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Characters and lengths of geodesics in hyperbolic 3-manifolds

by

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ABSTRACT

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According to the work by Randol, there exists pairs of closed curves on a surface S for which the geodesics in their respective homotopy classes have the same hyperbolic length, irrespective of the hyperbolic structure on S. In this work we look at this result in connection to hyperbolic 3-manifolds, and in particular the book of I-bundles manifold.

We consider the following problem.

Let $G = \pi_1(M)$, where M is a compact hyperbolizable 3-manifold, and consider all faithful representations of G into $SL_2(\mathbb{C})$. Find a topological condition P that can be imposed on the elements of G so the following is true. If $g \in G$ satisfies condition P, and $h \in G$ is any element such that $\chi[h] = \chi[g]$ then h is conjugate to $g^{\pm 1}$.

Here $\chi[g]$ is the character of g, which is defined in terms of the trace of the matrix representation of g in $SL_2(\mathbb{C})$. This problem can be translated into a question about the lengths of the geodesics in M by utilizing the connection between the character of an element of G, and the length of its geodesic representative in M. We therefore look for a property that gives some geometrical information about the manifold. For the purpose of this work the manifold M is a book of I-bundles.

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Chapter 1

Introduction

The question we are addressing concerns the characters and lengths of closed geodesics in hyperbolizable 3-manifolds. Let M be a compact hyperbolizable 3-manifold, let $G = \pi_1(M)$ and let $\mathcal{F}(G)$ be the space of all faithful representations of G into $SL_2(\mathbb{C})$. For each $\omega \in G$ we can define the character $\chi[\rho(\omega)]$ in terms of the trace of the matrix representation ρ in $SL_2(\mathbb{C})$. The most general question is to give a characterisation of elements ω and μ of G for which the characters are the same. Although some work has been done towards resolving this question, there is not a complete solution even in the simpler case when considering hyperbolic surfaces. However, there is more known in relation to proving almost the converse of this general question. The specific question to be considered here is to determine a reasonable property P such that if $g \in G$ satisfies condition P and if $\chi[g]=\chi[h]$ then h is conjugate to $g^{\pm 1}$. Equivalently, because there is a connection between the trace of a group element and the length of the corresponding geodesic in M, the problem is to look at what criteria ensure that two closed geodesics on a hyperbolic 3-manifold have the same hyperbolic length. More specifically, we want to find a natural property on the geodesics of M such that if two geodesics have the same length then they are essentially the same curve in M (up to orientation).

The initial chapters provide some background to hyperbolic manifolds and characters, and looks at what is already known in relation to this question.

In chapter two we build a picture of hyperbolic space, giving the basic models we will use, namely the upper half plane and upper half space models. We introduce the idea of a general Kleinian group and various properties and relevant results, which will be of use later.

In chapter three we move on to look at hyperbolic 2- and 3-manifolds and give some definitions and results from 3-dimensional topology. These become useful when addressing the question.

In chapter four we give the formal definition of a character of an element of a group, (as given by Horowitz in his papers as listed), and build the connection between this notion and the length of the equivalent closed geodesic in the hyperbolizable manifold. At the end of this section we give 1- and 2-dimensional results in relation to the question, including their proofs. These are due to Horowitz and McShane respectively. Note that these are included to show how the problem can be broken down, and elements of these proofs will be used in connection with the results later on. We also give a slight modification to one of these results by removing one of the redundant assumptions. There is an indication of other possible connected questions that may be considered.

In chapter five we look at the main problem in terms of the Book of I-bundles manifold. We define this manifold in general and then reduce to a specific case, the 3-page book, for which we consider the following results.

Theorem 5.7.1 *Let M be the specific book of I-bundles manifold with single solid torus binding and three pages. Let $g \in \pi_1(M)$, such that g is represented by the core curve of the solid torus in M. Then g is uniquely determined by $\chi[g]$. By this we mean that if $h \in \pi_1(M)$ with $\chi[g] = \chi[h]$ then h is conjugate to $g^{\pm 1}$ (so h is also represented by the core curve in M).*

Theorem 5.8.1 *Let M be the specific book of I-bundles with single solid torus binding and three pages. Let $g \in \pi_1(M)$ be represented by a geodesic γ that is contained in a component of the boundary of M ($\gamma \subseteq S_i \subseteq \partial M$). Let $h \in \pi_1(M)$ be represented by another curve γ' such that $\chi[h] = \chi[g]$. Then $\gamma' \subseteq S_i$ also.*

In chapter six we consider these results in connection with the general Book of I-bundles manifold, looking at how all the elements needed for the proofs extend to the more general setting.

In chapter seven, we consider the following idea.

Conjecture 7.0.3 *Let $G = \pi_1(M)$ where M is a book of I-bundles with single solid torus binding and three pages. Let $g \in G$ be represented by a geodesic in M which is uniquely projected onto a simple closed curve on F (where F is the spine of M). Let $h \in G$ such that $\chi[g] = \chi[h]$, then $h \cong g^{\pm 1}$.*

Chapter 2

Hyperbolic Space and Associated Groups

In this chapter we give the background material needed in the work. As we will be using both the hyperbolic plane, $\mathbb{H}^2$, and hyperbolic 3-space, $\mathbb{H}^3$, we give here the particular models we will use for each dimension, namely the *upper half plane* model for $\mathbb{H}^2$, and the *upper half space* model for $\mathbb{H}^3$. We will also introduce particular groups associated to $\mathbb{H}^2$ and $\mathbb{H}^3$, namely *Fuchsian Groups* and *Kleinian groups* respectively, and look at their actions on the boundary of the respective models. There are many references for this material, in particular, for hyperbolic 2-space and Fuchsian groups the majority of it may be found in [And99], [Kat92] or [JS87] and for hyperbolic 3-space and Kleinian groups, the majority may be found in [MT98] and [Mas88].

2.1 Hyperbolic n-space

In this section we will look at hyperbolic space, defining the models that will be used throughout. As this work is concerned with hyperbolic surfaces and hyperbolic manifolds, the focus will be on hyperbolic 2-space and hyperbolic 3-space. We will complete this section by looking at some hyperbolic trigonometry, and derive some results that will be used in later chapters of this work. In particular we look at hyperbolic triangles, including the statements of the hyperbolic sine and cosine rules.

2.1.1 The Upper Half Plane model

We start by defining a particular model used for the hyperbolic plane $\mathbb{H}^2$.

Definition 2.1.1 *The upper half plane model for $\mathbb{H}^2$, consists of all points in the top half of the complex plane, or more formally*

$$\mathbb{H}^2 = \{z \in \mathbb{C} | Im(z) > 0\}.$$

This is equipped with the hyperbolic metric $ds = \frac{\sqrt{dx^2+dy^2}}{y}$, where $x = Re(z)$ and $y = Im(z)$.

To measure the length of a given path in this space we integrate along the path over the metric ds. To be more precise, let $f : I = [0,1] \to \mathbb{H}^2$ be a piece-wise differentiable path with $f(t) = x(t) + iy(t)$. Then the hyperbolic length is given by

$$\int_I \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y(t)} dt = \int_f \frac{1}{Im(z)} |dz|.$$

The lines in the hyperbolic plane are the geodesics (the paths of shortest length) with respect to the hyperbolic metric. In the upper half plane model these consist of Euclidean straight lines and Euclidean semicircles orthogonal to the real axis. Any two points in $\mathbb{H}^2$ can be joined by a unique hyperbolic geodesic. If we consider the set of all piece-wise differentiable paths between two points x and y in $\mathbb{H}^2$, then we can define the hyperbolic distance to be the infimum of the lengths over this set of paths. It is known that this distance realizing path is a parameterization of the hyperbolic line segment joining x and y. Therefore, as a consequence, the hyperbolic distance between two points is the hyperbolic length of the hyperbolic line segment joining them.

Now consider the boundary of the model, by considering the 'end points' of the lines in $\mathbb{H}^2$. These either lie on the real axis or, in the case of the vertical Euclidean lines, lie at ∞. This gives the following definition,

Definition 2.1.2 *The boundary at infinity of the upper half plane model, denoted $\partial_\infty(\mathbb{H}^2)$, is given by $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. (This is the one point compactification of $\mathbb{R}$.)*

We refer to the points of $\partial_\infty(\mathbb{H}^2)$ as the *points at infinity* of $\mathbb{H}^2$. Although this boundary is important, we must note here that these points are not included in the upper half plane.

The geometry of $\mathbb{H}^2$ is determined by its congruent transformations, or its isometries, by which we mean the automorphisms of the model that preserve the hyperbolic distance and angles. Before we can consider the isometries of the model we must first take a look at a particular group of transformations called Möbius transformations, which are defined as follows.

Definition 2.1.3 *Möbius transformations are linear fractional transformations which map from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$ and have the form $z \rightarrow \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.*

Here $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and is the Riemann sphere. (This is the one-point compactification of the complex plane.)

Note that we can normalize any such transformation (by dividing a, b, c and d by $\sqrt{ad - bc}$) and get another representation of the same Möbius transformation which has determinant equal to 1. Therefore, without loss of generality, we may assume that $ad - bc = 1$ for all our Möbius transformations.

We can split Möbius transformations up into particular sets by giving a classification. We classify Möbius transformations by conjugating a given transformation by another appropriate Möbius transformation that puts it into a standard form. We then classify these standard forms. This gives the following classification.

Definition 2.1.4 *Let m be a Möbius transformation that is not equal to the identity transformation. Then*

- *If m is conjugate to $z \rightarrow z + 1$, then we call m parabolic.*
- *If m is conjugate to $z \rightarrow \lambda z$, $(\lambda \in \mathbb{C} \setminus \{0, 1\})$ and $|\lambda| = 1$, then we call m elliptic.*

- *If m is conjugate to $z \to \lambda z$, ($\lambda \in \mathbb{C} \setminus \{0,1\}$) and $|\lambda| \neq 1$, then we call m loxodromic.*
- *If m is conjugate to $z \to \lambda z$, ($\lambda \in \mathbb{C} \setminus \{0,1\}$) and $\lambda > 1$, $\lambda \in \mathbb{R}$, then we call m hyperbolic.*

(Note that from definition 2.1.4 we see that hyperbolic Möbius transformations are a particular type of loxodromic Möbius transformations.)

The above is a complete classification as these are the only possible standard forms. For a proof of this see the discussion given in [[And99] section 2.4] which goes through every possible Möbius transformation and shows that it must be conjugate to one of these standard forms.

Note that definition 2.1.4 also gives the action of each type of Möbius transformation. Parabolic elements act as translations, elliptic elements act as rotations about some origin and loxodromic (and hence hyperbolic elements) are a composition of a dilation and a rotation in $\widehat{\mathbb{C}}$.

For the purpose of this section, (while considering $\mathbb{H}^2$), we will consider the set of Möbius transformations with real coefficients (i.e. $a, b, c, d \in \mathbb{R}$). This particular subset forms a group under composition of functions. To see this note that the composition of any two transformations of this kind corresponds to the product of the corresponding matrices with real entries of the form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $ad - bc = 1$. The inverse transformation, which has the form $z \to \frac{dz-b}{-cz+a}$, corresponds to the inverse matrix g^{-1}. These matrices form the special linear group denoted $SL_2(\mathbb{R})$. There is a slight ambiguity here as in a given Möbius transformation we can multiply a, b, c and d by -1 and still have $ad - bc = 1$. Hence each Möbius transformation as defined is represented by a pair of matrices $\pm g$ in $SL_2(\mathbb{R})$, and therefore represented by a unique matrix in $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm I\}$.

This group $PSL_2(\mathbb{R})$ acts on $\mathbb{H}^2$ by homeomorphisms and so maps $\mathbb{H}^2$ onto $\mathbb{H}^2$ continuously. It can be shown that $PSL_2(\mathbb{R})$ is a subgroup of $Isom(\mathbb{H}^2)$ (the set of isometries of $\mathbb{H}^2$), and in fact $PSL_2(\mathbb{R})$ is equivalent to the orientation preserving half of the isometries of $\mathbb{H}^2$ denoted by $Isom^+(\mathbb{H}^2)$. With this in mind, the elements of $PSL_2(\mathbb{R})$ may be classified as follows.

Definition 2.1.5 *An element $T \neq id$ of $PSL_2(\mathbb{R})$ is distinguished by the value of the square of its trace, $tr^2(T) = (a+d)^2$, as follows:*

- *if $0 \leq tr^2(T) < 4$, then T is elliptic.*
- *if $tr^2(T) = 4$, then T is parabolic.*
- *if $tr^2(T) > 4$, then T is hyperbolic.*

(Note that apart from the hyperbolic elements there are not any other loxodromic elements in $PSL_2(\mathbb{R})$. Any loxodromic element that is not a hyperbolic element will have $tr^2(T)$ equal to either a negative or complex number. The trace of a matrix in $PSL_2(\mathbb{R})$ is always real, and hence the square of the trace will always be a non negative real number.)

The elements of $PSL_2(\mathbb{R})$ can also be classified by the number of fixed points that they have in the hyperbolic plane.

- An elliptic element has a pair of complex conjugate fixed points, so has one fixed point in $\mathbb{H}^2$.
- A parabolic element has one fixed point in $\mathbb{R} \cup \{\infty\} = \partial_\infty(\mathbb{H}^2)$
- A hyperbolic element has two fixed points in $\mathbb{R} \cup \{\infty\} = \partial_\infty(\mathbb{H}^2)$, which are joined by a hyperbolic geodesic called an *axis*.

The above summary gives a complete classification of the elements of $PSL_2(\mathbb{R})$, as shown by the following lemma which comes from [[And99] page 25].

Lemma 2.1.6 *If an element of $PSL_2(\mathbb{R})$ has three or more fixed points then it is the identity transformation and therefore fixes every point of $\mathbb{H}^2$.*

For a proof of this result see [And99].

To connect these together, let $M\ddot{o}b(\mathbb{H}^2)$ be the group of all Möbius transformations which map $\mathbb{H}^2$ onto itself. Then $M\ddot{o}b(\mathbb{H}^2)$ can be identified with $Isom^+(\mathbb{H}^2)$ which as we have seen can be identified with $PSL_2(\mathbb{R})$. Hence the elements of $PSL_2(\mathbb{R})$ give us the complete group of orientation preserving isometries for the upper half plane model. This completes the description of the model for $\mathbb{H}^2$.

2.1.2 The Upper Half Space Model

The *upper half space model* is the 3-dimensional analogue to the upper half plane model for $\mathbb{H}^2$. For completeness we define this model of hyperbolic 3-space.

Definition 2.1.7 *The upper half space model is the set*

$$\mathbb{H}^3 = \{(z,t) \in \mathbb{C} \times (0,\infty)\} = \mathbb{C} \times (0,\infty).$$

Visually we identify $\mathbb{C}$ with the xy-plane in $\mathbb{R}^3$ and then this model consists of all points above the xy-plane. (Note that the xy-plane itself is not included but forms part of the boundary at infinity.)

This model is equipped with the following hyperbolic metric

$$ds = \frac{\sqrt{|dz|^2 + dt^2}}{t}$$

As in the two dimensional model, to measure length in this space we integrate along the given path over the metric ds as given above. More precisely, let $f : I = [0,1] \to \mathbb{H}^3$ be piece-wise differentiable path then we define the hyperbolic length of f to be

$$length_{\mathbb{H}^3}(f) = \int_f ds.$$

The hyperbolic geodesics in this model, or paths of minimal length using the given hyperbolic metric, are either Euclidean lines or Euclidean semi-circles with centres on $\mathbb{C}$, which are perpendicular to $\mathbb{C}$. In the same way as in the 2-dimensional model, any two points can be joined by a unique hyperbolic geodesic. We define the hyperbolic distance in the same way (the infimum of the length over all paths between the two points) and as a consequence this is realized by the length of the hyperbolic line segment joining the two points.

By considering the 'endpoints' of the geodesics in $\mathbb{H}^3$, we see that the *boundary at infinity* can be identified with the Riemann sphere, $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

For this space it is also possible to define the geodesic planes or *hyperplanes* as either vertical Euclidean planes or hemi-spheres perpendicular to $\mathbb{C}$.

Considering the isometries of $\mathbb{H}^3$, denoted $Isom(\mathbb{H}^3)$, it is noted that they are generated by reflections in the geodesic planes of the model. More importantly from this we know that the orientation preserving isometries of $\mathbb{H}^3$, denoted by $Isom^+(\mathbb{H}^3)$, are generated by reflections in an even number of geodesic planes in $\mathbb{H}^3$.

These reflections can be extended onto the boundary at infinity in the following way. First look at how the geodesic planes in $\mathbb{H}^3$ intersect $\widehat{\mathbb{C}}$. A hyperplane in $\mathbb{H}^3$ is the intersection of either a sphere in $\mathbb{R}^3$ with centre in the xy-plane, or a vertical Euclidean plane in $\mathbb{R}^3$, with $\mathbb{H}^3$. The first case gives a circle in $\mathbb{C}$ and the second a line, which can be viewed as a circle through infinity. Hence both types of hyperplane give a circle in $\widehat{\mathbb{C}}$. Therefore $Isom(\mathbb{H}^3)$ extends maps from the Riemann sphere to itself which consist of compositions of reflections in circles in $\widehat{\mathbb{C}}$. Similarly $Isom^+(\mathbb{H}^3)$ extends maps which consist of compositions of reflections in an even number of circles in $\widehat{\mathbb{C}}$. The same is true in reverse. This extension of an element of $PSL_2(\mathbb{C})$ to an element of $M\ddot{o}b(\mathbb{H}^3)$ is called the *Poincaré extension.* (For more details on this see [MT98].)

Each element of $Isom^+(\mathbb{H}^3)$ can be expressed as a Möbius transformation, so has the form $\frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. Similar to the 2-dimensional case, this can be identified with $PSL_2(\mathbb{C})$. Hence we can view $PSL_2(\mathbb{C})$ as either the group of orientation preserving isometries of $\mathbb{H}^3$ or as the group of Möbius transformations of $\widehat{\mathbb{C}} = \partial_\infty(\mathbb{H}^3)$. In the same way as in $PSL_2(\mathbb{R})$, we can classify the elements of $PSL_2(\mathbb{C})$ by looking at the number of fixed points they have. They are also distinguished by the value of the square of the trace of the matrix T in $PSL_2(\mathbb{C})$. The only difference to the classification is that we have general loxodromic elements in $PSL_2(\mathbb{C})$ and not just hyperbolic elements.

A loxodromic element has two fixed points in the boundary of $\mathbb{H}^3$, (which can be joined by a hyperbolic geodesic called an axis), and $tr^2(T)$ has either non-zero imaginary part or it is real and lies in $(-\infty, 0) \cup (4, \infty)$. (Note the overlap with hyperbolic elements which have $tr^2(T) \in (4, \infty)$.)

2.1.3 Hyperbolic Trigonometry

To complete this section we now take a brief look at some hyperbolic trigonometry to highlight the points needed later. As in Euclidean geometry, a polygon is one of the basic objects in hyperbolic geometry. We will mainly be dealing with hyperbolic triangles, but we define a hyperbolic polygon as follows.

Definition 2.1.8 *A hyperbolic polygon is a closed convex set in the hyperbolic plane that can be expressed as the intersection of hyperbolic half planes, such that the vertices of the hyperbolic polygon do not accumulate (so the collection of half planes are locally finite).*

As an example of this a *hyperbolic triangle* is a hyperbolic polygon which can be realized as the intersection of three half-planes.

As in the case of a Euclidean triangle, there are trigonometric rules for hyperbolic triangles relating its interior angles to the hyperbolic lengths of its sides. These can be derived by linking Euclidean and hyperbolic distances between pairs of points and then making use of the Euclidean trigonometric rules (as the method of measuring angle is the same in both spaces).

Let T be a hyperbolic triangle with side lengths a, b, c and interior angles α, β, γ such that the side of length a is opposite angle α, the side of length b is opposite angle β and the side of length c is opposite angle γ. The following are three basic trigonometric rules in the hyperbolic plane.

The hyperbolic law of sines

$$\frac{sinh(a)}{sin(\alpha)} = \frac{sinh(b)}{sin(\beta)} = \frac{sinh(c)}{sin(\gamma)} \tag{2.1}$$

The hyperbolic law of cosines I

$$cosh(a) = cosh(b)cosh(c) - sinh(c)sinh(b)cos(\alpha) \tag{2.2}$$

The hyperbolic law of cosines II

$$cos(\gamma) = -cos(\alpha)cos(\beta) + sin(\alpha)sin(\beta)cosh(c) \tag{2.3}$$

Using these trig rules we can compare the lengths of the sides of a hyperbolic triangle and consider what happens when side length changes.

Proposition 2.1.9 *Let T be a hyperbolic triangle with side lengths a, b, c and interior angles α, β, γ as described. Let the sides of length a and b increase at the same rate (so $\frac{da}{dt} = \frac{db}{dt}$), and let the angle γ between them be fixed. Then the side of length c increases in length also.*

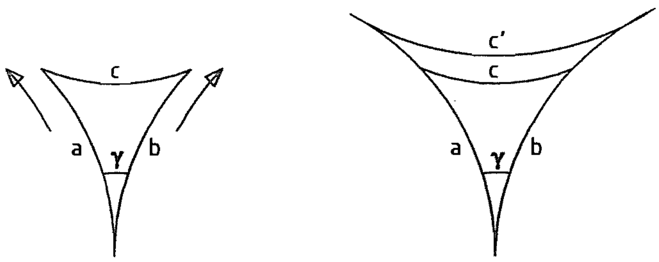


Fig. 2.1: hyperbolic triangles I

Proof: Let the sides of length a and b increase at the same rate (so assume $a, b \to \infty$ evenly, by which we mean that $\frac{da}{dt} = \frac{db}{dt}$ where t is measure of time), and keep the angle γ between them fixed. Then $sinh(a) \to \infty$ and $sinh(b) \to \infty$ (at the same rate).

We also know that α and β will lie between 0 and π (as they are angles in a standard hyperbolic triangle), and so $sin(\alpha) > 0$ and $sin(\beta) > 0$. Therefore,

$$\frac{sinh(a)}{sin(\alpha)} \to \infty$$

and similarly

$$\frac{sinh(b)}{sin(\beta)} \to \infty$$

Therefore by the hyperbolic law of sines (see equation 2.1),

$$\frac{sinh(a)}{sin(\alpha)} = \frac{sinh(c)}{sin(\gamma)} \longrightarrow \infty$$

As γ is fixed, $sinh(c) \longrightarrow \infty \Rightarrow c \longrightarrow \infty$. Therefore the side of length c must be increasing also. □

Note that we can get more from the proof of proposition 2.1.9

Corollary 2.1.10 *Let T be a hyperbolic triangle with side lengths a, b, c and interior angles α, β, γ as described. Let the sides of length a and b increase at the same rate (so $\frac{da}{dt} = \frac{db}{dt}$), and let the angle γ between them be fixed. Then angles β and α must decrease.*

Proof: As side lengths a and b increase at the same rate, the area of T is increasing. To see this note that at any point in time a piece has been added to T (see right hand picture of figure 2.1). The area of the new triangle will be equal to the area of T plus the area of this new piece. By the *Gauss-Bonnet formula*, $area(T) = \pi - (\alpha + \beta + \gamma)$. As $area(T)$ increases and γ is fixed, then either α or β (or both) must decrease. As a and b are increasing at the same rate then both α and β will decrease by the hyperbolic law of sines. □

We can say more than this.

Proposition 2.1.11 *Let T be a hyperbolic triangle with side lengths a,b,c and interior angles α, β, γ. Let the side of length a increase in length (so $a \longrightarrow \infty$) and let the side of length b be fixed and the angle γ between them be fixed. Assume that angle β (opposite the side of length b) be smaller than $\frac{\pi}{2}$. Then the side of length c increases.*

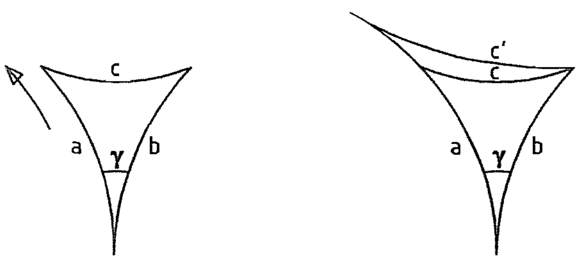


Fig. 2.2: hyperbolic triangles II

Proof: We have assumed $a \longrightarrow \infty$, which means that $sinh(a) \longrightarrow \infty$. As α will lie between 0 and π (as it is an interior angle of a hyperbolic triangle), then $0 < sin(\alpha) < 1$, and so

$$\frac{sinh(a)}{sin(\alpha)} \longrightarrow \infty$$

Hence by the hyperbolic law of sines (see equation 2.1),

$$\frac{sinh(a)}{sin(\alpha)} = \frac{sinh(c)}{sin(\gamma)} \longrightarrow \infty$$

As γ is fixed, this implies that $sinh(c) \longrightarrow \infty \Rightarrow c \longrightarrow \infty$ as required. □

Note that we assume that $\beta < \frac{\pi}{2}$ here so that c is always increasing. If $\beta \geq \frac{\pi}{2}$ then as the side of length a increases in length, the side of length c will decrease until $\beta = \frac{\pi}{2}$ and then increase after that. In the context in which we will be using this result, a will be increasing arbitrarily, and so this assumption is not necessary. We only require that c will ultimately increase (i.e. $c \longrightarrow \infty$ in the limit).

Hence if you have a hyperbolic triangle with one side length increasing arbitrarily and one side length fixed, and the angle between them fixed, then (assuming that the length changes enough) the remaining side length must will ultimately be increasing.

Proposition 2.1.12 *Let T be a hyperbolic triangle with side lengths a, b, c and interior angles α, β, γ as described. Let the sides of length a and b increase at the same rate (so $\frac{da}{dt} = \frac{db}{dt}$). Let the angle γ between them tend to an angle θ (for θ fixed such that $0 < \theta < \pi$). Then the side of length c increases in length also.*

Proof: As $a \longrightarrow \infty$ then $sinh(a) \longrightarrow \infty$. We know that $0 < \alpha < \pi$ (as α is an interior angle of a hyperbolic triangle) and so $0 < sin(\alpha) \leq 1$). Therefore (independent of what happens to α),

$$\frac{sinh(a)}{sin(\alpha)} \longrightarrow \infty$$

Hence, by the hyperbolic law of sines,

$$\frac{sinh(a)}{sin(\alpha)} = \frac{sinh(c)}{sin(\gamma)} \longrightarrow \infty$$

As $\gamma \longrightarrow \theta < \pi$, then γ is never equal to 0 or π, and so $sin(\gamma) \neq 0$ for all $\gamma \longrightarrow \theta$. Therefore $sinh(c) \longrightarrow \infty$ and hence $c \longrightarrow \infty$ as required.

□

These results will become useful in chapter 5.

2.2 General Kleinian Groups

In this section the aim is to consider the groups that can be associated to the two spaces already described in section 2.1. In the 2-dimensional case these are *Fuchsian Groups* and in the 3-dimensional case they are *Kleinian Groups.* Both groups display very similar properties which will be described here.

2.2.1 Fuchsian Groups

Fuchsian groups are a kind of group associated to $\mathbb{H}^2$, as they comprise of a particular group of isometries of the hyperbolic plane. They are also the fundamental groups of hyperbolic surfaces. Most of the material in this subsection may be found in [Kat92] and [JS87].

To define a Fuchsian group, we first require the following definitions.

Definition 2.2.1 *A subset of $\mathbb{H}^2$ is discrete if each point of the subset can be isolated from all other points in the subset. By this we mean that there exists an open neighbourhood around each point of the subset that does not contain any other point of the subset.*

A subgroup Γ of Möb($\mathbb{H}^2$) is discrete if the set $\Gamma(z) = \{\gamma(z)|\gamma \in \Gamma\}$ is discrete for every point $z \in \mathbb{H}^2$.

From section 2.1.1 we know that $M\ddot{o}b(\mathbb{H}^2)$ can be identified with $PSL_2(\mathbb{R})$, and we have the following definition for discreteness for a subgroup of this group.

Definition 2.2.2 *Let Γ be a subgroup of $PSL_2(\mathbb{R})$. We call Γ discrete if there does not exist a sequence $\{\gamma_n\}$ of distinct elements of Γ converging to the identity. This means that there does not exist a sequence $\left\{\gamma_n(z) = \frac{a_n z + b_n}{c_n z + d_n}\right\}$ of elements of $PSL_2(\mathbb{R})$ such that $a_n \to \pm 1$, $b_n \to 0$, $c_n \to 0$ and $d_n \to \pm 1$ as $n \to \infty$.*

We can now give the definition of a Fuchsian group in terms of discrete subgroups.

Definition 2.2.3 *A Fuchsian group is a discrete subgroup of the orientation preserving isometries of* $\mathbb{H}^2$. *Equivalently, a Fuchsian group is a discrete subgroup of* $PSL_2(\mathbb{R})$.

Example: The following are examples of Fuchsian groups

a) Hyperbolic cyclic subgroups generated by $z \longrightarrow \lambda z$ $(\lambda > 0)$. These subgroups consist of only hyperbolic elements and the identity.

b) Parabolic cyclic groups generated by a parabolic element, for example the standard form $z \longrightarrow z + 1$.

c) Elliptic cyclic groups, these subgroups are generated by an elliptic element and are Fuchsian groups if and only if they are finite. (For proof of this see [[JS87] section 5.7].)

These first three examples are elementary Fuchsian groups, which means that the limit set $\Lambda(\Gamma)$ of the Fuchsian group Γ consists of at most two points. (The limit set is defined to be the set of limit points - see definition 2.2.6.) The next example is non-elementary.

d) The Modular group

$$PSL_2(\mathbb{Z}) = \left\{ \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

is a Fuchsian group. This follows from the fact that $\mathbb{Z}$ is discrete in $\mathbb{R}$, which implies that $SL_2(\mathbb{Z})$ is a discrete subgroup of $SL_2(\mathbb{R})$ and consequently $PSL_2(\mathbb{Z})$ is a discrete subgroup of $PSL_2(\mathbb{R})$ as required.

Now consider the action of a Fuchsian group on $\mathbb{H}^2$. Fuchsian groups do not necessarily behave discontinuously in the normal sense. By 'discontinuously in the normal sense' we mean that every point of $\mathbb{H}^2$ has a neighbourhood which is carried off itself by all elements of the group except for the identity. (This definition comes from [JS87] page 232. They look at lattices, which have this discontinuity property.) In particular if a Fuchsian group contains elliptic elements then these have fixed points in $\mathbb{H}^2$ around which the element acts as a rotation. Therefore these fixed points cannot have such a neighbourhood required for discontinuity. However there does exist a 'weaker' notion of discontinuity that can be applied to Fuchsian groups.

Definition 2.2.4 *Let Γ be a subgroup of $PSL_2(\mathbb{R})$. We say Γ acts properly discontinuously on $\mathbb{H}^2$ if, for every compact subset K in $\mathbb{H}^2$, the set $\{\gamma \in \Gamma | \gamma(K) \cap K \neq \emptyset\}$ is finite.*

This is a property that all Fuchsian groups have irrespective of what type of elements they contain. In fact this is a necessary and sufficient condition for a group to be Fuchsian as detailed in the following theorem (found in [[Kat92] page 32]).

Theorem 2.2.5 *If $\Gamma \subset PSL_2(\mathbb{R})$, then Γ is a Fuchsian group if and only if Γ acts properly discontinuously on $\mathbb{H}^2$.*

(For the proof see [Kat92].)

Theorem 2.2.5 gives another way of defining a Fuchsian group as one that acts properly discontinuously on $\mathbb{H}^2$. Considering the way in which a Fuchsian group acts on $\mathbb{H}^2$ gives another important set.

Definition 2.2.6 *Let $z \in \mathbb{H}^2$ and let $\{\gamma_n\}$ be a sequence of distinct elements in a Fuchsian group $\Gamma \subset PSL_2(\mathbb{R})$. Then if $\{\gamma_n(z)\}$ tends towards a point α, this is called a limit point. The set of all possible limit points is called the limit set of Γ and denoted $\Lambda(\Gamma)$.*

Hence for a Fuchsian group, $\Lambda(\Gamma) \subseteq \mathbb{R} \cup \{\infty\} = \widehat{\mathbb{R}}$, and so from definition 2.1.2 the limit set is a subset of $\partial_\infty(\mathbb{H}^2)$.

We know more than this. The appearance of the limit set of a Fuchsian group depends upon how many points it contains. The details are given in the following two theorems.

Theorem 2.2.7 *If $\Lambda(\Gamma)$ contains more than one point then it is the closure of the set of fixed points of the hyperbolic transformations of Γ.*

Theorem 2.2.8 *If $\Lambda(\Gamma)$ contains more than two points, then either*

- $\Lambda(\Gamma) = \mathbb{R} \cup \{\infty\} = \partial_\infty(\mathbb{H}^2)$ *(hence a circle) or,*
- $\Lambda(\Gamma)$ *is a perfect nowhere dense subset of* $\partial_\infty(\mathbb{H}^2)$ *(hence a Cantor set).*

Here perfect means that every point of $\Lambda(\Gamma)$ is a limit point of the points in the set $\Lambda(\Gamma)$.

See [[Kat92] pages 65-67] for complete proofs for both of these theorems.

2.2.2 Kleinian Groups

To define a Kleinian group we need to extend the ideas of discreteness and the group being properly discontinuous given in section 2.2.1. The definition of a properly discontinuous subgroup of $PSL_2(\mathbb{R})$ extends directly to $PSL_2(\mathbb{C})$, so definition 2.2.4 applies here.

The definition of discreteness also extends to $PSL_2(\mathbb{C})$, but we give an equivalent definition here.

Definition 2.2.9 *Let Γ be a subgroup of $PSL_2(\mathbb{C})$, then Γ is discrete if there does not exist a sequence $\{\gamma_n\}$ of distinct elements of Γ converging to γ for any $\gamma \in PSL_2(\mathbb{C})$.*

Lemma 2.2.10 *Let Γ act on $\mathbb{H}^3$, then the notions of discrete and properly discontinuous are equivalent.*

Note that this equivalence does not completely extend to $\widehat{\mathbb{C}}$. Here all subgroups that act properly discontinuously are discrete, but there are examples that show the converse is false. For example the group $PSL_2(\mathbb{Z}[i])$ is a subgroup of $PSL_2(\mathbb{C})$ which is discrete but not properly discontinuous anywhere on $\widehat{\mathbb{C}}$.

Definition 2.2.11 *A Kleinian Group Γ is a discrete subgroup of $PSL_2(\mathbb{C})$.*

Referring back to the description of $\mathbb{H}^3$ in subsection 2.1.2, it is clear that this is equivalent to saying that a Kleinian group is a discrete subgroup of the orientation preserving isometries of $\mathbb{H}^3$. With the equivalent notions of discreteness and acting properly discontinuously on $\mathbb{H}^3$, it allows a third definition of a Kleinian group. (As given in [[MT98] page 26].)

Definition 2.2.12 *A subgroup* Γ *of Isom*$^+(\mathbb{H}^3)$ *is called a Kleinian group if* Γ *acts properly discontinuously on* $\mathbb{H}^3$.

The elements of a Kleinian group can be conjugated into standard forms for classification (in the same way as in definition 2.1.4). They may also be classified by their fixed points.

Lemma 2.2.13 *An element of a Kleinian group has finite order if and only if it has a fixed point in* $\mathbb{H}^3$.

Equivalently an element of a Kleinian group has finite order if and only if it is elliptic. Visually this is clear as elliptics act as rotations around their fixed points, whereas the parabolics and loxodromics act as translations and dilations.

Definition 2.2.14 *A Kleinian group* Γ *is torsion free if it has no elements of finite order other than the identity.*

The types of surfaces and 3-manifolds we will be considering have torsion-free fundamental groups. Hence they do not contain any elliptic elements (or equivalently any fixed points in $\mathbb{H}^3$).

2.2.3 Action of Γ on $\widehat{\mathbb{C}}$

A Kleinian group Γ acts on $\widehat{\mathbb{C}}$ by splitting it into two parts called the *limit set* and the *domain of discontinuity.*

Definition 2.2.15 *Let* $\Omega(\Gamma)$ *be the set of all points* $z \in \widehat{\mathbb{C}}$ *such that there exists a neighbourhood* U *of* z *so that* $\gamma(U) \cap U \neq \emptyset$ *for only finitely many* $\gamma \in \Gamma$. *We call* $\Omega(\Gamma)$ *the domain of discontinuity of* Γ.

Hence $\Omega(\Gamma)$ is the largest open set in $\widehat{\mathbb{C}}$ on which Γ acts properly discontinuously.

This set $\Omega(\Gamma)$ is open and in general will have many connected components. Since $\Omega(\Gamma)$ is open there are at most countably many connected components, and in fact it is known that $\Omega(\Gamma)$ has either 0, 1, 2 or countably infinitely many components. As a subset of $\widehat{\mathbb{C}}$ it is either dense or empty.

In the case when $\Omega(\Gamma)$ is empty, Γ is called a *Kleinian group of the first kind* and otherwise Γ is called a *Kleinian group of the second kind.*

For a Kleinian group Γ, consider the orbit of any point $p \in \widehat{\mathbb{C}}$ under the action of the group. By this we mean consider the set,

$$\Gamma(p) = \{\gamma(p) | \gamma \in \Gamma\}.$$

We know that Γ acts properly discontinuously on $\mathbb{H}^3$, and so has accumulation points only in $\widehat{\mathbb{C}}$. (This is because the starting point p is in $\widehat{\mathbb{C}}$, and therefore any other point in the sequence will remain in $\widehat{\mathbb{C}}$ under the action of Γ.)

Definition 2.2.16 *The accumulation points described above are called limit points of* Γ*, and the set of all these points is called the limit set of* Γ*, and is denoted by* $\Lambda(\Gamma)$*.*

Note that from definition 2.2.16, the limit set of a Kleinian group appears to depend upon a base point p, but this is not actually the case.

Lemma 2.2.17 *For any two points* p *and* $p' \in \mathbb{H}^3$*, the set of accumulation points of* $\Gamma(p)$ *and* $\Gamma(p')$ *are the same, and so the limit set does not depend on the choice of base point.*

(For proof of the above see [[MT98] page 41].)

From definition 2.2.16 it is clear that we know some of the points which must be contained in $\Lambda(\Gamma)$. Let $Fix(\gamma)$ be the set of points in $\widehat{\mathbb{C}}$ fixed by an element $\gamma \in Isom^+(\mathbb{H}^3)$. Then by definition 2.1.4, if γ is a loxodromic or parabolic element then $Fix(\gamma) \subset \Lambda(\Gamma)$. We know more than this.

Theorem 2.2.18 *Let* Γ *be a Kleinian group, then* $\widehat{\mathbb{C}}$ *is the disjoint union of* $\Lambda = \Lambda(\Gamma)$ *and* $\Omega = \Omega(\Gamma)$*.*

(For the proof of theorem 2.2.18 see [Mas88].)

As a corollary to the above, observe the following.

Corollary 2.2.19 *Let* Γ *be a Kleinian group, then* $\Lambda(\Gamma) = \widehat{\mathbb{C}} - \Omega(\Gamma)$.

By definition, $\Lambda(\Gamma)$ is a closed set of points. From the comments made about $\Omega(\Gamma)$ and corollary 2.2.19 we know that $\Lambda(\Gamma)$ is either nowhere dense or everything.

2.2.4 Convex Hulls and Convex Cores

In this subsection we define sets associated to $\Lambda(\Gamma)$.

Definition 2.2.20 *The convex hull of a Kleinian group* Γ*, denoted* $CH(\Gamma)$*, is the smallest non-empty closed convex subset of* $\mathbb{H}^3$ *that is invariant under* Γ.

For a more visual description take pairs of points in $\Lambda(\Gamma)$ and join them by hyperbolic lines with these end points. Then $CH(\Gamma)$ of $\Lambda(\Gamma)$ is the smallest convex set containing all of the hyperbolic lines.

Definition 2.2.21 *The convex core of* $\mathbb{H}^3/\Gamma$*, denoted by* $C(\Gamma) = CH(\Gamma)/\Gamma$*, is the smallest convex submanifold of* $\mathbb{H}^3/\Gamma$ *which has a fundamental group isomorphic to* $\pi_1(\mathbb{H}^3/\Gamma)$.

The convex core of a hyperbolic manifold contains all of its closed geodesics.

2.2.5 Finiteness conditions

To close this chapter we are going to highlight some of the useful properties that a Kleinian group can possess. These properties make the group easier to handle and to understand in many cases. For more details on these properties see [MT98] and [Mas88].

Kleinian groups were originally studied because of their connections to Riemann surfaces, and then later with hyperbolic 3-manifolds. More details about hyperbolic manifolds (including definitions) may be found in the next chapter and the references given there. For now we are going to define some manifolds to which we directly relate Kleinian groups, and which we need in order to discuss the finiteness conditions.

Definition 2.2.22 *Let Γ be torsion-free Kleinian group. The complete hyperbolic 3-manifold associated to Γ is the quotient space $\mathbb{H}^3/\Gamma$ with the quotient topology.*

Definition 2.2.23 *Let Γ be a torsion-free Kleinian group. The (possibly disconnected) Riemann surface associated to Γ is the surface $\Omega(\Gamma)/\Gamma$.*

The quotient surface $\Omega(\Gamma)/\Gamma$ has a complex structure induced from $\Omega(\Gamma)$ and hence $\Omega(\Gamma)/\Gamma$ is a countable union of Riemann surfaces lying at infinity of the complete hyperbolic manifold $\mathbb{H}^3/\Gamma$.

Definition 2.2.24 *Let Γ be a torsion-free Kleinian group. The topological manifold associated to Γ is the space $(\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$ (possibly with boundary). This is called the Kleinian manifold, and its interior $\mathbb{H}^3/\Gamma$ admits a hyperbolic structure.*

Note that if $\Omega(\Gamma)$ is empty (and so Γ is a Kleinian group of the first kind) the Kleinian manifold is just equal to the complete hyperbolic manifold $\mathbb{H}^3/\Gamma$.

Definition 2.2.25 *A Riemann surface S is analytically finite if it has finite topological type. This means that the surface is closed (compact without boundary) except for a finite number of punctures.*

We say that a non-elementary Kleinian group Γ is analytically finite if the Riemann surface, $\Omega(\Gamma)/\Gamma$, associated to Γ is analytically finite. This means that the space consists of a finite number of surfaces each of which is of finite genus with only a finite number of punctures. Equivalently Γ is analytically finite if area$(\Omega(\Gamma)/\Gamma) < \infty$.

(Note that here the area of $\Omega(\Gamma)/\Gamma$ is induced by the hyperbolic area on $\Omega(\Gamma) \subset \widehat{\mathbb{C}}$.)

Ahlfors proved that if Γ was finitely generated then it is analytically finite. The converse is not true. For a counterexample see 'Killing a component' in [[Mas88] page 175]. For a detailed discussion on Ahlfors finiteness theorem see [[MT98] sections 4.1 and 4.2].

By considering the fundamental polyhedron of Γ the next property can be considered.

Definition 2.2.26 *A Kleinian group Γ is geometrically finite if it has a fundamental polyhedron bounded by a finite number of convex sides.*

In the 2-dimensional case algebraic finiteness (by which we mean that Γ is finitely generated) is equivalent to geometric finiteness, but the concepts are not coincident in the 3-dimensional case. For more information on finitely generated Kleinian groups and their geometric properties see [[MT98] chapter 4, pages 102-130].

Chapter 3

Hyperbolic manifolds

The focus for this chapter is to provide the background to some basic properties of hyperbolic 2- and 3-manifolds. The initial section looks at hyperbolic surfaces, giving basic definitions and useful properties including the decomposition into pairs of pants by the *pants decomposition*. The most important result is the *collar lemma*, which is vital to later chapters. The other two sections concentrate on hyperbolic 3-manifolds, giving some particular examples and their properties, including the *Book of I-bundles* which is the manifold that is central to later chapters.

The material used in this chapter may be found in [Bus92], [Hem76], [MT98] and [Rat94]. More on Riemann surfaces may be found in [JS87].

3.1 Hyperbolic Surfaces

Primarily the interest is in hyperbolic surfaces (these are surfaces which have the hyperbolic plane as their universal cover). Initially here though we introduce Riemann surfaces and will then go on to look at some universal properties of hyperbolic surfaces.

Before hyperbolic 3-manifolds were studied extensively by Thurston, the study of Kleinian groups was important because of their connection to Riemann surfaces. We have already encountered the Riemann sphere $\widehat{\mathbb{C}}$, which is a particular example of a Riemann surface. The following gives the general definition.

Definition 3.1.1 *A Riemann surface R is a connected complex 1-manifold. By this we mean a connected Hausdorff space R where there exists a family of sets and maps (ϕ_j, U_j), for j = 1, 2, ..., called an atlas which satisfies the following conditions.*

- $\{U_j | j = 1, 2, ...\}$ *is an open cover of R (where U_j is an open subset in R).*
- *Each ϕ_j is a homeomorphism of U_j onto an open subset of the complex plane.*
- *If $U = U_i \cap U_j \neq \emptyset$ then $\phi_i \circ \phi_j^{-1} : \phi_j(U) \to \phi_i(U)$ is an analytic map.*

To clarify the above definition, the first condition states that the surface R is covered by a collection of open sets, such that each is homeomorphic to open subsets of $\mathbb{C}$ (by the second condition). It is possible that two of these open sets could overlap, and the homeomorphisms corresponding to these particular open sets are related by an analytic homeomorphism (as given in the third condition).

By definition 3.1.1, a Riemann surface does not have a boundary. A Riemann surface which is homeomorphic to a compact 2-manifold without boundary is *closed*. We call a Riemann surface that is homeomorphic to the interior of a compact 2-manifold with or without boundary *topologically finite*.

Each end of a topologically finite Riemann surface has a regular neighbourhood which is conformally equivalent to either a punctured disc or an annulus. If the neighbourhood of the end c is conformally equivalent to an annulus then we say that c bounds a disc or corresponds to a removed disc or *hole* in the surface. If the neighbourhood of the end c is conformally equivalent to a punctured disc, then we say that c corresponds to a *puncture* on the surface or that the surface is punctured at c. There does not exist a holomorphic homeomorphism from a punctured disc onto any annulus. If R is a topologically finite (possibly disconnected) Riemann surface, then R is analytically finite if each end of R has a regular neighbourhood conformally equivalent to the punctured disc.

Riemann surfaces, and in particular hyperbolic surfaces, are completely classified as detailed in the following *Uniformization theorem.* This theorem is stated in most of the references given at the start of this chapter, and may also be found in [And99].

Theorem 3.1.2 *Any Riemann surface R has the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the complex plane $\mathbb{C}$ or the upper half plane $\mathbb{H}^2$ as its universal covering.*

Hence every simply connected Riemann surface is conformally equivalent to either the sphere, the complex plane, or the hyperbolic plane. If R has the upper half plane $\mathbb{H}^2$ as its covering surface, then we can represent it as $\mathbb{H}^2/F$, where F is a Fuchsian group that is isomorphic to the fundamental group of R. When a Riemann surface is represented in this form, the surface inherits a natural hyperbolic structure from the covering $\mathbb{H}^2$, and can therefore be regarded as a complete 2-dimensional hyperbolic manifold or hyperbolic surface.

One way to consider hyperbolic surfaces is by decomposing them into smaller pieces. One such decomposition, called the *pants decomposition* (as given in definition 3.1.3 below), breaks the hyperbolic surface into genus 0 surfaces with three geodesic boundaries. These 'pieces' are called *pairs of pants* (or a three-holed sphere). These can be obtained by gluing together two copies of the same right-angled geodesic hexagon in the hyperbolic plane along every other side. Further descriptions of the pants decomposition may be found in [Bus92] and [Pau99].

Definition 3.1.3 *The pants decomposition of a compact surface S is a decomposition of the surface along simple closed curves into three-holed spheres.*

Such a decomposition is determined by a choice of a maximal collection P of simple closed curves on S such that

- *the elements of P are pair-wise disjoint*
- *each element α of P is homotopically essential and non-peripheral (so α is not homotopic into the boundary of S)*
- *no two elements of P are freely homotopic.*

In basic terms the pants decomposition involves cutting the surface along a collection of disjoint closed geodesics (defined according to the hyperbolic metric on the surface) until the only simple closed geodesics left are homotopic to the existing collection of closed geodesics.

Pairs of pants are the building blocks for all compact Riemann surfaces of genus greater than one, (see [Bus92]). Given any three lengths for the boundary geodesics, a unique pair of pants is determined, and all pairs of pants may be found in this way. Hence pairs of pants are uniquely determined by the lengths of their three boundary geodesics. (See [Pau99] and [[Bus92] theorem 3.1.7 page 65].) By the nature of the decomposition this set of three boundary geodesics will be equivalent to two or three simple closed curves on our hyperbolic surface (the number depends on whether the curve in the original surface is separating or non-separating). As pairs of pants are uniquely defined by their boundary geodesics, to understand more about them and the decomposition we look at what is known about simple closed geodesics in hyperbolic 2-manifolds. The following lemma, with its proof, can be found in [Pau99].

Lemma 3.1.4 *The free homotopy class of any homotopically non trivial simple loop in a hyperbolic surface contains a unique simple closed geodesic.*

(Note that this lemma is true for all Riemann surfaces and is stated in all generality in [Pau99].)

To put this into context, it is along a set of these unique geodesics that the hyperbolic surface is cut to get the pairs of pants in the decomposition. There is always more than one way to perform a pants decomposition for a particular compact hyperbolic surface, but what is common for all decompositions is the number of pieces the surface is decomposed into and the number of geodesics along which the surface is cut. It can be shown that a orientable, closed surface of genus g without boundary always contains $3g - 3$ disjoint closed curves along which to cut, and the surface decomposes into exactly $2g - 2$ pairs of pants. (The proof of this may be found in [Bus92].)

As we may apply the pants decomposition to any Riemann surface, it is possible to apply it in reverse and build all hyperbolic surfaces by gluing together pairs of pants along their boundary geodesics. Hence these boundary geodesics determine the complex structure of the entire surface, therefore parameterizing the hyperbolic surface using the pants decomposition. We know that we have $3g - 3$ geodesic length functions (one for each closed curve along which we cut), but we also have to consider how the pairs of pants are attached to one another. They can be glued together with any amount of twisting, so there will be $3g - 3$ 'twisting' parameters (which equate to the angle of the twist). To perform a twist, remove a collar around a simple closed geodesic that forms the boundary between two of the constituent pairs of pants. Then glue the collar back into the surface after rotating one of its boundary components by some angle.

Hence, all hyperbolic surfaces are uniquely determined by the lengths of the boundary components of their constituent pairs of pants (giving $3g - 3$ non-negative real numbers), as well as the angles of the twists between glued pairs of pants (giving a further $3g - 3$ real parameters). These $6g - 6$ lengths and angles which parameterize the hyperbolic surface are known as *Fenchel-Nielsen coordinates.* We will use the length parameters later on. In particular utilizing the fact that we can change the length of one of these geodesics whilst keeping the others fixed and still have a hyperbolic structure on the surface. In using this fact we note that we are able to deform an arbitrary curve on a hyperbolic surface by altering the lengths of the curves in the pants decomposition that it intersects. This uses the fact that the pattern of crossings over the pants decomposition does not change (for an arbitrary geodesic) as we change the lengths of the pants curves. This becomes particularly useful in section 5.5, when considering surfaces with boundary.

(We will not explicitly use the twist parameters, but more detail may be found in [[Bus92] pages 69-75]. For more on the pants decomposition and Fenchel-Nielsen coordinates see [Bus92], [Pau99] and [Mas01].)

As an aside here, note that although only compact hyperbolic surfaces have been discussed, the pants decomposition can be performed on non-compact hyperbolic surfaces. When considering non-compact surfaces a new type of neighbourhood occurs called a *cusp.* A cusp is an end of the hyperbolic surface which corresponds to a parabolic element of the fundamental group. (See [[MT98] pages 5 and 6].) One of the assumptions made in the work which follows is that the group does not contain parabolics, so we do not need to deal with cusps.

There exist several universal properties that hold for every hyperbolic surface. One such fact is that in the hyperbolic world things are curved in an opposite manner in transversal directions. This feature is highlighted in the following collar lemma, which essentially says that around short geodesics there exist long tubular neighbourhoods called *collars* whose width solely depends upon the length of the geodesic.

Lemma 3.1.5 *Let $m(l) = arcsinh\left(\frac{1}{sinh(\frac{l}{2})}\right)$ which tends to ∞ as $l \to 0$ (and which tends to 0 as $l \to \infty$). Then for a simple closed geodesic α of length l in an arbitrary hyperbolic surface R, the set $c(\alpha)=\{p \in R | d(p,\alpha) < m(l)\}$ is an embedded annular neighbourhood of α.*

The proof is not included here, but a full detailed proof using the pants decomposition can be found in [Bus92]. There are many references for the collar lemma. A statement of it can be found in [MT98] and a different proof can be found in [Hal81].

The collar lemma will become useful later when considering what happens on certain surfaces as lengths of curves change.

3.2 Hyperbolic 3-manifolds

The purpose of this section is to introduce 3-dimensional hyperbolic manifolds, and to see how they can be expressed as quotient spaces of $\mathbb{H}^3$ by a Kleinian group.

A *hyperbolic 3-manifold M* is a space which is locally modeled on $\mathbb{H}^3$. This means that in a small neighbourhood of a point on M it looks and behaves like $\mathbb{H}^3$. The precise definition of a hyperbolic 3-manifold follows, and is similar to definition 3.1.1 of a Riemann surface.

Definition 3.2.1 *A connected Hausdorff space M is called a hyperbolic 3-manifold if it has a family (U_j, ϕ_j), for j = 1, 2, ..., which satisfies the following conditions*

- *Each U_j is an open subset of M and $\{U_j\}$ covers M.*
- *Each ϕ_j is a homeomorphism of U_j onto an open subset of $\mathbb{H}^3$.*

- *If $U = U_i \cap U_j$ is non-empty, then it is connected and $\phi_i \circ \phi_j^{-1} : \phi_j(U) \to \phi_i(U)$ is an orientation preserving diffeomorphism which preserves the hyperbolic metric.*

For any complete hyperbolic 3-manifold M, we have a torsion-free Kleinian group Γ such that $M = \mathbb{H}^3/\Gamma$. Any such Γ is unique up to conjugation by elements of $Isom(\mathbb{H}^3)$. Conversely, for any torsion-free Kleinian group Γ, the manifold $M = \mathbb{H}^3/\Gamma$ is a complete hyperbolic 3-manifold.

As mentioned in section 2.2.5, there are two 3-manifolds associated to a torsion-free Kleinian group Γ, namely the hyperbolic manifold $\mathbb{H}^3/\Gamma$ as detailed above, and the topological manifold (possibly with boundary), namely $(\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$. The latter is the Kleinian manifold and its interior $\mathbb{H}^3/\Gamma$ admits a hyperbolic structure. Another way to view the hyperbolic manifold $\mathbb{H}^3/\Gamma$ is by constructing it from a fundamental polyhedron by pasting its sides according to the side-pairing transformations.

The manifolds of interest here are hyperbolizable 3-manifolds. These are defined to be 3-manifolds whose interior admits a hyperbolic structure or can be written as $\mathbb{H}^3/\Gamma$ where Γ is a torsion-free Kleinian group.

Note that the hyperbolic 3-manifolds given here and in the previous chapter can also be defined in terms of Kleinian groups with torsion. In this situation the manifold is not necessarily smooth and the quotient is an orbifold. For more details see [MT98] and [Rat94].

3.3 Properties of hyperbolic 3-manifolds

In this final section of chapter 3 the focus is to describe some properties that hyperbolic 3-manifolds may possess. Much of what is given here may be found in [Hem76] and [MT98].

First we formally define a hyperbolizable 3-manifold as in section 3.2.

Definition 3.3.1 *A compact 3-manifold M is hyperbolizable if there exists a Kleinian group Γ so that M is homeomorphic to $(\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$, or alternatively M is uniformized* *by Γ. These 3-manifolds have an interior that admits a hyperbolic structure and $int(M) = \mathbb{H}^3/\Gamma$.*

In this work it will be assumed that the 3-manifolds are hyperbolizable and hence it is possible to put a hyperbolic structure on the interior. The following set of definitions give properties of embedded surfaces in hyperbolizable 3-manifolds.

Definition 3.3.2 *A surface S in a compact hyperbolizable 3-manifold M is properly embedded if S is compact and orientable and if either S is contained in ∂M or $S \cap \partial M = \partial S$.*

Definition 3.3.3 *Let S be an embedded, orientable and compact surface in a compact hyperbolizable 3-manifold M (possibly with boundary), such that S is properly embedded in M. If S satisfies one of the following conditions then S is incompressible:*

- *S is a topological sphere which does not bound a ball*
- *S is a topological disk whose boundary is a non-trivial simple closed curve in the boundary of M*
- *S is a surface other than a sphere or a disk such that the homomorphism between fundamental groups induced by the inclusion map is injective.*

Otherwise it is compressible.

Definition 3.3.4 *A surface S is two-sided in a hyperbolizable 3-manifold M if there is an embedding $h : S \times [-1, 1] \to M$ such that;*

- *$h(x, 0) = x$ for any $x \in S$, and*
- *$h(S \times [-1, 1]) \cap \partial M = h(\partial S \times [-1, 1])$.*

The following definition describes a property which all hyperbolizable 3-manifolds have.

Definition 3.3.5 *We say that an orientable, compact, hyperbolizable 3-manifold M is irreducible if every embedded 2-sphere bounds a 3-ball in the manifold M.*

In particular, if M is an irreducible 3-manifold which is not a 3-ball, then every component of ∂M must have positive genus. As Γ acts freely on the space $\mathbb{H}^3 \cup \Omega(\Gamma)$ by orientation-preserving homeomorphisms, the Kleinian manifold $(\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$ is irreducible. For a proof of this see [[MT98] page 64]. Hence a hyperbolizable 3-manifold is necessarily irreducible.

As a corollary to this definition, note that if a topological 3-manifold M is irreducible then $\pi_2(M)$ is trivial. This follows directly from the sphere theorem. (We have not included this theorem in our discussion, but a statement may be found in [MT98] theorem 2.37.)

Definition 3.3.6 *A compact irreducible 3-manifold M is Haken if it contains a two-sided incompressible surface.*

It is known that a compact hyperbolizable 3-manifold with non-empty boundary is Haken. For a proof of this see [[Hem76] Lemma 6.8].

Definition 3.3.7 *We say that a topological 3-manifold M is aspherical if $\pi_2(M)$ is trivial.*

It follows from this that the Kleinian manifold is aspherical, and as any hyperbolizable 3-manifold is necessarily irreducible they are also aspherical. As a corollary to this it is known that as $\pi_2(M)$ is trivial then $\pi_i(M)$ for $i \geq 2$ is also trivial for these 3-manifolds.

Definition 3.3.8 *We say that a topological 3-manifold M is atoroidal if all embedded incompressible tori are peripheral.*

Peripheral means that the inclusion map of the torus T into the 3-manifold M is homotopic to a map $f : T \to M$ for which $f(T) \subset \partial M$, so any torus can be moved to the boundary of M by homotopy. (This definition holds for any embedded surface.)

A surface S in a compact hyperbolizable 3-manifold is essential if it is properly embedded, incompressible and non-peripheral.

Definition 3.3.9 *Let S be a properly embedded annulus in a 3-manifold M. Then S is essential if it is incompressible and not homotopic into the boundary of M.*

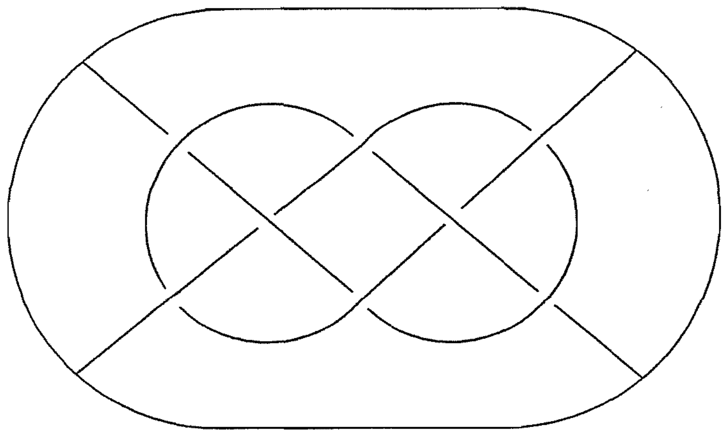
With all these properties in place, we complete this section by giving two examples of hyperbolizable 3-manifolds. They can be considered 'extreme' cases in terms of the number of essential annuli they contain.

Example:

- Let $F_1, F_2, ..., F_n$ be a collection of compact, orientable surfaces of positive genus with connected non-empty boundary. Let $B_i = F_i x I$ and let $\partial_0 B_i$ be the annulus $\partial F_i \times I$. Let T be a solid torus and let A_i be a family of disjoint parallel closed annuli on ∂T homotopically equivalent to the core curve on T. Form a manifold M from T and $\{B_i\}$ by identifying the boundary $\partial_0 B_i$ with A_i for all i by an orientation-reversing homeomorphism. M is called a *Book of I-bundles.* A Book of I-bundles contains lots of essential annuli. This manifold will be considered in further detail from chapter 5 onwards.

- Let M be a compact hyperbolizable 3-manifold with incompressible boundary, and let $S_1, ..., S_p$ be a collection of components of ∂M. The subset $S = S_1 \cup ... \cup S_p$ of ∂M is *an-annular* if each S_i is incompressible and if there does not exist an embedded essential annulus in M both of whose boundary curves lie in S. A compact hyperbolizable 3-manifold M is *acylindrical* if its entire boundary ∂M is an-annular.

  Acylindrical hyperbolizable 3-manifolds contain no essential annuli. A specific example of one of these manifolds is called the *Tripos Link* (and is equivalent to $S^3 - N(G)$ for G a specific graph). This particular example is easier to draw than explain without a picture. (See figure 3.1 below.) Such a family of examples are constructed by Paoluzzi and Zimmermann in [PZ96].



$S^3 - N(G)$

Fig 3.1: The Tripos Link

Chapter 4

Characters of Curves

The aim of this chapter is to lay the foundations and set the scene for the character problem. In the first section we introduce the idea of the character of an element of a group as given by Horowitz in [Hor72], including some character relations. We also look at the particular problem of determining when two elements have the same character. In section two we look at what is known in connection to this in terms of free groups. Although it appears to be an algebraic concept as it relates to representations of groups in $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$, there are connections to the geometrical world. In the third section this link is described in detail, which provides the information needed for McShane's result (as given in [McS93]) for characters of curves on surfaces as given in section 4.4. We close this chapter by looking at other questions with regard to characters of curves on surfaces. Most of this material may also be found in [And03] and [Hor75].

4.1 Characters

We begin by giving some definitions, including that of the character of an element of a group.

Let G be a finitely generated, torsion free group. (We assume G is torsion-free as the focus is on fundamental groups of hyperbolic 2- and 3-manifolds.)

Let $\mathcal{F}(G) = \{\rho : G \to SL_2(\mathbb{C}) \text{ faithful}\}$ be the space of all injective homomorphisms from G into $SL_2(\mathbb{C})$, where $SL_2(\mathbb{C})$ is the group of 2×2 matrices with complex entries and determinant 1.

Definition 4.1.1 *For an element ω of G, the character associated to ω is the function $\chi[\omega] : \mathcal{F}(G) \to \mathbb{C}$ given by $\chi[\omega](\rho) = tr(\rho(\omega))$, where $\rho(\omega)$ is the 2×2 matrix in $SL_2(\mathbb{C})$ that represents ω and tr is the usual 2×2 matrix trace.*

The following character relations come directly from definition 4.1.1 and the trace relations of 2×2 matrices with determinant one.

Proposition 4.1.2 *From definition 4.1.1 we have the following character relations.*

1) An element ω of G and its inverse ω^{-1} determine equal characters, so $\chi[\omega] = \chi[\omega^{-1}]$

2) For elements ν, ω of G, $\chi[\omega] = \chi[\nu\omega\nu^{-1}]$

3) For elements ν, ω of G, $\chi[\omega\nu] = \chi[\omega]\chi[\nu] - \chi[\omega\nu^{-1}]$.

Proof: Let A and B be arbitrary 2×2 matrices in $SL_2(\mathbb{C})$. Then the following trace relations can be verified by direct calculation.

1) $tr(A) = tr(A^{-1})$

2) $tr(A) = tr(BAB^{-1})$

3) $tr(AB) = tr(A)tr(B) - tr(AB^{-1})$

These combined with definition 4.1.1 establish the character relations. □

One possible problem is to try and determine when two distinct non-conjugate elements in the group G have the same character. An approach to this would be to use the character relations given in proposition 4.1.2 on distinct elements to show that they have the same (or different) characters.

If we consider $F_2 = free(a, b)$ (the free group on two elements), then we have the following as given in [Hor72].

Proposition 4.1.3 *Let $w = w(a, b)$ be any element of F_2. Then $\chi[w(a, b)] = P(\chi[a], \chi[b], \chi[ab])$, where P is a unique polynomial.*

This can be proven using the character relations of proposition 4.1.2. Horowitz proves this for a free group F_n on n elements in [Hor72].

With this in mind, the following example takes two non-conjugate elements of F_2, and uses the relations from proposition 4.1.2 to show they have the same character.

Example: Let $G = F_2 = free(a, b)$ and consider the two cyclically reduced words $g = a^2b^{-1}ab$ and $h = a^2bab^{-1}$. As these are cyclically reduced words, it is apparent that $g \not\cong h^{\pm 1}$, and so we apply the character relations to show $\chi[a^2b^{-1}ab] = \chi[a^2bab^{-1}]$.

$$\begin{aligned}
\chi[a^2b^{-1}ab] &= \chi[a]\chi[ab^{-1}ab] - \chi[ab^{-1}a^{-1}ba^{-1}] \\
&= \chi[a]\chi[ab^{-1}ab] - \chi[a^{-1}] \\
&= \chi[a]\chi[ab^{-1}ab] - \chi[a] \\
&= \chi[a](\chi[ab^{-1}]\chi[ab] - \chi[ab^{-1}b^{-1}a^{-1}]) - \chi[a] \\
&= \chi[a](\chi[ab]\chi[ab^{-1}] - \chi[ab^{-2}a^{-1}]) - \chi[a] \\
&= \chi[a](\chi[ab]\chi[ab^{-1}] - \chi[b^2]) - \chi[a] \\
&= \chi[a](\chi[ab]\chi[ab^{-1}] - \chi[ab^2a^{-1}]) - \chi[a] \\
&= \chi[a]\chi[abab^{-1}] - \chi[a] \\
&= \chi[a]\chi[abab^{-1}] - \chi[a^{-1}] \\
&= \chi[a]\chi[abab^{-1}] - \chi[aba^{-1}b^{-1}a^{-1}] \\
&= \chi[a^2bab^{-1}].
\end{aligned} \tag{4.1}$$

In fact for any representation $\rho : free(a, b) \to SL_2(\mathbb{C})$ it is known that

$$\chi_g(\rho) = \chi_a(\rho)[\chi_{ab}(\rho)\chi_{ab^{-1}}(\rho) - \chi_{b^2}(\rho)] - \chi_a(\rho) = \chi_h(\rho).$$

Note that this example is not just restricted to F_2. If $\langle a, b, \ldots\rangle$ is a free group on two or more generators then $\chi[a^2b^{-1}ab] = \chi[a^2bab^{-1}]$.

Using proposition 4.1.3, we have a way of constructing such examples of non-conjugate elements with the same character, as given in [Hor72].

Proposition 4.1.4 *Let u and v be elements of F_2 for which $\chi[u] = \chi[v]$ and let $w(u,v)$ be a word in u and v. Then $\chi[w(u,v)] = \chi[w(v,u)]$.*

Proof: Let $w = w(a,b)$ be any element of $F_2 = free(a,b)$. Then by proposition 4.1.3, $\chi[w]$ may be expressed as a polynomial $\chi[w] = P(\chi[a], \chi[b], \chi[ab])$.

Hence $\chi[w(u,v)] = P(\chi[u], \chi[v], \chi[uv]) = P(\chi[v], \chi[u], \chi[vu]) = \chi[w(v,u)]$.

Here the middle equality comes from the assumptions that $\chi[u] = \chi[v]$ and $\chi[uv] = \chi[vu]$.

□

Given two non-conjugate elements of a free group, the algorithm given does answer the question of whether they have the same character. There are two problems with this method. Firstly, if the words being considered are long it may take a long time to determine the polynomial and hence whether the two have the same character. Secondly, and more importantly, the algorithm provides no geometric information.

4.2 Free Group Result

In this section we focus on a result that gives some information in relation to the character problem for elements of a free group. The main result of this section (see theorem 4.2.2) is due to Horowitz in [Hor72], and gives conditions on the group elements that ensure that if they have the same character then they are conjugate elements. First the following definition is required.

Definition 4.2.1 *An element g of a free group G is primitive if there exists a free basis S for G containing g.*

The following result is due to Horowitz, who gave necessary conditions for an element of the free group to give rise to the same character as a specified element. (See [theorem 7.1 in [Hor72]].)

Theorem 4.2.2 *Let G be a free group of any countable rank, and let g be an element of G. If a is a primitive element of G and $\chi[g] = \chi[a^m]$, then $g \cong a^{\pm m}$.*

Before giving the proof of theorem 4.2.2, the following result from [Hor72] is required.

Lemma 4.2.3 *Let U and U^* be cyclically reduced words in the free group F_n such that $tr(U) = tr(U^*)$. Then every generator of F_n occurs exactly the same amount of times in U and U^*, although possibly with ± 1 in the exponent.*

Proof: The proof of this result follows by induction on n (the number of generators of F_n). Since F_1 can be embedded naturally into F_2, the lemma will follow for F_1 once solved for F_2.

Let U and U^* be cyclically reduced words in $F_2 = \langle a, b \rangle$ with $\text{tr}(U) = \text{tr}(U^*)$. (Note that as the trace of a matrix is invariant under conjugation, it is possible to cyclically permute the syllables of U without altering its character, i.e. $tr(u_1 u_2 ... u_i u_{i+1}) = tr(u_{i+1} .. u_s u_1 .. u_i)$, hence cyclically permute the syllables until the word is in its simplest form so U is cyclically reduced.) There are three possibilities for each.

Firstly U could be a power of a alone and so $U = a^\alpha$ for some $\alpha \in \mathbb{Z}$. Or U could be a power of b alone, so $U = b^\beta$ for some $\beta \in \mathbb{Z}$. Finally U could be a word in both generators, so $U = a^{\alpha_1} b^{\beta_1} a^{\alpha_2} b^{\beta_2} ... a^{\alpha_s} b^{\beta_s}$ for some $\alpha_1 ... \alpha_s \in \mathbb{Z}$ and $\beta_1 ... \beta_s \in \mathbb{Z}$.

Since we are dealing with free groups we may use the following representation of F_2 into $SL_2(\mathbb{R})$ given in [Hor72]. Let $\rho \in \mathcal{F}(F_2)$ be defined by

$$\rho(a) = \begin{pmatrix} \lambda & t \\ 0 & \lambda^{-1} \end{pmatrix} \rho(b) = \begin{pmatrix} \mu & 0 \\ t & \mu^{-1} \end{pmatrix}$$

It follows that

$$\rho(a^\alpha) = \begin{pmatrix} \lambda^\alpha & f_\alpha(t, \lambda) \\ 0 & \lambda^{-\alpha} \end{pmatrix} \rho(b^\beta) = \begin{pmatrix} \mu^\beta & 0 \\ f_\beta(t, \mu) & \mu^{-\beta} \end{pmatrix}$$

Hence if $U = a^\alpha$ then $tr(U) = \lambda^\alpha + \lambda^{-\alpha}$. If $U = b^\beta$ then $tr(U) = \mu^\beta + \mu^{-\beta}$. If U contains both generators then $tr(U)$ is a function of both α and β.

If $U^* = a^{\alpha^*}$, as $\text{tr}(U^*) = \text{tr}(U)$, then U must be of the form $U = a^\alpha$ otherwise $\text{tr}(U)$ would be a non-constant function of β contradicting the equality of the traces. Thus we have

$\lambda^{\alpha} + \lambda^{-\alpha} = \lambda^{\alpha^*} + \lambda^{-\alpha^*}$ for all λ. Hence $U^* = a^{\pm\alpha} = U^{\pm 1}$. Hence the generator $a^{\pm 1}$ occurs exactly the same amount of times in U and U^*.

The case that $U^* = b^{\beta^*}$ follows in a similar way. If both a and b occur non-trivially in U then they must occur non-trivially in U^*. Hence the lemma is proven for F_2.

To complete the proof, the induction step needs to be put in place, so if the lemma is true for F_n then this implies it is also true for F_{n+1} (for $n \geq 2$). This is done as follows.

Let U and U^* be cyclically reduced words in F_{n+1} with $\text{tr}(U) = \text{tr}(U^*)$. It is now necessary to show that if g is any generator of F_{n+1}, then $O(U, g) = O(U^*, g)$, where $O(\circ, g)$ denotes the number of times $g^{\pm 1}$ occurs in a given word.

Let $a, b \neq g$ be two other generators in F_{n+1}, and let P, Q be positive integers such that P is greater than any exponent of a in U or U^*. Take the homomorphism $\pi : F_{n+1} \longrightarrow F_n$ such that $g \longrightarrow a^P b^Q a^P$ and every other generator maps to itself.

By assumption $\text{tr}(U) = \text{tr}(U^*)$ in F_{n+1} and so $\text{tr}(\pi(U)) = \text{tr}(\pi(U^*))$ in F_n.

Let V and V^* be words formed from $\pi(U)$ and $\pi(U^*)$ after cyclic reduction. Hence $tr(V) = tr(V^*)$ in F_n. This means that V has at least as many syllables as U. Moreover, V will not contain $g^{\pm 1}$, and b^Q will occur in V for each occurrence of $g^{\pm 1}$ in U.

Therefore

$$O(V, b) = O(U, b) + QO(U, g)$$

and

$$O(V^*, b) = O(U^*, b) + QO(U^*, g)$$

Hence by rearranging and using $O(V, b) = O(V^*, b)$ we get

$$O(U^*, g) = O(U, g)$$

as required. □

We now use this to prove theorem 4.2.2

Proof: Choose a free basis for the free group G including a, and let U be the cyclically reduced word obtained from g. Then U equals a^m or a^{-m} by lemma 4.2.3. □

Note that this result only applies to free groups as the definition of primitive is restricted to these groups only.

4.3 Connection between characters and lengths of curves

According to the work by Randol (in [Ran80]) there exists pairs of closed curves on a surface S for which the geodesics in their respective homotopy classes have the same hyperbolic length irrespective of the hyperbolic structure on S. This comes from the link between the length of a curve and the character of the corresponding element of the fundamental group. It transpires that if two group elements have the same character then the corresponding geodesics in the surface must have equal length. In this section we build this geometrical connection between discrete, faithful representations of a group G into $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$ and the lengths of curves in hyperbolic 2- and 3-manifolds.

First we will solve a minor discrepancy between the definitions given. We will be considering the fundamental groups of hyperbolic manifolds, and these are defined to be discrete subgroups of $PSL_2(\mathbb{R})$ (for hyperbolic surfaces) and $PSL_2(\mathbb{C})$ (for hyperbolic 3-manifolds). From the definition of a character of an element (see definition 4.1.1), the representation of the group is mapped into $SL_2(\mathbb{R})$ or $SL_2(\mathbb{C})$.

(Note that the argument given below is for hyperbolic surfaces, but the same result can be used for hyperbolic 3-manifolds with the appropriate changes.)

Let $Q : SL_2(\mathbb{R}) \to PSL_2(\mathbb{R})$ be the usual quotient map, and let $\widehat{\rho}$ be a discrete faithful representation of a finitely generated, torsion-free group G into $PSL_2(\mathbb{R})$. Then $\widehat{\rho}$ can be lifted to a discrete faithful representation ρ of G into $SL_2(\mathbb{R})$. (See [Kra85] for details of this lifting.) By this we mean that $\widehat{\rho} = Q \circ \rho$.

Conversely if G is a finitely generated torsion-free group and ρ is a discrete faithful representation of G into $SL_2(\mathbb{R})$, then the composition $\widehat{\rho} = Q \circ \rho$ is necessarily a faithful representation of G into $PSL_2(\mathbb{R})$, because the image $\rho(G)$ of G into $SL_2(\mathbb{R})$ cannot contain -id (the non-trivial element of the kernel of Q).

Hence if we start with a discrete, faithful representation into $PSL_2(\mathbb{R})$ then we can find a discrete faithful representation into $SL_2(\mathbb{R})$ and vice versa. In particular if we start with a discrete, faithful representation ρ of a finitely generated torsion-free group G into $SL_2(\mathbb{R})$, we can compose with Q to find a discrete faithful representation of G into

$PSL_2(\mathbb{R})$, and this gives rise to an orientable hyperbolic surface $\mathbb{H}^2/\widehat{\rho}(G)$.

(Note for hyperbolic 3-manifolds we are mapping from $SL_2(\mathbb{C})$ to $PSL_2(\mathbb{C})$ and get an orientable hyperbolic 3- manifold $\mathbb{H}^3/\widehat{\rho}(G)$.)

Now this discrepancy has been dealt with, the aim of the rest of this section is to explore the connection between the character of an element of a fundamental group and the length of the corresponding geodesic in the manifold.

First we need the following definition.

Definition 4.3.1 *An element g of a group G is maximal if g generates a maximal cyclic subgroup of G. Equivalently g is not a proper power of any other element of the group.*

As an aside here, the following shows why it is possible to restrict the attention to maximal elements of a group. Let $w \in G$ be a maximal element of the group G. From the trace and character relations given in proposition 4.1.2, for $m \geq 2$,

$$tr(u^m) = tr(u^{m-1})tr(u) - tr(u^{m-2})$$

Putting $tr(u^m) = \tau_m(tr(u))$, we can define a family of polynomials. Here $\tau_m(x)$ is called the *Chebyshev polynomial* and is defined by the recursion

$$\tau_m(x) = x\tau_{m-1}(x) - \tau_{m-2}(x)$$

where $\tau_1(x) = x$ and $\tau_0(x) = 2$.

Using the identity $\chi[wu] = \chi[w]\chi[u] - \chi[wu^{-1}]$ we see that $\chi[w^m] = \tau_m(\chi[w])$, and hence the character $\chi[w^m]$ is a polynomial in $\chi[w]$. Hence we may restrict the focus to maximal elements of the group.

(Note that for free groups, if an element is primitive then it must be maximal, but the converse is false. This is only true for free groups as the term primitive does not apply to other groups, although the notion of maximality of an element of a general group still holds.)

The groups being considered are discrete torsion-free subgroups Γ of $PSL_2(\mathbb{R})$. There is a one-to-one correspondence between free homotopy classes of closed curves in the quotient manifold $\mathbb{H}^2/\Gamma$, and the conjugacy classes of maximal cyclic subgroups of Γ. This means we only need to look at the maximal elements of the group.

Looking at the classification of elements in $PSL_2(\mathbb{R})$ (see definition 2.1.4), there are three cases to consider. Let α be an element of $PSL_2(\mathbb{R})$.

- Let α be elliptic. There are no elliptics in our groups by assumption (as the groups are torsion free).
- Let α be parabolic. Parabolic elements correspond to cusps on the surface, and as a closed curve moves out along the cusp the length of it gets smaller and smaller. Hence for a maximal parabolic there are closed curves in the quotient manifold whose lengths tend to zero. With this in mind, define the length of this homotopy class of curves, and therefore the conjugacy class, to be 0.
- Let α be loxodromic. Loxodromic elements have two fixed points on $\partial\mathbb{H}^2$. They are conjugate to $z \to \lambda^2 z$ for some $\lambda > 1$, $\lambda \in \mathbb{R}$. Here λ^2 is called the multiplier of α. A loxodromic element has a hyperbolic line which joins its two fixed points called an axis, and the loxodromic acts as a translation along its axis. Without loss of generality, let the fixed points be at 0 and ∞. (If they are not then it is possible to find a Möbius transformation that takes them there, and these transformations are distance preserving.) It can be shown that the translation distance is $\ln(\lambda^2)$. For a maximal loxodromic element, the axis projects to a closed geodesic of length $\ln(\lambda^2)$ in the quotient manifold and among all closed curves in the free homotopy class determined by α, this closed curve will have minimal length. Hence define this to be the length of this homotopy class of curves and the equivalent conjugacy class.

The above gives a way of assigning a length to each type of group element, so it remains to make the connection between this length assignment and the idea of the character of the group element.

The character of an element of the group was defined in terms of the trace of the matrix representation of the element in $SL_2(\mathbb{R})$. The trace of an element in $SL_2(\mathbb{R})$ determines the multiplier λ^2 of the corresponding element in $PSL_2(\mathbb{R})$. From the discussion above we have assigned lengths to free homotopy classes of curves in terms of the multiplier, and therefore the multiplier of the element in $PSL_2(\mathbb{R})$ determines the length of the closed geodesic in the quotient manifold $\mathbb{H}^2/\widehat{\rho}(G)$.

Specifically, if $\omega \in G$ and $\widehat{\rho}(\omega)$ is a loxodromic with multiplier λ^2 then $t = tr(\widehat{\rho}(\omega))$ and $tr(\widehat{\rho}(\omega)) = \pm(\lambda + \lambda^{-1})$ and so $\lambda^2 = \frac{1}{2}(t^2 - 2 \pm t\sqrt{t^2-4})$ where the sign is chosen so $\lambda^2 > 1$. Hence if we take two elements ω and ν of G which satisfy $tr(\rho(\omega)) = tr(\rho(\nu))$, and $\rho(\omega)$ is loxodromic, then we know two things. Firstly $\rho(\nu)$ must also be loxodromic (otherwise traces would not be equal). Secondly that $\widehat{\rho}(\omega)$ and $\widehat{\rho}(\nu)$ must correspond to closed geodesics of equal length in the quotient manifold.

Being more specific, if G is any finitely generated, torsion-free group and if ω and ν are two elements of G which generate non-conjugate maximal cyclic subgroups and which have the same character, then we know that $tr(\rho(\omega)) = tr(\rho(\nu))$ for all $\rho \in \mathcal{F}(G)$. Hence the lengths of the free homotopy classes determined by ω and ν are equal in $\mathbb{H}^2/\widehat{\rho}(G)$ (where here we remember that $\widehat{\rho} = Q \circ \rho$ and $Q : SL_2(\mathbb{R}) \to PSL_2(\mathbb{R})$).

Hence our original problem of finding pairs of elements of G that give rise to the same character over the space of faithful representations of G into $SL_2(\mathbb{R})$ (and that generate non-conjugate maximal cyclic subgroups of G) is equivalent to finding pairs of closed curves on the quotient manifold whose geodesics have the same hyperbolic length over all hyperbolic structures.

This connection between character and length will now be used to prove McShane's Lemma.

4.4 McShane's Lemma

In this section we state and prove McShane's result for surfaces, as given in [McS93]. This lemma looks at fundamental groups of closed orientable surfaces, and says that if the group element is maximal and represents a simple closed curve on the surface, then it is uniquely determined by its character. (See [lemma 6.2 in [And03]].)

Lemma 4.4.1 *Let G_p be the fundamental group of a closed orientable surface S_p of genus $p \geq 2$. Let $g \in G_p$ be a maximal element that represents a simple closed curve on S_p. Then $\chi[g]$ determines g. By this we mean that if there exists another maximal element $h \in G_p$ with $\chi[h] = \chi[g]$ then $h \cong g^{\pm 1}$.*

The following gives a proof for McShane's result as given in [And03]. We give this proof because we will use elements of it in later sections of this work.

Proof: By the discussion in section 4.3 we may restrict attention to discrete faithful representations of G_p into $SL_2(\mathbb{R})$. We then get hyperbolic structures on S_p by taking the quotient of $\mathbb{H}^2$ by $Q \circ \rho(G_p)$. This uses the fact that $\rho(G_p)$ in $SL_2(\mathbb{R})$ is isomorphic to $Q \circ \rho(G_p)$ in $PSL_2(\mathbb{R})$ because G_p is torsion-free and hence has no 2-torsion.

Let $g, h \in G_p$ and we assume that $\chi[g] = \chi[h]$. Using the discussion in section 4.3 we know that g and h represent curves on S_p with the same hyperbolic length for every hyperbolic structure on S_p (i.e. the length of a closed geodesic on S_p determines the character of the corresponding element of G_p and vice versa).

Now we use a result about non-simple curves. If h represents a homotopically non-trivial non-simple curve c on S_p then there is a positive uniform lower bound for the length of the closed geodesic homotopic to c over all hyperbolic structures on S_p. This lower bounds depends on the number of self-intersections the geodesic has. (See [Bas93] corollary 1.2.) However if h represents a homotopically non-trivial simple closed curve no such lower bound is imposed on the length of the closed geodesic homotopic to it. In fact there exist hyperbolic structures for which the length of the closed geodesic goes to 0. Using this fact we conclude that as g represents a simple closed curve on S_p then h must also represent a simple closed curve on S_p.

Now we are reduced to considering two simple closed curves on S_p so that the lengths of their corresponding closed geodesics are equal independent of the hyperbolic structure. We are reduced to three cases.

Firstly the simple curves that g and h represent could intersect. Using the collar lemma (see lemma 3.1.5), as one of these curves increases in length the other will decrease. By our assumption that our curves have the same hyperbolic length this cannot occur.

Secondly the simple curves that g and h represent could be completely disjoint. We can eliminate this case by using Fenchel-Nielsen coordinates of Teichmuller space (as described in section 3.1). From the discussion in section 3.1, we are able to change the length of one of the two curves completely independently of the other. This again cannot occur by our assumption.

This leaves the final case, which is that the simple curves that g and h represent coincide, which means that they are conjugate, and this is the result we require. □

It is possible to remove one of the conditions given for g in lemma 4.4.1.

Lemma 4.4.2 *We can remove the maximal condition from McShane's lemma. By this we mean that we can restate the lemma 4.4.1 as follows;*

Let $G_p = \pi_1(S_p)$, where S_p is a closed orientable surface of genus $p \geq 2$. Let $g \in G_p$ represent a simple closed curve on S_p. Then $\chi[g]$ determines g. By this we mean that if there exists another element $h \in G_p$ with $\chi[h] = \chi[g]$ then $h \cong g^{\pm 1}$.

Proof: Let $g, h \in G_p$ and assume $\chi[g] = \chi[h]$. Let $g = c^{\pm k}$ (for $k \in \mathbb{N}$) where c represents a simple closed curve on S_p.

If $k = 1$ then $g = c$ and g is a maximal element representing a simple closed curve and we are done by lemma 4.4.1. Hence we may assume $k \geq 2$.

Removing the maximality condition on g does not effect the discussion from section 4.3 connecting character and length. Therefore we may still use the connection to length, and so if $\chi[g] = \chi[h]$ then we know that g and h represent curves on S_p with the same hyperbolic length for every hyperbolic structure on S_p.

Note also that the fact we used about non-simple closed curves does not rely on the fact that g is maximal and so as g is a power of a simple closed curve, then h must also represent a power of a simple closed curve.

We are again reduced to considering powers of simple closed curves in three cases. Firstly the powers of simple curves that g and h represent could intersect. The collar lemma does not rely on the maximality condition so we may still rule out this case in the same way as before. Secondly the powers of simple curves that g and h represent could be disjoint. Again we may use Fenchel-Nielsen coordinates and see that we can change the length of one without effecting the other. Hence the powers of simple curves that g and h represent must coincide, and so we know that h must be conjugate to the same simple closed curve representing c, but h may be equal to $c^{\pm l}$ where $l \neq k$. Hence we know that $h = qc^l q^{-1}$ for some $q \in G_p$.

Therefore $\chi[h] = \chi[qc^l q^{-1}] = \chi[c^l]$ (by character relations). Therefore as $\chi[g] = \chi[h]$ we know that $\chi[c^k] = \chi[c^l]$.

From the discussion at the start of the chapter, we know that $\chi[c^n]$ may be written as a polynomial in $\chi[c]$ of degree n. Therefore we have a polynomial in $\chi[c]$ of degree k equal to a polynomial in $\chi[c]$ of degree l. Hence $k = \pm l$. Hence $h \cong g^{\pm 1}$ as required. □

4.5 Other surface problems

There are other conditions for surfaces that may be considered in relation to giving a partial answer to the original character question. Firstly in section 4.4 we have only considered closed orientable surfaces of varying genus. Another question would be to find a similar condition for other hyperbolic surfaces, for example those with holes or possibly punctures, where the parabolic elements of the fundamental group come into play. We will be considering surfaces with non-empty boundary in section 5.5.

Alternatively we could look for other conditions. McShane's result told us that a condition on the group elements is that they had to be represented by simple closed curves on the surface. We could therefore consider non-simple closed curves. From the proof above we know that a simple closed curve and a non-simple closed curve may not have the same character because there is a lower bound on the length of a non-simple closed curve and no such lower bound exists for simple closed curves. The following conjecture says something about non-simple closed curves having the same character.

Conjecture 4.5.1 *Let $g, h \in G_p$ where G_p is the fundamental group of a hyperbolic surface. If $\chi[g] = \chi[h]$ then g and h are represented by curves on S_p which have the same number of self-intersections.*

To give an indication as to where the idea for this conjecture comes from, it is known that there is a universal lower bound for the length of a non-simple closed geodesic on a hyperbolic surface (as mentioned in the proof of 4.4.1). In [Bas93], Basmajian shows that this can be improved by considering the self-intersection number of the closed geodesic. He shows that there exists an increasing sequence M_k tending to infinity so that if ρ is a closed geodesic with self-intersection number k then $l_\rho > M_k$. (Here l_ρ denotes the length of ρ.) Hence the length of a closed geodesic gets arbitrarily large as its self-intersection gets large. (See corollary 1.2 of [Bas93].) Note that this result is a consequence of the stable neighbourhood theorem.

This result only gives a bound on the length, but it may be possible to utilise this to prove this conjecture.

From the results given in this chapter we have some information in relation to the character problem in 1- and 2-dimensions. The two theorems given provide conditions which ensure elements of particular groups have the same character only if they are conjugate, and so gives a partial solution to the original question.

Chapter 5

Special Books of I-bundles

Chapter 4 gave a summary of some of what is known in relation to the character question of how to determine when two non-conjugate elements of a group have the same character. The two results given did not answer this question, but a modified version, which forms a partial converse. They gave conditions on the group elements that ensured that if an element of the group satisfied a particular property, then if another element of the group had the same character then the two elements are conjugate. These results considered elements of free groups (1-dimensional) and fundamental groups of surfaces (2-dimensional). It is therefore natural to try and extend to 3-dimensions and look for possible conditions for elements of fundamental groups of 3-manifolds.

In this chapter we discuss this question in relation to a particular type of 3-manifold called a *Book of I-bundles*, reducing to a specific case on which we will build later. We discuss a construction which ensures the manifold is hyperbolizable and give ideas for possible properties satisfying the question. We then prove some results on surfaces with boundary that are required when considering these properties. The most important technical tool is the projection of geodesics in M onto its spine F. This is discussed in detail in section 5.6. The chapter closes with the proofs of two of the properties for the specific manifold.

5.1 A three dimensional question

In this section we give the generalized question on which this work is based. The ultimate aim would be to find a solution to the following problem about hyperbolic 3-manifolds.

Let $G = \pi_1(M)$, where M is a compact hyperbolizable 3-manifold. Consider all faithful representations of G into $SL_2(\mathbb{C})$. Find a topological condition P that can be imposed on the elements of G so that the following statement is true:

If $g \in G$ satisfies the condition P and if $h \in G$ is any element such that $\chi[h] = \chi[g]$, then h is conjugate to $g^{\pm 1}$. (If more than one such a condition P exists, then we want to find the weakest.)

As stated above, this is a difficult problem as the property will need to apply to many different manifolds. A particular property that fits the assumptions and holds for one manifold may not be true for another manifold. For this reason we reduce the scale of the question by considering just one particular family of hyperbolizable 3-manifolds called *Books of I-bundles.* (These will be described in detail in the next section.)

Following the example of McShane's lemma in the previous chapter (see section 4.4), the sort of properties that are interesting will be those that give some geometrical information, and hence relate to the geodesics in M. We will utilize the connection between the length of a geodesic in M and the character of the corresponding group element in $\pi_1(M)$, and look for a natural collection of curves in M. We look at examples of possible properties which fit in section 5.3.

5.2 Books of I-bundles

This section introduces the book of I-bundles manifold, giving the general definition for the family of manifolds and then reducing to the specific case which will be considered in relation to the character question in this chapter.

First we give a general definition that comes from [AC96]. To clarify this algebraic description, a more visual interpretation follows the definition.

Definition 5.2.1 *Let* $\{F_i : i = 1, .., n\}$*, be a collection of surfaces, each of which is a compact orientable surface minus an open disc (so has a connected non-empty boundary). Form* B_i *by 'thickening'* F_i*, so for each* i*, let* $B_i = F_i \times I$*. Let* $\partial_0 B_i$ *be the annulus* $\partial F_i \times I$*. (Note that* $\partial B_i = (\partial F_i \times I) \cup (F_i \times \partial I)$*.)*

Let T be a solid torus and let A_i $(i = 1, ..., n)$ be a family of disjoint parallel closed annuli on ∂T homotopically equivalent to the core curve on T. Hence the boundary of T is decomposed into the closed annuli A_i and the open annuli $\partial T - (A_1 \cup ... \cup A_n)$.

Form a manifold M from T and $\{B_i\}$ by identifying the boundary $\partial_0 B_i$ with A_i for all i by an orientation-reversing homeomorphism.

M is called a Book of I-bundles.

A loose translation of this definition is that M is obtained by gluing a collection of I-bundles (thickened surfaces) to a solid torus along a family of parallel annuli. To visualize, think of the solid torus as the binding and the I-bundles as the pages, and hence M is a 'book'.

Definition 5.2.1 gives the description of the basic book of I-bundles manifold which has a single solid torus binding. This can be extended to give more complicated and general books of I-bundles. The first way to extend the definition given is to consider the case where M contains multiple solid torus bindings $T_1, T_2, ..., T_m$. The boundary of each T_k $(k = 1, ..., m)$ is decomposed in the same way as described in definition 5.2.1. To ensure that this manifold is connected, the I-bundles may be glued to more than one solid torus (at least one I-bundle must be attached to two or more solid tori in M to ensure M is connected). Hence M will contain I-bundles with bases consisting of surfaces with multiple boundary components, so using the terminology given in definition 5.2.1, the collection of surfaces $\{F_i\}$ will each be compact, orientable minus one or more open discs.

In this more general setting, it is possible for M to contain a 'loop'. This means that M comes back and meets itself, so we have a loop of solid tori and I-bundles.

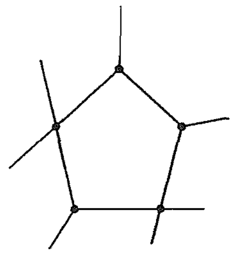


Fig 5.1: M containing a loop

(Figure 5.1 illustrates M containing a loop. Here the vertices represent solid tori and the edges are I-bundles.)

This is not possible in the basic book of I-bundles given in definition 5.2.1 as here there was only one solid torus binding, so would produce a 'tree-like' manifold (n-prong manifold), without any loops.

A further extension would be to remove the compactness condition and allow parabolic elements in the underlying fundamental group. In this situation a second kind of binding appears in M (i.e. a thickened torus) which corresponds to the rank 2 parabolic subgroups in $\pi_1(M)$. The I-bundles are still glued to the thickened tori along a union of annuli, but only one of the boundaries of the thickened torus participates in the gluing. This extension is beyond the scope of this work as parabolics are ruled out, but there is potential for future work in considering M with these bindings in relation to the character question.

Another consideration is how the gluing annuli are situated on each solid torus binding, and therefore how the I-bundles are glued to the bindings. We could consider (p, q) curves along which to glue, so the annuli wrap several times around the torus. For ease of exposition, in this work we will be gluing along $(0, 1)$ torus curves (as shown in figure 5.2), so the annuli do not wrap around the binding. There is potential here for future investigation in relaxing this assumption.

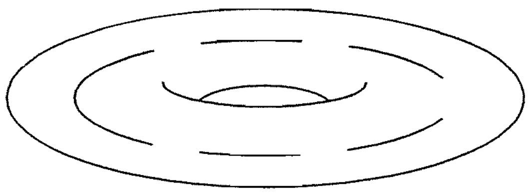


Fig 5.2: (0,1) torus curve

In addition to these extensions, two further properties that can differ between Books of I-bundles is the angles between the I-bundles on the solid torus, and the thickness of the I-bundles. Although initially it is assumed that the angles between the I-bundles are equal around the solid torus, this is relaxed in chapter 6 when considering more general Books of I-bundles.

The most general book of I-bundles manifold will be one that incorporates all the above extensions combined with the original basic definition. These are the only possibilities, as can be seen from the following description from [CMT99].

Definition 5.2.2 *A generalized book of I-bundles is a compact irreducible 3-manifold M with incompressible boundary, such that it is possible to find a disjoint collection A of essential annuli in M so that each component of M obtained by cutting along A is either a solid torus, a thickened torus or homeomorphic to an I-bundle.*

Considering the most general book of I-bundles which incorporates all of these components will make it more difficult to find a property to fit the question given in section 5.1. It may even be the case that such a property will not exist or if one does it will be very weak.

Initially we will focus on a particular book of I-bundles that fits in with definition 5.2.1. Let M be the book of I-bundles with one single solid torus binding and three 'pages', or three I-bundles which are attached to the solid torus by the gluing described in definition 5.2.1. For ease of exposition, the gluing occurs along (0, 1) torus curves (so the annuli do not wrap around the torus). It is possible to draw a picture of this (see figure 5.3 below) by representing the solid torus by a vertex and the I-bundles by three lines coincident with the vertex such that they are evenly spaced (so the angle between each pair is the same).

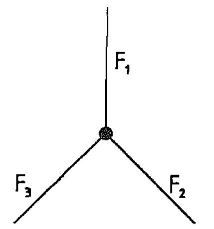


Fig 5.3: 3-prong case

Once the discussion for possible properties for this case is complete, the aim is to extend to the more general setting.

5.3 Possible properties

In this section the aim is to discuss possible properties that can be imposed on an element of $\pi_1(M)$ so that any other element with the same character is conjugate to it. (Here M is a general book of I-bundles manifold.) As discussed in section 4.3, there is a relationship between the character of an element of the fundamental group and the length of the corresponding geodesic in M. Note that although this is discussed in terms of surfaces in section 4.3, it is indicated there that this also applies in 3-dimensions, where the underlying fundamental group is Kleinian. Hence our problem of finding a property that ensures that two elements with the same character are conjugate translates to finding two curves whose geodesic representatives have the same length in the hyperbolic 3-manifold.

When considering potential properties what we are looking for is a natural collection of curves in M. In the surface case the property consisted of the geodesic being simple, and these curves form a natural family on any closed surface. We would like to find a similar property in 3-dimensions for the book of I-bundles manifold.

Let g, h be geodesics in M such that $\chi[g] = \chi[h]$. We want to find a property on g that implies that g and h are the same curve (up to homotopy and orientation). The following lists some ideas for such a property, with discussion as to why they may or may not hold true.

- An initial idea is to let g be a *filling curve* on one component of the boundary of M. (Note that g is a filling curve if it crosses every simple closed curve in that boundary component S, or alternatively if $S - g$ is a union of discs.) In order to deduce that this property gives the required result, it would first be necessary to show that any curve h with the same length is in the same boundary component that g is situated. Once this has been deduced then the situation would be reduced to a question about surfaces, and therefore it would be hoped that elements of the proof of the surface result (see lemma 4.4.1) could be used to complete a proof. The potential problem with this property in relation to the book of I-bundles manifold is that it has more than one component to its boundary. It would therefore be necessary to ensure that the two curves are not only both in the boundary of M, but that they both live in the same component of the boundary. In the paper by Leininger (see [Lei03]), an example is given of two closed geodesics on a closed orientable surface of genus $g > 1$ which both fill the surface and have the same intersection number with every simple closed curve on the surface, but are not hyperbolically equivalent. (By *not hyperbolically equivalent* we mean that the two curves do not have the same hyperbolic length over all structures on the surface.) In relation to the book of I-bundles, this shows that if g is a filling curve on one component of ∂M (the boundary of M), then even if it is possible to prove that h is also a filling curve on the same boundary component, it will not be enough to show $g \cong h^{\pm 1}$ by the counterexample in [Lei03].

(As an aside in connection to this it may be interesting to consider what properties would work for surfaces with Fuchsian or quasi-fuchsian fundamental groups, or even to consider this particular property in relation to these surfaces. This is an idea for future investigation and is not covered within the scope of this work, as we will focus on 3-manifold ideas.)

- A second possibility for a property considers a slightly different but related question to the character problem given in section 5.1. Let g be a geodesic contained in a component of the boundary of M (i.e $g \subseteq S \subseteq \partial M$) and let h be a curve in M such that $\chi[h] = \chi[g]$. Then an idea would be to show that h is not only contained in the boundary of M, but is contained in the same component of the boundary as g (i.e. $h \subseteq S$). This reduces the problem from an unanswered question about 3-manifolds to an unanswered surface question. This is a problem that can be looked at in relation to any hyperbolizable manifold. We will look at this question in relation to the book of I-bundles manifold later (see section 5.8 for details).
- A third possible property on the geodesics of M is to let g be the core curve of the solid torus (or in the general case one of the set of core curves). These curves form a natural collection of curves in M and this is therefore a desirable property to look at. As will be seen in the next section, the structure on M only cares about the lengths of the core curves and each set of lengths generates a family of manifolds. To show that this property holds means proving that the core curve is uniquely determined by its length. This property will be considered in more detail in sections 5.7 and 6.3.
- An idea for a fourth possible property comes from considering the surface result (see lemma 4.4.1). Let g be represented by a simple closed curve on the spine F of M. (The spine of M is the union of the I-bundle bases with their boundaries glued together inside the solid tori.) Considering the manifold as a whole, the concept of a closed curve being simple in M is not very useful. Unlike on a surface, there is more space in a three-dimensional object, and so the majority of the geodesics in M will

be simple closed curves. The spine of M is constructed from surfaces, and so the idea of a simple closed curve on F just involves extending the usual definition. Hence a simple closed curve on F will be made up of simple arcs meeting (without crossing) in the core curves.

There are some technical issues to be resolved when considering this property. The connection between the character of an element and the length of a closed curve applies to the geodesics in M. In general (although there are exceptions) these do not exist on the spine. It is therefore necessary to find a way of connecting a geodesic in M to a closed curve on F, and hence a way of projecting that pushes a geodesic onto F such that it gives a unique curve on the spine. To prove this property, a starting point would be to consider the surface result and try to extend it to fit this situation. This property will be considered in more detail in chapter 7.

It is important to point out that the idea of projecting the geodesics of M onto F is a major technical tool in proving any of the above properties. Hence this projection will be discussed in detail in section 5.6.

In this work the last three of these properties will be considered, first restricting to the specific 3-prong case, and then for a general book of I-bundles (without parabolics). Hence we have the following three statements.

Theorem 5.7.1 *Let M be the specific book of I-bundles manifold with single solid torus binding and three pages. Let $g \in \pi_1(M)$, such that g is represented by the core curve of the solid torus in M. Then g is uniquely determined by $\chi[g]$. By this we mean that if $h \in \pi_1(M)$ with $\chi[g] = \chi[h]$ then h is conjugate to $g^{\pm 1}$ (so h is also represented by the core curve in M).*

This theorem will be discussed in section 5.7.

Theorem 5.8.1 *Let M be the specific book of I-bundles with single solid torus binding and three pages. Let $g \in \pi_1(M)$ be represented by a geodesic γ that is contained in a component of the boundary of M ($\gamma \subseteq S_i \subseteq \partial M$). Let $h \in \pi_1(M)$ be represented by another curve γ' such that $\chi[h] = \chi[g]$. Then $\gamma' \subseteq S_i$ also.*

This theorem will be discussed in section 5.8.

Conjecture 7.0.3 *Let $G = \pi_1(M)$ where M is a book of I-bundles with single solid torus binding and three pages. Let $g \in G$ be represented by a geodesic in M which is uniquely projected onto a simple closed curve on F (where F is the spine of M). Let $h \in G$ such that $\chi[g] = \chi[h]$, then $h \cong g^{\pm 1}$.*

This conjecture will be discussed in chapter 7.

Before discussing these, we will look more carefully at the book of I-bundles manifold and the construction which will be used throughout.

5.4 The Canary, Minsky, Taylor construction

In this section we describe a construction for the book of I-bundles manifold that is due to Canary, Minsky and Taylor and follow the description they give in [CMT99]. In this paper they show that it is possible to put a family of convex co-compact hyperbolic structures on the interior of M (denoted $int(M)$), where M is a general book of I-bundles manifold, and so $int(M) = \mathbb{H}^3/\Gamma$ for Γ Kleinian. This construction proves that M is hyperbolizable.

Canary, Minsky and Taylor give the construction for a general book, including consideration of parabolic elements. In this section we will follow this construction in general, but will rule out the parabolic case as we require M to be compact. We will then highlight what is required for the initial 3-prong case. For details on this construction, including dealing with parabolics, see [[CMT99] section 4].

From the discussion in section 5.2, the general book of I-bundles (with no parabolic elements in its fundamental group) is comprised of solid tori and thickened surfaces or 'I-bundles', which are glued to the solid tori along families of annuli on the boundaries of the tori. (Equivalently, for each I-bundle, the subbundle over the boundary of its base surface is a union of annuli which are glued to the boundary of a solid torus.)

The union of the I-bundle bases (i.e. the surfaces which are thickened to make the I-bundles) with boundaries glued together inside each solid torus, define the '*spine*' for M. This spine is a 2-complex around which M is a regular neighbourhood. (N.B. The spine of M, which will be denoted F, is very important to this work. With reference to section 5.3, the property of a curve being simple on F is one which is mentioned and will be considered later in chapter 7.)

To start describing the construction of M, we begin by constructing F, and look in particular at the solid tori. For each solid torus in M, the cores of the annuli glued to it describe some number m of parallel (p, q) curves.

Let L be a geodesic in $\mathbb{H}^3$ which is also the boundary of mq half-planes, such that these half-planes are equally spaced around L. (By being *equally spaced* we mean that the angles between the planes are equal.) Let γ be the loxodromic element that has L as its axis, and such that the translation distance is $\frac{l_i}{q}$ (for some small $l_i > 0$) and rotation angle $\frac{2\pi p}{q}$.

Take an ϵ-neighbourhood of L and look at the quotient of this neighbourhood by γ. The result is a solid torus. The quotients of the mq half-planes meet in a collection of annuli with boundaries glued together at the core of the solid torus. The intersection of these annuli with the boundary of the solid torus give the m parallel $(p,\ q)$-curves required.

We complete this construction for each solid torus in M, and hence get a list of parameters $\{l_i\}$ (the translation distances of γ_i) for the solid tori in M.

Each I-bundle has a base surface F_j $(j = 1, ..., n)$, which is a compact, orientable surface of positive genus with non-empty (possibly disconnected) boundary. For each F_j, we choose a hyperbolic structure (so the surface has negative curvature) so that each boundary component of F_j that glues to a solid torus with parameter l_i is a geodesic of length l_i. It is important to note here that the lengths of the boundary curves are the only constraints on the choice of hyperbolic structure on F_j. (We highlight this here as we will use this fact when considering changing the lengths of other curves on F_j, or alternatively changing the hyperbolic structure, while keeping the length of the boundary fixed.) (N.B. The union F of these F_j with their boundaries glued together inside the solid tori comprise the spine described earlier.)

As each F_j has a hyperbolic structure, it is possible to find a Fuchsian group associated to it. The convex core of its quotient will realize the given hyperbolic structure. Note also that the boundary components of each F_j correspond to pure translations.

To piece the manifold together, for each solid torus, identify the neighbourhoods of the corresponding boundaries of the I-bundle bases to the annuli arranged around its core. This structure extends consistently to the thickenings of the I-bundle bases, so we get a hyperbolic structure on $int(M)$. Using this construction we obtain a hyperbolic structure for which each I-bundle base (F_j) is totally geodesic, and a set of parameters $\{l_i\}$ for each that correspond to the lengths of the core curves of the solid tori in M. Let $l_o = max\{l_i\}$ and $\theta_o = min\left\{\frac{2\pi}{q_i m_i}\right\}$, where $\{m_i\}$ and $\{(p_i, q_i)\}$ describe the gluings for the solid tori.

To complete this section on the construction, we show M is hyperbolizable and hence show that $N = \mathbb{H}^3/\Gamma$ is homeomorphic to $int(M)$, where Γ is discrete. To do this we consider the lift into $\mathbb{H}^3$. Each component of the lift $\widetilde{F_j}$ of a base surface F_j is a totally geodesic subset called a *flat*. For a given flat, at each lift of a geodesic boundary of its base surface, there is a collection of $m_i q_i - 1$ other flats equally spaced around it. (By *equally spaced* we mean that the angles between the flats are equal and are therefore evenly spaced around the lift of the geodesic boundary.) The flats are arranged in a tree, and each flat is contained in a half-plane in $\mathbb{H}^3$.

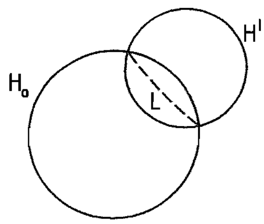


Fig 5.4: Child of H_0

This picture may be normalized as follows. Let H_0 be a flat chosen to be the *root* of the tree. Normalize so that the plane containing H_0 is a hemisphere meeting the complex plane in the unit circle. Each child H' of H_0 (i.e. an adjacent flat - these are the only ones that are not disjoint from H_0 - see figure 5.4) meets H_0 along a geodesic L and is contained in a half-plane which meets the complex plane in a circle.

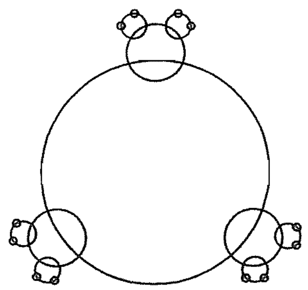


Fig 5.5: Tree of circles

If we look down on this picture in $\mathbb{H}^3$ from infinity, this tree of flats maps out a tree of circles in the complex plane (see figure 5.5). This normalized picture will be used later.

Let x, y be points in the two flats H and H' respectively. Let $H = H_1, ..., H_n = H'$ be the unique sequence of flats connecting H and H', where H_i and H_{i+1} share a geodesic boundary. Note that the sequence of flats is unique as the flats are arranged in a tree, and $H_i \neq H_j$ for $i \neq j$. There is a chain of geodesic arcs $\{\alpha_i\}$ connecting x to y such that $\alpha_i \subset H_i$ and so α_i meets α_{i+1} in the geodesic boundary shared by H_i and H_{i+1}. We denote this chain by $\gamma_{x,y}$. The following lemma looks at the uniqueness of this piece-wise geodesic path $\gamma_{x,y}$.

Lemma 5.4.1 *The piece-wise geodesic path $\gamma_{x,y}$, from x to y in the tree of flats, of shortest length is unique.*

Proof: Let x and y be points in the two flats H and H' respectively and let $H = H_1, H_2, ..., H_n = H'$ be the unique sequence of flats from H to H'. Each member of this sequence is contained in a half-plane of $\mathbb{H}^3$ such that the half-plane containing H_i will intersect the half-plane containing H_{i+1} at angle ϕ_i for $i = 1, ..., n-1$. (Note that as we are looking at the general case, all ϕ_i may be different.)

There will be several possible paths $\gamma_{x,y}$ from x to y, but what they will have in common is that they all will consist of a chain of geodesic arcs α_i, such that $\alpha_i \subset H_i$ and α_i meets α_{i+1} in the geodesic boundary shared by H_i and H_{i+1}.

Take any path $\gamma_{x,y}$ from x to y. The length of each segment α_i of which $\gamma_{x,y}$ is constructed, will not be altered if we change the angle between the adjacent planes (i.e. if we change ϕ_i), as each α_i is contained in a half-plane. Therefore 'flatten out' the planes so that each $\phi_i = \pi$. We are then considering a piece-wise geodesic path between two points (x and y) in $\mathbb{H}^2$.

Now consider all possible paths $\gamma_{x,y}$ between x and y in this 'flattened out' space, and find one of shortest length in $\mathbb{H}^2$.

In $\mathbb{H}^2$, there is a unique hyperbolic line between any two points, and it is the distance realizing path. Hence join x and y by the unique hyperbolic line that contains them both, and this path will be shortest among all paths between x and y in the 'flattened out' space (see figure 5.6).

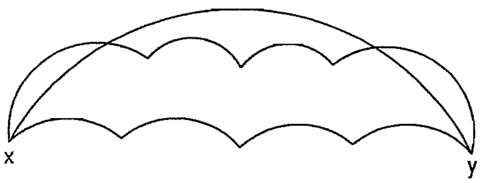


Fig 5.6: other paths in the 'flattened out' space

Therefore we have a unique shortest path in $\mathbb{H}^2$.

Now re-bend the planes back to their original positions. Again this can be done without altering the lengths of the segments α_i which define $\gamma_{x,y}$. Then the unique shortest path found in $\mathbb{H}^2$ will be the unique shortest piece-wise geodesic path in the tree of flats from x to y as required (see figure 5.7).

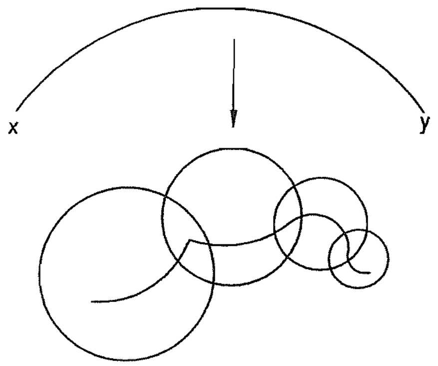


Fig 5.7: unique shortest path in the tree

(Note that any other path in the tree of flats will not correspond to the hyperbolic line segment when we bend the planes to angle π and hence will be longer in $\mathbb{H}^2$ and hence longer in the tree of flats - as the bending of the planes does not alter the lengths.) □

As a summary of the above result, to find the unique shortest path between two particular points x and y in the tree of flats, take the unique chain of flats between them and flatten out the hyperplanes containing the flats so the angle between each adjacent pair is π. Then join x and y by the unique hyperbolic line that contains them both. Finally re-bend the hyperplanes back to their original angle, giving the chain $\gamma_{x,y}$ of shortest length. Note that as there is a unique hyperbolic line between any two points, and it is the distance realizing path, the chain of shortest length between x and y is unique. Note also that this result is independent of the angle between the hyperplanes containing the flats, so applies to any F, and hence any M, where M is a book of I-bundles manifold.

As a brief aside here and in connection to lemma 5.4.1, it should be noted that the intersection angle of this unique shortest path found between x and y with the bending lines (along which the hyperplanes are bent) is not necessarily the same at each point of intersection. In [CMT99] it is stated that the uniqueness of this chain $\gamma_{x,y}$ comes from the assumption that this intersection angle is $\frac{\pi}{2}$. However, as the above result illustrates, this is not the case. If we make the restriction that each segment of the chain meets the geodesic boundary at a right-angle then the chain $\gamma_{x,y}$ would contain gaps, (i.e. see figure 5.8). These gaps would have to be filled by moving along the geodesic boundary and the proof of lemma 5.4.1 illustrates that this would not be the shortest path. However, as Canary, Minsky and Taylor do not use this intersection angle directly, this error does not effect the results or proofs given in [CMT99].

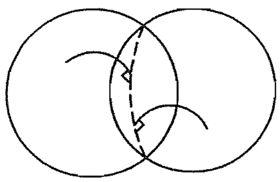


Fig 5.8: path in tree of flats that contains gaps

The μ-*thin* part of a flat H is defined to be the set of points where some element of the stabilizer of H acts with translation distance μ or less. The μ-*thick* part of a flat H is defined to be the complement of this. If μ is sufficiently small (i.e smaller than the Margulis constant - which we will assume to be the case. The Margulis constant is the smallest constant r_o such that for each discrete group G and each point x in $\mathbb{H}^3$, the group generated by the elements in G which move x less that r_o is elementary), then the μ-thin part consists of a union of disjoint pieces, each of which is a neighbourhood of an axis of translation. In terms of the tree of flats, the lift of the μ-thin parts will consist of neighbourhoods of the lifts of the geodesic boundaries of the base surfaces.

In terms of the spine F of the manifold M, the μ-thin parts will consist of neighbourhoods of the core curves of the solid tori, or equivalently of the geodesic boundaries of each F_j (as shown in figure 5.9). Geometrically, the μ-thin part of F_j consists of the subset of points p on F_j such that there is a non-trivial closed curve passing through p whose length is less than μ. (The μ-thick part is the complement of this.)

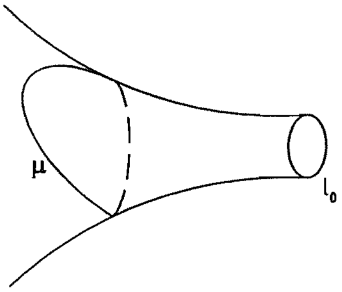


Fig 5.9: μ - thin part of F

Returning to the tree of flats picture, suppose x is in μ-thick part of H (so there is definite spacing between x and the lift of the appropriate geodesic boundaries). Then in [CMT99], Canary, Minsky and Taylor state that given any $k > 0$, each segment of the geodesic chain $\gamma_{x,y}$ (except possibly the last) has length at least k, with the assumption that the parameter $l_o = max\{l_i\}$ is *sufficiently short.* (Note that the first segment of $\gamma_{x,y}$, which contains x, will be bounded by the assumption that x is in the μ-thick part. The last segment may or may not dependent on whether y is in the μ-thick part of a flat.)

To see this is the case, look back at the spine. As F is comprised of surfaces F_j with geodesic boundary components, we may use the collar lemma around each boundary curve. As l_o is small, and as we have control over l_o, each geodesic boundary is short and hence has a wide collar around it. Another consequence of l_o being small and the control we have over l_o is that we may assume that $\mu > l_o$ (and hence $\mu > l_i$ for all i). Hence each geodesic boundary will be in the μ-thin part of the spine F. (See figure 5.9.)

In [CMT99], the bound stated is $c\ log\left(\frac{\mu}{l_o}\right)$, assuming l_o is sufficiently small. Hence by making l_o small, so $\mu > l_o$, the μ-thick part of each F_j is separated from its respective boundaries by at least $c\ log\left(\frac{\mu}{l_o}\right)$ for a fixed constant c.

Consider a point x' on F such that x' is in the μ-thick part of F, and consider any closed curve $\tilde{\gamma}$ through x' on F. Then $\tilde{\gamma}$ is comprised of geodesic segments, such that each segment (apart from the first and last which start and end at x' respectively) starts and finishes at a geodesic boundary of one of the F_j (i.e. each segment is contained in one F_j only). As x' is assumed to be in the μ-thick part of F, then the length of this first segment of $\tilde{\gamma}$ will obviously be bounded below by the bound stated above. All other segments of $\widetilde{\gamma}$ must pass into the μ-thick part of F before returning to a geodesic boundary, otherwise a shorter piece-wise geodesic closed curve could be found by homotopy back to the corresponding l_i. Even the last segment of $\widetilde{\gamma}$ will be bounded as it ends in the μ-thick part of F, namely at x'.

Referring to the lift to $\mathbb{H}^3$ and the tree of flats picture, as x is assumed to be in the μ-thick part of a flat, then the length of the first segment of $\gamma_{x,y}$ is bounded below. If we assume $\gamma_{x,y}$ is a lift of a closed curve $\widetilde{\gamma}$ on F (as it will be closed curves on F that will be of interest), then y will also be in the μ-thick part of a flat. Hence the last segments length will also be bounded below. All intermediary segments of $\gamma_{x,y}$ must cross a flat H' from the lift of one geodesic boundary to another lift of a geodesic boundary (note the two lifts may be of the same geodesic boundary of F). Hence they must pass into the μ-thick part of H', and therefore have length bounded below, (see figure 5.10). Hence every segment of $\gamma_{x,y}$ has length that is bounded below.

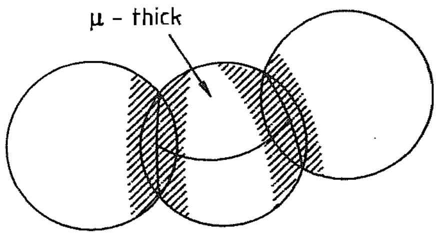


Fig 5.10: path segment with bounded length

The chain $\gamma_{x,y}$ is a broken geodesic in $\mathbb{H}^3$. From [lemma 4.2 in [CMT99]] we know more than this. As $\gamma_{x,y}$ is composed of geodesic arcs α_i $(i = 1, ..., n)$ each of which has length bounded below by a constant $k \geq 0$ (dependent on the value of l_o), then the orthogonal bisecting planes P_i to each α_i are all disjoint. Furthermore each plane P_j separates P_i from P_k and α_i from α_k whenever $i < j < k$. (Note that in proving this lemma, Canary, Minsky and Taylor choose k by the formula $cosh^2\left(\frac{k}{2}\right) = \frac{2}{1-cos(\theta)}$, where θ is a lower bound on the angles between the geodesic arcs α_i. Furthermore, they show that $dist(P_i, P_{i+1}) \geq \frac{1}{2}(l_{\alpha_i} + l_{\alpha_{i+1}}) - k$, where l_{α_i} is the length of α_i.)

These P_i give us a sequence of planes with definite spacing, and hence non-adjacent α_i are completely disjoint. As each α_i is contained in a flat, then these bisecting planes will give definite spacing between the flats. In particular any two non-adjacent flats in the tree are completely disjoint. (Note that this also shows that to get from one flat to another there is a unique sequence of flats to go through.) Therefore the entire tree is properly embedded in $\mathbb{H}^3$. Since a neighbourhood of this tree embeds it must be the homeomorphic developing image of a neighbourhood of the lift to the universal cover of F. Hence N is homeomorphic to int(M), where $N = \mathbb{H}^3/\Gamma$ for Γ discrete, and so M is hyperbolizable. (For more details on this construction and how they use it see [CMT99] and the references given there.)

This construction establishes that M is hyperbolizable with the only assumption that the core curves of the solid tori are *sufficiently short* to give the spacing required. By making the assumption that the core curves are short, it ensures that around the boundary components of the I-bundle bases there are long half-collars. This in turn ensures that the I-bundles do not intersect.

We make the following observation (which we set aside because of how important it is in this work).

Observation

The only requirement for this construction to hold is that F is constructed from surfaces which have short geodesic boundary curves. The lengths of these boundary geodesics provide the only constraint on both the choice of hyperbolic structure on each F_j and therefore M itself. The details of the construction given show that there are no further restrictions on both F and M to be hyperbolizable. In particular (and most importantly in the context we will use this construction) this implies that what occurs on the rest of the surface (for example in the μ-thick parts of each F_j) does not effect the construction on F, unless what occurs changes the lengths of the boundary geodesics, therefore maintaining a hyperbolic structure on M. For example, if we altered the lengths of other curves on F (not equal to one of the core curves), but managed to keep $\{l_i\}$ fixed length, then this would not effect the fact that M is hyperbolizable.

This observation will become important in section 5.5 when we start to manipulate curves on the F_j which are situated away from the core curves.

This completes the discussion on the general CMT construction on M. We now look at a specific case.

5.4.1 The 3-prong case

The objective of this part is to consider the CMT construction in terms of a particular book of I-bundles manifold on which we will initially focus. As described in section 5.2, the '3-prong' case consists of a single solid torus binding with three I-bundles glued to it. Hence F is constructed from three surfaces F_1, F_2, F_3, each of which has a non-empty connected geodesic boundary of length l_o. (N.B. the spine for the 3-prong case will be referred to as the 'fanblade' because of its appearance.)

For the solid torus, the cores of the disjoint annuli glued to it describe three parallel $(0,1)$-curves (so in terms of the variables from section 5.4, $m = 3, p = 0$ and $q = 1$). When considering the construction of the solid torus, start with a geodesic L in $\mathbb{H}^3$ with three half-planes equally spaced around it, so the angle between them is $\frac{2\pi}{3}$ (see figure 5.11 below). The loxodromic element γ which has L as its axis will act as a pure translation along L, so will act with translation distance l_o (for small $l_o > 0$) and zero rotational angle. When the quotient of a ϵ-neighbourhood of L is taken, the result is the solid torus with a collection of three annuli whose boundaries are glued together at the core of the solid torus.

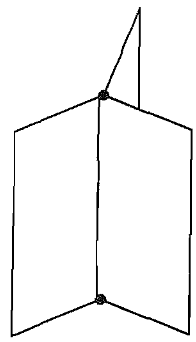


Fig 5.11: 3-prong construction

Hence for this particular case there is a single parameter l_o, which provides the only constraint imposed on the construction. The rest of the construction is the same as described in section 5.4.

Note that as lemma 5.4.1 is independent of the original angle between the planes (which in this particular case will be $\frac{2\pi}{3}$), it will apply to this particular case (and in fact any book of I-bundles manifold.)

As per the observation at the end of the previous section, the only requirement to be within the scope of the construction (and for M to be hyperbolizable) is that the fanblade is made up of surfaces whose geodesic boundaries are short. Note that there are no further restrictions. In particular the lengths of other closed curves on each surface component of F are not restricted, and hence we may manipulate these curves by changing their lengths, as long as this does not alter the length of the boundary geodesic. This is looked at in more detail in section 5.5.

5.5 Surfaces with boundary

Following on from the observations made in section 5.4, this section looks more closely at the spine of M. In particular, we consider how lengths of closed curves on F may be altered within the scope of the construction described. From the comments at the end of section 5.4, we know the only constraint on the construction is that F is comprised of surfaces with short geodesic boundary components. Hence we are working in the space of hyperbolic structures for which boundary geodesics are short. As long as this remains the case, then we stay within the construction. For this reason we consider how we may manipulate the lengths of other curves on F whilst keeping the lengths of these boundary geodesics (or equivalently the core curves) fixed, and hence staying within the family of hyperbolic structures.

To give an indication of why this is important, the reason for looking at this is that as F is constructed from surfaces, there is a greater possibility of being able to control the curves on the spine. If it is possible to manipulate and change the lengths of curves on F without interfering with the structure of the manifold, and we can find a way of projecting geodesics onto F, then it will provide a method of gaining some control in the 3-manifold. (The projection itself will be considered in section 5.6.) This will then provide the tools required to consider the character question (as given in section 5.1). Hence first we consider F and its component parts.

In section 5.4, we constructed the spine of M out of I-bundle bases. These consisted of surfaces with non-empty (possibly disconnected) geodesic boundary. For each base surface F_j we found a Fuchsian group such that the convex core of its quotient realized this hyperbolic structure (i.e. the space of hyperbolic structures being considered are those which give short boundary geodesics). Hence each I-bundle base is totally geodesic and the boundary components correspond to pure translations. (Essentially, we considered each F_j to be the convex core of a surface with ends, giving a surface with boundary - see figure 5.12. We cut off the end around the geodesic to give a surface with finite volume.) The spine was created by gluing these surfaces together along these geodesic boundaries.

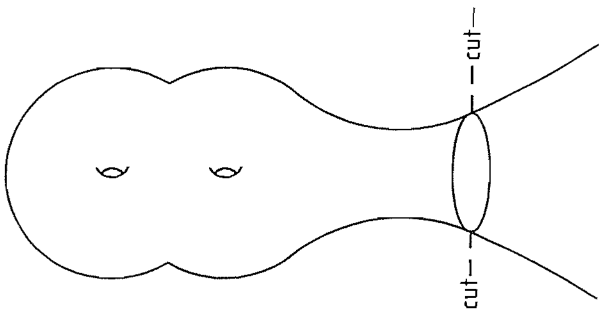


Fig 5.12: convex core of F_j

We can decompose F into its component parts by applying the latter part of this in reverse, so we have a set of surfaces with non-empty (possibly disconnected) geodesic boundary. (In terms of the initial 3-prong case, the spine will be decomposed into three surfaces with non-empty, connected geodesic boundary.) We now consider these components, and hence are returning to consider surfaces once more. In the previous chapter we considered only closed compact surfaces, so the 'nicest' possible in some sense. Now to be considered are surfaces with non-empty boundary.

Take one of the surfaces F_j from which F is constructed. Let α be a simple closed curve on F_j that is not homotopic into ∂F_j. We want to know that it is possible to change the length of α while the length(s) of the boundary geodesic(s) of F_j remain constant. We start by considering the simplest case, and prove the result for a surface of genus 1 with connected geodesic boundary. This is formalized into the following.

Lemma 5.5.1 *Let S be a compact oriented surface of genus 1 with connected non-empty boundary consisting of a geodesic c of length l_c. (Here we have assumed that S has hyperbolic structure with geodesic boundary - so the end has been cut off around this boundary.) Let $hyp(l_c)$ be the space of hyperbolic structures on S with l_c constant. Let p be a simple closed geodesic on S with length l_p. Then if $p \not\simeq c^{\pm k}$ for some integer k then l_p is non-constant on $hyp(l_c)$.*

From this it will be possible to change the length of any simple closed geodesic on S while keeping the boundary curve at constant length. The proof of this is as follows.

Proof: Let $g_a, g_b \in \pi_1(S)$ be represented by geodesics a and b on S of lengths l_a and l_b respectively, such that a and b generate S. Then, given the correct orientation and labeling, $c = aba^{-1}b^{-1}$. (See figure 5.13 below.)

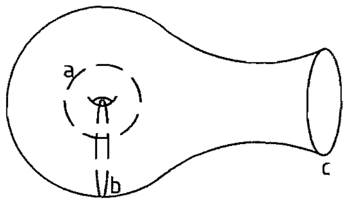


Fig 5.13: genus 1 surface

The first thing to show is that the lengths of a and b (denoted l_a and l_b) can be changed while the length of c (denoted l_c) is kept constant. It is also necessary to know how much the lengths l_a and l_b can vary for l_c fixed. To do this, we implement the connection between the length of a geodesic on S and the character of the corresponding fundamental group element (as given in section 4.3) and look at the character equations. Using these we show that $\chi[c]$ can stay constant while $\chi[a], \chi[b]$ and $\chi[ab]$ are variable.

As we know that $c = aba^{-1}b^{-1}$ (so $\chi[c] = \chi[aba^{-1}b^{-1}]$), using the character relations given in chapter 4 we can expand the right hand side to give

$$\chi[c] = \chi^2[a] + \chi^2[b] + \chi^2[ab] - \chi[a]\chi[b]\chi[ab] - 2$$

We need to know how much we can vary $\chi[a], \chi[b]$ and $\chi[ab]$ and keep $\chi[c]$ constant. First note that $\chi[a], \chi[b]$ and $\chi[ab]$ are independent variables. This independence can be seen by considering the traces of 2x2 matrices A and B in $SL_2(\mathbb{R})$. Although tr(AB) is dependent on A and B, it is independent of tr(A) and tr(B). To see this, let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

then

$$AB = \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix}.$$

From these matrices we see that tr(A) and tr(B) are each dependent on only two entries in the corresponding matrix. However tr(AB) is dependent on all eight entries from matrices A and B. In fact it is easy to construct examples where tr(A) and tr(B) remain fixed but tr(AB) changes depending on the other entries of A and B.

The above character equation therefore contains three independent variables, and with the substitutions $x = \chi[a], y = \chi[b], z = \chi[ab]$ and $C = \chi[c]$, is equivalent to;

$$x^2 + y^2 + z^2 - xyz - 2 = C \tag{5.1}$$

for C fixed.

We now use what we know about x, y and z to see that they can take a range of values while C remains fixed.

The group elements that the curves a, b and ab represent are all loxodromic. Therefore the value of the trace (and hence the character) must be greater than 2 or less than -2. (Equivalently the square of the trace must be greater than 4.) Note that these traces can get arbitrarily close to ± 2 (i.e. a curve can be shrunk to almost a point) and still be loxodromic.

We show that for any x and y we can find a value for z that satisfies equation 5.1. To do this first rearrange equation 5.1 and solve for z using the quadratic formula.

$$z = \frac{1}{2}\left(xy \pm \sqrt{x^2y^2 - 4(x^2 + y^2 - 2 - C)}\right) \tag{5.2}$$

As we are considering surfaces and so representatives into $SL_2(\mathbb{R})$ we want $x, y, z \in \mathbb{R}$. Hence z will only be well defined if the discriminant is non-negative. First consider where it is equal to zero.

$$x^2y^2 - 4(x^2 + y^2 - 2 - C) = 0 \Leftrightarrow 4x^2 + 4y^2 - x^2y^2 = 8 + 4C$$

As $x^2 > 4$ and $y^2 > 4$, then $4x^2 + 4y^2 - x^2y^2 > 16 + 16 - 16 = 16$, therefore $8 + 4C > 16 \Rightarrow C > 2$. (This is true as the geodesic boundary curve is loxodromic, although it is short so almost parabolic.)

In this case $z = \frac{1}{2}xy \Rightarrow z^2 = \frac{1}{4}x^2y^2 > 4$ as required.

Now assume that the discriminant is positive.

$$\begin{aligned} x^2y^2 - 4x^2 - 4y^2 + 8 + 4C &> 0 \\ 8 + 4C &> 4x^2 + 4y^2 - x^2y^2 > 16 \end{aligned} \tag{5.3}$$

(Again this is true as $C > 2$.)

In this case $z = \frac{1}{2}xy \pm \frac{\sqrt{discriminant}}{2}$, so for each x and y we have two possible values for z. However, as $x^2 > 4$ and $y^2 > 4$ then $\frac{1}{2}xy > 2$ or $\frac{1}{2}xy < -2$, and so at least one value will give $z > 2$ or $z < -2$ as required (i.e. if $\frac{1}{2}xy > 2$ take the positive square root, and if $\frac{1}{2}xy < -2$ take the negative square root to be certain).

To complete this part of the proof we show that we can make x and y arbitrarily close to 2 or -2 (from appropriate side) and still be able to find a value for z, all with C fixed. We split into two cases.

First let x and y both approach 2 from above (so $x, y > 2$), or both approach -2 from below (so $x, y < -2$). Then,

$$x^2 + y^2 + z^2 - xyz - 2 = C \rightarrow 4 + 4 + z^2 - 4z - 2 = C$$

so we get a quadratic in z,

$$z^2 - 4z + (6 - C) = 0 \Rightarrow z = \frac{4 \pm \sqrt{4C-8}}{2} = 2 \pm \sqrt{C-2}$$

If we take the positive square root (and assume $C > 2$), then $z > 2$ as required. Hence we can always find an appropriate value for z in this case.

Now let $x \rightarrow 2$ from above and $y \rightarrow -2$ from below (all with C fixed). Then,

$$x^2 + y^2 + z^2 - xyz - 2 = C \rightarrow 4 + 4 + z^2 + 4z - 2 = C$$

so again we get a quadratic in z,

$$z^2 + 4z + (6 - C) = 0 \Rightarrow z = \frac{-4 \pm \sqrt{4C-8}}{2} = -2 \pm \sqrt{C-2}$$

This time we take the negative square root (and assume $C > 2$), to guarantee $z < -2$ as required.

Therefore for every x and y (such that $x^2, y^2 > 4$) it is possible to find at least one value for z in the appropriate range. Hence there is a lot of scope for movement in these variables.

Referring back to lengths, from this we know that l_a and l_b can be changed as much as is required without altering the length of c. Now to complete the proof of this lemma, we need to show that the length of any other simple closed curve on S can vary unless it is homotopic to the boundary.

Any non-trivial simple closed geodesic p on S must either intersect a or b (or both) at some point, or will be homotopic to $c^{\pm k}$. (To see this is the case, we use the pants decomposition on S. Decompose S into a single pair of pants by cutting along b. There are two possibilities for a simple closed geodesic p on S when considered on the pair of pants. Either p is a simple closed geodesic on the pair of pants also or it is not. The only simple closed geodesics on a pair of pants are the boundary curves. Hence in this case either p is homotopic to c or p is homotopic to b, and hence intersects a. If p is a simple closed geodesic on S that is not a simple closed geodesic on the pair of pants created from S, then it must intersect the boundary components of the pair of pants, and hence the only possibility in this case is that p intersects b.) In the latter case, p will have the same length as c and l_p will be fixed. We need to know that this is the only case where p has fixed length. Therefore consider the other cases, i.e. where p is a closed curve that intersects at least one of the two generating curves a and b.

Let p be a simple closed geodesic that intersects b. (The case where it intersects a will have a similar argument.) From the above analysis l_b is variable, and can be made arbitrarily small whilst keeping c fixed length. Apply the collar lemma (see lemma 3.1.5) around b. This means that we are putting a collar or topological cylinder around b whose width ω depends only on l_b. Any geodesic intersecting b at some point must cross the entire width of the collar (otherwise a homotopically equivalent curve that does not intersect b can be found - see figure 5.14). Therefore this collar gives a lower bound on the length of the intersecting curve p. This lower bound is 2ω (multiplied by the number of intersection points between b and p.)

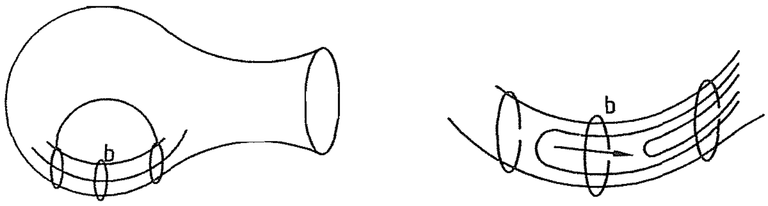


Fig 5.14: collar around b

Now let b decrease in length. By the collar lemma, the cylinder around b gets wider (as $l_b \to 0$ $\omega \to \infty$). From the above, b can be made arbitrarily short while l_c stays constant. Hence we can shrink b enough to force l_p to increase. (The lower bound will be greater than the original length of p at some point.) Therefore from this point onwards as we decrease l_b we are increasing l_p and hence the length of p can be forced to change without altering the length of the boundary geodesic.

We use a similar argument if p intersects a. Therefore the only closed curves with fixed length are those that are homotopic to c^k for any integer k. As p is simple, then if p has fixed length then $k = \pm 1$ as required. □

One thing to note at this point (in line with the observation at the end of section 5.4) is that the only specification on the hyperbolic structure on S is that it keeps the boundary curve short and at constant length. Therefore this result will be valid for all hyperbolic structures with this constraint, and in particular for the hyperbolic structure imposed on the surfaces of M by the Canary, Minsky, Taylor construction. The only constraint in the construction was a requirement for short boundary curves. It is therefore feasible that this manipulation of curves may be done within the scope of the bigger construction.

This result has so far only been shown for a surface of genus one, but it is necessary to extend this result to higher genus surfaces with connected non-empty boundary. One method would be to try and apply a similar idea as in the proof of lemma 5.5.1, but in higher genus surfaces it is apparent that this method would run into difficulties. Firstly the character equation for the boundary curve will become far more complicated, and the resulting equation will have considerably more variables to work with, which would make it hard to analyse in the same way. There is also the problem of looking at the separating geodesics of S, which do not necessarily intersect the generating curves of the surface. Hence it would be necessary to handle these cases separately. However instead of proceeding with this method, we find a new proof which tackles both of these problems and does it all in one.

Lemma 5.5.2 *Let S be a compact, oriented surface of genus $g \geq 2$ with connected non-empty boundary consisting of geodesic c of length l_c. (Here we have assumed that S has hyperbolic structure with geodesic boundary, so the end has been cut off around this boundary.) Let hyp(l_c) be the space of hyperbolic structures on S with l_c constant. Let p be a simple closed geodesic on S with length l_p. Then if $p \not\cong c^{\pm k}$ for some integer k, then l_p is non-constant on hyp(l_c).*

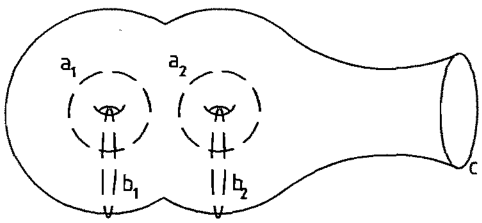


Fig 5.15: genus g surface

Proof: Let $a_1, b_1, a_2, b_2, ..., a_g, b_g \in \pi_1(S)$ generate S. Assume that p is a simple closed curve on S such that $p \not\cong c^{\pm k}$ for any integer k and show that the length of p (denoted l_p) can vary while l_c is fixed.

We utilise the pants decomposition as given in definition 3.1.3. Decompose S using the pants decomposition into pairs of pants. This will produce $2g-1$ pieces by cutting along $3g-2$ geodesics on S. Take an exact copy of S called S^* and glue the boundaries of S and S^* together. (Effectively reflect S across its boundary geodesic to get $S \cup S^*$, so we are taking the double of S - see figure 5.16.) This gives a closed surface without boundary with a marked curve c, of fixed length (the common boundary of S and S^*). Taking the double (and reflecting) ensures that our new surface has the same structure throughout (for consistency) and makes sure that no lengths are altered in the process. Complete the pants decomposition on this new surface (it does not matter what way this is done as the new half of the surface will not be used directly). Using Fenchel-Nielsen coordinates we know that we can change the lengths of a subset of these pants curves while keeping others of fixed length. In particular we can change the length of all the pants curves contained in S except c, and keep l_c fixed.

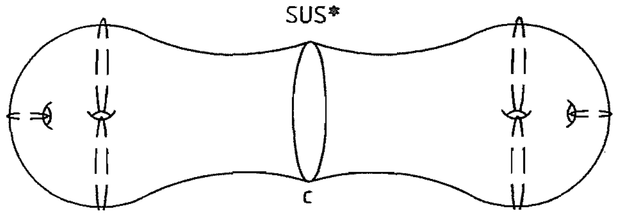


Fig 5.16: decomposition of the double of S

To complete the proof it is necessary to show that any other simple closed geodesic on S must vary in length as the curves in the pants decomposition change length.

First note that any simple closed geodesic on S must either intersect the geodesics in the decomposition or wrap around one of the pants curves.

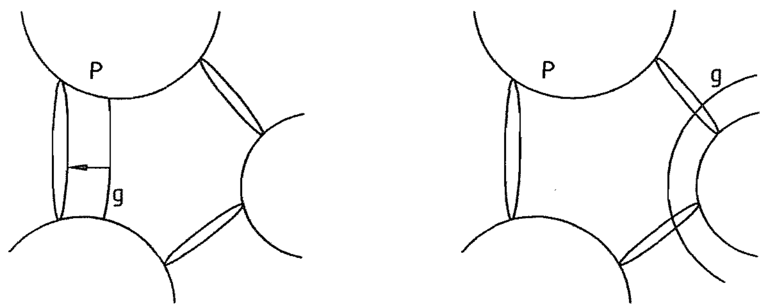


Fig 5.17: simple closed geodesics on P

To see this, take one pair of pants P in the decomposition and consider a simple closed geodesic g such that g and P have some points in common. Here there are two options, either g is contained entirely in P, or only part of g is contained in P (see figure 5.17).

If g is contained entirely in P, then g is a simple closed geodesic on a pair of pants, and the only such curves are equivalent to one of the boundaries of P. Hence g will wrap around one of the geodesics in the pants decomposition of S. If only part of g is contained in P, then as g is a simple closed geodesic on S, then it must intersect at least one of the boundary components of P, and hence will intersect at least one geodesic in the decomposition. (In fact it will intersect at least two.)

If the geodesic wraps around one of the pants curves, then it is homotopically equivalent to one of the simple closed geodesics in the decomposition that lies in S. Therefore as that pants curve varies in length so will the length of any geodesic homotopic to it or a power of it. (Note that if the geodesic is homotopic to c then it will have constant length as c has constant length. We want these to be the only such curves.)

The only simple closed geodesics left to consider are those intersecting at least one of the curves in the decomposition. We need to know that if we make small changes in the lengths of these curves in the decomposition then any curve crossing them will automatically change length.

(N.B. Geodesic lengths are real analytic in the real analytic structure of Teichmuller space.)

If we let the length of the geodesics in the decomposition shorten (so length tends to zero) then the length of any crossing simple closed geodesic will tend to infinity. This is because p must cross the collars of all of the pants curves it intersects, therefore the length of p is a non-constant function of the lengths of the geodesics it crosses in the pants decomposition, and since it is real analytic, then in any neighbourhood it will actually vary.

Hence if the length of any of the geodesics in the decomposition changes then the length of the intersecting geodesic will also change in length. This shows that it is possible to change the length of a simple closed geodesic on S without changing the length of the boundary geodesic. □

Note that this proof in the context in which it will be utilised (i.e. for looking at the spine of the Book of I-bundles) uses the fact that the pattern of crossings over the pants curves for a given geodesic does not change as we change the lengths of the boundary curves. (This was commented on in section 3.1.)

We take a moment here to note a few things from the proof of lemma 5.5.2. Firstly the new surface created, $S \cup S^*$, is a closed surface and so it would have been possible to use elements of McShane's surface proof on $S \cup S^*$ as follows. In the proof of lemma 4.4 it was noted that two disjoint simple closed geodesics on a closed surface do not have the same character because the length of one can be changed independently from the other. (This result was true for any hyperbolic structure on the surface.) Relating this to the proof of lemma 5.5.2, it means that the length of any simple closed geodesic p on $S \cup S^*$ which is disjoint from c can be changed while l_c remains fixed. Choose p so it is contained entirely in S. Then the hypotheses are valid and this gives an alternative proof to lemma 5.5.2.

The other thing to note is that the case where $g = 1$ (as covered in lemma 5.5.1) can be incorporated and proven in the same way. Hence lemma 5.5.1 and lemma 5.5.2 can be combined into the following.

Lemma 5.5.3 *Let S be a compact, oriented hyperbolic surface of genus g with connected non-empty boundary consisting of geodesic c of length l_c. Let $hyp(l_c)$ be the space of hyperbolic structures on S with l_c constant. Let p be a simple closed curve on S with length l_p. Then if $p \not\cong c^k$ $(k \in \mathbb{Z})$ then l_p is non-constant on $hyp(l_c)$.*

This result can be extended further, by noting that the condition that the boundary is connected is a redundant assumption and can be removed.

Lemma 5.5.4 *Let S be a compact, oriented surface of genus $g \geq 1$ with multiple boundary components consisting of geodesics $c_1, c_2, ..., c_m$ of lengths $l_{c_1}, l_{c_2}, ..., l_{c_m}$ respectively. Let $hyp(l_c)$ be the space of hyperbolic structures on S with $\{l_{c_i}\}$ $(i = 1, ..., m)$ constant. Let p be any simple closed geodesic on S with length l_p. Then if $p \not\cong c_i^k$ $(k \in \mathbb{Z}, i = 1, 2, .., m)$ then l_p is non-constant on $hyp(l_c)$.*

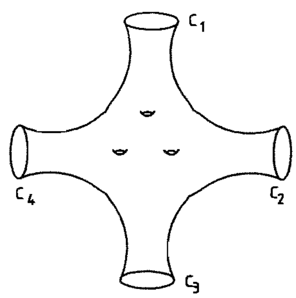


Fig 5.18: S with multiple boundary components

The proof follows the same method as lemma 5.5.2.

Proof: Decompose S using a pants decomposition and then take the double of S. By this we mean make an exact copy of S called S^* and glue it to the original surface along like boundaries. (Essentially reflect S across its boundaries, this ensures that the hyperbolic structure on S and the lengths of the curves are not altered.) The resulting surface $S \cup S^*$ is closed with m marked curves $c_1, c_2, ..., c_m$ (common boundary geodesics of S and S^*). (See figure 5.19 below for an example of this.)

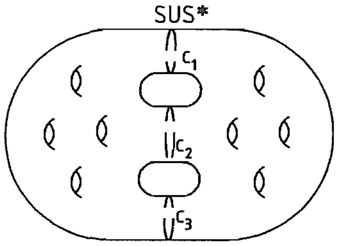


Fig 5.19: the double of S

Complete the pants decomposition on this new surface. The decomposition on the new half does not necessarily have to be the same as the first half, all that is required is a decomposition of the whole surface $S \cup S^*$ which includes the m marked geodesics $c_1, c_2, ..., c_m$. As before, the lengths of a subset of these pants curves may be changed while keeping the others constant length. Therefore let $c_1, c_2, ..., c_m$ be in the subset of fixed length geodesics in the decomposition, and let all the other pants curves contained in S be in the subset of geodesics of variable length. (As the pants curves in S^* are not required, they can be in either subset.)

Let p be a simple closed geodesic in S such that $p \not\cong c_i^k$. As p is contained in S, we know that p does not intersect any c_i on $S \cup S^*$. It therefore must be disjoint from all the boundary components of S. From the previous proof we know that l_p may be written as a non-constant function in the lengths of the pants curves it intersects, and will therefore have variable length as we change the lengths of the pants curves in the variable subset.

Hence we may vary the length of p and keep $l_{c_1}, l_{c_2}, ..., l_{c_m}$ fixed as required. □

The only hyperbolic surfaces that have been excluded from lemma 5.5.4 are those of genus 0 with multiple boundary components ($m \geq 3$). The same method of proof can be applied if $m \geq 4$ (sphere with at least four holes), and so this case can be incorporated into lemma 5.5.4. Hence all of the above can be incorporated into the following,

Theorem 5.5.5 *Let S be a compact, oriented surface of genus $g \geq 1$ with multiple boundary components consisting of geodesics $c_1, c_2, ..., c_m$ of lengths $l_{c_1}, l_{c_2}, ..., l_{c_m}$, or a surface of genus 0 with at least four boundary components. Let hyp(l_c) be the space of* *hyperbolic structures on S with l_{c_i} $(i = 1, ..., m)$ constant. Let p be any simple closed geodesic on S with length l_p. Then if $p \not\cong c_i^k$ $(k \in \mathbb{Z}, i = 1, 2, .., m)$ then l_p is non-constant on $hyp(l_c)$.*

The only other hyperbolic surface is a three-holed sphere. This is already a pair of pants and so is uniquely determined by its boundary curves. Hence changing the length of a simple closed curve on this surface will change the length of at least one of the boundary curves. It is also the case that the only simple closed geodesics on a pair of pants are the boundary curves themselves. Hence this particular surface does not fit in with these results. This should not cause a problem for general books of I-bundles, even though they may contain pair of pants pieces. To see why this is the case consider a book of I-bundles M that contains an I-bundle base P that is a pair of pants.

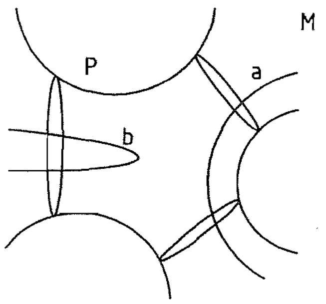


Fig 5.20: pair of pants in M

We will be using the results of this section to manipulate curves on F in such a way that it does not interfere with the hyperbolic structure on F and M. Any closed curve on F that passes through P must pass from one boundary component to another (see figure 5.20), such that the boundary components are distinct. (In figure 5.20 curve a is such a curve but curve b cannot occur as this will not be piece-wise geodesic as required, as the segment in P is trivial.) Hence there will be pieces of the closed curve on other component parts of F. We can therefore manipulate the curve there, and by theorem 5.5.5 we can do this whilst leaving the boundary components of P fixed. Hence if the general book of I-bundles has a spine containing a pair of pants it should not cause a problem.

For the specific book of I-bundles manifold that is initially to be considered (the 3-prong case), we only need to concern ourselves with surfaces with connected geodesic boundary. However the more general case will be required when extending to larger and more general books.

In this section we have established that it is possible to change the length of an arbitrary simple closed geodesic on a surface of genus g with non-empty (possibly disconnected) boundary, while keeping the length of the boundary geodesics fixed. Associating this to our manifold, as the spine is constructed from surfaces of this type, theorem 5.5.5 gives a way of manipulating curves on F. Note that neither of these results depended upon how long the boundary curve was.

In the CMT construction, what is essentially done is that a hyperbolic structure is put on each surface so that the ends are defined by geodesics. The ends are then cut off at this geodesic (so what is left is the convex core of the surface) and these surfaces are then glued together along these geodesics. As such these boundary geodesics are short, and this was the only constraint on the construction. Hence we are working in the space of hyperbolic structures for which boundary geodesics are short. The work in this section therefore can be applied to the surfaces that make up the spine F of M without interfering with the construction of M, or the hyperbolic structure. These results will therefore be of use later. First we need a way of connecting the geodesics in M to unique closed curves on F so that we may use these results about surfaces in relation to the character question.

5.6 Projecting geodesics onto F

The primary aim of this section is to build a link between the geodesics in M and closed curves on F. In particular we need to forge a connection between their lengths, and hence determining how much the curve on F needs to be manipulated in order for the geodesics length to change. Hence we wish to project the geodesics onto F in such a way that we get a unique closed curve on the spine.

The reason for building this projection is two-fold. Firstly we want to use the results from section 5.5, so that we may manipulate the closed curves and change their lengths on F without interfering with the family of hyperbolic structures or the construction of M. In the 3-manifold we do not have the same control over the geodesics, and so this link would give the ability to manipulate and have some control in M. Secondly, the connection between the character and the length of the curve in 3-dimensions only applies to the geodesics in M, and so (in general) this connection does not apply to closed curves on F. This is because any curve which exists on F and intersects the core curve will have 'corners' in M and hence will not be geodesic in the manifold. (There are exceptions, for example where the curve in F is a geodesic in M.) Therefore we need to build a connection between the lengths of the geodesic in M and the curve onto which it is projected on F.

Hence the projection of geodesics onto F is an important technical tool when considering the properties as described in section 5.3. As a starting point, we return to the description and construction of a book of I-bundles manifold as given in 5.4, and look at where the geodesics in M exist in relation to its spine. For the purpose of this section, we will be considering the initial 3-prong case. The projection will be extended to a general book of I-bundles in section 6.2.

5.6.1 CMT Revisited

Let M be a book of I-bundles manifold with single solid torus binding and three I-bundle pages. Let F be the spine of M, which is constructed from three base surfaces F_1, F_2, F_3 each of which has connected non-empty boundary.

In section 5.4, we gave a description of the lift of each base surface F_i to a totally geodesic subset or flat in $\mathbb{H}^3$. These flats are arranged in a tree such that at each lift of the geodesic boundary curve c (the core curve of the solid torus), there are two other flats which are equally spaced, so the angles between them are equal to $\frac{2\pi}{3}$ (and hence evenly spaced around the lift of c). We have a tree of flats here as non-adjacent flats are disjoint, and hence between any two flats there is a unique sequence of flats.

Viewing this picture in the complex plane (or alternatively looking down from infinity), this gives a tree of circles in $\mathbb{C}$. The limit set of the group Γ which determines this lift is contained in the closure of this tree of circles. (Remember each flat will determine a surface in F when moving back to the manifold.)

Normalize the tree so that the root is chosen to be the unit circle. (We may arbitrarily choose which flat will be the root.) Let C_O and C_I be outscribed and inscribed circles on this tree of circles. (See figure 5.21 for this tree of circles picture.)

These two circles are determined by the tree as they enclose the whole tree of circles between them. (As an aside here, note that these circles are not necessarily uniquely determined. We could force the uniqueness by insisting that both C_O and C_I are centred at (0,0) (or at the centre of the circle chosen to be the root of the tree). However the purpose of introducing C_O and C_I is to look at the limiting behaviour and monitor what happens to the tree as the core curve shrinks. This behaviour will be the same independent of the choice of C_O and C_I. Hence this is an arbitrary choice to make.)

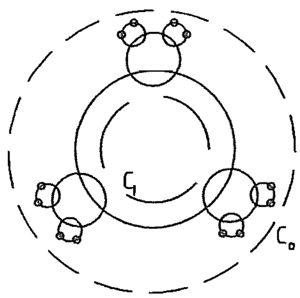


Fig 5.21: bounded tree of circles

As the limit set is contained in the closure of the tree of circles, it is sandwiched between C_O and C_I. Hence, by definition, the convex hull of the limit set of Γ is sandwiched between the hyperplanes H_O and H_I meeting the complex plane in the circles C_O and C_I respectively.

We now consider what happens to this picture as the length of the core curve is shortened.

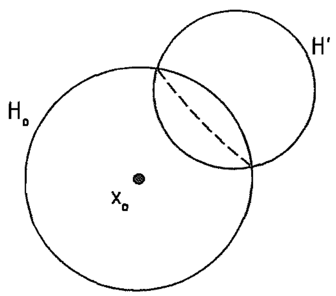


Fig 5.22: normalized tree

Let H_0 be the flat which is chosen to be the root of the tree, and choose a point x_0 in the μ-thick part of H_0. (Remember that μ is a constant that is smaller than the Margulis constant. Hence the μ-thin part of a flat consists of the union of disjoint pieces, each of which is a neighbourhood of the lift of a geodesic boundary. The μ-thick part is the complement of this.) Normalize so that H_0 is contained in the hemisphere which intersects the complex plane in the unit circle, and x_0 is the top most point (i.e. $x_0 = (0, 0, 1)$). Each child H' of H_0 meets it along a geodesic, and H' is contained in a half-plane which meets the complex plane in a circle (see figure 5.22).

Let C be one of the circles in the tree, and let $r(C)$ denote the Euclidean diameter of C. Let $S(C)$ denote the set of children of C (set of adjacent circles). Fix any positive $\rho < \frac{1}{2}$ and then, assuming c is sufficiently short, (depending on choice of ρ) then Canary, Minksy and Taylor (in [CMT99]) prove that the diameters of the circles in the tree are decreasing from the root outwards. Moreover they show that for any C and $D \in S(C)$ we have,

$$r(D) \leq \rho r(C) \tag{5.4}$$

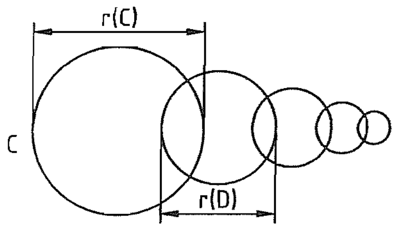


Fig 5.23: diameter of tree of circles

(Figure 5.23 shows the tree with the biggest overall diameter, as an illustration of how the circles are decreasing in size.)

We give an outline of part of the proof of equation 5.4 as explained in [CMT99] to illustrate why this is true.

Outline proof of equation 5.4

Let P and P' be hyperbolic planes such that $\delta_\infty(P) = C$ and $\delta_\infty(P') = D$. (Here δ_∞ denotes the boundary at infinity or equivalently where P and P' intersect $\mathbb{C}$.) Hence P and P' intersect in a geodesic α. Let $thin(P, P')$ denote the component of the μ-thin part associated to the intersection of P and P' (so $thin(P, P')$ will consist of a neighbourhood of α).

Let x be the top most point of P when looking in $\mathbb{H}^3$, and assume x is outside of $thin(P, P')$. Then the geodesic chain $\gamma_{x,y}$ for any point $y \in P'$ has initial segment with length γ_1 which is bounded below by k (which can be made arbitrarily large by shrinking the length of the core curve - see section 5.4 for details on this bound).

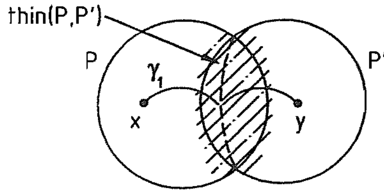


Fig 5.24: proof of equation 5.4

Hence all of P' is separated from x by the bisecting hemisphere of γ_1, which is at a distance at least $\frac{k}{2}$ from x. Therefore the diameter of P' is at most $ae^{\frac{-k}{2}} diam(P)$ for a fixed constant a. This gives

$$r(D) \leq ae^{\frac{-k}{2}} r(C) \tag{5.5}$$

and as long as k is chosen so that $ae^{\frac{-k}{2}} \leq \rho$ then we have the desired bound. We can ensure k satisfies this by making c sufficiently short.

Canary, Minsky and Taylor go on to complete this proof by showing that the assumption made that x is outside $thin(P, P')$ is true. This part of the proof is not included here as the details are not required for this work, but we refer the reader to [CMT99] for the details.

This outline proof shows that the diameters of the circles in the tree are getting smaller the further we move from the root. Moreover as c shrinks (so the core curve shortens), the diameters get smaller more rapidly. Hence the circles in the tree will become closer together (but still remaining disjoint if non-adjacent). To see this, look at equation 5.5.

$$r(D) \leq ae^{\frac{-k}{2}} r(C)$$

As $l_c \to 0$ (here l_c denotes the length of c), then $k \to \infty$ (see section 5.4 for this) and so $e^{\frac{-k}{2}} \to 0$. As we also know that $ae^{\frac{-k}{2}} \leq \rho \leq \frac{1}{2}$, then this shows that the diameter of the tree of circles decreases as c shrinks. (As an aside here, note that this will be in contrast to the geodesic chain $\gamma_{x,y}$ discussed in section 5.4. As c shrinks, the lengths of each segment of $\gamma_{x,y}$ will have an increasing lower bound, which implies that $\gamma_{x,y}$ will get longer.)

Apply this result to the outscribed and inscribed circles picture. As C_O and C_I are determined by the tree of circles, if the diameter of the tree decreases, then the choice of C_O and C_I will become closer to the root of the tree. (Note that this is independent of the choice of C_O and C_I.) The shorter c becomes, the smaller the diameter of the tree and the closer C_O and C_I become to the root of the tree (and consequently to each other - see figure 5.25).

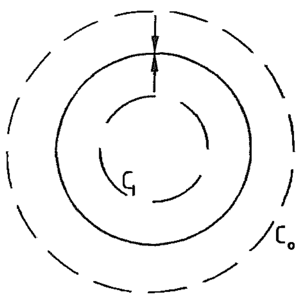


Fig 5.25: shrinking tree of circles

As the convex hull is sandwiched between the hyperplanes determined by C_O and C_I, then the convex hull is approaching the hyperplane meeting $\mathbb{C}$ in the unit circle. (Note that this limit is never reached as some space is needed for the tree.) The convex hull is therefore getting 'thinner' at the top (although the hyperplanes determining C_O and C_I will still be infinitely far apart where they intersect the complex plane).

The geodesics in M lie in its convex core, which is the quotient of the convex hull by the group Γ. The position of the geodesics in relation to the spine F depends upon their position in this tree of circles/hyperplanes picture. As long as the geodesics stay within a bounded region of the top of the hyperplane which is determined by the root of the tree, and therefore lie in a region where the convex hull is thinnest, then sections of the geodesic will be arbitrarily close to the spine (and will get closer as the length of the core curve reduces).

The potential problem that will cause this not to be true (i.e. the geodesics staying within a bounded region of the top of the hyperplane determined by the root of the tree) is if in the lift to $\mathbb{H}^3$, the end points of the loxodromic axis representing the geodesic are getting closer together. This would mean that the geodesic is moving away from the top of the hyperplane determined by the root of the tree of circles.

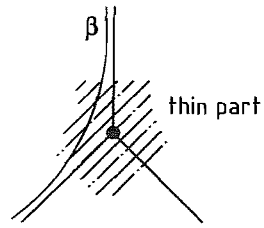


Fig 5.26: location of geodesics in M

To see that this cannot happen we return to looking at M and its convex core. A geodesic β in M will not be contained entirely in the thin part of the manifold, and hence the projection of β onto F will intersect the thick part of F. If the geodesic is contained entirely in the thin part then it will be trivial or will be homotopic to a power of the core curve. (This applies to a geodesic in M and after projecting onto F.) (Figure 5.26 illustrates the location of geodesic β in M such that β exists partly in the thin part of M. Note that the converse is false, as a geodesic in M does not have to pass into the thin part, but such a β will exist on F.)

Consider each base surface F_i of F and what happens as $l_c \to 0$ (here l_c is the length of the core curve c). Each base surface in the limit will be a surface with a puncture (see figure 5.27). The half collars around c on F (i.e. on each F_i, for $i = 1, 2, 3$) will get wider by the collar lemma, and this is dependent only on l_c. Therefore the distance between the thick part of each F_i and the geodesic boundary is getting bigger. Although in the limit each F_i has an infinite end, the thick part remains bounded (see shaded region in figure 5.27) because of this distance increasing.

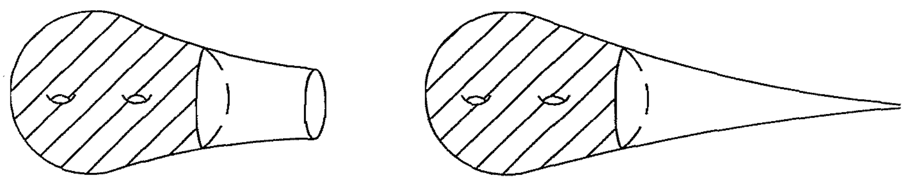


Fig 5.27: limiting behaviour as c shortens

These regions will contain all the information for $\pi_1(F)$.

If we lift back into $\mathbb{H}^3$, then we need to ensure that the thick part of the base surface lifted to the flat chosen to be the root of the tree, is a region around the top of the hyperplane H (where H is the hyperplane containing this flat and hence determined by the root of the tree). This will ensure that the geodesics pass through this part as required. However if this is not the case, we can conjugate by Möbius transformations that fix H, but move the interior of H. Pick the Möbius transformation so this region is moved to the top of H (as in figure 5.28).

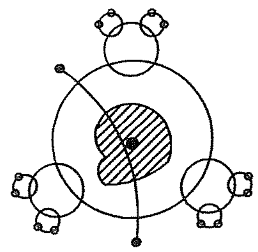


Fig 5.28: geodesic in lift

Any loxodromic axis representing a geodesic in M will pass through this shaded region (as shown in figure 5.28). Hence we can control the geodesics in the sense that we can ensure that they have sections which are arbitrarily close to F (i.e. the points in the thick parts).

Note that when looking in the lift to $\mathbb{H}^3$, the convex hull is not only thinnest at the top, but that this is also the furthest point away from any of the lifts of c. Hence an arbitrary geodesic in M will be closest to F in the regions furthest away from the neighbourhood of c (and furthest from F within a neighbourhood of c). (Figure 5.29 illustrates the convex core of M in relation to the spine of M.)

We can force a geodesic in M to become arbitrarily close to F by letting $l_c \to 0$, and by the construction described in section 5.4 we can do this and still have M hyperbolizable.

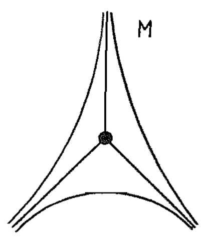


Fig 5.29: convex core of M

A final note to make is that a closed curve on F which is totally disjoint from c (so exists entirely on one surface component F_i for $i = 1, 2, 3$) will be a geodesic in M. (Hence these will be geodesics in M which lie on its spine.) As M is a regular neighbourhood of F, all geodesics will lie close to F in the above sense, assuming c is short.

5.6.2 The projection

From section 5.4 we know that M is a regular neighbourhood of its spine F. From section 5.6.1 we know that the geodesics of M lie close to the spine in the sense that we can ensure that they have sections that are arbitrarily close to F (i.e. in the lift to $\mathbb{H}^3$, the loxodromic axis representing a geodesic will pass within a bounded distance of the top of the hyperplane determined by the root of the tree). We can control how 'close' the geodesics get to F by letting the length of the core curve shorten. (In fact they become arbitrarily close as $l_c \to 0$.) Note also that a geodesic in M will be closer to F the further away we move from the core curve (as seen in figure 5.29 above).

The aim of this part is to put a measure on this by trying to connect the length of a geodesic in M to the length of a closed curve on F. To achieve this, we first need to define a projection from M onto F so that each geodesic β in M may be represented by a unique closed curve γ on F, and then compare the lengths of β and γ. We have information on the length of γ (and a way of controlling its length) coming from the change in length of the core curve. We want to relate this to the length of β and show that if $l_\gamma \to \infty$ then $l_\beta \nrightarrow 0$ (where l_γ denotes the length of γ).

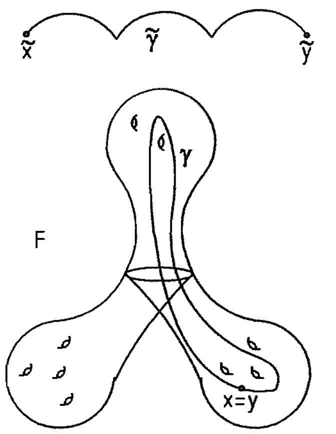


Fig 5.30: unique shortest path

In section 5.4, we showed that in the lift to $\mathbb{H}^3$, that between any two points $\widetilde{x}$ and $\widetilde{y}$ in any two distinct flats there is a unique shortest path $\widetilde{\gamma}$ (i.e. see lemma 5.4.1). Move this back to the spine F, and let x and y be the points lifted to $\widetilde{x}$ and $\widetilde{y}$ respectively, then if $x = y$ in F, then $\widetilde{\gamma}$ corresponds to a piece-wise geodesic closed curve γ on F that passes through $x = y$ and intersects the core curve c a specified number of times. (This is dependent on the number of segments in $\widetilde{\gamma}$ - see figure 5.30). Hence lemma 5.4.1 shows that there is a unique shortest closed curve on F that passes through a particular point and intersecting c a specific number of times. (By the nature of the proof of lemma 5.4.1, this is true for any base point x and any number of intersection points with c - i.e. these are the only two pieces of information required to find the unique shortest path.)

Let β be an arbitrary geodesic in M, and let γ be a closed curve on F (so γ is piece-wise geodesic in M). We want to push β onto γ such that γ is unique to β. This would give a one-to-one correspondence between geodesics in M and closed curves on F (so only β is projected onto γ).

As M is a regular neighbourhood of F, we have an inclusion map $F \hookrightarrow M$ and from this we get an isomorphism $\pi_1(F) \to \pi_1(M)$ between the fundamental groups. Hence for each element $g_\beta \in \pi_1(M)$ there is a unique element $g_\gamma \in \pi_1(F)$ such that $g_\gamma \to g_\beta$ under this isomorphism. Here both g_β and g_γ represent conjugacy classes in $\pi_1(M)$ and $\pi_1(F)$ respectively.

For each conjugacy class g_β in $\pi_1(M)$ there exists a unique geodesic β in M (so a unique path of shortest length, so all other curves are homotopic to β). When we lift to $\mathbb{H}^3$, the geodesic β will be represented by a hyperbolic line which corresponds to the axis of the appropriate loxodromic element $g_\beta \in \Gamma$, where Γ is the group determining the lift. (Remember we already know that this loxodromic axis can be forced to pass within a bounded region of the top of the hyperplane determined by the root of the tree of flats by the discussion in section 5.6.1.)

For each conjugacy class g_γ in $\pi_1(F)$ there will be a set of closed piece-wise geodesic curves on F. We need to show that there is a unique shortest curve amongst this set. Lift F to $\mathbb{H}^3$ as described in section 5.4. Choose a point y in one of the flats and look at all possible translates under the group action (of both y and the flat containing y). One potential problem is that we could get a path in the tree of flats which has back-tracking.

By this we mean that it doubles back on itself (retraces its steps). (See figure 5.31 which illustrates such a path. The concaternation of the paths xy and yz backtracks along the arcs py and yp.)

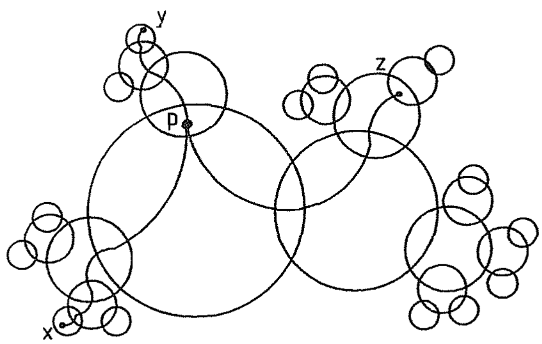


Fig 5.31: back-tracking path in tree

However for each path with back-tracking we can find one without back-tracking that is shorter. To do this we move back to the point before the piece of path that is doubled up - as marked in figure 5.31, and then look at the translates of that point. In this way we can eliminate all back-tracking.

Once we have a direct path, we then have two points y and $g_\gamma(y)$ in two distinct flats that represent the end points of γ. We then use lemma 5.4.1 to find the unique shortest path between these two points (i.e. we bend the hyperplanes containing the flats so that the angle between them is π, and take the hyperbolic line segment between y and $g_\gamma(y)$. Then re-bend the hyperplanes back to their original position, and this hyperbolic line segment gives us γ as required). We can normalize the tree so that a segment of γ passes through the flat chosen to be the root of the tree. Then γ will pass close to the top of the hyperplane containing this flat. (This mean through the shaded region marked in figure 5.28 in section 5.6.1.)

Let this unique shortest piece-wise geodesic closed curve γ on F be the representative of β on F. This projection then associates geodesic β to a closed curve γ, hence ensuring β is mapped onto a unique piece-wise geodesic on F.

Now we have made an association between geodesics in M and closed curves on the spine, we want to make a connection between their lengths.

From the argument made in section 5.6.1, it is known that β and consequently γ, must have sections in the μ-thick part of M and F (otherwise both β and γ will be homotopic to the core curve of the solid torus). (Here μ is a fixed constant that is smaller than the Margulis constant, as in section 5.4.) Let $\beta \rightarrow \gamma$ via the projection (so γ is the unique closed curve on F associated to the geodesic β in M). Choose a base point y on the piece-wise geodesic γ on F such that y is in the μ-thick part of F. This can be done by taking a point which is furthest away from the core curve c. (If there is more than one such point then an arbitrary choice can be made as the only requirement is that y is in the μ-thick part.) Then by the construction as given in section 5.4, each geodesic segment of γ will have length bounded below by k which depends only on the length of c.

In the lift to $\mathbb{H}^3$ we normalize the picture so that the root of the tree contains a lift of the point y, and then use Möbius transformations to ensure y is within a bounded distance of the top of the hyperplane determined by the root. (We can do this by the discussion at the end of section 5.6.1.) This means that the lift of geodesic β will be very close to F at the lift of the point y in γ.

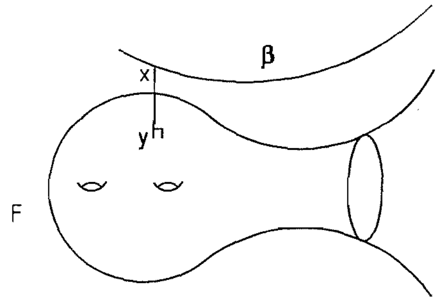


Fig 5.32: orthogonal projection

The shortest distance between y and the geodesic β will be along the line that meets F at y orthogonally. Let x be the point where this orthogonal line intersects β (see figure 5.32).

Now lift to $\mathbb{H}^3$ using the developing map given in section 5.4. The closed curve γ on F will lift to a piece-wise geodesic path from $\widetilde{y}$ to $g_\gamma(\widetilde{y})$ (where $g_\gamma \in \Gamma$, where Γ defines the lift), and the geodesic β will be represented by a hyperbolic line (the axis of the corresponding loxodromic element $g_\beta \in \Gamma$) with end points $\widetilde{x}$ and $g_\beta(\widetilde{x})$. As we have an orthogonal projection from x onto y ($\beta \rightarrow \gamma$) then $g_\beta(\widetilde{x}) \rightarrow g_\gamma(\widetilde{y})$ orthogonally also. (See figure 5.33.)

(Figure 5.33 shows β and γ lifted to $\mathbb{H}^3$ and the orthogonal projection from one to the other. We now use this picture to look at length.)

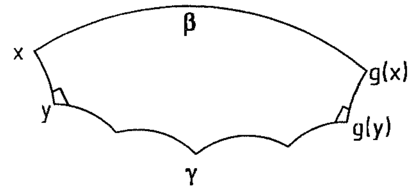


Fig 5.33: lifted orthogonal projection

As the core curve c shortens, from the discussion in section 5.4 we know that each segment of γ (including the first and last) will have length bounded below and that this bound increases as c shrinks. Hence $l_c \to 0 \Rightarrow l_\gamma \to \infty$. We now need to relate this to l_β and show that $l_\beta \nrightarrow 0$. To do this we find bounds on l_β in terms of l_γ.

We do this in two stages. First consider the case where β intersects F in the μ-thick part (but does not lie on F). In this situation the points x and y defined above will be equivalent, and in the lift to $\mathbb{H}^3$, the end points of the hyperbolic line representing β will be coincident with the end points of the broken geodesic path representing γ. We generalize this as follows.

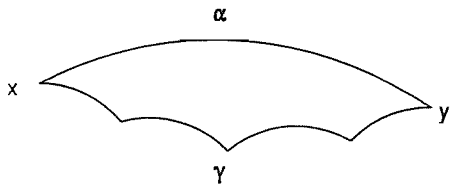


Fig 5.34: case 1 - geodesic intersects F

Let $\gamma_{x,y}$ be a broken geodesic in $\mathbb{H}^3$ with end points x and y, and consider the geodesic line segment $\alpha_{x,y}$ in $\mathbb{H}^3$ with the same end points (as in figure 5.34).

We will compare the lengths of γ and α. From the following lemma about broken geodesics as given in [Bis96] it is known that the two paths are 'close together'.

Lemma 5.6.1 *Given $\theta > 0$ there are $c, m < \infty$ so that the following is true. Suppose γ is a piece-wise geodesic path from a to b, by which we mean a union of disjoint (except for endpoints) geodesic arcs, each of hyperbolic length at least m and such that γ_i and γ_{i+1}*

meet at an angle $\geq \theta$. Then γ is within hyperbolic distance c of the geodesic arc δ connecting a and b.

From this we see that the two paths γ and α are within a bounded distance of each other, and in some sense the broken geodesic path γ cannot venture too far away from α, otherwise the end points will not meet up as required. This is not enough, as we require to know that as $l_\gamma \to \infty$ then $l_\alpha \nrightarrow 0$. Hence we look at estimating l_α in terms of l_γ.

First relate this to the situation being considered to find more information on γ. We know from section 5.4 that as the core curve is short, the length of each segment of a piece-wise geodesic path on F is bounded below, so has length at least m. As c gets smaller, the μ-thick part of each surface component of F is pushed away from its boundary, and so this lower bound m increases. The angle between the segments in the piece-wise geodesic (when looking in the lift to $\mathbb{H}^3$) will remain fixed at $\theta = \frac{2\pi}{3}$. (This is because the flats are evenly spaced around the lift of the geodesic boundary, and the initial manifold is a book of I-bundles with single solid torus binding and three I-bundle pages. Hence around each lift of the geodesic boundary will be three flats evenly spaced.) If we shrink the core curve (and increase m) we will not change this angle.

To start with we will look at what happens to l_α as m increases, and for simplicity we will assume for the moment that each segment of γ is a geodesic arc of length m. (We will relax this to $\geq m$ later.)

Hence $l_\gamma = nm$, where n is the number of segments in γ. (As an aside note that this also gives an indication of the number of times the corresponding closed curve on F intersects the core curve, i.e. $n - 1$ times). We start by considering the simplest case.

5.6.3 The case $n = 2$

Lemma 5.6.2 *Let γ be a broken geodesic path in $\mathbb{H}^3$ with two segments of length m, and angle $\theta = \frac{2\pi}{3}$ between them. Let α be the hyperbolic line segment with the same end points as γ. Then if $m \to \infty$, then $l_\alpha \nrightarrow 0$.*

Proof: Looking at the picture sideways on, γ and α together define a triangle with three hyperbolic line segments, two of length m (with angle fixed at $\frac{2\pi}{3}$ between them), and one of length l_α (see figure 5.35).

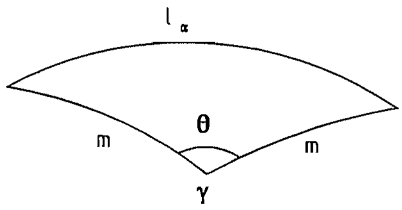


Fig 5.35: broken geodesic with n=2 segments

We use the hyperbolic law of cosines I (see equation 2.2 in section 2.1.3) to calculate l_α in terms of l_γ.

$$\begin{aligned} l_\alpha &= cosh^{-1}\left[cosh^2(m) - sinh^2(m)cos\left(\frac{2\pi}{3}\right)\right] \\ &= cosh^{-1}\left[cosh^2(m) + \frac{1}{2}sinh^2(m)\right] \end{aligned} \tag{5.6}$$

From equation 5.6, we see that l_α is dependent on m. To see what happens to l_α as m increases we look at the derivative of l_α with respect to m.

$$\frac{dl_\alpha}{dm} = \frac{3cosh(m)sinh(m)}{\sqrt{cosh^2(m) + \frac{1}{2}sinh^2(m) - 1}\sqrt{cosh^2(m) + \frac{1}{2}sinh^2(m) + 1}} \tag{5.7}$$

As $m > 0$ (measure of length), then $cosh(m) > 0$ and $sinh(m) > 0$, and so $\frac{dl_\alpha}{dm} \neq 0$. Hence a change in m, however small, will effect l_α. All that is left is to determine whether l_α increases or decreases with respect to m.

The numerator of equation 5.7 is always positive. In the denominator there are two positive square roots, and so the denominator will also be positive for all $m > 0$, assuming that the square roots are well defined (i.e. we are in $\mathbb{R}$, so do not want to take a square root of a negative number).

First consider $\sqrt{cosh^2(m) + \frac{1}{2}sinh^2(m) - 1}$. Using $cosh^2(m) = sinh^2(m) + 1$,

$$cosh^2(m) + \frac{1}{2}sinh^2(m) - 1 = \frac{3}{2}sinh^2(m) > 0 \tag{5.8}$$

for all $m > 0$. Hence this is always well defined.

Similarly, the other square root,

$$cosh^2(m) + \frac{1}{2}sinh^2(m) + 1 = \frac{3}{2}sinh^2(m) + 2 > 0 \tag{5.9}$$

for all $m > 0$.

Hence $\frac{dl_\alpha}{dm}$ is well defined for all positive m, and moreover $\frac{dl_\alpha}{dm} > 0$. Hence l_α is monotonically increasing with respect to m. Hence if $m \to \infty$ (and $l_\gamma \to \infty$) $\Rightarrow l_\alpha \to \infty$ and more importantly $l_\alpha \nrightarrow 0$ as required. □

From this proof we note that however small an increase in m, l_α will increase. (Hence a change in m gives a change to both l_γ and l_α.)

As an aside here, we are able to calculate how close l_α and l_γ can get by considering the limit of their difference as m increases. To do this use equation 5.6 and consider how l_α behaves first.

$$l_\alpha = cosh^{-1}\left[cosh^2(m) + \frac{1}{2}sinh^2(m)\right]$$

Now,

$$cosh(m) = \frac{e^m + e^{-m}}{2} \Rightarrow cosh^2(m) = \frac{1}{4}(e^{2m} + e^{-2m} + 2)$$

Similarly,

$$sinh(m) = \frac{e^m - e^{-m}}{2} \Rightarrow sinh^2(m) = \frac{1}{4}(e^{2m} + e^{-2m} - 2)$$

Therefore for m large, e^{2m} dominates as $e^{-2m} \to 0$, and so $cosh^2(m) \sim \frac{1}{4}e^{2m}$ and $sinh^2(m) \sim \frac{1}{4}e^{2m}$. Therefore for m large,

$$cosh^2(m) + \frac{1}{2}sinh^2(m) \sim \frac{3}{8}e^{2m}$$

Using $cosh^{-1}(x) = ln(x + \sqrt{x^2-1})$, if x is large then $x^2 - 1 \sim x^2 \Rightarrow \sqrt{x^2-1} \sim \sqrt{x^2} = x$. Therefore , $ln(x + \sqrt{x^2-1}) \sim ln(x + x) = ln(2x)$ for x large.

Applying all of this to equation 5.6,

$$l_\alpha \to ln\left[\frac{3}{4}e^{2m}\right] = ln\left[\frac{3}{4}\right] + 2m$$

and hence

$$\begin{aligned}
\lim_{m\to\infty}(l_\gamma - l_\alpha) &= \lim_{m\to\infty}\left(2m - cosh^{-1}\left[cosh^2(m) + \frac{1}{2}sinh^2(m)\right]\right) \\
&= 2m - ln\left[\frac{3}{4}\right] - 2m \\
&= ln\left[\frac{4}{3}\right] \qquad (5.10)
\end{aligned}$$

Hence for large m, the difference in lengths $|l_\alpha - l_\gamma|$ approaches $ln[\frac{4}{3}]$, and so α and γ get closer to being the same curve. Note that the limit is non-zero because there is a restriction in terms of the angle between the segments of γ.

To summarise, if $m \to \infty$, then both $l_\gamma \to \infty$ and $l_\alpha \to \infty$, and in particular $l_\alpha \nrightarrow 0$ as required. Note that we could have used proposition 2.1.9 to prove this result with a specified fixed angle of $\theta = \frac{2\pi}{3}$.

We now look at a more general case, i.e. where γ has n segments of length m.

5.6.4 The n segment case

At this point we have only considered the case where the broken geodesic γ has two segments of length $m \to \infty$. We can apply a similar method to analyse the case where γ has $n > 2$ segments of length m. However we will need to introduce an induction process to simplify the proof. As an initial illustration, consider the case where $n = 3$.

Lemma 5.6.3 *Let γ be a broken geodesic path in $\mathbb{H}^3$ with three segments of length m, and angle between each pair equal to $\theta = \frac{2\pi}{3}$. Let α be the hyperbolic line segment with the same end points as γ. Then if $m \to \infty$, then $l_\alpha \nrightarrow 0$.*

Proof: As in the proof of lemma 5.6.2, viewing the picture sideways on, γ and α together define a hyperbolic polygon with four sides, three sides of length m and one of length l_α (see figure 5.36). Divide this into two triangles as illustrated in figure 5.36, each of which is made up of three hyperbolic line segments.

Fig 5.36: broken geodesic with n=3 segments

(Figure 5.36 shows this in $\mathbb{H}^2$ to make it clear. In $\mathbb{H}^3$, the hyperbolic quadrilateral will not lie in a hyperbolic plane, but each of the two triangles will sit in a hyperbolic plane. It is these triangles on which we will be working.)

One of these triangles will have two sides of length m, and so will be equivalent to the $n = 2$ case. The other will have one side of length l_α, one of length m and one of length P. Using the hyperbolic law of cosines I (equation 2.2) we have,

$$l_\alpha = cosh^{-1}\left[\left(cosh^2(m) + \frac{1}{2}sinh^2(m)\right)cosh(m) - sinh(P)sinh(m)cos(\phi)\right]$$

where $P = cosh^{-1}[cosh^2(m) + \frac{1}{2}sinh^2(m)]$ (the length of α in the $n = 2$ case) and $\phi < \theta = \frac{2\pi}{3}$.

As in the proof of lemma 5.6.2, we have an equation for l_α which is dependent on m. One way to complete this proof would be to follow the method of lemma 5.6.2 and calculate the derivative of l_α with respect to m. However the resulting equation will be more complicated to analyse (and will be increasingly so as we increase n). Hence for this reason we find another method, utilizing both lemma 5.6.2 and the results from section 2.1.3.

To calculate the equation for l_α in terms of m we divided the polygon defined by γ and α into two triangles. The first is constructed from two sides of length m and one side of length P (equivalent to the triangle in $n = 2$ case). The angle between the two sides of length m is fixed at $\theta = \frac{2\pi}{3}$ (by the construction of the tree). Hence using lemma 5.6.2, it is known that P is monotonically increasing with respect to m. Also, using proposition 2.1.9, we know that if we have a hyperbolic triangle with two sides increasing in length at the same rate, and the angle between them fixed, then the third side will also have increasing length. Therefore P increases as m increases.

Now consider the other triangle. It is made up of one side of length m, one of length P, and one of length l_α. Both P and m are increasing with respect to m, i.e. as $m \to \infty, P \to \infty$. The angle between these two sides is equal to $\phi < \theta = \frac{2\pi}{3}$. By corollary 2.1.10 we know that $\phi \to \theta$ as m (and P) increase. Hence by proposition 2.1.12, l_α will increase with respect to m, so $\frac{dl_\alpha}{dm} > 0$, and in fact $l_\alpha \to \infty$ as $m \to \infty$

More importantly, $l_\alpha \nrightarrow 0$ as $m \to \infty$ as required. □

In the proof of lemma 5.6.3, we used the case $n = 2$ and the results from section 2.1.3 to prove the case for $n = 3$. This method of 'induction' can be used for the general case.

Lemma 5.6.4 *Let γ be a broken geodesic path in $\mathbb{H}^3$ with n segments of length m, and angle $\theta = \frac{2\pi}{3}$ between each pair of adjacent segments. Let α be the hyperbolic line segment with the same end points as γ. Then if $m \to \infty$, then $l_\alpha \nrightarrow 0$.*

Proof: By lemma 5.6.2 we know that if $n = 2$ then the result holds (i.e. as $m \to \infty$ $l_\alpha \nrightarrow 0$).

Now assume that the result is true when γ has $n - 1$ segments of length m, and consider the case when γ has n segments, so $l_\gamma = nm$.

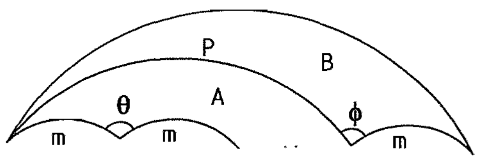


Fig 5.37: broken geodesic with n segments

Looking sideways on, we have a polygon with $n+1$ sides, n of length m and one of length l_α. We take a hyperbolic line segment that divides this polygon into two pieces. One piece (A in figure 5.37) is a polygon with n sides (i.e. the $n - 1$ case) and the second (marked B in figure 5.37) is a hyperbolic triangle with one side of length P, one side of length m and one side of length l_α.

By assumption that the result is true for γ with $n-1$ segments, we know that P is increasing with respect to m (i.e. as $m \to \infty, P \to \infty$). Hence triangle B has two sides whose length gets arbitrarily large as $m \to \infty$ with angle ϕ between them. By corollary 2.1.10, we know that as $m \to \infty$, $\phi \to \theta = \frac{2\pi}{3}$. Hence by proposition 2.1.12, l_α will increase with respect to m, so $\frac{dl_\alpha}{dm} > 0$ and in fact $l_\alpha \to \infty$ as $m \to \infty$. Hence as $m \to \infty$ then $l_\alpha \not\to 0$ as required. □

5.6.5 Segment lengths $\geq m$

In the previous sections (5.6.3 and 5.6.4) it was assumed that each segment of the curve on F when viewed in the lift had length m. In this section this is relaxed so each segment has length $\geq m$. This allows the curve on F to have segments of different lengths, but each still has a lower bound which is dependent on the length of the core curve. (As a reminder, this lower bound m can be made arbitrarily large by making the core curve short - and this we can do.) The analysis for $\geq m$ case is as follows.

Lemma 5.6.5 *Let γ be a broken geodesic in $\mathbb{H}^3$ with n segments of length $m + \lambda_i$, where $\lambda_i \geq 0$ for all $i = 1, ..., n$. Let the angle between adjacent segments be $\theta = \frac{2\pi}{3}$. Let α be the hyperbolic line segment with the same end points as γ (see figure 5.38). Then if $m \to \infty$ then $l_\alpha \not\to 0$.*

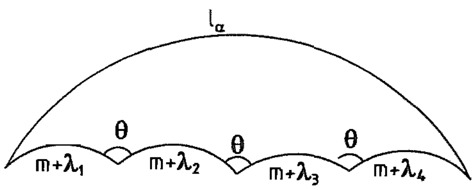


Fig 5.38: ≥ m case

Proof: Start by considering the case where $n = 2$. Hence γ has two segments of lengths $m + \lambda_i$ for $i = 1, 2$. Then $l_\gamma = 2m + \lambda_1 + \lambda_2$, and using the hyperbolic law of cosines I (as per equation 2.2),

$$l_\alpha = cosh^{-1}\left(cosh(m+\lambda_1)cosh(m+\lambda_2) + \frac{1}{2}sinh(m+\lambda_1)sinh(m+\lambda_2)\right) \qquad (5.11)$$

When $n = 2$, γ and α together comprise a triangle with one fixed angle $\theta = \frac{2\pi}{3}$ between the two segments of γ. The difference between this situation and that in lemma 5.6.2 is that here all three sides of the triangle can have different lengths. However, proposition 2.1.9 does not depend on the lengths of the sides being the same, but on the fact that two of the sides increase in length at the same rate, with the angle between them fixed. Here $\frac{d(m+\lambda_1)}{dm} = \frac{d(m+\lambda_2)}{dm}$ and so we can apply proposition 2.1.9 here to see that as $m \to \infty$, l_α will increase (and hence $l_\alpha \nrightarrow 0$) as required. Hence if $n = 2$ then l_α is an increasing function in m.

If $n > 2$ then we can use the same induction argument as in the proof of lemma 5.6.4. This induction proof did not depend on how long the sides were, but on the fact that each segment of γ had length increasing. This is still the case here, as $m \to \infty$. Therefore the assumption can be relaxed so each segment of γ is bounded below by m, and so as $m \to \infty$ (and $l_\gamma \to \infty$) then l_α increases, and hence $l_\alpha \nrightarrow 0$ as required. □

Before summarizing these results, note that the pictures drawn are in 2-dimensions, but in general the broken geodesic path and the hyperbolic line segment with the same end points will not sit in a hyperbolic plane (as we are in 3-dimensions). However, the induction proof works by decomposing the larger polygon into triangles. Each triangle will exist in a hyperbolic plane. Hence the only thing that could cause a problem is how the triangles fit together, and by this we mean the interior angles of the triangles. Although this highlights a discrepancy in the illustrations it does not effect the results. To see this note that what we have been showing here is that as the lengths of the segments of the broken geodesic get arbitrarily large, then the geodesic joining the end points cannot have length tending to zero, (or equivalently that the geodesic with the same end points has increasing length also). The only thing that could cause this not to be true is if the angles between the segments of the broken geodesic are tending towards zero. This cannot be the case because of the way the tree of flats is constructed.

To summarise the results from sections 5.6.3 to 5.6.5, if the geodesic β in M intersects F at some point (in the μ-thick part), then if the broken geodesic path γ on F representing

β increases in length (i.e. $m \to \infty$, which we can make arbitrarily large by shrinking the core curve), then β will increase in length also. More importantly, $l_\beta \nrightarrow 0$ as $l_\gamma \to \infty$.

5.6.6 Comparing l_β to l_α

To complete the projection analysis, this subsection looks at the geodesics in M which do not intersect F, or exist on F, and compares their lengths to the length of their representative γ on F after the projection.

Let β be a geodesic in M which is projected onto γ on F. We compare l_β to l_α (where α is the geodesic in the lift to $\mathbb{H}^3$ with the same end points as γ, as in the previous sections), and hence to l_γ. The aim is to obtain bounds on l_β in terms of l_α, and consequently in terms of l_γ (as $l_\alpha \to \infty$ as $l_\gamma \to \infty$), making them as tight as possible so that it is clear that $l_\beta \nrightarrow 0$ as $l_\gamma \to \infty$. This will show that a small change to the length of γ on F should effect l_β in M.

From section 5.6.1 and the nesting argument involving the outscribed and inscribed circles C_o and C_I, it was shown that the convex core of M consists of a neighbourhood close to F. In the lift to $\mathbb{H}^3$, we showed that we could force the lift of a geodesic to get arbitrarily close to the tree of flats by letting the core curve shrink, giving a way of controlling how 'close' the geodesics in M are to F. We did this by showing that the thick parts of F have a bounded diameter, and that this remains the case as $l_c \to 0$, and then using the fact that the projection of geodesic β onto F must intersect the thick part of F. This is because if the projection of β is in the thin part then it will be homotopic to a power of c.

Therefore in the lift to $\mathbb{H}^3$, the end points of the lift of β are close to the end points of α (which are equivalent to the end points of the lift of γ, where γ is the closed curve on F onto which β is projected). By 'close', we mean that the distance between the end points is equal to δ for some small $\delta > 0$, and as $l_c \to 0$, we have $\delta \to 0$. Hence this distance becomes insignificant as the core curve shortens.

This situation can be summarised as follows.

Consider two long geodesic arcs α and β in $\mathbb{H}^3$ with end points x_1, y_1 for α and x_2, y_2 for β, such that $d_{\mathbb{H}}(x_1, x_2) = \delta$ and $d_{\mathbb{H}}(y_1, y_2) = \rho$ for δ, $\rho > 0$ small. We find bounds on the length of β in terms of the length of α.

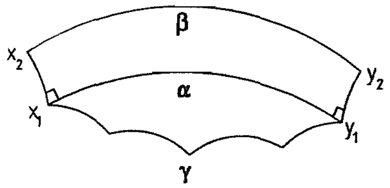


Fig 5.39: geodesics that don't intersect F

Let x_1 and x_2 be joined by the hyperbolic line segment and y_1 and y_2 be joined by the hyperbolic line segment between them, so x_1, x_2, y_2, y_1 determines a hyperbolic rectangle (see figure 5.39).

Join x_2 to y_1 by a hyperbolic line segment so $x_2x_1y_1$ is a hyperbolic triangle where the angle at x_1 is θ_x. Let l_α denote the length of α, and let l_I be the length of the hyperbolic line x_2y_1. The length of x_1x_2 is δ (see figure 5.40 below).

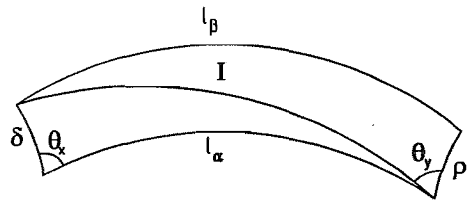


Fig 5.40: comparing lengths of β and α

Then by the triangle inequality,

$$l_\alpha - \delta \leq l_I \leq l_\alpha + \delta \tag{5.12}$$

Now applying similar analysis on the second hyperbolic triangle $x_2y_1y_2$ with angle at $y_1 = \theta_y$. Let l_β denote the length of β, then again by using the triangle inequality,

$$l_I - \rho \leq l_\beta \leq l_I + \rho \tag{5.13}$$

Combining inequality 5.12 with inequality 5.13, we have the following bounds on l_β.

$$l_\alpha - (\delta + \rho) \leq l_\beta \leq l_\alpha + (\delta + \rho) \tag{5.14}$$

Note that the difference between the lower and upper bounds is $2(\delta + \rho)$. As $\delta, \rho > 0$ and are small, the difference between the lower and upper bounds is small. Relating this to the situation in mind, note that these bounds are dependent on the distance between β and its representative on F (where this is measured in the thick part of the flat chosen to be the root of the tree - and hence close to the top of the hyperplane containing the flat). This distance is made arbitrarily small as $l_c \to 0$.

These bounds can be improved by making some basic observations from the particular situation we are considering.

Let β be a geodesic in M, which when looking in the lift into $\mathbb{H}^3$ has endpoints $\widetilde{y}$ and $\lambda\widetilde{y}$ (where $\lambda \in \Gamma$ is the loxodromic with axis β). Project β onto F so that the end points map to $\widetilde{x}$ and $\lambda\widetilde{x}$ on the tree of flats picture, as described in subsection 5.6.2, and let α be the hyperbolic line segment between these two points.

Firstly observe that as $x = \lambda x$ when looking back in F and $y = \lambda y$ when looking back in M, so $d_{\mathbb{H}}(\widetilde{x}, \widetilde{y}) = d_{\mathbb{H}}(\lambda\widetilde{x}, \lambda\widetilde{y})$, and hence $\delta = \rho$. (This is stating that $d_{\mathbb{H}^3}(\beta, \beta_F) = \delta$, where β_F is the representative of β on F.) Applying this to inequality 5.14, the new bounds are,

$$l_\alpha - 2\delta \leq l_\beta \leq l_\alpha + 2\delta \tag{5.15}$$

A second observation is that the projection from β onto F is orthogonal, so $\theta_x = \frac{\pi}{2}$. (Here θ_x is the angle as marked in figure 5.40.) Hence looking again at the first triangle and l_I, using the hyperbolic Pythagorean theorem,

$$cosh(l_I) = cosh(\delta)cosh(l_\alpha) \tag{5.16}$$

Hence $l_I = cosh^{-1}(cosh(\delta)cosh(l_\alpha))$. Note here that as $\theta_x = \frac{\pi}{2}$, this value for l_I is exactly halfway between the original bounds - see inequality 5.12.

Let θ_y be as indicated in figure 5.40. Then looking at the second (upper) triangle,

$$cosh(l_\beta) = cosh(l_I)cosh(\delta) - sinh(\delta)sinh(l_I)cos(\theta_y) \tag{5.17}$$

Note that bounds can be found for $cosh(l_\beta)$ as the right hand side is bounded above and below dependent on θ_y. Looking at the extremes,

if $cos(\theta_y) = -1$ then equation 5.17 becomes

$$cosh(l_\beta) = cosh(l_I)cosh(\delta) + sinh(\delta)sinh(l_I) = cosh(l_I + \delta)$$

This is the maximum value for $cosh(l_\beta)$.

If $cos(\theta_y) = 1$ then equation 5.17 becomes

$$cosh(l_\beta) = cosh(l_I)cosh(\delta) - sinh(\delta)sinh(l_I) = cosh(l_I - \delta)$$

This is the minimum value for $cosh(l_\beta)$.

Therefore

$$\begin{aligned} cosh(l_I - \delta) \leq cosh(l_\beta) \leq cosh(l_I + \delta) \\ \Rightarrow l_I - \delta \leq l_\beta \leq l_I + \delta \\ \Rightarrow cosh^{-1}(cosh(\delta)cosh(l_\alpha)) - \delta \leq l_\beta \leq cosh^{-1}(cosh(\delta)cosh(l_\alpha)) + \delta \end{aligned} \qquad (5.18)$$

These bounds on l_β are tighter than inequality 5.15.

Note that one more improvement can be made to the upper bound by observing that as $\lambda\widetilde{y}$ is pushed onto $\lambda\widetilde{x}$ orthogonally, then $0 \leq \theta_y \leq \frac{\pi}{2}$ and so $0 \leq cos(\theta_y) \leq 1$.

If $cos(\theta_y) = 0$ then

$$cosh(l_\beta) = cosh(l_I)cosh(\delta) \Rightarrow l_\beta = cosh^{-1}(cosh(l_\alpha)cosh^2(\delta))$$

which gives a new maximum value for l_β, which is smaller than before. Hence the new bounds are,

$$l_\alpha - 2\delta \leq cosh^{-1}(cosh(\delta)cosh(l_\alpha)) - \delta \leq l_\beta \leq cosh^{-1}(cosh(l_\alpha)cosh^2(\delta)) \leq l_\alpha + 2\delta$$

To complete this projection analysis, it is necessary to know what these bounds mean for l_β. First note that as $l_\alpha \to \infty$, $l_\beta \nrightarrow 0$ as required. In fact if l_α increases by more than $4\delta = 4d_{\mathbb{H}^3}(\beta, \gamma)$, (where $d_{\mathbb{H}^3}(a, b)$ is the distance from a to b in $\mathbb{H}^3$), then l_β is forced to increase. Combining this with the work of subsections 5.6.3 to 5.6.5, if $l_\gamma \to \infty$ then $l_\beta \nrightarrow 0$ and in fact l_β will be forced to increase if there is a big enough change to l_γ.

The tightness of these bounds is dependent on the parameter of the construction, i.e. the length of the core curve c. From the discussion at the beginning of this subsection and in subsection 5.6.1 when considering the nesting of circles C_o and C_I, as $l_c \to 0$, we can ensure that the geodesic gets close to F (as it is forced to pass through a bounded region of the top of the hyperplane containing the flat which defined to be the root of the tree in the lift), and so δ gets arbitrarily close to 0 as $l_c \to 0$. As $\delta \to 0$,

$$cosh^{-1}(cosh(l_\alpha)cosh^2(\delta)) \to cosh^{-1}(cosh(l_\alpha)) = l_\alpha$$

and

$$cosh^{-1}(cosh(\delta)cosh(l_\alpha)) \to l_\alpha$$

and so as expected $l_\beta \to l_\alpha$ as $\delta \to 0$ (because $\beta \to \alpha$).

Using the original bounds found (i.e. inequality 5.15), to ensure a change in l_β, then l_α needs to increase (or decrease) by at least 4δ. This is equivalent to four times the distance between β and β_F (where β_F is β projected onto F). The tighter bounds imply that an even smaller change would be sufficient.

Linking this to the previous sections (i.e. sections 5.6.3 to 5.6.5), we know that as $l_{\beta_F} \to \infty$ (where β_F is β projected onto F), then $l_\alpha \to \infty$ (where α is the geodesic with the same end points in the lift as β_F). In fact a small change in l_{β_F} gives a change in l_α. Therefore if the length of β_F changes by more than 4δ, where $\delta = d_{\mathbb{H}^3}(\beta, \beta_F)$, then l_β will also change. More importantly if $l_{\beta_F} \to \infty$ then $l_\beta \nrightarrow 0$.

This is all summarized as follows.

Lemma 5.6.6 *Let β and β' be two geodesics in M, where M is the book of I-bundles manifold with single solid torus binding and three I-bundle pages. Let β_F and β'_F be the representatives on F onto which β and β' are projected. Let l_{β_F} denote the length of β_F. Then if $|l_{\beta_F} - l_{\beta'_F}| > d_{\mathbb{H}^3}(\beta, \beta_F)$ then $l_\beta \neq l_{\beta'}$ and hence $\chi[\beta] \neq \chi[\beta']$.*

(Note that this lemma includes both the cases where β and β_F intersect and when they do not. If β and β_F intersect (the case considered in sections 5.6.3 to 5.6.5) then the lower bound on the difference in lengths required will be smaller.)

To rephrase in another way if $\beta \mapsto \beta_F$ and $\beta' \mapsto \beta'_F$ via the projection, then to show $\chi[\beta] \neq \chi[\beta']$ it is needed to show that $|l_{\beta_F} - l_{\beta'_F}| > 4d_{\mathbb{H}^3}(\beta, \beta_F)$. As $l_c \to 0$ we know that $d_{\mathbb{H}^3}(\beta, \beta_F) \to 0$, and hence changing lengths of curves in F is enough to alter lengths of geodesics in M, and gives the means to determine when two geodesics in M cannot have the same character.

This projection and lemma 5.6.6 will be used when considering the properties as discussed in section 5.3.

5.7 Core curve property

In this section we prove the core curve property as given in section 5.3 for the book of I-bundles with single solid torus and three I-bundle pages, as formalized in theorem 5.7.1. Sections 5.5 and 5.6 have provided the tools required to do this. We restate the theorem here.

Theorem 5.7.1 *Let M be the specific book of I-bundles manifold with single solid torus binding and three pages. Let $g \in \pi_1(M)$, such that g is represented by the core curve of the solid torus in M. Then g is uniquely determined by $\chi[g]$. By this we mean that if $h \in \pi_1(M)$ with $\chi[g] = \chi[h]$ then h is conjugate to $g^{\pm 1}$ (so h is also represented by the core curve in M).*

Proof: Let g be represented by the core curve γ, and let h be represented by another geodesic γ' in M such that $\chi[h] = \chi[g]$ (and therefore the geodesics γ and γ' representing g and h respectively have the same hyperbolic length in M). Project γ and γ' onto F via the projection in section 5.6, so γ is represented by the unique closed curve γ_F and γ' is represented by the unique closed curve γ'_F on F (where both γ_F and γ'_F are piece-wise geodesic on F). Note that as γ is equivalent to the core curve, this is a geodesic in M that is already on F, and so $\gamma = \gamma_F$. We now compare $\gamma_F = \gamma$ to γ'_F.

There are three possibilities for γ_F'. Firstly γ_F' could intersect γ in at least one place, or γ_F' could be completely disjoint from γ or γ_F' could coincide with γ. The aim is to rule out the first two possibilities.

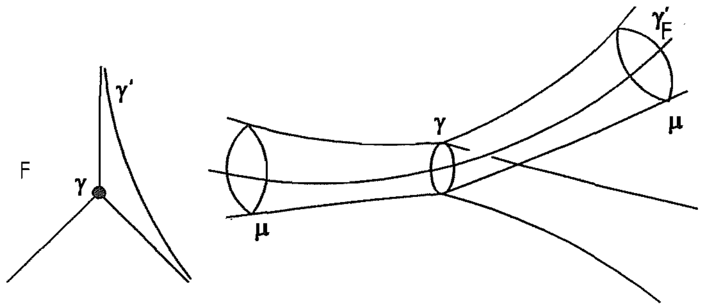


Fig 5.41: core curve property : γ_F' intersects γ

Let γ_F' intersect γ (see figure 5.41). As γ_F' is a closed curve it must intersect γ at least twice since γ separates. From the construction imposed on M (as given in section 5.4), we know that the core curve of the solid torus is short in length, so that there is a wide half-collar around the geodesic boundary γ on each F_i (for $i = 1, 2, 3$). This means that each segment of γ_F' (by segment of γ_F' we mean a geodesic contained entirely in one F_i) will have length bounded below by $c\ log\left(\frac{\mu}{l_\gamma}\right)$. (N.B. This bound comes from section 5.4, where c is a fixed constant, l_γ is the length of γ, and $\mu > l_\gamma$ is a constant smaller than the Margulis constant.) This lower bound increases as l_γ decreases. Therefore if γ is short then any curve intersecting it will be long in comparison.

This means that $|length_F(\gamma_F') - length_F(\gamma)|$ is large, and by lemma 5.6.6 this implies that $l_\gamma \neq l_{\gamma'}$ which contradicts the assumption that $\chi[h] = \chi[g]$. Hence γ_F' cannot intersect γ.

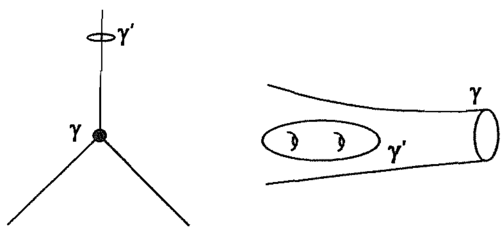


Fig 5.42: core curve property : γ' disjoint from γ

Let γ_F' be completely disjoint from γ (see figure 5.42). Hence γ_F' exists solely on one F_i (for $i = 1, 2, 3$) in F. Hence γ_F' is a geodesic in M which exists on F, and so $\gamma_F' = \gamma'$ (so we do not need to use the projection in this case). From lemma 5.5.3, it is possible to change the length of any arbitrary closed curve on F_i while keeping the length of the boundary geodesic γ fixed. Hence, this ensures that $l_\gamma \neq l_{\gamma'}$, which contradicts the assumption that $\chi[h] = \chi[g]$. Hence γ_F' cannot be completely disjoint from γ.

The only possibility left is that γ_F' coincides with γ and so $\gamma' = \gamma_F' = \gamma$, and so both g and h are represented by the core curve of the solid torus, and hence $\chi[h] = \chi[g] \Rightarrow h \cong g^{\pm 1}$ as required. □

Theorem 5.7.1 says that the core curve of the solid torus in the specific 3-prong case is uniquely determined by its character (and therefore by its length in M). We will consider extending this result to more general books of I-bundles in section 6.3.

5.8 The boundary property

Before moving on to consider more general books of I-bundles, in this section we consider another question that is closely related to the character problem, and therefore worth considering here. The aim is to show that if one geodesic in M is contained in one component of ∂M, (the boundary of M), then any other curve with the same character will exist in the same boundary component (but not necessarily that the curves are homotopic). In this section we consider this in relation to the 3-prong book of I-bundles, as formalized in theorem 5.8.1.

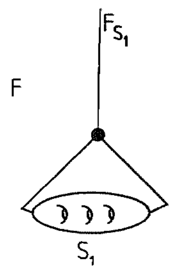


Fig 5.43: Boundary components of F

Let M be a '3-page' book of I-bundles and let F denote its spine. This spine F consists of three pieces, which we denote (for the purpose of this section) F_{S_1}, F_{S_2} and F_{S_3}. The boundary of M consists of three components, denoted S_1, S_2 and S_3, labeled such that F_{S_i} is the part of F that is 'disjoint' from S_i (for $i = 1, 2, 3$). (See figure 5.43 for clarification of this.)

We have the following.

Theorem 5.8.1 *Let M be the specific book of I-bundles with single solid torus binding and three pages. Let $g \in \pi_1(M)$ be represented by a geodesic γ that is contained in a component of the boundary of M ($\gamma \subseteq S_i \subseteq \partial M$). Let $h \in \pi_1(M)$ be represented by another curve γ' such that $\chi[h] = \chi[g]$. Then $\gamma' \subseteq S_i$ also.*

Proof: Let g be represented by a closed curve γ such that $\gamma \subseteq S_1$. (If $\gamma \subseteq S_2$ or $\gamma \subseteq S_3$ then the argument will be similar, just change the indexing.) This means that γ is disjoint from F_{S_1} (so no part of γ exists on F_{S_1} - by the notation given above).

Let h be represented by another closed curve γ' on F, and consider the possibilities for where γ' is situated. Either γ' exists partly in F_{S_1} or it has no part in F_{S_1} (and is therefore disjoint from F_{S_1}).

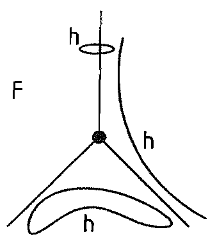


Fig 5.44: Possibilities for h

If γ' exists partly in F_{S_1}, then we may find a simple closed curve α that intersects γ' but is disjoint from γ (i.e. let α be a simple closed curve on F_{S_1}). By lemma 5.5.3, we can change the length of α while keeping the boundary geodesic of F_{S_1} at constant length. Hence we may change the length of α without effecting S_1 or γ. Shrink α enough to ensure $l_{\gamma'} \neq l_\gamma$ (where l_γ denotes the length of γ), and hence contradicting the assumption that $\chi[g] = \chi[h]$.

If γ' is disjoint from F_{S_1}, then by the notation given above, $\gamma' \subseteq S_1$ as required.

As these are the only two possibilities, if $\chi[g] = \chi[h]$, then $\gamma' \subseteq S_1$ as required. □

A similar method to this could be used in proving the same property for more general books of I-bundles.

Chapter 6

General Books of I-bundles

In this chapter the focus moves to considering the character question for more general books of I-bundles. We aim to extend the result from section 5.7 (the core curve property) to this more general setting. As an initial reminder, the problem is to find a topological condition on the geodesics in a compact hyperbolizable 3-manifold M, such that if two curves have the same length over all hyperbolic structures then they are the same geodesic (up to homotopy and orientation). To start we will briefly look again at the construction, as given in detail in section 5.4, and then extend the projection as described in section 5.6.

6.1 Definitions and the general CMT Construction

This section forms a recap of both the general definition and construction for the book of I-bundles to gather all the information to start extending the results in chapter 5. The details of the general construction are given in section 5.4, but here are the important points.

First note that for the purpose of this work, the books of I-bundles only have solid torus bindings. This is because we have ruled out having parabolics in the fundamental group.

Let M be a general book of I-bundles (without parabolics). The components of M will consist of a number of solid tori, denoted T_j (for $j \in \{1, ..., n\}$), and I-bundles, denoted B_i (for $i \in \{1, ...q\}$). Each solid torus will have a family of disjoint parallel closed annuli that are homotopically equivalent to the core curve on its boundary. Each I-bundle $B_i = F_i$ x $[0, 1]$, where F_i is a surface (or I-bundle base) which is a compact, orientable surface minus p open discs (so has non-empty possibly disconnected boundary). Form M by identifying the boundary $\partial_0 B_i$ with the annuli on the solid tori. (Note that if M contains more than one solid torus binding, then it must contain I-bundle bases with multiple boundary components, so at least one I-bundle is attached to more than one solid torus. Otherwise M will be disconnected.)

As for the 3-prong case, the union of the I-bundle bases with boundaries glued together inside each solid torus comprise a 'spine' F for M around which M is a regular neighbourhood.

In section 5.4, the CMT construction was given for the general case, which put a family of hyperbolic structures on M such that each I-bundle base is a totally geodesic surface with geodesic boundary (so we take the convex core of a surface with ends to give the I-bundle base). This construction gives a list of parameters $\{l_{c_j}\}$ for the solid tori in M, where l_{c_j} corresponds to the length of the core curve c_j of the solid torus T_j. Then each boundary component (of an I-bundle base) that glues to the solid torus T_j, with parameter l_{c_j}, is a geodesic of length l_{c_j}.

Following the description given in [CMT99], these parameters for the solid tori can be incorporated into a single parameter l_0, where $l_0 = max\{l_{c_j}\}$. Then assuming l_0 is sufficiently small, the manifold M is hyperbolizable (as seen in section 5.4). As $l_0 \geq l_{c_j}$ for all j, each core curve c_j is short. This is the only constraint imposed on M, (and consequently F), and the family of hyperbolic structures. (This was set aside as an important observation at the end of section 5.4.)

As described in detail in section 5.4, the developing map for this structure lifts the universal cover of M to $\mathbb{H}^3$ such that each base surface is mapped to a totally geodesic subset of $\mathbb{H}^3$ called a flat. For a specific flat, at each lift of a geodesic boundary c_j of its base surface F_i there are $p_j - 1$ other flats equally spaced. (Here p_j is the number of annuli glued to the corresponding torus binding T_j - or equivalently the number of geodesic boundaries glued to the solid torus T_j with parameter l_{c_j}.) By *equally spaced* we mean that the angles between the half-planes containing the flats are equal around the lift of c_j (e.g. in the 3-prong case, this angle was always equal to $\frac{2\pi}{3}$). Note that unlike the

3-prong example considered in chapter 5, this angle may be different for each solid torus binding (and hence for the lifts of each c_j), as it is dependent on the quantity of I-bundles that are attached to that specific solid torus.

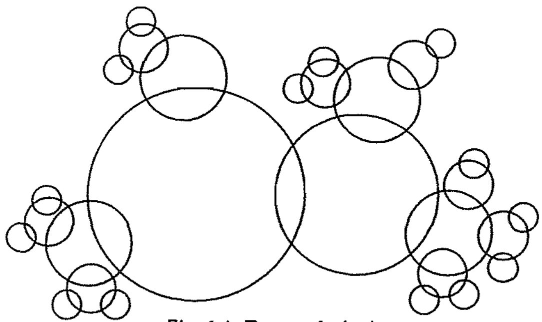


Fig 6.1: Tree of circles

The lift gives a tree of circles picture (when looking from infinity or considering where the half planes intersect the complex plane in $\mathbb{H}^3$ - see figure 6.1), however it is important in this more general setting to keep a note of which c_j is being lifted as they are not all equivalent in M (unlike the 3-prong case), and each distinct c_j equates to a different solid torus. (See figure 6.2 below.)

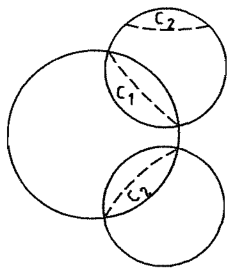


Fig 6.2: keeping note of the lift of the core curves

Recall from section 5.4 that the μ-thin part of a flat denotes the set of points where some element of the stabilizer of the flat acts with translation μ or less. Set μ to be less than the Margulis constant. Then the μ-thin parts consist of a union of disjoint pieces, each of which is a neighbourhood of an axis of translation - hence will be a neighbourhood of the lift of a geodesic boundary. (The μ-thick part is the complement of the μ-thin parts.)

Given any two points x, y in two flats H and H' in the tree, let $H = H_1, H_2, ..., H_n = H'$ be the unique sequence of flats between H and H'. Then there is a geodesic chain $\gamma_{x,y}$ from x to y, such that $\gamma_{x,y}$ is made up of geodesic arcs $\{\gamma_i\}$ where each $\gamma_i \subset H_i$ and γ_i

meets γ_{i+1} in the geodesic boundary shared by H_i and H_{i+1}. By lemma 5.4.1 there is a unique shortest path amongst all such piece-wise geodesic paths between x and y. (Note that lemma 5.4.1 was stated and proven in all generality so applies to this more general setting. The proof is independent of the angles between the planes, and only uses the tree-like structure of the flats, which we still have in this more general situation.)

Following the work of Canary, Minsky and Taylor (as given in [CMT99] and described in section 5.4), if x and y are located in the μ-thick part of their respective flats, then with the assumption that l_0 is small, then each segment of the unique shortest path $\gamma_{x,y}$ has length bounded below. This can be seen by looking back to the spine F of the manifold. If l_0 is small, then around each geodesic boundary of a base surface F_i there is a wide half-collar (by the collar lemma) whose width depends only on l_0. Hence the μ-thick part of F_i is separated from the geodesic boundary by a distance which is dependent on the length of the geodesic boundary and μ.

According to Canary, Minsky and Taylor in [CMT99], given any k and assuming l_0 is small, then each segment has length at least k. They give this bound to be $c\ log\left(\frac{\mu}{l_0}\right)$ where c is a fixed constant. Hence the smaller l_0 becomes, the larger this lower bound k becomes.

To summarise, the construction for a general book of I-bundles is the same, except the result is a set of parameters $\{l_{c_j}\}$ for the solid tori, corresponding to the lengths of the core curves. The only condition imposed to ensure M is hyperbolizable is that $l_{c_j} \leq l_0$ small for all c_j.

For more information on this construction see section 5.4 or see Canary, Minsky and Taylor's description in [CMT99].

6.2 Extending the projection

The aim of this section is to extend the work in section 5.6 to this more general book of I-bundles manifold. As with the 3-prong example, it is necessary to find a way of associating each geodesic β in M to a unique closed curve β_F on the spine F. Using this association we may approximate the length of β, denoted l_β, by the length of β_F, denoted

l_{β_F}, and show that if $l_{\beta_F} \longrightarrow \infty$ then $l_\beta \nrightarrow 0$.

There are two main differences (that are important) between the general book of I-bundles and the 3-prong case. Firstly we have a set of core curves c_j (and hence a set of geodesic boundaries), each of which is short in length (and can be shortened within the limitations of the CMT construction). Secondly, when considering the lift to $\mathbb{H}^3$, the angles between the flats may differ from $\frac{2\pi}{3}$, and may be different for each geodesic boundary in the lift. However, by the nature of the construction of the tree of flats, these angles between adjacent flats will be fixed (as in the 3-prong case) for a specific book of I-bundles.

In section 5.6, the initial association between a geodesic in M and a closed curve on F was independent of these angles between the flats (when looking in the lift to $\mathbb{H}^3$). In fact the projection relied on the fact that the geodesics stay within a bounded distance of the spine. This is still the case, and so we may use the same projection to push the geodesics onto F as described in section 5.6.

The second stage in section 5.6 was to find bounds on l_β in terms of l_{β_F}. This did depend upon the angles between the flats, and so we need to consider this in more detail here. As in subsections 5.6.3 to 5.6.6, this is done in two stages.

Let β be a geodesic in M, and let β_F be its representative on F (so $\beta \mapsto \beta_F$ under the projection). First assume that β intersects F at some point (in the thick part of F). Choose this point as the base point for β_F and lift to $\mathbb{H}^3$. In the lift β_F is represented by a broken piece-wise geodesic path $\widetilde{\beta_F}$, such that each segment of $\widetilde{\beta_F}$ is contained in a unique flat, and β will be represented by a hyperbolic line segment $\widetilde{\beta}$ with the same end points as $\widetilde{\beta_F}$. (See figure 6.3.) We compare the length of $\widetilde{\beta_F}$ to the length of $\widetilde{\beta}$.

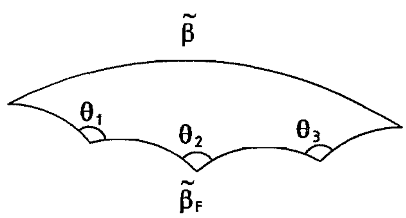


Fig 6.3: projection on to F

Each segment of $\widetilde{\beta_F}$ has length bounded below (i.e. $\geq m$), where m can be made arbitrarily large by making $l_0 = max\{l_{c_j}\}$ small (where c_j is the core curve associated to solid torus T_j), hence $m \longrightarrow \infty$. Unlike the 3-prong example, the angles θ_i between each pair of segments are not all necessarily equal, but they will be fixed as they are equivalent to the angle between adjacent flats. Hence let $\theta_1, \theta_2, ..., \theta_{n-1}$ be the angles between the n segments, where $0 < \theta_i < \pi$.

Let $\widetilde{\beta_F}$ have n segments of length $m + \lambda_i$ where $\lambda_i \geq 0$ for all $i = 1, ..., n$. As in subsection 5.6.5, an equation for $l_{\widetilde{\beta_F}}$ could be found in terms of m, λ_i and θ_j (for $i = 1, ..., n$ and $j = 1, ..., n-1$), however with far more unknowns to deal with, these equations would become complicated. Hence we adapt the work from subsection 5.6.5. As a reminder, the aim is to show that if $l_{\widetilde{\beta_F}} \longrightarrow \infty$ then $l_{\tilde{\beta}} \not\rightarrow 0$.

The hyperbolic line $\tilde{\beta}$ and the n segments of $\widetilde{\beta_F}$ construct a hyperbolic $n+1$-gon. We use an induction process as in section 5.6.

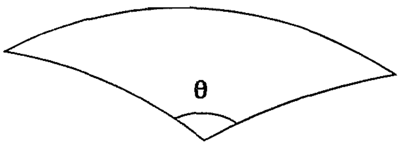


Fig 6.4: n=2 case

Let $n = 2$, then $\tilde{\beta}$ and $\widetilde{\beta_F}$ construct a hyperbolic triangle (see figure 6.4). This triangle has two sides increasing at the same rate (as $m \longrightarrow \infty$) with fixed angle between them. By proposition 2.1.9, we know that $\tilde{\beta}$ will increase in length also. Therefore for $n = 2$ if $m \longrightarrow \infty$ then $l_{\tilde{\beta}}$ will increase and hence $l_{\tilde{\beta}} \not\rightarrow 0$ as required. Hence it is solved for $n = 2$.

Now assume that if $\widetilde{\beta_F}$ has $n-1$ segments and if $m \longrightarrow \infty$ then $l_{\tilde{\beta}} \not\rightarrow 0$, and use this to prove the same for $\widetilde{\beta_F}$ with n segments.

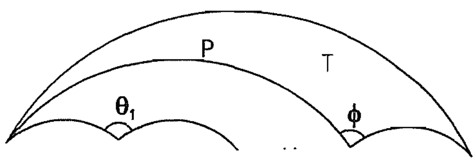


Fig 6.5: $\widetilde{\beta_F}$ with n segments

The hyperbolic line $\tilde{\beta}$ and the n segments of $\widetilde{\beta_F}$ define a hyperbolic $n+1$-gon. We take a hyperbolic line segment that divides this polygon into two pieces. One piece is a polygon with n sides (i.e. the $n-1$ case which we have assumed has the required result), and the second is a hyperbolic triangle with one side of length P, one side of length $m + \lambda_n$ and one of length $l_{\tilde{\beta}}$. (See figure 6.5 for illustration of this.)

By the assumption that the result is true for $\widetilde{\beta_F}$ with $n-1$ segments, we know that P is not decreasing as $m \to \infty$, and in fact using proposition 2.1.9 and proposition 2.1.12 successively we know that $P \to \infty$. Hence the hyperbolic triangle (marked T in figure 6.5), has two sides increasing in length with respect to m, with angle ϕ between them. By corollary 2.1.10, we know that as $m \to \infty, \phi \to \theta_{n-1}$ and θ_{n-1} is fixed such that $0 < \theta_{n-1} < \pi$. Hence by proposition 2.1.12, $l_{\tilde{\beta}}$ will increase with respect to m. Hence as $m \to \infty$ (and $l_{\widetilde{\beta_F}} \to \infty$), $l_{\tilde{\beta}} \nrightarrow 0$ as required.

Now assume β does not intersect F. The argument from section 5.6.6 extends directly to this more general setting, as this part of the projection analysis did not depend on the angles between adjacent flats. Hence

$$l_{\beta_F} - 2d_{\mathbb{H}^3}(\beta, \beta_F) \leq l_\beta \leq l_{\beta_F} + 2d_{\mathbb{H}^3}(\beta, \beta_F) \qquad (6.1)$$

where $d_{\mathbb{H}^3}(\beta, \beta_F)$ is the distance between β and its representative β_F on F.

Hence as $l_{\beta_F} \to \infty \Rightarrow l_\beta \nrightarrow 0$.

Hence lemma 5.6.6 extends to this general case.

6.3 The core curve property

In section 5.7 it was shown that in the 3-prong example the core curve was uniquely determined by its length (and the corresponding group element by its character). In this section this result is extended to the general setting described. First we comment on the case where M still has only one solid torus binding, but with n pages (see figure 6.6 for graphical representation of an example with $n = 6$). This is an almost direct extension of theorem 5.7.1.

Lemma 6.3.1 *Let $G = \pi_1(M)$ where M is the Book of I-bundles with single torus binding and n pages. Let $g \in G$ be represented the core curve α of the solid torus. Let $h \in G$ such that $\chi[g] = \chi[h]$, then $h \cong g^{\pm 1}$.*

To prove this the results on surfaces from section 5.5 and the described projection from section 5.6 are utilized. (Figure 6.6 illustrates an example of such a book of I-bundles, where the vertex is the solid tori, and the edges are the I-bundles.)

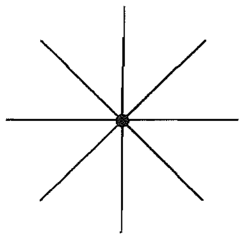


Fig 6.6 n-prong case

Proof: Let g be represented by the core curve α with length l_α. Then from the construction given in section 5.4, l_α is small (and can be made arbitrarily small within the limits of the construction - hence around α on each base surface F_i there will be a wide half collar, whose width depends on l_α and increases as $l_\alpha \to 0$).

Let h be represented by another geodesic β in M, with length l_β. Use the projection (see sections 5.6 and 6.2) to push β onto β_F on F. (Note that as α is the core curve, then α already exists on F and so $\alpha_F = \alpha$). Now consider the possibilities for β_F.

Firstly β_F could be completely disjoint from α, or β_F could intersect α in at least two places (as β_F is a closed curve on F), or β_F could coincide with α. We rule out the first two possibilities.

Let β_F be completely disjoint from α. Then β_F is a geodesic in M, and so $\beta_F = \beta$, (as β_F will lie solely on one base surface F_i of F). Hence β is a simple closed curve on a surface with non-empty connected boundary. By lemma 5.5.3, the length of β can be changed while the length of α remains constant. Hence ensuring that $l_\beta \neq l_\alpha \Rightarrow l_\beta \neq l_\alpha$ which contradicts $\chi[h] = \chi[g]$. Hence β_F cannot be completely disjoint from α.

Let β_F intersect α (this must be in at least two places as β_F is a closed curve). From section 6.2, we know that to ensure $l_\beta \neq l_\alpha$ we need to make sure that $|l_{\beta_F} - l_\alpha| > 4d_{\mathbb{H}^3}(\beta_F, \beta)$. However, as l_α gets smaller, $d_{\mathbb{H}^3}(\beta, \beta_F) \to 0$. More importantly the half collars around α on each F_i get wider, and so $|l_{\beta_F} - l_\alpha|$ gets larger (as β_F intersects α). By making α small enough, we can ensure that $|l_{\beta_F} - l_\alpha|$ is large enough to imply $l_\beta \neq l_\alpha$ and giving the contradiction to $\chi[h] = \chi[g]$ as required. Hence β_F cannot intersect α.

The only possibility left is α and β_F coincide, so $\alpha = \beta_F = \beta$ as required. □

Now let M be a book of I-bundles with q solid torus bindings T_j for $j = 1, .., q$, each with n_i pages. (N.B. some of these pages will be common to two or more of the solid tori.) Let α_j be the set of core curves, one for each solid torus.

Lemma 6.3.2 *Let $G = \pi_1(M)$ where M is the book of I-bundles with q solid tori bindings T_j such that each T_j has n_i pages $(i = 1, 2, ...m)$. Let $g \in G$ be represented by the core curve α_j (of length l_{α_j}) of the solid torus T_j. Let $h \in G$ be such that $\chi[g] = \chi[h]$. Then $h \cong g^{\pm 1}$.*

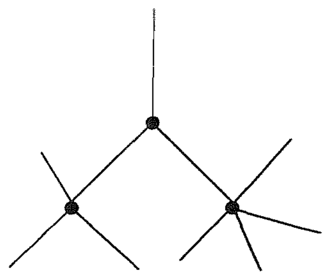


Fig 6.7: example of general case

Proof: Without loss of generality let g be represented by the core curve α_1 of torus T_1 (if not just re-label the solid tori accordingly). Let h be represented by geodesic β in M.

Apply projection to β, so that $\beta \longrightarrow \beta_F$ on F and then compare l_{β_F} to l_{α_1} over all possibilities for β_F.

As before we have three possibilities, i.e. β_F disjoint from α_1, β_F intersecting α_1, or β_F coincident with α_1. Take each in turn and break into cases.

First note that as we have the freedom to change the lengths of the core curves independently (as long as they are short) then for ease of exposition set $l_{\alpha_1} = l_{\alpha_2} = ... = l_{\alpha_q}$.

Let β_F intersect α_1. Here we have two cases, either β_F intersects other α_j (for $j = 2, ..., q$) also, or β_F only intersects α_1 (see figure 6.8). Assume that β_F intersects α_1 only. Then β_F remains on the base surfaces F_k that are glued to the solid torus T_1. As α_1 is short, then around it on each F_k is a wide half collar whose width increases as $l_{\alpha_1} \rightarrow 0$. Therefore let

$l_{\alpha_1} \to 0$ so $|l_{\beta_F} - l_{\alpha_1}|$ is large (i.e. larger than four times the distance between β and β_F, which is decreasing as l_{α_1} is decreasing). We can make α_1 arbitrarily small, so ensuring that $l_\beta \neq l_{\alpha_1}$ and contradicting $\chi[h] = \chi[g]$.

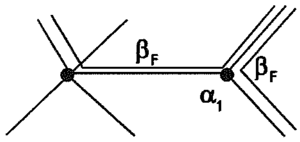


Fig 6.8: β_F intersecting α_1

Now assume that β_F intersects other α_j (for $j = 2, ..., q$) as well as α_1. By the same argument as above, as $l_{\alpha_1} \to 0$, the half collar around α_1 gets wider. If we assume $l_{\alpha_1} = l_{\alpha_2} = ... = l_{\alpha_q}$ then all the core curves will be shrinking, and so any curve intersecting one or more of them will have to be very long. Hence $|l_{\beta_F} - l_{\alpha_1}|$ is large, which implies that $l_\beta \neq l_{\alpha_1}$ as required. Hence β_F cannot intersect α_1.

Let β_F be disjoint from α_1. Then either $\beta_F = \alpha_j$ (for $j \neq 1$), or β_F is disjoint from all α_j ($j = 1, .., q$) or β_F intersects at least one $\alpha_j \neq \alpha_1$. (Figure 6.9 illustrates these possibilities.)

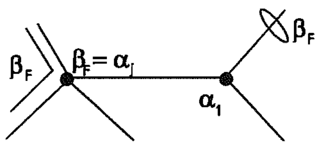


Fig 6.9: β_F disjoint from α_1

If $\beta_F = \alpha_j$ (for $j \neq 1$), then we have the freedom to change the lengths of the core curves independently (as long as they stay short). Hence shrink α_j so $l_{\alpha_j} \neq l_{\alpha_1}$ and so $l_\beta \neq l_{\alpha_1}$ as required.

If β_F is completely disjoint from all α_j then β_F lies purely on one of the base surfaces F_i of F, and so $\beta_F = \beta$. Hence β exists on a surface with non-empty (possibly disconnected) boundary. Using lemma 5.5.5, the length of β_F can be changed while the length(s) of the boundary curve(s) of F_i remain constant. If α_1 corresponds to one of these curves then we can ensure $l_\beta \neq l_{\alpha_1}$ as required. If α_1 is not one of these curves, then note that the rest of

F stays fixed as l_β changes, and so α_1 remains constant length. Therefore $l_\beta \neq l_{\alpha_1}$ as required.

If β_F intersects at least one of $\alpha_j \neq \alpha_1$, then, by the construction given in section 5.4, the length of any closed curve crossing a core curve will be made up of segments which have length bounded below by a function, which is dependent on the length of that core curve. As a core curve shrinks, any curve crossing will grow in length as this lower bound increases. As all core curves are short, any curve crossing them will be long. In fact any curve intersecting a core curve will be longer than $\max\{\alpha_i\}$ for all i. Hence in this situation we know that $l_{\beta_F} \neq l_{\alpha_1}$ and $|l_{\beta_F} - l_{\alpha_1}|$ is bigger than four times the distance between β and β_F, and so $l_\beta \neq l_{\alpha_1}$ as required.

Hence β_F cannot be disjoint from α_1.

The only other possibility for β_F is that it coincides with α_1, and hence $l_{\beta_F} = l_{\alpha_1}$ and as $\beta_F = \beta$ this implies $l_\beta = l_{\alpha_1}$ and so $\chi[h] = \chi[g] \Rightarrow h \cong g^{\pm 1}$ as required. □

Chapter 7

Simple on F property

This chapter discusses one of the other properties suggested in section 5.3. We will therefore be considering the following conjecture.

Conjecture 7.0.3 *Let $G = \pi_1(M)$ where M is a book of I-bundles with single solid torus binding and three pages. Let $g \in G$ be represented by a geodesic in M which is uniquely projected onto a simple closed curve on F (where F is the spine of M). Let $h \in G$ such that $\chi[g] = \chi[h]$, then $h \cong g^{\pm 1}$.*

7.1 Extending McShane's lemma

When looking for possible properties to fit in with the character question, it was important to find a set of natural curves in the manifold. In the 2-dimensional case, (see McShane's lemma in section 4.4 - lemma 4.4.1), the property was being a simple closed curve. The idea of a simple closed curve on a surface forms a natural divide for the geodesics (between simple and non-simple). In 3-dimensions, the idea of a geodesic being simple is not very interesting as the majority of the geodesics will be simple. (To see this is the case see [BW03]. In this paper, Basmajian and Wolpert show that for a 3-manifold which supports a geometrically finite hyperbolic structure, then either the generic hyperbolic structure has the spd-property or no hyperbolic structure has the spd-property. Here a hyperbolic 3-manifold is said to have the spd-property if all of its closed geodesics are simple and pair-wise disjoint. They prove that both cases occur.)

The spine F of M is constructed from surfaces, and so the concept of a closed curve on F being simple will form a natural divide for closed curves on F, and therefore form a natural set of geodesics in M (those that project onto simple closed curves and those that do not). This combined with the projection from section 5.6 means that the property of a geodesic in M being 'simple on F' is a natural extension to McShane's lemma.

As a starting point, look back at the proof of McShane's result (see the proof of lemma 4.4.1). This proof can be broken into two main pieces. Let S be a closed orientable surface of genus $p \geq 2$. Firstly we show that if $g \in \pi_1(S)$ is represented by a simple closed curve on the surface S and $\chi[h] = \chi[g]$ then $h \in \pi_1(S)$ must also be represented by a simple closed curve on S. (The proof of this part uses the fact that non-simple curves on S have a positive lower bound on their length, whereas sequences of simple curves have lengths which go to zero in certain hyperbolic structures.) Secondly we use the fact that both g and h are represented by simple closed curves on S, and shows that they must coincide (up to orientation).

In order to prove the simple on F property (for a book of I-bundles M) in relation to the character problem, the aim would be to extend both of these steps to M. For the purpose of this work we will assume that M is a book of I-bundles manifold with single solid torus binding and three I-bundle pages - as described in section 5.4.1.

We start by considering the second step of the McShane's lemma proof.

7.2 Both simple $\Rightarrow$ Same curve

In this section the aim is to extend the second part of the proof of McShane's lemma to M, where M is a book of I-bundles with single solid torus binding and three I-bundle pages. To do this we will be utilizing the projection described in section 5.6.

Hence the aim is to prove the following,

Lemma 7.2.1 *Let $g, h \in \pi_1(M)$ be such that $\chi[h] = \chi[g]$ and such that both g and h are represented by geodesics β and γ in M that are projected onto simple closed curves β_F and γ_F on F (i.e $\beta \to \beta_F$ and $\gamma \to \gamma_F$ via the projection in section 5.6). Then $h \cong g^{\pm 1}$.*

Note that this will also say that $l_\beta = l_\gamma$ (where l_β denotes the length of β).

In order to prove this lemma we have several intermediate steps to show a chain of implications which combined will lead to lemma 7.2.1. This chain of implications is as follows.

$$\chi[g] = \chi[h] \Rightarrow l_\beta = l_\gamma \Rightarrow i(\beta_F, \alpha_j) = i(\gamma_F, \alpha_j) \Rightarrow \beta_F = \gamma_F \Rightarrow \beta = \gamma$$

(Note here that $i(a, b)$ is the number of points of intersection between a and b, and α_j is a simple closed curve on F.)

The first implication (i.e. $\chi[g] = \chi[h] \Rightarrow l_\beta = l_\gamma$) follows from the connection between character and length as described in section 4.3. The last implication (i.e. $\beta_F = \gamma_F \Rightarrow \beta = \gamma$) comes from the uniqueness of the projection as given in section 5.6, and as $\beta \rightarrow \beta_F$ and $\gamma \rightarrow \gamma_F$ uniquely, then $\beta \rightarrow \beta_F = \gamma_F$ and $\gamma \rightarrow \gamma_F = \beta_F$ and so $\beta = \gamma$.

The rest of this section aims to show the other two implications are true (i.e. $l_\beta = l_\gamma \Rightarrow i(\beta_F, \alpha_j) = i(\gamma_F, \alpha_j)$ - see lemma 7.2.7, and $i(\beta_F, \alpha_j) = i(\gamma_F, \alpha_j) \Rightarrow \beta_F = \gamma_F$ - see lemma 7.2.6).

Before proving these we start by considering simple closed geodesics in M which lie on the spine F (i.e. these consist of simple closed curves on F that do not intersect the core curve, or equivalently that lie solely on one of F_1, F_2 or F_3). We show that such a pair of geodesics with the same length in M are the same curve up to homotopy and orientation. First we need the following which is based on an idea from the proof of the core curve property.

Lemma 7.2.2 *Let M be a book of I-bundles with single solid torus and three pages, and let F be the spine of M. Let c be the core curve of the solid torus, and let α_F be a simple closed curve on F. Let $n_c(\alpha)$ be the number of times α_F intersects c. Then if β_F is another simple closed curve on F such that $l_{\alpha_F} = l_{\beta_F}$ then $n_c(\alpha_F) = n_c(\beta_F)$. (Here l_{α_F} is the length of α_F.)*

Lemma 7.2.2 says that if two simple closed curves on F that have the same length as measured on F then each must intersect the core curve the same number of times. (N.B.

As α and β are simple closed curves on F they are not geodesics in M, unless $n_c(\alpha) = n_c(\beta) = 0$.)

Proof: We proceed by proving the contrapositive, so $n_c(\alpha) \neq n_c(\beta) \Rightarrow l_{\alpha_F} \neq l_{\beta_F}$.

First note that by the CMT construction (as given in section 5.4), the length of the core curve, which we will denote l_c, is short and can be made arbitrarily short whilst keeping M hyperbolizable.

As c is short there is a wide half collar around it on each base surface F_i (for $i = 1, 2, 3$) that determine F, whose width depends only on l_c. Consequently any simple closed curve α_F on F that intersects c (and hence will do so at least twice as it is closed) will have a lower bound of $2w(n_c(\alpha_F))$, where w is the width of the half collar around c (so w is dependent only on l_c).

As $l_c \to 0$, w will be increasing and hence so will this lower bound. (Note that this lower bound will also increase the more points of intersection there are between the simple closed curve on F and c.) Therefore if $n_c(\alpha_F) > n_c(\beta_F)$ then l_{α_F} has a lower bound which is greater than the lower bound on l_{β_F}, and this will remain so as $l_c \to 0$.

As $l_c \to 0$, the length of any simple closed curve will be dominated by the thin parts of F (i.e. neighbourhoods of the geodesic boundaries, or equivalently c) that it passes through. Hence if $n_c(\alpha_F) \neq n_c(\beta_F)$ then one of α_F or β_F will intersect c more times and hence have to cross the wide collar more times, and spend more time in the thin part of F.

Combining this with the lower bounds we see that if $n_c(\alpha) \neq n_c(\beta)$ then $l_{\alpha_F} \neq l_{\beta_F}$ as required. (In fact if $n_c(\alpha) > n_c(\beta)$ then $l_{\alpha_F} > l_{\beta_F}$.)

Therefore if α_F and β_F are simple closed curves on F such that $l_{\alpha_F} = l_{\beta_F}$ then $n_c(\alpha_F) = n_c(\beta_F)$. □

Lemma 7.2.2 allows the problem to be broken down into cases dependent on the number of points of intersection between a simple closed curve on F and c. We are considering those simple closed curves on F that are disjoint from c (so $n_c(*) = 0$). Note that such a closed curve on F will be geodesic in M. From lemma 7.2.2, if α_F is a simple closed curve on F such that $n_c(\alpha_F) = 0$, and β_F is such that $n_c(\beta_F) > 0$ then $l_{\alpha_F} \neq l_{\beta_F}$.

We now need to show that if α_F and β_F are simple closed curves on F such that $l_{\alpha_F} = l_{\beta_F}$

then α_F and β_F are equivalent up to homotopy and orientation. We do this in two steps. First consider the case where α_F and β_F exist on different base surfaces F_i (for $i = 1, 2, 3$).

Lemma 7.2.3 *Let F_1, F_2 and F_3 be compact orientable surfaces of positive genus with connected non-empty boundary, and let F be the spine constructed from these surfaces by connecting along their boundary geodesic c (with length l_c). If α_F is a simple closed geodesic on F_i and β_F is a simple closed geodesic on F_j, $j \neq i$ (for $i, j = 1, 2, 3$) then $l_{\alpha_F} \neq l_{\beta_F}$.*

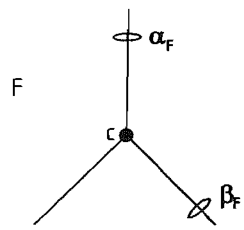


Fig 7.1: simple closed curves on F that do not intersect c

Proof: Without loss of generality let α_F be a simple closed geodesic on F_1. (If not then just re-label the surfaces accordingly.) Decompose the spine by cutting along c, giving the three surfaces F_i, for $i = 1, 2, 3$. As F_1 is a compact surface with connected non-empty boundary apply lemma 5.5.3. Hence it is possible to change the length of α_F while keeping l_c constant. Reconstruct F by gluing the three surfaces back together to see that changing length of α_F does not effect the lengths of the curves on F_2 or F_3. Therefore l_{α_F} varies while l_{β_F} constant and so $l_{\alpha_F} \neq l_{\beta_F}$ as required. □

Lemma 7.2.3 says that two geodesics α, β in M that exist on F (so $\alpha = \alpha_F$ and $\beta = \beta_F$) such that $l_\alpha = l_\beta$ must lie on the same base surface F_i of F (for $i = 1, 2, 3$). It remains to show that if α_F and β_F are on the same F_i and $\alpha_F \not\simeq \beta_F^{\pm 1}$ then $l_{\alpha_F} \neq l_{\beta_F}$.

Note that as both α_F and β_F exist purely on F_i, then the other two base surfaces, F_j for $j \neq i$, can be disregarded by lemma 5.5.3, as what happens on F_i will not effect F_j unless l_c changes. This part is therefore reduced to considering a surface with non-empty connected boundary.

Lemma 7.2.4 *Let F_1, F_2 and F_3 be compact orientable surfaces of positive genus with connected non-empty boundary of length l_c, and let F be the spine constructed from these* *surfaces by gluing along their boundaries. If α_F and β_F are simple closed curves on F_1, such that $l_{\alpha_F} = l_{\beta_F}$ then $\alpha_F \cong \beta_F^{\pm 1}$.*

Proof: Let α_F and β_F be simple closed curves on F_1 such that $l_{\alpha_F} = l_{\beta_F}$. There are three possibilities to consider. Either α_F and β_F intersect one another in at least one place, or α_F and β_F are completely disjoint, or α_F and β_F coincide (so $\alpha_F \cong \beta_F^{\pm 1}$).

Let α_F and β_F be such that they intersect. Apply the collar lemma around α_F to see that as $l_{\alpha_F} \to 0$ then $l_{\beta_F} \to \infty$. (Note that by lemma 5.5.3, l_{α_F} can be made smaller while keeping l_c at constant length and so $l_{\alpha_F} \to 0$ does not effect F_2 and F_3.) Hence $l_{\alpha_F} \neq l_{\beta_F}$ in this case, which gives the contradiction required. Therefore α_F and β_F do not intersect on F_1.

Let α_F and β_F be completely disjoint on F_1. Take the double of F_1 (so take another copy of F_1 and glue it to F_1 along the boundary curve c, or equivalently reflect F_1 across its boundary). The result is a closed surface, and by applying Fenchel-Nielsen coordinates to this new surface, we see it is possible to change l_{α_F} and keep l_{β_F} constant. Hence we can ensure that $l_{\alpha_F} \neq l_{\beta_F}$ as required. Therefore α_F and β_F are not completely disjoint.

The only other possibility is that α_F and β_F coincide, and so $\alpha_F \cong \beta_F^{\pm 1}$ as required. □

In connection to lemma 7.2.4, note that we assumed that both α_F and β_F were simple closed curves on F_1. We could equally have assumed that both α_F and β_F are simple closed curves on F_2 or F_3 and use the same proof (just re-label the base surfaces).

Combining lemma 7.2.3 and lemma 7.2.4 provides a proof of lemma 7.2.1 in the case where the geodesics in M exist on the spine F (and hence we do not need to use the projection). This can be summarised as follows.

Lemma 7.2.5 *Let M be the book of I-bundles with single solid torus binding and three I-bundle pages. Let α and β be geodesics in M such that $l_\alpha = l_\beta$, and let α_F and β_F be simple closed curves on F representing α and β respectively. Then if α_F and β_F do not intersect the core curve c (so $\alpha = \alpha_F$ and $\beta = \beta_F$), then $\alpha \cong \beta^{\pm 1}$ (up to homotopy and orientation).*

Proof: Combine lemmas 7.2.3 and 7.2.4 to show the above. □

We have therefore proven lemma 7.2.1 for a particular case. The other case to consider is where geodesics α and β are projected (via the projection given in section 5.6) onto simple closed curves α_F and β_F respectively, such that α_F and β_F do intersect the core curve. (Note that as the core curve c separates, α_F and β_F have to intersect c at least twice as they are closed curves.)

To consider this we return to the chain of implications stated earlier, and given again below as a reminder.

$$\chi[g] = \chi[h] \Rightarrow l_\beta = l_\gamma \Rightarrow i(\beta_F, \alpha_j) = i(\gamma_F, \alpha_j) \Rightarrow \beta_F = \gamma_F \Rightarrow \beta = \gamma$$

We start by considering $i(\beta_F, \alpha_j) = i(\gamma_F, \alpha_j) \Rightarrow \beta_F = \gamma_F$

From lemma 7.2.2, two simple closed curves on F with the same length both intersect the core curve the same number of times. We consider whether this is true for any other curves on F. From a statement made in [[Lei03] section 6], on a closed surface S, two simple curves with the same length, must cross a finite set of simple closed curves on S the same number of times. Now consider this in relation to the spine F.

Let $i(\alpha, \beta)$ be the number of times geodesics α and β intersect (i.e. their intersection number).

Lemma 7.2.6 *Let F_1, F_2 and F_3 be compact orientable surfaces of positive genus with connected non-empty boundary of length l_c, and let F be the spine constructed from these surfaces by gluing along their boundaries. If γ and γ' are two simple closed curves on F and $i(\gamma, \alpha_j) = i(\gamma', \alpha_j)$ for all simple closed curves α_j on F then $\gamma = \gamma'$.*

Proof: Let γ and γ' be simple closed curves on F such that $i(\gamma, \alpha_j) = i(\gamma', \alpha_j)$ for all simple closed curves α_j on F.

Let $\alpha_k = \gamma$ (this is valid as γ is a simple closed curve on F), then $i(\gamma, \alpha_k) = i(\gamma, \gamma) = 0$ as γ is simple. This implies that $i(\gamma', \alpha_k) = i(\gamma', \gamma) = 0$. Hence γ' is either completely disjoint from γ or coincides with γ (so $\gamma' = \gamma$).

To rule out the possibility that γ' is completely disjoint, we show that a simple closed curve on F can always be found such that it has a different intersection number with γ and γ'. We have two cases.

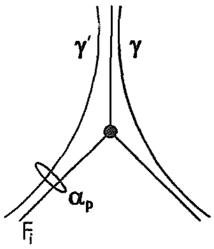


Fig 7.2: $i(\gamma', \alpha_p) > 0$

Firstly let γ and γ' be disjoint simple closed curves on F, such that γ' has an arc on F_i, but γ does not have an arc on F_i (see figure 7.2). Let α_p be a simple closed curve on F_i such that α_p intersects γ'. Then $i(\gamma', \alpha_p) > 0$ but $i(\gamma, \alpha_p) = 0$ and therefore giving the contradiction required.

The second case is where both γ and γ' are disjoint simple closed curves on F such that both have segments on the same F_i. The aim is to again find a simple closed curve α_p on F such that $i(\gamma, \alpha_p) \neq i(\gamma', \alpha_p)$. We proceed by finding a proof by contradiction.

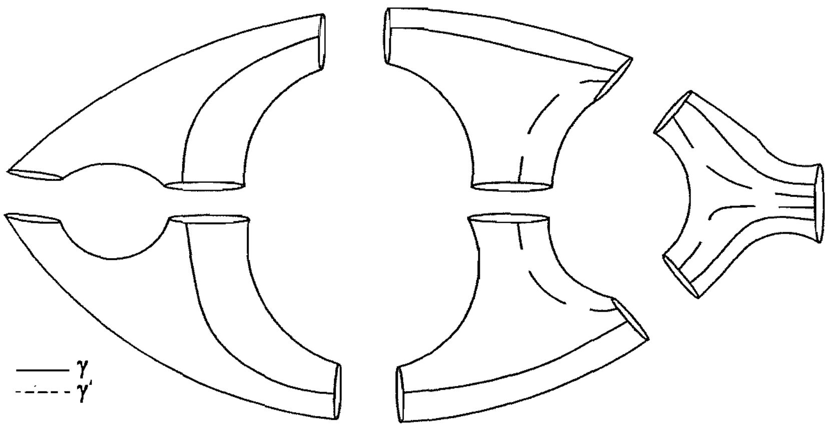


Fig 7.3: pants decomposition of F_i

Let γ and γ' be disjoint simple closed curves on F and assume $i(\gamma, \alpha_p) = i(\gamma', \alpha_p)$ for all simple closed curves α_p on F. Decompose F using a pants decomposition on each F_i $(i = 1, 2, 3)$, then each curve in the decomposition is a simple closed curve on F. We consider γ and γ' in relation to the set of pants curves $\{\alpha_j\}$. (Figure 7.3 shows such a decomposition and possible paths γ and γ'.)

We have assumed that $i(\gamma, \alpha_j) = i(\gamma', \alpha_j)$ for each α_j in the pants decomposition. However this implies that both γ and γ' must follow the same route around F. To see this look at one of the pair of pants pieces P of the decomposition (see figure 7.4).

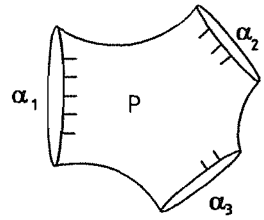


Fig 7.4: piece of the decomposition

By the assumption γ and γ' intersect each of α_1, α_2 and α_3 (as in figure 7.4) the same number of times (i.e. $i(\gamma, \alpha_1) = i(\gamma', \alpha_1)$ and $i(\gamma, \alpha_2) = i(\gamma', \alpha_2)$ and $i(\gamma, \alpha_3) = i(\gamma', \alpha_3)$). Hence we have a number associated to each α_i which is equal to this intersection number. Let $i(\gamma, \alpha_1) = a, i(\gamma, \alpha_2) = b$ and $i(\gamma, \alpha_3) = c$ and assume $a \leq b \leq c$ (if not then just reorder labels).

Then $a + b + c = 2n$, for $n \in \mathbb{N}$ (as each segment on P has two end points). We also know that each arc of γ on P must go from one boundary to another (otherwise it is trivial). Then,

$$N(\alpha_1 \text{ to } \alpha_2) + N(\alpha_1 \text{ to } \alpha_3) = a$$
$$N(\alpha_1 \text{ to } \alpha_2) + N(\alpha_2 \text{ to } \alpha_3) = b$$
$$N(\alpha_2 \text{ to } \alpha_3) + N(\alpha_1 \text{ to } \alpha_3) = c$$

where $N(\alpha_i \text{ to } \alpha_j)$ is the number of arcs of γ going between α_i and α_j on P.

Also,

$$N(\alpha_1 \text{ to } \alpha_3) < min(a, c) = a \Rightarrow N(\alpha_1 \text{ to } \alpha_3) = a - \delta_1$$
$$N(\alpha_1 \text{ to } \alpha_2) < min(a, b) = a \Rightarrow N(\alpha_1 \text{ to } \alpha_2) = a - \delta_2$$
$$N(\alpha_2 \text{ to } \alpha_3) < min(b, c) = b \Rightarrow N(\alpha_2 \text{ to } \alpha_3) = b - \delta_3$$

for $\delta_1, \delta_2, \delta_3 > 0$ and $\delta_1 < a, \delta_2 < a$ and $\delta_3 < b$.

Hence we have,

$$a - \delta_1 + a - \delta_2 = a \tag{7.1}$$

$$a - \delta_1 + b - \delta_3 = b \tag{7.2}$$

$$b - \delta_3 + a - \delta_2 = c \tag{7.3}$$

From equations 7.2 and 7.3, we see $\delta_2 - \delta_1 = b - c$, and from equations 7.1 and 7.3, $\delta_3 - \delta_1 = b - c$. Therefore,

$$\delta_2 = \delta_3 \tag{7.4}$$

From equation 7.1, $a - \delta_1 - \delta_2 = 0$, and so

$$a = \delta_1 + \delta_2 \Rightarrow a - \delta_2 = \delta_1 \tag{7.5}$$

Combining equations 7.5, 7.3 and 7.4, we have

$$b - \delta_2 + \delta_1 = c \Rightarrow \delta_1 - \delta_2 = c - b \tag{7.6}$$

Hence combining equations 7.5 and 7.6,

$$\delta_1 = \frac{a + c - b}{2} \tag{7.7}$$

We therefore can uniquely determine δ_1 (from equation 7.7) and hence δ_2 (from equation 7.5) and hence δ_3 (from equation 7.4) by knowing a, b and c.

This will be the case for each pair of pants in the decomposition. Hence the intersection numbers of γ (and equivalently γ') with each α_j in the pants decomposition determines the route around F. Therefore γ and γ' must follow the same path around F and hence γ' is homotopic to γ. Therefore γ and γ' are not disjoint on F. This gives the contradiction required.

Hence if γ and γ' are disjoint simple closed curves on F, then there exists a simple closed curve α_p on F such that $i(\gamma, \alpha_p) \neq i(\gamma', \alpha_p)$.

Therefore if for all simple closed curves α_j on F, $i(\gamma, \alpha_p) = i(\gamma', \alpha_p)$ then $\gamma = \gamma'$ on F. □

Lemma 7.2.6 shows that the idea of intersection numbers being equal is not restricted to the core curve, and in fact for two curves on F to be equal or coincident then they must intersect every simple closed curve on F the same number of times.

We now prove the other implication, i.e. $l_\beta = l_\gamma \Rightarrow i(\beta_F, \alpha_j) = i(\gamma_F, \alpha_j)$. This is formalized into the following.

Lemma 7.2.7 *Let M be a book of I-bundles with single solid torus binding and three pages, and let F be the spine of M. Let β and γ be two geodesics in M such that they are represented by two simple closed curves β_F and γ_F respectively on F. Then if $l_\beta = l_\gamma$ then $i(\beta_F, \alpha_j) = i(\gamma_F, \alpha_j)$ for all simple closed curves α_j on F.*

Proof: Let β and γ be two geodesics in M that are uniquely projected onto simple closed curves β_F and γ_F respectively on F using the projections described in section 5.6. Then we have bounds on l_β in terms of l_{β_F} and on l_γ in terms of l_{γ_F}. By lemma 5.6.6, if

$$|l_{\beta_F} - l_{\gamma_F}| > 4 \ max(d_{\mathbb{H}}(\gamma, \gamma_F), d_{\mathbb{H}}(\beta, \beta_F))$$

then $l_\beta \neq l_\gamma$. We also know that $max(d_{\mathbb{H}}(\gamma, \gamma_F), d_{\mathbb{H}}(\beta, \beta_F))$ is small and decreases as the length of the core curve shrinks. This we can do within the constraints of the construction of M (as seen in section 5.4). Therefore even if $|l_{\beta_F} - l_{\gamma_F}|$ is small then $l_\beta \neq l_\gamma$.

To prove the lemma we show the contrapositive, i.e. show that if $i(\beta_F, \alpha_j) \neq i(\gamma_F, \alpha_j)$ for at least one simple closed curve α_j on F, then $|l_{\beta_F} - l_{\gamma_F}|$ can be made arbitrarily large so that we ensure $l_\beta \neq l_\gamma$.

Let α_p be a simple closed curve on F such that $i(\beta_F, \alpha_p) \neq i(\gamma_F, \alpha_p)$. Without loss of generality assume $i(\beta_F, \alpha_p) > i(\gamma_F, \alpha_p) > 0$. (N.B. if $i(\gamma_F, \alpha_p) = 0$ then shrinking α_p will not effect γ_F but will effect β_F and so $|l_{\beta_F} - l_{\gamma_F}|$ will increase, and we shrink α_p enough so that this difference is large enough to get a contradiction.)

Note that as $i(\beta_F, \alpha_p) > i(\gamma_F, \alpha_p)$, l_{β_F} will increase at a faster rate than l_{γ_F} and therefore $|l_{\beta_F} - l_{\gamma_F}|$ will be increasing. We continue to shrink α_p until this difference is large enough to get a contradiction. Hence if $i(\beta_F, \alpha_p) \neq i(\gamma_F, \alpha_p)$ for some simple closed curve α_p on F, then $l_\beta \neq l_\gamma$.

Therefore if $l_\alpha = l_\beta$ then $i(\alpha_F, \gamma_j) = i(\beta_F, \gamma_j)$ for all simple closed curves γ_j on F. □

Combining lemma 7.2.6 and lemma 7.2.7, the chain of implications is shown to be true, hence giving a proof to lemma 7.2.1, which we restate below.

Lemma 7.2.1 *Let $g, h \in \pi_1(M)$ be such that $\chi[h] = \chi[g]$ and such that both g and h are represented by geodesics β and γ in M that are projected onto simple closed curves β_F and γ_F on F (i.e $\beta \rightarrow \beta_F$ and $\gamma \rightarrow \gamma_F$ via the projection in section* 5.6). *Then $h \cong g^{\pm 1}$.*

Proof: Lemma 7.2.6 and lemma 7.2.7 provide the following chain of implications.

$$\chi[g] = \chi[h] \Rightarrow l_\beta = l_\gamma \Rightarrow i(\beta_F, \alpha_j) = i(\gamma_F, \alpha_j) \Rightarrow \beta_F = \gamma_F \Rightarrow \beta = \gamma$$

□

Therefore if we have two geodesics in M that have the same length and are both projected onto simple closed curves on F, then they are the same curve (up to orientation and homotopy).

To complete the proof of the simple property the first step of showing if α is represented by a simple closed curve on F and $l_\alpha = l_\beta$ then β is also represented by a simple closed curve on F needs to be extended.

7.3 Simple $\Rightarrow$ Simple

Let $g \in \pi_1(M)$ be represented by a geodesic β_g which when projected onto γ_g on F is a simple closed curve. Let $h \in \pi_1(M)$ be such that $\chi[h] = \chi[g]$. We would like to show that h is also projected onto a simple closed curve on F.

The problem with extending McShane's result directly is that all closed curves on F that intersect the core curve have lengths with a positive lower bound. This comes from the fact that the length of each segment in the broken geodesic path when lifted to $\mathbb{H}^3$ has positive lower bound (as seen in section 5.4 and figure 7.5) which is dependent on the length of the core curve. Hence an alternative method needs to be found.

As an initial idea, consider what could go wrong, and how β_h could be projected onto a non-simple closed curve on F. For this to happen, the projected curve must either intersect itself in at least one of the pieces F_i of F, or intersect itself on the boundary or core curve of M.

It is necessary to show that a geodesic that projects onto a non-simple closed curve on F can never have the same length as one that projects onto a simple closed curve on F. By the analysis in section 5.6, it is necessary to show that the difference between the lengths is bigger than four times the distance between the geodesic and its representative on F.

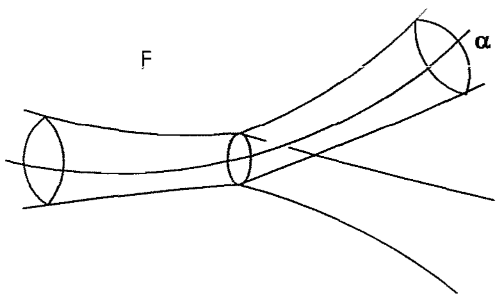


Fig 7.5: simple closed curve on F with lower bound on its length

As all closed curves on F have a positive lower bound on length, extending this part of McShane's lemma has proven to be extremely difficult, and is the reason that the simple property is just a conjecture. This part is still open for future work.

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