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LOCAL ROOT NUMBERS OF TWO-DIMENSIONAL SYMPLECTIC REPRESENTATIONS

by

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ABSTRACT

FACULTY OF MATHEMATICAL STUDIES

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Let F be a non-archimedean local field with residue field F_q and q odd. Consider A the unique quaternion division algebra over F. We prove the existence of a homomorphism of the form $\widehat{\Gamma}_F : RO(A^*/F^*) \longrightarrow \mu_4 = \{\pm 1, \pm i\}$ analogous to $\Gamma_F : RO(F) \longrightarrow \mu_4$ given in [31]. Using $\widehat{\Gamma}_F$ and the results of D. Prasad and D. Ramakrishnan [22] regarding the Langlands correspondence, we construct Υ_F, a map from two-dimensional symplectic Galois representations to fourth roots of unity. If σ is a two-dimensional symplectic Galois representation, this construction, when $q \equiv 1(mod.4)$, gives a formula for the local root number of σ in terms of $\Upsilon_F(\sigma)$.

Contents

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Chapter 1

Introduction

Let F be a non-archimedean local field with residue field F_q and q odd. V.P. Snaith in [31] shows how to construct a homomorphism

$$\Gamma_F : RO(F) = \lim_{\overrightarrow{K/F}} \{RO(G(K/F))\} \longrightarrow \mu_4 = \{\pm 1, \pm i\}$$

where, taking $\overline{F}$ as a fixed choice of separable closure of F, the limit is taken over finite Galois extensions with $F \subset K \subset \overline{F}$. $RO(G)$ is the free abelian group on the isomorphism classes, $[\rho]$, of irreducible orthogonal representations $\rho : G \longrightarrow O_n(\mathbb{R})$. The map Γ_F is given by the composition of maps of the following form

$$\begin{array}{ccccccc} RO(F) & \longrightarrow & Y_F & \stackrel{Ind_{F/\mathbb{Q}_p}}{\longrightarrow} & Y_{\mathbb{Q}_p} & \longrightarrow & \mu_4 \\ \rho & \rightsquigarrow & [\rho - n] & \rightsquigarrow & Ind_{F/\mathbb{Q}_p}[\rho - n] & \rightsquigarrow & \Gamma_F(\rho) \end{array}$$

where $n = dim\rho$.

$$Y_F = \lim_{\overrightarrow{K/F}} \{IO(G(K/F))/(J_{G(K/F)})\}$$

Here, $IO(G)$ is the augmentation ideal of $RO(G)$ and $J_G = \{x \in IO(G)/dim(x) = 0, SW_1(x) = 0 = SW_2(x)\}$.

This construction is important because, given $\rho : \Omega_F \longrightarrow O_n(\mathbb{R})$ a continuous orthogonal representation, $\Gamma_F(\rho) = W_F(\rho)$, the local root number of ρ.

Now, given A, a quaternion algebra with centre a local field F with odd order residue field F_q, we will prove, by construction, the existence of a homomorphism of the form $\widehat{\Gamma}_F : RO(A^*/F^*) \longrightarrow \mu_4$ analogous to the one above. This is studied in chapter 7.

Let $IO(A^*/F^*)$ be the augmentation ideal of $RO(A^*/F^*)$. Define $J = \{x \in RO(A^*/F^*)/dim(x) = 0, SW_1(x) = 0 = SW_2(x)\}$.

We define $\widehat{\Gamma}_F$ as the composition of the following maps,

$$\begin{array}{ccccccc} RO(A^*/F^*) & \longrightarrow & IO(A^*/F^*)/J & \stackrel{\pi}{\longrightarrow} & Y_F & \longrightarrow & \mu_4 \\ W & \rightsquigarrow & W-n & \rightsquigarrow & \pi(W-n) & \rightsquigarrow & \Gamma_F(W) \end{array}$$

where $n = dimW$ and π is a surjective homomorphism constructed in sections 7.2 and 7.3.

J is an ideal of $RO(A^*/F^*)$ contained in $IO(A^*/F^*)$. Therefore the elements of $IO(A^*/F^*)/J$ are faithfully detected by the first and second Stiefel-Whitney classes, SW_1 and SW_2. Taking this into account, formulae for the first and second Stiefel-Whitney classes of orthogonal representations of A^*/F^* in terms of their character values on elements of order two are given in chapter 8.

Using $\widehat{\Gamma}_F$ and the results of D.Prasad and D.Ramakrishnan [21] regarding the Langlands correspondence, we now construct a map from two-dimensional symplectic Galois representations to fourth roots of unity.

Let $\sigma = Ind_{K/F}(\chi)$ be a two-dimensional symplectic representation induced by χ a character . σ is mapped to $\Pi(\sigma)$, an orthogonal representation of A^*/F^* through the Langlands correspondence.

Define $\Upsilon_F(\sigma) = \widehat{\Gamma}(\Pi(\sigma)) \in \mu_4$.

The following Theorem is proved,

Theorem 9.4.2 *Let σ be a two-dimensional symplectic Galois representation, $\sigma = Ind_{K/F}(\chi)$ induced from a character $\chi : K^* \longrightarrow \mathbf{C}^*$, where F is a non-Archimedean local field of odd order residual characteristic field F_q, $q \equiv 1(mod.4)$ and K/F is a quadratic extension, then*

$$\Upsilon_F(\sigma) = (-1)^{\frac{q-1+2e}{2e}} W_F(\sigma) W_F(w_{K/F})^{-1} \in \mu_4$$

where e is the ramification index of the extension and $w_{K/F}$ is the non-trivial quadratic character of F^* given by class field theory.

This theorem gives a relation between $\Upsilon_F(\sigma)$ and the value of the local root number of σ.

As immediate corollaries we obtain the following results,

Corollary 9.5.2 *($q \equiv 1(mod.4)$) Let σ and σ' be two two-dimensional symplectic Galois representations, $\sigma = Ind_{K/F}(\chi)$ and $\sigma' = Ind_{K/F}(\chi')$ induced from characters $\chi,\chi' : K^* \longrightarrow \mathbb{C}$. Then, the following holds,* *(I) If K/F is ramified with $f(\chi) \neq f(\chi')$ and $min(f(\chi), f(\chi')) = 1$ (it is no restriction to assume $f(\chi') = 1$) then,*

$$\frac{\Upsilon_F(\sigma)}{\Upsilon_F(\sigma')} = (-1)^{\frac{q-1}{4}} \frac{W_F(\sigma)}{W_F(\sigma')} \frac{W_F(w_{K'/F})}{W_F(w_{K/F})}$$

where K'/F is the unique unramified quadratic extension over F.

(II) In any other case,

$$\frac{\Upsilon_F(\sigma)}{\Upsilon_F(\sigma')} = \frac{W_F(\sigma)}{W_F(\sigma')}$$

Corollary 9.5.3 *($q \equiv 1(mod.4)$) Let σ be a two-dimensional symplectic Galois representation, $\sigma = Ind_{K/F}(\chi)$ induced from a character $\chi : K^* \longrightarrow \mathbb{C}^*$. Consider the tame ramification of σ, this is, $\sigma_{tame} = Ind_{K/F}(\chi_1)$ where $\chi = \chi_1 \otimes \chi_2$, the order of χ_1 is coprime to q and the order of χ_2 is a power of q. Then,χ_1 is tamely ramified, and*

$$\frac{\Upsilon_F(\sigma)}{\Upsilon_F(\sigma_{tame})} = (-1)^{\frac{(e-1)(q-1)}{4}} \frac{W_F(\sigma)}{W_F(\sigma_{tame})} \frac{W_F(w_{K'/F})}{W_F(w_{K/F})}$$

where K'/F is the unique unramified quadratic extension over F and e is the ramification index of the extension K/F.

The case $q \equiv 3(mod.4)$ does not seem to give such a good relation as $q \equiv 1(mod.4)$. However, let us note that $q \equiv 1(mod.4)$ covers most cases. That is, if $q = p^d$ for some odd prime p and integer d, $q \equiv 1(mod.4)$ if $p \equiv 1(mod.4)$ or $p \equiv 3(mod.4)$ and d even.

To finish with, the case $q \equiv 3(mod.4)$ has been introduced for completeness. This is section 9.4.2.

Chapter 2

A very brief introduction to representation theory

In this chapter, we give a brief introduction to representation theory and Brauer Induction.

Here we present some basic definitions, omitting proof, that will be required in order to work with Explicit Brauer Induction.

REPRESENTATIONS:

Let us suppose that G is a finite group, K a field and V a finite-dimensional vector space over K. Let $GL(V)$ denote the group of K-linear automorphisms of V.

A homomorphism $\rho : G \to GL(V)$ gives an action of G on V by $g \cdot v = \rho(g)(v)$.

A finite-dimensional K-representation of G is the K-isomorphism class of such an action.

NOTE: We can think of ρ as a KG-module.

Very often an isomorphism between V and K^n is chosen ($dim_K(V) = n$), in this case we can write: $\rho : G \to GL(K^n) \equiv GL_n(K)$.

Usually, we will consider the case $K = \mathbb{C}$. In this case, $GL_n(\mathbb{C})$ has a number of compact subgroups of special interest:

(i) The Unitary group: $U(n) = \{X \in GL_n(\mathbb{C})/XX^* = I_n\}$. Basically, $U(n)$ is the subgroup of $GL_n(\mathbb{C})$ that preserve the semilinear product on $\mathbb{C}^n$ $\langle \underline{x}, \underline{y} \rangle = \sum_i x_i \overline{y_i}$. We will mean by an Unitary representation, a $U(n)$-conjugacy class of a homomorphism of the form $\rho : G \to U(n)$.

There is a one-one correspondence between $U(n)$-representations and n-dimensional $\mathbb{C}$-representations.

(ii) Similarly, if we consider $\mathbb{R}$, we define $O(n)$ the orthogonal group, as the subgroup of matrices that preserve the inner-product $\langle \underline{x}, \underline{y} \rangle = \sum_i x_i y_i$. So then $O(n) = GL_n(\mathbb{R}) \bigcap U(n)$.

We will mean by an orthogonal representation an $O(n)$-conjugacy class of a homomorphism of the form $\rho : G \to O(n)$.

As in the complex case, there is a one-one correspondence between $O(n)$-representations and n-dimensional $\mathbb{R}$-representations.

(iii) Denote by $\mathbb{H}$ the quaternion skew-field, i.e., if z is a quaternion $z = a+ib+jc+kd$ then $\overline{z} = a-ib-jc-kd$. On $\mathbb{H}^n$ we have an inner-product $\langle \underline{x}, \underline{y} \rangle = \sum_i x_i \overline{y_i}$. The subgroup of left $\mathbb{H}$-automorphisms of $\mathbb{H}^n$ that preserve this form is called the symplectic group, and denoted by $Sp(n)$. As a $\mathbb{C}$-

vector space $\mathbb{H} \cong \mathbb{C}^2$, so $Sp(n) \subset U(2n)$. A symplectic representation would be then, a conjugacy class of a homomorphism of the form $\rho : G \to Sp(n)$.

INDUCTION AND RESTRICTION:

ρ a complex representation $\rho : G \to GL_n(\mathbb{C})$. As a fact, we can express ρ as a sum of multiples of ρ_i, where ρ_i are irreducible representations [32](this is a finite set):

$$\rho = \sum_{Irr(G)} n_i \rho_i$$

We can define now the representation ring: $R(G) = \{\sum_{\nu \in Irr(G)} n_\nu \nu, n_\nu \in \mathbb{Z}\}$.

So then, if we have $H \leq G$, we can define induction and restriction:

$$Ind_H^G : R(H) \to R(G)$$

$$Res_H^G : R(G) \to R(H)$$

Restriction: Given $\rho \in R(G)$, one can define $Res_H^G(\rho) = \rho_{|H} \in R(H)$.

Induction: The definition of Ind_H^G is a little bit more complicated. Given $\rho \in R(H)$, we consider V the $\mathbb{C}H$-module associated to ρ, and then $Ind_H^G(\rho)$ will be the G-representations associated to the $\mathbb{C}G$-module $\mathbb{C}G \otimes_{\mathbb{C}H} V$.

BRAUER INDUCTION

Brauer (1947) proved a theorem called "The Brauer induction theorem", relating ρ with one-dimensional representations. The theorem is announced as follows [32]:

Theorem 2.0.1 *G a finite group. Given $\rho \in R(G)$ there exist one-dimensional representations of elementary subgroups H_i, $\phi_i : H_i \to \mathbb{C}^*$ and integers n_i*

such that:

$$\rho = \sum_i n_i Ind_{H_i}^G(\phi_i)$$

We should speak about Explicit Brauer Induction at this point. Explicit just means that we would like to specify canonically what n_i, H_i and ϕ_i are. These formulae were given by Snaith in 1986 and Boltje in 1989. We will speak about this later on, when we study Snaith's proof of existence of local root numbers.

THE TRANSFER

Given G a group, we denote by G^{ab} the quotient of G by its commutator subgroup. Let us consider now $H \leq G$ of finite index, and $\sigma : G/H \to G$ a set of representatives for the left cosets of G modulo H.

If $s \in G$ and $t \in G/H$ we define $a_{s,t}$ as:

$$s\sigma(t) = \sigma(st)a_{s,t}$$

Definition: Let $[s] \in G^{ab}$. The image in H^{ab} of $\prod_{t\in G/H} a_{s,t}$ is called the transfer of $[s]$ that we will denote by $Ver_G^H([s])$ or simply by $Ver(s)$. We can also define then a map from G^{ab} in H^{ab}, $Ver : G^{ab} \to H^{ab}$.

Usually we will deal with Galois groups. For convenience, we use infinite Galois groups.

We will need the following result about the transfer map [20],

Proposition 2.0.2

(a) Let E/K be a finite extension of local fields of finite degree over $\mathbb{Q}_p$ contained in an algebraic closure $\overline{\mathbb{Q}}_p$. The following diagram is commutative:

$$\begin{array}{ccc} & Ver & \\ Gal(\overline{\mathbb{Q}}_p/K)^{ab} & \rightarrow & Gal(\overline{\mathbb{Q}}_p/E)^{ab} \\ \uparrow & i & \uparrow \\ K^* & \rightarrow & E^* \end{array}$$

(b) Let E/K be a finite extension of number fields with I_K, I_E the corresponding idele groups. The following diagram is commutative:

$$\begin{array}{ccc} & Ver & \\ Gal(\overline{\mathbb{Q}}/K)^{ab} & \rightarrow & Gal(\overline{\mathbb{Q}}/E)^{ab} \\ \uparrow & i & \uparrow \\ I_K & \rightarrow & I_E \end{array}$$

where the vertical homomorphisms come from local and global class field theory [20].

Chapter 3

Character theory, Artin L-Functions and Artin root numbers

In this chapter, we recall the definition of L-Functions and the enlarged L-Function [20], allowing us to define the local root numbers in the character case, and the relationship between them and the Gauss sums.

The local root number is an invariant of a local Galois representation which is to be taken very seriously. For instance, in the theory of the structure of a ring of algebraic integers as a Galois module, the local root numbers determine whether this projective module is free, in the case of a tame extension [31]. In the Langlands programme, which speculates about bijections between Galois representations and other categories of representations, the local root number plays an important role as part of the detection machinery in the local conjecture. The Langlands correspondences have now been established in the local case by M. Harris and R. Taylor [11] in characteristic zero, by G. Laumon, M. Rapoport and U. Stuhler [19] in characteristic $p > 0$ and in the global case by L. Lafforgue.

In order to define local root numbers in the character case, firstly we must introduce all the ingredients required.

3.1 Frobenius substitution

In order to define the Frobenius substitution, let us suppose that we have E/K a finite normal extension of number fields with Galois group G, and $\mathcal{P}$ is a finite prime of K.

Also, let us assume that E/K is unramified at $\mathcal{P}$.

For every prime P lying above $\mathcal{P}$, there exists a unique element of $\sigma_P \in G$ such that $\sigma_P(x) \equiv x^{N(P)}(modP)$ with x any integer in E, and $N(P)$ the absolute norm of P. This is what is known as the "Frobenius substitution". It can be proved that for every cyclic group C of G there exist infinitely many primes P such that $C =< \sigma_P >$.

3.2 Weber

Let us take now b an ideal of K and let us define:

$$I_b = \{a \text{ ideal of } K \ s.t.a \text{ is prime to } b\}$$

$$P_b = \{a \in I_b \ s.t. \ a =< \alpha > \text{ with } \alpha \in K \text{ totally positive } \alpha \equiv 1(modb)\}$$

Now, we take $H \leq I_b$ such that $P_b \subset H$.

In this situation let's suppose that the abelian extension E of K is what Weber called a "class field for H" (i.e., the prime ideals of K that decompose completely in E are precisely those ones in H, and b is the smallest in some sense). In this case the prime divisors of b are precisely the prime ideals of K ramified in E.

Now, given a character $\chi : I_b/H \to \mathbb{C}^*$, there is an L-Function defined for $Re(s) > 1$ by:

$$L(s, \chi) = \prod_P \frac{1}{1 - \chi(P)N(P)^{-s}} \tag{3.1}$$

Notice that the product is over the P not dividing b, i.e.: the product is over the prime ideals of K unramified in E.

Now, we are prepared to introduce Artin's first definition of L-Functions [20].

3.3 Artin's first definition of L-Functions

This definition first appeared in 1922. Takagi in 1920 had established that "the Galois group G is isomorphic to I_b/H", but he did not give any canonical isomorphism. If we had one given it would be possible to define L-Functions for one-degree characters.

In any case, Artin gave a definition of an L-series, for one-degree characters of G by thinking of the above formula 3.1 in the following way:

$$L(s, \psi) = \prod_{P\ unramified} \frac{1}{1 - \psi(\sigma_P)N(P)^{-s}}$$

This led Artin to think that the isomorphism that Takagi spoke about could be built by sending the class of an unramified prime ideal P of I_b/H onto σ_P. This is known as "the general law of reprocity".

We can give now Artin's first definition of L-functions:

Definition 3.3.1 : *E/K finite normal extension of number fields with Galois group G. V finite dimensional complex vector space, and let $\rho : G \to Gl(V)$ be a representation. Denote χ the character of ρ (i.e., $\chi(s) = Tr(\rho(s))$). For a prime $\mathcal{P}$ in K, $det(1 - N(P)^{-s}\rho(\sigma_P))$ does not depend on the choice of P above $\mathcal{P}$ and takes the same value for two isomorphic representations, so we can define:*

$$L(s,\chi) = \prod_{P\ unramified} \frac{1}{det(1 - N(p)^{-s}\rho(\sigma_P))} \ for\ Re(s) > 1.$$

L, defined in this way, verifies the following properties:

(a) $L(s, \chi_1 + \chi_2) = L(s, \chi_1)L(s, \chi_2)$.

(b) H normal subgroup of G, ρ a representation of G/H and ρ' the lifting of ρ to G, then : $L(s, \chi') \sim L(s, \chi)$.

(c) H subgroup of G, χ a character of H that induces χ^* of G, then: $L(s, \chi^*) \sim L(s, \chi)$.

NOTE: What we mean by '$\sim$' is that they are equal up to a finite number of Euler-factors.

The problem we have with this definition is that we do not get equalities in (b) and (c), which would be likely to occur. So now we will study the general definition of non-abelian L-Functions where we will get equalities in these properties.

3.4 General definition of non-abelian L-function

We are going to define local factors at ramified primes, in such a way we can put equalities in the above formulae.

Let us denote for D_P and I_P the decomposition group and inertia group of P respectively. D_P/I_P is isomorphic to the Galois group of the residue extension, so we can define the Frobenius substitution $\sigma_P \in D_P/I_P$.

G acts on V via $\sigma x = \rho_\sigma(x)$ for all $x \in V$ and $\sigma \in G$.

Let us define V^{I_P} as: $V^{I_P} = \{x \in V s.t. \forall \sigma \in I_P, \sigma x = x\}$.

Now, we can define:

$$L(s,\chi) = \prod_{P \ finite} \frac{1}{det_{V^{I_P}}(1 - N(p)^{-s}\sigma_P)} \text{ for } Re(s) > 1.$$

With this definition we get equalities in all the above formulae (a), (b) and (c). We also get an extra one:

(d) χ one degree character, ψ the congruence class character, then: $L(s,\chi) = L(s,\psi)$.

At this point the definition of an enlarged L-function Λ arises, and this function will verify a functional equation which will allow us to define the Artin root number. We will not go deeply into this definition, since we are just interested in the functional equation verified. We will just say that this enlarged L-Function has the following shape [20],

$$\Lambda(s,\chi) = A(\chi)^{s/2}\gamma_\chi(s)L(s,\chi) \text{ for } Re(s) > 1.$$

For the definition of A, some new concepts are needed, and we will discuss them as well since they will play an important role in the definition of local root numbers:

For our P above $\mathcal{P}$, we consider G_i the corresponding ramification groups and let us denote by g_i the order of G_i. We define then:

$$n(\chi, \mathcal{P}) = \sum_{i=0}^{\infty} \frac{g_i}{g_0} codim V^{G_i}$$

. Artin proved that,in fact, this number (that is independent of the choice of P above $\mathcal{P}$) is an integer.

We are going to define as well what is known as "the Artin conductor", this is just a particular ideal of K defined by:

$$f(\chi, E/K) = f(\chi) = \prod_{\mathcal{P}} \mathcal{P}^{n(\chi,\mathcal{P})}$$

So, for example $A(\chi)$ is defined as $|d_K|N_{K/\mathbb{Q}}(f(\chi))$ where d_K is the discriminant of K.

As we said before, this enlarged L-Function verifies a functional equation in which the definition of Artin root numbers arise. The basic properties verified by this function are summarized in the following theorem:

Theorem 3.4.1 *[20] Λ possesses a meromorphic continuation in the whole complex plane and satisfies the functional equation $\Lambda(1-s,\chi) = W(\chi)\Lambda(s,\overline{\chi})$ for some constant $W(\chi)$ of absolute modulus 1.*

This number of modulus 1 is called the "ARTIN ROOT NUMBER".

3.5 More on the Artin conductor.

The Artin conductor can be defined for more general extensions than the ones we have looked at. Let us suppose that we have A a Dedekind ring with quotient field K, and E a finite normal extension of K with Galois group G.

Also, let us suppose that we have ρ a representation of G in a finite dimensional vector space with character χ.

Assume that all the residue class extensions are separable. To extend the conductor to the inseparable residue field case was a problem of J.P. Serre (Annals of Mathematics 1960), solved in chapter 6 of [32] (which is based on a paper of Boltje-Cram-Snaith).

Let us take $\mathcal{P}$ a prime ideal of K, and P a prime ideal of E lying above $\mathcal{P}$. Then we defined n as:

$$n(\chi,\mathcal{P}) = \sum_{i=0}^{\infty} \frac{g_i}{g_0} codim V^{G_i}$$

. where G_i are the ramification groups of P, and g_i its order.

This number is an integer. And it is obvious that in the case E/K unramified at $\mathcal{P}$, $n(\chi,\mathcal{P}) = 0$, and in the case E/K tamely ramified $n(\chi,\mathcal{P}) = codim V^{G_0}$.

We have defined as well the Artin conductor as:

$$f(\chi, E/K) = f(\chi) = \prod_{\mathcal{P}} \mathcal{P}^{n(\chi,\mathcal{P})}$$

This particular ideal of K verifies the following properties:

(i) $f(\chi+\chi') = f(\chi)f(\chi')$

(ii) If χ is lifted from a character χ' of a quotient H of G then: $f(\chi) = f(\chi')$

(iii) Let H be a subgroup of G. χ a character of H and χ^* the character of G induced by χ. Then:

$$f(\chi^*) = N_{F/K}(f(\chi))D(F/K)^{\chi(1)}$$

where D(F/K) is the discriminant (relative to A).

Now, we are going to define the conductor for an infinite extension. Let L be an infinite normal extension of K with Galois group G. We take a representation ρ of G (a homomorphism $\rho : G \to Gl(V)$ with open kernel, V a finite dimensional vector space). Such representation factors through the Galois group of a finite extension, so we define $f(\rho)$ to be the conductor of ρ', where ρ' is any representation of a finite Galois extension s.t. ρ is the lifting of ρ'.

3.6 Local Gauss sums

Now we are going to study the " Local Gauss sum" and the relationship between this and the "Local Root Numbers". For this purpose, let us suppose we are in the following situation:

p is a fixed prime, K a finite extension of $\mathbf{Q}_p$. We will denote by $\mathcal{O}_K$ the valuation ring of K. $\wp_K$ is the maximal ideal of $\mathcal{O}_K$. $\mathcal{D}_K$ is the different of the extension $K/\mathbf{Q}_p$. U_K is the group of units of $\mathcal{O}_K$. U_K^i is the subgroup of units of K that are congruent to $1(mod \wp_K^i)$. And π_K denotes $\wp_K = \pi_K \mathcal{O}_K = (\pi_K)$.

Firstly, we will define the non-trivial character $\psi : K \to \mathbf{C}^*$ as the composition of the following maps:

$$K \xrightarrow{Tr_{K/\mathbf{Q}_p}} \mathbf{Q}_p \underset{surjection}{\xrightarrow{canonical}} \mathbf{Q}_p/\mathbf{Z}_p \underset{injection}{\xrightarrow{canonical}} \mathbf{Q}/\mathbf{Z} \xrightarrow{e^{2\pi i\_}} \mathbf{C}^*$$

We can see easily that this character verifies $\psi(x+y) = \psi(x)\psi(y)$ and $\psi(-x) = \overline{\psi(x)}$.

REMARK: ψ is trivial on $\mathcal{D}_K^{-1}$ (the codifferent) and actually this is the greatest fractional ideal of K on which ψ is trivial.

Now, let $\theta : K^* \to \mathbf{C}^*$ be a character with open kernel. If we denote $n(\theta)$ by n, our conductor will be $f(\theta) = \wp_K^n$, and this integer n is the least one such that θ is trivial on U_K^n.

We will say that θ is unramified if $n(\theta) = 0$.

Let us define now the Gauss sum for this character θ,

Definition 3.6.1 *The local Gauss sum $\tau(\theta)$ is defined as:*

$$\tau(\theta) = \sum_{x \in U_K/U_K^n} \theta(\frac{x}{c})\psi(\frac{x}{c})$$

where c is a generator of the ideal $\mathcal{D}(\theta) = f(\theta)\mathcal{D}_K$.

REMARK : When θ is unramified, this sum will be just one term: $\tau(\theta) = \theta(\mathcal{D}_K^{-1})$.

Important properties about Gauss sums are the following:

Proposition 3.6.2 *θ a character of K^*:*

(i) $|\tau(\theta)| = \sqrt{N(f(\theta))}$

(ii) $\tau(\theta)\tau(\overline{\theta}) = \theta(-1)N(f(\theta))$

Proof:

(ii) Using the definition of τ we get $\tau(\overline{\theta}) = \theta(-1)\overline{\tau}(\theta)$, and now our result follows easily from (i).

(i) To prove this we can think first of the unramified case (then $N(f(\theta)) = 1$ and $\tau(\theta)\overline{\tau}(\theta) = \theta(\mathcal{D}_K^{-1})\overline{\theta}(\mathcal{D}_K^{-1}) = 1$) and then of the ramified one. In this case the proof is quite straightforward using the following Lemma and writing $\tau(\theta)\overline{\tau}(\theta) = \sum_x \theta(x) \sum_y \psi(y\frac{x-1}{c})\square$

Lemma 3.6.3 *[20] $n \geq 0$, d an element of $\mathcal{D}_K^{-1}N(\mathcal{P}_K)^{-n}$. S a set of representatives of $\mathcal{O}_K$ modulo $\mathcal{P}_K^n$. Then $\lambda = \sum_{y\in S}\psi(yd)$ does not depend on the choice of S and $\lambda = N(\mathcal{P}_K)^n$ if $d \in \mathcal{D}_K^{-1}$ and $\lambda = 0$ otherwise.*

Now we are prepared to define the LOCAL ROOT NUMBER. Let K be a local field of characteristic 0 and θ a character of K^*.

Definition 3.6.4

For $K = \mathbf{R}$ or $\mathbf{C}$ define $W(\theta) = i^{-n(\theta)}$.

For K non Archimedean, define:

$$W(\theta) = \frac{\tau(\overline{\theta})}{\sqrt{N(f(\theta))}} = N(f(\theta))^{-1/2} \sum_{x\in U_K/U_K^n} \overline{\theta}(\frac{x}{c})\psi(\frac{x}{c})$$

where $\sqrt{N(f(\theta))}$ is the positive square root.

Note that when θ is unramified:

$$W(\theta) = \frac{\tau(\overline{\theta})}{\sqrt{N(f(\theta))}} = \tau(\overline{\theta}) = \overline{\theta}(\mathcal{D}_K^{-1}) = \theta(\mathcal{D}_K)$$

REMARK: Using the above proposition, we can see that $|W(\theta)| = 1$ and $W(\theta)W(\overline{\theta}) = \theta(-1)$ and as an immediate corollary, when $\theta = \overline{\theta}$, $W(\theta)^4 = 1$, i.e, $W(\theta)$ is a fourth root of unity.

Furthermore, if we have K a number field and χ an idele class character (i.e., continuous on the group I_K of the ideles of K and trivial on the principal ideles), for every place v of K, the natural embedding $K_v^* \to I_K$ defines a character χ_v on K_v^*, and then we have the relation (Tate 1950):

$$W(\chi) = \prod_v W(\chi_v)$$

3.7 Local Galois Gauss sums.

Now, let us consider p a place of $\mathbf{Q}$, $\overline{\mathbf{Q}}_p$ an algebraic closure of $\mathbf{Q}_p$. By a local field we will mean a finite extension of $\mathbf{Q}_p$ that is contained in $\overline{\mathbf{Q}}_p$. Given a local field K, we consider virtual characters of $Gal(\overline{\mathbf{Q}}_p/K)$ (these are differences of two characters of representations of open kernel).

Let us take θ a virtual character of $Gal(\overline{\mathbf{Q}}_p/K)$. The local root number is well defined by the following properties:

(i) $W(\theta_1 + \theta_2) = W(\theta_1)W(\theta_2)$

(ii) Let θ be an irreducible character of degree one and θ' the character of K^* defined by θ in the Artin map (i.e, $K^* \to Gal(\overline{\mathbf{Q}}_p/K)$), then $W(\theta)$ is the local root number $W(\theta')$ defined in the section before.

(iii) $E : K$ a finite extension, θ a character of $Gal(E/K)$ of degree 0, and θ^* the character induced by θ in $Gal(\overline{\mathbf{Q}}_p/K)$. Then $W(\theta^*) = W(\theta)$.

Now we can define the Local Galois Gauss sum using local root numbers:

Definition 3.7.1 *K non-Archimedean local field, θ a character of $Gal(\overline{\mathbf{Q}}_p/K)$. We define:*

$$\tau(\theta) = W(\overline{\theta})\sqrt{N(f(\theta))}$$

We know that the conductor verifies: $f(\theta) = f(\overline{\theta})$, so $W(\theta) = \frac{\tau(\overline{\theta})}{\sqrt{N(f(\theta))}}$.

From the properties verified by $f(\theta)$ and $W(\theta)$, we can see that τ is well-defined by the same properties as W.

The local Galois Gauss sums verify similar properties to the Gauss sums:

Proposition 3.7.2 *K finite extension of $\mathbf{Q}_p$ and θ a character of $Gal(\overline{\mathbf{Q}}_p/K)$. Then:*

(i) $|\tau(\theta)| = \sqrt{N(f(\theta))}$

(ii) $\tau(\theta)\tau(\overline{\theta}) = N(f(\theta))det_\theta(-1)$

The proof of this proposition follows from the fact that τ is well defined by the properties mentioned above. So it is enough to prove that it is true for 1-degree irreducible characters (as done in the section before) and that both sides are invariant under induction for characters of degree 0 (which is true. It is enough to notice that $\overline{\theta^*} = \overline{\theta}^*$).

As a **COROLLARY** of this we can see that for K a local field $|W(\theta)| = 1$ and $W(\theta)W(\overline{\theta}) = det_\theta(-1)$.

Also, we can see as well from this second property verified by W that $W(\theta)\tau(\theta) = det_\theta(-1)\sqrt{N(f(\theta))}$.

Chapter 4

Existence of local constants or local root numbers

In this chapter we review the literature on the existence of local root numbers.

The existence of local root numbers in the non-abelian case is due to R. P. Langlands, but although his proof is purely local, it is extremely long and remains unpublished. Deligne (1973) [6] gave a very elegant existence proof by a global method which we will discuss below.

However, Langlands expresses a hope for an eventual, shorter, conceptual, local existence proof in his long essay "On the functional equation of Artin L-function" .

V. P. Snaith (1987) also gave such a proof of existence [29], but although his method of construction embodies many very good features, it is not the "type of local construction" envisioned by Langlands. Because it requires a global step to get the "inductive in dimension zero" axiom- what we will call axiom (b). But it does give the other properties plus factorisation of the Artin root number into a product of local root numbers. Nevertheless, Snaith uses Explicit Brauer Induction for his proof , which is a novel method of proof and to which we will also recall.

4.1 Deligne's proof of existence of local root numbers

In this section we will have a look at Deligne's proof of existence of local constants. For his proof, he needs a relation between the value taken at $\beta\alpha$ and at α by the local root number W where α and β are two characters of K^* and the ramification of β is relatively small compared to that of α. First of all, we will discuss this relation:

4.1.1 Abelian root numbers

When we have K a non-Archimedean local field of characteristic 0 and α a character of K^*, we have already seen in chapter 3 what the definition of $W(\alpha)$ is. The important part of this section (from which the result we are looking for will follow) is the following proposition that gives a new formulae for the local root numbers:

Proposition 4.1.1 *[35] Let K be a non-Archimedean local field of characteristic 0, and let α be a character of K^* of finite order. Let $\mathcal{A}$ be an ideal of $\mathcal{O}_K$ such that $\mathcal{A}^2/f(\alpha)$, and let us take $\mathcal{B} = \mathcal{A}^{-1}f(\alpha)$. Then there exists*

$c \in K$ verifying:

(i) $< c >= \mathcal{D}(\alpha)$.

(ii) $\alpha(1+y) = \psi(c^{-1}y)$ for all $y \in \mathcal{B}$.

(iii) $W(\alpha) = N(\mathcal{A}^{-1}\mathcal{B})^{-1/2}\sum_{x\in 1+\mathcal{A}/\mathcal{B}}\overline{\alpha}(\frac{x}{c})\psi(\frac{x}{c})$.

Proof: We are not going to give the complete proof, but will see how it follows once one has proved (ii). We know the definition of $W(\alpha)$:

$$W(\alpha) = N(f(\alpha))^{-1/2}\sum_{x\in\mathcal{O}_K^*/f(\alpha)}\overline{\alpha}(\frac{x}{c})\psi(\frac{x}{c})$$

As $\mathcal{B}/f(\alpha)$ we can split the sum into $x = z(1+y)$ where $z \in \mathcal{O}_K^*/\mathcal{B}$ and $y \in \mathcal{B}/f(\alpha)$. Using (ii) we get that $\overline{\alpha}(x) = \overline{\alpha}(\frac{z}{c})\psi(\frac{-y}{c})$ and of course $\psi(x) = \psi(\frac{z}{c})\psi(\frac{zy}{c})$, so we get:

$$N(f(\alpha))^{1/2}W(\alpha) = \sum_{z\in\mathcal{O}_K^*/\mathcal{B}(\alpha)}\overline{\alpha}(\frac{z}{c})\psi(\frac{z}{c})\sum_{y\in\mathcal{B}/f(\alpha)}\psi(\frac{y(z-1)}{c})$$

The inner sum is 0 unless $y \to \psi(\frac{y(z-1)}{c})$ is the trivial character of $\mathcal{B}/f(\alpha)$, that is, $z \equiv 1 \bmod \mathcal{A}$. We get then:

$$N(f(\alpha))^{1/2}W(\alpha) = N(\mathcal{A})\sum_{z\in 1+\mathcal{A}/\mathcal{B}}\overline{\alpha}(\frac{z}{c})\psi(\frac{z}{c})$$

giving us the result we were looking for. □

Now, we can give now the result needed by Deligne for his proof:

Corollary 4.1.2 *[35] Let β be a character of K^* of finite order s.t. $f(\beta)/\mathcal{A}$. Then:*

$$W(\beta\alpha) = \beta(c_\alpha)W(\alpha)$$

where c_α is the c we get in the above proposition, just denoted this way to make clear that it does depend on α.

Proof: By hypothesis, either $\wp_K f(\beta)/f(\alpha)$ or α and β are both non-ramified. Hence $f(\beta\alpha) = f(\alpha)$. So then, by the above proposition applied to $\beta\alpha$ and taking into account that $\overline{\beta\alpha}(\frac{x}{c}) = \overline{\beta}(c^{-1})\overline{\alpha}(\frac{x}{c}) = \beta(c)\overline{\alpha}(\frac{x}{c})$ when $x \equiv 1 \ \mathrm{mod} \mathcal{A}$ the corollary follows. □

Now we are prepared to study the proof of existence of local constants given by Deligne.

4.1.2 Proof of existence.

Throughout this section we will consider only local or global fields of characteristic 0. We will denote by $R(K)$ the set of pairs (L, ρ) where $K \subset L \subset \overline{K}$, L/K finite and ρ a virtual representation of $Gal(\overline{K}/L)$.

If E/K is a finite Galois extension contained in $\overline{K}/K$, $R(E/K)$ denotes all the pairs (L, ρ), $K \subset L \subset E$ and ρ a virtual representation of $Gal(E/L)$. It is obvious then in a natural way that:

$$R(K) = \bigcup_{E/K} R(E/K)$$

We will write $R_1(K)$ and $R_1(E/K)$ when we are in the character case.

Now we are going to discover when a function defined on $R_1(K)$ is extendible:

Definition 4.1.3 *Let us suppose that we have a function F defined on $R_1(K)$ taking values in some abelian group A. We say that F is* ***extendible*** *if F can be extended to an A-valued function on $R(K)$ satisfying:*

(a) $F(L, \rho_1 + \rho_2) = F(L, \rho_1)F(L, \rho_2)$ *for all* $(L, \rho_i) \in R(K)$.

(b) If $(L, \rho) \in R(K)$ *and* $dim(\rho) = 0$, *and* $L \supset L' \supset K$ *then:*

$$F(L, \rho) = F(L', Ind_{Gal(\overline{K}/L)}^{Gal(\overline{K}/L')}(\rho))$$

This definition would be analogous for E/K Galois finite and $R(E/K)$.

Now we give some basic properties about extendible functions:

(1) If F is extendible (or extendible in E/K) there is a unique extension of F to $R(K)$ (or $R(E/K)$). If we have two extensions F_1, F_2 and $(L, \rho) \in R(E/K)$, we could consider $\rho - dim(\rho)[1_L]$ of dim 0. Then:

$$\begin{aligned} F_i(L, \rho) &= F_i(L, \rho - dim(\rho)[1_L] + dim(\rho)[1_L]) \\ &= F_i(L, \rho - dim(\rho)[1_L])F_i(L, [1_L])^{dim(\rho)} \end{aligned}$$

Now using Explicit Brauer Induction, $\rho - dim(\rho)[1_L]$ could be expressed as a sum of multiples of $Ind_{Gal(E/L)}^{Gal(E/L_i)}(\chi_i - [1_{L_i}])$ where $(L_i, \chi_i) \in R_1(E/L)$. So using (a) and (b) from the definition, and the fact that over R_1 we know $F_1 = F_2 = F$, we can see that actually $F_1 = F_2$.

It is obvious now that F is extendible iff it is so for E/K for all E.

(2) Now, we will try to find a similar relation to the one given in (b) but without the hypothesis $dim(\rho) = 0$. From the properties verified by the extension of F we have:

$$\begin{aligned} F(L, \rho) &= F(L, \rho - dim(\rho)[1_L] + dim(\rho)[1_L]) \\ &= F(L, \rho - dim(\rho)[1_L])F(L, [1_L])^{dim(\rho)} \end{aligned}$$

$$= F(L', Ind_{Gal(\overline{K}/L)}^{Gal(\overline{K}/L')}(\rho - dim(\rho)[1_L])F(L, [1_L])^{dim(\rho)}$$
$$= F(L', Ind_{Gal(\overline{K}/L)}^{Gal(\overline{K}/L')}(\rho))F(L', Ind_{Gal(\overline{K}/L)}^{Gal(\overline{K}/L')}[1_L])^{-dim(\rho)}F(L, [1_L])^{dim(\rho)}$$

So then:

$$F(L', Ind_{Gal(\overline{K}/L)}^{Gal(\overline{K}/L')}(\rho)) = \lambda(F)^{dim(\rho)}F(L, \rho)$$

$$\text{where} \quad \lambda(F) = \frac{F(L', Ind_{Gal(\overline{K}/L)}^{Gal(\overline{K}/L')}[1_L])}{F(L, [1_L])}$$

When $\lambda(F) = 1$ for all $L \supset L' \supset K$ we say that F is **strongly extendible.**

It is time now to study some examples of extendible functions. In all these examples it is obvious that they verify (a) and (b) in the definition, and when we restrict ourselves to the case of R_1 one gets the original function:

(I) When K is global, $(L, \chi) \to \Lambda(s, \chi)$ is strongly extendible by $(L, \rho) \to \Lambda(s, \rho)$ given by Artin theory of non-abelian L-series.

(II) K global or local non-Archimedean, $(L, \chi) \to N(f(\chi))$ is extendible by $(L, \rho) \to N(f(\rho))$.

(III) K local and $c \in K^*$, $L(s, \chi) \to \chi(c)$ is extendible by $L(s, \rho) \to det_\rho(c)$.

(IV) If $F(L, \chi)$ depends only on L, let us say $F(L, \chi) = a(L)$, then this is extendible by $F(L, \rho) = a(L)^{dim(\rho)}$.

(V) K global, $(L, \chi) \to W(\chi) = \Lambda(1-s, \chi)\Lambda(s, \overline{\chi})^{-1}$ is strongly extendible by $(L, \rho) \to \Lambda(1-s, \rho)\Lambda(s, \overline{\rho})^{-1}$

At this point we are prepared to prove the existence of local constants:

Theorem 4.1.4 *(Langlands) If K is a local field of characteristic 0 (this is just to fix ideas, the result is true in any characteristic), then $(L, \chi) \to W(\chi)$ is extendible.*

The proof we are going to give is not that of Deligne. It is a modified version. In order to prove this theorem, we are going to need the following result (for proof look at [35]):

Lemma 4.1.5 *There exists a finite Galois extension e/k of global fields and a place v_0 of k such that:*

(i)There is a unique place u_0 of e lying over v_0 and the extension e_{u_0}/k_{v_0} is isomorphic to E/K.
(ii) k is totally complex (i.e., k has no real Archimedean place).

Now, if we take k, e, v_0 and u_0 as in the Lemma, we identify e_{u_0}/k_{v_0} and E/K. We have an isomorphism $Gal(E/K) \simeq Gal(e/k)$. Hence giving us a bijection between $R(e/k)$ and $R(E/K)$ given by:

$$(l, \rho) \to (l_{w_0}, \rho_{w_0})$$

where w_0 is the unique place of l above v_0 for $e \supset l \supset k$ and ρ_{w_0} is the restriction of ρ to $Gal(E/l_{w_0})$.

Now our problem is to prove that $(l, \chi) \to W(\chi_{w_0})$ is extendible in e/k. What we will try to do is to express this number $(W(\chi_{w_0}))$ as a product of things we know are extendible. To do this:

Let us take v a place of k and write u and w for primes of e and l respectively such that $u/w/v$. For each non-Archimedean $v \neq v_0$, let $\mathcal{A}_v$ be an ideal of $\mathcal{O}_v$ verifying $f(\beta)/\mathcal{A}_v$ for each $(F, \beta) \in R_1(e_u/k_v)$ and when v is non-ramified also verifying $\mathcal{A}_v = \mathcal{O}_v$.

Now, let α be a character of the idele class group of k such that $\alpha_{v_0} = 1$ and $\mathcal{A}_v^2 \mathcal{D}_{e_u/k_v}/f(\alpha_v)$.

We will construct at this moment an idele of k in the following way:

$$c = (c_v)$$

where $c_v = 1$ if v is Archimedean or $v = v_0$, and $c_v = c$ where c is the element associated to α_v and $\mathcal{A}_v$ in the corollary 4.1.2 for non-Archimedean $v \neq v_0$.

Now, let us take $(l, \chi) \in R(e/k)$ and $\alpha_l = \alpha \circ N_{l/k}$. Using corollary 4.1.2 we know:

$$W(\chi_w(\alpha_l)_w) = \begin{cases} \chi_w(c_v)W((\alpha_l)_w) & w \text{ is non-Archimedean, } w \neq w_0 \\ W(\chi_{w_0}) & \text{if } w = w_0 \\ 1 & w \text{ non-Archimedean} \end{cases}$$

At this stage, if we express our global root number as local root numbers we get:

$$\begin{aligned} W(\chi\alpha_l) &= \prod_w W(\chi_w(\alpha_l)_w) = W(\chi_{w_0}) \prod_{\substack{w \neq w_0 \\ w \text{ non-Archimedean}}} \chi_w(c_v)W((\alpha_l)_w) \\ &= W(\chi_{w_0})\chi(c)a(l) \end{aligned}$$

where $a(l) = \prod_{\substack{w \neq w_0 \\ w \text{ non-Archimedean}}} W((\alpha_l)_w).$

So then $W(\chi_{w_0}) = W(\chi\alpha_l)\chi(c)^{-1}a(l)^{-1}$ would be extendible since $(l,\chi) \to a(l),\ \chi(c),\ W(\chi\alpha_l)$ are extendible (this last just follows from example (V) extending by $(l,\rho) \to W(\rho\otimes\alpha_l))$. Thus the theorem then is proved. $\square$

Now that we have proved the existence of local root numbers, we can give some corollaries that follow immediately from the uniqueness of the extension of $W(\chi)$ to $W(\rho)$.

Corollary 4.1.6 *Let K be a local field of characteristic 0, and $(L,\rho) \in R(K)$, then,*

(i) $|W(\rho)| = 1$

(ii)$W(\rho)W(\overline{\rho}) = det_\rho(-1)$

(iii) if $\rho = \overline{\rho}$ (i.e., ρ is orthogonal) then $W(\rho)$ is a fourth root of unity.

Proof: (i) and (ii) come from extending $(L,\chi) \to |W(\chi)| = 1,\ W(\chi)W(\overline{\chi}) = \chi(-1)$ and using uniqueness of the extension we get $(L,\rho) \to |W(\rho)| = 1,\ W(\rho)W(\overline{\rho}) = det_\rho(-1)$.

(iii) just follows from (ii), if $\rho = \overline{\rho}$ then $W(\rho)^2 = det_\rho(-1) = \pm 1$ so then $W(\rho)^4 = 1$, as we wanted to prove. $\square$

Corollary 4.1.7 *K an algebraic number field, $(K,\rho) \in R(K)$. For each place v of K, ρ_v is the restriction of ρ to a decomposition group of v. Then, $(K_v,\rho_v) \in R(K_v)$ and:*

$$W(\rho) = \prod_v W(\rho_v)$$

Proof: The proof of this corollary is based on extending $(L,\chi) \to W(\chi) = \prod_w W(\chi_w)$ to $(L,\rho) \to W(\rho) = \prod_w W(\rho_w)$ $\square$

4.2 Snaith's proof of existence of local constants

The proof we now look at, is completely different to that of Deligne. As stated before, in order to prove this existence, Snaith uses a novel method of proof, applying Explicit Brauer Induction (E.B.I) [32]. First we should describe briefly the E.B.I. formulae as well as their basic properties:

4.2.1 E.B.I.

Let us consider G a finite group and let us denote by $R(G)$ the ring of isomorphism classes of finite-dimensional complex representations of G, which we can assume to be unitary:

$$\rho : G \to U(n)$$

If NT^n is the normaliser of the diagonal maximal torus in $U(n)$, then we can let G act upon the cosets $U(n)/NT^n$ by left multiplication via ρ. We will write $X = U(n)/NT^n$ and $M = G\backslash X$.

If H is a subgroup of G, (H) will denote the conjugacy class of H. For each of these we have associated a subspace of M, let us call it $M_{(H)}$, which consists of all the orbits which are isomorphic to G/H. Let $\chi^{\sharp}_{(H)}$ denote the Euler characteristic of $M_{(H)}$ with respect to singular rational cohomology with compact supports. We will not go deeper into these numbers, we just mention that they are integers which ultimately depend only on ρ.

Throughout this section, we will call the conjugacy class of a homomorphism from G to NT^n and a representation of the form $Ind_H^G(\varphi)$ where φ

is a character, a monomial homomorphism and a monomial representation respectively.

The connection between these is given by the well-known Lemma, which proof can be found in [32]:

Lemma 4.2.1

(a) Up to equivalence of monomial homomorphisms, $\nu : G \to NT^n$ *is the direct sum of monomial homomorphisms of the form:* $Ind_{H_i}^G(\nu_i : H_i \to U(1) = \mathbb{S}^1)$ *where the set of* $\{\nu_i\}$ *is well-defined up to permutation and conjugation within* G

(b) Consequently, as an element of $R(G)$, $\nu : G \to NT^n$ *has a canonical form:* $\nu = \sum_i Ind_{H_i}^G(\nu_i) \in R(G)$.

Now, we will state the weak form of E.B.I. [32]:

Theorem 4.2.2 $\rho : G \to U(n)$ *a representation of a finite group:*

(i) In $R(G)$ $1 = \sum_{(H)} \chi^{\sharp}_{(H)} Ind_H^G(1)$.

(ii) In $R(G)$ $\rho = \sum_{(H)} \chi^{\sharp}_{(H)} Ind_H^G(Res_H^G(\rho))$.

(iii) If $z \in NT^n$ *and* H *is the stabiliser of* zNT^n *then* $z^{-1}\rho(H)z$ *lies in* NT^n *(consequently, by the above Lemma,* $Ind_H^G(Res_H^G(\rho))$ *has a canonical form as a sum of monomial representations).*

Restriction and induction homomorphism.

In the general case, let us suppose we have a finite group G and a compact Lie group π. We will say that ρ is a subhomomorphism from G to π when it is a homomorphism from $H \leq G$ into π.

$$\rho : H \subset G \to \pi$$

Let us define now, $R_+(G,\pi)$ to be the free abelian group on equivalence classes of subhomomorphisms from G to π.

In the case $\pi = \mathbb{S}^1$ (i.e. the character case), $R_+(G,\pi)$ will be denoted by $R_+(G)$.

Definition 4.2.3 *Given $i : J \to G$ an inclusion, we can define:*

(i) The restriction homomorphism $Res_J^G : R_+(G,\pi) \to R_+(J,\pi)$ by:

$$Res_J^G(\rho : G \supset H \to \pi) = \sum_{x \in J \backslash G / H} (\rho(x_-^{-1}x) : J \supset J \cap (xHx^{-1}) \to \pi$$

(ii) the Induction homomorphism $Ind_J^H : R_+(J,\pi) \to R_+(G,\pi)$ by:

$$Ind_J^G(\rho : J \supset H \to \pi) = (\rho : G \supset H \to \pi)$$

After these definitions, we are going to define two new operators that we will call τ_G and B to follow Snaith's notation.

$$\tau_G : R(G) \to R_+(G, NT^n)$$

$$B : R_+(G, NT^n) \to R(G)$$

In order to do this, we consider $\rho : G \to U(n)$ a representation and the left action via ρ of G upon X. We will denote by $\{M_\alpha\}$ the set of connected components of $M_{(H)}$. And we will choose for each M_α and element $g_\alpha \in U(n)$ whose orbit lies in M_α. Now, if we denote by $\chi_\alpha^\sharp$ the compactly supported cohomology Euler characteristic of M_α we get: $\chi_{(H)}^\sharp = \sum_\alpha \chi_\alpha^\sharp$.

Definition 4.2.4 *We can define now:*

(i) $\tau_G(\rho) = \sum_\alpha \chi^\sharp_\alpha(g_\alpha^{-1}\rho g_\alpha : G \supset \rho^{-1}(gNT^n g^{-1}) \to NT^n) \in R_+(G, NT^n)$

(ii) $B(\rho : G \supset H \to NT^n) = Ind_H^G(\rho : H \to NT^n \subset U(n)) \in R(G)$

Properties: *We give some basic properties of these operators [32]:*

(I) $\tau_G(\rho) \in R_+(G, NT^n)$ *is well-defined, depending only on the equivalence class of* ρ *as a representation.*

(II) $B\tau_G(\rho) = \rho$.

(III) $\mu_i : G_i \to U(n_i)$ $i = 1, 2$ *representations, and let us denote by "$*$" the operation induced by the direct sum of matrices, then,*

$$\tau_{G_1 \times G_2}(\mu_1 \oplus \mu_2) = \tau_{G_1}(\mu_1) * \tau_{G_2}(\mu_2) \in R_+(G_1 \times G_2, NT^{n_1+n_2})$$

(IV) $i : J \to G$ *an inclusion,*

$$Res_J^G(\tau_G(\rho)) = \tau_J(Res_J^G(\rho))$$

A presentation for $R(G)$.

This is a problem which appears, for instance, in [28] (footnote, p.71). It originates with Brauer (c.1946).

In this section we will study a presentation for $R(G)$ in terms of monomial representations [32]. We will denote by $R_*(G) = \oplus_{n\geq 1} R_+(G, NT^n)$.

Now, if we have $\rho : H \to NT^n$ a monomial homomorphism ($H \leq G$), the matrix representation of the induced representation gives:

$$Ind_H^G(\rho) : G \to NT^{dn} \quad \text{where } d = [G : H]$$

We will define now relations on $R_*(G)$, so we will set Λ the subgroup of $R_*(G)$ generated by elements of the following three classes:

(a)$J \subset H \subset G$ and $d = [H : J]$, then,

$$\{(\rho : G \supset J \to NT^n) - (Ind_J^H(\rho) : G \supset H \to NT^{dn})\} \in \Lambda$$

(b) $\nu : G \to NT^n$ and $\mu : G \to NT^m$ then

$$\{(\nu \oplus \mu) - \nu - \mu\} \in \Lambda$$

(c)$\rho : G \to NT^n$ then $\{\tau_G(\rho) - \rho\} \in \Lambda$.

From its definition, it is clear that $\Lambda \subset KerB$, and in fact, by means of τ_G it can be shown that $B : R_+(G, NT^n) \to R(G)$ induces an isomorphism $B : R_*(G)/(\Lambda) \to R(G)$

4.2.2 Proof of existence.

We are now ready to study Snaith's proof of existence of Local Root Numbers. Firstly, we will recall the properties that characterize the Local Root Numbers denoted by $W_K(\rho)$.

Given L/K a finite Galois extension of local fields and $\rho : Gal(L/K) \to U(n)$ an n-dimensional unitary local Galois representation, the local root number $W_K(\rho)$ is a complex number of modulus 1, verifying the following axioms:

(a) If ν is another Galois representation the $W_K(\rho \oplus \nu) = W_K(\rho)W_K(\nu)$.

(b) If K/N is a finite extension with L/N Galois then $W_N(Ind_{Gal(L/K)}^{Gal(L/N)}(\rho)) = W_K(\rho)W_N(Ind_{Gal(L/K)}^{Gal(L/N)}(1))^n$

(c) $M \supset L \supset K$ a chain of finite Galois extensions and $\pi : Gal(M/K) \to Gal(L/K)$ the canonical isomorphism, then: $W_K(\rho\pi) = W_K(\rho)$.

(d) If ρ is a character, then $W_K(\rho)$ is given by the formula given in chapter 3 for local root numbers.

In other proofs of existence, the procedure was to verify all these properties, and from them to deduce that $W_K(\rho) = \prod_v W_{K_v}(\rho_v)$ (global/local factorisation of the Artin root number). But Snaith's proof proceeds quite the other way around. His method, however, has a handicap: his construction does not ensure the verification of (a) and (b). We will see later how this point could be overcome.

Speaking very generally, what he does in his proof is to define over $R_+(G)$ (where $G = Gal(L/K)$) w_K using the definition for local root numbers we know from chapter 3. Then he defines W_K using the diagram:

$$\begin{array}{ccc} & \tau_G & \\ R(G) & \to & R_+(G) \\ W_K & \searrow & \downarrow w_K \\ & & \mathbb{S}^1 \end{array}$$

Note that we have already seen that $R_*(G)/\Lambda \simeq R(G)$.

Let us study the proof more deeply:

For the case $\chi : Gal(L/K) \to U(1) = \mathbb{S}^1$ a character, we define $w_K(\chi)$ as in 3.7.1.

Now, we will construct now $w_K : R_*(Gal(L/K)) \to \mathbb{S}^1$. Let us suppose we have a subhomomorphism of the form $z = (\nu : Gal(L/K) \supset Gal(L/F) \to NT^n)$. Applying Lemma 4.2.1 we get a set of characters:

$$\{\nu_i : H_i = Gal(L/N_i) \to \mathbb{S}^1;\ 1 \le i \le u\}$$

and then we define:

$$w_K(z) = \prod_{1\le i\le u} w_{N_i}(\nu_i)[W_K(Ind_{Gal(L/N_i)}^{Gal(L/K)}(1))]^{n_i} \quad n_i = [N_i : K]$$

The one term that needs explanation in this formula is $W_K(Ind(1))$ since w_{N_i} is well-defined by the Lemma 4.2.1 (a). The definition of $W_K(Ind(1))$ comes from the fact that $Ind(1)$ is an orthogonal representation and in [30] we can find a simple, local construction of the local root number in this case. We are not going deeper into this definition, we just want to know that it is easily constructible.

Next, we notice that the construction that assigns $\{\nu_i : H_i = Gal(L/N_i) \to \mathbb{S}^1;\ 1 \le i \le u\}$ to the subhomomorphism $(\nu : Gal(L/K) \supset Gal(L/F) \to NT^n)$ defines a retraction:

$$\begin{array}{ccc} R_+(Gal(L/K), NT^n) & \to & R_+(Gal(L/K), \mathbb{S}^1) \\ z & \rightsquigarrow & \sum_{1\le i\le u} \nu_i \end{array}$$

So now, using τ_G, if we have $\rho : G \to U(n)$ we can define the local root number as:

$$W_K(\rho) = w_K(\tau_G(\rho)) \in \mathbb{S}^1$$

With this definition, one can check easily that $W_K(\rho) = \prod_v W_{K_v}(\rho_v)$ (global/local factorisation). Firstly, one would check that $w_K : R_* \to \mathbb{S}^1$ verifies it, but this is true simply because it is true for characters and the double coset formula holds in R_*. Then just writing $W_K(\rho) = w_K(\tau_G(\rho)) = \prod_v w_{K_v}(\tau_{Gal(L_w/K_v)}(\rho_v))$ (Using property (IV) of τ_G) one would get $\prod_v W_{K_v}(\rho_v)$.

At this point, we should check the rest of the axioms that the root numbers verify, the ones seen at the beginning of this section. (c) and (d) are immediate just using the properties verified by τ_G. The ones that cause trouble are (a) and (b). What we see is that (a) and (b) will be true modulo roots of unity. To prove this, we can look at the definition of Λ in section 4.2.1, page 38, *A presentation for* $R(G)$, and notice that to state that, it would be enough to prove that w_K annihilates Λ. And this is exactly what Snaith does. To prove this, first he needs the following proposition:

Proposition 4.2.5 *[29] Let $E/\mathbf{Q}_p$ be a finite Galois extension of local fields with $G = Gal(E/\mathbf{Q}_p)$. Then there exists a Galois extension of number fields $\widehat{E}/\widehat{K}$ such that:*
(i) $\widehat{E}$ is dense in E and $\widehat{E} \subset E$ is the unique place lying over $\widehat{K} \subset K$.
(ii) $Gal(\widehat{E}/\widehat{K}) \cong G$.
(iii) $\widehat{E}/\widehat{K}$ is tamely ramified at all finite primes not lying over p.

REMARK: In order to get the result one also needs to arrange that there is only one prime above p.

REMARK: The Explicit Brauer Induction formula [32] is a homomorphism, so the "Snaith method" using it, gives (a) at once but still not (b).

With this proposition, one can show that w_K annhilates Λ. In the tame case, we know that tame local root numbers exist [8], so in this case w_K

annhilates Λ. We also know that the Artin root number (as a function of R_*) does annhilates Λ. Then, by the global/local factorisation of $w_{\widehat{R}}$, the annhilation of Λ is true for the product of the remaining factors.

Finally, we have finished this novel proof of the existence of local root numbers.

Chapter 5

Orthogonal and Symplectic root numbers

This chapter concentrates on orthogonal root numbers, namely, when the representation considered is orthogonal, and the symplectic representation case.

The aim of section 1 is to give us some machinery enabling us to work with orthogonal representations. In the algebraic case, usually called the global case from the terminology of local fields and global fields (that is, when we are working with algebraic number fields), $W(\rho)$ is characterized by being always 1, when ρ is an orthogonal Galois representation. In the local case (i.e., when we are working with local fields), the only information known is $W(\rho)$ being a fourth root of unity. Nevertheless, Deligne ([5], (1976)) noticed that when $W(\rho)$ is thought of in cohomological terms, there is a close relation between $w(\rho)$ and the first and second Stiefel-Whitney classes of such representation. Namely, $W(\rho) = W(det_\rho)SW_2(\rho)$. Here, $SW_2(\rho)$ is thought under the image of $H^2(G; \mathbf{Z}/2) \cong \{\pm 1\}$. Two different proofs of such result can be found in the literature, including that of Snaith's ([30] and [31], (1988)), which differs from Deligne's original.

In section 2, we review some results concerning symplectic representations, giving us a characterization for such in the Quaternion case.

5.1 Orthogonal root numbers.

We are interested in orthogonal root numbers, i.e., when the representation ρ is an orthogonal one. At this stage, we can already state some results concerning these.

When E/K is a finite Galois extension of global fields, we know from Example (V) of section 2.1.2 that the root number is characterized by:

$$\Lambda(1-s,\rho) = W(\rho)\Lambda(s,\overline{\rho})$$

so, applying this formula to $\overline{\rho}$ and writing $t = 1-s$, we get:

$$\Lambda(1-t,\rho) = W(\overline{\rho})^{-1}\Lambda(t,\overline{\rho})$$

and then $W(\rho) = W(\overline{\rho})^{-1}$, i.e.: $W(\rho)W(\overline{\rho}) = 1$. So, when ρ is an orthogonal representation ($\rho = \overline{\rho}$) we get $W(\rho)^2 = 1$, i.e.: $W(\rho) = \pm 1$.

When we are in the local case, i.e.: E/K a finite Galois extension of local fields, we know from Corollary 4.1.6 that in this case $W_K(\rho)$ is going to be a fourth root of unity.

Below, we will study orthogonal root numbers in more depth, and will be able to see that in the algebraic case (algebraic number fields) $W(\rho) = 1$, this result originally due to Fröhlich-Queyrut [7].

In order to see this, let us suppose first, as usual, that E/K is a finite Galois extension of local or global fields of characteristic 0, $G = Gal(E/K)$ and ρ a representation of G. Let us define for ρ the following number:

$$c(\rho) = \frac{W(\rho)}{W(det_\rho)}$$

which verifies the following **properties** (easily checkable, some of them are trivial, and the rest just come from Corollary 4.1.6):

(I) $dim\rho = 1$ then $c(\rho) = 1$.

(II) $c(\rho_1 + \rho_2) = c(\rho_1)c(\rho_2)W(det_{\rho_1})W(det_{\rho_2})W(det_{\rho_1}det_{\rho_2})^{-1}$

(III) $c(\rho + \overline{\rho}) = det_\rho(-1)$.

(IV) $c(\overline{\rho}) = \overline{c(\rho)}$ and $|c(\rho)| = 1$.

(V) if $\rho = \overline{\rho}$ then $c(\rho) = \pm 1$

As we said above, we are interested in the case when ρ is an orthogonal representation. We already now by (v) that in this case $c(\rho) = \pm 1$. Let us suppose that ρ is an orthogonal representation, K is local non-Archimedean and G is a dihedral group. Then $E \supset L \supset K$, where E/L is cyclic, L/K quadratic and each element of $Gal(E/K)\backslash Gal(E/L)$ has order 2.

Then we take $\rho = Ind_{Gal(E/L)}^{Gal(E/K)}(\chi)$ for some character χ of $Gal(E/L)$.

The transfer map $ver_{L/K} : Gal(E/K) \to Gal(E/L)$ is trivial, so the character $\chi_{|K^*}$ of K^* corresponding to $\chi \circ ver_{L/K}$ will be also trivial.

We could write $L = K(\delta)$ for some δ s.t. $\delta^2 \in K^*$, then $Tr_{L/K}(\delta) = 0$ and $\chi(\delta) = \pm 1$ (independent on the choice of δ).

In the above situation, Fröhlich-Queyrut [7] proved the following theorem from which our results will follow:

Theorem 5.1.1 *In the above situation:* $c(\rho) = \chi(\delta)$. *One verifies easily that* $c(\rho) = W(\chi)$.

Now, we wish to study the global case when ρ is an orthogonal representation, but the global root number is invariant under induction, so due to the Induction Theorem for orthogonal representations [20] we know that we can restrict ourselves to the following cases:

(i) $\dim \rho = 1$.

(ii) $\rho = \theta + \overline{\theta}$ for some representation θ.

(iii) ρ dihedral (what we mean by ρ dihedral is that we are in a global analogous situation to the one in the theorem).

Let us study these three cases:

(i) When $dim\rho = 1$, ρ is either $[1_K]$ or the non-trivial character of $Gal(L/K)$ with L/K quadratic, in both cases $W(\rho) = 1$.

(ii) When $\rho = \theta + \overline{\theta}$, $W(\rho) = W(\theta)W(\overline{\theta}) = W(\theta)\overline{W(\theta)}$ since we are in the global case. And then $W(\rho) = |W(\theta)|^2 = 1$.

(iii) ρ dihedral. By (i) we already know that $W(det_\rho) = 1$, so $W(\rho) = c(\rho)$.

We are going to show now, that for each place v of K $c(\rho_v) = \prod_{w/v} \chi_w(\delta)$:

If v is non-Archimedean and undecomposed in L, this follows from the theorem.

If v is Archimedean and undecomposed in L, then L_w is complex and $\chi_w = 1$. On the other hand, $\rho_v = [1] + sgn$ so using properties (I) and (II) of c, we get $c(\rho_v) = 1$.

So now, we just have left the case when v splits in L. Here, the decomposition group of w/v is in $Gal(E/L)$, so $\rho_v = \chi_w + \overline{\chi_w}$ and then, using property (III) of c, we get $c(\rho_v) = c(\chi_w + \overline{\chi_w}) = \chi_w(-1)$. On the other hand, if w' is another place above v, $\chi_w(\delta)\chi_{w'}(\delta) = \chi_w(-1)$, since $\chi_{w'}(\delta) = \overline{\chi_w(-\delta)}$. We get in this way the result we were looking for.

So now, as $c(\rho_v) = \prod_{w/v} \chi_w(\delta)$ we can write:

$$c(\rho) = \prod_v c(\rho_v) = \prod_v \prod_{w/v} \chi_w(\delta) = \prod_w \chi_w(\delta) = \chi(\delta) = 1$$

since $\delta \in K^*$ and χ is an idèle class character.

So now, we can state the the following result [35],

Corollary 5.1.2 *If E/K is a finite Galois extension of algebraic number fields and ρ is an orthogonal representation of $Gal(E/K)$, then $W(\rho) = 1$.*

To finish this section, once we have studied the global case, we should study the local one. In this case there is an alternative interpretation of $c(\rho)$ for an orthogonal representation given by Deligne [5].

If we have G any finite group and ρ an orthogonal representation of G, we could consider $SW_i(\rho) \in H^i(G, \mathbf{Z}/2\mathbb{Z})$ the i^{th} Stiefel-Whitney invariant of ρ.

For low dimension i, SW_i is given algebraically:

$H^1(G;\mathbf{Z}/2) \simeq Hom(G,\{\pm 1\})$, and under this canonical isomorphism the image of $SW_1(\rho)$ is det_ρ.

If $SW_1(\rho)$ is trivial (i.e. $det_\rho = 1$), $SW_2(\rho) \in H^2(G;\{\pm 1\}) = H^2(G;\mathbf{Z}/2)$ is the inverse image under $\rho : G \to SO(n)$ of the class of the extension:

$$1 \to \{\pm 1\} \to Spin(n) \to SO(n) \to 1$$

considered as a class in $H^2(SO(n);\mathbf{Z}/2)$.

Now, if we take $G = Gal(E/K)$ (in the local case) using inflation (which is injective) we get:

$$H^2(G,\mathbf{Z}/2) \to H^2(Gal(\overline{\mathbf{Q}}/K),\mathbf{Z}/2) \simeq \{\pm 1\} \quad (K \neq \mathbf{C})$$

By writing $cl(SW_2(\rho))$ for the image of $SW_2(\rho)$ in $\{\pm 1\}$ under this maps, Deligne saw that $c(\rho) = cl(SW_2(\rho))$. Also proved in ([30] and [31], (1988)) in a completely different way.

Now, we proceed to outline the steps required to obtained Deligne's result. Deligne noticed that if we restrict ourselves to the case of orthogonal representations of $Gal(\overline{K}/L)$, $c(\rho)$ is uniquely defined by the properties,

(a) $dim\rho = 1$ then $c(\rho) = 1$ and $c(\rho + \overline{\rho}) = det_\rho(-1)$.

(b) $c(\rho_1 + \rho_2) = c(\rho_1)c(\rho_2)W(det_{\rho_1})W(det_{\rho_2})W(det_{\rho_1}det_{\rho_2})^{-1}$

(c) $K \subset L' \subset L$ and ρ an orthogonal representation of $Gal(\overline{K}/L)$ with $dim\rho = 0$ and $det_\rho = 1$ then $c(Ind_{Gal(\overline{K}/L)}^{Gal(\overline{K}/L')}(\rho)) = c(\rho)$

So it is enough to note that $cl(SW_2(\rho))$ verifies these properties (for details [5]), some of which are just re-arrangements of well-known results such as the Cartan formula for second Stiefel-Whitney classes. Therefore we can state his theorem [35],

Theorem 5.1.3 *(Deligne) Let E/K be a finite Galois extension of local fields, and let ρ be an orthogonal representation of $Gal(E/K)$. Then,*

$$W_K(\rho) = W_K(det_\rho)cl(SW_2(\rho))$$

5.2 A brief introduction to symplectic representations

The definition of symplectic representation has already been introduced in section 0.1. In this chapter we have a look at some properties of such representations, characterizing symplectic representations in the Quaternion case.

5.2.1 Real valued characters.

Let G be a finite group and K a subfield of the complex numbers. Let us consider as well, V a finite dimensional K-vector space and $\rho : G \to GL(V)$ a representation. Then, we can define $\rho' : G \to GL(\mathbf{C} \otimes_K V)$. Such a representation is called a K-representation.

We can define now $R_{ch}^K(G)$ as the set of characters of K-representations, which is a subring of $R_{ch}(G)$, the set of characters of G.

Clearly $\chi \in R_{ch}^K(G)$ takes values in K, however the converse is not true, that is why the definition of K-valued characters is introduced: $\overline{R}_{ch}^K(G)$ will be then, the subring of $R_{ch}(G)$ consisting of characters with values in K.

We are interested in the case $K = \mathbf{R}$, i.e., real valued characters. We can define now THE three types of irreducible real valued characters:

(1) $\chi = \phi + \overline{\phi}$, where ϕ is an irreducible character of G.

(2) χ is an irreducible orthogonal character.

(3) χ is an irreducible symplectic character.

These are the three types of irreducible real valued characters. In cases (2) and (3), χ is also irreducible as a complex representation, but not in case (1), since ϕ and $\overline{\phi}$ are complex representations.

5.2.2 Induction.

In this section we will recall some useful induction theorems for symplectic characters [20]. In order to do that, we need to introduce some definitions.

Definition 4.1: The Quaternion group Q_{4n} of order $4n$ is the group on two generators X, Y with relations: $X^n = Y^2$, $Y^4 = 1$, and $YXY^{-1} = X^{-1}$.

Note: Q_{4n} contains a unique element of order 2, i.e., Y^2. Q_4 is cyclic, and for $n > 1$, $\{1, Y^2\}$ is the center of Q_{4n}, and $Q_{4n}/\{1, Y^2\}$ is the dihedral group D_{2n}.

Q_{4n} has 4 characters of degree 1, and the other irreducible characters are real valued characters of degree 2. Those factorising through a dihedral quotient are orthogonal, and the rest symplectic. For instance, in the case of Q_8 there is only a 2-dimensional irreducible symplectic representation, that is why it is an useful group to work with.

Definition 5.2.1 *G a finite group. A "quaternion character" of G is an irreducible character of degree* 2 *of G which is lifted from a symplectic character of a quaternion quotient of G.*

We are ready now to recall induction theorems for symplectic representations. The proofs of which can be found in [20],

Theorem 5.2.2 *G a finite group and χ a symplectic character of G. Then χ is a $\mathbb{Z}$-linear combination of characters of the form $Ind_H^G(\phi)$ for some subgroup H of G, where:*

(1) either $\phi = \psi + \overline{\phi}$, ϕ irreducible character of degree 1 of H.

(2) or ϕ is a quaternion character of H.

Theorem 5.2.3 *G a finite supersolvable group (namely, there exists a sequence $\{e\} = G_0 \subset G_1 \subset ... \subset G_k = G$ of normal subgroups of G such that G_i/G_{i-1} is cyclic), and χ an irreducible symplectic character of G. Then one of the following holds:*

(1) $\chi = \phi + \overline{\phi}$, where ϕ is induced by an irreducible character of degree 1 *of some subgroup of G.*

(2) χ is induced by a quaternion character of some subgroup of G.

Thus, when G is supersolvable, we have a characterisation for irreducible symplectic representations. And let us just recall that the Quaternions are supersolvable.

Chapter 6

A construction with continuous, orthogonal, Galois representations.

In this chapter, we study V.P. Snaith's construction [31] with continuous, orthogonal, Galois representations.

In his book "*Topological methods in Galois Representation Theory*" [31], the construction of a map $\Gamma_F : RO(F) \longrightarrow \mu_4 = \{\pm 1, \pm i\}$ is given. This map is obtained from the composition of the following maps,

$$\begin{array}{ccccccc} RO(F) & \longrightarrow & Y_F & \overset{Ind_{F/\mathbb{Q}_p}}{\longrightarrow} & Y_{\mathbb{Q}_p} & \longrightarrow & \mu_4 \\ \rho & \rightsquigarrow & [\rho - n] & \rightsquigarrow & Ind_{F/\mathbb{Q}_p}[\rho - n] & \rightsquigarrow & \Gamma_F(\rho) \end{array}$$

with $n = dim\rho$ and Y_F defined by

$$Y_F = \lim_{\overrightarrow{K/F}} \{IO(G(K/F))/(J_{G(K/F)})\}$$

Here $IO(G)$ is the Augmentation ideal of $RO(G)$ and $J_G = \{x \in IO(G)/dim(x) = 0, SW_1(x) = 0 = SW_2(x)\}$.

This construction is important because, given a continuous orthogonal representation $\rho : \Omega_F \longrightarrow O_n(\mathbb{R})$, $W_F(\rho) = \Gamma_F(\rho)$. Therefore Γ_F gives an independent construction of the orthogonal root number of ρ. Furthermore Γ_F is easily shown to satisfy Deligne's formula [5]

$$W_F(\rho) = SW_2(\rho)W_F(det(\rho)) \qquad [31]$$

Imitating this construction, we prove, in chapter 7, that a similar homomorphism can be obtained in the case of orthogonal representations of division algebras.

Let us start by recalling some facts about quadratic characters and some of their properties.

6.1 Quadratic characters

In this section, we recall some well-known facts about quadratic characters [31] such as how to represent them as cohomology classes. We also tabulate their cup-products and local root numbers for the field $\mathbb{Q}_p$.

Let θ be a one-dimensional orthogonal Galois representation, that is, a homomorphism

$$\theta : \Omega_F \longrightarrow \{\pm 1\}$$

where throughout this chapter, F denotes a non-Archimedean local field.

Such θ represents a class

$$l(a) \in H^1(F; \mathbf{Z}/2) \cong Hom_{cts}(\Omega_F, \{\pm 1\})$$

where $a \in F^*/(F^*)^2$, and $l(a)$ is defined by $l(a)(g) = g(\sqrt{a})/\sqrt{a}$.

Alternatively, by class field theory [14], $l(a)$ may be described as a quadratic character $l(a) : F^* \longrightarrow \{\pm 1\} \subset \mathbf{C}^*$.

Now, via the isomorphism $H^2(F, \mathbf{Z}/2) \cong \{\pm 1\}$ any such character may also be given by the formula,

$$l(a)(b) = l(a) \cup l(b) = (a, b)$$

where $\cup$ denotes the cohomology cup-product and (a, b) denotes the Hilbert symbol [20].

Here, using the well-known fact [25] that $\mathbf{Q}_p^*/(\mathbf{Q}_p^*)^2 \cong \mathbf{Z}/2 \times \mathbf{Z}/2 < u, p >$ where u is a unit with Legendre symbol $[\frac{u}{p}] = -1$ (i.e., u is non-square $mod.p$) for $p \neq 2$ a prime, and $\mathbf{Q}_2^*/(\mathbf{Q}_2^*)^2 \cong \mathbf{Z}/2 \times \mathbf{Z}/2 \times \mathbf{Z}/2 < -1, 5, 2 >$, we have the following cup-products tables [31] in the case $F = \mathbf{Q}_p$.

Cup-products on $\mathbf{Q}_p^*/(\mathbf{Q}_p^*)^2$ if $p \neq 2$, where $\lambda = \left[\frac{-1}{p}\right]$

	1	u	p	up
1	1	1	1	1
u	1	1	-1	-1
p	1	-1	λ	$-\lambda$
up	1	-1	$-\lambda$	λ

Cup-products on $\mathbf{Q}_2^*/(\mathbf{Q}_2^*)^2$

	1	−1	5	−5	2	−2	10	−10
1	1	1	1	1	1	1	1	1
−1	1	−1	1	−1	1	−1	1	−1
5	1	1	1	1	−1	−1	−1	−1
−5	1	−1	1	−1	−1	1	−1	1
2	1	1	−1	−1	1	1	−1	−1
−2	1	−1	−1	1	1	−1	−1	1
10	1	1	−1	−1	−1	−1	1	1
−10	1	−1	−1	1	−1	1	1	−1

Now, as $l(a)$ is an orthogonal representation, we may evaluate

$$W_F(l(a)) \in \{\pm 1, \pm i\}$$

giving us the following tables [31],

$W_{Q_2}(l(a)) \in \{\pm 1, \pm i\}$. $a \in \mathbf{Q}_2^*/(\mathbf{Q}_2^*)^2$

a	1	−1	5	−5	2	−2	10	−10
$W_{Q_2}(l(a))$	1	i	1	i	1	i	−1	$-i$

$W_{Q_p}(l(a)) \in \{\pm 1, \pm i\}$. $a \in \mathbf{Q}_p^*/(\mathbf{Q}_p^*)^2$

a	1	u	p	up
$W_{Q_p}(l(a))$ $p \equiv 3(mod.4)$	1	1	$-i$	i
$W_{Q_p}(l(a))$ $p \equiv 1(mod.4)$	1	1	1	−1

6.2 The construction

This section explains how Snaith's construction [31] for orthogonal Galois representations is obtained. Let us start, by setting down the details required.

Given G a finite group, let $RO(G)$ denote the Grothendieck ring of finite-dimensional real representations of G. That is, the free abelian group on the isomorphism classes, $[\rho]$, of irreducible orthogonal representations $\rho : G \longrightarrow O_n(\mathbf{R})$, or alternatively, the quotient of the free abelian group on isomorphism classes of representations modulo the relation

$$[\rho_1 \oplus \rho_2] = [\rho_1] + [\rho_2].$$

This ring structure is defined via tensor product of representations.

Consider now, the augmentation ideal of $RO(G)$, denote it by $IO(G) \lhd RO(G)$ given by $Ker(dim : RO(G) \longrightarrow \mathbb{Z})$, and define

$$J_G = \{x \in IO(G)/SW_1(x) = 0 = SW_2(x)\}$$

J_G is an ideal of $RO(G)$ [31]. Therefore, $IO(G)/J_G$ is a ring, and so too is

$$Y_F = \lim_{\overrightarrow{K/F}} \left\{IO(G(K/F))/(J_{G(K/F)})\right\}$$

where, if $\overline{F}$ is a fixed choice of separable closure of F, the limit is taken over finite Galois extensions with $F \subset K \subset \overline{F}$.

As, by definition, any element $x \in IO(G)/J_G$ is detected by SW_1 and SW_2, we also see that any element $x \in Y_F$ is detected by SW_1 or SW_2 in $H^*(F; \mathbb{Z}/2)$.

Now, if $a \in F^*/(F^*)^2$, consider the orthogonal one-dimensional representation $l(a)$. Studying the definition of Y_F, it follows trivially that

$l(a) - dim(l(a)) = l(a) - 1$ lies in Y_F . Hence, we can define

$$P_F(a) = l(a) - 1 \in Y_F$$

P_F satisfies the following properties [31].

Theorem 6.2.1 *Let F be a non-Archimedean local field.*

(i) There is an exact sequence

$$H^2(F; \mathbf{Z}/2) \cong \{\pm 1\} \overset{i}{\hookrightarrow} Y_F \overset{\pi}{\twoheadrightarrow} F^*/(F^*)^2 \cong H^1(F; \mathbf{Z}/2)$$

(ii) $\pi(P_F(a)) = l(a)$ for all $a \in F^*/(F^*)^2$

(iii) If $a,\ b \in F^*/(F^*)^2$, then $P_F(ab) = P_F(a) + P_F(b) + i\,((a, b))$

We define a homomorphism $\varphi_p : Y_{\mathbf{Q}_p} \longrightarrow \mu_4$ given by

$$\varphi_p(P_{\mathbf{Q}_p}(a)) = W_{\mathbf{Q}_p}(l(a))$$

This gives a homomorphism as is seen by the local root number tables given in section 1 of this chapter. In fact, φ_p may be extended to Y_F, by using the induction homomorphism

$$Ind_{F/\mathbf{Q}_p} : Y_F \longrightarrow Y_{\mathbf{Q}_p}$$

Given $\rho : \Omega_F \longrightarrow O_n(\mathbf{R})$ a continuous, orthogonal, Galois representation, we define

$$\Gamma_F(\rho) = \varphi_p(Ind_{F/\mathbf{Q}_p}[\rho - n])$$

In order to prove that the value of this map on orthogonal representations is the local root number, we must understand the way this map acts, as some of its properties. If we denote by

$$RO(F) = \lim_{\overrightarrow{K/F}} \{RO(G(K/F))\}$$

$$IO(F) = \lim_{\overrightarrow{K/F}} \{IO(G(K/F))\}$$

we have the following result concerning Γ_F [31].

Theorem 6.2.2 *Let $\Gamma_F : RO(F) \longrightarrow \mu_4$ is defined as above,*

(i) Let K/F be a finite extension, and let $\rho : \Omega_K \longrightarrow O_n(\mathbf{R})$ be a continuous, orthogonal representation, then

$$\Gamma_F(Ind_{K/F}(\rho)) = \Gamma_K(\rho)\Gamma_F(Ind_{K/F}(1))^n \in \mu_4$$

(ii) If $a, b \in F^*$, then

$$\Gamma_F(l(ab)) = \Gamma_F(l(a))\Gamma_F(l(b)) \in \mu_4$$

(iii) For all continuous, orthogonal representations, $\rho : \Omega_F \longrightarrow O_n(\mathbf{R})$,

$$\Gamma_F(\rho) = \Gamma_F(det\rho)SW_2(\rho) \in \mu_4$$

where $SW_2(\rho) \in H^2(F; \mathbf{Z}/2) \cong \{\pm 1\}$.

Using these properties, and the axioms for local root numbers of orthogonal representations, the following result is proved in [31].

Theorem 6.2.3 *Following the notation above, on $RO(F)$, $W_F(\rho) = \Gamma_F(\rho)$*

Note that Snaith's proof did not do use of Deligne's formula, which is therefore a corollary.

We move on now, in chapter 7 to imitate this procedure in the case of orthogonal representations of division algebras.

Chapter 7

A construction with finite-dimensional orthogonal continuous representations of division algebras.

Let A be quaternion algebra with centre a local field F of odd residue characteristic. The aim of this chapter is to construct a homomorphism of the form $\widehat{\Gamma}_F : RO(A^*/F^*) \longrightarrow \mu_4$, analogous to that of chapter 6.

Consider $IO(A^*/F^*)$ the augmentation ideal of $RO(A^*/F^*)$. Define $J = \{x \in RO(A^*/F^*) | dim(x) = 0, SW_1(x) = 0 = SW_2(x)\}$, which is an ideal of $RO(A^*/F^*)$ contained in $IO(A^*/F^*)$. Therefore the elements of $IO(A^*/F^*)/J$ are faithfully detected by SW_1 and SW_2.

The goal of this chapter is to construct a surjective homomorphism (sections 7.2 and 7.3)

$$\pi : IO(A^*/F^*)/J \longrightarrow Y_F$$

where Y_F is as in section 6.2.

Once such a map is found, we can define $\widehat{\Gamma}_F$ as the following composition

$$\begin{array}{ccccccc} RO(A^*/F^*) & \longrightarrow & IO(A^*/F^*)/J & \stackrel{\pi}{\longrightarrow} & Y_F & \stackrel{W_F}{\longrightarrow} & \mu_4 \\ W & \rightsquigarrow & W-n & \rightsquigarrow & \pi(W-n) & \rightsquigarrow & W_F(\pi(W-n)) \end{array}$$

where $n = dimW$.

Due to the fact that

$$H^*(A^*/F^*; \mathbf{Z}/2) \cong \begin{cases} \mathbf{Z}/2[x_1, x_2, w_2]/(x_2^2 + x_1x_2) & \text{if q} \equiv 3 \text{ (modulo 4)}, \\ \\ \mathbf{Z}/2[x_1, x_2] & \text{if q} \equiv 1 \text{ (modulo 4)}. \end{cases}$$

where F_q is the residue field of F (section 7.1) we need to make different constructions depending on $q \equiv 1(mod.4)$ or $q \equiv 3(mod.4)$, these are studied in section 7.2 and 7.3 respectively.

Preliminaries

Let E/F be a quadratic extension of p-adic local fields and let $\theta : E^* \longrightarrow \mathbf{C}^*$ be a continuous character of finite order. We shall assume that p is odd. If $\{1, \tau\} = G(E/F)$ then we shall assume that $\tau^*(\theta) \neq \theta$ (i.e. θ is *regular* in the terminology of [9] p.157).

By local class field theory ([14], [32], [33]), there are isomorphisms of the form

$$G(E/F) \cong F^*/(N_{E/F}(E^*)) \cong (E^*)^{G(E/F)}/(N_{E/F}(E^*)) \cong H^2(G(E/F); E^*).$$

Therefore, the non-trivial cohomology class corresponds to any element $x \in F^* - N_{E/F}(E^*)$.

Suppose that $B_*G(E/F)$ is the bar-resolution [32]

$$\ldots \xrightarrow{d} B_1G(E/F) \xrightarrow{d} B_0G(E/F) \xrightarrow{\epsilon} \mathbf{Z} \longrightarrow 0$$

Therefore, since $B_2G(E/F)$ is the free $\mathbf{Z}[G(E/F)]$-module on $G(E/F) \times G(E/F)$, a $\mathbf{Z}[G(E/F)]$-module homomorphism of the form $f : B_2G(E/F) \longrightarrow E^*$ satisfying $0 = f \cdot d$ corresponds to a 2-cycle $f : G(E/F) \times G(E/F) \longrightarrow E^*$.

Lemma 7.0.4

The generator of $H^2(G(E/F); E^*)$ is represented by the 2-cocycle f given by

$$f(\tau, \tau) = x, 1 = f(1, \tau) = f(\tau, 1) = f(1, 1)$$

where $x \in F^* - N_{E/F}(E^*)$.

Proof

The most economical resolution is

$$\ldots \xrightarrow{\tau-1} \mathbf{Z}[G(E/F)] \xrightarrow{\tau+1} \mathbf{Z}[G(E/F)] \xrightarrow{\tau-1} \mathbf{Z}[G(E/F)] \xrightarrow{\epsilon} \mathbf{Z} \longrightarrow 0$$

and one may easily verify that the formulae below define $h_i : B_iG(E/F) \longrightarrow \mathbf{Z}[G(E/F)]$ for $i = 0, 1, 2$ which are part of a chain map from the bar-resolution to the more economical one:

$$h_0 = 1 : B_0G(E/F) \longrightarrow \mathbf{Z}[G(E/F)],$$

$$h_1(1) = 0, h_1(\tau) = 1,$$

$$h_2(\tau, \tau) = 1, 0 = h_2(1, \tau) = h_2(\tau, 1) = h_2(1, 1).$$

The result follows since the generator of $H^2(G(E/F); E^*)$ is given by the $\mathbf{Z}[G(E/F)]$-module homomorphism from $\mathbf{Z}[G(E/F)]$ to E^* sending 1 to x. $\square$

Next we recall the well-known association of a group extension to a normalised 2-cocycle of the form

$$f : G \times G \longrightarrow A$$

where G is a group and A is an abelian group, written additively, on which G acts.

Given f as above, define a 'product' map

$$(G \times A) \times (G \times A) \longrightarrow (G \times A)$$

by the formula $(g, g_1 \in G,\ a, a_1 \in A)$

$$(g, a) \cdot (g_1, a_1) = (gg_1, a + g(a_1) + f(g, g_1)).$$

Theorem 7.0.5 *(i) With this multiplication $G \times A$ is a group.*

(ii) The subset $\{(1, a) \in G \times A\}$ is a normal subgroup isomorphic to A. The conjugation action is given by

$$(g, a) \cdot (1, a_1) \cdot (g, a)^{-1} = (1, g(a_1)).$$

Proof

Recall that the cocycle condition states that

$$0 = g(f(g_1, g_2)) - f(gg_1, g_2) + f(g, g_1g_2) - f(g, g_1).$$

Therefore,

$$\begin{aligned}
&(g,a)\cdot((g_1,a_1)\cdot(g_2,a_2))\\
&=(g,a)\cdot(g_1g_2,a_1+g_1(a_2)+f(g_1,g_2))\\
&=(gg_1g_2,a+g(a_1)+gg_1(a_2)+g(f(g_1,g_2))+f(g,g_1g_2))\\
&=(gg_1g_2,a+g(a_1)+gg_1(a_2)+f(g,g_1)+f(gg_1,g_2)).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&((g,a)\cdot(g_1,a_1))\cdot(g_2,a_2)\\
&=(gg_1,a+g(a_1)+f(g,g_1))\cdot(g_2,a_2)\\
&=(gg_1g_2,a+g(a_1)+gg_1(a_2)+f(g,g_1)+f(gg_1,g_2)).
\end{aligned}$$

Hence the multiplication is associative.

The identity element is given by $(1,0)$ since

$$(1,0)\cdot(g,a)=(g,0+1(a)+f(1,g))=(g,a)$$

and

$$(g,a)\cdot(1,0)=(g,a+g(0)+f(g,1))=(g,a)$$

since f is a normalised 2-cocycle (i.e. $f(1,g)=f(g,1)=0$ for all $g\in G$).

The inverse of (g,a) is given by

$$(g,a)^{-1}=(g^{-1},-g^{-1}(a)-g^{-1}(f(g,g^{-1})))$$

since

$$\begin{aligned}
&(g,a)\cdot(g^{-1},-g^{-1}(a)-g^{-1}(f(g,g^{-1})))\\
&=(1,a-g(g^{-1}(a)+g^{-1}(f(g,g^{-1})))+f(g,g^{-1}))\\
&=(1,0)
\end{aligned}$$

and

$$
\begin{aligned}
&(g^{-1}, -g^{-1}(a) - g^{-1}(f(g, g^{-1}))) \cdot (g, a) \\
&= (1, -g^{-1}(a) - g^{-1}(f(g, g^{-1})) + g^{-1}(a) + f(g^{-1}, g)) \\
&= (1, 0).
\end{aligned}
$$

This proves part (i). For part (ii) we have

$$
\begin{aligned}
&(g, a) \cdot (1, a_1) \cdot (g^{-1}, -g^{-1}(a) - g^{-1}(f(g, g^{-1}))) \\
&= (g, a + g(a_1)) \cdot (g^{-1}, -g^{-1}(a) - g^{-1}(f(g, g^{-1}))) \\
&= (1, a + g(a_1) - a - f(g, g^{-1}) + f(g, g^{-1})) \\
&= (1, g(a_1))
\end{aligned}
$$

as required.

Example 7.0.6 *From the* 2*-cocycle of Lemma* 7.0.1 *we obtain a group structure on* $\{1, \tau\} \times E^*$ *given by*

$$
\begin{aligned}
(1, e)(1, e') &= (1, ee'), \\
(1, e)(\tau, e') &= (\tau, ee'), \\
(\tau, e)(1, e') &= (\tau, e\tau(e')), \\
(\tau, e)(\tau, e') &= (1, e\tau(e')x).
\end{aligned}
$$

This group is denoted by $W_{E/F}$ and sits in an extension of the form

$$
E^* \longrightarrow W_{E/F} \longrightarrow G(E/F)
$$

([9] p.158).

Let us recall now some facts from ([32], chapter 7). If f is the 2-cocycle of Lemma 7.0.2, then, the associated quaternion algebra, which is a division algebra with centre field F, is given by the left E-vector space on basis u_1 and u_τ where $u_g e u_g^{-1} = g(e)$ for all $e \in E^*$ and $u_g u_h = f(g,h)u_{gh}$. Hence, u_1 is the identity of A and $u_\tau^2 = x$ and $u_\tau e u_\tau = x\tau(e)$.

The subset

$$W = \{e, e'u_\tau \mid e, e' \in E^*\} \subset A^*$$

is a subgroup of A^*. In fact, the map

$$\lambda : W \longrightarrow W_{E/F}$$

given by $\lambda(e) = (1,e), \lambda(e'u_\tau) = (\tau, e')$ yields a group isomorphism since

$$\lambda(e)\lambda(e') = (1,e)(1,e') = (1,ee') = \lambda(ee'),$$

$$\lambda(e)\lambda(e'u_\tau) = (1,e)(\tau,e') = (\tau,ee') = \lambda(ee'u_\tau),$$

$$\lambda(eu_\tau)\lambda(e') = (\tau,e)(1,e') = (\tau,e\tau(e')) = \lambda(e\tau(e')u_\tau) = \lambda(eu_\tau e'),$$

$$\lambda(eu_\tau)\lambda(e'u_\tau) = (\tau,e)(\tau,e') = (1,e\tau(e')x) = \lambda(e\tau(e')x) = \lambda(eu_\tau e'u_\tau).$$

7.1 $H^*(A^*/F^*; \mathbf{Z}/2)$

Let E/F be a quadratic extension of p-adic local fields and we shall assume that p is odd. Let $\mathcal{O}_F$ denote the ring of integers of F, π_F a uniformiser of F. Hence, the residue field $\mathcal{O}_F/(\pi_F)$ is a finite field with q elements $\mathbf{F}_q$ for $q = p^d$, for some integer d.

By descent theory, quaternion algebras over F are classified by the non-trivial elements of $H^2(G(E/F); E^*) \cong \mathbf{Z}/2$ [31], so that, there is only one such quaternion algebra A, up to isomorphism. Also [24] any quadratic extension L/F is embeddable into A/F and the image of L is a maximal subfield of A.

An explicit description for A [31] is

$$A = F[X, Y]/(X^2 - a, Y^2 - b, XY + YX)$$

where $a, b \in F^*/((F^*)^2)$ are non-trivial and the cohomology classes $l(a), l(b) \in H^1(F; \mathbf{Z}/2)$ have a non-trivial cup-product in $H^2(F; \mathbf{Z}/2) \cong \mathbf{Z}/2$. This means that we may choose a, b to suit our purposes. That is, let us make the convention that $F(X)/F$ is the unique unramified quadratic extension and that $b = \pi_F$ so that $(b) \lhd \mathcal{O}_F$ is the maximal ideal.

7.1.1 *The reduced norm*

The reduced norm is a homomorphism of the form

$$N_{red} : A^* \longrightarrow F^*$$

which is defined in the following manner. Let $L = F(\sqrt{a})$, then we have an isomorphism of left L-algebras

$$L \otimes_F A \cong M_2(L)$$

$$z \otimes 1 \mapsto \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \text{ if z} \in \text{L},$$

$$1 \otimes X \mapsto \begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix} \text{ and}$$

$$1 \otimes Y \mapsto \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}.$$

where $M_2(L)$ denotes the 2×2 matrices with entries in L.

The reduced norm of $\alpha \in A$ is given by the determinant of the image of $1 \otimes \alpha$ in $M_2(L)$. That is, if $a_i \in F$,

$$N_{red}(a_0 + a_1X + a_2Y + a_3XY)$$

$$= det \begin{pmatrix} a_0 + a_1\sqrt{a} & a_2b + a_3b\sqrt{a} \\ \\ a_2 - a_3\sqrt{a} & a_0 - a_1\sqrt{a} \end{pmatrix}$$

$$= a_0^2 - a_1^2a - a_2^2b + a_3^2ab.$$

The reduced norm is surjective. To see this observe that

$$N_{red}(F(\sqrt{a})^*) = N_{F(\sqrt{a})/F}(F(\sqrt{a})^*)$$

which has index two in F^* and similarly $[F^* : N_{red}(F(\sqrt{b})^*)] = 2$. However, [31] $l(a) \cup l(b)$ is non-trivial if and only if b is not a norm from $F(\sqrt{a})$ so that

$$N_{red}(F(\sqrt{a})^*)N_{red}(F(\sqrt{b})^*) = F^*.$$

Now, we recall a few facts from ([2] §1.1). If $v_F : F^* \longrightarrow \mathbf{Z}$ is the valuation, normalised so that $v_F(\pi_F) = 1$, then

$$v_A = v_F \cdot N_{red} : A^* \longrightarrow \mathbf{Z}$$

gives a surjective valuation which extends $2v_F$ on F^*. Setting $v_A(0) = \infty$ then

$$\mathcal{O}_A = \{x \in A \mid v_A(x) \geq 0\}$$

is a subring, which is the unique maximal order in A whose unique maximal ideal is P_A^1 where

$$P_A^i = \{x \in A \mid v_A(x) \geq i\}.$$

and we can define

$$U_A^i = 1 + P_A^i.$$

In fact, P_A^1 is a 2-sided ideal and the quotient ring $\mathcal{O}_A/P_A^1$ is the field $\mathbf{F}_{q^2}$. Setting $U_A^i = 1+P_A^i \subseteq A^*$ for $i \geq 1$ we have a chain of compact, open normal subgroups of A^*

$$\ldots U_A^i \subseteq U_A^{i-1} \subseteq \ldots \subseteq U_A^1 \subseteq \mathcal{O}_A^* \subseteq A^*.$$

Each quotient U_A^i/U_A^{i+1} is an elementary abelian p-group and U_A^1 is a pro-p-group

$$U_A^1 \cong \lim_{\overleftarrow{n}} \ U_A^1/U_A^n.$$

Now we are going to calculate the mod 2 cohomology of the topological group A^*/F^*. This is continuous cohomology in the sense of ([26] Ch I, §2). This means that there is an isomorphism of the form

$$\lim_{\overrightarrow{n}} \ H^i(A^*/F^*U_A^n; \mathbf{Z}/2) \longrightarrow H^i(A^*/F^*; \mathbf{Z}/2).$$

The Serre spectral sequence for the extension

$$U_A^1/U_A^n \longrightarrow A^*/F^*U_A^n \longrightarrow A^*/F^*U_A^1$$

takes the form

$$E_2^{s,t} = H^s(A^*/F^*U_A^1; H^t(U_A^1/U_A^n; \mathbf{Z}/2) \Longrightarrow H^{s+t}(A^*/F^*U_A^n; \mathbf{Z}/2).$$

Since U_A^1/U_A^n is a finite p-group and $p \neq 2$ we have $E_2^{s,t} = 0$ when $t \neq 0$. Therefore

$$H^i(A^*/F^*; \mathbf{Z}/2) \cong \lim_{\overrightarrow{n}} \ H^i(A^*/F^*U_A^n; \mathbf{Z}/2) \cong H^s(A^*/F^*U_A^1; \mathbf{Z}/2)$$

for each i.

From the previous discussion the homomorphism v_A induces a surjection of the form

$$\tilde{v} : A^*/F^* \longrightarrow \mathbf{Z}/2$$

whose kernel contains the pro-p-group U_A^1 so that we obtain a surjection

$$v : A^*/F^*U_A^1 \longrightarrow \mathbf{Z}/2.$$

By ([2] (1.2.5)/(1.2.6)) the kernel of v is isomorphic to $\mathcal{O}_A^*/F^*U_A^1 \cong \mathbf{F}_{q^2}^*/\mathbf{F}_q^*$. Hence we have an extension of the form

$$\mathbf{F}_{q^2}^*/\mathbf{F}_q^* \longrightarrow A^*/F^*U_A^1 \xrightarrow{v} \mathbf{Z}/2.$$

If we can show that this extension is the dihedral group $D_{2(q+1)} = \mathbf{Z}/2 \propto (\mathbf{F}_{q^2}^*/\mathbf{F}_q^*)$, where $A \propto B$ denotes the semi-direct product and $\mathbf{Z}/2$ acts on $\mathbf{F}_{q^2}^*/\mathbf{F}_q^*$ as the Galois group $G(\mathbf{F}_{q^2}/\mathbf{F}_q)$, then we may read off the mod 2 cohomology from ([31] p.24). To see this let L/F be the unique unramified quadratic extension. Then, from the preliminaries, we have an extension

$$L^* \longrightarrow W_{L/F} \longrightarrow G(L/F)$$

where $W_{L/F} = N_{A^*}L^*$, the normaliser of L^* in A^*, (see [32] §7.1.25) and we have an inclusion $W_{L/F}/U_L^1F^* \subset A^*/F^*U_A^1$. Since L/F is unramified

$$L^*/U_L^1F^* \cong \mathcal{O}_L^*/U_L^1\mathcal{O}_F^* \cong \mathbf{F}_{q^2}^*/\mathbf{F}_q^*$$

and, by construction, the resulting extension

$$\mathbf{F}_{q^2}^*/\mathbf{F}_q^* \longrightarrow W_{L/F}/U_L^1F^* \longrightarrow G(L/F)$$

is the required dihedral extension. It is straightforward to verify that the canonical homomorphism

$$W_{L/F}/U_L^1F^* \longrightarrow A^*/F^*U_A^1$$

is an isomorphism.

Note that the 2-Sylow subgroup of $D_{2(q+1)}$ is isomorphic to $\mathbf{Z}/2 \times \mathbf{Z}/2$ if and only if $q \equiv 1$ (modulo 4) and is otherwise a non-abelian dihedral group.

Now we may describe the cohomology ring $H^*(A^*/F^*; \mathbf{Z}/2)$. Firstly we know that $H^1(A^*/F^*; \mathbf{Z}/2) \cong Hom_{conts}(A^*/F^*; \mathbf{Z}/2)$, the group of continuous homomorphisms. However, every continuous homomorphism from A^* to $\mathbf{Z}/2$ factorises through

$$A^* \xrightarrow{N_{red}} F^* \longrightarrow F^*/(F^*)^2.$$

Since F^* modulo squares is isomorphic to $\mathbf{Z}/2 \times \mathbf{Z}/2$ we have $H^1(A^*/F^*; \mathbf{Z}/2) \cong \mathbf{Z}/2 \times \mathbf{Z}/2$.

Let $x_1 : A^* \longrightarrow \mathbf{Z}/2$ be the non-trivial continuous homomorphism which annihilates the subgroup $L^* = F(X)^*$, where $F(X)/F$ is the unique unramified quadratic extension. Let $x_2 : A^* \longrightarrow \mathbf{Z}/2$ be one of the other two non-trivial continuous homomorphisms. If $q \equiv 3$ (modulo 4) $W_{L/F}/U_L^1F^* \cong A^*/F^*U_A^1$ is a non-abelian dihedral quotient of A^*, we have a faithful two-dimensional complex representation, which is the complexification of an orthogonal representation ρ given by $Ind_{\mathbf{F}_{q^2}^*/\mathbf{F}_q^*}^{W_{L/F}/U_L^1F^*}(\lambda)$. Set $w_2 = SW_2(\rho) \in H^2(A^*/F^*U_A^1; \mathbf{Z}/2)$, the second Stiefel-Whitney class of ρ.

Theorem 7.1.2 *([31] p.24 Theorem 4.6)*

$$H^*(A^*/F^*; \mathbf{Z}/2) \cong \begin{cases} \mathbf{Z}/2[x_1, x_2, w_2]/(x_2^2 + x_1x_2) & \text{if q} \equiv 3 \text{ (modulo 4)}, \\ \\ \mathbf{Z}/2[x_1, x_2] & \text{if q} \equiv 1 \text{ (modulo 4)}. \end{cases}$$

7.2 The construction on orthogonal representations of A^*/F^* when $q \equiv 1(mod.4)$

In the previous chapter, we introduced a construction with orthogonal Galois representations. In this section, we will imitate the procedure, obtaining a similar map, but in this case on orthogonal representations of A^*/F^* in the case $q \equiv 1(mod.4)$. The case $q \equiv 3(mod.4)$ is studied in section 7.3.

Let $RO(A^*/F^*)$ denote the representation ring of finite-dimensional, orthogonal continuous representations $\rho : A^*/F^* \longrightarrow O_n(\mathbf{R})$. Then

$$J = \{x \in RO(A^*/F^*) \mid \dim(x) = 0, SW_1(x) = 0 = SW_2(x)\}$$

is an ideal of $RO(A^*/F^*)$ contained in the augmentation ideal $IO(A^*/F^*)$. By construction, the elements of the ring $IO(A^*/F^*)/J$ are faithfully detected by SW_1 and SW_2.

Suppose that $a, b \in F^*/(F^*)^2$ then we have $l(a), l(b) \in H^1(F; \mathbf{Z}/2) \cong Hom_{cts}(\Omega_F, \{\pm 1\})$ defined by $l(a)(g) = g(\sqrt{a})/\sqrt{a}$ and $l(b)(g) = g(\sqrt{b})/\sqrt{b}$. If we consider $l(a), l(b), l(ab)$ as one-dimensional orthogonal representations then $l(a)l(b) = l(ab)$. Set $p(a) = l(a) - 1 \in IO(A^*/F^*)/J$. If $SW_1(l(a)) = x_1$ and $SW_1(l(b)) = x_2$, we can write the total Stiefel-Whitney class of the following elements

$$SW(p(a) + p(b) - p(ab)) = (1 + x_1 t)(1 + x_2 t) \quad (1 + \quad (x_1 + x_2)t)^{-1} =$$

$$= 1 + x_1 x_2 t^2 + \ldots,$$

$$SW(p(a) + p(ab) - p(b)) = 1 + (x_1^2 + x_1 x_2)t^2 \quad + \ldots,$$

$$SW(p(b) + p(ab) - p(a)) = 1 + (x_2^2 + x_1x_2)t^2 + \dots,$$

$$SW(2p(a)) = 1 + x_1^2t^2,$$

$$SW(2p(b)) = 1 + x_2^2t^2,$$

$$SW(2p(ab)) = 1 + (x_1^2 + x_2^2)t^2.$$

This shows that $p(a), p(b)$, and $p(ab)$ generate $IO(A^*/F^*)/J$. Let us recall that the elements of $IO(A^*/F^*)/J$ are detected by SW_1 and SW_2.

The additive group $IO(A^*/F^*)/J$ sits in a short exact sequence

$$0 \longrightarrow V \longrightarrow IO(A^*/F^*)/J \longrightarrow F^*/(F^*)^2 \longrightarrow 0$$

where V is the $\mathbf{F}_2$ vector space with basis x_1^2, x_2^2, x_1x_2.

In order to construct a homomorphism $\pi_1 : IO(A^*/F^*)/J \longrightarrow Y_F$ we want to map each $p(z)$ to the class denoted by $p(z) \in Y_F$ in section 6.2. We must show this is well-defined. Hence, if we map $p(a)$, $p(b)$, and $p(ab)$ into themselves via π_1, then x_1^2 and x_2^2 must be mapped trivially (as $q \equiv 1$ (modulo 4), -1 is a square in F and so $l(z) \cup l(z) = l(z) \cup l(-1) = 0$ for all $z \in F^*$. Therefore $\pi_1(x_1^2) = \pi_1(2p(a)) = l(a) \cup l(a) = l(a) \cup l(-1) = 0$, and $\pi_1(x_2^2) = \pi_1(2p(b)) = l(b) \cup l(-1) = 0$), and x_1x_2 must be mapped not trivially (as from the first total Stiefel-Whitney class, $\pi_1(x_1x_2) = l(a) \cup l(b)$ and that is non-trivial by choice in chapter 7.1). Therefore, mapping $p(a), p(b)$, and $p(ab)$ into themselves and mapping x_1^2 and x_2^2 trivially into $\mathbf{Z}/2$ and x_1x_2 non-trivially gives a well-defined surjective homomorphism

$$\pi_1 : IO(A^*/F^*)/J \longrightarrow Y_F$$

where Y_F is as in section 6.2 and sits in a short exact sequence of the form

$$0 \longrightarrow \mathbf{Z}/2 \longrightarrow Y_F \longrightarrow F^*/(F^*)^2 \longrightarrow 0.$$

As explained in section 6.2, sending $p(z)$ to the local root number (quadratic Gauss sum) $W_F(l(z))$ yields a homomorphism of the form

$$\varphi_F^1 : Y_F \longrightarrow \mu_4$$

from Y_F to fourth roots of unity.

Now, let W be an orthogonal representation of A^*/F^*, we may send it to $W - dimW \in IO(A^*/F^*)/J$. Then, this can be sent via $\pi_1 : IO(A^*/F^*)/J \longrightarrow Y_F$ to Y_F, where we can apply φ_F^1 and send it to μ_4. That is, we can define for an orthogonal representation of A^*/F^*

$$\widehat{\Gamma}_F(W) = \varphi_F^1(\pi_1(W - dimW)) \in \mu_4.$$

7.3 The construction on orthogonal representations of A^*/F^* when $q \equiv 3(mod.4)$

We proceed now to study the case $q \equiv 3(mod.4)$ in a similar way, in order to obtain $\widehat{\Gamma}_F(\pi(\sigma))$.

For $q \equiv 3(mod.4)$ we have a short exact sequence of the form

$$0 \longrightarrow W \longrightarrow IO(A^*/F^*)/J \longrightarrow F^*/(F^*)^2 \longrightarrow 0$$

where W is the $\mathbf{F}_2$ vector space with basis x_1^2, x_1x_2, w_2 because $x_2^2 = x_1x_2$ (see Theorem 7.1.2) .

We have $det(\rho) = x_1$, in terms of the preceding notation and so we see $p(\rho) = \rho - 2$ sits in $IO(A^*/F^*)/J$. Then the calculations of total Stiefel-Whitney classes for the following elements

$$SW(p(\rho) - p(a)) = (1 + x_1 t + w_2 t^2 + \ldots)(1 + x_1 t)^{-1} = 1 + w_2 t^2 + \ldots,$$

$$SW(p(a) + p(b) - p(ab)) = 1 + x_1 x_2 t^2 + \ldots,$$

$$SW(p(a) + p(ab) - p(b)) = 1 + (x_1^2 + x_1 x_2)t^2 + \ldots,$$

$$SW(p(b) + p(ab) - p(a)) = 1 + (x_2^2 + x_1 x_2)t^2 + \ldots,$$

$$SW(2p(a)) = 1 + x_1^2 t^2,$$

$$SW(2p(b)) = 1 + x_2^2 t^2,$$

$$SW(2p(ab)) = 1 + (x_1^2 + x_2^2)t^2$$

show that $p(\rho), p(a), p(b)$ generate $IO(A^*/F^*)/J$ and give the relations between these elements. Note that $p(b) + p(ab) = p(a)$ because $x_2^2 + x_1 x_2 = 0$.

Since $q \equiv 3$ (modulo 4), -1 is not a square in $\mathbf{F}_q$ and so is not a square in F. Therefore $L = F(\sqrt{-1})$ is the unramified quadratic extension and we may take $x_1 = l(-1)$ and $a = -1$ in the preceding formulae.

This case is more complicated than $q \equiv 1(mod.4)$, since now we also have to deal with the 2-dimensional representation ρ, but with a bit of care we define a surjective homomorphism of the form

$$\pi_3 : IO(A^*/F^*)/J \longrightarrow Y_F.$$

As in the previous section, when constructing this homomorphism, we must be careful with whichever choice we make for the image of $p(-1)$, $p(b)$, and $p(\rho)$. We shall map each of $p(-1), p(b), p(-b)$ to the elements of Y_F denoted by $p(-1), p(b), p(-b)$ respectively. Therefore x_1^2, x_2^2, x_1x_2 must map to $l(-1) \cup l(-1), l(-1) \cup l(b), l(-1) \cup l(b)$ respectively. Now, we have to find a sensible choice for the image of $p(\rho)$, keeping in mind that such choice will have to verify $\pi_3(p(\rho) - p(a)) = \pi_3(w_2)$, from the first total Stiefel-Whitney class listed above. Send $p(\rho)$ to the class of the Galois representation $Ind_{L/F}(\lambda) - 2$ which is defined in the following manner. By definition ρ is induced from a faithful character

$$\lambda : L^*/U_L^1 F^* \cong \mathcal{O}_L^*/U_L^1 \mathcal{O}_F^* \cong \mathbf{F}_{q^2}^*/\mathbf{F}_q^* \longrightarrow \mathbf{C}^*$$

which yields a character $\tilde{\lambda} : L^* \longrightarrow \mathbf{C}^*$. Since the restriction of $\tilde{\lambda}$ to F^* is trivial, the induced Galois representation

$$Ind_{L/F}(\tilde{\lambda}) : \Omega_F \longrightarrow GL_2\mathbf{C}$$

is dihedral [7].

In addition we send w_2 to $SW_2(Ind_{L/F}(\tilde{\lambda}))$

Since $L = F(\sqrt{-1})$ we have

$$\begin{aligned} SW_2(Ind_{L/F}(\tilde{\lambda}))W_F(l(-1)) &= W_F(Ind_{L/F}(\tilde{\lambda})) \\ &= W_L(\tilde{\lambda})W_F(Ind_{L/F}(1)) \\ &= W_L(\tilde{\lambda})W_F(l(-1)). \end{aligned}$$

Therefore

$$SW_2(Ind_{L/F}(\tilde{\lambda})) = W_L(\tilde{\lambda}) \in \{\pm 1\} = H^2(F; \mathbf{Z}/2).$$

Choose π_3 to satisfy the following formulae

$$\pi_3(p(\rho)) = Ind_{L/F}(\tilde{\lambda}) - 2,$$

$$\pi_3(p(z)) = p(z) \text{ for } z = a, b, ab,$$

$$\pi_3(x_1^2) = l(-1) \cup l(-1),$$

$$\pi_3(x_2^2) = \pi_3(x_1 x_2) = l(-1) \cup l(b),$$

$$\pi_3(w_2) = SW_2(Ind_{L/F}(\tilde{\lambda})),$$

there can only be at most one homomorphism satisfying these formulae.

These formulae give a well-defined surjective homomorphism $\pi_3 : IO(A^*/F^*)/J \longrightarrow Y_F$, provided that

$$\pi_3(p(\rho) - p(a)) = \pi_3(w_2).$$

However,

$$\pi_3(p(\rho) - p(a)) = Ind_{L/F}(\tilde{\lambda}) - 2 - l(a) + 1 = SW_2(Ind_{L/F}(\tilde{\lambda}) - l(a))$$

since $\det(Ind_{L/F}(\tilde{\lambda})) = \det(l(a)) = l(-1)$. However $SW_2(Ind_{L/F}(\tilde{\lambda}) - l(a))$ is the coefficient of t^2 in $SW(Ind_{L/F}(\tilde{\lambda}) - l(a))$,

$$(1 + l(-1)t + SW_2(Ind_{L/F}(\tilde{\lambda}))t^2)(1 + l(-1)t + l(-1)^2 t^2 + \ldots)$$

which is

$$SW_2(Ind_{L/F}(\tilde{\lambda})) = \pi_3(w_2),$$

as required.

So now, as in the previous section, sending $p(z)$ to the local root number $W_F(l(z))$ for $z = a, b, ab$, and sending $p(\rho)$ to the local root number $W_F(\rho)$, yields a homomorphism of the form

$$\varphi_F^3 : Y_F \longrightarrow \mu_4$$

from Y_F to fourth roots of unity. Now, as in the case $q \equiv 1(mod.4)$, we consider W an orthogonal representation of A^*/F^*, and define

$$\widehat{\Gamma}_F(W) = \varphi_F^3(\pi_3(W - dimW)).$$

Chapter 8

First and second Stiefel-Whitney classes for orthogonal representations of A^*/F^*

The aim of this chapter is to obtain formulae which allow us to calculate the first and second Stiefel-Whitney classes of orthogonal representations of A^*/F^* in terms of its character values on elements of order two. This will reduce obtaining Stiefel-Whitney classes to an easy algebra exercise. These formulae will be used in Chapter 9 to allow us to calculate the first and second Stiefel-Whitney classes of a special orthogonal representation of A^*/F^* constructed via the Langlands correspondence using the results of [21]. This calculation falls into two cases depending on $H^*(A^*/F^*; \mathbb{Z}/2)$ (see Theorem 7.1.2) according to the values of $q(mod.4)$, where F_q is the residue field of F and q is odd.

8.1 Formulae for the first and second Stiefel-Whitney classes, when $q \equiv 1(mod.4)$

In this section, we obtain a formula for the first and second Stiefel-Whitney classes for representations of A^*/F^* in the case $q \equiv 1(mod.4)$.

Recall from Theorem 7.1.2 which gives us that

$$H^*(A^*/F^*; \mathbf{Z}/2) \cong \begin{cases} \mathbf{Z}/2[x_1, x_2, w_2]/(x_2^2 + x_1 x_2) & \text{if q} \equiv 3 \text{ (modulo 4)}, \\ \\ \mathbf{Z}/2[x_1, x_2] & \text{if q} \equiv 1 \text{ (modulo 4)}. \end{cases}$$

When $q \equiv 1(mod.4)$ the Sylow 2-subgroup of $A^*/F^*U_A^1$ is $V = \{1, X, Y, XY\}$, an elementary abelian group of order four, where X, Y, and XY will be elements of order two . Since restriction

$$Res_V^{A^*/F^*} : H^*(A^*/F^*; \mathbf{Z}/2) \longrightarrow H^*(V; \mathbf{Z}/2)$$

is an isomorphism we may compute the Stiefel-Whitney classes of an orthogonal representation W by the formula

$$Res_V^{A^*/F^*}(SW_i(W)) = SW_i(Res_V^{A^*/F^*}(W)) \in \mathbf{Z}/2[x_1, x_2].$$

Suppose that

$$x_1(Y) \equiv 1,\ x_1(X) \equiv 0,\ x_2(Y) \equiv 0,\ x_2(X) \equiv 1 \text{ (modulo 2)}$$

where $F(\sqrt{a})/F$ is unramified and $X^2 = a \in F$. Define $\chi_i : A^*/F^* \longrightarrow \{\pm 1\}$ by $\chi_i(z) = (-1)^{x_i(z)}$. With this notation the orthogonal representation W restricts to $c_0 + c_1\chi_1 + c_2\chi_2 + c_3\chi_1\chi_2$ where $c_i \in \mathbf{Z}$ satisfies

$$Trace(W)(1) = c_0 + c_1 + c_2 + c_3 = dim(W),$$

$$Trace(W)(X) = c_0 + c_1 - c_2 - c_3,$$

$$Trace(W)(Y) = c_0 - c_1 + c_2 - c_3,$$

$$Trace(W)(XY) = c_0 - c_1 - c_2 + c_3.$$

Now, write the total Stiefel-Whitney class, $SW(W) = 1 + SW_1(W)t + SW_2(W)t^2 + \ldots$, a formal series which satisfies $SW(W_1 \oplus W_2) = SW(W_1)SW(W_2)$. We obtain

$$SW(Res_V^{A^*/F^*}(W))$$

$$= SW(c_0 + c_1\chi_1 + c_2\chi_2 + c_3\chi_1\chi_2)$$

$$= (1 + x_1 t)^{c_1}(1 + x_2 t)^{c_2}(1 + (x_1 + x_2)t)^{c_3}$$

$$= 1 + (c_1x_1 + c_2x_2 + c_3(x_1 + x_2))t$$

$$+[\begin{pmatrix} c_1 \\ 2 \end{pmatrix} x_1^2 + \begin{pmatrix} c_2 \\ 2 \end{pmatrix} x_2^2 + \begin{pmatrix} c_3 \\ 2 \end{pmatrix} (x_1 + x_2)^2$$

$$+c_1c_2x_1x_2 + c_1c_3x_1(x_1 + x_2) + c_2c_3x_2(x_1 + x_2)]t^2 \ldots$$

therefore, comparing this to the definition of $SW(W)$, yields the following result:

Proposition 8.1.1 *Let A be a quaternion algebra with centre a local field F with residue field F_q of odd order $q \equiv 1(mod.4)$, and W a continuous orthogonal representation of A^*/F^*. The first and second Stiefel-Whitney classes of W are given by the formulae,*

$$SW_1(W) = (c_1 + c_3)x_1 + (c_2 + c_3)x_2$$

$$SW_2(W) = \begin{pmatrix} c_1 + c_3 \\ 2 \end{pmatrix} x_1^2 + \begin{pmatrix} c_2 + c_3 \\ 2 \end{pmatrix} x_2^2 + [c_1c_2 + c_1c_3 + c_2c_3]x_1x_2.$$

with c_i and x_j as above and the coefficients lie in the integers modulo 2.

8.2 Formulae for the first and second Stiefel-Whitney classes, when $q \equiv 3(mod.4)$

In this section we calculate the formulae for the first and second Stiefel-Whitney classes in the case $q \equiv 3(mod.4)$. This case is more complicated than the case when $q \equiv 1(mod.4)$. This is because the 2-Sylow subgroup of $A^*/F^*U_A^1$ is no longer an elementary abelian group of order 4. When $q \equiv 3(mod.4)$ the Sylow 2-subgroup of $A^*/F^*U_A^1$ is non-abelian dihedral: that is, it is isomorphic to

$$D_{2^{n+1}} = \{x, y \mid x^{2^n} = y^2 = 1, yxy = x^{-1}\}$$

for some $n \geq 2$. In terms of X and Y in A with $a = X^2, b = Y^2$ and $F(\sqrt{a})/F$ unramified we may take Y to represent y and X to represent $x^{2^{n-1}}$.

The three non-conjugate elements of order two in this case will therefore be X, Y, and xY, and then we have two non-conjugate copies of $\mathbf{Z}/2 \times \mathbf{Z}/2$ in $D_{2^{n+1}}$

$$H_1 = \langle x^{2^{n-1}}, y\rangle \text{ and } \mathrm{H}_2 = \langle \mathrm{x}^{2^{n-1}}, \mathrm{xy}\rangle.$$

We shall evaluate the restriction homomorphism

$$H^*(D_{2^{n+1}};\mathbf{Z}/2) \stackrel{(i_1^*,i_2^*)}{\longrightarrow} H^*(H_1;\mathbf{Z}/2) \oplus H^*(H_2;\mathbf{Z}/2)$$

with particular interest in dimensions one and two. Define homomorphisms $u_1, u_2 : H_1 \longrightarrow \mathbf{Z}/2$ and $v_1, v_2 : H_2 \longrightarrow \mathbf{Z}/2$ by the formulae

$$u_1(y) \equiv 1,\ u_1(x^{2^{n-1}}) \equiv 0,\ u_2(y) \equiv 0,\ u_2(x^{2^{n-1}}) \equiv 1 \text{ (modulo 2)}$$

and

$$v_1(xy) \equiv 1,\ v_1(x^{2^{n-1}}) \equiv 0,\ v_2(xy) \equiv 0,\ v_2(x^{2^{n-1}}) \equiv 1 \text{ (modulo 2)}.$$

Therefore, we have $H^*(H_1;\mathbf{Z}/2) \cong \mathbf{Z}/2[u_1,u_2]$ and $H^*(H_2;\mathbf{Z}/2) \cong \mathbf{Z}/2[v_1,v_2]$.

Consider the composition $x_1 i_1$ which sends $x^{2^{n-1}}$ to X and then to zero (modulo 2) while sending y to Y and then to 1 (modulo 2). Therefore $i_1^*(x_1) = u_1$. Similarly, $x_2 i_1$ sends $x^{2^{n-1}}$ to X to zero and y to Y to zero (modulo 2) which yields $i_1^*(x_2) = 0$. If $\chi_i : H_1 \longrightarrow \{\pm 1\}$ given by $\chi_i = (-1)^{u_i}$ then the composition

$$H_1 \stackrel{i_1}{\longrightarrow} D_{2^{n+1}} \subset O_2(\mathbf{R})$$

is $\chi_2(1+\chi_1)$ so that $i_1^*(w_2) = SW_2(\chi_2(1+\chi_1))$ is the degree two term in $SW(\chi_2 + \chi_1\chi_2) = (1+u_2 t)(1+(u_1+u_2)t)$. Hence $i_1^*(w_2) = u_2^2 + u_1 u_2$.

Similarly, the composition $x_1 i_2$ which sends $x^{2^{n-1}}$ to X and then to zero (modulo 2) while sending xy to $x_1(xY) = x_1(Y) \equiv 1$ (modulo 2). Therefore $i_2^*(x_1) = v_1$. Similarly, $x_2 i_2$ sends $x^{2^{n-1}}$ to X to zero and xy to $x_2(xY) = x_2(x) \equiv 1$ (modulo 2) which yields $i_2^*(x_2) = v_1$. If $\chi_i : H_2 \longrightarrow \{\pm 1\}$ given by $\chi_i = (-1)^{v_i}$ then the composition

$$H_2 \stackrel{i_2}{\longrightarrow} D_{2^{n+1}} \subset O_2(\mathbf{R})$$

is again equal to $\chi_2(1+\chi_1)$ so that $i_2^*(w_2) = v_2^2 + v_1 v_2$.

Proposition 8.2.1

In the case $q \equiv 3(mod.4)$*, if* $n = 1, 2$ *the restriction homomorphism*

$$H^n(A^*/F^*; \mathbf{Z}/2) \overset{(i_1^*, i_2^*)}{\longrightarrow} H^n(H_1; \mathbf{Z}/2) \oplus H^n(H_2; \mathbf{Z}/2)$$

is injective.

Proof

By Theorem 7.1.2 we have only to verify injectivity on $H^*(D_{2^{n+1}}; \mathbf{Z}/2)$. In dimension one

$$0 = (i_1^*, i_2^*)(\alpha x_1 + \beta x_2) = (\alpha u_1, (\alpha + \beta) v_1)$$

implies that $\alpha \equiv \beta \equiv 0$ (modulo 2), as required. In dimension two

$$0 = (i_1^*, i_2^*)(\alpha x_1^2 + \beta x_2^2 + \gamma w_2) = (\alpha u_1^2 + \gamma(u_2^2 + u_1 u_2), (\alpha + \beta) v_1^2) + \gamma(v_2^2 + v_1 v_2)$$

implies that $\alpha \equiv \beta \equiv \gamma \equiv 0$ (modulo 2), as required. □

Corollary 8.2.2

In the situation of Proposition 8.2.1*,*

$$H^2(A^*/F^*; \mathbf{Z}/2) \longrightarrow H^2(\langle x^{2^{n-1}} \rangle; \mathbf{Z}/2) \oplus H^2(\langle y \rangle; \mathbf{Z}/2) \oplus H^2(\langle xy \rangle; \mathbf{Z}/2)$$

is injective.

Proof

From the formulae of the restriction of $aw_2 + bx_1^2 + cx_2^2$ to $H^2(H_1; \mathbf{Z}/2) \oplus H^2(H_2; \mathbf{Z}/2)$, that is, $i_1^*(w_2) = u_2^2 + u_1 u_2$, $i_1^*(x_1) = u_1$, $i_1^*(x_2) = 0$, $i_2^*(w_2) =$

$v_2^2 + v_1v_2$, $i_2^*(x_1) = v_1$, and $i_2^*(x_2) = v_1$, $aw_2 + bx_1^2 + cx_2^2$ is sent to $(a(u_2^2 + u_1u_2) + bu_1^2, a(v_2^2 + v_1v_2) + bv_1^2 + cv_1^2)$. Therefore, we will prove that, under the three restriction maps to the subgroups of order two generated by $x^{2^{n-1}}, y$ and xy, $aw_2 + bx_1^2 + cx_2^2$ maps to $(a, b, b + c)$. Thus, it is injective.

(i) Restriction to $H^2(\langle x^{2^{n-1}} \rangle; \mathbf{Z}/2)$. In this case, we have the two possible choices of restriction, but let us note that in both of them, we should get the same answer.

$$H^2(A^*/F^*; \mathbf{Z}/2) \xrightarrow{i_1^*} H^2(H_1; \mathbf{Z}/2) \longrightarrow H^2(\langle x^{2^{n-1}} \rangle; \mathbf{Z}/2)$$

given by

$$aw_2 + bx_1^2 + cx_2^2 \mapsto a(u_2^2 + u_1u_2) + bu_1^2 \mapsto (a(u_2^2 + u_1u_2) + bu_1^2)(x^{2^{n-1}}) = a$$

or

$$H^2(A^*/F^*; \mathbf{Z}/2) \xrightarrow{i_2^*} H^2(H_2; \mathbf{Z}/2) \longrightarrow H^2(\langle x^{2^{n-1}} \rangle; \mathbf{Z}/2)$$

given by

$$aw_2 + bx_1^2 + cx_2^2 \mapsto a(v_2^2 + v_1v_2) + bv_1^2 + cv_1^2 \mapsto (a(v_2^2 + v_1v_2) + bv_1^2 + cv_1^2)(x^{2^{n-1}}) = a$$

(ii) Restriction to $H^2(< y >; \mathbf{Z}/2)$.

$$H^2(A^*/F^*; \mathbf{Z}/2) \xrightarrow{i_1^*} H^2(H_1; \mathbf{Z}/2) \longrightarrow H^2(< y >; \mathbf{Z}/2)$$

given by

$$aw_2 + bx_1^2 + cx_2^2 \mapsto a(u_2^2 + u_1u_2) + bu_1^2 \mapsto (a(u_2^2 + u_1u_2) + bu_1^2)(y) = b$$

(iii) And finally, restriction to $H^2(< xy >; \mathbf{Z}/2)$.

$$H^2(A^*/F^*; \mathbf{Z}/2) \xrightarrow{i_2^*} H^2(H_2; \mathbf{Z}/2) \longrightarrow H^2(< xy >; \mathbf{Z}/2)$$

given by

$$aw_2+bx_1^2+cx_2^2 \mapsto a(v_2^2+v_1v_2)+bv_1^2+cv_1^2 \mapsto (a(v_2^2+v_1v_2)+bv_1^2+cv_1^2)(xy) = b+c$$

q.e.d

Remark 8.2.3 *When $q \equiv 3$ (modulo 4) then X is a square in A^*/F^* because it is a square in $A^*/F^*U_A^1$ and therefore $F(XY) = F(Y)$.*

We can study now the formulae for the Stiefel-Whitney classes.

Suppose that π is a representation of the cyclic group of order two generated by g and that the character values of π are $Trace\pi(1) = d_0 = dim_{\mathbf{C}}(\pi)$ and $Trace\pi(g) = d_1$. If $\pi = (d_0 - a) \cdot 1 + a \cdot L$ where L is the non-trivial one-dimensional representation then $d_0 - 2a = d_1$ and the Stiefel-Whitney classes satisfy

$$SW_1(\pi) = a = (d_0 - d_1)/2 \in \mathbf{Z}/2 \cong H^1(\langle g \rangle; \mathbf{Z}/2),$$

$$SW_2(\pi) = \left(\begin{array}{c} a \\ 2 \end{array} \right) = \left(\begin{array}{c} (d_0 - d_1)/2 \\ 2 \end{array} \right) \in \mathbf{Z}/2 \cong H^2(\langle g \rangle; \mathbf{Z}/2).$$

From this observation we can obtain the following result.

Proposition 8.2.4 *Let the three conjugacy classes of elements of order two in A^*/F^* be $X = x^{2^{n-1}}, Y = y$ and $xY = xy$. Suppose that W is a continuous, orthogonal representation of A^*/F^* as in section 8.1 with character values*

$$Tr_W(1) = d_0, \ Tr_W(X) = d_1,$$

$$Tr_W(Y) = d_2, \ Tr_W(xY) = d_3$$

then $d_0 \equiv d_1$ *(modulo 4). Furthermore*

$$SW_1(W) = ((d_0 - d_2)/2)x_1 + ((d_2 - d_3)/2)x_2$$

and

$$\begin{aligned} SW_2(W) &= \begin{pmatrix} (d_0 - d_1)/2 \\ 2 \end{pmatrix} w_2 + \begin{pmatrix} (d_0 - d_2)/2 \\ 2 \end{pmatrix} x_1^2 \\ &+ (\begin{pmatrix} (d_0 - d_3)/2 \\ 2 \end{pmatrix} - \begin{pmatrix} (d_0 - d_2)/2 \\ 2 \end{pmatrix}) x_2^2. \end{aligned}$$

Proof

From the formulae of Corollary 8.2.2 (proof) the restriction maps

$$H^i(D_{2^{n+1}}; \mathbf{Z}/2) \longrightarrow H^i(\langle X \rangle; \mathbf{Z}/2) \oplus H^i(\langle Y \rangle; \mathbf{Z}/2) \oplus H^i(\langle xY \rangle; \mathbf{Z}/2)$$

are given by

$$\alpha x_1 + \beta x_2 \mapsto (\alpha u_1, (\alpha + \beta)v_1) \mapsto (0, \alpha, \alpha + \beta)$$

and

$$a w_2 + b x_1^2 + c x_2^2 \mapsto (a, b, b + c).$$

.

Therefore, the required formulae follows by considering W on the $< X >$, $< Y >$ and $< xY >$, calculating SW_1 and SW_2 on each $H^i(< g >; \mathbf{Z}/2)$ ($i = 1, 2$, $g = X, Y, xY$) by the formulae given above when π is a representation of a cyclic group of order two, and finding the inverse image in $H^i(D_{2^{n-1}}; \mathbf{Z}/2)$ ($i = 1, 2$).

q.e.d.

Chapter 9

A construction with two-dimensional symplectic, Galois representations

The goal of this chapter is to construct a map from two-dimensional symplectic Galois representations to fourth roots of unity. This map will be constructed by combining $\widehat{\Gamma}_F$ defined in chapter 7 and the results of D. Prasad and D. Ramakrishnan [22] about the Langlands correspondence when the representation considered is two-dimensional symplectic and Galois. That is, let $\sigma = Ind_{K/F}(\chi)$ be a two-dimensional symplectic Galois representation induced by a character χ, σ is mapped to $\pi(\sigma)$ an orthogonal representation of A^*/F^* through the Langlands correspondence.

Let us define $\Upsilon_F(\sigma) = \widehat{\Gamma}_F(\pi(\sigma))$. The information obtained about $\widehat{\Gamma}_F$ in chapter 7 and the results of D. Prasad-D. Ramakrishnan [22] about $\pi(\sigma)$ will allow us to prove the following relation between the value of the local root number of σ and the value of $\Upsilon_F(\sigma)$,

Theorem 9.4.2 *Let σ be a two-dimensional symplectic Galois representation, $\sigma = Ind_{K/F}(\chi)$ induced from a character $\chi : K^* \longrightarrow \mathbf{C}^*$, where F is a non-Archimedean local field with residue field F_q of odd order $q \equiv 1(mod.4)$ and K/F is a quadratic extension. Then*

$$\Upsilon_F(\sigma) = (-1)^{\frac{q-1+2e}{2e}} W_F(\sigma) W_F(w_{K/F})^{-1} \in \mu_4$$

where e is the ramification index of the extension and $w_{K/F}$ is the quadratic character of F^* given by class field theory.

As immediate corollaries of this, we obtain the following results.

Corollary 9.5.2 *($q \equiv 1(mod.4)$) Let σ and σ' be two two-dimensional symplectic Galois representations, $\sigma = Ind_{K/F}(\chi)$ and $\sigma' = Ind_{K/F}(\chi')$ induced from characters χ,$\chi' : K^* \longrightarrow \mathbb{C}$. Then, the following holds,*

(I) If K/F is ramified with $f(\chi) \neq f(\chi')$ and $min(f(\chi), f(\chi')) = 1$ (it is no restriction to assume $f(\chi') = 1$) then,

$$\frac{\Upsilon_F(\sigma)}{\Upsilon_F(\sigma')} = (-1)^{\frac{q-1}{4}} \frac{W_F(\sigma)}{W_F(\sigma')} \frac{W_F(w_{K'/F})}{W_F(w_{K/F})}$$

where K'/F is the unique unramified quadratic extension over F.

(II) In any other case,

$$\frac{\Upsilon_F(\sigma)}{\Upsilon_F(\sigma')} = \frac{W_F(\sigma)}{W_F(\sigma')}$$

Corollary 9.5.3 *($q \equiv 1(mod.4)$) Let σ be a two-dimensional symplectic Galois representation, $\sigma = Ind_{K/F}(\chi)$ induced from a character $\chi : K^* \longrightarrow \mathbb{C}^*$. Consider $\sigma_{tame} = Ind_{K/F}(\chi_1)$ where $\chi = \chi_1 \otimes \chi_2$, the order of χ_1 is* *coprime to q and the order of χ_2 is a power of q. Then, χ_1 is tamely ramified, and*

$$\frac{\Upsilon_F(\sigma)}{\Upsilon_F(\sigma_{tame})} = (-1)^{\frac{(e-1)(q-1)}{4}} \frac{W_F(\sigma)}{W_F(\sigma_{tame})} \frac{W_F(w_{K'/F})}{W_F(w_{K/F})}$$

where K'/F is the unique unramified quadratic extension over F and e is the ramification index of the extension K/F.

The case $q \equiv 3(mod.4)$ will be studied in section 9.4. The information obtained about $\Upsilon_F(\sigma)$ in this case does not seem to give much information about the value of the local root number of σ. However, it has been introduced for completeness.

Let us note that although only the case $q \equiv 1(mod.4)$ gives us valuable information, this covers most cases. This is, if $q = p^d$ for some odd prime p and an integer d, $q \equiv 1(mod.4)$ if $p \equiv 1(mod.4)$ or $p \equiv 3(mod.4)$ and d even.

9.1 The Langlands correspondence

This section recalls the correspondence between two-dimensional, irreducible representations of the Weil group W_F of a local field F and irreducible representations of A^*.

Let F be a local field of residue characteristic $p \neq 2$. We are interested in two-dimensional, continuous Galois representations

$$\sigma : \Omega_F \longrightarrow GL_2\mathbf{C}.$$

Such a σ is a special case of a W_F-representation so that the bijection, established by combining the results of [15] and [18] and reiterated in ([22]

Theorem 1.5), yields an injective map, $\sigma \mapsto \pi(\sigma)$,

$$\pi : \left\{ \begin{array}{c} irreducible,\ 2-dimensional \\ \\ continuous\ Galois \\ \\ representations\ over\ \mathbf{C} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} irreducible,\ continuous \\ \\ finite-dimensional\ complex \\ \\ representations\ of\ \mathrm{A}^* \end{array} \right\}$$

where A is the unique quaternion algebra with centre F.

The map π is characterised by the following properties.

Via class field theory, if Ω_F is the absolute Galois group of F,

$$det(\sigma) : \Omega_F \xrightarrow{\sigma} GL_2\mathbf{C} \xrightarrow{det} \mathbf{C}^*$$

corresponds to a continuous character of finite order

$$det(\sigma) : F^* \longrightarrow \mathbf{C}^*.$$

The central character of $\pi(\sigma)$ is the continuous character of finite order such that $\pi(\sigma)(z)$ is multiplication by the scalar $\omega_{\pi(\sigma)}(z)$ for all $z \in F^* \subset A^*$ and

$$det(\sigma) = \omega_{\pi(\sigma)}. \qquad (1)$$

Let $f(\sigma)$ and $f(\pi(\sigma))$ denote the conductors of σ and $\pi(\sigma)$ as introduced in Chapter 3, then

$$f(\sigma) = f(\pi(\sigma)). \qquad (2)$$

If $V^\vee = Hom(V, \mathbf{C})$ denotes the contragredient of a representation V then

$$\pi(\sigma^\vee) = \pi(\sigma)^\vee. \qquad (3)$$

The epsilon factors defined in [15] and [34] satisfy

$$\epsilon(\sigma \otimes \mu, \psi, s) = \epsilon(\pi(\sigma) \otimes (\mu \cdot N_{red}), \psi, s) \qquad (4)$$

for all characters μ on F^* of finite order. The local root numbers are defined to be the value of the associated ϵ-factor at $s = 1/2$ so that

$$W_F(\sigma \otimes \mu) = W(\pi(\sigma) \otimes (\mu \cdot N_{red}))$$

for all characters μ on F^* of finite order.

Also each σ is writeable as an induced representation $\sigma = Ind_{\Omega_K}^{\Omega_F}(\chi)$ for some quadratic extension K/F and character $\chi : F^* \longrightarrow \mathbf{C}^*$. We have

$$\omega_{\pi(\sigma)} = (\chi \mid {}_{F^*}) \cdot \omega_{K/F} \qquad (5)$$

where $\omega_{K/F} : \Omega_F \longrightarrow G(K/F) \cong \{\pm 1\} \subset \mathbf{C}^*$ is non-trivial.

[22] observe that, when σ is symplectic, $\pi(\sigma)$ is an orthogonal representation of A^* which is trivial on F^*.

Let us restrict then, to the case when $\sigma = Ind_{\Omega_K}^{\Omega_F}(\chi)$ is a symplectic, irreducible two-dimensional representation. This implies (see [7] or use a simple transfer argument using the formula of ([31] Proposition 2.50 p.14)) that $\chi : K^* \longrightarrow \mathbf{C}^*$ restricts on F^* to a non-trivial character of order two. In fact, by (1) and (5), the fact that $det(\sigma)$ is trivial for symplectic representations implies that $1 = \omega_{\pi(\sigma)} = (\chi \mid {}_{F^*}) \cdot \omega_{K/F}$ so that

$$(\chi \mid {}_{F^*}) = \omega_{K/F}. \qquad (6)$$

Now, by ([22] Lemma 1.4), if K/F is a ramified quadratic extension and χ on K^* satisfies (6) then the Artin conductor of χ satisfies either $f(\chi) = 2m > 0$ or $f(\chi) = 1$ and $(\chi \mid \mathcal{O}_K^*)$ is given by the composition

$$\mathcal{O}_K^* \longrightarrow \frac{\mathcal{O}_K^*}{U_K^1} \cong \frac{\mathcal{O}_F^*}{U_F^1} \stackrel{\omega_{K/F}}{\longrightarrow} \{\pm 1\} \subset \mathbf{C}^*.$$

This means that, when K/F is ramified the conductor $f(\pi(\sigma))$ is odd unless χ is tamely ramified (χ cannot be unramified).

In [22], a table giving the dimension, conductor and trace values of $\pi(\sigma)$ on elements of order two in A^*/F^* is given (such an element of order two is associated uniquely to a quadratic extension K/F of F and viceversa).

Proposition 9.1.1 *Let F be a non-Archimedean local field with residue field F_q and q odd, K/F a quadratic extension and $\pi = \pi_\chi$ be the representation of A^*/F^* attached to χ a character of K^*. Then we have the following table,*

K/F	$f(\chi)$	$dim(\pi)$	$f(\pi)$
unramified	f	$2q^{f-1}$	$2f$
ramified	$2f$	$(q+1)q^{f-1}$	$2f+1$

Let L be any quadratic extension of F, and x the unique element of L^*/F^* of order two. Then we have:

(1) If $L \neq K$, $Tr_\pi(x) = 0$

(2) If $L = K$ and K/F is unramified, $Tr_\pi(x) = (-1)^{f+1}2\chi(x)$

(3) If $L = K$ and K/F is ramified, $Tr_\pi(x) = -2G_\chi w(2)w(-1)^{f-1}\chi(x)$, where

$$G_\chi = \frac{1}{\sqrt{q}} \sum_{x \in (\mathcal{O}_F/\varpi_F)^*} \chi(1 + \varpi_K^{2f-1}x)w(x)$$

Here, in the notation of [22], w is the unique quadratic character of $\mathbf{F}_q$, and ϖ_F, ϖ_K denote chosen uniformizers for F, K respectively.

Proposition 9.1.2 *All cases are covered in the proposition above.*

Proof:

The cases when K/F is unramified are clearly all covered by the table. When K/F is ramified the conductor of a character χ satisfying (6) is either $2m > 0$ (covered in the table) or 1, by [22], Lemma 1.4. However, if $f(\chi) = 1$ then $\sigma = Ind_{K/F}(\chi)$ may be written as $\sigma = Ind_{E/F}(\theta)$ with E/F unramified and $f(\theta) = 1$, which is included in the table. To see this observe that, if γ is the non-trivial F-automorphism of K and χ is tame, the characters χ and $\gamma(\chi)$ must agree on units. Therefore $\gamma(\chi) = \alpha\chi$ for some unramified character α on K^*. By Hilbert 90, we may write $\alpha = \phi N_{K/F}$ for some unramified, non-trivial character ϕ on F^*. Hence, $\sigma = \phi \otimes \sigma$ which implies, by the classification of admissible pairs $(E/F, \theta)$ given in ([2], pages $54 - 55$) that σ has the required form. □

The following result is ([22],Lemma 4.6),

Lemma 9.1.3 *We have the following table,*

K/F	$W(\pi)$
unramified	$(-1)^f \chi(x)$
ramified	$w(2)w(-1)^{f+1} G_\chi \chi(x)$

Note: It does not matter if we write $w(-1)^{f+1}$ or $w(-1)^{f-1}$, since w takes values in ± 1.

Corollary 9.1.4 *In the situation above, we have the following table:*

K/F	$f(\chi)$	*dim*(π)	$f(\pi)$
unramified	f	$2q^{f-1}$	$2f$
ramified	$2f$	$(q+1)q^{f-1}$	$2f+1$

Let L be any quadratic extension of F, and x the unique element of L^*/F^* *of order two. Then we have:*

(1) If $L \neq K$, $Tr_\pi(x) = 0$

(2) If $L = K$, $Tr_\pi(x) = -2W(\pi)$

By Chapter 8, we know that the first and second Stiefel-Whitney classes of an orthogonal representation of A^*/F^* can be obtained from its character values on elements of order two, and explicit formulae for them have been given. Therefore, using the information given by Corollary 9.1.4, the Stiefel-Whitney classes of $\pi(\sigma)$ can be calculated. This will be discussed in section 9.2 and 9.3 depending on whether $q \equiv 1(mod.4)$ or not .

9.2 First and second Stiefel-Whitney classes of $\pi(\sigma)$ when $q \equiv 1(mod.4)$

In this section we calculate the first and second Stiefel-Whitney classes of $\pi(\sigma)$ when $q \equiv 1(mod.4)$. We know from Proposition 8.1.1 that in this case, only the character values of $\pi(\sigma)$ on the elements 1, X, Y, and XY are needed, and these are given by Corollary 9.1.4.

There are three quadratic extensions over F, in terms of X, Y and XY. These are $K = F(X)$, $K = F(Y)$, and $K = F(XY)$, where $F(X)/F$ is the unramified extension. Therefore we are going to divide our study into three different cases depending on the extension we are dealing with.

To start with, let us remember the formulae given by Proposition 8.1.1, these are,

$$SW_1(\pi(\sigma)) = (c_1 + c_3)x_1 + (c_2 + c_3)x_2$$

$$SW_2(\pi(\sigma)) = \begin{pmatrix} c_1 + c_3 \\ 2 \end{pmatrix} x_1^2 + \begin{pmatrix} c_2 + c_3 \\ 2 \end{pmatrix} x_2^2 + [c_1c_2 + c_1c_3 + c_2c_3]x_1x_2.$$

where,

$$\begin{aligned} c_0 + c_1 + c_2 + c_3 &= dim\pi(\sigma) \\ c_0 + c_1 - c_2 - c_3 &= Tr_{\pi(\sigma)}(X) \\ c_0 - c_1 + c_2 - c_3 &= Tr_{\pi(\sigma)}(Y) \\ c_0 - c_1 - c_2 + c_3 &= Tr_{\pi(\sigma)}(XY) \end{aligned}$$

Therefore, we can write

$$c_2 + c_3 = \frac{dim\pi(\sigma) - Tr_{\pi(\sigma)}(X)}{2}$$
$$c_1 + c_3 = \frac{dim\pi(\sigma) - Tr_{\pi(\sigma)}(Y)}{2}$$

$$c_1 = \frac{dim\pi(\sigma) + Tr_{\pi(\sigma)}(X) - Tr_{\pi(\sigma)}(Y) - Tr_{\pi(\sigma)}(XY)}{4}$$
$$c_2 = \frac{dim\pi(\sigma) - Tr_{\pi(\sigma)}(X) + Tr_{\pi(\sigma)}(Y) - Tr_{\pi(\sigma)}(XY)}{4}$$
$$c_3 = \frac{dim\pi(\sigma) - Tr_{\pi(\sigma)}(X) - Tr_{\pi(\sigma)}(Y) + Tr_{\pi(\sigma)}(XY)}{4}$$

Let us start now our discussion

The case of $\chi : K^* = F(X)^* \longrightarrow \mathbf{C}^*$

In this case, K/F is unramified, and using Corollary 9.1.4 we can write:

$$\begin{aligned} dim\pi(\sigma) &= 2q^{f-1} \\ Tr_{\pi(\sigma)}(X) &= -2W(\pi(\sigma)) \\ Tr_{\pi(\sigma)}(Y) &= Tr_{\pi(\sigma)}(XY) = 0 \end{aligned}$$

Thus, in this case:

$$c_1 = \frac{q^{f-1} - W(\pi(\sigma))}{2}$$
$$c_2 = c_3 = \frac{q^{f-1} + W(\pi(\sigma))}{2}$$

We can see that $c_2 + c_3 = 2c_2 \equiv 0(mod.2)$ and $c_1 + c_3 = q^{f-1}$. As q is an odd integer, $q^{f-1} \equiv 1(mod.2)$. And we can conclude:

$$SW_1(\pi(\sigma)) = x_1$$

Let us start now the calculations for $SW_2(\pi(\sigma))$.

By using the fact that $c_2 = c_3$ we can write

$$SW_2(\pi(\sigma)) = \begin{pmatrix} c_1 + c_2 \\ 2 \end{pmatrix} x_1^2 + \begin{pmatrix} 2c_2 \\ 2 \end{pmatrix} x_2^2 + [2c_1c_2 + c_2^2]x_1x_2.$$

Now, as we are working modulo 2,

$$SW_2(\pi(\sigma)) = \begin{pmatrix} c_1 + c_2 \\ 2 \end{pmatrix} x_1^2 + c_2(x_2^2 + x_1x_2).$$

and

$$c_2 = \frac{q^{f-1} + W(\pi(\sigma))}{2} = \frac{q^{f-1} - 1}{2} + \frac{W(\pi(\sigma)) + 1}{2} \equiv \frac{W(\pi(\sigma)) + 1}{2}(mod.2)$$

since as $q \equiv 1(mod.4)$, then so too is $q^{f-1} \equiv 1(mod.4)$. Therefore, $\frac{q^{f-1}-1}{2} \equiv 0(mod.2)$.

Here,

$$\begin{pmatrix} c_1 + c_2 \\ 2 \end{pmatrix} = \begin{pmatrix} q^{f-1} \\ 2 \end{pmatrix} = q^{f-1}\frac{q^{f-1}-1}{2} \equiv 0(mod.2)$$

Therefore we can write,

$$SW_2(\pi(\sigma)) = \left(\frac{W(\pi(\sigma)) + 1}{2}\right)(x_2^2 + x_1x_2)$$

The case of $\chi : K^* = F(Y)^* \longrightarrow \mathbf{C}^*$

As above, using Corollary 7.4.3, we have

$$\begin{aligned} dim\pi(\sigma) &= (q+1)q^{f-1} \\ Tr_{\pi(\sigma)}(Y) &= -2W(\pi(\sigma)) \\ Tr_{\pi(\sigma)}(X) &= Tr_W(XY) = 0 \end{aligned}$$

where $f(\chi) = 2f$.

Therefore, in this case,

$$\begin{aligned} c_2 &= \frac{(q+1)q^{f-1} - 2W(\pi(\sigma))}{4} \\ c_1 = c_3 &= \frac{(q+1)q^{f-1} + 2W(\pi(\sigma))}{4} \end{aligned}$$

We can see that $c_1 + c_3 = 2c_1 \equiv 0(mod.2)$ and $c_2 + c_3 = \frac{(q+1)q^{f-1}}{2}$. An as $q \equiv 1(mod.4)$, $\frac{q+1}{2} \equiv 1(mod.2)$ and $q^{f-1} \equiv 1(mod.2)$, we can conclude:

$$SW_1(\pi(\sigma)) = x_2$$

Let us concentrate now on the calculations of $SW_2(\pi(\sigma))$.

As $c_1 = c_3$, we can write

$$SW_2(\pi(\sigma)) = \begin{pmatrix} 2c_1 \\ 2 \end{pmatrix} x_1^2 + \begin{pmatrix} c_1 + c_2 \\ 2 \end{pmatrix} x_2^2 + [2c_1c_2 + c_1^2]x_1x_2.$$

As in the unramified case, as we are working module 2

$$SW_2(\pi(\sigma)) = c_1(x_1^2 + x_1x_2) + \begin{pmatrix} c_1 + c_2 \\ 2 \end{pmatrix} x_2^2.$$

Now

$$c_1 = \frac{(q+1)q^{f-1} + 2W(\pi(\sigma))}{4} = \frac{(q+1)q^{f-1} - 2}{4} + \frac{W(\pi(\sigma)) + 1}{2}$$

And we know that $q \equiv 1(mod.4)$, and then so too is q^{f-1}. Therefore, we can write $q = 1 + 4t$ and $q^{f-1} = 1 + 4l$ for some positive integers t, l. Obtaining,

$$(q+1)q^{f-1} - 2 = 4t + 8l + 16tl \equiv 4t(mod.8)$$

which implies,

$$\frac{(q+1)q^{f-1} - 2}{4} \equiv t = \frac{q-1}{4}(mod.2)$$

Thus,

$$c_1 \equiv \frac{q-1}{4} + \frac{W(\pi(\sigma)) + 1}{2}(mod.2)$$

Here,

$$\begin{pmatrix} c_1 + c_2 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{(q+1)q^{f-1}}{2} \\ 2 \end{pmatrix} = \frac{(q+1)}{2}q^{f-1}\frac{(q+1)q^{f-1}-2}{4} \equiv \frac{q-1}{4}(mod.2)$$

since $\frac{q+1}{2} \equiv 1 \equiv q^{f-1}(mod.2)$ as $q \equiv 1(mod.4)$

Therefore, we can write

$$SW_2(\pi(\sigma)) = \left(\frac{q-1}{4}\right) x_2^2 + \left(\frac{q-1}{4} + \frac{W(\pi(\sigma))+1}{2}\right)(x_1^2 + x_1 x_2)$$

The case of $\chi : K^* = F(XY)^* \longrightarrow \mathbf{C}^*$

This case is trivial once we have dealt with the case $K = F(Y)$. We can just replace Y by XY in the definition of our quaternion algebra, obtaining $\widetilde{X} = X$ and $\widetilde{Y} = XY$.

Now, notice that x_1 and x_2 have to be rearranged too. We can choose $\widetilde{x}_1 = x_1$ and $\widetilde{x}_2 = x_1 + x_2$. These playing the role of x_1 and x_2 since (see section 1.2) $\widetilde{x}_1(\widetilde{X}) = x_1(X) \equiv 0(mod.2)$, $\widetilde{x}_1(\widetilde{Y}) = x_1(XY) = x_1(X) + x_1(Y) \equiv 1(mod.2)$, $\widetilde{x}_2(\widetilde{X}) = (x_1 + x_2)(X) = x_1(X) + x_2(X) \equiv 1(mod.2)$ and $\widetilde{x}_2(\widetilde{Y}) = (x_1 + x_2)(XY) = x_1(XY) + x_2(XY) = x_1(X) + x_1(Y) + x_2(X) + x_2(Y) \equiv 1 + 1 = 0(mod.2)$

Therefore, using the previous case

$$SW_1(\pi(\sigma)) = \widetilde{x}_2 = x_1 + x_2$$

and

$$\begin{aligned} SW_2(\pi(\sigma)) &= \left(\frac{q-1}{4}\right) \widetilde{x}_2^2 + \left(\frac{q-1}{4} + \frac{W(\pi(\sigma))+1}{2}\right)(\widetilde{x}_1^2 + \widetilde{x}_1 \widetilde{x}_2) \\ &= \left(\frac{q-1}{4}\right)(x_1^2 + x_2^2) + \left(\frac{q-1}{4} + \frac{W(\pi(\sigma))+1}{2}\right) x_1 x_2 \end{aligned}$$

We may now summarize all the information obtained, in the following Theorem,

Theorem 9.2.1 *($q \equiv 1(mod.4)$) In this situation the Sylow 2-subgroup of A^*/F^* is $V = \{1, X, Y, XY\}$ an elementary abelian group of order four. There are three quadratic extensions over F, and those are $F(X)/F$, $F(Y)/F$ and $F(XY)/F$, where the first is the unique unramified quadratic extension of F. Assuming that x_1, x_2 in the isomorphism $H^*(A^*/F^*; \mathbf{Z}/2) \cong \mathbf{Z}/2[x_1, x_2]$ verify:*

$$x_1(Y) \equiv 1,\ x_1(X) \equiv 0,\ x_2(Y) \equiv 0,\ x_2(X) \equiv 1 \text{ (modulo 2)}$$

Then if $\sigma = Ind_{\Omega_K}^{\Omega_F}(\chi)$, we can write the following table for the first and second Stiefel-Whitney classes of $\pi(\sigma)$:

K/F	$SW_1(\pi(\sigma))$	$SW_2(\pi(\sigma))$
$F(X)/F$	x_1	$\left(\frac{W(\pi(\sigma))+1}{2}\right)(x_2^2 + x_1x_2)$
$F(Y)/F$	x_2	$\left(\frac{q-1}{4}\right)x_2^2 + \left(\frac{q-1}{4} + \frac{W(\pi(\sigma))+1}{2}\right)(x_1^2 + x_1x_2)$
$F(XY)/F$	$x_1 + x_2$	$\left(\frac{q-1}{4}\right)(x_1^2 + x_2^2) + \left(\frac{q-1}{4} + \frac{W(\pi(\sigma))+1}{2}\right)x_1x_2$

9.3 First and second Stiefel-Whitney classes of $\pi(\sigma)$ when $q \equiv 3(mod.4)$

As it was done in the case $q \equiv 1(mod.4)$, we use the information given in Corollary 9.1.4 which gives the required character values . In order to obtain the first and second Stiefel-Whitney classes, let us recall the formulae obtained in Proposition 8.2.4. That is,

$$Tr_{\pi(\sigma)}(1) = d_0, \; Tr_{\pi(\sigma)}(X) = d_1,$$
$$Tr_{\pi(\sigma)}(Y) = d_2, \; Tr_{\pi(\sigma)}(xY) = d_3$$

then

$$SW_1(\pi(\sigma)) = ((d_0 - d_2)/2)x_1 + ((d_2 - d_3)/2)x_2$$

and

$$\begin{aligned} SW_2(\pi(\sigma)) &= \binom{(d_0 - d_1)/2}{2} w_2 + \binom{(d_0 - d_2)/2}{2} x_1^2 \\ &\quad + (\binom{(d_0 - d_3)/2}{2} - \binom{(d_0 - d_2)/2}{2})x_2^2. \end{aligned}$$

As it was done in the previous section, we split this study into three different cases, depending on the extension considered. Recall that the three quadratic extension over F in this case are $F(X)/F$, $F(Y)/F$ and $F(xY)/F$, where $F(X)/F$ is the unramified one.

Before starting the calculations of the first and second Stiefel-Whitney classes, we record some congruences modulo two which we shall need later.

(i) $q^{f-1} \equiv 1(mod.2)$, since q is an odd integer and therefore so too is q^{f-1}.

(ii) $W(\pi(\sigma)) = \pm 1 \equiv 1(mod.2)$.

(iii) $\frac{q+1}{2} \equiv 0(mod.2)$, since $q + 1 \equiv o(mod.4)$ as $q \equiv 3(mod.4)$.

(iv) $\frac{q^{f-1}-1}{2} \equiv f-1 \equiv \frac{(-1)^{f-1}-1}{2}(mod.2)$. This equivalence comes from the fact that when $f-1$ is even, as $q \equiv 3(mod.4)$, then $q^{f-1} \equiv 1(mod.4)$ and therefore $\frac{q^{f-1}-1}{2} \equiv 0 \equiv f-1 \equiv \frac{(-1)^{f-1}-1}{2}(mod.2)$. When $f-1$ is odd, $q^{f-1} \equiv 3(mod.4)$ and therefore, $\frac{q^{f-1}-1}{2} \equiv 1 \equiv f-1 \equiv \frac{(-1)^{f-1}-1}{2}(mod.2)$.

The case of $\chi : K^* = F(X)^* \longrightarrow \mathbf{C}^*$

In this case, K/F is unramified, $f = f(\chi)$, and using Corollary 9.1.4 we can write,

$$\begin{aligned} dim\pi(\sigma) &= 2q^{f-1} \\ Tr_{\pi(\sigma)}(X) &= -2W(\pi(\sigma)) \\ Tr_{\pi(\sigma)}(Y) &= Tr_{\pi(\sigma)}(xY) = 0 \end{aligned}$$

Thus, in this case:

$$d_0 = 2q^{f-1}$$
$$d_1 = -2W(\pi(\sigma))$$
$$d_2 = d_3 = 0$$

Therefore, $\frac{d_0-d_2}{2} = q^{f-1} \equiv 1(mod.2)$ and $\frac{d_2-d_3}{2} = 0$, and we can write,

$$SW_1(\pi(\sigma)) = x_1$$

For SW_2, we have,

$$SW_2(\pi(\sigma)) = \begin{pmatrix} q^{f-1} + W_F(\sigma) \\ 2 \end{pmatrix} w_2 + \begin{pmatrix} q^{f-1} \\ 2 \end{pmatrix} x_1^2$$

Now,

$$\binom{q^{f-1}+W_F(\sigma)}{2} = \frac{q^{f-1}+W(\pi(\sigma))}{2}(q^{f-1}+W(\pi(\sigma))-1)$$

$$\equiv \frac{W(\pi(\sigma))+(-1)^{f-1}}{2}(mod.2)$$

$$\binom{q^{f-1}}{2} = \frac{q^{f-1}-1}{2}q^{f-1} \equiv f-1(mod.2)$$

and we can conclude,

$$SW_2(\pi(\sigma)) = \frac{W(\pi(\sigma))+(-1)^{f-1}}{2}w_2 + (f-1)x_1^2$$

The case of $\chi : K^* = F(Y)^* \longrightarrow \mathbf{C}^*$

In this case, K/F is ramified, $f(\chi) = 2f$, and using Corollary 9.1.4 we can write,

$$\begin{aligned} dim\pi(\sigma) &= (q+1)q^{f-1} \\ Tr_{\pi(\sigma)}(Y) &= -2W(\pi(\sigma)) \\ Tr_{\pi(\sigma)}(X) &= Tr_{\pi(\sigma)}(xY) = 0 \end{aligned}$$

Thus, in this case:

$$d_0 = (q+1)q^{f-1}$$

$$d_2 = -2W(\pi(\sigma))$$

$$d_1 = d_3 = 0$$

Therefore, $\frac{d_0-d_2}{2} = \frac{q+1}{2}q^{f-1}+W(\pi(\sigma)) \equiv 1(mod.2)$, and $\frac{d_2-d_3}{2} = -W(\pi(\sigma)) \equiv 1(mod.2)$. Thus,

$$SW_1(\pi(\sigma)) = x_1 + x_2$$

For SW_2 we have,

$$\begin{aligned} SW_2(\pi(\sigma)) &= \begin{pmatrix} ((q+1)q^{f-1})/2 \\ 2 \end{pmatrix} w_2 + \begin{pmatrix} ((q+1)q^{f-1}+2W_F(\sigma))/2 \\ 2 \end{pmatrix} x_1^2 \\ &+ (\begin{pmatrix} ((q+1)q^{f)-1})/2 \\ 2 \end{pmatrix} - \begin{pmatrix} ((q+1)q^{f-1}+2W_F(\sigma))/2 \\ 2 \end{pmatrix}) x_2^2 \end{aligned}$$

Now,

$$\begin{pmatrix} \frac{q+1}{2}q^{f-1} \\ 2 \end{pmatrix} = \frac{q+1}{4}q^{f-1}(\frac{q+1}{2}q^{f-1} - 1) \equiv \frac{q+1}{4}(mod.2)$$

and

$$\begin{aligned} \begin{pmatrix} \frac{q+1}{2}q^{f-1} + W_F(\pi(\sigma)) \\ 2 \end{pmatrix} &= (\tfrac{q+1}{2}q^{f-1} + W(\pi(\sigma)))(\tfrac{q+1}{4}q^{f-1} + \tfrac{W(\pi(\sigma))-1}{2}) \\ &\equiv \tfrac{q+1}{4} + \tfrac{W(\pi(\sigma))-1}{2}(mod.2) \end{aligned}$$

Therefore, we can conclude,

$$SW_2(\pi(\sigma)) = \frac{q+1}{4}w_2 + \left(\frac{q+1}{4} + \frac{W(\pi(\sigma))-1}{2}\right)x_1^2 + \left(\frac{1-W(\pi(\sigma))}{2}\right)x_2^2$$

The case of $\chi : K^* = F(xY)^* \longrightarrow \mathbf{C}^*$

In this case, K/F is ramified, $f(\chi) = 2f$, and using Corollary 9.1.4 we can write,

$$\begin{aligned} dim\pi(\sigma) &= (q+1)q^{f-1} \\ Tr_{\pi(\sigma)}(xY) &= -2W(\pi(\sigma)) \\ Tr_{\pi(\sigma)}(X) &= Tr_{\pi(\sigma)}(Y) = 0 \end{aligned}$$

Thus, in this case:

$$\begin{aligned} d_0 &= (q+1)q^{f-1} \\ d_3 &= -2W(\pi(\sigma)) \\ d_1 &= d_2 = 0 \end{aligned}$$

Therefore $\frac{d_0-d_2}{2} = \frac{q+1}{2}q^{f-1}0 \equiv 1(mod.2)$, and $\frac{d_2-d_3}{2} = W(\pi(\sigma)) \equiv 1(mod.2)$. Thus,

$$SW_1(\pi(\sigma)) = x_2$$

Now, for SW_2,

$$\begin{aligned} SW_2(\pi(\sigma)) &= \begin{pmatrix} ((q+1)q^{f(x)-1})/2 \\ 2 \end{pmatrix} w_2 + \begin{pmatrix} ((q+1)q^{f(x)-1})/2 \\ 2 \end{pmatrix} x_1^2 \\ &+ (\begin{pmatrix} ((q+1)q^{f(x)-1})/2 + W_F(\sigma)) \\ 2 \end{pmatrix} - \begin{pmatrix} ((q+1)q^{f(x)-1})/2 \\ 2 \end{pmatrix})x_2^2. \end{aligned}$$

and let us notice that all these combinatorial numbers have already been obtained in the case $K = F(Y)$. Therefore we can conclude,

$$SW_2(\pi(\sigma)) = \frac{q+1}{4}(w_2 + x_1^2) + \left(\frac{W(\pi(\sigma)) - 1}{2}\right) x_2^2$$

Now, once the three cases have been solved, we conclude by giving a table of first and second Stiefel-Whitney classes as it was done in the case $q \equiv 1(mod.4)$, that is

Theorem 9.3.1 *($q \equiv 3(mod.4)$) In this situation the Sylow 2-subgroup of A^*/F^* is non-abelian dihedral and isomorphic to $D_{2^{n+1}} = \{x, y | x^{2^n} = y^2 = 1, yxy = x^{-1}\}$ for some $n \geq 2$, where Y is taken to represent y and X is taken to represent $x^{2^{n-1}}$.*

There are three quadratic extensions over F, and those are, $F(X)/F$, $F(Y)/F$ and $F(xY)/F$, where the first is the unique unramified quadratic extension over F. And the following table for the first and second Stiefel-Whitney classes of $\pi(\sigma)$ can be written

K/F	$SW_1(\pi(\sigma))$	$SW_2(\pi(\sigma))$
$F(X)/F$	x_1	$\left(\frac{W(\pi(\sigma))+(-1)^{f-1}}{2}\right) w_2 + (f-1)x_1^2$
$F(Y)/F$	$x_1 + x_2$	$\left(\frac{q+1}{4}\right) w_2 + \left(\frac{q+1}{4} + \frac{W(\pi(\sigma))-1}{2}\right) x_1^2 + \left(\frac{1-W(\pi(\sigma))}{2}\right) x_2^2$
$F(xY)/F$	x_2	$\left(\frac{q+1}{4}\right)(w_2 + x_1^2) + \left(\frac{W(\pi(\sigma))-1}{2}\right) x_2^2$

where $f = f(\chi)$ and $H^*(A^*/F^*; \mathbf{Z}/2) \cong \mathbf{Z}/2[x_1, x_2, w_2]/(x_1^2 + x_1x_2)$ (Theorem7.1.2)

9.4 The construction with two-dimensional symplectic, Galois representations

The aim of this section is to obtain a chapter 6-type map, but this time with symplectic representations, and see how the value of this map is related to the value of the local root number of the symplectic representation considered.

In order to construct such map, we combine the construction with orthogonal representations of A^*/F^* with the Langlands correspondence in the following way,

Definition 9.4.1 *Let σ be a two-dimensional symplectic, Galois representation, $\sigma = Ind_{K/F}(\chi)$ induced from a character $\chi : K^* \longrightarrow \mathbb{C}^*$, where F is a non-Archimedean local field of odd residual characteristic, and K/F is a quadratic extension. The map Υ_F is defined as,*

$$\Upsilon_F(\sigma) = \widehat{\Gamma}_F(\pi(\sigma))$$

,where $\widehat{\Gamma}_F$ was defined in sections 7.2 *and* 7.3.

As it was said in chapter 7, once the values of the first and second Stiefel-Whitney classes of W are obtained, the value of $\widehat{\Gamma}_F(W)$ is easily calculated. These haven been calculated in the last two sections for $\pi(\sigma)$.

We will demonstrate that in the case $q \equiv 1(mod.4)$, accurate information about the value of the local root number of σ is given in terms of $\Upsilon_F(\sigma)$ and the value of the local root number of a quadratic character only depending on the quadratic extension considered over F.

Due to the different values for the first and second Stiefel-Whitney classes obtained, depending on whether $q \equiv 1(mod.4)$, we will divide again our discussion into two different cases.

9.4.1 The value of Υ_F when $q \equiv 1(mod.4)$

Let us recall that $\widehat{\Gamma}_F(W) = \varphi_F^1(\pi_1(W - dimW))$ where φ_F^1 and π_1 where introduced in section 7.2

Let us start our discussion by obtaining the value of $\widehat{\Gamma}_F(l(z))$ for $z \in F^*/(F^*)^2$, where here $l(z)$ means

$$A^* \overset{N_{red}}{\longrightarrow} F^* \overset{l(z)}{\longrightarrow} \{\pm 1\}$$

Now, we know that $\pi_1(l(z)-1) = \pi_1(p(z)) = p(z)$ and therefore $\widehat{\Gamma}_F(l(z)) = W_F(l(z)) \in \mu_4$.

At this point, as done in previous chapters, we divide our study into three different cases depending on the quadratic extension over F considered, i.e., depending on if χ, the character that induces σ, is defined on $F(X)^*$, $F(Y)^*$ or $F(XY)^*$.

The case of $\chi : F(X)^* \longrightarrow \mathbf{C}^*$

In this case $det(\pi(\sigma)) = SW_1(\pi(\sigma)) = x_1$ by Theorem 9.2.1. And let us note that $SW_1(l(a)) = x_1$. Therefore, if we consider $\pi(\sigma) \oplus l(a)$ and use Cartan's formula for SW_1 (additive version), we obtain

$$SW_1(\pi(\sigma) \oplus l(a)) = x_1 + x_1 = 2x_1 \equiv 0(mod.2)$$

Therefore, the class of $\pi(\sigma)\oplus l(a)$ in Y_F is determined by $SW_2(\pi(\sigma)\oplus l(a))$. Using Cartan's formula for SW_2,

$$SW_2(\pi(\sigma)\oplus l(a)) = SW_2(\pi(\sigma)) + x_1^2 = \left(\frac{W(\pi(\sigma))+1}{2}\right)(x_2^2 + x_1x_2) + x_1^2$$

by Theorem 9.2.1.

Now, we know that $\pi_1(x_1^2) = \pi_1(x_2^2) = 1$ and $\pi_1(x_1x_2) = -1$ (written multiplicatively instead of additively). Hence, the coefficient of x_1x_2 is the only one that provides information. When this coefficient is $0(mod.2)$, $\widehat{\Gamma}_F(\pi(\sigma)\oplus l(a)) = 1$ and when this is $1(mod.2)$, $\widehat{\Gamma}_F(\pi(\sigma)\oplus l(a)) = -1$, namely,

$$\widehat{\Gamma}_F(\pi(\sigma)\oplus l(a)) = (-1)^{\frac{W(\pi(\sigma))+1}{2}} = -W(\pi(\sigma))$$

as $W(\pi(\sigma)) \in \{\pm 1\}$. And hence, as $\widehat{\Gamma}_F(l(a)) = W_F(l(a))$, we can write

$$\widehat{\Gamma}_F(\pi(\sigma)) = -W(\pi(\sigma))W_F(l(a))^{-1} \in \mu_4$$

Here, if we note that the extension considered is $K/F = F(X)/F$, the quadratic character of F^* given by class field theory $w_{K/F}$, is none other than $l(a)$, and we can conlude

$$\widehat{\Gamma}_F(\pi(\sigma)) = -W(\pi(\sigma))W_F(w_{K/F})^{-1} \in \mu_4$$

The case of $\chi : F(Y)^* \longrightarrow \mathbf{C}^*$

In this case $det(\pi(\sigma)) = SW_1(\pi(\sigma)) = x_2 = SW_1(l(b))$ by Theorem 9.2.1. Therefore

$$SW_1(\pi(\sigma)\oplus l(b)) = 2x_2 \equiv 0(mod.2)$$

So now, as in the case above, the class of $\pi(\sigma)\oplus l(b)$ in Y_F is detected by SW_2. The only coefficient that provides information is the coefficient of x_1x_2. This is, by Theorem 9.2.1

$$\frac{q-1}{4} + \frac{W(\pi(\sigma))+1}{2}$$

Hence,

$$\widehat{\Gamma}_F(\pi(\sigma) \oplus l(b)) = (-1)^{(\frac{q-1}{4} + \frac{W(\pi(\sigma))+1}{2})} = (-1)^{\frac{q+3}{4}} W(\pi(\sigma))$$

And as $\widehat{\Gamma}_F(l(b)) = W_F(l(b))$, we obtain

$$\widehat{\Gamma}_F(\pi(\sigma)) = (-1)^{\frac{q+3}{4}} W(\pi(\sigma)) W_F(w_{K/F})^{-1} \in \mu_4$$

since $w_{K/F} = l(b)$ as the extension considered is $K/F = F(Y)/F$.

The case of $\chi : F(XY)^* \longrightarrow \mathbf{C}^*$

Here, $det(\pi(\sigma)) = SW_1(\pi(\sigma)) = x_1 + x_2 = SW_1(l(ab))$ by Theorem 9.2.1, and therefore $SW_1(\pi(\sigma) \oplus l(ab)) \equiv 0(mod.2)$. Now, as in previous cases, the coefficient of x_1x_2 in $SW_2(\pi(\sigma) \oplus l(ab))$ is the only one giving information, this is,

$$\frac{q-1}{4} + \frac{W(\pi(\sigma))+1}{2}$$

and therefore

$$\widehat{\Gamma}_F(\pi(\sigma) \oplus l(ab)) = (-1)^{\frac{q+3}{4}} W(\pi(\sigma))$$

We can conclude then,

$$\widehat{\Gamma}_F(\pi(\sigma)) = (-1)^{\frac{q+3}{4}} W(\pi(\sigma)) W_F(w_{K/F})^{-1} \in \mu_4$$

Now that all the cases have been considered and taking into account that $W(\pi(\sigma)) = W_F(\sigma)$, we can state the main Theorem of this chapter.

Theorem 9.4.2 *Let σ be a two-dimensional symplectic Galois representation, $\sigma = Ind_{K/F}(\chi)$ induced from a character $\chi : K^* \longrightarrow \mathbf{C}^*$, where F is a non-Archimedean local field of odd residual characteristic $q \equiv 1(mod.4)$ and K/F is a quadratic extension, then*

$$\Upsilon_F(\sigma) = (-1)^{\frac{q-1+2e}{2e}} W_F(\sigma) W_F(w_{K/F})^{-1} \in \mu_4$$

where e is the ramification index of the extension and $w_{K/F}$ is the quadratic character of F^* given by class field theory.

Let us study now, the case $q \equiv 3(mod.4)$.

9.4.2 The value of Υ_F when $q \equiv 3(mod.4)$

Let us recall that in this case $\widehat{\Gamma}_F(W) = \varphi_F^3(\pi_3(W - dimW))$, where φ_F^3 and π_3 were defined in section 7.3

As for $q \equiv 1(mod.4)$,we divide our discussion in three different cases depending on the quadratic extension over F considered, i.e., depending on if χ, the character that induces σ, is defined on $F(X)^*$, $F(Y)^*$ or $F(xY)^*$.

The case of $\chi : F(X)^* \longrightarrow \mathbf{C}^*$

In this case $det(\pi(\sigma)) = SW_1(\pi(\sigma)) = x_1$ by Theorem 9.3.1. And we know that $SW_1(l(-1)) = x_1$. Therefore, if we consider $\pi(\sigma) \oplus l(-1)$ we obtain for SW_1

$$SW_1(\pi(\sigma) \oplus l(-1)) = x_1 + x_1 = 2x_1 \equiv 0(mod.2)$$

Hence, the class of $\pi(\sigma) \oplus l(-1)$ in Y_F is detected by SW_2.

$$SW_2(\pi(\sigma) \oplus l(-1)) = SW_2(\pi(\sigma)) + x_1^2 = \left(\frac{W(\pi(\sigma)) + (-1)^{f-1}}{2}\right) w_2 + f x_1^2$$

by Theorem 9.3.1, where $f = f(\chi)$.

Now, as $q \equiv 3(mod.4)$, we know that $\pi_3(x_1^2) = l(-1) \cup l(-1) = 1$, $\pi_3(x_2^2) = l(-1) \cup l(b) = -1$. Furthermore, $\pi_3(w_2) = SW_2(Ind_{L/F}(\widetilde{\lambda})) = W_L(\widetilde{\lambda}) \in \{\pm 1\}$. Therefore,

$$\widehat{\Gamma}_F(\pi(\sigma) \oplus l(-1)) = W_L(\widetilde{\lambda})^{\frac{W(\pi(\sigma))+(-1)^{f-1}}{2}}$$

and we can conclude

$$\widehat{\Gamma}_F(\pi(\sigma)) = W_L(\widetilde{\lambda})^{\frac{W(\pi(\sigma))+(-1)^{f-1}}{2}} W_F(l(-1))^{-1}$$

The case of $\chi : F(Y)^* \longrightarrow \mathbf{C}^*$

Here, $det(\pi(\sigma)) = SW_1(\pi(\sigma)) = x_1 + x_2 = SW_1(l(-b))$ using Theorem 9.3.1. Therefore, if we consider $\pi(\sigma) \oplus l(-b)$, $SW_1(\pi(\sigma) \oplus l(-b)) \equiv 0(mod.2)$, and the class of $\pi(\sigma) \oplus l(-b)$ in Y_F is detected by SW_2.

From Theorem 9.3.1, one can see that

$$SW_2(\pi(\sigma)\oplus l(-b)) = \frac{q+1}{4}w_2 + \left(\frac{q+1}{4} + \frac{W(\pi(\sigma))+1}{2}\right) x_1^2 + \left(\frac{1+W(\pi(\sigma))}{2}\right) x_2^2$$

and therefore

$$\widehat{\Gamma}_F(\pi(\sigma) \oplus l(-b)) = W_L(\widetilde{\lambda})^{\frac{q+1}{4}}(-1)^{\frac{1+W(\pi(\sigma))}{2}} = -W_L(\widetilde{\lambda})^{\frac{q+1}{4}} W(\pi(\sigma))$$

and we can conclude

$$\widehat{\Gamma}_F(\pi(\sigma)) = -W_L(\widetilde{\lambda})^{\frac{q+1}{4}} W(\pi(\sigma)) W_F(l(-b))^{-1}$$

The case of $\chi : F(xY)^* \longrightarrow \mathbf{C}^*$

To reach our aim, we study now the case of $K = F(xY)$. In this case, from Theorem 9.3.1, $det(\pi(\sigma)) = SW_1(\pi(\sigma)) = x_2 = SW_1(l(b))$. Therefore, $SW_1(\pi(\sigma) \oplus l(b)) \equiv 0(mod.2)$.

Now, using Theorem 9.3.1 again,

$$SW_2(\pi(\sigma) \oplus l(b)) = \frac{q+1}{4}(w_2 + x_1^2) + \left(\frac{W(\pi(\sigma))+1}{2}\right) x_2^2$$

and hence,

$$\widehat{\Gamma}_F(\pi(\sigma) \oplus l(b)) = -W_L(\widetilde{\lambda})^{\frac{q+1}{4}} W(\pi(\sigma))$$

and we can conclude

$$\widehat{\Gamma}_F(\pi(\sigma)) = -W_L(\widetilde{\lambda})^{\frac{q+1}{4}} W(\pi(\sigma)) W_F(l(b))^{-1}$$

In this case, $\Upsilon_F(\sigma)$ does not provide much information about $W_F(\sigma)(=W(\pi(\sigma)))$ as it would be desired. For instance, in the unramified case, if $W_L(\widetilde{\lambda}) = 1$, $\Upsilon_F(\sigma) = W_F(l(-1))^{-1}$, where $W_F(\sigma)$ does not appear. Although this case may lack of significance for the current study, it has been introduced for completeness.

9.5 Some easy applications of Theorem 9.4.2

In this section we concentrate on some easy applications of Theorem 9.4.2, this is when $q \equiv 1(mod.4)$

To begin with, $\Upsilon_F(\sigma)$ can be used to obtain $W_K(\chi)$ where χ is the character σ is induced from, i.e., $\sigma = Ind_{K/F}(\chi)$.

Then, we will use Theorem 9.4.2 to give relations between $\Upsilon_F(\sigma)$ and $\Upsilon_F(\sigma')$.

When σ is a two-dimensional symplectic representation, $\sigma = Ind_{K/F}(\chi)$, Number theorists consider what is called the tame ramification of σ and denoted by σ_{tame}, where σ_{tame} is induced from a tamely ramified character. Due to the difficulty of calculating $W_F(\sigma)$, it is helpful to have a way of obtaining this local root number in terms of $W_F(\sigma_{tame})$. This will be our third application.

The value of $W_K(\chi)$

Let us consider $\sigma = Ind_{K/F}(\chi)$ a two-dimensional symplectic representation induced from a character $\chi : K^* \longrightarrow \mathbf{C}^*$. Using inductivity in dimension zero for the local root number of σ we obtain,

$$W_F(\sigma) = W_K(\chi)W_F(Ind_{K/F}(1)) = W_K(\chi)W_F(1+w_{K/F}) = W_K(\chi)W_F(w_{K/F})$$

and now using Theorem 9.4.2

$$\Upsilon_F(\sigma) = \Upsilon_F(\sigma) = (-1)^{\frac{q-1+2e}{2e}} W_F(\sigma)W_F(w_{K/F})^{-1} = (-1)^{\frac{q-1+2e}{2e}} W_K(\chi)$$

namely

Corollary 9.5.1 *($q \equiv 1(mod.4)$) In the situation above,*

$$W_K(\chi) = (-1)^{\frac{q-1+2e}{2e}} \Upsilon_F(\sigma)$$

Note that when E/F is ramified and $\sigma = Ind_{E/F}(\theta)$ with $f(\theta) = 1$, χ in this corollary will be the one such that $\sigma = Ind_{K/F}(\chi)$ with K/F unramified following Proposition 9.1.2 (Proof).

Relations between $\Upsilon_F(\sigma)$ and $\Upsilon_F(\sigma')$

Corollary 9.5.2 *($q \equiv 1(mod.4)$) Let σ and σ' be two two-dimensional symplectic Galois representations, $\sigma = Ind_{K/F}(\chi)$ and $\sigma' = Ind_{K/F}(\chi')$ induced from characters χ,$\chi' : K^* \longrightarrow \mathbb{C}$. Then, the following holds,*

(I) If K/F is ramified with $f(\chi) \neq f(\chi')$ and $min(f(\chi), f(\chi')) = 1$ (it is no restriction to assume $f(\chi') = 1$) then,

$$\frac{\Upsilon_F(\sigma)}{\Upsilon_F(\sigma')} = (-1)^{\frac{q-1}{4}} \frac{W_F(\sigma)}{W_F(\sigma')} \frac{W_F(w_{K'/F})}{W_F(w_{K/F})}$$

where K'/F is the unique unramified quadratic extension over F.

(II) In any other case,

$$\frac{\Upsilon_F(\sigma)}{\Upsilon_F(\sigma')} = \frac{W_F(\sigma)}{W_F(\sigma')}$$

Proof:

(I) As stated above, we will assume $f(\chi') = 1$. Then, following Proposition 9.1.2 (Proof) σ' may be written as $\sigma' = Ind_{K'/F}(\theta)$ with K'/F unramified and $f(\theta) = 1$. And the above equality follows from Theorem 9.4.2.

(II) Follows trivially from Theorem 9.4.2 □

The tame ramification of σ

As in the above case, we consider $\sigma = Ind_{K/F}(\chi)$. Now, χ can be written as tensor product

$$\chi = \chi_1 \otimes \chi_2$$

where χ_1 is of order u and χ_2 is of order p^v with $(u, p) = 1$, where the residual characteristic of F is $q = p^d$ for some positive integer d.

Therefore, χ_1 is tamely ramified, and we can consider

$$\sigma_{tame} = Ind_{K/F}(\chi_1)$$

Now using Corollary 9.5.2 we have the following result,

Corollary 9.5.3 *($q \equiv 1(mod.4)$) In the situation above,*

$$\frac{\Upsilon_F(\sigma)}{\Upsilon_F(\sigma_{tame})} = (-1)^{\frac{(e-1)(q-1)}{4}} \frac{W_F(\sigma)}{W_F(\sigma_{tame})} \frac{W_F(w_{K'/F})}{W_F(w_{K/F})}$$

where K'/F is the unique unramified quadratic extension over F and e is the ramification index of the extension K/F.

Note that, although $q \equiv 1(mod.4)$ is the case dealt with, and this could seem rather restrictive, most q's will be in this case. Just recall that if $q = p^d$ for some positive integer d, then $q \equiv 1(mod.4)$ if $p \equiv 1(mod.4)$ or $p \equiv 3(mod.4)$ and d even.

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