UNIVERSITY OF SOUTHAMPTON

FACULTY OF ENGINEERING, SCIENCE AND MATHEMATICS

School of Mathematics

The Adams Operation ψ^3 as an Upper Triangular Matrix

by

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ABSTRACT

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In the 2-local stable homotopy category the group of left bu-module automorphisms of $bu \wedge bo$ which induce the identity on mod 2 homology is isomorphic to the group of infinite, invertible upper triangular matrices with entries in the 2-adic integers. After giving a survey of the required background material from stable homotopy theory, we identify the conjugacy class of the matrix corresponding to $1 \wedge \psi^3$, where ψ^3 is the Adams operation. We conclude by giving two applications of having knowledge of the identity of this matrix.

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INDEX OF NOTATION

$\mathbb{Z}/2$	cyclic group of order 2
$\mathbb{Z}_2$	2-adic integers
$\mathbb{Q}_2$	2-adic numbers
KU	complex periodic K-theory spectrum
KO	real periodic K-theory spectrum
bu	(2-local) complex connective K-theory spectrum
bo	(2-local) real connective K-theory spectrum
$\Sigma^\infty X$	suspension spectrum of X
$\mathscr{S}$	Boardman's stable homotopy category of spectra
$\mathscr{S}_2$	$\mathscr{S}$ localised with respect to mod 2 singular homology

Chapter 1

Introduction

1.0.1 Summary

I would like to begin by giving a very general summary of the results of this thesis aimed at a general mathematical audience. In [24] Snaith defined an isomorphism of groups which relates an automorphism group of a purely stable homotopy theoretic construction and a certain matrix group. The motivation for defining this isomorphism was to be able to turn difficult homotopy theoretic problems into simpler problems of matrix algebra. Within this automorphism group there exists an element which is particularly important in algebraic topology and in [24] a footnote was printed predicting the identity of the matrix corresponding to this automorphism. The bulk of the original work contained in this thesis proves this prediction, which turned out to take more work than expected. The final chapter of this thesis goes on to give two applications of the results obtained in proving the identity of the matrix.

1.0.2 Overview

I now wish to give a more detailed summary of the structure of this thesis aimed at an audience with some knowledge of algebraic topology and homotopy theory. In 1966 Boardman [7] introduced the stable homotopy category of spectra, which we denote $\mathscr{S}$. The objects in this category are spectra, introduced by Lima, which are sequences of topological spaces along with maps from the suspension of one space in the sequence to the next. The morphisms are complicated to define, but importantly are homotopy classes, in some appropriate sense, of maps of spectra. After giving a few topological preliminaries in §2.1, we give details of the construction of $\mathscr{S}$ in §2.2. In this thesis we work with a localisation of the category $\mathscr{S}$, with respect to mod 2 singular homology, in the sense of Bousfield [9]. The details of this localisation are given in §2.7.

The notion of spectra is very natural if one starts with a generalised cohomology theory, as every such cohomology theory defines a spectrum which represents it. Conversely, to every spectrum we can associate a generalised cohomology theory. The relationship between generalised (co)homology theories and spectra is discussed in §2.5. In this thesis we are particularly interested in the spectra bu and bo which represent 2-local complex and real connective K-theory respectively. After defining a suitable product of spectra we can introduce the notion of a ring spectrum and a module spectrum over a ring spectrum. The smash product of spectra $\wedge$, once correctly defined, makes $bu \wedge bo$ a left module spectrum over the ring spectrum bu. Ring and module spectra, and in particular bu and bo, are introduced in §2.6. After describing how to calculate the mod 2 singular homology of a spectrum

(§2.5) we can consider the set of left bu-module automorphisms of $bu \wedge bo$ which induce the identity map on mod 2 singular homology. This set actually forms a group, which we denote $Aut^0_{left-bu-mod}(bu \wedge bo)$. The main result of [24], Theorem 1.2, is a group isomorphism of the form

$$Aut^0_{left-bu-mod}(bu \wedge bo) \cong U_\infty \mathbb{Z}_2$$

where $U_\infty \mathbb{Z}_2$ represents the group of infinite upper triangular matrices with 2-adic integer entries. This isomorphism is defined up to inner automorphism in $U_\infty \mathbb{Z}_2$. As introduced in §2.8, by far the most important element in $Aut^0_{left-bu-mod}(bu \wedge bo)$ is the automorphism $1 \wedge \psi^3$, where $\psi^3 : bo \longrightarrow bo$ is the Adams operation on the real connective K-theory spectrum. The obvious question to ask is, what is the conjugacy class of matrices which represents $1 \wedge \psi^3$. In [24] a footnote appeared (page 1273) predicting that the matrix representing $1 \wedge \psi^3$ is conjugate in $U_\infty \mathbb{Z}_2$ to the matrix

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 9 & 1 & 0 & 0 & \dots \\ 0 & 0 & 9^2 & 1 & 0 & \dots \\ 0 & 0 & 0 & 9^3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

The main result of this thesis, Theorem 3.4.2, is a proof of this prediction. This result and its proof have been published in the collaborative paper [11].

As discussed in §3.5, an element of the group $Aut^0_{left-bu-mod}(bu \wedge bo)$ is determined by its effect on $\frac{\pi_*(bu \wedge bo) \otimes \mathbb{Z}_2}{Torsion}$, the homotopy group of the spectrum $bu \wedge bo$ tensored with the 2-adic integers and modulo torsion. The homotopy groups of spectra are introduced in §2.3.1. As explained in §3.1, Snaith's group isomorphism mentioned above makes use of a decomposition of the spectrum $bu \wedge bo$ given by Mahowald. We therefore find that if we wish to determine the effect of the map induced on homotopy by $1 \wedge \psi^3$ we need to work out its effect on basis elements, for the homotopy group as a module of the 2-adic integers, given in terms of this decomposition. These basis elements are discussed in §3.5. To enable us to do this we make use of a second, far more convenient, basis for the homotopy group shown above given by Clarke, Crossley and Whitehouse in [12] and described here in §3.6. The advantage of this basis is that the effect of the map induced by $1 \wedge \psi^3$ is well known. We therefore proceed to find the relationship between the two bases and translate the effect of the map induced by $1 \wedge \psi^3$ from Clarke, Crossley and Whitehouse's basis to the basis coming from Mahowald's decomposition. Once this relationship of bases is established we are able to use Snaith's isomorphism to calculate the matrix corresponding to $1 \wedge \psi^3$. We find that this matrix is

$$C = \begin{pmatrix} 1 & 1 & c_{1,3} & c_{1,4} & c_{1,5} & \dots \\ 0 & 9 & 1 & c_{2,4} & c_{2,5} & \dots \\ 0 & 0 & 9^2 & 1 & c_{3,5} & \dots \\ 0 & 0 & 0 & 9^3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

where the $c_{i,j}$ are arbitrary 2-adic integers. We conclude Chapter 3 by proving that the matrix B is in fact conjugate in $U_\infty\mathbb{Z}_2$ to the constructed matrix C, as predicted in [24].

In Chapter 4 we present two applications of the main results of Chapter 3. The first application (§4.1) describes how knowledge of the matrix corresponding to $1\wedge\psi^3 : bu\wedge bo \longrightarrow bu\wedge bo$ allows us to explicitely describe the left bu-module automorphism $1 \wedge \psi^3 : bu \wedge bu \longrightarrow bu \wedge bu$ in terms of morphisms arising from the Mahowald splitting of $bu\wedge bo$. The second application (§4.2) investigates the ring of left bu-module endomorphisms of $bu \wedge bo$. In particular the main result of this section reproves and slightly extends a 1974 result of Milgram ([19], Theorem B). Both of these applications are the result of the collaborative work of the author and his PhD supervisor Prof. V. Snaith being published in [11]. They are not presented as the original work of the author but they are included here as immediate examples of how knowledge of the matrix corresponding to $1 \wedge \psi^3$ can be put to good use.

1.0.3 Prerequisites and assumptions

The required knowledge to begin original research in stable homotopy theory is vast. There are certain substantial assumptions that needed to be made in the preparation of this thesis, as a complete account of the relevant background material would be impossible. We therefore assume a basic general knowledge of algebraic topology, specifically CW-complexes, basic singular (co)homology theory, generalised homology theories and basic homotopy theory. We also assume a basic working knowledge of homological algebra. Possibly our biggest assumption though is knowledge of the basic theory of spectral sequences and, in particular, the Adams spectral sequence.

Chapter 2

Background material

The aim of the following chapter is to provide the reader with enough background material, at an appropriate level of detail, so that they may feel comfortable with the concepts required to understand the original research contained in this thesis. References are given for more comprehensive accounts and the reference chosen usually reflects the publication the author found most useful in understanding the concept.

2.1 Topological preliminaries

Let $\mathscr{T}$ denote the category of topological spaces with basepoint. In this category the morphisms are base-point preserving continuous maps of topological spaces. Let $\mathscr{CW}$ denote the category of CW-complexes with base-point. In this category morphisms are base-point preserving cellular maps of CW-complexes.

Definition 2.1.1. Given $(X, x_0), (Y, y_0) \in \mathscr{T}$, we define the *smash product*

$(X \wedge Y, *) \in \mathscr{T}$ to be the quotient space

$$X \wedge Y = \frac{X \times Y}{X \vee Y}$$

with basepoint $* = p(X \vee Y)$, where $p : X \times Y \longrightarrow X \wedge Y$ is the projection.

Definition 2.1.2. Given $(X, x_0) \in \mathscr{T}$ we define the *suspension* $(\Sigma X, *) \in \mathscr{T}$ of X to be the smash product $(S^1 \wedge X, *)$ of X with the 1-sphere.

Definition 2.1.3. Given $(X, x_0) \in \mathscr{T}$, we define the *n-th suspension of* X $(\Sigma^n X, *)$ inductively as the smash product $(S^1 \wedge \Sigma^{n-1} X, *)$ of $\Sigma^{n-1} X$ with the 1-sphere, for $n \geq 1$.

Definition 2.1.4. $(X, x_0) \in \mathscr{T}$ is called *n-connected* if and only if $\pi_k(X, x) = 0$ for $0 \leq k \leq n$ and all $x \in X$.

Let σ denote the homomorphism induced on homotopy groups by suspension.

Theorem 2.1.5. Freudenthal Suspension Theorem *For every n-connected CW-complex X, $n \geq 0$, the homomorphism $\sigma : \pi_r(X, x_0) \longrightarrow \pi_{r+1}(\Sigma X, *)$ is an isomorphism for $1 \leq r \leq 2n$.*

Definition 2.1.6. Given $(Y, y_0) \in \mathscr{T}$, we define the *loop space* $(\Omega Y, \omega_0) \in \mathscr{T}$ of Y to be the function space

$$\Omega Y = (Y, y_0)^{(S^1, s_0)}$$

with the constant loop $\omega_0(s) = y_0$ for all $s \in S^1$ as base-point.

Definition 2.1.7. Given $(Y, y_0) \in \mathscr{T}$ we define the *n-th loop space* $(\Omega^n Y, \omega_0) \in \mathscr{T}$ inductively as the loop space $\Omega(\Omega^{n-1} Y)$, for $n \geq 1$.

Proposition 2.1.8. *The suspension and loop space functors are adjoint.*

For the remainder of this chapter we shall use the word *space* to mean an element of $\mathscr{T}$ and *map* of spaces to mean a morphism in $\mathscr{T}$. We shall omit basepoints unless explicitely required.

2.2 The stable homotopy category of spectra

In this section we shall describe the stable homotopy category of CW-spectra introduced by Boardman in his 1966 Warwick preprint [7]; but our description is due to Adams ([4]). We shall denote this category by $\mathscr{S}$. The account given here largely follows Adams's account but also includes explanations of some concepts inspired by the explanations given in [26] and [13].

2.2.1 Objects

Definition 2.2.1. A *CW-spectrum* E is a sequence of pointed CW-complexes E_n provided with structure maps

$$\epsilon_n : \Sigma E_n \longrightarrow E_{n+1}$$

such that each structure map ϵ_n is a homeomorphism from ΣE_n to a subcomplex of E_{n+1}.

It is equivalent to define a spectrum in terms of structure maps $\epsilon' : E_n \longrightarrow \Omega E_{n+1}$ as Σ and Ω are adjoint functors.

We could define a more general notion of spectra which drops the requirement that the sequence of spaces be CW-complexes, but for the purposes of this thesis this is unnecessary as all the spectra we will be using are CW-spectra. There is no loss in generality by restricting to CW-spectra in any case since it can be shown that any spectrum in the more general sense is weakly equivalent to some CW-spectrum. Therefore from this point on we shall simply say spectrum to mean CW-spectrum.

Example: Given a CW-complex X, we define the *suspension spectrum* $\Sigma^\infty X$ to be the spectrum with

$$(\Sigma^\infty X)_n = \begin{cases} \Sigma^n X & n \geq 0 \\ * & n < 0 \end{cases}$$

and structure maps the obvious maps $\epsilon_n : \Sigma(\Sigma^\infty X)_n = \Sigma\Sigma^n X \longrightarrow (\Sigma^\infty X)_{n+1} = \Sigma^{n+1} X$. In particular if we take $X = S^0$ we obtain a spectrum S called the *sphere spectrum.*

Example: We may also define a spectrum representing the *n-th desuspension* of a space X as

$$(\Sigma^{-n} X)_m = \begin{cases} S^0 & 0 \leq m < n \\ \Sigma^{m-n} X & m \geq n \end{cases}$$

with the obvious maps.

Example: As is usual, let $U(n)$ denote the $n \times n$ unitary group. This is the group of $n \times n$ complex matrices U satisfying the condition $U^*U = UU^* = I_n$, where I_n is the $n \times n$ identity matrix and U^* is the conjugate transpose of U. The group operation is matrix multiplication. Let BU denote the classifying space of the infinite unitary group (i.e. a $K(U,1)$ space where $U = \bigcup_{n\geq 1} U(n)$). Most proofs of the Bott Periodicity Theorem actually prove a stronger result, that there is a homotopy equivalence

$$\mathbb{Z} \times BU \simeq \Omega^2(\mathbb{Z} \times BU).$$

This result allows us to define the complex K-theory spectrum, denoted KU, by

$$KU_n = \begin{cases} \mathbb{Z} \times BU & \text{if } n \text{ is even,} \\ \Omega(\mathbb{Z} \times BU) & \text{if } n \text{ is odd.} \end{cases}$$

The structure maps ϵ_n

$$\Sigma(\mathbb{Z} \times BU) \longrightarrow \Omega(\mathbb{Z} \times BU)$$

and

$$\Sigma\Omega(\mathbb{Z} \times BU) \longrightarrow \mathbb{Z} \times BU$$

are given by the adjoints of the Bott periodicity homotopy equivalence and the identity map,

$$\mathbb{Z} \times BU \longrightarrow \Omega^2(\mathbb{Z} \times BU)$$

and

$$\Omega(\mathbb{Z} \times BU) \longrightarrow \Omega(\mathbb{Z} \times BU),$$

respectively. It is easy to see that all the structure maps $\epsilon'_n : E_n \longrightarrow \Omega E_{n+1}$ are weak equivalences. This property means that we call the spectrum KU an *Ω-spectrum*.

Example: Let BO and BSp denote the classifying spaces of the infinite orthogonal and symplectic groups respectively. The proof of the Bott Periodicity Theorem also yields the homotopy equivalences

$$\mathbb{Z} \times BO \simeq \Omega^4 BSp$$

and

$$\mathbb{Z} \times BSp \simeq \Omega^4 BO.$$

These results allow us to define the real K-theory spectrum, denoted KO. The spectrum KO is of period 8 (i.e. $KO_n{=}KO_{n+8}$) and is defined by

$$\begin{array}{cccccccc} KO_1 & KO_2 & KO_3 & KO_4 & KO_5 & KO_6 & KO_7 & KO_8 \\ \| & \| & \| & \| & \| & \| & \| & \| \\ \Omega^3 BSp & \Omega^2 BSp & \Omega BSp & \mathbb{Z} \times BSp & \Omega^3 BO & \Omega^2 BO & \Omega BO & \mathbb{Z} \times BO \end{array}$$

The structure maps $\epsilon'_{8n} : KO_{8m} \longrightarrow \Omega KO_{8m+1}$ are given by the first of the homotopy equivalences given above. Similarly the structure maps ϵ'_{8m+4} : $KO_{8m+4} \longrightarrow \Omega KO_{8m+5}$ are given by the second of the homotopy equivalences given above. All other structure maps are given by the identity map.

The following definitions relating to the objects will be required when we come to define the morphisms in $\mathscr{S}$.

Definition 2.2.2. A *subspectrum* E' of a spectrum E consists of subcomplexes $E'_n \subset E_n$ for each n such that the structure map $\epsilon_n : \Sigma E_n \longrightarrow E_{n+1}$ maps $\Sigma E'_n$ into E'_{n+1}.

Definition 2.2.3. A subspectrum E' of a spectrum E is said to be *cofinal* in E if for each n and each finite subcomplex $K \subset E_n$ there is an m, which depends on n and K, such that $\Sigma^m K$ maps into E'_{m+n} under the map

$$\Sigma^m E_n \xrightarrow{\Sigma^{m-1}\epsilon_n} \Sigma^{m-1}E_{n+1} \longrightarrow \cdots \longrightarrow \Sigma E_{m+n-1} \xrightarrow{\epsilon_{m+n-1}} E_{m+n}.$$

Intuitively, this definition says that given enough suspensions any cell in E_n gets mapped into E'.

2.2.2 Morphisms

We now wish to complete the construction of $\mathscr{S}$ by defining morphisms between spectra. Unfortunately the obvious definition, which we call a *function* of spectra, turns out to be inadequate. It can be shown that many reasonable morphisms you may expect to have in $\mathscr{S}$ cannot exist using this naïve definition. See [4] pages 141-2 for an example of such a morphism. Therefore we define the morphisms in a series of steps.

Definition 2.2.4. A *function* $f : E \longrightarrow F$ of CW-spectra of degree r is a collection of cellular maps $f_n : E_n \longrightarrow F_{n-r}$, $n \in \mathbb{Z}$, such that the following diagram is strictly commutative for each n:

$$\begin{array}{ccc} \Sigma E_n & \xrightarrow{\epsilon_n} & E_{n+1} \\ \downarrow{\scriptstyle \Sigma f_n} & & \downarrow{\scriptstyle f_{n+1}} \\ \Sigma F_{n-r} & \xrightarrow{\epsilon'_{n-r}} & F_{n-r+1} \end{array}$$

Remark 2.2.5. The grading of functions given here is designed to eventually give $\pi_r(F) = [S, F]_r$.

We now define an equivalence relation $\sim$ which will allow us to define *maps* of spectra.

Definition 2.2.6. Let E, F be spectra. Consider the set S of all pairs (E', f') such that $E' \subset E$ is a cofinal subspectrum and $f' : E' \longrightarrow F$ is a function of spectra. We introduce an equivalence relation $\sim$ on S by $(E', f') \sim (E'', f'')$ if and only if there is a pair (E''', f''') with $E''' \subset E' \cap E''$, E''' cofinal and $f'|E''' = f''' = f''|E'''$.

The proof that $\sim$ is an equivalence relation follows from the facts that intersections and arbitrary unions of cofinal subspectra are cofinal, and that if $G \subset F \subset E$ are subspectra such that F is cofinal in E and G is cofinal in F, then G if cofinal in E.

Definition 2.2.7. We call equivalence classes of $\sim$ *maps* from E to F

This definition of $\sim$ in terms of cofinal subspectra allows us to define a map on the suspension $\Sigma^n c \in E_{m+n}$ of a cell $c \in E_m$ rather than having to define the map on c itself.

A morphism in $\mathscr{S}$ will be defined as a homotopy class of maps, therefore we require the notion of a homotopy of maps of spectra.

Definition 2.2.8. Let I^+ denote the union of the unit interval $[0, 1]$ and a disjoint base-point. A *homotopy* is a map of spectra $g : E \wedge I^+ \longrightarrow F$, where $E \wedge I^+$ is defined to be the spectrum with $(E \wedge I^+)_n = E_n \wedge I^+$. There are two obvious morphisms of spectra $i_0 : E \longrightarrow E \wedge I^+$, $i_1 : E \longrightarrow E \wedge I^+$ induced by the inclusions of $0, 1$ in I^+. We say two maps of spectra $f_0, f_1 : E \longrightarrow F$ are *homotopic* if there is a homotopy $h : E \wedge I^+ \longrightarrow F$ with $h \circ i_0 = f_0$, $h \circ i_1 = f_1$.

Homotopy as defined here is an equivalence relation, which leaves us in a position to finally define the morphisms in our category. If E, F are spectra we write $[E, F]_r$ for the set of homotopy classes of maps of degree r from E to F.

Definition 2.2.9. A morphism of degree r in $\mathscr{S}$ is a homotopy class of maps of degree r.

Given the notion of homotopy we may define *fibre* and *cofibre* sequences for spectra in exactly the same way as on the space level so that they will enjoy all the same homotopical properties. In fact, for the CW-spectra we are considering the notion of fibre sequence and cofibre sequence coincide.

2.3 Homotopy groups of spectra

We shall now define *homotopy groups* of spectra in such a way that the homotopy groups of a suspension spectrum $\Sigma^\infty X$ will coincide with the stable homotopy groups of the space X.

Given any spectrum $E \in \mathscr{S}$ we have the following homomorphisms of homotopy groups of spaces

$$\pi_{n+r}(E_n) \xrightarrow{\sigma} \pi_{n+r+1}(\Sigma E_n) \xrightarrow{(\epsilon_n)_*} \pi_{n+r+1}(E_{n+1})$$

given by the Freudenthal suspension homomorphism ([26] Theorem 15.46) and the map induced by the structure map ϵ_n in the spectrum E. If we consider all such homomorphisms for $n \in \mathbb{Z}$ we obtain a direct system and hence we can take the direct limit of such a system, which leads us to the following definition:

Definition 2.3.1. The *r-th homotopy group of the spectrum E* is defined to be

$$\pi_r(E) = \lim_{n\to\infty} \pi_{n+r}(E_n).$$

Definition 2.3.2. A spectrum E is called *connective* if $\exists n_0 \in \mathbb{Z}$ such that $\pi_q(E) = 0$ for $q < n_0$.

Recall that a standard definition of a homotopy group of a space is given in terms of homotopy classes of maps by $\pi_n(X) = [S^n, X]$, obviously we are omitting writing base-points here. Therefore we may rewrite the homomorphisms above as

$$[S^{n+r}, E_n] \longrightarrow [S^{n+r+1}, \Sigma E_n] \longrightarrow [S^{n+r+1}, E_{n+1}].$$

Restating Definition 2.3.1 in this notation gives rise to the following proposition:

Proposition 2.3.3.

$$\pi_r(E) = \lim_{n\to\infty} [S^{n+r}, E_n] = [S, E]_r.$$

For a proof of this proposition see [4] Proposition 2.8.

Clearly we have yet to show that $[E, F]_r$ has the structure of a group for any spectra E, F. In fact it is possible to prove the stronger result that $[E, F]_r$ has the structure of an abelian group. For a proof of this fact see [26] Corollary 8.27.

When considering homotopy groups of suspension spectra it is important to make the distinction between the homotopy group of the space X and the homotopy group of its suspension spectrum as they may be quite different.

Definition 2.3.4. Any morphism of spectra $f : E \longrightarrow F$ induces a homomorphism $f_* : \pi_n(E) \longrightarrow \pi_n(F)$, $n \in \mathbb{Z}$. If f_* is an isomorphism for all $n \in \mathbb{Z}$ we call f a *weak homotopy equivalence.*

We state, without proof, the following propositions which give the relationship between weak homotopy equivalences of spectra, homotopy equivalences of spectra and homotopy classes of morphisms of spectra:

Proposition 2.3.5. *A morphism of spectra is a weak homotopy equivalence if and only if it is a homotopy equivalence.*

Proposition 2.3.6. *If $f : E \longrightarrow F$ is a morphism of spectra which is a weak homotopy equivalence, then $f_* : [G, E]_r \longrightarrow [G, F]_r$ is a bijection for any spectrum G.*

For proofs of these propositions see [26] pages 140 and 141 respectively.

Definition 2.3.7. Two spaces S, T are said to be *stably homotopy equivalent* if there exists a homotopy equivalence of spectra $s : \Sigma^\infty S \longrightarrow \Sigma^\infty T$.

In the spectra we are going to study it will sometimes be convenient to work only with the spaces E_{2n} to represent the spectrum E. There is no loss of generality in doing this providing we are given all the maps $\epsilon_{2n+1} \circ \Sigma\epsilon_{2n} : \Sigma^2 E_{2n} \longrightarrow E_{2n+2}$ as from this we may define a spectrum E' with $E'_{2n} = E_{2n}, E'_{2n+1} = \Sigma E_{2n}$ and structure maps $\epsilon'_{2n} = 1, \epsilon'_{2n+1} = \epsilon_{2n+1} \circ \Sigma\epsilon_{2n}$. Let $f_m : E' \longrightarrow E$ be the degree 0 function of spectra which is ϵ_m for m odd and equality for m even. The morphism of spectra obtained from f_m is a homotopy equivalence of spectra.

2.4 The smash product of spectra

We now wish to introduce a smash product of spectra which is compatible with the smash product we already have for CW-complexes and, given spectra $E, F, G \in \mathscr{S}$, satisfies the following properties, with each equivalence natural in E, F, G:

(i) $E \wedge F$ is a functor in two variables from $\mathscr{S}$ to $\mathscr{S}$

(ii) associativity, i.e. $\exists$ a homotopy equivalence $a : (E \wedge F) \wedge G \longrightarrow E \wedge (F \wedge G)$

(iii) commutativity, i.e. $\exists$ a homotopy equivalence $c : E \wedge F \longrightarrow F \wedge E$

(iv) the sphere spectrum S is a two sided unit, i.e. $\exists$ homotopy equivalences $l : S \wedge E \longrightarrow E$ and $r : E \wedge S \longrightarrow E$

The above list is not intended to be a complete list of the desired properties of such a product.

The basic strategy in constructing such a product is that we want $E \wedge F$ to be the spectrum you obtain from some limit of the spaces $E_m \wedge F_n$ as m, n tend to infinity. A naïve way to construct such a product would be to take $(E \wedge F)_n = E_{r(n)} \wedge F_{s(n)}$ for some functions $r(n)$ and $s(n)$ with $r(n) + s(n) = n$ and such that $r(n) \longrightarrow \infty$ and $s(n) \longrightarrow \infty$ as $n \longrightarrow \infty$. Unfortunately constructing smash products in this way requires the choice of functions $r(n)$

and $s(n)$ to be made and it is not immediately clear what is the correct choice as there are many such possibilities. Making a particular choice for $r(n)$ and $s(n)$ gives what Boardman [7] referred to as a "handicrafted smash product". Adams ([4] Theorem 4.1) proved that these different choices of product are in fact related by natural homotopy equivalences. I have chosen to omit the details of this proof here as they are unecessary for the rest of this thesis and instead direct the reader to [4] Part 3 Chapter 4 and [26] Chapter 13. In these references complete constructions are given of the smash product although it requires a significant amount of work to do so. The important point to note for the purposes of this thesis is that we only wish the smash product to be defined upto homotopy, hence any such "handicrafted" product will do.

2.5 Generalised homology theories

Given any spectrum $E \in \mathscr{S}$, we now show how to define the (reduced) homology and cohomology theories associated to E.

Definition 2.5.1. Let E, F be spectra. For each $n \in \mathbb{Z}$ we define the E-*homology* and E-*cohomology* to be

$$E_n(F) = [S, E \wedge F]_n = \pi_n(E \wedge F)$$

and

$$E^n(F) = [F, E]_{-n}$$

respectively.

We shall now list the properties that E-homology and E-cohomology are required to satisfy in order to be called a generalized (co)homology theory defined on spectra. These are the analogues for spectra of the Eilenberg-Steenrod axioms for spaces. Proofs that these properties are satisfied may be found in [26] §8.33.

1. $E_*(F)$ is a covariant functor of two variables from $\mathscr{S}$ to the category of graded abelian groups.

2. $E^*(F)$ is also a functor between $\mathscr{S}$ and the category of graded abelian groups but it is covariant in E and contravariant in F.

3. Given a cofiber sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

then

$$E_n(X) \xrightarrow{f_*} E_n(Y) \xrightarrow{g_*} E_n(Z)$$

and

$$E^n(Z) \xrightarrow{g^*} E^n(Y) \xrightarrow{f^*} E^n(X)$$

are exact.

4. Given a cofiber sequence

$$E \xrightarrow{i} F \xrightarrow{j} G$$

then

$$E_n(X) \xrightarrow{i_*} F_n(X) \xrightarrow{j_*} G_n(X)$$

and

$$E^n(X) \xrightarrow{i_*} F^n(X) \xrightarrow{j_*} G^n(X)$$

are exact.

5. We have natural isomorphisms

$$E_n(F) \cong E_{n+1}(S^1 \wedge F)$$

and

$$E^n(F) \cong E^{n+1}(S^1 \wedge F).$$

6. $E_n(S) = E^{-n}(S) = \pi_n(E)$.

Given a spectrum E and a CW-complex X we may also define the E-homology and E-cohomology, $\tilde{E}_n(X)$ and $\tilde{E}^n(X)$ respectively, for $n \in \mathbb{Z}$ by the E-homology and E-cohomology of $\Sigma^\infty X$, the suspension spectrum of X. Exactly the same properties as above are satisfied.

Example: An *Eilenberg-Mac Lane spectrum for the group* π is a spectrum $H\pi$ with n^{th} space the Eilenberg-Mac Lane space of type (π, n) and structure maps as described in [26] §10.2. In this case, $(H\pi)_*$ applied to the suspension spectrum of a complex X coincides with the ordinary singular homology of X with coefficients in π (see [4] Part 3, Chapter 6).

The above example motivates the following definition:

Definition 2.5.2. The *(co)homology with coefficients in* π of a spectrum $E \in \mathscr{S}$ is defined to be the $H\pi$-(co)homology of E.

Example: Consider the spectra KU and KO defined in §2.2.1. The (co)homology theories associated to these spectra are called the complex and real K-(co)homology respectively. If X is a finite-dimensional CW-complex, $[X, \mathbb{Z} \times BU]$ and $[X, \mathbb{Z} \times BO]$ agree with the classical definitions of complex and real topological K-theory (see [4] Part 3, Chapter 6).

2.6 Ring and module spectra

With a smash product defined on $\mathscr{S}$ we can define spectra with further structure in the following way:

Definition 2.6.1. A *ring spectrum* is a spectrum R with product, i.e. a morphism of spectra, $\mu : R \wedge R \longrightarrow R$ and identity $\eta : S \longrightarrow R$ such that the diagrams

$$\begin{array}{ccc} R \wedge R \wedge R & \xrightarrow{\mu \wedge 1} & R \wedge R \\ {\scriptstyle 1 \wedge \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ R \wedge R & \xrightarrow{\mu} & R \end{array} \qquad \begin{array}{ccc} S \wedge R & \xrightarrow{\eta \wedge 1} & R \wedge R \\ \simeq\downarrow & & \downarrow\mu \\ R & \xrightarrow{1} & R \\ \simeq\downarrow & & \uparrow\mu \\ R \wedge S & \xrightarrow{1 \wedge \eta} & R \wedge R \end{array}$$

commute. The product μ is commutative if

$$\begin{array}{ccc} R \wedge R & \xrightarrow{\mu} & R \\ {\scriptstyle c}\downarrow & \nearrow_{\mu} & \\ R \wedge R & & \end{array}$$

also commutes.

Given the notion of a ring spectrum we have the following obvious notion of a module spectrum:

Definition 2.6.2. Let R be a ring spectrum. Then a spectrum M is called a *left R-module spectrum* if there is a morphism $\nu : R \wedge M \longrightarrow M$ of degree 0, which we will sometimes refer to as the action morphism, such that the following diagrams commute upto homotopy

$$\begin{array}{ccc} R \wedge R \wedge M & \xrightarrow{\nu \wedge 1} & R \wedge M \\ {\scriptstyle 1 \wedge \nu}\downarrow & & \downarrow{\scriptstyle \nu} \\ R \wedge M & \xrightarrow{\nu} & M \end{array} \qquad \begin{array}{ccc} S \wedge M & \xrightarrow{\eta \wedge 1} & R \wedge M \\ {\scriptstyle \simeq}\downarrow & & \downarrow{\scriptstyle \nu} \\ M & \xrightarrow{1} & M \end{array}$$

It is in fact true that all the spectra we have already mentioned, and will mention, in this thesis are ring spectra. Explicit constructions of the products are given in [26] Chapter 13.

Example: Following the account of [4] there exists a spectrum bu, the unitary connective K-theory spectrum, which comes provided with a morphism $bu \longrightarrow KU$ and which is characterized by the following properties:

1. $\pi_r(bu) = 0$ for $r < 0$.
2. The induced map
$$\pi_r(bu) \longrightarrow \pi_r(KU)$$
is an isomorphism for $r \geq 0$.

The structure maps of bu are constructed using obstruction theory so as to make the following diagram commute upto homotopy.

$$\begin{array}{ccc} \Sigma^2 bu_{2m} & \xrightarrow{\epsilon_{2m}} & bu_{2m+2} \\ \downarrow{\scriptstyle f_{2m}} & & \downarrow{\scriptstyle f_{2m+2}} \\ \Sigma^2 KU_{2m} & \xrightarrow{\epsilon_{2m}} & KU_{2m+2} \end{array}$$

This diagram defines a canonical morphism of spectra $bu \longrightarrow KU$

In a similar way we can define bo, the orthogonal connective K-theory spectrum.

Proposition 2.6.3. *bu and bo are ring spectra.*

Proof. The product morphisms giving bu and bo their ring structure are pulled back from the ring spectrum structure of KU and KO respectively. The products in KU and KO, as shown in [26] Chapter 13, are defined at the level of vector bundles. □

Let μ denote the product $\mu : bu \wedge bu \longrightarrow bu$ then $bu \wedge bo$ is a left bu-module spectrum with left bu action defined by

$$\mu \wedge 1 : bu \wedge (bu \wedge bo) \longrightarrow bu \wedge bo.$$

Definition 2.6.4. Let E be a ring spectrum and F, G be E-module spectra. A *left-E-module spectrum morphism* is a morphism of spectra $f : F \longrightarrow G$

of degree 0 such that the following diagram commutes:

$$\begin{array}{ccc} E \wedge F & \xrightarrow{\quad \nu \quad} & F \\ {\scriptstyle 1\wedge f}\downarrow & & \downarrow{\scriptstyle f} \\ E \wedge G & \xrightarrow{\quad \nu \quad} & G \end{array}$$

Definition 2.6.5. A *left-E-module spectrum endomorphism of F* is a left-E-module morphism $g : F \longrightarrow F$.

Proposition 2.6.6. *The set $End_{left-E-mod}(F)$ of endomorphisms of the left-E-module spectrum F has the structure of a ring.*

Proof. The fact the set of endomorphisms of a spectrum F has the structure of a ring comes from the fact that $\mathscr{S}$ is an additive category, which is proven in [4] Part 3 Chapter 3. Composition of morphisms gives the multiplicative structure and Adams gives addition in [4] Part 3 Chapter 3. It is straightforward to show that $End_{left-E-mod}(F)$ is a subring. □

Definition 2.6.7. A left-E-module spectrum endomorphism $g : F \longrightarrow F$ is a *left-E-module spectrum automorphism* if there exists a left-E-module spectrum endomorphism $h : F \longrightarrow F$ such that $hg \simeq 1_F$ and $gh \simeq 1_F$.

Remark 2.6.8. By definition, a left E-module automorphism is the same as a homotopy equivalence of left E-module spectra.

Proposition 2.6.9. *The set of $Aut_{left-E-mod}(F)$ of automorphims of the left-E-module spectrum F is the group of units of $End_{left-E-mod}(F)$.*

2.7 Bousfield localisation

Recall from §2.5 that $H\mathbb{Z}/2$ represents the Eilenberg-Maclane spectrum related to the group $\mathbb{Z}/2$. A morphism $f : X \longrightarrow X'$ in $\mathscr{S}$ is a $(H\mathbb{Z}/2)_*$-*equivalence* if the induced homomorphism $f_* : (H\mathbb{Z}/2)_*(X) \longrightarrow (H\mathbb{Z}/2)_*(X')$ is an isomorphism.

In 1979, Bousfield ([9]) introduced a functor $L_{H\mathbb{Z}/2}$ from $\mathscr{S}$ to a new category, which we shall denote $\mathscr{S}_2$, in which we do not attempt to distinguish between two spectra if there is an $(H\mathbb{Z}/2)_*$-equivalence between them. The objects of $\mathscr{S}_2$ are the same as those of $\mathscr{S}$ and $L_{H\mathbb{Z}/2}$ is the identity on objects. If $e : X \longrightarrow Y$ is an $(H\mathbb{Z}/2)_*$-equivalence in $\mathscr{S}$, then $L_{H\mathbb{Z}/2}(e)$ is an actual equivalence in $\mathscr{S}_2$, i.e. it has an inverse. $L_{H\mathbb{Z}/2}$ is universal with respect to this property. To be more precise, we give the following definition and theorem.

Definition 2.7.1. A spectrum $Y \in \mathscr{S}$ is said to be $(H\mathbb{Z}/2)_*$*-local* if each $(H\mathbb{Z}/2)_*$-equivalence $f : X \longrightarrow X'$ induces a bijection $f^* : [X', Y]_* \longrightarrow [X, Y]_*$. This is equivalent to saying that Y is $(H\mathbb{Z}/2)_*$-local if $[X, Y]_* = 0$ whenever $(H\mathbb{Z}/2)_*(X) = 0$.

Theorem 2.7.2. *(Bousfield Localisation Theorem) There exists a functor* $L_{H\mathbb{Z}/2} : \mathscr{S} \longrightarrow \mathscr{S}_2$, *called the* $(H\mathbb{Z}/2)_*$-localisation, *such that* $L_{H\mathbb{Z}/2}(X)$ *is* $(H\mathbb{Z}/2)_*$*-local.* $L_{H\mathbb{Z}/2}(X)$ *is functorial in* X.

Proof. See [9] □

Localisation with respect to $H\mathbb{Z}/2$ may also be referred to as localisation with respect to mod 2 singular homology as this is the homology theory associated to $H\mathbb{Z}/2$.

In particular, we will be interested in the case of connective spectra. In this case Bousfield proved that localisation of $X \in \mathscr{S}$ with respect to $H\mathbb{Z}/2$ is equivalent to taking the 2-adic completion of X, which we shall not define here. But for this reason throughout this thesis we shall use the terms "2-adic completion" and "localisation with respect to $H\mathbb{Z}/2$" synonymously, and we shall refer to spectra which have undergone this process to be *2-local.*

As described in [9], $\mathscr{S}_2$ has a smash product which enjoys all the same properties as the smash product in $\mathscr{S}$. $(H\mathbb{Z}/2)_*$-localisation does not necessarily preserve smash products, but there is a canonical map $X_{H\mathbb{Z}/2} \wedge Y_{H\mathbb{Z}/2} \longrightarrow (X \wedge Y)_{H\mathbb{Z}/2}$ with the expected properties.

Proposition 2.7.3. *If R is a commutative ring spectrum, then so is $R_{H\mathbb{Z}/2}$.*

Proposition 2.7.4. *If R is a 2-local ring spectrum, then any module spectrum over R is 2-local.*

Both propositions are discussed in [9].

If the homotopy of a spectrum $E \in \mathscr{S}$ is finitely generated in every dimension then the n-th homotopy group of the $(H\mathbb{Z}/2)_*$-localisation of E is isomorphic to $\pi_n(E) \otimes \mathbb{Z}_2$, the n-th homotopy group of E tensored with the 2-adic integers. This is precisely the case for the spectra bu and bo (See [21] Chapter 3).

2.8 Cohomology Operations

2.8.1 $1 \wedge \psi^3 : bu \wedge bo \longrightarrow bu \wedge bo$

Definition 2.8.1. Given a spectrum $E \in \mathscr{S}$ and the corresponding cohomology theory E^* a *cohomology operation of type* (p, q) is a natural transformation $\theta : E^p(-) \longrightarrow E^q(-)$ between cofunctors regarded as taking values in sets.

Example: Let X be an element of $\mathscr{T}$. Recall the spectrum KU and the corresponding cohomology theory $KU^*(X)$, KU-cohomology. Adams showed ([5]) there exist cohomology operations (originally defined at the level of vector bundles for topological K-theory) of the form $\psi^k : KU^0(X) \longrightarrow KU^0(X)$ for $k \in \mathbb{Z}$ with the following properties:

1. ψ^k is a ring homomorphism $KU^0(X) \longrightarrow KU^0(X)$
2. $\psi^k \psi^l = \psi^{kl}$
3. If ξ is a line bundle then $\psi^k(\xi) = \xi^k$
4. If p is a prime then $\psi^p(x) \equiv x^p \bmod p$
5. On the reduced cohomology theory $\tilde{KU}^0(S^{2n}) = \mathbb{Z}$, ψ^k acts as multiplication by k^n

These operations are defined on $KU^0(X) = [X, \mathbb{Z} \times BU]$ by composition with a map $\psi^k : \mathbb{Z} \times BU \longrightarrow \mathbb{Z} \times BU$.

We wish to construct a morphism of spectra $\psi^3 : KU \longrightarrow KU$ which extends the map $\psi^3 : \mathbb{Z} \times BU \longrightarrow \mathbb{Z} \times BU$. For reasons related to the fifth property above, it is only possible to do this if we first 2-adically complete the spectrum KU, as discussed in §2.7, and work with the spectrum $(KU)_{H\mathbb{Z}/2}$. The morphism $\psi^3 : (KU)_{H\mathbb{Z}/2} \longrightarrow (KU)_{H\mathbb{Z}/2}$ is induced by the function of spectra which is $3^{-m}\psi^3 : \mathbb{Z} \times BU \longrightarrow \mathbb{Z} \times BU$ on the $2m$-th space of the spectrum for $m \geq 0$. The maps on the spaces of the spectrum $(KU)_{H\mathbb{Z}/2}$ which form the morphism ψ^3 induce maps on the spaces of the 2-adically complete spectrum $(bu)_{H\mathbb{Z}/2}$. Hence, we obtain a morphism of 2-adically complete spectra $\psi^3 : (bu)_{H\mathbb{Z}/2} \longrightarrow (bu)_{H\mathbb{Z}/2}$ so that the following diagram commutes:

$$\begin{array}{ccc} (bu)_{H\mathbb{Z}/2} & \xrightarrow{\psi^3} & (bu)_{H\mathbb{Z}/2} \\ \downarrow & & \downarrow \\ (KU)_{H\mathbb{Z}/2} & \xrightarrow{\psi^3} & (KU)_{H\mathbb{Z}/2} \end{array}$$

In the case of the real connective K-theory spectrum bo we can obtain a morphism $\psi^3 : (bo)_{H\mathbb{Z}/2} \longrightarrow (bo)_{H\mathbb{Z}/2}$ of 2-adically complete spectra in a similar way.

The homorphisms induced by $\psi^3 : (bo)_{H\mathbb{Z}/2} \longrightarrow (bo)_{H\mathbb{Z}/2}$ and $1 \wedge \psi^3 : (bu)_{H\mathbb{Z}/2} \wedge (bo)_{H\mathbb{Z}/2} \longrightarrow (bu)_{H\mathbb{Z}/2} \wedge (bo)_{H\mathbb{Z}/2}$ on mod 2 singular homology are the identity homomorphisms.

The significance of the left $(bu)_{H\mathbb{Z}/2}$-module automorphism $1 \wedge \psi^3 : (bu)_{H\mathbb{Z}/2} \wedge (bo)_{H\mathbb{Z}/2} \longrightarrow (bu)_{H\mathbb{Z}/2} \wedge (bo)_{H\mathbb{Z}/2}$ is shown by results given in [1] §6.3.

2.8.2 The mod 2 Steenrod Algebra

Definition 2.8.2. Given a spectrum $E \in \mathscr{S}$ and the corresponding cohomology theory E^*, a *stable cohomology operation of degree* q is a sequence of cohomology operations $\theta^n : E^n(-) \longrightarrow E^{n+q}(-)$, which commutes with the suspension isomorphisms given in §2.5 which we denote here by σ, i.e. $\theta^n \circ \sigma = \sigma \circ \theta^{n+1}$.

Let $A(E)^q$ denote the set of all stable cohomology operations of degree q for the cohomology theory E^*. $A(E)^q$ can be made into an abelian group by taking $(\theta + \phi)(x) = \theta(x) + \phi(x)$ for operations θ and ϕ, all $x \in E^*(X)$. Via composition of operations we can construct a pairing $A(E)^q \otimes A(E)^r \longrightarrow A(E)^{q+r}$. This makes $A(E)^* = \bigoplus_q A(E)^q$ a graded ring.

Proposition 2.8.3. *For any* X, $E^*(X)$ *is a graded module over* $A(E)^*$.

Example: The *mod 2 Steenrod Algrebra* is defined to be the algebra of operations for singular homology with $\mathbb{Z}/2$ coefficients. We denote this algebra by $\mathscr{A}$. $\mathscr{A}$ is an incredibly complicated algebra but, although it is highly non-trivial to prove, it can be expressed in terms of generators and relations in the following way: the generators are the Steenrod Squares, stable cohomology operations of the form $Sq^i : H^n(X;\mathbb{Z}/2) \longrightarrow H^{n+i}(X;\mathbb{Z}/2)$ and the relations are the Adém relations, which we omit here as their explicit detail is unnecessary for this thesis. Following [4] Part 3 Chapter 16 let $Sq^{0,1}$ denote the operation $Sq^1Sq^2 + Sq^2Sq^1$.

Proposition 2.8.4. *The operations* Sq^1 *and* $Sq^{0,1}$ *generate a subalgebra of* $\mathscr{A}$*, denoted* B.

Proof. See [4] Part 3 Chapter 16. □

For the purposes of this thesis, B can be thought of as the following algebra over $\mathbb{Z}/2$:

$$B = \frac{\mathbb{Z}/2[Sq^1, Sq^{0,1}]}{((Sq^1)^2, (Sq^{0,1})^2)} = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \cdot Sq^1 \oplus \mathbb{Z}/2 \cdot Sq^{0,1} \oplus \mathbb{Z}/2 \cdot Sq^1 \cdot Sq^{0,1}$$

2.9 A 2-local Adams spectral sequence

In this section we wish to merely introduce a particular case of the Adams spectral sequence. The details of this spectral sequence are well known, and the author recommends [18],[21] and [15] for full expositions.

The Adams spectral sequence was originally invented as a method of computing the stable homotopy groups of spheres, but in its full generality the Adams spectral sequence computes $[X, Y]$ for spectra X and Y. We specialise to a version of the spectral sequence which computes $\pi_*(Y)$ for a spectrum Y. This version of the spectral sequence is the classical Adams spectral sequence first introduced in [3].

Let X be a 2-local connective spectrum with finitely generated homotopy in every dimension. Note that, bu and bo, and $bu \wedge X$ for any other such spectrum X, satisfy this property (see [4] Part 3 Chapter 16). Recall from §2.8.2 that $H^*(X; \mathbb{Z}/2)$ is a module over the mod 2 Steenrod algebra. Adams ([3]) proved the following theorem:

Theorem 2.9.1. *There is a spectral sequence with E_2 term given by*

$$E_2^{s,t} = Ext_{\mathscr{A}}^{s,t}(H^*(X;\mathbb{Z}/2);\mathbb{Z}/2)$$

and converging to $\pi_*(X) \otimes \mathbb{Z}_2$ where $\mathbb{Z}_2$ denotes the 2-adic integers.

Remark 2.9.2. $Ext^{s,t}$ is defined in the following way. A projective resolution

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

of X is taken and $Ext_{\mathscr{A}}^{s,t}(H^*(X;\mathbb{Z}/2);\mathbb{Z}/2) = H^s(Hom_{\mathscr{A}}^t(P_*,\mathbb{Z}/2) fors \geq 0$, $t \in \mathbb{Z}$.

Adams ([4] Proposition 16.1) also showed that the E_2 term of this spectral sequence is isomorphic to $Ext_B^{s,t}(H^*(X;\mathbb{Z}/2);\mathbb{Z}/2)$. Therefore the main spectral sequence used in this thesis takes the form

$$E_2^{s,t} = Ext_B^{s,t}(H^*(X;\mathbb{Z}/2);\mathbb{Z}/2) \Longrightarrow \pi_{t-s}(bu \wedge X) \otimes \mathbb{Z}_2 \tag{2.1}$$

Proposition 2.9.3. *Let $f : E \longrightarrow F$ be a morphism of spectra which induces an isomorphism in mod 2 homology. Then f induces an isomorphism (from the E_2 page onwards) of Adams spectral sequences.*

Proof. See [21] Corollary 2.1.13 □

Chapter 3

ψ^3 as an upper triangular matrix

Throughout the remainder of this thesis we shall only consider the 2-adically complete spectra $(bu)_{H\mathbb{Z}/2}$ and $(bo)_{H\mathbb{Z}/2}$, and their smash product in $\mathscr{S}_2$. Therefore, for clarity we shall omit the completion notation and simply denote them by bu and bo.

3.1 A stable splitting of $bu \wedge bo$

We shall begin by recalling the 2-local homotopy decomposition of $bu \wedge bo$. This is one of a number of similar results discovered by Mahowald in the 1970's ([17]). We are referring to a proof of the result given in 2002 by Snaith in [24].

Consider the second loop space of the 3-sphere, $\Omega^2 S^3$. In the 1970's the work of Brown and Peterson ([10]) and separately Snaith ([22]) showed that there exists a filtration of $\Omega^2 S^3$ of the form

$$S^1 = F_1 \subset F_2 \subset F_3 \subset \ldots \subset \Omega^2 S^3 = \bigcup_{k \geq 1} F_k$$

where each F_i is a finite complex. Snaith also showed there is a stable homotopy equivalence, an example of the so-called Snaith splitting, of the form

$$\Omega^2 S^3 \simeq \vee_{k \geq 1} F_k / F_{k-1}.$$

There is a 2-local homotopy equivalence of left bu-module spectra (see [24] Theorem 2.3(ii)) of the form

$$\hat{L} : \vee_{k \geq 0} bu \wedge (F_{4k}/F_{4k-1}) \xrightarrow{\simeq} bu \wedge bo.$$

The important fact about this homotopy equivalence is that its induced homomorphism on mod 2 homology is a specific isomorphism which is described in [24] §2.2.

If we wish to study 2-local left bu-module morphisms of $bu \wedge bo$, the splitting $\hat{L}$ leads us to study 2-local left bu-module morphisms of the form

$$\phi_{k,l} : bu \wedge (F_{4k}/F_{4k-1}) \longrightarrow bu \wedge (F_{4l}/F_{4l-1})$$

for all k, l. A 2-local left bu-module morphism of this form is determined by its restriction to $S^0 \wedge (F_{4k}/F_{4k-1})$. To see this, consider the following homotopy commutative diagram:

$$\begin{array}{ccc}
bu \wedge (S^0 \wedge (F_{4k}/F_{4k-1})) & \xrightarrow{\;1 \wedge \phi_{k,l}(\eta \wedge 1) \simeq 1 \wedge \phi_{k,l}|_{S^0 \wedge F_{4k}/F_{4k-1}}\;} & bu \wedge bu \wedge (F_{4l}/F_{4l-1}) \\
\downarrow{\scriptstyle 1 \wedge \eta \wedge 1} & & \\
bu \wedge bu \wedge (F_{4k}/F_{4k-1}) & \xrightarrow{\;1 \wedge \phi_{k,l}\;} & bu \wedge bu \wedge (F_{4l}/F_{4l-1}) \\
\downarrow{\scriptstyle \mu \wedge 1} & & \downarrow{\scriptstyle \mu \wedge 1} \\
bu \wedge (F_{4k}/F_{4k-1}) & \xrightarrow{\;\phi_{k,l}\;} & bu \wedge (F_{4l}/F_{4l-1})
\end{array}$$

where the composition of the left hand vertical morphisms is homotopic to the identity morphism, μ is the product $\mu : bu \wedge bu \longrightarrow bu$ and η is the unit morphism $\eta : S^0 \longrightarrow bu$. This diagram implies $\phi_{k,l} \simeq (\mu \wedge 1)(1 \wedge (\phi_{k,l}|_{S^0 \wedge F_{4k}/F_{4k-1}})$ and hence $\phi_{k,l}$ is completely determined by its restriction to $S^0 \wedge F_{4k}/F_{4k-1}$.

Following the account of [26] §14.19, let X be any finite spectrum, then there exists a spectrum DX called the S-dual of X, characterized by the property that for any spectra U, V there exist isomorphisms of groups

$$\begin{aligned} D_\mu : [U, V \wedge DX]_* &\xrightarrow{\cong} [U \wedge X, V]_* \\ \text{and} \quad {}_\mu D : [U, X \wedge V]_* &\xrightarrow{\cong} [DX \wedge U, V]_* \end{aligned}$$

Hence, letting $D(F_{4k}/F_{4k-1})$ denote the S-dual of F_{4k}/F_{4k-1}, this means that the restriction of a 2-local morphism of the form $\phi_{k,l} : bu \wedge (F_{4k}/F_{4k-1}) \longrightarrow bu \wedge (F_{4l}/F_{4l-1})$ to $S^0 \wedge F_{4k}/F_{4k-1}$ is equivalent to a 2-local morphism $\phi'_{k,l} : S^0 \longrightarrow D(F_{4k}/F_{4k-1}) \wedge (F_{4l}/F_{4l-1}) \wedge bu$. A morphism of this form is a homotopy element

$$[\phi'_{k,l}] \in \pi_0(D(F_{4k}/F_{4k-1}) \wedge (F_{4l}/F_{4l-1}) \wedge bu) \otimes \mathbb{Z}_2.$$

This homotopy group is calculated using the (collapsed) Adams spectral sequence

$$\begin{aligned} \dot{E}_2^{s,t} &= Ext_B^{s,t}(H^*(D(F_{4k}/F_{4k-1};\mathbb{Z}/2) \otimes H^*(F_{4l}/F_{4l-1};\mathbb{Z}/2);\mathbb{Z}/2) \\ &\Longrightarrow \pi_{t-s}(D(F_{4k}/F_{4k-1}) \wedge (F_{4l}/F_{4l-1}) \wedge bu) \otimes \mathbb{Z}_2 \end{aligned} \tag{3.1}$$

Proposition 3.1.1. *The spectral sequence 3.1 has all differentials zero.*

Proof. A clear, concise proof that this spectral sequence collapses is given by Adams in [4] (Lemma 17.12). We follow his proof in the next section to show that a different spectral sequence, (3.2), collapses. □

3.2 The structure of B-modules

The following section uses techniques devised by Adams and Margolis ([6]) to study the structure of modules over exterior algebras. The notation used is from [4] Part 3 Chapter 16.

Definition 3.2.1. Two left B-modules M, N are said to be *stably isomorphic* if there exist free B-modules F, G such that $M \oplus F \cong N \oplus G$.

Proposition 3.2.2. *In this sense, stable isomorphism is an equivalence relation. Furthermore, for $s > 0$ the groups $Ext_B^{s,t}(M, \mathbb{Z}/2)$ depend only on the stable isomorphism class of M.*

Proof. See [4] Part 3 Chapter 16 □

Therefore, we may simplify the calculation of the E_2 term in the Adams spectral sequence (3.1)

$$\begin{aligned} \dot{E}_2^{s,t} &= Ext_B^{s,t}(H^*(D(F_{4k}/F_{4k-1};\mathbb{Z}/2)\otimes H^*(F_{4l}/F_{4l-1};\mathbb{Z}/2);\mathbb{Z}/2) \\ &\Longrightarrow \pi_{t-s}(D(F_{4k}/F_{4k-1})\wedge(F_{4l}/F_{4l-1})\wedge bu)\otimes\mathbb{Z}_2 \end{aligned}$$

by changing the B-module $H^*(D(F_{4k}/F_{4k-1}))\otimes H^*(F_{4l}/F_{4l-1})$ for some simpler stably isomorphic module. Since B is a Hopf algebra we can define the tensor product of M and N by giving $M\otimes N$ the diagonal action. The sum and product pass to stable isomorphism classes and the product has a unit given by the module 1 with $\mathbb{Z}/2$ in degree 0. A stable class P is invertible if there is a stable class Q such that $P\otimes Q\cong 1$.

Recall from [4] p.332 that Σ^a is the B-module given by $\mathbb{Z}/2$ in degree a and $\Sigma^{-a}=Hom(\Sigma^a,\mathbb{Z}/2)$. Σ^a and Σ^{-a} are inverse modules, i.e. $\Sigma^a\Sigma^{-a}\cong 1$. I is the augmentation ideal, $I=ker(\epsilon: B\longrightarrow\mathbb{Z}/2)$ with inverse module given by $I^{-b}=Hom(I^b,\mathbb{Z}/2)$, where I^b is the b-fold tensor product of I, for $b>0$.

In [4] (p.334 Theorem 16.3) Adams shows how to calculate the stable class of a B-module M in the form Σ^aI^b for unique $a,b\in\mathbb{Z}$. He then goes on to show (p.341) that the B-module given by

$$H^{-*}(D(F_{4k}/F_{4k-1});\mathbb{Z}/2)\cong H_*(F_{4k}/F_{4k-1};\mathbb{Z}/2)$$

is stably equivalent to $\Sigma^{2^{r-1}+1}I^{2^{r-1}-1}$ when $0<4k=2^r$. Therefore $H^*(D(F_{4k}/F_{4k-1});\mathbb{Z}/2)$ is stably equivalent to $\Sigma^{-(2^{r-1}+1)}I^{1-2^{r-1}}$ when $0<4k=2^r$. If k is not a power of two we may write $4k=2^{r_1}+2^{r_2}+\ldots+2^{r_t}$

with $2 \leq r_1 < r_2 < \ldots < r_t$. In this case

$$H_*(F_{4k}/F_{4k-1};\mathbb{Z}/2) \cong \otimes_{j=r_1}^{r_t} H_*(F_{2^j}/F_{2^j-1};\mathbb{Z}/2)$$

which is stably equivalent to $\Sigma^{2k+\alpha(k)}I^{2k-\alpha(k)}$, where $\alpha(k)$ equals the number of 1's in the dyadic expansion of k, as in Proposition A.0.4. Similarly, $H^*(D(F_{4k}/F_{4k-1});\mathbb{Z}/2)$ is stably equivalent to $\Sigma^{-2k-\alpha(k)}I^{\alpha(k)-2k}$.

Now, following the account of [24] p.1268, $Ext_B^{s,t}(\Sigma^a M,\mathbb{Z}/2) \cong Ext_B^{s,t-a}(M,\mathbb{Z}/2)$. The short exact sequence

$$0 \longrightarrow I \otimes M \longrightarrow B \otimes M \longrightarrow M \longrightarrow 0$$

induces a long exact sequence of the form

$$\cdots \longrightarrow Ext_B^{s,t}(B \otimes M,\mathbb{Z}/2) \longrightarrow Ext_B^{s,t}(I \otimes M,\mathbb{Z}/2)$$

$$\longrightarrow Ext_B^{s+1,t}(M,\mathbb{Z}/2) \longrightarrow Ext_B^{s+1,t}(B \otimes M,\mathbb{Z}/2) \longrightarrow \cdots$$

so that, for $s > 0$, there is an isomorphism

$$Ext_B^{s,t}(I \otimes M,\mathbb{Z}/2) \cong Ext_B^{s+1,t}(M,\mathbb{Z}/2).$$

Finally, for $s > 0$ we obtain an isomorphism of the form

$$E_2^{s,t} \cong Ext_B^{s,t}(\Sigma^{2l-2k+\alpha(l)-\alpha(k)}I^{2l-2k-\alpha(l)+\alpha(k)},\mathbb{Z}/2)$$

$$\cong Ext_B^{s+2l-2k-\alpha(l)+\alpha(k),t-2l+2k-\alpha(l)+\alpha(k)}(\mathbb{Z}/2,\mathbb{Z}/2).$$

3.3 Spectral sequence structure

A standard calculation shows there is also an algebra isomorphism of the form $Ext_B^{*,*}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2[a, b]$ where $a \in Ext_B^{1,1}$, $b \in Ext_B^{1,3}$. Clearly in the spectral sequence (3.1) the contributions to the groups $\pi_0(D(F_{4k}/F_{4k-1}) \wedge (F_{4l}/F_{4l-1}) \wedge bu) \otimes \mathbb{Z}_2$ come from the groups $\{\dot{E}_2^{s,s} | s \geq 0\}$. These groups correspond to $Ext_B^{S,T}(\mathbb{Z}/2, \mathbb{Z}/2)$ with $S = s + 2l - 2k - \alpha(l) + \alpha(k)$ and $T = s - 2l + 2k - \alpha(l) + \alpha(k)$, which implies that $T - S = 4(k - l)$.

Clearly for $k < l$, $\dot{E}_2^{s,s} = Ext_B^{S,T}(\mathbb{Z}/2, \mathbb{Z}/2) = 0$. Therefore $\pi_0(D(F_{4k}/F_{4k-1}) \wedge (F_{4l}/F_{4l-1}) \wedge bu) \otimes \mathbb{Z}_2 = 0$ for $k < l$.

For $k \geq l$, as shown in [24] p.1268, $Ext^{S,T}(\mathbb{Z}/2, \mathbb{Z}/2)$ is non-zero only for $s \geq 0$ if $k = l$ or for $s \geq 2(k - l) + 1$ if $k > l$. If this group is non-zero then it is cyclic of order two generated by $a^{(3S-T)/2} b^{(T-S)/2} = a^{s+4l-4k-\alpha(l)+\alpha(k)} b^{2(k-l)}$. In [4] Part 3 Lemma 17.11(i) Adams showed that multiplication by a in the spectral sequence, which maps the generator of $\dot{E}_2^{s,t}$ to $\dot{E}_2^{s+1,t+1}$ corresponds to left multiplication by 2 on $\pi_{t-s}(D(F_{4k}/F_{4k-1}) \wedge (F_{4l}/F_{4l-1}) \wedge bu) \otimes \mathbb{Z}_2$ and multiplication by b, which maps the generator of $\dot{E}_2^{s,t}$ to $\dot{E}_2^{s+1,t+3}$, corresponds to left multiplication by the generator u of $\pi_*(bu) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2[u]$.

Now again consider, for $k > l$, a non-trivial homotopy class of left bu-module morphisms

$$[\phi''_{k,l}] : bu \wedge (F_{4k}/F_{4k-1}) \longrightarrow bu \wedge (F_{4l}/F_{4l-1}).$$

In the spectral sequence these morphisms are represented by elements of $\dot{E}_2^{s,s} = \dot{E}_\infty^{s,s}$ for $s \geq 0$. In order that we can use the Adams-Margolis structure theory we shall restrict to such morphisms which also induce the zero homomorphism on mod 2 singular homology, as any morphism detected by the induced map on mod 2 homology is represented in $\dot{E}_\infty^{0,*}$. Any such morphism $\phi''_{k,l}$ is represented by the generator of a group $\dot{E}_\infty^{\epsilon+4(k-l)+\alpha(l)-\alpha(k),\epsilon+4(k-l)+\alpha(l)-\alpha(k)}$ for some $\epsilon \geq 0$. The generator of this group is a^ϵ times the generator of $\dot{E}_\infty^{4(k-l)+\alpha(l)-\alpha(k),4(k-l)+\alpha(l)-\alpha(k)}$. Since multiplication by a in the spectral sequence corresponds to multiplication by 2 on $\pi_0(D(F_{4k}/F_{4k-1})\wedge(F_{4l}/F_{4l-1})\wedge bu)\otimes\mathbb{Z}_2$ we find that

$$\phi''_{k,l} = \gamma 2^\epsilon \iota_{k,l}$$

for some 2-adic unit γ and $\epsilon \geq 0$, where

$$\iota_{k,l} : bu \wedge (F_{4k}/F_{4k-1}) \longrightarrow bu \wedge (F_{4l}/F_{4l-1})$$

is represented by the generator of $\dot{E}_\infty^{4(k-l)+\alpha(l)-\alpha(k),4(k-l)+\alpha(l)-\alpha(k)}$.

Similarly if $k = l$

$$\phi''_{k,l} = \gamma 2^\epsilon \iota_{k,k}$$

where $\iota_{k,k}$ denotes the identity map of $bu \wedge (F_{4k}/F_{4k-1})$.

3.4 An isomorphism of groups

Recall from §2.6 that $Aut_{left-bu-mod}(bu \wedge bo)$ the set of left bu-module automorphisms of $bu \wedge bo$, which are precisely the left bu-module homotopy equivalences of $bu \wedge bo$, has the structure of a group. Let $Aut^0_{left-bu-mod}(bu \wedge bo)$ denote the set of left bu-module automorphisms of $bu \wedge bo$ which induce the identity map on mod 2 singular homology.

Proposition 3.4.1. *$Aut^0_{left-bu-mod}(bu \wedge bo)$ is a subgroup of $Aut_{left-bu-mod}(bu \wedge bo)$.*

Proof. See [24]. □

Let $U_\infty \mathbb{Z}_2$ denote the group of infinite, invertible upper triangular matrices with entries in the 2-adic integers. That is, $X = (X_{i,j}) \in U_\infty \mathbb{Z}_2$ if $X_{i,j} \in \mathbb{Z}_2$ for each pair of integers $0 \leq i, j$ and $X_{i,j} = 0$ if $j > i$ and $X_{i,i}$ is a 2-adic unit. This upper triangular group is *not* equal to the direct limit $\lim_{\vec{n}} U_n \mathbb{Z}_2$ of the finite upper triangular groups. The main result of [24] is the existence of an isomorphism of groups

$$\Lambda : U_\infty \mathbb{Z}_2 \xrightarrow{\cong} \mathrm{Aut}^0_{left-bu-mod}(bu \wedge bo).$$

By the Mahowald decomposition of $bu \wedge bo$ the existence of Λ is equivalent to an isomorphism of the form

$$\Lambda : U_\infty \mathbb{Z}_2 \xrightarrow{\cong} \mathrm{Aut}^0_{left-bu-mod}(\vee_{k \geq 0} bu \wedge (F_{4k}/F_{4k-1})).$$

If we choose $\iota_{k,l}$ to satisfy $\iota_{k,l} = \iota_{l+1,l}\iota_{l+2,l+1} \ldots \iota_{k,k-1}$ for all $k - l \geq 2$ then, for $X \in U_\infty \mathbb{Z}_2$, we define ([24] §3.2)

$$\Lambda(X^{-1}) = \sum_{l \leq k} X_{l,k}\iota_{k,l} : bu \wedge (\vee_{k \geq 0} F_{4k}/F_{4k-1}) \longrightarrow bu \wedge (\vee_{k \geq 0} F_{4k}/F_{4k-1}).$$

The ambiguity in the definition of the $\iota_{k,l}$'s implies that Λ is defined up to inner automorphism in $U_\infty \mathbb{Z}_2$, i.e. conjugation by a matrix.

The obvious question to ask is, given an element of $Aut^0_{left-bu-mod}(bu \wedge bo)$ what is the corresponding conjugacy class of matrices in $U_\infty \mathbb{Z}_2$. As discussed in §2.8, by far the most important such element is $1 \wedge \psi^3$. The main result of this thesis is the following theorem:

Theorem 3.4.2. *Under the isomorphism Λ the automorphism $1 \wedge \psi^3 \in Aut^0_{left-bu-mod}(bu \wedge bo)$ corresponds to a matrix in the conjugacy class of*

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 9 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 9^2 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 9^3 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

3.5 The 2-adic homotopy of $bu \wedge bo$

The key observation in deducing the identity of the matrix corresponding to $1 \wedge \psi^3$ under the isomorphism Λ is the following proposition:

Proposition 3.5.1. *An automorphism in $Aut^0_{left-bu-mod}(bu \wedge bo)$ is determined by its effect on $\frac{\pi_*(bu \wedge bo) \otimes \mathbb{Z}_2}{Torsion}$.*

Proof. See [4] pg.355 et seq. □

We begin by calculating $\frac{\pi_*(bu \wedge bo) \otimes \mathbb{Z}_2}{Torsion}$ via the Mahowald decomposition of $bu \wedge bo$ of §3.1.

Let $G_{s,t}$ denote the 2-adic homotopy group modulo torsion

$$G_{s,t} = \frac{\pi_s(bu \wedge F_{4t}/F_{4t-1}) \otimes \mathbb{Z}_2}{\text{Torsion}}$$

so

$$G_{*,*} = \oplus_{s,t} \frac{\pi_s(bu \wedge F_{4t}/F_{4t-1}) \otimes \mathbb{Z}_2}{\text{Torsion}} \cong \frac{\pi_*(bu \wedge bo) \otimes \mathbb{Z}_2}{\text{Torsion}}.$$

This group is calculated by means of the Adams spectral sequence

$$(3.2) \qquad \begin{aligned} \ddot{E}_2^{s,t} &= Ext_B^{s,t}(H^*(F_{4k}/F_{4k-1};\mathbb{Z}/2);\mathbb{Z}/2) \\ &\Longrightarrow \pi_{t-s}(bu \wedge (F_{4k}/F_{4k-1})) \otimes \mathbb{Z}_2 \end{aligned}$$

From the discussion of §3.2, $H^*(F_{4k}/F_{4k-1};\mathbb{Z}/2)$ is stably isomorphic to $\Sigma^{2k+\alpha(k)}I^{2k-\alpha(k)}$, therefore

$$\begin{aligned} \ddot{E}_2^{s,t} &= Ext_B^{s,t}(\Sigma^{2k+\alpha(k)}I^{2k-\alpha(k)};\mathbb{Z}/2) \\ &\cong Ext_B^{S,T}(\mathbb{Z}/2;\mathbb{Z}/2) \end{aligned}$$

where $S = s + 2k - \alpha(k)$, $T = t - 2k - \alpha(k)$.

Since $Ext_B^{*,*}(\mathbb{Z}/2;\mathbb{Z}/2) \cong \mathbb{Z}/2[a,b]$ with $a \in Ext_B^{1,1}$ and $b \in Ext_B^{1,3}$ we must have $T - S = t - s - 4k \geq 0$ otherwise $\ddot{E}_2^{s,t} = 0$. Hence, the only non-zero $\ddot{E}_2^{s,t}$'s are in the region $t - s \geq 4k$, $s \geq 0$. Therefore we find that the $\ddot{E}_2$ page of the spectral sequence looks as follows:

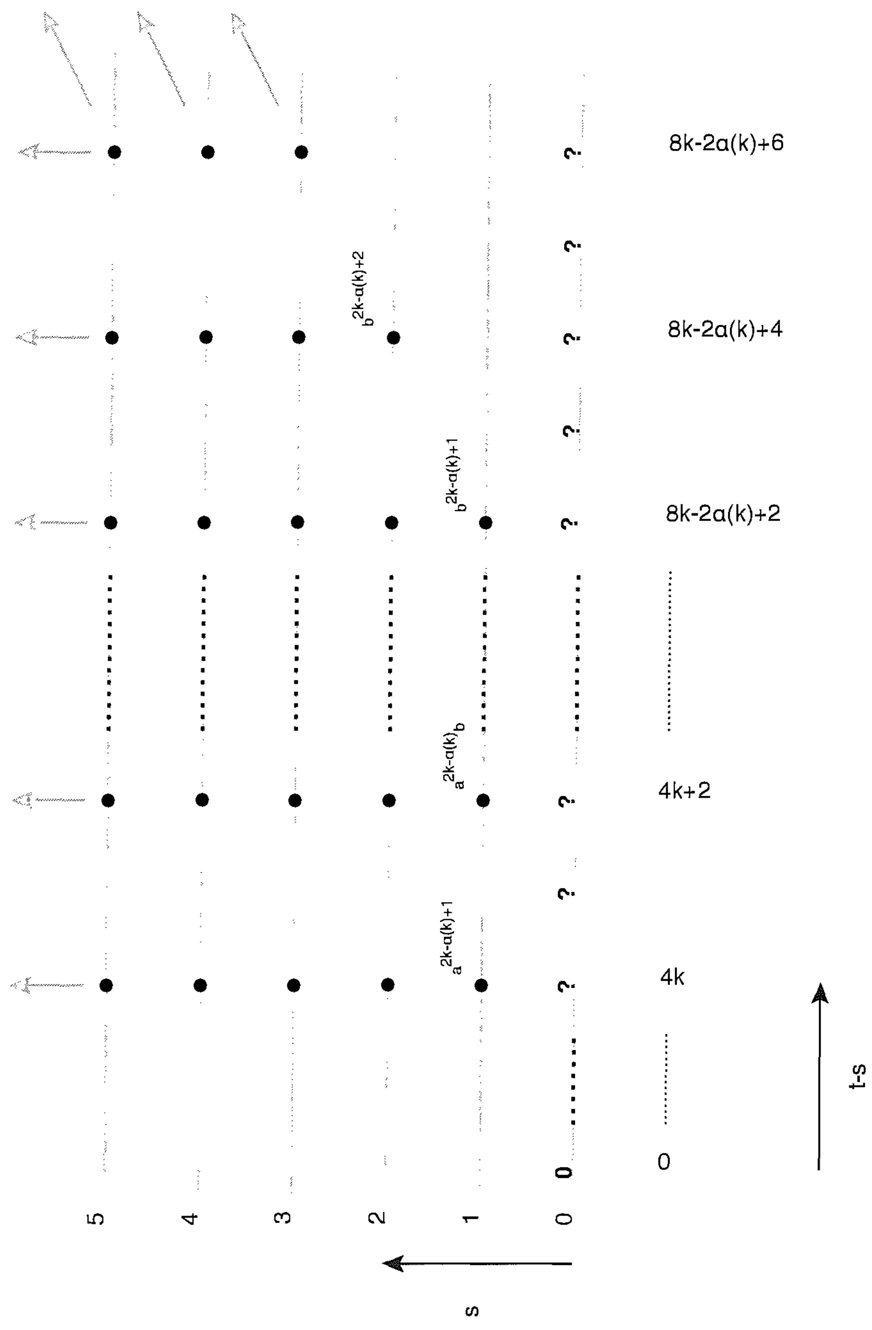

Figure 3.1: $\ddot{E}_2^{s,t} = Ext_B^{s,t}(H^*(F_{4k}/F_{4k-1}); \mathbb{Z}/2)$

Summarising, above the $s = 0$ line we have vertical towers of groups $\mathbb{Z}/2$ connected vertically by multiplication by a. The foot of each of these towers is given in the following table:

Generator	s	t	$t-s$
$a^{2k-\alpha(k)+1}$	1	$4k+1$	$4k$
$a^{2k-\alpha(k)}b$	1	$4k+3$	$4k+2$
$a^{2k-\alpha(k)-1}b^2$	1	$4k+5$	$4k+4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$ab^{2k-\alpha(k)}$	1	$8k-2\alpha(k)+1$	$8k-2\alpha(k)$
$b^{2k-\alpha(k)+1}$	1	$8k-2\alpha(k)+3$	$8k-2\alpha(k)+2$
$b^{2k-\alpha(k)+2}$	2	$8k-2\alpha(k)+6$	$8k-2\alpha(k)+4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$b^{2k-\alpha(k)+s}$	s	$8k-2\alpha(k)+3s$	$8k-2\alpha(k)+2s$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Recall that the differentials in the Adams spectral sequence are of the form $d_r : \ddot{E}_2^{s,t} \longrightarrow \ddot{E}_2^{s+r,t+r-1}$.

Proposition 3.5.2. *The spectral sequence 3.2 has all differentials zero*

Proof. The following proof is a specific case of the proof of [4] Lemma 17.12. Since $\ddot{E}_2^{s,t} = 0$ for $s > 0$ and $t - s$ odd, the same must hold for $\ddot{E}_r^{s,t}$, $r > 2$. Hence there can be no non-zero differentials with domain $\ddot{E}_r^{s,t}$ for $s > 0$. Therefore it is sufficient to check $d_r(e) = 0$, where $e \in \ddot{E}_r^{s,t}$ and $s = 0$, $t - s$ odd. We proceed by induction. First consider $r = 2$, any non-zero differential would be of the form $d_2 : \ddot{E}_2^{s,t} \longrightarrow \ddot{E}_2^{s+2,t+1}$ for $s = 0$, t odd, i.e. in the spectral sequence it would map the generator of a group on the $s = 0$ line with $t-s$ odd to a copy of $\mathbb{Z}/2$ two places up and one place to the left. Since multiplication by a in the spectral sequence is a monomorphism from $\ddot{E}_2^{s,t}$ to $\ddot{E}_2^{s+1,t+1}$ we must have that $ad_2(e) = d_2(ae) = 0 \implies d_2(e) = 0$. Now suppose that $d_m = 0$ for $m < r$ so that $\ddot{E}_r^{s,t} \cong \ddot{E}_2^{s,t}$. Again, any non-zero differential in $\ddot{E}_r^{*,*}$ would have to be of the form $d_r : \ddot{E}_r^{s,t} \longrightarrow \ddot{E}_r^{s+r,r+r-1}$ for $s = 0$, t odd, i.e in the spectral sequence it would map the generator of a group on the $s = 0$ line with $t - s$ odd to a copy of $\mathbb{Z}/2$ r places up and one place left. As before, since multiplication by a in the spectral sequence is a monomorphism $a : \ddot{E}_r^{s,t} \longrightarrow \ddot{E}_r^{s+1,t+1}$ we must have that $ad_r(e) = d_r(ae) = 0 \implies d_r(e) = 0$. This completes the induction. □

Proposition 3.5.3.

$$G_{s,t} \cong \begin{cases} \mathbb{Z}_2 & \text{if } s \text{ even, } s \geq 4t, \\ \\ 0 & \text{otherwise} \end{cases}$$

and if $\tilde{G}_{s,t}$ denotes $\pi_s(bu \wedge F_{4t}/F_{4t-1}) \otimes \mathbb{Z}_2$ then $\tilde{G}_{s,t} \cong G_{s,t} \oplus W_{s,t}$ where $W_{s,t}$ is a finite, elementary abelian 2-group.

Proof. Adams ([4] Lemma 17.1 (i)) showed that $H_*(bu \wedge F_{4k}/F_{4k-1})$ is a direct sum of a finite number of groups $\mathbb{Z}$, $\mathbb{Z}/2$ and $\mathbb{Z}/p$ for $p > 2$. He then went on to show ([4] Proposition 17.2 (i)) that the Hurewicz homomorphism

$$h : \pi_*(bu \wedge F_{4k}/F_{4k-1}) \longrightarrow H_*(bu \wedge F_{4k}/F_{4k-1})$$

is a monomorphism. Hence, it follows that $\pi_*(bu \wedge F_{4k}/F_{4k-1})$ is a direct sum of a finite number of groups $\mathbb{Z}$, $\mathbb{Z}/2$ and $\mathbb{Z}/p$ for $p > 2$. Therefore $\tilde{G}_{*,k}$ is a direct sum of a finite number of groups $\mathbb{Z}_2$ and $\mathbb{Z}/2$. Since

$$G_{*,*} = \oplus_{s,t} \frac{\pi_s(bu \wedge F_{4t}/F_{4t-1}) \otimes \mathbb{Z}_2}{\text{Torsion}},$$

$\tilde{G}_{s,t}$ must be a direct sum of a finite number of groups $\mathbb{Z}_2$ and $\mathbb{Z}/2$. Therefore, when we work with homotopy modulo torsion we find $G_{s,t}$ is at most a direct sum of a finite number of copies of $\mathbb{Z}_2$.

Recall from the construction of the Adams spectral sequence 3.2 there exists a filtration

$$\cdots \subset F^{s+1} \subset F^s \subset F^{s-1} \subset \cdots \subset F^1 \subset F^0 = \tilde{G}_{t-s,k}$$

such that

$$\ddot{E}_\infty^{s,t} \cong \frac{F^s\tilde{G}_{t-s,k}}{F^{s-1}\tilde{G}_{t-s,k}}.$$

By passing from the $\ddot{E}_\infty$ page of the spectral sequence to the filtration quotients, multiplying the generator of $\ddot{E}_\infty^{s,t}$ by a corresponds to multiplication by 2 on $\frac{F^s\tilde{G}_{t-s,k}}{F^{s-1}\tilde{G}_{t-s,k}}$. For $s > 0$ we only have non-zero entries in the spectral sequence for $t-s \geq 4k$ and $t-s$ even. In this case we have an infinite tower of $\mathbb{Z}/2's$ connected by multiplication by a, which corresponds to multiplication by 2 on homotopy elements. The foot of each tower is on or above $s = 1$. Any homotopy element represented at the foot of such a tower is of infinite order since, for all $j > 0$, 2^j times this element is non-zero. This element therefore either represents the generator of a copy of $\mathbb{Z}_2$, as a module over $\mathbb{Z}_2$, or if the foot of the tower is on $s = 1$ it may represent 2 times the generator of a copy of $\mathbb{Z}_2$ which is represented on $s = 0$. $\tilde{G}_{t-s,k}$ can in fact contain at most one copy of $\mathbb{Z}_2$ as a direct summand. This is due to the fact that at no point in the tower of $\mathbb{Z}/2$'s do we have a direct sum of more than one copy of $\mathbb{Z}/2$. Similarly, any 2-torsion in $\tilde{G}_{t-s,k}$ must be represented on the $s = 0$ line. Hence, modulo torsion, $G_{s,t} \cong \mathbb{Z}_2$ for $t - s \geq 4k$ and $t - s$ even. To see that $G_{s,t} = 0$ otherwise the same arguments show that $\tilde{G}_{s,t}$ can be at most a direct sum of a finite number of copies of $\mathbb{Z}/2$ being represented on $s = 0$. □

3.6 A basis for $\frac{\pi_*(bu\wedge bo)\otimes\mathbb{Z}_2}{\text{Torsion}}$

In [12] a $\mathbb{Z}_2$-basis is given for $G_{*,*}$ consisting of elements lying in the subring $\mathbb{Z}_2[u/2, v^2/4]$ of $\mathbb{Q}_2[u/2, v^2/4]$. One starts with the elements

$$c_{4k} = \Pi_{i=1}^{k}\left(\frac{v^2 - 9^{i-1}u^2}{9^k - 9^{i-1}}\right), \qquad k = 1, 2, \ldots$$

and "rationalises" them, after the manner of ([4] p.358), to obtain elements of $\mathbb{Z}_2[u/2, v^2/4]$. In order to describe this basis we shall require a few well-known preparatory results about 2-adic valuations. These results are Proposition A.0.4 and Lemmas A.0.1 to A.0.3 in Appendix A.

Now consider the elements $c_{4k} = \Pi_{i=1}^{k}\left(\frac{v^2-9^{i-1}u^2}{9^k-9^{i-1}}\right)$ for a particular $k = 1, 2, \ldots$. For completeness write $c_0 = 1$ so that $c_{4k} \in \mathbb{Q}_2[u/2, v^2/4]$. Since the degree of the numerator of c_{4k} is $2k$, Proposition A.0.4 implies that

$$f_{4k} = 2^{4k-\alpha(k)-2k}c_{4k} = 2^{2k-\alpha(k)}\Pi_{i=1}^{k}\left(\frac{v^2 - 9^{i-1}u^2}{9^k - 9^{i-1}}\right)$$

lies in $\mathbb{Z}_2[u/2, v^2/4]$ but $2^{4k-\alpha(k)-2k-1}c_{4k} \notin \mathbb{Z}_2[u/2, v^2/4]$. Similarly $(u/2)f_{4k} =$

$2^{4k-\alpha(k)-2k-1} u c_{4k} \in \mathbb{Z}_2[u/2, v^2/4]$ but $2^{4k-\alpha(k)-2k-2} u c_{4k} \notin \mathbb{Z}_2[u/2, v^2/4]$ and so on. This process is the "rationalisation yoga" referred to in §3.5. One forms $u^j c_{4k}$ and then multiplies by the smallest positive power of 2 to obtain an element of $\mathbb{Z}_2[u/2, v^2/4]$.

By Proposition A.0.4, starting with $f_{4l} = 2^{4l-\alpha(l)-2l} c_{4l}$ this process produces the following set of elements of $\mathbb{Z}_2[u/2, v^2/4]$

$$f_{4l},\ (u/2)f_{4l},\ (u/2)^2 f_{4l},\ \ldots\ ,\ (u/2)^{2l-\alpha(l)} f_{4l},$$

$$u(u/2)^{2l-\alpha(l)} f_{4l},\ u^2(u/2)^{2l-\alpha(l)} f_{4l},\ u^3(u/2)^{2l-\alpha(l)} f_{4l},\ \ldots .$$

As explained in ([4] p.352 et seq), the Hurewicz homorphism defines an injection of graded groups of the form

$$\frac{\pi_*(bu \wedge bo) \otimes \mathbb{Z}_2}{\text{Torsion}} \longrightarrow \mathbb{Q}_2[u/2, v^2/4]$$

which, by the main theorem of [12], induces an isomorphism between $\frac{\pi_*(bu \wedge bo) \otimes \mathbb{Z}_2}{\text{Torsion}}$ and the free graded $\mathbb{Z}_2$-module whose basis consists of the elements of $\mathbb{Z}_2[u/2, v^2/4]$ listed above for $l = 0, 1, 2, 3, \ldots$.

From this list we shall be particularly interested in the elements whose degree is a multiple of 4. Therefore denote by $g_{4m,4l} \in \mathbb{Z}_2[u/2, v^2/4]$ for $l \leq m$ the element produced from f_{4l} in degree $4m$. Hence, for $m \geq l$, $g_{4m,4l}$ is given by the formula

$$g_{4m,4l} = \begin{cases} u^{2m-4l+\alpha(l)} \left[\frac{u^{2l-\alpha(l)} f_{4l}}{2^{2l-\alpha(l)}}\right] & \text{if } 4l - \alpha(l) \leq 2m, \\ \\ \left[\frac{u^{2(m-l)} f_{4l}}{2^{2(m-l)}}\right] & \text{if } 4l - \alpha(l) > 2m. \end{cases}$$

Lemma 3.6.1. *Let P_k denote the projection*

$$P_k : \frac{\pi_*(bu \wedge bo) \otimes \mathbb{Z}_2}{\text{Torsion}} \cong G_{*,*} \longrightarrow G_{*,k} = \oplus_m \, G_{m,k}.$$

Then $P_k(g_{4k,4i}) = 0$ for all $i < k$.

Proof. Since $G_{m,k}$ is torsion free it suffices to show that $P_k(g_{4k,4i})$ vanishes in $G_{*,k} \otimes \mathbb{Q}_2$. When $i < k$, by definition

$$g_{4k,4i} \in u^{2k-2i} \frac{\pi_{4i}(bu \wedge bo) \otimes \mathbb{Z}_2}{\text{Torsion}} \otimes \mathbb{Q}_2 \subset \frac{\pi_{4k}(bu \wedge bo) \otimes \mathbb{Z}_2}{\text{Torsion}} \otimes \mathbb{Q}_2.$$

However P_k projects onto $\oplus_s \frac{\pi_s(bu \wedge F_{4k}/F_{4k-1}) \otimes \mathbb{Z}_2}{\text{Torsion}}$ and commutes with multiplication by u so the result follows from the fact that the homotopy of $bu \wedge F_{4k}/F_{4k-1}$ is trivial in degrees less than $4k$ (see [24] §3). □

3.7 A basis for $\frac{\pi_{4k}(bu \wedge F_{4k}/F_{4k-1}) \otimes \mathbb{Z}_2}{Torsion}$

Recall from §3.5 that $G_{4k,k} \cong \mathbb{Z}_2$ for $k = 0, 1, 2, 3, \ldots$ so we may choose a generator z_{4k} for this group as a module over the 2-adic integers (with the convention that $z_0 = f_0 = 1$). Let $\tilde{z}_{4k}$ be any choice of an element in the 2-adic homotopy group $\tilde{G}_{4k,k} \cong G_{4k,k} \oplus W_{4k,k}$ whose first coordinate is z_{4k}.

Lemma 3.7.1. *Let B denote the exterior subalgebra of the $\mathbb{Z}/2$ Steenrod algebra generated by Sq^1 and $Sq^{0,1}$. In the collapsed Adams spectral sequence (see [4] or [24])*

$$\ddot{E}_2^{s,t} \cong Ext_B^{s,t}(H^*(F_{4k}/F_{4k-1};\mathbb{Z}/2),\mathbb{Z}/2)$$

$$\Longrightarrow \pi_{t-s}(bu \wedge (F_{4k}/F_{4k-1})) \otimes \mathbb{Z}_2$$

the homotopy class $\tilde{z}_{4k}$ is represented either in $\ddot{E}_2^{0,4k}$ or $\ddot{E}_2^{1,4k+1}$.

Proof. Recall from §3.5 that $\pi_{4k}(bu\wedge(F_{4k}/F_{4k-1}))\otimes\mathbb{Z}_2 = \tilde{G}_{4k,k} \cong \mathbb{Z}_2 \oplus W_{4k,k}$. The following behaviour of the filtration coming from the spectral sequence is well-known, being explained in [4]. The group $\tilde{G}_{4k,k}$ has a filtration

$$\ldots \subset F^i \subset \ldots F^2 \subset F^1 \subseteq F^0 = \tilde{G}_{4k,k}$$

with $F^i/F^{i+1} \cong E_2^{i,4k+i}$ and $2F^i \subseteq F^{i+1}$. Also $2 \cdot W_{4k,k} = 0$, every non-trivial element of $W_{4k,k}$ being represented in $\ddot{E}_2^{0,4k}$. Furthermore for $i = 1,2,3,\ldots$ we have $2F^i = F^{i+1}$ and $F^1 \cong \mathbb{Z}_2$.

Now suppose that $\tilde{z}_{4k}$ is represented in $\ddot{E}_2^{j,4k+j}$ for $j \geq 2$ then $\tilde{z}_{4k} \in F^j$. From the multiplicative structure of the spectral sequence there exists a generator $\hat{z}_{4k}$ of F^1 such that $2^j\hat{z}_{4k}$ generates F^{j+1} and therefore $2^j\gamma\hat{z}_{4k} = 2\tilde{z}_{4k}$ for some 2-adic integer γ. Hence $2(2^{j-1}\gamma\hat{z}_{4k} - \tilde{z}_{4k}) = 0$ and so $2^{j-1}\gamma\hat{z}_{4k} - \tilde{z}_{4k} \in W_{4k,k}$ which implies the contradiction that the generator z_{4k} is divisible by 2 in $G_{4k,k}$. □

3.8 Relating the bases

Theorem 3.8.1. *In the notation of §3.5 and §3.7*

$$z_{4k} = \Sigma_{i=0}^{k} 2^{\beta(k,i)} \lambda_{4k,4i} g_{4k,4i} \in \frac{\pi_{4k}(bu \wedge bo) \otimes \mathbb{Z}_2}{\text{Torsion}}$$

with $\lambda_{s,t} \in \mathbb{Z}_2$, $\lambda_{4k,4k} \in \mathbb{Z}_2^*$ *and*

$$\beta(k,i) = \begin{cases} 4(k-i) - \alpha(k) + \alpha(i) & \text{if } 4i - \alpha(i) > 2k, \\ \\ 2k - \alpha(k) & \text{if } 4i - \alpha(i) \leq 2k. \end{cases}$$

Proof. From [12], as explained in §3.5, a $\mathbb{Z}_2$-module basis for $G_{4k,*}$ is given by $\{g_{4k,4l}\}_{0 \leq l \leq k}$. Hence there is a relation of the form

$$z_{4k} = \lambda_{4k,4k} g_{4k,4k} + \tilde{\lambda}_{4k,4(k-1)} g_{4k,4(k-1)} + \ldots + \tilde{\lambda}_{4k,0} g_{4k,0}$$

where $\tilde{\lambda}_{4k,4i}$ and $\lambda_{4k,4k}$ are 2-adic integers. Applying the projection $P_k : G_{4k,*} \longrightarrow G_{4k,k}$ we see that $z_{4k} = P_k(z_{4k}) = \lambda_{4k,4k} P_k(g_{4k,4k})$, by Lemma 3.6.1. Hence, if $\lambda_{4k,4k}$ is not a 2-adic unit, then z_{4k} would be divisible by 2 in $G_{4k,k}$ and this is impossible since z_{4k} is a generator, by definition.

Multiplying the relation

$$z_{4k} = \lambda_{4k,4k} P_k(g_{4k,4k}) = \lambda_{4k,4k} P_k(f_{4k}) \in G_{4k,k}.$$

by $(u/2)^{2k-\alpha(k)}$ we obtain $(u/2)^{2k-\alpha(k)} z_{4k} = \lambda_{4k,4k} P_k((u/2)^{2k-\alpha(k)} f_{4k})$, which lies in $G_{8k-2\alpha(k),k}$, by the discussion of §3.5. Therefore, in $G_{8k-2\alpha(k),k} \otimes \mathbb{Q}_2$

we have the relation

$$(u/2)^{2k-\alpha(k)} z_{4k} = \lambda_{4k,4k}(u/2)^{2k-\alpha(k)} f_{4k} + \sum_{i=0}^{k-1} \tilde{\lambda}_{4k,4i}(u/2)^{2k-\alpha(k)} g_{4k,4i}.$$

Since the left hand side of the equation lies in $G_{8k-2\alpha(k),k}$, the $\mathbb{Q}_2$ coefficients must all be 2-adic integers once we re-write the right hand side in terms of the basis of §3.6.

For $i = 0, 1, \ldots, k-1$

$$(u/2)^{2k-\alpha(k)} g_{4k,4i} = \begin{cases} \frac{u^{2k-\alpha(k)+2k-4i+\alpha(i)+2i-\alpha(i)}}{2^{2k-\alpha(k)-2i-\alpha(i)}} f_{4i} & \text{if } 4i-\alpha(i) \leq 2k, \\ \\ \frac{u^{2k-\alpha(k)+2k-2i}}{2^{2k-\alpha(k)+2k-2i}} f_{4i} & \text{if } 4i-\alpha(i) > 2k \end{cases}$$

$$= \begin{cases} \frac{u^{4k-2i-\alpha(k)}}{2^{2k+2i-\alpha(k)-\alpha(i)}} f_{4i} & \text{if } 4i-\alpha(i) \leq 2k, \\ \\ \frac{u^{4k-2i-\alpha(k)}}{2^{4k-2i-\alpha(k)}} f_{4i} & \text{if } 4i-\alpha(i) > 2k. \end{cases}$$

Now we shall write $(u/2)^{2k-\alpha(k)} g_{4k,4i}$ as a power of 2 times a generator derived from f_{4i} in §3.6 (since we did not define any generators called $g_{4k+2,4i}$ the generator in question will be $g_{8k-2\alpha(k),4i}$ only when $\alpha(k)$ is even).

Assume that $4i - \alpha(i) \leq 2k$ so that $2i - \alpha(i) \leq 4k - 2i - \alpha(k)$ and

$$\frac{u^{4k-2i-\alpha(k)}}{2^{2k+2i-\alpha(k)-\alpha(i)}} f_{4i} = \frac{1}{2^{2k-\alpha(k)}} u^{4k-4i-\alpha(k)+\alpha(i)} (u/2)^{2i-\alpha(i)} f_{4i}$$

which implies that $\tilde{\lambda}_{4k,4i}$ is divisible by $2^{2k-\alpha(k)}$ in the 2-adic integers, as required.

Finally assume that $4i-\alpha(i) > 2k$. We have $2i-\alpha(i) \leq 4k-2i-\alpha(k)$ also. To see this observe that $\alpha(i)+\alpha(k-i)-\alpha(k) \geq 0$ because, by Proposition A.0.4, this equals the 2-adic valuation of the binomial coefficient $\binom{k}{i}$. Therefore

$$\alpha(k) - \alpha(i) \leq \alpha(k-i) \leq k - i < 4(k-i).$$

Then, as before,

$$\frac{u^{4k-2i-\alpha(k)}}{2^{4k-2i-\alpha(k)}} f_{4i} = \frac{1}{2^{4k-4i-\alpha(k)+\alpha(i)}} u^{4k-4i-\alpha(k)+\alpha(i)} (u/2)^{2i-\alpha(i)} f_{4i}$$

which implies that $\tilde{\lambda}_{4k,4i}$ is divisible by $2^{4k-4i-\alpha(k)+\alpha(i)}$ in the 2-adic integers, as required. □

Theorem 3.8.2. *(i) In the collapsed Adams spectral sequence and the notation of Lemma 3.7.1 $\tilde{z}_{4k}$ may be chosen to be represented in $\ddot{E}_2^{0,4k}$.*

(ii) In fact, $\tilde{z}_{4k}$ may be taken to be the smash product of the unit η of the bu-spectrum with the inclusion of the bottom cell j_k into F_{4k}/F_{4k-1}

$$S^0 \wedge S^{4k} \xrightarrow{\eta \wedge j_k} bu \wedge F_{4k}/F_{4k-1}.$$

Proof. For part (i), suppose that $\tilde{z}_{4k}$ is represented in $\ddot{E}_2^{1,4k+1}$. By Lemma 3.7.1 we must show that this leads to a contradiction. From [24] we know that on the $s = 1$ line the non-trivial groups are precisely $\ddot{E}_2^{1,4k+1}$, $\ddot{E}_2^{1,4k+3}, \ldots$, $\ddot{E}_2^{1,8k+3-2\alpha(k)}$ which are all of order two. From the multiplicative structure of the spectral sequence, if a homotopy class w is represented $\ddot{E}_2^{j,4k+2j-1}$ and $\ddot{E}_2^{j,4k+2j+1}$ is non-zero then there is a homotopy class w' represented in $\ddot{E}_2^{j,4k+2j+1}$ such that $2w' = uw$. Applied to $\tilde{z}_{4k}$ this implies that the homotopy element $u^{2k-\alpha(k)+1}\tilde{z}_{4k}$ is divisible by $2^{2k-\alpha(k)+1}$. Hence $u^{2k-\alpha(k)+1}z_{4k}$ is divisible by $2^{2k-\alpha(k)+1}$ in $G_{*,*}$, which contradicts the proof of Theorem 3.8.1.

For part (ii) consider the Adams spectral sequence

$$\ddot{E}_2^{s,t} = Ext_B^{s,t}(H^*(F_{4k}/F_{4k-1};\mathbb{Z}/2),\mathbb{Z}/2) \Longrightarrow \pi_{t-s}(bu \wedge F_{4k}/F_{4k-1}) \otimes \mathbb{Z}_2.$$

We have an isomorphism

$$\ddot{E}_2^{0,t} = Hom(\frac{H^t(F_{4k}/F_{4k-1};\mathbb{Z}/2)}{Sq^1H^{t-1}(F_{4k}/F_{4k-1};\mathbb{Z}/2) + Sq^{0,1}H^{t-3}(F_{4k}/F_{4k-1};\mathbb{Z}/2)},\mathbb{Z}/2).$$

The discussion of the homology groups $H_*(F_{4k}/F_{4k-1};\mathbb{Z}/2)$ given in ([4] p.341; see also §3.9) shows that $\ddot{E}_2^{0,4k} \cong \mathbb{Z}/2$ generated by the Hurewicz image of $\eta \wedge j_k$. Therefore the generator of $\ddot{E}_2^{0,4k}$ represents $\eta \wedge j_k$. Since there is only one non-zero element in $\ddot{E}_2^{0,4k}$ it must also represent $\tilde{z}_{4k}$, by part (i), which completes the proof. □

3.9 The effect of ψ^3 on basis elements

In this section we wish to calculate the effect of the induced map $(1 \wedge \psi^3)_* : G_{*,*} \longrightarrow G_{*,*}$ on the basis elements $g_{4k,4l}$ introduced in §3.6 and the effect of the induced maps $(\iota_{k,l})_* : G_{4k,k} \longrightarrow G_{4k,l}$ on the basis elements z_{4k} introduced in §3.7. In order to do this we first need to recall the multiplicative pairing of Adams spectral sequences due to [3] and [20]. Let $\ddot{E}_r^{s,t}(l)$ denote the r-th page of the spectral sequence (3.2) with $k = l$, i.e. the spectral sequence

$$\ddot{E}_2^{s,t}(l) = Ext_B^{s,t}(bu \wedge F_{4l}/F_{4l-1}) \implies \pi_{t-s}(bu \wedge F_{4l}/F_{4l-1}) \otimes \mathbb{Z}_2.$$

Theorem 3.9.1. *There exists an associative pairing between the spectral sequences (3.1) and (3.2) of the form*

$$\dot{E}_r^{s,t} \otimes \ddot{E}_r^{s',t'}(k) \longrightarrow \ddot{E}_r^{s+s',t+t'}(l)$$

such that

(i) for r=2 the pairing agrees with the Yoneda composition pairing on Ext's

(ii) the pairings commute with the isomorphisms $\dot{E}_{r+1} \cong H(\dot{E}_r, \dot{d}_r)$ and $\ddot{E}_{r+1} \cong H(\ddot{E}_r, \ddot{d}_r)$

(iii) the pairings converge to a composition pairing of homotopy groups of the form

$$(\pi_*(D(F_{4k}/F_{4k-1}) \wedge (F_{4l}/F_{4l-1}) \wedge bu) \otimes \mathbb{Z}_2) \otimes \tilde{G}_{*,k} \longrightarrow \tilde{G}_{*,l},$$

In particular, the pairing of $\iota_{k,l} \in \pi_*(D(F_{4k}/F_{4k-1}) \wedge (F_{4l}/F_{4l-1}) \wedge bu) \otimes \mathbb{Z}_2$, represented by the generator of $\dot{E}_2^{4(k-l)-\alpha(k)+\alpha(l),4(k-l)-\alpha(k)+\alpha(l)}$, and $\tilde{z}_{4k} \in G_{*,k}$, represented by the generator of $\ddot{E}_2^{0,4k}(k)$, gives the image of the induced homomorphism $\iota_{k,l*}(\tilde{z}_{4k}) \in G_{*,l}$, represented by the generator of $\ddot{E}_2^{4(k-l)-\alpha(k)+\alpha(l),8k-4l-\alpha(k)+\alpha(l)}(l)$.

Proof. This is a particular case of the more general Theorem 9.27 of [18]. □

Proposition 3.9.2.

For $l < k$, in the notation of §3.3, the homomorphism

$$(\iota_{k,l})_* : G_{4k,k} \longrightarrow G_{4k,l}$$

satisfies $(\iota_{k,l})_*(z_{4k}) = \mu_{4k,4l} 2^{2k-2l-\alpha(k)+\alpha(l)} u^{2k-2l} z_{4l}$ *for some 2-adic unit* $\mu_{4k,4l}$.

Proof. Let $\tilde{z}_{4k} \in \tilde{G}_{4k,k}$ be as in §3.7 so that, proved in a similar manner to Lemma 3.7.1, $2\tilde{z}_{4k}$ is represented in $\ddot{E}_2^{1,4k+1}$ in the spectral sequence

$$\ddot{E}_2^{s,t} = Ext_B^{s,t}(H^*(F_{4k}/F_{4k-1}; \mathbb{Z}/2), \mathbb{Z}/2)$$

$$\Longrightarrow \pi_{t-s}(bu \wedge (F_{4k}/F_{4k-1})) \otimes \mathbb{Z}_2.$$

where, from §3.9, we have

$$\ddot{E}_2^{1,4k+1} \cong Ext_B^{1+2k-\alpha(k),4k+1-2k-\alpha(k)}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2 = \langle a^{2k+1-\alpha(k)} \rangle.$$

The multiplicative pairing between these spectral sequences shows that $(\iota_{k,l})_*(2\tilde{z}_{4k}) \in \tilde{G}_{4k,l}$ is represented in the spectral sequence

$$\ddot{E}_2^{s,t} = Ext_B^{s,t}(H^*(F_{4l}/F_{4l-1}; \mathbb{Z}/2), \mathbb{Z}/2)$$

$$\Longrightarrow \pi_{t-s}(bu \wedge (F_{4l}/F_{4l-1})) \otimes \mathbb{Z}_2.$$

by the generator of $\ddot{E}_2^{1+4k-4l-\alpha(k)+\alpha(l),1+8k-4l-\alpha(k)+\alpha(l)}$ because $a^{2k+1-\alpha(k)} b^{2k-2l}$

is the generator of

$$\ddot{E}_2^{1+4k-4l-\alpha(k)+\alpha(l),1+8k-4l-\alpha(k)+\alpha(l)} \cong Ext_B^{1+4k-2l-\alpha(k),1+8k-6l-\alpha(k)}(\mathbb{Z}/2,\mathbb{Z}/2).$$

Since multiplication by a and b in the spectral sequence corresponds to multiplication by 2 and u respectively on homotopy groups we have the following table of representatives in $\pi_*(bu \wedge (F_{4l}/F_{4l-1})) \otimes \mathbb{Z}_2$.

homotopy element	representative	dimension
$2z_{4l}$	$a^{2l-\alpha(l)+1}$	$4l$
$(u/2)(2z_{4l})$	$a^{2l-\alpha(l)}b$	$4l+2$
$(u/2)^2(2z_{4l})$	$a^{2l-\alpha(l)-1}b^2$	$4l+4$
$\vdots$	$\vdots$	$\vdots$
$(u/2)^{2l-\alpha(l)}(2z_{4l})$	$ab^{2l-\alpha(l)}$	$8l-2\alpha(l)$
$u(u/2)^{2l-\alpha(l)}(2z_{4l})$	$b^{2l-\alpha(l)+1}$	$8l-2\alpha(l)+2$
$u^2(u/2)^{2l-\alpha(l)}(2z_{4l})$	$b^{2l-\alpha(l)+2}$	$8l-2\alpha(l)+4$
$\vdots$	$\vdots$	$\vdots$

Therefore there are two cases for $(\iota_{k,l})_*(2\tilde{z}_{4k})$. If $2k-2l \geq 2l-\alpha(l)+1$ then b^{2k-2l} represents $u^{2k-2l-(2l-\alpha(l))}(u/2)^{2l-\alpha(l)}\tilde{z}_{4l} = u^{2k-4l+\alpha(l)}(u/2)^{2l-\alpha(l)}\tilde{z}_{4l}$ and, up to multiplication by 2-adic units, $(\iota_{k,l})_*(2\tilde{z}_{4k})$ is equal to $2^{1+2k-\alpha(k)}u^{2k-4l+\alpha(l)}(u/2)^{2l-\alpha(l)}\tilde{z}_{4l}$, as required. On the other hand, if $2k-2l \leq 2l-\alpha(l)$ then $a^{2l-\alpha(l)+1-(2k-2l)}b^{2k-2l} = a^{4l-2k-\alpha(l)+1}b^{2k-2l}$ represents $(u/2)^{2k-2l}(2\tilde{z}_{4l})$ which shows that, up to 2-adic units, $(\iota_{k,l})_*(2\tilde{z}_{4k})$ is equal to $2^{1+2k-\alpha(k)-(4l-2k-\alpha(l)+1)}(u/2)^{2k-2l}(2\tilde{z}_{4l}) = 2^{4k-\alpha(k)-4l+\alpha(l)}(u/2)^{2k-2l}(2\tilde{z}_{4l})$, as required. □

Proposition 3.9.3.

Let $\psi^3 : bo \longrightarrow bo$ denote the Adams operation, as usual. Then, in the notation of §3.5,

$$(1 \wedge \psi^3)_*(g_{4k,4k}) = \begin{cases} 9^k g_{4k,4k} + 9^{k-1}2^{\nu_2(k)+3} g_{4k,4k-4} & \text{if } k \geq 3, \\ 9^2 g_{8,8} + 9 \cdot 2^3 g_{8,4} & \text{if } k = 2, \\ 9 g_{4,4} + 2 g_{4,0} & \text{if } k = 1, \\ g_{0,0} & \text{if } k = 0. \end{cases}$$

Proof. The effect of the homomorphism $(1 \wedge \psi^3)_*$ on u and v is well known, being given in [8] Chapter 7, for example. $(1 \wedge \psi^3)_*$ multiplies v by 9, fixes u and is multiplicative. Therefore

$$\begin{aligned}
(1 \wedge \psi^3)_*(c_{4k}) &= \Pi_{i=1}^{k}\left(\tfrac{9v^2 - 9^{i-1}u^2}{9^k - 9^{i-1}}\right) \\
&= 9^{k-1}\left(\tfrac{(9v^2 - 9^k u^2 + 9^k u^2 - u^2)\Pi_{i=2}^{k}(v^2 - 9^{i-2}u^2)}{\Pi_{i=1}^{k}(9^k - 9^{i-1})}\right) \\
&= 9^{k-1}\left(\tfrac{(9v^2 - 9^k u^2)\Pi_{i=2}^{k}(v^2 - 9^{i-2}u^2)}{\Pi_{i=1}^{k}(9^k - 9^{i-1})}\right) + 9^{k-1}\left(\tfrac{(9^k u^2 - u^2)\Pi_{i=2}^{k}(v^2 - 9^{i-2}u^2)}{\Pi_{i=1}^{k}(9^k - 9^{i-1})}\right) \\
&= 9^{k}\left(\tfrac{(v^2 - 9^{k-1} u^2)\Pi_{i=2}^{k}(v^2 - 9^{i-2}u^2)}{\Pi_{i=1}^{k}(9^k - 9^{i-1})}\right) + 9^{k-1}\left(\tfrac{(9^k u^2 - u^2)\Pi_{i=2}^{k}(v^2 - 9^{i-2}u^2)}{\Pi_{i=1}^{k}(9^k - 9^{i-1})}\right) \\
&= 9^k c_{4k} + 9^{k-1}(9^k - 1)\left(\tfrac{u^2 \Pi_{i=1}^{k-1}(v^2 - 9^{i-1}u^2)}{(9^k - 1)\Pi_{i=1}^{k-1}(9^k - 9^{i-1})}\right)
\end{aligned}$$

$$= 9^k c_{4k} + 9^{k-1}u^2 c_{4k-4}.$$

Hence, for $k \geq 1$, we have

$$\begin{aligned}
(1 \wedge \psi^3)_*(f_{4k}) &= 2^{2k-\alpha(k)}(1 \wedge \psi^3)_*(c_{4k}) \\
&= 2^{2k-\alpha(k)} 9^k c_{4k} + 9^{k-1}u^2 2^{2k-\alpha(k)-2k+2+\alpha(k-1)} c_{4k-4} \\
&= 9^k f_{4k} + 9^{k-1}u^2 2^{2-\alpha(k)+\alpha(k-1)} f_{4k-4} \\
&= 9^k f_{4k} + 9^{k-1}u^2 2^{\nu_2(k)+1} f_{4k-4},
\end{aligned}$$

which yields the result, by the formulae of §3.6. □

Proposition 3.9.4.

When $k > l$

$$(1 \wedge \psi^3)_*(g_{4k,4l}) = \begin{cases} 9^l g_{4k,4l} + 9^{l-1} g_{4k,4l-4} & \text{if } 4l - \alpha(l) \leq 2k, \\ 9^l g_{4k,4l} + 9^{l-1} 2^{4l-\alpha(l)-2k} g_{4k,4l-4} & \text{if } 4l - \alpha(l) - \nu_2(l) - 3 \\ & \quad \leq 2k < 4l - \alpha(l), \\ 9^l g_{4k,4l} + 9^{l-1} 2^{3+\nu_2(k)} g_{4k,4l-4} & \text{if, } 2k < 4l - \alpha(l) - \nu_2(l) - 3 \\ & \quad < 4l - \alpha(l) \end{cases}$$

Proof. Suppose that $4l - \alpha(l) \leq 2k$ then, by Proposition 3.9.3 (proof),

$$(1 \wedge \psi^3)_*(g_{4k,4l})$$

$$= (1 \wedge \psi^3)_*(u^{2k-4l+\alpha(l)}\left[\tfrac{u^{2l-\alpha(l)}f_{4l}}{2^{2l-\alpha(l)}}\right])$$

$$= u^{2k-4l+\alpha(l)}\left[\tfrac{u^{2l-\alpha(l)}(9^l f_{4l}+9^{l-1}u^2 2^{\nu_2(l)+1}f_{4l-4})}{2^{2l-\alpha(l)}}\right]$$

$$= 9^l g_{4k,4l} + 9^{l-1}u^{2k-4l+\alpha(l)}\left[\tfrac{u^{2l-\alpha(l)}u^2 2^{\nu_2(l)+1}f_{4l-4}}{2^{2l-\alpha(l)}}\right]$$

$$= 9^l g_{4k,4l} + 9^{l-1}u^{2k-4l+\alpha(l)}\left[\tfrac{u^{2l+2-\alpha(l)}2^{\nu_2(l)+1}f_{4l-4}}{2^{2l-\alpha(l)}}\right].$$

Then, since $\nu_2(l) = 1 + \alpha(l-1) - \alpha(l)$,

$$4(l-1) - \alpha(l-1) = 4l - \alpha(l) + \alpha(l) - \alpha(l-1) - 4 = 4l - \alpha(l) - 3 - \nu_2(l) < 2k$$

so that

$$\begin{aligned} g_{4k,4l-4} &= u^{2k-4l+4+\alpha(l-1)}\left[\tfrac{u^{2l-2-\alpha(l-1)}f_{4l-4}}{2^{2l-2-\alpha(l-1)}}\right] \\ &= u^{2k-4l+\alpha(l)}\left[\tfrac{u^{2l+2-\alpha(l)}f_{4l-4}}{2^{2l-2-\alpha(l)+\alpha(l)-\alpha(l-1)}}\right] \\ &= u^{2k-4l+\alpha(l)}\left[\tfrac{u^{2l+2-\alpha(l)}f_{4l-4}}{2^{2l-\alpha(l)-\nu_2(l)-1}}\right] \end{aligned}$$

so that, for $0 < l < k$ suppose that $4l - \alpha(l) \leq 2k$,

$$(1 \wedge \psi^3)_*(g_{4k,4l}) = 9^l g_{4k,4l} + 9^{l-1}g_{4k,4l-4}.$$

Similarly, for $0 < l < k$ if $4l - \alpha(l) > 2k$ then, by Proposition 3.9.3 (proof),

$$\begin{aligned}
&(1 \wedge \psi^3)_*(g_{4k,4l}) \\
&= (1 \wedge \psi^3)_*\left(\left[\tfrac{u^{2(k-l)} f_{4l}}{2^{2(k-l)}}\right]\right) \\
&= \left[\tfrac{u^{2(k-l)} 9^l f_{4l} + 9^{l-1} u^2 2^{\nu_2(k)+1} f_{4k-4})}{2^{2(k-l)}}\right] \\
&= 9^l g_{4k,4l} + 9^{l-1}\left[\tfrac{u^{2k-2l+2} 2^{\nu_2(k)+1} f_{4k-4}}{2^{2(k-l)}}\right].
\end{aligned}$$

This situation splits into two cases given by

(i) $4l - \alpha(l) - \nu_2(l) - 3 \leq 2k < 4l - \alpha(l)$ or

(ii) $2k < 4l - \alpha(l) - \nu_2(l) - 3 < 4l - \alpha(l)$.

In case (i) $4l - 4 - \alpha(l-1) = 4l - \alpha(l) - \nu_2(l) - 3 \leq 2k$ and so again we have

$$\begin{aligned}
g_{4k,4l-4} &= u^{2k-4l+4+\alpha(l-1)}\left[\tfrac{u^{2l-2-\alpha(l-1)} f_{4l-4}}{2^{2l-2-\alpha(l-1)}}\right] \\
&= \tfrac{u^{2k-2l+2} f_{4l-4}}{2^{2l-1-\nu_2(l)-\alpha(l)}} \\
&= \tfrac{u^{2k-2l+2} 2^{1+\nu_2(l)} f_{4l-4}}{2^{2k-2l+4l-\alpha(l)-2k}}
\end{aligned}$$

so that

$$(1 \wedge \psi^3)_*(g_{4k,4l}) = 9^l g_{4k,4l} + 9^{l-1} 2^{4l-\alpha(l)-2k} g_{4k,4l-4}.$$

In case (ii)

$$g_{4k,4l-4} = \left[\tfrac{u^{2k-2l+2} f_{4k-4}}{2^{2(k-l+2)}}\right]$$

so that

$$(1 \wedge \psi^3)_*(g_{4k,4l}) = 9^l g_{4k,4l} + 9^{l-1}2^{3+\nu_2(k)}g_{4k,4l-4}.$$

□

3.10 What is the matrix?

In the notation of §3.1, suppose that $A \in U_\infty\mathbb{Z}_2$ satisfies

$$\Lambda(A^{-1}) = [1 \wedge \psi^3] \in \mathrm{Aut}^0_{left-bu-mod}(bu \wedge bo).$$

Therefore, by definition of Λ and the formula of Theorem 3.8.1

$$\begin{aligned}\sum_{l \leq k} A_{l,k}(\iota_{k,l})_*(z_{4k}) &= (1 \wedge \psi^3)_*(z_{4k}) \\ &= \Sigma_{i=0}^k 2^{\beta(k,i)}\lambda_{4k,4i}(1 \wedge \psi^3)_*(g_{4k,4i}).\end{aligned}$$

On the other hand

$$\sum_{l \leq k} A_{l,k}(\iota_{k,l})_*(z_{4k})$$

$$= A_{k,k}z_{4k} + \sum_{l<k} A_{l,k}\mu_{4k,4l}2^{2k-2l-\alpha(k)+\alpha(l)}u^{2k-2l}z_{4l}$$

$$= A_{k,k}\Sigma_{i=0}^k 2^{\beta(k,i)}\lambda_{4k,4i}g_{4k,4i}$$

$$+\sum_{l<k} \Sigma_{i=0}^l A_{l,k}\ \mu_{4k,4l}2^{2k-2l-\alpha(k)+\alpha(l)}u^{2k-2l}2^{\beta(l,i)}\lambda_{4l,4i}g_{4l,4i}.$$

In order to determine the $A_{k,l}$'s it will suffice to express $u^{2k-2l}g_{4l,4i}$ as a multiple of $g_{4k,4i}$ and then to equate coefficients in the above expressions. By definition

$$u^{2k-2l}g_{4l,4i} = \begin{cases} u^{2k-2l}u^{2l-4i+\alpha(i)}\left[\frac{u^{2i-\alpha(i)}f_{4i}}{2^{2i-\alpha(i)}}\right] & \text{if } 4i-\alpha(i) \leq 2l, \\ \\ u^{2k-2l}\left[\frac{u^{2(l-i)}f_{4i}}{2^{2(l-i)}}\right] & \text{if } 4i-\alpha(i) > 2l. \end{cases}$$

$$= \begin{cases} \frac{u^{2k-2i}f_{4i}}{2^{2i-\alpha(i)}} & \text{if } 4i-\alpha(i) \leq 2l, \\ \\ \frac{u^{2k-2i}f_{4i}}{2^{2l-2i}} & \text{if } 4i-\alpha(i) > 2l \end{cases}$$

while

$$g_{4k,4i} = \begin{cases} u^{2k-4i+\alpha(i)}\left[\frac{u^{2i-\alpha(i)}f_{4i}}{2^{2i-\alpha(i)}}\right] & \text{if } 4i-\alpha(i) \leq 2k, \\ \\ \left[\frac{u^{2(k-i)}f_{4i}}{2^{2(k-i)}}\right] & \text{if } 4i-\alpha(i) > 2k. \end{cases}$$

From these formulae we find that

$$u^{2k-2l}g_{4l,4i} = \begin{cases} g_{4k,4i} & \text{if } 4i-\alpha(i) \leq 2l \leq 2k, \\ \\ 2^{4i-\alpha(i)-2l}g_{4k,4i} & \text{if } 2l < 4i-\alpha(i) \leq 2k, \\ \\ 2^{2k-2l}g_{4k,4i} & \text{if } 2l < 2k < 4i-\alpha(i). \end{cases}$$

Now let us calculate $A_{l,k}$.

When $k = 0$ we have $z_0 = (1 \wedge \psi^3)_*(z_0) = A_{0,0}(\iota_{0,0})_*(z_0) = A_{0,0}z_0$ so that $A_{0,0} = 1$.

When $k = 1$ we have

$$\begin{aligned}
\sum_{l\leq 1} A_{l,1}(\iota_{1,l})_*(z_4) &= (1 \wedge \psi^3)_*(z_4) \\
&= \lambda_{4,4}(1 \wedge \psi^3)_*(g_{4,4}) + 2\lambda_{4,0}(1 \wedge \psi^3)_*(g_{4,0}) \\
&= \lambda_{4,4}(9g_{4,4} + 2g_{4,0}) + 2\lambda_{4,0}g_{4,0}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{l\leq 1} A_{l,k}(\iota_{1,l})_*(z_4) &= A_{1,1}z_4 + A_{0,1}\mu_{1,0}2g_{4,0} \\
&= A_{1,1}(2\lambda_{4,0}g_{4,0} + \lambda_{4,4}g_{4,4}) + A_{0,1}\mu_{1,0}2g_{4,0}
\end{aligned}$$

which implies that $A_{1,1} = 9$ and $A_{0,1} = \mu_{1,0}^{-1}(\lambda_{4,4} - 8\lambda_{4,0})$ so that $A_{0,1} \in \mathbb{Z}_2^*$.

When $k = 2$ we have

$$\begin{aligned}
\sum_{l\leq 2} A_{l,2}(\iota_{2,l})_*(z_8) &= (1 \wedge \psi^3)_*(z_8) \\
&= (1 \wedge \psi^3)_*(\lambda_{8,8}g_{8,8} + 2^3\lambda_{8,4}g_{8,4} + 2^3\lambda_{8,0}g_{8,0}) \\
&= \lambda_{8,8}(9^2g_{8,8} + 9\cdot 2^3g_{8,4}) + 2^3\lambda_{8,4}(9g_{8,4} + g_{8,0}) + 2^3\lambda_{8,0}g_{8,0}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{l\leq 2} A_{l,2}(\iota_{2,l})_*(z_8) &= A_{2,2}z_8 + A_{1,2}(\iota_{2,1})_*(z_8) + A_{0,2}(\iota_{2,0})_*(z_8) \\
&= A_{2,2}(\lambda_{8,8}g_{8,8} + 2^3\lambda_{8,4}g_{8,4} + 2^3\lambda_{8,0}g_{8,0})
\end{aligned}$$

$$+A_{1,2}(\mu_{8,4}2^2u^2z_4) + A_{0,2}(\mu_{8,0}2^3u^4z_0)$$

$$= A_{2,2}(\lambda_{8,8}g_{8,8} + 2^3\lambda_{8,4}g_{8,4} + 2^3\lambda_{8,0}g_{8,0})$$

$$+A_{1,2}\mu_{8,4}2^2(2\lambda_{4,0}g_{8,0} + \lambda_{4,4}u^2g_{4,4}) + A_{0,2}\mu_{8,0}2^3g_{8,0}$$

$$= A_{2,2}(\lambda_{8,8}g_{8,8} + 2^3\lambda_{8,4}g_{8,4} + 2^3\lambda_{8,0}g_{8,0})$$

$$+A_{1,2}\mu_{8,4}2^2(2\lambda_{4,0}g_{8,0} + \lambda_{4,4}2g_{8,4}) + A_{0,2}\mu_{8,0}2^3g_{8,0}.$$

Therefore we obtain

$$\lambda_{8,8}(9^2g_{8,8} + 9\cdot 2^3g_{8,4}) + 2^3\lambda_{8,4}(9g_{8,4} + g_{8,0}) + 2^3\lambda_{8,0}g_{8,0}$$

$$= A_{2,2}(\lambda_{8,8}g_{8,8} + 2^3\lambda_{8,4}g_{8,4} + 2^3\lambda_{8,0}g_{8,0})$$

$$+A_{1,2}\mu_{8,4}2^2(2\lambda_{4,0}g_{8,0} + \lambda_{4,4}2g_{8,4}) + A_{0,2}\mu_{8,0}2^3g_{8,0}$$

which yields

$$9^2 = A_{2,2},$$

$$\lambda_{8,8}\cdot 9 + \lambda_{8,4}(9 - 9^2) = A_{1,2}\mu_{8,4}\lambda_{4,4},$$

$$\lambda_{8,4} + \lambda_{8,0}(1 - 9^2) = A_{1,2}\mu_{8,4}\lambda_{4,0} + A_{0,2}\mu_{8,0}.$$

Hence $A_{1,2} \in \mathbb{Z}_2^*$.

Now assume that $k \geq 3$ and consider the relation derived above

$$\Sigma_{i=0}^{k} 2^{\beta(k,i)} \lambda_{4k,4i}(1 \wedge \psi^3)_*(g_{4k,4i})$$

$$= A_{k,k} \Sigma_{i=0}^{k} 2^{\beta(k,i)} \lambda_{4k,4i} g_{4k,4i}$$

$$+ \sum_{l<k} \Sigma_{i=0}^{l} A_{l,k} \ \mu_{4k,4l} 2^{2k-2l-\alpha(k)+\alpha(l)} u^{2k-2l} 2^{\beta(l,i)} \lambda_{4l,4i} g_{4l,4i}.$$

The coefficient of $g_{4k,4k}$ on the left side of this relation is equal to $\lambda_{4k,4k} 9^k$ and on the right side it is $A_{k,k}\lambda_{4k,4k}$ so that $A_{k,k} = 9^k$ for all $k \geq 3$. From the coefficient of $g_{4k,4k-4}$ we obtain the relation

$$\lambda_{4k,4k} 9^{k-1} 2^{\nu_2(k)+3} + 2^{3+\nu_2(k)} \lambda_{4k,4k-4} 9^{k-1}$$

$$= 9^k 2^{3+\nu_2(k)} \lambda_{4k,4k-4}$$

$$+ A_{k-1,k} \ \mu_{4k,4k-4} 2^{2-\alpha(k)+\alpha(k-1)} 2^2 \lambda_{4k-4,4k-4} 2^{3+\nu_2(k)} \lambda_{4k,4k-4} 9^{k-1}$$

$$= 9^k 2^{3+\nu_2(k)} \lambda_{4k,4k-4}$$

$$+ A_{k-1,k} \ \mu_{4k,4k-4} 2^{3+\nu_2(k)} \lambda_{4k-4,4k-4}$$

which shows that $A_{k-1,k} \in \mathbb{Z}_2^*$ for all $k \geq 3$. This means that we may conjugate A by the matrix $D = \mathrm{diag}(1, A_{1,2}, A_{1,2}A_{2,3}, A_{1,2}A_{2,3}A_{3,4}, \ldots) \in U_\infty \mathbb{Z}_2$ to obtain

$$DAD^{-1} = C = \begin{pmatrix} 1 & 1 & c_{1,3} & c_{1,4} & c_{1,5} & \dots \\ 0 & 9 & 1 & c_{2,4} & c_{2,5} & \dots \\ 0 & 0 & 9^2 & 1 & c_{3,5} & \dots \\ 0 & 0 & 0 & 9^3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

In the next section we examine whether we can conjugate this matrix further in $U_\infty \mathbb{Z}_2$ to obtain the matrix

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 9 & 1 & 0 & 0 & \dots \\ 0 & 0 & 9^2 & 1 & 0 & \dots \\ 0 & 0 & 0 & 9^3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

3.11 Conjugating the matrix

Let $B, C \in U_\infty\mathbb{Z}_2$ denote the upper triangular matrices which occurred in §3.10

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 9 & 1 & 0 & 0 & \dots \\ 0 & 0 & 9^2 & 1 & 0 & \dots \\ 0 & 0 & 0 & 9^3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, C = \begin{pmatrix} 1 & 1 & c_{1,3} & c_{1,4} & c_{1,5} & \dots \\ 0 & 9 & 1 & c_{2,4} & c_{2,5} & \dots \\ 0 & 0 & 9^2 & 1 & c_{3,5} & \dots \\ 0 & 0 & 0 & 9^3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

The following result is the main result of this section. Along with the discussion of §3.10 it completes the proof of Theorem 3.4.2 by explicitely constructing a matrix $U \in U_\infty\mathbb{Z}_2$ which satisfies $UCU^{-1} = B$.

Theorem 3.11.1. *Let $U = (u_{i,j})_{i,j\geq 1}$ be the upper triangular matrix whose entries satisfy*

$$u_{1,j} = \begin{cases} 1 & \text{if } j = 1 \text{ or } j = 2 \\ 0 & \text{if } j > 2 \end{cases}$$

and

$$u_{i+1,j} = (\sum_{r=i}^{j-2} u_{i,r}c_{i,r}) + u_{i,j-1} + (9^{j-1} - 9^{i-1})u_{i,j}.$$

Then (i). U is invertible, (ii). $UC = BU$.

Proof. (i). We shall prove, by induction, that $u_{i,i} \in \mathbb{Z}_2^\times$ for $i \geq 1$, which is sufficient to prove that U is invertible.

For $i = 1$, $u_{1,1} = 1$ which is clearly in $\mathbb{Z}_2^\times$. Now assume $u_{i,i} \in \mathbb{Z}_2^\times$ for all $1 \leq i < n$ where $n \in \mathbb{Z}^+$. We wish to show that this implies $u_{n,n} \in \mathbb{Z}_2^\times$. By definition

$$u_{n,n} = u_{n-1,n-1} + (9^{n-1} - 9^{n-2})u_{n-1,n}.$$

Since $9^{n-1} - 9^{n-2} = 9^{n-2}(9-1) \in 2\mathbb{Z}_2$ and, by the induction hypothesis, $u_{n-1,n-1} \in \mathbb{Z}_2^\times$ it follows that $u_{n,n} \in \mathbb{Z}_2^\times + 2\mathbb{Z}_2 \subset \mathbb{Z}_2^\times$. Hence, by induction, $u_{i,i} \in \mathbb{Z}_2^\times$ for $i \geq 1$, and therefore, U is invertible.

(ii). Since we have defined U to be upper triangular it is trivial to show that $(UC)_{i,j} = (BU)_{i,j}$ for $j > i$. The entries of UC and BU, for $i \geq j$ are given below:

$$\begin{aligned}
(UC)_{i,j} &= u_{i,i}c_{i,j} + u_{i,i+1}c_{i+1,j} + \cdots + u_{i,j-2}c_{j-2,j} + u_{i,j-1} + u_{i,j}9^{j-1} \\
&= \left(\textstyle\sum_{r=i}^{j-2} u_{i,r}c_{i,r}\right) + u_{i,j-1} + u_{i,j}9^{j-1} \\
(BU)_{i,j} &= 9^{i-1}u_{i,j} + u_{i+1,j}.
\end{aligned}$$

We shall now prove that these are equal for all $i, j \geq 1$ and $i \geq j$:

$$\begin{aligned}
(UC)_{i,j} &= \left(\textstyle\sum_{r=i}^{j-2} u_{i,r}c_{i,r}\right) + u_{i,j-1} + u_{i,j}9^{j-1} \\
&= u_{i+1,j} - (9^{j-1} - 9^{i-1})u_{i,j} + u_{i,j}9^{j-1} \\
&= u_{i+1,j} - 9^{j-1}u_{i,j} + 9^{i-1}u_{i,j} + u_{i,j}9^{j-1} \\
&= u_{i+1,j} + 9^{i-1}u_{i,j} \\
&= (BU)_{i,j}
\end{aligned}$$

as required. □

Chapter 4

Applications

The following Chapter contains two applications of the results of Chapter 3. The first application concerns using knowledge of the left bu-module automorphism $1 \wedge \psi^3 : bu \wedge bo \longrightarrow bu \wedge bo$ to explicitly describe the left bu-module automorphism $1 \wedge \psi^3 : bu \wedge bu \longrightarrow bu \wedge bu$. In the second application we use Theorem 3.4.2 to study the ring $End_{left-bu-mod}(bu \wedge bo)$ of left bu-module endomorphisms of $bu \wedge bo$. Both of these applications come from collaborative work of the author and his PhD supervisor Prof. V. Snaith. They are published in [11]. They are not presented here as original work of the author, but the description given here by the author serves to illustrate possible uses for the original results of Chapter 2.

4.1 $bu \wedge bu$

In this section we wish to use the knowledge of $1 \wedge \psi^3 : bu \wedge bo \longrightarrow bu \wedge bo$ from Theorem 3.4.2 to explicitly describe the left bu-module automorphism

$1 \wedge \psi^3 : bu \wedge bu \longrightarrow bu \wedge bu$, in terms of the morphisms $\iota_{k,k-1}$ of §3.3.

Recall that the group isomorphism Λ of §3.4 determines the matrix corresponding to an element of $Aut^0_{left-bu-mod}(bu \wedge bo)$ upto inner automorphism. Therefore, Theorem 3.4.2 implies that, in the 2-local stable homotopy category there exists an equivalence $C' \in \mathrm{Aut}^0_{left-bu-mod}(bu \wedge bo)$ such that

$$C'(1 \wedge \psi^3)C'^{-1} = \sum_{k \geq 0} 9^k \iota_{k,k} + \sum_{k \geq 1} \iota_{k,k-1}$$

where $\iota_{k,l}$ is as in §3.1, considered as left bu-endomorphism of $bu \wedge bo$ via the equivalence $\hat{L}$ of §3.1.

In [24] use is made of a homotopy equivalence of spectra of the form $bu \simeq bo \wedge \Sigma^{-2}\mathbb{CP}^2$, first noticed by Reg Wood (as remarked in [4] pg. 206) and independently by Don Anderson (both unpublished), where $\Sigma^{-2}\mathbb{C}P^2$ denotes the 2^{nd} desuspension of $\mathbb{C}P^2$ as given in §2.2.1.

Proposition 4.1.1. *In the* 2*-local stable homotopy category there is a morphism*

$$\Psi : \Sigma^{-2}\mathbb{CP}^2 \longrightarrow \Sigma^{-2}\mathbb{CP}^2$$

which satisfies $\Psi^*(z) = \psi^3(z)$ *for all* $z \in bu^0(\Sigma^{-2}\mathbb{CP}^2)$.

Proof. A construction of this morphism is given by Snaith in [23]. □

There is a commutative diagram in the 2-local stable homotopy category of the form

$$\begin{array}{ccc} bo \wedge \Sigma^{-2}\mathbb{C}P^2 & \xrightarrow{\psi^3 \wedge \Psi} & bo \wedge \Sigma^{-2}\mathbb{C}P^2 \\ \downarrow \simeq & & \downarrow \simeq \\ bu & \xrightarrow{\psi^3} & bu \end{array}$$

in which the vertical morphisms are equal, given by the Anderson-Wood equivalence.

Now suppose that we form the smash product with $\Sigma^{-2}\mathbb{CP}^2$ of the 2-local left bu-module equivalence $bu \wedge bo \simeq \vee_{k \geq 0} bu \wedge (F_{4k}/F_{4k-1})$ to obtain a left bu-module equivalence of the form

$$bu \wedge bu \simeq \vee_{k \geq 0} bu \wedge (F_{4k}/F_{4k-1}) \wedge \Sigma^{-2}\mathbb{CP}^2.$$

For $l \leq k$ set

$$\kappa_{k,l} = \iota_{k,l} \wedge \Psi : bu \wedge (F_{4k}/F_{4k-1}) \wedge \Sigma^{-2}\mathbb{CP}^2 \longrightarrow bu \wedge (F_{4l}/F_{4l-1}) \wedge \Sigma^{-2}\mathbb{CP}^2$$

then we obtain the following result.

Theorem 4.1.2.

In the notation of §4, in the 2-local stable homotopy category, there exists $C' \in \mathrm{Aut}^0_{left-bu-mod}(bu \wedge bo)$ *such that*

$$1 \wedge \psi^3 : bu \wedge bu \longrightarrow bu \wedge bu$$

satisfies

$$(C' \wedge 1)(1 \wedge \psi^3)(C' \wedge 1)^{-1} = \sum_{k \geq 0} 9^k \kappa_{k,k} + \sum_{k \geq 1} \kappa_{k,k-1}.$$

4.2 $End_{left-bu-mod}(bu \wedge bo)$

In this section we shall apply Theorem 3.4.2 to study the ring of left *bu*-module homomorphisms of $bu \wedge bo$. As usual we shall work in the 2-local stable homotopy category. Let $\tilde{U}_\infty \mathbb{Z}_2$ denote the ring of upper triangular, infinite matrices with coefficients in the 2-adic integers. Therefore the group $U_\infty \mathbb{Z}_2$ is the multiplicative group of units of $\tilde{U}_\infty \mathbb{Z}_2$. Choose a left *bu*-module homotopy equivalence of the form

$$\hat{L} : \vee_{k \geq 0} bu \wedge (F_{4k}/F_{4k-1}) \xrightarrow{\simeq} bu \wedge bo,$$

as in §3.1. For any matrix $A \in \tilde{U}_\infty \mathbb{Z}_2$ we may define a left *bu*-module endomorphism of $bu \wedge bo$, denoted by λ_A, by the formula

$$\lambda_A = \hat{L} \cdot (\sum_{0 \leq l \leq k} A_{l,k} \iota_{k,l}) \cdot \hat{L}^{-1}.$$

Incidentally here and throughout this section we shall use the convention that a composition of maps starts with the right-hand map, which is the *opposite* convention used in the definition of the isomorphism Λ of §3.1 and [24]. When $A \in U_\infty \mathbb{Z}_2$ we have the relation $\lambda_A = \Lambda(A^{-1})$. For $A, B \in \tilde{U}_\infty \mathbb{Z}_2$

we have

$$\begin{aligned}
\lambda_A \cdot \lambda_B &= (\hat{L} \cdot (\textstyle\sum_{0 \leq l \leq k} A_{l,k} \iota_{k,l}) \cdot \hat{L}^{-1}) \cdot (\hat{L} \cdot (\textstyle\sum_{0 \leq t \leq s} B_{t,s} \iota_{s,t}) \cdot \hat{L}^{-1}) \\
&= \hat{L} \cdot (\textstyle\sum_{0 \leq l \leq t \leq s} A_{l,t}\ B_{t,s} \iota_{s,l}) \cdot \hat{L}^{-1} \\
&= \hat{L} \cdot (\textstyle\sum_{0 \leq l \leq s} (AB)_{l,s} \iota_{s,l}) \cdot \hat{L}^{-1} \\
&= \lambda_{AB}.
\end{aligned}$$

By Theorem 3.4.2 there exists $H \in U_\infty \mathbb{Z}_2$ such that

$$1 \wedge \psi^3 = \lambda_{HBH^{-1}}$$

for

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 9 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 9^2 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 9^3 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Hence, for any integer $u \geq 1$, we have $1 \wedge (\psi^3 - 9^{u-1}) = \lambda_{HB_uH^{-1}}$ where $B_u = B - 9^{u-1} \in \tilde{U}_\infty \mathbb{Z}_2$ and 9^{u-1} denotes 9^{u-1} times the identity matrix. Following [19] write $\phi_n : bo \longrightarrow bo$ for the composition $\phi_n = (\psi^3 - 1)(\psi^3 - 9) \ldots (\psi^3 - 9^{n-1})$. Write $X_n = B_1 B_2 \ldots B_n \in \tilde{U}_\infty \mathbb{Z}_2$.

Theorem 4.2.1. *(i) In the notation of §4.2* $1 \wedge \phi_n = \lambda_{HX_nH^{-1}}$ *for* $n \geq 1$.

(ii) The first n-columns of X_n *are trivial.*

(iii) Let $C_n = Cone(\hat{L} : \vee_{0 \leq k \leq n-1} bu \wedge (F_{4k}/F_{4k-1}) \xrightarrow{\simeq} bu \wedge bo$, *which is a left bu-module spectrum. Then in the 2-local stable homotopy category there exists a commutative diagram of left bu-module morphisms of the form*

$$\begin{array}{ccc} bu \wedge bo & \xrightarrow{1 \wedge \phi_n} & bu \wedge bo \\ & {\scriptstyle \pi_n} \searrow \quad \nearrow {\scriptstyle \hat{\phi}_n} & \\ & C_n & \end{array}$$

where π_n *is the cofibre of the restriction of* $\hat{L}$. *Also* $\hat{\phi}_n$ *is determined up to homotopy by this diagram.*

(iv) For $n \geq 1$ *we have*

$$(X_n)_{s,s+j} = 0 \text{ if } j < 0 \text{ or } j > n$$

and the other entries are given by the formula

$$(X_n)_{s,s+t} = \sum_{1 \leq k_1 < k_2 < \ldots < k_t \leq n} A(k_1)A(k_2)\ldots A(k_t)$$

where

$$A(k_1) = \prod_{j_1=n-k_1+1}^{n} (9^{s-1} - 9^{j_1-1}),$$

$$A(k_2) = \prod_{j_2=n-k_2+1}^{n-k_1-1} (9^{s} - 9^{j_2-1}),$$

$$A(k_3) = \prod_{j_3=n-k_3+1}^{n-k_2-1} (9^{s+1} - 9^{j_3-1}),$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$A(k_t) = \prod_{j_t=1}^{n-k_t-1}(9^{s+t+1} - 9^{j_t-1}).$$

Proof. Part (i) follows immediately from the discussion of §4.2. Part (ii) follows from part (iii), but it is simpler to prove it directly. For part (ii) observe that the B_i commute, being polynomials in the matrix B so that $X_n = X_{n-1}B_n$. Since $(B_n)_{s,t}$ is zero except when $t = s, s+1$ so that $(X_n)_{i,j} = (X_{n-1})_{i,j}(B_n)_{j,j} + (X_{n-1})_{i,j-1}(B_n)_{j-1,j}$, which is zero by induction if $j < n$. When $j = n$ by induction we have $(X_n)_{i,j} = (X_{n-1})_{i,n}(B_n)_{n,n}$ which is trivial because $(B_n)_{n,n} = 9^{n-1} - 9^{n-1}$. In view of the decomposition of $bu \wedge bo$, part (iii) amounts to showing that HX_nH^{-1} corresponds to a left bu-module endomorphism of $\vee_{0 \leq k}\, bu \wedge (F_{4k}/F_{4k-1})$ which is trivial on each summand $bu \wedge (F_{4k}/F_{4k-1})$ with $k \leq n-1$. The (i,j)-th entry in this matrix is the multiple of $\iota_{j-1,i-1} : bu \wedge (F_{4j-4}/F_{4j-5}) \longrightarrow bu \wedge (F_{4i-4}/F_{4i-5})$ given by the appropriate component of the map. The first n columns are zero if and only if the map has no non-trivial components whose domain is $bu \wedge (F_{4j-4}/F_{4j-5})$ with $j \leq n$. Since H is upper triangular and invertible, the first n columns of X_n vanish if and only if the same is true for HX_nH^{-1}. Finally the formulae of part (iv) result from the fact that B_j has $9^{m-1} - 9^{j-1}$ in the (m,m)-th entry, 1 in the $(m, m+1)$-th entry and zero elsewhere. □

Remark 4.2.2. Theorem 4.2.1 is closely related to the main result of [19]. Following [19] let $bo^{(n)} \longrightarrow bo$ denote the map of 2-local spectra which is universal for all maps $X \longrightarrow bo$ which are trivial with respect to all higher $\mathbb{Z}/2$-cohomology operations of order less than n. Cf with [19] Theorem B.

Milgram shows that ϕ_{2n} factorises through a map of the form $\theta_{2n} : bo \longrightarrow \Sigma^{8n} bo^{(2n-\alpha(n))}$ and that ϕ_{2n+1} factorises through a map of the form $\theta_{2n+1} : bo \longrightarrow \Sigma^{8n+4} bsp^{(2n-\alpha(n))}$ and then uses the θ_m's to produce a left-bo-module splitting of $bo \wedge bo$. Using the homotopy equivalence $bu \simeq bo \wedge \Sigma^{-2}\mathbb{CP}^2$ mentioned in [24] one may pass from the splitting of $bu \wedge bo$ to that of $bo \wedge bo$ (and back again). In the light of this observation, Theorem 4.2.1 should be thought of as the upper triangular matrix version of the proof that the θ_n's exist. The advantage of the matrix version is that Theorem 4.2.1(iv) gives us every entry in the matrix X_n, not just the zeroes in the first n columns.

Appendix A

2-adic valuation results

The main result from this appendix is Proposition A.0.4. The proof requires Lemmas A.0.1-A.0.3.

Lemma A.0.1. *For any integer* $n \geq 0$, $9^{2^n} - 1 = 2^{n+3}(2s+1)$ *for some* $s \in \mathbb{Z}$.

Proof. We prove this by induction on n, starting with $9 - 1 = 2^3$. Assuming the result is true for n, we have

$$\begin{aligned} 9^{2^{(n+1)}} - 1 &= (9^{2^n} - 1)(9^{2^n} + 1) \\ &= (9^{2^n} - 1)(9^{2^n} - 1 + 2) \\ &= 2^{n+3}(2s+1)(2^{n+3}(2s+1) + 2) \\ &= 2^{n+4}(2s+1)\underbrace{(2^{n+2}(2s+1) + 1)}_{odd} \end{aligned}$$

as required. □

Lemma A.0.2. *For any integer $l \geq 1$, $9^l - 1 = 2^{\nu_2(l)+3}(2s+1)$ for some $s \in \mathbb{Z}$, where $\nu_2(l)$ denotes the 2-adic valuation of l.*

Proof. Write $l = 2^{e_1} + 2^{e_2} + \ldots + 2^{e_k}$ with $0 \leq e_1 < e_2 < \ldots < e_k$ so that $\nu_2(l) = e_1$. Then, by Lemma A.0.1,

$$\begin{aligned} 9^l - 1 &= 9^{2^{e_1}+2^{e_2}+\cdots+2^{2_k}} - 1 \\ &= ((2s_1+1)2^{e_1+3}+1))\ldots((2s_k+1)2^{e_k+3}+1)) - 1 \\ &\equiv (2s_1+1)2^{e_1+3} \text{ (modulo} 2^{e_1+4}) \\ &= 2^{e_1+3}(2s+1) \end{aligned}$$

as required. □

Lemma A.0.3. *For any integer $l \geq 1$, $\prod_{i=1}^{l} (9^l - 9^{i-1}) = 2^{\nu_2(l!)+3l}(2s+1)$ for some $s \in \mathbb{Z}$.*

Proof. By Lemma A.0.2 we have

$$\begin{aligned} \prod_{i=1}^{l} (9^l - 9^{i-1}) &= \prod_{i=1}^{l} (9^{l-i+1} - 1)9^{i-1} \\ &= \prod_{i=1}^{l} 2^{\nu_2(l-i+1)+3}(2t_i+1)9^{i-1} \\ &= (2t+1)2^{\nu_2(l!)+3l}, \end{aligned}$$

as required. □

Proposition A.0.4. *For any integer $l \geq 0$, $2^{\nu_2(l!)+3l} = 2^{4l-\alpha(l)}$ where $\alpha(l)$ is equal to the number of 1's in the dyadic expansion of l.*

Proof. Write $l = 2^{\alpha_1} + 2^{\alpha_2} + \ldots + 2^{\alpha_k}$ with $0 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_k$ so that $\alpha(l) = k$. $9^l - 1 = 2^{\nu_2(l)+3}(2s+1)$ for some $s \in \mathbb{Z}$, where $\nu_2(l)$ denotes the 2-adic valuation of l. Then

$$\begin{array}{rcccccccc} \nu_2(l!) & = & & 2^{\alpha_1-1} & + & 2^{\alpha_2-1} & + \cdots & + & 2^{\alpha_k-1} \\ & & + & 1 & + & 2^{\alpha_2-\alpha_1} & + \cdots & + & 2^{\alpha_k-\alpha_1} \\ & & & & + & 1 & + \cdots & + & 2^{\alpha_k-\alpha_2} \\ & & & & & & & + & 1 \end{array}$$

because the first row counts the multiples of 2 less than or equal to l, the second row counts the multiples of 4, the third row counts multiples of 8 and so on. Adding by columns we obtain

$$\nu_2(l!) = 2^{\alpha_1} - 1 + 2^{\alpha_2} - 1 + \cdots + 2^{\alpha_k} - 1 = l - k$$

which implies that $2^{3l+\nu_2(l!)} = 2^{3l+l-\alpha(l)} = 2^{4l-\alpha(l)}$, as required. □

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