only if $(x.z.y.w)_k$ or $(x.w.y.z)_k$ is satisfied. Moreover, $(x.y.z.w)_k$ is also equivalent to $(xy : zw)_k$, $(xz : yw)_k$ and $(xw : yz)_k$, all together. If x, y, z, w satisfies $(xy : zw)_k$ then either $(xy|zw) \gg_k 1$ or $(x.y.z.w)_k$ is satisfied. Evidently by (B1) if $(xy|zw) \gg_k 1$ then we have $(xy : zw)_k$.

Definition. A hyperbolic crossratio is a *path crossratio*, if for any distinct $x, y, z, w \in M$ and for any $0 < p \leq (xy|zw)$, there exists $u \in M$ such that $(xy : u : zw)$ and $(xy|zu) \simeq p$.

This last definition is relative to some implicit additive constant which can be assumed equal to the constant of hyperbolicity by increasing the latter if necessary.

2. SYSTEMS OF ANNULI

In this section we will see how to obtain a hyperbolic path crossratio on a compactum (compact hausdorff topological space), M, relative to a "system of annuli", which is a set of pairs of disjoint closed subsets of M and which satisfies certain axioms which will be given later in this section. Actually the main work in defining the axioms necessary to obtain such a system has been done by Bowditch in his article [Bo4]. We will be content with stating his results, with some explanation, and leave to reader to look up the details.

So, in this section, our main objective is to modify and to develop the original axioms so as to define a system of annuli on M which give rise a dichotomy on the points of M, and to give some reformulations of these axioms. We will also give some technical results about such systems of annuli under some conditions. [These will be used in the following sections.]

Firstly we need some definitions. We will use the notation and definitions of Bowditch given in [Bo4] for systems of annuli. Let M be a compactum.

Definition. An *annulus*, A, is an ordered pair, (A^-, A^+), of disjoint closed subsets of M such that $M \backslash (A^- \cup A^+) \neq \emptyset$.

A *system of annuli* $\mathcal{A}$ is a set of such annuli.

Let $A = (A^-, A^+)$ be an annulus. We write $-A = (A^+, A^-)$, and say the system of annuli $\mathcal{A}$ is *symmetric* if $-A \in \mathcal{A}$ for every $A \in \mathcal{A}$.

Let K be a closed subset in M. We write $K < A$ if $K \subseteq \operatorname{int} A^-$, and $K > A$ if $K \subseteq \operatorname{int} A^+$. Given two annuli A, B, we write $A < B$ if $M = \operatorname{int} A^+ \cup \operatorname{int} B^-$. We note that $A < B$ implies $-B < -A$ and that $A < B < C$ implies $A < C$.

Definition. Let $\mathcal{A}$ be a system of annuli and K, L, be closed subsets of M. We define $(K|L) \in \mathbf{N} \cup \{\infty\}$ to be the maximal number n such that we can find nested annuli, $A_1, \ldots, A_n$, in $\mathcal{A}$ that separate K and L, i.e $K < A_1 < \cdots < A_n < L$. If the maximum is not attained we adopt the convention that $(K|L) = \infty$. Note also that by our definition 0 is in $\mathbf{N}$.

We introduce the following property, which, when satisfied by a particular system of annuli, gives rise to a dichotomy between the points of M. We say that $\mathcal{A}$ satisfies Δ if:

(Δ) for every $x \in M$ and for every closed subset K of M satisfying $(x|K) = \infty$ there exists an infinite set of nested annuli in $\mathcal{A}$ that separates x and K.

Let $\mathcal{A}$ be a symmetric system of annuli defined on M. For any $K, L \subseteq M$ and any $K' \subseteq K$ closed subsets of M we have $(K|L) = (L|K)$ and $(K'|L) \geq (K|L)$. If $K = \{x_1, \ldots, x_n\}$ we abbreviate $(\{x_1, \ldots, x_n\}|L)$ as $(x_1, \ldots, x_n|L)$. In particular we define $(\{x, y\}|\{z, w\}) = (xy|zw)$ for any $x, y, z, w \in M$.

We also note that the map $(..|..) : \Theta_4(M) \to [0, \infty)$, $[(x, y, z, w) \mapsto (xy|zw)]$ defines a crossratio on M.

Let $\mathcal{A}$ be a system of annuli on M satisfying:

(A1) there is no set of four distinct points $x, y, z, w \in M$ such that $(xz|yw) > 0$ and $(xw|yz) > 0$, and

(A2) if $x \neq y$ and $z \neq w$ in M then $(xy|zw) < \infty$.

Firstly we note that the axiom (A1) is a recapitulation of the axiom (B1) of Section 1 with the constant $k = 0$. Secondly by Proposition 6.5 of [Bo4] [p. 659] we see that the axioms (A1) and (A2) are, indeed, sufficient to obtain a hyperbolic path crossratio on M:

Proposition 2.1. *Let M a compactum and $\mathcal{A}$ a symmetric annulus system defined on M satisfying (A1) and (A2). Then the map $(..|..) : \Theta_4(M) \to [0, \infty)$, $[(x, y, z, w) \mapsto (xy|zw)]$ defines a hyperbolic path crossratio on M.*

For future reference we need to add that, in the proposition above, the constant of hyperbolicity and the constant involved in the path crossratio are equal. Note that it can be shown, following the proof of the above proposition in [Bo4], that the implicit constant is equal to 2. However, for the rest of the work the precise value of this constant is completely irrelevant. Note also that the constant, thus determined, is universally fixed. We will denote it throughout by κ.

We will give, in addition to Proposition 2.1, a series of technical results, which are satisfied by such an annulus system but which are not relevant for this section. We shall need these results as a reference for the proofs in the following sections, which involve some arguments using annuli. The following lemmas correspond respectively to Lemmas 6.1, 6.2 and 6.3 of [Bo4](pp. 658-659). (Recall that we have set $k = 0$ in the notation of [Bo4]).

Lemma 2.2. *Given any closed subsets, $K, L \subseteq M$ and any $a \in M$, we have*

$$(K|L) \leq (K|L \cup \{a\}) + (K \cup \{a\}|L) + 1.$$

Lemma 2.3. *Given any closed nonempty subsets, $K, L \subseteq M$ and any $a \in M$, we have*

$$(K|L \cup \{a\}) + (K \cup \{a\}|L) \leq (K|L) + 2.$$

As a consequence we obtain:

Lemma 2.4. *Given any closed subsets, $K, L \subseteq M$ and any $a \in M$, we have*

$$(K|L) \simeq_2 (K|L \cup \{a\}) + (K \cup \{a\}|L).$$

Let M be a metrisable compactum and let δ be any metric on M inducing the topology. Let $\mathcal{A}$ be a symmetric annulus system defined on M satisfying (A1), (A2). Suppose that $\mathcal{A}$ also satisfies the following property:

(A3) for every $x, y, z \in M$ where $x \notin \{y, z\}$ there exists a neighbourhood N_x^{yz} of x such that for every $w \in N_x^{yz}$ we have $(xy|zw) = 0$ and $(xz|wy) = 0$.

Our main objective in this section is to obtain an annulus system which gives rise to a hyperbolic path crossratio on $\Theta_4(M)$ and which satisfies, at the same time, the property (Δ). As we already see by Proposition 2.1, the system above gives a κ-hyperbolic path crossratio on M for some $\kappa \geq 0$. We thus need to show that such a system also satisfies (Δ).

We will firstly show that the axioms (A3) and (A2), together, give rise to the property:

(A4) There exists ξ such that for every $a \neq b, c \neq d$ and for every sequence $\{a_k\}_k \subseteq M$ tending to a, we have $(a_k b|cd) \simeq_\xi (ab|cd) < \infty$ for all sufficient large k.

Lemma 2.5. *(A2) and (A3) imply (A4) with* $\xi = 2$.

Proof. Suppose we have $a \neq b, c \neq d$, and a sequence $\{a_k\}_k \in M$ such that $a_k \to a$. We put $(a_k b|cd) = n_k$. By (A2) we know that $m = (ab|cd) < \infty$. So there exists a sequence of annuli $\{B_j\}_{j \in \{1,\ldots,m\}}$ with $\{c,d\} < B_1 < \cdots < B_m < \{a,b\}$. Clearly for k large enough $a_k \in (B_m)^+$. Thus we obtain $n_k = (ba_k|cd) \geq (aba_k|cd) \geq m = (ab|cd)$.

Now we will show that $n_k \leq m+1$. For that, we argue by contradiction. Suppose that $n_k \geq m+2$ for infinitely many k. In addition we can find, for every such k, a sequence of annuli $\{A_i^k\}_{i \in \{1,\ldots,n_k\}}$ with $\{c,d\} < A_1^k < \cdots < A_{n_k}^k < \{a_k, b\}$. Let $i_k \in \{1,\ldots,n_k\}$ be the maximal index such that $A_{i_k}^k < \{a\}$. But, since $m < \infty$, i_k cannot be more than m. As a result $n_k - i_k \geq n_k - m > 1$ and we have $\{a\} < A_{i_k+2}^k$. So $(a_k b|ac) \geq n_k - i_k - 1 > 0$.

On the other hand we know by (A3) that for k large enough $a_k \in N_a^{bc}$ where N_a^{bc} is defined as in (A3). Thus $(a_k b|ac) = 0$. This gives a contradiction with $(a_k b|ac) > 0$. As a consequence, for all sufficiently large k we have $n_k \leq m+1$.

We showed that $n_k \geq m$ and $n_k \leq m+1$. This means we have $(a_k b|cd) \simeq_2 (ab|cd)$ and we can choose $\xi = 2$ in (A4).

$\diamondsuit$

In the next section, it will also be shown, using the properties of a hyperbolic crossratio on M, that an annulus system satisfying (A1), (A2) and (A3) must satisfy, as a consequence, the property:

(A5) There exists ξ such that for every $a \neq b, c \neq d$ and for every sequence $\{a_i\}, \{b_j\}, \{c_k\}, \{d_l\} \in M$ satisfying $a_i \to a, b_j \to b, c_k \to c$ and $d_l \to d$ we have for i,j,k,l large enough $(a_i b_j|c_k d_l) \simeq_\xi (ab|cd) < \infty$. (See Lemma 3.7)

We note that the additive constants of properties (A4) and (A5) need not be the same. However they could be chosen identical by increasing one of them if necessary.

We give the following theorem which will achieve the aim of this section and show that an annulus system satisfying (A1), (A2), (A3) satisfies also (Δ).

Theorem 2.6. *Let (M, δ) be a metrisable compactum and $\mathcal{A}$ a symmetric annulus system defined on M satisfying (A1), (A2) and (A3). For any distinct $x, a, b \in M$ satisfying $(x|ab) = \infty$ there exists an infinite set of nested annuli that separate x and $\{a, b\}$.*

Proof. Since $(x|ab) = \infty$ there exist infinitely many sequences of nested annuli which separate a, b and x such that the cardinality of these sequences tends to infinity. That means there are annuli $\{(A_i^k)_{i\in\{1,\ldots,n_k\}}\}_{k\in\mathbf{N}}$ where $n_k \to \infty$ as $k \to \infty$ such that $\{a, b\} < A_1^k < \cdots < A_{n_k}^k < \{x\}$ for every $k \in \mathbf{N}$.

To simplify the notation we will denote $\left(A_{n_k}^k\right)^+$ by P^k. We will show firstly that $\operatorname{diam} P^k \to 0$ when $k \to \infty$. Suppose to the contrary that there is a constant $\epsilon > 0$, such that, for every $k \in \mathbf{N}$, $\operatorname{diam} P^k \geq \epsilon$. As a consequence, for every k there exists a point $y_k \in P^k$, such that $\delta(y_k, x) \geq \epsilon/2$. Since M is compact, by passing to a subsequence, we can suppose that $y_k \to y \in M$. Moreover since $\delta(y_k, x) \geq \epsilon/2$ we see $x \neq y$. On the other hand since $y_k \in P^k$ we have $(y_k x|ab) = n_k$. But $a \neq b$ and $x \neq y$, so by property (A4) we see that $(y_k x|ab)$ is bounded. This give us a contradiction with $n_k \to \infty$.

We have shown that $\operatorname{diam} P^k \to 0$ when $k \to \infty$. Now we will construct an infinite sequence of nested annuli which separate $\{x\}$ and $\{a, b\}$.

We will also denote $\left(A_{n_k-1}^k\right)^+$ by R^k. Evidently, by the same argument which showed that $\operatorname{diam} P^k \to 0$ when $k \to \infty$, we can show that $\operatorname{diam} R^k \to 0$ when $k \to \infty$. Now, fix any index $k_0 \in \mathbf{N}$. We have $\{a, b\} < A_{n_{k_0}}^{k_0} < \{x\}$. Since $x \in \operatorname{int} P^{k_0}$, $\delta(x, \overline{M \backslash P^{k_0}}) = \zeta > 0$. Moreover $\operatorname{diam} R^k \to 0$. So there exists $k_1 > k_0$ such that $\operatorname{diam} R^{k_1} \leq \zeta/2$. As a consequence $R^{k_1} \subseteq \operatorname{int} P^{k_0}$ and we obtain that $\{a, b\} < A_{n_{k_0}}^{k_0} < A_{n_{k_1}}^{k_1} < \{x\}$.

Now, with applying same argument to k_1 we can find an index k_2 such that $\{a, b\} < A_{n_{k_0}}^{k_0} < A_{n_{k_1}}^{k_1} < A_{n_{k_2}}^{k_2} < \{x\}$. Thus, an inductive argument gives us an infinite sequence of nested annuli which separate $\{x\}$ and $\{a, b\}$ as required.

$\diamond$

Lemma 2.7. *Given $x \in M$, if there exist distinct $a, b \in M \backslash \{x\}$ such that $(x|ab) = \infty$ then for any K compact in $M \backslash \{x\}$ we have $(K|x) = \infty$.*

Proof. This result is a consequence of the last proof. In fact we observed that if $(x|ab) = \infty$ then there exists an infinite sequence of nested annuli $(A_i)_{i \in \mathbb{N}}$ which separate $\{x\}$ and $\{a, b\}$ such that $\{a, b\} < A_1 < A_2 < A_3 < \cdots < \{x\}$. Therefore $\operatorname{diam}(A_i)^+ \to 0$ when $i \to \infty$. As a consequence for every $K \subset M \backslash \{x\}$ there is an index i_0 such that $A_{i_0} < K$, i.e. $K < A_{i_0} < A_{i_0+1} < A_{i_0+2} < \cdots < \{x\}$. Thus $(x|K) = \infty$.

$\diamondsuit$

Definition. We shall call a point x a *conical point* if it satisfies $(x|ab) = \infty$ for any two distincts points a, b in M. We denote the set of conical points by Ξ.

The choice of this terminology will be justified later in Section 5.

As a direct result of the Lemma 2.7 we note that if $x \in M$ satisfies $(L|x) < \infty$ for some compact set, L, of $M \backslash \{x\}$ then for any compact non singleton $K \in M \backslash \{x\}$ we have $(K|x) < \infty$. We will denote $M \backslash \Xi$ by Π. We note by Lemma 2.4 that if x and y are in Π then x, y satisfy $(x|y) < \infty$.

Lemma 2.8. *Given $x \in \Pi$ we can find a compact set K in $M \backslash \{x\}$ such that $(K|x) = 0$.*

Proof. We notice that any point $x \in \Pi$ satisfies $n = (K'|x) < \infty$ for a compact set K' in $M \backslash \{x\}$. So we consider a nested annuli system $(A_i)_{i \in \{1, \ldots, n\}}$ of maximal cardinality which separates K' from x such that $\{x\} < A_1 < \cdots < A_n < K'$. Thus it is easy to see by the maximality of n that $K = M \backslash \operatorname{int}(A_1^-)$ satisfies $(K|x) = 0$.

$\diamondsuit$

3. APPROXIMATING TREES

In this section we will explain how one can approximate by trees the structure of a finite set on which a hyperbolic crossratio is defined. Many of the following results will be proved by using these tree approximations. So we begin by introducing some notation and terminology related to trees.

Let $\mathcal{T}$ be a simplicial tree. We denote by $V(\mathcal{T})$ and $E(\mathcal{T})$ respectively the vertex set and the edge set of $\mathcal{T}$. By a *terminal vertex* we mean a vertex of degree 1 and an *internal vertex*, a vertex of degree at least 3. For the rest of this paper we shall assume that $\mathcal{T}$ has no vertex of degree 2. $V_T(\mathcal{T})$ will be the set of terminal vertices and $V_I(\mathcal{T})$ will be the set of internal vertices. We say an edge is *terminal* if one of its end points is terminal; otherwise it is *internal.*

Given $x, y \in V(\mathcal{T})$ we denote by $[x, y]$ the unique arc connecting x and y. If $x, y, z \in V(\mathcal{T})$ we write $\mathrm{med}(x, y, z)$ to mean the *median* of x, y, z, which is the unique intersection of the arcs $[x, y], [y, z]$ and $[z, x]$.

A *metric tree* $(\mathcal{T}, \sigma)$ is a simplicial tree associated to a metric σ, which is defined by assigning to each edge a value in $(0, \infty)$. We note that a metric tree $(\mathcal{T}, \sigma)$ satisfies $\sigma(x, y) + \sigma(z, w) \leq \max\{\sigma(x, z) + \sigma(y, w), \sigma(x, w) + \sigma(y, z)\}$ for all distinct $x, y, z, w \in V(\mathcal{T})$, i.e. $\mathcal{T}$ is 0-hyperbolic in the sense of Gromov. (In fact any finite 0-hyperbolic space can be embedded isometrically in a metric tree [Gro2]) .

We can define on a metric tree $(\mathcal{T}, \sigma)$ a 0-hyperbolic path crossratio $(..|..)_{\mathcal{T}}$ such that $(xy|zw)_{\mathcal{T}} = \frac{1}{2}\max\{0, \sigma(x, z) + \sigma(y, w) - \sigma(x, y) - \sigma(z, w)\}$, i.e. the distance between the segments $[x, y], [z, w]$. Thus we notice that a hyperbolic crossratio defined on a set of 5 elements is, in fact, derived from some metric tree up to an additive constant. Moreover by Theorem 3.1, which is shown by Bowditch (Theorem 2.1 of [Bo4]), we see this is true for any finite set on which we have defined a hyperbolic crossratio.

A generalisation of the notion of hyperbolicity for a "quasi-metric" space has been given, with the equality above satisfied up to an additive constant, by Bowditch in [Bo4] Section 3. We will also use his generalisation by adapting it to our case. [Section 4].

Theorem 3.1. *For all $n \in \mathbf{N}$, there exists a constant $\nu_0(n)$, which depends only on n, such that if $(..|..)$ is a κ-hyperbolic crossratio defined on a set F of cardinality n then we can embed F in a metric tree, $(\mathcal{T}, \sigma)$, such that for all distinct $x, y, z, w \in F$ we have $|(xy|zw) - (xy|zw)_{\mathcal{T}}| \leq \kappa\nu_0(n)$.*

We refer to the tree, $\mathcal{T}$, obtained by Theorem 3.1 as an *approximating tree* for the set F. By the construction of $\mathcal{T}$, we can assume that F is precisely the set $V_T(\mathcal{T})$ and only the lengths of the internal edges of $\mathcal{T}$ are relevant. Later in this section, a general version of this theorem will be given by Corollary 3.5, where we will also determine by definition the lengths of some final edges.

Definition. $\mathcal{T}$ is called *efficient* with respect to κ if for all distinct $x, y, z, w \in F$ satisfying $(x.y.z.w)_\kappa$ the segments $[x, y]$ and $[z, w]$ intersect in a single internal vertex of $\mathcal{T}$, i.e. $[x, y] \cap [z, w] = \{u\}$ where $u \in V_I(\mathcal{T})$. Such a quadruple will be called a *star* in $\mathcal{T}$.

Corollary 3.2. *For all $n \in \mathbf{N}$ there exists a constant $\nu_0'(n)$, which depends only on n, such that if $(..|..)$ is a κ-hyperbolic crossratio defined on a set F of cardinality n then we can embed F in an efficient metric tree (with respect to κ), $(\mathcal{T}', \sigma')$, such that for all distinct $x, y, z, w \in F$ we have $|(xy|zw) - (xy|zw)_{\mathcal{T}'}| \leq \kappa\nu_0'(n)$.*

Proof. We know by Theorem 3.1 that there is a constant $\nu_0(n)$ such that F can be embedded in a metric tree $(\mathcal{T}, \sigma)$ satisfying $|(xy|zw) - (xy|zw)_{\mathcal{T}}| \leq \kappa\nu_0(n)$ for all x, y, z, w in $\mathcal{T}$.

Let x, y, z, w be 4 points of F satisfying $(x.y.z.w)_\kappa$. We notice that $\sigma([x,y],[z,w]) \leq (xy|zw) + \kappa\nu_0(n)$. So, since $(x.y.z.w)_\kappa$ implies $(xy|zw) \simeq_\kappa 0$, we obtain $\sigma([x,y],[z,w]) \preceq_\kappa \kappa\nu_0(n)$. For the same reason, $\sigma([x,z],[y,w])$ and $\sigma([x,w],[y,z])$ are at most $\kappa(\nu_0(n)+1)$.

We define $\mathcal{T}'$ as the simplicial metric tree constructed by collapsing each internal edge of length at most $\kappa(\nu_0(n)+1)$ in $E(\mathcal{T})$ to an internal vertex. In particular we notice that the number of edges which have this property cannot be more than $n-3$, which is the maximal number of internal edges of $\mathcal{T}$. We denote by $\mathcal{E}$ the set of such edges in $\mathcal{T}$.

The metric σ' is defined on $\mathcal{T}'$ so that for each $e \in E(\mathcal{T}) \setminus \mathcal{E}$ we put $\text{length}_{\sigma'}(e) = \text{length}_\sigma(e)$. That means the metric σ is conserved for all edges not collapsed.

By the above remark, we notice also that if x, y, z, w are distinct points of $V_T(\mathcal{T}')$ satisfying $(x.y.z.w)_\kappa$ then x, y, z, w are in a star in $\mathcal{T}'$, i.e $\mathcal{T}'$ is efficient.

It remains to verify that $|(xy|zw) - (xy|zw)_{\mathcal{T}'}| \leq \kappa\nu_0'(n)$. Let x, y, z, w be 4 distinct points of $V_T(\mathcal{T}')$. And let α be the arc which joins $[x,y]$ and $[z,w]$ in $\mathcal{T}'$. Evidently $\text{length}_{\sigma'}(\alpha) \leq \sigma([x,y],[z,w]) = (xy|zw)_{\mathcal{T}}$. On the other hand $(xy|zw)_{\mathcal{T}} \leq \text{length}_{\sigma'}(\alpha) + (n-3)\kappa(\nu_0(n)+1)$. Together these give $|(xy|zw)_{\mathcal{T}} - (xy|zw)_{\mathcal{T}'}| \leq (n-3)\kappa(\nu_0(n)+1)$. Hence we obtain $|(xy|zw) - (xy|zw)_{\mathcal{T}'}| \leq (n-3)\kappa(\nu_0(n)+1) + \kappa\nu_0(n) = \kappa\nu_0'(n)$ where $\nu_0'(n) = (n-2)\nu_0(n) + n - 3$.

$\diamondsuit$

Convention. Each time that we consider an approximating tree $\mathcal{T}$ for a set F on which a κ-hyperbolic crossratio defined, we will choose it to be efficient with respect to κ. That ensures that if $x, y, z, w \in F$ satisfy $(x.y.z.w)_\kappa$ then $(xy|zw)_{\mathcal{T}} = 0$, $(xz|yw)_{\mathcal{T}} = 0$ and $(xw|yz)_{\mathcal{T}} = 0$.

Let M a metrisable compactum. Suppose that we have a symmetric annulus system $\mathcal{A}$ satisfying properties (A1), (A2) and (A3) defined on M. As explained in Section 2, $\mathcal{A}$ gives rise a κ-hyperbolic path crossratio on M and satisfies (Δ). So we can introduce the sets $\Xi = \{\text{conical points}\}$ and $\Pi = M \setminus \Xi$.

Lemma 3.3. *Fix $k \in \mathbf{N}$. If $x \in \Pi$ and $y, z, w \in M \backslash \{x\}$ are distinct points of M satisfying $(xz|yw) \simeq_k 0$ and $(xw|yz) \simeq_k 0$, (i.e. $(xy : zw)_k$) then we have*

$$(x|yz) \simeq_{k+1} (x|yzw) \text{ and } (x|yw) \simeq_{k+1} (x|yzw).$$

In particular $(x|yz) - k - 1 \leq (x|yzw) \leq (x|yw) \leq (x|yzw) + k + 1 \leq (x|yz) + k + 1$. So $(x|yz) \simeq_{k+1} (x|yw)$.

Proof. Put $n = (x|yz)$. Choose a sequence of annuli $\{A_i\}_{i \in \{1 \dots n\}}$ such that $\{x\} < A_n < \cdots < A_1 < \{y, z\}$. We remark that $\{x\} < A_n < \cdots < A_{k+2} < \{w\}$. Because, otherwise, $w \notin (A_{k+2})^+$ and since $(A_{k+2})^+ \cup (A_{k+1})^- = M$, we obtain $\{w\} < A_{k+1}$, i.e. $\{x, w\} < A_{k+1} < \cdots < A_1 < \{y, z\}$. That implies $(xw|yz) \geq k+1$, which is not possible since $(xw|yz) \simeq_k 0$.

So we obtain $n = (x|yz) \geq (x|yzw) \geq n - k - 1$. Therefore $(x|yz) \simeq_{k+1} (x|yzw)$. By interchanging z and w we can obtain likewise $(x|yw) \simeq_{k+1} (x|yzw)$.

$\diamondsuit$

Lemma 3.4. *Fix $k \in \mathbf{N}$. If $x \in \Pi$ and $y, z, w, u \in M \backslash \{x\}$ are distinct points of M satisfying $(yz : x : uw)_k$ then all the following assertions are true*

$$(x|yzwu) \simeq_{k+1} (x|yw),$$
$$(x|yzwu) \simeq_{k+1} (x|yu),$$
$$(x|yzwu) \simeq_{k+1} (x|zw),$$
$$(x|yzwu) \simeq_{k+1} (x|zu).$$

In addition, for any $b, d \in \{u, w\}$ (possibly $b = d$) and for any $a, c \in \{y, z\}$(possibly $a = c$) we obtain $(x|ab) - k - 1 \leq (x|yzwu) \leq (x|cd) \leq (x|yzwu) + k + 1 \leq (x|ab) + k + 1$, i.e. $(x|ab) \simeq_{k+1} (x|cd)$.

Proof. Put $n = (x|yw)$. Choose a sequence of annuli $\{A_i\}_{i \in \{1 \dots n\}}$ such that $\{x\} < A_n < \cdots < A_1 < \{y, w\}$. Then, z satisfies $\{x\} < A_n < \cdots < A_{k+2} < \{z\}$. Because, otherwise, $z \notin (A_{k+2})^+$ and since $(A_{k+2})^+ \cup (A_{k+1})^- = M$, we obtain $\{z\} < A_{k+1}$, i.e. $\{x, z\} < A_{k+1} < \cdots < A_1 < \{y, w\}$. That implies $(xz|yw) \geq k+1$, which is not possible since $(xz|yw) \simeq_k 0$. By exchanging z and u we can obtain likewise $\{x\} < A_n < \cdots < A_{k+2} < \{u\}$.

So we obtain $n = (x|yw) \geq (x|yzwu) \geq n - k - 1$. Therefore $(x|yw) \simeq_{k+1} (x|yzwu)$. We can argue similarly for $(x|yu)$, $(x|zu)$, $(x|zw)$ and $(x|yw)$ to obtain the required relations by interchanging (w and u), etc.

$\diamondsuit$

We will give a more general version of Corollary 3.2, in the case of a metrisable compactum M on which a κ-hyperbolic path crossratio is defined and where a dichotomy between its points is achieved, i.e $M = \Xi \cup \Pi$. But before setting out this result, we give a bit more notation and make some remarks.

Let $\mathcal{T}$ be a tree. For every $K \subset V(\mathcal{T})$ we denote by ω_K the sub-tree spanned by K in $\mathcal{T}$, which is the intersection of connected subsets of $\mathcal{T}$ whose vertex set contains K. Given $K, L \subset \mathcal{T}$ we write $(K : L)_{\mathcal{T}}$ to mean $\omega_K \cap \omega_L$ contains at most one vertex. If two subsets K, L of $\mathcal{T}$ satisfy $(K : L)_{\mathcal{T}}$ then we shall denote by $\alpha(K, L)$ the unique minimal path which joins ω_K and ω_L in $\mathcal{T}$.

Let x, y, z, w be distinct points of M and $\mathcal{T}$ be the approximating tree of $\{x, y, z, w\}$. We notice that $(xy : zw)_\kappa$ is satisfied only if $(\{x, y\} : \{z, w\})_{\mathcal{T}}$ holds and $(\{x, y\} : \{z, w\})_{\mathcal{T}}$ implies $(xy : zw)_{\kappa\nu_0'(4)}$. Therefore, we will use $(xy : zw)_{\mathcal{T}}$ and $(x : yz)_{\mathcal{T}}$ to mean respectively $(\{x, y\} : \{z, w\})_{\mathcal{T}}$ and $(\{x\} : \{yz\})_{\mathcal{T}}$. We note also that every $x \in V_T(\mathcal{T})$ satisfies $(\{x\} : V_T(\mathcal{T})\backslash\{x\})_{\mathcal{T}}$. Let K, L and $K' \subset K$ be subsets of $\mathcal{T}$. If we have $(K : L)_{\mathcal{T}}$ then we have also $(K' : L)_{\mathcal{T}}$.

Corollary 3.5. *Let $(..|..)$ be a κ-hyperbolic crossratio defined on M. For all $n \in \mathbf{N}$ there exists a constant $\nu(n)$, which depends only n, such that any subset $F \subset M$ of cardinality n can be embedded in an efficient metric tree $(\mathcal{T}, \sigma)$ such that for all $a \neq b$, $c \neq d \in F$ and $x, y \in \Pi \cap F$ we have*

$$|(ab|cd) - (ab|cd)_{\mathcal{T}}| \leq \kappa\nu(n),$$
$$|(x|cd) - (x|cd)_{\mathcal{T}}| \leq \kappa\nu(n),$$
$$|(x|y) - (x|y)_{\mathcal{T}}| \leq \kappa\nu(n).$$

Proof. Consider the approximating tree $(\mathcal{T}, \sigma)$ of F. We have $V_T(F) = F$, and $|(ab|cd) - (ab|cd)_{\mathcal{T}}| \leq \kappa\nu_0'(n)$ for all distinct $a, b, c, d \in F$, where $\nu_0'(n)$ is given by Corollary 3.2.

Given $x \in \Pi \cap F$, let $v \in V_I(F)$ be the internal vertex adjacent to x. So $[x, v]$ is a terminal edge. We choose 2 points $a, b \in F\backslash\{x\}$ such that a and b belong to different components of $\mathcal{T}\backslash[x, v]$. We will assign a length to $[x, v]$ by putting $\sigma(x, v) = (x|ab)$.

To prove the corollary we need to verify that $|(x|cd) - (x|cd)_{\mathcal{T}}| \leq \kappa\nu(n)$ for any $c, d \in F\backslash\{x\}$ and $|(x|y) - (x|y)_{\mathcal{T}}| \leq \kappa\nu(n)$ for any $y \in \Pi \cap F\backslash\{x\}$. Since these assertions can be verified using similar arguments, here we will only make explicit the former, i.e we verify $|(x|cd) - (x|cd)_{\mathcal{T}}| \leq \kappa\nu(n)$ for any $c, d \in F\backslash\{x\}$. For this, we will proceed in two different cases, $v \in [c, d]$ and $v \notin [c, d]$.

Suppose that $v \in [c,d]$. It could happen that $c \in \{a,b\}$ or $d \in \{a,b\}$. In this case, without loss of generality, we can suppose that $c = a$. So $(bd : xc)_{\mathcal{T}}$ holds. This implies $(bd : xc)_{\kappa\nu_0'(n)}$, and by applying Lemma 3.3 we obtain $|(x|ab) - (x|cd)| = |(x|bc) - (x|cd)| \le \kappa\nu_0'(n) + 1$. Now, suppose that $\{c,d\} \cap \{a,b\} = \emptyset$. So either $(ac : x : bd)_{\kappa\nu_0'(n)}$ or $(ad : x : bc)_{\kappa\nu_0'(n)}$ holds. But, in each possibility, by Lemma 3.4 we obtain $|(x|ab) - (x|cd)| \le \kappa\nu_0'(n) + 1$. (Figure 1.3.1 illustrates two possibilities for $\mathcal{T}$).

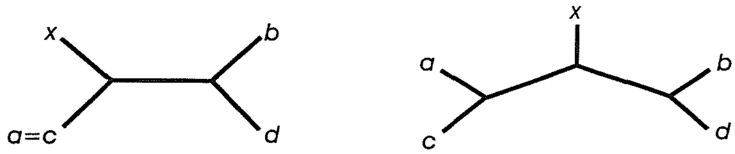

Figure 1.3.1

Moreover, whether $\{c,d\}$ is different from $\{a,b\}$ or not, we have $(x|ab) = (x|ab)_{\mathcal{T}} = (x|cd)_{\mathcal{T}}$. So we obtain $|(x|cd) - (x|cd)_{\mathcal{T}}| \le \kappa\nu_0'(n) + 1$.

Now, suppose that $v \notin [c,d]$. So $[c,d]$ lies in one component of $\mathcal{T}\backslash\{v\}$. Without loss of generality, we can say that $[c,d]$ does not belong to the component which contains a, otherwise we exchange a and b and apply the same reasoning. Hence, we have $(\{x,a\} : \{b,c,d\})_{\mathcal{T}}$ where b could be equal to c or d. As a consequence, we see $(x|cd)_{\mathcal{T}} = \sigma(x,v) + \sigma(v,[c,d]) = (x|ab)_{\mathcal{T}} + (xa|cd)_{\mathcal{T}}$. (One of the possible forms for $\mathcal{T}$ is as Figure 1.3.2).

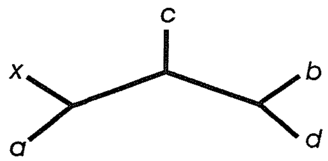

Figure 1.3.2

Suppose that $b \in \{c,d\}$. In this case without loss of generality we can suppose $b = c$, and so $(xa : bd)_{\mathcal{T}}$ holds. Thus we have $(xa : bd)_{\kappa\nu_0'(n)}$ and by Lemma 3.3 we obtain $|(x|ab) - (x|abd)| = |(x|ab) - (x|acd)| \le \kappa\nu_0'(n) + 1$. Now if we suppose that $b \notin \{c,d\}$ then $(\{x,a\} : \{b,c,d\})_{\mathcal{T}}$ implies $(xa : bc)_{\mathcal{T}}$, which gives $(xa : bc)_{\kappa\nu_0'(n)}$, and so by Lemma 3.3 $(x|ab) \simeq_{\kappa\nu_0'(n)+1} (x|ac)$. Likewise we have $(xa : cd)_{\mathcal{T}}$, which implies $(xa : cd)_{\kappa\nu_0'(n)}$ and so by Lemma 3.3 $(x|ac) \simeq_{\kappa\nu_0'(n)+1} (x|acd)$. Thus, together, these give $(x|ab) \simeq_{2\kappa\nu_0'(n)+2} (x|acd)$. So, in both cases, we obtain $(x|ab) \simeq_{2\kappa\nu_0'(n)+2} (x|acd)$.

In addition we have $(xa|cd)_{\mathcal{T}} \simeq_{\kappa\nu_0'(n)} (xa|cd)$. Thus, if we put together the relations $(x|ab) \simeq_{2\kappa\nu_0'(n)+2} (x|acd)$ and $(xa|cd)_{\mathcal{T}} \simeq_{\kappa\nu_0'(n)} (xa|cd)$ with $(x|cd)_{\mathcal{T}} = (x|ab) + (xa|cd)_{\mathcal{T}}$, we obtain $(x|cd)_{\mathcal{T}} \simeq_{3\kappa\nu_0'(n)+2} (x|acd) + (xa|cd)$.

Moreover, by Lemma 2.4, we know that $(x|cd) \simeq_2 (x|acd) + (xa|cd)$. As a result we have shown $(x|cd)_{\mathcal{T}} \simeq_{3\kappa\nu_0'(n)+4} (x|cd)$.

As a summary we see that if $\kappa > 0$ we can choose $\kappa\nu(n) \geq 3\kappa\nu_0'(n) + 4$, and we obtain, for any $c, d \in F\backslash\{x\}$, that $|(x|cd) - (x|cd)_{\mathcal{T}}| \leq \kappa\nu(n)$. Now, as any 0-hyperbolic space can be embedded in a metric tree [Gro2], if $\kappa = 0$ then we have immediately the required result with $\nu(n) = 0$, which completes the proof.

$\diamondsuit$

Definition. In following sections, we will refer to the approximating tree introduced by Corollary 3.5 as "a ζ-approximating tree" where the constant ζ is equal to $\kappa\nu(n)$.

To finish this section, we prove a result (Lemma 3.7) that we already stated in Section 2 and which shows us that property (A5) could be obtained as a consequence of properties (A1), (A2) and (A3). we begin with a preliminary lemma.

Lemma 3.6. *Let K, L be subsets of $\mathcal{T}$ with $(K : L)_{\mathcal{T}}$ and x be an element of $V(\mathcal{T})$. If there exists $y \in L$ satisfying $(K : \{x, y\})_{\mathcal{T}}$ then $(K : L \cup \{x\})_{\mathcal{T}}$ also holds.*

Proof. We argue by contradiction. Suppose that $(K : L \cup \{x\})_{\mathcal{T}}$ does not hold. So $\omega_K \cap \omega_{L\cup\{x\}}$ contains an edge e.

On the other hand we can write $\omega_{L\cup\{x\}} = \omega_L \cup [x, y]$. Thus either $e \subset \omega_K \cap \omega_L$ or $e \subset \omega_K \cap [x, y]$. But both are impossible since we have $(K : L)_{\mathcal{T}}$ and $(K : \{x, y\})_{\mathcal{T}}$.

$\diamondsuit$

Lemma 3.7. *There exists ξ which depends only on κ such that for all $a \neq b, c \neq d$ and for all sequences $\{a_i\}$, $\{b_j\}$, $\{c_k\}$, $\{d_l\} \subseteq M$ with $a_i \to a, b_j \to b, c_k \to c$ and $d_l \to d$ we have for all i, j, k, l large enough $(a_ib_j|c_kd_l) \simeq_\xi (ab|cd) < \infty$.*

Proof. Here there are three 3 possibilities to study. The first possibility is when a, b, c, d are all distinct, the second case is when precisely two of the points $\{a, b, c, d\}$ are identical and the last case is when either $a = c$ and $b = d$ or $a = d$ and $b = c$. All these cases can be treated by similar arguments and methods so we will deal explicitly only with the first.

Suppose that a, b, c, d are all distinct. We define $N_a = N_a^{bc} \cap N_a^{cd}$ where N_a^{bc} and N_a^{cd} are given by property (A3). So, for large enough i, we have $a_i \in N_a$ and so $(aa_i : bc)$, $(aa_i : cd)$ hold. Likewise we define $N_b = N_b^{ac} \cap N_b^{cd}$, $N_c = N_c^{ab} \cap N_c^{bd}$, $N_d = N_d^{ac} \cap N_d^{cb}$, and observe that for large enough j, k, l we have $(bb_j : ac)$, $(bb_j : cd)$, $(cc_k : ab)$, $(cc_k : bd)$, $(dd_l : ac)$ and $(dd_l : cb)$.

We consider an efficient approximating tree $(\mathcal{T}, \sigma)$ of $\{a, b, c, d, a_i, b_j, c_k, d_l\}$. So, by efficiency, the relations $(aa_i : bc)$, $(aa_i : cd)$ imply that $(aa_i : bc)_{\mathcal{T}}$, $(aa_i : cd)_{\mathcal{T}}$ hold in $\mathcal{T}$. Therefore by applying Lemma 3.6 we obtain $(\{a, a_i\} : \{b, c, d\})_{\mathcal{T}}$. In the same way, we obtain also $(\{b, b_j\} : \{a, c, d\})_{\mathcal{T}}$, $(\{c, c_k\} : \{a, b, d\})_{\mathcal{T}}$ and $(\{d, d_l\} : \{a, b, c\})_{\mathcal{T}}$. (See Figure 1.3.3 for an illustration).

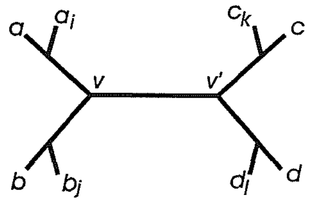

Figure 1.3.3

In addition, without loss of generality, we can suppose $(ab : cd)_{\mathcal{T}}$. Thus let $\{v, v'\}$ be the endpoints of the arc, $\alpha([a, b], [c, d])$, that joins $[a, b]$ and $[c, d]$ in $\mathcal{T}$ so that $v \in [a, b]$ and $v' \in [c, d]$. Moreover, by using the relations $(\{a, a_i\} : \{b, c, d\})_{\mathcal{T}}$ and $(\{b, b_j\} : \{a, c, d\})_{\mathcal{T}}$ it is easy to see that $v \in [a_i, b_j]$. Likewise, we can also see that $v' \in [c_k, d_l]$. Consequently we obtain, for large enough i, j, k, l, $\sigma([a, b], [c, d]) = \sigma([a_i, b_j], [c_k, d_l])$. But, since $(ab|cd) \simeq_{\kappa\nu(8)} \sigma([a, b], [c, d])$ and $(a_i b_j | c_k d_l) \simeq_{\kappa\nu(8)} \sigma([a_i, b_j], [c_k, d_l])$ we have $(a_i b_j | c_k d_l) \simeq_{2\kappa\nu(8)} (ab|cd)$; and by property (A2) we obtain $(a_i b_j | c_k d_l) \simeq_{2\kappa\nu(8)} (ab|cd) < \infty$. Hence clearly if we choose $\xi \geq 2\kappa\nu(8)$ the lemma is verified.

$\diamondsuit$

4. QUASIMETRICS

We will see how a system of annuli defined on a metrisable compactum M and satisfying properties (A1), (A2) and (A3) gives rise to a "hyperbolic quasimetric" on the set of distinct triples union the set of non-conical points.

A quasimetric fulfills the same axioms as a metric does, except that the triangle inequality holds only up to an additive constant and we allow the existence of two distinct points of zero distance apart. We will see that on the large scale a quasimetric behaves as a metric. Therefore in the case of a quasimetric space, we get the standard results about quasi-isometry, hyperbolicity, etc.

Definition. A *k-quasimetric* ρ on a set Q is a map $\rho : Q^2 \to [0, \infty)$ such that $\rho(x, x) = 0$, $\rho(x, y) = \rho(y, x)$ and $\rho(x, z) \le \rho(x, y) + \rho(y, z) + k$ for all $x, y, z \in Q$.

We will say that ρ is a *quasimetric* if it is a k-quasimetric for some k.

We will refer to k as the *quasimetric constant.* Given $x \in Q$ and $r \ge 0$ we write $N_\rho(x, r) = \{y \in Q | \rho(x, y) \le r\}$. Also given P, a subset of Q, we write $N_\rho(P, r) = \bigcup_{x \in P} N_\rho(x, r)$. We will say that P is *r-quasidense* if $Q = N_\rho(P, r)$.

Definition. A *k-pseudogeodesic segment* is a finite sequence of points $\{x_i\}_{i \in \{0,\ldots,n\}}$ satisfying $\rho(x_i, x_j) \simeq_k |i - j|$ for all $i, j \in \{0, \ldots, n\}$.

We note that a pseudogeodesic is a quasigeodesic with the multiplicative constant equal 1. We will say that a k-pseudogeodesic segment $\{x_i\}_{i \in \{0,\ldots,n\}}$ connects a and b if $x_0 = a$ and $x_n = b$. We can similarly define pseudogeodesic rays and bi-infinite pseudogeodesics.

Actually the notions above, such as quasigeodesic segments and quasigeodesic rays have been used by, for example, Bowditch in [Bo4] and Ghys and de la Harpe in [GhH], so also has the notion of "quasi-isometry" between two quasimetric spaces. A quasi-isometry between two quasimetric spaces (Q, ρ) and (Q', ρ') is defined exactly as for metric spaces. It is a map $\varphi : (Q, \rho) \to (Q', \rho')$ which changes distances at most by a linearly bounded amount, i.e there exist constants K and λ satisfying, for every $x, y \in Q$, that $\frac{1}{K}\ \rho(x, y) - \lambda \le \rho'(\varphi(x), \varphi(y)) \le K\rho(x, y) + \lambda$, and such that $\varphi(Q)$ is a quasidense set in Q'. We notice that if P is a quasidense subset of Q then the inclusion of (P, ρ) in (Q, ρ) is a quasi-isometry.

Suppose (Q, ρ) is a k-quasimetric space, then as we mentioned in the previous section we can define a crossratio on the set (Q, ρ) by $(xy|zw)_\rho = \frac{1}{2}\{\max\{\rho(x, y) + \rho(z, w), \rho(x, z) + \rho(y, w), \rho(x, w) + \rho(y, z)\} - \{\rho(x, y) + \rho(z, w)\}\}$ for any $x, y, z, w \in Q$.

Evidently $(..|..)_\rho$ is symmetric and its restriction to $\Theta_4(Q)$ defines a crossratio. We will refer to this crossratio as "the crossratio induced from ρ" or simply the "induced crossratio".

Definition. We say that a quasimetric space is hyperbolic if there exists η such that the induced crossratio $(..|..)_\rho$ satisfies (B1) for the constant η.

We will refer to the constant involved as the "constant of hyperbolicity" and denote it by η. This constant is not a-priori equal to the constant of quasimetric but it can always be taken to be equal to it.

We notice that if ρ is a metric then the fact that the induced crossratio satisfies property (B1) is equivalent to say that ρ satisfies the four points characterisation of hyperbolicity in the sense of Gromov [Gro2]. In other words in the case of a metric space the hyperbolicity gives the four points characterisation. This latter notion was introduced by Gromov and the generalisation to the quasimetric case has been described without any essential change by, for instance, Bowditch [Bo4]. Now, we will give a proposition which is stated in one of the references [Gro2], [Bo1], [GhH] for the metric space case. This proposition helps to understand the connection between the hyperbolicity of a quasimetric and the hyperbolicity of the crossratio induced from it and which is in fact an analogue of Theorem 3.1. Here we allow us to state it for quasimetric case without given a proof as one can generalise the metric space case without any essential changes to the quasimetric space case.

Proposition 4.1. *For all $n \in \mathbf{N}$ there exist some constant, $\nu(n)$, depending only on n, such that if (F, ρ) is a η-hyperbolic η-quasimetric space of cardinality n, then we can embed F in a metric tree $(\mathcal{T}, \sigma)$, such that $|\rho(x,y) - \sigma(x,y)| \leq \eta\nu(n)$ for all $x, y \in F$.*

As a result, if (Q, ρ) is an η-hyperbolic η-quasimetric space, i.e $(..|..)_\rho$ satisfies (B1) for the constant η, then by applying Proposition 4.1 to a set of five elements we see that $(..|..)_\rho$ satisfies (B2) for the constant $\eta\nu(5)$. So $(..|..)_\rho$ is a $\eta\nu(5)$-hyperbolic crossratio defined on Q.

Definition. A quasimetric is a *path quasimetric* if there is a $k \geq 0$ such that every two points can be connected by a k-pseudogeodesic segment.

The constant involved by this definition is, in fact, independent from the constant of the quasimetric. But, for simplicity, it can always be taken to be equal to the constant of quasimetric. Also in the case of a hyperbolic path quasimetric space, we adopt the convention that all constants in question (i.e the constant of quasimetricity, hyperbolicity and the path quasimetricity), will be chosen to be equal. We will work generally in the case of a quasimetric hyperbolic space, so we give in this case an alternative formulation of the property of having the path property. A quasimetric hyperbolic space, (Q, ρ), is a path quasimetric space if and only if there is a $k \geq 0$ such that for any $x, y \in Q$ and $p \leq \rho(x,y)$ there is $z \in Q$ such that $\rho(x,z) \simeq_k p$ and $\rho(x,z) + \rho(z,y) \simeq_k \rho(x,y)$. We note that since a quasi-isometry distorts the distance at most by a linearly bounded amount, the property of being a hyperbolic path quasimetric space is also a quasi-isometry invariant. The proof of this result can be found for instance in [Bo1].

Let M be a metrisable compactum. Suppose that we have a symmetric system of annuli $\mathcal{A}$ on M which satisfies (A1), (A2) and (A3). So we know that it defines a κ-hyperbolic path crossratio on M and satisfies (Δ). Now, we will see how we can define a hyperbolic path quasimetric on, $\Theta_3(M) \cup \Pi$, the set of distinct triples union the set of non-conical points. Bowditch gave the construction of a hyperbolic path quasimetric on $\Theta_3(M)$ when a hyperbolic crossratio is defined on M ([Bo4],Section 4). In our case in order to obtain such a quasimetric we will use his work as a background. The main argument will be to use tree approximations of the quasimetric which will be defined on $\Theta_3(M) \cup \Pi$.

We define the quasimetric on $\Theta_3(M) \cup \Pi$ by

$$\begin{aligned} &\rho(a, b) = (a|b), \\ &\rho(a, X) = \rho(X, a) = \max\{(a|x_i x_j) \text{ where } i \neq j\}, \\ &\rho(X, Y) = \max\{(x_i x_j|y_k y_l) \text{ where } i \neq j \text{ and } k \neq l\}. \end{aligned}$$

where $X = (x_1, x_2, x_3), Y = (y_1, y_2, y_3) \in \Theta_3(M)$ and $a, b \in \Pi$.

Thus, we have $\rho(x, x) = 0$ and $\rho(x, y) = \rho(y, x)$ for every $x, y \in \Theta_3(M) \cup \Pi$. We need also to verify the triangle inequality, up to an additive constant. To do this, we will use approximating tree to find a geometric interpretation of this quasimetric. Let F be finite subset of M of cardinality n so that $\{a, b, x_1, x_2, x_3, y_1, y_2, y_3\} \subseteq F$ We consider a ζ-approximating tree $(\mathcal{T}, \sigma)$ of F defined as in Corollary 3.5 (so that $\zeta = \kappa\nu(n)$) and we denote $x = \text{med}(x_1, x_2, x_3)$ and $y = \text{med}(y_1, y_2, y_3)$. Then,

Lemma 4.2.
$$\begin{aligned} &\rho(a, b) \simeq \sigma(a, b), \\ &\rho(a, X) \simeq \sigma(a, x), \\ &\rho(X, Y) \simeq \sigma(x, y). \end{aligned}$$

Proof. By the construction of $(\mathcal{T}, \sigma)$ we know that $\sigma(a, b) \simeq (a|b)$. So we have $\sigma(a, b) \simeq \rho(a, b)$.

Now, we will show $\sigma(x, y) \simeq \rho(X, Y)$. For every $i \neq j$ and $k \neq l$ in $\{1, 2, 3\}$ we have $x \in [x_i, x_j]$ and $y \in [y_k, y_l]$. So $(x_i x_j|y_k y_l) \simeq (x_i x_j|y_k y_l)_{\mathcal{T}} \leq \sigma(x, y)$. As a consequence we obtain $\rho(X, Y) \preceq \sigma(x, y)$. On the other hand we can find $i_0 \neq j_0$ and $k_0 \neq l_0$ in $\{1, 2, 3\}$ such that $\sigma(x, y) = \sigma([x_{i_0}, x_{j_0}], [y_{k_0}, y_{l_0}]) = (x_{i_0} x_{j_0}|y_{k_0} y_{l_0})_{\mathcal{T}}$. But we have $(x_{i_0} x_{j_0}]y_{k_0} y_{l_0})_{\mathcal{T}} \simeq (x_{i_0} x_{j_0}|y_{k_0} y_{l_0})$. So we obtain $\sigma(x, y) \simeq (x_{i_0} x_{j_0}|y_{k_0} y_{l_0}) \leq \rho(X, Y)$.

Finally we can see, by a similar argument to the one above, that $\sigma(a,x) \simeq \rho(a,X)$ which completes the proof.

$\diamondsuit$

Here the approximating constants depend only on κ and the cardinality n of the set F. In the application below, $\zeta = \kappa\nu(n)$ will depend only on κ since all the entries of any triple or quadruple of $\Theta_3(M) \cup \Pi$ belong to a subset of M of cardinality at most 16 and hence we can fix n to be 16.

Proposition 4.3. *$(\Theta_3(M) \cup \Pi, \rho)$ is a hyperbolic quasimetric space.*

Proof. To see that ρ is a quasimetric, we need the verify the triangle inequality. Let X,Y and Z be three points in $\Theta_3(M) \cup \Pi$. We will verify the triangle inequality only in the case where $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3)$ and $Z = (z_1, z_2, z_3)$ are in $\Theta_3(M)$. Other cases can be verified in similar manner. Consider an approximating tree $(\mathcal{T}, \sigma)$ of $\{x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3\}$ defined as in Corollary 3.5 (so that $\zeta = \kappa\nu(9)$) and we denote x, y and z respectively $\mathrm{med}(x_1, x_2, x_3)$, $\mathrm{med}(y_1, y_2, y_3)$ and $\mathrm{med}(z_1, z_2, z_3)$. Then, by Lemma 4.2, we obtain $\rho(X,Y) \simeq \sigma(x,y)$, $\rho(Y,Z) \simeq \sigma(y,z)$ and $\rho(X,Z) \simeq \sigma(x,z)$. We know $\sigma(x,y) \leq \sigma(x,z) + \sigma(z,y)$, and so we obtain $\rho(X,Y) \preceq \rho(X,Z) + \rho(Z,Y)$.

Now, we will prove that ρ is hyperbolic. Let W be another point in $\Theta_3(M) \cup \Pi$. We will again give a proof only in the case where X, Y, Z and $W = (w_1, w_2, w_3)$ are in $\Theta_3(M)$. So, we consider an approximating tree $(\mathcal{T}, \sigma)$ of $\{x_1,\ x_2,\ x_3,\ y_1,\ y_2,\ y_3,\ z_1,\ z_2,\ z_3,\ w_1,\ w_2,\ w_3\}$ defined as in Corollary 3.5. (so that $\zeta = \kappa\nu(12)$). We know, by a remark we mentioned in the previous section, that $(\mathcal{T}, \sigma)$ is 0-hyperbolic and following this we obtain the result required.

$\diamondsuit$

We have shown that $(\Theta_3(M) \cup \Pi, \rho)$ is quasimetric hyperbolic space. In fact, the restriction on $\Theta_3(M)$ of the quasimetric ρ is exactly the quasimetric defined by Bowditch in Section 4 of [Bo4]. Therefore, by Lemma 4.3 of [Bo4], we can also say that the restriction of ρ on $\Theta_3(M)$ is path quasimetric. The proof of this result uses basically approximating trees and the path property of the crossratio induced from $\mathcal{A}$. We will use this result to show that $(\Theta_3(M) \cup \Pi, \rho)$ is, in addition to its hyperbolicity, a path quasimetric space.

Lemma 4.4. *ρ on $\Theta_3(M) \cup \Pi$ is a path quasimetric.*

Proof. We already know that the restriction of of ρ on $\Theta_3(M)$ is path quasimetric, so we need to take into account the set Π. We will prove that for any $x \neq y \in \Pi$ and $p \leq (x|y)$ there exists an element W of $\Theta_3(M) \cup \Pi$ satisfying $\rho(x,W) \simeq p$ and $\rho(x,W)+\rho(W,y) \simeq \rho(x,y)$. We need a similar result for any a in Π, $X = (x_1, x_2, x_3)$ in $\Theta_3(M)$ and $p \leq (a|X)$ there exists an element W of $\Theta_3(M) \cup \Pi$ satisfying $\rho(a,W) \simeq p$ and $\rho(a,W) + \rho(W,X) \simeq \rho(a,X)$. But the latter case is an analogue of the first one, so we will only demonstrate the assertion for the former.

Let $x \neq y \in \Pi$ and $p \leq (x|y) = n$. So there exists a finite sequence of annuli $\{A_i\}_{i\in\{1\dots n\}}$ such that $\{x\} < A_1 < \dots < A_n < \{y\}$. We choose a point $z \in M \backslash \{(A_p)^- \cup (A_p)^+\}$ and denote by W the triple (x,z,y). We note that $\rho(x,W) = (x|zy)$ and $\rho(y,W) = (xz|y)$. In addition, by Lemma 2.4, we have that $(x|y) \simeq_2 (x|zy) + (xz|y)$. In other words $\rho(x,y) \simeq_2 \rho(x,W) + \rho(y,W)$. On the other hand we can also notice that since $A_{p-1} < \{z\}$ we have $(x|yz) \geq p-1$, as well as $(xz|y) \geq n-p$ since $\{z\} < A_{p+1}$. As a consequence we obtain $(x|yz) \leq n+2-(xz|y) \leq p+2$ and so $\rho(x,W) \simeq_2 p$, which completes the demonstration.

$\diamondsuit$

Lemma 4.5. *If Π is quasi-dense in $\Theta_3(M) \cup \Pi$ then we have $(\Pi, \rho|_\Pi)$ is a path quasimetric hyperbolic space.*

Proof. As we mentioned in the beginning of the section, if Π is a quasidense subset of $\Theta_3(M) \cup \Pi$, the inclusion of (Π, ρ) in $(\Theta_3(M) \cup \Pi, \rho)$ is a quasi-isometry. Also we noted that the property of being hyperbolic path quasimetric is a quasimetric invariant. Now if Π is quasi-dense in $\Theta_3(M) \cup \Pi$, since we already proved that $(\Theta_3(M) \cup \Pi, \rho)$ is a hyperbolic path quasimetric space, we see that $(\Pi, \rho|_\Pi)$ is also a hyperbolic path quasimetric space, which gives the result needed.

$\diamondsuit$

5. CONVERGENCE GROUPS

In this section we will link systems of annuli defined on a compactum and convergence groups acting on a compactum. We will then describe which hypotheses for systems of annuli allow us to obtain the set of bounded parabolic points as a hyperbolic path quasimetric space.

The study of convergence groups was introduced by Gehring and Martin [GeM1] in order to describe the dynamical properties of a Kleinian group acting on a standard sphere of $\mathbf{R}^n$. The notion has been generalised to compact Hausdorff spaces by Tukia and Freden [Tu1, Fr]. Their motivation came by observing that the action of an isometry group of a Gromov hyperbolic metric space can be extended as the action of a convergence group to its Gromov boundary. There are mainly two equivalent definitions of convergence groups. Let M be a metrisable compactum and Γ be a group which acts by homeomorphisms on M.

Definition 1. Γ is a *convergence group* if, for any sequence $\{g_i\}_i$ of distinct elements of Γ, there are two points $a, b \in M$ and a subsequence $\{g_{i_n}\}_n$ such that $\{g_{i_n}\}_n$ converges to b locally uniformly on $M\backslash\{a\}$ as n tends to ∞.

Definition 2. Γ is a *convergence group* if its action on the space of distinct triples, $\Theta_3(M)$, is properly discontinuous (i.e for any compact subset $K \subseteq \Theta_3(M)$ the set $\{g \in \Gamma \mid gK \cap K \neq \emptyset\}$ is finite).

The equivalence of these definitions is proved for group actions on spheres in [GeM2] and for the general case (actions on compacta) in [Bo5]. We note that the first definition gives a dynamical characterisation of convergence groups, while the second one is more natural topologically given that $\Theta_3(M)$ can be compactified by adding a copy of M. This compactification can be described by presenting $\Theta_3(M) \sqcup M$ as a quotient of $M \times M \times M$ where the quotient map is the identity on $\Theta_3(M)$ and sends a triple (x, y, z) to $a \in M$ if at least two of x, y, z are equal to a. We will refer to the topology thus defined on $\Theta_3(M) \sqcup M$ as the *topology of compactification.* In this topology we can see that if $(x_i, y_i, z_i)_i$ is a sequence in $\Theta_3(M)$ with x_i and y_i tending to a then (x_i, y_i, z_i) converge to a in $\Theta_3(M) \sqcup M$. The converse is also true up to permuting x_i, y_i, z_i. Therefore the action of a convergence group Γ on M can be extended to $\Theta_3(M)$ with $(x, y, z) \mapsto (gx, gy, gz)$ for every $g \in \Gamma$.

In this paper we need the dynamical properties of "bounded parabolic points" and a geometrical characterisation of "conical limit points". Before giving an explicit definition of these, we will briefly outline the basic elements of the theory of convergence groups acting on a compactum. Suppose Γ is a convergence group acting on a compactum M. We write $\Lambda\Gamma$ for the limit set of Γ. The limit set can be described as the set of accumulation points of any Γ-orbit. So, $\Lambda\Gamma$ is a closed subset of M. In fact, unless Γ is finite or virtually cyclic, it is the unique, minimal, non-empty, closed

Γ-invariant subset of M. We say that the action of Γ is *minimal* if $\Lambda\Gamma = M$, which will be the case in this paper from our definition of "geometrically finite" convergence groups.

Definition. A convergence group, Γ, is *geometrically finite* if every point of M is either a conical limit point or a bounded parabolic point.

The notion of geometrical finiteness has been described by various authors for Möbius groups [Bea,Bo2,] and for Riemannian spaces in [Bo3] and its extension to convergence groups has been done by Tukia in [Tu2]. We say that a subgroup $G \leq \Gamma$ is *parabolic* if it is infinite, fixes some point of M and contains no loxodromic. (A loxodromic element g of Γ is an element with infinite order and with $\mathrm{card}(\mathrm{fix}(g)) = 2$.) In this case the fixed point, x, of G is unique and called a *parabolic point.* Moreover G acts on $M\backslash\{x\}$ properly discontinuously. We note that a parabolic group can be an infinite torsion group, so by our definition a parabolic point does not need to be fixed by a parabolic element of G. It can be shown that $\mathrm{Stab}_\Gamma(x)$ is necessarily a parabolic group for any parabolic point x. Therefore, there is a bijection between maximal parabolic subgroups of Γ and parabolic points of M. We will say that a parabolic group G with fixed point x is *bounded* if the quotient $(M\backslash\{x\})/G$ is compact. Now we can give the definitions of bounded parabolic points and conical limit points.

Definition. A *parabolic point* x is a bounded parabolic point if and only if $\mathrm{Stab}_\Gamma(x)$ is bounded, i.e., $(M\backslash\{x\})/\,\mathrm{Stab}_\Gamma(x)$ is compact.

In fact this definition arises from the extension, to convergence groups, of the notion introduced in the case of Kleinian groups, called "standard parabolic region" by Bowditch or "cusp neighbourhood" by Tukia. For a deeper study of bounded parabolic points see [Tu2, Bo2]. The definition of conical limit points for convergence groups agrees with one for the Kleinian groups.

Definition. A point $x \in M$ is a *conical limit point* if there exist a sequence $\{g_i\}_i \subseteq \Gamma$ and distinct points $a, b \in M$ such that $g_i x \to a$ and $g_i y \to b$ for all $y \neq x \in M$.

In this definition the convergence of g_i on $M\backslash\{x\}$ can assumed locally uniform.

In this section we will need an equivalent and more geometric definition of conical limit points given by Tukia in [Tu2]. We define a *line* $L(a, b)$, with endpoints a and

$b \neq a$, of $\Theta_3(M)$ as a set $\{(a,b,z) : z \in M\backslash\{a,b\}\}$. This definition respects the orientation and we refer to a as the initial point.

Definition. A point $x \in M$ is a *conical limit point* if and only if any line $L(x,y)$ of $\Theta_3(M)$ with initial point x contains a sequence $\{X_i = (x,y,x_i)\}_i \subset L(x,y)$ such that x_i converges to x in M and $(x,y,x_i) \in \Gamma K$ for some compact set K in $\Theta_3(M)$.

The following result is a standard result due to Beardon and Maskit in the case of Kleinian groups. This has been proved for the case of convergence groups by Tukia [Tu2].

Proposition 5.1. *A conical limit point cannot be a parabolic point*

Let Γ be a convergence group acting on a metrisable compactum M. Suppose that M contains only conical limit points and bounded parabolic points and that the quotient by Γ of the set of bounded parabolic points is finite. Let $\mathcal{A}$ be a Γ-invariant symmetric annulus system constructed on M, satisfying (A1), (A2), (A3) such that the set of conical points, Ξ, is exactly the set of conical limit points. This justifies the choice of the terminology for "conical points" in Section 2. Thus $\Pi = M\backslash\Xi = \{\text{bounded parabolic points}\}$. Suppose also that there exists $\chi > 0$ such that every pair of distinct points $x, y \in \Pi$ satisfies $(x|y) \geq \chi$. We saw in Section 2 that the constant of path hyperbolicity, κ, of the induced crossratio, is universally defined and fixed for such a system of annuli (Proposition 2.1). Here, we assume that χ can be chosen independently from κ. Later in Proposition 6.5, the system of annuli, $\mathcal{A}$, will be constructed to enable us to choose χ large enough to obtain the results required. Moreover, by Lemma 3.7, $\mathcal{A}$ satisfies (A5), i.e. there exists ξ which depends only o κ, and thus which is also universally defined, such that for every $a \neq b, c \neq d$ and for all sequences $\{a_i\}$, $\{b_j\}$, $\{c_k\}$, $\{d_l\} \subseteq M$ satisfying $a_i \to a, b_j \to b, c_k \to c$ and $d_l \to d$ we have for all i,j,k,l sufficient large $(a_i b_j | c_k d_l) \simeq_\xi (ab|cd) < \infty$. In Section 4 we constructed, by Lemmas 4.3 and 4.4, a hyperbolic path quasimetric, ρ, on $\Theta_3(M) \cup \Pi$ with the property that if (Π, ρ) is quasidense in $(\Theta_3(M) \cup \Pi, \rho)$, then (Π, ρ) is also a η-hyperbolic path quasimetric space (Lemma 4.5), where η depends on the constant of quasidensity. From the definition of ρ we note that since $\mathcal{A}$ is Γ-invariant, ρ is also Γ-invariant.

Lemma 5.2. *Given a sequence, $\{X_i\}_i$, and an element, X, of $\Theta_3(M) \cup \Pi$, if $\rho(X, X_i)$ tends to ∞ then, after passing to a subsequence of $\{X_i\}_i$, we can find an element x of M such that $\{X_i\}_i$ converges to x in the topology of compactification.*

Proof. We can suppose without loss of generality that $\{X_i\}_i \subset \Theta_3(M)$ or $\{X_i\}_i \subset \Pi$. We will prove this lemma in the case that $X = (a,b,c)$ and $X_i = (a_i, b_i, c_i)$, for all i, are in $\Theta_3(M)$. In fact there are three more possible cases, which are $X, X_i \in \Pi$, $X \in \Theta_3(M)$ while $X_i \in \Pi$ and, the last one, $X \in \Pi$ while $X_i \in \Theta_3(M)$. These other possible cases can be proved similarly. We can suppose without loss of generality that $\rho(X, X_i) = (ab|a_ib_i)$. Because, if not, we can exchange a or b with c, as well as a_i or b_i with c_i and change notation. Thus $(ab|a_ib_i) \to \infty$. Since M is a compactum we can also suppose, by passing to a subsequence, that $a_i \to x$ and $b_i \to y$ in M. Now if $x \neq y$, as a consequence of property (A5), we obtain $(ab|a_ib_i) \simeq (ab|xy) < \infty$, which gives a contradiction with $(ab|a_ib_i) \to \infty$. So $x = y$ and X_i converges to x in $\Theta_3(M) \cup M$ in the topology of compactification.

$\diamondsuit$

Lemma 5.3. *Any compact subset of $\Theta_3(M)$, in the topology of compactification, is bounded in $(\Theta_3(M) \cup M, \rho)$.*

Proof. We argue by contradiction. Let K be a compact subset of $\Theta_3(M)$ such that K is not bounded in $(\Theta_3(M) \cup M, \rho)$, i.e., there exist a sequence $\{X_i\}_i \subseteq K$ and an element $X \in K$ such that $\rho(X, X_i) \to \infty$. Thus by Lemma 5.2, after passing to a subsequence we can assume that there exists x in M such that X_i converges to x in the topology of compactification. So, this gives us a contradiction with the fact that M and K are disjoint compact sets of $\Theta_3(M) \cup M$ in the topology of compactification.

$\diamondsuit$

The following two lemmas will be used several times in our proofs of this section, (specifically to prove the principal result, namely the quasidensity of Π in $\Theta_3(M)$), as well as in section 7.

Lemma 5.4. *Let x be a bounded parabolic point. If x_i and y_i are two sequences tending respectively to x and $y \in M \backslash \{x\}$, then for all large enough i we have $(x|x_iy_i) \leq \xi + 1$ where ξ is the constant defined by property (A5).*

Proof. We choose a sequence of nested annuli $\{B^i_j\}_{j \in \{1,\dots,r_i\}}$ of maximal cardinality such that $r_i = (x|x_iy_i)$ and $\{x_i, y_i\} < B^i_1 < \cdots < B^i_{r_i} < \{x\}$ and we consider the compact set K in $M \backslash \{x\}$ given by Lemma 2.8 such that $(x|K) = 0$. Figure 1.5.1 gives an illustration for the compact K and the annuli B^i_j, where any annulus B corresponds to a grey domain which represents $M \setminus B^+ \cup B^-$.

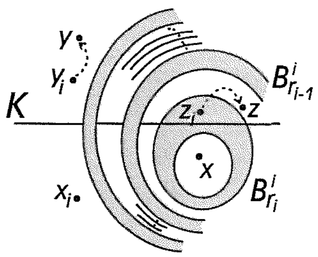

Figure 1.5.1

We note that there is no i and $j \in \{1, \dots, r_i\}$ such that $K < B^i_j < \{x\}$. Because if there were, we would obtain $(K|x) \geq 1$, which is not possible by the choice of K. So, there exists a sequence $\{z_i\}_i$ such that for all i, $z_i \in (M \setminus \operatorname{int}(B^i_{r_i})^-) \cap K$. Moreover we notice that for every $j \leq r_i - 1$ we have $z_i \in \operatorname{int}(B^i_j)^+$, i.e. $\{x_i, y_i\} < B^i_1 < \dots < B^i_{r_i-1} < \{x, z_i\}$. This implies $r_i - 1 \leq (x_iy_i|xz_i)$.

On the other hand for i large enough we have $(x_iy_i|xz_i) \leq \xi$. For, suppose that $(x_iy_i|xz_i) > \xi$ for infinitely many i. By passing to a subsequence we can suppose that z_i converges to $z \in M$ and $z \neq x$ since $z_i \in K \subseteq M\setminus\{x\}$. So, by property (A5) we obtain $(x_iy_i|xz_i) \simeq_\xi (xy|xz) = 0$, which gives a contradiction with $(x_iy_i|xz_i) > \xi$. Consequently there exists i_0 such that for all $i \geq i_0$, $(x_iy_i|xz_i) \leq \xi$ and so $(x|x_iy_i) = r_i \leq (x_iy_i|xz_i) + 1 \leq \xi + 1$.

$\diamondsuit$

For the next two lemmas, let x, y be two distinct bounded parabolic points and let $n = (x|y)$. Choose a sequence of nested annuli $\{A_k\}_{k\in\{1,\dots,n\}}$ and $\{x\} < A_1 < \dots < A_n < \{y\}$. Let y_i be a sequence of elements of Π tending to y. Then $(y|y_i)$ is finite and we can suppose that $y_i \in (A_n)^+$. For all i, we choose a sequence of nested annuli $\{B^i_j\}_{j\in\{1,\dots,r_i\}}$ of maximal cardinality such that $\{y_i\} < B^i_1 < \dots < B^i_{r_i} < \{y\}$ (so that $r_i = (y|y_i)$). We denote by s_i the maximal index such that $(B^i_{s_i})^- \subseteq \operatorname{int}(A_n)^+$. (Figure 1.5.2).

Lemma 5.5. *For i sufficiently large we have $(y|y_i) \leq s_i + \xi + 1$ and $(x|y_i) \geq n + s_i - 1$, where ξ is the constant defined by property (A5).*

Proof. We will prove $r_i - s_i \leq \xi + 1$. We can suppose $s_i < r_i$ since $r_i = s_i$ implies the inequality required. We know that for all $j \geq s_i + 1$, $(B^i_j)^- \cap (M \setminus \operatorname{int}(A_n)^+) \neq \emptyset$ and so there exists a sequence $\{w_i\}_i$ such that for all i, $w_i \in (B^i_{s_i+1})^- \cap (M \setminus \operatorname{int}(A_n)^+)$. We notice that $\{y_i, w_i\} < B^i_{s_i+1} < \dots < B^i_{r_i} < \{y\}$, i.e. $r_i - s_i \leq (y_iw_i|y)$. By passing to a subsequence we can suppose that w_i converges to w, where $w \in M\setminus\{y\}$ since $w_i \in M \setminus \operatorname{int}(A_n)^+$. Thus, by Lemma 5.4 applied to y, y_i and $w_i \to w \in M\setminus\{y\}$

in place of respectively x, x_i and $y_i \to y \in M \backslash \{x\}$, for large enough i we obtain $(y_i w_i|y) \leq \xi + 1$, that is to say $(y|y_i) = r_i \leq (y_i w_i|y) + s_i \leq \xi + 1 + s_i$.

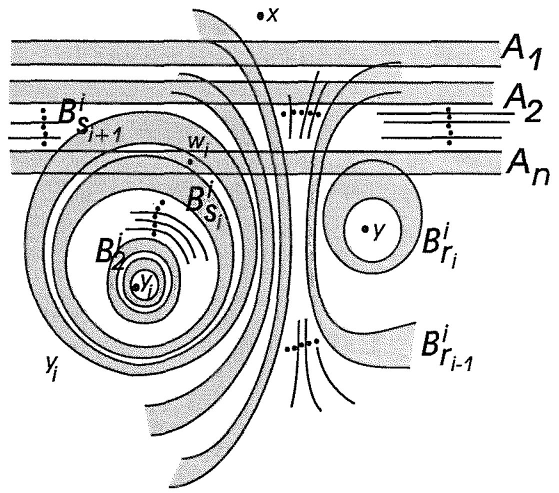

Figure 1.5.2

Now, we will show that $(x|y_i) \geq n + s_i - 1$. In fact, since $(B^i_{s_i})^- \subseteq \text{int}(A_n)^+$, for all $j \leq s_i - 1$, we have $M \backslash \text{int}(B^i_j)^+ \subset \text{int}(A_n)^+$. So $\{y_i\} < B^i_1 < \cdots < B^i_{s_i-1} < A_n < \cdots < A_1 < \{x\}$, which implies $n + s_i - 1 \leq (x|y_i)$.

$\diamondsuit$

Given y_i, z_i two sequences tending to y let $\{B^i_j\}_{j \in \{1,\ldots,r_i\}}$ be a sequence of nested annuli of maximal cardinality such that $r_i = (y|y_i z_i)$ and $\{y_i, z_i\} < B^i_1 < \cdots < B^i_{r_i} < \{y\}$. We denote by s_i the maximal index such that $(B^i_{s_i})^- \subseteq \text{int}(A_n)^+$. By the same argument of Lemma 5.5 we can see:

Lemma 5.6. *For i sufficiently large we have $(y|y_i z_i) \leq s_i + \xi + 1$ and $(x|y_i z_i) \geq n + s_i - 1$, where ξ is the constant defined by property (A5).*

Proposition 5.7. *Let $\mathcal{A}$ be a Γ-invariant symmetric annulus system constructed on $M = \Xi \cup \Pi$, satisfying (A1), (A2), (A3) such that the set of conical points, Ξ, is exactly the set of conical limit points. Assume that $\Pi \neq \emptyset$ and there are only finitely many Γ-orbit in Π. Then Π is quasidense in $\Theta_3(M) \cup \Pi$ provided that $(x|y) \geq \chi$ for all distinct $x, y \in \Pi$ and $\chi \geq \xi + 3$.*

We will see that taking the constant χ (as in A5) greater than $\xi + 3$ will be sufficient for the proof of Proposition 5.7. This also will be enough to prove Theorem 7.1. Later, it will be shown that we need to take it greater than $2\xi + 6$ to prove Theorem 8.2.

Proof. We suppose to the contrary that Π is not quasidense in $\Theta_3(M) \cup \Pi$, i.e, there exists a sequence $\{X_i\}_i$ in $\Theta_3(M)$ such that $\rho(X_i, \Pi) \to \infty$. Set $X_i = (a_i, b_i, c_i)$. Since ρ is Γ-invariant and there are only finite many Γ-orbit of bounded parabolic points, without loss of generality we can suppose that $\rho(X_i, \Pi) = \rho(X_i, x)$ where $x \in \Pi$ is fixed. Again without loss of generality we can suppose that $\rho(X_i, \Pi) = (a_i b_i | x)$. (If not we can change notation for $X_i = (a_i, b_i, c_i)$.) Now, x is a bounded parabolic point, so there exist $g_i \in \mathrm{Stab}_\Gamma(x)$ such that the sequence $g_i a_i$ stays in a compact set, K, of $M \backslash \{x\}$. Thus after passing to a subsequence we can suppose that $g_i a_i \to a \in K \subset M\backslash\{x\}$ and $g_i b_i \to b \in M$. In fact $a = b$ because otherwise, $(ab|x)$ being finite, we would obtain $(\{g_i a_i, g_i b_i\}|x)$ bounded, which would give a contradiction. (To see this we can use an efficient approximating tree $\mathcal{T}$ of $\{a, b, g_i a_i, g_i b_i, x\}$. For i large enough we will have $(\{a, g_i a_i\} : \{b, g_i b_i, x\})_\mathcal{T}$ and $(\{b, g_i b_i\} : \{a, g_i a_i, x\})_\mathcal{T}$, i.e. $(ab|x) \simeq (\{g_i a_i, g_i b_i\}|x)$.) Thus we set $a = b = y$ and suppose $g_i a_i$ and $g_i b_i$ converge to y. Since $g_i x = x$ and $g_i \Pi = \Pi$ we can simplify notation by replacing $g_i a_i, g_i b_i$ by a_i, b_i respectively. Thus a_i and b_i converge to y.

From here, we have two possibilities, either y is a conical limit point or it is a bounded parabolic point.

We suppose first that y is a conical limit point. We consider the line $L(y, x) = \{(y, x, z)|_{z \in M}\}$ in $\Theta_3(M)$.

Firstly using the definition of conical limit point given by Tukia [Tu2] , we can find a sequence $\{Y_j = (y, x, y_j)\}_j$ in $L(y, x)$ with $y_j \to y$ and $\gamma_j \in \Gamma$ such that for all j, $\gamma_j^{-1} Y_j$ stays a compact set K of $\Theta_3(M)$. So by Lemma 5.3 there is some $\zeta \geq 0$ such that for all j, $\rho(\gamma_j x, Y_j) = \rho(x, \gamma_j^{-1} Y_j) \leq \zeta$. (The equality is satisfied since ρ is Γ-invariant.)

Secondly, since y is a conical point, we know $(y|xy_0) = \infty$, i.e. there exists an infinite sequence of nested annuli which separate y and $\{x, y_0\}$. Thus since y_i converges to y, we can choose j_0 large enough to obtain $\rho(x, Y_{j_0}) = (yy_{j_0}|x) \gg \zeta$. To simplify notation we suppose $j_0 = 0$ and write $z = \gamma_0 x$. So we obtain $\rho(x, Y_0) \gg \zeta \geq \rho(\gamma_0 x, Y_0) = \rho(z, Y_0)$.

We aim to show that $\rho(x, X_i) \gg \rho(z, X_i)$ in order to obtain a contradiction with the fact that for all i $\rho(X_i, \Pi) = \rho(X_i, x)$ and therefore to prove that y cannot be a conical limit point.

Since a_i and b_i converge to y, by property (A3) we have $(ya_i : xy_0)$, $(ya_i : xz)$, $(yb_i : xy_0)$ and $(yb_i : xz)$ for i large enough. We consider an efficient approximating tree $(\mathcal{T}, \sigma)$ of $\{a_i, b_i, c_i, x, y, z, y_0\}$. Therefore by applying a few times Lemma 3.6 and using the efficiency of $\mathcal{T}$ we obtain $(\{y, a_i, b_i\} : \{y_0, x, z\})_\mathcal{T}$. (Figure 1.5.3 illustrates two of the possibilities for $\mathcal{T}$).

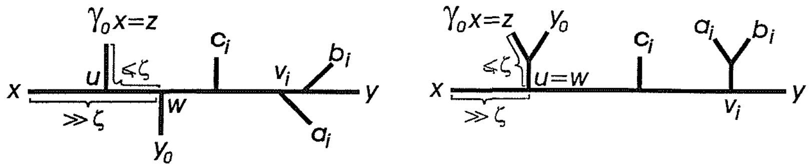

Figure 1.5.3

We write $v_i = \text{med}(a_i, b_i, c_i)$, $u = \text{med}(x, y, z)$ and $w = \text{med}(y_0, x, y)$. Using Lemma 4.2 we see that $\rho(x, X_i) \simeq \sigma(x, v_i)$ and $\rho(z, X_i) \simeq \sigma(z, v_i)$. (This also clarifies why the position of c_i in Figure 5.3 is of no importance and only the positions of x, z, u, w and v_i relative to each other are relevant to compare $\rho(x, X_i)$ and $\rho(z, X_i)$) But $v_i \in [a_i, b_i]$ and a_i, b_i satisfies $(\{y, a_i, b_i\} : \{y_0, x, z\})_{\mathcal{T}}$ in $(\mathcal{T}, \sigma)$. So we obtain $\sigma(x, v_i) = \sigma(x, w) + \sigma(w, v_i)$ and $\sigma(z, v_i) = \sigma(z, u) + \sigma(u, v_i)$.

On the other hand $\rho(z, Y_0) \leq \zeta$, which implies $\rho((x, y, z), Y_0) \leq \zeta$. In fact, $\rho((x, y, z), Y_0) = \max\{(xz|y_0 y), (yz|xy_0)\} \leq \max\{(z|y_0 y), (z|xy_0)\} \leq \zeta$. Therefore we obtain $\rho((x, y, z), Y_0) \simeq \sigma(u, w) \preceq \zeta$.

But, by the choice of Y_0 we have $\rho(x, Y_0) \gg \rho(z, Y_0)$, i.e., $\sigma(x, w) \gg \sigma(z, w)$ and, since $\sigma(u, w)$ is bounded we have also $\sigma(z, w) \simeq \sigma(z, u)$ and $\sigma(w, v_i) \simeq \sigma(u, v_i)$. Consequently we obtain $\rho(x, X_i) \simeq \sigma(x, v_i) = \sigma(x, w) + \sigma(w, v_i) \gg \sigma(z, w) + \sigma(w, v_i) \simeq \sigma(z, u) + \sigma(u, v_i) = \sigma(z, v_i) \simeq \rho(z, X_i)$. This concludes the proof for the case where y is a conical limit point, since we found $z = \gamma_0 x \neq x \in \Pi$ such that $\rho(x, X_i) \gg \rho(z, X_i)$ and this gives the required contradiction.

Now, we suppose y is a bounded parabolic point. We will compare $\rho(X_i, x)$ and $\rho(X_i, y)$. So we are interested in the quantities $(a_i b_i|y)$,$(a_i c_i|y)$ and $(b_i c_i|y)$. We have two cases to deal with $c_i \to y$ and $c_i \to c \neq y$

Since x and y are bounded parabolic points we have $(x|y) < \infty$. We choose a sequence of nested annuli $\{A_k\}_{k \in \{1,\ldots,n\}}$ such that $n = (x|y)$ and $\{x\} < A_1 < \cdots < A_n < \{y\}$.

Firstly we suppose that $c_i \to y$. Now, $\rho(X_i, y) = (y_i z_i|y)$ where $y_i \neq z_i \in \{a_i, b_i, c_i\}$. Since y_i, z_i converge to y, for large enough i, we can suppose that y_i and $z_i \in (A_n)^+$. For all such i, we choose a sequence of nested annuli $\{B^i_j\}_{j \in \{1,\ldots,r_i\}}$ of maximal cardinality such that $r_i = (y|y_i z_i)$ and $\{y_i, z_i\} < B^i_1 < \cdots < B^i_{r_i} < \{y\}$. We denote by s_i the maximal index such that $(B^i_{s_i})^- \subseteq \text{int}(A_n)^+$. Thus by applying Lemma 5.6 we see that for large enough i we obtain $(y|y_i z_i) \leq s_i + \xi + 1$ and $(y_i z_i|x) \geq n + s_i - 1$.

On the other hand $n = (x|y) \geq \chi \geq \xi+3$. Thus $(y_i z_i|x) \geq \xi+2+s_i > \xi+1+s_i \geq (y_i z_i|y)$ and $\rho(X_i, x) \geq (y_i z_i|x) > (y_i z_i|y) = \rho(X_i, y)$. Since $y \in \Pi$ this contradicts the fact that $\rho(X_i, \Pi) = \rho(X_i, x)$.

The second case is when $c_i \to c \neq y$. In this case by Lemma 5.4 applied to y, $a_i \to y$ and $c_i \to c \neq y$ in place of respectively, x, $x_i \to x$ and $y_i \to y \neq x$, we obtain $(a_i c_i|y) \leq \xi + 1$ for sufficiently large i. Likewise again by applying Lemma 5.4 we obtain $(b_i c_i|y) \leq \xi + 1$. So there remains only the quantity $(a_i b_i|y)$ to consider. But this is the same situation as in the first case since a_i and b_i converge to y. Thus we can apply the same argument of first case for $y_i = a_i$ and $z_i = b_i$. As a result we obtain $(a_i b_i|x) > (a_i b_i|y)$ and also $(a_i b_i|x) > \xi + 1$. Therefore, $\rho(X_i, x) = (a_i b_i|x) > \max\{\xi + 1, (a_i b_i|y)\} \geq \rho(X_i, y)$. This again gives a contradiction to the fact that $\rho(X_i, \Pi) = \rho(X_i, x)$.

$\diamondsuit$

6. THE CONSTRUCTION OF A SYSTEM OF ANNULI

In this section we describe a construction of a system of annuli on a metrisable compactum, M, using the action of a convergence group, Γ, on M. We use the notation of Section 5, namely $\Xi = \{\text{conical points}\}$ and $\Pi = M \backslash \Xi$. This system will satisfy all the properties required in Section 5 to enable us to obtain Π as a quasidense subset of $\Theta_3(M) \cup \Pi$. Let (M, δ) be a compact metric space and Γ be a convergence group acting on M such that M consists of only conical limit points and bounded parabolic points. Suppose that the quotient by Γ of the set of bounded parabolic points is finite. Let $\{x_1, \ldots, x_p\}$ be a set of orbit representatives of the set of bounded parabolic points.

Lemma 6.1. *If $\mathcal{A}$ is a Γ-invariant, symmetric system of annuli defined on M such that $\mathcal{A}/\Gamma$ is finite, then there are only finitely many annuli which separate any parabolic point x from any compact subset, K, of $M \backslash \{x\}$ of cardinality at least 2.*

Proof. Suppose to the contrary that there exists a parabolic point x, a compact set, K, of $M \backslash \{x\}$ and an infinite sequence of distinct annuli $A_i \in \mathcal{A}$ such that for all i, $K < A_i < \{x\}$. Since $\mathcal{A}/\Gamma$ is finite we can find an annulus A and $g_i \in \Gamma$ such that, after passing to a subsequence of $\{A_i\}$, we have $A_i = {g_i}^{-1} A$. In other words $g_i K < A < \{g_i x\}$ for all i. On the other hand since Γ is a convergence group, on passing to a subsequence, we can suppose that there are two points $a, b \in M$ such that g_i converges to b locally uniformly on $M\backslash\{a\}$ when i tends to ∞.

As the cardinality of K is more than two without loss of generality we can choose an element y of K such that $y \neq a$. Therefore $g_i y$ converges to b, which belongs to A^- since $g_i y \in A^-$. Now, we distinguish two possible cases, either $a = x$ or $a \neq x$. If $a \neq x$ then we have $g_i x$ converges to b, which belongs to A^+ since $g_i x \in A^+$. Thus we obtain $b \in A^+ \cap A^-$ which is impossible. So the only possible case is when $a = x$. Now since M is compact we can also suppose that $g_i x$ converges to z where $z \in A^+$ as $g_i x$ belongs A^+. Thus $z \neq b$ because $b \in A^-$ and $z \in A^+$. In summary we see that g_i converges to b locally uniformly on $M\backslash\{x = a\}$ and $g_i x \to z \neq b$. But this is exactly the property that x is a conical limit point, and by Proposition 5.1 we know that a conical limit point can not be a parabolic point, which gives us the contradiction.

$\diamondsuit$

The next lemma is a corollary of Lemma 7.2 given in [Bo4].

Lemma 6.2. *If $\mathcal{A}$ is a Γ-invariant, symmetric system of annuli defined on M such that $\mathcal{A}/\Gamma$ finite then there are only finitely many annuli which separate x, y from z, w where $x \neq y, z \neq w$ are four points of M.*

Lemma 6.3. *There exists a compact set K_0 of $M\backslash\{x_1, \dots, x_p\}$ of cardinality at least 2 such that $\mathrm{Stab}_\Gamma(x_i)K_0 = M\backslash\{x_i\}$ for all i.*

Proof. Since x_1 is bounded parabolic point we can find a compact set $L_1 \subset M\backslash\{x_1\}$ such that $M\backslash\{x_1\} = \mathrm{Stab}_\Gamma(x_1)L_1$. Moreover we know that $\mathrm{Stab}_\Gamma(x_1)$ acts on $M\backslash\{x_1\}$ properly discontinuously. We consider the compact set $L_1' = L_1 \cup \{x_2, \dots, x_p\} \subset M\backslash\{x_1\}$. We can find a $g \in \mathrm{Stab}_\Gamma(x_1)$ such that $gL_1' \cap L_1' = \emptyset$. Hence we find a compact subset, $K_1 = gL_1'$, of $M\backslash\{x_1, \dots, x_p\}$ such that $M\backslash\{x_1\} = \mathrm{Stab}_\Gamma(x_1)K_1$. Likewise, we perform the same construction for all $i \in \{2, \dots, p\}$ to obtain the compact sets $K_i \in M\backslash\{x_1, \dots, x_p\}$ such that $M\backslash\{x_i\} = \mathrm{Stab}_\Gamma(x_i)K_i$. Consequently the compact set $K_0 = \bigcup_i K_i$ of $M\backslash\{x_1, \dots, x_p\}$ satisfies $\mathrm{Stab}_\Gamma(x_i)K_0 = M\backslash\{x_i\}$ for all i.

$\diamondsuit$

Let $A = (A^-, A^+)$ be an annulus. We write $\lambda(A) = \min\{\mathrm{diam}(A^-), \mathrm{diam}(A^+)\}$ and $\mu(A) = \delta(A^-, A^+)$. Now, $\Theta_2(M)$ is the set of distinct pairs of M, i.e M^2 minus the diagonal. We denote by $\Theta_2(M, n)$ the set $\{(x, y) \in \Theta_2(M) \mid \delta(x, y) \geq 1/n\}$. We note that $\Theta_2(M) = \bigcup_n \Theta_2(M, n)$. Thus $\{\Theta_2(M, n)\}_n$ gives a exhaustion of $\Theta_2(M)$ by compact sets.

Lemma 6.4. *If $\mathcal{A}$ is a Γ-invariant system of annuli defined on M such that $\mathcal{A}/\Gamma$ is finite, then for all $\epsilon > 0$ the set $\{A \in \mathcal{A} \mid \lambda(A) \geq \epsilon\}$ is finite.*

Proof. Suppose to the contrary that there exist an infinite sequence of distinct annuli $\{A_i\}_i$ such that $\lambda(A_i) \geq \epsilon$. Since $\mathcal{A}/\Gamma$ is finite after passing to a subsequence we can suppose that $A_i = g_i A$ where $\{g_i\}_i$ is a sequence of distinct element of Γ and A is a fixed annulus of $\mathcal{A}$. Moreover, after passing to a subsequence of $\{g_i\}_i$, we can find $a, b \in M$ such that g_i converges to b uniformly on $M \backslash \{a\}$ as i tends to ∞. Without loss of generality we can suppose that $a \notin A^-$. Thus $g_i A^- \to b$, which implies that for i sufficiently large, $\operatorname{diam} g_i A^- < \epsilon$. This gives a contradiction with $\lambda(A_i) = \lambda(g_i A) \geq \epsilon$

$\diamondsuit$

The following gives an annulus sytem of the type requierd in Section 5.

Proposition 6.5. *Suppose a group Γ acts as a convergence group on a metrisable compactum M such that the action is geometrically finite. Suppose that the quotient of the set of bounded parabolic points under Γ is finite. We fix a $\chi > 0$. Then there exists a symmetric, Γ-invariant annulus system defined on M such that $\mathcal{A}$ satisfies (A1), (A2), (A3) with $\Xi = \{\text{conical limit points}\}$, $\Pi = \{\text{bounded parabolic points}\}$ and such that $(x|y) \geq \chi$ for every distinct $x, y \in \Pi$.*

Proof. The construction is in three steps. The first step is the construction of an annulus system with finite quotient which will ensure that any two distinct bounded parabolic points are separated at least by χ annuli. The second step will be the construction, using an inductive argument, of another annulus system. This system will ensure that any two points of M are separated at least by one annulus. In addition, for K a fixed compact set of M of cardinality at least 2, no annuli separate K from any point in some fixed orbit transversal. In the last step we put together the systems constructed in first and second steps and the annulus system thus obtained has the properties required.

Step 1:

Let $\{x_1, \dots, x_p\}$ be a fixed set of orbit representatives of the set of bounded parabolic points. Let K_0 be a compact set defined as in Lemma 6.3. We choose an annulus A_1^1 such that $K_0 < A_1^1 < \{x_1\}$ and A_1^1 satisfies $\lambda(gA_1^1) \leq \mu(A_1^1)$ for all $g \in \Gamma$. The choice of A_1^1 can be justified by using Lemma 6.4. In fact if $g \in \Gamma$ satisfies $\lambda(gA_1^1) > \mu(A_1^1)$, then we can choose an open neighbourhood S of gx_1 such that $\overline{S} \subset g(A_1^1)^+$ and $\operatorname{diam}(\overline{S}) \leq \mu(A_1^1)$. Thus, replacing A_1^1 with $((A_1^1)^-, g^{-1}\overline{S})$ and denoting it by A_1^1 we obtain $\lambda(gA_1^1) \leq \mu(A_1^1)$. Note that $\mu(A_1^1)$ can only increase in this process. Moreover, since by Lemma 6.4 the set $\{g \in \Gamma \mid \lambda(gA_1^1) > \mu(A_1^1)\}$ is finite, we need to do the same construction only finitely many times to obtain A_1^1 as required.

Now we choose another annulus A_2^1 such that $K_0 < A_1^1 < A_2^1 < \{x_1\}$ and A_2^1 satisfies $\lambda(gA_2^1) \leq \min\{\mu(A_1^1), \mu(A_2^1)\}$ for all $g \in \Gamma$. In fact the same argument of the previous construction gives the annulus A_2^1, where $M \backslash \operatorname{int}((A_1^1)^+)$ plays the role of K_0 (Noting that we only required that $x_1 \notin K_0$).

We can continue to construct such annuli until we have $K_0 < A_1^1 < \cdots < A_\chi^1 < \{x_1\}$ with $\lambda(gA_j^1) \leq \min\{\mu(A_k^1) \mid k \leq j\}$ for all $j \in \{1, \ldots, \chi\}$ and all $g \in \Gamma$.

We perform the same construction, inductively, for all $x_i \in \{x_1, \ldots, x_p\}$. In other words, for all i, we construct a sequence of annuli $\{A_j^i\}_{j \in \{1,\ldots,\chi\}}$ with $K_0 < A_1^i < \cdots < A_\chi^i < \{x_i\}$ such that for all $g \in \Gamma$,

$$\lambda(gA_j^i) \leq \min(\{\mu(A_k^i) \mid k \leq j\} \cup \{\mu(A_k^l) \mid 1 \leq k \leq \chi,\ l < i\}). \quad (\nabla)$$

We write $\mathcal{C} = \{gA_j^i, -gA_j^i \mid i \in \{1, \ldots, p\}, j \in \{1, \ldots, \chi\}, g \in \Gamma\}$. Clearly $\mathcal{C}$ is Γ-invariant, symmetric and $\mathcal{C}/\Gamma$ is finite. We will write, for P and R compact subsets of M, $(P|R)_{\mathcal{C}}$ to mean $(P|R)$ defined with respect to $\mathcal{C}$.

Step 2:

We write $\mu = \min\{\mu(A_j^i) \mid i \in \{1, \ldots, p\}, j \in \{1, \ldots, \chi\}\}$ and fix a compact subset K of $M \backslash \{x_1, \ldots, x_p\}$, which need not be equal to K_0 for this construction but can be chosen to be equal.

We will use a recursive argument on $n \in \mathbf{N}$ to construct a sequence of Γ-invariant, symmetric annulus systems, $\mathcal{B}(n)$, with $\mathcal{B}(n)/\Gamma$ finite. For $n \geq 1$ let μ_n' be the minimal value of $\{\max \mu(gB) \mid g \in \Gamma\}$ as B varies in $\bigcup_{m=1}^n \mathcal{B}(m)$. We set $\mu_0' = 1$. Note that since $\bigcup_{m=1}^n \mathcal{B}(m)/\Gamma$ is finite μ_n' is strictly greater than 0 for all n.

For all $n \geq 1$, we will assume, inductively, that $\mathcal{B}(n)$ satisfies

$(*)$ there exists a finite set, $\{B_i^n\}_{i=1}^{q_n}$, of Γ-orbit representative of $\mathcal{B}(n)$ such that $(\forall i)$ $\mu(B_i^n) \geq 1/(n+1)$.

$(**)$ $\forall B \in \mathcal{B}(n)$ $\lambda(B) \leq \min\{\mu, \mu_{n-1}', 1/(n+1)\}$.

$(***)$ $\forall i \in \{1, \ldots, p\}$ there is no element of $\mathcal{B}(n)$ which separates x_i and K.

$(****)$ if $(x, y) \in \Theta_2(M, n)$ then some element of $\mathcal{B}(n)$ separates x and y.

We suppose that we have achieved this up to a fixed n. We describe the construction for $\mathcal{B}(n+1)$.

Given an element $\omega = (x, y)$ of $\Theta_2(M, n+1)$ we can choose an annulus $B(\omega)$ with $x < B(\omega) < y$ such that $B(\omega)$ satisfies:

(a) $\mu(B(\omega)) \geq 1/(n+2)$,

(b) $\forall g \in \Gamma \ \ \lambda(gB(\omega)) \leq \min\{\mu, \mu'_n, 1/(n+2)\}$,

(c) $\forall i \in \{1, \ldots, p\}$ there is no $g \in \Gamma$ such that $gB(\omega)$ separates x_i and K.

The choice of $B(\omega)$ can be justified by the similar arguments as used in the first step. Clearly, ω being an element of $\Theta_2(M, n+1)$, $B(\omega)$ can be chosen to satisfy $\mu(B(\omega)) \geq 1/(n+2)$. Moreover, noting by Lemma 6.4 that the set $\{g \in \Gamma \mid \lambda(gB(\omega)) \geq \epsilon\}$ for $\epsilon > 0$ is finite, we can choose $B(\omega)$ satisfying (b) in the same manner described for the construction of A_1^1 in the first step. We still need to verify (c). To do this we use Lemma 6.1. Let g be an element of Γ such that $gB(\omega)$ separates x_i and K. Thus we have either $K \cup \{gx\} < gB(\omega) < \{x_i, gy\}$ or $\{x_i, gx\} < gB(\omega) < K \cup \{gy\}$. Without loss of generality we can suppose the former holds. We choose an open neighbourhood S of gx such that $\overline{S} \subset g(B(\omega))^-$ and $K \not\subseteq \overline{S}$. Now, replacing $B(\omega)$ with $(g^{-1}\overline{S}, B(\omega)^+)$ and denoting it by $B(\omega)$ we see that $gB(\omega)$ does not separate x_i and K. On the other hand, for all $i \in \{1, \ldots, p\}$, by Lemma 6.1, the set $\{g \in \Gamma \mid gB(\omega)$ separates x_i and $K\}$ is finite. So considering only finitely many $i \in \{1, \ldots, p\}$ and $\{g \in \Gamma \mid gB(\omega)$ separates x_i and $K\}$ we can obtain $B(\omega)$ as required. Evidently, after these corrections, $B(\omega)$ still satisfies (a) and (b).

We can write $\Theta_2(M, n+1) = \bigcup_{\omega \in \Theta_2(M,n+1)} (\operatorname{int} B(\omega)^- \times \operatorname{int} B(\omega)^+)$. Therefore since $\Theta_2(M, n+1)$ is compact, there is a finite set $\{w_1, \ldots, w_{q_{n+1}}\} \subseteq \Theta_2(M, n+1)$, so that $\Theta_2(M, n+1) = \bigcup_{i=1}^{q_{n+1}} (\operatorname{int} B(\omega_i)^- \times \operatorname{int} B(\omega_i)^+)$. We write $B_i^{n+1} = B(\omega_i)$ for all $i \in \{1, \ldots, q_{n+1}\}$.

Let $\mathcal{B}(n+1) = \{gB_i^{n+1}, -gB_i^{n+1} \mid i \in \{1, \ldots, q_{n+1}\}, g \in \Gamma\}$. Clearly $\mathcal{B}(n+1)$ is Γ-invariant, symmetric and $\mathcal{B}(n+1)/\Gamma$ is finite. Also as a direct consequence of the properties $(a), (b), (c)$ and the construction, $\mathcal{B}(n+1)$ satisfies all the properties $(*)$–$(* * **)$ demanded.

So, after setting $B(0) = \emptyset$ this inductive construction proceeds for all n. We write $\mathcal{B} = \bigcup_n \mathcal{B}(n)$. If P and R are compact sets of M, denote $(P|R)$ defined with respect to $\mathcal{B}$ by $(P|R)_{\mathcal{B}}$. We note that if $(x, y) \in \Theta_2(M)$ then there is an annulus of $\mathcal{B}$ (specifically an annulus of $\mathcal{B}(n)$ where $n > 1/\delta(x, y)$) which separates x and y. Also we note that for all $i \in \{1, \ldots, p\}$ there is no annulus of $\mathcal{B}$ which separates x_i and K.

Step 3:

We consider the system of annuli $\mathcal{A} = \mathcal{C} \cup \mathcal{B}$ which is Γ-invariant and symmetric.

We write $(P|R) = (P|R)_{\mathcal{A}}$ where P and R are compact sets of M. We will verify respectively that every pair of distinct bounded parabolic points x, y satisfy $(x|y) \geq \chi$, $\Pi = \{\text{bounded parabolic points}\}$, $\Xi = \{\text{conical limit points}\}$ and that $\mathcal{A}$ satisfies (A1), (A2) and (A3).

1. $(x|y) \geq \chi$ for all $x \neq y \in \{\text{bounded parabolic points}\}$.

Let x, y be two distinct bounded parabolic points. There are $g \in \Gamma$ and x_i in the fixed orbit transversal, $\{x_1, \ldots, x_p\}$, such that $x = gx_i$. Since $M \backslash \{x_i\} = \mathrm{Stab}_\Gamma(x_i)K_0$, there is $h \in \mathrm{Stab}_\Gamma(x_i)$ such that $h^{-1}g^{-1}y \in K_0$. But by the construction of $\mathcal{C}$ we have $K_0 < A_1^i < \cdots < A_\chi^i < \{x_i\}$. As a consequence $(x_i|h^{-1}g^{-1}y) \geq \chi$ which implies $(hx_i|g^{-1}y) = (x_i|g^{-1}y) \geq \chi$ and so $(gx_i|y) = (x|y) \geq \chi$, (noting that $\mathcal{A}$ is Γ-invariant).

2. $\mathcal{A}$ satisfies (A1), i.e., there do not exist four distinct points $x, y, z, w \in M$ such that $(xy|zw) > 0$ and $(xz|yw) > 0$.

Suppose to the contrary that there are four distinct points $x, y, z, w \in M$ such that $(xy|zw) > 0$ and $(xz|yw) > 0$, i.e, there are two annuli A, B of $\mathcal{A}$ such that $\{x, y\} < A < \{z, w\}$ and $\{x, z\} < B < \{y, w\}$. There are three possible cases to deal with.

The first case is when A, B are both in $\mathcal{C}$. Hence, since $\mathcal{C}/\Gamma = \{A_j^i, -A_j^i \mid i \in \{1, \ldots, p\}, j \in \{1, \ldots, \chi\}\}$ we can find $j, k \in \{1, \ldots, \chi\}$, $i, l \in \{1, \ldots, p\}$ and $g \in \Gamma$ such that after translating A by an element of Γ we can assume $A = A_j^i$ or $A = -A_j^i$ and $B = hA_k^l$ or $B = -hA_k^l$. Now, either $i = l$ and in this case we can suppose without loss of generality that $j \leq k$, or $i \neq l$ and in this case we can suppose without loss of generality that $i < l$. In both case the reasoning will be same. By (∇) we know that $\lambda(hA_k^l) = \lambda(-hA_k^l) \leq \mu(A_j^i) = \mu(-A_j^i)$. Now, if $B = hA_k^l$ without loss of generality we can suppose that $\lambda(hA_k^l) = \mathrm{diam}(h(A_k^l)^-)$ and similarly if $B = -hA_k^l$ we can suppose that $\lambda(-hA_k^l) = \mathrm{diam}(-h(A_k^l)^-)$. In both cases we obtain $\delta(x, z) < \lambda(B) \leq \mu(A) < \delta(x, z)$, which gives a contradiction.

The second case is when A, B are both in $\mathcal{B}$. Thus there is $n, m \in \mathbf{N}$ such that $A \in \mathcal{B}(n)$ and $B \in \mathcal{B}(m)$. Without loss of generality we can suppose that $m \leq n$. Also after translating B by an element of Γ we can suppose, depending on whether $m < n$ or $m = n$, that either $\mu(B) = \max\{\mu(gB) \mid g \in \Gamma\}$ or $B \in \{B_i^n\}_{i=1}^{q_n}$. In the former case we have $\mu(B) \geq \mu'_m \geq \mu'_{n-1}$. In the latter case, by $(*)$, $\mu(B) \geq 1/(n+1)$. It follows that $\mu(B) \geq \min\{\mu'_{n-1}, 1/(n+1)\}$. Moreover since $A \in \mathcal{B}(n)$ we have also, by $(**)$, that $\lambda(A) \leq \min\{\mu, \mu'_{n-1}, 1/(n+1)\}$. Without loss of generality we can suppose that $\lambda(A) = \mathrm{diam}(A^-)$. We obtain $\delta(x,y) < \lambda(A) \leq \min\{\mu, \mu'_n, 1/(n+1)\} \leq \mu(B) < \delta(x,y)$, which gives a contradiction.

The third case is either $A \in \mathcal{C}$ and $B \in \mathcal{B}$ or else $A \in \mathcal{C}$ and $B \in \mathcal{B}$. Without loss of generality we can assume that A is in $\mathcal{C}$ and B is in $\mathcal{B}$. Thus after translating A by an element of Γ we can suppose that there are $i \in \{1, \ldots, p\}$, $j \in \{1, \ldots, \lambda\}$ such that $A = A^i_j$ or $A = -A^i_j$ and also there is $n \in \mathbf{N}$ such that $B \in B(n)$. Moreover without loss of generality, we can suppose that $\lambda(B) = \mathrm{diam}(B^-)$. Therefore $\delta(x,z) < \lambda(B) \leq \mu \leq \mu(A) < \delta(x,z)$, which gives a contradiction.

3. $\mathcal{A}$ satisfies (A2), i.e., If $x \neq y$ and $z \neq w$ in M then $(xy|zw) < \infty$.

We are interested only the case when x, y, z, w are distinct. Let ϵ be the minimum of $\delta(x,y)$ and $\delta(z,w)$. We choose an $n \in \mathbf{N}$ such that $n > 1/\epsilon$. We notice that if $A \in \mathcal{A}$ satisfies $\{x,y\} < A < \{z,w\}$ then $\lambda(A) \geq \epsilon > 1/n$ and so $A \in \bigcup_{m=1}^n \mathcal{B}(m) \cup \mathcal{C}$. But $(\bigcup_{m=1}^n \mathcal{B}(m) \cup \mathcal{C})/\Gamma$ is finite and so by Lemma 6.2 we can find only finitely many annuli separating $\{x,y\}$ and $\{z,w\}$. i.e $(xy|zw) < \infty$.

4. $\mathcal{A}$ satisfies (A3), i.e. for every $x, y, z \in M$ where $x \notin \{y, z\}$ there exists a neighbourhood N_x^{yz} of x such that for every $w \in N_x^{yz}$ we have $(xy|wz) = 0$ and $(xz|wy) = 0$.

Suppose to the contrary that there is a sequence, w_i, in $M \backslash \{x,y,z\}$ converging to x such that for all i either $(xy|w_iz) \neq 0$ or $(xz|w_iy) \neq 0$. Thus without loss of generality after passing to a subsequence we can assume that for all i, $(xy|w_iz) \neq 0$. In other words there is an infinite sequence of annuli $\{A_i\}_i$ such that $\{x,y\} < A_i < \{z, w_i\}$ for all i. Let ϵ be the minimum of $\delta(x,y)$ and $\delta(x,z)$. We choose an $n \in \mathbf{N}$ such that $n > 2/\epsilon$. We notice that for large enough i, $\delta(x, w_i) \leq \epsilon/2$ thus $\delta(z, w_i) \geq \epsilon/2$ and so $\lambda(A_i) \geq \epsilon/2 > 1/n$. This implies by Property (**) that for all large enough i, $A_i \in \bigcup_{m=1}^n \mathcal{B}(m) \cup \mathcal{C}$. Now, since $(\bigcup_{m=1}^n \mathcal{B}(m) \cup \mathcal{C})/\Gamma$ is finite, we can find an annulus A and $g_i \in \Gamma$ such that, again after passing to a subsequence of $\{A_i\}$, we have $A_i = {g_i}^{-1}A$. In other words $\{g_ix, g_iy\} < A < \{g_iw_i, g_iz\}$ for all i. But we can also suppose that there are two points $a, b \in M$ such that g_i converges to b locally uniformly on $M \backslash \{a\}$ when i tends to ∞. Thus if $x = a$ then $g_iy \to b$ and $g_iz \to b$. As a result since $g_iy \in A^-$ and $g_iz \in A^+$, we see that b belongs to $A^- \cap A^+$, which is impossible. So, $x \neq a$ and then $g_ix \to b$. Since $w_i \to x$, we have $g_iw_i \to b$. Again we obtain a contradiction, because $g_ix \in A^-$ and $g_iw_i \in A^+$, which implies b belongs to $A^+ \cap A^-$.

5. {bounded parabolic points} $\subseteq \Pi$

Let x be a bounded parabolic point. We will show that x cannot be a conical point with respect to the annulus system $\mathcal{A}$. Firstly since $\{\text{bounded parabolic points}\}/\Gamma$ is finite there exist $g \in \Gamma$ and x_i in the fixed orbit transversal, $\{x_1, \dots, x_p\}$, such that $x = g^{-1}x_i$. Suppose, for contradiction, that there is a compact set L of cardinality at least 2 of M such that $(x|L) = \infty$. Since $\mathcal{A}$ is Γ-invariant, this implies $(x_i|gL) = \infty$. In addition, since $\mathcal{A} = \mathcal{C} \cup \mathcal{B}$, either $(x_i|gL)_{\mathcal{C}} = \infty$ or $(x_i|gL)_{\mathcal{B}} = \infty$. But $\mathcal{C}/\Gamma$ is finite and by Lemma 6.1 we see that $(x_i|gL)_{\mathcal{C}} < \infty$. Thus $(x_i|gL)_{\mathcal{B}} = \infty$ holds, and using Lemma 2.7 we see that every compact L' of $M\backslash\{x\}$ satisfies $(x_i|L')_{\mathcal{B}} = \infty$. In particular $(x_i|K)_{\mathcal{B}} = \infty$ where K is the compact fixed in the construction of $\mathcal{B}$ in Step 2. This gives a contradiction with the fact that for all $i \in \{1, \dots, p\}$ there is no annulus of $\mathcal{B}$ which separates x_i and K.

6. $\{\text{conical limit points}\} \subseteq \Xi$

Let x be a conical limit point. We will show that for any compact set L of $M\backslash\{x\}$ we have $(x|L) > 0$. In fact by the definition of conical limit points there are two distinct points, a, b, in M and a sequence, $\{g_i\}_i \subseteq \Gamma$ such that $g_i x$ converges to a and g_i converges locally uniformly to b on $M\backslash\{x\}$. Also we know that there is an annulus A of $\mathcal{B}$ such that $a < A < b$, (namely, an annulus of $\mathcal{B}(n)$ where $n > 1/\delta(a,b)$). Hence for large enough i we obtain $g_i x < A < g_i L$ and so $x < g_i^{-1}A < L$. In other words, for all L compact subsets of $M\backslash\{x\}$ we have $(x|L) > 0$. Thus, as a direct result of Lemma 2.8, we obtain $x \in \Xi$, i.e., x is conical point.

As a consequence of the last two verifications, we note that since $M = \Xi \cup \Pi = \{\text{conical limit points}\} \cup \{\text{bounded parabolic points}\}$ we have in fact $\{\text{conical limit points}\} = \Xi$ and $\{\text{bounded parabolic points}\} = \Pi$.

$\diamondsuit$

7. RELATIVELY HYPERBOLIC GROUPS

The notion of "a relatively hyperbolic group" was defined by Gromov [Gro2]. This is a group which is word hyperbolic relative to some infinite subgroups, namely "peripheral subgroups" in the terminology of Bowditch in [Bo7]. We gave in the Preface two equivalent definitions of relative hyperbolicity. The equivalence of these definitions has been proved by Bowditch [Bo7]. The first definition is a modified formulation of the original definition introduced by Gromov. It gives a dynamical characterisation of relatively hyperbolic groups in terms of a group action on a hyperbolic space, while the second definition characterises relative hyperbolicity in terms of a group action on a "fine hyperbolic (Γ, V)-graph". We mean by a *(Γ, V)-graph* a connected Γ-invariant graph with vertex set V and with finite quotient under the action of Γ. We say that a graph is *fine* if there are only finitely many circuits of a given length containing any given edge. Thus, the second definition of relative hyperbolicity can be alternatively formulated in terms of an action on a fine hyperbolic (Γ, V)-graph with finite edge stabilisers and finitely generated vertex stabilisers. For further discussion of these notions, such as fineness and a hyperbolic Γ-set, see [Bo7]. These notions were introduced by Bowditch in order to prove the equivalence of Definition 1 and Definition 2 and to analyse further the theory of relatively hyperbolic groups.

The main aim of this section is to prove the following theorem.

Theorem 7.1. *Suppose that M is a metrisable compactum, and Γ is a convergence group acting on M such that the action is geometrically finite. Suppose also that the quotient of the set of bounded parabolic points under Γ is finite and the stabiliser of each bounded parabolic point is finitely generated. Then Γ is hyperbolic relative to the set of its maximal parabolic subgroups.*

In the rest of this section we give a construction of a connected graph $\mathcal{K}$ such that Γ acts on $\mathcal{K}$ satisfying all the properties required in definition 2. In other words we will obtain the set of bounded parabolic points as a hyperbolic Γ-set. The hypothesis that the stabiliser of each bounded parabolic point is finitely generated does not play any role in the construction of the graph, it is there merely to satisfy hypothesis (4).

Before we start the construction of $\mathcal{K}$, we recapitulate the results and the constants under discussion in the preceding sections, in order to make clear the choice of the constants in the following arguments. In section 6, under the hypotheses of Theorem 7.1, we constructed, for any given $\chi > 0$, a Γ-invariant symmetric annulus system $\mathcal{A}$ satisfying (A1), (A2), (A3) with $\Xi = \{\text{conical limit points}\}$ and $\Pi = \{\text{bounded parabolic points}\}$ such that each distinct pair x, y of Π are separated by at least χ nested annuli. Thus, by Lemma 3.7, $\mathcal{A}$ satisfies property (A5) for the constant ξ. We saw by Proposition 2.1 that the constant of path hyperbolicity, κ, of the induced crossratio, is universally defined and fixed for such a system of annuli. The same applies to the constant ξ of property (A5) since it depends only κ (Lemma 3.7). In other words κ, ξ are universal constants. We note that we have no reason to suppose that the neighbourhood given in property (A3) (for any $x \in M$

and $y \neq z \in M\{x,y\}$) is uniformly defined, so it may depend on the constant χ. However this will not effect the reasoning since the constant ξ is universal, and by construction we can set χ sufficiently large, namely ($\geq \xi + 3$), to obtain (Π, ρ) as a quasidense subset of $(\Theta_3(M) \cup \Pi, \rho)$ (see Lemma 5.7). Thus, by Lemma 4.5, (Π, ρ) is η-hyperbolic path quasimetric space, where η depends, a priori, on the constant of quasidensity and hence χ. The constant η is only used in Lemma 7.2, Lemma 7.5 and case 2 of Lemma 7.6 where in each case its size relative to χ (or to any other constant) is not important.

We define $\mathcal{K}$ equipped with a combinatorial metric, *dist*, as follows. Let $V(\mathcal{K}) = \Pi$ and let $x, y \in V(\mathcal{K})$ be the endpoints of an edge in $E(\mathcal{K})$ if and only if there is no $z \in \Pi \backslash \{x, y\}$ such that $\rho(x,z) < \rho(x,y)$ and $\rho(y,z) < \rho(x,y)$.

Lemma 7.2. *There exists a constant $\zeta > 0$ such that if x, y are adjacent in $E(\mathcal{K})$ then $\rho(x,y) \leq \zeta$.*

(Here ζ depends on η and hence may depend on χ.)

Proof. Given two distinct points $x, y \in \Pi$ with $\rho(x,y) = p > 2\eta + 1$, we choose r such that $\eta < r < p - \eta$. Since (Π, ρ) is an η-hyperbolic path quasimetric space, there exists $z \in \Pi$ such that $\rho(x,z) = (x|z) \simeq_\eta r$ and $\rho(y,z) = (y|z) \simeq_\eta p - r$. So $1 \leq r - \eta \leq (x|z)$, which implies $x \neq z$, and $1 \leq p - r - \eta \leq (y|z)$, which implies $y \neq z$. Moreover $\rho(x,z) \leq r + \eta < p$ and $\rho(y,z) \leq p - r + \eta < p$. Hence we find an element, namely z, of Π satisfying $\rho(x,z) < \rho(x,y)$ and $\rho(y,z) < \rho(x,y)$. As a result x, y are not adjacent in $\mathcal{K}$ and we can set $\zeta = 2\eta + 1$.

$\diamondsuit$

Given an edge e of a graph $\mathcal{K}$ we will denote by $\mathrm{end}(e)$ the extremities of e. Given $x, y \in \Pi$, $[x, y]$ refers to a geodesic connecting x, y in $\mathcal{K}$.

Lemma 7.3. *For all $r > 0$ if $\rho(x,y) \leq r$ where $x, y \in \Pi$, then $dist(x,y) \leq 2^r$.*

Proof. We will proceed by induction. Suppose that $\rho(x,y) \leq r$. Then either $x, y \in \mathrm{end}(e)$ where $e \in E(\mathcal{K})$ or there exists $z_1 \in \Pi \backslash \{x, y\}$ such that $\rho(x, z_1) < \rho(x,y)$ and $\rho(y, z_1) < \rho(x,y)$, which implies in particular $\rho(x, z_1) \leq r - 1$ and $\rho(y, z_1) \leq r - 1$. In the second case we argue similarly for $\{x, z_1\}$ and $\{y, z_1\}$. Thus either $x, z_1 \in \mathrm{end}(e)$ where $e \in E(\mathcal{K})$ or there exists $z_2 \in \Pi \backslash \{x, z_1\}$ such that $\rho(x, z_2) < \rho(x, z_1)$ and $\rho(z_1, z_2) < \rho(x, z_1)$, which implies that $\rho(x, z_2) \leq r - 2$ and $\rho(z_1, z_2) \leq r - 2$. Likewise either $y, z_1 \in \mathrm{end}(e)$ where $e \in E(\mathcal{K})$ or there exists $z_3 \in \Pi \backslash \{y, z_1\}$ such that $\rho(y, z_3) < \rho(y, z_1)$ and $\rho(z_1, z_3) < \rho(y, z_1)$. Again we note in the latter case that $\rho(y, z_3) \leq r - 2$ and $\rho(z_1, z_3) \leq r - 2$.

We can continue this inductive process, decreasing the distance in each step. In fact this process ends when two points are adjacent or equal. Thus, we can have at most 2^r steps, and x, y are connected in $\mathcal{K}$ by an arc of length at most 2^r. In other words $dist(x, y) \leq 2^r$.

◇

As a direct result of the previous lemma we obtain:

Lemma 7.4. *$\mathcal{K}$ is connected.*

Lemma 7.5. *The inclusion $(\Pi, \rho) \hookrightarrow (\mathcal{K}, dist)$ is a quasi-isometry.*

Proof. Since (Π, ρ) is an η-path quasimetric, for every pair of distinct points x, y of Π, we can find a finite sequence $\{z_i\}_{i\in\{0,\ldots,n\}}$ where $n = (x|y)$ such that $z_0 = x$, $z_n = y$ and for all $i \in \{1, \ldots, n\}$, $\rho(z_{i-1}, z_i) \simeq_\eta 1$. In particular $\rho(z_{i-1}, z_i) \leq 1 + \eta$. But by Lemma 7.3, we know that there is an arc, α_i, in $\mathcal{K}$ of length at most $2^{1+\eta}$ which connects z_{i-1} and z_i. As a result x, y are connected in $\mathcal{K}$ by the arc, $\alpha_1\alpha_2 \ldots \alpha_n$, of length less than $n2^{\eta+1}$. In other words, $dist(x, y) \leq 2^{\eta+1}\rho(x, y)$.

On the other hand, for all $x, y \in V(\mathcal{K})$ we consider a finite sequence $\{z_i\}_{i\in\{0,\ldots,n\}} = V(\mathcal{K}) \cap [x, y]$ where $n = dist(x, y)$ such that $z_0 = x$, $z_n = y$ and z_i and z_{i+1} are adjacent in $\mathcal{K}$ for all i . Now by Lemma 7.2, for all i, we have $\rho(z_i, z_{i+1}) \leq \zeta$. Thus since ρ is a η-quasimetric, we obtain that $\rho(x, y) \leq \sum_{i=0}^{n-1} \rho(z_i, z_{i+1}) + (n-1)\eta \leq n\zeta + (n-1)\eta$. Thus $\rho(x, y) \leq dist(x, y)(\zeta + \eta)$.

We denote by K the maximum of the quantities $\zeta + \eta$ and $2^{\eta+1}$. We obtain $(1/K)\rho(x, y) \leq dist(x, y) \leq K\rho(x, y)$. In addition Π is a quasidense subset of $\mathcal{K}$. Consequently these two last assertions together give that $(\Pi, \rho) \hookrightarrow (\mathcal{K}, dist)$ is a quasi-isometry.

◇

Corollary 7.6. *$(\mathcal{K}, dist)$ is hyperbolic*

Lemma 7.7. *$\mathcal{K}/\Gamma$ is finite.*

Proof. By hypotheses of the Theorem 7.1, we have $\Pi/\Gamma = V(\mathcal{K})/\Gamma$ finite. So we need to prove only $E(\mathcal{K})/\Gamma$ is finite. Suppose, to the contrary, that $E(\mathcal{K})/\Gamma$ is not finite. Thus there exists an infinite sequence of edges, $\{e_i\}_i$, of $\mathcal{K}$ such that for all distinct $i, j \in \mathbf{N}$, e_i and e_j belong to different Γ-orbits of $E(\mathcal{K})$. We write (x_i, y_i) for the end points of e_i. Then since Π/Γ is finite, after translating by an element of Γ we can suppose for all i that $x_i = x$, where $x \in \Pi$. Moreover since x is a bounded parabolic point we can also suppose that $y_i \to y \neq x$ by replacing y_i by a suitable $\mathrm{Stab}_\Gamma(x)$-image and passing to a subsequence. From here, there are two possible cases: either y is a bounded parabolic point or y is a conical limit point.

Case 1:

Firstly we suppose that y is a bounded parabolic point. Thus, $\chi \leq (x|y) = n < \infty$. We choose a sequence of nested annuli $\{A_k\}_{k \in \{1,\dots,n\}}$ such that $n = (x|y)$ and $\{x\} < A_1 < \cdots < A_n < \{y\}$.

Since y_i converge to y, for large enough i, we have $y_i \in (A_n)^+$. For all such i, we choose a sequence of nested annuli $\{B^i_j\}_{j \in \{1,\dots,r_i\}}$ of maximal cardinality such that $r_i = (y|y_i)$ and $\{y_i\} < B^i_1 < \cdots < B^i_{r_i} < \{y\}$. We denote by s_i the maximal index such that $(B^i_{s_i})^- \subseteq \mathrm{int}(A_n)^+$. Thus by applying Lemma 5.5 we see that for large enough i we obtain $(y|y_i) \leq s_i + \xi + 1$ and $(x|y_i) \geq n + s_i - 1$ where ξ is the constant defined by property (A5).

On the other hand, since $y, y_i \in \Pi$ for all i, we have $(y|y_i) \geq \chi$. Thus if χ is big enough ($\geq \xi + 3$) then $s_i \geq 2$ and so $(x|y_i) \geq n + s_i - 1 > n = (x|y)$. Moreover $(x|y_i) \geq n + s_i - 1 \geq \chi + s_i - 1 > (y|y_i)$ since $n \geq \chi$.

As a result if we fix an i big enough we see that $y \in \Pi \backslash \{x, y_i\}$ satisfies $\rho(x,y) < \rho(x, y_i)$ and $\rho(y, y_i) < \rho(x, y_i)$. This gives a contradiction with the fact that x, y_i are adjacent in $\mathcal{K}$.

Case 2:

Now, we suppose that y is conical limit point. Thus since $y_i \to y$ we obtain $(x|y_i) \to \infty$. But since x and y_i are adjacent in $\mathcal{K}$ by Lemma 7.2, there exists a constant $\zeta > 0$ such that $\rho(x, y_i) \leq \zeta$, which gives a contradiction.

$\diamondsuit$

The following lemma is not directly related to the construction of $\mathcal{K}$. It is a more general result using only dynamical properties of a convergence group which acts on a connected graph with finite quotient. It has been proved as Lemma 4.2 in [Bo8].

Lemma 7.8. *Suppose that Γ acts on a compactum M as a convergence group, and that $\Pi \subseteq M$ is a Γ-invariant subset. Suppose that no point of Π is a conical limit point. If Π can be Γ-equivariantly embedded in a (connected) graph, $\mathcal{K}$, with vertex set Π, and with finite quotient under the action of Γ, then $\mathcal{K}$ is fine.*

As a summary we see that the graph $\mathcal{K}$ thus constructed is connected by Lemma 7.4. Also we obtain that $(\mathcal{K}, dist)$ is hyperbolic since it is quasi-isometric to (Π, ρ). In addition, since $\mathcal{K}/\Gamma$ is finite, by Lemma 7.7 there are only finitely many Γ-orbits of edges and $\mathcal{K}$ is fine by Lemma 7.8. We denote by $\mathcal{G}$ the set of infinite vertex stabilisers of $\mathcal{K}$, which is exactly the set of maximal parabolic subgroups of $\mathcal{K}$. The intersection of any two distinct maximal parabolic subgroups is finite, and so each edge stabiliser is finite. Finally by the inert hypothesis, namely that every element of $\mathcal{G}$ is finitely generated, we have that the elements of $\mathcal{G}$ are finitely generated. These remarks prove Theorem 7.1.

8. THE BOUNDARY OF Γ

This section can be viewed as the continuation of Section 7. The main objective in this section is to identify homeomorphically "the boundary of Γ" with M under the hypotheses of Theorem 7.1.

The boundary of a relatively hyperbolic group, Γ, will be defined in the same way as by Bowditch in his paper [Bo7]. Given a fine hyperbolic graph $(\mathcal{K}, dist)$ we write $\Delta\mathcal{K} = V_\infty(\mathcal{K}) \cup \partial\mathcal{K}$, where $\partial\mathcal{K}$ is the Gromov boundary of $\mathcal{K}$ and $V_\infty(\mathcal{K})$ is the vertex set of infinite valence of $\mathcal{K}$. Given a function $f : \mathbf{N} \to \mathbf{N}$ we say that f is *bounded above by a linear function* if there exists a linear function $g : \mathbf{N} \to \mathbf{N}$ such that $f(n) \leq g(n)$ for all n. Let f be a function bounded above by a linear function with $f(n) \geq n$ for all n. An "f-quasigeodesic arc" in $(\mathcal{K}, dist)$ is an arc β such that length$(\alpha) \leq f(dist(x, y))$ for any subarc, α, of β where x, y are the endpoints of α. Similarly we can define an "f-quasigeodesic ray" in $(\mathcal{K}, dist)$. Clearly a geodesic is $1_{\mathbf{N}}$-quasigeodesic where $1_{\mathbf{N}}$ is the identity function. It has been shown in [Bo7] that one can define a topology on $\Delta\mathcal{K}$ as follows. Given a function, f, as above, an element, a, of $\Delta\mathcal{K}$ and a subset, A, of $V_\infty(\mathcal{K})$, let $M_f(a, A)$ be the set of points $b \in \Delta\mathcal{K}$ such that any f-quasigeodesic from b to a meets A, if at all, only in the point a. Hence a set $O \subseteq \Delta\mathcal{K}$ is open if for all $a \in O$ there is a finite set $A \subseteq \Delta\mathcal{K}$ such that $M_f(a, A) \subseteq O$. It has been shown in Section 8 of [Bo7] that the topology thus defined does not depend to the choice of the function f and hence it is well defined. Alternative formulations of this topology can be found in the same paper. Bowditch proved also, in his paper, that $\Delta\mathcal{K}$ with its topology is hausdorff and compact. (For details see Section 8 of [Bo7]). Moreover he showed that given two fine hyperbolic graphs $\mathcal{K}$ and $\mathcal{L}$, with same vertex set V such that the identity on V extends to a quasi-isometry, there is a natural homeomorphism from $\Delta\mathcal{K}$ to $\Delta\mathcal{L}$, which is the identity on V. In other words $\Delta\mathcal{K}$ is canonically defined for fine hyperbolic graphs with same vertex set up to quasi-isometry. From this, there follows a natural definition of the boundary of a relatively hyperbolic group:

Definition. Given a relatively hyperbolic group $(\Gamma, \mathcal{G})$ and the graph $\mathcal{K}$, featuring in the second definition of relative hyperbolicity, *the boundary* $\partial(\Gamma, \mathcal{G})$ *of* $(\Gamma, \mathcal{G})$ is defined as $\Delta\mathcal{K}$ with its topology defined as above. (We abbreviate this to $\partial\Gamma$ when the peripheral structure $\mathcal{G}$ is assumed)

For further discussion concerning this definition see Sections 8 and 9 of [Bo7].

Given a hyperbolic path quasimetric space, (Q, ρ), we will say two pseudogeodesic rays in Q are *parallel* if they remain a bounded distance apart. Note that this defines an equivalence relation on the set of pseudogeodesic rays. We note also that two parallel pseudogeodesics are eventually uniformly parallel. Following on from this, we define the *boundary,* ∂Q, of Q as the set of parallel classes of pseudogeodesic rays. We will refer to ∂Q as the *Gromov boundary* of Q since this definition was introduced for hyperbolic spaces by Gromov in [Gro2]. We fix some r much greater than the constant of path hyperbolicity of ρ. For $x \in \partial Q$ we choose $\{x_i\}_i$ a pseudogeodesic ray in the parallel class of x. For all $n \in \mathbf{N}$ we define $S(n)$ as the set of $y \in Q \cup \partial Q$ such that some pseudogeodesic (ray or segment depending, respectively, on whether $y \in \partial Q$ or $y \in Q$) connecting x_0 and y intersects $N_\rho(x_n, r)$. The topology on $Q \cup \partial Q$ is defined such that it is discrete on Q and for any $x \in \partial Q$ the set $\{S(n)\}_n$ is a base of neighbourhoods of x in $Q \cup \partial Q$. We note that this topology is well defined and does not depend on the choice of the pseudogeodesic ray $\{x_i\}_i$ in the parallel class of x. In the rest of this paper, we will refer to the topology, thus defined on $Q \cup \partial Q$ as the *usual boundary topology.*

The usual boundary topology on $\mathcal{K} \cup \partial K$ is clearly different than the topology given on $\Delta\mathcal{K}$. However we see from the following proposition, which is given as Proposition 8.5 in [Bo7] that a sequence $\{x_n\}_n$ in $\Delta\mathcal{K}$ converges to a point x of $\partial\mathcal{K}$ if and only if it converges to x in the usual boundary topology on $\Delta\mathcal{K}$.

Proposition 8.1. *The subspace topology on $\partial\mathcal{K}$ induced from $\Delta\mathcal{K}$ agrees with the usual boundary topology of $\partial\mathcal{K}$.*

Now, we impose the same hypotheses and notation of Section 7, i.e that M is a metrisable compactum, and that Γ is a convergence group acting on M such that the action is geometrically finite. Suppose also that the quotient of the set of bounded parabolic points by Γ is finite and the stabiliser of each bounded parabolic point is finitely generated. By Theorem 7.1, Γ is hyperbolic relative to the set, $\mathcal{G}$, of its maximal parabolic subgroups. Let $(\mathcal{K}, dist)$ be the graph constructed in Section 7. This satisfies all the properties required by definition 2 of relative hyperbolicity (Theorem 7.1). Thus $\Delta\mathcal{K}$ defines the boundary $\partial(\Gamma, \mathcal{G})$ of Γ. Note that we have also $V_\infty(\mathcal{K}) = \mathcal{V}(\mathcal{K})$, therefore $\Delta\mathcal{K} = V(\mathcal{K}) \cup \partial\mathcal{K}$. The main objective of this section is to prove:

Proposition 8.2. *There is an equivariant homeomorphism, Ψ, from $\Delta\mathcal{K}$ to M where M is considered with its given topology.*

Consider the Γ-invariant system of annuli, $\mathcal{A}$, as constructed in Section 6, under the same hypotheses of Theorem 7.1. Thus A satisfies (A1), (A2), (A3) and hence (A5) with $\Xi = \{\text{conical limit points}\}$, $\Pi = \{\text{bounded parabolic points}\}$. Also any two distinct points x, y of Π are separated at least by χ annuli, where the constant χ can be chosen, by the construction of $\mathcal{A}$, independently from the constant, ξ, of property (A5). The latter is universally defined for such an annulus system. Moreover (Π, ρ), where ρ the quasimetric introduced in Section 4, is an η-hyperbolic path quasimetric space. We already observed that η may depend to the constant χ. But this dependence is not relevant for the following results, as it will be explained later. To prove Proposition 8.2 we will firstly find a bijection between the "Gromov boundary" of (Π, ρ) and the set Ξ.

Proposition 8.3. *There is a bijective map $\Phi : \partial\Pi \to \Xi$ such that given $X \in \partial\Pi$ and $\{x_i\}_i$ a pseudogeodesic ray in the parallel class of X, if $\Phi(X) = x$ then x_i converges to x for the given topology on M.*

Before we start to prove this proposition we need some preliminary lemmas.

Lemma 8.4. *Given $X \in \partial\Pi$ and $\{x_i\}_i$ a pseudogeodesic ray in the parallel class of X, there exists $x \in M$ such that x_i converges to x in the given topology of M.*

Proof. Firstly we note that after passing to a subsequence, x_i converges to some point x in M by the compactness of M. Thus, we need to prove that the point $x \in M$ does not depend to the choice of subsequence.

Suppose to the contrary that we can find two subsequences x_{i_n} and x_{i_m} such that x_{i_n} converges to $x \in M$ and x_{i_m} converges to $y \neq x \in M$ in the topology of M. Note that $x \neq y$ implies that $(x_0|xy) < \infty$

By property (A3) for large enough n and m, we have $(xx_{i_n} : x_0y)$ and $(yx_{i_m} : x_0x)$. We consider an efficient approximating tree $(\mathcal{T}, \sigma)$ for $\{x_0, x, y, x_{i_n}, x_{i_m}\}$. We write $u = \text{med}(x_0, x_{i_n}, x_{i_m})$. Therefore, by efficiency, $(xx_{i_n} : x_0y)_{\mathcal{T}}$ and $(yx_{i_m} : x_0x)_{\mathcal{T}}$ hold. Thus $u = \text{med}(x_0, x, y)$ and we obtain $(x_0|xy)_{\mathcal{T}} = \sigma(x_0, u)$. Moreover we note that $\rho(x_0, x_{i_n}) \simeq \sigma(x_0, x_{i_n}) = \sigma(x_0, u) + \sigma(u, x_{i_n})$, $\rho(x_0, x_{i_m}) \simeq \sigma(x_0, x_{i_m}) = \sigma(x_0, u) + \sigma(u, x_{i_m})$ and $\rho(x_{i_n}, x_{i_m}) \simeq \sigma(x_{i_n}, x_{i_m}) = \sigma(u, x_{i_n}) + \sigma(u, x_{i_m})$. (See Figure 1.8.1).

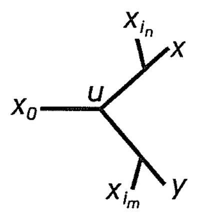

Figure 1.8.1

On the other hand, since $\{x_i\}$ is a pseudogeodesic ray in (Π, ρ) we have $(x_0|x_{i_n}) \simeq i_n \to \infty$. Similarly $(x_0|x_{i_m}) \to \infty$. Thus we can choose i_n and $i_m \gg i_n$ such that $\rho(x_0, x_{i_n}) \simeq i_n \gg (x_0|xy)$. Now, using the pseudogeodesic property of $\{x_i\}$, we see that $\rho(x_0, x_{i_m}) \simeq \rho(x_0, x_{i_n}) + \rho(x_{i_n}, x_{i_m})$. Therefore, putting together this last equality with the earlier ones, we obtain $2\sigma(u, x_{i_n}) \simeq 0$. This give a contradiction since $\sigma(u, x_{i_n}) = \sigma(x_0, x_{i_n}) - \sigma(x_0, u) \simeq \rho(x_0, x_{i_n}) - (x_0|xy) \gg 0$.

$\diamondsuit$

Lemma 8.5. *Given $X \in \partial\Pi$ and $\{x_i\}_i$ a pseudogeodesic ray in the parallel class of X there exists $x \in \Xi$ such that x_i converges to x for the given topology of M.*

Proof. We saw by Lemma 8.4 that there exists $x \in M$ such that x_i converges to x for the given topology of M. We will see that there exist sequences of nested annuli, with arbitrarily large cardinality, which separate x_0 and x. It follows by Theorem 2.6 that there exists, in fact, an infinite sequence of nested annuli which separate x and x_0. In other words, x is a conical point, hence, a conical limit point.

Suppose to the contrary that the maximal number of nested annuli which separate x_0 and x is finite. Denote it by p. We fix an x_j in $\{x_i\}_i$. Since x_i converge to x by property (A3) $(x_0x_j : xx_i)$ is satisfied for i large enough.

We consider an efficient approximating tree $(\mathcal{T}, \sigma)$ of $\{x_0, x, x_j, x_i\}$. So we have $(x_0x_j : xx_i)_{\mathcal{T}}$. We write $u = \text{med}(x_0, x_j, x_i)$. Thus $p = \sigma(x_0, x) \geq (x_0|x_ix_j)_{\mathcal{T}} = \sigma(x_0, u)$. Moreover, we note that $\rho(x_0, x_j) \simeq \sigma(x_0, x_j) = \sigma(x_0, u) + \sigma(u, x_j)$, $\rho(x_0, x_i) \simeq$

$\sigma(x_0, x_i) = \sigma(x_0, u) + \sigma(u, x_i)$ and $\rho(x_j, x_i) = \sigma(x_j, x_i) = \sigma(u, x_j) + \sigma(u, x_i)$. (See Figure 1.8.2 for an illustration of $\mathcal{T}$).

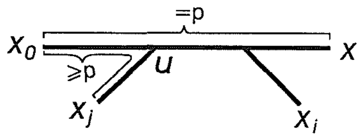

Figure 1.8.2

Moreover since $\{x_i\}$ is a pseudogeodesic ray in (Π, ρ) we have $(x_0|x_i) \simeq i \to \infty$. So we can choose j and $i \gg j$ such that $\rho(x_0, x_j) \simeq j \gg p = \sigma(x_0, x) \geq \sigma(x_0, u)$. Thus, using the pseudogeodesic property of $\{x_i\}$, we see that $\rho(x_0, x_i) \simeq \rho(x_0, x_j) + \rho(x_j, x_i)$. Therefore, putting together this last equation with earlier ones, we obtain $2\sigma(u, x_j) \simeq 0$. This give a contradiction since $\sigma(u, x_j) = \sigma(x_0, x_j) - \sigma(x_0, u) \simeq \rho(x_0, x_j) - \sigma(x_0, u) \gg 0$.

$\diamondsuit$

Lemma 8.6. *Given $X \in \partial\Pi$, let $\{x_i\}_i$ and $\{y_j\}_j$ be two pseudogeodesic rays in the parallel class of X. Then $\{x_i\}_i$ and $\{y_j\}_j$ converge to the same conical point x for the given topology of M.*

Proof. This result can be proved by exactly the same argument as in the proof of Lemma 8.4. In fact let $\{x_i\}_i$ and $\{y_j\}_j$ be two pseudogeodesic rays in the parallel class of X. Without loss of generality, we can suppose that $\rho(x_0, y_0) \simeq 0$ since $\{x_i\}_i$ and $\{y_j\}_j$ are uniformly parallel for i,j large enough. Also since $\{x_i\}_i$ and $\{y_j\}_j$ remain a bounded distance from each other, after passing to a subsequence of $\{y_j\}_j$, we can suppose that if $m \geq n$ then $\rho(x_0, x_n) + \rho(x_n, y_m) \simeq \rho(x_0, y_m)$. Thus, by replacing $\{x_{i_n}\}_n$ and $\{x_{i_m}\}_m$ in the proof of Lemma 8.4 respectively by $\{x_i\}_i$ and $\{y_j\}_j$ and by supposing that x_i converges to $x \in M$ and y_j converges to $y \neq x \in M$ for the topology of M, we arrive at the same contradiction.

$\diamondsuit$

Proof. (of Proposition 8.3) From the previous lemmas, the construction of the identification map $\Phi : \partial\Pi \to \Xi$ is fairly obvious. Given $X \in \partial\Pi$, we set $\Phi(X) = x$ where $x \in \Xi$ is the point defined by Lemmas 8.5 and 8.6. We need to verify that Φ thus constructed is bijective.

Firstly we show that Φ is injective. Let X, Y be two distinct points of $\partial\Pi$. Consider $x = \Phi(X)$ and $y = \Phi(Y)$ in Ξ. We will prove that $x \neq y$. Suppose for contradiction that $x = y$. Since (Π, ρ) is hyperbolic there exists a bi-infinite pseudogeodesic, $\{x_i\}_{i\in\mathbf{Z}}$, such that the pseudogeodesic rays, $\{x_i\}_{i\in\mathbf{N}}$ and $\{x_{-i}\}_{i\in\mathbf{N}}$, lie respectively in the parallel class of X and Y. By the definition of Φ we know that $x_i \to x$ and $x_{-i} \to y = x$ in M (Lemma 8.4). Without loss of generality we suppose that $x_0 \neq x$.

We consider an efficient approximating tree $(\mathcal{T}, \sigma)$ of $\{x_0, x, x_i, x_{-i}\}$. We write $u_i = \mathrm{med}(x_0, x_i, x_{-i})$. Thus we have $(x_0|x_ix_{-i})_{\mathcal{T}} = \sigma(x_0, u_i)$. We note that $\rho(x_0, x_i) \simeq \sigma(x_0, x_i) = \sigma(x_0, u_i) + \sigma(u_i, x_i)$, $\rho(x_0, x_{-i}) \simeq \sigma(x_0, u_i) + \sigma(u_i, x_{-i})$ and $\rho(x_i, x_{-i}) \simeq \sigma(u_i, x_i) + \sigma(u_i, x_{-i})$. Using the bi-infinite pseudogeodesic property of $\{x_i\}_{i\in\mathbf{Z}}$, we see that x_0, x_i and x_{-i} satisfy $\rho(x_{-i}, x_i) \simeq \rho(x_0, x_{-i}) + \rho(x_0, x_i)$. Therefore, putting together this last equation with the earlier ones, we obtain $2\sigma(x_0, u_i) \simeq 0$.

Moreover, since x is a conical point and x_i and x_{-i} converge to x we have $(x_0|x_ix_{-i}) \to \infty$. So we can choose an $i \in \mathbf{N}$ such that $\sigma(x_0, u_i) \simeq (x_0|x_ix_{-i})_{\mathcal{T}} \gg 0$. This give a contradiction with $2\sigma(x_0, u_i) \simeq 0$. Thus we conclude that $x \neq y$ and hence Φ is injective.

Now, we prove that Φ is surjective. In other words we will show that for all $x \in \Xi$ there is a pseudogeodesic ray, $\{y_i\}_i$, such that $y_i \to x$ in M.

Let x be a conical point, and $\{x_i\}_i \subset M$ be a sequence converging to x in M. Without loss of generality on passing to a subsequence we can suppose that $x_0 \neq x$ and $(x_0x_1|x_ix) \ll (x_0x_1|x_jx)$ whenever $i < j$ as $(x_0|x) \to \infty$. It follows easily that for all $i < j$ we have $(x_0x_i|x_jx) \gg 0$. We know that the crossratio induced from the annulus system, $\mathcal{A}$ is a hyperbolic path crossratio. So we can use the path property to interpolate between $\{x_0, x_i\}$ and $\{x_j, x\}$ with $i < j$, so as to find in $\Theta_3(M)$ a pseudogeodesic ray, $\{Z_i\}_i = \{(z_0, z_i, x)\}_i$, with $\{z_i\}_i$ converging to x. Moreover, since (Π, ρ) is quasidense in $(\Theta_3(M) \cup \Pi, \rho)$ (Proposition 5.7), we can find a sequence $\{y_i\}_i$ such that $\{y_i\}_i$ is a pseudogeodesic ray in (Π, ρ). Now using similar arguments as before one can easily check that y_i converges to x in M, which completes the proof.

$\diamondsuit$

Thus, we have proved Proposition 8.3. We note that we can extend Φ, constructed as in Proposition 8.3, to $\Phi : \Pi \cup \partial\Pi \to M$ by setting $\Phi(x) = x \in M$ for all $x \in \Pi$. Here $\Pi \cup \partial\Pi$ is considered with its usual boundary topology. Therefore the map Φ, as extended above, still satisfies the property that given $X \in \partial\Pi$ and $\{x_i\}_i$ a pseudogeodesic ray in parallel class of X then x_i converges to $\Phi(X)$ for the given topology on M. In the remainder of this section, each time we will use the map, Φ, we will mean the extended map.

We know by Lemma 7.5 that $(\mathcal{K}, dist)$ is quasi-isometric to (Π, ρ). So the Gromov boundary, $\partial\mathcal{K}$, of $\mathcal{K}$ is homeomorphic to the Gromov boundary, $\partial\Pi$, of Π. Therefore the usual boundary topology of $\mathcal{K} \cup \partial\mathcal{K}$ restricted to $V(\mathcal{K}) \cup \partial\mathcal{K}$ agrees with the usual boundary topology of $\Pi \cup \partial\Pi$. Thus for the remainder of the section, we will identify $\partial\mathcal{K}$ and $\partial\Pi$, and we will say that a sequence $\{x_i\}_i \subset V(\mathcal{K}) \cup \partial\mathcal{K}$ converges to $X \in \partial\mathcal{K}$ for the usual boundary topology of $\mathcal{K} \cup \partial\mathcal{K}$ if and only if it converges to $X \in \partial\Pi$ for the usual boundary topology of $\Pi \cup \partial\Pi$. We define a bijective map $\Psi = \Phi$ from $\Delta\mathcal{K}$ on M. Note that the only reason for using a different notation for Ψ and Φ is to make clear the use of different topologies on $V(\mathcal{K}) \cup \partial\mathcal{K}$. We will prove in what follows that the bijection Ψ, thus defined, is in fact an homeomorphism between $\Delta\mathcal{K}$ and M.

Lemma 8.7. *Let X be a point of $\partial\Pi$ and $\{z_n\}_n$ be a sequence in $\Pi \cup \partial\Pi$ converging to X for the usual boundary topology of $\Pi \cup \partial\Pi$. Then $\Phi(z_n) \to \Phi(X) \in \Xi$ in the given topology of M.*

Note that the sequence, z_n, does not necessarily remain within a bounded distance of a pseudogeodesic. This therefore strengthens Proposition 8.3.

Proof. Let $x = \Phi(X)$. Suppose for contradiction that $\Phi(z_n)$ does not converge to x in M. Thus after passing to a subsequence we assume that $\Phi(z_n) \to z \neq x$ in M. Let $\{x_n\}_n$ be a pseudogeodesic ray in the parallel class of X and set $x_0 = w$. We know that z_n converges to X for the usual boundary topology of $\Pi \cup \partial\Pi$. Thus, for all n there exist $\{y_i^n\}_i \subset \Pi$ a pseudogeodesic connecting $w = y_0^n$ and z_n, and an index i^n such that $y_{i^n}^n \in N_\rho(x_n, r) \cap \{y_i^n\}_i$. In particular $y_{i^n}^n$ remains within a bounded distance of x_n for all n. But $\{x_n\}_n$ and $\{y_i^n\}_i$ are pseudogeodesic in Π, and so $\rho(y_{i^n}^n, w) \simeq \rho(x_n, w) \simeq n$. Thus we can suppose $i^n = n$. We write $y_n^n = a_n$. Since a_n remains within a bounded distance of x_n for all n, $\{a_n\}_n$ defines a pseudogeodesic ray in the parallel class of X. (See Figure 1.8.3). It follows that $a_n \to \Phi(X) = x$ in M by the construction of Φ. (See Lemma 8.6)

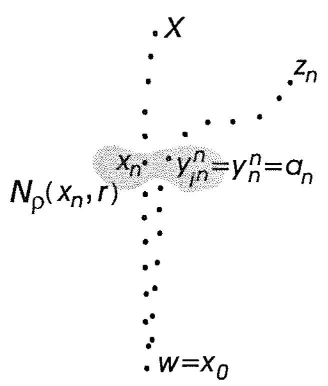

Figure 1.8.3

We note that since x is a conical point, there exists an infinite sequence of nested annuli which separate $\{x\}$ and $\{z, w\}$. Thus, since $a_n \to x$, we have that $(a_n|zw) \to \infty$. We choose $y^n_m \in \{y^n_i\}_i$ with $m \geq n$ such that $y^n_m = z_n$ if $z_n \in \Pi$ and $(y^n_m|xw) \gg 0$ if $z_n \in \partial\Pi$. The latter is possible since, in this case, $\{y^n_i\}_i$ is a pseudogeodesic in the parallel class of $z_n \in \partial\Pi$, and, by construction of Φ, $\Phi(y^n_i) = y^n_i$ converge, as $i \to \infty$, to $\Phi(z_n) \in \Xi$, which is a conical point. Moreover $\Phi(z_n) \to z \neq x$ in M. Thus there exists an infinite sequence of nested annuli which separate $\{\Phi(z_n)\}$ and $\{x, w\}$. In particular we can choose $y^n_m \in \{y^n_i\}_i$ as required. We denote y^n_m by b_n.

Now, by property (A3), for large enough n, $(a_n x : zw)$ holds since $a_n \to x$. Also for n large enough with $z_n \in \Pi$, $(b_n z : xw)$ since $\Phi(z_n) = z_n = b_n \to z$. We consider an efficient approximating tree $(\mathcal{T}, \sigma)$ of $\{w, x, z, a_n, b_n\}$. Thus, we obtain by efficiency $(a_n x : zw)_{\mathcal{T}}$ for large enough n and $(b_n z : xw)_{\mathcal{T}}$ for all n large enough with $z_n \in \Pi$. Moreover, since $(b_n|xw) \gg 0$ for all n with $z_n \in \partial\Pi$, we obtain in $\mathcal{T}$, $(\{b_n x\} : \{xw\})_{\mathcal{T}}$ for such n. (For an illustration of $\mathcal{T}$, see Figure 1.8.4).

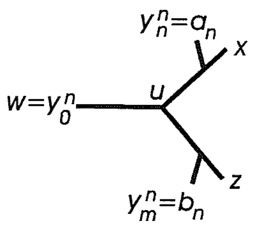

Figure 1.8.4

Thus $u = \mathrm{med}(w, a_n, b_n) = \mathrm{med}(w, x, z)$. These relations, together, imply $(a_n|zw) \simeq (a_n|zw)_{\mathcal{T}} = \sigma(a_n, u)$, i.e, $\sigma(a_n, u) \to \infty$. We note also $\rho(y^n_0, y^n_n) \simeq \sigma(w, u) + \sigma(u, a_n)$, $\rho(y^n_n, y^n_m) \simeq \sigma(a_n, u) + \sigma(u, b_n)$ and $\rho(y^n_0, y^n_m) \simeq \sigma(w, u) + \sigma(u, b_n)$. In addition, since $n \leq m$ and $a_n = y^n_n$, $b_n = y^n_m$ are the elements of the pseudogeodesic $\{y^n_i\}_i$, we have $\rho(y^n_0, y^n_n) + \rho(y^n_n, y^n_m) \simeq \rho(y^n_0, y^n_m)$. It follows that $2\sigma(a_n, u) \simeq 0$, which gives a contradiction with $\sigma(a_n, u) \to \infty$.

◇

Lemma 8.8. *Suppose that $\{x_i\}_i, \{y_i\}_i$ are two sequences of points of Π with the properties that x_i, y_i are adjacent in $\mathcal{K}$ for all i and $y_i \neq y_j$ for all $i \neq j$. If $x_i \to x$ and $y_i \to y$ in M then $x = y$.*

Proof. Since $E(\mathcal{K})/\Gamma$ is finite and x_i and y_i are adjacent in $\mathcal{K}$ with $y_i \neq y_j$ for all $i \neq j$, we can find, after passing to a subsequence, two distinct points, x_0, y_0, of Π and an infinite sequence, $\{g_i\}_i$, such that $x_i = g_i x_0$ and $y_i = g_i y_0$ for all i. We know there exist two points a, b of M such that g_i converges to b locally uniformly on $M \backslash \{a\}$.

Thus either x_0 and y_0 are both different from a or one of them is equal to a. In the former case $g_i x_0 = x_i$ and $g_i y_0 = y_i$ both converge to b. As a result $x = y = b$, which gives the required result. In the latter case we can suppose without loss of generality that $x_0 = a$. Thus $g_i y_0$ converges to b. Now if $g_i x_0$ also converges to b, we have again the required result. If not, after passing to a subsequence, $g_i x_0$ converges to $c \neq b$, i.e. g_i converges to b locally uniformly on $M \backslash \{x_0\}$ and $g_i x_0$ converges to $c \neq b$. But, this says exactly that x_0 is a conical limit point. Thus, we obtain a contradiction, since x_0 is a bounded parabolic point and we know, by Proposition 5.1, that a conical limit point cannot be a parabolic point.

$\diamondsuit$

Lemma 8.9. *Suppose $x \in V(\mathcal{K}) = \Pi$ and suppose that $\{x_n\}_n$ is a sequence in $V(\mathcal{K}) = \Pi$ converging to x in $\Delta\mathcal{K}$. Then $\{x_n\}_n$ cannot converge to a conical limit point in M.*

Proof. Suppose to the contrary that x_n converges to a conical limit point y in M in the given topology of M. We denote by Y the element of $\partial\Pi$ with $\Phi(Y) = y$. We consider a pseudogeodesic ray, $\{y_i\}_i$, connecting $x = y_0$ and Y in (Π, ρ). By the construction of Φ, we know that y_i converges to y in M with its given topology. For each n, we denote by y_{i_n} an element of $\{y_i\}_i$ satisfying $\rho(x_n, \{y_i\}_i) = \rho(x_n, y_{i_n})$.

Firstly we prove that $i_n \to \infty$ as $n \to \infty$. To show this, we argue by contradiction. Thus we suppose that $i_n \leq s$ for all n.

First we note that since $\{y_i\}_i$ is a pseudogeodesic connecting x and Y in (Π, ρ) for all i, $(xy|y_i)$ is bounded. This can be proved by similar arguments used in proof of Proposition 8.3. In other words using property (A3) and efficient approximating trees in an argument by contradiction.

We know that there exists an infinite sequence of nested annuli which separate y and the set $\{y_0 = x, y_1, \ldots, y_s\}$. So, since y_i converges to y, we can choose $y_m \in \{y_i\}_i$ with $(\{y_0 = x, \ldots, y_s\}|y_m y) \gg 0$. In particular this implies $(y_0 y_j : y_m y)$ for all $j \leq s$. We define $N_y = \bigcap_{j \leq s} N_y^{xy_j} \cap N_y^{xy_m}$ where $N_y^{xy_j}$ and $N_y^{xy_m}$ are defined by property (A3). Now since $x_n \to y$ in M, for large enough n, we have $x_n \in N_y$, and so the relations $(xy_m : yx_n)$ and $(xy_j : yx_n)$ hold for all $j \leq s$. We consider an efficient approximating tree $(\mathcal{T}, \sigma)$ of $\{y_0 = x, \ldots, y_s, y, y_m, x_n\}$. Then by efficiency we obtain $(xy_m : yx_n)_{\mathcal{T}}$, $(xy_j : yx_n)_{\mathcal{T}}$ and $(xy_j : y_m y)_{\mathcal{T}}$ for all $j \leq s$. This implies that $(\{xy_j\} : \{y_m y x_n\})_{\mathcal{T}}$ for all $j \leq s$. We write $u_j = \mathrm{med}(x, x_n, y_j)$ for all $1 \leq j \leq s$ and we set $u_0 = x$. Also we write $w = \mathrm{med}(x, x_n, y_m)$. (For a possible form of $\mathcal{T}$, see Figure 1.8.5)

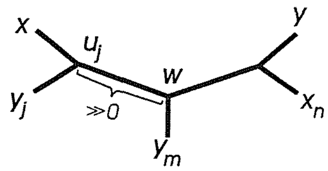

Figure 1.8.5

We note that for all large enough n, $\sigma(x_n, y_m) = \sigma(x_n, w) + \sigma(w, y_m)$, and $\sigma(x_n, y_{i_n}) = \sigma(x_n, u_{i_n}) + \sigma(u_{i_n}, y_{i_n})$, since $i_n \in \{0, \ldots, s\}$. In addition $(\{xy_j\} : \{y_m y x_n\})_\mathcal{T}$, where $j \leq s$, and $(xy_m : yx_n)_\mathcal{T}$ together give that $\sigma(x_n, u_{i_n}) = \sigma(x_n, w) + \sigma(w, u_{i_n})$. Thus $\sigma(x_n, y_{i_n}) = \sigma(x_n, w) + \sigma(w, u_{i_n}) + \sigma(u_{i_n}, y_{i_n})$. Moreover the same relations imply $(xy|y_m)_\mathcal{T} = \sigma(w, y_m)$, $(xy|y_j)_\mathcal{T} = \sigma(u_j, y_j)$ and $(xy_j|y_m y)_\mathcal{T} = \sigma(u_j, w)$ for all $j \leq s$. Therefore since for all i, $(xy|y_i)$ is bounded we have $\sigma(x_n, y_m) = \sigma(x_n, w) + \sigma(w, y_m) \simeq \sigma(x_n, w) + (xy|y_m) \simeq \sigma(x_n, w)$ and $\sigma(x_n, y_{i_n}) = \sigma(x_n, w) + \sigma(w, u_{i_n}) + \sigma(u_{i_n}, y_{i_n}) \simeq \sigma(x_n, w) + \sigma(w, u_{i_n}) + (xy|y_{i_n}) \simeq \sigma(x_n, w) + \sigma(w, u_{i_n})$. So, since $(xy_j|y_m y) \simeq \sigma(u_j, w) \gg 0$ for all $j \leq s$, we have $\sigma(x_n, y_m) \simeq \sigma(x_n, w) \ll \sigma(x_n, w) + \sigma(w, u_{i_n}) \simeq \sigma(x_n, y_{i_n})$, i.e., $\rho(x_n, y_m) \simeq \sigma(x_n, y_m) \ll \sigma(x_n, y_{i_n}) \simeq \rho(x_n, y_{i_n})$. This gives a contradiction with the fact that $\rho(x_n, y_{i_n}) = \rho(x_n, \{y_i\}_i)$, and so $i_n \to \infty$ as $n \to \infty$ as required.

Now, we consider the geodesic, $\{z_i\}_i$, connecting $x = y_0$ and Y in $(\mathcal{K}, dist)$. We denote by z_{j_n} an element of $\{z_i\}_i$ satisfying $dist(x_n, \{z_i\}_i) = dist(x_n, z_{j_n})$. We will show that $j_n \to \infty$ as $n \to \infty$, using the fact that $i_n \to \infty$ as $n \to \infty$. In fact this follows some basic results of theory of hyperbolic spaces.

Given a hyperbolic path quasi-metric space one can define the notion of centre for a triple of distinct points and develop the theory considering pseudogeodesics in place of geodesics. Moreover the notion of center can be extended to the boundary of a hyperbolic path quasimetric space using the usual boundary topology. (For a complete explanation see for example [Bo4]). It follows that in our case, y_{i_n} is a centre for x, x_n, Y in (II, ρ) since it is the nearest point of x_n on a pseudogeodesic connecting x and Y and so does z_{j_n} in $(\mathcal{K}, dist)$. Now, since (II, ρ) and $\mathcal{K}$ are quasi-isometric, y_{i_n} define also a centre for x, x_n, Y in $(\mathcal{K}, dist)$. Therefore since y_{i_n}, z_{j_n} are both centers for x, x_n, Y in $(\mathcal{K}, dist)$, we obtain $dist(y_{i_n}, z_{j_n}) \simeq 0$. This, together with the fact that $i_n \to \infty$ and $\{z_i\}_i$ is a geodesic in $(\mathcal{K}, dist)$, gives that j_n also tends to ∞.

We denote by α^n the segment with vertex set $\{z_i\}_{i=0}^{j_n}$ connecting $x = y_0$ and z_{j_n} in $(\mathcal{K}, dist)$. Thus since $j_n \to \infty$, for enough large n we have $[y_0 = x, z_1] \subset \alpha^n$. We consider a geodesic β^n connecting z_{j_n} and x_n in $(\mathcal{K}, dist)$. Since z_{j_n} is the nearest point to x_n on α^n, we see that $\alpha^n \cup \beta^n$ is a uniform quasigeodesic in $\mathcal{K}$. (For a proof of this fact see Section 8 of [Bo7].) In other words we can find a function, f_0, bounded above by a linear function, with $f_0(n) \geq n$ for all n, such that $\alpha^n \cup \beta^n$ is a f_0-quasigeodesic in $(\mathcal{K}, dist)$. Now consider the neighbourhood $M_{f_0}(x, y_1)$ of x in $\Delta\mathcal{K}$. For large enough n, $[y_0 = x, y_1] \subset \alpha^n \cup \beta^n$ and so $x_n \notin M_{f_0}(x, y_1)$. This gives a contradiction with the fact that x_n converges to x in $\Delta\mathcal{K}$.

$\diamondsuit$

Proof. (of Theorem 8.2) We have already given a bijection Ψ between $\Delta\mathcal{K}$ and M, which is identity on $V(\mathcal{K}) = \Pi$ and with $\Psi(X) = \Phi(X) \in \Xi$ for any given $X \in \partial\mathcal{K}$. Both $\Delta\mathcal{K}$ and M being compact hausdorff, it will be enough to prove the continuity of Ψ only in one direction to obtain that $\Delta\mathcal{K}$ and M are homeomorphic.

Step 1

We will prove that Ψ is continuous at a boundary point X. Let $\{x_n\}_n$ a sequence of $V(\mathcal{K}) \cup \partial\mathcal{K}$ converging to $X \in \partial\mathcal{K}$ in $\Delta\mathcal{K}$. In fact we know, as a result of Proposition 8.1, that $\{x_n\}_n$ converges to X for the usual boundary topology of $\mathcal{K} \cup \partial\mathcal{K}$ and hence it converges to X for the usual boundary topology of $\Pi \cup \partial\Pi$. Thus it is sufficient to prove that if $\{x_n\}_n \subset \Pi \cup \partial\Pi$ converges to $X \in \partial\Pi$ then $\Psi(x_n) = \Phi(x_n)$ converges to $x = \Psi(X) = \Phi(X)$ in M. But this is exactly what Lemma 8.7 says, and we obtain the continuity of Ψ at X.

Step 2

We will prove that Ψ is continuous at a bounded parabolic point x. Let $\{x_n\}_n$ be a sequence in $V(\mathcal{K}) \cup \partial\mathcal{K}$ converging to $x \in V(\mathcal{K})$ in $\Delta\mathcal{K}$. Let α_n be a geodesic connecting x and x_n in $\mathcal{K}$. Without loss of generality we can assume that $\alpha_n \cap \alpha_m = \{x\}$ for all $n \neq m$. Suppose for contradiction that $\Psi(x_n)$ does not converge to $\Psi(x) = x$ in M. We can suppose after passing to a subsequence that $\Psi(x_n)$ converges to $y \neq x$ in M.

Now, Π is countable. So, there exist two closed subsets, L, P, of M such that $x \in \mathrm{int}(L)$, $y \in \mathrm{int}(P)$, $M = L \cup P$ and $L \cap P \subseteq \Xi$. To see this we note that the set $\{r | r = \delta(x, p) \text{ where } p \in \Pi\}$ is countable and the interval $(0, \delta(x, y))$ is uncountable. (Here δ is the given metric on M.) So $N = (0, \delta(x, y)) \setminus \{r | r = \delta(x, p) \text{ where } p \in \Pi\} \neq \emptyset$. It follows that for any $r \in N$ taking $L = \{z | \delta(x, z) \leq r\}$ and $P = \overline{M \setminus L}$ we obtain the closed subsets, L, P, as required. Now for n large enough we have $\Psi(x_n) \in P$. Denote by $\{w_i^n\}_i$ the vertex set of α_n where $w_0^n = x$ and w_i^n, w_{i+1}^n are adjacent in $\mathcal{K}$. Let i_n be an index such that $\Psi(w_{i_n}^n) = w_{i_n}^n \in L$ and $\Psi(w_{i_n+1}^n) = w_{i_n+1}^n \in P$. We write $w_{i_n}^n = y_n$ and $w_{i_n+1}^n = z_n$. Evidently y_n and z_n are adjacent in $\mathcal{K}$. Moreover they converge to x in $\Delta\mathcal{K}$. To see this, suppose for contradiction y_n does not converge to x in $\Delta\mathcal{K}$. We can find a finite set, A, of $V(\mathcal{K})$ and a neigbourhood $M_{1_{\mathbf{N}}}(x, A)$ of x in $\Delta\mathcal{K}$ such that $y_n \notin M_{1_{\mathbf{N}}}(x, A)$. It follows that $x_n \notin M_{1_{\mathbf{N}}}(x, A)$ since $y_n \in \alpha_n$, which is not possible. Now, there is two possible cases:

The first case is when there exists an n_0 such that for all $n \geq n_0$ we have either $\{y_n\} = \{y_{n_0}\}$ or $\{z_n\} = \{z_{n_0}\}$. Suppose without without loss of generality, that for all $n \geq n_0$, we have $\{z_n\} = \{z_{n_0}\}$. But this is not possible since it holds we obtain $[z_{n_0}] \in V(\alpha^n)$ for all $n \geq n_0$. It follows that for all $n \geq n_0$, $x_n \notin M_{1_{\mathbf{N}}}(x, \{z_{n_0}\})$, which gives a contradiction with x_n converge to x in $\Delta\mathcal{K}$.

The second case is when for all $n_0 \in \mathbf{N}$ there exist $n, m \geq n_0$ such that $y_n \neq y_m$ and $z_n \neq z_m$. Thus after passing to a subsequence, we can suppose, without loss of generality, that for all $n \neq m$, we have $y_n \neq y_m$ and $z_n \neq z_m$. In addition again after passing to a subsequence, we can suppose that $z_n \to a$ and $y_n \to b$. Then applying Lemma 8.8, for $z_n \to a$, $y_n \to b$ (in place of, respectively, $x_n \to x$, $y_n \to y$) we obtain that $a = b$. But for all n, $y_n \in L$ and $z_n \in P$, and so $a \in L \cap P \subset \Xi$. In other words a is a conical limit point. Now we use Lemma 8.9 to obtain a contradiction since we have found a sequence $\{y_n\}_n \subset V_\infty(\mathcal{K}) = \Pi$ converging to $x \in V_\infty(\mathcal{K}) = \Pi$ in $\Delta\mathcal{K}$ and converging to a conical limit point, a, in M. This concludes the proof.

$\diamondsuit$

9. CUSP UNIFORM GROUPS

In this section we will give the main ideas of the proof of the proposition 9.1, which shows that, under the hypotheses of Theorem 0.1, we obtain also the result of Tukia, namely the "only if" direction of Theorem 0.2. Tukia in his paper [Tu2] goes further than this result and proves that the quotient of the set of bounded parabolic points by Γ is finite.

Proposition 9.1. *Suppose that Γ is a convergence group acting on a metrisable compactum, M, such that M consists of only conical limit points and bounded parabolic points. Suppose also that the quotient of the set of bounded parabolic points by Γ is finite and the stabiliser of each bounded parabolic point is finitely generated. Then $\Theta_3(M)/\Gamma$ is "cusp uniform".*

We saw in Section 5 that $\Theta_3(M)$ can be compactified by adding a copy of M, and we called this topology "the topology of compactification". In this section, $\Theta_3(M) \cup M$

is always considered with its topology of compactification. Given a bounded parabolic point, p, let K be a compact subset of $M \setminus \{p\}$ such that $\mathrm{Stab}_\Gamma(p)K = M \setminus \{p\}$. Consider a compact neighbourhood, W, of K in $\Theta_3(M) \cup M$ not containing p.

Definition 1. A *cusp neighbourhood* of p is an open subset of $\Theta_3(M)$ which has the form $\Theta_3(M) \setminus \mathrm{Stab}_\Gamma(p)(W)$ for some such W.

Definition. We say that Γ is *cusp uniform* if $\Theta_3(M)/\Gamma$ is the union of a compact set and a finite number of Γ-quotients of cusp neighbourhood of bounded parabolic points.

The notions of cusp neighbourhood and of cusp uniformity were introduced by Tukia in [Tu2] in order to prove Theorem 0.2. We will later adapt the definition of cusp neighbourhood in the metric case to prove Proposition 9.1.

Now, we place ourselves under the hypotheses of Proposition 9.1. Hence, suppose that (M, δ) is a metrisable compactum, and that Γ is a convergence group acting on M such that M consists of only conical limit points and bounded parabolic points. As in previous sections we denote by Π the set of bounded parabolic points. Suppose also that the quotient of Π by Γ is finite and the stabiliser of each bounded parabolic point is finitely generated. Then, Theorem 0.1 says that Γ is hyperbolic relative to the set of its maximal parabolic subgroups. Also, M and the boundary of Γ are equivariantly homeomorphic. Thus, using the first definition of relative hyperbolicity we see that there is a proper hyperbolic path-metric space (Σ, m) on which Γ admits a properly discontinuous isometric action. This hyperbolic path-metric space (Σ, m) satisfies the property that every point in Σ is (close to) a centre of a triple of $\partial\Sigma$. Moreover, the boundary of Γ can be defined as the Gromov boundary, $\partial\Sigma$, of Σ. In fact, Bowditch proved that the boundary of Γ, thus defined, is homeomorphic to the one defined in Section 8. (For details see Proposition 9.1 of [Bo7]). Hence it follows that $\partial\Sigma$ is equivariantly homeomorphic to M. For the rest of this section, by abuse of notation, we will identify $\partial\Sigma$ and M.

Bowditch gives a method of construction of a Γ-invariant system of "horoball" indexed by Π in the space (Σ, m). (Section 6 of [Bo7]). This construction is based essentially on defining a system of "horofunctions" for Π. Let p be an element of $\partial\Sigma$. A *horofunction* about p is a function $h : \Sigma \to \mathbf{R}$ such that given $x, y \in \Sigma$ and a geodesic, α, connecting x and p, if $m(y, \alpha) \simeq 0$ then $h(y) \simeq h(x) + m(x, y)$. A closed subset, B, of Σ is a *horoball* about p if there exists a horofunction about p such that

$B = h^{-1}[0, \infty)$. Given a horoball B, we denote by $\text{int}(B)$ the interior of B and by $\text{Fr}(B)$ the set $B \setminus \text{int}(B)$. The following lemma has been shown by Bowditch. (For the proof see Lemma 6.3 of [Bo7]).

Lemma 9.2. *Given a bounded parabolic point, p, the quotient of the boundary of a horoball about p by* $\text{Stab}_\Gamma(p)$ *is compact.*

We want to show that if p is a bounded parabolic point then the interior of a horoball about p is a "cusp neighbourhood" in the sense of the definition below. In fact this is an adaptation to the metric case, of Definition 1 given by Tukia in [Tu2]. Given a bounded parabolic point, p, let K be a compact subset of $\partial\Sigma \setminus \{p\}$ such that $\text{Stab}_\Gamma(p)K = \partial\Sigma \setminus \{p\}$ and let W be the compact neighbourhood of K in $\Sigma \cup \partial\Sigma$ not containing p.

Definition 2. A *cusp neighbourhood* of p in Σ is an open subset of Σ at the form $\Sigma \setminus \text{Stab}_\Gamma(p)(W)$ for some such W.

Lemma 9.3. *Given a bounded parabolic point, p, and a horoball, B, about p,* $\text{int}(B)$ *is a cusp neighbourhood of p in the sense of Definition 2.*

Proof. Since p is a bounded parabolic point there is a compact set K of $\partial\Sigma \setminus \{p\}$ such that $\text{Stab}_\Gamma(p)K = \partial\Sigma \setminus \{p\}$. Consider a compact set, K', of $\partial\Sigma \setminus \{p\}$ such that $K \subset \text{int}\, K'$. (Finding such a compact neighbourhood of K is possible since $\partial\Sigma \setminus \{p\}$ is locally compact).

We note that for large enough r, every point of Σ is within an r-neighbourhood of a geodesic with one endpoint p. We fix r and consider the set $S = \{x \in \Sigma \setminus \text{int}(B)$ such that x lies within an r-neighbourhood of a geodesic connecting p and a for some $a \in K'\}$. One can easily check that given an element x of $\Sigma \setminus \text{int}(B)$ there exists a $g \in \text{Stab}_\Gamma(p)$ with $gx \in S$. In other words $\Sigma \setminus \text{int}(B) = \text{Stab}_\Gamma(p)S$.

We denote by W the closure of S in $\Sigma \cup \partial\Sigma$. It is easy to see that $K' \cup S \subset W$. Moreover one can show using the definition of S and the hyperbolicity of Σ that W is a compact neighbourhood of K in $\Sigma \cup \partial\Sigma$ and that W does not contain p.

In summary we find, for the compact set K satisfying $\text{Stab}_\Gamma(p)K = \partial\Sigma \setminus \{p\}$, a compact neighbourhood, W, in $\Sigma \cup \partial\Sigma \setminus \text{int}\, B$, such that W satisfies $\Sigma \setminus \text{Stab}_\Gamma(p)W = \text{int}(B)$ and it does not contain p. All these facts together give the result required.

$\diamondsuit$

Definition. Let p be a bounded parabolic point. A *cusp region* is a set of the form $B/\,\text{Stab}_\Gamma(p)$ where B is a $\text{Stab}_\Gamma(p)$-invariant horoball about p in Σ.

Bowditch proved in [Bo7] that the space Σ/Γ can be expressed as a union of a compact set together with a finite number of pairwise disjoint cusp regions (see section 6 of [Bo7]). In other words, there is a finite set I such that $\Sigma/\Gamma = C_0 \cup \bigcup_{i\in I} B_i/\operatorname{Stab}_\Gamma(p_i)$ where C_0 is a compact set and B_i are horoballs about some bounded parabolic point p_i. We know, by Lemma 9.2, that $\operatorname{Fr}(B_i)/\operatorname{Stab}_\Gamma(p_i)$ is compact. Thus by replacing B_i by $\operatorname{Fr}(B_i) \cup \operatorname{int}(B_i)$ and by rearranging we obtain $\Sigma = \Gamma C \cup \bigcup_{i\in I} \Gamma \operatorname{int}(B_i)$ where C is compact subset of Σ.

Now we can give the main idea of the proof of Proposition 9.1.

Proof. (of Proposition 9.1) There is a natural correspondence between Σ and $\Theta_3(M)$ determined by the set $\mathcal{Q} = \{(x,(a,b,c)) \in \Sigma \times \Theta_3(M) \mid x \text{ is a centre of } a,b,c\} \cup \{(x,x) \in \partial\Sigma \times M\}$. We can prove that $\mathcal{Q}$ is a compact subset of $(\Sigma \cup \partial\Sigma) \times (\Theta_3(M) \cup M)$ and the projection maps $\pi_1 : \mathcal{Q} \to \Sigma \cup \partial\Sigma$ and $\pi_2 : \mathcal{Q} \to \Theta_3(M) \cup M$ are continuous (hence proper, since $\mathcal{Q}$ is compact), surjective, Γ-equivariant. Moreover $\pi_2(\pi_1^{-1}\Sigma) = \Theta_3(M)$, $\pi_1(\pi_2^{-1}\Theta_3(M)) = \Sigma$, and $\pi_2(\pi_1^{-1}a) = a = \pi_1(\pi_2^{-1}a)$ for all $a \in \partial\Sigma \equiv M$.

Note that since π_1 is proper and π_2 is continuous, compact subsets of $\Sigma \cup \partial\Sigma$ are sent by $\pi_2\pi_1^{-1}$ onto compacts subset of $\Theta_3(M) \cup M$. The key idea is to use this fact to show that a cusp neighbourhood of a parabolic point in Σ (in sense of Definition 2) is sent by $\pi_2\pi_1^{-1}$ to a cusp neighbourhood of a parabolic point in $\Theta_3(M)$ (in sense of Definition 1). Therefore for all i, $\pi_2(\pi_1^{-1}\operatorname{int}(B_i))$ is a cusp neighbourhood in $\Theta_3(M)$ as well as $\pi_2(\pi_1^{-1}C)$ is compact. This gives that $\Theta_3(M)/\Gamma$ is a union of a compact set together with a finite number of Γ-quotients of cusp neighbourhoods (in the sense of Definition 1), which is the result required.

$\diamondsuit$

Part 2

SYMBOLIC DYNAMICS AND RELATIVELY HYPERBOLIC GROUPS

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0.INTRODUCTION

The main objective of this work is to prove the following theorem:

Theorem 0.1. *Let $(\Gamma, \mathcal{G})$ be a relatively hyperbolic group, and $\partial\Gamma$ be its boundary. If for each $G \in \mathcal{G}$, the action of G on its one-point compactification $G \cup \{\infty\}$ is finitely presented with special character, then the action of Γ on its boundary $\partial\Gamma$ is finitely presented.*

The notion of relatively hyperbolic groups was first introduced by Gromov in [Gro2]. Later on several authors like B.Bowditch [Bo7] and B.Farb [Farb] developed the theory. One can see relatively hyperbolic groups as a generalization of geometrically finite Kleinian groups (See [Bo6]). We will use for this work the definition of relatively hyperbolic groups given by Bowditch in [Bo7]. A group $(\Gamma, \mathcal{G})$ is *hyperbolic relative* to the family, $\mathcal{G}$, of finitely generated subgroups of Γ if it acts on an hyperbolic "fine graph" $\mathcal{K}$, with finite stabilizers of edges, finitely many orbits of edges, and such that the stabilizers of infinite valence vertices are exactly the elements of $\mathcal{G}$ (see Section 3). If one replaces the property of fineness by the property of locally finiteness in the above definition, then the family $\mathcal{G}$ is empty and the group is hyperbolic. In [Bo7] Bowditch also describes a boundary for a relatively hyperbolic group. This is a compact hausdorff space on which the group acts so that the elements of the family $\mathcal{G}$ are exactly "parabolic subgroups".

Let Γ be a group acting on a compact hausdorff space, K. The dynamical system arising from this action is *of finite type* if there exists a finite alphabet $\mathcal{A}$, and a "subshift of finite type" $\Phi \subset \mathcal{A}^{\Gamma}$, together with a continuous equivariant, surjective map $\Phi \to \partial\Gamma$, which codes the action of Γ on K. We say that the dynamical system is *finitely presented* if it is of finite type and if the action of Γ on K is "expansive". One can describe expansivity for compact metric spaces as follows. The action of a group Γ on a compact metric space, K is *expansive* if there exists $\varepsilon > 0$ such that any pair of distinct points in K can be taken at distance at least ε from each other by an element of Γ. We give in Section 1 the definition of expansivity for general case, i.e compact hausdorff spaces. Here we also give a variation of finitely presented dynamical system for the action of groups on their minimal compactifications. We say that the action of a group G on its one-point compactification $G \cup \{\infty\}$ is *finitely presented with special character* if there exists an alphabet, and a subshift, Φ, through which the left action of G on $G \cup \{\infty\}$ can be factorised with the map $\Pi : \Phi \to (G \cup \{\infty\})$ so that Π satisfies the additional condition that there exists a special character \$ in the alphabet such that $\sigma \in \Phi$ maps by Π on $g \in G$ if and only if $\sigma(g) = \$$.

In this paper we also try to understand this property described above (See Section 7). We prove the following theorem in order to give a restriction on groups satisfying this property and give some examples of such groups.

Theorem 0.2. *If a group has its one point compactification finitely presented with special character, then it is finitely generated.*

Finitely generated virtually polycyclic groups have their one-point compactification finitely presented with special character. (This includes abelian, and virtually nilpotent groups).

The notion of a finitely presented dynamical system was first introduced as a hyperbolic system by Gromov in [Gro1] and in [Gro2] the notion appears in a more general context. In [Gro2] Gromov describes consequences of such a presentation, like the rationality of some counting functions. An interesting dynamical system is the one arising from the action of a hyperbolic group Γ on its Gromov boundary $\partial\Gamma$. Already in [Gro2], Gromov uses methods of symbolic dynamics for the study of this action, and in [CP], Coornaert, and Papadopoulos explain a way to factorise such a dynamical system through a subshift of finite type. It is also well known that the action of an hyperbolic group Γ on $\partial\Gamma$ is expansive. This property, together with the existence of the coding given in [CP], makes the action of a hyperbolic group, Γ, on its boundary, $\partial\Gamma$, finitely presented (see [Gro2], [CP]).

In this work we generalise the result given in [CP] for relatively hyperbolic groups. Coornaert, and Papadopoulos capture the local properties of busemann functions on balls with fixed radius (say $R > 0$) in the Cayley graph of a hyperbolic group Γ. They define the alphabet $\mathcal{A}$ whose character are the maps from the ball, B_R, of radius R centered at Id_Γ onto $\mathbf{Z}$ satisfying these local properties of busemann functions. As the Cayley graph is locally finite there are only finitely many such maps and hence the alphabet is finite. The existence of a Rips complex for hyperbolic groups allow them to extend these local properties defined on the balls of radius R onto the whole Cayley graph, and therefore obtain a family of "cocycle" and "geodesic flows" associated to each cocycle. They define the subshift $\Phi \subset \mathcal{A}^\Gamma$ to be the family of maps sending each $\gamma \in \Gamma$ to the restriction of a cocycle on γB_R and they show using the local properties that the geodesic flows associated to a cocycle necessarily converge to a unique point on $\partial\Gamma$. Hence they obtain a natural coding of the action of Γ on its boundary by a subshift of finite type. To prove Theorem 0.1 we prove that, despite the parabolic subgroups, the action of a relatively hyperbolic group on its boundary is expansive (Proposition 6.12), and we try to realise the same construction for the "graph $\mathcal{K}$" (see Section 3) associated to a relatively hyperbolic group Γ. Although the construction of the subshift of finite type given by Coornaert and Papadopoulos will not work properly for the case of relatively hyperbolic groups (either one would need an infinite alphabet, or the map $\Phi \to \partial\Gamma$ would not be well defined) the property of special character for the maximal parabolic subgroups allows us to make successful modifications. In fact in our case $\mathcal{K}$ is not locally finite, hence balls of radius $R > 0$ are not finite. Therefore we use "cones" in $\mathcal{K}$ as the analogue of balls in the Cayley graph of a hyperbolic group. Moreover as the boundary of Γ is $\partial\mathcal{K} \cup \mathcal{V}_\infty(\mathcal{K})$ we consider distance functions as well as busemann functions which induce the same properties on cones as on balls in a Cayley graph. So similarly using a "relative Rips complex" we obtain a family of cocycle and geodesic flows associated to each cocycle. Unfortunately the local properties given on cones are not enough to control the geodesic flows around vertices of infinite valences, and geodesic flows associated to a cocycle can converge to more than one point in $\partial\Gamma$. Thus we use the property of special character for the maximal parabolic subgroups as an indicator showing where geodesic flows associated to a cocycle should escape from a vertex of infinite valence.

The structure of this paper, in outline, is as follows. We give in section 1 definitions on symbolic dynamics. In section 2 we introduce a notion called "angle", and gives some interesting results using this notion about Gromov hyperbolic graphs. We also introduce an interesting tool called "cones". In Section 3, we give the definition of a relatively hyperbolic group $(\Gamma, \mathcal{G})$ and its boundary, $\partial\Gamma = \partial\mathcal{K} \cup \mathcal{V}_\infty(\mathcal{K})$. In the same section we also introduce the Relative Rips Complex. In Section 4 we define a subshift of finite type, which will be the main ingredient of the proof of Theorem 0.1. The subshift that we construct will produce "cocyles" which have the same local properties as "Busemann functions" or "distance functions" on a $\mathcal{K}$ associated to a relatively hyperbolic group, Γ. In order to associate a point in the boundary to an element of the subshift, we will consider the "gradient lines" associated to these cocyles. In Section 5 we will prove that "gradient lines" associated to a cocyle converge to a unique point at $\partial\mathcal{K}$ or to a unique vertex of infinite valence of $\mathcal{K}$. For this we use the property of special character for each stabilizer of infinite valence vertex. In Section 6 we will finish the proof of the theorem 0.1, defining the surjection of the subshift of finite type constructed in Section 4 on the boundary of Γ and showing that the action of Γ on its boundary its expansive. In the last section we study the property of special character, and in particular, prove Theorem 0.2.

1. DYNAMICAL SYSTEMS AND Γ-ACTIONS

In this section we give definitions related to finite presentations of dynamics arising from group actions. In other words we define the basic vocabulary which will be used throughout of this paper.

Definition. (Shift, subshift, cylinder, subshift of finite type)

If $\mathcal{A}$ is a finite alphabet and Γ is a group, $\mathcal{A}^\Gamma$, with the product topology, is the *total shift* of Γ on $\mathcal{A}$. It admits a natural left Γ-action given by $(\gamma\sigma)(g) = \sigma(\gamma^{-1}g)$ for all $g \in \Gamma$ and $\sigma \in \mathcal{A}^\Gamma$.

A closed Γ-invariant subset of $\mathcal{A}^\Gamma$ is called a *subshift.*

A *cylinder* $\mathcal{C}$ in the total shift is a subset of the total shift such that there exists a finite set $F \subset \Gamma$, and a family of maps $M \subset \mathcal{A}^F$ with

$$\mathcal{C} = \{\sigma \in \mathcal{A}^\Gamma \ s.t.\ \sigma|_F \in M\}$$

A subshift Φ is of *finite type* if there exists a cylinder $\mathcal{C}$ such that $\Phi = \bigcap_{\gamma\in\Gamma} \gamma^{-1}\mathcal{C}$.

Note that the finite set $F \subset \Gamma$ introduced in the definition of a subshift of finite type can be chosen so that $Id_\Gamma \in F$. Note also that subshifts of finite type are subshifts, but in general, cylinders are not Γ invariant.

The purpose of this machinery is to study dynamical systems.

Definition. (Dynamical systems of finite type)

Let Γ act on a compact set K. The dynamical system is *of finite type* if there exists a finite alphabet, $\mathcal{A}$, a subshift of finite type, $\Phi \subset \mathcal{A}^\Gamma$, and a continuous, surjective, Γ-equivariant map $\pi : \Phi \to K$.

We next illustrate the definitions with a simple example.

Consider $\Gamma = \mathbf{Z}$, and the alphabet $\mathcal{A} = \{a, b, \$\}$. Consider the cylinder $\mathcal{C}$ defined by the finite subset F of $\mathbf{Z}$, $F = \{0, 1\}$, and let M be the set of maps $\{m_i\}_{i=1}^4$ so that $m_1(0) = a$, $m_1(1) = a$, $m_2(0) = a$, $m_2(1) = \$$, $m_3(0) = \$$, $m_3(1) = b$ and $m_4(0) = b$, $m_4(1) = b\}$.

The elements of $\mathcal{C}$ are all the bi-infinite words that agree with one of the m_i on the subset F. According to the action of $\mathbf{Z}$, the elements of $n+\mathcal{C}$ are all the bi-infinite words σ such that $\sigma(x-n)$ agrees with one of the m_i on F. Let Φ be the subshift of finite type defined by $\mathcal{C}$: $\Phi = \bigcap_{n\in\mathbf{Z}}(n+\mathcal{C})$.

One can check that the elements of Φ are all the bi-infinite words in the alphabet $\mathcal{A}$ satisfying :

· after an a is either an a or a \$,

· after a \$ comes a b,

· after a b comes a b.

Therefore, the elements of Φ are (*...aaaa...*), the constant word on a, (*...bbbb...*), the constant word on b and all the words (*...aaa\$bbb...*) beginning by a, until there is a \$ on the n^{th} letter ($n \in \mathbf{Z}$) and then b.

Now consider the one point compactification $\mathbf{Z}\cup\{\infty\}$ of $\mathbf{Z}$ with the obvious order topology. There is a natural action of $\mathbf{Z}$ on it, fixing the infinity. Consider the map $\pi : \Phi \to \mathbf{Z} \cup \{\infty\}$ that sends the constant words on ∞ and (*...aaa\$bbb...*) on $n \in \mathbf{Z}$ where n is the index of the letter \$. The map π is onto, continuous and equivariant, and therefore proves that the action of $\mathbf{Z}$ on its one point compactiication is of finite type.

Similarly we can factorise the action of $\mathbf{Z}$ on its two point compactification $K = \mathbf{Z} \cup \{+\infty, -\infty\}$ with the order topology. Here the natural action of $\mathbf{Z}$ on K fixes the two additional points. We consider the map $\pi : \Phi \to K$ that sends (*...aaaa...*) on $-\infty$, (*...bbbb...*) on $+\infty$, and (*...aaa\$bbb...*) on $n \in \mathbf{Z}$ where n is the index of the letter \$. The map π is onto, continuous and equivariant, and this therefore proves that the action of $\mathbf{Z}$ on K is of finite type.

One can refine the property of being a dynamical system of finite type with the following definition.

Definition. (Expansivity)

The action of a group Γ on a compact space K is *expansive* if there exists an open subset $U \subset K \times K$ containing the diagonal Δ of $K \times K$ such that $\Delta = \bigcap_{\gamma \in \Gamma} \gamma^{-1} U$.

The following proposition gives an equivalent formulation of expansivity in the case of compact metric spaces. (see for example [CP] Chapter 2 Proposition 2.3).

Proposition 1.2. *Let K be a compact metric space with metric d and Γ be a group acting on K. The action of Γ on K is expansive if there exists $\epsilon > 0$ such that for all distinct $x_1, x_2 \in K$ there exists $\gamma \in \Gamma$ satisfying $d(\gamma x_1, \gamma x_2) > \epsilon$.*

Now we can define a finitely presented dynamical system.

Definition. (Finitely presented dynamical systems)

Let Γ act on a compact space K. The dynamical system is *finitely presented* if it is both of finite type and expansive.

If one has a subshift of finite type $\Phi \subset \mathcal{A}^\Gamma$ and a surjective continuous equivariant map $\pi : \Phi \to K$, the expansivity of the action of Γ on K turns out to be equivalent to the fact that the subshift $\Psi \subset (\mathcal{A} \times \mathcal{A})^\Gamma$ defined by $[(\sigma_1 \times \sigma_2) \in \Psi] \Leftrightarrow [\pi(\sigma_1) = \pi(\sigma_2)]$, is of finite type (see [CP]).

If Γ is an infinite discrete group, it acts on its minimal Alexandrov compactification, $\Gamma \cup \{\infty\}$ so that the action is by translations on Γ and fixes ∞. If Γ is finite, its minimal compactification is itself.

Definition. (Finite presentation with special character)

A minimal compactification of a discrete group Γ is said to be *finitely presented with special character* if the Γ-action is finitely presented by a subshift $\Phi \subset \mathcal{A}^\Gamma$ and if there exists a map $\pi : \Phi \to \Gamma \cup \{\infty\}$ with the following property (condition of special character):

$$(\exists \$ \in \mathcal{A}) \; s.t. \; (\forall \gamma \in \Gamma)(\forall \sigma \in \Phi) \quad (\pi(\sigma) = \gamma) \Leftrightarrow (\sigma(\gamma) = \$)$$

By the following lemma we note that in this case the property of expansivity is always satisfied.

Lemma 1.3. The action of a discrete group Γ on its minimal compactification $\Gamma \cup \{\infty\}$ is expansive.

Proof. Denote $K = \Gamma \cup \{\infty\}$ and consider the open set $O = K \backslash \{Id_\Gamma\}$ and the subset $U = \{(Id_\Gamma, Id_\Gamma)\} \cup O \times O$ of $K \times K$.

We first show that U is an open neighbourhood of $K \times K$ containing the diagonal. This is to say that an infinite sequence $(x_i, y_i) \in (K \times K) \backslash U$ cannot converge to an element of U. Note that if $(x_i, y_i) \in (K \times K) \backslash U$ then either $x_i = Id_\Gamma$ or $y_i = Id_\Gamma$ for all i. Therefore if (x_i, y_i) converges to an element of U then this element can only be (Id_Γ, Id_Γ). On the other hand by discreteness of Γ there is no sequence in K converging to Id_Γ, which completes the argument.

We can also easily see that the diagonal is the intersection of Γ-translates of U. Let x, y be two distinct elements of K. We can assume without loss of generality that $x \neq \infty$. Thus $x = \gamma \in \Gamma$ and $\gamma^{-1}(x, y) = (Id_\Gamma, z)$ where $z \in O$ and (x, y) does not belong to γU.

$\diamondsuit$

Note also that finite groups which are already compact admit a trivial finite presentation with special character. Let Γ be a finite group. Consider the alphabet $\mathcal{A} = \{\$, 0\}$ and the elements of the cylinder which is also the subshift in this case is defined by the finite set Γ and the set M of maps from Γ onto $\mathcal{A}$ which satisfies that there exists a unique $\gamma \in \Gamma$ with $\sigma(\gamma) = \$$.

The example of dynamical system of finite type described previously is a finite presentation with special character of $\mathbf{Z}$. We give in Section 7 several examples of groups with a compactification finitely presented with special character, as well as the basic fact that these groups have to be finitely generated.

2. ANGLES

In this section we prepare the graph theoretical background of this work. We introduce the notion of "angle" between two adjacent paths in a graph and we give some general results related to this notion. We also define the notion of "cone" in a graph and prove that cones are quasi-convex (Lemma 2.7). Moreover we give a criterion on the geodesicity of a path in a hyperbolic graph as well as a criterion on the intersection of two geodesic paths with the same end points (Lemma 2.8 and Lemma 2.9).

A *graph* is a pair of two sets $(\mathcal{V}, \mathcal{E})$ where the set $\mathcal{V}$ of vertices, and the set $\mathcal{E}$ of edges. For the rest of this work we will follow the convention that there are no edge loops or multiple edges in any given graph. So one can associate to a graph its geometrical realisation, as a 1-dimensional simplicial complex. We will not make any distinction between them. Hence, a graph has a natural structure of a metric space so that each edge has unit length. For the remaining of this work we adopt the convention that 0 is an element of $\mathbf{N}$.

Definition. A *geodesic* path in a graph is an injective simplicial path which has the shortest length among all the other path connecting its end points.

A *circuit* in a graph is an injective simplicial loop. Its length is the number of edges it contains.

Given two adjacent point a and b in a graph we denote by (a, b) the edge connecting these two points.

Given a path α in a graph and two points a, b on α we denote by $[a, b]_\alpha$ the subpath of α connecting a and b.

The *valence* of a vertex is the number (in $\mathbf{N} \cup \{\infty\}$) of edges containing this vertex.

If $\mathcal{K}$ is a graph (respectively oriented graph), we denote by $\mathcal{V}(\mathcal{K})$ its set of vertices, and by $\mathcal{E}(\mathcal{K})$ its set of edges. We will denote by $\mathcal{V}_\infty(\mathcal{K})$ the set of vertices having infinite valence. But we will often omit $\mathcal{K}$ in these notation when there is no ambiguity.

For any graph, one can define a notion of *angle* as follow.

Definition. (Angles)

Let $\mathcal{K}(\mathcal{V}, \mathcal{E})$ be a graph, and let $e_1 = (a, b)$ and $e_2 = (a, c)$ be two edges in $\mathcal{K}$. The *angle* between e_1 and e_2 at vertex a, $\mathrm{Ang}_a(e_1, e_2)$ is the length of a shortest path from b to c in $\mathcal{K} \setminus \{a\}$ ($+\infty$ if there is none). We will frequently omit the subscript a since there is no ambiguity about which point the angle is defined.

Similarly one can define the angle between two paths. Let α_1 and α_2 two paths in $\mathcal{K}$ connecting the vertices a, b and a, c. We chose the orientation of α_1 and α_2 so that $\alpha_1(0) = \alpha_2(0) = a$, $\alpha_1(1) = b$ and $\alpha_2(1) = c$. Then the angle between α_1 and α_2 at the point a, $\mathrm{Ang}_a(\alpha_1, \alpha_2)$ is the angle $\mathrm{Ang}_a(e_1, e_2)$ where e_1, e_2 are the first edges of α_1 and α_2 adjacent to a.

Given a path α connecting b, c in $\mathcal{K}$ and a vertex a on α, the angle at the vertex a, $\mathrm{Ang}_a(\alpha)$ is $\mathrm{Ang}_a([b,a]_\alpha, [a,c]_\alpha)$.

Let α a path in $\mathcal{K}$. The *maximal angle* in α, $\mathrm{Maxang}(\alpha)$, is the maximum of the angles between two consecutive edges of α.

Given a path α connecting b, c in $\mathcal{K}$ we say that α is *θ-straight* if $\mathrm{Maxang}(\alpha) \leq \theta$, and *$\theta$-bent at a* if $\mathrm{Ang}_a([b,a]_\alpha, [a,c]_\alpha) > \theta$. Similarly we say that α is *r-short* if its length is less than r.

The following remarks will be useful

Proposition 2.1. *Given three edges e_1, e_2 and e_3 with the same end point a in a graph $\mathcal{K}$ one has*

- $\mathrm{Ang}_a(e_1, e_2) = \mathrm{Ang}_a(e_2, e_1)$,
- $\mathrm{Ang}_a(e_1, e_3) \leq \mathrm{Ang}_a(e_1, e_2) + \mathrm{Ang}_a(e_2, e_3)$.

Proposition 2.2. *Given $\eta \geq 2$, any circuit of length η has a maximal angle less than $\eta - 2$.*

Proof. Indeed, if e_1 and e_2 are two consecutive edges in the circuit, the circuit itself gives a path of length $\eta - 2$ from b to c.

$\diamondsuit$

Given a constant $\rho > 0$. If α is a circuit of length at most 8ρ, then α is $(8\rho - 2)$-straight. We choose a constant $\theta \geq 8\rho$, as in next sections we will need the results which follows only for the paths of length less than 8ρ. In fact the following arguments can be done independently of the choice of 8ρ by considering the paths of different length.

Given three geodesic paths, $\alpha_1 = [a_2, a_3]$, $\alpha_2 = [a_3, a_1]$ and $\alpha_3 = [a_1, a_2]$ so that the concatenation $\delta = \alpha_1.\alpha_2.\alpha_3$ is 8ρ-short, denote $\eta_3 = \mathrm{Ang}_{a_3}(\alpha_1, \alpha_2)$.

Lemma 2.3. *If $\eta_3 > \theta$ then $a_3 \in \alpha_3$.*

Proof. Since $\eta_3 = \mathrm{Ang}_{a_3}(\alpha_1, \alpha_2) > \theta$ and δ is 8ρ-short, δ cannot be a circuit in $\mathcal{K}$ by the choice of θ. So δ passes through a_3 at least twice. But, since α_1 and α_2 are geodesics, they cannot pass through a_3 more than once. Therefore $a_3 \in \alpha_3$.

$\diamondsuit$

When α_1 and α_2 have the same end points, i.e α_3 is an empty path in the above lemma, we obtain as an immediate consequence the following result replacing $a_1 = a_2$, a_3 of the lemma by a_1 and a_2:

Corollary 2.4. *Given two distinct elements a_1, a_2 of $\mathcal{V}$ at distance at most 4ρ in $\mathcal{K}$, if α_1, α_2 are two geodesic paths connecting a_1 and a_2 then $\mathrm{Ang}_{a_1}(\alpha_1, \alpha_2) \leq \theta$ and $\mathrm{Ang}_{a_2}(\alpha_1, \alpha_2) \leq \theta$.*

We assume the same notations and hypothesis as in Lemma 2.3 and we also set $\eta_1 = \mathrm{Ang}_{a_1}(\alpha_2, \alpha_3)$.

Lemma 2.5. 1) *If α_3 is s_3-straight then $\eta_3 \leq s_3 + 2\theta$. Moreover*

2) *If α_2 is s_2-straight then* $\mathrm{Maxang}(\alpha_1) \leq \mathrm{Max}\{s_3 + s_2 + 3\theta, \eta_1 + 2\theta\}$.

Proof. First we prove that if α_3 is s_3-straight then $\eta_3 \leq s_3 + 2\theta$. If $\eta_3 \leq \theta \leq s_3 + 2\theta$ then the assertion is trivial. So we suppose that $\eta_3 > \theta$ and show that $\eta_3 \leq s_3 + 2\theta$. In this case, by Lemma 2.3, $a_3 \in \alpha_3$. Moreover, since α_3 is a geodesic path, we notice that α_3 can pass through a_3 only once.(See Figure 2.2.1). Now, by Corollary 2.4 we have $\mathrm{Ang}_{a_3}(\alpha_1, [a_3, a_2]_{\alpha_3}) \leq \theta$ and $\mathrm{Ang}_{a_3}([a_3, a_1]_{\alpha_3}, \alpha_2) \leq \theta$. Therefore $\eta_3 \leq \mathrm{Ang}_{a_3}(\alpha_1, [a_3, a_2]_{\alpha_3}) + \mathrm{Ang}([a_3, a_1]_{\alpha_3}, \alpha_2) + \mathrm{Ang}_{a_3}(\alpha_3) \leq 2\theta + s_3$.

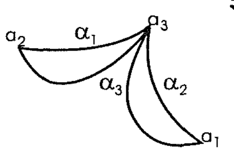

Figure 2.2.1

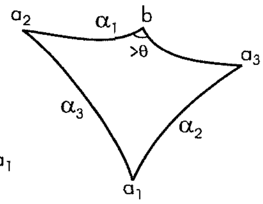

Figure 2.2.2

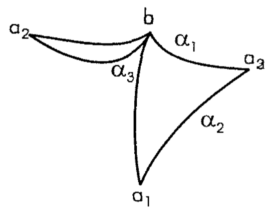

Figure 2.2.3

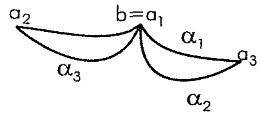

Figure 2.2.4

Now we will show that $\mathrm{Maxang}(\alpha_1) \leq \mathrm{Max}\{s_3 + s_2 + 3\theta, \eta_1 + 2\theta\}$. If $\mathrm{Maxang}(\alpha_1) \leq \theta$ there is nothing to prove. Suppose that α_1 is θ-bent at b, i.e, there exists $b \in \alpha_1$ so that $\mathrm{Ang}_b(\alpha_1) > \theta$. (See Figure 2.2.2). Now, since δ is 8ρ-short, δ passes through b at least twice. But, since α_1 is geodesic, it cannot pass through b twice. Therefore $b \in \alpha_2.\alpha_3$ and again since α_2 and α_3 are geodesics they cannot pass through b twice. Now there are two possibilities. If $b = a_1$ as in Figure 2.2.4 then by Corollary 2.4 we have $\mathrm{Ang}_b([b, a_2]_{\alpha_1}, \alpha_3) \leq \theta$ and $\mathrm{Ang}_b([b, a_3]_{\alpha_1}, \alpha_2) \leq \theta$. Therefore $\mathrm{Ang}_b(\alpha_1) \leq \mathrm{Ang}_b([b, a_2]_{\alpha_1}, \alpha_3) + \eta_1 + \mathrm{Ang}_b([b, a_3]_{\alpha_1}, \alpha_2) \leq \eta_1 + 2\theta$. If $b \neq a_1$ then without loss of generality we can suppose that $b \in \alpha_3$ (if not $b \in \alpha_2$ and the argument works similarly)(See Figure 2.2.3). Therefore we obtain $\mathrm{Ang}_b(\alpha_1) \leq \mathrm{Ang}_b([b, a_2]_{\alpha_1}, [b, a_2]_{\alpha_3}) + \mathrm{Ang}_b(\alpha_3) + \mathrm{Ang}_b([b, a_1]_{\alpha_3}, [b, a_3]_{\alpha_1})$. Now by applying the first part of Lemma 2.5 to the geodesic paths α_2, $[b, a_1]_{\alpha_3}$ and $[b, a_3]_{\alpha_1}$ instead of respectively α_3, α_1 and α_2 we obtain $\mathrm{Ang}_b([b, a_1]_{\alpha_3}, [b, a_3]_{\alpha_1}) \leq \mathrm{Maxang}(\alpha_2) + 2\theta$. Also by Corollary 2.4 we have $\mathrm{Ang}_b([b, a_2]_{\alpha_1}, [b, a_2]_{\alpha_3}) \leq \theta$. This implies $\mathrm{Ang}_b(\alpha_1) \leq \theta + \mathrm{Maxang}(\alpha_3) + \mathrm{Maxang}(\alpha_2) + 2\theta \leq s_3 + s_2 + 2\theta$. So in all the possible cases we have $\mathrm{Ang}_b(\alpha_1) \leq \mathrm{Max}\{s_3 + s_2 + 3\theta, \eta_1 + 2\theta\}$.

By repeating the above argument for all $b \in \alpha_1$ with $\mathrm{Ang}_b(\alpha_1) > \theta$, we obtain $\mathrm{Maxang}(\alpha_1) \leq \mathrm{Max}\{s_3 + s_2 + 3\theta, \eta_1 + 2\theta\}$, which is the result required.

$\diamondsuit$

Suppose we have three geodesic paths $\alpha_1 = [a_2, a_3]$, $\alpha_2 = [a_3, a_1]$ and $\alpha_3 = [a_1, a_2]$. Suppose that α_2 and α_1 are 2ρ-short.

Lemma 2.6. *Let $b \in \mathcal{V}$ be a vertex of α_3 so that $a_3 \neq b$, and let β be a geodesic path connecting a_3 and b. Then either $\mathrm{Ang}_{a_3}(\alpha_1, \beta) \leq \theta$ or $\mathrm{Ang}_{a_3}(\alpha_2, \beta) \leq \theta$.*

Proof. Suppose that both angles are strictly bigger than θ as in Figure 2.2.5. We note that α_3 is 4ρ-short and we can assume without loss of generality that $\mathrm{Dist}(b, a_2) \leq 2\rho$. Thus $\mathrm{Dist}(a_3, b) \leq 4\rho$. Therefore the path $\alpha_1.\beta.[a_2, b]_{\alpha_3}$ is 8ρ-short.

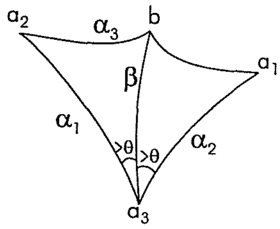

Figure 2.2.5

By applying Lemma 2.3 to the angle $\mathrm{Ang}_{a_3}(\alpha_1, \beta) > \theta$ we obtain $a_3 \in [a_2, b]_{\alpha_3}$. This implies that $\mathrm{Dist}(b, a_3) + \mathrm{Dist}(a_3, a_2) = \mathrm{Dist}(b, a_2) \leq 2\rho$. So $\mathrm{Dist}(a_3, b) \leq 2\rho$, and $\mathrm{Dist}(a_1, b) \leq 4\rho$. Therefore the path $\alpha_2.[b, a_1]_{\alpha_3}.\beta$ is 8ρ-short, and by applying again Lemma 2.6 we have $a_3 \in [b, a_1]_{\alpha_3}$ since $\mathrm{Ang}_{a_3}(\beta, \alpha_2) > \theta$. As a conclusion we obtain $a_3 \in [b, a_2]_{\alpha_3} \cap [b, a_1]_{\alpha_3}$, which gives a contradiction since α_3 is a geodesic path.

$\diamondsuit$

Let us give another definition.

Definition. (Cones)

Let $\mathcal{K} = (\mathcal{V}, \mathcal{E})$ be a graph, and let $A > 0$ and $B > 0$. For each edge $e = (x, y)$ the *cone* of radius A and angle B, $Cone_{A,B}(e, x)$ is the set of vertices v such that there exists an A-short, B-straight geodesic from x to v meeting e at an angle at most B.

$$\begin{aligned} Cone_{A,B}(e, x) = \{ & v \in \mathcal{V} \mid \exists\, \alpha \text{ a } A\text{-short, } B\text{-straight geodesic path from } x \text{ to } v \\ & \text{so that } \mathrm{Ang}_x(e, \alpha) \leq B \} \end{aligned}$$

The following lemma will be used later.

Let $\mathcal{K}(\mathcal{V}, \mathcal{E})$ be a graph. Given an edge $e = (a_3, v)$ in $\mathcal{E}$ and constants $A \geq 4\rho$, $B \geq 7\theta$, consider two points a_1, a_2 of the cone $Cone_{A,B}(e, a_3)$. Let α_2, α_1 be two geodesic paths connecting respectively a_1, a_3 and a_2, a_3. Fix an $\eta > 0$ such that $3\eta \le B - 6\theta$. Suppose that the following properties are satisfied:

1) $\mathrm{Ang}_{a_3}(\alpha_1, \alpha_2) \le \eta$.
2) $\mathrm{Ang}_{a_3}(\alpha_1, e) \le B - \theta$ and $\mathrm{Ang}_{a_3}(\alpha_2, e) \le B - \theta$.
3) α_1 and α_2 are 2ρ-short.
4) α_1 and α_2 are η-straight.

Lemma 2.7. (lemma of quasi-convexity of cones)

Under the above hypotheses if $b \in \mathcal{V}$ is an element of a geodesic path α_3, connecting a_1 and a_2, then $b \in Cone_{A,B}(e, a_3)$.

Proof. First since $\mathrm{Dist}(a_1, a_2) \le 4\rho$ we note that $\mathrm{Dist}(a_3, b) \le 4\rho$.

First we need to check that if β is a geodesic path connecting b and a_3 then β is B-straight. But α_1 and α_2 are η-straight and $\mathrm{Ang}_{a_3}(\alpha_1, \alpha_2) \le \eta$. So by applying Lemma 2.5.(2) we obtain $\mathrm{Maxang}(\alpha_3) \le \mathrm{Max}\{2\eta + 3\theta, \eta + 2\theta\} = 2\eta + 3\theta$. Also, by 2.5.(1) we obtain $\mathrm{Ang}_{a_1}(\alpha_2, \alpha_3) \le \eta + 2\theta$. Therefore we have $\mathrm{Maxang}([b, a_1]_{\alpha_3}) \le 2\eta + 3\theta$, $\mathrm{Maxang}(\alpha_2) \le \eta$ and $\mathrm{Ang}_{a_1}(\alpha_2, [a_1, b]_{\alpha_3}) \le \eta + 2\theta$. So again by Lemma 2.5.(2) applied to α_2, $[a_1, b]_{\alpha_3}$ and β instead of α_2, α_3 and α_1 we obtain $\mathrm{Maxang}(\beta) \le \mathrm{Max}\{3\eta + 6\theta, \eta + 4\theta\} = 3\eta + 6\theta$. But $3\eta \le B - 6\theta$. Thus β is B-straight.

It remains to prove that $\mathrm{Ang}_{a_3}(e, \beta) \le B$. But this is an immediate consequence of Lemma 2.6 since either $\mathrm{Ang}_{a_3}(\alpha_1, \beta) \le \theta$ or $\mathrm{Ang}_{a_3}(\beta, \alpha_2) \le \theta$. Without loss of generality we suppose that $\mathrm{Ang}_{a_3}(\alpha_1, \beta) \le \theta$. This implies $\mathrm{Ang}_{a_3}(e, \beta) \le \mathrm{Ang}_{a_3}(\alpha_1, \beta) + \mathrm{Ang}(\alpha_1, e) \le B$.

$\diamondsuit$

The next two results accentuate the importance of the notion of angle as a useful tool in δ-hyperbolic graphs. We will need them for the appropriate choice of the constants in next sections. So suppose $\mathcal{K}$ is a δ-hyperbolic graph.

Lemma 2.8. *There exists a constant μ depending only the constant of hyperbolicity δ of $\mathcal{K}$, so that given two geodesic paths (possibly geodesic rays) α and β connecting a and b in $\mathcal{K}$ (or with an end point a and converging to the same boundary point in $\partial\mathcal{K}$), if α is μ-bent at a vertex x then β passes through x.*

Proof. Denote by $\{a_i\}_i$ and $\{b_i\}_i$ the consecutive vertices of α and β so that $a_0 = b_0 = a$. We know by hyperbolicity of $\mathcal{K}$ that α and β are a uniformly bounded distance from each other. In other words there exists a constant κ which depends only the constant of hyperbolicity δ so that $\mathrm{Dist}(a_i, b_i) \le \kappa$ for all i . Suppose that $x = a_{i_0}$.

Denote $i_0 + \kappa + 1 = i_1$ and consider a path ω_{i_1} connecting a_{i_1} and b_{i_1}. Now ω_{i_1} is κ-short. Moreover note that ω_{i_1} cannot pass through $x = a_{i_0}$ since if it did we will have $\kappa + 1 = \mathrm{Dist}(a_{i_0}, a_{i_1}) \leq \mathrm{Dist}(a_{i_1}, b_{i_1}) \leq \kappa$, which is a contradiction. Similarly denote $i_0 - \kappa - 1 = i_2$ and consider a path ω_{i_2} connecting a_{i_2} and b_{i_2}. Then the previous argument works to prove that ω_{i_2} does not pass through x. Note that it is possible that a_{i_1} do not exist. In this case we replace a_{i_1} and b_{i_1} by b and consider ω_{i_1} as an empty path. Similarly if a_{i_2} does not exist we replace a_{i_2} and b_{i_2} by a and consider ω_{i_2} as an empty path. Therefore the concatenation $\omega_{i_2}.[a_{i_2}, a_{i_1}]_\alpha.\omega_{i_1}.[b_{i_2}, b_{i_1}]_\beta$ is $(6\kappa+4)$-short. So if α is $(6\kappa+4)$-bent at a vertex x then the concatenation cannot be a circuit in $\mathcal{K}$ by Proposition 2.2. So it passes through x at least twice. But, since ω_{i_2} and ω_{1_1} do not pass through x, $x \in [b_{i_2}, b_{i_1}]_\beta$. Hence any $\mu \geq 6\kappa + 4$ gives the result required.

$\diamondsuit$

In the next lemma the constant involved is not the same as the lemma above but certainly can be taken to be equal. So from now on we will suppose that they are the same.

Lemma 2.9. *There exist a constant μ depending only the constant of hyperbolicity δ of $\mathcal{K}$ so that given two geodesic paths (or geodesic rays) α and β connecting respectively x, y and x, z (or with an end point x) in $\mathcal{K}$ if $\mathrm{Ang}_x(\alpha, \beta) > \mu$ then the concatenation $\alpha.\beta$ is a geodesic path.*

Proof. Suppose that $\alpha.\beta$ is not a geodesic. So in particular $\mathrm{Ang}_x(\alpha, \beta) < \infty$ by definition. Denote by a the vertex adjacent to x in α and b the vertex adjacent to x in β. Thus $\mathrm{Ang}((x, a), (x, b)) < \infty$. Moreover by hyperbolicity of $\mathcal{K}$ there exists a constant κ which depends only the constant of hyperbolicity of $\mathcal{K}$ and two vertices a_1, b_1 respectively on α and β so that $[x, a_1]_\alpha$ and $[x, b_1]_\beta$ remain uniformly κ-distant from each other. In particular $\mathrm{Ang}((x, a), (x, b)) \leq \kappa$, which ends the proof choosing $\mu = \kappa$.

3. RELATIVELY HYPERBOLIC GROUPS

In this section we will define "relative hyperbolicity" and introduce the "boundary of a relatively hyperbolic group" as well as the "relative Rips complex" associated to a relatively hyperbolic group. We will also see cones in a fine graph are finite.

The notion of "a relatively hyperbolic group" was defined by Gromov [Gro2]. This is a group which is word hyperbolic relative to some infinite subgroups, namely "peripheral subgroups" in the terminology of Bowditch in [Bo7]. We gave in the introduction in Part 1 two equivalent definitions of relative hyperbolicity. The equivalence of these definitions has been proved by Bowditch [Bo7]. The first definition is a modified formulation of the original definition introduced by Gromov. It gives a dynamical characterisation of relatively hyperbolic groups in terms of a group action on a hyperbolic space, while the second definition characterises relative hyperbolicity in terms of a group action on a "fine hyperbolic (Γ, V)-graph". For this work we will use the second definition that we state below. First we have to define fineness. For further discussion of these notions, such as fineness and a hyperbolic Γ-set, see [Bo7].

Definition. (Fineness)

A graph $\mathcal{K}$ is *fine* if for all $r > 0$, for all $e \in \mathcal{E}(\mathcal{K})$, the set of all the circuits of length less than r and containing e is finite.

Definition. (Relatively Hyperbolic Groups)

Let Γ be a finitely generated group, and let $\mathcal{G}$ be a collection of finitely generated subgroups. Γ is *hyperbolic relative* to $\mathcal{G}$ if it acts on a graph $\mathcal{K}$, such that

- for all $e \in \mathcal{E}$, the stabiliser of e is finite and there are only finitely many orbits of edges
- $\mathcal{K}$ is hyperbolic
- $\mathcal{K}$ is fine
- the stabilisers of the vertices of infinite valence are exactly the elements of $\mathcal{G}$

We will say that such a graph is *associated* to the relatively hyperbolic group Γ.

It follows from this definition that relative hyperbolictity can be understood as a generalisation of Gromov hyperbolic groups. In fact in the case of absence of peripheral subgroups we obtain exactly the definition of a hyperbolic group. In fact if one replaces the third point by "$\mathcal{K}$ is locally finite", which is obviously a stronger condition, one obtains the definition of a hyperbolic group.

We say that a graph $\mathcal{K}$ is 2-*vertex connected* if for every vertex v the graph $\mathcal{K}\backslash(v)$ is connected. Recall that for a graph $\mathcal{K}$, $\mathcal{V}(\mathcal{K})$ and $\mathcal{E}(\mathcal{K})$ denote respectively the set of vertices and the set of edges. Given $x \in \mathcal{V}(\mathcal{K})$ we denote by $\mathcal{V}(\mathcal{K})(x)$ and $\mathcal{E}(\mathcal{K})(x)$ respectively the set of adjacent vertices to x and the set of adjacent edges to x in $\mathcal{K}$.

We will also denote by $\mathcal{V}_{\mathcal{N}}(\mathcal{K})$ and $\mathcal{V}_{\mathcal{S}}(\mathcal{K})$ respectively the set of vertices in $\mathcal{V}(\mathcal{K})$ with non-trivial stabiliser and the set of vertices in $\mathcal{V}(\mathcal{K})$ with trivial stabiliser in Γ.

Lemma 3.1. *Let Γ a relatively hyperbolic group, relative to a family of subgroups. Then the associated graph $\mathcal{K}$ can be chosen 2-vertex connected and so that the action of Γ is free on the set of edges and for all vertex $x \in \mathcal{V}_{\mathcal{N}}(\mathcal{H})$ the stabiliser* $\mathrm{Stab}(x)$ *acts on $\mathcal{V}(\mathcal{H})(x)$ transitively.*

Proof. First up to replacing each edge by several edges joining the same two vertices, one can change the graph $\mathcal{K}$ into another graph $\mathcal{H}$ for which the action is free on edges. As the stabilizer of each edge of $\mathcal{K}$ is finite, each edge is split into only finitely many new ones. Now after taking the barycentric subdivision, we can still assume that the graph has no multiple edges. Moreover we can assume that there are not two adjacent vertices x, y in $\mathcal{H}$ with non-trivial stabilisers by taking the barycentric subdivision of the edge connecting x and y. Note that the only vertices added to $\mathcal{K}$ are the ones which come from barycentric subdivision and that the stabilisers of such vertices are trivial.

Now we will modify $\mathcal{H}$ in order to ensure that for all $x \in \mathcal{V}_{\mathcal{N}}(\mathcal{H})$, $\mathrm{Stab}(x)$ acts on $\mathcal{E}(\mathcal{H})(x)$ transitively. Given an $x \in \mathcal{V}_{\mathcal{N}}(\mathcal{H})$ we take the barycentric subdivision of the edges in $\mathcal{E}(\mathcal{H})(x)$. We denote the new graph by $\mathcal{H}'$. We index the edges in $\mathcal{E}(\mathcal{H}')(x)$ $\mathrm{Stab}(x)$-equivariantly by the elements of $\mathrm{Stab}(x)$ and we collapse the edges indexed by a same element of $\mathrm{Stab}(x)$ to a unique edge. Note that since there are only finitely many orbits of vertices one can perform this construction Γ-equvariantly for all $x \in \mathcal{V}_{\mathcal{N}}(\mathcal{H})$. The graph thus obtained satisfies the above property and the only vertices added to $\mathcal{H}$ are ones with trivial stabilisers. We continue to denote the final graph by $\mathcal{H}$.

For all vertices $x \in \mathcal{V}_{\mathcal{N}}(\mathcal{H})$ the stabiliser $\mathrm{Stab}(x)$ acts on $\mathcal{V}(\mathcal{H})(x)$ freely and transitively by construction of $\mathcal{H}$. Since for all vertices x in $\mathcal{H}$, $\mathrm{Stab}(x)$ is finitely generated (by definition of $\mathcal{K}$) we can find a connected graph, $\mathcal{L}(x)$ on which $\mathrm{Stab}(x)$ acts freely with vertex set $\mathcal{V}(\mathcal{H})(x)$ and with $\mathcal{E}(\mathcal{L}(x))/\mathrm{Stab}(x)$ finite. Now let $\mathcal{T}$ be the graph with vertex set $\mathcal{V}(\mathcal{H})$ and the edge set $\mathcal{E}(\mathcal{T}) = \mathcal{E}(\mathcal{H}) \cup \bigcup_{x \in \mathcal{V}_{\mathcal{N}}(\mathcal{H})} \mathcal{E}(\mathcal{L}(x))$. We see that $\mathcal{V}_{\mathcal{N}}(\mathcal{H}) = \mathcal{V}_{\mathcal{N}}(\mathcal{T})$. Hence no vertex in $\mathcal{V}_{\mathcal{N}}(\mathcal{T})$ can be a cut point. Moreover Γ acts on $\mathcal{E}(\mathcal{T})$ freely and for each $x \in \mathcal{V}_{\mathcal{N}}(\mathcal{T})$, $\mathrm{Stab}(x)$ acts transitively on $\mathcal{V}(\mathcal{T})(x)$ and $\mathcal{E}(\mathcal{T})(x)$. Note that in order to obtain $\mathcal{T}$ from $\mathcal{H}$ no new vertices and only finitely many orbit of edges have been added.

We want to modify $\mathcal{T}$ in order to obtain a 2-vertex connected graph. Let $x \in \mathcal{V}_{\mathcal{S}}(\mathcal{T})$. Given $y \in \mathcal{V}(\mathcal{T})(x)$, a_y denotes the vertex y if $y \in \mathcal{V}_{\mathcal{S}}(\mathcal{T})$ and denotes the mid-vertex that we add to $\mathcal{T}$ by taking the barycentric subdivision of the edge $e = (x, y)$ if $y \in \mathcal{V}_{\mathcal{N}}(\mathcal{T})$. Now for all pairs of vertices $y, z \in \mathcal{V}(\mathcal{T})(x)$ we connect a_y to a_z by an edge. Note that for all vertices x in $\mathcal{V}_{\mathcal{S}}(\mathcal{T})$ the set $\mathcal{V}(\mathcal{T})(x)$ is finite and so there are only finitely many pairs of vertices in $\mathcal{V}(\mathcal{T})(x)$. Therefore only finitely many orbits of edges have been added and they have all trivial stabilisers. Moreover none of the added edges is adjacent to an element of $\mathcal{V}_{\mathcal{N}}(\mathcal{T})$. Hence the final graph still satisfies the properties that for all vertices $x \in \mathcal{V}_{\mathcal{N}}(\mathcal{H})$ the stabiliser $\mathrm{Stab}(x)$ acts on $\mathcal{V}(\mathcal{H})(x)$ transitively and Γ acts on the set of edges freely. Note also the graph thus obtained is 2-vertex connected.

Finally we note that throughout the modifications of the initial graph $\mathcal{K}$ only finitely many orbits of vertices and edges have been added and all of them have trivial stabilisers. Therefore, the final graph remains hyperbolic and fine.

$\diamondsuit$

Henceforth we will assume that each graph associated to a relatively hyperbolic group fulfills the hypothesis of Lemma 3.1.

We can see that the third point of the definition of a relatively hyperbolic group can be easily expressed in terms of angles :

Proposition 3.2. *Let $\mathcal{K}$ be a fine graph. Given $e = (x, y) \in \mathcal{E}$ and $\theta > 0$, there exists only finitely many edges $e' = (x, z)$ such that $\mathrm{Ang}_x(e, e') \leq \theta$.*

Proof. There exists only finitely many circuits of length θ containing e.

$\diamondsuit$

The following is a direct corollary of the previous proposition.

Corollary 3.3. *Let Γ be a relatively hyperbolic group. In a graph associated to a relatively hyperbolic group, the cones are finite subsets of $\mathcal{V}$.*

Therefore, in a graph associated to a relatively hyperbolic group, we see cones as analogue of the balls in a locally finite hyperbolic graph.

Definition. (Boundary of Relatively Hyperbolic groups)

Given a relatively hyperbolic group, Γ, let $\mathcal{K}$ be a graph associated to Γ. Then the *boundary* $\partial\Gamma$ of Γ is $\partial\mathcal{K} \cup \mathcal{V}_\infty$ where $\partial\mathcal{K}$ is the Gromov boundary of the hyperbolic graph $\mathcal{K}$, and $\mathcal{V}_\infty\mathcal{K}$ is the set of vertices of infinite valence in $\mathcal{K}$.

This definition of the boundary for relatively hyperbolic groups is given by Bowditch in [Bo7]. He shows in this paper that this boundary admits a natural topology as a metrisable compactum as it shall be described below. Later different authors use this boundary in order to develop the theory (see [Y], and also [D2]).

Given a fine hyperbolic graph $(\mathcal{K}, dist)$ we write $\Delta\mathcal{K} = V_\infty(\mathcal{K}) \cup \partial\mathcal{K}$, where $\partial\mathcal{K}$ is the Gromov boundary of $\mathcal{K}$ and $V_\infty(\mathcal{K})$ is the vertex set of infinite valence of $\mathcal{K}$. Given a function $f : \mathbf{N} \to \mathbf{N}$ we say that f is bounded above by a linear function if there exists a linear function $g : \mathbf{N} \to \mathbf{N}$ such that $f(n) \leq g(n)$ for all n. Let f be a function bounded above by a linear function with $f(n) \geq n$ for all n. An "f-quasigeodesic arc" in $(\mathcal{K}, dist)$ is an arc β such that length$(\alpha) \leq f(dist(x, y))$ for any subarc, α, of β where x, y are the endpoints of α. Similarly we can define an "f-quasigeodesic ray" in $(\mathcal{K}, dist)$. Clearly a geodesic is $1_{\mathbf{N}}$-quasigeodesic where $1_{\mathbf{N}}$ is the identity function. It has been shown in [Bo7] that one can define a topology on $\Delta\mathcal{K}$ as follows. Given a function, f, as above, an element, a, of $\Delta\mathcal{K}$ and a subset, A, of $V_\infty(\mathcal{K})$, let $M_f(a, A)$ be the set of points $b \in \Delta\mathcal{K}$ such that any f-quasigeodesic from b to a meets A, if at all, only in the point a. Hence a set $O \subseteq \Delta\mathcal{K}$ is open if for all $a \in O$ there is a finite set $A \subseteq \Delta\mathcal{K}$ such that $M_f(a, A) \subseteq O$. It has been shown in Section 8 of [Bo7] that the topology thus defined does not depend to the choice of the function f and hence it is well defined. Alternative formulations of this topology can be found in the same paper. Bowditch proved also, in his paper, that $\Delta\mathcal{K}$ with its topology is hausdorff and compact. (For details see Section 8 of [Bo7]). Moreover he showed that given two fine hyperbolic graphs $\mathcal{K}$ and $\mathcal{L}$, with same vertex set V such that the identity on V extend to a quasi-isometry, there is a natural homeomorphism from $\Delta\mathcal{K}$ to $\Delta\mathcal{L}$, which is the identity on V. In other words $\Delta\mathcal{K}$ is canonically defined for fine hyperbolic graphs with same vertex set up to quasi-isometry. For further discussion concerning this definition see Sections 8 and 9 of [Bo7].

We will use the following theorem in section 4. This is a reformulation of a result in [D1], which ensures the existence of a"relative Rips complex" for relatively hyperbolic groups.

Theorem 3.4. *Let Γ be a relatively hyperbolic group and $\mathcal{K}$ be an associated graph, which is δ-hyperbolic. There exists an aspherical simplicial complex such that its vertex set is the one of $\mathcal{K}$, and such that for each simplex there exists cone of $\mathcal{K}$, of radius $10\delta + 10$, and angle $100\delta + 30$ so that the simplex has all its vertices in this cone.*

In [D1], the first author defines the relative Rips complex $P_{d,r}(\mathcal{K})$ for a relatively hyperbolic group. It is the maximal complex on the set of vertices of $\mathcal{K}$ such that an edge is between two vertices if a geodesic of length less than d and maximal angle less than r joins them in $\mathcal{K}$. Although in [D1], the notion of angle is replaced by "length of traveling in cosets", the proof of Theorem 6.2 remains the same, and gives the asphericity of $P_{d,r}(\mathcal{K})$ for large d and r.

4. A SUBSHIFT OF FINITE TYPE

In this section, in order to construct a subshift of finite type we will introduce a family of cocycles, Υ, defining their local properties (Part 4.1). Under the hypotheses of Theorem 0.1 we will then give in Part 4.3 a subshift, Φ, of finite type coding the action of Γ on its boundary $\partial\Gamma$.

Let Γ be a group hyperbolic relative to the family $\mathcal{G}$. We fix $G_1, \ldots, G_n$ as an orbit transversal of conjugacy classes in $\mathcal{G}$ and we consider $\mathcal{K}(\mathcal{V}, \mathcal{E})$, a fine δ-hyperbolic graph introduced by definition of relatively hyperbolic groups with its metric "Dist". For all i we denote by $p_i \in \mathcal{V}_\infty$ the fixed point of G_i. For each i, we choose an arbitrary edge e_i adjacent to p_i.

4.1 Local Properties of a Cocycle System

We choose the constants $\rho \geq 64\delta + 3$ and $\theta \geq 8\rho$ where δ is the constant of hyperbolicity of $\mathcal{K}$. The choice of ρ will be justified later in section 5. We also assume that no pair of edges at a vertex of finite valence have an angle more than θ. This is possible as $\mathcal{K}$ is 2-vertex connected (see Lemma 3.1). We fix an orbit transversal $\{r_j\}_j$ of $\mathcal{V}$ under the action of Γ so that $\{p_i\}_i \subseteq \{r_j\}_j$, and for each j we choose an arbitrary edge $\{e'_j\}_j$ adjacent to r_j such that $\{e_i\}_i \subseteq \{e'_j\}_j$.

We fix a vertex v_0 and an edge $e_0 = (v_0, v)$. For $k \geq 1000$, we choose $\boldsymbol{R}$ and Θ such that $Cone_{R,\Theta}(e_0, v_0)$ contains $\mathrm{Cone}_{k\theta,k\theta}(e'_j, r_j)$ for all j, and so that if r_j is a vertex of finite valence then $\mathcal{V}(r_j)$ is contained in $Cone_{R,\Theta}(e_0, v_0)$ where $\mathcal{V}(x)$ denotes the set of adjacent vertices to a vertex x in $\mathcal{K}$. We denote $\mathrm{Cone}_{R,\Theta}(e_0, v_0)$ by N. Note that by Corollary 3.3, N is finite and $\mathcal{V} \subset \bigcup_{\gamma\in\Gamma} \gamma(N)$.

We define Ψ to be the family of maps $\psi : N \times N \to \mathbf{Z}$ satisfying the properties below:

(A1) The property of integer valuation:
$\psi(x, y) \in \{-1, 0, 1\}$ $\forall x, y \in N$ that are adjacent in $\mathcal{K}$.

(A2) The property of cocycle:

$\psi(x,y)+\psi(y,z)+\psi(z,x)=0 \;\; \forall x,y,z \in N$.

(A3) The exit property :

If x has finite valence and if $\mathcal{V}(x)$ contained in N then there exists at least one $y \in N$, adjacent to x in $\mathcal{K}$ so that $\psi(x,y)=1$.

(A4) The property of quasi-convexity:

All vertices $z \in N$ lying on a geodesic path connecting two vertices $x, y \in N$ satisfy $t\psi(z,y)+(1-t)\psi(z,x)-4\delta \leq 0$ where $t = \mathrm{Dist}(x,z)/\mathrm{Dist}(x,y)$.

Remark. (inclusion in a cone) Given $x, y \in \mathrm{Cone}_{k\theta,k\theta}(e'_j, r_j)$ for some j, consider the geodesic paths α connecting y to r_j, β connecting x to r_j and ω connecting x to y. Denote $\theta_1 = (k\theta - 6\theta)/3$ and $\theta_2 = k\theta - \theta$. Suppose the following properties hold:

· $\mathrm{Ang}(\alpha,\beta) \leq \theta_1$.

· $\mathrm{Ang}(\alpha, e_i) \leq \theta_2$ and $\mathrm{Ang}(\beta, e_i) \leq \theta_2$.

· α and β are 2ρ-short.

· α and β are θ_1-straight.

Then by Lemma 2.7, replacing a_1, a_2, a_3 and b respectively by x, y, r_j and z and the constants $A(\geq 4\rho), B(\geq 7\theta), \eta(>0)$ by $k\theta, k\theta, \theta_1 = (k\theta - 6\theta)/3(>0)$, we obtain that if $z \in \mathcal{V}$ lies on ω then it is also in $Cone_{k\theta,k\theta}(e'_j, r_j)$ and therefore is in N.

Moreover note that if $t\psi(z,y)+(1-t)\psi(z,x)-4\delta \leq 0$ then also $\psi(r_j,z) \geq t\psi(r_j,y)+(1-t)\psi(r_j,x)-4\delta$ by the Property of cocycle (A2).

4.2 Globalisation of local properties

In this section we will give a system of maps Υ from $\mathcal{V}\times\mathcal{V}$ on $\mathbf{Z}$ which satisfies properties (A1), ..., (A4) on $\gamma(N\times N)$ for all $\gamma \in \Gamma$. Moreover we will show that we can extend some of local properties of such a map on $\mathcal{K}$, in other words we will show that this map arises from a global cocycle on $\mathcal{K}$.

We consider a contractible simplicial complex $\mathcal{P}(\mathcal{K})$ given by Theorem 3.4 on $\mathcal{K}$. In $\mathcal{K}$ there are only finitely many Γ-orbits of $Cone_{10\delta+10,100\delta+30}(e'_j, r_j)$. Moreover note that $k\theta > 10\delta + 10$ and $k\theta > 100\delta + 30$, hence each orbit representative lies in $Cone_{k\theta,k\theta}(e'_j, r_j)$ for some e'_j. Therefore by Theorem 3.4 each simplex of $\mathcal{P}(\mathcal{K})$ lies in γN for some $\gamma \in \Gamma$.

For all $\psi \in \Psi$ and for all $\gamma \in \Gamma$ we define the map $\gamma_*\psi : \gamma(N\times N) \to \mathbf{Z}$ so that $\gamma_*\psi(\gamma x, \gamma y) = \psi(x,y)$ for all $(x,y) \in N \times N$. Set Υ to be the set of maps $\varphi : \bigcup_{\gamma\in\Gamma} \gamma(N\times N) \to \mathbf{Z}$ so that for all γ there exists $\psi \in \Psi$ satisfying $\varphi_{|\gamma(N\times N)} = \gamma_*\psi$.

Proposition 4.1. *Given $\varphi \in \Upsilon$ there exists a map $\overline{\varphi}$ defined on the vertex set $\mathcal{V}$ of $\mathcal{P}(\mathcal{K})$ so that for all γ if x, y in $\gamma(N \times N)$ then $\varphi(x,y) = \overline{\varphi}(x) - \overline{\varphi}(y)$. Moreover $\overline{\varphi}$ is unique up to an additive constant.*

Proof. Given $\varphi \in \Upsilon$ for all γ it gives rise to a unique 1-cochain on the set of edges of $\gamma(N \times N)$. In fact if x, y are two points of $\mathcal{V}$, adjacent in $\gamma(N \times N)$, we know that $\varphi(x,y) = \psi(\gamma^{-1}x, \gamma^{-1}y)$ for some $\psi \in \Psi$. But since ψ satisfies property (A1), $\varphi(x,y)$ can take only the values 1, -1 or 0. Thus if $\varphi(x,y) = 1$ (respectively $= -1$ or $= 0$) one can set $\varphi(x,z) = \mathrm{Dist}(x,z)$ (respectively $= -\mathrm{Dist}(x,z)$ or $= 0$) for all z lying on the edge connecting x and y in $\mathcal{K}$. So in fact φ is a simplicial 1-cochain in $\mathcal{P}(\mathcal{K})$. Moreover it is closed by property (A2) of ψ. As $\mathcal{P}(\mathcal{K})$ is contractible (simply connected is enough) and each simplex of $\mathcal{P}(\mathcal{K})$ lies in γN for some $\gamma \in \Gamma$ this cochain is indeed a coboundary. Thus we define $\overline{\varphi}$ to be the 0-cochain of which φ is the coboundary. The uniqueness comes from the fact that two 0-cochains of which φ is the coboundary differ by a constant.

$\diamondsuit$

Given $\varphi \in \Upsilon$ we will refer to $\overline{\varphi}$ as a primitive of φ, which is well defined up to an additive constant. As a result of the following proposition, from now on we assume that each $\varphi \in \Upsilon$ is defined on $\mathcal{V} \times \mathcal{V}$ entirely.

Proposition 4.2. *If φ is an element of Υ then it admits a unique extension, that we still continue to denote by φ, and $\varphi : \mathcal{V} \times \mathcal{V} \to \mathbf{Z}$ satisfies the following properties:*

(C1) $\varphi(x,y) + \varphi(y,z) + \varphi(z,x) = 0 \;\; \forall x, y, z \in \mathcal{V}$.

(C2) $\varphi(x,y) = -\varphi(y,x) \;\; \forall x, y \in \mathcal{V}$.

(C3) $|\varphi(x,y)| \leq \mathrm{Dist}(x,y) \;\; \forall x, y \in \mathcal{V}$.

Proof. The globalisation of φ on $\mathcal{V} \times \mathcal{V}$ follows Proposition 4.1 and it is easy to see that the first two properties are the direct result of the globalisation. Let us prove property (C3). For this we consider a sequence $\{a_i\}_{i \in \{0,\ldots,r\}} \in \alpha \cap \mathcal{V}$ so that $r = \mathrm{Dist}(x,y)$, $a_0 = x$, $a_r = y$ and for all $i \in \{0, \ldots, r-1\}$, (a_i, a_{i+1}) are the distinct end points of an edge in $\mathcal{K}$. By the first property, we have $\varphi(x,y) = \sum_{i=0}^{r-1} \varphi(a_i, a_{i+1})$. But since $|\varphi(a_i, a_{i+1})| \leq 1$ we obtain $|\varphi(x,y)| \leq r = \mathrm{Dist}(x,y)$.

$\diamondsuit$

Definition. (Gradient Line)

Given an element φ in Υ, a *gradient line* for φ is a path $l : I \subseteq [0, \infty) \to \mathcal{K}$, parameterized by arc length in $\mathcal{K}$, satisfying for all $l(t), l(t') \in \mathcal{V}$, $\varphi(l(t), l(t')) = t' - t$.

Note that a local gradient line is also a global one since φ satisfies the cocycle property (A2). We say that a gradient line l is *issued* from x if $\varphi(x, y) = \text{Dist}(x, y)$ for all $y \in l$. Given a gradient line l issued from x and joining x and y we say that it is *extendable forwards* if there exists $z \in \mathcal{V}$ adjacent to y so that $\varphi(y, z) = 1$. Hence the concatenation $l.(y, z)$ is a gradient line. Similarly we say that it is *extendable backwards* if there exists $z \in \mathcal{V}$ adjacent to x so that $\varphi(z, x) = 1$ and hence the concatenation $(z, x).l$ is also a gradient line. We say that a gradient line l is *maximal* if it is not extendable forwards. A vertex x is a *landing point* if there does not exist y with $\varphi(x, y) = 1$. Therefore the final vertex of a maximal gradient line is a landing point. We say that two gradient lines are *parallel* if they remain a bounded distance apart (always) from each other, and they are *coterminal* if they are parallel gradient lines or maximal gradient lines landing at the same point.

Lemma 4.3. *If l is a gradient line for $\varphi \in \Upsilon$ then l is a geodesic path in $\mathcal{K}$.*

Proof. Since φ satisfies property (C3) for all $l(t), l(t') \in \mathcal{V}$ we have $t' - t = \varphi(l(t), l(t')) \leq \text{Dist}(l(t), l(t'))$. But l is a path parametrised by arc length. So length of the subpath of l laying between $l(t)$ and $l(t') = t' - t \geq \text{Dist}(l(t), l(t'))$.

Therefore for all $l(t), l(t') \in \mathcal{V}$ we obtain that the length of the subpath of l lying between $l(t)$ and $l(t')$ is equal to $\text{Dist}(l(t), l(t'))$, which is to say that l is a geodesic path.

$\diamondsuit$

As a result of Lemma 4.3 the gradient lines associated to a given global cocycle can be understood as a geodesic flow defined on $\mathcal{K}$.

4.3 Construction of a Subshift

Our main purpose in the next two sections is to construct a subshift of finite type which will induce a family of "gradient lines" and therefore a system of "geodesic flow" on $\mathcal{K}$ so that each geodesic flow converges to a unique point of $\partial\mathcal{K}$ or sinks at a vertex of infinite valence. In fact, in Section 4.1 and Section 4.2 we gave the definition of a cocycle system, Υ, on $\mathcal{K}$ defining their local properties on each translate of N. Each element of this cocycle system will give rise to a "geodesic flow" on $\mathcal{K}$, as we will examine in the next section. A priori we can also construct a subshift of finite type so that an element of this subshift maps each $\gamma \in \Gamma$ to the restriction on γN of the geodesic flow obtained from an element of Υ. But, as the link of a vertex of infinite valence can only be covered by infinitely many of the translates of N, two "gradient lines" obtained from elements of Υ may diverge at the vertices of infinite valence. In other words we will not be able to include enough information in local coding of Υ, in order to have control over the gradient lines. So in this part we will correct these errors around infinite vertex set using the another hypothesis which are given by Theorem 0.1.

As in Theorem 0.1, we assume that for each $i \in \{1, \dots, n\}$, the minimal compactification of G_i is finitely presented with special character (see Section 1 for definition). In other words, for all i there exists a cylinder $\mathcal{C}_i$, defined by a finite alphabet $\mathcal{A}_i$ containing a special character $\$_i$, a finite subset F_i containing the identity element 1_{G_i} of G_i and a family of maps $M_i \subset \mathcal{A}_i^{F_i}$. For all i, the cylinder $\mathcal{C}_i$ gives a subshift of finite type, Φ_i, which surjects onto $G_i \cup \{\infty\}$ and satisfies the condition of special character. Note that from now on we will not make a distinction between different special characters and denote all special characters by \$. In this section we adapt also the notations of the preceding section. So recall that for all i we denote by $p_i \in \mathcal{V}_\infty$ the fixed point of G_i and by e_i the arbitrary edge adjacent to p_i chosen at the beginning of the section. Recall also that $\rho \geq 64\delta + 3$ and $\theta \geq 8\rho$. Here, in addition, we suppose that $\theta/2$ to be strictly greater than $\mathrm{Max}_{g \in F_i}\{\mathrm{Ang}(e_i, ge_i)\}$ for all $i \in \{1, \dots, n\}$. This is possible, since F_i is finite for all i and $\mathcal{K}$ is 2-vertex connected. Moreover we assume that $\theta > 4\mu$ where μ is the constant introduced by Lemma 2.8 and Lemma 2.9.

Let F be the set of elements γ of Γ such that $\gamma N \cap N$ contains two distinct element of $\mathcal{V}$.

Lemma 4.4. *The set F is finite.*

Proof. Suppose not. Then without loss of generality we can suppose that there exists an infinite sequence γ_m of Γ so that for all m, $\gamma_m N \cap N$ contains two distinct points a_m, b_m. Since N is finite, without loss of generality we can suppose that for all m, $a_m = a$ and $b_m = b$ in $\gamma_m N \cap N$ and $\gamma_m^{-1} a = x, \gamma_m^{-1} b = y$ in N. It follows that there exists an infinite sequence, γ'_m, with $\gamma'_m x = x$ and $\gamma'_m y = y$. Moreover, since $\mathcal{K}$ is fine, there are only finitely many geodesic paths connecting x and y. So without loss of generality we can suppose that there is a geodesic path, α, connecting x and y and fixed by γ'_m for all m. But this gives a contradiction since the edge stabilisers of $\mathcal{K}$ are finite.

$\diamondsuit$

Let $\mathcal{A}$ be the alphabet given by $\bigsqcup_{i=1}^{n}(\Psi \times \mathcal{A}_i)$.

Define the cylinder $\mathcal{C}$ to be the set of $\sigma \in \mathcal{A}^\Gamma$ so that $\sigma_{|F} \in M$ where M is the set of maps, $f : F \to \mathcal{A}$, $\gamma \mapsto (f_1(\gamma), f_2(\gamma))$ satisfying the following properties:

(B1) $\forall\gamma \in F$, $f(\gamma) = (f_1(\gamma), f_2(\gamma)) \in \Psi \times \mathcal{A}_i$ for some i

(B2) $\forall\gamma \in F$ if $x, y \in \gamma N \cap N$ then $f_1(\gamma)(\gamma^{-1}x, \gamma^{-1}y) = f_1(1_\Gamma)(x, y)$

(B3) $\forall i \in \{1, \dots, n\}$ the map $F_i \to \mathcal{A}_i$, $g \mapsto f_2(g)$ is in M_i.

(B4) $\forall g \in F_i$, $\forall x, y \in \mathcal{V}(\mathcal{K})$, with $ge_i = (p_i, x)$ and so that there exists a θ-straight, $k\theta$-short geodesic path, $[p_i, y]$, containing x, one has $f_1(g)(p_i, y) \geq 1 - \mathrm{Dist}(p_i, y)$ only if there exists $g' \in G_i$ satisfying $\mathrm{Ang}(g'e_i, e_i) \leq \theta/2$ and $f_2(gg') = \$$.

(B5) $\forall i \in \{1, \dots, n\}$ if there exists a $g_0 \in F_i$ satisfying $f_2(g_0) = \$$ then $f_1(1_{G_i})(p_i, x) = 1$ where x is such that $g_0 e_i = (p_i, x)$.

Remark. The above conditions concern only finitely many elements of Γ. In fact in Property B4) there are only finitely many $y \in \mathcal{V}$ such that a θ-straight, $k\theta$-short geodesic path, $[p_i, y]$ contains ge_i as g varies in the finite set F_i.

They are also well defined. Let $g \in F_i$. Then the vertices of ge_i are in $\mathrm{Cone}_{k\theta,k\theta}(e_i, p_i)$ and so they are in $N \cap gN$. Thus property (B3) is well defined because if $g \in F_i$ then $g \in F$. Similarly let $g \in F_i$ and $g' \in G_i$ so that $\mathrm{Ang}(g'e_i, e_i) \leq \theta/2$. Then gg' is in F since $\mathrm{Ang}(gg'e_i, e_i) \leq \theta$. Hence property (B4) is well defined.

We define $\Phi \subseteq \mathcal{A}^\Gamma$, the subshift given with alphabet $\mathcal{A}$, as the intersection $\Phi = \bigcap_{\gamma\in\Gamma} \gamma^{-1}\mathcal{C}$ and denote an element of Φ by $\sigma = (\sigma_1, \sigma_2)$ where $\sigma_1 : \Gamma \to \Psi$ and $\sigma_2 : \Gamma \to \bigsqcup_{i=1}^n \mathcal{A}_i$.

The following lemma proves that for each $\sigma \in \Phi$ one can associate a global cocycle $\varphi \in \Upsilon$ defined on $\mathcal{V} \times \mathcal{V}$.

Lemma 4.5. *Given a $\sigma \in \Phi$ there exists a $\varphi_\sigma \in \Upsilon$ so that $\sigma_1(\gamma) = \varphi_{\sigma|\gamma(N\times N)}$ for all $i \in \{1, \dots, n\}$ and $\gamma \in \Gamma$.*

Proof. We define the map $\varphi_\sigma : \bigcup_{\gamma\in\Gamma}\gamma(N \times N) \to \mathbf{Z}$ associated to σ as follows:

Given two distinct points x, y of $\mathcal{V}$, so that $(x, y) \in \gamma(N \times N)$ for some $\gamma \in \Gamma$, we define $\varphi_\sigma(x, y) = \sigma_1(\gamma)(\gamma^{-1}x, \gamma^{-1}y)$. We also set $\varphi_\sigma(x, x) = 0$ for all $x \in \mathcal{V}$.

The definition of φ_σ does not depend on the choice of γ. In fact given two distinct points x, y of $\mathcal{V}$, if $(x, y) \in \gamma(N \times N)$ and $(x, y) \in \gamma'(N \times N)$ for $\gamma, \gamma' \in \Gamma$ then $x, y \in \gamma N \cap \gamma' N$. Thus $\gamma^{-1}\gamma' \in F$. Moreover $\sigma(\gamma) = \gamma^{-1}\sigma(1_\Gamma)$ for all i and γ and $\gamma^{-1}\sigma \in \mathcal{C}$, so it satisfies property (B2). Therefore

$$\begin{aligned}
&\sigma_1(\gamma)(\gamma^{-1}x, \gamma^{-1}y) = \gamma^{-1}\sigma_1(1_\Gamma)(\gamma^{-1}x, \gamma^{-1}y)\\
&= \gamma^{-1}\sigma_1(\gamma^{-1}\gamma')((\gamma^{-1}\gamma')^{-1}\gamma^{-1}x, (\gamma^{-1}\gamma')^{-1}\gamma^{-1}y)\\
&= \gamma^{-1}\sigma_1(\gamma^{-1}\gamma')(\gamma'^{-1}x, \gamma'^{-1}y).
\end{aligned}$$

Finally since $\gamma^{-1}\sigma(\gamma^{-1}\gamma') = \sigma(\gamma')$ we obtain

$$\sigma_1(\gamma)(\gamma^{-1}x, \gamma^{-1}y) = \sigma_1(\gamma')(\gamma'^{-1}x, \gamma'^{-1}y).$$

We also know now that for all $\gamma \in \Gamma$, $\sigma_1(\gamma) = \psi$ where $\psi \in \Psi$. So it satisfies properties (A1),...,(A4). Thus $\varphi_\sigma(\gamma x, \gamma y) = \sigma_1(\gamma)(x, y) = \psi(x, y)$ for all $(x, y) \in N \times N$. In other words for all γ, $\varphi_{\sigma|\gamma(N\times N)} = \gamma_*\psi$. Thus $\varphi_\sigma \in \Upsilon$, and as we remarked after Proposition 4.1 we can consider φ_σ as defined globally on $\mathcal{V} \times \mathcal{V}$.

$\diamondsuit$

By Lemma 4.5 we associated to any given σ a global cocycle φ_σ. Note that although we use only σ_1 in order to obtain φ_σ from σ, φ_σ depends on the properties of σ_2. So we are interested in understanding the behaviour of this cocycle φ in relation to σ_2. The following lemmas help to make explicit some of these relations.

Lemma 4.6 shows that given a vertex, v, of infinite valence in the graph $\mathcal{K}$ there is at most one edge with initial point v which takes the special character \$ under σ_2.

Lemma 4.6. *Given $\sigma \in \Phi$, for all $i \in \{1, \ldots, n\}$ and for all $\gamma \in \Gamma$, there is at most one $g \in G_i$ so that $\sigma_2(\gamma g) = \$$.*

Note that if there is such $g \in G_i$ then g depends the choice of γ.

Proof. Fix $i \in \{1, \ldots, n\}$ and $\gamma \in \Gamma$. Now, $\sigma \in \Phi$ implies that for all $g \in G_i$, $g^{-1}\gamma^{-1}\sigma \in \Phi$. Thus $g^{-1}\gamma^{-1}\sigma$ satisfies property (B3). Therefore the map $F_i \to \mathcal{A}_i$ defined by $h \mapsto g^{-1}\gamma^{-1}\sigma_2(h)$ is in M_i, and $g^{-1}\gamma^{-1}\sigma_{2|G_i}$ is in the cylinder $\mathcal{C}_i$. It follows that, for all $g \in G_i$, $\gamma^{-1}\sigma_{2|G_i} \in g\mathcal{C}_i$, and so $\gamma^{-1}\sigma_{2|G_i} \in \bigcap_{g\in G_i} g\mathcal{C}_i = \Phi_i$. Hence since Φ_i is a subshift of finite type surjecting onto $G_i \cup \infty$ with a special character \$, by definition there exists at most one $g \in G_i$ with $\gamma^{-1}\sigma_2(g) = \$$. This gives the result since $\gamma^{-1}\sigma_2(g) = \sigma_2(\gamma g) = \$$ for at most one $g \in G_i$.

$\diamondsuit$

The next lemma play a key role in the main proof of section 5. It shows that one can extends a gradient line backwards by a $k\theta$-short geodesic path, if the angle between the path and the gradient line is greater than θ. Similarly Corollary 4.8 shows that if x is a landing point then any $k\theta$-short geodesic path ending at x will be a gradient line with landing point x.

We justify the choice of the name for the following lemma, noting that since $\theta > 4\mu$, where μ is the constant given by Lemma 2.9, if two geodesic path meets at an angle at least θ then their concatenation is a geodesic.

Lemma 4.7. (Geodesic Extension) *Let α be a gradient line issued from x and β be a $k\theta$-short geodesic path connecting z and x. If $\mathrm{Ang}_x(\alpha, \beta) > \theta$ then the concatenation $\beta.\alpha$ is a gradient line issued from z.*

Proof. We will use property (B4) of the subshift Ψ which is given on translates of N. Therefore we will split β into subpaths whose end points belong to a translate of N and apply the property recursively for each subpath.

Note that as we have an angle greater than θ at x, x is not a vertex of finite valence by the choice of θ. So there exist i and γ so that $\gamma e_i = (x, y)$ is the edge of α adjacent to x. Now as $\gamma e_i \in \gamma N$ it satisfies property (B4). In other words as by hypothesis $\varphi_\sigma(x, y) = \varphi_\sigma(\gamma p_i, y) = \sigma_1(\gamma)(p_i, \gamma^{-1}y) = 1 \geq 0$ there exists $g_0 \in G_i$ with $\mathrm{Ang}(g_0 e_i, e_i) \leq \theta/2$ and $\sigma_2(\gamma g_0) = \$$. Moreover by Lemma 4.6, $g_0 \in G_i$ with $\sigma_2(\gamma g_0) = \$$ is unique.

Now denote by $\{a_i\}_{i \in \{0,\dots,m\}}$ the consecutive vertices where β is θ-bent and suppose $a_0 = x$. As G_i acts transitively on $\mathcal{V}(p_i)$ (See Lemma 3.1) there exists a $g \in G_i$ so that $\gamma g e_i$ is the edge of $[a_i, x]_\beta$ adjacent to x (See Figure 2.4.1). We see that as $\mathrm{Ang}_x(\beta, \alpha) > \theta$, no $g' \in G_i$ with $\mathrm{Ang}(g' e_i, e_i) \leq \theta/2$ satisfies $\sigma_2(\gamma g g') = \$$. If not we would have $gg' = g_0$ and $\mathrm{Ang}(\gamma g e_i, \gamma e_i) \leq \mathrm{Ang}(g e_i, g g' e_i) + \mathrm{Ang}(g_0 e_i, e_i) \leq \theta$. This is a contradiction with $\mathrm{Ang}_x(\beta, \alpha) > \theta$. Now as the θ-straight, $k\theta$-short path $[a_1, x]_\beta$ lies in $\gamma g\, \mathrm{Cone}_{k\theta, k\theta}(e_i, p_i) \subset \gamma g N$, it satisfies property (B4). Thus $\varphi_\sigma(x, a_1) = \varphi_\sigma(\gamma g p_i, a_1) = \sigma_1(\gamma g)(p_i, g^{-1}\gamma^{-1}a_1) < 1 - \mathrm{Dist}(p_i, g^{-1}\gamma^{-1}a_1)$, i.e $\varphi_\sigma(x, a_1) \leq -\mathrm{Dist}(x, a_1)$. This together with $|\varphi_\sigma(x, a_1)| \leq \mathrm{Dist}(x, a_1)$ implies $\varphi_\sigma(a_1, x) = \mathrm{Dist}(a_1, x)$.

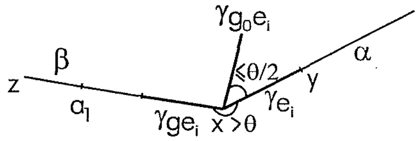

Figure 2.4.1

Now applying the above argument recursively but only finitely many times for each v_i where β is θ-bent we obtain that $\varphi_\sigma(a_i, a_{i-1}) = \mathrm{Dist}(a_i, a_{i-1})$. This ends the proof as $\varphi_\sigma(z, x) = \sum_{i=0}^{m} \mathrm{Dist}(a_i, a_{i-1}) = \mathrm{Dist}(z, x)$.

◇

The following result is in fact a corollary of the proof of Lemma 4.7.

Corollary 4.8. (Geodesic Extension) *Given a $k\theta$-short geodesic path α connecting z and x suppose that x is a landing point for φ_σ then α is a gradient line issued from z.*

Corollary 4.9. *Given a geodesic path α connecting z and y so that $\varphi_\sigma(z,y)=0$, suppose for some $x\in\alpha$, that $[x,y]_\alpha$ is a gradient line issued from x and $[z,x]_\alpha$ is $k\theta$-short. Then α cannot be θ-bent at x.*

Proof. Suppose that α is θ-bent at x. As we have $\varphi_\sigma(x,y)=\mathrm{Dist}(x,y)$, Lemma 4.7 applied to $[x,y]_\alpha$ and $[z,x]_\alpha$ instead of α and β gives $\varphi_\sigma(z,x)=\mathrm{Dist}(z,x)$. But this together with $\varphi_\sigma(x,y)=\mathrm{Dist}(x,y)$ give the contradiction since $\varphi_\sigma(z,y)=0$.

$\diamondsuit$

The next lemma shows that a cocycle associated to σ gives a dichotomy on the vertices in $\mathcal{K}$, and it assures that the geodesic flow obtained by φ_σ does not sink at a vertex of finite valence.

Lemma 4.10. *For every vertex x in $\mathcal{K}$ either there exists a gradient line issued from x or x is a landing point. Moreover if x is a vertex of finite valence then there is always a gradient line issued from x.*

Proof. The result follows from properties (B5) and (B4) of σ.

First suppose x is of infinite valence, i.e $x=\gamma p_i$ for some $i\in\{1,\dots,n\}$. Denote $e_i=(p_i,q_i)$. Suppose there is $g\in G_i$ so that $\sigma_2(\gamma g)=\$$. Since $\gamma^{-1}\sigma\in\Phi$ it satisfies property (B5). Thus $\gamma^{-1}\sigma_1(1_{G_i})(p_i,gq_i)=1$. Hence if we take $y=\gamma gq_i$ we obtain $\varphi_\sigma(x,y)=\sigma_1(\gamma)(p_i,gq_i)=1$ by definition of φ_σ. Now suppose there is no $g\in G_i$ so that $\sigma_2(\gamma g)=\$$. Hence this time, by property (B4) and by the fact that G_i acts transitively on $\mathcal{V}(p_i)$, $\gamma^{-1}\sigma_1(1_{G_i})(p_i,y)=-1$ for every vertex y adjacent to x.

Now suppose x is of finite valence. As no pair of edges at a vertex of finite valence have angle more than θ, all the edges with initial point x belongs in the same translate of N. Therefore as φ_σ satisfies the exit property (A4) on this translate of N, there exists a vertex adjacent y so that $\varphi_\sigma(x,y)=1$.

$\diamondsuit$

5. COTERMINAL GRADIENT LINES

In the preceding section under hypothesis of Theorem 7.1 we constructed a subshift of finite type, Φ with alphabet $\mathcal{A}$. We denoted an element of Φ by σ. In this section we will see that given a $\sigma\in\Phi$ and its associated map, $\varphi_\sigma\in\Upsilon$, defined by Lemma 4.7, one can associate to φ_σ, and therefore to σ via φ_σ, a unique point of

$\partial\mathcal{K} \cup \mathcal{V}_\infty$ where $\partial\mathcal{K}$ is the hyperbolic boundary of $\mathcal{K}$ and $\mathcal{V}_\infty$ is the set of vertices of infinite valence.

We adopt the notation of the previous section. Γ is a hyperbolic group relative to the family of subgroups $\mathcal{G}$, $\mathcal{K}(\mathcal{V},\mathcal{E})$ is a δ-hyperbolic graph given by the definition of Γ with its metric "Dist", and $G_1, ..., G_n$ is a set of representatives of the conjugacy classes in $\mathcal{G}$. We assume that for each $i \in \{1,\dots,n\}$, the minimal compactification of G_i is finitely presented with special character. For all $i \in \{1,\dots,n\}$ we denote by $p_i \in \mathcal{V}_\infty$ the fixed point of G_i and by $e_i = (p_i, q_i)$ the arbitrary edge adjacent to p_i chosen at the beginning of Section 4.

Recall that $\rho \geq 64\delta + 3$ and $\theta \geq 8\rho$, no pair of edges at a vertex of finite valence have an angle more than θ and $\theta/2$ is strictly greater then $\mathrm{Max}_{g\in F_i}\{\mathrm{Ang}(e_i, ge_i)\}$ for all $i \in \{1,\dots,n\}$. As before we choose θ to be greater than 4μ, where μ is the constant given by Lemma 2.8 and Lemma 2.9. Recall also that $\{r_j\}_j$ is an orbit transversal of $\mathcal{V}$ under the action of Γ so that $\{p_i\}_i \subseteq \{r_j\}_j$, and for each j we have an arbitrary edge $\{e'_j\}_j$ such that it has an end point r_j and $\{e_i\}_i \subseteq \{e'_j\}_j$. We fix a vertex v_0 and an edge $e_0 = (v_0, v)$ and we choose for $k \geq 1000$ the constants R and Θ such that for all j, $Cone_{R,\Theta}(e_0, v_0)$ contains $Cone_{k\theta,k\theta}(e'_j, r_j)$. We denote $Cone_{R,\Theta}(e_0, v_0)$ by N.

One can ask whether a geodesic flow obtained by a global cocycle converges to a unique boundary point or possibly to a unique vertex of infinite valence. If we can answer positively this question we will obtain a natural map from Φ into $\partial\mathcal{K} \cup \mathcal{V}_\infty = \partial\Gamma$. Hence for the remainder of this section we fix an element σ of Φ and denote by φ its associated cocycle $\varphi_\sigma \in \Upsilon$. The main objective of this section is to prove the following lemma,

Lemma 5.1. *Given two vertices a and b consider two gradient lines l_a and l_b for φ issued respectively from a and b. Then l_a and l_b parallel and coterminal.*

Before proving this lemma we will need a theorem and some preliminary lemmas. Using these results we will try to understand the behaviour of the geodesic flow associated to a global cocycle in specific cases.

The following theorem is a particular case of a well known theorem about hyperbolic space (see for example [GhH], Theorem 2.1). We will need it later for one of the lemmas used to prove the main result.

Theorem 5.2. (of approximating trees) *Let $\mathcal{K}$ be a (Gromov) hyperbolic space. Given a geodesic path α and two points x, y in $\mathcal{K}$ there exists a real tree τ so that*

$\alpha \cup \{x, y\}$ can be embedded in τ so that for all $z, z' \in \alpha \cup \{x, y\}$ we have $\mathrm{Dist}(z, z') - 6\delta \leq \mathrm{Dist}_\tau(z, z') \leq \mathrm{Dist}(z, z')$.

The next lemma proves that given two gradient lines issued from two close enough vertices if one of them is 4θ-bent at a vertex then they intersect at this vertex.

Recall that $\rho \geq 64\delta + 3$ and $\theta \geq 8\rho$. We fix a constant $s = 32\delta + 1 < \rho/2$. Given two vertices x, y suppose that $\varphi(x, y) = 0$ and that there exists a θ-straight s-short geodesic path α_1 connecting x and y. Note that we allow the possibility $x = y$ since in this case the hypotheses are satisfied and we can consider α_1 as the empty path for the following arguments. Let α_2 and α_4 be two gradient lines issued respectively from x and y and connecting x, x' and y, y'. Suppose that α_2 and α_4 are ρ-short and $\varphi(x', y') = 0$. Consider also a geodesic path α_3 connecting x' and y'.

Lemma 5.3. *If there exists $z \neq y' \in \alpha_4$ so that α_4 is 4θ-bent at z then $z \in \alpha_2$.*

Proof. To prove the lemma we will show that we have control over the angles of α_3. This, together with Corollary 2.4 and the hypotheses on α_1 will give the result required.

Since $\delta = \alpha_1.\alpha_2.\alpha_3.\alpha_4$ is 8ρ-short ($s < \rho/2$), by the choice of θ, δ passes through z at least twice. So either $z \in \alpha_2$ or $z \in \alpha_3$ or $z \in \alpha_1$.

If $z \in \alpha_2$ then we have the result required.

Suppose $z \in \alpha_3$ as in Figure 2.5.1. Since $z \in \alpha_4$ we have $\varphi(z, y') = \mathrm{Dist}(z, y')$. Now as $[x', z]_{\alpha_3}$ is $k\theta$-short, by Corollary 4.9 (replacing α, x, y and z respectively by $[x', z]_{\alpha_3}.[z, y']_{\alpha_4}$, z, y' and x'), we see that $\mathrm{Ang}_z([x', z]_{\alpha_3}.[z, y']_{\alpha_4}) \leq \theta$. So we have $\mathrm{Ang}_z([x', z]_{\alpha_3}, [z, y]_{\alpha_4}) > 3\theta$ since $4\theta < \mathrm{Ang}_z(\alpha_4) \leq \mathrm{Ang}_z([x', z]_{\alpha_3}.[z, y']_{\alpha_4}) + \mathrm{Ang}_z([z, x']_{\alpha_3}, [z, y]_{\alpha_4}) \leq \mathrm{Ang}_z([x', z]_{\alpha_3}, [z, y]_{\alpha_4}) + \theta$.

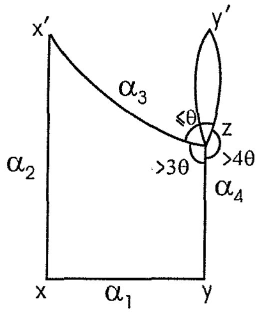

Figure 2.5.1

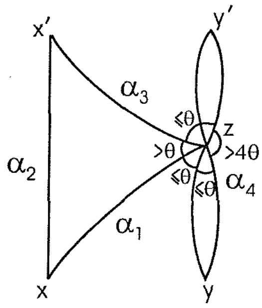

Figure 2.5.2

Now again since $\delta_1 = \alpha_1.\alpha_2.[x', z]_{\alpha_3}.[z, y]_{\alpha_4}$ is 8ρ-short and $\mathrm{Ang}_z([x', z]_{\alpha_3}, [z, y]_{\alpha_4}) > \theta$, δ_1 passes through z twice. In other words either $z \in \alpha_1$ or $z \in \alpha_2$. But if $z \in \alpha_2$ we again obtain the result required. So suppose $z \in \alpha_1$ (See Figure 2.5.2). Then by

Corollary 2.4 we have $\mathrm{Ang}_z([z,y]_{\alpha_1},[z,y]_{\alpha_4}) \leq \theta$, moreover by hypothesis we have α_1 is θ-straight. Thus $3\theta < \mathrm{Ang}_z([x',z]_{\alpha_3},[z,y]_{\alpha_4}) \leq \mathrm{Ang}_z([x',z]_{\alpha_3},[z,x]_{\alpha_1}) + \mathrm{Ang}_z(\alpha_1) + \mathrm{Ang}_z([z,y]_{\alpha_1},[z,y]_{\alpha_4}) \leq \mathrm{Ang}_z([x',z]_{\alpha_3},[z,x]_{\alpha_1}) + 2\theta$, i.e $\mathrm{Ang}_z([x',z]_{\alpha_3},[z,x]_{\alpha_1}) > \theta$. This implies by Lemma 2.3 that $z \in \alpha_2$.

It remains the case where $z \in \alpha_1$ but this case can be treated as the case above using exactly same arguments in different order in order to obtain $z \in \alpha_2$.

◊

Lemma 5.4 says that given two vertices close enough and two gradient lines issued from these vertices if we have a control over the angles of these gradients lines then they stay in a bounded distance.

As in the previous lemma fix $s = 32\delta + 1 < \rho/2$ where $\rho \geq 64\delta + 3$ and $\theta \geq 8\rho$. Denote $\theta_1 = (k\theta - 6\theta)/3$, $\theta_2 = k\theta - \theta$, which are the constants used in the remark following the quasi convexity property (A4) at section 4.1. Note that since $k \geq 1000$ these constants are positive. Given two vertices x, y suppose that $\varphi(x,y) = 0$. Denote by α_1 a geodesic path connecting x and y. Let α_2 and α_4 be two gradient lines issued respectively from x and y. Suppose that α_2 and α_4 connect respectively x, x' and y, y' where $\mathrm{Dist}(x,x') = \mathrm{Dist}(y,y') = \rho$. Let α_3 be a geodesic path connecting x' and y'.

Lemma 5.4. *If α_1 is θ-straight s-short and if α_2 and α_4 are 4θ-straight then*

(1) *α_3 is s-short*

(2) *α_3 is θ-straight*

Note that for application it is important that the same constant s and θ appear in hypotheses and conclusion. Lemma 5.4 is applied in an induction argument in Lemma 5.5 and Lemma 5.6.

Proof. As in the previous proof here the proof is given in two cases where $x \neq y$ and $x = y$. But in both cases the steps and subcases which are treated are the same and the arguments are similar except that the case $x = y$ is simpler. So here to simplify the prove we will only make explicit the case where $x \neq y$.

So we suppose that $x \neq y$. We divide the proof into three steps.

Step 1

In step 1 we prove that x, x', y' and the vertices of α_3 stay in a same translate of N and satisfy the property of quasi-convexity. Similarly we prove this statement for y, x', y' and the vertices of α_3. (see Figure 2.5.3). In other words we will prove that for every vertex z of α_3 we have $\varphi(x,z) \geq t\varphi(x,y') + (1-t)\varphi(x,x') - 4\delta$ and $\varphi(y,z) \geq t\varphi(y,y') + (1-t)\varphi(y,x') - 4\delta$ where $t = \mathrm{Dist}(x',z)/\mathrm{Dist}(x',y')$. In fact it will be enough to prove this only for x, x' and y' since x and y have symmetric roles and so the same reasoning will give us the result required for y, x', y'. Hence we prove the statement only for x, x' and y'.

Consider a geodesic path β connecting x and y'. Denote by e the edge of β adjacent to x and consider $\mathrm{Cone}_{k\theta,k\theta}(e,x)$. (See Figure 2.5.3). We will prove that x, x', y' and the vertices of α_3 stay in $\mathrm{Cone}_{k\theta,k\theta}(e,x)$. For this, it is enough to show that the hypothesis of the remark (inclusion in a cone) which follows the cocycle property (A4) in section 4.1 holds under our assumptions, i.e:

- β and α_2 are θ_1-straight
- $\mathrm{Ang}_x(\beta,\alpha_2) \leq \theta_1$.
- $\mathrm{Ang}_x(e,\beta) \leq \theta_2$ and $\mathrm{Ang}_x(e,\alpha_2) \leq \theta_2$.
- β and α_2 are 2ρ-short.

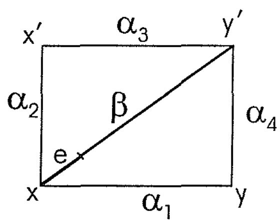

Figure 2.5.3

To check the above four assertions first note that if y has infinite valence then $\mathrm{Ang}_y(\alpha_1,\alpha_4) \leq \theta$. Because if not by the lemma of geodesic extension (Lemma 4.7) applied to α_4 and α_1 instead of α and β we obtain $\varphi(x,y) = \mathrm{Dist}(x,y)$, which is a contradiction with $\varphi(x,y) = 0$. Also if y has finite valence then again $\mathrm{Ang}_y(\alpha_1,\alpha_4) \leq \theta$ since no pair of edges at a vertex of finite valence have an angle more than θ. Hence in both case $\mathrm{Ang}_y(\alpha_1,\alpha_4) \leq \theta$. We also note using the same arguments applied to x, α_1 and α_2 instead of respectively y, α_1 and α_4, that $\mathrm{Ang}_x(\alpha_1,\alpha_2) \leq \theta$.

Note that $\alpha_1.\alpha_4.\beta$ is 3ρ-short and hence by Lemma 2.5.(2) (replacing α_1, α_2, α_3 in the lemma respectively by β, α_1 and α_4) we obtain $\mathrm{Maxang}(\beta) \leq \mathrm{Max}(\mathrm{Maxang}(\alpha_4) + \mathrm{Maxang}(\alpha_1) + 3\theta, \mathrm{Ang}_y(\alpha_1,\alpha_4) + 2\theta)$. Thus β is 8θ-straight, hence θ_1-straight. That gives the first assertion since α_2 also is θ_1-straight by hypothesis. Note that the last assertion is also direct from the hypotheses. So it remains to prove the second and the third assertions.

Now we have $\mathrm{Ang}_x(\alpha_1,\alpha_2) \leq \theta$. Moreover replacing α_1, α_2, α_3 of Lemma 2.5.(1) respectively by α_1, β and α_4 we obtain $\mathrm{Ang}_x(\alpha_1,\beta) \leq \mathrm{Maxang}(\alpha_4) + 2\theta \leq 6\theta$. Therefore $\mathrm{Ang}_x(\beta, \alpha_2) \leq \mathrm{Ang}_x(\beta, \alpha_1) + \mathrm{Ang}_x(\alpha_1, \alpha_2) \leq 7\theta \leq \theta_1, \theta_2$. This gives the second and third assertions since e is an edge of β.

Hence we showed that x, x', y' and the vertices of α_3 are lying in $\mathrm{Cone}_{k\theta,k\theta}(e, x) \subseteq \gamma N$ for some $\gamma \in \Gamma$. Since for all γ, $\varphi_{|\gamma N} = \gamma_*(\psi)$ where $\psi \in \Psi$, φ satisfies property (A4) of quasi convexity on $\mathrm{Cone}_{k\theta,k\theta}(e, x)$. Therefore for all vertices z of α_3 we have $t\varphi(z, y') + (1-t)\varphi(z, x') - 4\delta \leq 0$ where $t = \mathrm{Dist}(x', z)/\mathrm{Dist}(x', y')$.

Step 2

In this step we will prove that x' and y' stay a bounded distance from each other.

Note that $\varphi(x, x') = \mathrm{Dist}(x, x') = \rho$, $\varphi(y, y') = \mathrm{Dist}(y, y') = \rho$ since α_2 and α_4 are gradient lines. So in particular, since $\varphi(x, y) = 0$, $\varphi(x, y') = \varphi(y, x') = \rho$. Moreover from Step 1 we obtain that if z is a vertex of α_3 with $t = \mathrm{Dist}(x', z)/\mathrm{Dist}(x', y')$ then

$\varphi(x, z) \geq t\varphi(x, y') + (1-t)\varphi(x, x') - 4\delta$ and

$\varphi(y, z) \geq t\varphi(y, y') + (1-t)\varphi(y, x') - 4\delta$.

Thus for all $z \in \alpha_3$, $\varphi(x, z) \geq \rho - 4\delta$ and $\varphi(y, z) \geq \rho - 4\delta$. This together with Property C3) we obtain $\mathrm{Dist}(x, z) \geq \varphi(x, z) \geq \rho - 4\delta$ and $\mathrm{Dist}(y, z) \geq \rho - 4\delta$.

Now we consider the approximating tree τ given by Theorem 5.2 where α_3, x and y play respectively the role of α, x and y. Now, the tree τ has one of the combinatorial forms below:

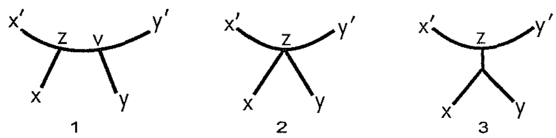

Figure 2.5.4

Suppose that τ has the form 1 or 2 of Figure 2.5.4 and denote by z and v respectively the projection of x and y onto α_3 in τ. Thus $\mathrm{Dist}_\tau(z, v) \geq 0$. So $s \geq \mathrm{Dist}(x, y) \geq \mathrm{Dist}_\tau(x, y) = \mathrm{Dist}_\tau(x, z) + \mathrm{Dist}_\tau(z, v) + \mathrm{Dist}_\tau(v, y) \geq \mathrm{Dist}(x, z) - 6\delta + \mathrm{Dist}(y, v) - 6\delta$. Now as z and v are vertices of α_3 we obtain $s \geq \rho - 4\delta + \rho - 4\delta - 12\delta = 2\rho - 20\delta$. But $s = 32\delta + 1$ so we obtain $32\delta + 1 \geq 2\rho - 20\delta$, which gives the contradiction since $\rho \geq 64\delta + 3$.

It follows that we have the combinatorial form 3 for the tree τ. Denote by z the projection of x and y onto α_3 in τ. We have $\mathrm{Dist}(x', z) \leq \mathrm{Dist}_\tau(x', z) + 6\delta = \mathrm{Dist}_\tau(x', x) - \mathrm{Dist}_\tau(x, z) + 6\delta \leq \mathrm{Dist}(x', x) - \mathrm{Dist}(x, z) + 12\delta \leq 16\delta$. Similarly $\mathrm{Dist}(y', z) \leq 16\delta$. So $\mathrm{Dist}(x', y') \leq 32\delta + 1 = s$, i.e α_3 is s-short. This ends the proof of point (1) of Lemma 5.4.1.

Step 3

In this step we will prove that α_3 is θ-straight.

We argue by contradiction and we suppose that α_3 is θ-bent at vertex z. Now since $\delta = \alpha_1.\alpha_2.\alpha_3.\alpha_4$ is 8ρ-short, by the choice of θ, δ passes at least twice through z. So either $z \in \alpha_1$ or $z \in \alpha_2$ or $z \in \alpha_4$.

Suppose $z \in \alpha_1$. Then $\text{Dist}(x, z) \leq \text{Dist}(x, y) \leq s$ by hypothesis and $\text{Dist}(x', z) \leq \text{Dist}(x', y') \leq s$ by Step 2. Thus $\rho = \text{Dist}(x, x') \leq \text{Dist}(x, z) + \text{Dist}(z, x') \leq 2s$, which gives a contradiction since $\rho \geq 64\delta + 3$ and $s = 32\delta + 1$. Thus the only possibilities remaining are either $z \in \alpha_4$ or $z \in \alpha_2$.

Now suppose that $z \in \alpha_4$, then as α_4 is a gradient line we have $\varphi(z, y') = \text{Dist}(z, y')$. Therefore by Corollary 4.9 α_3 cannot be θ-bent at z. (Considering α_3, x', y' and z instead of α, y, z and x of Corollary 4.9). So $\text{Ang}_z(\alpha_3) \leq \theta$.

Similarly one can prove that if $z \in \alpha_2$ then $\text{Ang}_z(\alpha_3) \leq \theta$. That gives that α_3 is θ-straight.

$\diamondsuit$

We consider the same constant s, ρ and θ and k as in previous lemma. Given two vertices x, y suppose that $\varphi(x, y) = 0$ and there exists a θ-straight s-short geodesic path α connecting x and y. Let l_x and l_y be two gradient lines issued respectively from x and y.

Lemma 5.5. *l_x and l_y are parallel and coterminal.*

Proof. Consider maximal sequences $\{x_i\}_i \subseteq l_x$ and $\{y_i\}_i \subseteq l_y$ so that $x_0 = x$, $y_0 = y$ and for all $i \geq 1$

$*$ $\varphi(x_i, y_i) = 0$

$*$ $\text{Dist}(x_i, x) > \text{Dist}(x_{i-1}, x)$ and $\text{Dist}(y_i, x) > \text{Dist}(y_{i-1}, x)$

$*$ either $\text{Dist}(x_i, x_{i-1}) = \rho, \text{Dist}(y_i, y_{i-1}) = \rho$ and $[x_{i-1}, x_i]_{l_x}$, $[y_{i-1}, y_i]_{l_y}$ are 4θ-straight,

or $\text{Dist}(x_i, x_{i-1}) < \rho, \text{Dist}(y_i, y_{i-1}) < \rho$ and either l_x is 4θ-bent at x_i or l_y is 4θ-bent at y_i.

For all i let α_i be a geodesic path connecting x_i and y_i so that $\alpha_0 = \alpha$. We will apply a recursive argument on the indices of sequences to prove that for all i, α_i is s-short and θ-straight. Hence we will obtain that l_x and l_y are parallel and that if they are maximal then they are also coterminal.

Thus for $x_0 = x$ and $y_0 = y$ the hypotheses of recursion is satisfied by hypotheses of the lemma. Suppose now that α_{i-1} is s-short and θ-straight. Now there are two possibilities. Either there is no x_i and y_i with the above properties (in this case one of the gradient lines l_x, l_y is maximal), or there are such x_i and y_i.

In first case we can suppose without loss of generality that l_x lands to a vertex a where $\mathrm{Dist}(x_{i-1}, a) \leq \rho$ and if l_y also lands to a vertex b' then $\mathrm{Dist}(x_{i-1}, a) \leq \mathrm{Dist}(y_{i-1}, b')$. (If not, we exchange x and y and apply the same argument). So consider the vertex b of l_y so that $\varphi(a, b) = 0$. Suppose that $b \neq a$ and consider a geodesic path β connecting a and b. Since β is $k\theta$-short and a is a landing point by Corollary 4.8 applied to β, a and b instead of α, x and z we obtain $\varphi(b, a) = \mathrm{Dist}(a, b)$. But this is impossible since $\varphi(a, b) = 0$. Therefore $a = b$ and it follows that l_y lands also at a. So l_x and l_y stay $(\rho + s)$- distant from each other, hence they are parallel and coterminal.

In second case there are x_i and y_i satisfying the properties required above. Then again there are two possible cases. First case is when $\mathrm{Dist}(x_i, x_{i-1}) = \rho$, $\mathrm{Dist}(y_i, y_{i-1}) = \rho$ and $[x_{i-1}, x_i]_{l_x}$, $[y_{i-1}, y_i]_{l_y}$ are 4θ-straight and second case is when $\mathrm{Dist}(x_i, x_{i-1}) < \rho, \mathrm{Dist}(y_i, y_{i-1}) < \rho$ and either l_x is 4θ-bent at x_i or l_y is 4θ-bent at y_i.

First suppose the former holds for x_i and y_i. Thus we are under the hypothesis of Lemma 5.4 replacing x, y, x', y', α_1, α_2, α_3, α_4 by respectively x_{i-1}, y_{i-1}, x_i, y_i, α_{i-1}, $[x_{i-1}, x_i]_{l_x}$, α_i and $[y_{i-1}, y_i]_{l_y}$. So α_i is s-short and θ-straight.

Now suppose that the second case holds. Denote by x_i' the vertex adjacent to x_i in l_x with $\mathrm{Dist}(x_i', x_0) > \mathrm{Dist}(x_i, x_0)$. Similarly we define y_i' and consider a geodesic path β_i connecting x_i' and y_i'. Note that $[x_{i-1}, x_i']_{l_x}$ and $[y_{i-1}, y_i']_{l_y}$ are ρ-short, $\varphi(x_i', y_i') = 0$ and $\mathrm{Dist}(y_{i-1}, y_i) < \mathrm{Dist}(y_{i-1}, y_i')$. Now we can suppose without loss of generality that l_y is 4θ-bent at y_i. (In fact if not we exchange the roles of x and y). Hence we are under the hypothesis of Lemma 5.3 considering x_{i-1}, y_{i-1}, x_i', y_i' and y_i instead of x, y, x', y' and z. Thus $y_i \in l_x$ which implies $y_i = x_i$ since $\varphi(x_i, y_i) = 0$, and so α_i is an empty path and satisfies the inductive hypotheses.

$\diamondsuit$

Now we have all the necessary tools to prove the main result Lemma 5.1.

Proof. (of Lemma 5.1.) Let α be a geodesic path connecting a and b. Consider the vertex set $\{x_i\}_{i \in \{0,\ldots,n\}}$ of α so that $x_0 = a$, $x_n = b$ and x_i and x_{i+1} are adjacent in α. For all i consider a gradient line l_i issued from x_i so that $l_0 = l_a$ and $l_n = l_b$. To prove that l_a and l_b are parallel and coterminal we will prove that l_i and l_{i+1} are parallel and coterminal. For each i, we have three possibilities, $\varphi(x_i, x_{i+1}) = 0$, $\varphi(x_i, x_{i+1}) < 0$ and $\varphi(x_i, x_{i+1}) > 0$. Now note that by the choice of x_i, (x_i, x_{i+1}) is s-short and θ-straight. Therefore in the first case we are under the assumptions of Lemma 5.5. Note also that considering $(x_{i+1}, x_i).l_i$ in the second case (or $(x_i, x_{i+1}).l_{i+1}$ in the third case) we brought the problem to the question whether $(x_{i+1}, x_i).l_i$ and l_{i+1} (or respectively $(x_i, x_{i+1}).l_{i+1}$ and l_i) are parallel and coterminal as in figure Figure 2.5.5.

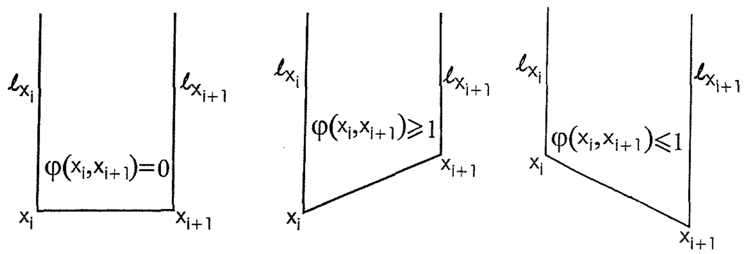

Figure 2.5.5

But these last two cases also satisfy the assumptions of Lemma 5.5, replacing $(x_{i+1}, x_i).l_i$, l_{i+1} and x_i (or respectively $(x_i, x_{i+1}).l_{i+1}$, l_i and x_{i+1}) by l_x, l_y and $y = x$ of Lemma 5.5.

$\diamondsuit$

6. A FINITE PRESENTATION FOR RELATIVELY HYPERBOLIC GROUPS

Let Γ be a hyperbolic group relative to the family of subgroups $\mathcal{G}$, $\mathcal{K}(\mathcal{V}, \mathcal{E})$ be a δ-hyperbolic graph given by the definition of Γ with its metric "Dist", and $G_1, ..., G_n$ be a set of representatives of the conjugacy classes in $\mathcal{G}$. As in Theorem 0.1, for the rest of this section we assume that for each i the minimal compactification of G_i is finitely presented with special character \$.

For all $i \in \{1, \ldots, n\}$, $p_i \in \mathcal{V}_\infty$ denotes the fixed point of G_i and $e_i = (p_i, q_i)$ the arbitrary edge adjacent to p_i chosen at the beginning of Section 4. Recall that $\rho \geq 64\delta+3$ and $\theta \geq 8\rho$, no pair of edges at a vertex of finite valence have an angle more than θ and $\theta/2$ is strictly greater than $\mathrm{Max}_{g \in F_i}\{\mathrm{Ang}(e_i, ge_i)\}$ for all $i \in \{1, \ldots, n\}$. As before we choose θ to be greater than 4μ, where μ is the constant given by Lemma 2.8 and Lemma 2.9. Recall also that $\{r_j\}_j$ is an orbit transversal of $\mathcal{V}$ under the action of Γ so that $\{p_i\}_i \subseteq \{r_j\}_j$, and for each j we have an arbitrary edge $\{e'_j\}_j$ such that it has an end point r_j and $\{e_i\}_i \subseteq \{e'_j\}_j$. We fix a vertex v_0 and an edge $e_0 = (v_0, v)$ and we choose for $k \geq 1000$ the constants R and Θ such that for all j, $Cone_{R,\Theta}(e_0, v_0)$ contains $Cone_{k\theta,k\theta}(e'_j, r_j)$. We denote $Cone_{R,\Theta}(e_0, v_0)$ by N.

6.1 Busemann Functions and Distance Functions

Definition. (Busemann Function)

Given a geodesic ray α in $\mathcal{K}$, denote by $\{a_i\}_{i\in\mathbf{N}}$, its vertex set. Recall that the *busemann function* associated to α is

$$h_\alpha : \mathcal{V} \to \mathbf{Z}, \;\; h_\alpha(x) = \lim_{i\to\infty}(i - \mathrm{Dist}(x, a_i)).$$

Note that the sequence $i - \mathrm{Dist}(x, a_i)$ is an non decreasing sequence in $\mathbf{Z}$. Moreover it is bounded above by $\mathrm{Dist}(a_0, x)$. Therefore it converges.

Definition. (Distance Function)

Given a vertex a in $\mathcal{K}$ recall that the *distance function* associated to a is

$$d_a : \mathcal{V} \to \mathbf{Z}, \;\; d_a(x) = \mathrm{Dist}(a, x).$$

We will often omit the subscript α of the busemann function h_α and the subscript a of the distance function d_a since there is no ambiguity once the geodesic ray α or the vertex a is fixed.

The next two lemmas that we mention without giving a proof will be used in the main result of this part. A detailed proof can be found for example in [GhH,CP].

Lemma 6.1. *Given a vertex a, the distance function associated to a is 1-lispchitz. Similarly given a geodesic ray α, the busemann function associated to α is 1-lipschitz.*

Let X be a geodesic space with metric Dist_X. We say that a function $f : X \to \mathbf{R}$ is *ϵ-convex* if it satisfies the following property:

If z lies on a geodesic path connecting two points x and y then $f(z) \le tf(y) + (1-t)f(x) + \epsilon$ where $t = \mathrm{Dist}_X(z, x)/\mathrm{Dist}_X(x, y)$.

Lemma 6.2. *For all a in $\mathcal{V}(\mathcal{K})$, the distance function d_a is 4δ-convex. Similarly for all geodesic rays α, the function $-h_\alpha$ is 4δ-convex, where h_α is the busemann function.*

Given a geodesic ray α we define a map $\varphi : \mathcal{V}\times\mathcal{V} \to \mathbf{Z}$ associated to the busemann function h_α so that $\varphi(x, y) = h_\alpha(y) - h_\alpha(x)$. Similarly given a vertex a we define a map φ associated to the distance function d_a by $\varphi(x, y) = d_a(x) - d_a(y)$.

Lemma 6.3. *φ associated to the busemann function h_α or to a distance function d_a satisfies the properties below:*

1) *If x, y are two adjacent vertices in $\mathcal{K}$ then $\varphi(x, y) \in \{-1, 0, 1\}$.*

2) *For all $x, y, z \in \mathcal{V}(\mathcal{K})$, $\varphi(x, y) + \varphi(y, z) + \varphi(z, x) = 0$*

3) *If z belongs to a geodesic path connecting x and y then $t\varphi(z, y) + (1 - t)\varphi(z, x) - 4\delta \leq 0$ where $t = \mathrm{Dist}(x, z)/\mathrm{Dist}(x, y)$.*

4) *For all vertices x in $\mathcal{K}$, except the vertex a in the case of the distance function d_a, there exists an adjacent vertex y to x so that $\varphi(x, y) = 1$.*

Proof. The property 2) is immediate by definition of h_α and d_a.

The property 1) follows from Lemma 6.1 since $|\varphi(x, y)| \leq \mathrm{Dist}(x, y)$ for all $x, y \in \mathcal{K}$ and so when x, y are adjacent vertices we obtain $|\varphi(x, y)| \leq 1$. Moreover $\varphi(x, y) \in \mathbf{Z}$ for all x, y. Hence $\varphi(x, y) \in \{-1, 0, 1\}$.

To prove Property 3) we use lemma 6.2. We give the proof only for $h = h_\alpha$ since the case of d_a can be done similarly. Suppose z belongs to a geodesic path connecting two vertices x and y. Then $t\varphi(z, y) + (1-t)\varphi(z, x) - 4\delta = th(y) + (1-t)h(x) - 4\delta - h(z) \leq h(z) - h(z) = 0$.

In the case of the distance function property 4) is immediate since we can take as y the vertex adjacent to x in the geodesic connecting a and x, while the case of busemann function is a bit more complicate. This time we have the geodesic ray α with vertex set $\{a_i\}_{i \in N}$. Denote by β_i a geodesic path connecting x and a_i and denote also the vertex adjacent to x in β_i by y_i.(See Figure below).

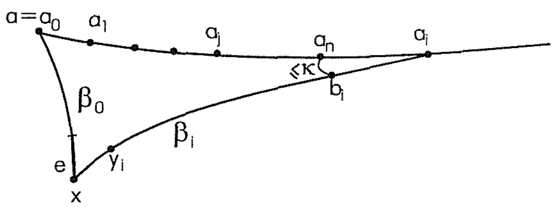

Figure 2.6.1

Since $\mathcal{K}$ is hyperbolic there exists a constant κ depending only the constant of hyperbolicity δ and an index n so that $\mathrm{Dist}(a_n, \beta_i) \leq \kappa$ for all i. Suppose that $\mathrm{Dist}(a_n, \beta_i) = \mathrm{Dist}(a_n, b_i)$ where $b_i \in \beta_i$. Now note that $\mathrm{Dist}(b_i, x)$ is bounded by $\kappa + n + \mathrm{Dist}(x, a_0) = \lambda$ and so $\mathrm{Ang}_x(\beta_0, [x, b_i]_{\beta_i})$ is bounded by $2\lambda - 2$ by Proposition 2.2. Therefore for all i, y_i belongs to the cone $\mathrm{Cone}_{1,2\lambda}(e, x) = C$ where e is the edge adjacent to x in β_0. Thus by Corollary 3.3, C is finite and after passing to subsequence we have $y_i = y$ for all i. Thus $1 = \varphi(x, y_i) = \varphi(x, y)$.

$\diamond$

We see that φ defined as above is in fact a global cocycle since it satisfies Property 2 of Lemma 6.3. We will refer to φ associated to a busemann function as a *busemann cocycle* and associated to a distance function as a *radial cocycle.* Moreover we see by the properties 4) of Lemma 6.3 for every vertex of $\mathcal{K}$, except the vertex a in the case of a distance function d_a, there exists a gradient line issued from this vertex.

We note that if φ is a busemann cocycle associated to h_α then by definition, α itself is a gradient line. The next lemma shows that in fact all the gradient lines associated to φ are asymptotic to α.

Lemma 6.4. *If φ is a busemann cocycle associated to h_α then gradient lines associated to φ remains a bounded distance from each other, hence define the same point of the boundary.*

Proof. To prove the lemma we use the property of convexity of $\varphi(x, y)$. Let l_x, l_y be two gradient lines issued from x and y. Suppose that l_x and l_y do not converge to the same boundary point of α. Without loss of generality suppose that $\varphi(x, y) = 0$. In fact if not either we can change x by $x' \in l_x$ or y by some $y' \in l_y$ and argue similarly. Let $\{x_i\}_i$ and $\{y_i\}_i$ be vertex sets of l_x and l_y so that $\varphi(x_i, y_i) = 0$ for all i. For all i we consider a geodesic path β_i connecting x_i and y_i and a geodesic path ω_i connecting x and y_i. We know by hyperbolicity of $\mathcal{K}$ that there exists a constant μ depending only the constant of hyperbolicity so that for all i there exists a vertex $z_i \in \beta_i$ which stays μ-distant from $[x, x_i]_{l_x}$ and ω_i. Moreover as i tends to ∞ z_i stays in a bounded distant from x in $\mathcal{K}$ since x_i and y_i converges to distinct points of boundary. But by the property 3) of Lemma 6.3 we have $\varphi(x, z_i) \geq (1 - t)\varphi(x, x_i) + t\varphi(x, y_i) - 4\delta$ where $t = \mathrm{Dist}(x_i, z_i)/\mathrm{Dist}(x_i, y_i)$. We also know that $\varphi(x, x_i) = i$ and $\varphi(x, y_i) = \varphi(x, y) + \varphi(y, y_i) = i$ since $\varphi(x, y) = 0$. In other words $\varphi(x, z_i) \geq i - 4\delta$ But this gives a contradiction for i large enough since $i - 4\delta \leq \varphi(x, z_i) \leq \mathrm{Dist}(x, z_i)$ and $\mathrm{Dist}(x, z_i)$ is bounded.

$\diamondsuit$

Similarly we see that if φ is a radial cocycle associated to d_a then every geodesic path connecting the vertex a and a vertex x of $\mathcal{K}$ is a maximal gradient line issued from x landing at a. The next lemma assures that all the gradient lines of φ are maximal and landing at a.

Lemma 6.5. *If φ is the radial cocycle associated to the distance function d_a then all the gradient lines associated to φ are the maximal gradient lines landing at a*

Proof. Let l be a gradient line issued from a vertex x. Consider a geodesic path α connecting x and a. This is a maximal gradient line landing at a. For all $y \in l$ consider the geodesic path β_y connecting a and y again this is a maximal gradient line landing at a. Hence we have $\mathrm{Dist}(x,a) = \varphi(x,a) = \varphi(x,y) + \varphi(y,a) = \mathrm{Dist}(x,y) + \mathrm{Dist}(y,a)$. But this is exactly to say that for all $y \in l$ the concatenation $[x,y]_l.\beta_y$ is a geodesic path. But there are only finitely many such geodesic paths. Therefore l is a maximal gradient line landing at a.

$\diamondsuit$

The next lemma is one of the key points of this piece of work as it gives the motivation. We will use it in order to associate to a busemann cocycle or a radial cocycle an element of the subshift Φ constructed in the previous sections. Recall that μ is the constant given in Lemma 2.9.

Lemma 6.6. *Given a busemann cocycle or radial cocyle, if β is a gradient line issued from x and ω is a geodesic line connecting x and z so that $\mathrm{Ang}_x(\beta,\omega) > 2\mu$ then the concatenation $\omega.\beta$ is a gradient line issued from z.*

Proof. As $\mathrm{Ang}_x(\beta,\omega) > 2\mu$ where μ is the constant given by Lemma 2.9, the concatenation $\omega.\beta$ is a geodesic line.

If φ is the radial cocycle associated to the distance function, d_a, then $\varphi(z,x) = \mathrm{Dist}(z,a) - \mathrm{Dist}(x,a)$. But as $\omega.\beta$ is a geodesic line $\mathrm{Dist}(z,a) = \mathrm{Dist}(z,x) + \mathrm{Dist}(x,a)$, which implies $\varphi(z,x) = \mathrm{Dist}(z,x)$.

Now suppose that φ is the busemann cocycle associated to the busemann function h_α. Denote the vertex set of α by $\{a_n\}_{n\in\mathbf{N}}$, and the geodesic paths connecting x to a_n by α_n. By lemma 6.4 we know that β and α converge to the same boundary point. Thus, a_n converge to the boundary point of β and $\mathrm{Dist}(a_n,\beta)$ is bounded by a constant. Now by Lemma 2.9, we see that $\mathrm{Ang}_x(\beta,\alpha_n) \le \mu$ for large n, since the concatenation $\alpha_n.\beta$ cannot be a geodesic line. Therefore for all large n, we have $2\mu < \mathrm{Ang}_x(\omega,\beta) \le \mathrm{Ang}_x(\omega,\alpha_n) + \mathrm{Ang}_x(\beta,\alpha_n) \le \mathrm{Ang}_x(\omega,\alpha_n) + \mu$, and so $\mathrm{Ang}_x(\omega,\alpha_n) > \mu$. Then again by Lemma 2.9 we see that the concatenation $\omega.\alpha_n$ is a geodesic path and therefore $\mathrm{Dist}(z,a_n) = \mathrm{Dist}(z,x) + \mathrm{Dist}(x,a_n)$. Hence by the definition of h_α we obtain $\varphi(z,x) = \mathrm{Dist}(z,x)$.

$\diamondsuit$

Recall that p_i is the fixed point of G_i in $\mathcal{K}$ and $e_i = (p_i, q_i)$ is the arbitrary edge adjacent to p_i.

Lemma 6.7. *Given φ associated to a busemann function or a distance function landing at a vertex a of infinite valence there exists a $\sigma \in \Phi$ so that φ is the global cocycle associated to σ by Lemma 4.5.*

Proof. We know by property 4) of Lemma 6.3 that for all vertices x in $\mathcal{K}$, except the vertex a in the case of the distance function, d_a, there exists at least one vertex y so that $\varphi(x, y) = 1$. For each vertex of infinite valence, x, we fix such a vertex and denote it by y_x. Moreover since the action of Γ is free on the edge set of $\mathcal{K}$ (by Lemma 3.1), there exists a unique $\gamma_x \in G_i$ so that $\gamma_x e_i = (x, y_x)$.

Now we define the map $\sigma : \Gamma \to \mathcal{A}$ so that

- For all $\gamma \in \Gamma$ and for all $x, y \in N \times N$, $\sigma_1(\gamma)(x, y) = \varphi(\gamma x, \gamma y)$.

- For all $i \in \{1, \ldots, n\}$, for all $\gamma \in \Gamma$, the map $G_i \to A_i$, $g \mapsto \sigma_2(\gamma g)$ is an element of Φ_i satisfying:

if $\gamma p_i = a$ where φ is associated to a distance function landing at a then $\sigma_2(\gamma g) \neq \$$ for all $g \in G_i$

otherwise $\sigma_2(\gamma_x) = \$$ where $x = \gamma p_i$.

Note that φ satisfies properties (A1),..., (A4) on every γN by Lemma 6.3. In other words φ is an element of Υ. Therefore σ satisfies properties (B1), (B2), (B3) and (B5) of subshift Φ by definition.

To prove that σ is in Φ it remains to check that σ satisfies property (B4). Note that if γp_i is the landing point a of the distance function d_a then $\sigma_2(\gamma g) \neq \$$ for all $g \in G_i$ by definition of σ, and every geodesic path connecting any given vertex y to a is a gradient line issued from y. Therefore $\sigma_1(\gamma)(p_i, \gamma^{-1} y) = \varphi(\gamma p_i, y) = -\operatorname{Dist}(a, y) < 1 - \operatorname{Dist}(a, y)$. So suppose that $\gamma p_i = x$ is not a landing point. Thus $\sigma_2(\gamma_x) = \$$. Now let α be a geodesic path connecting x and y so that $\varphi(x, y) \geq 1 - \operatorname{Dist}(x, y)$. Consider $\gamma' \in G_i$ so that $\gamma' e_i$ is the edge adjacent to x in α. We will show that $\operatorname{Ang}(\gamma_x e_i, \gamma' e_i) \leq \theta/2$. In fact if not $\operatorname{Ang}(\gamma_x e_i, \gamma' e_i) > 2\mu$, moreover by definition (x, y_x) is a gradient line issued from x. So by Lemma 6.6 applied to α, (x, y_x) we see that the concatenation $\alpha.(x, y_x)$ is a gradient line issued from y. Hence $\varphi(x, y) = -\operatorname{Dist}(y, x)$, which is a contradiction with $\varphi(x, y) \geq 1 - \operatorname{Dist}(x, y)$.

$\diamondsuit$

6.2 End of the proof of Theorem 0.1

In this section we will define a map Π from Ψ on $\partial\Gamma$ and prove that Π is surjective, continuous and Γ-equivariant.

We saw in the section 4 that given an element σ of Φ we can associate a unique element φ_σ of Υ (Lemma 4.5). Moreover in section 5 by Lemma 5.1 we prove that any two gradient lines associated to a given φ_σ are coterminal and parallel. This together with Lemma 4.10 implies that either the gradient lines associated to φ_σ converge to a unique point of $\partial\mathcal{K}$ or land at a point of $\mathcal{V}_\infty$. Therefore we can define a function $\Pi : \Phi \to \partial\mathcal{K} \cup \mathcal{V}_\infty = \partial\Gamma$ so that

$\Pi(\sigma) = x \in \partial\mathcal{K}$ if the gradient lines of φ_σ converge to x

$\Pi(\sigma) = x \in \mathcal{V}_\infty$ if the gradient lines of φ_σ sink at x.

We will prove that Π thus defined is surjective continuous and Γ-equivariant. The next result is given for fine hyperbolic graphs in [Bo1] (Proposition 8.9). The proof uses strongly fineness of the graphs which is satisfies in our case as $\mathcal{K}$ is fine.

Lemma 6.8. *For every quasigeodesic ray α issued from a point a in $\mathcal{K}$ there exists a geodesic ray issued from a remaining in a bounded distance of α.*

Lemma 6.9. Π *is surjective*

Proof. By Lemma 6.8 we see that for every point X of $\partial\mathcal{K}$ we have a geodesic ray α converging to X. So we can consider the busemann function h_α associated to this geodesic ray. By lemma 6.4 we know that every gradient line of the cocycle φ associated to h_α converges to X. Moreover by Lemma 6.7 there exists a $\sigma \in \Phi$ so that φ is the global cocycle associated to σ by Lemma 4.5. Thus $\Pi(\sigma) = X$. Similarly given $x \in \mathcal{V}_\infty$ the distance function d_x gives rise to a cocycle $\varphi \in \Upsilon$ so that all gradient lines associated to φ land at x. Again considering $\sigma \in \Phi$ given by Lemma 6.7 so that φ is the global cocycle associated to σ we obtain $\Pi(\varphi) = x \in \mathcal{V}_\infty$.

$\diamondsuit$

Lemma 6.10. Π *is continuous*

Proof. We will prove that if $\sigma_i \in \Phi$ converges to $\sigma \in \Phi$ then $\Pi(\sigma_i)$ converges to $\Pi(\sigma)$ in $\partial\mathcal{K}$.

First we note that if $\sigma_i \in \Phi$ converges to $\sigma \in \Phi$ then φ_{σ_i} will coincide on the large finite sets of $\mathcal{K}$ with the ones associated to φ_σ. Therefore the gradient lines associated to φ_{σ_i} will coincide with the ones associated to φ_σ on the large finite sets of $\mathcal{K}$.

Now assume that $\sigma_i \in \Phi$ converges to $\sigma \in \Phi$ and $\Pi(\sigma_i) = a_i$ converges to $a \neq b = \Pi(\sigma)$ in $\partial\Gamma$. We will treat two possible cases.

First case is when $a \in \mathcal{V}_\infty$. Consider a gradient line associated to φ_σ issued from a (the existence is justified by Lemma 4.10). Denote it by α. Hence α either lands at b or converges to b depending on whether $b \in \mathcal{V}_\infty$ or $b \in \partial\mathcal{K}$. We choose $c \in \mathcal{V}(\mathcal{K})$ so that if $b \in \partial\mathcal{K}$ then $c \in \alpha$ with $\mathrm{Dist}(c, a) \geq 1$ and if not $c = b$. As $\sigma_i \in \Phi$ converges to $\sigma \in \Phi$ we see by the above note that for all large i, $[a, c]_\alpha$ is a gradient line associated to φ_{σ_i}. We consider for all i a gradient line β_i associated to φ_{σ_i} issued from c. Hence either β_i lands at a_i or converges to a_i depending on $a_i \in \mathcal{V}(\mathcal{K})$ or $a_i \in \partial\mathcal{K}$. In either case for all large i the concatenation $[a, c]_\alpha.\beta_i$ is a gradient line associated to φ_{σ_i}. Therefore it is a geodesic line connecting a to a_i. Now denote by x the vertex adjacent to a in α and consider the neighbourhood $M_{1_\mathrm{N}}(a, \{x\})$ of a in $\partial\Gamma$. We see that the geodesics $[a, c]_\alpha.\beta_i$ connecting a and a_n are not in $M_{1_\mathrm{N}}(a, \{x\})$ for all large i, but this gives a contradiction with a_i converges to a in $\partial\Gamma$.

Now suppose that $a \in \partial\mathcal{K}$. Consider a geodesic line α connecting a and b. We know that there exists a constant κ depending only the constant of hyperbolicity of $\mathcal{K}$ so that any two geodesic rays with the same end point and converging to the same boundary point stay uniformly κ-distant from each other. Choose a vertex x on α so that $\mathrm{Dist}(x, b) > \kappa$ if $b \in \mathcal{V}_\infty$. There exists a gradient line β associated to φ_σ issued from x. Hence β either lands at b or converges to b depending on $b \in \mathcal{V}_\infty$ or $b \in \partial\mathcal{K}$. Choose a vertex $c \in \beta$ so that if $b \in \partial\mathcal{K}$ then $c \in \beta$ with $\mathrm{Dist}(c, x) > \kappa$ and if not $c = b$. As σ_i converges to σ, for all large i, $[x, c]_\beta$ is a gradient line associated to φ_{σ_i} issued from x. For all i we consider a gradient line β_i associated to φ_{σ_i} issued from c. Thus for all large i the concatenation $[x, c]_\beta.\beta_i$ is a gradient line associated to φ_{σ_i} issued from x, and hence a geodesic line connecting x to a_i. Denote the components of α remaining between x and a, and between x and b respectively by α_a and α_b. As a_i converges to a we see that α_a and $[x, c]_\beta.\beta_i$ stay uniformly κ-distant from each other. In other words for $y \in \alpha_a$ with $\mathrm{Dist}(x, y) = \mathrm{Dist}(x, c)$ we have $\mathrm{Dist}(y, c) \leq \kappa$. Moreover similarly as α_b and β are uniformly κ-distant we see that for $z \in \alpha_b$ with $\mathrm{Dist}(z, x) = \mathrm{Dist}(c, x)$ we have $\mathrm{Dist}(z, c) \leq \kappa$. Therefore $\mathrm{Dist}(y, z) \leq \kappa$. But z, x, y are in α and $\mathrm{Dist}(x, c) = \mathrm{Dist}(x, z) = \mathrm{Dist}(x, y) > \kappa$. Thus $\mathrm{Dist}(y, z) > 2\kappa$, which gives a contradiction.

$\diamondsuit$

Lemma 6.11. Π *is* Γ*-equivariant*

Proof. We first note that given a $\sigma \in \Phi$ and $\gamma \in \Gamma$ the gradient lines associated to $\varphi_{\gamma\sigma}$ are exactly γ-translations of the gradient lines associated to φ_σ. In fact given $x, y \in \gamma_0 N$ for some γ_0 we have $\varphi_\sigma(x, y) = \sigma_1(\gamma_0)({\gamma_0}^{-1}x, {\gamma_0}^{-1}y) =$

$\gamma\sigma_1(\gamma\gamma_0)(\gamma_0{}^{-1}\gamma^{-1}(\gamma x), \gamma_0{}^{-1}\gamma^{-1}(\gamma y)) = \varphi_{\gamma\sigma}(\gamma x, \gamma y)$ for all $\gamma \in \Gamma$. Hence if $\varphi_\sigma(x, y) = 1$ then $\varphi_{\gamma\sigma}(\gamma x, \gamma y) = 1$. Then the Γ-equivariance follows from the definition of Π.

$\diamondsuit$

Lemma 6.12. *The action of Γ on its boundary is expansive.*

Proof. If Δ is the diagonal of $(\partial\Gamma) \times (\partial\Gamma)$, then we have to find a neighborhood U of Δ such that $\Delta = \bigcap_{\gamma\in\Gamma} \gamma U$.

Recall that $e'_1, \dots, e'_m$ is a set of orbit representatives of the edges in $\mathcal{K}$. Let X be the set of pairs of points $(x_1, x_2) \in (\partial\mathcal{K})^2$ such that there is a bi-infinite geodesic between x_1 and x_2 passing through one of the e'_j. Now as $p_1, \dots, p_n$ are bounded parabolic points, for all i, the stabilizer G_i of p_i acts on $\partial\Gamma \setminus \{p_i\}$ with compact quotient. We can assume without loss of generality that there exists compact subset $K \subset \partial\Gamma \setminus \{p_1, \dots, p_n\}$ so that $\partial\Gamma \setminus \{p_i\} = \cup_{g\in G_i} gK$ for all $i \in \{1, \dots, n\}$ (See Lemma 6.3 Part1). Let then Y be the set of pairs of points (p_i, K). We choose $U = (\partial\Gamma \times \partial\Gamma) \setminus (X \cup Y)$.

First we show that $\Delta = \bigcap_{\gamma\in\Gamma} \gamma U$.

Clearly if there exists γ so that $\gamma(x_1, x_2)$ is not in U then (x_1, x_2) is not in Δ. Hence we have the direct inclusion. Now let x_1 and x_2 two distinct points of $\partial\Gamma$, and so (x_1, x_2) is not in Δ. We will show that it is not in all translates of U. We consider two cases, either they are both in $\partial\mathcal{K}$, or one them, say x_1, is a vertex of $\mathcal{K}$ of infinite valence. In the first case, there is a bi-infinite geodesic from one point to another, and it can be translated so that its image passes by one of the e'_j. Therefore, there is γ such that $\gamma(x_1, x_2)$ is in X, hence not in U. In the second case, there is $\gamma \in \Gamma$ such that γx_1 is one of the p_i. Now there is $\gamma' \in G_i$ such that $\gamma'\gamma(x_1, x_2)$ is in Y, hence not in U. This proves that the intersection of the translates of U is equal to the diagonal set.

Now we have to show that U is a neighborhood of Δ. That is to say that a sequence of elements in $X \cup Y$ cannot converge to a point of Δ.

Let $(X_n = (x_1^n, x_2^n))_n$ be a convergent sequence of elements of X. After passing to a subsequence, one can assume that, for all n, there is a bi-infinite geodesic between x_1^n and x_2^n passing through the same edge e'_j. If x_1^n converges to a_1 and x_2^n converges to a_2, we see that a_1 and a_2 are linked by a geodesic passing through e'_j, hence non-trivial. Therefore $a_1 \neq a_2$.

Let now $(Y_n = (x_1^n, x_2^n))_n$ be a convergent sequence of elements of Y. After passing to a subsequence, and without loss of generality, one can assume that $x_1^n = p_i$, for all n, and for some i. Then, x_2^n is in the compact set K, and therefore does not converge to p_i. This finally proves that U is a neighborhood of Δ, and ends the proof of Proposition 6.12.

$\diamondsuit$

7. GROUPS ADMITTING A COMPACTIFICATION FINITELY PRESENTED WITH SPECIAL CHARACTER

In this section we give examples of groups admitting a compactification finitely presented with special character, and we give a condition for this. We will concentrate mostly on the case of the minimal compactification, by which we mean the trivial one for finite groups or the one-point compactification otherwise.

Let us begin with a necessary condition.

Proposition 7.1. *If Γ has a compactification finitely presented with special character, then Γ is finitely generated.*

Proof. Let $\pi : \Phi \to \Gamma \cup \{\infty\}$ be a finite presentation with special character. Let $\mathcal{A}$ be the alphabet. Let $\mathcal{C}$ be a cylinder defining Φ, and itself defined by a finite subset, F, of Γ and a set M of maps from F to $\mathcal{A}$. The set of translates of F is a covering of Γ. Let P be the nerve of the covering. As F is finite, P is a finite dimensional, locally finite polyhedron on which Γ acts properly discontinuously and cocompactly. The set of vertices of P is naturally identified with Γ. The claim is that P is connected. If it was not, there would be distinct connected components, C_i. Let $\gamma_i \in C_i$, and consider $\sigma_i \in \Phi$ such that $\pi(\sigma_i) = \gamma_i$. Let $\sigma \in \mathcal{A}^\Gamma$ such that $\sigma|_{C_i} \equiv \sigma_i|_{C_i}$. Now, σ has several special characters (one in each C_i). On the other hand all the cylinder conditions defining Φ are satisfied, as, by definition, they are read on the connected components of P. This is a contradiction, and it proves the claim. Therefore, Γ is generated by F which is a finite set.

$\diamondsuit$

We state the next proposition, which in fact is a slight variation of a theorem of Gromov. A detailed proof of this theorem can be found in [CP].

Proposition 7.2. *If Γ is a hyperbolic group, then Γ action on $\Gamma \cup \{\infty\}$ is finitely presented with special character. Its minimal compactification is finitely presented with special character.*

We give a idea of the proof. Consider the proof of Theorem 0.1, considering Γ relatively hyperbolic relative to the trivial subgroup $\{1\}$. A Cayley graph plays the role of $\mathcal{K}$, and we consider the same cocycles. They can define either a point in the hyperbolic boundary and hence ∞, or a vertex of the graph. We obtain our presentation choosing the special character to be the restriction of a radial cocycle.

In fact as we will elaborate in Appendix A), we can in fact define the notion of finite presentation with a special character for the action of a group on a compactification $\Gamma \cup M$. We also prove in Appendix A) that this property passes to the quotients of $\Gamma \cup M$ which have the property of expansivity. This gives us another proof of the above proposition.

◇

Although it could be seen as a consequence of the proposition above, the example in part 1 already gave the basic examples of $\mathbf{Z}$ and of finite groups. Most of our remaining examples come from the following remark.

Proposition 7.3. *If a group Γ splits in a short exact sequence $\{1\} \to N \to \Gamma \to H \to \{1\}$, and if both N and H have their minimal compactification finitely presented with special character, then the minimal compactification of Γ is finitely presented with special character.*

Proposition 7.4. *Let G be a subgroup of finite index of a group Γ. The group G has its minimal compactification finitely presented with special character if, and only if, the minimal compactification of Γ is finitely presented with special character.*

Before giving the proofs, we give some consequences. A group said to be *poly*-hyperbolic if there is a sequence of subgroups $\{1\} = N_0 \lhd N_1 \lhd \ldots \lhd N_{k-1} \lhd N_k = \Gamma$ with all the quotients N_{i+1}/N_i are hyperbolic.

Corollary 7.5. *Any polyhyperbolic group has its one point compactification finitely presented with special character.*

In particular this includes virtually polycyclic(hence vitually nilpotent) groups

Proof. If Γ is polyhyperbolic , there is a sequence of subgroups $\{1\} = N_0 \lhd N_1 \lhd \ldots \lhd N_{k-1} \lhd N_k = \Gamma$, with all the quotients N_{i+1}/N_i hyperbolic. Using Proposition 7.3, and the fact that hyperbolic groups have their one-point compactification finitely presented with special character, we see by induction on i that all N_i have their minimal compactification finitely presented with special character.

◇

Corollary 7.6. *Any free-by-cyclic, or free-by-finite group has its one-point compactification finitely presented with special* character.

Proof. If $\Gamma = F \ltimes N$, where F is free, finitely generated, and N is cyclic or finite, then from Proposition 7.2, F has its one-point compactification finitely presented with special character, and Proposition 7.3 ends the proof.

$\diamondsuit$

Proof. **(of Prop 7.3.)** Let us denote by $\mathcal{A}_N$, $\mathcal{A}_H$, $\$_N$, $\$_H$, $\mathcal{C}_N$, $\mathcal{C}_H$, Φ_N, Φ_H, the alphabets, special characters, cylinders, and subshifts of finite type for the presentations of $N \cup \{\infty\}$ and $H \cup \{\infty\}$. Let F_N, F_H, M_N and M_H be the finite subsets of N and H, and the sets of maps defining the two given cylinders. From Proposition 7.1, N is finitely generated, then up to enlarging F_N, we can assume that F_N generates N (in fact, in the proof of Proposition 7.1, it is proved that necessarly, F_N generates N).

Let $\mathcal{A} = \mathcal{A}_H \times \mathcal{A}_N$. Since we have the surjection from $\pi : \Gamma \to H$ choose one element of $\pi^{-1}(h)$ for each $h \in H$ and denote it by $\tilde{h}$. Let $\widetilde{H}$ be the set of $\tilde{h}$, and let F be the finite subset of Γ defined by $F = \{\tilde{h}.n \,|\tilde{h} \in \widetilde{H}, h \in F_H, n \in F_N\}$.

Let M be the following set of maps. $M = \{m : F \to \mathcal{A} | \exists m_H \in M_H \textit{ s.t. } \forall n \in F_N, m(\cdot.n)_1 = m_H \ ; \ \forall h \in F_H, m(\tilde{h}.\cdot)_2 \in M_N\}$, where the subscripts 1 and 2 denote the coordinates in the product $\mathcal{A} = \mathcal{A}_H \times \mathcal{A}_N$. Consider the cylinder defined by F and M, and the associated subshift of finite type, Φ. We need the following lemma.

We claim that for any $\sigma \in \Phi$, there is at most one element $\gamma \in \Gamma$ such that $\sigma(\gamma) = (\$_H, \$_N)$. We first prove that for any $\sigma \in \Phi$, there is at most one left coset of N, $\tilde{h}N$, such that $\forall n \in N$, $\sigma(\tilde{h}.n)_1 = \$_H$. By definition of M, if $n \in F_N$, $n_0 \in N$, then $\sigma(\tilde{h}.n_0.n)_1$, the first coordinate of $\sigma(\tilde{h}.n_0.n)$ only depends on $\tilde{h}$ and n_0. But F_N was chosen generating N, hence $\sigma(\tilde{h}.n_0.n)_1$ only depends on $\sigma(\tilde{h})$. But, by definition of M, the map $h \in H \mapsto \sigma(\tilde{h})_1$ is in Φ_H, and therefore, by the special character property, there is at most one value of $\tilde{h}$ where it takes the value $\$_H$, this proves the subclaim. Now we need to prove that if $\tilde{h}$ is such that $\sigma(\tilde{h}.n)_1 = \$_H$ then there is at most one $n \in N$ such that $\sigma(\tilde{h}.n)_2 = \$_N$. This is because of the definition of M, as the map $n \in n \mapsto \sigma(\tilde{h}.n)_2$ is in Φ_N. This proves the claim.

Now, the map π so that it sends an element $\sigma \in \Phi$ on the point at infinity, if σ does not contain the character $(\$_H \$_N)$, and on $\gamma \in \Gamma$ if $\sigma(\gamma) = (\$_H \$_N)$. This is well defined, and gives a finite presentation with special character of $\Gamma \cup \{\infty\}$.

$\diamondsuit$

Proof. **(of Prop. 7.4)** Assume that Γ has its one point compactification finitely presented with special character and let us denote by $\mathcal{A}_\Gamma$, $\$_\Gamma$, C_Γ, Φ_Γ, the alphabet, special character, cylinder, and subshift of finite type for the presentation of $G \cup \{\infty\}$.

Let $F_\Gamma \subseteq \Gamma$ and $M_\Gamma \subseteq \mathcal{A}_\Gamma^{F_\Gamma}$ be the finite subset of G , and the set of maps defining the given cylinder. We choose $\{g_1, ..., g_n\}$ a set of orbit transversal of left cosets of G in Γ and denote $F = (\bigcup_{i=1}^n g_i^{-1} F_\Gamma) \cap G$ a finite subset of G. We set $\mathcal{A} = (\mathcal{A}_\Gamma)^n$ and $M \subset \mathcal{A}^F$ be the set of maps m from F to $(\mathcal{A}_\Gamma)^n$ such that there exists $m_\Gamma \in M_\Gamma$ whose translates in $g_i^{-1} m_\Gamma$ coincide with the i-th coordinate of an element σ on the coset $g_i G$. Those three choices define a finite subshift of finite type $\Phi \subseteq \mathcal{A}^G$. By definition of M one sees that there is natural map from Φ to Φ_Γ which consists of pushing the i-th coordinate of an element σ on the coset $g_i G$. This map is continuous G-equivariant, and it is a bijection, its inverse being the map that associates to $\varphi \in \Phi_\Gamma$ the element $\sigma \in \Phi$ whose i-th coordinate coincide with $\gamma_i^{-1} \varphi$. Therefore, one has a map $\Phi \to \Gamma \cup \{\infty\} \to G \cup \{\infty\}$, the second map being identity on G and sending each γ_i to 1. At this point we do not have a special symbol, but, by property of Φ_Γ, an element of Φ can take a value in $\mathcal{A}$ which has $\$_\Gamma$ among its coordinates, only once. Hence, by renaming each of those symbol by a single one \$, we get the expected presentation with special symbol.

Conversely, it suffices to see that the intersection of all the conjugates of G is of finite index in G, hence it has its one point compactification finitely presented with special character by the first part of the proof. Moreover it is normal and of finite index in Γ, and we can apply the Proposition 7.3 to the intersection of all the conjugates of G, Γ and Γ/G.

$\diamondsuit$

APPENDIX

Appendix A

A GENERALISATION FOR SECOND PART

The main result (Theorem 0.1) of Part 2 can be given for a more general case. In fact we will see below that for a relatively hyperbolic group Γ, one can associate a class of boundaries, and moreover we claim that a generalisation of Theorem 0.1 given in Part 2 gives a condition for its action on a boundary in this class to be finitely presented.

We will introduce this class of boundaries for a relatively hyperbolic group. But first we will need the following definition, used by many authors [FarrH], [FeW], [Bes].

Definition. (finite sets fade at infinity)

If G is a discrete group, and $G \cup K$ is a compactification, we say that *finite sets fade at infinity* if for all finite subset F of G and for any open cover $\mathcal{U}$ of $G \cup K$, all but finitely many translates of F are contained in some element of $\mathcal{U}$.

This condition is obviously satisfied for the one-point compactification of any infinite discrete group. It is required for Bestvina's $\mathcal{Z}$-structures (see for example [Bes], [D1]).

Let Γ be a hyperbolic group relative to the family of subgroups $\mathcal{G}$, and let $\mathcal{K}$ be an associated graph. Choose $G_1, ..., G_n$ a set of orbit representatives of conjugacy classes in $\mathcal{G}$. Assume that there is a metrisable compactification $G_i \cup K_i$, in which "finite sets fade at infinity" holds for each i. Let $\widetilde{\Gamma/G_i}$ be a set of orbit representatives of the action of G_i in Γ. Denote by Ω the set of all the translates of each K_i, i.e

$$\Omega = \bigcup_{i=1..n} \{\tilde{\gamma} K_i \mid \tilde{\gamma} \in \widetilde{\Gamma/G_i}\}.$$

Definition. The *boundary of* Γ associated to the set of boundaries of the elements of $\mathcal{G}$ is

$$\partial_\Omega \Gamma = \partial\mathcal{K} \cup \Omega$$

where $\partial\mathcal{K}$ denote the Gromov boundary of the hyperbolic graph $\mathcal{K}$.

In the case where all K_i are singletons, Ω is exactly the set of vertices of infinite valence in $\mathcal{K}$, and in this case, $\partial_\Omega \Gamma$ is the usual boundary defined by Bowditch in [Bo7] (See definition Part 1 Section 8 or Part 2 Section 3). If each K_i gives a $\mathcal{Z}$ structure in the sense of Bestvina (an aspherical, equivariant compactification of a finite dimensional $E\Gamma$, cf [Bes]), then $\partial_\Omega \Gamma$ is the boundary described in [D1], where it is proved that it gives a $\mathcal{Z}$ structure to Γ. Note also that the boundary $\partial\Gamma$ defined by Bowditch is always an equivariant quotient of any boundary, $\partial_\Omega \Gamma$. The following result is proved in [D1].

Theorem Let Γ be a hyperbolic group relative to the family of subgroups $\mathcal{G}$. Assume that for all i "finite sets fade at infinity" in $G_i \cup K_i$. Then there exists a topology on $\Gamma \cup \partial_\Omega \Gamma$ so that with this topology it is compact and metrisable.

Before we announce the main result of Appendix A we need to introduce a general formulation of the property of special character for a compactification of group G.

Definition. Let G be a discrete group and $G \cup K$ be a compactification which induces the discrete topology on G, and on which G acts. We say that the action of G on $G \cup K$ is *finitely presented with special character* if the action of G on $G \cup K$ is expansive and there exists a subshift of finite type $\Phi \subset \mathcal{A}^G$ and a continuous g-equivariant surjective map $\pi : \Phi \to G \cup K$ with the following property (condition of special character):

$$(\exists \$ \in \mathcal{A})\ s.t.\ (\forall g \in G)(\forall \sigma \in \Phi) \quad (\pi(\sigma) = g) \Leftrightarrow (\sigma(g) = \$)$$

As we already observed (Section 9 Part 2) the basic examples of such presentation occurs when we consider the one-point compactification of G, i.e when $K = \{\infty\}$. One can actually see that the special character property defined as above is stable under continuous surjections of K, hence the case of $K = \{\infty\}$ is of special interest, in which case, as we explained in Lemma 1.3 the property of expansivity is always satisfied.

Lemma A.1. Let $G \cup K_1$ and $G \cup K_2$ be two compactifications of a discrete group G which induce the discrete topology on G, and on which G acts. Assume that $G \cup K_2$ is a G-equivariant continuous quotient of $G \cup K_1$. If the action of G on $G \cup K_1$ admits a finite presentation with special character and the action of G on $G \cup K_2$ is expansive then the action of G on $G \cup K_2$ is also finitely presented with special character.

The proof of the above is direct. In fact let Φ be the finite subshift and π : $\Phi \to G \cup K_2$ is the map given by finite presentation of $G \cup K_1$. By considering the composition of π and the quotient map we have the result.

Now we will announce the result of Appendix A.

Theorem A.2. Let $(\Gamma, \mathcal{G})$ be a relatively hyperbolic group, and $G_1, ..., G_n$ be a set of orbit representatives of conjugacy classes in $\mathcal{G}$. Suppose that for all i, G_i admits a compactification $G_i \cup K_i$ where finite sets fade at infinity. If for each i, the action of G_i on $G_i \cup K_i$ is finitely presented with special character, then the action of Γ on the boundary $\partial_\Omega \Gamma$ is finitely presented.

We will not give a proof for this result as it is mainly the same as that of Theorem 0.1 of Part 2. In fact the construction of subshift remains exactly the same, but one has to modify the definition of the map Π given in Section 6.2. Note that if the gradients lines associated to an element σ, of the subshift Φ, sink at a vertex of infinite valence, $x = \tilde{\gamma} p_i$, then as $\sigma_{2|\tilde{\gamma} G_i}$ is in Φ_i, where Φ_i is the subshift given by the finite presentation of the action of G_i on $G_i \cup K_i$, it surjects by Φ_i onto an element k, of K_i. Thus in such case one can define $\Pi(\sigma)$ to be $\tilde{\gamma} k$. It is easy to check the surjectivity of Π with this definition, but for continuity and expansivity one has to use the topology of $G_i \cup K_i$.

Appendix B

COMPACTIFICATION OF DISCRETE GROUPS

We will give a condition on discrete groups to admit a compactification where finite sets fade at infinity (see Appendix A for the definition). In fact we will see that if we have a compactification, $\Gamma \sqcup K$, of a discrete group Γ where finite sets fade at infinity then every Γ-set S on which γ admits a "finite action" can be naturally compactified by adding a copy of K. Moreover we will show that this compactification does not depend to choice of the Γ-set and that in $S \sqcup K$ finite sets fade also at infinity.

Definition. (Γ-sets, finite actions on Γ-sets)

A Γ-set is a set on which Γ acts. We say that the action of Γ on a set S is finite if there are only finitely many Γ-orbits in S and the stabiliser of an element of S in Γ is finite.

Our main objective in this section is to prove the following proposition.

Proposition B.1. If Γ is a discrete group with a hausdorff compactification $\Gamma \sqcup K$ where finite sets fade at infinity then every Γ-set, S, on which Γ has a finite action, admits a natural hausdorff compactification, $S \sqcup K$ where finite sets fade at infinity.

Here the main work will be to introduce a topology on $S \sqcup K$ and show that $S \sqcup K$ is a compact hausdorff space where finite sets fade at infinity.

Let Γ a discrete group. Assume that Γ admits a hausdorff compactification $\Gamma \sqcup K$ where finite sets fade at infinity. Fix an $n \in \mathbf{N}$ and consider the space $X = \bigsqcup_{i=1}^{n} \Gamma^i \sqcup K^i$ where each $\Gamma^i \sqcup K^i$ is a copy of $\Gamma \sqcup K$. For each subset U of $\Gamma \sqcup K$ we will denote by U^i its copy in $\Gamma^i \sqcup K^i$. In particular for all $\gamma \in \Gamma$, for all $k \in K$ and for all $i \in \{1, \ldots, n\}$ we denote the copy of γ by γ^i and the copy of k by k^i in $\Gamma^i \sqcup K^i$. For the rest of this section, by abuse of notation, for each i and for each subset F of X we will not make a difference between $F \cap (\Gamma^i \sqcup K^i)$ and its representative in $\Gamma \sqcup K$.

Now X has a natural topology induced from the topology of $\Gamma \sqcup K$, where a subset O is open in X if and only if $O \cap (\Gamma^i \sqcup K^i)$ is open in $\Gamma \sqcup K$ for all $i \in \{1, \ldots, n\}$. Note that this topology restricted to $\bigsqcup_{i=1}^{n} \Gamma^i$ is discrete. Note also that with this topology X is compact and hausdorff but finite sets do not fade at infinity in X. We will give next some preliminary lemmas for a better understanding of this topology, which will also be used later in the section. But first, in order to facilitate the notations in following arguments, we introduce the quotient map $\pi : X \to \Gamma \sqcup K$. Therefore for each subset U of $\Gamma \sqcup K$, $\pi^{-1}(U)$ denotes $\bigsqcup_{i=1}^{n} U^i$.

Lemma B.2. Let O be an open set in X and $k \in K$. If O contains k^i for all i then there exists an open set U in $\Gamma \sqcup K$ so that $\pi^{-1}(U) \subseteq O$ and $k^i \in \pi^{-1}(U)$ for all i.

Proof. For all i, $O \cap (\Gamma^i \sqcup K^i)$ is an open set containing k^i, hence it is an open set in $\Gamma \sqcup K$ containing k. Denote $U = \bigcap_{i=1}^{n} O \cap (\Gamma^i \sqcup K^i)$. Thus $\pi^{-1}(U)$ is an open set in X containing k^i for all i and contained in O.

$\diamondsuit$

Lemma B.3. Let $\mathcal{O} = \{O_l\}_{l \in L}$ be an open cover of X so that for each $l \in L$, $O_l = \pi^{-1}(U_l)$ for some open U_l in $\Gamma \sqcup K$. For all finite sets F in X all but finitely many translates of F lies in elements of $\mathcal{O}$.

Proof. Denote $F_\Gamma = \pi^{-1}(F)$. Note that $\mathcal{O}^i = \{(U_l)^i\}_{l\in L}$ is an open cover of $\Gamma^i \sqcup K^i$. In other words $\mathcal{O}^i$ is an open cover of $\Gamma \sqcup K$. But since in $\Gamma \sqcup K$ finite sets fade at infinity all but finitely many translates of F_Γ lies in elements of $\mathcal{O}^i$. Note also that $F \subseteq \pi^{-1}(F_\Gamma)$. Hence for all but finitely many $\gamma \in \Gamma$, $\gamma F \subseteq \pi^{-1}(\gamma F_\Gamma) \subseteq \pi^{-1}(U_l) = O_l$ for some $l \in L$.

$\diamondsuit$

Let S be a Γ-set with a finite action of Γ. Assume that $|S/\Gamma| = n$. Denote by $\{s_i\}_{i=1}^n$ an orbit transversal of S. We consider the map $\Pi : X \to S \sqcup K$ given by:

$\Pi(k^i) = k$ for all $k \in \Gamma$

$\Pi(\gamma^i) = \gamma s_i$ for all $\gamma \in \Gamma$.

Note that $\Pi_{|(\gamma\,\mathrm{Stab}(s_i))^i} = \gamma s_i$. Moreover $\gamma s_i = \gamma' s_j$ if and only if $\gamma\,\mathrm{Stab}(s_i) = \gamma'\,\mathrm{Stab}(s_j)$. So $\{(\gamma\,\mathrm{Stab}(s_i))^i\}_{\gamma,i}$ as i varies in $\{1,\ldots,n\}$ and γ varies in Γ forms a partition of $\bigsqcup_{i=1}^n \Gamma^i$. So we can see $S \sqcup K$ as a quotient of X equipped with the quotient topology. Our objective will be to prove that $S \sqcup K$ equipped with this quotient topology is a hausdorff compact space where finite sets fade at infinity. The next two lemmas are immediate since X is a compact space and its topology restricted to $\bigsqcup_{i=1}^n \Gamma^i$ is discrete.

Lemma B.4. $S \sqcup K$ is compact.

Lemma B.5. The topology of $S \sqcup K$ restricted to S is discrete.

Before we prove that $S \sqcup K$ is hausdorff and in $S \sqcup K$ finite sets fade at infinity, we will give some preliminary lemmas.

Lemma B.6. If O is an open set in X satisfying the following property:

($\triangle$) If $k^i \in O$ for some $k \in K$ and $i \in \{1,\ldots,n\}$ then for all $j \in \{1,\ldots,n\}$, $k^j \in O$

Then $\Pi(O)$ is open in $S \sqcup K$.

Proof. We have $\Pi(O) = \{\gamma s_i \in \bigsqcup_{i=1}^n \Gamma^i \mid \gamma^i \in O\} \cup \{k \in K \mid k^j \in O \text{ for some } j \in \{1,\ldots,n\}\}$. Since O has the property ($\triangle$) we obtain $\Pi^{-1}(\Pi(O)) = \{(\gamma g)^i \mid g \in \mathrm{Stab}(s_i) \text{ and } \gamma^i \in O\} \cup (O \cap K)$. Note also that $O \subseteq \Pi^{-1}(\Pi(O))$. Therefore $\Pi^{-1}(\Pi(O)) = \{(\gamma g)^i \mid g \in \mathrm{Stab}(s_i) \text{ and } \gamma^i \in O\} \cup O$, which is an open set in X since it is union of open sets.

$\diamondsuit$

Lemma B.7. $S \sqcup K$ is hausdorff.

Proof. By discreteness it is immediate that for every γ, γ' and for every $i, j \in \{1, \dots, n\}$, γs_i and γs_j can be separated by two disjoint open sets.

Fix $k \in K$ and $\gamma s_i \in S$. Since X is compact hausdorff, hence normal, we know that there exists an open set O in X so that for all i, $k^i \in O$ and for all $g \in \mathrm{Stab}(s_i)$, $(\gamma g)^i \notin O$. Thus by Lemma B.2 there exists an open set U in $\Gamma \sqcup K$ so that $\pi^{-1}(U) \subseteq O$ and $k^i \in \pi^{-1}(U)$ for all i. Note that $\pi^{-1}(U)$ satisfies the property $(\triangle)$, therefore by Lemma B.6, $\Pi(\pi^{-1}(U))$ is an open set in $S \sqcup K$ containing k. Moreover it does not contain γs_i since for all $g \in \mathrm{Stab}(s_i)$, $\gamma g \notin \pi^{-1}(U)$.

Now fix $k_1, k_2 \in K$. As $\Gamma \sqcup K$ is hausdorff we can find two disjoint open sets U_1 and U_2 of $\Gamma \sqcup K$ so that for $j \in \{1, 2\}$ and for all $i \in \{1, \dots, n\}$, $(k_j)^i \in \pi^{-1}(U_j)$. We can also assume without loss of generality that $\overline{U_1} \cap \overline{U_2} = \emptyset$.

Note that if there exists $k \in \Pi(\pi^{-1}(U_1)) \cap \Pi(\pi^{-1}(U_2))$ then $k^i \in \pi^{-1}(U_1) \cap \pi^{-1}(U_2)$ for all i, which is impossible since $\pi^{-1}(U_1) \cap \pi^{-1}(U_2) = \emptyset$. So $\Pi(\pi^{-1}(U_1)) \cap \Pi(\pi^{-1}(U_2)) \subseteq S$. Denote $F = \bigcup_{i=1}^{n} \mathrm{Stab}(s_i)$. Suppose that there exists $\gamma \in \Gamma$ so that $\gamma s_i \in \Pi(\pi^{-1}(U_1)) \cap \Pi(\pi^{-1}(U_2))$ for some i, then $\gamma \pi^{-1}(F) \cap \pi^{-1}(U_j) \neq \emptyset$ for $j \in \{1, 2\}$. We will show that there are in fact only finitely many such γ in Γ.

Since $\Gamma \sqcup K$ is a normal space and $\overline{U_1} \cap \overline{U_2} = \emptyset$ we can find two open set U_1' and U_2' in $\Gamma \sqcup K$ so that $\overline{U_j} \cap ((\Gamma \sqcup K) \setminus U_j) \subseteq U_j'$ for $j \in \{1, 2\}$, $U_1' \cap U_2 = \emptyset$ and $U_2' \cap U_1 = \emptyset$. Denote $O = (\Gamma \sqcup K) \setminus (\overline{U_1} \cup \overline{U_2})$. Now consider the set $\{U_j, U_j'\}_{j \in \{1,2\}} \cup \{O\}$. This is an open cover of $\Gamma \sqcup K$. So the set $\mathcal{O} = \{\pi^{-1}(U_j), \pi^{-1}(U_j')\}_{j \in \{1,2\}} \cup \{\pi^{-1}(O)\}$ is an open cover of X satisfying the hypotheses of Lemma B.3. Now as $\pi^{-1}(F)$ is a finite set in X all but finitely many translate of lies in an element of $\mathcal{O}$. But since there is no open in $\mathcal{O}$ intersecting both $\pi^{-1}(U_1)$ and $\pi^{-1}(U_2)$ there are only finitely many $\gamma \in \Gamma$ so that $\gamma \pi^{-1}(F) \cap \pi^{-1}(U_j) \neq \emptyset$ where $j \in \{1, 2\}$. Denote this finite set in Γ by Q and consider the set $V = \pi^{-1}(U_1) \setminus \cup_{\gamma \in Q} (\gamma \pi^{-1}(F))$. We see that V and $\pi^{-1}(U_2)$ are open sets in X satisfying the property $(\triangle)$, hence by Lemma B.6, $\Pi(V)$ and $\Pi(\pi^{-1}(U_2))$ are open sets in $S \sqcup K$ containing respectively k_1 and k_2, since $k_j^i \in \pi^{-1}(U_j)$ for all i and for $j \in \{1, 2\}$. Moreover by the construction of V we have $\Pi(V) \cap \Pi(\pi^{-1}(U_2)) = \emptyset$. That completes the proof as we found two disjoint open sets in $S \sqcup K$ separating k_1 and k_2.

$\diamondsuit$

Lemma B.8. In in $S \sqcup K$, finite sets fade at infinity.

Proof. Let $\{V_l\}_{l \in L}$ be an open cover of $S \sqcup K$ and F be a finite set in $S \sqcup K$. Consider the set $\{\Pi^{-1}(V_l)\}_{l \in L}$, which is an open cover of X where each $\Pi^{-1}(V_l)$ satisfies the property $(\triangle)$. Thus for each $k \in K$ we can find an open set U_k in $\Gamma \sqcup K$

so that $k^i \in \pi^{-1}(U_k)$ for all i and $\pi^{-1}(U_k) \subseteq \Pi^{-1}(V_l)$ for some $l \in L$ (See Lemma B.2). Note that $\bigcup_{k\in K} U_k$ is an open set in $\Gamma \sqcup K$, containing K. In other words its complement is a closed set in $\Gamma \sqcup K$ not intersecting K. As K is also a closed set in $\Gamma \sqcup K$ and $\Gamma \sqcup K$ is normal we an find two open sets W_1 and W_2 so that W_1 contains the complement of $\bigcup_{k\in K} U_k$, W_2 contains K and $W_1 \cap W_2 = \emptyset$. Without loss of generality we can also assume that $W_2 \subseteq \bigcup_{k\in K} U_k$ since if not we can take as W_2 their intersection.

The set $\{U_k\}_k \cup \{W_1\}$ is an open cover of $\Gamma \sqcup K$. Moreover we can easily see that W_1 is finite by discreteness of the topology on Γ and by compactness of $\Gamma \sqcup K$. Now consider $\mathcal{O} = \{\pi^{-1}(W_1)\} \cup \{\pi^{-1}(U_k)\}_{k\in K}$. By Lemma B.3, as $\Pi^{-1}(F)$ is finite, we see that all but finitely many translate of $\Pi^{-1}(F)$ lies in elements of $\mathcal{O}$. But since W_1 is finite only finitely many translate of $\Pi^{-1}(F)$ can intersect to $\pi^{-1}(W_1)$. Thus all but finitely many translate of $\Pi^{-1}(F)$ lies in $\pi^{-1}(U_k)$ for some $k \in K$, and therefore in $\Pi^{-1}(V_l)$ for some $l \in L$ by the choice of $\pi^{-1}(U_k)$. That completes the proof as if $\gamma\Pi^{-1}(F) \subseteq \Pi^{-1}(V_l)$ for some $l \in L$ then $\Pi(\gamma\Pi^{-1}(F)) = \gamma(F) \subseteq V_l$ for some $l \in L$.

$\diamondsuit$

We will now show that the compactification given on a Γ-set is natural. In other words we will introduce a topology on $\Gamma \sqcup K$ induced from the topology of $S \sqcup K$, and then we will show that in fact this topology coincides with the initial one. Suppose that Γ admits an hausdorff compactification $\Gamma \sqcup K$ where finite sets fade at infinity. We will denote the topology initial on $\Gamma \sqcup K$ by τ_1. Fix a Γ-set S on which Γ admits a finite action. Let $|S/\Gamma| = n$ and $\{s_i\}_{i=1}^n$ be an orbit representative of S. Consider the quotient topology defined on $S \sqcup K$ at the beginning of section.

We introduce a topology that we will denote by τ_2, on $\Gamma \sqcup K$ by giving a basis, $\mathcal{B}$. So U is an element of $\mathcal{B}$ if:

either, $U = U_\gamma = \{\gamma\}$ for some $\gamma \in \Gamma$,
or, there exists an open set O of $S \sqcup K$ so that $U = U_O = \{\gamma \in \Gamma \mid \exists i \in \{1, \ldots, n\}$ with $\gamma s_i \in O\}$.

Clearly the elements of $\mathcal{B}$ cover $\Gamma \sqcup K$. Note that if $O_1 \subseteq O_2$ in $S \sqcup K$ then $U_{O_1} \subseteq U_{O_2}$ and note also that $U_{O_1 \cap O_2} \subseteq U_{O_1} \cap U_{O_2}$. It follows that $\mathcal{B}$ is a topology basis. We also see that the topology thus defined on $\Gamma \sqcup K$ is discrete on Γ.

Lemma B.9. Let U be an open set in $(\Gamma \sqcup K, \tau_1)$. Denote $F = \bigcup_{i=1}^n \mathrm{Stab}(s_i)$, and let $W_U = U \backslash \{\gamma \mid \gamma F \not\subseteq U\} = \{\gamma \mid \gamma F \subseteq U\} \sqcup (U \cap K)$. Then W_U is an open set in $(\Gamma \sqcup K, \tau_1)$.

Proof. We will show that for all $x \in W_U$ there exists O_x open in $(\Gamma \sqcup K, \tau_1)$ containing x and contained in V. The result follows for every $\gamma \in W_U$ by discreteness of τ_1 on Γ. So let $k \in K \cap W_U$. Then $k \in U$. Since U is open in τ_1 and since $(\Gamma \sqcup K, \tau_1)$ is normal there exist two open U_k, U' so that $k \in U_k \subseteq U$, $\overline{U} \cap (\Gamma \sqcup K \backslash U) \subseteq U'$ and $U' \cap U_k = \emptyset$.

Now we see that $\mathcal{U} = \{U, U_k, U', \Gamma \sqcup K \backslash U\}$ is an open cover of $\Gamma \sqcup K$. Moreover since finite sets fade at infinity in $(\Gamma \sqcup K, \tau_1)$ all but finitely many translate of F lie in a open set of $\mathcal{U}$. Note that for all $\gamma \in U_k$, γ does not belong to U' or the complement of U. Thus for all but finitely many γ in U_k, $\gamma F \subseteq U$. Denote this finite subset of Γ by Q and consider $U_k' = U_k \backslash \bigcup_{\gamma \in Q} \gamma F$. Since Q is finite U_k' is an open in τ_1. Moreover by the construction $U_k' \subseteq W_U$. Thus we have found an open set, U_k' of τ_1, contained in W_U and containing k.

$\diamondsuit$

Lemma B.10. The topologies τ_1 and τ_2 on $\Gamma \sqcup K$ coincide.

Proof. We will show that $\mathcal{B}$ is also a topology basis for τ_1. Thus we will show that every $U \in \mathcal{B}$ is a open set in $(\Gamma \sqcup K, \tau_1)$ and every open in $(\Gamma \sqcup K, \tau_1)$ is an union of elements of $\mathcal{B}$.

By discreteness of τ_1 the first assertion is satisfies for all $U = U_\gamma = \gamma$ in $\mathcal{B}$. Fix an element $U_O \in \mathcal{B}$ where O is an open set in $S \sqcup K$. We will show that for every $x \in U_O$ there exists an open set in $(\Gamma \sqcup K, \tau_1)$ containing x and contained in U_O. Again by discreteness that is satisfied for all $x = \gamma \in U_O$. So let $x = k \in K \cap U_O$. So $k \in O$ by definition of U_O. Since O is an open set in $S \sqcup K$, $\Pi^{-1}(O)$ is an open set in X containing k^i for all i. Now, by lemma B.2, there exists an open set U_k in $\Gamma \sqcup K$ so that $k^i \in \pi^{-1}(U_k)$ and $\pi^{-1}(U_k) \subseteq \Pi^{-1}(O)$. Moreover as $\pi^{-1}(U_k)$ satisfies the property $(\triangle)$ we see that $\Pi(\pi^{-1}(U_k))$ is an open set in $S \sqcup K$ contained in O. Therefore $U_{\Pi(\pi^{-1}(U_k))} \subseteq U_O$. We will in fact show that $U_k \subseteq U_{\Pi(\pi^{-1}(U_k))}$. Let γ, (respectively $k \in K$) be an element of U_k. Then for all i, γ^i (resp. k^i) is in $\in \pi^{-1}(U_k)$, which implies γs_i (resp. k) is in $\Pi(\pi^{-1}(U_k))$ and so $\gamma \in U_{\Pi(\pi^{-1}(U_k))}$ (resp. $k \in U_{\Pi(\pi^{-1}(U_k))}$). So we have found an open set U_k of τ_1, containing k and contained in $U_{\Pi(\pi^{-1}(U_k))} \subseteq U_O$.

It remains to show that each open set U in the topology τ_1 can be written as union of the elements of $\mathcal{B}$. For this it is enough to show that for each k in $K \cap U$ there exists an open set O in $S \sqcup K$ so that $k \in U_O$ and $U_O \subseteq U$. Consider W_U as defined in Lemma B.9 and note that $\pi^{-1}(W_U)$ is an open set in $(\Gamma \sqcup K, \tau_1)$ (See Lemma B.9). Recall that $F = \bigcup_{i=1}^{n} \mathrm{Stab}(s_i)$. Then $\Pi(\pi^{-1}(W_U)) = \{\gamma s_i \in \Gamma \mid \gamma F \subseteq U\} \cup (U \cap K)$. Since

$\pi^{-1}(W_U)$ satisfies $(\triangle)$ by Lemma 1.6, $\Pi(\pi^{-1}(W_U))$ is open in $S \sqcup K$ containing k. Now note that $U_{\Pi(\pi^{-1}(W_U))} = \{\gamma \in \Gamma \mid \exists j \in \{1, \ldots, n\}$ so that $\gamma s_i \in \Pi(V)\} \cup (U \cap K) = \{\gamma \in \Gamma \mid \gamma F \subseteq U\} \cup (U \cap K) = W_U$. Thus $U_{\Pi(\pi^{-1}(W_U))} = W_U \subseteq U$ and it is an element of $\mathcal{B}$ containing k.

$\diamondsuit$

Appendix C

SYMBOLIC DYNAMICS FOR Γ-SETS

In this section we will try to understand dynamical systems of finite type and to bring some new approach to the theory by some remarks and questions.

We have already given definitions of subshift, cylinder, subshift of finite type and dynamical system of finite type in Part 2 Section 1. Here we will give a generalisation of these notions using Γ-sets, and try to question in order to obtain equivalent definitions.

Definition. (Subshifts through Γ-sets)

Let Γ be a group acting on a set S and let $\mathcal{A}$ be a finite alphabet. The set $\mathcal{A}^S$, with the product topology, is the total shift of Γ on $\mathcal{A}$ through S. It admits a natural left action of Γ.

A closed invariant set $\Phi \subset \mathcal{A}^S$ is a *subshift through* S.

A set $\mathcal{C}$ is a *cylinder through* S if there exists F a finite subset of S and M a subset of $\mathcal{A}^F$ with $\mathcal{C} = \{\sigma \in \mathcal{A}^\Gamma$ such that $\sigma|_F \in M\}$.

A subshift of *finite type through* S is the intersection of all the translates of a cylinder.

We now see that we can reformulate a dynamical system of finite type in term of Γ-sets.

Definition. (Dynamical systems of finite type through a Γ-set)

Let Γ be a group acting on a compact space K. We say that the dynamical system is of finite type through a Γ-set, S if there exists a finite alphabet, $\mathcal{A}$, a subshift of finite type through S, $\Phi_S \subset \mathcal{A}^S$, and a continuous, surjective, Γ-equivariant map $\pi : \Phi_S \to K$.

My coauthor, Dahmani, and I have noticed the following result while working for the proof of Theorem 0.1 of in Part 2. This result is in fact the motivation of the above definitions.

Proposition C.1. If the action of Γ on a compact space Q is of finite type through S then the action of Γ on Q is of finite type.

Proof. The action is of finite type through S. So there exists a cylinder $\mathcal{C}_S$ defined by a finite alphabet $\mathcal{A}_S$, a finite set $F_S \subset S$ together with a subset $M_S \subset \mathcal{A}_S^{F_s}$ so that the subshift $\Phi_S = \bigcap_{\gamma\in\Gamma} \gamma^{-1}\mathcal{C}_S$ surjects by the map π_S onto Q, Γ-equivariantly and continuously.

Let $\mathcal{A} = (\mathcal{A}_S)^n$. Let $F = \bigcup_{s\in F_S} \mathrm{Stab}(s)$. By hypothesis, F is a finite subset of Γ. Let $M = \{m : F \to \mathcal{A}, \gamma \mapsto (m'(\gamma(s_1)), ..., m'(\gamma(s_n))), m' \in M_S\}$. Now let $\mathcal{C}$ be the cylinder defined by $\mathcal{A}, F, M$, and let $\Phi = \bigcap_{\gamma\in\Gamma} \gamma^{-1}\mathcal{C}$.

Now we map Φ into Φ_S by φ as follows. If $\sigma \in \Phi$, let $[\Psi(\sigma)](\gamma s_i)$ be the i^{th} coordinate of $\sigma(\gamma)$. This is well defined because it does not depend of the choice of γ. In fact, if $\gamma' s_i = \gamma s_i$, then $\gamma^{-1}\gamma' \in \mathrm{Stab}(s_i)$, hence, by definition of M, i^{th} coordinate of $\sigma(\gamma')$ is the same as i^{th} coordinate of $\sigma(\gamma)$. We also see by definition of M that $\Psi(\sigma)$ is in Φ_S.

Note also that φ is surjective. In fact for $\varphi \in \Phi_S$, let $\sigma(\gamma) = (\varphi(\gamma s_1), ..., \varphi(\gamma s_n))$. By its definition σ is in Φ and $\Psi(\sigma) = \varphi$. Moreover, Ψ is continuous and Γ-equivariant. Hence, $\pi = \pi_S \circ \varphi$, shows that the dynamical system is of finite type.

$\diamondsuit$

There are natural questions that follows from the above result.

Question C.1. Let Γ be a discrete group with a hausdorff compactification $\Gamma \sqcup K$ where finite sets fade at infinity, and let S be a set on which Γ has a finite action. Can we say that the following are equivalent:

- The action of Γ on $\Gamma \sqcup K$ is of finite type.
- The action of Γ on $S \sqcup K$ is of finite type through S.

Proposition C.1 gives one way of the question. Unfortunately I was not able to provide a satisfactory result for the other way. I also try to strengthen the hypothesis in order to obtain a partial result for a particular case. So here the second question.

Question C.2. If the action of Γ on a compact space Q is of finite type then the action of Γ on Q/H is of finite type through S where $H = \bigcap_{s\in S} \mathrm{Stab}(s)$.

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