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Classicalness and the Hausdorff Dimension of Limit sets of Divergent sequences of genus two Schottky Groups

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ABSTRACT

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CLASSICALNESS AND THE HAUSDORFF DIMENSION OF LIMIT SETS OF DIVERGENT SEQUENCES OF GENUS TWO SCHOTTKY GROUPS

by Thomas Higgens

We use Patterson-Sullivan Theory to show that there is a universal lower bound on the Hausdorff dimension of a genus two non-classical Schottky group. This provides a partial complement to the question by Schottky, that was answered by Doyle, that states that there is a universal upper bound on the Hausdorff dimension of a classical Schottky group.

Contents

List of Figures

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Chapter 1

Introduction and Basic Definitions

1.1 Introduction

Every compact Riemann Surface can be uniformised by a Schottky group, so to study the space of Riemann Surfaces we can study Schottky space, see Bers [Ber61]. They are also an extremely nice example of a geometrically finite Kleinian groups.

Schottky groups are naturally classified as to when their defining curves can be circles or not. If a Schottky group has defining curves which are circles then it is classical, otherwise we say it is non-classical. Marden [Mar74] was the first to show that non-classical Schottky groups do in fact exist. Yamamoto [Yam91] gave an explicit example of a family of non-classical Schottky groups, in Chapter 4 we generalise this family to give a family of non-classical Schottky groups that leaves $PSL_2(\mathbb{C})$.

The Hausdorff dimension of the limit set of a Kleinian group has been of interest for many years. It was studied by Ahlfors and his conjecture that the limit set of a finitely generated Kleinian group is either $\widehat{\mathbb{C}}$ or has 0 2-dimensional Hausdorff measure is still open.

Patterson [Pat76a], [Pat76b] in the Fuchsian case and Sullivan [Sul79], [Sul84] in the Kleinian case studied the connection between the exponent of convergence of the Poincaré series and the Hausdorff dimension of the limit set. We use these results in an essential way when finding bounds on the Hausdorff dimension of the limit set of a Schottky group.

Bowen [Bow79] characterised the Hausdorff dimension of the limit set of a Fuchsion or Schottky group in terms of the Pressure function. Ruelle [Rue82] used this fact and his analysis of the dynamical zeta function to show that the Hausdorff dimension of the limit set is a real analytic function over the appropriate deformation space.

An old question (from Schottky) that was answered by Doyle [Doy88] states that there is a universal upper bound on the Hausdorff dimension of the limit set of a classical Schottky group.

We provide a partial complement to this result and prove that there is a universal positive lower bound on the Hausdorff dimension of the limit set of a non-classical genus two Schottky group.

We prove the above result by contradiction. We show that if a sequence of Schottky groups has Hausdorff dimension tending to 0 then given any set of generators at least one of the sequence of generators must converge. So to study sequences of Schottky groups that might have Hausdorff dimension $\to 0$ we study divergent sequence of genus two Schottky groups. We classify what can happen to such a sequence in terms of its generators. In each case we show that either the sequence is eventually classical or there is a lower bound on the Hausdorff dimension in the limit.

Key to showing that the groups are eventually classical is the result that we can take the centres of the isometric circles of the groups "close" relative to the multiplier of the other generator, see Lemma 6.1.3. From this result it is a case of checking that the groups are indeed classical.

Finding lower bounds on the Hausdorff dimension of the limit sets is more sophisticated.

The main result is a refinement of the Poincaré Series, see Lemma 5.4.20, that allows us to sum over a set that reflects the "density" of the generators in the limit set. The technical result, Lemma 5.4.11, allows us to control the growth rate of this set. This result allows us to bound the exponent of convergence of the Poincaré series.

This thesis is split into six chapters.

The first, this one, is composed of two parts; an overview of the thesis and a section on the basic notation we will use.

In the second chapter we introduce standard results on Möbius transformations and Kleinian groups. The main theme we shall concentrate on is the relationship between the ball model and the upper half space model of hyperbolic space $\mathbb{H}^3$. There are three results we use repeatedly. Namely, that stereographic projection restricted to a closed subset of $\overline{B_1 0} - j$ is a bi-Lipshitz map and that it induces a homomorphism from Mob$(B_1 0)$ to Mob(H^3) via conjugation. We will often use the useful equality that given a Möbius transformation γ then

$$|\gamma(z) - \gamma(w)| = |\gamma'(z)|^{\frac{1}{2}} |\gamma'(w)|^{\frac{1}{2}} |z - w|$$

for $z, w \in \widehat{\mathbb{C}}$. Lastly we study what can happen to a sequence of loxodromics, in particular the manner in which they can diverge. This will be useful when we study divergent sequences of Schottky groups in Chapter 6.

Chapter three is composed of the definition of a Schottky group Γ, the notation we will use and the main known results. It is worth noting that $\mathbb{H}^3/\Gamma$ is a handlebody, that $\Omega(\Gamma)/\Gamma$ is a Riemann Surface and that every Riemann Surface is uniformised by a Schottky group. The main results we use are that the limit set of a Schottky group is a cantor set homeomorphic to its shift space. We also discuss Schottky space and its boundary.

In Chapter four we look at the different types of Schottky groups and give a family of non-classical Schottky groups for which a subsequence diverges. This family is an example of a sequence of Schottky groups which diverge but that the Hausdorff dimension of their limit sets does not vanish.

In Chapter five we introduce the two main measures, Hausdorff measure and Patterson-Sullivan measure, on the limit set of a Kleinian group. We state the important result that for a convex cocompact group they are the same up to multiplication by a constant. In fact for a geometrically finite group the Hausdorff dimension and exponent of convergence of the group coincide. We use this to show that if the Hausdorff dimension of the limit sets of a sequence of Schottky groups vanishes then the groups diverge. Chapter 6 will be spent showing to what extent the converse of this statement holds. In the second part of this chapter we show that when calculating the exponent of convergence of the Poincaré series you only need to sum over a certain subset of Γ and not the whole group. This result will allow us to give estimates for the limits of the Hausdorff dimension of various sequences of Schottky groups.

Chapter six contains the bulk of the calculations. We consider a divergent sequence of Schottky groups $< g(n), h(n) >$, where $g(n)$ diverges. We then split the problem into various cases depending on what $h(n)$ converges to and how $g(n)$ diverges. For each case we discuss on what conditions the sequence is eventually classical and whether the Hausdorff dimension of the limit sets vanish. We use this classification to show that there is a lower bound on the Hausdorff dimension of a non-classical Schottky group although we do not find an explicit bound.

1.2 Basic Definitions

In this Section we introduce the basic topological and notational conventions we will use.

Definition 1.2.1 For any $n \geq 1$ we denote the **one point compactification** of $\mathbb{R}^n$ by $\widehat{\mathbb{R}}^n$.

Definition 1.2.2 The Riemann Sphere $\mathbb{C} \cup \{\infty\}$ is denoted by $\widehat{\mathbb{C}}$.

In this thesis we will often use a generalised metric, this is a very natural notion when considering path metric spaces. In a generalised metric we allow the distance between any two points to be infinite. We immediately see that $\widehat{\mathbb{R}}^n$ and $\widehat{\mathbb{C}}$ are examples of spaces with generalised metrics. We will keep the normal norm notation $|\cdot|$ and allow it to take values in $[0, \infty]$. As usual, to allow arithmetical operations we define $|\infty - \infty|$ to be 0. It still makes sense to talk about divergence in a generalised metric, for instance in $\widehat{\mathbb{C}}$ it just means convergence to ∞. We restrict the use of divergence to spaces such as $PSL_2(\mathbb{C})^g$ where divergence cannot include convergence (to infinity). In spaces such as $\widehat{\mathbb{R}}^n$ and $\widehat{\mathbb{C}}$ we will talk about convergence to ∞.

We will need to discuss the distances between sets, to do this we make the following definition.

Definition 1.2.3 Given sets $X, Y \subset Z$ we let

$$\operatorname{dist}_{\sup}(X, Y) = \sup_{x \in X, y \in Y} |x - y|$$

and

$$\operatorname{dist}_{\inf}(X, Y) = \inf_{x \in X, y \in Y} |x - y|.$$

We should note that except in very special circumstance neither of these is a metric on the space of all subsets of Z.

Definition 1.2.4 Let $B_r(x) = \{y \in \widehat{\mathbb{R}}^3 | |x - y| < r\}$ denote an **open ball**.

Definition 1.2.5 Given any set E in a topological space, let $\overline{E}$ denote its closure and $int(E)$ its interior.

Definition 1.2.6 We let $j \in \mathbb{R}^3$ be the point $(0, 0, 1)$.

Definition 1.2.7 We let 0 denote the points $(0, 0)$ and $(0, 0, 0)$.

Chapter 2

Hyperbolic space and Kleinian groups

In this chapter we look at the relationship between the ball model and the upper-half space model of Hyperbolic space $\mathbb{H}^3$. In both cases the isometries $\mathrm{Isom}(\mathbb{H}^3)$ of $\mathbb{H}^3$ are restrictions of Möbius transformations of $\widehat{\mathbb{R}}^3$, to $B_1(0)$ in the ball model and $H^3 = \{(x, y, z) | z > 0\}$ in the upper half space model. A homomorphism from $\mathrm{Mob}(B_1(0))$ to $\mathrm{Mob}(H^3)$ is induced by conjugation by stereographic projection so it makes sense to study this map. We show that stereographic projection is bi-Lipshitz away from j and ∞. The sphere at infinity S^∞ of $\mathbb{H}^3$ is $S_1(0)$ in the ball model and $\widehat{\mathbb{C}}$ in the upper-half space model. One of the reasons the ball model is useful is that the metric on $S_1(0)$ is compact while $\widehat{\mathbb{C}}$ has a generalised metric. However the action of $\mathrm{Isom}^+(\mathbb{H}^3)$ extends to S^∞ in a particularly nice way in the upper-half space model; the set of orientation preserving Möbius transformations that preserve

H^3 $\text{Mob}^+(H^3)$ is homomorphic to $PSL_2(\mathbb{C})$ and acts on $\widehat{\mathbb{C}}$ by

$$\begin{pmatrix} \pm a & \pm b \\ \pm c & \pm d \end{pmatrix}(z) = \frac{az+b}{cz+d}$$

which is obviously well defined.

In this Chapter we essentially follow the above discussion. Firstly we introduce Möbius transformations and a useful Lemma concerning them. Then we define the two models. We next give an explicit formula for stereographic projection that allows us to show that it is bi-Lipshitz when restricted to compact sets not containing j. The next step is to analyse the action of $PSL_2(\mathbb{C})$ on $\widehat{\mathbb{C}}$.

We then classify types of orientation preserving isometries of $\mathbb{H}^3$ in terms of their fixed points. We introduce the convex hull of a subset of $\mathbb{H}^3 \cup \widehat{\mathbb{C}}$. We then look at types of Kleinian groups and the objects associated to them.

Lastly we look at sequences of loxodromics that converge pointwise to some function.

2.1 Möbius Transformations

In this Section we give the definition of a Möbius Transformations and state a useful Lemma relating the Jacobian of a Möbius Transformations to the distance it moves points.

Definition 2.1.1 A **generalised sphere** of $\widehat{\mathbb{R}}^3$ is either a sphere $S_r(x) = \{y \in \mathbb{R}^3 | |x-y| = r\}$ or a plane union infinity $P_r(x) = \{y \in \mathbb{R}^3 | y \cdot x = r\} \cup \{\infty\}$.

We see that a generalized sphere is a topological sphere.

Definition 2.1.2 A **reflection** in a generalized sphere S is the map

$$v \mapsto \begin{cases} \frac{r^2(v-u)}{|v-u|^2} + u & \text{if} \quad S = S_r(u) \\ v + 2(r - v.u)u & \text{if} \quad S = P_r(u) \end{cases}$$

which is the normal plane reflection if $S = P_r(u)$. If $S = P_r(u)$ then ∞ is fixed and if $S = S_r(u)$ then $\infty \mapsto u$ and $u \mapsto \infty$.

Definition 2.1.3 A **Möbius transformation** $\gamma : \widehat{\mathbb{R}}^3 \to \widehat{\mathbb{R}}^3$ is a composition of reflections in finitely many generalised spheres. We let $\mathrm{Mob}(\widehat{\mathbb{R}}^3)$ denote the group of all such transformations.

The Jacobian of a Möbius Transformations is orthogonal up to multiplication by a constant see [Rat94] so we make the following definition.

Definition 2.1.4 Given a Möbius transformation γ then we define the **conformal dilation** at (x, y, z) to be the number $|\gamma'((x, y, z))|$ such that

$$\frac{1}{|\gamma'((x, y, z))|}\gamma'((x, y, z))$$

is orthogonal where γ' is the Jacobian of γ.

The conformal dilation is related to the distance that γ moves points in the following nice way.

Lemma 2.1.5 *Given a Möbius transformation γ we have that*

$$|\gamma(u) - \gamma(v)| = |\gamma'(u)|^{1/2}|\gamma'(v)|^{1/2}|u - v|$$

for $u, v \in \widehat{\mathbb{R}}$.

For a proof of this see [Nic89].

Möbius transformations are differentiable conformal homeomorphisms. A Möbius transformation is orientation preserving if it is a composition of reflections in an even number of distinct spheres. We let $\mathrm{Mob}^+(\widehat{\mathbb{R}}^3)$ be the group of all orientation preserving Möbius transformations.

In a similar manner we make the following definition.

Definition 2.1.6 Given $E \subset \widehat{\mathbb{R}}^3$ let $\mathrm{Mob}(E)$ be the set of Möbius transformations that preserve E, so $\mathrm{Mob}(E) = \{\gamma \in \mathrm{Mob}(\widehat{\mathbb{R}}^3) | \gamma(E) = E\}$. Define $\mathrm{Mob}^+(E)$ similarly.

We now look at the set on which γ acts as an Euclidean isometry.

Definition 2.1.7 Given $\gamma \in \mathrm{Mob}(\widehat{\mathbb{R}}^3)$ that does not fix infinity then the **isometric sphere** S_γ of γ is the unique sphere on which γ acts as an Euclidean isometry. For existence see p.117 - 120 [Rat94].

It is worth pointing out that the image, under γ, of the isometric sphere of γ is the isometric sphere of γ^{-1}.

Definition 2.1.8 Given $\gamma \in \mathrm{Mob}(B_1((0,0,0)))$ that does not fix infinity then the intersection of the isometric sphere S_γ with $S_1((0,0,0))$ is the **isometric circle** I_γ of γ.

2.2 The models of hyperbolic Space

We first define 3–dimensional hyperbolic space.

Definition 2.2.1 Hyperbolic 3–space $\mathbb{H}^3$ is the unique simply connected complete Riemannian manifold of constant curvature -1.

2.2.1 The models

Definition 2.2.2 We define the **unit ball model** of $\mathbb{H}^3$ to be $B_1(0)$ with the metric defined by

$$d_{\mathbb{H}}(p,q) = \inf_{\alpha} \int_{\alpha} \frac{2}{1-|x|^2}|dx|$$

where $p, q \in B_1(0)$ and the infimum is taken over all differentiable paths α in $B_1(0)$ from p to q.

Definition 2.2.3 We define the **upper-half space model** of $\mathbb{H}^3$ to be $H^3 = \{(x,y,z)|z > 0\}$ with the metric defined by

$$d_{\mathbb{H}}(p,q) = \inf_{\alpha} \int_{\alpha} \frac{1}{z}|dr|$$

where $p, q \in B_1(0)$ and the infimum is taken over all differentiable paths α in H^3 from p to q.

That these are equivalent spaces and that $\mathrm{Isom}(\mathbb{H}^3)$, $\mathrm{Mob}(B_1(0))$ and $\mathrm{Mob}(H^3)$ are all isomorphic can be found in [Rat94].

2.2.2 The boundaries of the models

The boundary of $\mathbb{H}^3$ is sometimes called the **sphere at infinity** or **visual boundary** S^{∞}.

The boundary of $B_1(0)$ is $S_1(0)$ and the action of $\mathrm{Mob}(B_1(0))$ extends naturally. This model is useful as it is conformal and the metric on the boundary of $B_1(0)$ is not a generalised metric.

The boundary of H^3 is $\widehat{\mathbb{C}}$ and the action of $\mathrm{Mob}^+(H^3)$ can be defined in the following nice way as $\mathrm{Mob}^+(H^3)$ is isomorphic to $PSL_2(\mathbb{C})$ as topological groups see [Rat94].

Definition 2.2.4 We define an action of $PSL_2(\mathbb{C})$ on $\widehat{\mathbb{C}}$ by, given $\gamma \in PSL_2(\mathbb{C})$ such that

$$\gamma = \begin{pmatrix} \pm a & \pm b \\ \pm c & \pm d \end{pmatrix}$$

and $z \in \widehat{\mathbb{C}}$ then

$$\gamma(z) = \begin{cases} \frac{az+b}{cz+d} & z \in \mathbb{C} \\ \frac{a}{c} & z = \infty \end{cases}$$

this is obviously well defined.

We can restrict Lemma 2.1.5 to $\widehat{\mathbb{C}}$ in the following way.

Lemma 2.2.5 *Given $\gamma \in PSL_2(\mathbb{C})$ and $z, w \in \mathbb{C}$ then*

$$|\gamma(z) - \gamma(w)| = |\gamma'(z)|^{1/2}|\gamma'(w)|^{1/2}|z - w|.$$

Proof: We know

$$\gamma(z) = \frac{az+b}{cz+d} \text{ with } ad - bc = 1 \text{ so } \gamma'(z) = \frac{1}{(cz+d)^2}$$

and

$$\gamma(z) - \gamma(w) = \frac{az+b}{cz+d} - \frac{aw+b}{cw+d} = \frac{z-w}{(cz+d)(cw+d)}$$

So when we take norms and get the result. □

We can extend the above result to $z, w \in \widehat{\mathbb{C}}$ if we allow $|\cdot - \cdot|$ to take values in $\widehat{\mathbb{R}}$ and define $|\infty - \infty|$ to be zero.

Definition 2.2.6 Given $\gamma \in PSL_2(\mathbb{C})$ that does not fix ∞ then the **isometric circle** I_γ of γ is the circle in $\mathbb{C}$ on which γ acts as a Euclidean isometry.

The following Lemma gives an algebraic formula for the isometric circle.

Lemma 2.2.7 *Given $\gamma \in PSL_2(\mathbb{C})$ that does not fix ∞ then*

$$I_\gamma = \{z \in \mathbb{C} | |cz + d| = 1\}$$

where I_γ is the isometric circle of γ.

Proof: By Lemma 2.2.5 we see

$$|\gamma(z) - \gamma(w)| = |\gamma'(z)|^{1/2} |\gamma'(w)|^{1/2} |z - w|$$

so γ acts on the set $\{z | |\gamma'(z)| = 1\}$ as a Euclidean isometry. But $|\gamma'(z)| = 1$ iff

$$|\gamma'(z)| = \frac{1}{|cz + d|^2} = 1$$

which is a circle and therefore the isometric circle. □

2.2.3 Stereographic projection

Definition 2.2.8 Stereographic projection is the map ϕ from the unit ball $B_1(0)$ to upper half space H^3 defined by

$$\phi(w) = \left(\frac{2x}{x^2 + y^2 + (z-1)^2}, \frac{2y}{x^2 + y^2 + (z-1)^2}, \frac{2(1-z)}{x^2 + y^2 + (z-1)^2} - 1 \right)$$

and has inverse defined by

$$\phi^{-1}(w) = \left(\frac{2x}{x^2 + y^2 + (z+1)^2}, \frac{2y}{x^2 + y^2 + (z+1)^2}, \frac{2(-1-z)}{x^2 + y^2 + (z+1)^2} + 1 \right)$$

where $w = (x, y, x)$.

Stereographic projection is a Möbius transformation of $\widehat{\mathbb{R}}^3$ see [Rat94].

Lemma 2.2.9 *The inverse of stereographic projection restricted to a bounded subset $X \subset H^3 \cup \mathbb{C}$ is bi-Lipshitz with constants*

$$K = \max_{w \in X} \frac{2}{|w + j|^2} \text{ and } K' = \min_{w \in X} \frac{2}{|w + j|^2}$$

where K is the constant corresponding to the upper bound and K' the lower bound.

Proof: The Jacobian of ϕ^{-1} at (x, y, z) is

$$\begin{pmatrix} -2\,\frac{x^2-y^2-z^2-2\,z-1}{(x^2+y^2+z^2+2z+1)^2} & -4\,\frac{x\,y}{(x^2+y^2+z^2+2z+1)^2} & -4\,\frac{x\,(z+1)}{(x^2+y^2+z^2+2z+1)^2} \\ 4\,\frac{x\,y}{(x^2+y^2+z^2+2z+1)^2} & 2\,\frac{x^2-y^2+z^2+2\,z+1}{(x^2+y^2+z^2+2z+1)^2} & -4\,\frac{y\,(z+1)}{(x^2+y^2+z^2+2z+1)^2} \\ 4\,\frac{x\,(z+1)}{(x^2+y^2+z^2+2z+1)^2} & 4\,\frac{y\,(z+1)}{(x^2+y^2+z^2+2z+1)^2} & -2\,\frac{x^2+y^2-z^2-2\,z-1}{(x^2+y^2+z^2+2z+1)^2} \end{pmatrix}$$

so we can calculate $|(\phi^{-1})'((x, y, z))|$ to get

$$|(\phi^{-1})'((x, y, z))| = \frac{2}{(x^2 + y^2 + z^2 + 2z + 1)}.$$

So given $u, v \in X \subset H^3 \cup \mathbb{C}$ then

$$|\phi^{-1}(u) - \phi^{-1}(v)| = |(\phi^{-1})'(u)|^{1/2}|(\phi^{-1})'(v)|^{1/2}|u - v|$$

by Lemma 2.1.5 so define

$$K = \max_{w \in X} \frac{2}{|w + j|^2} \text{ and } K' = \min_{w \in X} \frac{2}{|w + j|^2}$$

both of which are finite and non zero so we have that

$$K'|u - v| \leq |\phi^{-1}(u) - \phi^{-1}(v)| \leq K|u - v|$$

as required. $\square$

2.3 Types of isometries of $\mathbb{H}^3$

We classify elements of $\gamma \in \text{Isom}^+\mathbb{H}^3$ by their fixed points (they have at least one by the Brouwer fixed point theorem as they are homeomorphisms of the closed unit ball, see p.14 [Mil65] for an elegant proof of this.

Given $\gamma \in \text{Isom}^+(\mathbb{H}^3)$ then we say that

- γ is **elliptic** if it fixes at least one point of $\mathbb{H}^3$.
- γ is **loxodromic** if it fixes 2 points of S^∞ and no points of $\mathbb{H}^3$.
- γ is **parabolic** in any other case.

This classification is possible because of the fact that if $\gamma \in \text{Isom}^+(\mathbb{H}^3)$ fixes 3 or more points of S^∞ then γ is the identity see [And99].

If we work in the upper-half space model we can make the following algebraic classification.

Let $\gamma \in PSL_2(\mathbb{C})$ then

- If γ is elliptic then it is conjugate in $PSL_2(\mathbb{C})$ to the map $z \mapsto \lambda z$ where $|\lambda| = 1$. So an elliptic element is characterised by its 2 fixed points and the amount by which it rotates $\mathbb{C}$.
- If γ is loxodromic then it is conjugate to the map $z \mapsto \lambda z$ where $|\lambda| > 1$. So a loxodromic element is characterised by its 2 fixed points and its multiplier λ.
- If γ is parabolic then it is conjugate to the map $z \mapsto z + 1$. In fact a parabolic element is characterised by its fixed point and a constant related to the action of γ on $\mathbb{H}^3$.

We can express γ in terms of these constants, if γ if loxodromic or elliptic with fixed points $x, y \neq \infty$ and multiplier λ then

$$\gamma(z) = \frac{(x - y\lambda)z + xy(\lambda - 1)}{(1 - \lambda)z + x\lambda - y}$$

if γ is parabolic with fixed point $x \neq 0, \infty$ then

$$\gamma(z) = \frac{2z - \tau x}{\frac{1}{x}z}$$

where $\tau \in \mathbb{C}$.

2.4 Convex hulls

We will work in upper half space model for $\mathbb{H}^3$ although the following definitions and results can be made intrinsically.

Definition 2.4.1 Given any set $E \subset \mathbb{H}^3$ then the **convex hull** $\mathcal{C}(E)$ of E is the intersection of all closed convex sets in $\mathbb{H}^3$ containing E. Now suppose that $E \subset \mathbb{H}^3 \cup \widehat{\mathbb{C}}$ then $\mathcal{C}(E)$ is the intersection of all closed convex sets in $\mathbb{H}^3$ whose Euclidean closure contains E.

Example: If $z, w \in \widehat{\mathbb{R}}^2$ then $\mathcal{C}(\{z, w\})$ is the geodesic in $\mathbb{H}^3$ whose end points in $\widehat{\mathbb{R}}^2$ are z and w.

The following lemma is well known but we give its proof for completeness.

Lemma 2.4.2 *Let $E \subset \mathbb{H}^3 \cup \widehat{\mathbb{C}}$ and $\gamma \in \mathrm{Isom}(\mathbb{H}^3)$ then $\gamma(\mathcal{C}(E)) = \mathcal{C}(\gamma(E))$.*

Proof: Closure is taken in $\widehat{\mathbb{R}}^3$.

By definition

$$\gamma(\mathcal{C}(E)) = \gamma \bigcap X \text{ where } X \text{ convex and } E \subset \overline{X}$$

$$= \bigcap \gamma(X) \text{ where } X \text{ convex and } E \subset \overline{X}$$

since γ is a bijection.

$$= \bigcap X' \text{ where } X' \text{ convex and } E \subset \overline{\gamma^{-1}(X')}$$

where we let $X' = \gamma(X)$ and note that γ preserves convexity

$$= \bigcap X' \text{ where } X' \text{ convex and } \gamma(E) \subset \gamma(\overline{\gamma^{-1}(X')})$$

$$= \bigcap X' \text{ where } X' \text{ convex and } \gamma(E) \subset \overline{X'} = \mathcal{C}(\gamma(E))$$

since γ is a homeomorphism. □

2.5 Kleinian groups

Definition 2.5.1 A **Kleinian group** is a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$, where $\mathrm{Isom}^+(\mathbb{H}^3)$ is given the topology of pointwise convergence.

It is worth noting that the following result, Selberg's Lemma [Sel60], applies to any subgroup of $\mathrm{Isom}(\mathbb{H}^3)$.

Lemma 2.5.2 *Given any finitely generated subgroup of* $\mathrm{Isom}(\mathbb{H}^3)$ *then it has a torsion free normal subgroup of finite index.*

We will wish to exclude the most basic type of Kleinian group so we make the following definition.

Definition 2.5.3 A subgroup of $PSL_2(\mathbb{C})$ is **elementary** if every two elements of infinite order in the group share a fixed point in common. Equivalent definitions for a Kleinian group are that its limit set is finite or that it is a virtually abelian group.

We now give a necessary conditions for discreteness.

Lemma 2.5.4 *A non-elementary Kleinian group* $\Gamma =< g, h >$ *satisfies* ***Jørgensen's inequality***

$$|\mathrm{tr}(g)^2 - 4| + |\mathrm{tr}([g, h]) - 2| \geq 1$$

where $\mathrm{tr}(\gamma)$ *is the trace of a lift of* $\gamma \in PSL_2(\mathbb{C})$ *to* $SL_2(\mathbb{C})$ *and* $[g, h]$ *is the commutator of* g *and* h.

This was proved by Jørgensen [Jør76].

2.6 Fundamental domains

Fundamental domains are essential to understand the action of a Kleinian group on $\mathbb{H}^3$.

Definition 2.6.1 A **fundamental domain** for a Kleinian group Γ acting on $\mathbb{H}^3$ is an open set $D \subset \mathbb{H}^3$ with the following properties,

- $\Gamma\overline{D} = \mathbb{H}^3$
- $\gamma D \cap D = \emptyset \ \forall \gamma \in \Gamma - \{id\}$.
- The hyperbolic volume measure of $\partial D = 0$.

A fundamental region "tiles" $\mathbb{H}^3$ under Γ.

Definition 2.6.2 A fundamental domain D is **locally finite** if every compact set intersects only a finite number of translates of D.

We now give an example of a fundamental domain for any Kleinian group.

Definition 2.6.3 Given a Kleinian group Γ and a point $p \in \mathbb{H}^3$ not fixed by any element of Γ then the **Dirichlet domain** with centre p is

$$D(p) = \{q \in \mathbb{H}^3 | d_{\mathbb{H}}(p,q) < d_{\mathbb{H}}(\gamma p, q) \quad \forall \gamma \in \Gamma - id\}.$$

The Dirichlet domain is a locally finite convex fundamental domain bounded by hyperbolic planes meeting along geodesics see p.233-245 [Rat94].

Lemma 2.6.4 *Let Γ be a non-elementary Kleinian group with a locally finite fundamental domain D in the ball model for $\mathbb{H}^3$ then if $\{\gamma_n\} \subset \Gamma$ is a sequence of distinct elements then the hyperbolic distance from $\gamma_n(D)$ to 0 tends to ∞.*

Proof: We prove this by contradiction. Suppose there is a $K > 0$ and $p_n \in D$ such that $d(\gamma(p_n), 0) < K$ for an infinite number of γ_n. So an infinite number of $\gamma_n p_n$ lie in the compact ball $\overline{B_K(0)}$, so some subsequence converges to a point $p \in \mathbb{H}^3$. But then an infinite number of $\gamma_n(p_n)$ lie in the hyperbolic ball $B_K(p)$, so an infinite number of images of D lie in a compact set which violates the fact the collection ΓD is locally finite. $\square$

Lemma 2.6.5 *Let Γ be a non-elementary Kleinian group with a locally finite convex fundamental region D in the ball model for $\mathbb{H}^3$ then if $\{\gamma_n\} \subset \Gamma$ is* *any sequence of distinct elements then the Euclidean diameter* $\mathrm{diam}(\gamma_n(D))$ *of* $\gamma_n(D)$ *tends to* 0.

Proof:

We prove this by contradiction. Suppose that the Euclidean diameter of $\gamma_n(D)$ is greater than K for an infinite number of n. So there are $p_n, q_n \in D$ such that $|\gamma_n p_n - \gamma_n q_n| > K$ for all n.

So a subsequence of p_n converges to $p \in S_1(0)$ and a subsequence of q_n converges to $q \in S_1(0)$, such that $|p - q| \geq K$.

Since D is convex and γ_n is an isometry then $\gamma_n D$ is convex for all n. By the convexity of D the geodesic segment α_n from p_n to q_n is contained in D.

So a subsequence of $\gamma_n \alpha_n$ converges (in the Euclidean Hausdorff topology) to α the geodesic from p to q.

But this contradicts Lemma 2.6.4 as the hyperbolic distance from 0 to $\gamma_n \alpha_n$ goes to ∞ but for large n the hyperbolic distance from 0 to $\gamma_n \alpha_n$ is close to $\mathrm{dist}_{\mathrm{inf}}(0, \alpha)$ which is finite. $\square$

2.7 Spaces associated to a Kleinian group

Associated to a Kleinian group there are lots of topological objects with interesting properties; we introduce the most well known of them.

Definition 2.7.1 The **limit set** $\Lambda(\Gamma)$ of a Kleinian group Γ is the closure of the set of accumulation points in $\widehat{\mathbb{R}}^3$ of Γp for any $p \in \mathbb{H}^3$.

$\Lambda(\Gamma)$ is independent of the p chosen and is contained in $\widehat{\mathbb{C}}$ [Rat94].

Lemma 2.7.2 *The limit set of a non-elementary Kleinian group can be characterised in the following ways:*

- *the closure of the fixed points of all the loxodromic elements of* Γ,
- *if* Γ *contains a parabolic element then* $\Lambda(\Gamma)$ *is the closure of all the fixed points of all the parabolic elements,*
- *the set of accumulation points* Γz *for any* $z \in \widehat{\mathbb{C}}$,
- $\Lambda(\Gamma)$ *is the smallest non-empty closed subset of* $\widehat{\mathbb{C}}$ *invariant under* Γ.

A proof of this and that $\Lambda(\Gamma)$ is perfect can be found in [Rat94].

Definition 2.7.3 The **ordinary set** $\Omega(\Gamma)$ of a Kleinian group Γ is $\widehat{\mathbb{C}} - \Lambda(\Gamma)$.

Definition 2.7.4 Given any group Γ acting on a topological space X then we say that the action is **properly discontinuous** if given any compact set K then $\{\gamma \in \Gamma | \gamma K \cap K \neq \emptyset\}$ is finite.

The largest subset of $\mathbb{H}^3 \cup \widehat{\mathbb{C}}$ on which a Kleinian group Γ acts properly discontinuously is $\mathbb{H}^3 \cup \Omega(\Gamma)$ see p.579-580 [Rat94].

Definition 2.7.5 A **fundamental region** for a Kleinian group Γ acting on $\Omega(\Gamma)$ is an open set $D \subset \Omega(\Gamma)$ with the following properties:

- $\Gamma\overline{D} = \Omega(\Gamma)$,
- $\gamma D \cap D = \emptyset \ \forall \gamma \in \Gamma$,

- The spherical measure of $\partial D = 0$.

Definition 2.7.6 The **hyperbolic manifold** associated to a torsion-free Kleinian group Γ is the space $\mathbb{H}^3/\Gamma$ with the quotient topology.

This is an "extrinsic" definition for an intrinsic definition see p23-25 [MT98] and a general discussion see [Rat94].

Definition 2.7.7 The possibly disconnected **Riemann surface** associated to a torsion-free Kleinian group Γ is the space $\Omega(\Gamma)/\Gamma$.

Definition 2.7.8 The **closed topological manifold** associated to a torsion-free Kleinian group Γ is the space $(\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$.

We can extend the above definitions to Kleinian groups with torsion however the resulting manifolds are no longer smooth and are called orbifolds see chap.13 [Rat94] or [Thu80].

2.8 Types of Kleinian groups

We have already defined a non-elementary Kleinian group.

Definition 2.8.1 A Kleinian group Γ is **analytically finite** if $\Omega(\Gamma)/\Gamma$ is of finite analytic type, in other words $\Omega(\Gamma)/\Gamma$ consists of a finite number of surfaces each of which is of finite genus with only a finite number of punctures.

Ahlfors [Ahl64] showed that every finitely generated Kleinian group is analytically finite.

Definition 2.8.2 A Kleinian group Γ is **geometrically finite** if it has a convex fundamental domain which is a polyhedron which is bounded by a finite number of planes.

There are many other equivalent definitions of geometrically finite, see [Bow93]. Not every finitely generated group is geometrically finite see [Gre66].

Definition 2.8.3 A Kleinian group Γ is **convex cocompact** if the quotient of the convex hull of the limit set of Γ, the **convex core**, is compact in $\mathbb{H}^3/\Gamma$.

The above definition is well defined since $\Lambda(\Gamma)$ is invariant under Γ so $\mathcal{C}(\Lambda(\Gamma))$ is also invariant. An alternative definition is that the intersection of $\mathcal{C}(\Lambda(\Gamma))$ with any locally finite fundamental region for Γ is compact. A non-elementary convex cocompact group contains no parabolic elements see p.57-59 [MT98].

2.9 The Loxodromic

In this Section we collect a selection of formula involving loxodromics which we will use throughout this thesis and classify the limits of loxodromics.

Lemma 2.9.1 *Given γ a loxodromic such that*

$$\gamma = \begin{pmatrix} \pm a & \pm b \\ \pm c & \pm d \end{pmatrix}$$

then

$$\gamma = \begin{pmatrix} \pm\frac{x-y\lambda}{\sqrt{\lambda}(x-y)} & \pm\frac{xy(\lambda-1)}{\sqrt{\lambda}(x-y)} \\ \pm\frac{1-\lambda}{\sqrt{\lambda}(x-y)} & \pm\frac{x\lambda-y}{\sqrt{\lambda}(x-y)} \end{pmatrix}$$

where γ fixes x and y both not ∞ and has multiplier λ. We next look at the Isometric circle of γ

$$\mathrm{rad}I_\gamma = \frac{\sqrt{|\lambda|}|y-x|}{|\lambda-1|}$$

and

$$\mathrm{cen}I_\gamma = \frac{\lambda x - y}{\lambda - 1} \textit{ and } \mathrm{cen}I_{\gamma^{-1}} = \frac{y\lambda - x}{\lambda - 1}.$$

If γ does not fix ∞ then

$$\gamma = \begin{pmatrix} \pm\frac{-w(\lambda+1)}{\sqrt{\lambda}(z-w)} & \pm\frac{(w\lambda+z)(w+z\lambda)}{\sqrt{\lambda}(\lambda+1)(z-w)} \\ \pm\frac{-(\lambda+1)}{\sqrt{\lambda}(z-w)} & \pm\frac{(\lambda+1)z}{\sqrt{\lambda}(z-w)} \end{pmatrix}$$

where $z = \mathrm{cen}I_\gamma$ and $w = \mathrm{cen}I_{\gamma^{-1}}$.

Proof: The first equation is seen by conjugating γ so that its repulsive fixed point is 0 and its attractive fixed point is ∞ then the image of 1 is its multiplier. In fact it is the map $z \mapsto \lambda z$ and on conjugating back we have the above form.

The description of the isometric circles comes from the above formula and Lemma 2.2.7.

The last formula comes from solving for the fixed points in terms of the centres of the isometric circles and substituting this into the first formula. $\square$

We next give a Lemma relating the multiplier of a loxodromic to its isometric circles.

Lemma 2.9.2 *A loxodromic γ has disjoint isometric circles if $|\lambda| > 3+2\sqrt{2}$ where $\lambda = \mathrm{mult}(\gamma)$.*

Proof: Let γ fix x and y then

$$\text{rad}I_\gamma = \frac{\sqrt{|\lambda|}|x-y|}{|1-\lambda|}$$

by Lemma 2.9.1 and

$$\text{cen}I_\gamma = -\frac{x\lambda - y}{1-\lambda} \text{ and } \text{cen}I_{\gamma^{-1}} = \frac{x - y\lambda}{1-\lambda}$$

also by Lemma 2.9.1. We need that

$$|\text{cen}I_\gamma - \text{cen}I_{\gamma^{-1}}| = \frac{|x-y||1+\lambda|}{|1-\lambda|} \geq 2\text{rad}I_\gamma = \frac{2\sqrt{|\lambda|}|x-y|}{|1-\lambda|}$$

which is true if

$$1 \geq \frac{2\sqrt{|\lambda|}}{|1-\lambda|}$$

which is always satisfied for $|\lambda| > 3 + 2\sqrt{2}$. □

We classify what can happen to a sequence of loxodromics.

Lemma 2.9.3 *Suppose that $g(n)$ is a sequence of loxodromics such that the fixed points $x(n)$ and $y(n)$ converge to x and y respectively and $\lambda(n)$ the multiplier converges to λ. Then what $g(n)$ can converge to is collected in the following table.*

Figure 2.1: Limits of Loxodromics

	$\lambda = \infty$	$\|\lambda\| \neq \infty, 1$	$\|\lambda\| = 1$
$x \neq y$	Diverges	Loxodromic	Identity or elliptic
$x = y$	Diverges	Diverges	Diverges or parabolic

Proof: Recall that a sequence $g(n)$ diverges if any of its entries when lifted to a matrix in $SL_2(\mathbb{C})$ diverges, i.e if

$$g(n) = \pm \begin{pmatrix} a(n) & b(n) \\ c(n) & d(n) \end{pmatrix}$$

diverges then at least one of $a(n), b(n), c(n)$ or $d(n)$ diverges.

We shall work through the cases from left to right and top to bottom.

In the first case we look at $c(n)$ which is equal to

$$\frac{1 - \lambda(n)}{\sqrt{\lambda(n)}(x(n) - y(n))}$$

which converges to ∞ unless $|x(n) - y(n)| \to \infty$. However if $|x(n) - y(n)| \to \infty$ we look at $b(n)$ which is

$$\frac{x(n)y(n)(\lambda(n) - 1)}{\sqrt{\lambda(n)}(x(n) - y(n))}$$

which diverges as $\lim_{n\to\infty} \frac{|x(n)y(n)|}{|x(n)-y(n)|} \geq 1$. So in either case we have that at least one of the matrix entries of γ diverges.

In Case 2 we see that $g(n)$ converges to some element of $PSL_2(\mathbb{C})$ since all the matrix entries converge. It can be seen to be loxodromic as it fixes distinct points x and y so it cannot be parabolic. On conjugating so that it fixes 0 and ∞ we see that it has multiplier λ which has absolute value greater that 1 so it cannot be elliptic or the identity.

Case 3 is seen in a similar way to Case 2 we see that the matrix entries converge and so $g(n)$ converges in $PSL_2(\mathbb{C})$. It can be seen to be either elliptic or the identity by conjugating it so that it fixes 0 and ∞.

The bottom left hand case, Case 4 is seen to diverge by considering $c(n)$ which clearly diverges.

Case 5 is also seen to diverge by considering $c(n)$.

We shall now look at Case 6. If $\lambda = \exp(i\theta)$ where $\theta \neq 0$ then $c(n)$ clearly diverges. However if $\lambda = 1$ the rate of convergence comes into play. We can express this by looking at the isometric circles of $g(n)$. For convenience we shall look at the case that $x = y \neq \infty$ the case that $x = y = \infty$ is similar.

Since $x = y \neq \infty$ we have that for large n $g(n)$ does not fix ∞ so we can lift it to

$$\begin{pmatrix} \frac{-w(n)(\lambda(n)+1)}{\sqrt{\lambda(n)}(z(n)-w(n))} & \frac{(w(n)\lambda(n)+z(n))(w(n)+z(n)\lambda(n))}{\sqrt{\lambda(n)}(\lambda(n)+1)(z(n)-w(n))} \\ \frac{-(\lambda(n)+1)}{\sqrt{\lambda(n)}(z(n)-w(n))} & \frac{(\lambda(n)+1)z(n)}{\sqrt{\lambda(n)}(z(n)-w(n))} \end{pmatrix}$$

where $z(n) = \mathrm{cen}I_{g(n)}$ and $w(n) = \mathrm{cen}I_{g(n)^{-1}}$. We shall show that if $|z(n) - w(n)| \not\to 0$ then $g(n)$ converges in $PSL_2(\mathbb{C})$ but this means that it has to converge to a parabolic as it only fixes one point. We have that $\lambda(n) \to 1$ so that

$$\lim g(n) = \lim \begin{pmatrix} \frac{-w(n)2}{(z(n)-w(n))} & \frac{(w(n)+z(n))(w(n)+z(n))}{2(z(n)-w(n))} \\ \frac{-2}{(z(n)-w(n))} & \frac{2z(n)}{(z(n)-w(n))} \end{pmatrix}$$

if $|z(n) - w(n)| \to 0$ then $c(n)$ diverges so $g(n)$ diverges. However if $|z(n) - w(n)| \not\to 0$ then $a(n), c(n)$ and $d(n)$ obviously do not diverge. So we will concentrate on $b(n)$.

$$\lim b(n) = \lim \frac{(w(n) + z(n))^2}{2(z(n) - w(n))}$$

however we have assumed that $|z(n) - w(n)| \not\to 0$ and $w(n) + z(n) = x(n) + y(n)$ by the expressions for the centres of the isometric circles in terms of the fixed points. But we are in the case that $x(n), y(n) \to x \neq \infty$. This means that $b(n)$ converges and so $g(n)$ converges to a parabolic.

To summarise we see that $g(n)$ converges to a parabolic if its multiplier converges to 1, its fixed points converge to each other but the centres of its isometric circles do not converge to each other. $\square$

Chapter 3

Schottky groups

In this chapter we define Schottky groups and the various topological objects associated to them. We also give the some known results concerning them.

3.1 Definition and basic results

Definition 3.1.1 Given a finitely generated group Γ with generating set $\{g_1, \ldots, g_n\}$ we define its **symmetric generating set** $G(\Gamma)$ to be $\{g_1, \ldots, g_n, g_1^{-1}, \ldots, g_n^{-1}\}$.

Definition 3.1.2 Given a finitely generated group Γ with generating set $\{g_1, \ldots, g_n\}$ then $\zeta_1 \ldots \zeta_k$ is a **reduced word** if $\zeta_l \zeta_{l+1} \neq g_i g_i^{-1}, g_i^{-1} g_i$ for all l and i.

Definition 3.1.3 Given any finitely generated group Γ with generating set $\{g_1, \ldots, g_n\}$ we define the **length** $l(\gamma)$ of $\gamma \in \Gamma$ to be the minimal n such that $\gamma = \zeta_1 \ldots \zeta_n$, $\zeta_i \in G(\Gamma)$.

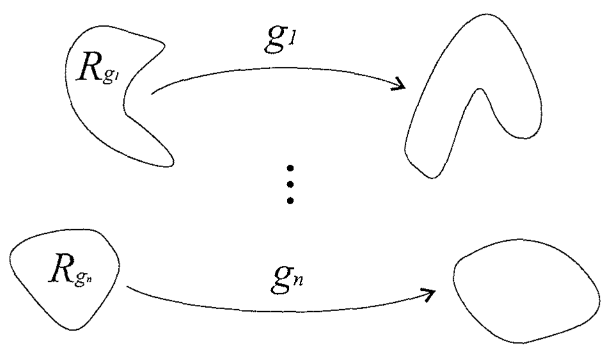
A free group is one with no relations. Given a generating set of a free group then every reduced word is unique.

Definition 3.1.4 A non-elementary sub-group $\Gamma = < g_1, \ldots, g_n >$ of $PSL_2(\mathbb{C})$ is a **Schottky group** if there are $2n$ analytic Jordan curves L_ξ, $\xi \in G(\Gamma)$ that bound an open $2n$ connected region D with the property that

$$\xi(R_\xi) = \widehat{\mathbb{C}} - int(R_{\xi^{-1}}) \qquad \forall \xi \in G(\Gamma)$$

where R_ξ is the closed component of $\widehat{\mathbb{C}} - D$ that contains L_ξ. We say that the curves L_ξ are **defining curves** for the group Γ.

Figure 3.1: Schottky group



The condition that the defining curves are analytic is not too strong, since if we are given Jordan curves paired up by $G(\Gamma)$, we can find analytic curves sufficiently close to the Jordan curves so that these are also paired up by $G(\Gamma)$. We use this fact in Chapter 4 when we construct a family of non-classical Schottky groups.

Chuckrow [Chu68] showed that every set of generators for a Schottky group has associated defining curves.

The lemmas in the rest of this section are well known probably going back to Schottky.

Lemma 3.1.5 *Given a Schottky group Γ with region D as above and $\gamma \in \Gamma - id$ then $\gamma(D) \subset int(R_{\zeta_1^{-1}})$, where $\zeta_1 \ldots \zeta_n$ is a reduced word for γ.*

Proof: We prove this by induction on the length of γ.

Base case: $l(\gamma) = 1$.

So $\gamma \in G(\Gamma)$ so $\gamma(R_\gamma) = \widehat{\mathbb{C}} - int(R_{\gamma^{-1}})$ and $\gamma(D) \subset R_{\gamma^{-1}}$ since $D \cap R_\gamma = \emptyset$.

Inductive step: Assume the result is true for all $\gamma \in \Gamma - id$ such that $l(\gamma) \leq n$.

Let $l(\gamma) = n + 1$ then $\gamma = \zeta_1 \ldots \zeta_{k+1}$ with $\zeta_1 \ldots \zeta_{k+1}$ a reduced word. Then $l(\zeta_2 \ldots \zeta_{k+1}) = n$ so we can apply the inductive hypothesis to say $\zeta_2 \ldots \zeta_{k+1}(D) \subset int(R_{\zeta_2^{-1}})$.

Since $\zeta_1\zeta_2 \neq id$ then $R_{\zeta_1} \cap R_{\zeta_2^{-1}} = \emptyset$ so that $\zeta_1(R_{\zeta_2^{-1}}) \subset int(R_{\zeta_1^{-1}})$ and therefore $\zeta_1 \ldots \zeta_{k+1}(D) \subset int(R_{\zeta_1^{-1}})$. $\square$

Lemma 3.1.6 *Given a Schottky group Γ and $\gamma \in \Gamma - id$ then the attractive fixed point r of γ lies in $R_{\zeta_1^{-1}}$ where $\zeta_1 \ldots \zeta_k$ is a reduced word for γ.*

Proof: The attractive fixed point a is defined by $\lim \gamma^n(z) = a$ for z any point in $\widehat{\mathbb{C}}$ not the repulsive fixed point of γ. By Lemma 3.1.5 we know that $\gamma^m(D) \subset R_{\zeta_1^{-1}}$ for all $m > 0$. So choose $z \in D$ not the repulsive point of γ then

$$a = \lim \gamma^m(z) \in R_{\zeta_1^{-1}}$$

as required. $\square$

Lemma 3.1.7 *Given a Schottky group Γ and $\gamma \in \Gamma$ with γ not fixing ∞ then $cen(I_\gamma)$ lies in R_{ζ_k}, where $\zeta_1 \dots \zeta_k$ is a reduced word for γ.*

Proof: Let

$$\gamma(z) = \frac{az+b}{cz+d}$$

then $cen(I_\gamma) = -d/c$ but $\gamma(cen(I_\gamma)) = \gamma(-d/c) = \infty$ so $cen(I_\gamma) = \gamma^{-1}(\infty) \in \gamma^{-1}(D)$ and by Lemma 3.1.5, we are done. $\square$

In the next four Lemmas we show that a Schottky group is discrete, free, purely loxodromic and $\Omega(\Gamma) \neq \emptyset$. Maskit [Mas67] proved that if a subgroup of $PSL_2(\mathbb{C})$ satisfies these conditions then it is a Schottky group.

Lemma 3.1.8 *A Schottky group is a discrete group*

Proof: We will prove this by contradiction. Given a Schottky group Γ imagine that there is a sequence of distinct elements $\gamma_n \in \Gamma$ such that $\gamma_n \to id$. Pick $z \in D$ then $\gamma_n(z) \to z$ but for large n this means that $\gamma_n(D) \cap D \neq \emptyset$ which contradicts Lemma 3.1.5. $\square$

Lemma 3.1.9 *A Schottky group Γ is a free group.*

Proof: We will prove this by contradiction. Imagine there is a word $\zeta_1 \dots \zeta_k \in \Gamma$ with $\zeta_i \in G(\Gamma)$ and $\zeta_i\zeta_{i+1} \neq id$ but $\zeta_1 \dots \zeta_k = id$. In the proof of Lemma 3.1.5 we only used the fact that $\zeta_i\zeta_{i+1} \neq id$ so that for any sequence $\zeta_1 \ldots \zeta_k$ such that $\zeta_i\zeta_{i+1} \neq id$ we have $\zeta_1 \ldots \zeta_k(D) \subset R_{\zeta_1^{-1}}$ but then $\zeta_1 \ldots \zeta_k(D) \cap D = \emptyset$ so $\zeta_1 \ldots \zeta_k \neq id$. □

In a free group reduced words are unique so from now on we will talk about the reduced word $\zeta_1 \ldots \zeta_k$ for γ.

Recall that a loxodromic or parabolic element γ has 2 fixed points a, r, an attractive one a such that $\gamma^n z \to a$ for all $z \in \widehat{\mathbb{C}} - \{r\}$ and a repulsive one r such that $\gamma^{-n} z \to r$ for all $z \in \widehat{\mathbb{C}} - \{a\}$, for a loxodromic these points are distinct and for a parabolic they coincide.

Lemma 3.1.10 *A Schottky group Γ is purely loxodromic, in other words every element apart from the identity is loxodromic.*

Proof: Since Γ is free we know it is torsion free so any elliptic elements are of infinite order but this violates discreteness.

We need to show that there are no parabolic elements of Γ, it is sufficient to prove that every element $\gamma \in \Gamma$ has distinct fixed points.

Given $\gamma \in \Gamma$ where $\gamma \neq id$ write γ as $\zeta f_1 \ldots f_k \zeta^{-1}$ where $f_1 f_k \neq id$ and $l(f_i) = 1$.

As ζ is a bijection we have that γ has two fixed points iff $\zeta^{-1}\gamma\zeta$ does.

But $(\zeta^{-1}\gamma\zeta)^m(D) \subset R_{f_1^{-1}}$ for all m by Lemma 3.1.5 so we have that the attractive fixed point of $\zeta^{-1}\gamma\zeta$ is in $R_{f_1^{-1}}$ and by the same argument the repulsive fixed point is in R_{f_k} and since these are disjoint the two fixed points must be distinct. □

Lemma 3.1.11 *The ordinary set of a Schottky group* Γ *is non-empty, in fact* $\overline{D} \subset \Omega(\Gamma)$

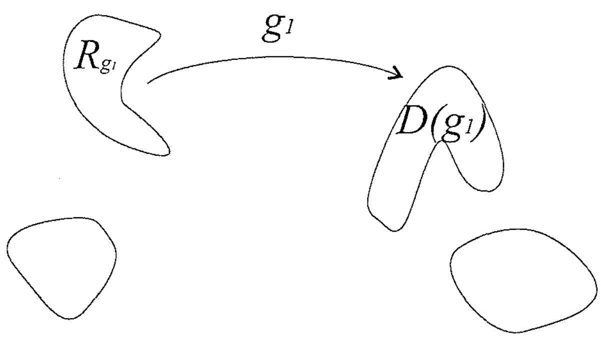
Proof: The limit set of Γ is the closure of all the fixed points of all the elements of Γ. But by Lemma 3.1.5 we see that all the fixed points lie outside D so $D \subset \Omega(\Gamma)$. In fact by changing the defining curves for a Schottky group a little at each point we see that $\overline{D} \subset \Omega(\Gamma)$. $\square$

A Schottky group is convex co-compact so is geometrically finite see p.118 [MT98].

Definition 3.1.12 Given a Schottky group $\Gamma =< g_1, \ldots, g_n >$ and $\gamma \in \Gamma$ we write $\gamma = \zeta_1 \ldots \zeta_k$ as a reduced word and define $D(\gamma) = D(\zeta_1 \ldots \zeta_k) = \zeta_1 \ldots \zeta_{k-1} R_{\zeta_k^{-1}}$.

So for $\zeta \in G(\Gamma)$ we see that $D(\zeta) = R_{\zeta^{-1}}$, we can think of $D(\zeta)$ to be what ζ maps D into and R_ζ to be what ζ maps over D.

Figure 3.2: Domains of generators



We collect the basic facts about $D(\gamma)$ in the following two Lemmas.

Lemma 3.1.13 *Given a Schottky group* Γ *and* $\zeta_1, \zeta_2 \in G(\Gamma)$ *such that* $\zeta_1\zeta_2 \neq id$ *then* $\zeta_1(D(\zeta_2)) \subset D(\zeta_1)$.

Proof: We have that

$$\zeta_1^{-1} D(\zeta_1) = D \cup \bigcup_{\zeta \in G(\Gamma) - \{\zeta_1\}} D(\zeta)$$

and since $\zeta_1\zeta_2 \neq id$ we see that $D(\zeta_2) \subset \bigcup_{\zeta \in G(\Gamma)-\{\zeta_1\}} D(\zeta)$ as required. □

Lemma 3.1.14 *Given a Schottky group* Γ *then:*

1. $\gamma(D) \subset D(\gamma)$,
2. *if* $\infty \in D$ *then* $\gamma(\infty) = \mathrm{cen} I_{\gamma^{-1}} \in D(\gamma)$ *and*
3. *the attractive fixed point of* γ *lies in* $\zeta D(f^m)$ *for all* m, *where* $\gamma = \zeta f \zeta^{-1}$ *and* f *is cyclically reduced, i.e* $l(f^2) = 2l(f)$

for all $\gamma \in \Gamma$.

Proof:

1. Write γ as the reduced word $\zeta_1 \dots \zeta_k$ then $\gamma(D) = \zeta_1 \dots \zeta_k(D)$ and $D(\gamma) = \zeta_1 \dots \zeta_{k-1} D(\zeta_k)$. So $\gamma(D) \subset D(\gamma)$ iff $\zeta_k(D) \subset D(\zeta_k) = R_{\zeta_k^{-1}}$ which is true by Lemma 3.1.5.

2. If $\infty \in D$ then $\gamma(\infty) \in \gamma(D) \subset D(\gamma)$ by 1.

3. Write γ as $\zeta f_1 \dots f_k \zeta^{-1}$ where $l(f_i) = 1$ and $f_1 f_k \neq id$. These two conditions are equivalent to $f = f_1 \dots f_k$ where $l(f^2) = 2l(f)$.

Let a_γ be the attractive fixed point of γ and a_f be the attractive fixed point of f. Then $a_\gamma = \zeta a_f$.

We shall show that $a_f \subset D(f)$ which will prove the Lemma.

If we can show that $fD(f) \subset D(f)$ then $f^n D(f) \subset D(f)$ for all n and so $a_f \subset D(f)$ as required.

Now $fD(f) \subset D(f)$ iff $f_1 \dots f_k D(f) \subset f_1 \dots f_{k-1} D(f_k)$ by definition. This is true if $f_k D(f) \subset D(f_k)$. By the definition of a Schottky group this holds if $D(f) \subset \widehat{\mathbb{C}} - D(f_k^{-1})$. However $D(f) \subset D(f_1)$ and $f_1 \neq f_k^{-1}$ so $D(f_k^{-1}) \cap D(f_1) = \emptyset$ so $D(f)$ is indeed in $\widehat{\mathbb{C}} - D(f_k^{-1})$ which proves the Lemma. $\square$

It is worth noting that given $\gamma \in \Gamma$ then, although $D(\gamma^m)$ converges to the attractive fixed point a of γ, it is not necessarily true that $a \in D(\gamma^m)$. It is this obstruction that means that we cannot include the multiplier of γ in the various bi-Lipshitz inequalities in Section 3.8.

Definition 3.1.15 Given a Schottky group Γ and $\gamma \in \Gamma$ then define $P(\gamma)$ to be the set

$$P(\gamma)\mathbb{H}^3 - int(con.hull(\widehat{\mathbb{C}} - D(\gamma))).$$

Lemma 3.1.16 *Given a Schottky group Γ and $\gamma \in \Gamma$ then $P(\gamma)$ is the union of all closed hyperbolic half spaces whose boundary in S^∞ is contained in $D(\gamma)$.*

Proof: Recall that the convex hull of X is

$$con.hull(X) = \bigcap_{X \subset \partial H} H$$

where H is a closed half space.

For convenience we will work in the upper half space model and will assume that $\infty \in D$, we can conjugate so this is true as conjugation will not affect the statement of the Lemma.

Then

$$\mathbb{H}^3 - int(con.hull(\widehat{\mathbb{C}} - D(\gamma))) = \overline{\mathbb{H}^3 - con.hull(\widehat{\mathbb{C}} - D(\gamma))}$$

$$= \overline{\mathbb{H}^3 - \bigcap_{\widehat{\mathbb{C}} - D(\gamma) \subset \partial H} H} = \overline{\bigcup_{\widehat{\mathbb{C}} - D(\gamma) \subset \partial H} \mathbb{H}^3 - H}.$$

We now let the complements cancel each other out, so the above is equal to

$$\overline{\bigcup_{\partial O \subset D(\gamma)} O}$$

where O is an open half space.

We shall show that

$$\overline{\bigcup_{\partial O \subset D(\gamma)} O} = \bigcup_{\partial H \subset D(\gamma)} H$$

where O and H are open and closed half spaces respectively.

Let $p \in \overline{\bigcup_{\partial O \subset D(\gamma)} O}$ then there are $p_n \in \bigcup_{\partial O \subset D(\gamma)} O$ such that $p_n \to p$. Now each $p_n \in O_n$ where $\partial O_n \subset D(\gamma)$.

Since $D(\gamma)$ is closed then $H_n = \overline{O_n} \subset D(\gamma)$. This means that $p_n \in H_n \subset \bigcup_{\partial H \subset D(\gamma)} H$ for all n. We will have that $p \in \bigcup_{\partial H \subset D(\gamma)} H$ if we can show that this set is closed.

Let $q_n \in \bigcup_{\partial H \subset D(\gamma)} H$ and $q_n \to q \in \mathbb{H}^3$. There are H_n such that $q_n \in H_n$ and $\partial H_n \subset D(\gamma)$, let the radius of ∂H_n be denoted r_n and its centre c_n.

Since $\infty \in D$ and $q \in \mathbb{H}^3$ we have that $r_n \not\to 0, \infty$. Since $D(\gamma)$ is compact we can choose a subsequence n_m such that both c_{n_m} and r_{n_m} converge. This means that ∂H_{n_m} converges to ∂H_∞ where H_∞ is a closed half space. However $\partial H_\infty \subset D(\gamma)$ as $D(\gamma)$ is closed. So $q \in H_\infty \subset \bigcup_{\partial H \subset D(\gamma)} H$ as required.

We now do the other direction. Let $q \in \bigcup_{\partial H \subset D(\gamma)} H$ then $q \in H$ where H is a closed half space and $H \subset D(\gamma)$. Let ∂H have radius r and centre c.

Consider O_n the open half space with centres c and radius $r_n = r - \frac{1}{n}$. Then there are $q_n \in O_n$ such that $q_n \to q \subset \overline{\bigcup_{\partial O \subset D(\gamma)} O}$ and we are done. $\square$

Lemma 3.1.17 *Given a Schottky group Γ and $p \in \mathbb{H}^3$ then $\gamma(p) \in P(\gamma)$ for every $\gamma \in \Gamma$ iff p lies on a hyperbolic plane whose boundary is contained in $\overline{D}$.*

Proof: Let $p \in Q$ where Q is a hyperbolic plane such that $\partial Q \subset \overline{D}$.

Then

$$\gamma(p) \in \gamma(Q) = con.hull(\gamma \partial Q)$$

and as $D(\gamma)$ is simply connected one of the hyperbolic half spaces which has $\gamma(Q)$ as its hyperbolic boundary must have its boundary at infinity contained in $D(\gamma)$ so we are done by Lemma 3.1.16.

Now suppose that $\gamma(p) \in P(\gamma)$ for every $\gamma \in \Gamma$. Then $\zeta(p) \in P(\zeta)$ for every $\zeta \in G(\Gamma)$.

So there are discs $B_\zeta \subset D(\zeta)$ such that $\zeta(p) \in con.hull(B_\zeta)$ for every $\zeta \in G(\Gamma)$ by Lemma 3.1.16.

So that

$$p \in \bigcap_{\zeta \in G(\Gamma)} \zeta^{-1} con.hull(B_\zeta) \subset \bigcap_{\zeta \in G(\Gamma)} \zeta^{-1} P(\zeta) = \bigcap_{\zeta \in G(\Gamma)} \zeta^{-1} \bigcup_{\partial H \subset D(\zeta)} H$$

by Lemma 3.1.16.

Now

$$\begin{aligned} \zeta^{-1} \bigcup_{\partial H \subset D(\zeta)} H &= \bigcup_{\partial H \subset D(\zeta)} \zeta^{-1} H \\ &= \bigcup_{\partial H \subset \zeta^{-1} D(\zeta)} H \end{aligned}$$

as ζ is a bijection.

However $\bigcap_{\zeta \in G(\Gamma)} \zeta^{-1} D(\zeta) = \overline{D}$ so that

$$p \in \bigcap_{\zeta \in G(\Gamma)} \zeta^{-1} \bigcup_{\partial H \subset D(\zeta)} H = \bigcup_{\partial H \subset \overline{D}} H$$

as required. □

Lemma 3.1.18 *Given a Schottky group Γ with $\infty \in D$ and $p \in H$ where H is a hyperbolic plane such that $\partial H \subset D$ then the vertical projection of $\gamma(p)$ to $\mathbb{C}$ is contained in $D(\gamma)$ for every $\gamma \in \Gamma$.*

Proof: This is obvious by Lemma 3.1.17 and Lemma 3.1.16. □

3.2 Types of Schottky groups

In this section we classify Schottky groups in terms of their geometry.

Example: Let $g_1, \dots, g_n$ be loxodromic with isometric circles $I_{g_1}, \dots, I_{g_n}$, $I_{g_1^{-1}}, \dots, I_{g_n^{-1}}$, disjoint and bounding a $2n$ connected region of $\widehat{\mathbb{C}}$. Then $< g_1, \dots, g_n >$ is a Schottky group as $g_j I_{g_j} = I_{g_j^{-1}}$ and $g_j(cenI_{g_j}) = \infty$ for all j.

Definition 3.2.1 A Schottky group Γ is an **isometric** Schottky group if it has some generating set $\{g_1, \dots, g_n\}$ such that the isometric circles of all the $g \in G(\Gamma)$ are disjoint and bound a $2n$ connected region.

Definition 3.2.2 A Schottky group Γ is a **classical** Schottky group if it has a generating set whose defining curves are circles.

So any isometric Schottky group is classical.

Definition 3.2.3 A Schottky group Γ is a **non-classical** Schottky group if it is not classical.

Marden in [Mar74] proved the existence of non-classical Schottky groups and Yamamoto [Yam91] gave an explicit example of one.

Lemma 3.2.4 *The property of being classical is independent of conjugation by elements of* $PSL_2(\mathbb{C})$.

Proof: Let $\Gamma =< g_1, \dots, g_n >$ be a classical Schottky group with defining circles $L_1, \dots, L_{2n}$ and $\phi \in PSL_2(\mathbb{C})$ then $\phi\Gamma\phi^{-1} =< \phi g_1 \phi^{-1}, \dots, \phi g_n \phi^{-1} >$ and has generating curves $\phi L_1, \dots, \phi L_{2n}$, which are circles since elements of $PSL_2(\mathbb{C})$ are conformal. $\square$

3.3 Schottky manifolds

The manifolds associated to a Schottky group are topologically very simple.

Lemma 3.3.1 *Given a Schottky group* Γ *and a sequence of distinct elements* $\gamma_k \in \Gamma$ *then* $\text{diam}(\gamma_k D) \to 0$ *where distance is taken in the ball model.*

Proof: By Lemma 3.1.11 we have that $\partial D \subset \Omega(\Gamma)$. Let P be a convex locally finite fundamental region for the action of Γ on $\Omega(\Gamma) \cup \mathbb{H}^3$. Since Γ acts properly discontinuously on $\Omega(\Gamma) \cup \mathbb{H}^3$ and $con.hull(\overline{D}) \cup \overline{D}$ is compact we have that $con.hull(\overline{D}) \cup \overline{D}$ is contained in the union of a finite number of images of P, $\zeta_1 P, \ldots, \zeta_m P$.

Let α be a geodesic with endpoints in D then $\alpha \subset con.hull(D)$. By Lemma 2.6.4 we know that

$$\text{dist}_{\text{inf}}(0, \gamma_k \bigcup \zeta_i P) \to 0$$

where distance is Euclidean and taken in the ball model. But this means that $\gamma_k(\alpha)$ converges to a point and so $\text{diam}(\gamma_k D) \to 0$ as required. □

Lemma 3.3.2 *If* $\Gamma =< g_1, \ldots, g_n >$ *is a Schottky group then* $\Omega(\Gamma)/\Gamma$ *is a genus* n *surface.*

Proof: We shall prove this by showing that D is a fundamental region for the action of Γ on $\Omega(\Gamma)$. For this we need three conditions to be satisfied:

1. the Lebesgue measure of ∂D is 0,

2. $\gamma(D) \cap D = \emptyset$ for all $\gamma \in \Gamma - \{id\}$,

3. $\Gamma\overline{D} = \Omega(\Gamma)$.

The first comes from the fact that ∂D is a collection of analytic curves, the second from Lemma 3.1.5. We will now prove the third statement.

We will work in the ball model so we can apply Lemma 3.3.1.

As $\overline{D} \subset \Omega(\Gamma)$ and $\Omega(\Gamma)$ is Γ invariant we have that $\Gamma\overline{D} \subset \Omega(\Gamma)$. We shall now prove the reverse inclusion.

Let $z_0 \in D$ be fixed and choose any $z \in \Omega(\Gamma)$ then there are two options either the number of simple curves in $\Gamma\partial D$ that separate z_0 and z is finite or infinite.

Suppose that the number of curves is finite. We shall prove the result by induction on the number of curves n.

Base Case: $n = 0$ then $z \in D$ along with z_0.

Inductive step: Suppose the result is true for w separated by n curves from z_0.

Let the number of curves in $\Gamma\partial D$ separating z_0 and z be $n+1$. Suppose the curves are $C_1, \ldots, C_{n+1}$ ordered so that the component of $\widehat{\mathbb{C}} - C_i$ contains z_0 and C_{i-1}. Then there are γ_i such that $\gamma_i(C_i) \subset \overline{D}$. If $\gamma_{n+1}z \in \overline{D}$ we are done. If not then there is some $\zeta \in G(\Gamma)$ such that ζz is only separated by n curves from z_0. We can take ζ so that it interchanges the two components of $\Omega(\Gamma) - \Gamma\partial D$ that bound C_{n+1}. Then $\zeta z \in \Gamma\overline{D}$ by the inductive hypothesis. Since $\Gamma\overline{D}$ is invariant under Γ we see that $z \in \Gamma\overline{D}$ as required.

We shall now show that if there are an infinite number of curves separating z and z_0 then $z \in \Lambda(\Gamma)$. Let the curves be $\{C_n\}$. By Lemma 3.3.1 we know that $\operatorname{diam} C_n \to 0$. Let D_n be the component of $\widehat{\mathbb{C}} - C_n$ that contains z_0 then $\operatorname{diam} D_n \not\to 0$ as it contained D so that $C_n \to z$. There are γ_n such that $C_n \subset \gamma_n \overline{D}$ this means that $\gamma_n \overline{D} \to z$ as $\gamma_n D \cap D = \emptyset$ by Lemma 3.1.5. But this means that $z \in \Lambda(\Gamma)$ as required.

□

The following proof is a consequence of [Ber61] and the fact that a quasi-conformal map extends to a homeomorphic quasi-isometry of $\mathbb{H}^3$ see [Thu80].

Lemma 3.3.3 *Given a Schottky group of genus n then $\mathbb{H}^3/\Gamma$ is genus n handlebody.*

3.4 The limit set of a Schottky group

We will reprove the following lemmas later in this section with a more dynamical flavour. We give the following elementary proofs for completeness.

Lemma 3.4.1 *The limit set of a Schottky group is totally disconnected.*

Proof: We recall that a set is totally disconnected iff for any two distinct points there are two disjoint open sets with one point in each set and whose union is the original set.

Conjugate Γ to the ball model of hyperbolic space, since elements of $Mob(\widehat{\mathbb{R}}^3)$ are homeomorphisms they preserve total disconnectedness.

Let $z, w \in \Lambda(\Gamma)$. Note that $\Lambda(\Gamma)$ has the subspace topology so that a set is open in $\Lambda(\Gamma)$ if it is the intersection of an open set in $\widehat{\mathbb{C}}$ with $\Lambda(\Gamma)$.

If z, w are in different $D(\gamma)$ then without loss of generality suppose that $z \in D(\gamma)$ and $w \notin D(\gamma)$. Then $D(\gamma) \cap \Lambda(\Gamma)$ and $(\widehat{\mathbb{C}} - D(\gamma)) \cap \Lambda(\Gamma)$ are both open sets as $\Lambda(\Gamma) \cap \Gamma\partial D = \emptyset$ so that $D(\gamma) \cap \Lambda(\Gamma) = intD(\gamma) \cap \Lambda(\Gamma)$. For the same reason the union of the two is $\Lambda(\Gamma)$.

Now assume that they lie in the same $D(\gamma_k)$ for some sequence of distinct elements γ_k such that $z, w \in D(\gamma_k)$ for all k but by Lemma 3.3.1 we have that $z = w$ as required. □

Lemma 3.4.2 *The limit set of a Schottky group is a Cantor set.*

Proof: Recall that a Cantor set is a metrisable, compact, perfect and totally disconnected set. The first and second conditions come from the fact it is a closed set subset of $\widehat{\mathbb{C}}$, the third from Section 2.7 and the fourth from the above lemma. □

3.5 Schottky space

We now investigate the collection of all Schottky groups.

Definition 3.5.1 Fix a genus $n > 1$ then **marked Schottky space** MS_n is the subspace of $PSL_2(\mathbb{C})^n$ such that $(g_1, \ldots, g_n) \in MS_n$ generates a genus n Schottky group.

Definition 3.5.2 Fix a genus $n > 1$ then **Schottky space** S_n is the quotient of MS_n by conjugation by elements of $PSL_2(\mathbb{C})$.

Chuckrow and Marden proved that MS_n and S_n are path connected and open see [Chu68] and [Mar74].

Note that different points of MS_n or S_n do not necessarily generate different Kleinian groups.

Definition 3.5.3 Marked Classical Schottky space MCS_n is the collection of elements of MS_n that generate a classical Schottky group.

Definition 3.5.4 Classical Schottky space CS_n is the collection of elements of S_n whose lift to MS_n is a classical Schottky group. Classical Schottky space is well defined as conjugation by $PSL_2(\mathbb{C})$ preserves classicalness see Lemma 3.2.4.

Lemma 3.5.5 *Classical Schottky space is open in S_n.*

Proof: Let $[\Gamma] \in S_n$ then $\Gamma = (g_1, \ldots, g_n) \in [\Gamma]$ is uniquely defined by requiring that g_1 fixes 0 and ∞ and the attractive fixed point of g_2 is 1.

Now let $[\Gamma(k)] \to [\Gamma]$ then $\Gamma(k) \to \Gamma$ where the $\Gamma(k) = (g_1(k), \ldots, g_n(k))$ are uniquely defined by letting $g_1(k)$ fix 0 and ∞ and the attractive fixed point of $g_2(k)$ be 1. So we have that $g_i(k) \to g_i$ for all i.

Now suppose that Γ has generators $\{h_1, \ldots, h_n\}$ on which it is classical. Each h_i can be expressed as a word in $\{g_1, \ldots, g_k\}$. Define $h_i(k)$ to be the same word with g_i replaced by $g_i(k)$. Then $h_i(k) \in \Gamma(k)$ and $\Gamma(k)$ is generated by $\{h_1(k), \ldots, h_n(k)\}$.

As $g_i(k) \to g_i$ we see $h_i(k) \to h_i$. Then for large k the circles paired up by h_i are almost paired up by $h_i(k)$ and by the openness of the fundamental domain of Γ we see that $\Gamma(k)$ is a classical Schottky group for large k and so

$[\Gamma(k)] \in CS_n$ for large k. $\square$

There is a nice discussion of Schottky space in [Mar74].

3.6 The boundary of Schottky space

Given a sequence $\{(g_1(k), \ldots, g_n(k))\}$ of MS_n, we ask the question "what can this sequence converge to?".

We first note that it is possible that one of the $g_i(k)$ leaves $PSL_2(\mathbb{C})$. If this happens then the objects associated to the groups degenerate fairly severely. We investigate this for genus 2 Schottky groups in Chapter 6.

We now look at what can occur if all the $g_i(k)$ converge in $PSL_2(\mathbb{C})$.

They can of course converge to a Schottky group.

They cannot converge to a non-discrete group, in other words leave the space of all Kleinian groups by a theorem of Jøergensen [JK82].

Chuckrow [Chu68] showed that if $\lim \Gamma_m$ is a Kleinian group then it must be free and of the same genus thus torsion free.

$\lim \Gamma_m$ can contain parabolics and the subset of ∂MS_n for which $\lim \Gamma_m$ has a parabolic element is of at most codimension 1 [Chu68] although it is a dense set see [RCS03] where they attribute the statement in the case of Schottky groups to Sullivan.

So the following case must occur.

They converge to a free, purely loxodromic Kleinian group that is not a Schottky group. By Maskit's classification of Schottky groups [Mas67] we see that $\Lambda(\Gamma) = \widehat{\mathbb{C}}$. This means that $\lim \Gamma_m$ is geometrically infinite.

3.7 Dynamics and Schottky groups

In this section we introduce the shift space of a free group Γ which is the set of sequences (ζ_i) where $\zeta_i \in G(\Gamma)$ and $\zeta_i \zeta_{i+1} \neq id$. This set can be viewed as the boundary at infinity of the Cayley graph of the group or for a Schottky group its limit set. We then project the left shift acting on the shift space to the limit set.

The statement of the following proofs can be found in [Bow79] where he proves it in the case of quasi-Fuchsian groups. The quasi-Fuchsian case is more complicated as the shift space is no longer in one to one correspondence with the limit set.

Definition 3.7.1 Given a Schottky group Γ let the **shift space** Σ_n of Γ be the subset of $\prod_1^\infty G(\Gamma)$ such that for all $(x_i) \in \Sigma_n$, $x_i \neq x_{i+1}^{-1}$. We give Σ_n the topology generated by setting a basis to be the **cylinder sets** $C(x_1, \ldots, x_i) = \{(y_1, y_2, \ldots) \in \Sigma_n | y_1 = x_1, \ldots, y_i = x_i\}$.

With this topology Σ_n is a Cantor set.

Definition 3.7.2 Given a Schottky group $\Gamma =< g_1, \ldots, g_n >$ define the map π from Σ_n to closed sets of $\widehat{\mathbb{C}}$ by

$$\pi((x_i)) = \bigcap_{i \geq 1} D(x_1 \ldots x_i).$$

Lemma 3.7.3 *π is a bijection from Σ_n to $\Lambda(\Gamma)$.*

Proof: We will first of all prove that π maps into $\Lambda(\Gamma)$.

Let $(x_i) \in \Sigma_n$ then $D(x_1 \dots x_{i+1}) \subset D(x_1 \dots x_i)$ as $x_i x_{i+1} \neq id$.

Because the intersection of an infinite number of nested compact sets is non-empty and Lemma 3.3.1 we have that $\bigcap_{i \geq 1} D(x_1 \dots x_i)$ is a single point.

Let $w \in D$ then by definition $x_1 \dots x_i(w) \in D(x_1 \dots x_i)$ for all i so that $\bigcap_{i \geq 1} D(x_1 \dots x_i)$ is an accumulation point of Γw so is in the limit set.

We now prove injectivity. Let $(x_i), (y_i) \in \Sigma_n$ such that $(x_i) \neq (y_i)$ then there is some minimum k such that $x_k \neq y_k$. We shall show that $\pi(x_i) \neq \pi(y_i)$. We have that

$$\pi(x_i) = \cap_i D(x_1 \dots x_i) \text{ and } \pi(y_i) = \cap_i D(y_1 \dots y_i).$$

However $x_1 \dots x_k \neq y_1 \dots y_k$ so that $D(x_1 \dots x_k)$ and $D(y_1 \dots y_k)$ are disjoint but by the definition of π this means that $\pi(x_i)$ and $\pi(y_i)$ are distinct as required.

We now prove surjectivity. Let $z \in \Lambda(\Gamma)$ then for each n there is some γ_n such that $z \in D(\gamma_n)$ and $l(\gamma_n) = n$. Now $l(\gamma_n^{-1}\gamma_{n+1}) = 1$ otherwise $D(\gamma_{n+1}) \not\subset D(\gamma_n)$. This means that $(\gamma_n^{-1}\gamma_{n+1}) \in \Sigma_n$ and so $\pi((\gamma_n^{-1}\gamma_{n+1})) = z$ as required. $\square$

To show that π is a homeomorphism we will need the following topological Lemma.

Lemma 3.7.4 *Two sets are homeomorphic if there is a bijection between the sets that induces a bijection between open bases for the two sets.*

Proof: Suppose that $\phi : X \to Y$ is a bijection that induces a bijection $\widetilde{\phi} B \to C$ where B and C are bases for X and Y respectively.

Given an open set $U \subset X$ we shall show that $\phi(U)$ is open.

As B is an open basis there are $b_i^j \in B$ such that $U = \cap_j \cup_i b_i^j$ where the union is over arbitrarily many elements while the intersection is over a finite number.

Now

$$\phi(U) = \phi \cap_j \cup_i b_i^j = \cap_j \cup_i \phi b_i^j$$

as ϕ is a bijection. This means that $\phi(U)$ is open as it is the union and finite intersection of open sets, the $\phi(b_i)$.

The reverse direction is done by considering ϕ^{-1} instead of ϕ and we are done. □

Lemma 3.7.5 *The map π is a homeomorphism.*

Proof:

We will use Lemma 3.7.4 so we only need to consider the cylinder sets.

π induces a bijection between the cylinder sets and the open sets $\Lambda(\Gamma) \cap int(D(\gamma))$. So we need to show that the sets $\Lambda(\Gamma) \cap int(D(\gamma))$ form a basis for $\Lambda(\Gamma)$.

Let U be an open subset of $\Lambda(\Gamma)$ and $z \in U$ then $z = \bigcap_{i \geq 1} D(x_1 \ldots x_i)$ for some $(x_i) \in \Sigma_n$ by Lemma 3.7.3. By Lemma 3.3.1 we have that $\text{diam}(D(x_1 \ldots x_i)) \to 0$ so that for some k $D(x_1 \ldots x_k) \subset U$ as required.

□

Definition 3.7.6 We define $z \simeq \{\gamma_n\}$ to mean that the sequence $\{\gamma_n\}$ satisfies $z \in D(\gamma_n)$, $l(\gamma_n) = n$ and $l(\gamma_n^{-1}\gamma_{n+1}) = 1$ for all n. Lemma 3.7.3 shows that such a sequence exists and is unique.

Definition 3.7.7 Define the **left shift** $\tau : \Sigma_n \to \Sigma_n$ by

$$\tau((x_1, x_2, x_3, \ldots)) = (x_2, x_3, \ldots).$$

We now project the left shift τ to $\Lambda(\Gamma)$.

Definition 3.7.8 Given a Schottky group $\Gamma =< g_1, \ldots, g_n >$ define the map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by

$$f(z) = \begin{cases} \zeta^{-1}(z) & z \in D(\zeta) \quad \forall \zeta \in G(\Gamma) \\ z & z \in D. \end{cases}$$

Lemma 3.7.9 *The following diagram commutes*

$$\begin{array}{ccc} \Sigma_n & \xrightarrow{\tau} & \Sigma_n \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \Lambda(\Gamma) & \xrightarrow{f} & \Lambda(\Gamma) \end{array}$$

Proof: Let $(x_i) \in \Sigma_n$ then $f(\pi((x_i))) =$

$$f\left(\bigcap_{i\geq 1} D(x_1 \ldots x_i)\right) = \bigcap_{i\geq 1} f(D(x_1 \ldots x_i)) = \bigcap_{i\geq 1} f x_1 \ldots x_{i-1} D(x_i)$$

$$= \bigcap_{i \geq 1} x_2 \dots x_{i-1} D(x_i) = \bigcap_{i \geq 2} D(x_2 \dots x_i)) = \pi(\tau((x_i)))$$

as required. □

3.8 Estimates for Schottky groups

In this section we give various inequalities that are specific to a particular Schottky group. All of the results in this section are well known.

The following Lemma is proved by Bowen [Bow79].

Lemma 3.8.1 *Given a Schottky group Γ with $\infty \in D$ then there are constants $K, K' > 0$ and $\rho, \sigma \in (0,1)$ such that*

$$K' \sigma^{l(\gamma)} \leq \mathrm{diam} D(\gamma) \leq K \rho^{l(\gamma)}$$

for every $\gamma \in \Gamma$.

Definition 3.8.2 Given a Schottky group Γ and $\gamma \in \Gamma$, we define the **contracting set** $\mathrm{con}(\gamma)$ of γ to be the set

$$\mathrm{con}(\gamma) = \bigcup_{\zeta \in G(\Gamma), \zeta \neq g_n^{-1}} D(\zeta)$$

where $\gamma = g_1 \dots g_n$ as a reduced word.

For an isometric Schottky group the set $\mathrm{con}(\gamma)$ contains the subset of $\Lambda(\Gamma)$ on which γ acts as a contraction, in the general case, given a Schottky group there is an N such that γ acts as a contraction on $\mathrm{con}(\gamma)$ as long as $l(\gamma) \geq N$, this can be seen by Lemma 3.8.1 and the following inequalities.

Lemma 3.8.3 *Given a Schottky group Γ with $\infty \in D$ and fixed $w \in D$ not ∞ then there are positive constants K_1', K_1 such that*

$$K_1'|\gamma'(w)| \leq |\gamma'(z)| \leq K_1|\gamma'(w)|$$

for all $\gamma \in \Gamma$ and $z \in \operatorname{con}(\gamma)$.

Proof: Fix $w \in D$ and let $\gamma \in \Gamma$ and $z \in \operatorname{con}(\gamma)$ be given. Then

$$\gamma(z) = \frac{az+b}{cz+d}$$

so that

$$\frac{|\gamma'(z)|}{|\gamma'(w)|} = \frac{|w+\frac{d}{c}|^2}{|z+\frac{d}{c}|^2}$$

where $-\frac{d}{c} = \operatorname{cen} I_\gamma = \gamma^{-1}(\infty)$.

Now define

$$L = \operatorname{dist}_{\inf}(\partial D, \bigcup_{\gamma\in\Gamma} \operatorname{cen} I_\gamma)$$

this is positive as the set of accumulation points of $\bigcup_{\gamma\in\Gamma} \operatorname{cen} I_\gamma$ is the limit set of Γ see Lemma 2.7.2, ∂D and $\Lambda(\Gamma)$ are closed and disjoint and $\operatorname{cen} I_\gamma \in D(\gamma)$ for all $\gamma \in \Gamma$ by Lemma 3.1.14.

We have that

$$\frac{|w - \operatorname{cen} I_\gamma|^2}{|z - \operatorname{cen} I_\gamma|^2} \leq \frac{\operatorname{dist}_{\sup}(w, \partial D)^2}{\min_{\zeta,\xi\in G(\Gamma)} \operatorname{dist}_{\inf}(D(\zeta), D(\xi))^2}$$

as z and $\operatorname{cen} I_\gamma$ lie in different components $\widehat{\mathbb{C}} - D$ by the definition of $\operatorname{con}(\gamma)$ and we define

$$K_1 = \frac{\operatorname{dist}_{\sup}(w, \partial D)}{\min_{\zeta,\xi\in G(\Gamma)} \operatorname{dist}_{\inf}(D(\zeta), D(\xi))}.$$

The lower bound is

$$\frac{|w - \operatorname{cen} I_\gamma|^2}{|z - \operatorname{cen} I_\gamma|^2} \geq \frac{L^2}{\operatorname{diam}(\partial D)^2}$$

and we let $K_1' = \frac{L}{\text{diam}(\partial D)}$. $\square$

Lemma 3.8.4 *Given a Schottky group Γ with $\infty \in D$ and fixed $w \in D$ then there are positive constants K_2 and K_2' such that*

$$K_2'|\gamma'(w)| \leq \text{diam}(D(\gamma)) \leq K_2|\gamma'(w)|$$

for all $\gamma \in \Gamma$.

Proof: Fix $w \in D$.

Given $\gamma \in \Gamma$ write γ as the reduced word $\zeta_1 \ldots \zeta_k$.

By definition

$$D(\gamma) = \zeta_1 \ldots \zeta_{k-1} D(\zeta_k)$$

so

$$\begin{aligned}\text{diam}(D(\gamma)) &= \text{diam}\left(\zeta_1 \ldots \zeta_{k-1} D(\zeta_k)\right) \\ &= \max_{z_1, z_2 \in D(\zeta_k)} |\zeta_1 \ldots \zeta_{k-1}(z_1) - \zeta_1 \ldots \zeta_{k-1}(z_2)| \\ &= \max_{z_1, z_2 \in D(\zeta_k)} |(\zeta_1 \ldots \zeta_{k-1})'(z_1)|^{1/2} |(\zeta_1 \ldots \zeta_{k-1})'(z_2)|^{1/2} |z_1 - z_2| \qquad (3.8.1)\end{aligned}$$

by Lemma 2.2.5.

We first do the upper bound

$$\text{diam}(D(\gamma)) \leq \max_{z \in D(\zeta_k)} |(\zeta_1 \ldots \zeta_{k-1})'(z)| \text{diam} D(\zeta_k)$$

by taking the maximum of each term of equation 3.8.1 and since $z \in D(\zeta_k) \subset \text{con}(\zeta_1 \ldots \zeta_{k-1})$ we can apply Lemma 3.8.3 to find the constant K such that

$$\text{diam}(D(\gamma)) \leq K \text{diam} D(\zeta_k) |(\zeta_1 \ldots \zeta_{k-1})'(w)|$$

$$\leq K \mathrm{diam}(\partial D)|(\zeta_1 \dots \zeta_{k-1})'(w)|$$

for some $w \in D$.

Now we have that

$$\frac{|(\zeta_1 \dots \zeta_{k-1})'(w)|}{|(\zeta_1 \dots \zeta_k)'(w)|} = \frac{|(\zeta_1 \dots \zeta_{k-1})'(w)|}{|(\zeta_1 \dots \zeta_{k-1})'(\zeta_k(w))||\zeta_k'(w)|}$$

by the chain rule. Since $\zeta_k(w) \in \mathrm{con}(\zeta_1 \dots \zeta_{k-1})$ we can apply Lemma 3.8.3 to get

$$\frac{|(\zeta_1 \dots \zeta_{k-1})'(w)|}{|(\zeta_1 \dots \zeta_k)'(w)|} \leq \frac{K}{|\zeta_k'(w)|} \leq \frac{K}{\min_{\zeta \in G(\Gamma)} |\zeta'(w)|}$$

and note that this is finite as $G(\Gamma)$ is a finite set and $w \in D$.

So putting it together we have

$$\mathrm{diam}(D(\gamma)) \leq \frac{K^2 \mathrm{diam}(\partial D)}{\min_{\zeta \in G(\Gamma)} |\zeta'(w)|} |\gamma'(w)|$$

so define

$$K_2 = \frac{K^2 \mathrm{diam}(\partial D)}{\min_{\zeta \in G(\Gamma)} |\zeta'(w)|}$$

which does not depend on the particular $\gamma \in \Gamma$ but does depend on w and the defining curves.

We now do the lower bound.

Let $w_1, w_2 \in D(\zeta_k)$ be such that $|w_1 - w_2| = \mathrm{diam} D(\zeta_k)$ then

$$\mathrm{diam}(D(\gamma)) = \max_{z_1, z_2 \in D(\zeta_k)} |(\zeta_1 \dots \zeta_{k-1})'(z_1)|^{1/2} |(\zeta_1 \dots \zeta_{k-1})'(z_2)|^{1/2} |z_1 - z_2|$$

$$\geq |(\zeta_1 \dots \zeta_{k-1})'(w_1)|^{1/2} |(\zeta_1 \dots \zeta_{k-1})'(w_2)|^{1/2} \mathrm{diam} D(\zeta_k)$$

and by Lemma 3.8.3 twice this is

$$\geq K' |(\zeta_1 \dots \zeta_{k-1})'(w)| \mathrm{diam} D(\zeta_k) \geq K' |(\zeta_1 \dots \zeta_{k-1})'(w)| \left(\min_{\zeta \in G(\gamma)} \mathrm{diam} D(\zeta) \right)$$

for the fixed $w \in D$.

In the same way as for the upper bound we have

$$\frac{|(\zeta_1 \ldots \zeta_{k-1})'(w)|}{|(\zeta_1 \ldots \zeta_k)'(w)|} \geq \frac{K'}{|\zeta_k'(w)|} \geq \frac{K'}{\min_{\zeta \in G(\Gamma)} |\zeta'(w)|}.$$

So putting it all together we have

$$\mathrm{diam} D(\gamma) \geq \frac{K'^2 \left(\min_{\zeta \in G(\gamma)} \mathrm{diam} D(\zeta)\right)}{\min_{\zeta \in G(\Gamma)} |\zeta'(w)|} |\gamma'(w)|$$

and we define

$$K_2 = \frac{K'^2 \left(\min_{\zeta \in G(\gamma)} \mathrm{diam} D(\zeta)\right)}{\min_{\zeta \in G(\Gamma)} |\zeta'(w)|}$$

which does not depend on the particular $\gamma \in \Gamma$ but does depend on the fixed $w \in D$. □

We need the following Lemma to relate distances in $\mathbb{H}^3$ to distances in $\widehat{\mathbb{C}}$.

Lemma 3.8.5 *Given two vertical geodesics H and H' in $\mathbb{H}^2$ whose non-infinite end-points are separated by a Euclidean distance d and a further geodesic segment α of length l whose endpoints lie on the two vertical geodesics then the lowest possible height that one of the endpoints of α can be is $\frac{d}{\sinh(l)}$.*

Proof: Without loss of generality let $p \in H$ be the lowest possible endpoint of α.

We shall prove by contradiction that α is perpendicular to H'. Assume not then there is some $q' \in H'$ such that the geodesic α' from p to q' realises the shortest path from p to H'. So we have that $d(p, q') < l$ and so there is some point q below p such that $d(q, q') = l$ and we have the contradiction.

Translation parallel to $\mathbb{R}$ is both a Euclidean and a hyperbolic isometry so preserves the setup, the same is true of a reflection in any line perpendicular to $\mathbb{R}$. So we can give coordinates to $\mathbb{H}^2$ such that $p = (d, y)$ for $d > 0$ and $p' = (0, y')$ but as α is perpendicular to H' we have that $y'^2 = d^2 + y^2$ so that

$$l = \log\left(\frac{\sqrt{d^2 + y^2} - d}{y}\right)$$

by the formula in [And99]. We can solve this for y to get

$$y = \frac{d}{\sinh(l)}$$

as required. □

Lemma 3.8.6 *Given a Schottky group* Γ *with* $\infty \in D$ *and fixed* $w \in D$ *then there are positive constants* K_3 *and* K_3' *such that*

$$K_3'|\gamma'(w)| \leq h_\gamma \leq K_3|\gamma'(w)|$$

for all $\gamma \in \Gamma$, *where* h_γ *is the Euclidean height of* $\gamma(j)$.

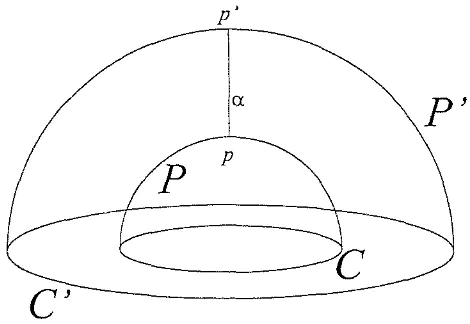
Proof: We shall prove the existence of a lower bound first by showing that there is a lower bound for the height of $\gamma(p)$ for a particular $p \in \mathbb{H}^3$ then extending this bound to any $q \in \mathbb{H}^3$.

Let C be a circle such that the disc bounded by C is contained in D then let p be the point that lies on the top of P, the hyperbolic plane whose boundary is C. We see that p satisfies the conditions of Lemma 3.1.17. Note that there are many choices for C.

Now choose another circle C' such that C and C' have the same centres and the annulus bounded by C and C' contains ∂D in its interior. Again there are many choices for C'.

Define P to be the hyperbolic plane that bounds C and P' to be the hyperbolic plane which bounds C'. Let α be the geodesic segment from P to P'. Define p and p' to be the endpoints of α which are in P and P' respectively. Note that α is a vertical line segment.

Figure 3.3: Setup for height derivative equivalence



Given $\gamma \in \Gamma$ we shall apply it to this setup, then find a lower bound on the height of $\gamma(p)$.

Project $\gamma(\alpha)$ vertically to $\mathbb{C}$; then this is a Euclidean line v that intersects $\gamma(C)$ and $\gamma(C')$ in z and z' say. Let H be the unique hyperbolic plane which contains ∞ and whose boundary is tangent to $\gamma(C)$ and intersects v at right angles, define H' similarly except let its boundary be tangent to $\gamma(C')$.

Now consider the hyperbolic sphere S of radius $l(\alpha)$ centred at $\gamma(p)$ and the points q and q' which are the unique points vertically below $\gamma(\alpha)$ such that $q \in S \cap H$ and $q' \in S \cap H'$.

We shall bound the height of q and then the height of $\gamma(p)$.

We have the following estimates

$$d(q, q') \leq 2l(\alpha) \text{ so that } \mathrm{dist}_{\inf}(q, H') \leq 2l(\alpha)$$

Figure 3.4: Setup under γ

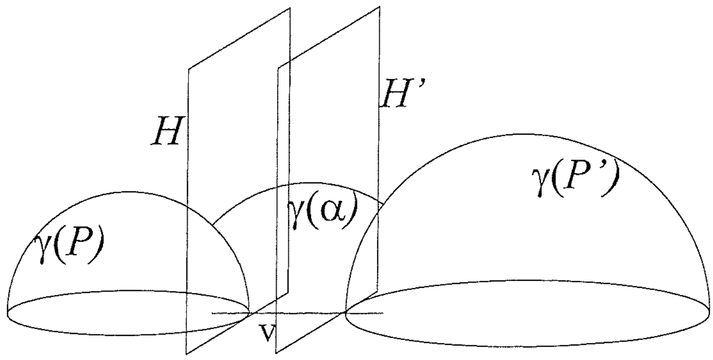
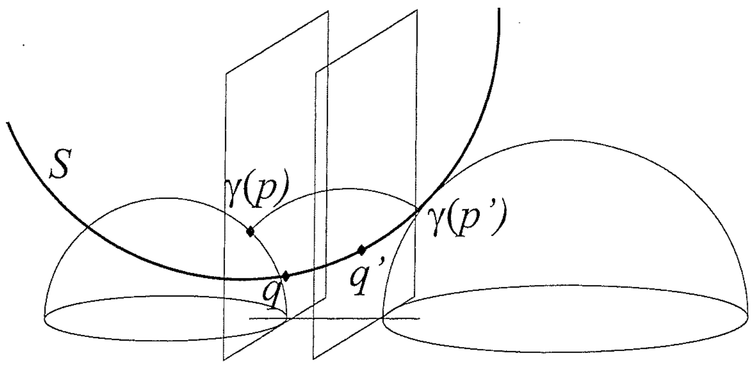


Figure 3.5: Lower bound on the distance



and

$$\begin{aligned}\operatorname{dist}_{\inf}(\gamma(C), \gamma(C')) &= \min_{w_1 \in C, w_2 \in C'} |\gamma'(w_1)|^{1/2} |\gamma'(w_2)|^{1/2} |w_1 - w_2| \\ &\geq K_1' |\gamma' w| \min_{z \in C, w \in C'} |z - w|\end{aligned}$$

for some $w \in D$ by Lemma 2.2.5 and then Lemma 3.8.3. We define $K_2' = K_1' \min_{z \in C, w \in C'} |z - w|$ and note that it is independent of γ.

By Lemma 3.8.5 we have that the height of q is greater than

$$\frac{\text{dist}_{\text{inf}}(\gamma(C), \gamma(C'))}{\sinh(d(q,q'))} \geq \frac{K_2'}{\sinh(2l(\alpha))}|\gamma' w|$$

where we note that $l(\alpha)$ is a constant independent of γ.

We have that $d(\gamma(p), q) = l(\alpha)$ so that the lowest that $\gamma(p)$ can be below q is the hyperbolic distance $l(\alpha)$ but this corresponds to $\gamma(p)$ being at the height of q divided by $\exp(l(\alpha))$.

Similarly if we let $l' = d(j,p)$ then h_γ can be at most $\frac{1}{\exp(l')}$ times the height of $\gamma(p)$ so in conclusion we have

$$h_\gamma \geq \frac{1}{\exp(l')} \frac{1}{\exp(l(\alpha))} \frac{K_2'}{\sinh(2l(\alpha))}|\gamma' z|$$

so define

$$K_3' = \frac{1}{\exp(l')} \frac{1}{\exp(l(\alpha))} \frac{K_2'}{\sinh(2l(\alpha))}.$$

The other side of the proof is simpler. Let p be the same p as in the first half of the proof then $\gamma(p) \in P(\gamma)$ for every $\gamma \in \Gamma$ by Lemma 3.1.17. Since $P(\gamma) \subset Con.Hull(D(\gamma))$ we have that the height of $\gamma(p)$ is less than $\text{diam}(D(\gamma))$ and so by Lemma 3.8.4 this is less than $K|\gamma'(w)|$ for some $K > 0$ and fixed $w \in D$.

As above if we let $l' = d(j,p)$ then we have that

$$h_\gamma \leq \exp(l')K|\gamma'(z)|$$

so define

$$K_3 = \exp(l')K$$

as required, note that the constant depends on w. $\square$

Lemma 3.8.7 *Given a Schottky group Γ with $\infty \in D$ then there are positive constants K_4 and K_4' such that*

$$K_4' \exp(-d(j, \gamma j)) \leq h_\gamma \leq K_4 \exp(-d(j, \gamma j))$$

for all $\gamma \in \Gamma$.

Proof: We first do the upper bound.

Given $\gamma \in \Gamma$ let α be the geodesic ray from j through $\gamma(j)$ with endpoint z and let $d = |z - \gamma(j)|$ then $d \geq h_\gamma$.

Pull everything back to the ball model by the inverse of stereographic projection ϕ^{-1} then $\phi^{-1}(j) = 0$.

Let $r = |\phi^{-1}(\gamma(j)) - \phi^{-1}(z)|$ we can then calculate to get that

$$\tanh \frac{d(j, \gamma j)}{2} = 1 - r$$

which means that

$$r = \frac{2}{\exp(d(j, \gamma j)) + 1}.$$

We can apply Lemma 2.2.9 since $\infty \in \Omega(\Gamma)$ to get a constant L' such that

$$h_\gamma < d < \frac{r}{L'} < \frac{2}{L' \exp(d(j, \gamma j)) + 1} < \frac{2 \exp(-d(j, \gamma j))}{L'}$$

so define

$$K_4 = \frac{2}{L'}.$$

We now do the lower bound.

Given $\gamma \in \Gamma$ then the lowest that h_γ can be is directly below j at hyperbolic distance $d(j, \gamma j)$ and we can calculate this explicitly as

$$d(j, \gamma j) = \int_{h_\gamma}^{1} \frac{1}{y} dy = \log \frac{1}{h_\gamma}$$

so that the lowest that h_γ can be is $\exp(-d(j, \gamma j))$ and we have the lower bound with $K_4' = 1$. $\square$

Chapter 4

Types and Examples of Schottky Groups

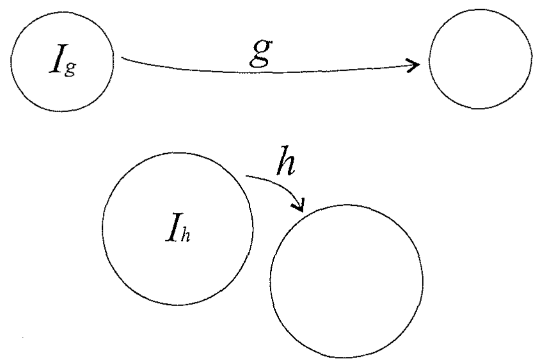
In this chapter we give various examples of Schottky groups. For simplicity we shall consider only two generator groups.

4.1 Isometric Schottky groups

Definition 4.1.1 Perhaps the simplest example of a Schottky group is an **isometric** Schottky group. A group Γ is isometric if it has generators $\{g, h\}$ such that $I_g, I_{g^{-1}}, I_h$ and $I_{h^{-1}}$ are all disjoint and not nested. The isometric circles then form a set of defining curves for Γ.

Example: Given four distinct points x_1, x_2, y_1, y_2 in $\mathbb{C}$ then for $\lambda, \mu \in \mathbb{C}$ sufficiently large, $< g, h >$ is an isometric Schottky group where g fixes x_1, x_2 with multiplier λ and h fixes y_1, y_2 with multiplier μ.

Figure 4.1: Isometric Schottky group



The proof is essentially Lemma 2.9.1 and the expression for the radius of the isometric circle.

Since the isometric circle of a Möbius transformation is not preserved by conjugation the property of being an isometric Schottky group is not invariant under conjugation. It is easy to see this if one of the fixed points is sent to ∞. A non-isometric Schottky group is a Schottky group that is not an isometric Schottky group.

4.2 Classical Schottky groups

Definition 4.2.1 A Schottky group is **classical** if it has defining curves which are circles for a set of generators.

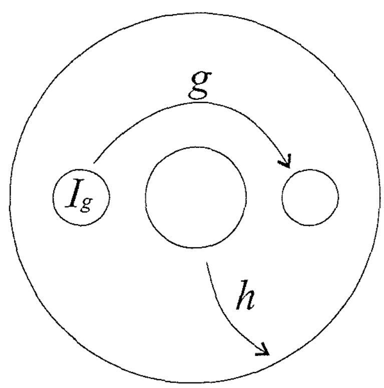
It is worth noting that classicalness is invariant by conjugation by Möbius transformations.

Example: The obvious example of a classical non-isometric Schottky group on a set of generators is

$$g(z) = \frac{(\lambda+1)^2 z + -(\lambda-1)^2}{-(\lambda+1)^2 z + (\lambda+1)^2} \text{ and } h(z) = \mu z$$

where $|\mu| \geq \max\{7, |\lambda|^2\}$ and g fixes ± 1 with multiplier λ.

Figure 4.2: Classical non-isometric



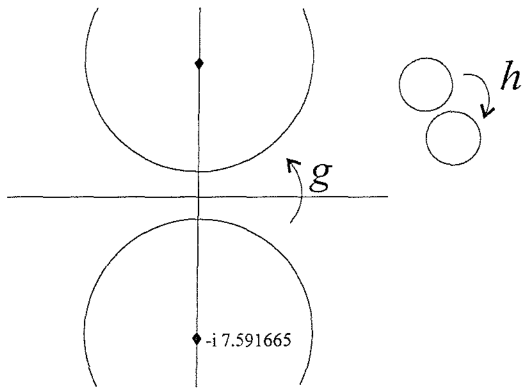
Example: A slightly more interesting example of a classical Schottky group is $< g, h >$ where $\text{cen}I_g = -i$, $\text{cen}I_{g^{-1}} = i$ and g has multiplier -2 then

$$g(z) = \frac{-i.7071067810z + 2.121320338}{-.7071067810z - i.7071067810}$$

so that g has fixed points $i1.732050806$ and $-i1.732050806$. The circles $S_{6.741665}(-i7.591665)$ and $g(S_{6.741665}(-i7.591665))$ are disjoint and as $\text{cen}I_g \in B_{6.741665}(-i7.591665)$ we have that $< g, h >$ is classical as long as the isometric circles of h do not intersect each other and $B_{6.741665}(-i7.591665)$ and $g(B_{6.741665}(-i7.591665))$.

Note that $< g, h >$ is non-isometric on these generators.

Figure 4.3: Non-isometric with bounded Limit set



All Fuchsian Schottky groups are classical see [Mar74] although he attributes it to Jørgensen.

We can define $[\Gamma]$ in Schottky space to be isometric if there is some $\Gamma \in [\Gamma]$ such that Γ is isometric. I believe it is an open question whether the isometric subset of Schottky space is equal to the classical subset. I conjecture that this is not the case, i.e that there are classical Schottky groups that are not isometric up to conjugation.

4.3 Non-classical Schottky groups

Definition 4.3.1 A **non-classical** Schottky group is a Schottky group for which no set of generators is classical. Non-classicalness is invariant by conjugation under Möbius transformations for the same reason that classicalness is.

Marden [Mar74] proved the existence of Non-classical Schottky groups by showing that if the limit of a sequence of classical Schottky groups is a

Kleinian group then it has non-empty domain of discontinuity. We know by a result of Chuckrow [Chu68] that there are limits of Schottky groups for which this does not happen.

Yamamoto [Yam91] gave the first explicit example of a non-classical Schottky group. In the following section we generalize this example to a family which contains a sequence of non-classical Schottky groups which do not converge to a subgroup of $PSL_2(\mathbb{C})$. Our proof follows very closely his proof except that we allow the multiplier of h_l to vary.

4.4 An Example of a sequence of Non-classical Schottky groups

We shall show that the following group is Schottky and non-classical for $l \in (1, \sqrt{2})$ and ϵ small depending on l.

Definition 4.4.1 We let $\Gamma_{l,\epsilon} = < g_{l,\epsilon}, h_l >$ where

$$h_l(z) = ilz$$

and

$$g_{l,\epsilon}(z) = \frac{az + c}{cz + a}$$

such that $a = \frac{l^2+1}{l^2-1} + \epsilon$ and $c = -\sqrt{a^2 - 1}$. We assume that $l \in (1, \sqrt{2})$ and $\epsilon \in (0, 1)$, this means that $a > 1$.

Lemma 4.4.2 *$\Gamma_{l,\epsilon}$ is a Schottky group.*

Proof: Consider $I_{g_{l,\epsilon}}$ and $I_{g_{l,\epsilon}^{-1}}$ then we have

$$\mathrm{cen}(I_{g_{l,\epsilon}}) = \frac{a}{|c|} = \frac{a}{\sqrt{a^2 - 1}}, \ \mathrm{cen}(I_{g_{l,\epsilon}^{-1}}) = -\frac{a}{|c|} = \frac{a}{\sqrt{a^2 - 1}}$$

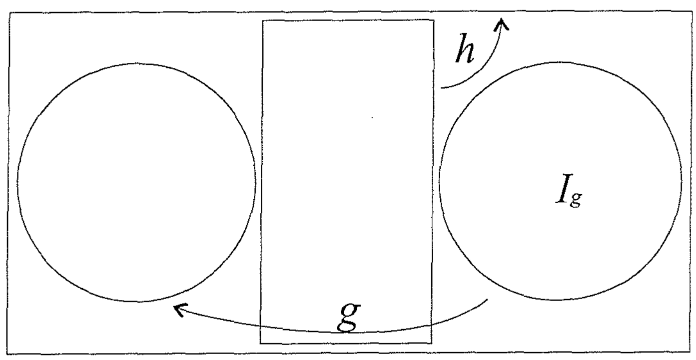
$$\text{and } \operatorname{rad}(I_{g_{l,\epsilon}}) = \frac{1}{|c|} = \frac{1}{\sqrt{a^2-1}}$$

since $l > 1$ and $\epsilon > 0$. Note that $a \to \infty$ as $l \to 1$ so that $\operatorname{cen}(I_{g_{l,\epsilon}^{\pm 1}}) \to \pm 1$ and $\operatorname{rad}(I_{g_{l,\epsilon}}) \to 0$.

Given $\delta, \delta' > 0$ then define $R_{\delta,\delta'}$ to be the boundary of a rectangle such that $R_{\delta,\delta'} \cap (\mathbb{R} \cup i\mathbb{R}) = \{\frac{a-1}{|c|} - \delta, -(\frac{a-1}{|c|} - \delta), il(\frac{a-1}{|c|} + \delta'), -il(\frac{a-1}{|c|} + \delta')\}$.

We shall show that $I_{g_{l,\epsilon}}, I_{g_{l,\epsilon}^{-1}}, R_{\delta,\delta'}$ and $h_l(R_{\delta,\delta'})$ do not intersect and form defining curves for $\Gamma_{l,\epsilon}$, for δ, δ' small.

Figure 4.4: Non-classical Schottky group

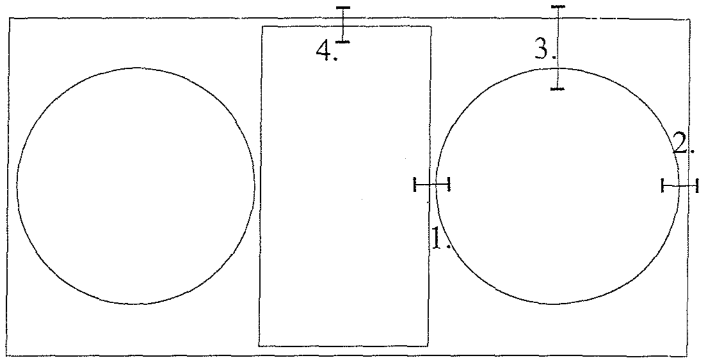


By symmetry we only need to check that:

1. $\frac{a-1}{|c|} - \delta < \operatorname{cen}(I_{g_{l,\epsilon}}) - \operatorname{rad}(I_{g_{l,\epsilon}})$,
2. $h_l(-il(\frac{a-1}{|c|} + \delta')) > \operatorname{cen}(I_{g_{l,\epsilon}}) + \operatorname{rad}(I_{g_{l,\epsilon}})$,
3. $h_l(\frac{a-1}{|c|} - \delta) > i\operatorname{rad}(I_{g_{l,\epsilon}})$,
4. $h_l(\frac{a-1}{|c|} - \delta) > il(\frac{a-1}{|c|} - \delta')$.

1. $\operatorname{cen}(I_{g_{l,\epsilon}}) - \operatorname{rad}(I_{g_{l,\epsilon}}) = \frac{a}{|c|} - \frac{1}{|c|}$ which is greater than 0 for all $\delta > 0$.

Figure 4.5: Distances to check



2. $h_l(-il(\frac{a-1}{|c|}+\delta')) > \mathrm{cen}(I_{g_{l,\epsilon}}) + \mathrm{rad}(I_{g_{l,\epsilon}})$ iff $l^2(a-1)+l^2\delta'|c| > a+1$ if $|c|l^2\delta' > 0$ which is true for all $\delta' > 0$.

3. $h_l(\frac{a-1}{|c|}-\delta) > i\mathrm{rad}(I_{g_{l,\epsilon}})$ iff $\frac{l^2+1}{l^2-1}+\epsilon-1-\delta|c| > \frac{1}{l}$ since $\frac{l^2+1}{l^2-1}-1 > \frac{1}{l}$ for $l \in (1, 1+\sqrt{/}2)$ we need to check $\epsilon > \delta|c|$ and for all small δ this is true.

4. $h_l(\frac{a-1}{|c|}-\delta) > il(\frac{a-1}{|c|}-\delta')$ iff $\delta < \delta'$ but we are allowed to choose this since the previous cases only needed δ less than some fixed small value.

□

Definition 4.4.3 Let $G_{l,\epsilon} =< {h_l}^2, g_{l,\epsilon} >$.

Lemma 4.4.4 *$G_{l,\epsilon}$ is a classical Schottky group. In fact it's extended Fuchsian.*

Proof: Since $l > 1$ and $\epsilon > 0$ we have that $a, c \in \mathbb{R}$. The generators of $G_{l,\epsilon}$ have real entries in their matrix forms so $G_{l,\epsilon}$ preserves $\mathbb{R}$ and since it is a subgroup of a Schottky group it is also a Schottky group [Chu68].

In fact we can give it explicit defining curves.

We have

$$l^2(|\mathrm{cen} I_{g_{l,\epsilon}}| - \mathrm{rad} I_{g_{l,\epsilon}}) > |\mathrm{cen} I_{g_{l,\epsilon}}| + \mathrm{rad} I_{g_{l,\epsilon}}$$

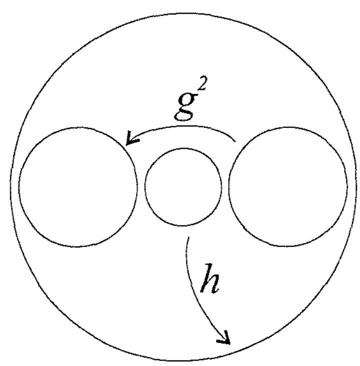
since

$$l^2 > \frac{l^2+1+(\epsilon+1)(l^2-1)}{l^2+1+(\epsilon-1)(l^2-1)} = \frac{a+1}{a-1}$$

for all $l > 1$ and $\epsilon > 0$.

So we can find a $\delta > 0$ such that the curves $I_{g_{l,\epsilon}}, I_{g_{l,\epsilon}^{-1}}, S_{\frac{a-1}{|c|}-\delta}(0)$ and $S_{l^2(\frac{a-1}{|c|}-\delta)}(0)$ are all disjoint which proves that the group is classical.

Figure 4.6: Subgroup is Classical



□

It is useful to give a short overview of the proof. In the first part of the proof we provide an upper bound on the length of the components of $\Omega(G_{l,\epsilon}) \cap \mathbb{R}$.

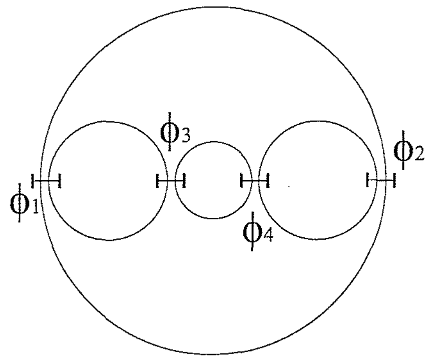
We then, for contradiction, assume that $\Gamma_{l,\epsilon}$ is classical. We use this and the bounds on $\Omega(G_{l,\epsilon}) \cap \mathbb{R}$ to bound $\Omega(\Gamma_{l,\epsilon}) \cap (\mathbb{R} \cup i\mathbb{R})$. Once we have this bound we show that there is some image of a defining circle under $\Gamma_{l,\epsilon}$ that intersects another image of a defining circle.

4.4.1 Investigate the lengths of components $\Omega(G_{l,\epsilon})$

Definition 4.4.5 Define ϕ_1 to be the interval in $\mathbb{R}$ bounded by the fixed points of $g_{l,\epsilon}{h_l}^2 g_{l,\epsilon}{h_l}^{-2}$. Define ϕ_2, ϕ_3 and ϕ_4 by $\phi_2 = {g_{l,\epsilon}}^{-1}(\phi_1)$, $\phi_3 = {h_l}^{-2}(\phi_2)$ and $\phi_4 = {h_l}^{-2}(\phi_1)$.

By the definition we see that $g_{l,\epsilon}{h_l}^2 g_{l,\epsilon}{h_l}^{-2}\phi_1 = \phi_1$ so ${h_l}^{-2}{g_{l,\epsilon}}^{-1}\phi_1 = g_{l,\epsilon}{h_l}^{-2}\phi_1$ which means $g_{l,\epsilon}(\phi_4) = \phi_3$.

Figure 4.7: The order of the ϕ_i



We will now show that the longest component of $(D(g_{l,\epsilon}) \cup D({g_{l,\epsilon}}^{-1})) \cap \Omega(G_{l,\epsilon})$ is shorter than $\max_i \phi_i$.

Lemma 4.4.6 *If I is a component of $(D(g_{l,\epsilon}) \cup D({g_{l,\epsilon}}^{-1})) \cap \Omega(G_{l,\epsilon})$ then $I \subset \gamma(\phi_i)$ for some i and some $\gamma \in G_{l,\epsilon}$.*

Proof: All components of $\Omega(G_{l,\epsilon}) \cap \mathbb{R}$ are equivalent under $G_{l,\epsilon}$ and each

ϕ_i contains at least one component as it has its endpoints in $\Lambda(\Gamma)$. $\square$

Lemma 4.4.7 *Given I as above and γ of minimal length such that $I \subset \gamma(\phi_i)$,*

$$\text{if } \phi_i = \begin{cases} \phi_1 & \text{then} \quad m_k > 0 \text{ or } m_k = 0 \text{ and } n_k > 0 \\ \phi_2 & \text{then} \quad m_k > 0 \text{ or } m_k = 0 \text{ and } n_k < 0 \\ \phi_3 & \text{then} \quad m_k < 0 \text{ or } m_k = 0 \text{ and } n_k > 0 \\ \phi_4 & \text{then} \quad m_k < 0 \text{ or } m_k = 0 \text{ and } n_k < 0 \end{cases}$$

where $\gamma = {g_{l,\epsilon}}^{n_1} {h_l}^{m_1} \dots {g_{l,\epsilon}}^{n_k} {h_l}^{m_k}$ such that $m_1, \dots, n_k \neq 0$.

Proof: If $\phi_i = \phi_1$ and $m_k < 0$ then we can reduce the length of γ by considering $\gamma {h_l}^2 \phi_4$ and if $m_k = 0$ and $n_k < 0$ we can reduce the length of γ by considering $\gamma g_{l,\epsilon} \phi_2$.

The other cases follow by the same argument on noting that $g_{l,\epsilon}(\phi_4) = \phi_3$. $\square$

Lemma 4.4.8 *Given I and minimal γ such that $I \subset \gamma(\phi)$ then*

$${g_{l,\epsilon}}^{n_t} {h_l}^{m_t} \dots {g_{l,\epsilon}}^{n_k} {h_l}^{m_k}(z) \in D(g_{l,\epsilon}) \cup D({g_{l,\epsilon}}^{-1})$$

for all $t \leq k$.

Proof: We prove this by induction on $k - t$.

Base Case: $t = k$ then consider ${g_{l,\epsilon}}^{n_k} {h_l}^{m_k}$ and by the table in Lemma 4.4.7 we have the result.

Inductive step: Assume the result is true for $t+1 \leq k$ then

$$g_{l,\epsilon}{}^{n_{t+1}} h_l{}^{m_{t+1}} \dots g_{l,\epsilon}{}^{n_k} h_l{}^{m_k}(z) \in D(g_{l,\epsilon}) \cup D(g_{l,\epsilon}{}^{-1})$$

but as the group is classical on the defining curves in Lemma 4.4.4 we have that $h_l{}^{m_t}(D(g_{l,\epsilon}) \cup D(g_{l,\epsilon}{}^{-1}))$ is disjoint from $D(g_{l,\epsilon}) \cup D(g_{l,\epsilon}{}^{-1})$ as $m_t \neq 0$.

If $t \neq 1$ then $n_t \neq 0$ so that $g_{l,\epsilon}{}^{n_t} h_l{}^{m_t}(D(g_{l,\epsilon}) \cup D(g_{l,\epsilon}{}^{-1})) \subset D(g_{l,\epsilon}) \cup D(g_{l,\epsilon}{}^{-1})$ as required.

If $t = 1$ then n_t might be 0 but we have just shown that

$$h_l{}^{m_1} \dots g_{l,\epsilon}{}^{n_k} h_l{}^{m_k}(z) \notin D(g_{l,\epsilon}) \cup D(g_{l,\epsilon}{}^{-1})$$

so that $\gamma(z) \notin D(g_{l,\epsilon}) \cup D(g_{l,\epsilon}{}^{-1})$ but this contradicts that $I \subset D(g_{l,\epsilon}) \cup D(g_{l,\epsilon}{}^{-1})$.

□

Lemma 4.4.9 $n_1 \neq 0$

Proof: This is the last part of Lemma 4.4.8. □

We now prove a technical Lemma.

Lemma 4.4.10 *We have*

$$|c| l \min_{i=1,2} \mathrm{dist}_{\inf}(0, \phi_i) - \frac{a}{l} - 1 \geq 0$$

for $l > 1$ *and* ϵ *small.*

Proof: By calculation we have

$$\min_{i=1,2} \mathrm{dist}_{\mathrm{inf}}(0, \phi_i) \to l$$

as $\epsilon \to 0$.

We will show that

$$\lim_{\epsilon=0} |c| l \min_{i=1,2} \mathrm{dist}_{\mathrm{inf}}(0, \phi_i) - \frac{a}{l} - 1 > 0$$

then for small ϵ we will have the result. Expressing this in terms of l we have that this inequality is satisfied if

$$l^4 \left(\left(\frac{l^2+1}{l^2-1} \right)^2 - 1 \right) > \left(\frac{l^2+1}{l(l^2-1)} + 1 \right)^2$$

and by calculation we see that this is true for $l > 1$. □

Lemma 4.4.11 *Given I and minimal γ such that $I \subset \gamma(\phi_i)$ then $|\gamma'(z)| \leq 1$ for all $z \in \phi_i$ and ϵ sufficiently small.*

Proof: Write γ as ${g_{l,\epsilon}}^{n_1} {h_l}^{m_1} \ldots {g_{l,\epsilon}}^{n_k} {h_l}^{m_k}$ such that $m_1, \ldots, n_k \neq 0$.

We shall prove that $|({g_{l,\epsilon}}^{n_t} {h_l}^{m_t} \ldots {g_{l,\epsilon}}^{n_k} {h_l}^{m_k})'(z)| \leq 1$ for every t by induction on $k - t$.

Base case: $k - t = 0$.

We split this into 3 cases depending on the value of m_k.

Case 1. $m_k = 0$, then ϕ_i lies outside $I_{{g_{l,\epsilon}}^{n_k}}$ by the table in Lemma 4.4.7 we have $|({g_{l,\epsilon}}^{n_k})'(z)| < 1$.

Case 2. $m_k < 0$, then

$$|({g_{l,\epsilon}}^{n_k}{h_l}^{m_k})'(z)| = |({g_{l,\epsilon}}^{n_k})'({h_l}^{m_k}z)||({h_l}^{m_k})(z)| = |({g_{l,\epsilon}}^{n_k})'({h_l}^{m_k}z)|\frac{1}{l^{|m_k|}}$$

which is less than 1 as ${h_l}^{m_k}(z)$ is outside $D(g_{l,\epsilon}) \cup D({g_{l,\epsilon}}^{-1})$ because $\phi_i = \phi_3$ or ϕ_4 by the table in Lemma 4.4.7.

Case 3. $m_k > 0$, then $\phi_i = \phi_1$ or ϕ_2 then

$$|{g_{l,\epsilon}}^{n_k}{h_l}^{m_k})'(z)| = |{g_{l,\epsilon}}^{n_k-1}(g_{l,\epsilon}{h_l}^{m_k}z)||{g_{l,\epsilon}}'({h_l}^{m_k})'(z)|l^{m_k}$$

now $|{g_{l,\epsilon}}^{n_k-1}(g_{l,\epsilon}{h_l}^{m_k}z)| \leq 1$ as the isometric circles of $g_{l,\epsilon}$ are disjoint so we will look at the last two terms. They are

$$|{g_{l,\epsilon}}'({h_l}^{m_k})'(z)|l^{m_k} = \frac{l^{m_k}}{|c(il)^{m_k}z + a|^2} = \frac{1}{|c(il)^{m_k}l^{-\frac{m_k}{2}}z + al^{-\frac{m_k}{2}}|^2}$$

which is less than or equal to 1 iff

$$|c(il)^{m_k}l^{-\frac{m_k}{2}}z + al^{-\frac{m_k}{2}}| \geq 1.$$

Using the triangle inequality we get that this is true is

$$|c|l^{\frac{m_k}{2}}|z| - al^{-\frac{m_k}{2}} \geq 1$$

but $|z|$ is greater than $\min_i \mathrm{dist}_{\inf}(0, \phi_i)$ so the above inequality is satisfied if

$$|c|l^{\frac{m_k}{2}} \min_{i=1,2} \mathrm{dist}_{\inf}(0, \phi_i) - al^{-\frac{m_k}{2}} \geq 1.$$

But

$$|c|l^{\frac{m_k}{2}} \min_{i=1,2} \mathrm{dist}_{\inf}(0, \phi_i) - al^{-\frac{m_k}{2}} \geq |c|l \min_{i=1,2} \mathrm{dist}_{\inf}(0, \phi_i) - a^{-1}l^{-1}$$

as $|c| \min_{i=1,2} \mathrm{dist}_{\inf}(0, \phi_i), a > 0$ so by Lemma 4.4.10 we are done.

The inductive step is done in exactly the same way. $\square$

Lemma 4.4.12 *The length of I is less than $L = \max_i \operatorname{diam}(\phi_i)$.*

Proof: Find minimal γ such that $I \subset \gamma(\phi_i)$ then by Lemma 2.2.5

$$\operatorname{diam}(I) \leq \operatorname{diam}(\gamma(\phi_i)) \leq \max_{z,w\in\phi_i} |\gamma'(z)|^{1/2}|\gamma'(z)|^{1/2}|z-w| \leq \operatorname{diam}(\phi_i)$$

by Lemma 4.4.11. □

4.4.2 Using bounds on $\Omega(G_{l,\epsilon})$ to get bounds on $\Omega(\Gamma_{l,\epsilon})$

For contradiction assume that $\Gamma_{l,\epsilon}$ is classical and let $\mathcal{C}$ be a set of defining curves which are circles. Note that the circles $\Gamma\mathcal{C}$ are pairwise disjoint.

Definition 4.4.13 Let $\{C_i\}_{i=1\ldots n}$ be the complete collection of circles in $\Gamma\mathcal{C}$ that separate 0 and ∞ and cut $\mathbb{R}$ inside $R(g_{l,\epsilon})$. We give the collection $\{C_i\}_{i=1\ldots n}$ the order such that C_i separates 0 and C_{i+1}. Let C_0 be the unique circle in $\Gamma\mathcal{C}$ that separates 0 and ∞ with the property C_0 intersects $(0, \frac{a-1}{|c|})$ and $C_0 \cap (0, \frac{a-1}{|c|})$ is greater than $C \cap (0, \frac{a-1}{|c|})$ for any other circle C that separates 0 and ∞. We define C_{n+1} to be the unique circle in $\Gamma\mathcal{C}$ that separates 0 and ∞ and intersects $(\frac{a+1}{|c|}, \infty)$ at the lowest point out of all circles in $\Gamma\mathcal{C}$ that separate 0 and ∞.

Lemma 4.4.14 *The collection $\{C_i\}_{i=0\ldots n+1}$ always contains at least two elements and if we let $z_k^j = j^{k-1}(0,\infty) \cap C_j$ then*

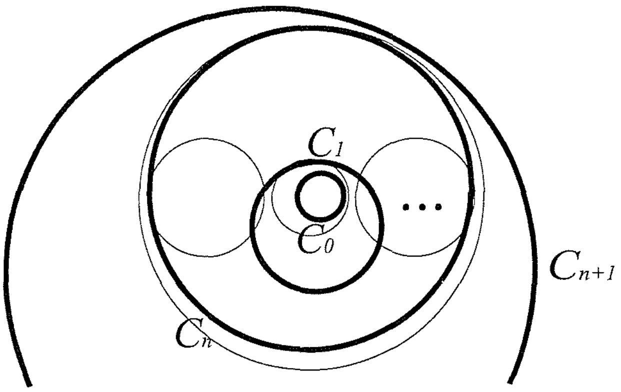
$$\frac{l^{-7}(a-1)}{|c|} \leq |z_k^j| \leq \frac{l^7(a+1)}{|c|}$$

for all k and j. Also note that

$$\operatorname{rad} C_j \leq 2\frac{l^7(a+1)}{|c|}$$

for all j.

Figure 4.8: The Images of the Defining Circles that separate 0 and ∞



Proof: By Lemma 4.1 of [Mar74] we know that there is a fundamental domain D for $\Gamma_{l,\epsilon}$ bounded by circles in $\Gamma_{l,\epsilon}\mathcal{C}$ such that D separates 0 and ∞ so there are at least two circles in $\Gamma_{l,\epsilon}\mathcal{C}$ that separate 0 and ∞ and we have that C_0 and C_{n+1} must exist.

Since the circles are disjoint, for the upper bound, we only need to check the assertion for $i = n+1$.

If $z_1^{n+1} > l^4 \frac{a+1}{|c|}$ then $h_l{}^{-4}C_{n+1}$ is in $\mathcal{C}$ and contradicts the definition of C_{n+1} so $z_1^{n+1} < l^4 \frac{a+1}{|c|}$.

Since the circles $\Gamma_{l,\epsilon}\mathcal{C}$ are disjoint, preserved by $\Gamma_{l,\epsilon}$ and in particular h_l we have

$$z_2^{n+1} \leq l z_1^{n+1} \leq l^5 \frac{a+1}{|c|}$$

and

$$z_3^{n+1} \leq l^6 \frac{a+1}{|c|}$$

and

$$z_4^{n+1} \leq l^7 \frac{a+1}{|c|}$$

by the same argument.

The lower bound works in exactly the same way.

The bound on the radius is obvious. $\square$

Definition 4.4.15 As C_i and C_{i+1} are adjacent in $\Gamma_{l,\epsilon}\mathcal{C}$ we have that they bound a fundamental region which we denote D_i.

Lemma 4.4.16 *The lengths of at least two components of $D_i \cap (\mathbb{R} \cup i\mathbb{R})$ are less than $l^7 L$.*

Proof: Let the boundary circles of D_i be C_i, C_{i+1}, C and C' say, then C and C' intersect at most two half axes. The other two half axis intersect D_i in α_1 and α_2 say.

We will consider α_1 and the same argument will work for α_2.

Let the furthest point on α_1 from 0 be denoted x_1.

By Lemma 4.4.14 the distance from 0 to x_1 is less than $\frac{l^7(a+1)}{|c|}$ so we can find $k \geq -7$ such that $|0 - {h_l}^k x_1| \in \left(\frac{(a-1)}{|c|}, \frac{(a+1)}{|c|}\right)$. There may be more than one possible value of k as $\left(\frac{(a-1)}{|c|}, \frac{(a+1)}{|c|}\right)$ is smaller than the increase in l, otherwise the group would obviously be classical.

If ${h_l}^k x_1$ is in $\mathbb{R}$ then α_1 it intersects $D(g_{l,\epsilon}) \cup D({g_{l,\epsilon}}^{-1})$ and by Lemma 4.4.12 we have that

$$l^k \mathrm{diam}(\alpha_1) = \mathrm{diam}({h_l}^k \alpha_1) \leq \max_i \mathrm{diam}(\phi_i) = L$$

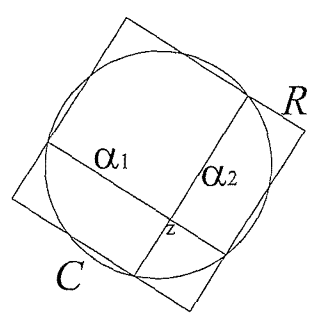
so that $\mathrm{diam}(\alpha_1) \leq l^7 L$.

If ${h_l}^k x_1$ is in $i\mathbb{R}$ then $h_l {h_l}^k x_1$ lies on $\mathbb{R}$. Consider $\Omega(G_{l,\epsilon})$ then $h_l {h_l}^k x_1$ either lies inside $D(g_{l,\epsilon}) \cup D({g_{l,\epsilon}}^{-1})$ or outside it away from 0. This means that either α_1 intersects $D(g_{l,\epsilon}) \cup D({g_{l,\epsilon}}^{-1})$ or is contained in one of the ϕ_i's. In either case we have the bound. $\square$

Lemma 4.4.17 *Given a point z in a circle C and two chords α_1, α_2 that meet at right angles and the rectangle R with the property that each side of R meets an endpoint of one of the chords at right angles then the circumference of R is greater than twice the diameter of C.*

Proof:

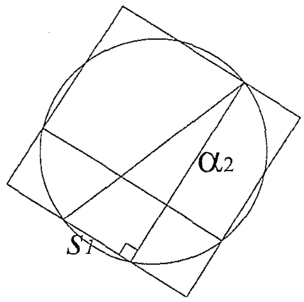
Figure 4.9: Setup for big rectangle and small circle



Let x_1 be an endpoint of α_1 and e_1 the edge of R that contains x_1. Consider the segment s_1 of e_1 that is contained in C, let y_1 be the other endpoint of this segment.

Now s_1 and α_2 meet at right angles and are both chords so the hypotenuse of this triangle is a diameter.

Figure 4.10: Proof that the rectangle is larger than the circle



So the diameter of the circle is less than the diameter of the rectangle which is less than the length of two incident sides of R. $\square$

Lemma 4.4.18 *Given circles C_j and C_{j+1} let $x_k = C_j \cap i^{k-1}(0,\infty)$ and $y_k = C_{j+1} \cap i^{k-1}(0,\infty)$ then if $|x_m - y_m|$ and $|x_n - y_n|$ for $m \neq n$ are both less than η then*

$$|x_k - y_k| \leq \eta 8282555$$

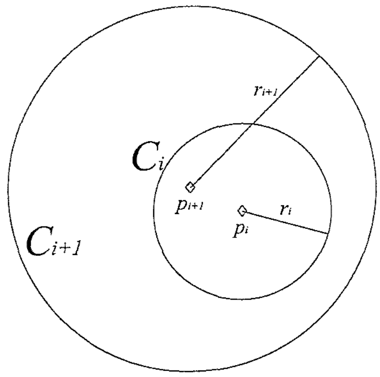
for all k such that $l \in (1,2)$ and $\epsilon > 0$.

Proof: Let p_i be the centre of C_i and r_i the radius and let $S_{r_{i+1}}(p_{i+1}) = C_{i+1}$.

This proof works by bounding $|x_k - y_k|$ in terms of $|p_i - p_{i+1}|$. First we bound $|p_i - p_{i+1}|$ in terms of an angle and then bound this angle in terms of l.

Construct a new circle C with centre p_i and radius $|p_{i+1} - p_i|$ then C touches C_{i+1} at a point z, let z' be the antipode of z then p_{i+1}, z', p_i and z all lie on a line. Let $z_k = C \cap i^{k-1}(0,\infty)$.

Figure 4.11: Initial Setup



Define θ_n to be the angle $\angle z' p_i z_n$ and θ_m to be the angle $\angle z' p_i z_m$.

We split the problem into two cases.

Case 1. $\theta_n, \theta_m \geq \frac{\pi}{2}$.

We now prove a series of inequalities relating these constants.

Let α_n and α_m be the angles $\angle p_i z' z_n$ and $\angle p_i z' z_m$ respectively then

$$\sin(\theta_n) + \sin(\theta_m) \geq \sin(\alpha_n) + \sin(\alpha_m) \tag{4.4.1}$$

by the sine rule as we are in Case 1.

We let r be the radius of C then

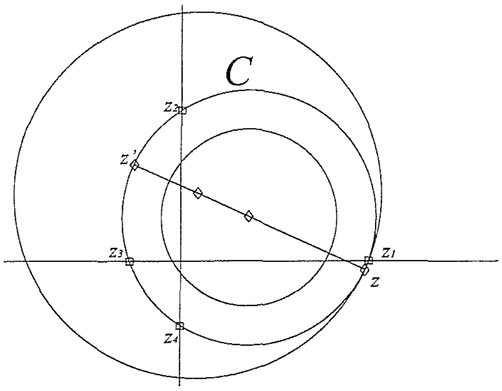
$$r = r_{i+1} - |p_i - p_{i+1}| \tag{4.4.2}$$

since $|p_{i+1} - z| = r_{i+1} = |p_{i+1} - p_i| + r$ as all 4 points lie on a straight line.

We have

$$\sin(\alpha_n) = \frac{|z_n - z|}{2r} \text{ and } \sin(\alpha_m) = \frac{|z_m - z|}{2r} \tag{4.4.3}$$

Figure 4.12: Construct the intermediate circle



since $\alpha_n = \angle zz'z_n$ and z to z' is a diameter so the angle $\angle zz_nz'$ is a right angle, the same is true for α_m.

We now find a lower bound for $\sin(\theta_n) + \sin(\theta_m)$ using 4.4.1 to 4.4.3,

$$\frac{|z_n - z|}{2r} + \frac{|z_m - z|}{2r} \geq \frac{|z_n|}{\sum_k |z_k|} \tag{4.4.4}$$

as $|z_n - z| + |z_m - z| \geq |z_n - z_m| \geq |z_n|$ or $|z_m|$ as $\angle z_n 0 z_m$ is a right angled triangle. We have $2r \leq \sum_k |z_k|$ by Lemma 4.4.17.

As the circles C_i and C_{i+1} are part of the boundary of a fundamental domain D_i for the action of $\Gamma_{l,\epsilon}$ on $\Omega(\Gamma_{l,\epsilon})$ we must have that $h_l D_i \cap D_i = \emptyset$ so that

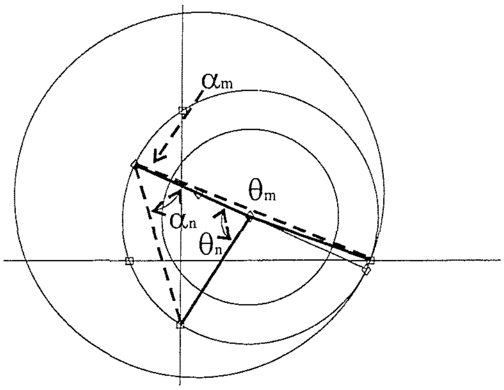
$$l|z_k| > |z_{k+1}| \tag{4.4.5}$$

for all k.

We have that $\sin(\theta_n) + \sin(\theta_m) \geq \frac{|z_n|}{\sum_k |z_k|}$ by 4.4.1 to 4.4.4. Without loss of generality we assume that $\theta_n \leq \theta_m$ then

$$2\sin(\theta_n) \geq \sin(\theta_n) + \sin(\theta_m) \geq \frac{1}{1 + l + l^2 + l^3}$$

Figure 4.13: Bound the angles



by 4.4.5. We can use this to get that

$$\frac{1}{1+\cos(\theta_n)} \leq \frac{1}{1-\sqrt{1-\frac{1}{4(1+l+l^2+l^3)^2}}} \tag{4.4.6}$$

by Pythagorases Theorem and as $\theta_n \in [\frac{\pi}{2}, \pi]$ we have that $\cos(\theta_n)$ is negative.

Case 2. Assume that at least one of $\theta_n, \theta_m < \frac{\pi}{2}$, without loss of generality assume that $\theta_n < \frac{\pi}{2}$ then

$$\frac{1}{1+\cos(\theta_n)} \leq 1 \leq \frac{1}{1-\sqrt{1-\frac{1}{4(1+l+l^2+l^3)^2}}}$$

which is the same equation as 4.4.6.

Consider the triangle $p_{i+1}p_iz_n$ then

$$|p_{i+1}-z_n|^2 = |p_i - p_{i+1}|^2 + r^2 - 2|p_i - p_{i+1}|r\cos(\theta_n)$$

by the cosine rule which implies

$$(r_{i+1} - |x_n - y_n|)^2 < |p_i - p_{i+1}|^2 + r^2 - 2|p_i - p_{i+1}|r\cos(\theta_n) \tag{4.4.7}$$

as $|p_{i+1} - z_n| > r_{i+1} - |y_n - z_n| > r_{i+1} - |x_n - y_n|$ by the triangle inequality applied to $p_{i+1}y_nz_n$ and the fact the C lies between C_i and C_{i+1}.

Applying 4.4.2 to 4.4.7 we have

$$(r_{i+1} - |x_n - y_n|)^2$$

$$< |p_i - p_{i+1}|^2 + (r_{i+1} - |p_i - p_{i+1}|)^2 - 2|p_i - p_{i+1}|(r_{i+1} - |p_i - p_{i+1}|)\cos(\theta_n)$$

so

$$\begin{aligned}(r_{i+1} - \eta)^2 < |p_i - p_{i+1}|^2 + (r_{i+1} - |p_i - p_{i+1}|)^2 \\ -2|p_i - p_{i+1}|(r_{i+1} - |p_i - p_{i+1}|)\cos(\theta_n)\end{aligned} \tag{4.4.8}$$

since $|x_n - y_n| \leq \eta$.

From 4.4.8 we get that

$$\frac{r_{i+1}\eta}{(1+\cos(\theta_n))\frac{l^{-7}(a-1)}{2|c|}} > |p_i - p_{i+1}| \tag{4.4.9}$$

using the fact that $r_{i+1} - |p_i - p_{i+1}| = r > r_i > \frac{l^{-7}(a-1)}{2|c|}$ since C_i contains 0 and $C_i \cap (0, \infty) \geq \frac{l^{-7}(a-1)}{|c|}$ by Lemma 4.4.14.

We apply 4.4.6 to 4.4.9 to get that

$$\eta\frac{\sqrt{2}\frac{l^7(a+1)}{|c|}}{\left(1 - \sqrt{1 - \frac{1}{4(1+l+l^2+l^3)^2}}\right)\frac{l^{-7}(a-1)}{2|c|}}$$

$$= \eta\frac{2\sqrt{2}l^{14}(a+1)}{\left(1 - \sqrt{1 - \frac{1}{4(1+l+l^2+l^3)^2}}\right)(a-1)} > |p_i - p_{i+1}| \tag{4.4.10}$$

as $r_{i+1} \leq \sqrt{2}\frac{l^7(a+1)}{|c|}$ by Lemma 4.4.14. This is our bound on $|p_i - p_{i+1}|$.

We prove the following for $k = 3$ so that $x_k, y_k \in (-\infty, 0)$ although the proof works for any k. We have that

$$(x_3 - re(p_i))^2 + (im(p_i))^2 = r_i^2$$

and

$$(y_3 - re(p_{i+1}))^2 + (im(p_{i+1}))^2 = r_{i+1}^2$$

so that

$$x_3 = re(p_i) - \sqrt{r_i^2 - (im(p_i))^2}$$

and

$$y_3 = re(p_{i+1}) - \sqrt{r_{i+1}^2 - (im(p_{i+1}))^2}$$

as $x_3 \leq re(p_i)$ and $y_3 \leq re(p_{i+1})$.

So we have that

$$|x_3 - y_3| = \left| re(p_{i+1}) - \sqrt{r_{i+1}^2 - (im(p_{i+1}))^2} - re(p_i) + \sqrt{r_i^2 - (im(p_i))^2} \right|$$

$$\leq |re(p_{i+1}) - re(p_i)| + \left| \sqrt{r_{i+1}^2 - (im(p_{i+1}))^2} - \sqrt{r_i^2 - (im(p_i))^2} \right|$$

$$\leq |p_{i+1} - p_i| + |r_{i+1}^2 - (im(p_{i+1}))^2 - r_i^2 + (im(p_i))^2|$$

$$\leq |p_{i+1} - p_i| + |r_{i+1} - r_i||r_{i+1} + r_i| + |im(p_{i+1}) - im(p_i)||im(p_{i+1}) + im(p_i)|$$

$$\leq |p_{i+1} - p_i| + |r_{i+1} - r_i|^2 + |im(p_{i+1}) - im(p_i)|^2$$

$$\leq |p_{i+1} - p_i| + |r_{i+1} - r_i||r_{i+1} + r_i| + |im(p_{i+1}) - im(p_i)||im(p_{i+1}) + im(p_i)|$$

$$\leq |p_{i+1} - p_i| + \frac{2\sqrt{2}l^7(a+1)}{|c|}|r_{i+1} - r_i| + \frac{2\sqrt{2}l^7(a+1)}{|c|}|p_{i+1} - p_i|$$

by Lemma 4.4.14. Using the fact that $|r_{i+1} - r_i| < |p_{i+1} - p_i| + \eta$ we get that

$$|x_3 - y_3| \leq |p_{i+1} - p_i| + \frac{4\sqrt{2}l^7(a+1)}{|c|}|p_{i+1} - p_i| + \frac{2\sqrt{2}l^7(a+1)}{|c|}\eta$$

$$= |p_{i+1} - p_i| \left(1 + \frac{4\sqrt{2}l^7(a+1)}{|c|} \right) + \eta \frac{2\sqrt{2}l^7(a+1)}{|c|}.$$

On applying 4.4.10 we get that

$$|x_3 - y_3| \leq \eta \frac{2\sqrt{2}l^7(a+1)}{|c|} +$$

$$\eta\frac{2\sqrt{2}l^{14}(a+1)}{\left(1-\sqrt{1-\frac{1}{4(1+l+l^2+l^3)^2}}\right)(a-1)}\left(1+\frac{4\sqrt{2}l^7(a+1)}{|c|}\right) \tag{4.4.11}$$

which is a linear bound in η.

However for $l \in (1, \sqrt{2})$ and $\epsilon > 0$ we get that

$$\frac{2\sqrt{2}l^7(a+1)}{|c|} \leq \sqrt{2}^{11}$$

$$\frac{a+1}{a-1} \leq 2$$

and

$$1-\sqrt{1-\frac{1}{4(1+l+l^2+l^3)^2}} \geq .008$$

by calculation, plugging these all in to 4.4.11 we get

$$|x_3 - y_3| \leq \eta\left(\frac{4\sqrt{2}^{1}5}{0.008}\left(\sqrt{2}^{13}+1\right)+\sqrt{2}^{1}1\right) \leq \eta 8282555$$

as required. ☐

Definition 4.4.19 Let C be the unique image of C_{n+1} under $< {h_l}^4 >$ that intersects $[-l^4\frac{a+1}{|c|}, -\frac{a+1}{|c|})$.

Lemma 4.4.20 *$\Gamma_{l,\epsilon}$ is non-classical for fixed $l \in (1, \sqrt{2})$ and $\epsilon > 0$ sufficiently small.*

Proof: The contradiction we shall derive is that $g_{l,\epsilon}(C)$ must intersect at least one of the C_i, which contradicts the fact that they are all disjoint.

Let $C \cap \mathbb{R} = \{x, y\}$ where $y < 0 < x$ then

$$|g_{l,\epsilon}(x) - g_{l,\epsilon}(y)| \geq |g_{l,\epsilon}(\infty) - g_{l,\epsilon}\left(-l^4\frac{a+1}{|c|}\right)| = \frac{1}{|c||l^4(2+\epsilon)+2l^2+1-\epsilon|}$$

as the component of $\widehat{\mathbb{C}} - C$ that contains ∞ does not contain $\text{cen} I_{g_{l,\epsilon}}$.

By Lemma 4.4.16 we can apply Lemma 4.4.18 with $\eta = l^7 L$ to get that

$$|x_i - x_{i+1}| \leq l^7 L 8282555$$

where $x_i = C_i \cap (0, \infty)$.

We have

$$L = \frac{\sqrt{(2la + c + cl^2)(-2la + c + cl^2)}}{a}$$

by calculation and $L \to 0$ as $\epsilon \to 0$, which was the point of setting up ϵ in the first place.

To get a contradiction we need

$$\frac{1}{|c||l^4(2+\epsilon) + 2l^2 + 1 - \epsilon|} \geq l^7 L 8282555$$

since C is outside $I_{g_{l,\epsilon}}$ so is contained in $D(g_{l,\epsilon})$.

But

$$\frac{1}{|c||l^4(2+\epsilon) + 2l^2 + 1 - \epsilon|} \to \frac{l^2 - 1}{2l(l^2+1)^2}$$

which is greater than 0 for all $l > 1$ so we have the desired contradiction for L small enough. □

Chapter 5

Dynamics

This chapter is split into four sections. The first three introduce the main definitions and tools we will use. In the first section we define Hausdorff dimension and give some basic invariance results. In section two we define the exponent of convergence of the Poincaré series and give the important result that for geometrically finite Kleinian groups it is equal to the Hausdorff dimension of the groups limit set. Using this we prove that if the Hausdorff dimension of the limit sets of a sequence of Schottky groups $\Gamma(n) =< g(n), h(n) >$ vanishes then at least one of the generators, $g(n)$ or $h(n)$, leaves $PSL_2(\mathbb{C})$. In chapter 6 we examine to what extent the converse of this statement is true. In section 3 we state the famous Birkhoff Ergodic Theorem. In the last section we prove technical results that allow us to analyse the Hausdorff dimension of a Schottky group. To do this we introduce the full measured set $L(g) \subset \Lambda(\Gamma)$ and the set $\widetilde{L}_\epsilon(t) \subset \Gamma$. We then bound the growth rate of $\widetilde{L}_\epsilon(t)(j)$ for small t. In the next part of this section we describe the relationship between $L(g)$ and $\widetilde{L}_\epsilon(t)$ when $t = m(D(g) \cup D(g^{-1}))$, specifically we show that $L(g)$ can be viewed as the "conical boundary" of $\widetilde{L}_\epsilon(t)(j)$. This with a result on the way that the embedded Cayley tree of Γ

approaches its limit set allows us to prove that when calculating the exponent of convergence of the Poincaré Series of the group Γ we only need to consider $\widetilde{L}_\epsilon(t)$ and not the whole group Γ. It is this result and the bound on the growth of $\widetilde{L}_\epsilon(t)$ which are the technical tools we use in Chapter 6.

5.1 Hausdorff Measure

In this section we give the definition of the Hausdorff dimension of a subset of $\widehat{\mathbb{C}}$. We also state that Hausdorff dimension is invariant under elements of $PSL_2(\mathbb{C})$.

Definition 5.1.1 Given a set $X \subset \widehat{\mathbb{C}}$ and $d \geq 0$ then the d**-dimensional Hausdorff measure** of E is

$$\mathcal{H}^d(X) = \limsup_{\epsilon \to 0} \inf \sum_i diam(U_i)^d$$

where the infimum is taken over all ϵ open covers of X, in other words countable collections of open sets U_i that cover X such that $diam(U_i) \leq \epsilon$ for all i.

Note that $\mathcal{H}^d$ takes values in $[0, \infty]$.

Lemma 5.1.2 *Given a set $X \subset \widehat{\mathbb{C}}$ then there is a unique number $\mathcal{H}(X) \in [0, \infty]$ such that $\mathcal{H}^d(X)$ is 0 for $d > \mathcal{H}(X)$ and ∞ for $d < \mathcal{H}(X)$.*

Proof: Take $\epsilon < 1$ then $diam(U_i)^d$ is a decreasing function of d so we have that $\mathcal{H}^d(X) \leq \mathcal{H}^{d'}(X)$ for $d > d'$.

Suppose that $\mathcal{H}^d(E) \in (0, \infty)$ and $\delta > 0$ then

$$\mathcal{H}^{d+\delta}(X) = \limsup_{\epsilon \to 0} \inf \sum_i diam(U_i)^{d+\delta} \leq \limsup_{\epsilon \to 0} \epsilon^\delta \sum_i diam(U_i)^d$$

$$\leq \limsup_{\epsilon \to 0} \epsilon^{\delta} \mathcal{H}^d(E) = 0.$$

Suppose that $\mathcal{H}^d(E) \in (0, \infty)$ and $\delta > 0$ then

$$\mathcal{H}^{d-\delta}(X) = \limsup_{\epsilon \to 0} \inf \sum_i diam(U_i)^{d-\delta} \geq \limsup_{\epsilon \to 0} \epsilon^{-\delta} \sum_i diam(U_i)^d$$

$$\geq \limsup_{\epsilon \to 0} \epsilon^{-\delta} \mathcal{H}^d(E) = \infty.$$

Next suppose that $\mathcal{H}^d(X) \notin (0, \infty)$ for any d. Then either $\mathcal{H}^d(X)$ is constant and equal to 0 or ∞ or there is some jump point d where $\mathcal{H}^{\delta}(X) = \infty$ for $\delta < d$ and $\mathcal{H}^{\delta}(X) = 0$ for $\delta > d$. At d $\mathcal{H}^d(X)$ may be 0 or ∞. This is the only jump point since $\mathcal{H}^d(X) \leq \mathcal{H}^{d'}(X)$ for $d > d'$.

□

This definition can be generalised to any metric space see [Fal97]. If $X \subset \mathbb{R}^m$ then $\mathcal{H}^n(X)$ is a constant times the n-dimensional Lebesgue measure of X [Fal97].

Definition 5.1.3 We define the **Hausdorff dimension of a Kleinian group** $\mathcal{H}(\Gamma)$ to be the Hausdorff dimension of its limit set.

The Hausdorff dimension of a Kleinian group is invariant under conjugation by Möbius transformations, this follows from the fact that the limit set is closed and that Hausdorff dimension is invariant under bi-Lipshitz maps see [Fal97].

Ruelle [Rue82] using techniques developed by Bowen [Bow79] showed that the Hausdorff dimension is a real-analytic function over either Quasi-Fuchsian or Schottky Space. Anderson and Rocha [AR97] extended this result to a wider class of Kleinian groups.

5.2 Patterson-Sullivan Theory

In this section we introduce the Poincaré series and its exponent of convergence which are the main tools in Patterson-Sullivan Theory. We state the important result that the exponent of convergence of a geometrically finite group is the same as the Hausdorff dimension of its limit set. We also prove that if a sequence of Schottky groups satisfies $\mathcal{H}(< g(n), h(n) >) \to 0$ then one of the generators leaves $PSL_2(\mathbb{C})$. It is worth noting that this is true for any set of generators.

Definition 5.2.1 Given a Kleinian group Γ and $p \in \mathbb{H}^3$ and $s > 0$ we define the **Poincaré series** to be

$$\sum_{\gamma \in \Gamma} \exp(-sd(p, \gamma p)).$$

The **Poincaré series** is independent of the base point p chosen, for a proof of this in the Fuchsian case see [Pat76a].

Definition 5.2.2 The **exponent of convergence** $\delta(\Gamma)$ of the Poincaré series of a Kleinian Γ is the infimum over all $s > 0$ such that

$$\sum_{\gamma \in \Gamma} \exp(-sd(p, \gamma p)).$$

converges.

A proof of the following theorem can be found in [Sul84]

Theorem 5.2.3 *Given a geometrically finite Kleinian group Γ then $\delta(\Gamma) = \mathcal{H}(\Gamma)$.*

As an application of Patterson-Sullivan theory we prove the following.

Lemma 5.2.4 *Given a sequence of Schottky groups $\Gamma(n)$ and any set of generators $\{g(n), h(n)\}$ such that $\mathcal{H}(\Gamma(n)) \to 0$ then either $g(n)$ or $h(n)$ leaves $PSL_2(\mathbb{C})$.*

Proof: We shall prove this Lemma by contradiction. We shall show that if both the generators do not diverge then $\liminf \mathcal{H}(\Gamma(n)) > 0$.

This proof is split into two parts. First we show that we can assume that the generators converge then the second part shows that in this case we can find a uniform lower bound on the exponent of convergence of the Poincaré Series.

Consider the accumulation points A of $\mathcal{H}(\Gamma(n))$. Now $A \subset [0, \infty]$ (in fact $A \subset [0, 2]$) so we can split this into two cases, either $A \subset [\epsilon, \infty]$ for some $\epsilon > 0$ or there is some subsequence $\mathcal{H}(\Gamma(n_m))$ of $\mathcal{H}(\Gamma(n))$ such that $\mathcal{H}(\Gamma(n_m)) \to 0$.

Suppose that for every subsequence n_m there is a subsubsequence n_{m_k} such that $\liminf \mathcal{H}(\Gamma(n_{m_k})) > 0$. Then we cannot be in the second case as in this case there is some subsequence n_m with the property that $\mathcal{H}(\Gamma(n_m)) \to 0$, but every subsubsequence n_{m_k} of n_m satisfies $\liminf \mathcal{H}(\Gamma(n_{m_k})) = 0$ which is a contradiction. This means that we are in the first case, so there is some uniform lower bound on $\mathcal{H}(\Gamma(n))$.

This means that if we can show that given any subsequence n_m we can find a subsubsequence n_{m_k} such that $\liminf \mathcal{H}(\Gamma(n_{m_k})) > 0$ then we will have proved the Lemma.

Given a subsequence n_m of n we can find a subsubsequence n_{m_k} such that $g(n_{m_k}) \to g$ and $h(n_{m_k}) \to h$ where $g, h \in PSL_2(\mathbb{C})$. We shall show that $\liminf \mathcal{H}(< g(n_{m_k}), h(n_{m_k}) >) > 0$ which by the above argument is enough to show that the original sequence has a lower bound.

Given $\gamma \in \Gamma(n_{m_k})$ write γ as a reduced word $\zeta_1 \ldots \zeta_{l(\gamma)}$.

By the triangle inequality we have

$$d(p, \gamma p) \leq \sum_{i=1}^{l(\gamma)} d(p, \zeta_i p) \leq l(\gamma) \max_{\xi \in G(\Gamma(n_{m_k}))} d(p, \xi p).$$

We apply this bound to the Poincaré series with exponent α

$$\sum_{\gamma \in \Gamma(n_{m_k})} \exp(-\alpha d(p, \gamma p)) \geq \sum_{\gamma \in \Gamma(n_{m_k})} \exp\left(-\alpha l(\gamma) \max_{\xi \in G(\Gamma(n_{m_k}))} d(p, \xi p)\right)$$

$$= 4 \sum_k 3^{k-1} \exp\left(-\alpha k \max_{\xi \in G(\Gamma(n_{m_k}))} d(p, \xi p)\right)$$

since the number of elements in a genus 2 free group of length k is $4\ 3^{k-1}$.

This diverges iff

$$3 \exp\left(- \max_{\xi \in G(\Gamma(n_{m_k}))} d(p, \xi p)\right)^{\alpha} \geq 1$$

or

$$\alpha \leq \frac{\log(3)}{\max_{\xi \in G(\Gamma(n_{m_k}))} d(p, \xi p)}.$$

So let $\alpha = \frac{\log(3)}{\max_{\xi \in G(\Gamma(n_{m_k}))} d(p, \xi p)}$ then the Poincaré series diverges but this means that the Hausdorff dimension of $\Gamma(n_{m_k})$ is greater than or equal to α by Theorem 5.2.3.

In conclusion we have

$$\mathcal{H}(\Gamma(n_{m_k})) \geq \frac{\log(3)}{\max_{\xi \in G(\Gamma(n_{m_k}))} d(p, \xi p)}$$

and so

$$\liminf \mathcal{H}(\Gamma(n_{m_k})) \geq \liminf \frac{\log(3)}{\max_{\xi \in G(\Gamma(n_{m_k})))} d(p, \xi p)}$$

but since $\max_{\xi\in G(\Gamma(n_{m_k}))} d(p,\xi p) \to \max_{\xi\in\{g^{\pm1},h^{\pm1}\}} d(p,\xi p)$ we have that

$$\lim \mathcal{H}(\Gamma(n_{m_k})) \geq \frac{\log 3}{\max_{\xi\in\{g^{\pm1},h^{\pm1}\}} d(p,\xi p)}$$

which is strictly greater than 0. □

5.3 The Birkhoff Ergodic Theorem

In this section we introduce the Birkhoff Ergodic Theorem along with a the statement that we can apply it to the ergodic map $f : \Lambda(\Gamma) \to \Lambda(\Gamma)$ as in Definition 3.7.8.

Definition 5.3.1 Given a probability space (X, m) and a map $\tau : X \to X$ then τ is **measurable** if $\tau^{-1}E$ is a measurable set for every measurable set E.

Definition 5.3.2 Given a probability space (X, m) and a measurable map $\tau : X \to X$ then τ is **m-preserving** if $m(\tau^{-1}E) = m(E)$ for every measurable set E.

Definition 5.3.3 Given a probability space (X, m) and a measurable m-preserving map $\tau : X \to X$ then τ is **ergodic** if $\tau^{-1}E = E$ implies that $m(E) = 0$ or 1.

Definition 5.3.4 Given a measurable m-preserving map $\tau : X \to X$ and $\phi : X \to \mathbb{R}$ we define the function $S_n : X \to \mathbb{R}$ by

$$S_n\phi(x) = \sum_{i=0}^{n-1} \phi(\tau^i x).$$

The following Theorem was proved by Birkhoff [Bir31].

Theorem 5.3.5 *Given a probability space (X, m) and $\tau : X \to X$ an ergodic measurable m-preserving surjective map then*

$$\lim_{n\to\infty} \frac{1}{n} S_n\phi(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(\tau^i x) = \int_X \phi \, dm$$

for every $m-$measurable function ϕ and m almost every $x \in X$.

The following Lemma was proved by Bowen [Bow79].

Lemma 5.3.6 *Given a Schottky group Γ and f as in Definition 3.7.8 then f is an ergodic $\mathcal{H}^d$-preserving surjective map on $\Lambda(\Gamma)$, where $d = \mathcal{H}(\Gamma)$.*

5.4 The density of g in the limit set

In this section we define the sets $L(g) \subset \Lambda(\Gamma)$ and $\widetilde{L}_\epsilon(t) \subset \Gamma$. We then bound the growth rate of $\widetilde{L}_\epsilon(t)$ in terms of t. We then prove the technical Lemma that if $\zeta_l(j) \to z \in \Lambda(\Gamma)$ conically then $\{\zeta_l\}$ stays close to $\{\gamma_n\}$ where $\{\gamma_n\} \simeq z$. This results allows us to define $L(g)$ in terms of $\widetilde{L}_\epsilon(t)$. This allows us to show that when calculating the exponent of convergence we only need to consider $\widetilde{L}_\epsilon(t)$ and not the whole group Γ. It is this result and the bound on the growth of $\widetilde{L}_\epsilon(t)$ which are the tools we use in Chapter 6 to investigate what happens to the Hausdorff dimension of a divergent sequence of Schottky groups.

Lemma 5.4.1 *Given a Schottky group $\Gamma =< g, h >$ then the indicator function $1_{D(g)\cup D(g^{-1})} : \Lambda(\Gamma) \to \mathbb{R}$ restricted to $\Lambda(\Gamma)$ is measurable for any Borel* *measure and is equal to*

$$1_{D(g)\cup D(g^{-1})}(z) = \begin{cases} 1 & \text{if } \gamma_1 = g \text{ or } g^{-1} \\ 0 & \text{otherwise} \end{cases}$$

where $z \simeq \{\gamma_n\}$.

Proof: Given $z \in \Lambda(\Gamma)$ then $z = \cap_n D(\gamma_n)$ by Lemma 3.7.5 so that $z \in D(g) \cup D(g^{-1})$ iff γ_1 is g or g^{-1}.

The function is measurable as both $[D(g)\cup D(g^{-1})]\cap\Lambda(\Gamma)$ and $\widehat{\mathbb{C}}-[D(g)\cup D(g^{-1})]\cap\Lambda(\Gamma) = [D(h)\cup D(h^{-1})]\cap\Lambda(\Gamma)$ are open in the subspace topology so Borel and therefore measurable. $\square$

Definition 5.4.2 Given a Schottky group $\Gamma =< g,h >$ and $\gamma \in \Gamma$ define $g(\gamma)$ to be the number of times the letters g or g^{-1} appear in γ when it is written as a reduced word. Define $h(\gamma)$ similarly.

Definition 5.4.3 Given a Schottky group Γ then $\mathcal{H}^{\mathcal{H}(\Gamma)}(\Gamma) \in (0,\infty)$ see [Bow79] so we can define the normalised $\mathcal{H}^{\mathcal{H}(\Gamma)}$ measure m by

$$m(E) = \frac{\mathcal{H}^{\mathcal{H}(\Gamma)}(E)}{\mathcal{H}^{\mathcal{H}(\Gamma)}(\Lambda(\Gamma))}$$

where E is a $\mathcal{H}^{\mathcal{H}(\Gamma)}$ measurable set.

Lemma 5.4.4 *Given a Schottky group* $\Gamma =< g,h >$ *then for* m*-almost every* $\{\gamma_n\} \simeq z \in \Lambda(\Gamma)$ *we have*

$$\lim_{n\to\infty} \frac{g(\gamma_n)}{n} = m[D(g) \cup D(g^{-1})]$$

where m *is defined above.*

Proof: Consider the probability measure $(\Lambda(\Gamma), m)$ and the function $f : \Lambda(\Gamma) \to \Lambda(\Gamma)$ as in Definition 3.7.8.

Apply S_k to this setup with $\phi = 1_{D(g)\cup D(g^{-1})}$ then

$$S_k(1_{D(g)\cup D(g^{-1})})z = \sum_{i=0}^{k-1} 1_{D(g)\cup D(g^{-1})} f^i z$$

$$= \sum_{i=0}^{k-1} 1_{D(g)\cup D(g^{-1})} \left(f^i \pi((\gamma_n^{-1}\gamma_{n+1})_{n\geq 0})\right)$$

where π is defined in Definition 3.7.2, $z \simeq \{\gamma_n\}$ and we define γ_0 to be the identity.

By Lemma 3.7.9 this is

$$\sum_{i=0}^{k-1} 1_{D(g)\cup D(g^{-1})} \left(\pi(\tau^i(\gamma_n^{-1}\gamma_{n+1})_{n\geq 0})\right) = \sum_{i=0}^{k-1} 1_{D(g)\cup D(g^{-1})} \left(\pi((\gamma_n^{-1}\gamma_{n+1})_{n\geq i})\right)$$

by the definition of τ.

Now $\pi((\gamma_n^{-1}\gamma_{n+1})_{n\geq i}) \in D(g) \cup D(g^{-1})$ iff $\gamma_i^{-1}\gamma_{i+1} = g$ or g^{-1} by the definition of π. So we are counting the number of times the letters g or g^{-1} appear in γ_k when it is written as a reduced word, which is $g(\gamma_k)$.

The result now follows directly from Theorem 5.3.5 by Lemma 5.3.6. $\square$

Since $\mathcal{H}^{\mathcal{H}(\Gamma)}(\Gamma)$ and m have the same 0-sets we can replace m in the above Lemma by $\mathcal{H}^{\mathcal{H}(\Gamma)}$.

Definition 5.4.5 Let $L(g)$ be the set of points $z \in \Lambda(\Gamma)$ for which $\frac{g(\gamma_n)}{l(\gamma_n)} \to m(D(g) \cup D(g^{-1}))$ where $z \simeq \{\gamma_n\}$.

$L(g)$ has full measure by Lemma 5.4.4.

The following is a purely combinatorial discussion and the results depend only on the properties of a free group and not on the particular generators of a Schottky group.

Definition 5.4.6 Given a free group $F_2 =< g, h >$ and constants $t \in [0, 1]$ and $\epsilon \in (0, \frac{t}{2})$ define

$$\widetilde{L}_\epsilon(t) = \left\{ \gamma \in \Gamma \left| \frac{g(\gamma)}{l(\gamma)} \in (t - \epsilon, t + \epsilon) \right. \right\}$$

We will link the growth rate of $\widetilde{L}_\epsilon(t)$ to the constant t. Before we can do this we need the following Lemmas.

Lemma 5.4.7 *Given $a \in \mathbb{N}$ then $\binom{a-b}{b-1}$ is an increasing function of b for $b < \frac{a}{4}$.*

Proof: We shall show that

$$\binom{a-b}{b-1} \leq \binom{a-b-1}{b}$$

for $b < \frac{a}{4}$.

Now

$$\frac{(a-b)!}{(b-1)!(a-2b+1)!} \leq \frac{(a-b-1)!}{b!(a-2b-1)!}$$

iff

$$1 \leq \frac{(a-2b)(a-2b+1)}{b(a-b)}$$

or that

$$a^2 - 5ba + a + 5b^2 - 2b \geq 0.$$

This equation is satisfied for

$$b \geq \frac{5a + 2 + \sqrt{5a^2 + 4}}{10} \text{ and } b \leq \frac{5a + 2 - \sqrt{5a^2 + 4}}{10}.$$

Now $b - 1 < a - b$ otherwise the binomial is invalid so $b \not\geq \frac{5a+2+\sqrt{5a^2+4}}{10}$. Which means we are left with $b \leq \frac{5a+2-\sqrt{5a^2+4}}{10}$. However

$$\frac{a}{4} < \frac{5a + 2 - \sqrt{5a^2 + 4}}{10}$$

for all $a > 0$ so if $b < \frac{a}{4}$ then $b < \frac{5a+2-\sqrt{5a^2+4}}{10}$ and we have the result. □

Lemma 5.4.8 *Given $n, m > 0$ then*

$$\frac{1}{m(n-m)} \left(\frac{n}{n-m}\right)^n \left(\frac{n-m}{m}\right)^m \leq \binom{n}{m} \leq n \left(\frac{n}{n-m}\right)^n \left(\frac{n-m}{m}\right)^m$$

for $n > 7$, $m > 7$ and $n - m > 7$.

Proof: We coarsen the following formula found in [Rob55],

$$\sqrt{2}\sqrt{\pi} n^{n+1/2} \exp(-n + 1/(12n + 1)) \leq n! \leq \sqrt{2}\sqrt{\pi} n^{n+1/2} \exp(-n + 1/(12n))$$

to

$$\exp(-n)n^n \leq n! \leq n \exp(-n)n^n$$

for $n > 7$.

So

$$\begin{aligned}\frac{n!}{(n-m)!m!} &\leq \frac{n \exp(-n)n^n}{\exp(-m)m^m \exp(-(n-m))(n-m)^{(n-m)}} \\ &= n \left(\frac{n}{n-m}\right)^n \left(\frac{n-m}{m}\right)^m\end{aligned}$$

as required.

The lower bound is done in the same way.

$\square$

Lemma 5.4.9 *Given $T \in \mathbb{R}$ then*

$$\frac{\lceil Tn \rceil}{n}, \frac{\lfloor Tn \rfloor}{n} \to T$$

as $n \to \infty$, where $\lceil n \rceil$ is the smallest integer greater than n and $\lceil n \rceil$ is the greatest integer less than n.

Proof: We have that

$$\frac{Tn-1}{n} \leq \frac{\lfloor Tn \rfloor}{n} \leq \frac{\lceil Tn \rceil}{n} \leq \frac{Tn+1}{n}$$

and so they all converge to T as $n \to \infty$. $\square$

The following Lemma will allow us to pull back a result that holds in the limit.

Lemma 5.4.10 *Given a sequence $a_n \in \mathbb{R}$ such that $\frac{\log a_n}{n} \to a \in \mathbb{R}$ and $b > a$ then there is an $N > 0$ such that $a_n \leq \exp(b)^n$ for all $n > N$.*

Proof: We shall prove this by contradiction. Assume that

$$a_n > \exp(b)^n \text{ for all } n$$

then

$$\frac{\log(a_n)}{n} > b \text{ for all } n$$

but this means that $\lim \frac{\log(a_n)}{n} \geq b$ which is the desired contradiction. It is worth pointing out that the inequality must be strict. $\square$

Lemma 5.4.11 *Given a free group $F_2 =< g, h >$ then*

$$t^{-\frac{tn}{2}} \leq \mathrm{card}\{\gamma \in \widetilde{L}_\epsilon(t) | l(\gamma) = n\} \leq t^{-6tn}$$

for $t > 0$ sufficiently small, $\epsilon \in (0, \frac{t}{2})$ and n sufficiently large depending on t and ϵ.

Proof: This proof is split into three parts. In the first part we express $\mathrm{card}\{\gamma \in \widetilde{L}_\epsilon(t) | l(\gamma) = n\}$ as a summation over k where $\gamma = g^{n_1} \ldots h^{m_k}$. The second part is the upper bound. We solve for the maximum value that k can be. Then we express $\mathrm{card}\{\gamma \in \Gamma | l(\gamma) = n, k = i, m_j, n_j \geq 0\}$ as a binomial. These two results allow us to find an upper bound that only involves t. We then take limits and apply Lemma 5.4.10. For the lower bound we show that $\{\gamma \in \Gamma | l(\gamma) = n, k = i, n_j = 1, m_j > 0, \lceil n(t-\epsilon) \rceil \leq k \leq \lfloor n(t+\epsilon) \rfloor\}$ is contained in the set we are considering. For fixed k we express this as a binomial. We then remove the dependence on k and use Lemma 5.4.7 to remove the dependence on ϵ. So once again we have a formula involving just t and n and we take limits and apply Lemma 5.4.10.

Before we start on the upper and lower bounds we express

$$\mathrm{card}\{\gamma \in \widetilde{L}_\epsilon(t) | l(\gamma) = n\}$$

as a summation.

Given $\gamma \in \Gamma$ express it as $g^{n_1} h^{m_1} \ldots g^{n_k} h^{m_k}$ where $n_j, m_j \neq 0$ except for n_1 and m_k which may be 0.

We have

$$\operatorname{card}\{\gamma \in \widetilde{L}_\epsilon(t) | l(\gamma) = n\}$$

$$= \sum_{i=1}^{K} 2^i 2^i \operatorname{card}\{\gamma \in \widetilde{L}_\epsilon(t) | l(\gamma) = n, k = i, m_j, n_j > 0\}$$

$$+ \sum_{i=1}^{K} 2^{i-1} 2^i \operatorname{card}\{\gamma \in \widetilde{L}_\epsilon(t) | l(\gamma) = n, k = i, m_j, n_j > 0 \text{ except } n_1 = 0\}$$

$$+ \sum_{i=1}^{K} 2^i 2^{i-1} \operatorname{card}\{\gamma \in \widetilde{L}_\epsilon(t) | l(\gamma) = n, k = i, m_j, n_j > 0 \text{ except } m_k = 0\}$$

$$+ \sum_{i=1}^{K} 2^{i-1} 2^{i-1} \operatorname{card}\{\gamma \in \widetilde{L}_\epsilon(t) | l(\gamma) = n, k = i, m_j, n_j > 0 \text{ except } n_1, m_k = 0\}$$

where K is the maximum possible value of k over all the 4 options, this does not change the summation as if a particular i is too large then $\operatorname{card}\{\gamma \in \widetilde{L}_\epsilon(t) | \ldots\} = 0$. We have taken the case that n_1, m_k are 0 or not separately then split it up into sums depending on the value of k. Finally we note that given the absolute values $|n_1|, \ldots, |m_k|$ there are $2^{i-1} 2^{i-1} 2^{\operatorname{card}\{n_1, m_k \neq 0\}}$ possible elements of the group that can have these absolute values. We are using the fact that the inclusion of $\gamma \in \widetilde{L}_\epsilon(t)$ depends only on the absolute values, $|n_1|, \ldots, |m_k|$, and note on the signs of the exponents.

The upper bound.

The upper sum we use is

$$\operatorname{card}\{\gamma \in \widetilde{L}_\epsilon(t) | l(\gamma) = n\}$$

$$\leq 4 \sum_{i=1}^{K} 2^i 2^i \operatorname{card}\{\gamma \in \widetilde{L}_\epsilon(t) | l(\gamma) = n, k = i, m_j, n_j > 0 \text{ except } n_1, m_k \geq 0\}$$

where we have taken the greatest generality of the elements to consider and also taken an upper bound on the component coming from the fact in the card we are only considering absolute values i.e the 2^is.

Next we find an upper bound on K.

We first look at the inequalities coming from the ranges of ϵ

$$t < t + \epsilon < \frac{3t}{2} \tag{5.4.1}$$

and

$$\frac{t}{2} < t - \epsilon < t \tag{5.4.2}$$

Given $\gamma \in \widetilde{L}_\epsilon(t)$ such that $l(\gamma) = n$ and $\gamma = g^{n_1}h^{m_1}\ldots g^{n_k}h^{m_k}$ then

$$t - \epsilon < \frac{\sum_{i=1}^{k}|n_i|}{n} < t + \epsilon$$

and an upper bound on k is when all the $|n_i| = 1$ except for n_1 which is 0 and in this case we have

$$k - 1 \leq n(t + \epsilon) \Rightarrow K \leq \frac{3tn}{2} + 1 \tag{5.4.3}$$

by equation 5.4.1.

We now bound

$$\text{card}\{\gamma \in \widetilde{L}_\epsilon(t)|l(\gamma) = n, k = i, m_j, n_j > 0 \text{ except } n_1, m_k \geq 0\}$$

by

$$\text{card}\{\gamma \in \Gamma|l(\gamma) = n, k = i, m_j, n_j \geq 0\}.$$

To calculate this we set up a bijection from C which is the set of all ways we can place $2k-1$ objects into $n+1$ slots to $\{\gamma \in \Gamma|l(\gamma) = n, k = i, m_j, n_j \geq 0\}$.

Elements of C are sets $\{s_1, \ldots s_{2k-1}\}$ where $s_j \in \{1, \ldots n+1\}$ and $s_j < s_{j+1}$ for all j.

We define the map

$$\kappa : C \to \{\gamma \in \Gamma | l(\gamma) = n, k = i, m_j, n_j \geq 0\}$$

as follows, given $\{s_1, \ldots {}_{2k-1}\} \in C$ then

$$\kappa(\{s_1, \ldots s_{2k-1}\}) = \gamma = g^{n_1} h^{m_1} \ldots g^{n_k} h^{m_k}$$

where

$$n_1 = s_1 - 1, m_1 = s_2 - s_1, \ldots, n_k = s_{2k-1} - s_{2k-2}, m_k = n + 1 - s_{2k-1}$$

We see that $\gamma \in \{\gamma \in \widetilde{L}_\epsilon(t) | l(\gamma) = n, k = i, m_j, n_j > 0$ except $n_1, m_k \geq 0\}$ as

$$l(\gamma) = \sum_{j=1}^{k} m_j + n_j = -1 + s_1 - s_1 + s_2 - \ldots - s_{2k-2} + s_{2k-1} - s_{2k-1} + n + 1 = n$$

and $m_j, n_j > 0$ except for n_1 and m_k which are greater than or equal to 0.

To show that κ is a bijection we define the map

$$\nu : \{\gamma \in \Gamma | l(\gamma) = n, k = i, m_j, n_j \geq 0\} \to C$$

as follows, given $\gamma \in \{\gamma \in \widetilde{L}_\epsilon(t) | l(\gamma) = n, k = i, m_j, n_j > 0$ except $n_1, m_k \geq 0\}$ define

$$s_1 = 1 + n_1, s_2 = 1 + n_1 + m_1, \ldots, s_{2k-1} = 1 + n_1 + m_1 + \ldots + n_k$$

it is worth noting that m_k does not appear in the definition of ν.

We shall now check that ν is well defined, i.e that $\{s_1, \ldots, s_{2k-1}\} \in C$. First we see that $s_j \in \{1, \ldots, n+1\}$ as $1 + n_1 + m_1 + \ldots n_k \leq n + 1$ and $n_j, m_j \geq 0$ for all j. Since $n_2, \ldots, n_{2k-1}, m_1, \ldots, m_{2k-2} \geq 1$ we have that $s_j < s_{j+1}$ for all j.

We complete the proof by showing that $\kappa\nu = \nu\kappa = id$.

Given $\gamma \in \{\gamma \in \widetilde{L}_\epsilon(t) | l(\gamma) = n, k = i, m_j, n_j > 0 \text{ except } n_1, m_k \geq 0\}$ then $\nu(\gamma) = \{s_1, \ldots, s_{2k-1}\}$ is defined by

$$s_1 = 1 + n_1, s_2 = 1 + n_1 + m_1, \ldots, s_{2k-1} = 1 + n_1 + m_1 + \ldots + n_k.$$

Now $\kappa(\{s_1, \ldots, s_{2k-1}\})$ is defined by

$$n_1' = s_1 - 1, m_1' = s_2 - s_1, \ldots, n_k' = s_{2k-1} - s_{2k-2}, m_k' = n + 1 - s_{2k-1}$$

but this means that

$$n_1' = n_1, m_1' = m_1, \ldots, n_k' = n_k, m_k' = n + 1 - (1 + n_1 + m_1 + \ldots + n_k) = m_k$$

as required.

The case of $\nu\kappa$ is similar.

So the cardinality of $\text{card}\{\gamma \in \Gamma | l(\gamma) = n, k = i, m_j, n_j \geq 0\}$ is equal to the number of ways to place $2k - 1$ objects into $n + 1$ places which is

$$\binom{n+1}{2k-1}.$$

We have shown that

$$K \leq \left\lceil \frac{3tn}{2} \right\rceil + 1$$

and

$$\text{card}\{\gamma \in \Gamma | l(\gamma) = n, k = i, m_j, n_j \geq 0\} \leq \binom{n+1}{2i-1}$$

so that

$$\text{card}\{\gamma \in \widetilde{L}_\epsilon(t) | l(\gamma) = n\} \leq 4 \sum_{i=1}^{\lceil \frac{3tn}{2} \rceil + 1} 4^i \binom{n+1}{2i-1}.$$

We wish to find an upper bound for this that does not involve i.

If $\frac{n+1}{2} > 2i - 1$ then we can maximise the binomial by maximising the i but we have that $i \leq \left\lceil \frac{3tn}{2} \right\rceil + 1$ so we need that

$$\frac{n+1}{2} > 2\left\lceil \frac{3tn}{2} \right\rceil + 2 - 1$$

which is true if

$$n > \frac{5}{1-6t}$$

but for fixed $t < \frac{1}{6}$ this is satisfied for n large enough.

So under these conditions we have that

$$\mathrm{card}\{\gamma \in \widetilde{L}_\epsilon(t) | l(\gamma) = n\} \leq 4\left(\left\lceil \frac{3tn}{2} \right\rceil + 1\right) 4^{\left\lceil \frac{3tn}{2} \right\rceil + 1} \binom{n+1}{2(\left\lceil \frac{3tn}{2} \right\rceil + 1) - 1}$$

by taking the maximum value $\left\lceil \frac{3tn}{2} \right\rceil + 1$ of i for each i.

We wish to bound this above using Lemma 5.4.8 so to apply this Lemma we need that

$$n + 1 > 7 \ , \ 2\left\lceil \frac{3tn}{2} \right\rceil + 1 > 7 \text{ and } n + 1 - \left(2\left\lceil \frac{3tn}{2} \right\rceil + 1\right) > 7$$

which are satisfied for $t < \frac{1}{3}$, so t small and n large enough.

So for n large enough we apply Lemma 5.4.8 to get that

$$\mathrm{card}\{\gamma \in \widetilde{L}_\epsilon(t) | l(\gamma) = n\} \leq 4\left(\left\lceil \frac{3tn}{2} \right\rceil + 1\right) 4^{\left\lceil \frac{3tn}{2} \right\rceil + 1}$$

$$\left(\frac{n+1}{n+1-(2\left\lceil \frac{3tn}{2} \right\rceil + 1)}\right)^{n+1} \left(\frac{n+1-(2\left\lceil \frac{3tn}{2} \right\rceil + 1)}{2\left\lceil \frac{3tn}{2} \right\rceil + 1}\right)^{2\left\lceil \frac{3tn}{2} \right\rceil + 1} . \qquad (5.4.4)$$

We wish to show that this is less than $(t^{-6t})^n$. We take the *log* of the righthand side of the inequality in equation 5.4.4 and divide by n then the limit of this as $n \to \infty$ is

$$\frac{3t\log(4)}{2} + \log\left(\frac{1}{1-3t}\right) + 3t\log\left(\frac{1-3t}{3t}\right) \qquad (5.4.5)$$

by Lemma 5.4.9.

We shall show that for small t that this is strictly less that $-6t\log(t)$.

We will deal with equation 5.4.5 term by term, the first term satisfies

$$\frac{3t\log(4)}{2} < -\frac{6t\log(t)}{4}$$

for $t < \frac{1}{4}$.

For the second term we want that

$$\log\left(\frac{1}{1-3t}\right) < -\frac{6t\log(t)}{4}.$$

Both sides of this inequality converge to 0 as $t \to 0$. As both are left differentiable as $t \to 0$ we have that the inequality is satisfied, for small t, if it is satisfied for the derivatives of the functions, i.e if

$$\frac{3}{1-3t} < \frac{3}{2}\left(\log\frac{1}{t} - 1\right)$$

as $t \to 0$, but the left hand side converges to 3 and the right hand side converges to ∞ so for small enough t the equation must be satisfied.

The third term,

$$3t\log\left(\frac{1-3t}{3t}\right) < -\frac{6t\log(t)}{2}$$

iff $\frac{1-3t}{3t} < \frac{1}{t}$ or that $1-3t < 3$ which is satisfied for all positive t.

We put this all together to get that equation 5.4.5 is strictly less than $-\frac{6t\log(t)}{4} - \frac{6t\log(t)}{4} - \frac{6t\log(t)}{2} = -6t\log(t)$ for small t. And so equation 5.4.4 is strictly less than t^{-6tn} for large n by Lemma 5.4.10.

We now do the lower bound.

We have that

$$\{\gamma \in \Gamma | l(\gamma) = n, k = i, n_j = 1, m_j > 0, \lceil n(t-\epsilon) \rceil \leq k \leq \lfloor n(t+\epsilon) \rfloor\}$$

$$\subset \{\gamma \in \widetilde{L}_\epsilon(t) | l(\gamma) = n, k = i, n_j = 1, m_j > 0 \text{ except } m_k \geq 0\}$$

as if $\lceil n(t-\epsilon) \rceil \leq k \leq \lfloor n(t+\epsilon) \rfloor$ then

$$n(t-\epsilon) \leq k \leq n(t+\epsilon)$$

and so

$$t - \epsilon \leq \frac{\sum_{i=1}^{k} |n_i|}{l(\gamma)} \leq t + \epsilon$$

since $n_i = 1$ for all i which means that $\gamma \in \widetilde{L}_\epsilon(t)$. Note that for small n there may be no k that satisfy $\lceil n(t-\epsilon) \rceil \leq k \leq \lfloor n(t+\epsilon) \rfloor\}$.

So we have that

$$\text{card}\{\gamma \in \widetilde{L}_\epsilon(t) | l(\gamma) = n\}$$

$$\geq \sum_{i=\lceil n(t-\epsilon) \rceil}^{\lfloor n(t+\epsilon) \rfloor} 2^i \text{card}\{\gamma \in \Gamma | l(\gamma) = n, k = i, n_j = 1, m_j > 0 \text{ except } m_k \geq 0\}$$

where we have absorbed the bounds on k into the summation. The component coming from the fact that we are only considering positive exponents is 2^i since the exponents of g_j is 1 for all j.

We now find a formula for $\text{card}\{\gamma \in \Gamma | l(\gamma) = n, k = i, n_j = 1, m_j > 0 \text{ except } m_k \geq 0\}$.

To do this we shall set up a bijection κ' from C' the set of all ways to place $k-1$ objects into $n-k$ slots to $\{\gamma \in \Gamma | l(\gamma) = n, k = i, n_j = 1, m_j > 0 \text{ except } m_k \geq 0\}$.

As in the upper bound an element of C' is $\{s_1, \ldots, s_{k-1}\}$ where $s_j \in \{1, \ldots, n-k\}$ and $s_j < s_{j+1}$ for all j.

We define κ' as follows, given $\{s_1, \ldots, s_{k-1}\}$ let $\kappa'(\{s_1, \ldots, s_{k-1}\}) = \gamma = g^{n_1} \ldots h^{m_k}$ where

$$m_1 = s_1, m_2 = s_2 - s_1, \ldots, m_k = n - k - s_{k-1}$$

and $n_j = 1$ for all j.

We now check that this is well defined i.e that $\gamma \in \{\gamma \in \Gamma | l(\gamma) = n, k = i, n_j = 1, m_j > 0 \text{ except } m_k \geq 0\}$. We see that $l(\gamma) = \sum n_j + m_j = k + s_1 - s_1 + s_2 - \ldots - s_k + n - k = n$ as required, $n_j = 1$ for all j by definition and m_j are all greater than 1 except for m_k which may be 0, so the map is well defined.

To show that κ' is a bijection we define

$$\nu' : \{\gamma \in \Gamma | l(\gamma) = n, k = i, n_j = 1, m_j > 0 \text{ except } m_k \geq 0\} \to C'$$

by, given $\gamma = gh^{m_1} \ldots gh^{m_k} \in \{\gamma \in \Gamma | l(\gamma) = n, k = i, n_j = 1, m_j > 0 \text{ except } m_k \geq 0\}$ then $\nu'(\gamma) = \{s_1, \ldots, s_{k-1}\}$ where

$$s_1 = m_1, s_2 = m_1 + m_2, \ldots, s_{k-1} = m_1 + \ldots + m_{k-1}.$$

This is well defined as $s_j \in \{1, \ldots, n-k\}$ since $m_1 \geq 1$ and $m_1 + \ldots + m_{k-1} \leq n - k$ and $s_j < s_{j+1}$ as $m_j \geq 1$ for $j < k$.

We shall now show that κ' is a bijection by showing that $\kappa'\nu' = id$ and $\nu'\kappa' = id$.

Given $gh^{m_1} \ldots gh^{m_k} = \gamma \in \{\gamma \in \Gamma | l(\gamma) = n, k = i, n_j = 1, m_j > 0 \text{ except } m_k \geq 0\}$ then $\nu'(\gamma) = \{s_1, \ldots, s_{k-1}\}$ is defined by

$$s_1 = m_1, s_2 = m_1 + m_2, \ldots, s_{k-1} = m_1 + \ldots + m_{k-1}.$$

Now $\kappa'(\{s_1, \ldots, s_{k-1}\}) = g^{n'_1} \ldots h^{m'_k}$ is

$$m'_1 = s_1 = m_1, m'_2 = s_2 - s_1 = m_2, \ldots, m'_k = n - k - s_{k-1} = m_k$$

and $n'_j = 1$ for all j this means that $\kappa'\nu'(\gamma) = \gamma$ as required.

The case of $\nu'\kappa'$ is similar.

This means that

$$\text{card}\{\gamma \in \Gamma | l(\gamma) = n, k = i, n_j = 1, m_j > 0 \text{ except } m_k \geq 0\} = \binom{n-i}{i-1}$$

and we have that

$$\text{card}\{\gamma \in \widetilde{L}_\epsilon(t) | l(\gamma) = n\} \geq \sum_{i=\lceil n(t-\epsilon) \rceil}^{\lfloor n(t+\epsilon) \rfloor} 2^i \binom{n-i}{i-1}$$

as long as $n - \lfloor n(t+\epsilon) \rfloor \geq \lfloor n(t+\epsilon) \rfloor - 1$, otherwise the binomial is not valid. But this inequality is satisfied for $t < \frac{1}{3}$.

Assume that n is large enough so that there is at least one summand in the sum, i.e let $\lfloor n(t+\epsilon) \rfloor - \lceil n(t-\epsilon) \rceil > 0$ which depends only on t and ϵ. There always is a lower bound on the n as $\epsilon > 0$.

For n sufficiently large we are then free to choose a particular i so let $i = \lceil n(t-\epsilon) \rceil$ and we have that

$$\text{card}\{\gamma \in \widetilde{L}_\epsilon(t) | l(\gamma) = n\} \geq 2^{\lceil n(t-\epsilon) \rceil} \binom{n - \lceil n(t-\epsilon) \rceil}{\lceil n(t-\epsilon) \rceil - 1}$$

by only considering one summand.

We wish to get rid of the dependence on ϵ. To do this we will show that

$$\binom{n - \lceil n(t-\epsilon) \rceil}{\lceil n(t-\epsilon) \rceil - 1}$$

is a decreasing function of ϵ.

We can apply Lemma 5.4.7 as long as $\frac{n}{4} > \lceil n(t-\epsilon) \rceil$ but this is true as long as $\frac{n}{4} > \lceil nt \rceil$ which is true for $n > 4nt + 4$ which is satisfied for $t < \frac{1}{5}$ and $n > 20$.

So under these conditions we can minimise

$$2^{\lceil n(t-\epsilon)\rceil}\binom{n-\lceil n(t-\epsilon)\rceil}{\lceil n(t-\epsilon)\rceil-1}$$

by maximising ϵ, but $\epsilon < \frac{t}{2}$ so we have

$$\mathrm{card}\{\gamma \in \widetilde{L}_\epsilon(t)|l(\gamma)=n\} \geq 2^{\lceil \frac{nt}{2}\rceil}\binom{n-\lceil\frac{nt}{2}\rceil}{\lceil\frac{nt}{2}\rceil-1}.$$

We wish to bound this using Lemma 5.4.8 to do this we need that

$$n-\left\lceil\frac{tn}{2}\right\rceil > 7\,,\ \left\lceil\frac{tn}{2}\right\rceil - 1 > 7 \text{ and } n-\left\lceil\frac{tn}{2}\right\rceil - \left(\left\lceil\frac{tn}{2}\right\rceil-1\right) > 7$$

which are satisfied for $t<1$ and n sufficiently large.

So we have that

$$\mathrm{card}\{\gamma \in \widetilde{L}_\epsilon(t)|l(\gamma)=n\} \geq 2^{\lceil \frac{tn}{2}\rceil-1}\frac{1}{(\lceil\frac{tn}{2}\rceil-1)(n-\lceil\frac{tn}{2}\rceil-(\lceil\frac{tn}{2}\rceil-1))}$$

$$\left(\frac{n-\lceil\frac{tn}{2}\rceil}{n-\lceil\frac{tn}{2}\rceil-(\lceil\frac{tn}{2}\rceil-1)}\right)^{n-\lceil\frac{tn}{2}\rceil}\left(\frac{n-\lceil\frac{tn}{2}\rceil-(\lceil\frac{tn}{2}\rceil-1)}{\lceil\frac{tn}{2}\rceil-1}\right)^{\lceil\frac{tn}{2}\rceil-1}. \tag{5.4.6}$$

We wish to show that this is greater than $(t^{-\frac{t}{2}})^n$. We take the *log* of the right hand side of the inequality in equation 5.4.6 and divide by n then the limit of this as $n\to\infty$ is

$$\frac{t}{2}\log(2)+\left(1-\frac{t}{2}\right)\log\left(\frac{1-\frac{t}{2}}{1-t}\right)+\frac{t}{2}\log\left(\frac{1-t}{\frac{t}{2}}\right) \tag{5.4.7}$$

by Lemma 5.4.9.

This is

$$\frac{t}{2}\log\left(4\frac{(1-t)^2}{1-\frac{t}{2}}\right)+\log\left(\frac{1-\frac{t}{2}}{1-t}\right)+\frac{t}{2}\log\frac{1}{t}$$

but

$$\frac{t}{2}\log\left(4\frac{(1-t)^2}{1-\frac{t}{2}}\right)+\log\left(\frac{1-\frac{t}{2}}{1-t}\right)>0$$

for small t as its gradient at $t=0$ is $\frac{3\log 2+1}{2}$.

In conclusion we have that equation 5.4.7 is greater that $\frac{t}{2}\log\left(\frac{1}{t}\right)$ for t small and so equation 5.4.6 is greater than $(t^{-\frac{t}{2}})^n$ for t small and n sufficiently large by Lemma 5.4.10. $\square$

We now look at the relationship between $\widetilde{L}_\epsilon(t)$ and the $L(g)$. To do this we prove that if $\{\gamma_n\} \simeq z \in \Lambda(\Gamma)$ then $\gamma_n(j) \to z$ conically.

Given an abstract free group $F_2 =< a, b >$, we let T be its Cayley Tree. We define a metric on T by letting each edge be isometric to the unit interval.

Definition 5.4.12 Given a Schottky group $\Gamma =< g, h >$ we define the **immersed Cayley tree** $T(p)$ at p to be the immersed tree in $\mathbb{H}^3$ whose vertices are $\Gamma(p)$ and whose edges are geodesic segments such that two points $\gamma(p)$ and $\zeta(p)$ are connected iff $l(\gamma\zeta^{-1}) = 1$.

There are three choices of metric we could put on $T(p)$, we could give each edge length 1 and then there would be an isometry from T to $T(p)$ or we could give $T(p)$ the metric coming from $\mathbb{H}^3$ but we shall choose the induced path metric on $T(p)$ coming from $\mathbb{H}^3$.

It is well known that all of these metrics are quasi-isometric, for example see Hamenstädt [Ham02].

Definition 5.4.13 We define a **branch** of $T(p)$ to be a ray of $T(p)$ that originates at p.

Given a branch R then the vertices $\{\gamma_n(p)\}$ of R satisfy $l(\gamma_n) = n$, $l(\gamma_n^{-1}\gamma_{n+1}) = 1$ and $\gamma_n(p) \to z \in \Lambda(\Gamma)$ for some $z \in \Lambda(\Gamma)$ so $z \simeq \{\gamma_n\}$ c.f Definition 3.7.6 for the converse.

Lemma 5.4.14 *Given a Schottky group Γ with $\infty \in D$ and $\{\gamma_n\} \simeq z \in \Lambda(\Gamma)$ with corresponding branch R in the embedded Cayley tree $T(j)$ then the sequence $\{\xi_n(j)\} \subset \Gamma(j)$ converges to z conically iff $\{\xi_n(j)\}$ lies in $N_{T(j)}(R,t)$, the t neighbourhood of R in $T(j)$ for some t.*

Proof:

We shall prove that $\xi_n(j)$ approaching z conically implies that $\{\xi_n(j)\} \subset N_{T(j)}(R,t)$ for some t by contradiction.

So assume that we have a sequence $\{\xi_n(j)\}$ that converges to z conically but does not lie in $N_{T(j)}(R,t)$ for any t. So for every $t > 0$ there is some $\xi_{n(t)}$ such that $\xi_{n(t)} \notin N_{T(j)}(R,t)$.

This means that $\{\xi_{n(t)}\} \cap \{\gamma_m\} = \emptyset$ so $D(\xi_{n(t)}) \cap D(\gamma_{m(t)}) = \emptyset$ where $m(t)$ is defined by $l(\zeta_{n(t)}) = l(\gamma_{m(t)}) = m(t)$.

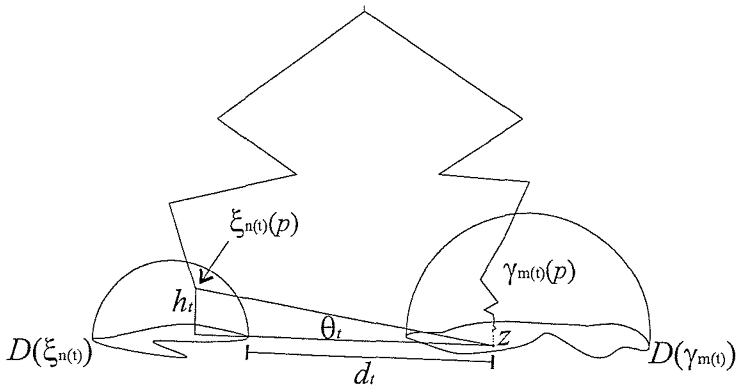
We first of all look at the setup in $\mathbb{H}^3$ then use estimates in $\widehat{\mathbb{C}}$ to get the contradiction.

As $\xi_{n(t)}(j)$ converges conically this means that $\xi_{n(t)}(p)$ converges conically for any $p \in \mathbb{H}^3$, we choose a $p \in H$ such that H is a hyperbolic plane with $\partial H \subset D$. Note that $\xi_{n(t)}(p)$ may or may not lie in $T(j)$.

Let θ_t be the angle that the Euclidean line from z to $\xi_{n(t)}(p)$ makes with $\mathbb{C}$.

Then $\tan(\theta_t) \leq \frac{h_t}{d_t}$ where h_t is the height of $\xi_{n(t)}(p)$ and d_t is the minimum distance from z to $D(\xi_{n(t)})$. Note that $d_t > 0$ by Lemma 3.1.18 and the fact that $D(\xi_{n(t)}) \cap D(\gamma_{m(t)}) = \emptyset$ where $z \in D(\gamma_{m(t)})$ by Lemma 3.7.5.

Figure 5.1: Conical Convergence means near the Immersed Cayley Tree



We shall show that $\theta_t \to 0$ so that $\{\xi_{n(t)}(p)\}$ cannot lie in a cone.

Let ζ_t be the unique maximal element of Γ such that there are elements $\widetilde{\gamma_{m(t)}}$ and $\widetilde{\xi_{n(t)}}$ where $\gamma_{m(t)} = \zeta_t\widetilde{\gamma_{m(t)}}$ and $\xi_{n(t)} = \zeta_t\widetilde{\xi_{n(t)}}$ such that $m(t) = l(\zeta_t) + l(\widetilde{\gamma_{m(t)}})$ and $m(t) = l(\xi_{n(t)}) = l(\zeta_t) + l(\widetilde{\xi_{n(t)}})$.

As $t \to \infty$ the distance in $T(j)$ from $\xi_{n(t)}(j)$ to R diverges, but this implies that $l(\widetilde{\xi_{n(t)}})$ diverges as $t \to \infty$ as it represents the geodesic in $T(j)$ from R to $\xi_{n(t)}(j)$. We shall use this fact to force $\frac{h_t}{d_t}$ to converge to 0 and so get a contradiction.

We get bounds on h_t and d_t in terms of the group elements.

Fix $w \in D$, then we can apply Lemma 3.8.6 to get that

$$h_t \leq K|\xi'_{n(t)}(w)|$$

for some $K > 0$ independent of $\xi_{n(t)}$.

Now

$$d_t \geq \mathrm{dist}_{\inf}(D(\gamma_{m(t)}), D(\xi_{n(t)})).$$

because $z \in D(\gamma_{m(t)})$.

There are $x \in D(\widetilde{\gamma_{m(t)}})$ and $y \in D(\widetilde{\xi_{n(t)}})$ such that

$$\mathrm{dist}_{\inf}(D(\gamma_{m(t)}), D(\xi_{n(t)})) = |\zeta_t x - \zeta_t y| = |\zeta_t'(y)|^{1/2}|\zeta_t'(x)|^{1/2}|x - y|$$

by Lemma 2.2.5.

By Lemma 3.8.3 we have that

$$d_t \geq K'|\zeta_t'(w)||x - y|$$

however x and y are in different components of $\widehat{\mathbb{C}} - D$ since ζ_t is maximal. So we have that $|x - y| \geq \min_{g_1,g_2 \in G(\Gamma)} \mathrm{dist}_{\inf}(D(g_1), D(g_2))$, we let $d = \min_{g_1,g_2 \in G(\Gamma)} \mathrm{dist}_{\inf}(D(g_1), D(g_2))$ then

$$d_t \geq K'd|\zeta'(w)|$$

So in conclusion we have that

$$\begin{aligned} \frac{h_t}{d_t} &\leq \frac{K|\xi_{n(t)}'(w)|}{K'd|\zeta_t'(w)|} = \frac{K}{K'd}\frac{|\zeta_t'(\widetilde{\xi_{n(t)}}w)||\widetilde{\xi_{n(t)}}'(w)|}{|\zeta_t'(w)|} \\ &\leq \frac{KK''}{K'd}|\widetilde{\xi_{n(t)}}'(w)| \end{aligned}$$

again by Lemma 3.8.3.

By our assumption we have that $l(\widetilde{\xi_{n(t)}}) \to \infty$ and by Lemma 3.8.1 and Lemma 3.8.4 this means that $|\widetilde{\xi_{n(t)}}'(z)| \to 0$ and we have the required contradiction.

We now prove the other direction. We wish to show that if $\{\xi_n(j)\} \subset N_{T(j)}(R, t)$ for some t then $\xi_n(j) \to z$ conically but this is equivalent to

$N_{T(j)}(R,t)$ being contained in a cone for any t which is equivalent to R lying in a cone. We shall prove this by contradiction so assume that some infinite subsequence $\{\gamma_{n(m)}(j)\} \subset \{\gamma_n(j)\}$ does not approach z conically.

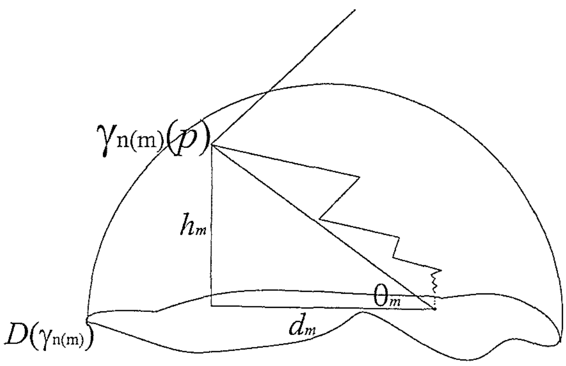
We choose a point p with the same properties as in the first part of the proof and note that $\{\gamma_n(j)\}$ tends to z conically iff $\{\gamma_n(p)\}$ tends to z conically.

This means that the Euclidean angle θ_m from z to $\gamma_{n(m)}(p)$ converges to 0 so that

$$\tan(\theta_m) = \frac{h_m}{d_m} \to 0$$

where h_m is the Euclidean height from $\mathbb{C}$ to $\gamma_{n(m)}(p)$ and d_m the distance from z to the vertical projection of $\gamma_{n(m)}(p)$ to $\mathbb{C}$.

Figure 5.2: Immersed Cayley Tree approches boundary Conically



Now $d_m \leq diamD(\gamma_{n(m)})$ as the projection of $\gamma_{n(m)}(p)$ to $\mathbb{C}$ lies in $D(\gamma_{n(m)})$ by Lemma 3.1.18 and z also lies in $D(\gamma_{n(m)})$ by Lemma 3.7.5. So $d_m \leq K|\gamma'_{m(t)}(w)|$ for some $w \in D$ by Lemma 2.2.5 and Lemma 3.8.4.

Now $h_m \geq K'|\gamma'_{m(t)}(w)|$ by Lemma 3.8.6 for some $w \in D$ and so

$$\frac{h_m}{d_m} \geq \frac{K'|\gamma'_{m(t)}(w)|}{K|\gamma'_{m(t)}(w)|} = \frac{K'}{K}$$

and so cannot converge to 0. □

Corollary 5.4.15 *Given $\{\gamma_n\} \simeq z \in \Lambda(\Gamma)$ then the corresponding branch lies in a cone centered at z with radius independent of z.*

Proof: Apply Lemma 5.4.14 to $\{\gamma_n(j)\}$ which trivially lies in $N_{T(j)}(R,t)$ for any $t > 0$. □

Definition 5.4.16 Given a set $X \subset \mathbb{H}^3$ then we define the **conical boundary** $\mathcal{C}(X)$ of X to be the set of all $z \in \widehat{\mathbb{C}}$ such that there are $x_n \in X$ such that $x_n \to z$ conically.

For a Kleinian group Γ then $\mathcal{C}(\Gamma(j))$ is the conical limit set of Γ see [Nic89].

Definition 5.4.17 Given $p, q \in \mathbb{H}^3$ and $c > 0$ then the **shadow** $b(p; q, c)$ of the hyperbolic ball $B_c(q)$ from p is the set of endpoints at infinity of all rays starting at p and intersecting $B_c(q)$. This shadow is a ball contained in $\widehat{\mathbb{C}}$.

The following Lemma gives an alternative formulation for $\mathcal{C}(X)$.

Lemma 5.4.18 *Given a set $X \subset \mathbb{H}^3$ then*

$$\mathcal{C}(X) = \bigcup_{c} \bigcap_{n \geq 1} \bigcup_{d(j,x)>n} b(j; x, c)$$

where $x \in X$.

A proof of the above Lemma can be found in Theorem 1.2.4 of [Nic89].

We now give an alternative definition of $L(g)$, see Definition 5.4.5.

Lemma 5.4.19 *$L(g)$ is equal to $\cap_{\epsilon>0}\mathcal{C}(\widetilde{L}_\epsilon(t)(j))$ where $t = m[D(g) \cup D(g^{-1})]$.*

Proof: We first show that $L(g) \subset \cap_{\epsilon>0}\mathcal{C}(\widetilde{L}_\epsilon(t)(j))$.

Let $\{\gamma_n\} \simeq z \in L(g)$ then $\frac{g(\gamma_n)}{l(\gamma_n)} \to t$ so for all $\epsilon > 0$ there is an N such that

$$t - \epsilon \leq \frac{g(\gamma_n)}{l(\gamma_n)} \leq t + \epsilon$$

for all $n > N$. This means that $\{\gamma_n\}_{n>N} \subset \widetilde{L}_\epsilon(t)$ and as $\gamma_n(j) \to z$ conically by corollary 5.4.15 we see that $z \in \mathcal{C}(\widetilde{L}_\epsilon(t)(j))$. Since this is true for all ϵ we have $z \in \cap_{\epsilon>0}\mathcal{C}(\widetilde{L}_\epsilon(t)(j))$ as required.

We now prove that $\cap_{\epsilon>0}\mathcal{C}(\widetilde{L}_\epsilon(t)(j)) \subset L(g)$.

Let $z \in \cap_{\epsilon>0}\mathcal{C}(\widetilde{L}_\epsilon(t)(j))$ then for every $\epsilon > 0$ there is a sequence $\{\zeta_l\} \subset \Gamma$ such that $\zeta_l(j)$ lies in a cone based at z and satisfies

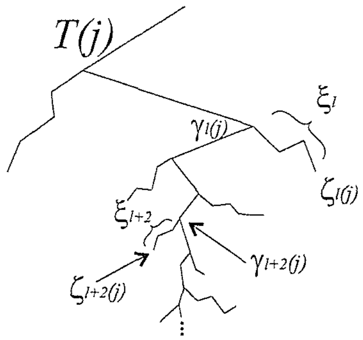
$$t - \epsilon \leq \frac{g(\zeta_l)}{l(\zeta_l)} \leq t + \epsilon$$

for all l.

We shall show that the above inequalities almost hold for γ_n where $\{\gamma_n\} \simeq z$.

By Lemma 5.4.14 the distance from $\{\zeta_l(j)\}$ to $\{\gamma_n(j)\}$ is bounded in the Cayley tree.

Figure 5.3: Finite elements at the end of long elements



So for each ζ_l there is some γ_l such that $\zeta_l = \gamma_l \xi_l$ and $l(\xi_l)$ is uniformly bounded. Since $l(\xi_l)$ is bounded over l there are only a finite number of choices for ξ_l. Pick any infinite subsequence l_m such that $\xi_{l_m} = \xi$ for all l_m.

Then

$$\frac{g(\zeta_{l_m})}{l(\zeta_{l_m})} = \frac{g(\gamma_{l_m}\xi)}{l(\gamma_{l_m}\xi)} \in [t-\epsilon, t+\epsilon]$$

and

$$\frac{g(\gamma_{l_m}) - g(\xi)}{l(\gamma_{l_m}) + l(\xi)} \leq \frac{g(\gamma_{l_m}\xi)}{l(\gamma_{l_m}\xi)} \leq \frac{g(\gamma_{l_m}) + g(\xi)}{l(\gamma_{l_m}) - l(\xi)}$$

for $l(\gamma_{l_m}) > l(\xi)$. So that

$$\frac{g(\gamma_{l_m})}{l(\gamma_{l_m})} \in [t-2\epsilon, t+2\epsilon]$$

for large l_m. But since the length of ξ_l is uniformally bounded the above equation holds for a finite number of infinite subsequences of l, so we have that

$$\frac{g(\gamma_l)}{l(\gamma_l)} \in [t-2\epsilon, t+2\epsilon]$$

for all l large enough. But as ϵ is arbitrary we have that

$$\frac{g(\gamma_l)}{l(\gamma_l)} \to t$$

and $z \in L(g)$ as required. $\square$

The next result is the main result of this part of the section. We show that when calculating the exponent of convergence we only need to consider $\widetilde{L}_\epsilon(t)$ where $t = m[D(g) \cup D(g^{-1})]$.

We state and prove the result in the ball model $\mathbb{H}^3$. Although the result holds in the upper half space model on conjugating the group by stereographic projection and replacing 0 by j.

Lemma 5.4.20 *Given a Schottky group Γ in the ball model for $\mathbb{H}^3$ and $\epsilon > 0$ then*

$$\sum_{\gamma \in \widetilde{L}_\epsilon(t)} \exp(-\delta d(0, \gamma 0)) = \infty$$

where $\widetilde{L}_\epsilon(t)$ is as in Definition 5.4.6, $t = m[D(g) \cup D(g^{-1})]$ and $\delta = \mathcal{H}(\Gamma)$.

Proof: We will prove this by contradiction, assume that

$$\sum_{\gamma \in \widetilde{L}_\epsilon(t)} \exp(-\delta d(0, \gamma 0)) < \infty.$$

Given $\eta > 0$ then there is an $N > 0$ such that

$$\sum_{\gamma \in \widetilde{L}_\epsilon(t), l(\gamma) \geq N} \exp(-\delta d(0, \gamma 0)) < \eta$$

and by Lemma 4.4.1 of [Nic89], there is an $A > 0$ such that

$$\sum_{\gamma \in \widetilde{L}_\epsilon(t), l(\gamma) \geq N} \mathcal{H}^\delta(b(0; \gamma(0), c)) < A\eta$$

for any fixed $c > 0$.

Now conjugate the group to the upper-half space model with $\infty \in D$ then choose $L > 0$ such that $d(j, \gamma j) > L \Rightarrow l(\gamma) > N$ which is possible by Lemma 3.8.1 and the various estimates in that section. However this is equivalent to $d(0, \gamma 0) > L \Rightarrow l(\gamma) > N$ on conjugating back so

$$\sum_{\gamma \in \widetilde{L}_\epsilon(t), d(0,\gamma 0) > L} \mathcal{H}^\delta(b(0; \gamma(0), c)) < A\eta.$$

By Lemma 5.4.18 we have

$$C(\widetilde{L}_\epsilon(t)) = \bigcup_{c>0} \bigcap_{l>0} \bigcup_{\gamma \in \widetilde{L}_\epsilon(t), d(0,\gamma 0) > l} b(0; \gamma(0), c)$$

so that

$$\mathcal{H}^\delta(C(\widetilde{L}_\epsilon(t))) \leq \limsup_{c \to \infty} \mathcal{H}^\delta \left(\bigcup_{\gamma \in \widetilde{L}_\epsilon(t), d(0,\gamma 0) > L} b(0; \gamma(0), c) \right)$$

$$\leq \limsup_{c \to \infty} \sum_{\gamma \in \widetilde{L}_\epsilon(t), d(0,\gamma 0) > L} \mathcal{H}^\delta(b(0; \gamma(0), c)) \leq A\eta$$

and since this is true for all $\eta > 0$ we have $\mathcal{H}^\delta(C(\widetilde{L}_\epsilon(t))) = 0$ but this contradicts the fact that $C(\widetilde{L}_\epsilon(t)) = L(g)$ by Lemma 5.4.19 which has full measure by Lemma 5.4.4. $\square$

Chapter 6

The Geometry and Dimension of sequences of Schottky groups that leave $PSL_2(\mathbb{C})$

In this chapter we classify what can happen to the classicalness and Hausdorff dimension of a sequence of divergent Schottky groups.

We define exactly what we mean by a divergent sequence of Schottky groups and then pick certain generators. We then define the cases that we will be looking at.

Recall that a sequence of Möbius transformations $\gamma(n)$ **leaves** $PSL_2(\mathbb{C})$ if it is unbounded as a set of $PSL_2(\mathbb{C})$ with the Euclidean norm see [Rat94]. We have classified how a sequence of loxodromics can leave $PSL_2(\mathbb{C})$ see Lemma 2.9.3.

Definition 6.0.21 Given a sequence of 2 generator Schottky groups $\Gamma(n)$ we say that $\Gamma(n)$ **leaves** $PSL_2(\mathbb{C})$ if given any set of generators $\{g(n), h(n)\}$

for $\Gamma(n)$ at least one of $g(n)$ or $h(n)$ diverges, see Lemma 2.9.3.

The reason for taking any set of generators is the example of $\Gamma = \Gamma(n) =< h, h^n g >$ where $h^n g$ leaves $PSL_2(\mathbb{C})$ while the group is constant.

We will be looking at the Hausdorff dimension of a Schottky group and whether or not it is classical, it is worth reiterating that both of these properties are invariant under conjugation by Möbius transformations see Lemma 3.2.4 and [Fal97].

This chapter is split into four sections. We first pick generators $\{g(n), h(n)\}$ for the groups $\Gamma(n)$ and define the cases we will work with. The next two sections deal with the cases. The last section shows that no sequence of non-classical Schottky groups can have vanishing Hausdorff dimension.

6.1 The Setup

6.1.1 Standard Generators

In this section we choose a particular set of generators for the groups $\Gamma(n)$ up to conjugation by a sequence of Möbius transformations.

To find the generators we shall need the following lemma.

Lemma 6.1.1 *Given γ and ζ Möbius transformations such that γ does not fix ∞ while ζ does fix ∞ then*

$$\mathrm{cen} I_{\gamma\zeta} = \zeta^{-1} \mathrm{cen} I_{\gamma}$$

and

$$\text{cen}I_{\zeta\gamma} = \text{cen}I_{\gamma}$$

we note here that the fact that γ does not fix ∞ guarantees the existence of the isometric circles.

Proof: The proof can either be done by direct calculation on the entries of γ and ζ or by noting that $\text{cen}I_{\gamma} = \gamma^{-1}(\infty)$ so that

$$\text{cen}I_{\gamma\zeta} = \zeta^{-1}\gamma^{-1}(\infty) = \zeta^{-1}\text{cen}I_{\gamma}$$

and

$$\text{cen}I_{\zeta\gamma} = \gamma^{-1}\zeta^{-1}(\infty) = \text{cen}I_{\gamma}$$

as ζ preserves ∞. $\square$

Definition 6.1.2 Given a Schottky group $\Gamma =< g, h >$ we say that g and h are **standard generators** if:

1. h fixes 0 and ∞,
2. the multiplier μ of h satisfies $|\mu| > 1$,
3. h has minimum absolute value of multiplier over all possible generators,
4. $\text{cen}I_g = 1$,
5. $1 \leq |\text{cen}I_{g^{-1}}| \leq \sqrt{|\mu|}$.

We have there is a minimum by the discreteness of the group.

It is possible that Standard generators are not unique, however the following Lemma shows that they always exist.

Lemma 6.1.3 *Given a Schottky group Γ then there is a Möbius transformation ϕ such that $\phi^{-1}\Gamma\phi$ has a set of standard generators.*

Proof: Find $h \in \Gamma$ such that h has minimum absolute value of multiplier over all possible generators of Γ. Then conjugate Γ by a Möbius transformation so that h fixes 0 and ∞ with multiplier μ such that $|\mu| > 1$. We will suppress this conjugation.

Now choose g to be another generator of Γ such that $\Gamma = < g, h >$.

Note that the centre of the isometric circle of g and its inverse are not 0. Assume that $I_g = 0$ then ghg^{-1} fixes ∞ but since the group is free we know that $ghg^{-1} \neq h^m$ for any m. This is a contradiction as it shows that the group is no longer discrete. The same argument works for g^{-1}.

We will be conjugating the group by Möbius transformations that fix 0 and ∞ so they leave h unchanged.

Find k and l such that

$$|\mu|^{-k} \leq |\text{cen}I_g| \leq |\mu|^{-k+1} \text{ and } |\mu|^{-l} \leq |\text{cen}I_{g^{-1}}| \leq |\mu|^{-l+1}$$

then

$$1 \leq |\mu|^k |\text{cen}I_g| \leq |\mu| \text{ and } 1 \leq |\mu|^l |\text{cen}I_{g^{-1}}| \leq |\mu|.$$

By Lemma 6.1.1 we have that

$$1 \leq |\text{cen}I_{gh^{-k}}| \leq |\mu| \text{ and } 1 \leq |\text{cen}I_{g^{-1}h^{-l}}| \leq |\mu|$$

so that

$$1 \leq |\text{cen}I_{h^l gh^{-k}}| \text{ , } |\text{cen}I_{(h^l gh^{-k})^{-1}}| \leq |\mu|$$

again by Lemma 6.1.1.

Let $z_1 = \text{cen}I_{h^lgh^{-k}}$ and $z_2 = \text{cen}I_{(h^lgh^{-k})^{-1}}$, if $|z_1| > |z_2|$ then consider $(h^lgh^{-k})^{-1}$ instead of h^lgh^{-k} so we will assume that $|z_1| \leq |z_2|$.

Now conjugate the group by $\psi(w) = z_1 w$ so that

$$\text{cen}I_{\psi^{-1}h^lgh^{-k}\psi} = 1 \text{ and } 1 \leq |\text{cen}I_{(\psi^{-1}h^lgh^{-k}\psi)^{-1}}| = \frac{|z_2|}{|z_1|} \leq |\mu|$$

by Lemma 6.1.1.

Let $z = \frac{z_2}{z_1}$, if $|z| \leq \sqrt{|\mu|}$ then we are done so assume that $|z| > \sqrt{|\mu|}$.

Conjugate the group by χ where $\chi(w) = zw$ then

$$\text{cen}I_{\chi^{-1}\psi^{-1}h^lgh^{-k}\psi\chi} = \frac{1}{z} \text{ and } \text{cen}I_{(\chi^{-1}\psi^{-1}h^lgh^{-k}\psi\chi)^{-1}} = 1$$

where $\frac{1}{|\mu|} \leq \frac{1}{|z|} \leq \frac{1}{\sqrt{|\mu|}}$. Look at

$$\chi^{-1}\psi^{-1}h^lgh^{-k}\psi\chi h^{-1}$$

then

$$\text{cen}I_{\chi^{-1}\psi^{-1}h^lgh^{-k}\psi\chi h^{-1}} = \frac{\mu}{z}$$

and

$$\text{cen}I_{(\chi^{-1}\psi^{-1}h^lgh^{-k}\psi\chi h^{-1})^{-1}} = 1$$

by Lemma 6.1.1, but $1 \leq \frac{|\mu|}{|z|} \leq \sqrt{|\mu|}$ and we are done. □

6.1.2 The Cases

We will investigate what can happen to the Hausdorff dimension and the classicalness of a sequence of Schottky groups $\Gamma(n)$ that leaves $PSL_2(\mathbb{C})$. To do this we will split the problem up into a number of cases.

By Lemma 6.1.3 we can assume that the Schottky groups $\Gamma(n)$ have standard generators $\{g(n), h(n)\}$. We are investigating the situation that $\Gamma(n)$ leaves $PSL_2(\mathbb{C})$ so by definition one of the generators $g(n)$ or $h(n)$ leaves $PSL_2(\mathbb{C})$.

We shall show that $g(n)$ always leaves $PSL_2(\mathbb{C})$.

If $h(n)$ leaves $PSL_2(\mathbb{C})$ then either $\mu(n)$ the multiplier of $h(n)$ is unbounded or the fixed points of $h(n)$ converge to each other. However we have chosen standard generators so that the fixed points of $h(n)$ are 0 and ∞ so the fixed points of $h(n)$ cannot converge to each other. We are therefore in the case that $\mu(n)$ is unbounded but we have chosen $\mu(n)$ to have minimal absolute value over all generators of $\Gamma(n)$ which means that the multiplier of $\lambda(n)$ of $g(n)$ is also unbounded. So $g(n)$ leaves $PSL_2(\mathbb{C})$.

We shall split the problem up into various cases depending on what $g(n)$ and $h(n)$ converge/diverge to, see Lemma 2.9.3.

To help the calculations we will assume that various objects associated to $g(n)$ and $h(n)$ converge (in $\widehat{\mathbb{C}}$). In particular we will assume that the multipliers and fixed points of $g(n)$ and $h(n)$ converge. Also that the centre and radius of the isometric circles of $g(n)$ and $g(n)^{-1}$ converge. In the infinite case (to be defined) we will need that $\frac{|z(n)|}{\sqrt{|\mu(n)|}}$ and $\frac{|\lambda(n)|}{|\mu(n)|}$ converge. We are talking about convergence since we are working with a divergent sequence of loxodromics we say that the various objects converge in $\widehat{\mathbb{C}}$ or $\widehat{\mathbb{R}}$ whichever is appropriate. If we are given a sequence for which these objects do not converge then we can always pass to a subsequence as we are only considering a finite number of objects.

We are now ready to define the various cases.

Definition 6.1.4 We define the following cases:

A. if the multiplier $\lambda(n)$ of $g(n)$ diverges

1. and $|\mu(n)|$ the multiplier of $h(n)$ diverges then this is the **infinite case**,
2. and $\lim |\mu(n)| \in (1, \infty)$ then this is the **loxodromic case**,
3. and $|\mu(n)| \to 1$ then this is the **identity/elliptic case**,

B. if the fixed points of $g(n)$ converge to the same point and $\lambda(n) \not\to \infty$

1. and $\lim |\mu(n)| \in (1, \infty)$ then this is the **bounded case**,
2. and $|\mu(n)| \to 1$ then this is the **identity/elliptic converging case**.

We note that the cases are mutually exclusive. We have chosen these particular cases as they imply information when combined with standard generators. For instance $|\mu(n)| \to 1$ implies that $|\text{cen} I_{g(n)}| \to 1$ which gives useful control.

Lemma 6.1.5 *Given any sequence of Schottky groups $\Gamma(n)$ with standard generators $\{g(n), h(n)\}$ that leaves $PSL_2(\mathbb{C})$ then there is a subsequence of $\Gamma(n)$ that is in one of the above cases.*

Proof: We choose a subsequence such that all the various objects we have discussed converge. This is possible because we are only considering a finite number of objects.

By the above discussion we know that $g(n)$ leaves $PSL_2(\mathbb{C})$. If $\lambda(n)$ the multiplier of $g(n)$ is unbounded then $|\lambda(n)|$ converges to ∞ and $\lim |\mu(m_n)| \in [1, \infty]$ which means that we are in one of the first 3 cases.

If the multiplier of $g(n)$ is bounded then the fixed points of $g(n)$ must both converge to the same point by Lemma 2.9.3 and we are in one of the last two cases. $\square$

6.2 The Multiplier Diverges

We shall often use the following Lemma to prove classicalness.

Lemma 6.2.1 *Given a Schottky group Γ with standard generators $\{g, h\}$ then the group is classical if*

$$|\mu|(1 - \mathrm{rad} I_g) > |\mathrm{cen} I_{g^{-1}}| + \mathrm{rad} I_g$$

where μ is the multiplier of h and $|\lambda| > 3 + 2\sqrt{2}$.

Proof: Consider the circles $S_\delta(0)$, $S_{|\mu|\delta}(0), I_g$ and $I_{g^{-1}}$. I_g and $I_{g^{-1}}$ are disjoint by Lemma 2.9.2 so we just need to check that the isometric circles are disjoint from $S_\delta(0)$ and $S_{|\mu|\delta}(0)$ for some δ.

Let $\delta = 1 - \mathrm{rad} I_g - \epsilon$ for small ϵ then the circles are all disjoint if

$$1 - \mathrm{rad} I_g > \delta \text{ and } |\mathrm{cen} I_{g^{-1}}| - \mathrm{rad} I_g > \delta$$

and

$$1 + \mathrm{rad} I_g < \delta|\mu| \text{ and } |\mathrm{cen} I_{g^{-1}}| + \mathrm{rad} I_g < \delta|\mu|.$$

Since $|\mathrm{cen} I_{g^{-1}}| > 1$ as we have standard generators we only need to check that

$$1 - \mathrm{rad} I_g > \delta \text{ and } |\mathrm{cen} I_{g^{-1}}| + \mathrm{rad} I_g < \delta|\mu|.$$

The first inequality is automatically satisfied by our choice of δ.

We shall now show that the second inequality is satisfied. We are assuming that

$$|\mu|(1 - \mathrm{rad}I_g) > |\mathrm{cen}I_{g^{-1}}| + \mathrm{rad}I_g$$

so that $\delta|\mu| > |\mathrm{cen}I_{g^{-1}}| + \mathrm{rad}I_g$ if

$$\delta|\mu| > |\mu|(1 - \mathrm{rad}I_g)$$

however this is true by our choice of ϵ. So that both inequalities are satisfied and the group is classical. $\square$

6.2.1 The Loxodromic case

This is the case that $\mu(n) \to \mu$ so that $h(n)$ converges to a loxodromic and $\lambda(n) \to \infty$.

We will first prove that the groups are eventually classical and then show that the Hausdorff dimension tends to 0.

Lemma 6.2.2 *Given a sequence of Schottky groups $\Gamma(n)$ in the loxodromic case with standard generators $\{g(n), h(n)\}$ then $\Gamma(n)$ is eventually classical.*

Proof: As $h(n)$ fixes 0 and ∞ for all n and $\lim |\mathrm{mult}(h(n))| \in (0, \infty)$ we know that $h(n) \to h$ some loxodromic and if we let $\mu(n) = \mathrm{mult}(h(n))$ and $\mu = \mathrm{mult}(h)$ then $\mu(n) \to \mu$.

Since we have standard generators we have that $cenI_{g(n)} = 1$ for all n, we let $I_{g(n)^{-1}} = z(n)$. We denote $\mathrm{mult}g(n)$ by $\lambda(n)$.

The group $\Gamma(n)$ is classical if

$$|z(n)| + \text{rad}I_{g(n)} < |\mu(n)|(1 - \text{rad}I_{g(n)})$$

by Lemma 6.2.1.

By Lemma 2.9.1 we know that

$$\text{rad}I_{g(n)} = \frac{\sqrt{|\lambda(n)|}|1 - z(n)|}{|1 + \lambda(n)|}$$

which converges to 0 as $n \to \infty$ since $|\lambda(n)| \to \infty$ and $|z(n)| \leq \sqrt{|\mu(n)|} \to \sqrt{|\mu|}$ which is finite.

Since we have standard generators we know that $|z(n)| \leq \sqrt{|\mu(n)|}$ so that the group is classical if

$$\sqrt{|\mu(n)|} + \text{rad}I_{g(n)} < |\mu(n)|(1 - \text{rad}I_{g(n)})$$

but this is eventually satisfied as $\text{rad}I_{g(n)} \to 0$. $\square$

We will now show that the Hausdorff dimension converges to 0. To do this we need the following result.

Definition 6.2.3 Given a sequence of classical Schottky groups $\Gamma(n) =< g(n), h(n) >$ with fundamental domains $D(n)$ then we say that the sequence is **nicely bounded** if:

1. $\infty \in D(n)$ for all n,

2. $\infty \in D$,

3. $D(h) \cap D(h^{-1}) = \emptyset$ and

4. $(D(g) \cup D(g^{-1})) \cap (D(h) \cup D(h^{-1})) = \emptyset$

where D is defined by $\widehat{\mathbb{C}} - D = \lim \widehat{\mathbb{C}} - D(n)$ in the Hausdorff topology on closed sets and $D(\zeta) = \lim D(\zeta(n))$ for $\zeta \in \{g, g^{-1}, h, h^{-1}\}$.

Lemma 6.2.4 *Given a sequence of Schottky groups $\Gamma(n)$ in the loxodromic case there is a ϕ such that $\phi\Gamma(n)\phi^{-1}$ is nicely bounded for large enough n.*

Proof: Choose standard generators $\{g(n), h(n)\}$ for $\Gamma(n)$. Then $\Gamma(n)$ is classical on these generators by Lemma 6.2.2 for large n. By the proof of Lemma 6.2.2 we have that the defining circles for $\Gamma(n)$ are $I_{g(n)}, I_{g(n)^{-1}}, S_{\delta(n)}(0)$ and $S_{|\mu(n)|\delta(n)}(0)$ for specific $\delta(n) > 0$ and n sufficiently large.

In fact we can take $\delta(n) = \frac{\sqrt{|\mu(n)|}+1}{2\sqrt{|\mu(n)|}}$ for large enough n. By the same reasoning as in Lemma 6.2.2 the group is classical if

$$1 - \mathrm{rad} I_{g(n)} > \frac{\sqrt{|\mu(n)|}+1}{2\sqrt{|\mu(n)|}}$$

and

$$\sqrt{|\mu(n)|} + \mathrm{rad} I_{g(n)} < |\mu(n)| \frac{\sqrt{|\mu(n)|}+1}{2\sqrt{|\mu(n)|}} (1 - \mathrm{rad} I_{g(n)}).$$

Which are both satisfied as $|\mu(n)| \to |\mu| > 1 > 1$ and $\mathrm{rad} I_{g(n)} \to 0$.

We are now ready to define ϕ, let $B_r(c)$ be a closed ball in D then for large n $B_r(c)$ is a closed ball in $D(n)$. Define ϕ to be the Möbius transformation that takes $B_r(c)$ to $\widehat{\mathbb{C}}$ with the open unit ball removed. Note that there are many choices of $B_r(c)$ and ϕ, none of these choices will affect the result.

We shall now check that $\phi\Gamma(n)\phi^{-1}$ is nicely bounded with fundamental domain $\phi D(n)$. Note that $\phi D(n)$ converges to ϕD in the Hausdorff topology.

Property 1 is satisfied as $\infty \in \phi B_r(c) \subset \phi D(n)$ for all large n. Property 2 is satisfied by the choice of $B_r(c)$.

We note that $D(\phi h(n)\phi^{-1}) = \phi D(h(n))$ so that $\lim D(\phi h(n)\phi^{-1}) = \phi D(h)$ and the same holds for h^{-1}. We have that $\phi D(h)$ and $\phi D(h^{-1})$ are disjoint as $D(h) \cap D(h^{-1}) = \emptyset$ since the boundary of $D(h)$ and $D(h^{-1})$ are the circles $S_{\frac{\sqrt{|\mu|}+1}{2\sqrt{|\mu|}}}(0)$ and $S_{|\mu|\frac{\sqrt{|\mu|}+1}{2\sqrt{|\mu|}}}(0)$ by our choice of $\delta(n)$. So we have property 3.

Property 4 is done in the same way by noticing that $1 \leq |\mathrm{cen} I_{g(n)^{-1}}| \leq \sqrt{|\mu(n)|}$ as we have standard generators and so

$$\frac{\sqrt{|\mu|}+1}{2\sqrt{|\mu|}} < 1 \leq \lim |\mathrm{cen} I_{g(n)^{-1}}| \leq \sqrt{|\mu|} < |\mu| \frac{\sqrt{|\mu|}+1}{2\sqrt{|\mu|}}.$$

$\square$

Definition 6.2.5 Given a sequence $\Gamma(n)$ of Schottky groups in the loxodromic case and ϕ such that $\phi\Gamma(n)\phi^{-1}$ is nicely bounded then we say that the sequence $\phi\Gamma(n)\phi^{-1}$ is **nicely bounded in the loxodromic case.**

The tool we shall use to show that the Hausdorff dimension vanishes is Patterson-Sullivan theory i.e we have that if

$$\sum_{\gamma \in \Gamma} |\gamma'(z)|^{\delta} = \infty$$

then δ is less than or equal to the Hausdorff dimension of Γ. In fact we shall use Lemma 5.4.20 and restrict ourselves to the subset $\widetilde{L}_{\epsilon}(t)$. This will mean that we only need to consider γ with a certain ratio of $g(n)$'s to $h(n)$'s. Although this ratio will change for the various groups $\Gamma(n)$ we will still be able to extract a contradiction if the Hausdorff dimension does not vanish.

Given $\gamma \in \Gamma(n)$ we look at

$$|\zeta_i'(\zeta_{i+1} \dots \zeta_{l(\gamma)} z)|$$

where $\gamma = \zeta_1 \ldots \zeta_{l(\gamma)}$ as a reduced word. We distinguish between the cases that $\zeta_i = g(n)^{\pm 1}$, $\zeta_i = h(n)^{\pm 1}$ is not followed followed by a sequence of $h(n)^{\pm 1}$s and $\zeta_i = h(n)^{\pm 1}$ followed by a sequence of $h(n)^{\pm 1}$s. The length of the sequence of $h(n)^{\pm 1}$s will be given later.

What we show is that if $\zeta_i = g(n)^{\pm 1}$ then the derivative of ζ_i converges to 0 uniformly over all the possible $\zeta_{i+1} \ldots \zeta_{l(\gamma)}$s as $n \to \infty$. This will force the Hausdorff dimension to 0 as long as the derivative of $h(n)$ is not too large.

There are only a finite number of options in the second case, this and the fact we are nicely bounded allows us to show that there is a uniform bound over all the derivatives of the ζ_i.

We show that, in the third case, there is some uniform bound s for which the derivative of ζ_i is less than this bound for large enough n.

The proof works by contradiction. We show that if the Hausdorff dimension does not vanish then $t(n)$ the density of $g(n)$ in the limit set must vanish. This forces the ratio of $h(n)^{\pm 1}$ to $g(n)^{\pm 1}$ in $\widetilde{L}_\epsilon(t(n))$ to tends to 1. This means that the proportion of $\zeta_i = h(n)^{\pm 1}$ such that $\zeta_{i+1} \ldots \zeta_{l(\gamma)}$ has a long string of $h(n)^{\pm 1}$'s at the start increases. This pulls the Hausdorff dimension down as the derivative of these ζ_is is bounded above. This gives us our contradiction.

Definition 6.2.6 Given a sequence of Schottky groups $\Gamma(n) = < g(n), h(n) >$ in the nicely bounded loxodromic case we define the following constants

$$\text{gmax}(n) = \max\{|g(n)'(z)|, |(g(n)^{-1})'(w)|\}$$

where $z \in \text{con}(g(n))$ and $w \in con(g(n)^{-1})$. Given $L \geq 0$ define

$$\text{hmax}^{>L}(n) = \max\{|h(n)'(h(n)^m z)|, |(h(n)^{-1})'(h(n)^{-m} w)|\}$$

where the maximum is taken over $z \in \mathrm{con}(h(n))$, $w \in \mathrm{con}(h(n)^{-1})$ and $m > L$. Given $L \geq 0$ define

$$\mathrm{hmax}^{\leq L}(n) = \max_{m \leq L} \max_{z,w} \{|h(n)'(h(n)^m z)|, |(h(n)^{-1})'(h(n)^{-m} w)|\}$$

where the second maximum is over $z \in \mathrm{con}(h(n))$, $w \in \mathrm{con}(h(n)^{-1})$.

Lemma 6.2.7 *Given a sequence of Schottky groups $\Gamma(n) =< g(n), h(n) >$ that is nicely bounded in the loxodromic case then $gmax(n) \to 0$.*

Proof:

This proof works by showing that $|g(n)'(z)|$ vanishes as $n \to \infty$. This will be true as $\lambda(n) \to \infty$ as long as z is not too close to the centre of the isometric circle of $g(n)$. The first part of the proof uses the fact that we are nicely bounded to show that this is the case. The second part of the proof is just the calculation.

We know by Lemma 6.2.2 that $\Gamma(n)$ is eventually classical. In fact choose defining curves as given in this Lemma, so that the defining curves for $g(n)$ and $g(n)^{-1}$ are eventually their isometric circles.

We will prove the result for $|g(n)'(z)|$ but the same technique works for $|(g(n)^{-1})'(w)|$.

Let $z \in \mathrm{con}(g(n))$ then, by the definition of $\mathrm{con}(g(n))$, z is either in $D(h(n)) \cup D(h(n)^{-1})$ or in $D(g(n))$. We shall deal with these two cases separately.

We have that $D(h(n)) \to D(h)$ some disk as the sequence is nicely bounded. The distance from this disk to the limit of fixed points of $g(n)$

is bounded below. The same is true for $D(h(n)^{-1})$. This means that there is an $N > 0$ such that for

$$\mathrm{dist}_{\inf}\left(\bigcup_{n>N} D(h(n)) \cup D(h(n)^{-1}), \bigcup_{n>N} x(n) \cup y(n)\right) > 0$$

where $x(n)$ is the attractive fixed point of $g(n)$ and $y(n)$ is the repulsive fixed point.

Let $\lambda(n)$ be the multiplier of $g(n)$ then by Lemma 2.9.1

$$\max |g(n)'(z)|^{1/2} = \max \frac{\sqrt{|\lambda(n)|}|x(n) - y(n)|}{|(1 - \lambda(n))z + x(n)\lambda(n) - y(n)|}$$

where the maximum is taken over $z \in \bigcup_{n>N} D(h(n)) \cup D(h(n)^{-1})$.

Now $|(1 - \lambda(n))z + x(n)\lambda(n) - y(n)| \geq |\lambda(n)||x(n) - z| - |y(n) - z|$ by the triangle inequality.

For $n > N$ we see that $|x(n) - z|$ is uniformly bounded away from 0. Since the sequence is nicely bounded so that $\infty \in \lim intD(n)$ we have that $|y(n) - z|$ is uniformly bounded away from ∞.

Since we are in the loxodromic case we have that $\lambda(n) \to \infty$ so that $\max |g(n)'(z)|^{1/2} \to 0$ for $z \in \bigcup_{n>N} D(h(n)) \cup D(h(n)^{-1})$.

The next case is that of $z \in D(g(n))$. We let $c(n) = \mathrm{cen}I_{g(n)}$ and $d(n) = \mathrm{cen}I_{g(n)^{-1}}$ then by Lemma 2.9.1 we have

$$\max |g(n)'(z)|^{1/2} = \max \frac{\sqrt{|\lambda(n)|}}{|\lambda(n) + 1|} \frac{|c(n) - d(n)|}{|z - c(n)|}$$

where we are taking the maximum over all $z \in D(g(n))$.

The extremal value is at the boundary of $D(g(n))$ but for large n we have chosen this to be $I_{g(n)^{-1}}$, which is possible by the proof of Lemma 6.2.2.

Let $r(n) = \text{rad} I_{g(n)}$ then

$$\max |g(n)'(z)|^{1/2} = \max_{\theta} \frac{\sqrt{|\lambda(n)|}}{|\lambda(n)+1|} \frac{|c(n)-d(n)|}{|d(n)+r(n)\exp(i\theta)-c(n)|}$$

which is

$$\max_{\theta} \frac{\sqrt{|\lambda(n)|}}{||1+\lambda(n)| + \exp(i\theta)\sqrt{|\lambda(n)|}|}$$

as $r(n) = \frac{|d(n)-c(n)|\sqrt{|\lambda(n)|}}{|1+\lambda(n)|}$. So we have that

$$\max |g(n)'(z)|^{1/2} = \max_{\theta} \frac{\sqrt{|\lambda(n)|}}{||1+\lambda(n)| + \exp(i\theta)\sqrt{|\lambda(n)|}|}$$

which is less than

$$\frac{\sqrt{|\lambda(n)|}}{|\lambda(n)| - \sqrt{|\lambda(n)|} - 1}$$

for $|\lambda(n)|$ large enough and converges to 0 as $|\lambda(n)| \to \infty$. $\square$

Lemma 6.2.8 *Given a sequence of Schottky groups $\Gamma(n) =< g(n), h(n) >$ that is nicely bounded in the loxodromic case then there is a constant $L > 0$ such that*

$$hmax^{>L}(n) \leq s$$

where $s \in (0,1)$ for all n large enough.

Proof: We will prove the result for $h(n)$, the proof for $h(n)^{-1}$ is essentially the same.

This proof works by showing that the result is true in the limit as $n \to \infty$, we then pull back the inequality for n large enough.

Since we are in the nicely bounded loxodromic case we have $h(n) \to h$ some loxodromic. Let the repulsive fixed point of h be y and the attractive fixed point be x.

Then for any compact set $C \subset \widehat{\mathbb{C}} - \{y\}$ we have $h^m(C) \to x$ in the Hausdorff metric.

Let C be $\lim \mathrm{con}(h(n))$ in the Hausdorff metric, then C is compact and disjoint from y.

Since the attractive fixed point x of h lies outside the isometric circle I_h of h we have that there exists an L such that $h^m(C)$ lies outside I_h for all $m > L$. This means that there is some $s \in (0,1)$ such that

$$|h'(h^m(z))| \leq s$$

for all $z \in C$ and $m > L$.

We now extend the bound to $h(n)$ for n large enough. We have 2 things converging, we have the convergence of $h(n)$ to h and the convergence of $C(n) = \mathrm{con}(h(n))$ to C.

Consider the set $\overline{\bigcup_{n>N} C(n)}$ which converges to C respectively as $N \to \infty$. This means that for large fixed N there is some s' such that

$$|h'(h^m(z))| \leq s' < 1$$

for all $z \in \overline{\bigcup_{n>N} C(n)}$ and $m > L$.

Now as $h(n) \to h$ it is easy to see that for fixed m we can find an N' such that for $n > N'(m)$

$$|h(n)'(h(n)^m(z))| \leq s'' < 1$$

for $z \in \overline{\bigcup_{n>N} C(n)}$ and $m > L$. Since $\overline{\bigcup_{n>N} C(n)}$ is compact. However this is not strong enough for us as we wish to have an N' that is valid for all m.

We shall accomplish this by showing that $h(n)\overline{\bigcup_{n>N} C(n)}$ is contained inside $\overline{\bigcup_{n>N} C(n)}$ for large n.

As we are nicely bounded we can find N large enough so that $\overline{\bigcup_{n>N} C(n)}$ is disjoint from $D(h^{-1})$.

Which means that

$$h\overline{\bigcup_{n>N'} C(n)} \subset D(h) \subset \overline{\bigcup_{n>N'} C(n)}$$

so that if

$$|h(n)'(h(n)^L z)|$$

is uniformly bounded over all $z \in \overline{\bigcup_{n>N} C(n)}$ then so is

$$|h(n)'(h(n)^m z)|$$

for all $m \geq L$.

So we can find $s'' < 1$ such that

$$|h(n)'(h(n)^m z)| \leq s''$$

for all $z \in \overline{\bigcup_{n>N} C(n)}$, $m \geq L$ and n sufficiently large as required. □

Lemma 6.2.9 *Given a sequence of Schottky groups $\Gamma(n) =< g(n), h(n) >$ that is nicely bounded in the loxodromic case and fixed $L > 0$ then $hmax^{\leq L}(n)$ is uniformly bounded above over all n large enough.*

Proof: Let $\{a, r\}$ be the limit of the fixed points of $g(n)$, note that these may not be disjoint.

We will do the case for $h(n)$, the case for $h(n)^{-1}$ is similar.

We have that $h(n) \to h$ and $\text{con}(h(n)) \to D(h^{-1}) \cup \{a, r\}$ in the Hausdorff topology as we are in the nicely bounded loxodromic case.

As $\text{con}(h(n))$ converges to a compact set we have that

$$\max_{z \in \text{con}(h(n))} |h(n)'(h(n)^m z)| \to \max_{z \in D(h^{-1}) \cup \{a,r\}} |h'(h^m z)|$$

for fixed m. This is bounded as the fixed repulsive fixed point of h is a bounded distance away from $\{a, r\}$. But this argument works for each $m \leq L$ so we have an overall bound. $\square$

Lemma 6.2.10 *Given a Schottky group $\Gamma(n) =< g(n), h(n) >$ that is nicely bounded in the loxodromic case then $\mathcal{H}(\Gamma(n)) \to 0$.*

Proof: We have already discussed the outline of the proof at the beginning of this section, however we will give a short outline of the end of the proof here.

Given arbitrary $\gamma \in \widetilde{L(n)}_\epsilon(t(n))$, see Definition 5.4.6, we find an upper bound for

$$|\gamma' z| = \prod_{i=1}^{l(\gamma)} |\zeta_i'(\zeta_{i+1} \dots \zeta_{l(\gamma)}(z)|.$$

The upper bound is expressed using the 3 constants we have defined, namely $\text{gmax}(n), \text{hmax}^{\leq L}(n)$ and $\text{hmax}^{>L}(n)$. It is obvious that to find an upper bound we shall have to estimate the number of times the appropriate ζ_i appears in $\gamma = \zeta_1 \dots \zeta_{l(\gamma}$ to replace it by one of the 3 constants. To do this we shall use the properties of $\widetilde{L(n)}_\epsilon(t(n))$.

We then use Lemma 5.4.11 to find an upper bound on the growth rate of $\widetilde{L(n)}_\epsilon(t(n))$. We assume that the Hausdorff dimension does not vanish. Then the fact that $\mathrm{gmax}(n) \to 0$ implies that $t(n) \to 0$ otherwise the influence of $g(n)$ on the derivatives is too large which will force the Hausdorff dimension to vanish. If $t(n) \to 0$ then the proportion of $\mathrm{hmax}^{>L}(n)$ to $\mathrm{hmax}^{\leq L}(n)$ in the upper bound on the derivative increases. However as $\mathrm{hmax}^{>L}(n) \leq s < 1$ for all n this forces the Hausdorff dimension to vanish as required.

Recall that $g(n)(\gamma)$ is the number of times the letters $g(n)$ or $g(n)^{-1}$ appear in the reduced word for $\gamma \in \Gamma(n)$ and similarly for $h(n)(\gamma)$.

By Lemma 5.4.4 there is a set $L(g(n))$ such that $\frac{g(n)(\gamma_k)}{l(\gamma_k)} \to m[D(g(n)) \cup D(g(n)^{-1})]$ for every $\{\gamma_k\} \simeq z \in L(g(n))$ see Definition 5.4.5.

Lemma 5.4.20 states that

$$\sum_{\gamma \in \widetilde{L(n)}_\epsilon(t(n))} |\gamma'(w)|^{\delta(n)} = \infty$$

where $t(n) = m[D(g(n)) \cup D(g(n)^{-1})]$ and $\delta(n)$ is the Hausdorff dimension of $\Lambda(\Gamma(n))$.

Given $\gamma \in \widetilde{L(n)}_\epsilon(t(n))$ then

$$t(n) - \epsilon \leq \frac{g(n)(\gamma)}{l(\gamma)} \leq t(n) + \epsilon$$

so that

$$(t(n) - \epsilon)l(\gamma) \leq g(n)(\gamma) \leq (t(n) + \epsilon)l(\gamma)$$

and

$$(1 - t(n) - \epsilon)l(\gamma) \leq h(n)(\gamma) \leq (1 - t(n) + \epsilon)l(\gamma). \tag{6.2.1}$$

For any $\gamma \in \Gamma$ we have that

$$|\gamma'(w)| = \prod_{i=1}^{l(\gamma)} |\zeta_i'(\zeta_{i+1} \cdots \zeta_{l(\gamma)}(w))|$$

where $\zeta_1 \ldots \zeta_{l(\gamma)}$ is a reduced word for γ.

We cut the word $\zeta_1 \ldots \zeta_{l(\gamma)}$ up into the pieces such that $\zeta_i = g(n)^{\pm 1}$, $\zeta_i = h(n)^{\pm 1}$ followed by L $h(n)^{\pm 1}$s and $\zeta_i = h(n)^{\pm 1}$ not followed by L $h(n)^{\pm 1}$s.

So we have that

$$|\gamma'(w)| = \prod_{\zeta_i = g(n) \text{ or } \zeta_i = g(n)^{-1}} |\zeta_i'(\zeta_{i+1} \ldots \zeta_{l(\gamma)}(w))|$$

$$\prod_{\zeta_i = h(n)^{\pm 1}\ ,\ \zeta_{i+1} \ldots \zeta_{i+L} = h(n)^{\pm L}} |\zeta_i'(\zeta_{i+1} \ldots \zeta_{l(\gamma)}(w))|$$

$$\prod_{\zeta_i = h(n)^{\pm 1}\ ,\ \zeta_{i+1} \ldots \zeta_{i+L} \neq h(n)^{\pm L}} |\zeta_i'(\zeta_{i+1} \ldots \zeta_{l(\gamma)}(w))|$$

which is less than

$$\prod_{\zeta_i = g(n) \text{ or } \zeta_i = g(n)^{-1}} \mathrm{gmax}(n)$$

$$\prod_{\zeta_i = h(n)^{\pm 1}\ ,\ \zeta_{i+1} \ldots \zeta_{i+L} = h(n)^{\pm L}} \mathrm{hmax}^{>L}(n) \prod_{\zeta_i = h(n)^{\pm 1}\ ,\ \zeta_{i+1} \ldots \zeta_{i+L} \neq h(n)^{\pm L}} \mathrm{hmax}^{\leq L}(n)$$

by Definition 6.2.6 which gives us the upper bounds.

We know bound the size of these products.

The first is the number of ζ_i equal to $g(n)^{\pm 1}$. We write $\gamma \in \widetilde{L}_\epsilon(t(n))$ such that $l(\gamma) = n$ as $\gamma = g(n)^{n_1} h(n)^{m_1} \ldots g(n)^{n_k} h(n)^{m_k}$ then

$$t(n) - \epsilon \leq \frac{\sum_{i=1}^{k} |n_i|}{l(\gamma)} \leq t(n) + \epsilon$$

which means that

$$\mathrm{card}\{i | \zeta_i = g(n)^{\pm 1}\} = g(n)(\gamma) \geq (t(n) - \epsilon) l(\gamma).$$

The second we bound is the case that $\zeta_i = h(n)^{\pm 1}$ and $\zeta_{i+1} \dots \zeta_{i+L} = h(n)^{\pm 1}$. Because we are looking for an upper bound on the derivative of γ and by Lemma 6.2.8 we know that for large n $\mathrm{hmax}^{>L}(n)$ is less than 1 so we need to find a lower bound on the size of the product.

We bound

$$\mathrm{card}\{i | \zeta_i = h(n)^{\pm 1} \text{ and } \zeta_{i+1} \dots \zeta_{i+L} = h(n)^{\pm L}\}$$

by saying it is greater than a lower bound on the number of $\zeta_i = h(n)^{\pm 1}$ minus an upper bound on the number of

$$\mathrm{card}\{i | \zeta_i = h(n)^{\pm 1} \text{ and } \zeta_{i+1} \dots \zeta_{i+L} \neq h(n)^{\pm L}\}$$

which is the size of the last product. We will return to this product later and find an upper bound on the last product.

The bound on the size of the last product is slightly complicated as we do not know if $\mathrm{hmax}^{\leq L}(n)$ is greater than 1 or not. The way we get around this is to consider $\max\{1, \mathrm{hmax}^{\leq L}(n)\}$ in the place of $\mathrm{hmax}^{\leq L}(n)$, if we do this the product is obviously larger than if we only consider $\mathrm{hmax}^{\leq L}(n)$ so we still have an upper bound. By considering $\max\{1, \mathrm{hmax}^{\leq L}(n)\}$ we need to find an upper bound on the size of the product. If we write γ as $\gamma = g(n)^{n_1} \dots h(n)^{m_k}$ then

$$\mathrm{card}\{i | \zeta_i = h(n)^{\pm 1} \text{ and } \zeta_{i+1} \dots \zeta_{i+L} \neq h(n)^{\pm L}\} \leq Lk.$$

By considering the case that $n_i = 1$ for all i except $i = 1$ which is 0 we see that the maximum that k can be is $l(\gamma)(t(n) + \epsilon) + 1$ since $\gamma \in \widetilde{L}_\epsilon(t(n))$. This means that

$$\mathrm{card}\{i | \zeta_i = h(n)^{\pm 1} \text{ and } \zeta_{i+1} \dots \zeta_L \neq h(n)^{\pm L}\} \leq L(l(\gamma)(t(n) + \epsilon) + 1).$$

We now return to the size of the second product. A lower bound on the number of ζ_i equal to $h(n)^{\pm 1}$ is

$$l(\gamma)(1-t(n)-\epsilon)$$

so that

$$\begin{aligned}&\operatorname{card}\{i|\zeta_i = h(n)^{\pm 1} \text{ and } \zeta_{i+1}\dots\zeta_{i+L} = h(n)^{\pm L}\}\\ &\geq l(\gamma)(1-t(n)-\epsilon) - L(l(\gamma)(t(n)+\epsilon)+1)\end{aligned}$$

and we have bounds on the size of all 3 products.

So we bound $|\gamma'(w)|$ from above by

$$\operatorname{gmax}(n)^{(t(n)-\epsilon)l(\gamma)}\operatorname{hmax}^{>L}(n)^{l(\gamma)(1-t(n)-\epsilon)-L(l(\gamma)(t(n)+\epsilon)+1)}$$
$$\max\{1,\operatorname{hmax}^{\leq L}(n)\}^{(l(\gamma)(t(n)+\epsilon)+1)}$$

in fact for ease of reading we shall write $\max\{1,\operatorname{hmax}^{\leq L}(n)\}$ as $\operatorname{hmax}^{\leq L}(n)$.

Having a bound on the derivatives of the individual $\gamma \in \widetilde{L}_\epsilon(t(n))$ we find a bound on the number of such γ.

Define the supremum growth rate of $\widetilde{L}_\epsilon(t(n))$ to be

$$\overline{gr(n)} = \limsup_m \frac{\log\operatorname{card}\{\gamma \in \widetilde{L}_\epsilon(t(n))|l(\gamma)=m\}}{m}$$

this is finite, since if we consider the whole group, $\Gamma(n)$, then its growth rate is 3.

Then for all $\eta > 0$ we can find a M such that

$$\operatorname{card}\{\gamma \in \widetilde{L}_\epsilon(t(n))|l(\gamma)=m\} \leq \exp(\overline{gr(n)}+\eta)^m$$

for $m > M$.

Putting this together we have that

$$\infty = \sum_{\gamma\in\widetilde{L(n)}_\epsilon(t(n))} |\gamma'(w)|^{\delta(n)} \leq \sum_{m>M} \Big(\exp(\overline{gr(n)}+\eta)\operatorname{gmax}(n)^{\delta(n)(t(n)-\epsilon)}$$

$$\text{hmax}^{>L}(n)^{\delta(n)(1-t(n)(L+1)+\epsilon(1-L))+\frac{\delta(n)L}{m}}\text{hmax}^{\leq L}(n)^{L\delta(n)(t(n)+\epsilon)+\frac{\delta(n)L}{m}}\Big)^m.$$

Note that

$$\infty = (\text{hmax}^{>L}(n)\text{hmax}^{\leq L}(n))^{\delta(n)L}\sum_{m>M}\Big(\exp(\overline{gr(n)}+\eta)\text{gmax}(n)^{\delta(n)(t(n)-\epsilon)}$$
$$\text{hmax}^{>L}(n)^{\delta(n)(1-t(n)(L+1)+\epsilon(1-L))}\text{hmax}^{\leq L}(n)^{L\delta(n)(t(n)+\epsilon)}\Big)^m$$

iff

$$\exp(\overline{gr(n)}+\eta)\text{gmax}(n)^{\delta(n)(t(n)-\epsilon)}$$
$$\text{hmax}^{>L}(n)^{\delta(n)(1-t(n)(L+1)+\epsilon(1-L))}\text{hmax}^{\leq L}(n)^{L\delta(n)(t(n)+\epsilon)} \geq 1.$$

This equation holds for all $\epsilon, \eta > 0$ so let $\epsilon, \eta \to 0$ and we have

$$\exp(\overline{gr(n)})\text{gmax}(n)^{\delta(n)t(n)}\text{hmax}^{>L}(n)^{\delta(n)(1-t(n)(L+1))}\text{hmax}^{\leq L}(n)^{L\delta(n)t(n)} \geq 1$$

and

$$\exp(\overline{gr(n)})\text{gmax}(n)^{\delta(n)t(n)}s^{\delta(n)(1-t(n)(L+1))}\text{hmax}^{\leq L}(n)^{L\delta(n)t(n)} \geq 1$$

since $\text{hmax}^{>L}(n) \leq s < 1$.

We shall now prove the Lemma by contradiction. Assume that $\delta(n) \to \delta > 0$. We know that $\text{gmax}(n) \to 0$ by Lemma 6.2.7 so we must have $t(n) \to 0$ as $(\text{gmax}(n)^\delta)^{t(n)}$ does not converge to 0. This means that for small $t(n)$ we have

$$t(n)^{-6t(n)}\text{gmax}(n)^{\delta(n)t(n)}s^{\delta(n)(1-t(n)(L+1))}\text{hmax}^{\leq L}(n)^{L\delta(n)t(n)} \geq 1$$

by Lemma 5.4.11.

But $t(n)^{-6t(n)} \to 1$ and $\limsup \text{gmax}(n)^{\delta(n)t(n)} \leq 1$ so on letting $t(n) \to 0$ we have that

$$s^\delta \geq 1$$

but this contradicts Lemma 6.2.8 as it states that $s < 1$ and we are done. □

6.2.2 The Infinite case

This is the case that both multipliers diverge.

We will first prove that the groups are eventually classical and then show that the Hausdorff dimension of the groups converges to 0.

Lemma 6.2.11 *Given a sequence of Schottky groups $\Gamma(n)$ in the infinite case with standard generators $\{g(n), h(n)\}$ then $\Gamma(n)$ is classical for n sufficiently large.*

Proof:

Recall that $\text{mult}g(n) = \lambda(n)$, $\text{mult}h(n) = \mu(n)$ and $z(n) = \text{cen}I_{g(n)^{-1}}$.

We have chosen a sequence for which various objects associated to the generators converge. In particular, for certain cases, in this proof we shall need that $\frac{|z(n)|}{\sqrt{|\mu(n)|}}$ and $\frac{|\mu(n)|}{|\lambda(n)|}$ converge.

We split the problem up into two cases:

1. $\frac{|z(n)|}{\sqrt{|\mu(n)|}} \to t < 1$,
2. $\frac{|z(n)|}{\sqrt{|\mu(n)|}} \to 1$.

Case 1.

We have that

$$\text{rad}I_{g(n)} = \frac{\sqrt{|\lambda(n)|}|1 - z(n)|}{|1 + \lambda(n)|} \leq \frac{\sqrt{|\lambda(n)|}(1 + |z(n)|)}{|1 + \lambda(n)|}$$

and as $\frac{|z(n)|}{\sqrt{|\mu(n)|}} \to t < 1$ for all $\epsilon > 0$ there is an N such that $n > N$ implies $|z(n)| \leq (t+\epsilon)\sqrt{|\mu(n)|}$. So for $n > N$

$$\text{rad}I_{g(n)} \leq \frac{\sqrt{|\lambda(n)|}(1+(t+\epsilon)\sqrt{|\mu(n)|})}{|1+\lambda(n)|}$$

and by our assumption of the minimality of $\mu(n)$ we have that

$$\text{rad}I_{g(n)} \leq \frac{\sqrt{|\lambda(n)|}(1+(t+\epsilon)\sqrt{|\lambda(n)|})}{|1+\lambda(n)|}$$

which converges to $t+\epsilon$ as $\lambda(n) \to \infty$ which occurs as $n \to \infty$. So there is an $N' > N$ such that $\text{rad}I_{g(n)} < t + 2\epsilon$ for $n > N'$. Since ϵ is arbitrary we choose it so that $t + 2\epsilon < 1$.

The group is classical if

$$|\mu(n)|(1 - \text{rad}I_{g(n)}) > |z(n)| + \text{rad}I_{g(n)}$$

by Lemma 6.2.1.

Which is true if

$$|\mu(n)|(1-(t+2\epsilon)) > \sqrt{|\mu(n)|} + t + 2\epsilon$$

by our bound on $\text{rad}I_{g(n)}$ of $t+2\epsilon$. But we have chosen ϵ such that $1-(t+2\epsilon) > 0$, so this inequality is eventually satisified as $|\mu(n)| \to \infty$.

Case 2.

We have that $\frac{|\mu(n)|}{|\lambda(n)|}$ converges in particular as $|\mu(n)| \leq |\lambda(n)|$ for all n it converges to s say, such that $s \in [0,1]$.

We have two cases:

a. $\frac{|\mu(n)|}{|\lambda(n)|} \to s < 1$,

b. $\frac{|\mu(n)|}{|\lambda(n)|} \to 1$.

Case a.

For each $\epsilon > 0$ we can find an N such that $|z(n)| \leq (1+\epsilon)\sqrt{|\mu(n)|}$ for all $n > N$ because we are in case 2 and similarly for all $\delta > 0$ we can find an N' such that $\sqrt{|\mu(n)|} \leq (\sqrt{s}+\delta)\sqrt{|\lambda(n)|}$ for all $n > N'$ as we are in Case a.

We can use this to find an upper bound on $\mathrm{rad}I_{g(n)}$ namely

$$\mathrm{rad}I_{g(n)} = \frac{\sqrt{|\lambda(n)|}|1-z(n)|}{|1+\lambda(n)|} \leq \frac{\sqrt{|\lambda(n)|}(1+|z(n)|)}{|1+\lambda(n)|}$$

$$\leq \frac{\sqrt{|\lambda(n)|}(1+(1+\epsilon)\sqrt{|\mu(n)|})}{|1+\lambda(n)|} \leq \frac{\sqrt{|\lambda(n)|}(1+(1+\epsilon)(\sqrt{s}+\delta)\sqrt{|\lambda(n)|})}{|1+\lambda(n)|}$$

which converges to $(1+\epsilon)(\sqrt{s}+\delta)$ as $n \to \infty$. If we choose ϵ and δ small enough so that $(1+\epsilon)(\sqrt{s}+\delta) < 1$ we have that the groups are eventually classical by the same reasoning as in case 1.

Case b.

In this case our bounds on $\mathrm{rad}I_{g(n)}$ are not strong enough so we consider a circle of radius less than 1, centred at 1.

Let $\delta \in (0,1)$ be given and consider the circle $S_{1-\delta}(1)$. This shall be one of the defining curves. Obviously $\mathrm{cen}I_{g(n)} = 1$ is inside this circle so we need to check that the repulsive fixed point is inside. We shall so this by showing that $x(n)$ the repulsive fixed point converges to 1.

Since we are in this case we know that

$$\frac{|z(n)|}{\sqrt{|\lambda(n)|}} \to 1$$

and as we have standard generators and the formula for the centre of the isometric circle in Lemma 2.9.1

$$\lambda(n) = \frac{1 - y(n)}{1 - x(n)}.$$

We can express $z(n)$ as $\frac{y(n)\lambda(n)-x(n)}{\lambda(n)-1}$ by Lemma 2.9.1. As $|x(n)|$ is uniformly bounded over n, we see that

$$\frac{|y(n)|}{\sqrt{|\lambda(n)|}} \to 1$$

where $y(n)$ is the attractive fixed point of $g(n)$.

But the formula $\lambda(n) = \frac{1-y(n)}{1-x(n)}$ shows that $y(n)$ does not diverge sufficiently fast so $1 - x(n)$ must converge to 0 as required.

Let $w \in S_{1-\delta}(1)$ then by Lemma 2.9.1

$$\frac{g(n)(w)}{z(n)} = \frac{1}{z(n)} \frac{z(n)(\lambda(n)+1)^2 w - (1 + z(n)\lambda(n))(\lambda(n) + z(n))}{(\lambda(n)+1)^2(w-1)}$$

$$= \frac{1}{w-1}\left(w - \frac{(z(n)\lambda(n)+1)(z(n)+\lambda(n))}{z(n)(\lambda(n)+1)^2}\right).$$

As we are in case 2 and case b we have that $\frac{|z(n)|}{\sqrt{|\lambda(n)|}} \to 1$ as $n \to \infty$. Which means that

$$\frac{(z(n)\lambda(n)+1)(z(n)+\lambda(n))}{z(n)(\lambda(n)+1)^2} \to 1$$

as $n \to \infty$. So in conclusion

$$\frac{|g(n)(w)|}{|z(n)|} \to 1$$

for $w \in S_{1-\delta}(1)$.

$\Gamma(n)$ is classical for small δ and large n if the circles $S_{\delta/2}(0)$, $S_{|\mu(n)|\delta/2}(0)$, $S_{1-\delta}(1)$ and $g(n)(S_{1-\delta}(1))$ are all disjoint as the interior of $S_{1-\delta}(1)$ is mapped to the exterior of $g(n)\,(S_{1-\delta}(1))$ since the repulsive fixed point $x(n)$ of $g(n)$ lies in $B_{1-\delta}(1)$ as $|x(n)-1| \to 0$ as $n \to \infty$ see Lemma 2.9.1.

$S_{\delta/2}(0)$, $S_{|\mu(n)|\delta/2}(0)$ and $S_{1-\delta}(1)$ are all disjoint as $\delta > 0$ and $|\mu(n)|\delta/2 > 2-\delta$ for n large enough. We need that $g(n)(S_{1-\delta}(1))$ is contained in the annulus bounded by $S_{\delta/2}(0)$ and $S_{|\mu(n)|\delta/2}(0)$ and is disjoint from $S_{1-\delta}(1)$.

We need to show that

$$\frac{|\mu(n)|\delta}{2} > |g(n)(w)| > \max\{\frac{\delta}{2}, 2+\delta\} = 2+\delta$$

for every $w \in S_{1-\delta}(1)$ and sufficiently large n.

Since $\frac{|g(n)(w)|}{|z(n)|} \to 1$, for all $\eta > 0$ we can find an N such that

$$(1-\eta)|z(n)| < |g(n)(w)| < (1+\eta)|z(n)|$$

for $n > N$ and all w.

Since we are in case 2 and b there is a constant N' such that

$$(1-\eta)\sqrt{|\mu(n)|} \leq |z(n)| \leq (1+\eta)\sqrt{|\mu(n)|}$$

for $n > N'$.

By the above inequalities we have that

$$(1-\eta)^2\sqrt{|\mu(n)|} < (1-\eta)|z(n)| < |g(n)(w)| < (1+\eta)|z(n)| < (1+\eta)^2\sqrt{|\mu(n)|}$$

so if we can show that

$$2+\delta < (1-\eta)^2\sqrt{|\mu(n)|} \text{ and } (1+\eta)^2\sqrt{|\mu(n)|} < \frac{|\mu(n)|\delta}{2}$$

then we are done. But these inequalities are obviously satisfied as $|\mu(n)| \to \infty$ so we are done. $\square$

We now show that the Hausdorff dimension tends to 0. Given $\gamma \in \Gamma(n)$ such that $\gamma = \zeta_1, \ldots \zeta_{l(\gamma)}$ we show that $|(\zeta_i \zeta_{i+1})'(w)|$ converges to as $n \to \infty$ for every i. This forces the Hausdorff dimension to vanish.

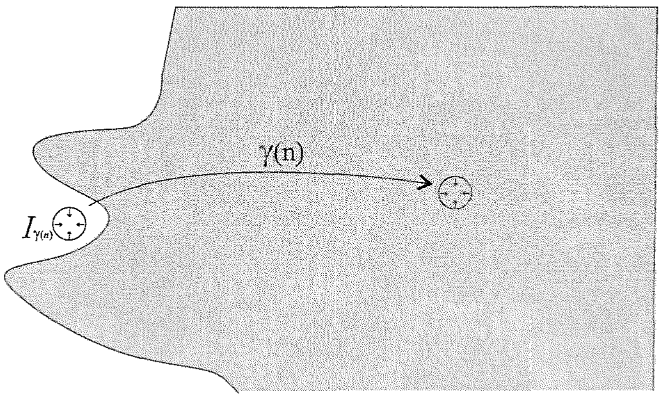
We shall need the following Lemma which gives conditions for the derivative to vanish.

Lemma 6.2.12 *Given a sequence of loxodromics $\gamma(n)$ with attractive fixed point $x(n)$ that converges to x, repulsive fixed point $y(n)$ that converges to y such that $x, y \neq \infty$ and multipliers $\lambda(n)$ such that $\lambda(n) \to \infty$ then given any set $X \subset \widehat{\mathbb{C}}$ we have*

$$\max_{w \in X} |\gamma(n)'(w)| \to 0$$

as long as the closure of X does not contain x.

Figure 6.1: Vanishing Derivative



Proof: The idea of this proof is captured in Diagram 6.1; since the isometric circle converges to the repulsive fixed point and X is away from this fixed point, eventually $I_{\gamma(n)}$ is disjoint from X. This means that the derivative is less than 1 and as the isometric circle shrinks the derivative vanishes.

By Lemma 2.9.1 we have

$$|\gamma(n)'(w)|^{\frac{1}{2}} = \frac{\sqrt{|\lambda(n)|}|x(n)-y(n)|}{|(1-\lambda(n))w + x(n)\lambda(n) - y(n)|} = \frac{\frac{\sqrt{|\lambda(n)|}}{|1-\lambda(n)|}|x(n)-y(n)|}{\left|w - \frac{x(n)-\frac{y(n)}{\lambda(n)}}{1-\frac{1}{\lambda(n)}}\right|}$$

which converges to 0 as $n \to \infty$ as long as the denominator does not converge to 0.

Since $\lambda(n) \to \infty$ as $n \to \infty$ choose $\lambda(n)$ large enough so that

$$\left|w - \frac{x - \frac{y}{\lambda(n)}}{1 - \frac{1}{\lambda(n)}}\right|$$

is bounded away from 0 for all $w \in X$. Then for large n

$$\left|w - \frac{x(n) - \frac{y(n)}{\lambda(n)}}{1 - \frac{1}{\lambda(n)}}\right|$$

is also bounded away from 0. □

The following Lemma gives conditions for the Hausdorff dimension of a sequence of Schottky groups to vanish. The condition first is that ∞ stays away from the limit set of the groups. We can always satisfy this by conjugating the group. The second condition is that the derivatives all the elements in the groups of a certain length vanish.

Lemma 6.2.13 *Given a sequence of Schottky groups* $\Gamma(n)$ *with* $\widehat{\mathbb{C}} - D(n) \subset B_R(0)$ *for some* $R > 0$ *and the property that*

$$\max_{\xi \in \Gamma(n), l(\xi)=K} |\xi'(z)| \to 0$$

for some fixed $K > 0$ *and any* $z \in \operatorname{con}(\xi)$ *as* $n \to \infty$ *then the Hausdorff dimension of* $\Gamma(n)$ *tends to* 0.

Proof: Consider the Poincaré series of $\Gamma(n)$

$$\sum_{\gamma \in \Gamma(n)} |\gamma'(z)|^{\delta},$$

if $\sum_{\gamma \in \Gamma} |\gamma'(z)|^{\delta} < \infty$ for some δ then $\delta > \mathcal{H}(\Gamma(n))$ see Theorem 5.2.3.

Given $\delta > 0$ then choose $\epsilon < \frac{1}{3}^{\frac{K}{\delta}}$. By the assumptions of the Lemma we can find an $N > 0$ so that for $n > N$

$$|\xi'(z)| < \epsilon$$

for all $\xi \in \Gamma(n)$ where $l(\xi) = K$ and $z \in \operatorname{con}(\xi)$.

Given $\gamma \in \Gamma(n)$ write it as $\xi_1 \ldots \xi_k \zeta$ where $l(\xi_i) = K$, $l(\xi_i \xi_{i+1}) = l(\xi_i) + l(\xi_{i+1})$ and $l(\zeta) < K$.

By the conditions on the defining curves of the group we can find a $w \neq \infty \in D(n)$ that remains a fixed distance away from the limit set of every group, fix this w.

We shall now bound the size of $|\zeta'(w)|$ over all $\zeta \in \Gamma(n)$ such that $l(\zeta) < K$. Suppose that there are $\zeta_n \in \Gamma(n)$ such that $|\zeta_n'(w)|$ is unbounded. Let x_n and y_n be the fixed points of ζ_n and λ_n its multiplier then

$$|\zeta_n'(w)| = \frac{|\lambda_n||x_n - y_n|^2}{|(w - x_n)\lambda_n - w + y_n|^2}.$$

However this cannot converge to ∞ as $|x_n - y_n|$ is bounded above and $(w - x_n)$ is bounded above and below so even if $|\lambda_n| \to \infty$ the denominator will dominate.

All of this means that there is some upper bound on $|\zeta'(w)|$ over all $\Gamma(n)$ call this L say.

We now find an upper bound on the Poincaré series,

$$\sum_{\gamma \in \Gamma(n)} |\gamma'(z)|^\delta = |\zeta'(w)|^\delta \sum_{\gamma \in \Gamma(n)} \sum_{i=1}^{k} |\xi_i'(\xi_{i+1} \dots \xi_k \zeta(w)|^\delta |\zeta'(w)|^\delta$$

since $\xi_{i+1} \dots \xi_k \zeta(w) \in \mathrm{con}(\xi)$ for every i and by our bound on ζ we have that his is less than

$$L^\delta \sum_{\gamma \in \Gamma(n)} \epsilon^{k\delta}.$$

We now calculate k in terms of $l(\gamma) = m$ and express this as a sum over m

$$L^\delta \sum_{\gamma \in \Gamma(n)} \epsilon^{\lfloor \frac{l(\gamma)}{K} \rfloor \delta} = 4L^\delta \sum_m 3^m \epsilon^{\lfloor \frac{m}{K} \rfloor \delta} \leq 4L^\delta \sum_m 3^m \epsilon^{\frac{m}{K} + 1\delta}$$

assuming that $\epsilon < 1$. The above converges if

$$3\epsilon^{\frac{\delta}{K}} < 1$$

but this is exactly what we have assumed that ϵ satisfies so the Poincaré Series converges at δ for $n > N$.

However δ was arbitrary so we have that the Hausdorff dimension vanishes. $\square$

Lemma 6.2.14 *Given a sequence of Schottky groups $\Gamma(n)$ in the infinite case then $\mathcal{H}(\Gamma(n)) \to 0$.*

Proof: We split the proof of this Lemma into various cases we will define later. Generally we will be applying Lemma 6.2.12 and then Lemma 6.2.13 to show that the Hausdorff dimension vanishes.

Pick standard generators $\zeta(n)$ and $\xi(n)$ for $\Gamma(n)$ where $\xi(n)$ fixes 0 and ∞ for all n. Let $\zeta(n)$ have multiplier $\lambda(n)$ and $\xi(n)$ have multiplier $\mu(n)$. Let the fixed points of $\zeta(n)$ be $x(n)$ and $y(n)$.

We are assuming that $x(n), y(n)$ and the multipliers converge.

Conjugate the group by $\phi(n)(w) = \frac{w-x(n)}{w+x(n)}$. Let $h(n) = \phi(n)\xi(n)\phi(n)^{-1}$ then $h(n)$ fixes -1 and 1. Let $g(n) = \phi(n)\zeta(n)\phi(n)^{-1}$ then $g(n)$ fixes 0 and $p(n)$ say. Note that conjugation does not change the multipliers.

We split the proof into four cases, depending on where $p(n)$ converges to:

1. $p(n) \to p \neq \infty, 0, 1$,
2. $p(n) \to \infty$,
3. $p(n) \to 0$,
4. $p(n) \to 1$.

Case 1.

As the isometric circles of $g(n)$ and $h(n)$ shrink to points eventually they must be disjoint so we have a sequence of isometric Schottky groups and we use these as defining curves.

We wish to apply Lemma 6.2.12. Let $\chi \in \{g^{\pm 1}, h^{\pm 1}\}$ then consider $\chi(n)$ with $X_N = \cup_{n>N}\mathrm{con}(\chi(n))$. Since $p(n) \to p$ away from the other fixed points we see that $\cup_{n>N}\mathrm{con}(\chi(n))$ is a bounded distance away from $\mathrm{cen}I_{\chi(n)}$ for N large enough.

This means that for n sufficiently large we can apply Lemma 6.2.12 to $\chi(n)$ with X containing $\mathrm{con}(\chi(n))$ for each n.

So we can apply Lemma 6.2.13 as the condition on ∞ is obviously satisfied to get that the Hausdorff dimension vanishes.

Case 2.

This is the case that $p(n) \to \infty$. We shall first of all show that this means that $x(n) \to 1$ and $y(n) \to -1$ then we shall conjugate the group $\Gamma(n)$ by a different Möbius transformation so we can apply similar reasoning as in case 1.

We have that $p(n) = \phi(n)(y(n)) = \frac{y(n)-x(n)}{y(n)+x(n)}$ which converges to ∞. By Lemma 2.9.1 we see that

$$1 = \mathrm{cen} I_{\zeta(n)} = \frac{\lambda(n)x(n) - y(n)}{\lambda(n) - 1}$$

which means that

$$\lambda(n) = \frac{1 - y(n)}{1 - x(n)}.$$

We shall consider the 4 case that $y(n)$ and $x(n)$ do or do not converge to ∞.

Case $x(n), y(n) \to \infty$.

Consider $\frac{y(n)}{x(n)}$ which either is or is not unbounded. If $\frac{y(n)}{x(n)}$ is unbounded then we know that

$$\frac{y(n) - x(n)}{y(n) + x(n)} = \frac{1 - \frac{x(n)}{y(n)}}{1 + \frac{x(n)}{y(n)}} \to \infty$$

so that $\frac{x(n)}{y(n)} \to -1$. However if $\frac{y(n)}{x(n)}$ is bounded then once again we must have $\frac{x(n)}{y(n)} \to -1$ by the above limit. But we get a contradiction as this forces

$$\lambda(n) = \frac{1 - y(n)}{1 - x(n)}$$

to converge to -1 but this is not the case. So this case cannot occur.

Case $x(n) \to \infty$, $y(n) \not\to \infty$.

This forces

$$\lambda(n) = \frac{1 - y(n)}{1 - x(n)}$$

to tend to 0 which is a contradiction.

Case $x(n) \not\to \infty$, $y(n) \to \infty$.

This forces

$$p(n) = \frac{y(n) - x(n)}{y(n) + x(n)}$$

to tend to 1 which is a contradiction.

Case $x(n), y(n) \not\to \infty$. In this case by the formula for $\lambda(n)$ we have that $x(n) \to -1$ and by the formula for $p(n)$ we get that $y(n) \to 1$ as required.

Instead of conjugation the groups by $\phi(n)$ we will conjugate them by $\psi(w) = \frac{w-i}{w+i}$.

Then $\psi\zeta(n)\psi^{-1}$ has fixed points $\frac{x(n)-i}{x(n)+i}$ that converges to $-i$ and $\frac{y(n)-i}{y(n)+i}$ that converges to i.

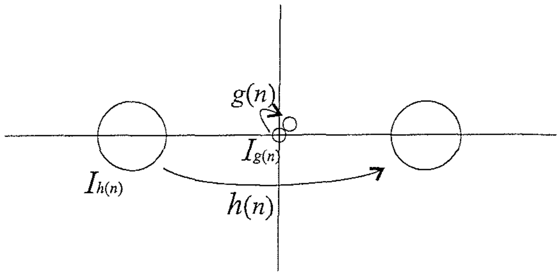
While $\psi\xi(n)\psi^{-1}$ has fixed point 1 and -1 as before.

An argument as in Case 1. will show that the Hausdorff dimension vanishes.

Case 3.

Once again we have a sequence of isometric Schottky groups as the isometric circles of $g(n)$ are disjoint by Lemma 2.9.2, we will use these as defining curves.

Figure 6.2: Infinite Case 3.



We wish to apply Lemma 6.2.13 to each sequence $g(n)^{\pm 1}, h(n)^{\pm 1}$. So we have 4 sequences to check, in fact the cases of the inverse are very similar so we shall just do the two cases $g(n)$ and $h(n)$. In most cases we will be able to apply Lemma 6.2.12 with X the contracting set.

Case $h(n)$.

In this case we let $X = \cup_{n>N}\mathrm{con}(h(n))$, for N large enough so that this set remains a bound distance away from -1. We can then apply Lemma 6.2.12 to get that the derivative vanishes.

Case $g(n)$.

This case is a little more complicated. Recall that

$$\mathrm{con}(g(n)) = D(g(n)) \cup D(h(n)) \cup D(h(n)^{-1})$$

so for $D(h(n)) \cup D(h(n)^{-1})$ we can apply Lemma 6.2.12 with $X = \cup_{n>N} D(h(n)) \cup D(h(n)^{-1})$ for N large enough.

However we will not be able to apply Lemma 6.2.12 for $D(g(n))$ as 0 is contained in the closure of $\cup_{n>N} D(g(n))$ for any N.

We will have to do this case by hand. Let $z \in D(g(n))$

By calculation and Lemma 2.9.1

$$|g(n)'(z)|^{\frac{1}{2}} = \frac{\sqrt{|\lambda(n)|}|p(n)|}{|z(\lambda(n)-1)+p(n)|}.$$

We shall now bound $|z|$ in terms of $|p(n)|$.

By Lemma 2.9.1

$$\mathrm{rad} I_{g(n)} = \frac{\sqrt{|\lambda(n)|}|p(n)|}{|1-\lambda(n)|}$$

and as $p(n)$ and z are both in $D(h(n))$ we have that

$$|p(n)| \leq |z| + 2\mathrm{rad} I_{g(n)} = |z| + 2|p(n)|\frac{\sqrt{|\lambda(n)|}}{|1-\lambda(n)|} \leq |z| + \frac{|p(n)|}{2}$$

for $|\lambda(n)| > 9+4\sqrt{5}$. So $|z| \geq \frac{|p(n)|}{2}$ for $|\lambda(n)|$ sufficiently large.

We now go back to the derivative

$$\frac{\sqrt{|\lambda(n)|}|p(n)|}{|z(\lambda(n)-1)+p(n)|} \leq \frac{\sqrt{|\lambda(n)|}|p(n)|}{||z||\lambda(n)-1|-|p(n)||} \leq \frac{\sqrt{|\lambda(n)|}|p(n)|}{|z||\lambda(n)-1|-|p(n)|}$$

for $|\lambda(n)-1| > 4$ as $|z| \geq \frac{|p(n)|}{2}$. So we can plug in our estimate for $|z|$ to get that

$$|g(n)'(z)|^{\frac{1}{2}} \leq \frac{\sqrt{|\lambda(n)|}|p(n)|}{|p(n)|(\frac{|\lambda(n)-1|}{2}-1)} = \frac{\sqrt{|\lambda(n)|}}{(\frac{|\lambda(n)-1|}{2}-1)}$$

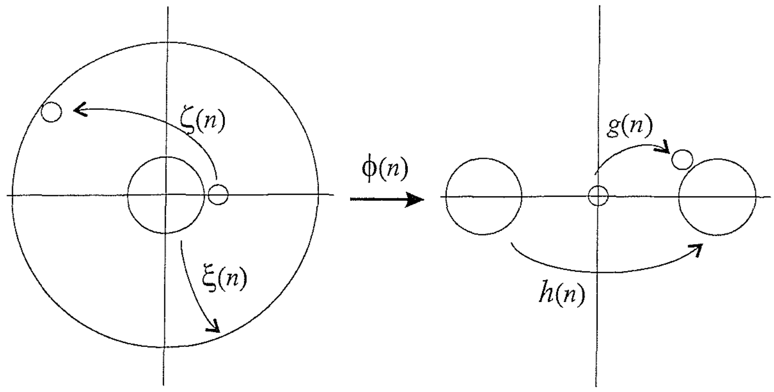
which converges to 0 as $\lambda(n) \to \infty$.

So we have that $\max_{z \in \mathrm{con}(g(n))} |g(n)(z)| \to 0$ as $n \to \infty$. Trivially we have the other condition of Lemma 6.2.13 so we can apply it and this case is done.

Case 4.

Previously the defining curves have been isometric circles, in this case the defining curves will be the images of the defining curves of the standard generators under ϕ.

Figure 6.3: The Conjugation of Infinite Case 4.



We shall once again consider the sequence $g(n)^{\pm 1}$, $h(n)^{\pm 1}$. However in this case we shall run into a problem as one of the sequences of derivatives does not converge to 0. We shall get around this problem by considering words of length 2 when it comes applying Lemma 6.2.13 instead of words of length 1.

As before there are four sequences to check, we shall check them in the order $g(n), h(n), h(n)^{-1}$ and $g(n)^{-1}$.

Case $g(n)$.

In this case we can apply Lemma 6.2.12 using a similar argument as case 1.

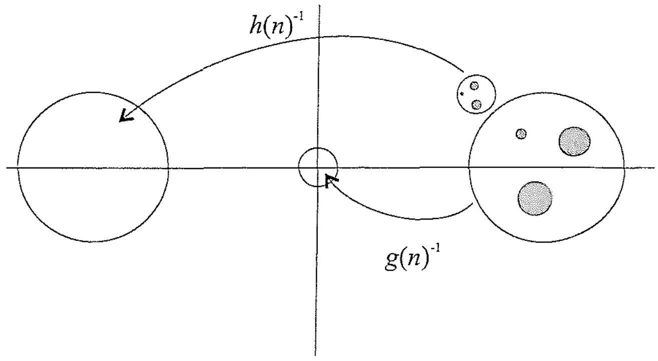
Case $h(n)$.

The same holds true in this case.

Case $h(n)^{-1}$.

The contracting set of $h(n)^{-1}$ is the set $D(h(n)^{-1}) \cup D(g(n)) \cup D(g(n)^{-1})$.

Figure 6.4: Tricky stages in Infinite Case 4.



Once again we can apply Lemma 6.2.12 to the set $D(h(n)^{-1} \cup D(g(n)^{-1})$ so the set we need to check by hand is the set $D(g(n))$ see Diagram 6.4.

First we bound the distance from $D(g(n))$ to 1. Let $w \in \phi^{-1}D(g(n))$ then the largest that $|w|$ can be is in case 2b of Lemma 6.2.11 but by the analysis of that section we have that

$$\frac{|\zeta(n)(w')|}{|\mathrm{cen}I_{\zeta(n)^{-1}}|} \to 1$$

for $w' \in S_{1-\delta}(1)$.

For all $\eta > 0$ there is an $N > 0$ such that $n > N$ implies

$$|w| \leq \max_{w' \in S_{1-\delta}(1)} |\zeta(n)(w')| \leq (1+\eta)|\mathrm{cen}I_{\zeta(n)^{-1}}| \leq (1+\eta)\sqrt{|\mu(n)|}$$

for $n > N$ large enough since we have standard generators.

The closest that $D(g(n))$ can be to 1 is greater than

$$\min_t \left|1 - \phi\left(\exp(it)(1+\eta)\sqrt{|\mu(n)|}\right)\right| = \min_t \left|1 - \frac{\exp(it)(1+\eta)\sqrt{|\mu(n)|} - 1}{\exp(it)(1+\eta)\sqrt{|\mu(n)|} + 1}\right|$$

$$= \min_t 2 \frac{1}{\sqrt{1 + 2((1+\eta)\sqrt{|\mu(n)|})\cos(t) + ((1+\eta)\sqrt{|\mu(n)|})^2}}$$

$$= \frac{2}{(1+\eta)\sqrt{|\mu(n)|}}.$$

By the above discussion we have that

$$\max_{|z-1| = \frac{2}{(1+\eta)\sqrt{|\mu(n)|}}} \left|(h(n)^{-1})' z\right| \geq \max_{w \in D(g(n))} \left|(h(n)^{-1})' w\right|$$

so we shall attempt to bound

$$\left| (h(n)^{-1})' \left(1 + \frac{2}{(1+\eta)\sqrt{\mu(n)}} \exp(is)\right) \right|$$

for any s. This is

$$\frac{2|\mu(n)|}{\left|(\mu(n)-1)\left(1+\frac{2}{(1+\eta)\sqrt{\mu(n)}}\exp(is)\right) - \mu(n) - 1\right|^2}$$

$$= \frac{2|\mu(n)|}{\left|(\mu(n)-1)\left(\frac{2}{(1+\eta)\sqrt{\mu(n)}}\exp(is)\right) - 2\right|^2}$$

which converges to

$$\frac{2}{\left|\left(\frac{2}{(1+\eta)}\exp(is)\right)\right|^2} = (1+\eta)^2$$

as $\mu(n) \to \infty$.

So we see that $|h(n)^{-1}z|$ does not vanish for $z \in D(g(n))$ but it is bounded.

Case $g(n)^{-1}$.

As before we can apply Lemma 6.2.12 for the sets $D(g(n)^{-1})$ and $D(h(n)^{-1})$. We shall have to do the case of $D(h(n))$ by hand see Diagram 6.4.

Consider $\phi(n)^{-1}D(h(n))$ which contains ∞ and is bounded by a circle of radius $\delta|\mu(n)|$ for some δ in every case of Lemma 6.2.11.

The maximum that the derivative of $g(n)^{-1}$ over $D(h(n))$ can be is at the boundary of $D(h(n))$ which is $\phi(S_{\delta|\mu(n)|}(0))$.

Let $z \in D(h(n))$ then

$$(g(n)^{-1})'(z) = \frac{p(n)^2\lambda(n)}{((1-\lambda(n))z + p(n)\lambda(n))^2}.$$

We are in the case that $p(n) \to 1$ and $z \in D(h(n))$ is bounded for all n so

$$\lim |(g(n)^{-1})'(z)| = \lim \frac{|\lambda(n)|}{|p(n)-z|^2|\lambda(n)|^2} = \lim \frac{1}{|p(n)-z|^2|\lambda(n)|}$$

we shall find an upper bound for this.

In fact we shall show that

$$\lim \frac{|p(n)-z|\sqrt{|\lambda(n)|}}{\eta} \geq 1$$

for some $\eta > 0$. This will show that

$$\lim |(g(n)^{-1})'(z)| \leq \frac{1}{\eta^2}$$

which will be our upper bound.

We will pull everything back to the standard generators and then use Lemma 2.2.5. We have that $\phi(n)^{-1}(p(n)) = y(n)$ and we let $\phi(n)^{-1}z = w$.

By Lemma 2.2.5 we see that

$$|p(n)-z| = |\phi(n)y(n) - \phi(n)w| = |\phi(n)'y(n)|^{1/2}|\phi(n)'w|^{1/2}|y(n)-w|$$

so we will bound the expression on the right.

We express the derivatives first,

$$|\phi(n)'y(n)| = \frac{\sqrt{|2x(n)|}}{|y(n)+x(n)|}$$

and

$$|\phi(n)'w)| = \frac{\sqrt{|2x(n)|}}{|w+x(n)|}.$$

So we will bound $|x(n)|, |y(n)|$ and $|w|$.

First of all we shall bound $|x(n)|$. Recall from Lemma 2.9.1 that

$$\mathrm{rad}I_{g(n)} = \frac{\sqrt{|\lambda(n)|}|1-z(n)|}{|1+\lambda(n)|}$$

$$\leq \frac{\sqrt{|\lambda(n)|}(1+|z(n)|)}{|1+\lambda(n)|} \leq \frac{\sqrt{|\lambda(n)|}(1+\sqrt{|\lambda(n)|})}{|1+\lambda(n)|}$$

since we have standard generators.

This means that $\lim \mathrm{rad}I_{g(n)} \leq 1$, in other words that

$$0 \leq |x(n)| \leq 2.$$

However in each case of Lemma 6.2.11 we have defining curves one of which is a circle centred at 0, since $x(n)$ is not within this defining curve we have that there is some $\eta > 0$ such that $|x(n)| > \eta$ for all n.

We now can bound $y(n)$. Since

$$p(n) = \frac{y(n)-x(n)}{y(n)+x(n)}$$

converges to 1 as we are in case 4 we see that $y(n) \to \infty$.

With the same reasoning as for $x(n)$ we have a lower bound on $|y(n)|$ of η. As we have standard generators we have that $|z(n)| \leq \sqrt{|\mu(n)|}$ and if we express $z(n)$ using Lemma 2.9.1 we get that

$$|z(n)| = |\mathrm{cen}I_{g(n)^{-1}}| = \frac{|y(n)\lambda(n)-x(n)|}{|\lambda(n)-1|} \leq \sqrt{|\mu(n)|}.$$

This means that for large n we have that

$$|y(n)| \leq 2\sqrt{|\mu(n)|}.$$

We now bound w. Since $w \in S_{\delta|\mu(n)|}(0)$ fairly obviously $|w| = \delta|\mu(n)|$.

We now use all the bounds,

$$\begin{aligned}|p(n) - z| = |\phi(n)y(n) - \phi(n)w| &= |\phi(n)'y(n)|^{1/2}|\phi(n)'w|^{1/2}|y(n) - w| \\ &= \frac{\sqrt{|2x(n)|}}{|y(n) + x(n)|}\frac{\sqrt{|2x(n)|}}{|w + x(n)|}|y(n) - w| \\ &\geq \frac{|2x(n)|(|w| - |y(n)|)}{(|y(n)| + |x(n)|)(|w| + |x(n)|)}\end{aligned}$$

by the triangle inequality.

We now plug in the bounds we have to get that the above is greater than

$$\frac{2\eta(\delta|\mu(n)| - \sqrt{|\mu(n)|})}{(\sqrt{|\mu(n)|} + 2)(\delta|\mu(n)| + 2)} \geq \frac{\eta}{\sqrt{|\mu(n)|}} \geq \frac{\eta}{\sqrt{|\lambda(n)|}}$$

for large n as we have standard generators.

We will now show that we can apply Lemma 6.2.13 with the length of the words being 2.

If you check each of the 12 combinations it is easy to see. As if the derivative of one of the elements does not vanish then the derivative of the other must vanish and since the derivative of first is bounded the derivative of the two multiplied together must vanish.

We shall give an example. Consider the case of $g(n)^{-1}h(n)^{-1}$.

We wish to show that

$$\max_{w \in \mathrm{con}(h(n)^{-1})} |(g(n)^{-1}h(n)^{-1})'w|$$

vanishes.

By the chain rule we have that

$$\max_{w\in \mathrm{con}(h(n)^{-1})} |(g(n)^{-1}h(n)^{-1})'w|$$

$$\leq \max_{z\in D(h(n)^{-1})} |(g(n)^{-1})'z| \max_{w\in \mathrm{con}(h(n)^{-1})} |(h(n)^{-1})'w|$$

which is less than

$$(1+\eta)^2 \max_{z\in D(h(n)^{-1})} |(g(n)^{-1})'z|$$

by our case by case discussion.

But if we consider the derivative of $g(n)^{-1}$ on $D(h(n)^{-1})$ we see that this does tend to 0, so we have the result. □

6.2.3 The Identity/Elliptic case

This is the case that $|\mu(n)| \to 1$ so $h(n)$ converges to either the identity or an elliptic. The multiplier $\lambda(n)$ of $g(n)$ diverges.

We prove two lemmas that give conditions for a sequence of Schottky groups in the identity/elliptic case to be eventually classical. The first shows that if $|\lambda(n)|$ is large compared to how small $|\mu(n)|-1$ is, then the sequence is eventually classical. The second compares the distance between the fixed points of $g(n)$ or equivalently the distance from j to the axis of $g(n)$ to $|\mu(n)|-1$. We then show that if these conditions are not satisfied for a sequence of Schottky groups then the Hausdorff dimension of the limit sets of these groups cannot converge to 0.

Lemma 6.2.15 *Given a sequence of Schottky groups* $\Gamma(n)$ *in the identity/elliptic case with standard generators* $\{g(n), h(n)\}$ *such that*

$$|\lambda(n)| > \frac{30^2}{(|\mu(n)|-1)^2} \textit{ for all } n$$

where $\mathrm{mult}(h(n)) = \mu(n)$ *and* $\mathrm{mult}(g(n)) = \lambda(n)$ *then the groups are eventually classical.*

Proof: Our groups are classical if

$$|\mu(n)|(1-\mathrm{rad}I_{g(n)}) > |z(n)| + \mathrm{rad}I_{g(n)}$$

by Lemma 6.2.1.

By Lemma 2.9.1

$$\mathrm{rad}I_{g(n)} = \frac{\sqrt{|\lambda(n)|}|1-z(n)|}{|1+\lambda(n)|} \le \frac{3\sqrt{|\lambda(n)|}}{|1+\lambda(n)|} \le \frac{6}{\sqrt{|\lambda(n)|}}$$

as $|z(n)| \le \sqrt{|\mu(n)|}$ since we have standard generators, for $|\mu(n)| < 2$ and $|\lambda(n)| > 2$.

So the group is classical if

$$|\mu(n)| - |\mu(n)|\mathrm{rad}I_{g(n)} > |\mu(n)|^{1/2} + \mathrm{rad}I_{g(n)}$$

which is implied by

$$\frac{|\mu(n)| - |\mu(n)|^{1/2}}{|\mu(n)|+1} > \frac{6}{\sqrt{|\lambda(n)|}}. \tag{6.2.2}$$

Now

$$\frac{|\mu(n)| - |\mu(n)|^{1/2}}{(|\mu(n)|-1)(|\mu(n)|+1)} \to \frac{1}{4}$$

as $|\mu(n)| \to 1$ so there is some $U > 1$ such that $|\mu(n)| < U$ means that

$$\frac{|\mu(n)| - |\mu(n)|^{1/2}}{(|\mu(n)| - 1)(|\mu(n)| + 1)} > \frac{1}{5}.$$

Combined with equation 6.2.2 we see that the group is eventually classical if

$$\frac{|\mu(n)| - 1}{5} > \frac{6}{\sqrt{|\lambda(n)|}}$$

or

$$\sqrt{|\lambda(n)|} > \frac{30}{|\mu(n)| - 1}$$

and we are done. □

We need the following technical lemma relating lengths in $\widehat{\mathbb{C}}$ to distances in $\mathbb{H}^3$.

Lemma 6.2.16 *Given a geodesic $\alpha \subset \mathbb{H}^3$ with endpoints $z, w \in \widehat{\mathbb{C}}$ such that $\frac{1}{2} \leq |z|, |w| \leq 2$ then*

$$\frac{1}{\exp(l)} \leq |z - w| \leq \frac{18}{\exp(l)}$$

where l is the hyperbolic distance from j to α.

Proof: We pull everything back to the ball model using the inverse of stereographic projection $\phi^{-1} : H^3 \to B^3$. We let d be the Euclidean distance from 0 to $\phi(\alpha)$ then $d = \tanh\left(\frac{l}{2}\right)$.

If α contains j then $\phi(\alpha)$ is a straight segment. The circle $S_{\frac{1}{2}}$ must be contained under α so $|z - w| > 1$ and the inequalities are satisfied.

If α does not contain j then $\phi(\alpha)$ is contained in a circle with radius r such that this circle is tangent to the unit sphere and so we have that

$$r = \frac{1-d^2}{2d}$$

and

$$|\phi^{-1}(z) - \phi^{-1}(w)| = 2\frac{1-d^2}{1+d^2} = \frac{4}{\exp(l)+\exp(-l)}.$$

By Lemma 2.2.9 we have constants K, K' such that

$$K'|z-w| \leq |\phi^{-1}(z) - \phi^{-1}(w)| \leq K|z-w|$$

so

$$\frac{1}{K}\frac{4}{\exp(l)+\exp(-l)} \leq |z-w| \leq \frac{1}{K'}\frac{4}{\exp(l)+\exp(-l)}.$$

We have that $\frac{1}{2} \leq |z|, |w| \leq 2$ so we can find explicit values for K and K' namely

$$K = \frac{8}{5} \text{ and } K' = \frac{2}{9}$$

so that

$$\frac{5}{8}\frac{4}{\exp(l)+\exp(-l)} \leq |z-w| \leq \frac{9}{2}\frac{4}{\exp(l)+\exp(-l)}$$

and because $l > 0$ we have

$$\frac{1}{\exp(l)} \leq \frac{5}{4\exp(l)} \leq |z-w| \leq \frac{18}{\exp(l)}$$

as required. $\square$

In the next Lemma we show that if the multiplier of a loxodromic is large then the distance between the isometric circles is approximately the distance between its fixed points.

Lemma 6.2.17 *Given a loxodromic g that fixes x and y such that* $\mathrm{cen}I_g = 1$, $1 \le |\mathrm{cen}_{g^{-1}}| = |z| \le 2$ *and λ the multiplier of g satisfies $|\lambda| > 3$ then*

$$\frac{1}{2}|x-y| \le |1-z| \le 2|x-y|.$$

Proof: We have that

$$|1-z| = |\mathrm{cen}I_g - \mathrm{cen}I_{g^{-1}}| = |x-y|\left|\frac{1+\lambda}{1-\lambda}\right|$$

by Lemma 2.9.1. Since

$$\frac{1}{2} \le \left|\frac{1+\lambda}{1-\lambda}\right| \le 2$$

for $|\lambda| > 3$ the Lemma is proved. □

Lemma 6.2.18 *Given a sequence of Schottky groups $\Gamma(n)$ in the identity/elliptic case with standard generators $\{g(n), h(n)\}$ then the groups are eventually classical if*

$$\exp(l(n)) > \frac{37}{|\mu(n)|-1} \text{ for all } n$$

where $l(n)$ is the hyperbolic distance from j to the axis of $g(n)$ and $\mu(n) =$ $\mathrm{mult}(h(n))$.

Proof: The group $\Gamma(n)$ is classical if

$$|\mu(n)|(1-\mathrm{rad}I_{g(n)}) > |z(n)| + \mathrm{rad}I_{g(n)}$$

by Lemma 6.2.1

By Lemma 2.9.1

$$\mathrm{rad}I_{g(n)} = \frac{\sqrt{|\lambda(n)|}|1-z(n)|}{|1+\lambda(n)|}$$

which we shall now bound.

We have

$$\frac{\sqrt{\lambda(n)}}{|1+\lambda(n)|} \leq \frac{2}{\sqrt{|\lambda(n)|}}$$

for $|\lambda(n)| > 2$ and

$$|1 - z(n)| \leq 2|x(n) - y(n)| \leq \frac{36}{\exp(l(n))}$$

by Lemma 6.2.17 and Lemma 6.2.16 for $|\mu(n)| < 4$ and $|\lambda(n)| > 3$.

So $\Gamma(n)$ is classical if

$$|\mu(n)|\left(1 - \frac{72}{\exp(l(n))\sqrt{|\lambda(n)|}}\right) > 1 + \frac{36}{\exp(l(n))} + \frac{72}{\exp(l(n))\sqrt{|\lambda(n)|}}$$

and on rearranging we have that the group is classical if

$$\sqrt{|\lambda(n)|} > \frac{72(|\mu(n)|+1)}{(|\mu(n)|-1)\exp(l(n)) - 36}$$

as $|\lambda(n)| \to \infty$ the above inequality is satisfied if the right hand side is finite which is implied by $(|\mu(n)|-1)\exp(l(n)) - 36 > 1$ so

$$\exp(l(n)) > \frac{37}{|\mu(n)|-1}$$

suffices to show that the sequence is eventually classical. □

Lemma 6.2.19 *Given a sequence of Schottky groups $\Gamma(n)$ in the identity/elliptic case with standard generators $\{g(n), h(n)\}$ such that*

$$\exp(l(n)) \leq \frac{37}{|\mu(n)|-1} \textit{ and } |\lambda(n)| \leq \frac{30^2}{(|\mu(n)|-1)^2}$$

where $l(n)$ is the distance from j to the axis of $g(n)$, $\mathrm{mult}(g(n)) = \lambda(n)$ *and* $\mathrm{mult}(h(n)) = \mu(n)$ *then* $\liminf \mathcal{H}(\Gamma(n)) \geq \frac{1}{8}$.

Proof: This proof works by finding lower bounds for the Poincaré Series that diverge, this forces the Hausdorff dimension to have a lower bound. We wish to estimate the Poincaré Series by a geometric sum. To do this we have two things to estimate, the growth rate of the group or by Lemma 5.4.20 the growth rate of some subset and the size of $d(p(n), \gamma p(n))$. Using geometric estimate we shall estimate $d(p(n), \gamma p(n))$ in terms of the generators of the group. Once we we have the geometric sum we can decide if it converges. We will manipulate this formula and show that in the limit for $\delta \geq \frac{1}{8}$ that the geometric sum diverges. This forces the Poincaré Series to diverge for large n and δ close to $\frac{1}{8}$, which by Lemma 5.2.3 means that the Hausdorff dimension does not vanish.

By Lemma 5.2.3 we will have the result if we can show that

$$\sum_{\gamma \in \Gamma(n)} \exp(-\delta d(p(n), \gamma p(n))) = \infty$$

for $p(n) \in \mathbb{H}^3$, δ arbitrarily close to $\frac{1}{8}$ and all n sufficiently large.

Choose $p(n)$ to be the point on the axis of $h(n)$ that is closest to the axis of $g(n)$. Then

$$d(p(n), h(n)p(n)) = \log |\mu(n)|$$

and

$$d(p(n), g(n)p(n)) \leq d(j, g(n)p(n)) \leq 2l(n) + \log |\lambda(n)|$$

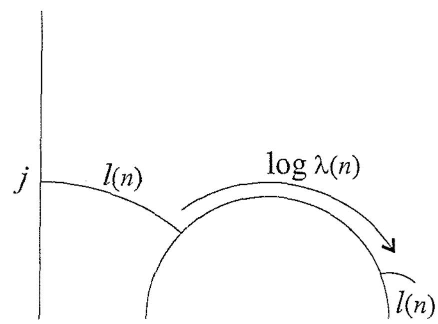
by the triangle inequality.

By the triangle inequality

$$\sum_{\gamma \in \Gamma(n)} \exp(-\delta d(p(n), \gamma p(n))) \geq \sum_{\gamma \in \Gamma(n)} \prod_{i} \exp(-\delta d(p(n), \zeta_i p(n)))$$

where $\zeta_1 \ldots \zeta_n$ is the reduced word for γ.

Figure 6.5: Bound on the generators



Given $\gamma \in \Gamma(n)$ recall the definition that $h(n)(\gamma) = card\{\zeta_i = h(n)^{\pm 1}\}$ and $g(n)(\gamma) = \mathrm{card}\{\zeta_i = g(n)^{\pm 1}\}$ where $\gamma = \zeta_1 \dots \zeta_{l(\gamma)}$ as a reduced word.

We have

$$\sum_{\gamma \in \Gamma(n)} \prod_i \exp(-\delta d(p(n), \zeta_i p(n)))$$

$$\geq \sum_{\gamma \in \Gamma(n)} |\mu(n)|^{-\delta h(n)(\gamma)} (|\lambda(n)| \exp(2l(n)))^{-\delta g(n)(\gamma)}$$

by the bounds on $g(n)$ and $h(n)$.

Given $t(n)$ and $\epsilon > 0$, recall the Definition 5.4.6. We have

$$\sum_{\gamma \in \Gamma(n)} |\mu(n)|^{-\delta h(n)(\gamma)} (|\lambda(n)| \exp(2l(n)))^{-\delta g(n)(\gamma)} = \infty$$

iff

$$\geq \sum_{\gamma \in \widetilde{L}_\epsilon(t(n))} |\mu(n)|^{-\delta(1-t(n)+\epsilon)l(\gamma)} (|\lambda(n)| \exp(2l(n)))^{-\delta(t(n)+\epsilon)l(\gamma)} = \infty$$

by Lemma 5.4.20. This is the vital result as it allows us to estimate $g(n)(\gamma)$ and $h(n)(\gamma)$ in terms of $l(\gamma)$. Since we also have a bound on the growth rate of $\widetilde{L}_\epsilon(t(n))$ we can find a lower bound on the above sum in terms of a geometric sum, for which we can determine whether it converges.

We shall now bound $g(n)(\gamma)$ and $h(n)(\gamma)$ in terms of $t(n), \epsilon$ and $l(\gamma)$. Since both $|\mu(n)|$ and $(|\lambda(n)|\exp(2l(n)))$ are greater than 1 and we are looking for a lower bound on the sum we need to find upper bounds on $g(n)(\gamma)$ and $h(n)(\gamma)$. Since $\gamma \in$ we know that

$$l(\gamma)(t(n)-\epsilon) \leq g(n)(\gamma) \leq l(\gamma)(t(n)+\epsilon)$$

this means that

$$l(\gamma)(1-t(n)-\epsilon) \leq h(n)(\gamma) \leq l(\gamma)(1-t(n)+\epsilon)$$

which are the bounds we shall use.

Note that $t(n)$ is arbitrary so choose $t(n)$ small enough so we can apply Lemma 5.4.11. We then have that the above sum is greater than

$$\sum_n t(n)^{-t(n)n/2}|\mu(n)|^{-\delta(1-t(n)+\epsilon)n}(|\lambda(n)|\exp(2l(n)))^{-\delta(t(n)+\epsilon)n}$$

by Lemma 5.4.11.

This sum is a geometric sum so it diverges iff

$$t(n)^{-t(n)/2}|\mu(n)|^{-\delta(1-t(n)+\epsilon)}(|\lambda(n)|\exp(2l(n)))^{-\delta(t(n)+\epsilon)} \geq 1.$$

Since ϵ is arbitrary we have that the above is true if

$$t(n)^{-t(n)/2}|\mu(n)|^{-\delta(1-t(n))}(|\lambda(n)|\exp(2l(n)))^{-\delta t(n)} > 1.$$

By our bounds on $l(n)$ and $|\lambda(n)|$ we have the above is true if

$$t(n)^{-t(n)/2}|\mu(n)|^{-\delta(1-t(n))}\left(\frac{366300}{(|\mu(n)|-1)^4}\right)^{-\delta t(n)} > 1$$

We isolate δ and get that the above inequality is satisfied iff

$$\delta > \frac{\frac{t(n)}{2}\log(t(n))}{\log\left(|\mu(n)|^{t(n)-1}\right) - t(n)\log\left(\frac{366300}{(|\mu(n)|-1)^4}\right)}.$$

To apply Lemma 5.4.11 we only need that t is sufficiently small. This value is a property of the free group and independent of the Schottky group. As $|\mu(n)| - 1 \to 0$ we can set $t(n) = |\mu(n)| - 1$ and apply Lemma 5.4.11 for all n large enough .

So we need that

$$\delta > \frac{\frac{|\mu(n)|-1}{2}\log(|\mu(n)|-1)}{\log\left(|\mu(n)|^{|\mu(n)|-2}\right) - (|\mu(n)|-1)\log\left(\frac{366300}{(|\mu(n)|-1)^4}\right)}$$

however $(|\mu(n)|-1)\log(|\mu(n)|-1) \to 0$ more slowly than $\log|\mu(n)|$ or $|\mu(n)|-1$, so

$$\lim \frac{\frac{|\mu(n)|-1}{2}\log(|\mu(n)|-1)}{\log\left(|\mu(n)|^{|\mu(n)|-2}\right) - (|\mu(n)|-1)\log\left(\frac{366300}{(|\mu(n)|-1)^4}\right)}$$
$$= \lim \frac{1}{2}\frac{(|\mu(n)|-1)\log(|\mu(n)|-1)}{4(|\mu(n)|-1)\log(|\mu(n)|-1)} = \frac{1}{8}$$

as required.

□

6.3 The Fixed Points Converging

We recall that this is the case where the fixed points of $g(n)$ converge to the same point and its multiplier does not converge to ∞. We first of all prove two Lemmas that we will use in both cases. The first shows that the fixed points of $g(n)$ converge to 1 and the second gives us defining circles for $g(n)$ that will be used, in some circumstances, to show that the sequence is eventually classical.

Lemma 6.3.1 *Given a sequence of Schottky groups $\Gamma(n)$ with standard generators $g(n)$ and $h(n)$ in either of the two fixed points converging cases case then the fixed points $x(n)$ and $y(n)$ of $g(n)$ converge to* 1.

Proof: Let $x(n)$ and $y(n)$ converge to x.

By Lemma 2.9.1 and as we have standard generators

$$1 = \frac{\lambda(n)x(n) - y(n)}{\lambda(n) - 1} \text{ and } z(n) = \text{cen} I_{g(n)^{-1}} = \frac{y(n)\lambda(n) - x(n)}{\lambda(n) - 1}$$

so that

$$x(n) = \frac{z(n) + \lambda(n)}{\lambda(n) + 1} \text{ and } y(n) = \frac{z(n)\lambda(n) + 1}{\lambda(n) + 1}.$$

We first note that as we have standard generators

$$1 \leq \lim |z(n)| \leq \lim \sqrt{|\mu(n)|} \leq \lim \sqrt{|\lambda(n)|} < \infty$$

so that $\lim z(n) \neq \infty$.

We shall now show that this means that $x \neq \infty$. Let $\lim \lambda(n) = \lambda$ then

$$x = \lim x(n) = \lim \frac{z(n) + \lambda(n)}{\lambda(n) + 1} = \frac{\lim z(n) + \lambda}{\lambda + 1}$$

as $|\lambda| \neq \infty$ since we have standard generators and are in the one of the fixed points converging case.

Consider $x(n) - y(n)$ this satisfies

$$x(n) - y(n) = \frac{z(n) + \lambda(n)}{\lambda(n) + 1} - \frac{z(n)\lambda(n) + 1}{\lambda(n) + 1} = \frac{(z(n) - 1)(1 - \lambda(n))}{\lambda(n) + 1}$$

and converges to 0 so either $z(n) \to 1$ or $\lambda(n) \to 1$.

We shall now show that $z(n) \to 1$, if $z(n) \not\to 1$ then $\lambda(n) \to 1$ and by Lemma 2.9.3 this means that $g(n)$ converges to a parabolic. However this contradicts [JK82] since the group generated in the limit is either elementary or is not free.

By the expression for $z(n)$ we see that

$$1 = \lim z(n) = \lim \frac{y(n)\lambda(n) - x(n)}{\lambda(n) - 1} = \frac{x\lambda - x}{\lambda - 1} = x$$

as required.

□

We will not be able to use the isometric circles as defining curves so we find disjoint circles and bound their size.

Lemma 6.3.2 *Suppose that g is a loxodromic that fixes x and y both not ∞ and has multiplier λ such that $|\lambda| > 1$ then there are disjoint circles C, C' such that $g(int(C)) = extC'$ and* $\operatorname{diam}(C \cup C') = |x - y|\frac{\sqrt{|\lambda|}+1}{\sqrt{|\lambda|}-1}$.

Proof: Conjugate g to g_1 by $w \mapsto \frac{2w}{x-y}$ then g_1 fixes $\frac{2x}{x-y}$ and $\frac{2y}{x-y}$ and the distance between the fixed points is 2.

Next conjugate g_1 to g_2 by $w \mapsto w - \frac{x+y}{x-y}$ so that g_2 fixes ± 1. This is a Euclidean isometry.

We now conjugate g_2 by $w \mapsto \frac{w+1}{-w+1}$ to g_3 where g_3 fixes 0 and ∞.

Now g_3 has disjoint circles $S_{\sqrt{|\lambda|}^{-1}}(0)$ and $S_{\sqrt{|\lambda|}}(0)$ satisfying the first property.

We will conjugate these circles back to find circles with the required properties.

Conjugate these circles back so they are paired up by g_2. Then the diameter of there union is $\left|\frac{-\sqrt{|\lambda|}^{-1}-1}{-\sqrt{|\lambda|}^{-1}+1} - \frac{-\sqrt{|\lambda|}-1}{-\sqrt{|\lambda|}+1}\right| = 2\frac{\sqrt{|\lambda|}+1}{\sqrt{|\lambda|}-1}$.

Now when we conjugate them so they are paired up by g_1 this does not change their diameter as it is a Euclidean isometry.

We conjugate them back so that they are paired up by g.

These last two maps are both similarities so they preserve extreme points this means that the diameter of the circles is $|x-y|\frac{\sqrt{|\lambda|}+1}{\sqrt{|\lambda|}-1}$ as required. $\square$

6.3.1 The Bounded case

This is the case that the multiplier $\mu(n)$ of $h(n)$ converge to μ such that $|\mu| \in (1, \infty)$. This means the $h(n)$ converges to a loxodromic. The fixed points of $g(n)$ converge to each other.

We shall first show that the sequence is eventually classical. As $|\mu(n)| \not\to 1$ we will easily be able to find defining curves for $\Gamma(n)$ that are circles.

Lemma 6.3.3 *Given a sequence of Schottky groups $\Gamma(n)$ in the bounded case with standard generators $\{g(n), h(n)\}$ then the groups are eventually classical.*

Proof: Let $x(n), y(n)$ be the fixed points of $g(n)$ then $x(n), y(n) \to 1$ by Lemma 6.3.1.

The sequence is eventually classical as the circles $S_{\sqrt{|\mu(n)|}^{-1}}(0)$, $S_{\sqrt{|\mu(n)|}}(0)$ and the circles $C(n)$ and $C'(n)$ which are paired up by $g(n)$ as in Lemma 6.3.2 are disjoint for large n.

This is because $|\mu(n)| \not\to 1$ so the circles $S_{\sqrt{|\mu(n)|}^{-1}}(0)$ and $S_{\sqrt{|\mu(n)|}}(0)$ remain a bounded distance away from 1.

And $\mathrm{diam}(C(n) \cup C'(n)) = |x(n) - y(n)| \frac{\sqrt{|\lambda(n)|}+1}{\sqrt{|\lambda(n)|}-1} \to 0$ as $n \to \infty$ since $\lambda(n)$ is bounded away from 1. This means that the circles $C(n)$ and $C'(n)$ are disjoint from $S_{\sqrt{|\mu(n)|}^{-1}}(0)$ and $S_{\sqrt{|\mu(n)|}}(0)$ for large n.

The group is seen to be Schottky by Lemma 6.3.2 as $g(n)(int(C(n))) = extC'(n)$. $\square$

6.3.2 The Identity/Elliptic Converging case

This is the case that $|\mu(n)| \to 1$ and the fixed points of $g(n)$ converge to the same point.

We shall first give conditions for a sequence to be classical and then show if this does not happen that the Hausdorff dimension must vanish.

Lemma 6.3.4 *Given a sequence of Schottky groups $\Gamma(n)$ with standard generators $g(n), h(n)$ in the identity/elliptic converging case such that*

$$(|\mu(n)| - 1)(|\lambda(n)| - 1) > 18|\lambda||x(n) - y(n)|$$

for all n, where $\lambda(n) \to \lambda$, then the sequence is eventually classical.

Proof: By Lemma 6.3.2 there are defining circles for $g(n)$ whose combined diameter is

$$|x(n) - y(n)|\frac{\sqrt{|\lambda(n)|} + 1}{\sqrt{|\lambda(n)|} - 1}$$

where $x(n)$ and $y(n)$ are the fixed points of $g(n)$.

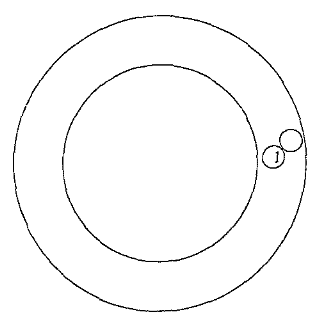
If $|\mu(n)|$ is large enough we can find circles centred at 0 that are defining curves for $h(n)$ and which are disjoint from the defining curves for $g(n)$ and we will have shown that the groups are classical.

Let $d(n) = |x(n) - y(n)|\frac{\sqrt{|\lambda(n)|+1}}{\sqrt{|\lambda(n)|-1}}$ then we can find circles centred at 0 that will suffice if

$$|\mu(n)|(1 - d(n)) > 1 + d(n) \tag{6.3.3}$$

as 1 is inside the defining curves for $g(n)$.

Figure 6.6: Conditions for classicalness when fixed points converge



We rearrange equation 6.3.3 to get

$$\frac{|\mu(n)| - 1}{|\mu(n)| + 1} > |x(n) - y(n)|\frac{\sqrt{|\lambda(n)|} + 1}{\sqrt{|\lambda(n)|} - 1} = d(n) \tag{6.3.4}$$

which we will now simplify.

Choose n large enough so that

$$|\mu(n)| + 1 < 3 \text{ and } |\lambda(n)| < \frac{3}{2}|\lambda|$$

then

$$\frac{|\mu(n)|-1}{|\mu(n)|+1} > \frac{|\mu(n)|-1}{3} \tag{6.3.5}$$

and

$$\frac{\sqrt{|\lambda(n)|}+1}{\sqrt{|\lambda(n)|}-1} \leq \frac{4|\lambda(n)|}{|\lambda(n)|-1} < \frac{6|\lambda|}{|\lambda(n)|-1} \tag{6.3.6}$$

as $(3\sqrt{|\lambda(n)|}+1)(\sqrt{|\lambda(n)|}-1)^2 > 0$ for $|\lambda(n)| > 1$ then applying the bound $|\lambda(n)|$.

Putting equation 6.3.5 and 6.3.6 into equation 6.3.4 we get that the group is classical if

$$\frac{|\mu(n)|-1}{3} > |x(n)-y(n)|\frac{6|\lambda|}{|\lambda(n)|-1}$$

or

$$|\mu(n)|-1 > 18|x(n)-y(n)|\frac{|\lambda|}{|\lambda(n)|-1}$$

as required. □

Lemma 6.3.5 *Given a geodesic $\alpha \subset \mathbb{H}^3$ with endpoints $z, w \in \widehat{\mathbb{C}}$ such that $\frac{1}{2} \leq |z|, |w| \leq 2$ then*

$$\frac{1}{\exp(l)} \leq |z-w| \leq \frac{18}{\exp(l)}$$

where l is the hyperbolic distance from j to α.

Proof: We pull everything back to the ball model using the inverse of stereographic projection $\phi^{-1} : H^3 \to B^3$. We let d be the Euclidean distance from 0 to $\phi(\alpha)$ then $d = \tanh\left(\frac{l}{2}\right)$.

If α contains j then $\phi(\alpha)$ is a straight segment. The circle $S_{\frac{1}{2}}$ must be contained under α so $|z-w| > 1$ and the inequalities are satisfied.

If α does not contain j then $\phi(\alpha)$ is contained in a circle with radius r such that this circle is tangent to the unit sphere and so we have that

$$r = \frac{1-d^2}{2d}$$

and

$$|\phi^{-1}(z) - \phi^{-1}(w)| = 2\frac{1-d^2}{1+d^2} = \frac{4}{\exp(l)+\exp(-l)}.$$

By Lemma 2.2.9 we have constants K, K' such that

$$K'|z-w| \leq |\phi^{-1}(z) - \phi^{-1}(w)| \leq K|z-w|$$

so

$$\frac{1}{K}\frac{4}{\exp(l)+\exp(-l)} \leq |z-w| \leq \frac{1}{K'}\frac{4}{\exp(l)+\exp(-l)}.$$

We have that $\frac{1}{2} \leq |z|, |w| \leq 2$ so we can find explicit values for K and K' namely

$$K = \frac{8}{5} \text{ and } K' = \frac{2}{9}$$

so that

$$\frac{5}{8}\frac{4}{\exp(l)+\exp(-l)} \leq |z-w| \leq \frac{9}{2}\frac{4}{\exp(l)+\exp(-l)}$$

and because $l > 0$ we have

$$\frac{1}{\exp(l)} \leq \frac{5}{4\exp(l)} \leq |z-w| \leq \frac{18}{\exp(l)}$$

as required. $\square$

Lemma 6.3.6 *Given a sequence of Schottky groups* $\Gamma(n)$ *in the identity/elliptic converging case with standard generators* $\{g(n), h(n)\}$ *such that*

$$(|\mu(n)| - 1)^2 < 18|\lambda||x(n) - y(n)|$$

then $\liminf \mathcal{H}(\Gamma(n)) \geq \frac{1}{4}$.

Proof: This proof is somewhat similar to the identity/elliptic case see Lemma 6.2.19.

By Lemma 5.2.3 we wish to show that

$$\sum_{\gamma \in \Gamma(n)} \exp(-\delta d(j, \gamma j)) = \infty$$

for δ arbitrarily close to $\frac{1}{4}$ and n sufficiently large.

We have

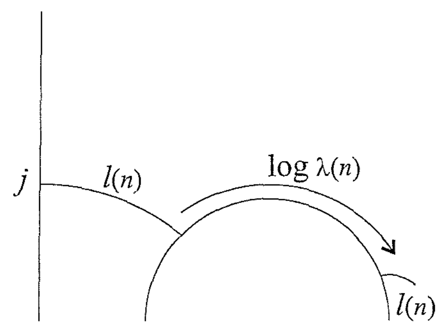
$$d(j, h(n)j) = \log|\mu(n)|$$

and

$$d(j, g(n)j) \leq \log|\lambda(n)| + 2l(n)$$

by the triangle inequality, where $l(n)$ is the distance from j to the axis of $g(n)$.

Figure 6.7: Bound on the generators



By the triangle inequality

$$\sum_{\gamma \in \Gamma(n)} \exp(-\delta d(j, \gamma j)) \geq \sum_{\gamma \in \Gamma(n)} \prod_i \exp(-\delta d(j, \zeta_i j))$$

where $\zeta_1 \ldots \zeta_n$ is the reduced word for γ.

Given $\gamma \in \Gamma(n)$ recall the definition that $h(n)(\gamma) = card\{\zeta_i | \zeta_i = h(n)^{\pm 1}\}$ and $g(n)(\gamma)$ is defined similarly, where $\gamma = \zeta_1 \ldots \zeta_{l(\gamma)}$ as a reduced word.

We have

$$\sum_{\gamma \in \Gamma(n)} \prod_i \exp(-\delta d(j, \zeta_i j)) \geq \sum_{\gamma \in \Gamma(n)} |\mu(n)|^{-\delta h(n)(\gamma)} (|\lambda(n)| \exp(2l(n)))^{-\delta g(n)(\gamma)}$$

by the bounds on $d(j, g(n)j)$ and $d(j, h(n)j)$.

Given $t(n)$ and $\epsilon > 0$, recall the Definition 5.4.6. We have

$$\sum_{\gamma \in \Gamma(n)} |\mu(n)|^{-\delta h(n)(\gamma)} (|\lambda(n)| \exp(2l(n)))^{-\delta g(n)(\gamma)} = \infty$$

iff

$$\sum_{\gamma \in \widetilde{L}_\epsilon(t(n))} |\mu(n)|^{-\delta h(n)(\gamma) l(\gamma)} (|\lambda(n)| \exp(2l(n)))^{-\delta g(n)(\gamma) l(\gamma)} = \infty$$

by Lemma 5.4.20. This is the vital result as it allows us to estimate $g(n)(\gamma)$ and $h(n)(\gamma)$ in terms of $l(\gamma)$. Since we also have a bound on the growth rate of $\widetilde{L}_\epsilon(t(n))$ we can find a lower bound on the above sum in terms of a geometric sum, for which we can determine whether it converges.

We shall now bound $g(n)(\gamma)$ and $h(n)(\gamma)$ in terms of $t(n)$, ϵ and $l(\gamma)$. Since both $|\mu(n)|$ and $(|\lambda(n)| \exp(2l(n)))$ are greater than 1 and we are looking for a lower bound on the sum we need to find upper bounds on $g(n)(\gamma)$ and $h(n)(\gamma)$. Since $\gamma \in \widetilde{L}_\epsilon(t(n))$ we know that

$$l(\gamma)(t(n) - \epsilon) \leq g(n)(\gamma) \leq l(\gamma)(t(n) + \epsilon)$$

this means that

$$l(\gamma)(1 - t(n) - \epsilon) \leq h(n)(\gamma) \leq l(\gamma)(1 - t(n) + \epsilon)$$

which are the bounds we shall use.

Note that $t(n)$ is arbitrary so choose $t(n)$ small enough so we can apply Lemma 5.4.11. We then have that the above sum is greater than

$$\sum_m t(n)^{-t(n)n/2}|\mu(n)|^{-\delta(1-t(n)+\epsilon)m}(|\lambda(n)|\exp(2l(n)))^{-\delta(t(n)+\epsilon)m}$$

by 5.4.11.

This sum is a geometric sum so it diverges iff

$$t(n)^{-t(n)/2}|\mu(n)|^{-\delta(1-t(n)+\epsilon)}(|\lambda(n)|\exp(2l(n)))^{-\delta(t(n)+\epsilon)} \geq 1.$$

Since ϵ is arbitrary we have that the above is true if

$$t(n)^{-t(n)/2}|\mu(n)|^{-\delta(1-t(n))}(|\lambda(n)|\exp(2l(n)))^{-\delta t(n)} > 1.$$

We now isolate δ

$$\delta > \frac{1}{2}\frac{-t(n)\log(t(n))}{t(n)\log|\lambda(n)| + 2t(n)l(n) + (1-t(n))\log|\mu(n)|}.$$

We let $x(n)$ and $y(n)$ be the fixed points of $g(n)$. By Lemma 6.3.1 we have $|x(n)-y(n)| \to 0$. So for n sufficiently large we can let $t(n) = \sqrt{|x(n)-y(n)|}$ and still apply Lemma 5.4.11.

We wish to show that the above equation cannot converge to 0, which occurs iff the inverse does not converge to ∞, i.e

$$\frac{t(n)\log|\lambda(n)| + 2t(n)l(n) + (1-t(n))\log|\mu(n)|}{-t(n)\log(t(n))} < \infty.$$

We shall look at each term in the inequality.

Firstly

$$\frac{\sqrt{|x(n)-y(n)|}|\log|\lambda(n)|}{-\sqrt{|x(n)-y(n)|}\log\sqrt{|x(n)-y(n)|}} = \frac{\log|\lambda(n)|}{-\log\sqrt{|x(n)-y(n)|}} \to 0$$

as $|\lambda(n)| \to |\lambda| \neq 0$.

Next

$$\frac{2\sqrt{|x(n)-y(n)|}l(n)}{-\sqrt{|x(n)-y(n)|}\log\sqrt{|x(n)-y(n)|}}$$

$$= \frac{2l(n)}{-\log\sqrt{|x(n)-y(n)|}} < \frac{2\left(\log\frac{18}{|x(n)-y(n)|}\right)}{-\log\sqrt{|x(n)-y(n)|}}$$

by Lemma 6.3.5 for $\frac{1}{2} \leq |x(n)|, |y(n)| \leq 2$ which occurs for large n by Lemma 6.3.1. This means

$$\frac{4\left(\log\frac{18}{|x(n)-y(n)|}\right)}{-\log|x(n)-y(n)|} \to 4$$

as $|x(n)-y(n)| \to 0$.

Lastly

$$\frac{(1-\sqrt{|x(n)-y(n)|})\log|\mu(n)|}{-\sqrt{|x(n)-y(n)|}\log\sqrt{|x(n)-y(n)|}}$$

$$\leq \frac{\log|\mu(n)|}{-\sqrt{|x(n)-y(n)|}\log\sqrt{|x(n)-y(n)|}}$$

$$\leq \frac{2\sqrt{18|\lambda||x(n)-y(n)|}}{-\sqrt{|x(n)-y(n)|}\log\sqrt{|x(n)-y(n)|}}$$

as $\log(r) < 2(r-1)$ for $r \in (1, e^2)$ and by our assumption on $\mu(n)$. We see that this is

$$\frac{2\sqrt{18|\lambda|}}{-\log|x(n)-y(n)|}$$

which converges to 0 as $|x(n)-y(n)| \to 0$.

So we have shown that

$$\lim \frac{1}{2}\frac{-t(n)\log(t(n))}{t(n)\log|\lambda(n)| + 2t(n)l(n) + (1-t(n))\log|\mu(n)|} \geq \frac{1}{4}$$

and we have a lower bound on the Hausdorff dimension in the limit.

□

6.4 Non-classical Schottky groups

We are now ready to show that there exists a universal lower bound on the Hausdorff dimension of the limit set of a non-classical Schottky group.

Theorem 6.4.1 *There are no non-classical Schottky groups of genus* 2 *with arbitrarily small Hausdorff dimension.*

Proof: We shall prove this by contradiction. If there is no universal lower bound then there exists some sequence $\Gamma(n)$ of non-classical Schottky groups such that $\mathcal{H}(\Gamma(n)) \to 0$.

Conjugate $\Gamma(n)$ so that it has standard generators this does not change its classicalness or its Hausdorff dimension by Lemma 3.2.4 and [Fal97]. Since the Hausdorff dimension of the sequence still vanishes we know that one of the generators diverges by Lemma 5.2.4.

By Lemma 6.1.5 there is a subsequence $\Gamma(n_m)$ of $\Gamma(n)$ that is in one of the 5 cases in definition 6.1.4. This subsequence has vanishing Hausdorff dimension and every group is non-classical.

If we are in the loxodromic, infinite or bounded case then the groups are eventually classical by Lemmas 6.2.2, 6.2.11 and 6.3.3.

The only other two cases are the identity/elliptic or identity/elliptic converging case.

In the identity/elliptic case the groups are either eventually classical by Lemmas 6.2.15 and 6.2.18 or $\liminf \mathcal{H}(\Gamma(n_m)) \geq \frac{1}{8}$ by Lemma 6.2.19.

The only case left is the identity/elliptic converging case. But in this case either the groups are eventually classical by Lemma 6.3.4 or the Hausdorff dimension does not converge to 0 by Lemma 6.3.6. Here we have used that if the assumption in Lemma 6.3.4, that $(|\mu(n)| - 1)(|\lambda(n)| - 1) > 18|\lambda||x(n) - y(n)|$, does not hold then the condition in Lemma 6.3.6, that $(|\mu(n)| - 1)^2 < 18|\lambda||x(n) - y(n)|$ does hold as $|\mu(n)| \leq |\lambda(n)|$ since we have standard generators.

So none of the cases can occur and we have a contradiction. This means that there is a universal lower bound on the Hausdorff dimension of a non-classical Schottky group. $\square$

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