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Epimorphic images of simplicial Coxeter groups and some associated hyperbolic manifolds

by

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ABSTRACT

FACULTY OF ENGINEERING, SCIENCE AND MATHEMATICS
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EPIMORPHIC IMAGES OF SIMPLICIAL COXETER GROUPS AND SOME ASSOCIATED HYPERBOLIC MANIFOLDS.

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Abstract

In this work torsion-free normal subgroups of the Lannér groups Γ_i are studied. All torsion-free normal subgroups of the orientation-preserving subgroups Γ_i^+ whose factor groups have the form $L_2(q)$, where $q = p^n$ and p is a prime, are classified. In the case of each group, some examples of manifolds are constructed and their homology is computed. Minimal index torsion free subgroups of each Lannér group are also constructed.

Computational techniques are developed to construct complete lists of conjugacy classes of subgroups of low index in these groups. These lists are then used to test the theoretical results proved in this thesis and also to search for specific subgroups. Computational techniques are also developed to calculate the action of the isometries of these manifolds $\mathcal{M}$ on their homology groups. This gives $H_1(\mathcal{M})$ the structure of an Isom($\mathcal{M}$)-module, which allows for the construction of arbitrarily large manifolds exhibiting a high degree of symmetry.

The computational techniques developed in this work are applied to the 4-dimensional Coxeter group $[5, 3, 3, 3]$ and a detailed study of the low index subgroups of this group has been implemented. The existence of torsion free subgroups of index 115200 is established and a possible approach towards determining the minimal index torsion-free subgroups of this group is outlined.

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Chapter 1

Introduction

1.1 Introduction

The classification and characterisation of topological 3-manifolds has had a powerful influence on mathematical research over the past hundred years. Starting with the work of Henri Poincaré [Po] and continuing up to the present-day work of William Thurston [Th1], Richard Hamilton [Ha] and Grigory Perelman [Pe1] [Pe2], the question of classifying all compact 3-dimensional manifolds has yielded many fruitful avenues of research.

In 1982, Thurston proposed his *Geometrization conjecture*. In the case of closed oriented 3-manifolds, the conjecture can be stated as follows:

GEOMETRIZATION CONJECTURE [Th1]
Let M be a closed, oriented, 3-manifold. Then there is a finite collection of disjoint, embedded tori T_i^2 in M, and a finite collection of disjoint, embedded spheres S_i^2 in M, such that each component of the complement $(M \setminus \bigcup S_i^2) \setminus \bigcup T_i^2$ admits a geometric structure, i.e., a complete, locally homogeneous Riemannian metric.

There are a total of eight 3-dimensional model geometries and they are listed as follows:

1) Spherical geometry $\mathbb{S}^3$, with constant positive curvature $+1$.

2) Euclidean geometry $\mathbb{R}^3$, with constant curvature 0.

3) Hyperbolic geometry $\mathbb{H}^3$, with constant negative curvature -1.

4) The geometry of $\mathbb{S}^2 \times \mathbb{R}$.

5) The geometry of $\mathbb{H}^2 \times \mathbb{R}$.

6) The geometry of the universal cover $\widetilde{SL_2}(\mathbb{R})$ of the Lie group $SL_2(\mathbb{R})$.

7) Nil geometry - a left invariant metric on the Heisenberg group

$$\left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

8) Sol geometry - the split extension $\mathbb{R} \ltimes (\mathbb{R} \times \mathbb{R})$.

Six of these eight geometries (1, 2, 4, 5, 6 and 7) are now well understood - geometric manifolds modelled on these geometries are Seifert fibered. Geometric manifolds modelled on Sol geometry are torus fibered over the circle. Since most 3-manifolds do not admit such fibrations, it is reasonable to say that "most" 3-manifolds are hyperbolic [An2].

In this work, one particular class of hyperbolic manifolds will be investigated in detail. These are the manifolds that arise as covers of the hyperbolic orbifolds associated to the Lannér groups. A Lannér group is a Coxeter group acting on $\mathbb{H}^3$ whose fundamental region is a compact simplex. The manifolds constructed have many interesting and useful geometric properties and can often be constructed so as to admit a high degree of symmetry. Furthermore, the manifolds inherit several possible tessellations derived from the original group. These tessellations can be used to provide explicit combinatorial constructions for the manifolds.

1.2 Aims of this work

This thesis aims to investigate simplicial Coxeter groups, in particular the Lannér groups. All maps from these groups to the classical family of simple groups $L_2(q)$ are classified and a comprehensive study of the low index subgroups of some of these groups is provided. Results on some simplicial Coxeter groups acting on $\mathbb{H}^4$ are presented as well.

As a consequence of Selberg's Lemma, since Coxeter groups Γ admit faithful representations into $GL(n, \mathbb{C})$ ([HW]) for a suitable value of n, they contain finite index torsion free subgroups. Let $\mathcal{L}(\Gamma)$ be the least common multiple of the orders of all the finite subgroups of Γ. Then a minimal index torsion free subgroup of Γ must have index divisible by $\mathcal{L}(\Gamma)$ [CFJR]. Jones and Reid have shown in [JR] that for any k there exist Kleinian groups whose minimal index torsion free subgroups has index greater than $k \times \mathcal{L}(\Gamma)$. It will be shown in this thesis that Lannér groups, by contrast, contain a torsion free subgroup of index $\mathcal{L}(\Gamma)$ or $2 \times \mathcal{L}(\Gamma)$.

Using techniques from group theory and the classification of finite simple groups, this work aims to provide a construction for families of manifolds exhibiting a high degree of symmetry. In some interesting cases, further results will be deduced by considering the action of the isometry group of the manifold on the first homology.

Additionally, in the last ten years advances in computational resources have enabled an explosion in the computational-aided research of low index subgroups of discrete groups acting on spaces. This, coupled with a resurgent interest in Coxeter groups, has led to several interesting publications and significant progress in the classification of interesting families of Coxeter groups ([CM], [E2], [ERT]). A computing cluster was designed in this work and sophisticated algorithms were used to classify all subgroups of the Lannér groups, up to a given index. A summary of these results is also provided in this thesis. This adds significantly to the previous work carried out by B. Everitt ([EMc], [E1], [E2]).

1.3 Structure of this Thesis

Chapter 2 introduces definitions and previously known results important for this thesis. Abstract properties of Coxeter groups are introduced. Key properties of these groups, including the structure of conjugacy classes of torsion elements in the Lannér groups and the discrete extensions of these groups in $\mathrm{Isom}(\mathbb{H}^3)$, as classified in [DM2], are introduced. The computational methods used in this thesis are also discussed in this chapter.

Chapter 3 focuses on the construction of a family of torsion free normal subgroups of the orientation preserving subgroup of the $[5, 3, 5]$ Coxeter group whose factor groups are projective special linear groups $L_2(q)$ over a field of $q = p^n$ elements. The extensions of these results to the full Coxeter group are also characterised. In particular, the case where $p = 5$ is discussed due to the fact that it yields a chiral pair of Seifert - Weber manifolds.

Chapter 4 analyses the other Lannér groups using the same techniques as used in Chapter 3. The case $\Gamma = [3, 5, 3]$ is omitted since it has already been studied by Anna Torstensson in her PhD thesis [To]. In each case all normal subgroups N of the orientation preserving subgroup Γ^+ of Γ with $\Gamma^+/N \cong L_2(q)$ are classified.

Chapter 5 investigates some of the manifolds arising from the maps constructed in chapter 3. A chiral pair of Seifert-Weber manifolds are investigated and all symmetric abelian covers (covering manifolds whose deck transformations form an abelian group and on which the factor group acts by automorphisms) are characterised. Other manifolds whose isometry group is $L_2(19)$ are also constructed.

In Chapter 6, some of the manifolds arising from the epimorphisms described in chapter 4 are constructed and their properties are investigated. In a few special cases, the action of the isometries on homology is characterised. This work unifies and extends many previously known constructions, such as Zimmerman's manifolds, as described in [Z], as well as creating many new and interesting examples.

Generalizing to higher dimensions, Chapter 7 focuses on the $[5, 3, 3, 3]$ Coxeter group. This is a simplicial Coxeter group acting on $\mathbb{H}^4$. The construction of a torsion-free subgroup of index 14400 in this group will provide an example of a smallest volume hyperbolic 4-manifold. Subgroups of index 8×14400 are constructed using a combination of computational and group theoretic techniques. Some interesting results on the existence of subgroups of this group are proved.

In chapter 8 the main results of this work are restated, some additional concluding remarks are provided and futher avenues of study are suggested. A summary of the low index subgroups of the Lannér groups is provided in Appendix A.

Chapter 2

Background

This chapter introduces the relevant background material required for this thesis. Coxeter groups are introduced in both an algebraic and a geometric setting and the equivalence of the definitions is shown. Special properties of Coxeter groups are defined, as well as some general terminology to be used throughout the thesis. The classification theorem for finite Coxeter groups is stated [Hu] and the conjugacy classes of elements in certain finite Coxeter groups are listed. The classification of all finite extensions in $\mathrm{Isom}(\mathbb{H}^3)$ of certain hyperbolic Coxeter groups (the Lannér groups), due to Derevnin and Mednykh [DM2], is stated. A general overview of some special computational techniques in determining conjugacy classes of subgroups in given finitely presented groups is also discussed in this chapter.

2.1 Introduction to Coxeter groups

Definition 2.1 *A Coxeter group Γ is a group with generating set $S = \{s_1, \ldots, s_n\}$, together with a set of defining relators $(s_i s_j)^{m_{ij}}$ where $m_{ii} = 1$, $m_{ij} \geq 2$ if $i \neq j$ and $m_{ij} = m_{ji}$ for all $i \neq j$.*

Associated with such a group is a graph called the *Coxeter graph* or *Coxeter scheme* for the group. It consists of a set of vertices v_i (one for each generator $s_i \in S$), connected by (labeled) edges under the following conditions:

- For $i \neq j$, if $m_{ij} > 3$ then v_i is connected to v_j by an edge labeled by m_{ij}
- For $i \neq j$, if $m_{ij} = 3$ then v_i is connected to v_j by an unlabeled edge
- If $i = j$ or $m_{ij} = 2$ then no edge is drawn.

Definition 2.2 *A Coxeter group* Γ *with a connected Coxeter scheme is called* irreducible.

For any subset $S' \subseteq S$, the special subgroup of Γ generated by the $s \in S'$ is also a Coxeter group with Coxeter scheme obtained from that of Γ by deleting the vertices whose corresponding generators lie in $S \backslash S'$. If a Coxeter scheme is disconnected, then $\Gamma = \Gamma_1 \times \ldots \times \Gamma_k$, where each Γ_i is an irreducible Coxeter group.

Definition 2.3 *Let* Γ *be a Coxeter group with generating set* S*. Then a* maximal special subgroup *is a subgroup generated by* $S' = S \backslash \{s\}$ *for any* $s \in S$.

In order to determine the torsion elements in a Coxeter group, some information is needed on the finite Coxeter groups. The finite irreducible Coxeter groups are the Weyl groups of simple Lie algebras over $\mathbb{C}$, the dihedral groups, the group of symmetries of a regular dodecahedron and the group of symmetries of the 120-cell. In the Killing-Cartan notation, these are the three infinite classical families, A_n with $n \geq 1$, B_n with $n \geq 2$ and D_n with $n \geq 4$, as well as the five exceptional groups $G_2 \cong I_2(6)$, F_4, E_6, E_7 and E_8. The non-Weyl groups are the dihedral groups $I_2(n)$ for $n \geq 3$, the symmetry group of the dodecahedron H_3 and the symmetry group of the 120-cell, H_4. The Coxeter scheme for any finite Coxeter group is a disjoint union of the connected Coxeter schemes listed in Table 2.1. In the cases of A_n, B_n and D_n, the subscript n is the number of generators of the group.

The irreducible parabolic Coxeter groups (affine groups) are listed in Table 2.2. These are the affine Weyl groups and they form four infinite families and five exceptional groups. In the Killing-Cartan notation, the four infinite families are $\widetilde{A_n}$ with $n \geq 1$, $\widetilde{B_n}$ with $n \geq 3$, $\widetilde{C_n}$ with $n \geq 3$ and $\widetilde{D_n}$ with $n \geq 4$. The five exceptional groups are $\widetilde{G_2}$, $\widetilde{F_4}$, $\widetilde{E_6}$, $\widetilde{E_7}$ and $\widetilde{E_8}$. A Coxeter group is parabolic if and only if it is the direct product of finite and parabolic irreducible Coxeter groups. In the cases of $\widetilde{A_n}$, $\widetilde{B_n}$, $\widetilde{C_n}$ and $\widetilde{D_n}$, the number of generators of the group is $n+1$.

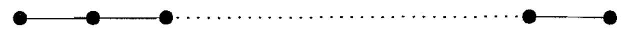
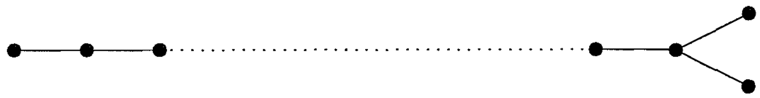
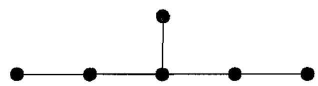
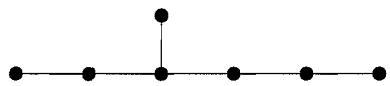
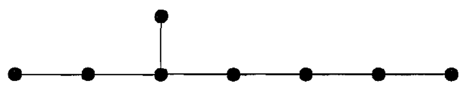
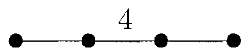
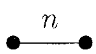
	A_n
	B_n
	D_n
	E_6
	E_7
	E_8
	F_4
	H_3
	H_4
	$I_2(n)$

Table 2.1: Finite irreducible Coxeter groups

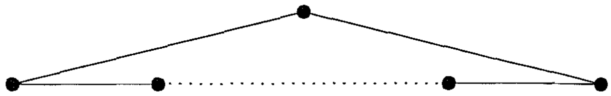
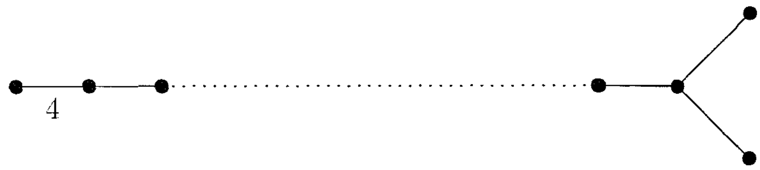
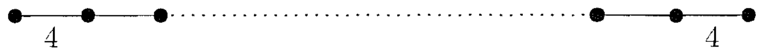
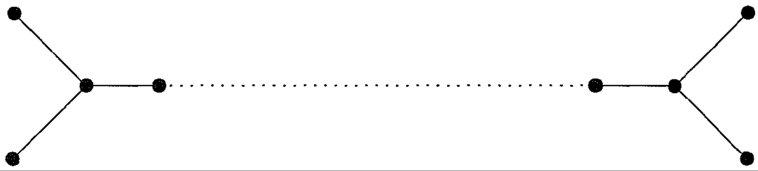
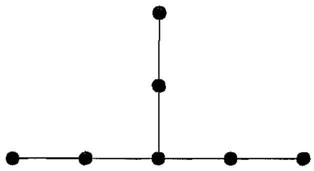
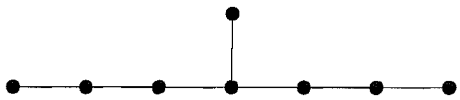
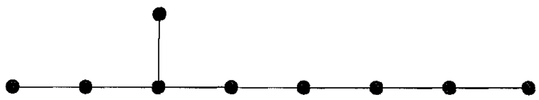
	$\widetilde{A}_n$
	$\widetilde{B}_n$
	$\widetilde{C}_n$
	$\widetilde{D}_n$
	$\widetilde{E}_6$
	$\widetilde{E}_7$
	$\widetilde{E}_8$
	$\widetilde{G}_2$
	$\widetilde{F}_4$

Table 2.2: Parabolic irreducible Coxeter groups

A convex polytope $\mathcal{P}$ in $\mathbb{H}^n$ is the intersection $\cap_{s \in S} \overline{H_s}$ of closed halfspaces bounded by hyperplanes H_s. A Coxeter group is hyperbolic of dimension n if and only if there is some polyhedron $\mathcal{P} \in \mathbb{H}^n$ such that, assigning each $s_i \in S$ to the reflection in the hyperplane H_{s_i}, an isomorphism is induced between Γ and the group generated by reflections in the faces of $\mathcal{P}$. $\mathcal{P}$ is called a *fundamental region* for Γ.

The Gram matrix $G(\Gamma)$ for a Coxeter group Γ is a matrix with entries $g_{ij} = -\cos(\frac{\pi}{m_{ij}})$, where the m_{ij} are as above. The *signature* of a symmetric matrix is a pair (p, q) where p is the number of negative eigenvalues and q is the number of positive eigenvalues. The following theorem is due to Vinberg [V2]

Theorem 2.1 *Let $G(\Gamma)$ be an indecomposable symmetric matrix of signature $(n, 1)$ with 1's along the diagonal and non-positive entries off it. Then there is a convex polytope P in $\mathbb{H}^n$ whose Gram matrix is G. The polytope is uniquely determined up to a motion in $\mathbb{H}^n$.*

Let Γ be a hyperbolic Coxeter group. Then Γ is co-compact if it has a compact fundamental polyhedron $\mathcal{P}$. Otherwise Γ is non-co-compact. Γ is co-finite if it has a finite volume fundamental polyhedron $\mathcal{P}$.

Let $\mathcal{F}$ be the collection of finite subgroups of Γ generated by subsets $S' \subset S$. Partially order $\mathcal{F}$ by inclusion. Similarly, let $\overline{\mathcal{F}}$ be the partially ordered collection of finite subgroups or parabolic subgroups of Γ generated by subsets $S' \subset S$. Then the following result is due to Vinberg [V2]

Proposition 2.1.1 *An n-dimensional hyperbolic Coxeter group Γ is co-compact (co-finite) if and only if $\mathcal{F}$ ($\overline{\mathcal{F}}$) is isomorphic as a partially ordered set to the poset of some n-dimensional abstract polytope.*

In particular, if Γ is a hyperbolic Coxeter group, then Γ is co-compact if and only if every maximal special subgroup is finite, and Γ is co-finite if and only if, for every maximal special subgroup, every factor is either finite or is an irreducible parabolic Coxeter group.

2.2 Conjugacy classes in finite Coxeter groups

The main focus of this thesis is on the nine co-compact simplicial hyperbolic Coxeter groups, first described by Lannér [La] and listed here in Table 2.3:

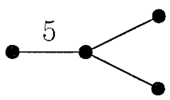
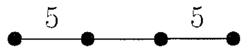
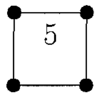
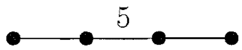
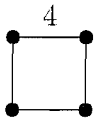
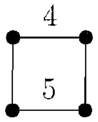
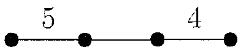
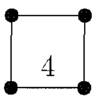
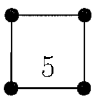
		
$\Gamma_1 = T_1[2,2,3;3,5,2]$	$\Gamma_4 = T_4[2,2,5;2,3,5]$	$\Gamma_7 = T_7[2,3,3;2,5,3]$
		
$\Gamma_2 = T_2[2,2,3;2,5,3]$	$\Gamma_5 = T_5[2,3,3;2,4,3]$	$\Gamma_8 = T_8[2,4,3;2,5,3]$
		
$\Gamma_3 = T_3[2,2,4;2,3,5]$	$\Gamma_6 = T_6[2,3,4;2,3,4]$	$\Gamma_9 = T_9[2,3,5;2,3,5]$

Table 2.3: The nine Lannér groups

These nine groups form a complete list of all Coxeter groups which have a 3-dimensional compact hyperbolic simplex as a fundamental region. Each group is described by a 6-tuple $[\lambda_1, \lambda_2, \lambda_3; \mu_1, \mu_2, \mu_3]$, of numbers, where π/λ_i are the angles between a chosen base face and the other three faces of T, and π/μ_i is the dihedral angle of the edge opposite that labelled with π/λ_i.

The finite Coxeter groups play a key role in the construction of manifolds arising as the covers of the fundamental regions of the simplicial Coxeter groups. If Γ is a Coxeter group acting discretely on $\mathbb{H}^3$ with the natural action and if $\mathcal{M}$ is an orbifold of the form $\mathcal{M} = \mathbb{H}^3/H$, for some $H < \Gamma$, then $\mathcal{M}$ is a manifold precisely when H is torsion free. It remains to determine all representatives for torsion elements in Γ. The following theorem is well known. It is exercise V.4.2 in [B], and is proved in [BH].

Theorem 2.2 *Let Γ be a Coxeter group with generating set S. Then any element of finite order in Γ is conjugate to an element of a finite subgroup generated by some $S' \subseteq S$.*

The maximal special subgroups of the Lannér groups are all of types $A_1 \times A_1 \times$

A_1, $A_1 \times A_2 \cong A_1 \times I_2(3)$, A_3, $A_1 \times B_2 \cong A_1 \times I_2(4)$, $A_1 \times I_2(5)$, B_3 or H_3. Hence, to understand the torsion elements in each of the Lannér groups, the conjugacy classes of torsion elements in each of these finite groups are needed. A Coxeter scheme and presentation for each of these seven groups is provided in Table 2.4

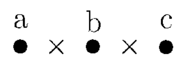
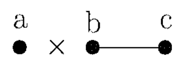
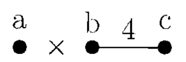
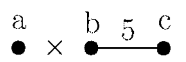
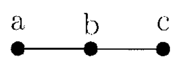
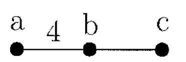
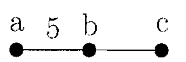
Group	Coxeter Scheme	Presentation
$A_1 \times A_1 \times A_1$		$\langle a,b,c \mid a^2 = b^2 = c^2 = (ab)^2 = (ac)^2 = (bc)^2 = 1\rangle$
$A_1 \times I_2(3)$		$\langle a,b,c \mid a^2 = b^2 = c^2 = (ab)^2 = (ac)^2 = (bc)^3 = 1\rangle$
$A_1 \times I_2(4)$		$\langle a,b,c \mid a^2 = b^2 = c^2 = (ab)^2 = (ac)^2 = (bc)^4 = 1\rangle$
$A_1 \times I_2(5)$		$\langle a,b,c \mid a^2 = b^2 = c^2 = (ab)^2 = (ac)^2 = (bc)^5 = 1\rangle$
A_3		$\langle a,b,c \mid a^2 = b^2 = c^2 = (ab)^3 = (ac)^2 = (bc)^3 = 1\rangle$
B_3		$\langle a,b,c \mid a^2 = b^2 = c^2 = (ab)^4 = (ac)^2 = (bc)^3 = 1\rangle$
H_3		$\langle a,b,c \mid a^2 = b^2 = c^2 = (ab)^5 = (ac)^2 = (bc)^3 = 1\rangle$

Table 2.4: Finite Coxeter groups whose conjugacy class structure is required

2.2.1 Conjugacy classes in $A_1 \times A_1 \times A_1$

The Coxeter group $A_1 \times A_1 \times A_1$ has presentation $\langle a,b,c \mid a^2 = b^2 = c^2 = (ab)^2 = (ac)^2 = (bc)^2 = 1\rangle \cong C_2 \times C_2 \times C_2$. The orientation preserving subgroup $(A_1 \times A_1 \times A_1)^+$ has presentation $\langle \alpha, \beta \mid \alpha^2 = \beta^2 = (\alpha\beta)^2 = 1\rangle$, where $\alpha = ab$ and $\beta = bc$ and is isomorphic to the Klein 4-group V_4. Representatives for the conjugacy classes of non-identity elements of $A_1 \times A_1 \times A_1$ and $(A_1 \times A_1 \times A_1)^+$ are given in Table 2.5:

$A_1 \times A_1 \times A_1$		$(A_1 \times A_1 \times A_1)^+$	
Order 2:	a, b, c, ab, ac, bc	Order 2:	α, β, $\alpha\beta$

Table 2.5: Conjugacy classes in $A_1 \times A_1 \times A_1$ and $(A_1 \times A_1 \times A_1)^+$

2.2.2 Conjugacy classes in $A_1 \times I_2(3)$

The Coxeter group $A_1 \times A_2$ has presentation $\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^2 = (ac)^2 = (bc)^3 = 1\rangle \cong C_2 \times D_3$. The orientation preserving subgroup $(C_2 \times I_2(3))^+$ has presentation $\langle \alpha, \beta \mid \alpha^2 = \beta^3 = (\alpha\beta)^2 = 1\rangle$, where $\alpha = ab$ and $\beta = bc$ and is isomorphic to the symmetric group S_3. Representatives for the conjugacy classes of non-identity elements of $A_1 \times I_2(3)$ and $(A_1 \times I_2(3))^+$ are given in Table 2.6:

$A_1 \times I_2(3)$		$(A_1 \times I_2(3))^+$	
Order 2:	a, b, ab	Order 2:	α
Order 3:	bc	Order 3:	β
Order 6:	abc		

Table 2.6: Conjugacy classes in $A_1 \times I_2(3)$ and $(A_1 \times I_2(3))^+$

2.2.3 Conjugacy classes in $A_1 \times I_2(4)$

The Coxeter group $A_1 \times I_2(4)$ has presentation $\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^2 = (ac)^2 = (bc)^3 = 1\rangle \cong C_2 \times D_4$. The orientation preserving subgroup $(C_2 \times I_2(4))^+$ has presentation $\langle \alpha, \beta \mid \alpha^2 = \beta^4 = (\alpha\beta)^2 = 1\rangle$, where $\alpha = ab$ and $\beta = bc$ and is isomorphic to the dihedral group D_4 of order 8. Representatives for the conjugacy classes of non-identity elements of $A_1 \times I_2(4)$ and $(A_1 \times I_2(4))^+$ are given in Table 2.7:

$A_1 \times I_2(4)$		$(A_1 \times I_2(4))^+$	
Order 2:	a, b, c, ab, ac, $(bc)^2$	Order 2:	α, $\alpha\beta$, β^2
Order 4:	bc, abc	Order 4:	β

Table 2.7: Conjugacy classes in $A_1 \times I_2(4)$ and $(A_1 \times I_2(4))^+$

2.2.4 Conjugacy classes in $A_1 \times I_2(5)$

The Coxeter group $A_1 \times I_2(5)$ has presentation $\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^2 = (ac)^2 = (bc)^3 = 1\rangle \cong C_2 \times D_5$. The orientation preserving subgroup $(C_2 \times I_2(5))^+$ has presentation $\langle \alpha, \beta \mid \alpha^2 = \beta^4 = (\alpha\beta)^2 = 1\rangle$, where $\alpha = ab$ and $\beta = bc$

and is isomorphic to the dihedral group D_5 of order 10. Representatives for the conjugacy classes of non-identity elements of $A_1 \times I_2(5)$ and $(A_1 \times I_2(5))^+$ are given in Table 2.8:

$A_1 \times I_2(5)$		$(A_1 \times I_2(5))^+$	
Order 2:	a, b, ac	Order 2:	$\alpha\beta$
Order 5:	bc, $(bc)^2$	Order 5:	β, β^2
Order 10:	abc, $a(bc)^2$		

Table 2.8: Conjugacy classes in $A_1 \times I_2(3)$ and $(A_1 \times I_2(3))^+$

2.2.5 Conjugacy classes in A_3

The Coxeter group A_3 has presentation $\langle a, b, c | a^2 = b^2 = c^2 = (ab)^3 = (bc)^3 = (ac)^2 = 1\rangle$. That this is a presentation for the symmetric group Sym_4 can be shown by the identifications: $a \leftrightarrow (1,2)$, $b \leftrightarrow (2,3)$ and $c \leftrightarrow (3,4)$. The orientation preserving subgroup A_3^+ has presentation $\langle \alpha, \beta | \alpha^3 = \beta^3 = (\alpha\beta)^2 = 1\rangle$, with $\alpha = ab$ and $\beta = bc$, and is isomorphic to the alternating group Alt_4. Representatives for the conjugacy classes of non-identity elements of A_3 and A_3^+ are as shown in Table 2.9:

A_3		A_3^+	
Order 2:	a, ac	Order 2:	$\alpha\beta$
Order 3:	bc	Order 3:	β, β^{-1}
Order 4:	abc		

Table 2.9: Conjugacy classes in A_3 and A_3^+

2.2.6 Conjugacy classes in B_3

B_3 has presentation $\langle a, b, c | a^2 = b^2 = c^2 = (ab)^4 = (bc)^3 = (ac)^2 = 1\rangle \cong S_4 \times C_2$. The orientation preserving subgroup B_3^+ has presentation $\langle \alpha, \beta | \alpha^4 = \beta^3 = (\alpha\beta)^2 = 1\rangle$, where $\alpha = ab$ and $\beta = bc$, and is isomorphic to S_4. Representatives for the conjugacy classes of non-identity elements of B_3 and B_3^+ are as shown in Table 2.10:

B_3		B_3^+	
Order 2:	a, c, ac, $(ab)^2$, $(abc)^3$	Order 2:	α^2, $\alpha\beta$
Order 3:	bc	Order 3:	β
Order 4:	ab, $c(ab)^2$	Order 4:	α
Order 6:	abc		

Table 2.10: Conjugacy classes in B_3 and B_3^+

2.2.7 Conjugacy classes in H_3

H_3 has presentation $\langle a, b, c | a^2 = b^2 = c^2 = (ab)^5 = (bc)^3 = (ac)^2 = 1\rangle \cong Alt_5 \times C_2$. The orientation preserving subgroup H_3^+ has presentation $\langle \alpha, \beta | \alpha^5 = \beta^3 = (\alpha\beta)^2 = 1\rangle$, where $\alpha = ab$ and $\beta = bc$, and is isomorphic to the alternating group Alt_5. Then representatives for the conjugacy classes of non-identity elements of H_3 and H_3^+ are as shown in Table 2.11:

H_3		H_3^+	
Order 2:	a, ac, $(abc)^5$	Order 2:	$\alpha\beta$
Order 3:	bc	Order 3:	β
Order 5:	ab, $(ab)^2$	Order 5:	α, α^2
Order 6:	$c(ab)^2$		
Order 10:	abc, $(abc)^3$		

Table 2.11: Conjugacy classes in H_3 and H_3^+

2.3 Discrete extensions of the 3-dimensional simplicial Coxeter groups

In [La] Lannér listed all the tetrahedra in $\mathbb{H}^3$ whose dihedral angles are of the form π/n, for some integer n. Associated with these tetrahedra are the reflection groups generated by reflections in the faces of the tetrahedra and referred to either as Coxeter groups or as Lannér groups. The angle between any pair of faces is an integer submultiple of π, hence the group generated by reflections across the faces of the tetrahedron forms a discrete subgroup of Isom($\mathbb{H}^3$).

Much of this thesis is devoted to the classification of torsion free normal subgroups N of the orientation preserving subgroup Γ_i^+ of the Lannér groups Γ_i, as listed in Table 2.3. In particular a complete classification of all $N \lhd \Gamma_i^+$ satisfying $\Gamma_i^+/N \cong L_2(p^n)$, where p is a prime, is given. Hyperbolic manifolds with a large number of symmetries are also constructed. A manifold $\mathcal{M}$ is said to have a large number of symmetries if $\mathrm{vol}(\mathcal{M}/\mathrm{Isom}(\mathcal{M}))$ is small. Let $\tilde{\Gamma}_i = N_{\mathrm{Isom}(\mathbb{H}^3)}(\Gamma_i)$ be the normalizer of Γ_i in $\mathrm{Isom}(\mathbb{H}^3)$ and $\mathcal{M} = \mathbb{H}^3/K$, where K is a torsion free subgroup of Γ_i. Then $\mathrm{Isom}(\mathcal{M})$ is the normalizer $N_{\tilde{\Gamma}_i}(K)$ of K in $\tilde{\Gamma}_i$. Hence it is important to know the normaliser $N_{\mathrm{Isom}(\mathbb{H}^3)}(\Gamma_i)$ of Γ_i in $\mathrm{Isom}(\mathbb{H}^3)$. In [DM2] Derevnin and Mednykh classified all extensions of the Lannér groups:

Theorem 2.3 *([DM2]) Let Γ_i be a one of the nine Lannér groups listed in Table 2.3 and let Γ_i^+ be the orientation preserving subgroup of Γ_i. If Γ_i^+ is a subgroup of a discrete group $G \subset Isom(\mathbb{H}^3)$, then Γ_i^+ is normal in G.*

As a consequence of this theorem, $N_{\mathrm{Isom}(\mathbb{H}^3)}(\Gamma_i)$ can be identified with $\mathrm{Aut}(\Gamma_i^+)$ (since the Γ_i^+ have trivial center) and $\mathrm{Inn}(\Gamma_i^+) = \Gamma_i^+$. The outer automorphism group of Γ_i^+ is then one of the following types:

If $i \in \{6, 9\}$, then $\mathrm{Aut}(\Gamma_i^+)/\mathrm{Inn}(\Gamma_i^+) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

If $i \in \{1, 2, 4, 5, 7, 8\}$, then $\mathrm{Aut}(\Gamma_i^+)/\mathrm{Inn}(\Gamma_i^+) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

If $i = 3$, then Γ_3 is the unique non-trivial discrete extension of Γ_3^+ in $\mathrm{Isom}(\mathbb{H}^3)$ and $\mathrm{Aut}(\Gamma_3^+)/\mathrm{Inn}(\Gamma_3^+) \cong \mathbb{Z}_2$.

The outer automorphisms of Γ_i can also be considered as the *graph automorphisms* of the Coxeter scheme for Γ_i, acting on the generators s_i corresponding to the vertices v_i. This induces an action on Γ_i^+, and together with conjugation by elements of $\Gamma_i \backslash \Gamma_i^+$ they generate the outer automorphism group described in the remarks following the statement of Theorem 2.3.

2.4 Determining low index subgroups of finitely presented groups

As is mentioned in the previous section, much of this thesis is spent characterizing families of normal subgroups of the Lannér groups. To better understand these groups, computer-aided constructions were employed to classify all conjugacy classes of low index subgroups. In order to construct complete lists of low index subgroups of G the program Lowx [Lx] was used. All conjugacy classes of subgroups up to index 60 were computed for Γ_1, Γ_2, Γ_3 and Γ_4. All conjugacy classes of subgroups up to index 20 were computed for Γ_5, Γ_6, Γ_7 and Γ_8, while all conjugacy classes of subgroups up to index 24 were computed for Γ_9.

Using this data, further subgroups could then be determined by letting a Lannér group Γ_i^+ act on the cosets of a representative H of a conjugacy class. The action induces a permutation representation $\phi : \Gamma_i^+ \to S_n$ of Γ_i in some symmetric group S_n, where n is the index of H in Γ_i. The core of a representative, $K(H)$, is the kernel of this representation. In many cases, provided that n is not too big, it is possible to determine a presentation for $K(H)$. Various invariants of the associated manifold, $\mathcal{M}_\phi = \mathbb{H}^3/K(H)$, such as its fundamental group and its homology, can then be determined.

Given a permutation representation $\phi(\Gamma_i)$, the combinatorial structure of the associated manifold can then be deduced by careful analysis of the group structure. As a result, a complete combinatorial description of the manifold $\mathcal{M}_\phi$, the action of the isometry group on $\mathcal{M}_\phi$, a presentation for its fundamental group, a complete description of its homology and its volume can all be provided.

The method of determining all conjugacy classes of subgroups up to a given index is a modification of the *coset enumeration* algorithm ([HEB], [Ne]). The basic coset enumeration algorithm constructs, for a given finitely generated subgroup H of a finitely presented group G, a coset table for H in G. One of the most important facts about the algorithm is that, for any finitely presented G and H of finite index in G, the algorithm will eventually terminate.

The low index search algorithm works as follows: In order to find all subgroups of index $\leq N$ in a group G. the algorithm starts with the trivial group and proceeds to enumerate cosets of it using the Todd-Coxeter (TC) algorithm. The number of cosets allowed is bounded by some function $f(N)$. It is generally assumed that $N < f(N) \leq 2N$. The TC algorithm then either returns the statement that G has some order $\leq f(N)$ or halts with an incomplete coset table. If G has order less than $f(N)$ then its coset table (modulo the unit subgroup) is printed as a first table. If, however, G has order bigger than $f(N)$, the TC algorithm stops with an incomplete coset table. In this case, pairs of forced coincidences (i_l, i_k) are then systematically introduced between the incomplete coset rows i_l and i_k in the table. After applying a coincidence, the TC procedure is then restarted on the modified coset table. In theory this procedure will continue until there are no further incomplete tables. In practise, however, if the initial bound N is even moderately large, the procedure is limited by machine constraints.

The above algorithm describes a method for finding all subgroups H of index less than N in G. In general it is more efficient to find conjugacy class representatives of subgroups. This is how the algorithm is implemented in both GAP and Lowx. Furthermore, the TC algorithm does not provide a generating set for each representative. There are a number of methods available to compute a presentation for H. One such method is to use the completed coset table for H in G to construct a set of *Schreier generators* for H([HEB] §2.5). Once a set of generators for H has been constructed, a *modified Todd-Coxeter procedure* can be used to find a presentation for H as an abstract group. The resulting presentation is generally rather cumbersome and can be simplified using a sequence of *Tietze transformations.*

Chapter 3

Epimorphic images of the $[5,3,5]$ Coxeter group

3.1 The Coxeter group $[5,3,5]$

Let T be a tetrahedron in $\mathbb{H}^3$ with vertices A, B, C, D, and with dihedral angles $\pi/5$, $\pi/3$, $\pi/5$, $\pi/2$, $\pi/2$ and $\pi/2$ along its edges CD, AD, AB, BD, BC, AC, as shown in Figure 3.1. The standard generators a, b, c and d of the Coxeter group $\Gamma = [5,3,5]$ are the reflections of $\mathbb{H}^3$ in the sides of T opposite A, B, C and D.

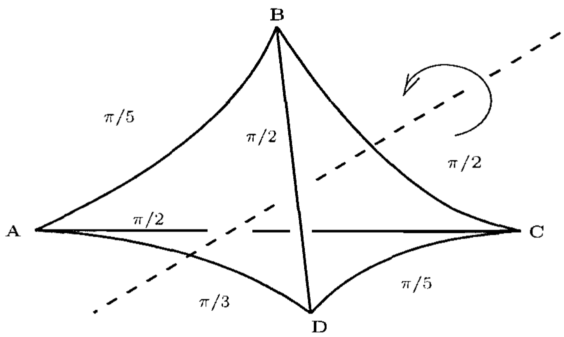


Figure 3.1: A hyperbolic tetrahedron T

Thus Γ is a discrete group of isometries of $\mathbb{H}^3$, and this space carries a tessellation $\mathcal{T}$ by the images of T, permuted regularly by Γ. The normaliser Ω of Γ in $\operatorname{Isom}\mathbb{H}^3$ is a semidirect product of Γ by a cyclic group of order 2, generated by a half-turn r of $\mathbb{H}^3$ about an axis through the mid-points of the edges AD and BC of T; this preserves the tessellation $\mathcal{T}$, and acts by conjugation on Γ as $a^r = d$, $b^r = c$, $c^r = b$ and $d^r = a$.

The subgroup $\Gamma_0 = \langle a, b, c\rangle$ of Γ is the [5, 3] Coxeter group. This group is the isometry group of a dodecahedron and is isomorphic to $A_5 \times C_2$. The 120 images of T under Γ_0 form a hyperbolic dodecahedron $\mathbb{D}$, with dihedral angles $2\pi/5$, and the images of $\mathbb{D}$ under Γ form a dodecahedral tessellation $\mathcal{D}$ of $\mathbb{H}^3$; there are five dodecahedra around every edge, so the vertex figure of $\mathcal{D}$ is an icosahedron. The isometry group of $\mathcal{D}$ is Γ, while r sends $\mathcal{D}$ to its dual tessellation $\mathcal{D}^*$, which is isomorphic to but distinct from $\mathcal{D}$.

The orientation-preserving subgroup Γ^+ of Γ has a presentation

$$\Gamma^+ = \langle \alpha, \beta, \gamma \mid \alpha^5 = \beta^3 = \gamma^5 = (\alpha\beta)^2 = (\alpha\beta\gamma)^2 = (\beta\gamma)^2 = 1\rangle,$$

where $\alpha = ab$, $\beta = bc$ and $\gamma = cd$ are rotations of $\mathbb{H}^3$ through $2\pi/5, 2\pi/3$ and $2\pi/5$ about the edges CD, AD and AB of T. It has an associated fundamental region $T' = T \cap a(T)$ formed by doubling T across the face BCD.

Lemma 3.1.1 *The torsion elements of Γ^+ are the conjugates in Γ^+ of the powers of α, β, γ and $\alpha\beta$.*

Proof: Any isometry g of $\mathbb{H}^3$ of finite order has a fixed point $p \in \mathbb{H}^3$. Let $g \in \Gamma^+$ be an element of finite order. Then, since T is a fundamental region for Γ, conjugating g by a suitable element of Γ, it may be assumed that $p \in T$. Suppose that $g \neq 1$, so p is in the boundary ∂T of T. If p is in the interior of an edge or face of ∂T then g fixes that entire edge or face, including any incident vertices; it can therefore be assumed that p is a vertex of T.

The stabiliser in Γ of the vertex D of T is the rank three Coxeter group $\Gamma_0 = \langle a, b, c\rangle$, so if g fixes D then g lies in the orientation-preserving subgroup $\Gamma_0^+ = [5,3]^+ = \langle \alpha, \beta\rangle \cong A_5$ of Γ_0; the non-identity elements of this group, of orders 2, 3 or 5, are conjugate to the powers of $\alpha\beta$, β or α respectively, as required. Similarly if g fixes A then it lies in the orientation-preserving subgroup $\Gamma_1^+ = \langle\beta,\gamma\rangle \cong A_5$ of $\Gamma_1 = \langle b,c,d\rangle$, and is conjugate to a power of $\beta\gamma$, β or γ. If g fixes C then it lies in the orientation-preserving subgroup $\langle\alpha,\beta\gamma\rangle \cong D_5$ of $\langle a,b,d\rangle \cong D_5 \times C_2$, and is therefore conjugate to a power of α or $\beta\gamma$; similarly if g fixes B it is conjugate in $\langle\alpha\beta,\gamma\rangle \cong D_5$ to a power of γ or $\beta\gamma$. The involution $\alpha\beta\gamma$ is conjugate in $\langle\alpha\beta,\gamma\rangle$ to $\alpha\beta$, and in $\langle\alpha,\beta\gamma\rangle$ to $\beta\gamma$, so these three involutions are all conjugate in Γ^+. ■

Note: Lemma 3.1.1 can be extended to determine all the conjugacy classes of torsion elements of Ω.

Corollary 3.1.1 *Each proper normal subgroup of Γ^+ is torsion-free.*

Proof: It follows easily from the presentation of Γ^+ that if any non-identity power of α, β, γ or $\alpha\beta$ is mapped to the identity then the group collapses to the trivial group. Thus the only normal subgroup containing non-identity torsion elements is Γ^+. ■

Hence, if N is a proper normal subgroup of Γ^+, then the corresponding quotient space $\mathbb{H}^3/N$ of $\mathbb{H}^3$ is a manifold, and it is compact if and only if N has finite index in Γ^+. The presentation given above shows that Γ^+ is perfect, so its maximal normal subgroups have nonabelian simple groups as quotients. The family $L_2(q)$ of finite simple groups will be considered as possible quotients since in a certain sense "most" nonabelian finite simple groups have the form $L_2(q)$. For technical reasons it is necessary to deal first with quotients isomorphic to the alternating group $A_5 \cong L_2(4) \cong L_2(5)$.

3.2 Quotients of Γ^+ isomorphic to A_5

Lemma 3.2.1

(i) *There are two normal subgroups N_1 and N_2 of Γ^+ with $\Gamma^+/N_i \cong A_5$, namely the kernels of the epimorphisms $\theta_i : \Gamma^+ \longrightarrow A_5$ $(i = 1,2)$ given by*

$$\alpha \mapsto (13524),\ \beta \mapsto (123),\ \gamma \mapsto (14352) \text{ or } (13425).$$

(ii) *N_1 and N_2 are conjugate in Γ.*

(iii) *Each N_i is normal in the orientation preserving subgroup Ω^+ of Ω, with Ω^+/N_i isomorphic to the symmetric group S_5.*

Proof: (i) The normal subgroups with quotient A_5 are the kernels of epimorphisms $\Gamma^+ \to A_5$. Two epimorphisms have the same kernel if and only if they differ by an automorphism of A_5. Any epimorphism θ must map α and β to elements of order 5 and 3 in A_5 and any pair of elements of this type generate A_5. By composing θ with a unique automorphism of A_5, it may be assumed that θ maps α and β to (13524) and (123), since their product is an involution. It remains to find all possible images of γ in A_5, preserving the relations of Γ^+ involving γ. It is easily seen that the only possibilities are (14352) and (13425), both of which are conjugate to α^2. This gives two inequivalent epimorphisms θ_1 and θ_2. Define $N_i = \ker\theta_i$ for $i = 1, 2$.

(ii) Since Γ normalises Γ^+, its action by conjugation permutes the normal subgroups of Γ^+ with a given quotient, so N_1 and N_2 must be either normal in Γ or conjugate in Γ. If either N_i were normal in Γ, then b would induce an automorphism of $\Gamma^+/N_i \cong A_5$ imitating the effect it has by conjugation on Γ^+. This automorphism must invert the images (13524) and (123) of α and β, and the only element of Aut $A_5 \cong S_5$ doing this is the permutation (13)(45). Mapping b to (13)(45), then $c = b\beta$ is mapped to (23)(45). Then $d = c\gamma$ is mapped to (14253) or (13524) for $i = 1, 2$, which is clearly not to an involution in either case. Thus the subgroups N_i are not normal in Γ, so they are conjugate to each other. (Alternatively, note that the element $(abc)^5$ of Γ, which represents the antipodal involution in the dodecahedral group $\Gamma_0 \cong A_5 \times C_2$ and hence commutes with α and β, transposes θ_1 and θ_2.)

(iii) The action of r by conjugation on Γ^+ is to transpose α and γ^{-1} and to invert β, and for $i = 1, 2$ the permutations (23) and (13) in S_5 respectively have the same effect on the images of these generators under θ_i; this shows that each θ_i extends to an epimorphism $\Omega^+ = \langle\Gamma^+, r\rangle \to S_5$, so each N_i is normal in Ω^+ with $\Omega^+/N_i \cong S_5$. ∎

The quotient manifolds $\mathbb{H}^3/N_i$ form a chiral pair of Seifert-Weber spaces, so it is useful to obtain presentations for their fundamental groups N_i. These were obtained by using GAP to implement the Reidemeister-Schreier process. The epimorphism θ_1 sends $abcd$ to $(13524)(14352) = (15)(23)$. Adjoining the relation $(abcd)^2 = 1$ to the presentation for Γ^+ gives a presentation for A_5. Hence N_1 is the normal closure of $(abcd)^2$ in Γ^+ ([C1, pp 23-33]). Using GAP a presentation for N_1 as an abstract group was found. This presentation is given in Table 3.1.

$$\begin{aligned}\langle F_1, F_2, F_3, F_4 \mid\ & F_1F_4^{-1}F_2F_3^{-1}F_1F_2F_3F_4^{-1}F_1^{-1}F_3F_1^{-1}F_2 = 1,\\ & F_3^{-1}F_1F_4^{-1}F_2F_3^{-1}F_1F_4F_3^{-1}F_2^{-1}F_4F_1^{-1}F_2F_1^{-1} = 1,\\ & F_1^{-1}F_2F_1^{-1}F_2^{-1}F_4F_1^{-1}F_2F_1^{-1}F_3^{-1}F_2^{-1}F_4^{-1}F_1^{-1}F_3 = 1,\\ & F_1F_4^{-1}F_2F_3^{-1}F_1F_4^{-1}F_1^{-1}F_3F_1^{-1}F_2F_1^{-1}F_3^{-1}F_1F_4^{-1} = 1\,\rangle\end{aligned}$$

Table 3.1: A presentation for N_1 as an abstract group

The generators for N_1 as a subgroup of Γ^+ can be described by writing F_i as follows: $F_1 = abacbdcd$, $F_2 = ababacbdcdba$, $F_3^{-1}F_1 = acbadcdb$ and $F_4^{-1}F_1^{-1}F_3 = bcbabdcdca$. Rewriting these gives $F_1 = \alpha^2\beta^2\gamma^2$, $F_2 = \alpha(\alpha\gamma)^2\alpha^{-1}$, $F_3 = \alpha\gamma^2\alpha^3\beta$ and $F_4 = \alpha^3\gamma^3\beta$. Similarly the epimorphism θ_2 sends $abcd$ to $(13524)(13425) = (143)$. Adjoining the relation $(abcd)^3 = 1$ to the presentation for Γ^+ gives a presentation for A_5, so N_2 is the normal closure of $(abcd)^3$ in Γ^+ ([C1, pp 23-33]). Using GAP a presentation for N_1 as an abstract group was found. This presentation is given in Table 3.2.

$$\begin{aligned}\langle F_1, F_2, F_3, F_4 \mid\ & F_1F_3^{-1}F_2^{-1}F_4^{-1}F_1F_4^{-1}F_1F_3^{-2}F_4F_2^{-1} = 1,\\ & F_2F_1F_3F_2^{-1}F_1F_3^{-1}F_4F_2F_3^{-1}F_4F_1^{-1}F_4 = 1,\\ & F_2F_4^{-1}F_3F_1^{-1}F_2F_3^{-1}F_1F_3^{-1}F_2^{-1}F_4^{-1}F_1 = 1,\\ & F_2F_3^{-1}F_4F_3^{-1}F_1^{-1}F_2F_3^{-1}F_4F_1^{-1}F_4F_2F_4 = 1\,\rangle\end{aligned}$$

Table 3.2: A presentation for N_2 as an abstract group

The generators for N_2 as a subgroup of Γ^+ can be described by writing F_i as follows: $F_1 = abadcdbc$, $F_3F_2^{-1}F_1F_3^{-1}F_4 = babcbcdcda$, $F_3F_2^{-1}F_1F_3^{-1}F_4F_2^{-1} = ababacbcdcdb$ and $F_1F_3^{-1} = ababadcdbcba$. Rewriting these gives $F_1 = (\alpha\gamma^{-1})^2$, $F_2 = (\alpha^{-1}\gamma^2)^2$, $F_3 = \alpha\beta^2\alpha\gamma^3$ and $F_4 = (\alpha\gamma)^2\gamma(\beta\alpha^{-1}\beta\alpha\beta)^2[\gamma, \beta]\alpha^{-1}$.

3.3 Some facts about the simple groups $L_2(q)$

In order to study normal subgroups of Γ or Γ^+ with quotients isomorphic to $L_2(q)$, some useful facts about these groups will now be presented. A more extensive treatment can be found in [D], Chapter XII.

The groups $L_2(q)$ are simple for all prime powers $q = p^n$, with the exception of $L_2(2) \cong S_3$ and $L_2(3) \cong A_4$. They are mutually non-isomorphic, with the exception of $L_2(4)$ and $L_2(5)$, both of which are isomorphic to A_5.

By Corollary 3.1.1, any nontrivial homomorphism $\Gamma^+ \to L_2(q)$ must send the non-identity torsion elements of Γ^+, as classified in Lemma 3.1.1, to elements of order $2, 3$ or 5, so it is useful to be able to identify the elements of these orders in $L_2(q)$. Regarding an element of $L_2(q) = SL_2(q)/\{\pm I\}$ as a pair $\pm A$ of matrices in $SL_2(q)$, then its trace is well-defined up to multiplication by -1. For each q the elements of order 2 form a single conjugacy class, consisting of the non-identity elements of trace 0. The elements of order 3 also form a single class, and these are the non-identity elements of trace ± 1. There are elements of order 5 in $L_2(q)$ if and only if $p = 5$ or $q \equiv \pm 1 \pmod 5$, in which case they form two conjugacy classes, each inverse-closed and consisting of the squares of the elements of the other class; these classes consist of the non-identity elements of traces $\pm(-1 \pm \sqrt{5})/2$, where $\sqrt{5}$ lies in a field $\mathbb{F}_p$ iff $p \cong \pm 1$ mod 5 or else $q = p^2$.

Since Γ^+ is perfect, the image of any nontrivial homomorphism $\Gamma^+ \to L_2(q)$ must be a perfect subgroup of $L_2(q)$. These are all isomorphic to A_5 (icosahedral subgroups) or to $L_2(q')$ where $F_{q'}$ is a subfield of F_q, so that $q' = p^m$ for some m dividing n. If $p = 2$ there is a single conjugacy class of icosahedral subgroups when n is even, and there are none when n is odd. If $p = 5$ there are two classes of such subgroups if n is even, or one class if n is odd. For odd $q \equiv \pm 1 \pmod 5$ there are two conjugacy classes, merging to form a single class in $PGL_2(q)$, whereas for odd $q \equiv \pm 2 \pmod 5$ there are none.

3.4 Quotients of Γ^+ isomorphic to $L_2(q)$, $p \notin \{2, 5\}$

Let $\overline{F}_p$ be the algebraic closure of the field F_p of order p, where p is prime. This is the union of the finite fields F_q for all powers q of p, with the natural inclusions, so the group $\overline{L} := L_2(\overline{F}_p)$ is the union of the corresponding groups $L_2(F_q) = L_2(q)$, with the induced inclusions. It follows that any epimorphism $\Gamma^+ \to L_2(q)$ can be regarded as a homomorphism $\Gamma^+ \to \overline{L}$, by composition with the natural embedding $L_2(q) \to \overline{L}$. Conversely, since Γ^+ is finitely generated, the image of any homomorphism $\Gamma^+ \to \overline{L}$ is contained in a subgroup $L_2(K)$ for some finite subfield K of $\overline{F}_p$; since Γ^+ is perfect, the image (if nontrivial) must be isomorphic to A_5 or to $L_2(q)$ for some power q of p. Thus we can find all quotients of Γ^+ isomorphic to $L_2(q)$ by considering all nontrivial homomorphisms $\Gamma^+ \to \overline{L}$, and excluding those with image isomorphic to A_5 if $q \neq 4, 5$. For technical reasons, it is assumed here that $p \neq 2$ or 5. These exceptional primes will be considered in § 3.5 of this chapter.

3.4.1 The restriction to Γ_0^+

If $\theta : \Gamma^+ \to \overline{L}$ is any nontrivial homomorphism, then its restriction ψ to $\Gamma_0^+ = \langle \alpha, \beta \rangle \cong A_5$ must be an isomorphism with a subgroup $G \cong A_5$ of $\overline{L}$. There is a single conjugacy class of such subgroups G in $\overline{L}$. This is because, being finite, any pair of such subgroups are both contained in a subgroup $L_2(K)$ for some finite subfield K of $\overline{F}_p$; now $L_2(K)$ has two conjugacy classes of such subgroups (as previously shown in § 3.3), and these are all conjugate in the subgroup $PGL_2(K) \leq L_2(\tilde{K}) \leq \overline{L}$, where $\tilde{K}$ is the quadratic extension of K in $\overline{F}_p$. Let us define two embeddings $\Gamma_0^+ \to \overline{L}$ to be equivalent if they differ by an inner automorphism of $\overline{L}$. Given any pair of embeddings ψ_i $(i = 1, 2)$, the conjugacy of their images implies that ψ_2 is equivalent to an embedding ψ_2' with the same image as ψ_1, so ψ_2' differs from ψ_1 by an automorphism of A_5. Since $|\mathrm{Out}\, A_5| = 2$ it follows that there are at most two equivalence classes of embeddings. Indeed, since the outer automorphisms of A_5 transpose its two conjugacy classes of elements of order 5, and these are distinguished by the traces $(-1 \pm \sqrt{5})/2$ of their images in $\overline{L}$, which are invariant under conjugation, it follows that there are exactly two equivalence classes of embeddings $\Gamma_0^+ \to \overline{L}$.

Representatives of these two classes can be constructed as follows. Define $F = \mathbb{F}_p$ or $\mathbb{F}_{p^2}$, depending on whether $p \equiv \pm 1$ or $\pm 2 \bmod (5)$, so F is the smallest subfield of $\overline{F}_p$ containing a square root of 5, or equivalently for which $L := L_2(F)$ contains elements of order 5. A simple counting argument shows that in any finite field, each element is a sum of two squares, so for each $t = t_i = (-1 \pm \sqrt{5})/2$ $(i = 1, 2)$ in F, elements $e, f \in F$ (depending on i) can be chosen such that $e^2 + f^2 + 3 = t^2$, or equivalently

$$e^2 + f^2 + t + 2 = 0 \tag{3.1}$$

since $t^2 = 1 - t$. For instance, if -3 is a square in F, then taking $e = \sqrt{-3}$ and $f = -t$ (or vice versa), Equation 3.1 is satisfied. The case $e = f = 0$ can not happen, since this gives $t = -2$ and so $4 = t^2 = 1 - t = 3$, which is impossible; thus, without loss of generality, it can be assumed that $e \neq 0$. Now define ψ_i : $\Gamma_0^+ \to L \leq \overline{L}$, for $i = 1, 2$, by taking $t = t_i$ and sending

$$\alpha \mapsto \frac{1}{2}\begin{pmatrix} t-f & e+1 \\ e-1 & t+f \end{pmatrix}, \ \beta \mapsto \frac{1}{2}\begin{pmatrix} e+1 & t+f \\ f-t & 1-e \end{pmatrix} \tag{3.2}$$

so $\alpha\beta \mapsto \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Since α, β and $\alpha\beta$ are sent to elements with traces $\pm t, \pm 1$ and 0 it follows that each ψ_i is a homomorphism and hence an embedding by the simplicity of Γ_0^+. Since the image of α under ψ_i has trace $\pm t_i$, the embeddings ψ_1 and ψ_2 are not equivalent, so every embedding is equivalent to precisely one of them.

3.4.2 Extension to Γ^+

A mapping

$$\gamma \mapsto \begin{pmatrix} w & x \\ y & z \end{pmatrix} \in \overline{L}$$

with $wz - xy = 1$ extends ψ_i to a homomorphism $\Gamma^+ \to \overline{L}$ if and only if it preserves the relations $\gamma^5 = (\beta\gamma)^2 = (\alpha\beta\gamma)^2 = 1$. Multiplying the above matrix by -1 if necessary, the first condition can be rewritten as $u = w + z = t_1$ or t_2. The second and third conditions are equivalent to $(e+1)w + (t+f)y + (f-t)x + (1-e)z = y - x = 0$, giving $y = x$ and $2ew + 2fx + (1-e)u = 0$, that is:

$$2ew = (e-1)u - 2fx \tag{3.3}$$

and so $2ez = 2eu - (e-1)u + 2fx = (e+1)u + 2fx$. The equation $wz - xy = 1$ then gives:

$$\begin{aligned} 4e^2 &= 2ew.2ez - 4e^2xy \\ &= \big((e-1)u - 2fx\big)\big((e+1)u + 2fx\big) - 4e^2x^2 \\ &= e^2u^2 - (u+2fx)^2 - 4e^2x^2, \end{aligned}$$

so

$$4(e^2+f^2)x^2 + 4fux + 4e^2 - (e^2-1)u^2 = 0 \qquad (3.4)$$

This is a quadratic equation for x, and extensions $\Gamma^+ \longrightarrow \overline{L}$ of ψ_i are in one-to-one correspondence with its solutions in $\overline{F}_p$. Thus, for each possible choice of the pair t and u from $\{t_1, t_2\}$, there are at most two such homomorphisms. We will denote such an extension of ψ_i by θ_{ij} if $t = t_i$ and $u = t_j$. From § 3.2 it is known that any isomorphism $\psi : \Gamma_0^+ \longrightarrow A_5$ has two extensions to epimorphisms $\theta : \Gamma^+ \longrightarrow A_5$, with the image of γ in each case conjugate in A_5 to that of α^2. It follows that if $i \neq j$ then the extensions $\theta_{ij} : \Gamma^+ \longrightarrow \overline{L}$ of ψ_i have images isomorphic to A_5, and hence not isomorphic to $L_2(q)$ for any power q of p, since $p \neq 2, 5$.

From now on it will be assumed that $i = j$. Thus $t = u$, so Equation 3.4 can be rewritten as:

$$(e^2+f^2)x^2 + ftx + e^2 - \frac{1}{4}(e^2-1)t^2 = 0 \qquad (3.5)$$

a quadratic equation with discriminant

$$\begin{aligned} D &= f^2t^2 + (e^2+f^2)((e^2-1)t^2 - 4e^2) \\ &= (e^2+t+2)(t-1) - (t+2)((e^2-1)(1-t) - 4e^2) \\ &= e^2(t-1) + (t+2)(t-1) + (e^2-1)(t+2)(t-1) + 4e^2(t+2) \\ &= e^2(t-1) + e^2(t+2)(t-1) + 4e^2(t+2) \\ &= e^2(5t+6). \end{aligned} \qquad (3.6)$$

Now $e \neq 0$, so D is a square in F if and only if $5t+6$ is. If D is a square, then provided $D \neq 0$ there are two solutions for x in F, giving two homomorphisms $\theta : \Gamma^+ \longrightarrow L < \overline{L}$. Since $t = u$ their images cannot be isomorphic to A_5, so these must be epimorphisms onto L. The exceptional case $D = 0$ corresponds to

$t = -6/5$; then $36/25 = t^2 = 1 - t = 1 + 6/5$, that is, $36 = 25 + 30 = 55$, so $p = 19$ and $t = -5$; in this case, there are two epimorphisms $\Gamma^+ \to L = L_2(19)$ for $t = u = 4$ (since $5t + 6 = 8^2$), and one for $t = u = -5$ corresponding to the unique solution x of Equation 3.5. For the moment, let us assume that $p \neq 19$, so that $D \neq 0$.

For $t = u = t_i$ $(i = 1, 2)$, with corresponding values $e = e_i$, the discriminants $D_i = e_i^2(5t_i + 6) = e_i^2(7 \pm 5\sqrt{5})/2$ satisfy $D_1 D_2 = -19(e_1 e_2)^2$. Suppose first that -19 is a square in F; equivalently, either $p \equiv \pm 2 \bmod (5)$, so $|F| = p^2$, or $p \equiv \pm 1 \bmod (5)$ and $p \equiv 1, 4, 5, 6, 7, 9, 11, 16$ or $17 \bmod (19)$. Then either both or neither of D_1, D_2 are squares, giving four epimorphisms onto L or none respectively. Suppose first that D_1 and D_2 are both squares, so that there are four epimorphisms; since $PGL_2(F)$ preserves traces, and only its identity element centralises an A_5 in L, it acts trivially on these four epimorphisms. It follows that when $p \equiv \pm 1 \bmod (5)$, so $|F| = p$ and hence $\mathrm{Aut}\, L = PGL_2(F)$, there are four normal subgroups in Γ^+ with quotient $L = L_2(p)$; when $p \equiv \pm 2 \bmod (5)$, however, so that $|F| = p^2$, the Galois group of the field F transposes t_1 and t_2. In this case $\mathrm{Aut}\, L = P\Gamma L_2(F)$ has two orbits on the four epimorphisms and hence there are just two normal subgroups with quotient $L = L_2(p^2)$. If neither D_1 nor D_2 is a square, so that there are no epimorphisms onto L, then, since D_1 and D_2 are both squares in the quadratic extension K of F of order $|F|^2$, four epimorphisms onto $L_2(K)$ are obtained. If $p \equiv \pm 1 \bmod (5)$, so that $|F| = p$, then the Galois group $\mathrm{Gal}\, K/F$ of this extension transposes the two epimorphisms corresponding to each discriminant, so there are two distinct normal subgroups with quotient $L_2(K) = L_2(p^2)$. If, however, $|F| = p^2$ then $\mathrm{Gal}\, F/\mathbb{F}_p$ transposes t_1 and t_2, while $\mathrm{Gal}\, K/F$ transposes the two epimorphisms corresponding to each t_i, so a unique normal subgroup with quotient $L_2(K) = L_2(p^4)$ is obtained.

Now suppose that -19 is not a square in F, that is, $p \equiv \pm 1 \bmod (5)$, so $|F| = p$, and $p \equiv 2, 3, 8, 10, 12, 13, 14, 15$ or $18 \bmod (19)$. In this case, exactly one D_i is a square in F, giving two epimorphisms onto $L = L_2(p)$ with distinct kernels; by adjoining the square root of the other discriminant two epimorphisms onto $L_2(K) = L_2(p^2)$ are obtained, and they are equivalent under $\mathrm{Gal}\, K/F$.

Similar arguments show that when $p = 19$ three normal subgroups with quotient $L_2(19)$ are obtained: two of these correspond to epimorphisms with $t = u = 4$ and

$D = 1$, and one to an epimorphism with $t = u = -5$ and $D = 0$. These three subgroups, and their associated manifolds, will be considered in more detail in § 5.11 of Chapter 5.

3.5 The cases $p = 2$ and $p = 5$

This section studies the exceptional primes $p \in \{2, 5\}$ as specified in section § 3.4.

3.5.1 $p = 5$

$\overline{L}$ has a single conjugacy class of icosahedral subgroups, since any two such subgroups of $L_2(q)$ are conjugate in $L_2(q^2)$. Take $L_2(5)$ as a representative of these subgroups. Now the normalizer of $L_2(5)$ in $\overline{L}$ is $PGL_2(5) \cong S_5 \cong \mathrm{Aut}(A_5)$, so it follows that in this case there is a single equivalence class of embeddings $\Gamma_0^+ \to \overline{L}$. Representatives for α and β are given by:

$$\alpha \mapsto \frac{1}{2}\begin{pmatrix} t-f & e+1 \\ e-1 & t+f \end{pmatrix}, \ \beta \mapsto \frac{1}{2}\begin{pmatrix} e+1 & t+f \\ f-t & 1-e \end{pmatrix} \tag{3.7}$$

with $t = 2$, $e = 1$ and $f = 0$. Then, arguing as in § 3.4.2 there are two choices for γ, namely $\gamma \mapsto \left(\begin{smallmatrix} 0 & 2 \\ 2 & \pm 2 \end{smallmatrix}\right)$. Hence there are two normal subgroups with quotient $L_2(5)$, namely N_1 and N_2, and none with quotient $L_2(5^n)$ for $n > 1$.

3.5.2 $p = 2$

If $p = 2$ then $L_2(2^n)$ has a subgroup $G \cong A_5$ if and only if n is even, in which case all such subgroups are conjugate to $L_2(2^2)$. Thus $\overline{L}$ has a single conjugacy class of such subgroups.

As discussed in § 3.4.1, there are two equivalence classes of embeddings $\Gamma_0^+ \to \overline{L}$, distinguished by the traces of the images of α. These are the elements $t_i \in F_4 \setminus F_2$, that is, the primitive cube roots of 1. Define $\psi_i : \Gamma_0^+ \to L \leq \overline{L}$, for $i = 1, 2$, by taking $t = t_i$ and sending

$$\alpha \mapsto \begin{pmatrix} t & t+1 \\ t & 0 \end{pmatrix}, \quad \beta \mapsto \begin{pmatrix} t+1 & 0 \\ t & t \end{pmatrix}, \quad \text{so} \quad \alpha\beta \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If

$$\gamma \mapsto \begin{pmatrix} w & x \\ y & u+w \end{pmatrix}$$

where $u = t_j$ for some $j = 1, 2$ then the relations $(\beta\gamma)^2 = (\alpha\beta\gamma)^2 = 1$ give $(t+1)w + tx + t(u+w) = y + x = 0$, so $w = t(x+u)$ and $y = x$. Thus $1 = w(u+w) - xy = t(x+u)(u+t(x+u)) + x^2 = (t^2+1)x^2 + tux + tu(u+tu) = tx^2 + tux + u^2$, so $tx^2 + tux + u = 0$, that is, $x^2 + ux + ut^2 = 0$. If $u = t$ this gives $x^2 + tx + 1 = 0$, which is irreducible over F_4 so it has two roots in F_{16} for each value of t; the Galois group $\mathrm{Gal}\, F_4/F_2$ transposes the two possible values t_1, t_2 of t, and for each t the Galois group $\mathrm{Gal}\, F_{16}/F_4$ transposes the two roots x, so all four roots x are conjugate under $\mathrm{Gal}(F_{16}/F_2)$ and hence one normal subgroup of Γ^+ with quotient $L_2(2^4)$ is obtained. If $u \neq t$ then $x^2 + ux + u^2 = 0$ with roots $x = 1$ and $x = u^2 = t$, yielding four epimorphisms $\Gamma^+ \longrightarrow L_2(2^2)$; these four choices of the pair t, x form two orbits under $\mathrm{Gal}\, F_4/F_2$, with $x = 1$ and $x = t$ respectively, so there are two normal subgroups of Γ^+ with quotient $L_2(2^2)$. Since $L_2(2^2)$ is isomorphic to both $L_2(5)$ and A_5, this is consistent with the earlier results on these two quotients.

3.6 The main result

The results obtained in § 3.4 and § 3.5 can be summarised in the following theorem:

Theorem 3.1 *Γ^+ has only the following normal subgroups with quotient isomorphic to $L_2(q)$ where q is a power of a prime p:*

a) for $p = 2$ there are two normal subgroups with quotient $L_2(4)$, and one with quotient $L_2(16)$;

b) for $p = 5$ there are two normal subgroups with quotient $L_2(5)$, namely those with quotient $L_2(4)$ appearing in (a)*;*

c) for $p = 19$ there are three normal subgroups with quotient $L_2(19)$;

d) for each $p \equiv \pm 1$ mod (5) with $p \equiv 2, 3, 8, 10, 12, 13, 14, 15$ or 18 mod (19) *there are two normal subgroups with quotient $L_2(p)$ and one with quotient $L_2(p^2)$;*

e) for each $p \equiv \pm 1$ mod (5) with $p \equiv 1, 4, 5, 6, 7, 9, 11, 16$ or 17 mod (19), there are either four normal subgroups with quotient $L_2(p)$ or two with quotient $L_2(p^2)$, as $(7 \pm 5\sqrt{5})/2$ are both squares or both non-squares in F_p;

f) for each odd $p \equiv \pm 2$ mod (5) there are either two normal subgroups with quotient $L_2(p^2)$ or one with quotient $L_2(p^4)$, as $(7 \pm 5\sqrt{5})/2$ are both squares or both non-squares in $F_p(\sqrt{5}) = F_{p^2}$.

3.7 Extending to Γ

Recall that we have

$$\alpha \mapsto \frac{1}{2}\begin{pmatrix} t-f & e+1 \\ e-1 & t+f \end{pmatrix} \quad ; \quad \beta \mapsto \frac{1}{2}\begin{pmatrix} e+1 & t+f \\ f-t & 1-e \end{pmatrix} \quad ; \quad \gamma \mapsto \begin{pmatrix} w & x \\ x & t-w \end{pmatrix}$$

and equations $e^2 + f^2 + 3 = t^2 = 1 - t$, $2ew = (e-1)t - 2fx$ (since $\mathrm{tr}(\alpha\beta\gamma) = 0$) and $4(e^2 + f^2)x^2 + 4fux + 4e^2 - (e^2 - 1)u^2 = 0$ (since $\det(\gamma) = 1$). Now consider extending a surjection $\phi_+ : \Gamma^+ \to L_2(q)$ to $\phi : \Gamma \to L_2(q)$. Let $g = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ be the image of a under ϕ. Note first that, since the subgroup $< a, b, d \mid a^2 = b^2 = d^2 = (ab)^5 = (ad)^2 = (bd)^2 = 1 >$ must map faithfully under such a ϕ, we require $20 \mid q(q^2 - 1)$. Since the order of g is 2, we have $x_4 = -x_1$, while $\phi(a\alpha\beta) = \phi(c)$ forces $x_2 = x_3$. Next,

$$\phi(a\alpha) = \phi(b) = \begin{pmatrix} x_1(t-f) + x_2(e-1) & * \\ * & x_2(e+1) - x_1(t+f) \end{pmatrix}$$

so $\mathrm{tr}(a\alpha) = 2ex_2 - 2fx_1 = 0$ and so $g = \begin{pmatrix} x_1 & fx_1/e \\ fx_1/e & -x_1 \end{pmatrix}$, while

$$\phi(a\alpha\beta\gamma) = \phi(d) = \begin{pmatrix} xx_1 - wfx_1/e & * \\ * & fx_1(t-w)/e + xx_1 \end{pmatrix}$$

so $x_1\left(2xe - 2wf + ft\right) = \mathrm{tr}(a\alpha\beta\gamma) = 0$. Since we cannot have $x_1 = 0$ (since then $g = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right)$, this forces $x = \frac{f(2w-t)}{2e}$. Finally, $\det(g) = 1$ yields the equation $-x_1^2e^2 - f^2x_1^2 = e^2$ so $x_1^2 = \frac{-e^2}{f^2+e^2}$. To summarise, we have four equations in four unknowns:

(I) $e^2 + f^2 + 3 = 1 - t$

(II) $2ew = (e-1)t - 2fx$

(III) $x = \frac{f(2w-t)}{2e}$

(IV) $4(e^2+f^2)x^2 + 4fux + 4e^2 - (e^2-1)u^2 = 0$

Now (II) and (III) gives $(e^2+f^2)(t-2w) - et = 0$. From (I) we get the equation $w = \frac{t+1+et}{4+2t}$. Hence we can write $x = \frac{ft}{2(2+t)}$ (IIIa). Next, (I), (III) and (IV) combine to give $\frac{f^2t^2}{2+t} + 4e^2 - (e^2-1)t^2 = 0$, so $(1-t)f^2 + (7+4t)e^2 + (4t-3) = 0$ (V). Then (V) and (I) gives $e^2 = \frac{4-4t}{6+5t}$, so $f^2 = \frac{-21-7t}{6+5t}$ and $x_1^2 = \frac{4t-4}{11t+17}$.

Theorem 3.2 *A map $\phi_+ : \Gamma^+ \longrightarrow L_2(q)$ can be extended to a map $\phi : \Gamma \longrightarrow L_2(q)$ with*

$$\alpha \mapsto \frac{1}{2}\begin{pmatrix} t-f & e+1 \\ e-1 & t+f \end{pmatrix} \quad ; \quad \beta \mapsto \frac{1}{2}\begin{pmatrix} e+1 & t+f \\ f-t & 1-e \end{pmatrix}$$

$$\gamma \mapsto \frac{1}{4+2t}\begin{pmatrix} (1+e)t+1 & ft \\ ft & (1-e)t+1 \end{pmatrix} \quad ; \quad a \mapsto \frac{1}{\sqrt{2+t}}\begin{pmatrix} e & f \\ f & -e \end{pmatrix}$$

provided there exists $f, e \in \mathbb{F}_p$ with $f^2 + e^2 + 3 = t^2$ and $(1-t)f^2 + (7+4t)e^2 + (4t-3) = 0$. If such a solution does not exist in $\mathbb{F}_p$, then one exists in $\mathbb{F}_{p^2}$ and our representation is given by

$$\phi(a) = g \quad ; \quad \phi(b) = g\alpha \quad ; \quad \phi(c) = g\alpha\beta \quad ; \quad \phi(d) = g\alpha\beta\gamma$$

Chapter 4

Further maps to linear groups

In chapter 3, all torsion free normal subgroups of the orientation preserving subgroup Γ^+ of the the simplicial Coxeter group $\Gamma = [5, 3, 5]$ (the Lannér group T_4) whose factor group is of the form $L_2(q)$ were classified. As a result, a family of hyperbolic manifolds with a "large" isometry group was constructed. This aim of this chapter is to extend this classification to the other 8 simplicial co-compact Coxeter groups acting on $\mathbb{H}^3$. For ease of exposition, the notation developed by Lannér [La] is used.

The complete list of compact simplicial hyperbolic Coxeter groups in dimension 3 was enumerated by Lannér [La] and is described in detail in § 2.2 of Chapter 2. The nine groups are listed here in Table 4.1.

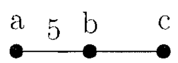
$$\begin{array}{lll} T_1[2,2,3;3,5,2], & T_4[2,2,5;2,3,5], & T_7[2,3,3;2,5,3], \\ T_2[2,2,3;2,5,3], & T_5[2,3,3;2,4,3], & T_8[2,4,3;2,5,3], \\ T_3[2,2,4;2,3,5], & T_6[2,3,4;2,3,4], & T_9[2,3,5;2,3,5]. \end{array}$$

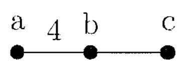
Table 4.1: The nine Lannér groups

Each group is described by a 6-tuple $[\lambda_1, \lambda_2, \lambda_3; \mu_1, \mu_2, \mu_3]$, of numbers, where π/λ_i are the angles between a chosen base face and the other three faces of T, and π/μ_i is the dihedral angle of the edge opposite that labeled with π/λ_i. Recall that these groups can be described as follows: T_1 is the $[5, [3]^2]$ Coxeter group whose Coxeter scheme has one vertex of valence three connected to the three vertices of valence one, one edge with label 5 and the other two edges unlabeled. T_2, T_3 and T_4 have linear Coxeter schemes. In this case the labeling for these schemes can be described as a list. The Coxeter scheme for T_2 has edge labels $3, 5, 3$. The Coxeter scheme for T_3 has edge labels $4, 3, 5$. The Coxeter scheme for T_5 has edge labels $5, 3, 5$. The remaining five Lannér diagrams correspond to Coxeter groups whose Coxeter scheme is a square: T_5 is the Coxeter scheme one of whose edges is labeled 4, T_6 is the Coxeter scheme with two opposite edges labeled 4, T_7 is the Coxeter scheme one of whose edges is labeled 5, T_8 has one edge labeled 4 and the opposite edge labeled 5 and T_9 has two opposite edges labeled 5.

4.1 Preliminaries

For each Lannér group T_i, the torsion free normal subgroups of T_i whose factor group has the form $L_2(q)$ will be determined in the same way as done for T_4 in chapter 3. All of the groups T_i contain one of the following two Coxeter schemes as a sub-diagram:

$[5, 3]$: 

$[4, 3]$ 

To classify all torsion-free normal subgroups N of $T_i = \Gamma^+$ that satisfy $\Gamma^+/N \cong L_2(q)$, representations of $[5, 3]^+$ and $[4, 3]^+$ in $L_2(q)$ are constructed. These representations are then extended to full epimorphisms $\Gamma^+ \twoheadrightarrow L_2(q)$ by determining the conditions under which a suitable image for $\gamma = cd$ can map to $g \in L_2(q)$ satisfying the relations in the presentation for Γ^+. In all of the Lannér groups, ensuring that all defining relators are mapped faithfully to $L_2(q)$ is sufficient to ensure that the kernel of the map is torsion free.

4.1.1 Coxeter schemes containing $[5, 3]$

First suppose that the group contains the scheme $[5, 3]$. Then write $\Gamma_0 = [5, 3] \cong A_5 \times C_2$, and $\Gamma_0^+ = [5, 3]^+ = \langle \alpha,\ \beta \mid \alpha^5 = \beta^3 = (\alpha\beta)^2 = 1 \rangle \cong A_5$. Working as in

§ 3.4.1, take $\psi_i : \Gamma_0^+ \mapsto A_5$ to be defined by

$$\alpha \mapsto \frac{1}{2}\begin{pmatrix} t-f & e+1 \\ -1+e & t+f \end{pmatrix} \text{ and } \beta \mapsto \frac{1}{2}\begin{pmatrix} e+1 & t+f \\ f-t & 1-e \end{pmatrix}$$

where $e^2 + f^2 + 3 = t^2$, $t = \frac{-1\pm\sqrt{5}}{2}$, $\alpha = ab$ and $\beta = bc$. Then

$$\alpha\beta \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Note that if $t_i \doteq \frac{-1+(-1)^i\sqrt{5}}{2}$, then $t_{1+i} = -1 - t_i$, where the subscripts are taken mod 2. Let

$$\gamma \mapsto \begin{pmatrix} w & x \\ y & z \end{pmatrix} \in \overline{L},$$

with determinant $wz - xy = 1$. Then ψ_i can be extended to a homomorphism $\Gamma^+ \to \overline{L}$ if and only if it preserves the relations $\gamma^{v_1} = (\beta\gamma)^{v_2} = (\alpha\beta\gamma)^{v_3} = 1$, where the $v_i \in \{2,3,4,5\}$. Recall that, from quadratic reciprocity, $\sqrt{5} \in F$ if $p \equiv \pm 1$ mod 5 or $|F| = p^2$.

4.1.2 Coxeter schemes containing $[4, 3]$

Suppose instead that Γ contains the special subgroup $\Gamma_0 = [4,3] \cong S_4 \times C_2$. Let $\overline{\mathbb{F}}_p$ be the algebraic closure of the field $\mathbb{F}_p$ of order p, where p is prime. This is the union of the finite fields $\mathbb{F}_q$ for all powers q of p, with the natural inclusions, so the group $\overline{L} := L_2(\overline{\mathbb{F}}_p)$ is the union of the corresponding groups $L_2(\mathbb{F}_q) = L_2(q)$, with the induced inclusions. It follows that any epimorphism $\Gamma^+ \longrightarrow L_2(q)$ can be regarded as a homomorphism $\Gamma^+ \longrightarrow \overline{L}$, by composition with the natural embedding $L_2(q) \longrightarrow \overline{L}$. Conversely, since Γ^+ is finitely generated, the image of any homomorphism $\Gamma^+ \longrightarrow \overline{L}$ is contained in a subgroup $L_2(K)$ for some finite subfield K of $\overline{F}_p$; from Dickson, if $p \equiv \pm 1$ mod 8, then the group S_4 is in $L_2(p)$. If $p \equiv \pm 3$ mod 8, then S_4 is in $PGL_2(p)$ and hence lies in $L_2(p^2)$.

If $\theta : \Gamma^+ \longrightarrow \overline{L}$ is any nontrivial homomorphism, then its restriction ψ to $\Gamma_0^+ = \langle \alpha, \beta \rangle \cong S_4$ must be an isomorphism with a subgroup $G \cong S_4$ of $\overline{L}$. There is a single conjugacy class of such subgroups G in $\overline{L}$. This is because, being finite, any pair of such subgroups are both contained in a subgroup $L_2(K)$ for some finite subfield

K of $\overline{F}_p$; now $L_2(K)$ has two conjugacy classes of such subgroups, and these are all conjugate in the subgroup $PGL_2(K) \leq L_2(\widetilde{K}) \leq \overline{L}$, where $\widetilde{K}$ is the quadratic extension of K in $\overline{F}_p$. Let us define two embeddings $\Gamma_0^+ \to \overline{L}$ to be equivalent if they differ by an inner automorphism of $\overline{L}$. Given any pair of embeddings ψ_i $(i = 1, 2)$, the conjugacy of their images implies that ψ_2 is equivalent to an embedding ψ_2' with the same image as ψ_1, so ψ_2' differs from ψ_1 by an automorphism of S_4. Since $|\mathrm{Out}\, S_4| = 1$ it follows that there is one equivalence classes of embeddings.

A representative for this class can be constructed as follows: Elements of $L_2(q)$ of order 4 form a single conjugacy class and have trace $\pm\sqrt{2}$. Let $\Gamma_0^+ = \langle \alpha,\, \beta \mid \alpha^4 = \beta^3 = (\alpha\beta)^2 = 1\rangle \cong S_4$, and then take

$$\alpha \mapsto \frac{1}{2}\begin{pmatrix} t-f & e+1 \\ -1+e & t+f \end{pmatrix} \text{ and } \beta \mapsto \frac{1}{2}\begin{pmatrix} e+1 & t+f \\ f-t & 1-e \end{pmatrix}$$

where $e^2 + f^2 = -1$, $t = \pm\sqrt{2}$, $\alpha = ab$ and $\beta = bc$. Then, again,

$$\alpha\beta \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

has order 2, as required. Letting

$$\gamma \mapsto \begin{pmatrix} w & x \\ y & z \end{pmatrix} \in \overline{L},$$

again with determinant $wz - xy = 1$, then the map ψ_i can be extended to a homomorphism $\Gamma^+ \to \overline{L}$ if and only if it preserves the relations $\gamma^{v_1} = (\beta\gamma)^{v_2} = (\alpha\beta\gamma)^{v_3} = 1$. Gauss' lemma for quadratic residues gives $\sqrt{2} \in F$ if and only if

$p \equiv \pm 1 \bmod 8$, or if $|F| = p^2$.

4.1.3 Extending $\Gamma^+ \twoheadrightarrow L_2(q)$ to $\Gamma \twoheadrightarrow L_2(q)$

Suppose now that there exists a normal subgroup N of Γ^+ with $\Gamma^+/N \cong L_2(q)$. A natural question is to ask under what conditions does the map Γ^+/N extend to a map $\Gamma/\widetilde{N}$ with $\Gamma/\widetilde{N} \cong L_2(q)$ and $N < \widetilde{N}$. Let

$$a \mapsto \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

Then $|a| = 2$ gives $x_4 = -x_1$ and $c = a\alpha\beta$ with $|c| = 2$ gives $x_2 = x_3$. Also

$$b = a\alpha \mapsto \frac{1}{2}\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\begin{pmatrix} t-f & e+1 \\ -1+e & t+f \end{pmatrix}$$

with trace $\frac{1}{2}(x_1(t-f) + x_2(e-1) + x_3(e+1) + x_4(t+f)) = 0$. $x_4 = -x_1$ and $x_2 = x_3$ gives $-x_1 f + x_2 e = 0$. Assuming that $e \neq 0$, then $x_2 = \frac{x_1 f}{e}$. Now $\det(a) = 1$ gives $-x_1^2(e^2+f^2) = e^2$. If the Coxeter scheme contains the subscheme $[5,3]$ then $e^2 + f^2 = -2-t$ and so $x_1^2 = \frac{e^2}{2+t}$. If the Coxeter scheme contains the subscheme $[4,3]$ then $e^2+f^2 = -1$ and so $x_1^2 = e^2$. Finally,

$$d = a\alpha\beta\gamma = \begin{pmatrix} x_1(y - \frac{fw}{e}) & x_1(z - \frac{fx}{e}) \\ x_1(w + \frac{fy}{e}) & x_1(x + \frac{fz}{e}) \end{pmatrix} \tag{4.1}$$

and this has trace $x_1(x + y + f\frac{z-w}{e}) = 0$. Assuming that $x_1 = 0$ give a as the zero matrix, a contradiction. Consequently the equation

$$ex + ey + fz - fw = 0 \tag{4.2}$$

must hold. Equation (4.2) can then be combined with the conditions derived for the entries in the matrices α, β and γ to solve for e and f explicitly. If such a solution exists, then the associated normal subgroup $\widetilde{N} < \Gamma$ satisfies the subgroup diagram given in Table 4.2

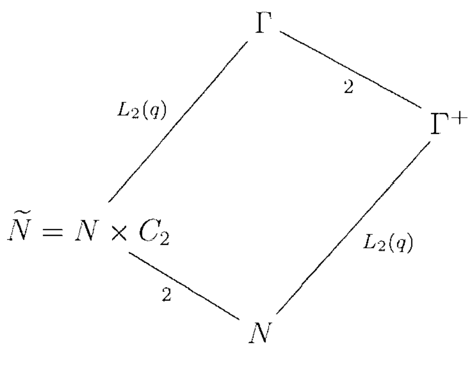


Table 4.2: Subgroup structure in Γ

4.1.4 A note on the order of elements in $L_2(2^n)$:

Let $L_2(2^n)^{(\infty)}$ be the subgroup of $L_2(2^n)$ fixing ∞ and let $Syl_2^{(\infty)}$ be the Sylow 2-subgroup of $L_2(2^n)^{(\infty)}$, so $Syl_2^{(\infty)} = \left\{\left(\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right)\right\}$ where $\mu \in \mathbb{F}_{2^n}$. Then from chapter

12 of Dickson [Di] the following three results can be obtained:

Lemma 4.1.1 *Every nonidentity element of $Syl_2^{(\infty)}$ has order 2. If $L_2(2^n)^{(k)}$ is the subgroup of $L_2(2^n)$ fixing any other element $k \in \mathbb{F}_{2^n} \bigcup \{\infty\}$, then its Sylow 2-subgroup $Syl_2^{(k)}$ is conjugate in $L_2(2^n)$ to $Syl_2^{(\infty)}$. These groups have trivial pairwise intersection and together contain all elements of $L_2(2^n)$ of order 2.*

Lemma 4.1.2 *$L_2(2^n)$ contains $2^{n-1}(2^n+1)$ cyclic groups of order $2^n - 1$. These groups are all conjugate in $L_2(2^n)$ and together contain $2^{n-1}(2^n+1)(2^n-2)$ non-trivial elements of order dividing $2^n - 1$.*

Lemma 4.1.3 *$L_2(2^n)$ contains $2^{n-1}(2^n-1)$ cyclic groups of order $2^n + 1$. These groups are all conjugate in $L_2(2^n)$ and together contain $2^{2n-1}(2^n-1)$ nontrivial elements of order dividing $2^n + 1$.*

This forms a complete list of all non-trivial elements of $L_2(2^n)$. As a result, the following results can be stated:

Theorem 4.1 *Let n be any positive integer. Then $L_2(2^n)$ contains no element of order 4. Therefore the groups $T_3[2,2,4;2,3,5]$, $T_5[2,3,3;2,4,3]$, $T_6[2,3,4;2,3,4]$ and $T_8[2,4,3;2,5,3]$ contain no torsion free normal subgroup whose factor group is isomorphic to $L_2(2^n)$.*

4.2 $\Gamma = T_1[2,2,3;3,5,2]$

4.2.1 Γ^+ to $L_2(q)$

Let $v_1 = 2$, $v_2 = 3$ and $v_3 = 2$, so $\gamma^2 = (\beta\gamma)^3 = (\alpha\beta\gamma)^2 = 1$. Then $|\gamma| = 2$ gives $w + z = 0$ (1), while $|\beta\gamma| = 3$ and $|\alpha\beta\gamma| = 2$ give $(e+1)w + (t+f)y + (f - t)x + (1-e)z = \pm 1$ (2) and $y - x = 0$ (3), respectively. Since $\det(\gamma) = 1$, then $w^2 + x^2 + 1 = 0$. substituting conditions (1) and (3) back into condition (2) gives $2ew = (u - 2fx)$, where $u = \pm 1$. Using this and the equation from the determinant gives the quadratic expression:

$$4(e^2 + f^2)x^2 - 4fux + (4e^2 + 1) \quad = \quad 0. \tag{4.3}$$

This has discriminant $D = 16f^2 - 16(e^2+f^2)(1+4e^2) = \ldots = (7+4t)(4e)^2$, which is a square in F when $(7+4t)$ is a square.

Write $D_i = (7+4t_i)(4e_i)^2$. Then $D_1D_2 = 5(16e_1e_2)^2$ which is a square in F if 5 is a square. 5 is a square mod p if $p \equiv \pm 1$ mod 5 or if $|F| = p^2$. Because $\mathrm{tr}(\alpha) = \frac{-1\pm\sqrt{5}}{2}$, then either both or neither D_1 and D_2 are squares in F. First suppose that they both are. Then if $p \equiv \pm 1$ mod 5, for each $(t_i,\ u_j)$, where $i,j \in \{1,2\}$, two epimorphisms onto $L_2(p)$ are obtained, giving a total of 8 epimorphisms. If $p \equiv \pm 2$ mod 5, then $|F| = p^2$, and the Galois automorphism $\phi\ \in$Gal$(F/\mathbb{F}_p)$ interchanges t_1 and t_2, giving 4 distinct non-conjugate epimorphisms.

Now suppose that neither D_i is a square in F. If $p \equiv \pm 1$ mod 5, then adjoining $\sqrt{D_1}$ to $\mathbb{F}_p$ 8 epimorphisms are obtained. Now the non-trivial Galois automorphism $\phi \in \mathrm{Gal}(\mathbb{F}_p(\sqrt{D_1})/\mathbb{F}_p)$ swaps $\sqrt{D_1}$ and $-\sqrt{D_1}$, so for any pair (t_i, u_j), where $i,j \in \{1,2\}$, ϕ swaps the two solutions for x. Hence 4 distinct epimorphisms $\Gamma \twoheadrightarrow L_2(p^2)$ are obtained. If $p \equiv \pm 2$ mod 5, then adjoining $\sqrt{D_1}$ to $\mathbb{F}_p^2$, 4 epimorphisms are obtained, and again the Galois automorphism swaps $\sqrt{D_1}$ and $-\sqrt{D_1}$, so 2 distinct epimorphisms $\Gamma \twoheadrightarrow L_2(p^4)$ are obtained.

Now the exceptional cases correspond to the discriminant $D = (7+4t)4e^2 = 0$. Because it was assumed that $e \neq 0$, then either $4 = 0$, so $p = 2$, or $(7+4t) = 0$, in which case $44 = 0$, so $p \in \{2, 11\}$. Now Γ is an index two subgroup of the $[5,3,4]$ group studied in § 4.4.1. Because $[5,3,4]$ has a unique conjugacy class of subgroups of index 2, Γ must be isomorphic to $[5,3,4]^+$ therefore by § 4.4.1 a torsion free epimorphism $\Gamma^+ \twoheadrightarrow L_2(2^n)$ cannot be constructed. There remains the case $p = 11$. In this case, for each set of values t_i, u_j, where $i,j \in \{1,2\}$, a unique map $\Gamma^+ \twoheadrightarrow L_2(11)$ is constructed.

Theorem 4.2 *Let $\Gamma = T_1$ and let $D_i = (4e_i)^2(7+4t_i)$ be the discriminant of equation 4.3. Then Γ^+ has torsion free normal subgroups N with $\Gamma^+/N \cong L_2(q)$ under the following conditions:*

1) If $p \equiv \pm 1$ mod 5 then either both discriminants D_i are squares in $\mathbb{F}_p$ or neither are. If they are both squares then Γ^+ has 8 distinct torsion-free normal subgroups N with factor group $\Gamma^+/N \cong L_2(p)$. If neither discriminant is a *square then Γ^+ has 4 distinct torsion-free normal subgroups N with factor group $\Gamma^+/N \cong L_2(p^2)$.*

2) *If $p \equiv \pm 2$ mod 5 then either both discriminants D_i are squares in $\mathbb{F}_{p^2}$ or neither are. If they are both squares then Γ^+ has 4 distinct torsion-free normal subgroups N with factor group $\Gamma^+/N \cong L_2(p^2)$. If neither discriminant is a square then Γ^+ has 2 distinct torsion-free normal subgroups N with factor group $\Gamma^+/N \cong L_2(p^4)$.*

4.2.2 Extending to $\Gamma \twoheadrightarrow L_2(q)$

Let $e^2 + f^2 = -2 - t$, where $t = \frac{-1\pm\sqrt{5}}{2}$and $e \neq 0$, $y = x$, $w = -z$ and $u = \pm 1$ be as in the previous section. Suppose that Γ^+ has a normal subgroup N with $\Gamma^+/N \cong L_2(q)$. From equation 4.2 in § 4.1.3 the condition

$$ex + ey + fz - fw = 0 \tag{4.4}$$

is obtained. From the results obtained in the previous section, the two equations

$$2ew = u - 2fx \tag{4.5}$$

and

$$4(e^2 + f^2)x^2 - 4fux + (4e^2 + 1) = 0 \tag{4.6}$$

are obtained. Since $y = x$ and $w = -z$, equation 4.4 becomes $x = \frac{fw}{e}$. Substituting this into equation 4.5 gives $w = \frac{eu}{-2(2+t)}$. Substituting in for x and w in equation 4.6 gives

$$f^2 + (1 + 4e^2)(2 + t) = 0$$

Since $e^2 + f^2 = -2 - t$, this can be rewritten to give

$$(7 + 4t)e^2 = 0$$

Since it is assumed that $e \neq 0$, $7 + 4t = 5 \pm 2\sqrt{5} = 0$. Solving give $5 = 0$, so the only possible case with $\Gamma/\widetilde{N} \cong L_2(q)$ is for $q = 5$.

Theorem 4.3 *There exists a normal subgroup $\widetilde{N} \lhd \Gamma$ with $\Gamma/\widetilde{N} \cong L_2(q)$ and $\Gamma^+/\Gamma^+ \cap \widetilde{N} \cong L_2(q)$ if and only if $p = 5$.*

4.3 $\Gamma = T_2[2,2,3;2,5,3]$

This group has been already extensively studied in the work of [JM]. The classification of all torsion-free normal subgroups N of Γ^+ whose factor group $\Gamma/N \cong L_2(q)$ has been done by Anna Torstensson in her PhD thesis [To], [CMT]. The statement is for completeness sake.

Theorem 4.4 *For any prime p there exists a positive integer k such that either $L_2(q)$ or $PGL_2(q)$ is a quotient of Γ^+ by some torsion-free normal subgroup.*

4.4 $\Gamma = T_3[2,2,4;2,3,5]$

4.4.1 Γ^+ to $L_2(q)$

Following the notation of section § 4.1.1, let $v_1 = 4$, $v_2 = 2$ and $v_3 = 2$, so $\gamma^4 = (\beta\gamma)^2 = (\alpha\beta\gamma)^2 = 1$. Then $|\gamma| = 4$ gives $u = \pm\sqrt{2} = w + z$. $|\beta\gamma| = 2$ and $|\alpha\beta\gamma| = 2$ give $(e+1)w + (t+f)y + (f-t)x + (1-e)z = y - x = 0$, respectively, giving $y = x$ and $2ew + 2fx + (1-e)u = 0$, that is, $2ew = (e-1)u - 2fx$ and so $2ez = 2eu - (e-1)u + 2fx = (e+1)u + 2fx$. The determinant $wz - xy = 1$ then gives

$$\begin{aligned}
4e^2 &= 2ew.2ez - 4e^2xy \\
&= \big((e-1)u - 2fx\big)\big((e+1)u + 2fx\big) - 4e^2x^2 \\
&= e^2u^2 - (u+2fx)^2 - 4e^2x^2,
\end{aligned}$$

so

$$2(e^2+f^2)x^2 + 2fux + (e^2+1) \;=\; 0. \tag{4.7}$$

Equation 4.7 is a quadratic equation for x, and extensions $\Gamma^+ \to L_2(F) < \overline{L}$ of ψ_i are in one-to-one correspondence with its solutions in $\overline{F}_p$, therefore for each possible choice of the pair t from $\{t_1, t_2\}$ and u from $\{\pm\sqrt{2}\}$ there are at most two such homomorphisms. Such an extension of ψ_i will be denoted by θ_{ij} if $t = t_i$ and $u = u_j$. Now equation (4.7) has discriminant $D = 4f^2u^2 - 8(e^2+f^2)(e^2+1) = 8f^2 - 8(e^2+f^2)(e^2+1) = 4e^2(2t+2)$. Since $e^2 \neq 0$, this is a square if and only if

$2t+2 = 1 \pm \sqrt{5}$ is a square. If D is a square, then for a choice u_i (provided $D \neq 0$) there are two solutions for x in F, giving two homomorphisms $\theta_i : \Gamma^+ \twoheadrightarrow L_2(F) < L_2(\overline{F})$, so a total of two epimorphisms is obtained. The exceptional case $D = 0$ corresponds to $p = 2$ which has already been excluded in § 4.1.4.

For $t = t_i$ the discriminants $D_i = 4e_i^2(2t_i + 2)$ satisfy $D_1 D_2 = -(8e_2e_2)^2$. If $-1 \in F$, so $p \equiv 1 \bmod 4$ or $|F| = p^2$, then either both or neither of D_1 and D_2 are squares in F. First suppose that they are both squares. Then there exists four epimorphisms $\Gamma^+ \twoheadrightarrow L$. Now $PGL_2(F)$ preserves traces and $B \in PGL_2(F)$ centralises $A_5 \cong \langle \alpha,\ \beta \mid \ \dots \ \rangle$ iff $B = Id$. Let $p \equiv 1 \bmod 8$. If $p \equiv \pm 1 \bmod 5$, then $|F| = p$ and Aut(L) =PGL$_2(F)$, so there exists 8 distinct epimorphisms. If, however, $p \equiv \pm 2 \bmod 5$ then there exists four epimorphisms $\phi : \Gamma^+ \twoheadrightarrow L_2(F')$, where F' is a degree two extension of F, forming four conjugacy classes of size two under the action of the Galois automorphism, giving two distinct automorphisms.

Now let $p \equiv -3 \bmod 8$. If $p \equiv \pm 1 \bmod 5$, then there are four epimorphisms $\phi : \Gamma^+ \twoheadrightarrow L_2(F)$ forming two conjugacy classes of size two under the action of the Galois automorphism, giving two distinct automorphisms. If $p \equiv \pm 2 \bmod 5$ then there exists $w \in \mathbb{F}_p$ such that $\sqrt{2} = w\sqrt{5}$. As a result the Galois automorphism simultaneously swaps the traces t_1, t_2 and the traces u_1, u_2, giving two distinct epimorphisms.

Now suppose that neither D_1 nor D_2 are squares in F. Then they are squares in the extension $F(\sqrt{D_i})$. There are several cases to consider:

a) If $p \equiv \pm 1 \bmod 5$ and $p \equiv 1 \bmod 8$, so $|F| = p$, then the Galois group of the extension transposes the two epimorphisms corresponding to each discriminant, so up to conjugacy there exists two epimorphisms onto $L_2(p^2)$.

b) If $p \equiv \pm 1 \bmod 5$ and $p \equiv \pm -3 \bmod 8$ then $F = \mathbb{F}(\sqrt{2})$ and the Galois group of the extension $K = F(D_1)$ transposes the two epimorphisms corresponding to each discriminant, while the Galois group of $F/\mathbb{F}$ transposes the traces u_1 and u_2, so up to conjugacy there exists a unique epimorphism onto $L_2(p^4)$.

c) If $p \equiv \pm 2 \bmod 5$ and $p \equiv \pm 1 \bmod 8$, then, arguing as in b) above, there exists, up to conjugacy, two distinct epimorphisms onto $L_2(p^4)$.

d) If $p \equiv \pm 2 \bmod 5$ and $p \equiv -3 \bmod 8$ write $F = \mathbb{F}_p(\sqrt{5})$. Then $\sqrt{2} = b'\sqrt{5}$ for some $b' \in \mathbb{F}_p$ and up to conjugacy there exists a unique epimorphism onto $L_2(p^4)$.

If $\sqrt{-1} \notin F$, then precisely one of D_1 or D_2 is a square in $L_2(F)$. Now, $\sqrt{-1} \notin F$ if and only if $F = \mathbb{F}_p$ and $p \equiv -1 \bmod 4$. Without loss of generality it can be assumed that D_1 is a square. As a result, $p \equiv \pm 1 \bmod 5$. Now $p \equiv -1 \bmod 4$ so either $p \equiv -1 \bmod 8$ or $p \equiv 3 \bmod 8$. If $p \equiv 3 \bmod 8$ then there are no epimorphisms onto $L_2(p)$, while if $p \equiv -1 \bmod 8$ then, up to conjugation by an automorphism of $L_2(q)$, there are 4 distinct epimorphisms. Adjoining the square root of the other discriminant gives four epimorphisms onto $L_2(K)$ (where $K = F(\sqrt{D_2})$), forming two conjugacy classes of epimorphisms. The exceptional case where the discriminant equals zero can only happen if $p = 2$, which is excluded in § 4.1.4.

The results obtained can be summarised in the following theorem:

Theorem 4.5 *Let $\Gamma^+ = T_3^+$ and let D_i be a discriminant for equation 4.7. Then*

1) If $p \equiv \pm 1$ mod 5, $p \equiv 1$ mod 8, then either Γ^+ has four distinct normal subgroups N with $\Gamma^+/N \cong L_2(p)$ or it has none. In the second case it has two distinct distinct normal subgroups N with $\Gamma^+/N \cong L_2(p^2)$

2) If $p \equiv 1$ mod 4 and

- *$p \equiv \pm 2$ mod 5 and $p \equiv 1$ mod 8, or*
- *$p \equiv \pm 1$ mod 5 and $p \equiv -3$ mod 8, or*
- *$p \equiv \pm 2$ mod 5 and $p \equiv -3$ mod 8*

then either Γ^+ admits two distinct distinct normal subgroups N with $\Gamma^+/N \cong L_2(p^2)$ or none. If there are no such normal subgroups N, then there is a unique normal subgroup N with $\Gamma^+/N \cong L_2(p^4)$.

3) If $p \equiv -1$ mod 4, $p \equiv \pm 1$ mod 5 and $p \equiv -1$ mod 8 then there are two distinct normal subgroups N with $\Gamma^+/N \cong L_2(p)$ and there is a uniequ normal subgroup N with $\Gamma^+/N \cong L_2(p^2)$

4.4.2 Extending to $\Gamma \twoheadrightarrow L_2(q)$

Let $e^2 + f^2 = -2 - t$, where $t = \frac{-1 \pm \sqrt{5}}{2}$and $e \neq 0$, $y = x$, $z = u - w$ and $u = \pm\sqrt{2}$ be as in the previous section. Suppose that Γ^+ has a normal subgroup N with $\Gamma^+/N \cong L_2(q)$. From equation 4.2 in § 4.1.3 the condition

$$ex + ey + fz - fw \;=\; 0 \tag{4.8}$$

is obtained. From the results obtained in the previous section, the two equations

$$2ew \;=\; eu - u - 2fx \tag{4.9}$$

and

$$2(e^2 + f^2)x^2 + 2fux + (e^2 + 1) \;=\; 0 \tag{4.10}$$

are obtained. Since $y = x$ and $z = u - w$, equation 4.8 becomes $x = \frac{2fw - fu}{2e}$. Substituting this into equation 4.9 gives $w = \frac{u(2+t)+eu}{2(2+t)}$. Substituting in for x and w in equation 4.10 gives

$$2f^4 + (2t + 4 + 2e^2)f^2 + 5 + 5e^2 + 3e^2t + 3t = 0$$

Since $e^2 + f^2 = -2 - t$, this can be rewritten to give

$$1 + e^2 = 0$$

so $e = \sqrt{-1}$ and hence either $p \equiv 1 \bmod 4$ or $q = p^{2n}$. Assume then that $p \equiv 1 \bmod 4$ or $q = p^{2n}$. Writing $e^2 = -1$ gives $f^2 = -1 - t$, so $f = \pm\sqrt{-1-t}$. If $t = \frac{-1 \pm \sqrt{5}}{2}$, then $-1 - t = \frac{-1 \mp \sqrt{5}}{2}$. Since $\sqrt{2}$ is already assumed to be in any field with $\Gamma^+ \twoheadrightarrow L_2(q)$, f is a square if and only if $\sqrt{-1 \pm \sqrt{5}}$ is.

Theorem 4.6 *Suppose there exists a normal subgroup $N \lhd \Gamma^+$ with factor group $\Gamma^+/N \cong L_2(q)$ and let t be the trace of the image of α in the quotient group. Then there exists an extension $\widetilde{N}$ of N with $\Gamma/\widetilde{N} \cong L_2(q)$ if and only if $q \cong 1 \bmod 4$ and $\sqrt{-1-t}$ lies in $\mathbb{F}_q$.*

4.5 $\Gamma = T_4[2, 2, 5; 2, 3, 5]$

Epimorphic images of this group have been extensively studied in Chapter 3.

4.6 $\Gamma = T_5[2,3,3;2,4,3]$

In this case $t = \pm\sqrt{2}$ and $e^2 + f^2 = -1$.

4.6.1 Maps from Γ^+ to $L_2(q)$

Following the notation of section § 4.1.2, let $v_1 = 3$, $v_2 = 2$ and $v_3 = 3$, so $\gamma^3 = (\beta\gamma)^2 = (\alpha\beta\gamma)^3 = 1$. Then $|\gamma| = 3$ gives $\mathrm{tr}(\gamma) = \pm 1 = u$, while $|\beta\gamma| = 2$ and $|\alpha\beta\gamma| = 3$ become the conditions $\mathrm{tr}(\beta\gamma) = 0$ and $\mathrm{tr}(\alpha\beta\gamma) = \pm 1 = k$, respectively. $\mathrm{tr}(\gamma) = u$ forces $z = u - w$ and $\mathrm{tr}(\alpha\beta\gamma) = k$ gives $y = x + k$. Section § 4.1.4 precludes the exceptional prime $p = 2$. Write

$$\gamma \mapsto \begin{pmatrix} w & x \\ x+k & u-w \end{pmatrix}$$

The condition $\mathrm{tr}(\beta\gamma) = 0$ gives $\frac{1}{2}(2ew+2fx+tk+fk+u-eu) = 0$ while $\det(\gamma) = 1$ gives $wu - w^2 - x^2 - xk = 1$. Rewriting $\det(\gamma) = 1$ as $4e^2\det(\gamma) = 4e^2$, and letting $2ew = eu - u - fk - tk - 2fx$, the following quadratic expression is obtained for x:

$$4x^2 + (-4fkt - 4fu + 4k)x + (-2ft + 2f^2 - 2fuk - 2utk) = 0 \quad (4.11)$$

Equation 4.11 has discriminant $16(1+f^2)(1+2utk) = -16e^2(1+2utk)$. Write $D_i = -16e_i^2(1+2t_i)$. Then $D_1D_2 = -7(16e_1e_2)^2$. Now this is a square if $\sqrt{7}$ lies in F, so either $|F| = p^2$ (equivalence of field extensions of $\mathbb{F}_p$), or, by quadratic reciprocity, $p \equiv 1, 2, 4$ or $4 \bmod 7$. Since $k, u \in \{\pm 1\}$, there are two cases to consider: $k = u$ and $k = -u$.

Case I: $k = u$

Equation 4.11 becomes $4x^2 + 4k(1 - f - tf)x + (-2ft + 2f^2 - 2f - 2t)$ with discriminant $D = -16e^2(1+2t)$. If $p \cong 1, 2$ or $4 \bmod 7$ or if $|F| = p^2$, then either both D_i are squares in F or neither are. Suppose first that they both are. Then it follows that the choice of t will not affect D_i. WLOG choose $t = \sqrt{2}$ and write $D = -16e^2(1+2\sqrt{2})$. Then for each value ± 1 of $u = k$ there are two solutions for x and these solutions are equivalent. This can be seen by writing $x = \frac{k}{2}((f + tf - 1) \pm \frac{e}{k}\sqrt{-1-2t}$ and $2ew = ek - k - fk - tk - 2fx$ becomes $w = \frac{k}{2}((1+e+et) \pm \frac{-f}{k}\sqrt{-1-2t})$. Then clearly the solution for γ does not depend on the value of k. So there are 4 distinct normal subgroups in total.

If neither D_i is a square in F, then they both are in $F(\sqrt{D})$. Now, if $p \cong \pm 1$ mod 8, then $F(\sqrt{D}) \equiv \mathbb{F}_{p^2}$. Let $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$ be the images of α, β and γ under the epimorphism constructed. The maximal subgroups of $L_2(p^2)$ containing $\langle \overline{\alpha}, \overline{\beta} \rangle \cong S_4$ are $L_2(p)$ or $PGL_2(p)$. Since Γ^+ contains no subgroup of index 2, and since the image of γ must lie in $L_2(p^2) \backslash L_2(p)$, it cannot be $PGL_2(p)$ and so, for each value $u = \pm 1$ and each D_i, a unique epimorphism $\Gamma^+ \twoheadrightarrow L_2(p^2)$ is constructed, giving a total of two distinct normal subgroups with factor group $L_2(p^2)$.

If $p \cong \pm 3$ mod 8, then S_4 is not a subgroup of $L_2(p)$, but $S_4 < PGL_2(p) < L_2(p^2)$. The subgroup $\langle \overline{\alpha}, \overline{\beta} \rangle$ of Γ^+ then embeds into $PGL_2(p) < L_2(p^2)$. From Dickson, [Di, §255], $PGL_2(p)$ is maximal in $L_2(p^2)$, so adjoining $\overline{\gamma}$ to $\langle \overline{\alpha}, \overline{\beta} \rangle$, generates the full group $L_2(p^4)$ provided $\gamma \notin \langle \overline{\alpha}, \overline{\beta} \rangle$. Since neither D_i is a square in $\mathbb{F}_{p^2}$ it follows that $\gamma \in L_2(p^4) \backslash L_2(p^2)$, so $\langle \overline{\alpha}, \overline{\beta}, \overline{\gamma} \rangle \cong L_2(p^4)$. The non-trivial Galois automorphism of $\mathbb{F}_{p^2}$ swaps $\sqrt{2}$ and $-\sqrt{2}$ while $\phi \in \mathrm{Gal}(\mathbb{F}_{p^4}/\mathbb{F}_{p^2})$ swaps D_1 and D_2. Hence, up to conjugacy, there is a unique epimorphism $\Gamma^+ \twoheadrightarrow L_2(p^4)$.

If $p \cong 3, 5$ or 6 mod 7, then precisely one of D_1, D_2 is a square in $\mathbb{F}_p$. By changing labels if necessary, assume that it is D_1. Then arguing as above, there are two distinct normal subgroups of Γ^+ whose factor group is $L_2(p)$. Adjoining $\sqrt{D_2}$ to F gives a map into $L_2(p^2)$, and again, by the arguments similar to those above, this map is onto.

Case II: $k = -u$

In this case Equation 4.11 becomes $4x^2 + 4k(1 + f - tf)x + (-2ft + 2f^2 + 2f + 2t)$, an equation with discriminant $D = -16e^2(1-2t)$. The analysis of the solutions of this equation is identical to that for the previous case $k = u$..

Finally, there remains to check when the solutions obtained for $k = u$ and for $k = -u$ coincide. If $p \cong \pm 3$ mod 8 then, under the action of $\mathrm{Gal}(F/\mathbb{F}_p)$, the solutions for $u = k$ and $u = -k$ coincide.

Theorem 4.7 *Let Γ be the Lannér group $T_5[2,3,3;2,4,3]$ and let Γ^+ be the orientation preserving subgroup.*

If $p \equiv \pm 1$ mod 8 and if $p \equiv 1$, 2 or 4 mod 7, then either both or neither of the discriminants D_i are squares in F. If they both are then, up to automorphisms, there are 4 distinct normal subgroups with factor group $L_2(p)$. If neither are then there are 2 distinct normal subgroups with factor group $L_2(p^2)$.

If $p \equiv \pm 3$ mod 8 then either both or neither of the discriminants D_i are squares in F. If they both are then there are 2 distinct normal subgroups with factor group $L_2(p^2)$. If neither are then there is a unique normal subgroup with factor group $L_2(p^4)$.

If $p \equiv \pm 1$ mod 8 and if $p \equiv 3$, 5 or 6 mod 17, then there are 2 distinct normal subgroups with factor group $L_2(p)$ and a unique normal subgroup with factor group $L_2(p^2)$.

4.6.2 The exceptional case $p = 7$

In the case $p = 7$, write $D = e^2(1 + 2utk)$. In $\mathbb{F}_7$, $\sqrt{2} = \pm 3$. Now if $t = 3$ then $D = 0$ mod 17 if $u = k$, while if $t = -3$ then $D = 0$ if $u = -k$. Choose $t = 3$. If $D \neq 0$ mod 7, then $u = -k$ and $D = -16e^2(1 + 2ukt) = e^2 \times 5$. Since 5 is not a square in $\mathbb{F}_{17}$, there are no epimorphisms $\Gamma \twoheadrightarrow L_2(7)$ for $D \neq 0$. However, there exists an epimorphism $\Gamma^+ \twoheadrightarrow L_2(49)$, corresponding to the quotient Γ^+/K, where K is the unique normal subgroup with this quotient. We now look at the two cases where $D = 0$ mod 7:
Equation 4.11 becomes $4x^2 = 0$, so $x = 0$. Since $-1 \equiv 6$ mod 7 is not a square in $\mathbb{F}_7$, choose $e = \sqrt{4}$ and $f = 2$. Then $e^2 + f^2 = -1$, while $2ew = eu - u - fk - tk - 2fx$ gives $2ew = -3u \equiv 4u$ mod 7, or $w = 2u/3 \equiv 3u$ mod 7. Then substituting in the values for e, f and t into α, β and γ gives

$$\alpha \mapsto \begin{pmatrix} 4 & 2 \\ 1 & 6 \end{pmatrix} \quad \beta \mapsto \begin{pmatrix} 2 & 3 \\ 6 & 6 \end{pmatrix} \quad \gamma \mapsto k \begin{pmatrix} 4 & 3 \\ 2 & 4 \end{pmatrix}$$

Since $k = \pm 1$ and the image $\overline{\gamma_1}$ of γ lies in $L_2(7)$ both maps are equivalent.

Theorem 4.8 *In the exceptional case where $p = 7$ and $D_i = 0$ a unique epimorphism $\Gamma^+ \twoheadrightarrow L_2(7)$ is recovered.*

4.6.3 Extending to $\Gamma \twoheadrightarrow L_2(q)$

Let $e^2 + f^2 = -1$, where $t = \sqrt{2}$and $e \neq 0$, $y = x + k$ where $k = \pm 1$ and $z = u - w$ where $u = \pm 1$ be as in the previous section. Suppose that Γ^+ has a normal subgroup N with $\Gamma^+/N \cong L_2(q)$. From equation 4.2 in § 4.1.3 the condition

$$ex + ey + fz - fw = 0 \tag{4.12}$$

is obtained. From the results obtained in the previous section, the two equations

$$2ew = eu - u - fk - tk - 2fx \tag{4.13}$$

and

$$4x^2 + (-4fkt - 4fu + 4k)x + (-2ft + 2f^2 - 2fuk - 2utk) = 0 \tag{4.14}$$

are obtained. Since $y = x + k$ and $z = u - w$, equation 4.12 becomes $x = \frac{2fw-fu-ek}{2e}$. Substituting this into equation 4.13 gives $w = \frac{u+eu+etk}{2}$. Substituting in for x and w in equation 4.14 gives

$$(1 + 2tuk)f^2 + 1 + tuk = 0$$

implying that $f^2 = -1$ and consequently $e^2 = 0$, a contradiction, or $1 + tuk = 0$. If $1 + tuk = 0$, then $1 = 4t^2u^2k^2 = 8$ and so $7 = 0$, so $p = 7$. In this case there is a unique normal subgroup K of Γ with factor group $L_2(7)$. Using GAP, it was shown that K is not torsion free - it contains the two conjugacy classes $\mathcal{C}((abc)^3)$ and $\mathcal{C}((bad)^3)$ of elements of order 2.

Theorem 4.9 *Suppose there exists a normal subgroup $N \lhd \Gamma^+$ with factor group $\Gamma^+/N \cong L_2(q)$. Then there can exist an extension $\widetilde{N}$ of N with $\Gamma/\widetilde{N} \cong L_2(q)$ if and only if $p = 7$. The subgroup $\widetilde{N}$ is not torsion free.*

4.7 $\Gamma = T_6[2,3,4;2,3,4]$

The orientation preserving subgroup of Γ has the following presentation:

$$\Gamma^+ = \langle \alpha, \beta, \gamma | \alpha^4 = \beta^3 = \gamma^4 = (\alpha\beta)^2 = (\beta(\gamma)^2 = (\alpha\beta\gamma)^3 = 1 \rangle$$

The subgroup Γ^+ is not perfect: If $N = \langle\langle \beta \rangle\rangle$ is the normal closure in Γ^+ of the subgroup $\langle \beta | \beta^3 = 1 \rangle$, then the factor group Γ^+/N is isomorphic to the dihedral group D_3. Consequently, Γ^+ has an index two subgroup.

4.7.1 Normal subgroups with quotient S_4

Lemma 4.7.1 *There are precisely two distinct normal subgroups of Γ^+ whose factor group is isomorphic to S_4.*

Proof: The proof amounts to constructing two non-conjugate epimorphisms $\Gamma^+ \twoheadrightarrow S_4$. Let $\alpha \mapsto \overline{\alpha} = (1,3,2,4)$ and $\beta \mapsto \overline{\beta} = (1,2,3)$. Then for $\overline{\gamma} \in \{(1,3,2,4), (1,3,4,2)\}$ we get $|\gamma| = 4$, $|\beta\gamma| = 2$ and $|\alpha\beta\gamma| = 3$. This proves existence of the two maps. If $\gamma \mapsto \overline{\gamma} = (1,3,2,4)$ then $\overline{\alpha\gamma} = \overline{\alpha}^2$ has order 2, while if $\gamma \mapsto \overline{\gamma} = (1,3,4,2)$ then $\overline{\alpha\gamma} = (1,4,3)$ has order 3. Hence the maps are not conjugate. Finally it is useful to note that $(1,3,4,2) = (1,3,2,4)^{(2,4)} = \overline{\beta}^{-1}\overline{\alpha}\overline{\beta}$. ■

4.7.2 $\Gamma^+ \twoheadrightarrow L_2(q)$

Write α and β as in section 4.1.2. Then $e^2 + f^2 = -1$ and $t = \pm\sqrt{2}$. Let

$$\gamma \mapsto \begin{pmatrix} w & x \\ y & z \end{pmatrix}.$$

Then $|\gamma| = 4$, $|\beta\gamma| = 2$ and $|\alpha\beta\gamma| = 3$ become the conditions $\mathrm{tr}(\gamma) = \pm\sqrt{2} = u$, $\mathrm{tr}(\beta\gamma) = 0$ and $\mathrm{tr}(\alpha\beta\gamma) = \pm 1 = k$, respectively. $\mathrm{tr}(\gamma) = u$ gives $z = u - w$ while $\mathrm{tr}(\alpha\beta\gamma) = k$ gives $y = x + k$. Again, by § 4.1.4, the case $p = 2$ is excluded. So

$$\gamma \mapsto \begin{pmatrix} w & x \\ x+k & u-w \end{pmatrix}.$$

Now $\mathrm{tr}(\beta\gamma) = 0$ gives $\frac{1}{2}(2ew + 2fx + tk + fk + u - eu) = 0$ while $\det(\gamma) = 1$ gives $wu - w^2 - x^2 - xk = 1$. Rewriting $\det(\gamma) = 1$ as $4e^2\det(\gamma) = 4e^2$, and letting $2ew = eu - u - fk - tk - 2fx$, then x satisfies the quadratic equation

$$4x^2 + (-4ftk - 4fu + 4k)x + (-2 + f^2 - 2ft - 2ufk - 2utk) = 0 \quad (4.15)$$

This quadratic has determinant $D = -16e^2(3 + 2utk)$, with $t, k \in \{\pm\sqrt{2}\}$. If $\mathrm{sign}(u) = \mathrm{sign}(t)$ and $k = -1$ or if $\mathrm{sign}(u) = -\mathrm{sign}(t)$ and $k = 1$ then $D = (4e)^2$. Let $\mathrm{sign}(u) = \mathrm{sign}(t)$ and $k = -1$. Then x satisfies the quadratic equation

$$4x^2 - 4x + 2 + f^2 = 0$$

and $D = (4e)^2$. Hence $x = \frac{4\pm 4e}{8} = \frac{1}{2}(1 \pm e)$. If $x = \frac{1}{2}(1+e)$ then $w = \frac{1}{2}(t - f)$ and γ can be written as

$$\overline{\gamma_1} = \frac{1}{2}\begin{pmatrix} t - f & 1 + e \\ e - 1 & t + f \end{pmatrix} = \overline{\alpha}$$

while if $x = \frac{1}{2}(1 - e)$, then $w = \frac{1}{2}(f + t)$ and γ has the form

$$\overline{\gamma_2} = \frac{1}{2}\begin{pmatrix} f + t & 1 - e \\ e - 1 & t - f \end{pmatrix} = \overline{\beta}^{-1}\overline{\alpha}\overline{\beta}$$

so we recover the two epimorphisms $\Gamma^+ \twoheadrightarrow S_4$ constructed in the previous section. If $\mathrm{sign}(u) = -\mathrm{sign}(t)$ and $k = 1$ then the same result is obtained.

Now suppose that $\mathrm{sign}(u) = \mathrm{sign}(t)$ and $k = 1$, so x satisfies the quadratic equation

$$4x^2 + (-8tf + 4)x - e^2 - 4tf - 7$$

This has discriminant $D = -7(4e)^2$. Write $D_i = -16e^2(3 + 2ukt_i)$. If $p \equiv 1, 2$ or $4 \bmod 7$, then either both or neither D_i are squares in $\mathbb{F}_p$. Since one discriminant $D = (4e)^2$ is always a square, it follows that they both are. Hence for $p \equiv \pm 1 \bmod 8$ and $p \equiv 1, 2$ or $4 \bmod 7$, there are two distinct normal subgroups whose factor group is $L_2(p)$. If, however, $p \equiv \pm 1 \bmod 8$ and $p \equiv 3, 5$ or $6 \bmod 7$, then precisely one discriminant, $D = (4e)^2$, is a square in $\mathbb{F}_p$. The second one, $D = -7(4e)^2$, is a square in the quadratic extension $\mathbb{F}_{p^2}$ of $\mathbb{F}_p$, and so the induced map sends Γ^+ onto a subgroup of $L_2(p^2)$. Since Γ^+ has a subgroup of index 2, it is possible that the image is $PGL_2(p)$.

Elements of $PGL_2(p)\backslash L_2(p)$ have non-square determinant. Since it has been assumed that $p \equiv 3, 5$ or $6 \bmod 7$, it follows from quadratic reciprocity that 7 is not a square in the field $\mathbb{F}_p$. To characterize maps $\Gamma^+ \twoheadrightarrow PGL_2(p)$, it is sufficient to consider the images $\overline{\gamma}$ of γ whose determinant is 7. The set of equations $\det(\overline{\gamma}) = 7$, $\mathrm{trace}(\alpha\beta\gamma) = \pm 1$, $\mathrm{trace}(\beta\gamma) = 0$ and $\mathrm{trace}(\overline{\gamma}) = \pm\sqrt{2}$ together yield the following quadratic equation for x:

$$4x^2 + (-4fu - 4ftk + 4k)x + (-25e^2 - 2ft - 3 - 2utk - 2fuk) = 0$$

This equation has discriminant $D = 16e^2(21 - 2utk)$. Since it is assumed that $\mathrm{sign}(u) = \mathrm{sign}(t)$ and that $k = 1$ (since otherwise epimorphisms onto S_4 are recovered), then $D = 17(4e)^2$. Then D is a square if and only if $p \equiv 1, 2, 4, 8, 9, 13, 15$ or 16 mod 17, and in this case two distinct normal subgroups with factor group $PGL_2(p)$ are recovered.

Hence, if $p \equiv \pm 1$ mod 8, $p \equiv 3, 5$ or 6 mod 7 and $p \equiv 1, 2, 4, 8, 9, 13, 15$ or 16 mod 17, then two distinct there are two distinct normal subgroups whose factor group is $PGL_2(p)$. Otherwise there is a unique normal subgroup with factor group $L_2(p^2)$.

Now suppose that $p \equiv \pm 3$ mod 8. Then $S_4 \not\subset L_2(p)$, but $S_4 \subset PGL_2(p) \subset L_2(p^2)$. Since $PGL_2(p)$ is maximal in $L_2(p^2)$ and S_4 is maximal in $PGL_2(p)$, there are two cases to consider: either Γ^+ has a normal subgroup whose factor group is $PGL_2(p)$, or it has a normal subgroup with factor group $L_2(p^2)$. By construction, the image of γ is a matrix with square determinant and so lies in $L_2(p)$ or in $L_2(p^2) \backslash PGL_2(p)$. If the image of γ lies in $L_2(p)$ then Γ^+ maps to $PGL_2(p)$, while if If the image of γ lies in $L_2(p^2) \backslash PGL_2(p)$, then Γ^+ maps to $L_2(p^2)$.

Theorem 4.10 *Let $\Gamma = T_6[2,3,4;2,3,4]$ and let D be the discriminant of Equation 4.15. Let t be the trace of α, u be the trace of γ and k be the trace of $\alpha\beta\gamma$. Then*

1) *Suppose that sign(t) = sign(u) and $k = -1$, or sign(t) = $-$sign(u) and $k = 1$. Then two distinct epimorphisms $\Gamma^+ \twoheadrightarrow S_4$ are recovered.*

2) *Suppose that sign(t) = sign(u) and $k = 1$, or sign(t) = $-$sign(u) and $k = -1$. If $p \equiv 1$, 2 or 4 mod 7 and $p \equiv \pm 1$ mod 8 then there are 4 distinct epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p)$. If $p \equiv \pm 3$ mod 8 or $p \equiv 3$, 5 or 6 mod 7 then there are 2 distinct epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p^2)$.*

4.7.3 Extending to $\Gamma \twoheadrightarrow L_2(q)$

Let $e^2+f^2=-1$, where $t=\sqrt{2}$ and $e \neq 0$, $y=x+k$ where $k=\pm 1$ and $z=u-w$ where $u=\pm\sqrt{2}$ be as in the previous section. Suppose that Γ^+ has a normal subgroup N with $\Gamma^+/N \cong L_2(q)$. From equation 4.2 in § 4.1.3 the condition

$$ex+ey+fz-fw = 0 \tag{4.16}$$

is obtained. From the results obtained in the previous section, the two equations

$$2ew = eu-u-fk-tk-2fx \tag{4.17}$$

and

$$4x^2+(-4ftk-4fu+4k)x+(-2+f^2-2ft-2ufk-2utk) = 0 \tag{4.18}$$

are obtained. Since $y=x+k$ and $z=u-w$, equation 4.16 becomes $x=\frac{2fw-fu-ek}{2e}$. Substituting this into equation 4.17 gives $w=\frac{u+eu+tek}{2}$. Substituting in for x and w in equation 4.18 gives

$$(3+2tuk)f^2+3+2tuk = 0 \tag{4.19}$$

implying that $f^2=-1$ and consequently $e^2=0$, a contradiction, or $3+2tuk=0$. If $3+2tuk=0$, then $9=4t^2u^2k^2=16$ and so $7=0$, so $p=7$. In this case there is a unique normal subgroup K of Γ with factor group $L_2(7)$. Using GAP, it was shown that K is not torsion free - it contains the four conjugacy classes $\mathcal{C}((dab)^3)$, $\mathcal{C}((abc)^3)$, $\mathcal{C}((bcd)^3)$ and $\mathcal{C}((cda)^3)$ of elements of order 2.

Theorem 4.11 *Suppose there exists a normal subgroup $N \lhd \Gamma^+$ with factor group $\Gamma^+/N \cong L_2(q)$. Then there can exist an extension $\widetilde{N}$ of N with $\Gamma/\widetilde{N} \cong L_2(q)$ if and only if $p=7$. The subgroup $\widetilde{N}$ is not torsion free.*

4.8 $\Gamma = T_7[2,3,3;2,5,3]$

4.8.1 $\Gamma^+ \twoheadrightarrow L_2(q)$

As defined in section 4.1.1, let $v_1=v_3=3$ and $v_2=2$, so $\gamma^3=(\beta\gamma)^2=(\alpha\beta\gamma)^3=1$. Then $|\gamma|=3$ gives $z=u-w$, $u=\pm 1$, while $|\alpha\beta\gamma|=3$ gives $y=x+k$, with

$k = \pm 1$. So

$$\gamma \mapsto \begin{pmatrix} w & x \\ x+k & u-w \end{pmatrix} \in \overline{L},$$

and $|\beta\gamma| = 2$ gives $2ew + 2fx + tk + fk + u - eu = 0$. $\det(\gamma) = 1$ gives $wu - w^2 - x^2 - xk = 1$. Writing this as $4e^2\det(\gamma) = 4e^2$, the following equality

$$(2ew)2u - (2ew)^2 - (2ex)^2 - (2ex)(2ek) = 4e^2 \tag{4.20}$$

can be obtained. Substituting in for $2ew$ gives

$$\begin{aligned} &-(8+4t)x^2 + (4ftk + 4fu - 8k - 4tk)x + \\ &(-4 - 4t - 2f^2 + 2ft + 2fuk + 2utk) = 0 \end{aligned} \tag{4.21}$$

a polynomial with discriminant $D = 16e^2(2 + 3t - 2ukt)$. Let $D_i = 16e_i^2(2 + 3t_i - 2kut_i)$, where $t_i = \frac{-1+(-1)^i\sqrt{5}}{2}$ as usual. Since $k, u \in \{\pm 1\}$, the two cases $k = u$ and $k = -u$ are considered separately.

Case I $k = u$:

Then $D_i = 16e_i^2(2 + t_i)$ and $D_1D_2 = (16e_1e_2)^2$ and either both or neither D_i are squares in F, giving 8 maps $\Gamma^+ \mapsto L_2(q)$ or none, respectively. If $q = p$ there are 8 distinct maps, while if $q = p^2$ then the 8 images form two conjugacy classes, so 4 distinct maps $\Gamma^+ \mapsto L_2(p^2)$ are recovered. If neither D_i is a root, then adjoining $\sqrt{D_1}$ to F gives maps $\Gamma^+ \mapsto L_2(F(\sqrt{D_1}))$. If $F = \mathbb{F}_p$, $K = \mathbb{F}_p(\sqrt{D_1})$ then the Galois automorphism $\phi \in \mathrm{Gal}(K/F)$ swaps the two solutions $a + b\sqrt{D_1}$, $a - b\sqrt{D_1}$ so 4 distinct maps $\Gamma^+ \mapsto L_2(F(\sqrt{D_1}))$ are constructed. If $F = \mathbb{F}_{p^2}$ so $K = F(\sqrt{D_1})$, 2 epimorphisms $\Gamma^+ \mapsto L_2(F(\sqrt{D_1}))$ are constructed.

Case II $k = -u$:

Then $D_i = 16e_i^2(2 + 5t_i)$ and $D_1D_2 = -31(16e_1e_2)^2$, so both D_1 and D_2 are squares in F if both 5 and -31 are squares in F. Now -31 is a square in F if either $|F| = p^2$ or if $p \equiv 1, 2, 4, 5, 7, 8, 9, 10, 14, 16, 18, 19, 20, 25$ or 28 mod 31, and 5 is a square in F if $p \equiv \pm 1$ mod 5 or $|F| = p^2$. First suppose that $p \equiv \pm 1$ mod 5 and $p \equiv 1, 2, 4, 5, 7, 8, 9, 10, 14, 16, 18, 19, 20, 25$ or 28 mod 31. Then there are either zero or 8 epimorphisms $\Gamma^+ \mapsto L_2(p)$, depending on whether or not D_i are squares or not in F. If they are not, then adjoining, say, $\sqrt{D_1}$ to F gives pairs of conjugate maps $\Gamma^+ \mapsto L_2(F(\sqrt{D_1}))$, giving four distinct epimorphisms $\Gamma^+ \mapsto L_2(p^2)$. If, however, $|F| = p^2$, then if the D_i are squares in F four epimorphisms $\Gamma^+ \mapsto L_2(p^2)$ are constructed. If the D_i are not squares, then adjoining, say, $\sqrt{D_1}$ to F gives maps $\Gamma^+ \mapsto L_2(F(\sqrt{D_1}))$, giving two epimorphisms $\Gamma^+ \mapsto L_2(p^2)$.

Now suppose that -31 is not a square in F, so $p \equiv \pm 1 \bmod 5$ and $p \equiv 3, 6, 11, 12, 13, 15, 17, 21, 22, 23, 24, 26, 27, 29, 30 \bmod 31$. Then precisely one of D_1, D_2 is a root in F (without loss of generality say D_1) and 4 distinct maps $\Gamma^+ \mapsto L_2(p)$ are recovered. Adjoining the square root $\sqrt{D_2} \notin F$ gives 4 epimorphisms $\Gamma^+ \mapsto L_2(p^2)$ forming two conjugacy classes of maps under the action of the Galois automorphism $\mathrm{Gal}(K/F)$, where $K = F(\sqrt{D_2})$

The following theorems have thus been proved:

Theorem 4.12 *Let $k = u$. Then either both or neither of the discriminants D_i are squares in F. If they both are, then if $p \equiv \pm 1$ mod 5 gives 8 distinct epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p)$, while if $p \equiv \pm 2$ mod 5 then 4 distinct epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p^2)$ are obtained. If neither D_i are squares in F, then they both are in $F(\sqrt{D_i})$. If $p \equiv \pm 1$ mod 5 then there are 4 distinct epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p^2)$, while if $p \equiv \pm 2$ mod 5 there are 2 distinct epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p^4)$.*

Theorem 4.13 *Let $k = -u$. Let $\Omega_1 = \{1, 2, 4, 5, 7, 8, 9, 10, 14, 16, 18, 19, 20, 25, 28\}$ be the squares in $\mathbb{F}_{31}$. Then the following cases hold:*

1) *If $p \equiv \pm 1$ mod 5 and $p \equiv i$ mod 31, where $i \in \Omega$, then either both discriminants or neither lie in $\mathbb{F}$. If they both do there are 8 epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p)$, while if they do not then there are 4 epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p^2)$.*

2) *If $p \equiv \pm 1$ mod 5 then either both discriminants or neither lie in $\mathbb{F}_{p^2}$. If they both do then there are 4 epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p)$, while if they do not then there are 2 epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p^2)$.*

3) *If $p \equiv \pm 1$ mod 5 and $p \equiv i$ mod 31, where $i \in \mathbb{F}_{31} \backslash \Omega$. Then precisely one discriminant is a square in $\mathbb{F}_p$ and there are 4 epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p)$. Adjoining the square root of the other discriminant gives 2 epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p^2)$.*

4.8.2 Case $p = 2$

As specified in § 3.5 α and β can be written as

$$\alpha \mapsto \begin{pmatrix} t & t+1 \\ t & 0 \end{pmatrix}, \quad \beta \mapsto \begin{pmatrix} t+1 & 0 \\ t & t \end{pmatrix}, \quad \text{so} \quad \alpha\beta \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

where $t \in \mathbb{F}_{p^2}/\mathbb{F}_p$ is a primitive cube root of 1. Writing

$$\gamma \mapsto \begin{pmatrix} w & x \\ y & z \end{pmatrix}$$

and noting that $|\gamma| = 3$ and $|\alpha\beta\gamma| = 3$ gives

$$\gamma \mapsto \begin{pmatrix} w & x \\ 1+x & 1+w \end{pmatrix}$$

while $|\beta\gamma| = 2$ gives $t+w+tx = 0$, so $w = tx+t$. Now $\det(\gamma) = (w+1)^2+(x+1)^2 = 1$ and so Equation 4.22 is recovered:

$$tx + t + t^2x^2 + t^2 + x + x^2 = 1 \tag{4.22}$$

Since $t^2 = 1 + t$ Equation 4.22 can be rewritten as $tx + tx^2 + x = 0$, so $x = 0$ or $tx = t + 1 = t^2$ so $x = t$.

$$x = 0 \implies \gamma \mapsto \begin{pmatrix} t & 0 \\ 1 & 1+t \end{pmatrix} \qquad x = t \implies \gamma \mapsto \begin{pmatrix} 1 & t \\ 1+t & 0 \end{pmatrix} \tag{4.23}$$

Consequently, there is no map $\Gamma^+ \twoheadrightarrow L_2(2^n)$ for $n > 2$.

Theorem 4.14 *There exists no epimorphism* $\Gamma^+ \twoheadrightarrow L_2(2^n)$ *for* $n > 2$. *For* $n = 2$, *there are two distinct epimorphisms.*

4.8.3 Extending to $\Gamma \twoheadrightarrow L_2(q)$

Let $e^2 + f^2 = -2 - t$, where $t = \frac{-1\pm\sqrt{5}}{2}$ and $e \neq 0$, $y = x + k$ where $k = \pm 1$ and $z = u - w$ where $u = \pm 1$ be as in the previous section, and $p > 2$. Suppose that Γ^+ has a normal subgroup N with $\Gamma^+/N \cong L_2(q)$. From equation 4.2 in § 4.1.3 the condition

$$ex + ey + fz - fw \;=\; 0 \tag{4.24}$$

is obtained. From the results obtained in the previous section, the two equations

$$2ew = eu - u - fk - tk - 2fx \tag{4.25}$$

and

$$\begin{aligned} &-(8+4t)x^2 + (4ftk + 4fu - 8k - 4tk)x + \\ &(-4 - 4t - 2f^2 + 2ft + 2fuk + 2utk) = 0 \end{aligned} \tag{4.26}$$

are obtained. Since $y = x+k$ and $z = u-w$, equation 4.24 becomes $x = \frac{2fw-fu-ek}{2e}$. Substituting this into equation 4.25 gives $w = \frac{(2+t)u+eu+tek}{2(2+t)}$. Substituting in for x and w in equation 4.26 gives

$$(-2-3t+2utk)f^2 - 7 - 5t + 2uk + 2utk = 0 \tag{4.27}$$

so $f = \frac{7+5t-2uk-2utk}{-2-3t+2utk}$. It now remains to consider the two cases, $\text{sign}(u) = \text{sign}(k)$ and $-\text{sign}(u) = \text{sign}(k)$.

Sign(u) =Sign(k): Write $u = k$. Then $f^2 = -\frac{5+3t}{2+t}$ and hence $e^2 = -2-t-f^2 = 0$, contradicting the assumption that $e \neq 0$, unless $f^2 = 0$. Now $f^2 = 0$ if and only if $5+3t = 5+3\frac{-1\pm\sqrt{5}}{2} = 0$. Solving $5+3t = 0$ gives $4 = 0$. So for all primes $p > 2$, if $\text{sign}(u) = \text{sign}(k)$ then there is no normal subgroup $\widetilde{N}$ of Γ with factor group $L_2(q)$.

Sign(u) = −Sign(k): Write $u = -k$. Then $f^2 = -\frac{9+7t}{2+5t}$ and hence $e^2 = 0$, contradicting the assumption that $e \neq 0$, unless $f^2 = 0$. Now $f^2 = 0$ if and only if $9+7t = 0$. Solving this gives $124 = 0$ so $p = 2$ (a contradiction) or $p = 31$. If $p = 31$ there is a unique normal subgroup K of Γ whose factor group $\Gamma/K \cong L_2(31)$. Since Γ contains conjugacy classes $C(abc)$, $C((abc)^3)$, $C(dab)$ and $C((dab)^3)$ of elements of order 10 and since $L_2(31)$ has no elements of order 10, K cannot be torsion free.

Theorem 4.15 *Suppose there exists a normal subgroup $N \lhd \Gamma^+$ with factor group $\Gamma^+/N \cong L_2(q)$. Then there exists an extension $\widetilde{N}$ of N with $\Gamma/\widetilde{N} \cong L_2(q)$ if and only if $p = 31$. In this case there is a unique normal subgroup $\widetilde{N}$ in Γ with $\Gamma/\widetilde{N} \cong L_2(31)$ and $\widetilde{N}$ is not torsion free.*

4.9 $\Gamma = T_8[2,4,3;2,5,3]$

4.9.1 $\Gamma^+ \twoheadrightarrow L_2(q)$

Write α and β as they are defined in section § 4.1. Then $e^2 + f^2 + 2 + t = 0$ and $t = \frac{-1\pm\sqrt{5}}{2}$. Let

$$\gamma \mapsto \begin{pmatrix} w & x \\ y & z \end{pmatrix}$$

Then $|\gamma| = 4$, $|\beta\gamma| = 2$ and $|\alpha\beta\gamma| = 3$ become the conditions $\mathrm{tr}(\gamma) = \pm\sqrt{2} = u$, $\mathrm{tr}(\beta\gamma) = 0$ and $\mathrm{tr}(\alpha\beta\gamma) = \pm 1 = k$, respectively. $\mathrm{tr}(\gamma) = u$ forces $z = u - w$ and $\mathrm{tr}(\alpha\beta\gamma) = k$ gives $y = x + k$. Again we cannot have $p = 2$. So

$$\gamma \mapsto \begin{pmatrix} w & x \\ x+k & u-w \end{pmatrix}$$

Now $\mathrm{tr}(\beta\gamma) = 0$ gives $\frac{1}{2}(2ew + 2fx + tk + fk + u - eu) = 0$ while $\det(\gamma) = 1$ gives $wu - w^2 - x^2 - xk = 1$. Rewriting $\det(\gamma) = 1$ as $4e^2\det(\gamma) = 4e^2$, and letting $2ew = eu - u - fk - tk - 2fx$ gives the following quadratic expression for x:

$$\begin{aligned} p(x)_{t_i,u_j,k_s} &= (8+4t)x^2 + (-4fu - 4tfk + 4tk + 8k)x \\ &+(1 + 3t + f^2 - 2tku - 2tf - 2fku) = 0. \end{aligned} \tag{4.28}$$

This has discriminant $D = 16[(1 - 2t + 2ukt)f^2 + (-t + 2ku + 2ktu)]$. Write $e^2 = -2 - t - f^2$ in D. Simplifying the resulting expression gives

$$D = 16e^2(2t - 1 - 2ktu).$$

Let $D_j = 16e^2[(2t-1) - 2ktu_j]$. Then $D_1D_2 = -3(16e^2)^2$ and this is a square in F if and only if -3 is is a square in F. (note that this number still depends on t, since t is expressed in terms of e and f). By Gauss' lemma for quadratic residues, $\sqrt{-3} \in \mathbb{F}_p$ if and only if $p \equiv 1 \bmod 6$.

First suppose that both D_1 and D_2 are squares in F. Then either $p \equiv 1 \bmod 6$ or $|F| = p^2$.

- If $p \equiv \pm 1 \bmod 5$, $p \equiv \pm 1 \bmod 8$ and $p \equiv 1 \bmod 6$, then there are 16 distinct epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p)$.

- If $p \equiv \pm 1 \bmod 5$ and $p \equiv \pm 3 \bmod 8$, then there are 8 distinct epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p^2)$.
- If $p \equiv \pm 2 \bmod 5$ and $p \equiv \pm 1 \bmod 8$, then there are 8 distinct epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p^2)$.
- If $p \equiv \pm 2 \bmod 5$ and $p \equiv \pm 3 \bmod 8$, then there are 8 distinct epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p^2)$.

Now suppose that neither D_1 nor D_2 are squares in F. Then they both are in $F(D_1)$ and the nontrivial Galois automorphism of $F(D_i)/F$ swaps $\sqrt{D_j}$ with $-\sqrt{D_j}$, so the Galois automorphism swaps the two solutions to the polynomial $p(x)_{t_i,u_j,k_s}$, for each i, s in $\{0,1\}$. Then one of the following cases holds:

- If $p \equiv \pm 1 \bmod 5$, $p \equiv \pm 1 \bmod 8$ and $p \equiv 1 \bmod 6$, then there are 8 distinct epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p^2)$
- If $p \equiv 1 \bmod 6$, and suppose one of the following cases holds:

  $p \equiv \pm 1 \bmod 5$ and $p \equiv \pm 3 \bmod 8$, or

  $p \equiv \pm 2 \bmod 5$ and $p \equiv \pm 1 \bmod 8$, or

  $p \equiv \pm 2 \bmod 5$ and $p \equiv \pm 3 \bmod 8$

Then there are 4 distinct epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p^2)$

Finally, suppose that precisely one of D_1, D_2 is a square in F. Then $p \equiv \pm 1 \bmod 5$, $p \equiv \pm 1 \bmod 8$ and $p \not\equiv 1 \bmod 6$. Without loss of generality suppose that D_1 is a square in $F = \mathbb{F}_p$. Then there are precisely 8 epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p)$. Adjoining $\sqrt{D_2}$ to $\mathbb{F}$ to get $F = \mathbb{F}(\sqrt{D_2})$ gives 4 distinct epimorphisms, one for each pair t_i, k_s, for $i, s \in \{0,1\}$.

Theorem 4.16 *Let $p(x)_{i,j,s}$ be as above and let $D_j = 16e^2(2t - 1 - 2ktu_j)$.*

1) Suppose that $|F| = p$. Then for $p \equiv \pm 1 \bmod 5$, $p \equiv \pm 1 \bmod 8$ and $p \equiv 1 \bmod 6$, either both or neither D_j are squares in F. If they both are then for each triple (i,j,s) two solutions are obtained for Equation 4.28 giving a total of 16 epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p)$. If neither D_j is a square in F, then they both are in $F(\sqrt{D_1})$ and there are 8 distinct epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p^2)$.

2) Suppose that $|F| = p^2$. Then either both or neither D_j are squares in F. If they both are then for each triple (i,j,s) there is one solution for Equation 4.28 giving a total of 8 epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p^2)$. If neither D_j is a square in F, then they both are in $F(\sqrt{D_1})$ and there are 4 distinct epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p^4)$.

3) Finally, suppose $p \equiv \pm 1 \bmod 5$, $p \equiv \pm 1 \bmod 8$ and $p \not\equiv 1 \bmod 6$. Then precisely one of D_1, D_2 is a square in F. In this case there are 8 epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p)$. Adjoining $\sqrt{D}$ to F, where D is the non-square discriminant gives 4 epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p^2)$.

4.9.2 Extending to $\Gamma \twoheadrightarrow L_2(q)$

Let $e^2 + f^2 = -2 - t$, where $t = \frac{-1\pm\sqrt{5}}{2}$ and $e \neq 0$, $y = x + k$ where $k = \pm 1$ and $z = u - w$ where $u = \pm\sqrt{2}$ be as in the previous section. Suppose that Γ^+ has a normal subgroup N with $\Gamma^+/N \cong L_2(q)$. From equation 4.2 in § 4.1.3 the condition

$$ex + ey + fz - fw = 0 \tag{4.29}$$

is obtained. From the results obtained in the previous section, the two equations

$$2ew = eu - u - fk - tk - 2fx \tag{4.30}$$

and

$$\begin{aligned}(8+4t)x^2 + (-4fu - 4tfk + 4tk + 8k)x \\ +(1 + 3t + f^2 - 2tku - 2tf - 2fku) = 0\end{aligned} \tag{4.31}$$

are obtained. Since $y = x+k$ and $z = u-w$, equation 4.29 becomes $x = \frac{2fw-fu-ek}{2e}$. Substituting this into equation 4.30 gives $w = \frac{(2+t)u+eu+tek}{2(2+t)}$. Substituting in for x and w in equation 4.31 gives

$$(-1 + 2t - 2tuk)f^2 - 2uk - 2tuk + t = 0 \tag{4.32}$$

so $f^2 = \frac{2uk+2tuk-t}{-1+2t-2tuk}$. Then $e^2 = -2 - t - f^2 = 0$, contradicting the assumption that $e \neq 0$, unless $f^2 = 0$. Now $f^2 = 0$ if and only if $2uk + 2tuk - t = 0$. Write

$t = 2uk(1+t)$. Then $t^2 = 4u^2k^2(1+t)^2$ so $1-t = 16+8t$, giving $17 = -9t$. Solving this gives $44 = 0$, so $p = 2$ or $p = 11$. By the previous section $p \neq 2$ and since $11 \equiv 3 \bmod 8$, $L_2(11)$ contains no element of order 4. Hence there is no torsion free normal subgroup N of Γ^+ whose factor group is $L_2(11)$.

Theorem 4.17 *Suppose there exists a normal subgroup $N \lhd \Gamma^+$ with factor group $\Gamma^+/N \cong L_2(q)$. Then there exist no extensions $\widetilde{N}$ of N with $\Gamma/\widetilde{N} \cong L_2(q)$.*

4.10 $\Gamma = T_9[2,3,5;2,3,5]$

4.10.1 $\Gamma^+ \twoheadrightarrow L_2(q)$

Let α, β and γ be as defined in § 4.1.1. Then $|\gamma| = 5$ and $|\alpha\beta\gamma| = 3$ give $y = x+k$ and $z = u - w$, where $k = \pm 1$ and $u = \frac{-1\pm\sqrt{5}}{2}$. $|\beta\gamma| = 2$ gives $2ew + 2fx + tk + fk + u - eu = 0$. From $\det(\gamma) = 1$, $wu - w^2 - x^2 - xk = 1$. Multiplying this by $4e^2$ gives

$$(2ew)(2eu) - (2ew)^2 - 4e^2x^2 - 4e^2xk - 4e^2 = 0 \tag{4.33}$$

Substituting for $2ew = eu - u - fk - tk - 2fx$ in Equation 4.33 and simplifying gives the quadratic equation

$$\begin{gathered}(8+4t)x^2 + (-4ftk - 4fu + 8k + 4tk)x + \\ 4 + 3u + 4t + tu + 2f^2 + uf^2 - 2fuk - 2tuk - 2ft = 0\end{gathered} \tag{4.34}$$

which has the rather unwieldy discriminant $D = 16e^2(2 + 3t + tu - 2tuk + 3u)$. There are two cases: either both α and γ have the same trace, in which case $u = t$, or they have different traces, in which case $u = -1 - t$.

Case I $u = t$:

Then Equation 4.34 becomes

$$\begin{gathered}(8+4t)x^2 + (-4ftk - 4ft + 8k + 4tk)x + \\ -2ft + f^2t + 5 + 6t + 2f^2 - 2kft - 2k + 2kt = 0\end{gathered} \tag{4.35}$$

and this has discriminant

$$D = 16e^2\left((5+2k)t+3-2k\right)$$

Writing $D_i = 16e_i^2\left((5+2k)t_i+3-2k\right)$ gives $D_1D_2 = (16e_1e_2)^2(-27-28k)$, so

$$D_1D_2 = \begin{cases} -55(16e_1e_2)^2, \text{ if } k=1 \\ (16e_1e_2)^2, \text{ if } k=-1 \end{cases}$$

Notice that, if $k=-1$, then Equation 4.36 is recovered

$$(4t+8)x^2-(4t+8)x+(7+4t+2f^2+f^2t)=0 \tag{4.36}$$

This can be written as

$$(4t+8)x^2-(4t+8)x+\frac{1}{4}(4t+8)(3+t+f^2)=0 \tag{4.37}$$

Substituting in for e^2 in Equation 4.37 and dividing by $(4t+8)$ gives

$$x^2-x+\frac{1}{4}(1-e^2)=0 \tag{4.38}$$

a quadratic with discriminant $D=e^2$. Hence $x=\frac{1\pm e}{4}$ and so, for each value of $t\in\{t_i\}_{i=1}^2$ there are two solutions, since $e\neq 0$. As a result, if $k=-1$ there are 4 epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p)$, if $p\equiv\pm1 \bmod 5$, or 2 epimorphisms $\Gamma^+\twoheadrightarrow L_2(p^2)$, if $p\equiv\pm2 \bmod 5$. If $k=1$ then for -55 a square in F, either both or neither D_1 and D_2 are squares in F. If they both are and if $p\equiv\pm1 \bmod 5$ and $p\equiv 1,3,4,5,9 \bmod 11$, then 4 distinct epimorphisms $\Gamma^+\twoheadrightarrow L_2(p)$ are constructed. If $p\equiv\pm2 \bmod 5$ or $p\equiv 2,6,7,8,10 \bmod 11$, then there are 2 distinct epimorphisms $\Gamma^+\twoheadrightarrow L_2(p^2)$. If neither discriminant is a square in F, then if $p\equiv\pm1 \bmod 5$ and $p\equiv 1,3,4,5,9 \bmod 11$, then there are 2 distinct epimorphisms $\Gamma^+\twoheadrightarrow L_2(p^2)$ while if $p\equiv\pm2 \bmod 5$ or $p\equiv 2,6,7,8,10 \bmod 11$ then there is a unique epimorphism $\Gamma^+\twoheadrightarrow L_2(p^4)$. If $p\equiv\pm1 \bmod 5$ and $p\equiv 2,6,7,8,10 \bmod 11$, then precisely one of the discriminants is a square in F and the other is not. In this case $p\equiv\pm1 \bmod 5$, $p\equiv 2,6,7,8,10 \bmod 11$ and there are 2 distinct epimorphisms $\Gamma^+\twoheadrightarrow L_2(p)$. Adjoining the square root of the non-square discriminant to F gives a unique epimorphism $\Gamma^+\twoheadrightarrow L_2(p^2)$. These results are summarised as follows:

Theorem 4.18 *Let $tr(\alpha) = tr(\gamma)$.*

1) *Suppose that $tr(\alpha\beta\gamma) = -1$. Then if $p\equiv\pm1 \bmod 5$ there are 4 distinct epimorphisms $\Gamma^+\twoheadrightarrow L_2(p)$. If $p\equiv\pm2 \bmod 5$ then there are 2 distinct epimorphisms $\Gamma^+\twoheadrightarrow L_2(p^2)$.*

2) Suppose that $tr(\alpha\beta\gamma) = 1$. Then if -55 is a square in $\mathbb{F}_p$ either either both or neither D_1, D_2 are squares in F. If they both are squares in F, then

If $p \equiv \pm 1$ mod 5 and $p \equiv 1,3,4,5,9$ mod 11 then there are 4 distinct epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p)$.

If $p \equiv \pm 2$ mod 5 or $p \equiv 2,6,7,8,10$ mod 11, then there are 2 distinct epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p^2)$.

If neither are squares in F, then

If $p \equiv \pm 1$ mod 5 and $p \equiv 1,3,4,5,9$ mod 11 then there are 2 distinct epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p^2)$.

If $p \equiv \pm 2$ mod 5 or $p \equiv 1,3,4,5,9$ mod 11 then there is a unique epimorphism $\Gamma^+ \twoheadrightarrow L_2(p^4)$

3) If $p \equiv \pm 1$ mod 5 and $p \equiv 2,6,7,8,10$ mod 11 then precisely one of D_1, D_2 is a square in F. In this case there are 2 distinct epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p)$. Adjoining the square root of the non-square discriminant to F, gives a unique epimorphism $\Gamma^+ \twoheadrightarrow L_2(p^2)$.

Case II $u = -1 - t$:

Then 4.34 becomes

$$\begin{aligned}(8+4t)x^2 + (-4kft + 4ft + 4f + 8k + 4tk)x + \\ 2k - 2ft + 2fk + t + f^2 - f^2t + 2kft = 0\end{aligned} \tag{4.39}$$

and this has discriminant $D = 32e^2(k-1)$ and so

$$D = \begin{cases} 0, \text{ if } k = 1 \\ -(8e)^2, \text{ if } k = -1 \end{cases}$$

So if $k = 1$ there is a unique epimorphism. Now, letting $k = -1$, if $p \equiv \pm 1$ mod 5 then there are 2 distinct epimorphisms onto $L_2(p)$ if $p \equiv 1$ mod 4. Otherwise there is a unique epimorphism $\Gamma^+ \twoheadrightarrow L_2(p^2)$.

Theorem 4.19 *Suppose that $tr(\alpha) \neq tr(\gamma)$. Then*

1) Suppose that $tr(\alpha\beta\gamma) = 1$. Then if $p \equiv \pm 1$ mod 5 there are two epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p)$. If $p \equiv \pm 2$ mod 5 there is a unique epimorphism $\Gamma^+ \twoheadrightarrow L_2(p^2)$.

2) Suppose that $tr(\alpha\beta\gamma) = -1$. If $p \equiv \pm 1$ mod 5 and $p \equiv 1$ mod 4 then there are 2 epimorphisms $\Gamma^+ \twoheadrightarrow L_2(p)$. Otherwise there is a unique epimorphism onto $L_2(p^2)$.

4.10.2 Exceptional cases

In the previous section, the discriminant $D = 16e^2((5+2k)t + 3 - 2k)$ if $u_i = t_i$, while $D = 32e^2(k-1)$ if $u_j = t_i$. In the case $u = t$ $D = 0$ if $(5+2k)t + 3 - 2k = 0$. If $k = 1$ then solving this gives $220 \equiv 0$ mod p, or $p \in \{2, 5, 11\}$. If $k = -1$ then $D = 0$ if and only if $p = 2$. If $u_j = t_i$ then $D = 0$ if and only if $k = 1$, giving one unique solution for each value of t_i.

4.10.3 Extending to $\Gamma \twoheadrightarrow L_2(q)$

Let $e^2 + f^2 = -2 - t$, where $t = \frac{-1\pm\sqrt{5}}{2}$ and $e \neq 0$, $y = x + k$ where $k = \pm 1$ and $z = u - w$ where $u = \frac{-1\pm\sqrt{5}}{2}$ be as in the previous section. Suppose that Γ^+ has a normal subgroup N with $\Gamma^+/N \cong L_2(q)$. From equation 4.2 in § 4.1.3 the condition

$$ex + ey + fz - fw = 0 \tag{4.40}$$

is obtained. From the results obtained in the previous section, the two equations

$$2ew = eu - u - fk - tk - 2fx \tag{4.41}$$

and

$$\begin{gathered}(8+4t)x^2 + (-4ftk - 4fu + 8k + 4tk)x + \\ 4 + 3u + 4t + tu + 2f^2 + uf^2 - 2fuk - 2tuk - 2ft = 0\end{gathered} \tag{4.42}$$

are obtained. Since $y = x+k$ and $z = u-w$, equation 4.40 becomes $x = \frac{2fw-fu-ek}{2e}$. Substituting this into equation 4.41 gives $w = \frac{(2+t)u+eu+tek}{2(2+t)}$. Substituting in for x and w in equation 4.42 gives

$$(2t - 2tuk + u)f^2 - 2uk - 2tuk + t = 0 \tag{4.43}$$

so $f^2 = \frac{2uk+2tuk-t}{2t-2tuk+u}$ and $e^2 = -2 - t - f^2 = \frac{t+2u+2+ut}{2t-2tuk+u}$.

Case I: $u_i = t_i,\ i \in \{1,2\}$

Then $f^2 = \frac{2k-t}{3t+2kt-2k}$ and $e^2 = \frac{-(2t+3)}{3t+2kt-2k}$. If $k = 1$ then $e^2 = \frac{2t+3}{2-5t}$ and $f^2 = \frac{1+2t}{2t-3}$ while if $k = -1$ then $e^2 = -1 - t$ and $f^2 = -1$.

Case II: $u_j = t_i,\ i \neq j$

Then $f^2 = \frac{4k+t+2kt}{t+2k-1}$ and $e^2 = \frac{1+t}{t+2k-1}$. If $k = 1$ then $e^2 = 1$ and $f^2 = -3 - t$ while if $k = -1$ then $e^2 = \frac{1+t}{t-3}$ and $f^2 = \frac{t+4}{t-3}$.

Theorem 4.20 *Suppose there exists a normal subgroup $N \lhd \Gamma^+$ with factor group $\Gamma^+/N \cong L_2(q)$. Then there exists an extension $\widetilde{N}$ of N with $\Gamma/\widetilde{N} \cong L_2(q)$ if and only if the following conditions hold:*

	$u = t$		$u = -1 - t$	
	$k = 1$	$k = -1$	$k = 1$	$k = -1$
e^2	$\frac{2t+3}{2-5t}$	$-1 - t$	1	$\frac{t+1}{t-3}$
f^2	$\frac{1+2t}{2t-3}$	-1	$-t - 3$	$\frac{t+4}{t-3}$

4.11 Conclusion

In this chapter, all torsion free normal subgroups N of the orientation preserving subgroups Γ^+ of the Lannér groups Γ, whose factor group is a simple group $L_2(q)$, are classified. These subgroups are constructed by classifying the equivalence classes of epimorphisms $\Gamma^+ \to L_2(q)$. For each such subgroup $N \lhd \Gamma^+$, necessary and sufficient conditions under which the result extends to torsion free normal subgroups $\widetilde{N}$ of Γ are investigated.

Chapter 5

Manifolds from the $[5, 3, 5]$ group

5.1 Introduction

In this chapter, various manifolds arising from the action of torsion-free subgroups of the $[5, 3, 5]$ Coxeter group will be studied. Several invariants (homology, fundamental group, combinatorial structure) of the manifolds will be constructed to distinguish between them.

5.2 The quotient manifolds M_i.

Recall that in Chapter 3 § 3.1 a tetrahedron T in $\mathbb{H}^3$ with vertices A, B, C, D, and with dihedral angles $\pi/5$, $\pi/3$, $\pi/5$, $\pi/2$, $\pi/2$ and $\pi/2$ along its edges CD, AD, AB, BD, BC, AC was introduced. The natural action of Γ on this tetrahedron equipped $\mathbb{H}^3$ with a tessellation $\mathcal{T}$ by tetrahedra. Recall further the subgroup $\Gamma_0 = \langle a, b, c\rangle$ of Γ. This group is the isometry group of a dodecahedron, isomorphic to $A_5 \times C_2$. The 120 images of T under Γ_0 form a hyperbolic dodecahedron $\mathbb{D}$, with dihedral angles $2\pi/5$, and the images of $\mathbb{D}$ under Γ form a dodecahedral tessellation $\mathcal{D}$ of $\mathbb{H}^3$.

A flag $\phi = (v, e, f, c)$ of $\mathcal{D}$, and its associated tetrahedron T in $\mathcal{T}$, is *right-* or *left-handed* if moving away from the vertex v along the edge e while rotating around e from the face f into the interior of the cell c represents a right- or left-handed screw motion, that is, a clockwise or anticlockwise rotation when viewed from v. Identify a right-handed flag with $1 \in \Gamma$. Then the right- and left-handed flags ϕ_g and tetrahedra T_g correspond to the even and odd elements g of Γ, forming the two cosets Γ^+ and $\Gamma \setminus \Gamma^+$ of Γ^+ in Γ.

In Lemma 3.2.1 in Chapter 3 it was shown that there are two distinct normal subgroups N_1 and N_2 of Γ^+ whose factor group is A_5.

Theorem 5.1 *The quotient manifolds $M_i = \mathbf{H}^3/N_i$ $(i = 1, 2)$ are a chiral pair of Weber-Seifert spaces, formed by identifying opposite faces of a hyperbolic dodecahedron after rotating them through $3\pi/5$ in the positive and negative senses respectively.*

Proof: Since Γ_0^+ is mapped isomorphically onto A_5 by each θ_i, we have $\Gamma^+ = N_i\Gamma_0^+ = \Gamma_0^+N_i$ with $\Gamma_0^+ \cap N_i = 1$, so $\Gamma = N_i\Gamma_0 = \Gamma_0 N_i$ with $\Gamma_0 \cap N_i = 1$. Since the dodecahedron D_1 consists of the 120 tetrahedra in $\mathcal{T}$ forming an orbit of Γ_0, this shows that D_1 is a fundamental region for the action of each N_i on $\mathbb{H}^3$. Each M_i can be formed from D_1 by identifying pairs of boundary points which are equivalent under N_i. Elements of N_i can be regarded as automorphisms, not monodromy permutations. As a result flags ϕ_g and ϕ_h in $\mathcal{D}$ are identified if and only if $N_i g = N_i h$, a condition which implies that g and h are both even or both odd. If g and h are even then they normalise N_i, so $N_i g = gN_i$ and $N_i h = hN_i$, which means that we may equivalently use the condition $gN_i = hN_i$ (which identifies the right-handed flags ϕ_g and ϕ_h under the monodromy action). Using this action has the advantage that if g is a word $g_1 \dots g_n$ in the generators $g_i \in \{a, b, c, d\}$ of Γ, then successive images $\phi, \phi g_1, \dots, \phi g_1 \dots g_n = \phi g$ of a flag ϕ correspond to adjacent tetrahedra in $\mathcal{T}$; this makes it easier to determine the action of g as a monodromy permutation than as an automorphism, where successive images could be increasingly far apart. However, if g and h are odd, then $N_i g = gN_i^g = gN_{i'}$, where $i' = 3 - i$, and similarly for $N_i h$. Therefore, in constructing M_i the monodromy action of this conjugate subgroup $N_{i'}$, rather than N_i, needs to be used to identify left-handed flags ϕ_g and ϕ_h. Fortunately, in order to determine the identifications of faces which result in M_i, it is sufficient to restrict attention to right-handed flags ϕ_g and ϕ_h, which are identified if and only if g and h have the same image under θ_i.

The element $(abc)^5$ of Γ represents the antipodal involution in Γ_0, so as a monodromy permutation it sends the flag ϕ_1 of D_1 to its antipodal flag $-\phi_1$. It follows that the element $g = (abc)^5 d = (\alpha\beta^{-1}\alpha^{-1}\beta)^2\alpha\gamma$ of Γ^+ sends ϕ_1 to the flag $\phi_g = (-\phi_1)d$ in the adjacent dodecahedron, sharing the same vertex, edge and face as $-\phi_1$. Under θ_1 and θ_2 this element g has the same image in A_5 as $h = \alpha^{-1}$ and α respectively, so the flags ϕ_g and ϕ_h are identified in M_1 and M_2. Now α^{-1} and α rotate right-handed flags through $2\pi/5$ around their faces in the positive and negative directions (when viewing D_1 from outside). Hence, in M_1 and M_2, the flag ϕ_g is obtained from ϕ_h by applying a screw motion with a rotation in the positive sense through $\pi - 2\pi/5 = 3\pi/5$ or through $\pi + 2\pi/5 = 7\pi/5 \equiv -3\pi/5 \bmod (2\pi)$ respectively. Figures 5.1 shows a dodecahedron with one pair of antipodal faces identified by a $+\frac{3\pi}{5}$ identification. Since Γ_0^+ acts transitively (by automorphisms) on the right-handed flags of D_1, the same identification applies to every such flag ϕ_h. Consequently, each face of D_1 is identified with its opposite face after a rotation through $3\pi/5$ or $-3\pi/5$. This is the rule for constructing the two Weber-Seifert spaces.

The subgroups N_1 and N_2 are conjugate in Ω, so the manifolds M_1 and M_2 are isometric. However, the only conjugating elements are orientation-reversing elements of $\Omega \setminus \Omega^+$, hence each M_i is the mirror-image of the other. ∎

Note that in the above proof, the monodromy permutations α^{-1} and α rotate left-handed flags in the opposite direction to right-handed flags; however, the earlier remarks imply that in constructing each M_i the roles of θ_1 and θ_2 must be transposed when considering left-handed flags, so all flags are rotated through the same angle.

Each manifold M_i has isometry group $\operatorname{Iso} M_i \cong N_\Omega(N_i)/N_i = \Omega^+/N_i \cong S_5$, consisting entirely of orientation-preserving elements. The subgroup $\Gamma^+/N_i \cong A_5$ of index 2 preserves the tessellation $\mathcal{D}/N_i$, acting as the rotation group of the single dodecahedral cell, while r sends $\mathcal{D}/N_i$ to its dual tessellation $\mathcal{D}^*/N_i \cong \mathcal{D}/N_i$. It can be verified, either by hand or by using GAP [GAP], that N_1 and N_2 are the normal closures in Γ^+ of the elements $(abcd)^t = (\alpha\gamma)^t$ for $t = 2$ and 3. Hence $\mathcal{D}/N_1$ and $\mathcal{D}/N_2$ are examples $\{5,3,5\}_2$ and $\{5,3,5\}_3$ of Coxeter's twisted honeycombs

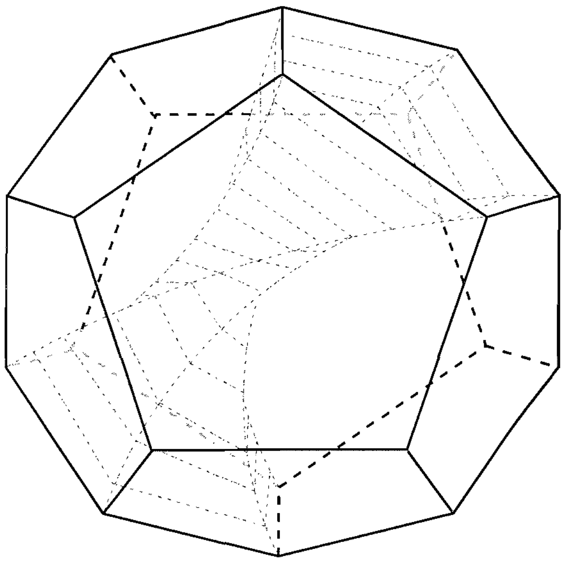


Figure 5.1: Identification of two faces of a dodecahedron

$\{p, q, r\}_t$ ([C1], pp. 32–33 and Fig. 23).

The tessellation $\mathcal{D}/N_1$ consists of one vertex, six edges e_i (all loops), six pentagonal faces, and one dodecahedral cell. Figure 5.2 illustrates the edges in D_1. Taking the single vertex as base-point, the six loops corresponding to the e_i can be treated as generators of the fundamental group $\pi_1(M_1) \cong N_1$. The abelianisation N_1/N_1', where N_1' is the commutator subgroup of N_1, is the first integer homology group $H_1(M_1)$. Let f_1 be the face of D_1 incident with ϕ_1, and let the edges of f_1, cyclically ordered and directed in the positive orientation, be $e_1, \ldots, e_5$. The sixth edge e_6 appears in the 'equatorial' Petrie polygon of D_1, disjoint from f_1 and its antipodal face $-f_1$: after identifications of faces, this polygon becomes the dotted path $e_1, e_6, e_2, e_6, e_3, e_6, e_4, e_6, e_5, e_6$, as shown in Figure 5.2. (The Petrie polygons of a dodecahedron have length 10, and each edge is identified with four others in the right-left Petrie polygon containing it).

The face f_1 gives rise to the homology relation $e_1 + \cdots + e_5 = 0$. Its neighbour across e_1 is the face adjacent to f_1 containing the edge e_1. Reading the edges of this second face in the positive orientation gives $-e_1 - e_2 + e_5 + e_6 + e_3 = 0$. The other four neighbours of f_1 across the edges e_2, e_3, e_4 and e_5 give relations formed from (1) by cyclically permuting $e_1, \ldots, e_5$; the relations obtained from the remaining six faces of D_1 are simply the negatives of the relations constructed, so

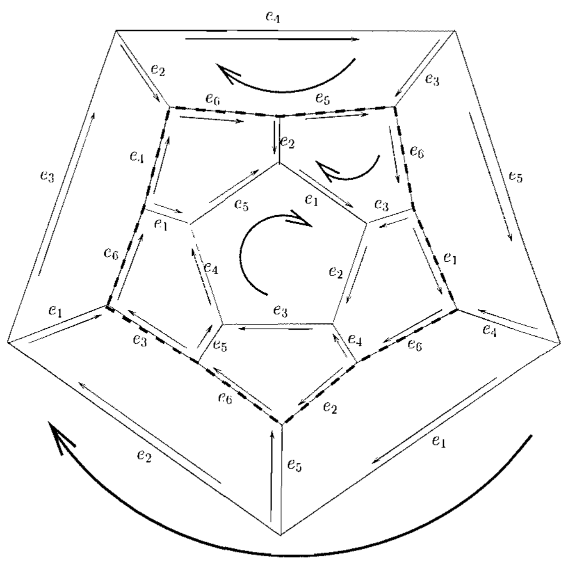


Figure 5.2: $+\frac{3\pi}{5}$ identification with Petrie polygon marked (dashed line)

they can be ignored. Eliminating e_4 $(= e_1 + 2e_2 + 3e_3)$, e_5 $(= 3e_1 + 2e_2 + e_3)$ and e_6 $(= 3e_1 - e_2 + 3e_3)$ gives $H_1(M_1)$ generated by e_1, e_2 and e_3 with defining relations $5e_1 = 5e_2 = 5e_3 = 0$. Hence $H_1(M_1) \cong \mathbf{Z}_5 \oplus \mathbf{Z}_5 \oplus \mathbf{Z}_5$ (see the comment at the end of [WS]). A presentation for the fundamental group of M_1 has been computed by Lorimer [Lo], and abelianising it confirms this result. The fundamental groups are also computed in Chapter 3 § 3.2 where the presentations are described in detail.

Conjugating by an element of $\Gamma \backslash \Gamma^+$ transposes N_1 and N_2, so M_2 and its tessellation $\mathcal{D}/N_2$ are obtained as the mirror images of M_1 and $\mathcal{D}/N_1$. Petrie polygons on D_1 can be used to illustrate the chirality of these two tessellations: in each case these closed paths all have length 10, with one of the six edges appearing five times and the other five edges once each (as previously shown in Figure 5.2). Figure 5.2 shows a view of $\mathcal{M}_1 = \mathcal{D}/N_1$ with the outside face in front. In $\mathcal{M}_1$ the repeated edge of the Petrie path is always followed by a right turn, while in M_2 it is always followed by a left turn. Figures 5.3 and 5.4 illustrate the

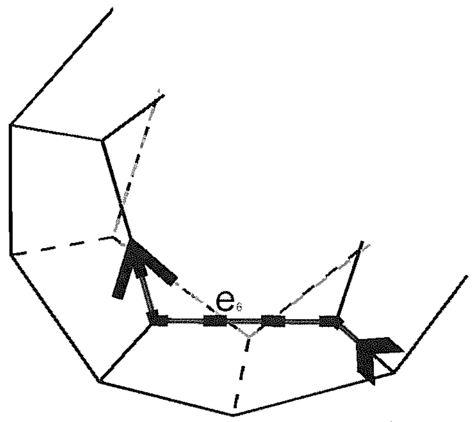


Figure 5.3: Right-handed petric path

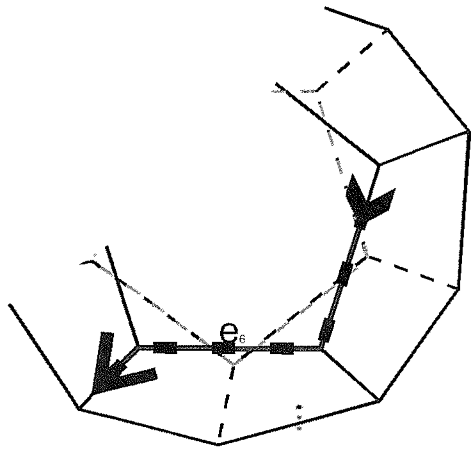


Figure 5.4: Left-handed petrie path

5.3 The tessellations $\mathcal{D}/N_i'$ by 125 dodecahedra

The group $\Gamma^+/N_1 \cong A_5$ of symmetries of $\mathcal{D}/N_1$ is generated by a rotation through $2\pi/5$ about the center of the face f_1, induced by α, and a rotation through $2\pi/3$ about the vertex v_1, induced by β; their product is a rotation through π about the mid-point of the edge e_1. These rotations act on $H_1(M_1)$, regarded as a 3-dimensional vector space over the field $GF(5)$, as the matrices

$$\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \alpha\beta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$$

with respect to the basis $\{e_1, e_2, e_3\}$. It is easily seen from this that $H_1(M_1)$ is an irreducible module for Γ^+/N_1, hence also for the full isometry group $\Omega^+/N_1 \cong S_5$ of M_1. This module is, up to isomorphism, the unique 3-dimensional irreducible module for S_5 over $GF(5)$: it can be identified with the unique nonprincipal irreducible constituent of the natural permutation module for S_5 over this field, or equivalently with the natural module for the special orthogonal group $SO_3(5) \cong S_5$ (this last equivalence can be read off from the Atlas of Finite Groups [Atlas], or equivalently by considering the natural representation of S_5 over the field of five elements. Since 5 divides the order of S_5, the representation splits as two 1-dimensional representations and an irreducible 3-dimensional representation). As discussed in the previous section, conjugating by an element of $\Gamma \setminus \Gamma^+$ gives the same conclusion for $H_1(M_2)$.

Since $H_1(M_i)$ is irreducible as an $\mathrm{Isom}(M_i)$–module, the only nontrivial regular abelian covering of M_i is the manifold $M_i' = \mathbf{H}^3/N_i'$. This has a tessellation $\mathcal{D}/N_i'$ consisting of 125 dodecahedra, in one-to-one correspondence with the elements of $H_1(M_i)$. The symmetry group $G = \Gamma^+/N_i'$ of this tessellation is a semidirect product of an elementary abelian normal subgroup $T = N_i/N_i' \cong H_1(M_i)$ by a complement $H = \Gamma_0^+ N_i'/N_i' \cong A_5$ stabilising a dodecahedron. Similarly, as will be shown later, the full isometry group Ω^+/N_i' of M_i' is a semidirect product of T by S_5, with the odd permutations in S_5 transposing the tessellation and its dual.

Proposition 5.3.1 *The subgroup H of G stabilising a dodecahedron of $\mathcal{D}/N_i'$ has two orbits each of lengths* 12, 20 *and* 30 *on the remaining dodecahedra.*

Proof: As in the proof of Theorem 5.1, the right-handed flags of $\mathcal{D}/N_i'$ can be identified with the elements of G, and the dodecahedral cells with the cosets gH of its subgroup H. As coset representatives for H in G choose the elements $t \in T$ and write D_t for the dodecahedron corresponding to tH. Each element $g \in G$ has a unique factorisation $g = th$, where $t \in T$ and $h \in H$, and the corresponding flag ϕ_g lies in the dodecahedron D_t. In particular, the flag ϕ_1 corresponding to the identity element of G lies in the dodecahedron D_0 corresponding to the coset $tH = H$ with $t = 0$ (here it is convenient to use additive notation for the abelian group T, and to write D_0 rather than D_1 for the dodecahedral cell incident with ϕ_1).

Each element $g \in G$ acts as an automorphism of $\mathcal{D}/N_i'$ by left multiplication, sending a coset tH to $g^{-1}tH$, and hence acting on dodecahedra by $D_t \mapsto D_{g^{-1}t}$. The stabiliser of D_0 is H, and its elements h act on cosets by $tH \mapsto h^{-1}tH = t^h H$, so its action on dodecahedra is equivalent to its action by conjugation on T, as a group of linear transformations of this vector space over $GF(5)$. It fixes 0, and by a routine calculation involving eigenspaces and the orbit-stabiliser theorem it is found that H has two orbits each of lengths 12, 20 and 30 on the remaining elements of T (since the eigenspaces of α and β are one-dimensional while that of $\alpha\beta$ is two-dimensional), so the same applies to its action on dodecahedra. ■

If $t \neq 0$ in T then, arguing as in Proposition 5.3.1, it can be shown that t and $-t$ are in the same orbit of H, while the elements $\pm 2t$ are in the other orbit of the same length. The actions of H on its orbits in T of lengths 12, 20 and 30 are equivalent to its actions on the faces, vertices and edges of D_0, since the stabilisers are the same in each case.

In the action of H on dodecahedra, one orbit of length 12 consists of the dodecahedra D_f meeting D_0 across a common face f (where, from Proposition 5.3.1, f corresponds as above to some element of T); the other orbit consists of dodecahedra D_{2f} meeting D_f across its face antipodal to f. Continuing like this 'bracelets' D_0, D_f, D_{2f}, $D_{3f} = D_{-2f}$, $D_{4f} = D_{-f}$ of five dodecahedra are obtained, with $-f$ denoting the face of D_0 antipodal to f: each dodecahedron meets its two neighbours across a pair of antipodal faces. After five steps any walk through these dodecahedra, passing through the shared faces, returns to $D_{5f} = D_0$ with a twist through π.

The vertex figure at a vertex v of D_0 is an icosahedron $\{3, 5\}$, with each of its 20 faces corresponding to a corner of a dodecahedron incident with v. These corners come in antipodal pairs, so there is a dodecahedron D_v meeting D_0 antipodaly at v. Such dodecahedra D_v form an orbit of H of length 20, and the other orbit of this length consists of dodecahedra D_{2v} meeting D_v antipodaly at its vertex antipodal to v. Continuing in this fashion, 'necklaces' D_0, D_v, D_{2v}, $D_{3v} = D_{-2v}$, $D_{4v} = D_{-v}$ of five dodecahedra, where $-v$ is the vertex of D_0 antipodal to v, are obtained: each dodecahedron meets its two neighbours antipodaly at a pair of antipodal vertices. A geodesic traveling through a pair of antipodal vertices in D_0 will meet four other dodecahedra, and the five dodecahedra join up to form a necklace of five dodecahedra each fixed setwise by an order three rotation fixing the geodesic.

If e is any edge of D_0, then a geodesic from the center of D_0 to the midpoint of e continues across a face separating two dodecahedra (since there are five dodecahedra about any edge, such a configuration is possible), then passes through the vertex of this face opposite e, and continues across another face, passes through the midpoint of an edge, entering a dodecahedron D_e. Continuing along this geodesic, a 'galaxy' D_0, D_e, D_{2e}, $D_{3e} = D_{-2e}$, $D_{4e} = D_{-e}$ of five dodecahedra is obtained, where $-e$ is the edge of D_0 antipodal to e: each dodecahedron is separated from its two neighbours by a pair of faces as above, and after five steps the path returns to $D_{5e} = D_0$. The orbits of H of length 30 consist of the dodecahedra D_e and D_{2e}, where e ranges over the edges of D_0.

This accounts for all 124 dodecahedra $D \neq D_0$ in $\mathcal{D}/N_i'$, but does not account for all of their incidences with D_0: there are two dodecahedra meeting D_0 along each edge e, in addition to the dodecahedra D_f and $D_{f'}$ corresponding to the faces f and f' of D_0 incident with e, and there are nine incidences with D_0 at each vertex v, in addition to those already described.

Proposition 5.3.2 *H has two orbits each of lengths 20 and 30 on the vertices of $\mathcal{D}/N_i'$, one orbit of length 10 and one orbit of length 15.*

Proof: There are 125 vertices in $\mathcal{D}/N_i'$. The vertices v of D_0 form an orbit of length 20, as do those joining dodecahedra D_v and D_{2v} in necklaces containing D_0. The galaxies provide two orbits of length 30, consisting of vertices between D_0 and D_e, and between D_e and D_{2e}. This leaves 25 vertices to be accounted for. Since each axis of 5-fold rotation of D_0 is contained in a bracelet and therefore passes through no vertices, there cannot be any orbits of length dividing 12. Hence the only possible lengths (not exceeding 25) are 5, 10, 15 and 20. The vertex joining D_{2v} and D_{-2v} in a necklace is invariant under the subgroup of order 6 in H preserving the pair of vertices $\pm v$, so it lies in an orbit of length dividing 10. This orbit, together with the two of length 20, accounts for all the vertices invariant under a 3-fold rotation of D_0. As a result there can be no other orbits of length dividing 20. Similarly, the vertex between D_{2e} and D_{-2e} in a galaxy is invariant under the subgroup of order 4 in H preserving the pair of edges $\pm e$, so it lies in an orbit of length dividing 15, and there are no other orbits of length dividing 30. It follows that these two orbits have lengths 10 and 15. ■

Since $\mathcal{D}/N_i'$ is isomorphic to its dual, the stabiliser in G of a vertex, isomorphic to A_5, permutes the vertices and dodecahedra in the same way as H permutes the dodecahedra and vertices. In fact the first cohomology group $H^1(A_5, T)$ is a 1-dimensional vector space over $GF(5)$ ([CPS], Theorem 4.2(c)), so there are five conjugacy classes of complements for T in G, two of which consist of the stabilisers of dodecahedra and of vertices. Arguments similar to those in Proposition 5.3.2 show that the complements in the other three classes have orbits of lengths 10, 15, 20, 20, 30 and 30 on the vertices and on the dodecahedra. The half-turn $r \in \Omega^+ \backslash \Gamma^+$ transposes the stabilisers of vertices and of dodecahedra, and also transposes two other classes of complements, leaving the fifth class invariant. Therefore the groups in this last class are contained in complements for T in $\Omega^+/N_i' = \mathrm{Iso}\, M_i'$, isomorphic to S_5.

5.4 The tessellation $\mathcal{D}/N$ by 60 dodecahedra

Since N_1 and N_2 are normal subgroups of Ω^+, so is their intersection $N = N_1 \cap N_2$. Further, as N_1 and N_2 are conjugate by elements of $\Omega \setminus \Omega^+$, N is normal in Ω. Since N_1 and N_2 are distinct maximal normal subgroups of Γ^+, $N_1N_2 = \Gamma^+$, and therefore $N_i/N \cong \Gamma^+/N_i \cong A_5$ for each i. This shows that the manifold $M = \mathbf{H}^3/N$ is a 60-sheeted regular unbranched covering of each M_i, with covering group A_5. There is a tessellation $\mathcal{D}/N$ of M by 60 dodecahedra; this has symmetry group Γ/N isomorphic to the wreath product $A_5 \wr C_2$, with the base group $\Gamma^+/N = N_1/N \times N_2/N \cong A_5 \times A_5$ as the orientation-preserving subgroup and $\Gamma_0 N/N \cong A_5 \times C_2$ as the subgroup stabilising a dodecahedron. The isometry r transforms $\mathcal{D}/N$ to its dual tessellation $\mathcal{D}^*/N \cong \mathcal{D}/N$. The normaliser of N in $\mathrm{Iso}\,\mathbf{H}^3$ is Ω, so M has isometry group Ω/N; this is isomorphic to the subgroup of index 2 in the wreath product $S_5 \wr C_2$ generated by the complement C_2 and the subgroup of the base group $S_5 \times S_5$ consisting of pairs of permutations with equal parity.

Since N_1 and N_2 are the normal closures in Γ^+ of the elements $(abcd)^t = (\alpha\gamma)^t$ for $t = 2$ and 3, their intersection N contains the normal closure in Γ^+ of $(abcd)^6$. Indeed, being normal in Γ, N contains the normal closure of $(abcd)^6$ in Γ, and this is also the normal closure in Γ^+ of $(abcd)^6$ and its conjugate $(bcda)^6 = (bacd)^6$. A computer calculation ([C1], p. 45) shows that N contains the normal closure of $(abcd)^6$ in Γ with index 2^9, so $\langle\langle (abcd)^6 \rangle\rangle_\Gamma$ has index 1843200 in Γ. By contrast, it should be noted that the normal closure of $(abcd)^6$ in Γ^+ has index $58,982,400$. Thus $\mathcal{D}/N$ has a 2^9-sheeted covering by Coxeter's tessellation $\{5,3,5\}_{6,6}$ ([C1], p. 45); this is formed from $\mathcal{D}$ by identifying flags which are equivalent under the monodromy permutations $(abcd)^6$ or $(bacd)^6$.

5.5 Subgroups of direct squares

In order to understand the structure of $\mathcal{D}/N$ and its symmetry group, some general facts about certain subgroups of direct squares of groups, such as $A_5 \times A_5$, are required. The following results are all easily verified.

Let S be any group, and let S^2 denote its direct square $S \times S$. For each $\alpha \in \mathrm{Aut}\, S$ there is a subgroup $S_\alpha = \{(s, s\alpha) \mid s \in S\}$ of S^2 which projects isomorphically onto both direct factors; conversely every such subgroup has this form for some automorphism α, so the number of such subgroups is $|\mathrm{Aut}\, S|$. Two such subgroups S_α and S_β are conjugate in S^2 if and only if $\alpha^{-1}\beta \in \mathrm{Inn}\, S$, therefore each conjugacy class contains $|\mathrm{Inn}\, S| = |S : Z(S)|$ subgroups, where $Z(S)$ is the center of S, and there are $|\mathrm{Out}\, S| = |\mathrm{Aut}\, S : \mathrm{Inn}\, S|$ conjugacy classes of them, one for each coset $[\alpha] = \alpha.\mathrm{Inn}\, S$ of $\mathrm{Inn}\, S$ in $\mathrm{Out}\, S$. Each conjugacy class of subgroups S_α is the set of point stabilisers in a transitive representation of degree $|S|$ of S^2. The kernel of this representation is $Z_\alpha = \{(z, z\alpha) \mid z \in Z(S)\}$, the induced permutation group is S^2/Z_α (a central product of two copies of S), and the direct factors of S^2 act as commuting regular normal subgroups, which can be identified with the regular representations of S on itself by left and right multiplication. In particular, if α is taken to be the identity automorphism then the corresponding permutation group is the holomorph $\mathrm{Hol}\, S$ of S, a semidirect product of a regular normal subgroup S by the diagonal subgroup $S_\alpha = \{(s, s) \mid s \in S\}$, acting by conjugation on S as the stabiliser of the identity element.

5.6 The structure of the tessellation $\mathcal{D}/N$

Consider again the tessellation $\mathcal{D}/N$. The epimorphism $\theta = (\theta_1, \theta_2) : \Gamma^+ \to A_5 \times A_5$ given by

$$\alpha \mapsto ((13524), (13524)), \quad \beta \mapsto ((123), (123)), \quad \gamma \mapsto ((14352), (13425))$$

has kernel N, and this can be used to identify the orientation-preserving symmetry group $G := \Gamma^+/N$ of $\mathcal{D}/N$ with $A_5 \times A_5$.

Proposition 5.6.1 *The subgroup H of G stabilising a dodecahedron of $\mathcal{D}/N$ has* *two orbits of length* 12*, and one orbit each of length* 15 *and* 20 *on the remaining dodecahedra.*

Proof: By taking $S = A_5$ in the previous section, so that $\mathrm{Aut}\, S \cong S_5$ and $Z(S) = 1$, it can be seen that in G there are $|\mathrm{Out}\, A_5| = 2$ conjugacy classes each consisting of $|\mathrm{Inn}\, A_5| = 60$ non-normal subgroups isomorphic to A_5; these are the stabilisers in G of dodecahedral cells and of vertices in $\mathcal{D}/N$, represented by the images under θ of $\Gamma_0^+ = \langle \alpha, \beta \rangle$ and of $\Gamma_1^+ = \langle \beta, \gamma \rangle$, where $\Gamma_1 = \langle b, c, d \rangle$. In particular, since θ_1 and θ_2 agree on α and β, it follows that the stabiliser of the dodecahedron D_1 is the diagonal subgroup $H = \{(g, g) \mid g \in A_5\}$. The direct factors of $G = A_5 \times A_5$ commute, and act regularly on the 60 dodecahedra, which can therefore be labeled with elements of A_5 so that the two factors act as the left and right regular representations of A_5. It follows that H acts by conjugation, so its orbits on dodecahedra correspond to the conjugacy classes of A_5. The dodecahedron D_1 corresponds to the class $\mathcal{C}_1$, which consists of the identity element, and the other orbits have lengths 15 (corresponding to the class $\mathcal{C}_2$ of involutions), 20 (corresponding to the class $\mathcal{C}_3$ of 3-cycles), 12 and 12 (corresponding to the two classes $\mathcal{C}_5$ and $\mathcal{C}_5'$ of 5-cycles, containing (12345) and its square). ■

The above five orbits will be denoted by $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{O}_5$ and $\mathcal{O}_5'$ respectively. The subgroups of H stabilising dodecahedra in these orbits are the centralisers of the corresponding elements of A_5, isomorphic to A_5, V_4, C_3, C_5 and C_5 respectively. This will help to determine how D_1 (and, by regularity, every dodecahedron in $\mathcal{D}/N$) is related geometrically to the 59 others. By self-duality, similar conclusions will apply to the vertices.

The flags of $\mathcal{D}$ and the tetrahedra of $\mathcal{T}$ are labeled with the elements of Γ, with this group acting regularly by right or left multiplication as a group of monodromy permutations or automorphisms. The flags of $\mathcal{D}/N$ and tetrahedra of $\mathcal{T}/N$ can therefore be labeled with the elements of Γ/N; here either action of N can be used since this is a normal subgroup of Γ. As a consequence the technical difficulties encountered earlier with the non-normal subgroups N_i can be avoided. In particular, the right-handed flags and tetrahedra can be identified with the elements of G, through its isomorphism with Γ^+/N induced by θ. Those in a given dodecahedral cell form a coset gH of the diagonal subgroup H in G, so the cells of $\mathcal{D}/N$ can be identified with these cosets of H in G. As coset representatives the elements $(g, 1)$ with $g \in A_5$ are chosen, since there are 60 of these, all in different cosets; then write D_g for the corresponding dodecahedral cell of $\mathcal{D}/N$. A general element $(g_1, g_2) \in G$ then lies in the coset containing $(g_1, g_2)(g_2^{-1}, g_2^{-1}) = (g_1g_2^{-1}, 1)$, represented by $(g, 1)$ where $g = g_1g_2^{-1} \in A_5$, so the flag $\phi = (g_1, g_2)$ lies in the corresponding cell D_g. In particular, the flag $\phi_1 = (1, 1)$ corresponding to the identity element of G lies in the cell D_1 corresponding to the coset $gH = H$. Let v_1, e_1 and f_1 denote the vertex, edge and face of D_1 incident with ϕ_1.

Proposition 5.6.2 *Dodecahedra D_g and D_h in $\mathcal{D}/N$ meet across a common face if and only if $g^{-1}h \in \mathcal{C}_5$. In these circumstances, this face is unique, and across its antipodal face D_g meets the dodecahedron $D_{h'}$ satisfying $g^{-1}h' = (g^{-1}h)^{-1} \in \mathcal{C}_5$.*

Proof: The element γ, acting as a monodromy permutation, rotates flags of $\mathcal{D}$ around their incident edges, so its image $\gamma\theta = ((14352), (13425)) \in G$ has this effect on the flags of $\mathcal{D}/N$. Applying the powers of this element to ϕ_1 the flags $((14352)^i, (13425)^i)$ are obtained for $i = 1, \ldots, 4$, and these lie in the dodecahedra D_g for $g = (14352)^i(13425)^{-i} = (13254), (152), (134), (15432)$ respectively. These cells are those that meet D_1 around e_1. In particular, the last of these cells meets D_1 across f_1, and the first meets D_1 across the other face of D_1 incident with e_1. Applying an arbitrary element $(h, h) \in H$ as an automorphism preserving D_1, it can be seen that the flags $\phi_h = (h^{-1}(14352)^i, h^{-1}(13425)^i)$ are all obtained by rotating the flag (h^{-1}, h^{-1}) of D_1 around its incident edge; these flags ϕ_h lie in the dodecahedra D_g for $g = (13254)^h, (152)^h, (134)^h, (15432)^h$. Thus D_1 meets these dodecahedra D_g around their common edge in the same way as it meets the first four dodecahedra, indexed by $(14352)^i(13425)^{-i}$, around e_1. In particular, it meets a dodecahedron D_g across a face if and only if g is in the conjugacy class $\mathcal{C}_5$ of A_5 containing (15432), or equivalently its inverse (12345). There are 12 faces of D_1, and 12 elements $g \in \mathcal{C}_5$, both permuted transitively by H, so each D_g indexed by $g \in \mathcal{C}_5$ meets D_1 across a single face. These 12 dodecahedra D_g form the orbit $\mathcal{O}_5$ of H on cells. More generally, given any pair of dodecahedra D_g and D_h, by applying an automorphism sending D_g to D_1, it can be seen that D_g and D_h meet across a common face (which is unique) if and only if $g^{-1}h \in \mathcal{C}_5$.

The monodromy permutation $(abc)^5$, representing the central involution in Γ_0, sends ϕ_1 to its antipodal flag $-\phi_1$ in D_1, so $(abc)^5 d$ sends ϕ_1 to the adjacent flag $(-\phi_1)d$ in the dodecahedron D_g meeting D_1 across its face $-f_1$ opposite f_1. Since θ maps $(abc)^5 d$ to $((13524)^{-1}, (13524))$, it follows that $g = (13524)^{-2} = (12345)$. This is the inverse of the element (15432) of $\mathcal{C}_5$ labeling the dodecahedron meeting D_1 across f_1. It follows that a general dodecahedron D_g meets neighbours D_h and $D_{h'}$ across opposite faces if and only if $g^{-1}h$ and $g^{-1}h'$ are mutually inverse elements of $\mathcal{C}_5$. ■

This result shows that the dual graph of $\mathcal{D}/N$ is isomorphic to the Cayley graph for A_5 with respect to its generating set $\mathcal{C}_5$, and by self-duality the same applies to the 1-skeleton of $\mathcal{D}/N$.

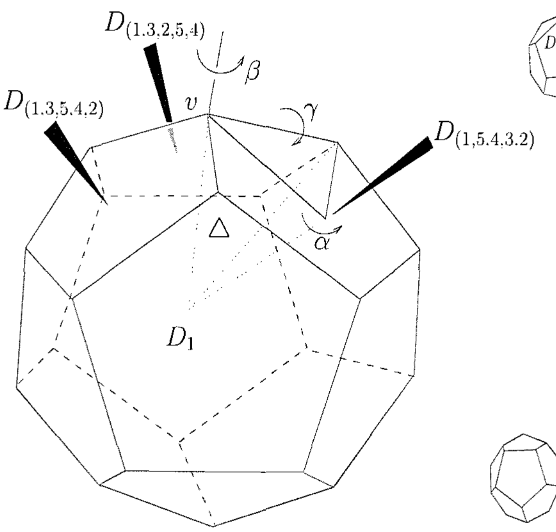


Figure 5.5: Local adjacency about v

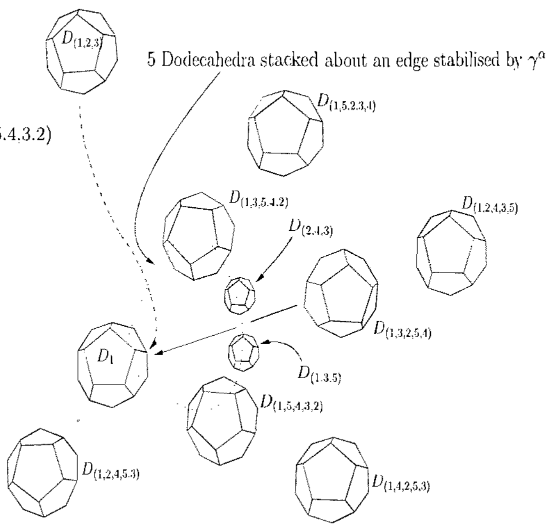


Figure 5.6: Partial schematic at v

As seen in the proof of Proposition 5.6.2, D_1 meets $D_{(15432)}$ across f_1, so this is the dodecahedron D_{f_1} in the notation of § 5.3; it is contained in the orbit of H consisting of the 12 dodecahedra D_f where f is a face of D_1. Since (15432) is in the conjugacy class $\mathcal{C}_5$ this is the orbit $\mathcal{O}_5$. At the face opposite f_1, D_{f_1} meets $D_{(14253)}$ where $(14253) = (15432)^2$ is a member of the other class $\mathcal{C}_5'$ of 5-cycles; this lies in a second orbit $\mathcal{O}_5'$, consisting of 12 dodecahedra, denoted D_{2f} in the earlier notation. By iterating this argument bracelets of five dodecahedra are obtained, each meeting its two neighbours across a pair of opposite faces; starting with D_1 these dodecahedra lie in the orbits $\mathcal{O}_1$, $\mathcal{O}_5$, $\mathcal{O}_5'$, $\mathcal{O}_5'$ and $\mathcal{O}_5$ of H.

Proposition 5.6.3 *Dodecahedra D_g and D_h in $\mathcal{D}/N$ meet antipodaly at a common vertex if and only if $g^{-1}h \in \mathcal{C}_3$. In these circumstances, this vertex is unique, and at its antipodal vertex D_g meets $D_{h'}$ antipodaly with $g^{-1}h' = (g^{-1}h)^{-1} \in \mathcal{C}_3$.*

Proof: The subgroup $\Gamma_1 = \langle b, c, d \rangle$, in its monodromy action, preserves the set of flags incident with v_1. Its central involution $(bcd)^5$ sends ϕ_1 to the antipodal flag at v_1, incident with the dodecahedron D_{v_1} antipodal to D_1 at v_1, so the element $(bcd)^5 b$ of Γ^+ also sends ϕ_1 to a flag in D_{v_1}. Now θ maps $(bcd)^5 b = \gamma^{-1}(\gamma\beta\gamma^{-1}\beta^{-1})^3$ to $((23)(45),(12)(45))$, and $(23)(45)((12)(45))^{-1} = (123) \in \mathcal{C}_3$, so $D_{v_1} = D_{(123)}$. D_1 has 20 vertices, which is the number of elements in $\mathcal{C}_3$, so the dodecahedra D_v meeting D_1 antipodaly across a common vertex v form an orbit $\mathcal{O}_3$ of H, each meeting D_1 at a unique vertex. As in the case of adjacency across faces, it can be seen , by applying automorphisms, that a pair of dodecahedra D_g and D_h meet antipodaly across a single vertex if and only if $g^{-1}h \in \mathcal{C}_3$. In this case D_g meets $D_{h'}$ antipodaly across the antipodal vertex of D_g if and only if $g^{-1}h$ and $g^{-1}h'$ are mutually inverse elements of $\mathcal{C}_3$. ■

By iterating this result, it follows that dodecahedra in $\mathcal{D}/N$ form necklaces of length 3, each dodecahedron meeting its two neighbours antipodaly at an antipodal pair of its vertices. These dodecahedra and incidences can be represented as the vertices and edges of the Cayley graph of A_5 with respect to its generating set $\mathcal{C}_3$.

As seen in the proof of Proposition 5.6.2, the dodecahedra $D_{(152)}$ and $D_{(134)}$ meet D_1 along its edge e_1, and therefore the dodecahedra in $\mathcal{O}_3$ all meet D_1 across an edge, and more generally D_h meets D_g across an edge if and only if $g^{-1}h \in \mathcal{C}_3$. Now D_g has 30 edges, each corresponding to two such dodecahedra D_h. Consequently, it has 60 such incidences. Since $|\mathcal{C}_3| = 20$, it follows that each of these dodecahedra D_h meets D_g three times along edges, in addition to its antipodal incidence at a vertex v of D_g. Let $\overline{v}$ be the vertex in D_g that is antipodal to v. For any vertex v in a dodecahedron D, the edges opposite v are defined as follows: take a geodesic from v through the center of the face containing v. Then this arc cuts an edge e across from v. Call this edge the edge opposite v. Calculations similar to those above show that these three edges are opposite $\overline{v}$. Of the two dodecahedra which meet D_g along such an edge, D_h is the nearer to v.

The vertex figure of $\mathcal{D}/N$ at v_1 is an icosahedron $\{3,5\}$, its 20 faces corresponding to the 20 corners of dodecahedra meeting at v_1. So far 11 of these dodecahedra have been accounted for: there is D_1 itself, three dodecahedra $D_{(15432)}$, $D_{(13254)}$ and $D_{(13542)}$ in $\mathcal{O}_5$ meeting D_1 across faces incident with v_1, six dodecahedra $D_{(152)}$, $D_{(134)}$, $D_{(253)}$, $D_{(142)}$, $D_{(135)}$ and $D_{(243)}$ in $\mathcal{O}_3$ meeting it along edges incident with v_1, and $D_{(123)} \in \mathcal{O}_3$ meeting D_1 antipodaly at v_1. Calculations with monodromy permutations show that, of the remaining nine dodecahedra meeting D_1 at v_1, three of them, $D_{(13452)}$, $D_{(14532)}$ and $D_{(13245)}$ in $\mathcal{O}'_5$, meet $D_{(123)}$ across a face, while the remaining six dodecahedra, $D_{(15)(23)}$, $D_{(13)(25)}$, $D_{(12)(35)}$, $D_{(12)(34)}$, $D_{(13)(24)}$ and $D_{(14)(23)}$ in $\mathcal{O}_2$, meet $D_{(123)}$ along an edge. Figures 5.7, 5.8 and 5.9 Illustrate the geometrical structure at v. More generally, as described above, two dodecahedra D_g and D_h meet at a common vertex if and only if $g^{-1}h$ is one of the corresponding permutations listed here. Since any meeting between two dodecahedra must be across a common face, along a common edge, or at a common vertex, all meetings between pairs of dodecahedra in $\mathcal{D}/N$ have been accounted for.

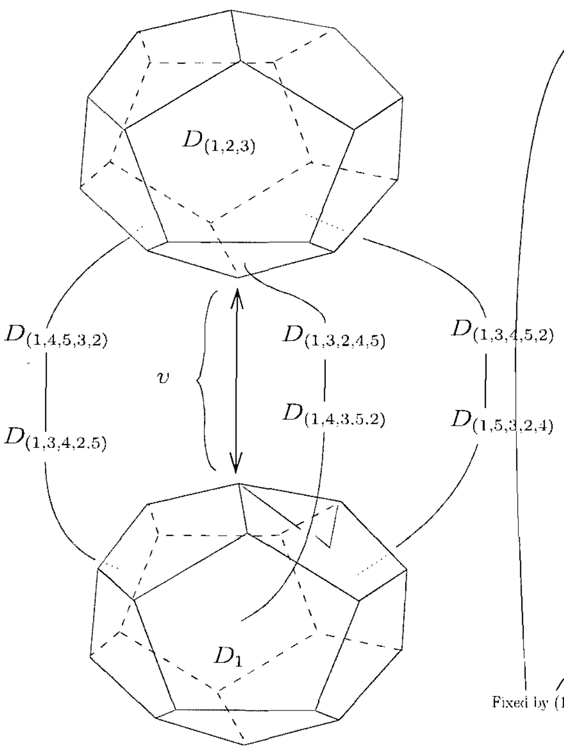


Figure 5.7: Vertex structure at v: I

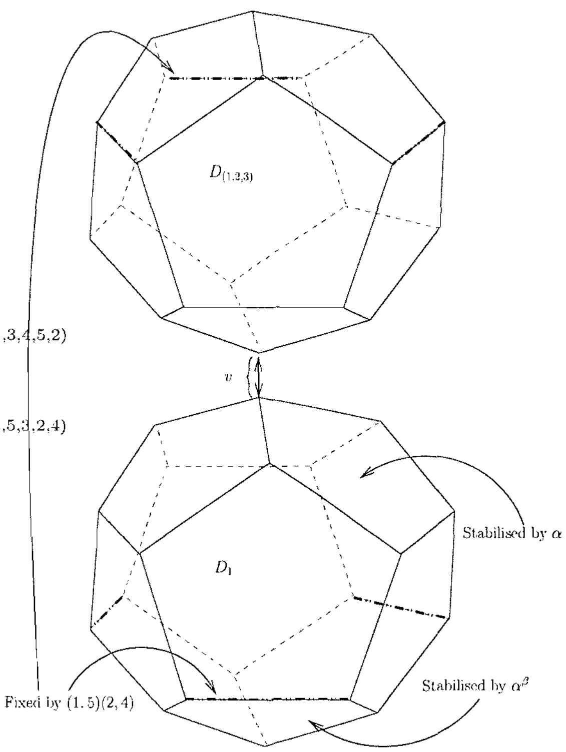


Figure 5.8: Vertex structure at v: II

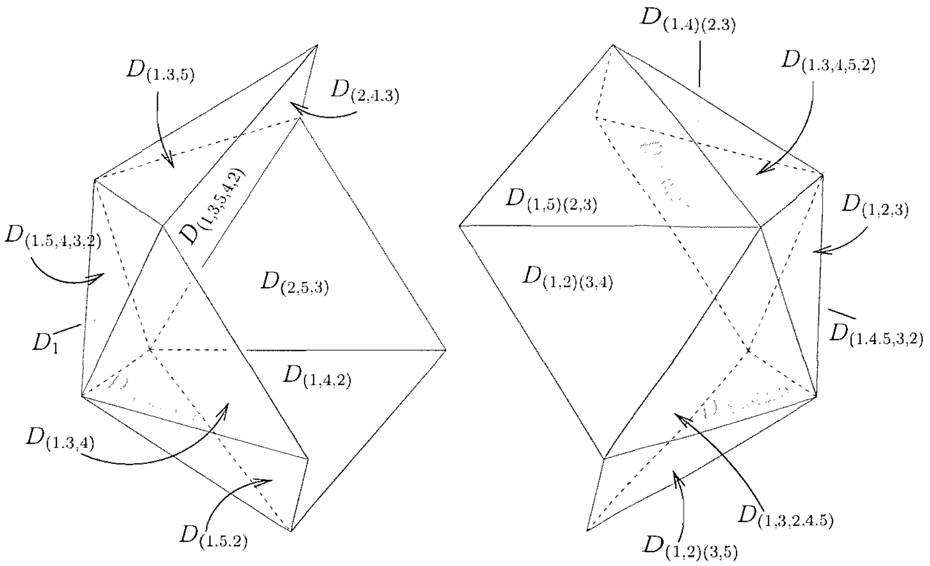


Figure 5.9: Vertex structure at v: III

5.7 The Poincaré dodecahedral spaces

The chirality exhibited by the Weber-Seifert space also arises in connection with Poincaré's dodecahedral space, where antipodal pairs of faces of a dodecahedron in S^3, with dihedral angles $2\pi/3$, are identified after a twist through $\pm\pi/5$ to produce a compact 3-manifold, Poincaré's homology sphere. As before, the two possible directions of twisting yield a chiral pair of oriented manifolds P_1 and P_2, each having only orientation-preserving isometries, and each manifold is the mirror-image of the other. The details of the construction are rather better-known than in the hyperbolic case ([Cox RP, Ch. VIII], [Mon, §3.13], [ST, §62] and [Thu, §1.4.4]), so they will just be outlined here.

Stereographic projection, which allows rotations of S^2 to be represented as Möbius transformations of the Riemann sphere $\mathbf{C} \cup \{\infty\}$, gives an isomorphism $SO(3) \cong PSU(2)$. The natural projection $SU(2) \to PSU(2)$ provides a double covering of $SO(3)$ by

$$SU(2) = \left\{ \begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix} \mid u, v \in \mathbf{C},\ u\overline{u} + v\overline{v} = 1 \right\}.$$

By taking real and imaginary parts of u and v as coordinates, $SU(2)$ can be identified with the 3-sphere $S^3 \subset \mathbf{R}^4$, and hence with the multiplicative group Q of unit quaternions. The actions of this group on itself by left and by right multiplication give two subgroups $Q_L, Q_R \cong Q$ in the orientation-preserving isometry group $SO(4)$ of S^3; these commute with each other, intersect in their common center of order 2, and generate $SO(4)$, so that $SO(4)$ is the central product $Q_L Q_R$ of Q_L and Q_R, isomorphic to the quotient of $Q \times Q \cong SU(2) \times SU(2)$ obtained by identifying each element of $Q \times Q$ with its negative. The full isometry group $O(4)$ of S^3 is an extension of $SO(4)$ by an orientation-reversing involution induced by conjugation of quaternions, sending (x_1, x_2, x_3, x_4) to $(x_1, -x_2, -x_3, -x_4)$; this inverts each element of Q, so it transposes the two subgroups Q_L and Q_R of $SO(4)$.

There is a single conjugacy class of icosahedral (or dodecahedral) rotation groups $I \cong A_5 \cong L_2(5)$ in $SO(3)$, each I lifting in $SU(2)$ to a binary icosahedral group $\hat{I} \cong \hat{A}_5 \cong SL_2(5)$, an extension of I by a center of order 2. The resulting embedding of $\hat{I}$ in Q yields a pair of subgroups $\hat{I}_L, \hat{I}_R \cong \hat{I}$ of Q_L and Q_R, which generate their central product $\hat{I}_L \hat{I}_R$ of order $120^2/2 = 7200$ in $SO(4)$; extending this by conjugation of quaternions, which transposes $\hat{I}_L$ and $\hat{I}_R$, gives a subgroup Δ of order 14400 in $O(4)$. This is the Coxeter group $[5,3,3]$, the symmetry group of the 120-cell $\{5,3,3\}$, a tessellation $\mathcal{E}$ of S^3 by 120 dodecahedra with dihedral angles $2\pi/3$; the orientation-preserving subgroup $\Delta^+ = [5,3,3]^+$ of Δ is $\hat{I}_L \hat{I}_R$.

One can construct $\mathcal{E}$ as the Dirichlet (or Voronoi) tessellation of S^3 corresponding to its discrete subgroup $\hat{I}$. For each $g \in \hat{I}$ define the Dirichlet region E_g to be the set of points in S^3 which have g as a nearest element of $\hat{I}$. These 120 sets are permuted regularly by $\hat{I}_L$ and $\hat{I}_R$, with the diagonal subgroup $H \cong A_5$ of the central product $\Delta^+ = \hat{I}_L \hat{I}_R$ stabilising E_1 and permuting the sets E_g in the same way as it acts by inner automorphisms on the elements $g \in \hat{I}$; the antipodal involution stabilising E_1 acts on these sets by inverting their labels g.

In order to determine the shape of each set E_g, it is convenient to choose the subgroup I of $SO(3)$ to contain the rotation $z \mapsto e^{2\pi i/5} z$ of the Riemann sphere, which lifts to the pair of elements

$$\pm \begin{pmatrix} e^{\pi i/5} & 0 \\ 0 & e^{-\pi i/5} \end{pmatrix} \in \hat{I}$$

in $SU(2)$ corresponding to the points $\pm(\cos \pi/5, \sin \pi/5, 0, 0) \in S^3$. The elements of $\hat{I}$ closest to the identity $(1,0,0,0)$ are those maximising their first coordinate, and with this choice of I these are the 12 members of the conjugacy class of $\hat{I}$ containing the element $(\cos\pi/5, \sin\pi/5, 0, 0)$. It follows that each E_g is a regular dodecahedron centered at g, meeting E_h across a face if and only if $g^{-1}h$ is conjugate to $(\cos\pi/5, \sin\pi/5, 0, 0)$. These 120 dodecahedra E_g have dihedral angles $2\pi/3$, and are the cells of a tessellation $\mathcal{E}$ of S^3 with Schläfli symbol $\{5,3,3\}$ and isometry group Δ. This tessellation has 600 vertices, 1200 edges and 720 faces, and its dual is the 600-cell $\{3,3,5\}$. The barycentric subdivision of $\mathcal{E}$ is a tessellation of S^3 by 14400 tetrahedra, each having as its vertices a vertex of $\mathcal{E}$ and the midpoints of an edge, face and cell of $\mathcal{E}$, all mutually incident. The reflections $\underline{a}$, $\underline{b}$, $\underline{c}$ and $\underline{d}$ of S^3 in the sides of a tetrahedron opposite these vertices generate Δ, giving a presentation

$$\begin{aligned}\Delta = \langle \underline{a}, \underline{b}, \underline{c}, \underline{d} \mid \underline{a}^2 = \underline{b}^2 = \underline{c}^2 = \underline{d}^2 = 1,\\ (\underline{ab})^5 = (\underline{bc})^3 = (\underline{cd})^3 = (\underline{ac})^2 = (\underline{ad})^2 = (\underline{bd})^2 = 1\rangle\end{aligned}$$

of Δ as a Coxeter group. Setting $\underline{\alpha} = \underline{ab}$, $\underline{\beta} = \underline{bc}$ and $\underline{\gamma} = \underline{cd}$ as before, the elements $\underline{\alpha}$, $\underline{\beta}$ and $\underline{\gamma}$ generate the index 2 subgroup Δ^+ of Δ. Let $\hat{I}_R =<< (abcd)^5 >>_{\Delta^+}$ and $\hat{I}_L =<< (abcd)^3 >>_{\Delta^+}$. Then $\hat{I}_R$ and $\hat{I}_R$ generate Δ^+ as a central product and $\hat{I}_R$ and $\hat{I}_R$ have order 120.

The quotients P_1 and P_2 of S^3 by $\hat{I}_R$ and $\hat{I}_L$ are obtained by identifying equivalent pairs of faces of the dodecahedron E_1, using right multiplication by the conjugates h of $(\cos\pi/5, \sin\pi/5, 0, 0)$ or left multiplication by h^{-1}; these identify antipodal pairs of faces of E_1 by means of left- or right-handed screw motions with angle $\pi/5$, so each P_i is a Poincaré dodecahedral space. In each case, the isometry group of P_i, induced by the action of the other copy of $\hat{I}$ on S^3, is isomorphic to I and contains only orientation-preserving elements. The involution which transposes $\hat{I}_R$ and $\hat{I}_L$ induces an orientation-reversing isometry between P_1 and P_2, so these two spaces form a chiral pair, as noted by Montesinos [Mon, Ch. 3]. Since $\hat{I}_R$ and $\hat{I}_L$ both act freely on the simply connected space S^3, it follows that P_1 and P_2 have fundamental groups isomorphic to $\hat{I}$; this is a perfect group, so their first (and, by duality, second) homology groups are trivial, that is, they are homology spheres.

The quotients of $\mathcal{E}$ by $\hat{I}_R$ and $\hat{I}_L$ are a chiral pair of tessellations of P_1 and P_2, each consisting of five vertices, ten edges, six faces and one cell. These are Coxeter's twisted honeycombs $\{5,3,3\}_t$ for $t = 5$ and $t = 3$ respectively, corresponding to the fact that $\hat{I}_R$ and $\hat{I}_L$ are the normal closures in Δ^+ of $(\underline{abcd})^5$ and $(\underline{abcd})^3$ ([Cox TH, Fig. 22], [Mon, Fig 29]). The identification $\{5, 3, 3\}_3$ is illustrated in Figure 5.10

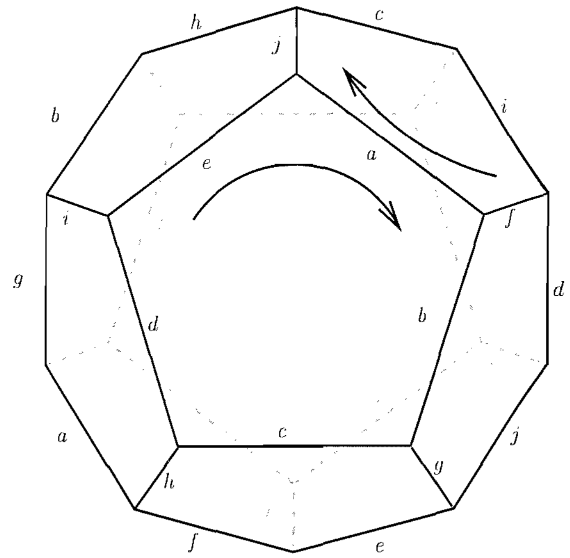


Figure 5.10: Coxeter's $\{5, 3, 3\}_3$

5.8 The structure of the 120-cell

The structure of $\mathcal{E}$, the 120-cell, can be studied in the same way as were the tessellations $\mathcal{D}/N_i'$ and $\mathcal{D}/N$, by identifying the flags with the elements of Δ and the dodecahedral cells with the cosets of $\Delta_0 = \langle \underline{a}, \underline{b}, \underline{c} \rangle \cong I \times C_2$ in Δ, or more conveniently of $\Delta_0^+ \cong I$ in Δ^+. The dodecahedra E_g are labeled by the elements g of $\hat{I}_L \cong \hat{I}$, acting as coset representatives of Δ_0^+ in Δ^+, so that the orientation-preserving stabiliser Δ_0^+ of D_1 permutes them in the same way as it acts as inner automorphisms on $\hat{I}$. In particular, the orbits of Δ_0^+ on dodecahedra correspond to the conjugacy classes of $\hat{I}$. Each conjugacy class $\mathcal{C} = \mathcal{C}_1$, $\mathcal{C}_3$, $\mathcal{C}_5$ or $\mathcal{C}_5'$ of elements of odd order $n = 1$, 3 or 5 in I lifts to two classes $\hat{\mathcal{C}}$ and $-\hat{\mathcal{C}}$ in $\hat{I}$, each inverse-closed and of size $|\mathcal{C}|$, containing elements of order n and $2n$ respectively. On the other hand, the class $\mathcal{C}_2$ in I containing the 15 involutions lifts to two mutually inverse classes of 15 elements of order 4 in $\hat{I}$. The antipodal involution $(\underline{abc})^5$ in $\Delta_0 \setminus \Delta_0^+$ acts by inverting the labels of the dodecahedra, so the orbits of Δ_0 on dodecahedra are the same as those of Δ_0^+, except that the two orbits of Δ_0^+ of length 15 labeled by the elements of order 4 form a single orbit of Δ_0 of length 30.

Proceeding as in § 5.3 and § 5.6, it is shown that bracelets of dodecahedra in $\mathcal{E}$ have length 10, each bracelet through E_1 being labeled with successive powers $g, g^2, \ldots, g^{10} = 1$ of an element $g \in -\hat{C}_5'$, lying in the classes $-\hat{C}_5'$, $\hat{C}_5$, $-\hat{C}_5$, $\hat{C}_5'$, $-\hat{C}_1$, $\hat{C}_5'$, $-\hat{C}_5$, $\hat{C}_5$, $-\hat{C}_5'$ and $\hat{C}_1$.

Every element of $\hat{I}$ whose order divides 10 appears as a label of such a dodecahedron, and in particular the antipodal dodecahedron E_{-1} appears opposite E_1 in each of the six bracelets containing E_1. Necklaces have length 3, the labels of those through E_1 accounting for the elements of order 3, while their antipodal necklaces each consist of E_{-1} and two dodecahedra labeled by mutually inverse elements of order 6.

The element $\underline{abcd}$ of Δ has order 30, with the involution $(\underline{abcd})^{15}$ corresponding to $(-1, 0, 0, 0) \in S^3$ and generating the center $\hat{I}_L \cap \hat{I}_R \cong C_2$ of Δ [C1, §11]. The antipodal quotient of $\mathcal{E}$, by this center, is a tessellation $\{5, 3, 3\}_{15}$ of $SO(3)$, or equivalently of real projective 3-space $\mathbf{P}^3(\mathbf{R})$, by 60 dodecahedra; it can be constructed as the Dirichlet tessellation of $SO(3)$ corresponding to its discrete subgroup I. In this case each bracelet of dodecahedra has length 5, lifting to a bracelet of length 10 in $\mathcal{E}$, while each necklace has length 3, lifting to an antipodal pair of necklaces of length 3 in $\mathcal{E}$.

5.9 The 120-cell and $\mathcal{D}/N$.

There is a connection between the spherical tessellation $\mathcal{E} = \{5, 3, 3\}$ and the hyperbolic tessellation $\mathcal{D}/N = \{5, 3, 5\}/N$ studied earlier. As shown by Lorimer [Lor], the Coxeter group $\Delta = [5, 3, 3]$ is generated by its elements $\underline{a}$, $\underline{b}$, $\underline{c}$ and $\underline{e} = \underline{d}(\underline{abc})^5\underline{d}(\underline{abc})^5\underline{d}$; these satisfy the relations

$$\underline{a}^2 = \underline{b}^2 = \underline{c}^2 = \underline{e}^2 = (\underline{ab})^5 = (\underline{bc})^3 = (\underline{ce})^5 = (\underline{ac})^2 = (\underline{ae})^2 = (\underline{be})^2 = 1,$$

which correspond to the standard defining relations of Γ with $\underline{a}$, $\underline{b}$, $\underline{c}$ and $\underline{e}$ replacing a, b, c and d, so there is an epimorphism $\varphi : \Gamma \to \Delta$ given by $a \mapsto \underline{a}$, $b \mapsto \underline{b}$, $c \mapsto \underline{c}$ and $d \mapsto \underline{e}$. (Actually, this is the dual of Lorimer's epimorphism $\Gamma \to \Delta^* = [3, 3, 5]$.)

To study all possible rewritings of the presentation of $[5, 3, 3]$, the following geometric description of the 120-cell is considered. In [C1, §9] the 120-cell is decomposed into two linked tori each consisting of 60 dodecahedra. Cutting one of these tori along a meridinal curve (as illustrated in Figure 5.11) gives a tower of

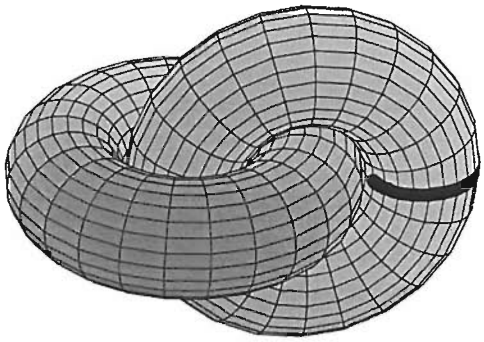


Figure 5.11: two linked tori with meridian path of one marked

dodecahedra, with a core tower of 10 stacked dodecahedra. Adjacent dodecahedra share a horizontal pentagonal face, and at each edge of this face a further dodecahedron is attached. The outer surface of the tower consists of the 200 free faces of the 50 dodecahedra wrapped around the central tower. Let A and B be planes in S^4 through the horizontal face of D_1 representing the reflecting planes for a and b, and consider how a plane representing d could be constructed. Such a plane must be orthogonal to both A and B in S^3 and so projects (under orthogonal projection) to either a horizontal plane in $\mathbb{R}^3$ or to a sphere in $\mathbb{R}^3$ orthogonal to both A and B. By looking in the 120-cell it can be seen that any reflection r whose plane R is orthogonal to A and B is going to be a symmetry of the meridian curve illustrated in Figure 5.12. The stabiliser of this curve is D_{20} (in fact the stabiliser of this curve will act transitively on the dodecahedra of the core of the second torus and so it is the stabiliser of a stalk), so there are 11 possible involutions $\{w_i\}$ whose product with a and b have the required properties. A quick computer check of these elements reveals that the product cw_i has order 2 (three of these), 3 (two of these), 5 (two of these), 6 (two of these) or 10 (two of these). Of the 11 w_i, eight of them (those with cw_i of order 3, 5 or 6 and two of those with cw_i of order 2) form a generating set $[a, b, c, w_i]$ for Δ. This also proves the existence of an epimorphism $\theta : [5, 3, 6] \to [5, 3, 3]$.

To find a more useful description of φ , its restriction to Δ^+ is first constructed. This group is the central product $\hat{I}_L\hat{I}_R$ of two copies of $\hat{I} \cong SL_2(5)$, that is, the

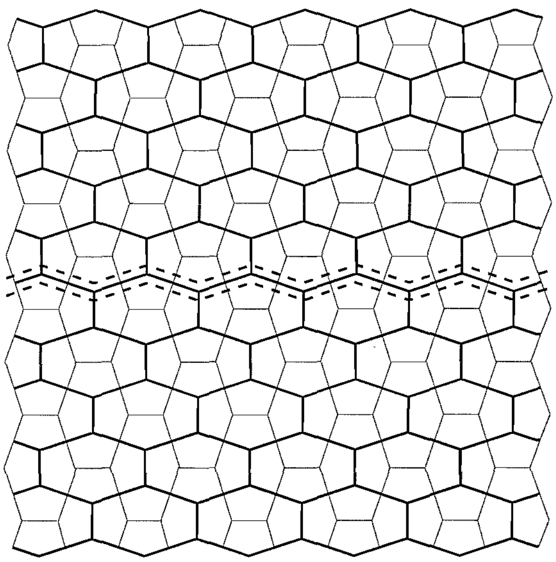


Figure 5.12: Two hundred faces of $\{5, 3, 3\}$ forming the boundary torus of a dodecahedral tower. Meridian path bounded by dotted lines

quotient of $SL_2(5) \times SL_2(5)$ obtained by identifying each pair of matrices with its negative. The center of this group is generated by the involution $\pm(I_2, -I_2)$, where I_2 is the identity matrix in $SL_2(5)$, and the central quotient, obtained by identifying each matrix with its negative, is $L_2(5) \times L_2(5) \cong A_5 \times A_5$. Define $\varphi^+ : \Gamma^+ \to \Delta^+$ by

$$\alpha \mapsto \pm(A, A), \quad \beta \mapsto \pm(B, B), \quad \gamma \mapsto \pm(C_1, C_2)$$

where

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & -2 \\ -2 & 2 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}$$

are elements of $SL_2(5)$. It is routine to check that these images of α, β and γ generate Δ^+ and satisfy the defining relations of Γ^+, so φ^+ is an epimorphism. Moreover, the epimorphism $SL_2(5) \to A_5$ given by $A \mapsto (13524)$ and $B \mapsto (123)$ sends $C_1 = [B^{-1}AB, A]^2$ to (14352) and $C_2 = BC_1B^{-1}$ to (13425), so φ^+ is a lifting of the epimorphism $\theta : \Gamma^+ \to A_5 \times A_5$ considered in § 5.7. Now Δ is an extension of Δ^+ by the involution $\pm(M_1, M_2) \mapsto \pm(M_2, M_1)$ which interchanges the two copies of $SL_2(5)$. Therefore $\varphi^+ : \Gamma^+ \to \Delta^+$ can be extended to an epimorphism $\varphi : \Gamma \to \Delta$ by sending $(abc)^5$, an involution which commutes with α and β and conjugates γ to

$[\alpha^{-1},\beta]^2\alpha\gamma^{-1}\beta^{-1}\alpha^{-1}\beta[\alpha^{-1},\beta]^2$, to this involution, since a calculation shows that this has the corresponding effect on the images of α, β and γ in Δ^+.

Theorem 5.2 *The kernel $K = \ker\varphi$ is the unique normal subgroup of Γ with $\Gamma/K \cong \Delta$. It is a subgroup of index 2 in N.*

Proof: Since $\varphi : \Gamma \to \Delta$ is an epimorphism, the first isomorphism theorem gives $\Gamma/K \cong \Delta$. Since Γ^+ is perfect and Δ is not, any normal subgroup L with $\Gamma/L \cong \Delta$ must be contained in Γ^+, and hence in N since it follows from Lemma 3.2.1 that N is the only normal subgroup of Γ^+ with quotient $A_5 \times A_5$. This implies that the epimorphism $\theta : \Gamma^+ \to \Gamma^+/N \cong A_5 \times A_5$ must lift to an epimorphism $\hat{\theta} : \Gamma^+ \to \Delta^+$ with kernel L. Now any element g of odd order n in $A_5 \cong L_2(5) \cong I$ lifts to two elements $\hat{g}$ and $-\hat{g}$ of orders n and $2n$ in $\hat{A}_5 \cong SL_2(5) \cong \hat{I}$, so any element (g,h) of odd order n in $A_5 \times A_5$ lifts to two elements $\pm(\hat{g},\hat{h})$ and $\pm(\hat{g},-\hat{h})$ of orders n and $2n$ in Δ^+. Since Γ^+ is generated by its elements α, β and γ of odd orders $n = 5, 3$ and 5, any lifting $\hat{\theta}$ of θ must send these generators to the unique elements of the same orders n covering their images in $A_5 \times A_5$. This shows that θ has only one lift $\hat{\theta}$, namely φ, so $L = K$, a subgroup of index 2 in N. ∎

Using GAP, a presentation was obtained for N. It has a presentation on 53 generators and 57 relations. The abelianisation $N/N' \cong \mathbb{Z}^{41} \oplus \mathbb{Z}_2^{12}$ shows that N cannot be presented with fewer generators.

In the action of Γ as a monodromy group on the flags of any quotient of $\mathcal{D}$, one can regard the first three generators a, b and c as describing how flags fit together to form dodecahedral cells, and d as describing how these cells meet across faces; there is a similar interpretation for the generators $\underline{a}, \ldots, \underline{d}$ of Δ, acting on quotients of $\mathcal{E}$. Now the epimorphism $\varphi : \Gamma \to \Delta$ sends the first three generators of Γ to those of Δ, so that under the induced isomorphism $\Gamma/K \to \Delta$, dodecahedra of $\mathcal{D}/K$ are matched up with dodecahedra of $\mathcal{E}$; however, φ sends d to $\underline{e}$ rather than $\underline{d}$, so that adjacency of dodecahedra in $\mathcal{D}/K$ corresponds to the effect of $\underline{e}$ in identifying faces of dodecahedra in $\mathcal{E}$.

Now consider the action of the element $\underline{e}$ of Δ as a monodromy permutation on a flag $\phi = (v, e, f, E)$ incident with a face f on a dodecahedron E of the 120-cell $\mathcal{E}$. Reading $\underline{e} = \underline{d}(\underline{abc})^5\underline{d}(\underline{abc})^5\underline{d}$ from left to right, it can be seen that $\underline{d}$ sends ϕ to the adjacent flag (v, e, f, E_f) in the dodecahedron E_f which meets E across their common face f; then $(\underline{abc})^5$ sends this to the antipodal flag of E_f, on the face f' of E_f antipodal to f; applying $\underline{d}$ again, the adjacent flag in the dodecahedron E_{2f} which meets E_f across f' is obtained; then applying $(\underline{abc})^5$ the antipodal flag of E_{2f}, on the face f'' of E_{2f} antipodal to f' is obtained; finally applying $\underline{d}$ the adjacent flag in the dodecahedron E_{3f} which meets E_{2f} across f'' is recovered. This means that, if $\underline{d}$ is replaced with $\underline{e}$ as a monodromy permutation and leaving $\underline{a}$, $\underline{b}$ and $\underline{c}$ unchanged, then the 120 dodecahedra E of $\mathcal{E}$ can be reassembled with this new rule for adjacency: each face f of E is identified, not with the corresponding face f of E_f, but with the face f'' of E_{3f}, three steps rather than one around a bracelet E, E_f, E_{2f}, $E_{3f}, \ldots, E_{-f}$. Combinatorially, this is possible, but since $\underline{ce}$ has order 5 these new identifications would require each edge to be surrounded by five dodecahedra, rather than three. Therefore spherical dodecahedra E in $\mathcal{E}$ need to be replaced with hyperbolic dodecahedra D, so that they can have dihedral angles $2\pi/5$. Doing this, and making the corresponding identifications of each face f of D with the face f'' of D_{3f}, the tessellation $\mathcal{D}/K$ of $\mathbf{H}^3/K$ by 120 dodecahedra corresponding to the normal subgroup K of Γ is obtained. This has symmetry group $\Gamma/K \cong \Delta$, and its quotient by the center $N/K \cong C_2$ is the tessellation $\mathcal{D}/N$ studied earlier. Alternatively, $\mathcal{D}/N$ can be obtained directly by applying the above reassembling process to the 60 dodecahedra in the antipodal quotient of $\mathcal{E}$ discussed in the preceding section. (This reassembling process can be seen as a 3-dimensional analogue of the operations on maps considered in dimension 2 by Wilson [Wil] and Jones and Thornton [JT]).

5.10 The tessellation $\mathcal{D}/K$

The tessellation $\mathcal{D}/K$ is an unbranched double covering of $\mathcal{D}/N$, each dodecahedral cell in the latter lifting to two in the former. The local properties of these two tessellations are similar, but globally they differ. As in the case of $\mathcal{D}/N$, but with $SL_2(5)$ and Δ^+ replacing A_5 and $A_5 \times A_5$, the right-handed flags and the dodecahedra of $\mathcal{D}/K$ can be labeled with the elements of Δ^+ and $SL_2(5)$ respectively, so that a flag $\pm(M_1, M_2)$ is in the dodecahedron labeled with the (well-defined) element $M_1M_2^{-1}$ of $SL_2(5)$. As in the case of $\mathcal{E}$, the diagonal subgroup $H \cong A_5$

of the central product Δ^+ stabilises a dodecahedron D_1, and its action on the 120 dodecahedra corresponds to the action of $\operatorname{Inn} SL_2(5) \cong L_2(5) \cong A_5$ on the elements of $SL_2(5)$. In particular, the orbits of H on the dodecahedra correspond to the conjugacy classes of $SL_2(5)$, as described in § 5.9. It follows that each orbit $\mathcal{O}$ of H on dodecahedra in $\mathcal{D}/N$ lifts to two orbits of size $|\mathcal{O}|$ on dodecahedra in $\mathcal{D}/K$, and that the labeling of these dodecahedra is compatible with the lifting of elements of A_5 to $SL_2(5)$.

By considering monodromy permutations of flags, as in the case of $\mathcal{D}/N$, the possible adjacencies between pairs of dodecahedra D_P and D_Q labeled with matrices $P, Q \in SL_2(5)$ can be determined: starting with the neighbours of the dodecahedron D_I labeled with the identity matrix $I = I_2$, apply automorphisms to consider general dodecahedra. For instance, the element γ^{-1} of Γ^+ has image $\pm(\left(\begin{smallmatrix}2&2\\2&0\end{smallmatrix}\right), \left(\begin{smallmatrix}2&3\\3&0\end{smallmatrix}\right))$ in Δ^+, so D_I meets D_M across a common face where

$$M = \begin{pmatrix}2 & 2\\ 2 & 0\end{pmatrix}\begin{pmatrix}2 & 3\\ 3 & 0\end{pmatrix}^{-1} = -A^2 \in SL_2(5),$$

and it follows from this that D_P meets D_Q across a common face if and only if $P^{-1}Q$ is in the conjugacy class $-\hat{C}_5$ of $SL_2(5)$ containing $-A^2$. In fact, whereas the element $(abc)^5 d$ of Γ^+, which sends flags one step around a bracelet of dodecahedra, has an image of order 5 in $A_5 \times A_5$, its image $\pm(-A^{-1}, A)$ in Δ^+ has order 10: its fifth power is mapped to the central involution $\pm(-I_2, I_2)$, which is not in the image of Γ_0, so each bracelet of five dodecahedra in $\mathcal{D}/N$ lifts to a bracelet of ten dodecahedra in $\mathcal{D}/K$. (This is obtained, as in the preceding section, from a bracelet of ten dodecahedra in $\mathcal{E}$, with dodecahedra three steps apart in $\mathcal{E}$ giving rise to adjacent dodecahedra in $\mathcal{D}/N$.) On the other hand, the element $(bcd)^5(abc)^5$ of Γ^+, which sends flags one step around a necklace of dodecahedra, has an image $\pm(B^{-1}, B)$ of order 3 in Δ^+; it follows from this that D_P and D_Q meet antipodaly across a vertex if and only if $P^{-1}Q \in \hat{C}_3$, and that each necklace of length 3 in $\mathcal{D}/N$ lifts to a pair of necklaces of length 3 in $\mathcal{D}/K$. The dodecahedron $D_{(123)}$ in $\mathcal{D}/N$ meets D_1 antipodaly at v_1 and also along three edges; this lifts to two dodecahedra in $\mathcal{D}/K$, namely D_B meeting D_I antipodaly at a vertex, and D_{-B} meeting it along three edges. Thus D_P and D_Q meet along an edge if and only if $P^{-1}Q \in -\hat{C}_3$, so it is no longer possible in $\mathcal{D}/K$ for a pair of dodecahedra to meet in both ways, as happened in $\mathcal{D}/N$.

5.11 Quotients of Γ^+ isomorphic to $L_2(19)$

After $L_2(4) \cong L_2(5) \cong A_5$, the next smallest epimorphic image of Γ^+ is $L_2(19)$ (in Theorem 3.1 of § 3.6 it was shown that no epimorphisms onto $L_2(7)$, $L_2(8)$, $L_2(9)$, $L_2(11)$, $L_2(13)$ or $L_2(17)$). As is shown in section § 3.4, there are three normal subgroups with quotient $L_2(19)$. In this section, these three subgroups K_i will be considered in more detail, together with their associated tessellations $\mathbb{H}^3/K_i$.

5.11.1 The three normal quotients and their extensions

Recall in § 3.4 of Chapter 3 that maps $\Gamma^+ \twoheadrightarrow L_2(q)$ were characterised by matrices with entries in terms of e, f, t, u, w and x, where $e^2 + f^2 = -2 - t$, $u, t = \frac{-1\pm\sqrt{5}}{2}$. When $p = 19$ take $u = t$, $e = 4$ and $f = -t$, giving epimorphisms $\Gamma^+ \to L = L_2(19)$ defined by

$$\alpha \mapsto \begin{pmatrix} t & 12 \\ 11 & 0 \end{pmatrix}, \quad \beta \mapsto \begin{pmatrix} 12 & 0 \\ -t & 8 \end{pmatrix}; \quad \gamma \mapsto \begin{pmatrix} w & x \\ y & z \end{pmatrix},$$

where $t = t_i = 4, -5$ for $i = 1, 2$. Here $y = x$, $w = ((e-1)u - 2fx)/2e = (5x-2)t$, $z = u - w = (3-5x)t$ and x is a root of $(e^2+f^2)x^2 + ftx + e^2 - \frac{1}{4}(e^2-1)t^2 = 0$, that is, $(t+2)x^2 - (t-1)x + (t+2) = 0$ or equivalently $x^2 + (2-3t)x + 1 = 0$. When $t = -5$ this quadratic equation becomes $x^2 - 2x + 1 = 0$, with a single root $x = 1$, so there is an epimorphism $\theta_0 : \Gamma^+ \to L$ defined by

$$\gamma \mapsto \begin{pmatrix} 4 & 1 \\ 1 & 10 \end{pmatrix}.$$

When $t = 4$ we have $x^2 + 9x + 1 = 0$, which has two roots $x = -4, -5$, so there are epimorphisms $\theta_1, \theta_2 : \Gamma^+ \to L$ defined by

$$\gamma \mapsto \begin{pmatrix} 7 & -4 \\ -4 & -3 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 6 & -5 \\ -5 & -2 \end{pmatrix}$$

respectively. Therefore, there are three normal subgroups $K_i = \ker\theta_i$ of Γ^+ with $\Gamma^+/K_i \cong L_2(19)$ for $i = 0, 1, 2$.

The epimorphism $\theta_0 : \Gamma^+ \to L$ can be extended to an epimorphism $\theta_0 : \Gamma \to L$

by sending

$$a \mapsto \begin{pmatrix} 1 & 6 \\ 6 & -1 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 4 & -7 \\ -3 & -4 \end{pmatrix}, \quad c \mapsto \begin{pmatrix} -6 & 1 \\ 1 & 6 \end{pmatrix}, \quad d \mapsto \begin{pmatrix} -4 & 4 \\ -9 & 4 \end{pmatrix},$$

so the kernel is a normal subgroup N_0 of Γ with $\Gamma/N_0 \cong L$. This shows that $K_0 = N_0^+ = N_0 \cap \Gamma^+$ is normal in Γ, with $\Gamma/K_0 \cong L \times C_2$. This is because N_0 contains the conjugacy classes with representatives $(abc)^5$ and $(bcd)^5$. These representatives are central in $\langle a, b, c\rangle$ and $\langle b, c, d\rangle$, respectively. The two epimorphisms θ_1 and θ_2, on the other hand, cannot be extended to Γ: when $t = 4$ the only possible image of b in $PGL_2(19)$, inverting the images of α and β, is $\left(\begin{smallmatrix} 2 & 2 \\ 7 & 17 \end{smallmatrix}\right)$, but then $d = b\beta\gamma$ does not map to an involution; thus K_1 and K_2 are not normal in Γ. Letting d act on the generators α, β and γ_1 by conjugation,

If N is any normal subgroup of Γ with quotient L, then N^+ is a normal subgroup of Γ with $\Gamma/N^+ \cong L \times C_2$ and $\Gamma^+/N^+ \cong L$. This last isomorphism implies that $N^+ = K_i$ for $i = 0, 1$ or 2, and since K_1 and K_2 are not normal in Γ, then $N^+ = K_0$. Thus N contains K_0 with index 2, so it maps onto a normal subgroup of order 2 in $\Gamma/K_0 \cong L \times C_2$; the only such subgroup is the direct factor $N_0/K_0 \cong C_2$, so N is its inverse image N_0 in Γ. Thus N_0 is the only normal subgroup of Γ with quotient L. Using GAP, a presentation for N_0 and K_0 was recovered. The abelianisations $N_0/N_0' \cong \mathbb{Z}_2^{57}$ and $K_0/K_0' \cong \mathbb{Z}^{53}$ were also computed. These give the first homology of the associated manifolds $\mathbb{H}^3/N_0$ and $\mathbb{H}^3/K_0$, respectively.

The involution $\left(\begin{smallmatrix} 9 & 8 \\ 9 & 10 \end{smallmatrix}\right)$ in $PGL_2(19)\backslash L$ induces an automorphism of L transposing the images of a and d, and also those of b and c, so K_0 is normal in Ω, with $\Omega/K_0 \cong PGL_2(19) \times C_2$. This is the isometry group of the manifold $\mathbb{H}^3/K_0$, with orientation-preserving subgroup $\Omega^+/K_0 \cong PGL_2(19)$; the tessellation $\mathcal{K}_0 = \mathcal{D}/K_0$ which it carries is reflexible and self-dual, with symmetry group $\Gamma/K_0 \cong L \times C_2$ and orientation-preserving subgroup $\Gamma^+/K_0 \cong L$. As already noted, the direct factor C_2 is generated by the common image of the involutions $(abc)^5$ and $(bcd)^5$ in $\Gamma\backslash\Gamma^+$, which applies the antipodal symmetry to each dodecahedral cell; factoring this out, it can be seen that the orbifold $\mathbb{H}^3/N_0$ is tessellated by the antipodal quotients of these dodecahedra.

The effect of the half-turn $r \in \Omega^+$ on Γ^+ is to transpose α and γ^{-1}, and to invert β. Conjugation by the element $\left(\begin{smallmatrix} 10 & 10 \\ 7 & 9 \end{smallmatrix}\right) \in L$ has the same effect on the images of these elements under θ_1, so K_1 is normalised by r and is thus normal in Ω^+ with $\Omega^+/K_1 \cong L \times C_2$. This is the isometry group of the manifold $\mathbb{H}^3/K_1$, consisting entirely of orientation-preserving transformations. The subgroup $\Gamma^+/K_1 \cong L$ is the symmetry group of the tessellation $\mathcal{K}_1 = \mathcal{D}/K_1$, which is chiral and self-dual. The same conclusions apply to K_2 and its quotient manifold and tessellation, with the element $\left(\begin{smallmatrix} 7 & 7 \\ 1 & 12 \end{smallmatrix}\right) \in L$ imitating r in this case. Since K_1 and K_2 are conjugate under elements of $\Gamma \setminus \Gamma^+$, their corresponding quotient manifolds and tessellations are mirror images of each other.

5.11.2 Factorisation in $L_2(19)$

The following method allows the structure of each of the three tessellations $\mathcal{K}_i$ associated with $L = L_2(19)$ to be determined, by identifying which dodecahedron contains the flag labeled by any given element of L. The 57 dodecahedra may be identified with the cosets gG in L of the subgroup $G \cong A_5$, which is the stabiliser of one dodecahedron D_0 in the monodromy representation of L on flags. Let B denote a Borel subgroup of L, that is, the normaliser $C_{19} : C_9$ of a Sylow 19-subgroup of L. Then $|B \cap G|$ divides $\gcd(171, 60) = 3$, so B has an orbit of length divisible by $|B|/3 = 57$ on the dodecahedra, and therefore acts transitively on them. Thus $L = BG$ and $C := B \cap G \cong C_3$, so each element $h \in L$ has the form $h = bg$ where $b \in B$ and $g \in G$. A second factorisation $h = b'g'$, for $b' \in B$ and $g' \in G$, exists if and only if $b' = bc$ and $g' = c^{-1}g$ where $c \in C$. Therefore the cosets bC of C in B can be used to represent the cosets bG of G in L, and hence to represent the dodecahedra in $\mathcal{K}_i$. (Unfortunately, there is no subgroup of L which acts regularly on the dodecahedra and whose elements could therefore provide coset representatives for G in L.)

In order to locate flags within specific dodecahedra, the coset bG of G which contains a given element $h \in L$ needs to be determined. That is, h needs to be factorised as a product $h = bg$ where $b \in B$ and $g \in G$. B can be assumed to be the stabiliser of ∞ in the natural representation of L as a group of Möbius transformations of the projective line $P^1(19) = F_{19} \cup \{\infty\}$, so that g maps ∞ to the same point $p \in P^1(19)$ as h does. A list of 20 elements $g_p \in G$, each sending ∞ to a different point p, is constructed (these will be coset representatives for C in G, so there are three arbitrary choices for each p). Given any $h \in L$, write it as the image under $\theta_i : \Gamma^+ \to L$ of a word w in the generators α, β and γ of Γ^+, and compute the image p of ∞ under h by composing the Möbius transformations corresponding to the generators appearing in w. Then take g to be the corresponding element $g_p \in G$ from the list, and write $b = hg^{-1} \in B$, so that $h = bg$ as required, and the flag corresponding to h lies in the dodecahedron corresponding to the coset bC.

5.11.3 A note on the above method

Factorisation in $L_2(19)$ has no nice extension to a more general case $L_2(q)$. For $q = 19$ the subgroup $B = N_{L_2(19)}(C_{19})$, where C_{19} is the abelian subgroup stabilising the point ∞ in the action of $L_2(19)$ on the projective line $\mathbb{F}_{19} \cup \infty$, was used. This subgroup has index $q+1 = 20$ in $L_2(19)$. For all but finitely many q, $q+1 > |G| = 60$, so the decomposition $L = BG$ cannot arise for large q.

5.11.4 Structure of the quotient tessellations for $L_2(19)$

In each of the cases $i = 0, 1$ or 2, the group $L = L_2(19)$ permutes the $|L|/|A_5| = 57$ dodecahedra in $\mathcal{K}_i = \mathcal{D}/K_i$ transitively, with each dodecahedron stabilised by a subgroup $G \cong A_5$. In order to understand how these dodecahedra are joined together, more information is needed on how the stabiliser of one of the dodecahedra permutes the others. Equivalently, the suborbits for the action of L on the cosets of a subgroup $G \cong A_5$ are needed. To do this, a result due to Jones and Zvonkin [JZ] is used:

Lemma 5.11.1 (Jones-Zvonkin) *If $\mathcal{G}$ and $\mathcal{H}$ are finite conjugacy classes of subgroups of a group S, then the number ν of groups $G \in \mathcal{G}$ containing a particular $H \in \mathcal{H}$ is given by*

$$\nu = |\tilde{H} : H| \sum_{i=1}^{t} \frac{1}{m_i}$$

with $\tilde{H} = N_S(H)$, and the subgroups $H \in \mathcal{H}$ contained in G form t conjugacy classes under the action of $\tilde{G} = N_S(G)$, with the groups H in the i-th class satisfying $|N_{\tilde{G}}(H) : H| = m_i$.

The proof of the Lemma in [JZ] shows that the ν groups $G \in \mathcal{G}$ containing H form t orbits of lengths $|\tilde{H} : H|/m_i$ under the action of $\tilde{H}$. In particular, if all subgroups $H \in \mathcal{H}$ contained in G are conjugate in $\tilde{G}$, then $t = 1$ and $\nu = |\tilde{H} : H|/m$ where $m = m_1 = |N_{\tilde{G}}(H) : H|$.

This can be applied to find the suborbit-lengths for the action of $S = L$ on the cosets of a subgroup $G \cong A_5$. There are two conjugacy classes of such subgroups, which fuse in $\operatorname{Aut} G = PGL_2(19)$, so $\mathcal{G}$ can be taken to be either of these classes. They are maximal subgroups, so $\tilde{G} = G$ and the action of S can be identified with its action by conjugation on $\mathcal{G}$. This action, which is primitive, has degree $|S : G| = \frac{1}{2}q(q^2 - 1)/60 = 3420/60 = 57$.

Let H to be an arbitrary subgroup of G. Then the number of points fixed by H is the number of conjugates of G in S that contain H. By comparing this with the corresponding numbers for supergroups of H in G, the number of suborbits of G with H as a point stabiliser can be determined.

1. If $H \cong A_4$ then $\tilde{H} = H$, $t = 1$ and $m_1 = 1$, so the Lemma gives $\nu = 1$, that is, H is contained in no other conjugate of G. Thus there is no suborbit with A_4 stabilisers. This argument applies for all $q \equiv \pm 3 \bmod (8)$; however, if $q \equiv \pm 1 \bmod (8)$ then $\tilde{H} \cong S_4$ with $t = 1$ and $m_1 = 1$ so $\nu = 2$ giving a suborbit of length $|G : H| = 5$.

2. If $H \cong D_5$ then $\tilde{H} \cong D_{10}$, $t = 1$ and $m_1 = 1$, so the Lemma gives $\nu = 2$, and hence there is one suborbit of length 6 with D_5 stabilisers, G acting as $L_2(5)$, or equivalently on antipodal pairs of faces of a dodecahedron. This applies for all $q \equiv \pm 1 \bmod (20)$, since $\tilde{H} = D_{10}$; if $q \equiv \pm 11 \bmod (20)$ then $\tilde{H} = H$ so there are no such suborbits.

3. If $H \cong D_3$ then $\tilde{H} = H$, $t = 1$ and $m_1 = 1$, so the Lemma gives $\nu = 1$, that is, there is no suborbit with stabiliser D_3. This applies for all $q \equiv \pm 5 \bmod (12)$, since $\tilde{H} = H$; if $q \equiv \pm 1 \bmod (12)$ then $\tilde{H} \cong D_6$ giving $\nu = 2$, so there is a suborbit of length 10 with D_3 stabilisers.

4. If $H \cong C_5$ then $\tilde{H} = D_{10}$, $t = 1$ and $m_1 = 2$ since $N_{\tilde{G}}(H) = N_G(H) \cong D_5$, so $\nu = 2$. The two conjugates of G containing H also contain its supergroup D_5, as shown in (2), so there is no suborbit with C_5 stabilisers.

5. If $H \cong V_4$ then $\tilde{H} \cong A_4$, $t = 1$ and $m_1 = 3$ since $N_{\tilde{G}}(H) = N_G(H) \cong A_4$, so $\nu = 1$ and there is no suborbit with V_4 stabilisers. More generally, if $q \equiv \pm 1 \mod 8$ then $N_S(H) = S_4$ so $\nu = 2$, so there is get 1 suborbit of length 15. If $q \equiv \pm 3 \mod 8$ then $N_S(H) = A_4$ and $\nu = 1$, there is no suborbit.

6. If $H \cong C_3$ then $\tilde{H} \cong D_9$, $t = 1$ and $m_1 = 2$ since $N_{\tilde{G}}(H) = N_G(H) \cong D_3$, so $\nu = 3$. The two conjugates other than G itself are not fixed by any supergroup of H in G, so there is one suborbit of length 20 with C_3 stabilisers; G acts as on the vertices of a dodecahedron, each stabiliser fixing two points. More generally, $\tilde{H} = D_{q\mp 1}$ depending on whether $q \equiv \pm 1 \mod 3$. Also, $N_{\tilde{G}}(H) = N_G(H) = D_3$, $t = 1$ so $m_1 = 2$. Then $\nu = \frac{q\mp 1}{6}$. If $q \equiv \pm 5 \mod 12$ then there is one suborbit of length 2 with C_3 stabilizers. If $q \equiv \pm 1 \mod 12$ then there are two suborbits of length 10 with D_3 stabilizers.

7. If $H \cong C_2$ then $\tilde{H} \cong D_{10}$, $t = 1$ and $m_1 = 2$ since $N_{\tilde{G}}(H) = N_G(H) \cong V_4$, so $\nu = 5$, giving four conjugates other than G fixed by H. Now there are 15 subgroups $H \cong C_2$ in G, each fixing four points other than G, giving $15 \times 4 = 60$ fixed points. There are six subgroups D_5 in G, each containing five subgroups C_2, so there are $6 \times 5 = 30$ pairs $D_5 > C_2$; since the 15 subgroups C_2 are all conjugate in G, each is contained in $30/15 = 2$ subgroups D_5 of G. It follows that H fixes two points in the orbit of length 6 given in (2), so it has $4 - 2 = 2$ points other than G outside that orbit. There are no other supergroups of H in G which can arise as stabilisers for these points, so G has an orbit of length $|G : H| = 30$ with H acting as the stabiliser of two points; this is equivalent to the action of G on edges of a dodecahedron.

8. There cannot be $H = 1$ since then $|G : H| = 60 > 57$, so G could not have a suborbit of length 60.

5.11.5 The three tessellations

To summarise, it was shown that G splits $\mathcal{G}$ into orbits of lengths $1, 6, 20$ and 30 with stabilisers G, D_5, C_3 and C_2. Since $1+6+20+30 = 57$ is the number of cells in the tessellations, this forms a complete list. This will now be interpreted in terms of the tessellations $\mathcal{K} = \mathcal{K}_i = \mathcal{D}/K_i$ for $i = 0, 1, 2$. General features common to all three tessellations will be described. More detailed features which distinguish between the three tessellations require a more precise examination of how L is generated as a quotient of Γ^+ in each case.

If D_0 is a dodecahedral cell in $\mathcal{K}$, then there are at most 12 dodecahedra which meet D_0 across common faces; this set of dodecahedra must be invariant under the stabiliser $G \cong A_5$ of D_0 in L, so it is a union of orbits of G. None of these dodecahedra can be equal to D_0, otherwise regularity and connectedness would imply that all dodecahedra in $\mathcal{K}$ would be equal to each other. Since there is only one orbit of G of length between 1 and 20, it follows that these dodecahedra form the orbit of length 6, each meeting D_0 twice; since this pair of common faces must be invariant under the stabiliser D_5 of the neighbouring dodecahedron, they must be an antipodal pair $\pm f$. Denoting this neighbour by $D_{\pm f}$, and applying the same argument to it, it can be seen that $D_{\pm f}$ meets D_0 across its faces $\pm f$, so these two dodecahedra form a bracelet of length 2. Note that this gives a $3\pi/5$ "twist" to the tessellation.

The tessellation from θ_0

For a dodecahedron D_0, let e and $-e$ be a pair of antipodal edges stabilized setwise by an element of $\langle \alpha , \beta \rangle$ of order two. Let ϕ be a flag in D_0 sharing an edge with e, and let v be the vertex common to them both. Under the epimorphism θ_0 the antipodal flag at the vertex v, $\phi\gamma^2\beta^2\gamma^2\beta^2\gamma\beta^2$, is also the flag $\phi\alpha^3\beta\alpha^4\beta\alpha^2$ at the antipodal vertex $-v$ of D_0. So the dodecahedron antipodal to D_0 across v is D_0 and v coincides with $-v$ under the identification, as illustrated in Figure 5.13.

If D_i, D_j meet D_0 only along an edge of D_0, then they also meet D_0 across the edge $-e$ as shown in Figure 5.14. This contributes 30 dodecahedra to the tessellation, forming a single orbit of G with C_2 stabilizers. So far 37 dodecahedra have been counted, leaving 20 as yet unaccounted for. These form a single orbit of dodecahedra disjoint from D_0. These 20 dodecahedra come in ten pairs $\{D_v, D_{-v}\}$, each pair fixed setwise by a rotation of order 3 fixing two antipodal vertices of D_0. Let v, $-v$ be such a pair of vertices in D_0, and let D_f meet D_0 along a face containing v. Then D_f has two additional faces f_1 and f_2 which also contain v. These faces are not shared with D_0. Let e_{f_i} be the edge in f_i opposite v. Then D_v

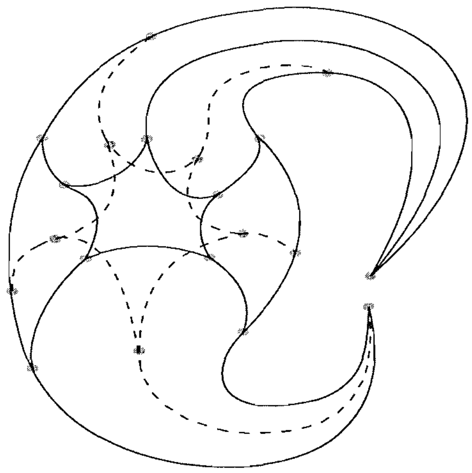


Figure 5.13: Vertex identification for $\mathcal{M}_0$

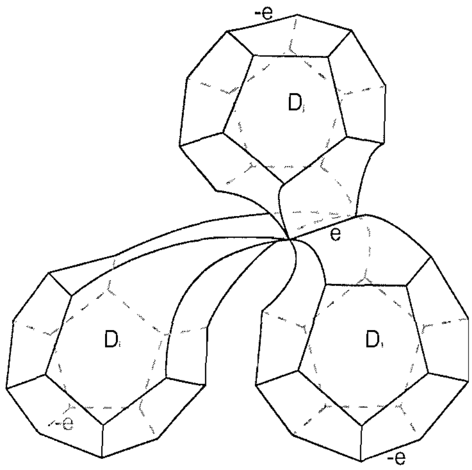


Figure 5.14: Edge pairings for $\mathcal{M}_0$

meets one of these edges and D_{-v} meets the other. Of the 6 dodecahedra meeting D_v across a face, three of them are dodecahedra sharing an edge with D_0 and three are vertex dodecahedra disjoint from D_0. Since the automorphisms β preserves the properties of the tessellation, this forces the three vertex dodecahedra to share a common vertex with D_v.

The tessellations from θ_1 and θ_2

For the epimorphisms θ_1 and θ_2, D_0 meets another dodecahedron D_v at v so that D_0 and D_v correspond to antipodal faces of the icosahedral vertex figure $I = \{3, 5\}$ at v. Since D_0 has 20 vertices v these dodecahedra D_v form an orbit of G of length 20, with D_v and D_{-v} having the same stabiliser C_3 where $-v$ denotes the vertex of D_0 antipodal to v. Since there are no further orbits with this stabiliser, D_v and D_{-v} meet at a common vertex, antipodal in each to v and $-v$ respectively, so that D_0, D_v and D_{-v} form a necklace of length 3.

This accounts for $1 + 6 + 20 = 27$ of the 57 dodecahedra, so 30 dodecahedra remain, forming a single orbit of G with C_2 stabilisers. Now D_0 has 30 edges e, each surrounded by five dodecahedra; one of these is D_0, and two are of the form D_f for the two faces f of D_0 incident with e, so two others meet D_0 along e. This gives $30 \times 2 = 60$ such incidences, two for each edge, so each of the 30 remaining dodecahedra meets D_0 along a pair of edges, which must be invariant under the stabiliser $H \cong C_2$. These incidences can be used to distinguish between the two tessellations: if p is a Petrie path on D_0 containing both e and $-e$, and let D_{e_1} and D_{e_2} be the two dodecahedra incident with e. Then the D_{e_i} meet D_0 along the edges adjacent to $-e$ in p. In the case of θ_1, p is a right-left-right Petrie path; in the case of θ_2, p is a left-right-left Petrie path. Figure 5.15 shows the edge identification for the manifold $\mathcal{M}_2$ associated with the kernel of the map $\theta_2 : \Gamma^+ \twoheadrightarrow L_2(19)$.

This accounts for all the dodecahedra, although not all incidences with D_0: at each of the 20 vertices v, the vertex figure is an icosahedron whose 20 faces represent the corners of dodecahedra meeting D_0 at v: one of these dodecahedra is D_0, three further dodecahedra have the form D_f for faces f incident with v, while six dodecahedra meet D_0 along the three edges incident with v. A further dodecahedron is D_v. There are nine remaining dodecahedra unaccounted for. Of these nine, six are antipodal dodecahedra, three of which are antipodal to the vertices bounding the edges containing v. Figure 5.16 illustrates this. The three dodecahedra meeting D_v across a face and which are not antipodal to D_0 meet D_0 across three of the six edges left after deleting all edges bounding a face containing v or $-v$. The other three edges meet dodecahedra which are face adjacent to D_{-v}. The three edges are also permuted regularly by the automorphism action of β.

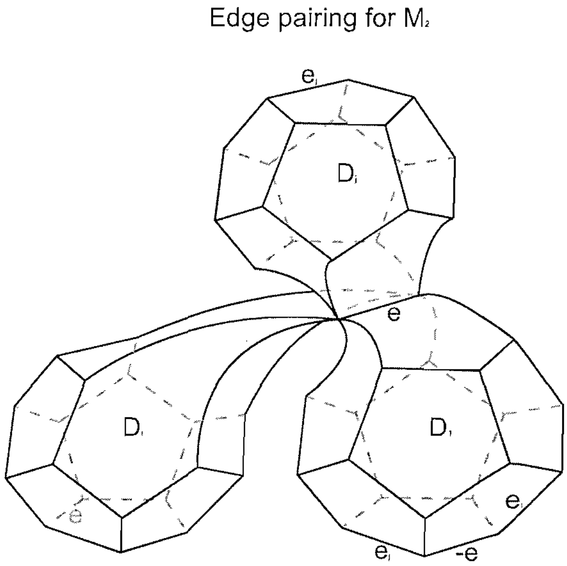


Figure 5.15: Edge pairings for $\mathcal{M}_2$

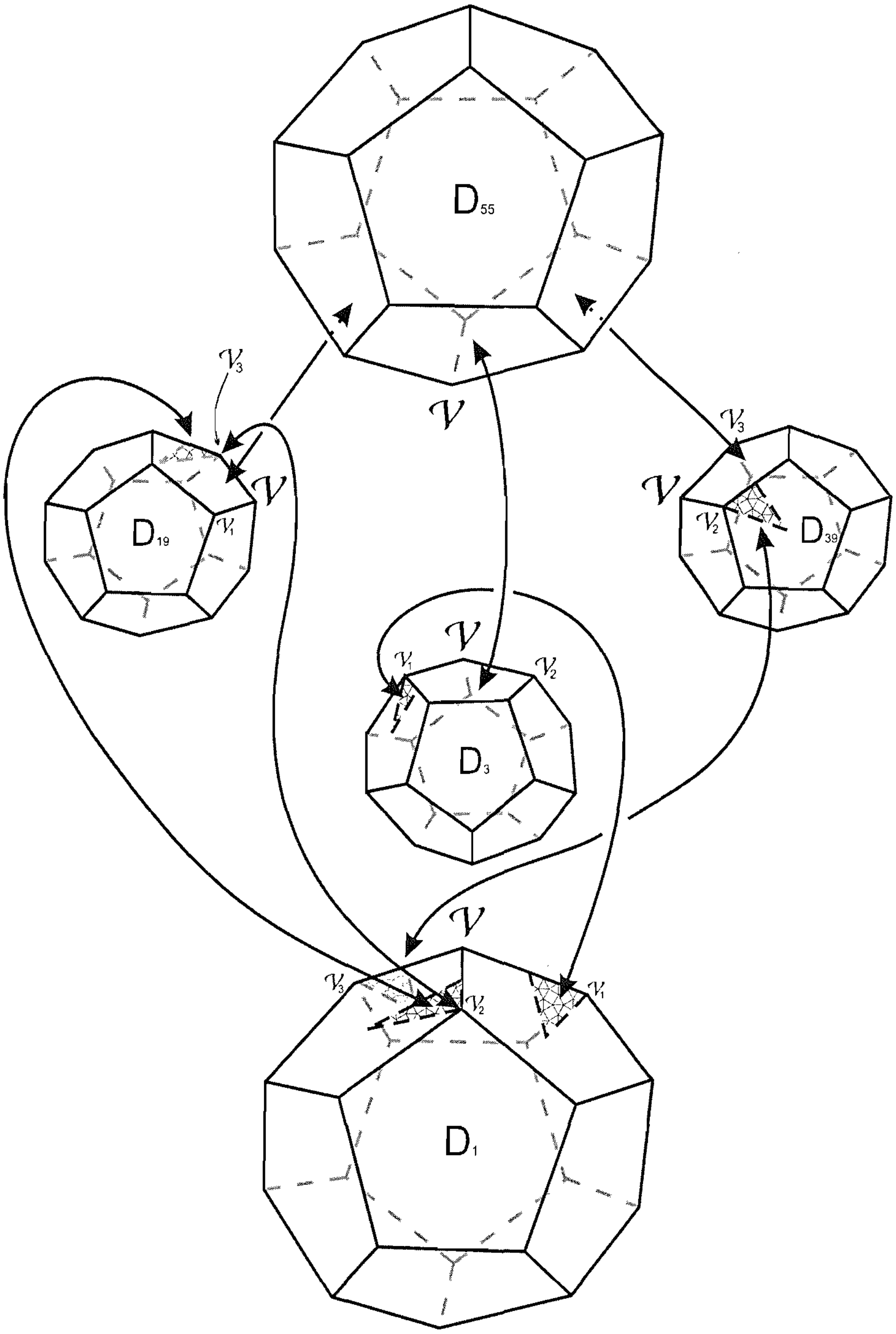


Figure 5.16: Vertex identification for $\mathcal{M}_1$

Chapter 6

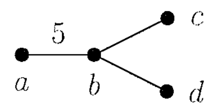
Manifolds from the other groups

In this chapter minimal index torsion free normal subgroups and minimal index torsion free subgroups of the eight Lannér groups T_1, T_2, T_3, T_5, T_6, T_7, T_8 and T_9, introduced in Table 2.3 of § 2.2, will be constructed. Manifolds arising from the group T_2 have been extensively studied by Jones and Mednykh [JM]. Their main results are included here for completeness only. The group T_4 is studied extensively in Chapter 5.

Let $\mathcal{L}(G)$ be the lowest common multiple of the orders of all finite subgroups of an arbitrary group G and let $\mathcal{M}(G)$ be the minimum index of a torsion-free subgroup of G. $\mathcal{L}(G)$ will divide $\mathcal{M}(G)$ since, for any torsion free subgroup H of G, the action of a representative g of a conjugacy class $\mathcal{C}(g)$ of torsion elements on the cosets of H in G fails to fix any coset. Hence $|g|$ divides the index of H in G. It is known that, for Fuchsian groups G, $\mathcal{M}(G)/\mathcal{L}(G)$ is either 1 or 2 ([EEK]). Jones and Reid [JR] have demonstrated that, for Kleinian groups G there is no global bound on the ratio $\mathcal{M}(G)$ $/\mathcal{L}(G)$ by constructing a sequence of groups Γ_k with $\mathcal{M}(\Gamma_k)/\mathcal{L}(\Gamma_k) > k$, for any $k \in \mathbb{N}$. It is interesting to note that, by contrast, several of the orientation preserving subgroups Γ^+ of the Lannér groups Γ have a minimal index torsion free normal subgroup with index equal to $\mathcal{L}(\Gamma^+)$.

6.1 $\Gamma = T_1[2,2,3;3,5,2]$

Recall that $T_1[2,2,3;3,5,2]$ is the Coxeter group with Coxeter diagram



where a, b, c and d are Coxeter generators of Γ. Let $\alpha = ab$, $\beta = bc$ and $\gamma = cd$. Then the orientation preserving subgroup Γ^+ has the following presentation:

$$\Gamma^+ = \langle \alpha, \beta, \gamma \mid \alpha^5 = \beta^3 = \gamma^2 = (\alpha\beta)^2 = (\beta\gamma)^3 = (\alpha\beta\gamma)^2 = 1 \rangle$$

6.1.1 Torsion in Γ and in Γ^+

Conjugacy classes of torsion elements in Γ are listed in Table 6.1. The restriction of Table 6.1 to representatives of conjugacy classes lying in Γ^+ is given in Table 6.2

Order	Representative
Order 2	$a, ac, ad, cd, acd, (abc)^5, (abd)^5$
Order 3	bc
Order 4	cbd
Order 5	$ab, (ab)^2$
Order 6	$c(ab)^2, d(ab)^2$
Order 10	$abc, (abc)^3, abd, (abd)^3$

Table 6.1: Conjugacy class representatives for elements of finite order in Γ

Order	Representative
Order 2	ac, ad, cd
Order 3	bc
Order 5	$ab, (ab)^2$

Table 6.2: Conjugacy class representatives for elements of finite order in Γ^+

6.1.2 Minimal index torsion free subgroups

Lemma 6.1.1 *Γ^+ has a unique normal subgroup K with $\Gamma^+/K \cong A_5$. The kernel of the map contains the conjugacy class $\mathcal{C}(\gamma)$ of torsion elements of Γ^+.*

Proof: Adjoining the relation $\gamma = 1$ to the presentation for Γ^+ gives a group with presentation $\langle \alpha, \beta, \gamma \mid \alpha^5 = \beta^3 = (\alpha\beta)^2 = 1 \rangle$, so $\Gamma^+/\langle\langle \gamma \rangle\rangle_{\Gamma^+} \cong A_5$. Let $\alpha \mapsto (1,3,5.2,4) = \overline{\alpha}$ and $\beta \mapsto (1,2,3) = \overline{\beta}$. Since A_5 contains no elements g of order 2 which satisfy the conditions $(\overline{\beta}g)^3 = 1$ and $(\overline{\alpha}\overline{\beta}g)^2 = 1$, this completes the proof. ■

Corollary 6.1.1 *There exists a minimal index torsion free subgroup of Γ^+ of index 120.*

Proof: This result was obtained computationally. The unique normal subgroup K of index 60 in Γ^+ has 2047 conjugacy classes of subgroups of index 2. 1024 of the class representatives avoid the conjugacy class $\mathcal{C}(\gamma)$. ■

The representatives of subgroups avoiding $\mathcal{C}(\gamma)$ can be further partitioned by their abelian invariants - this partitions the set of representatives into 10 classes, four of which have positive first Betti number. The normalisers of the subgroups of K have indices in $\{6, 10, 12, 15, 20, 30, 60\}$. Of the 1024 conjugacy class representatives avoiding $\mathcal{C}(\gamma)$, 8 have normalizer of index 6 in Γ^+.

Lemma 6.1.2 *There are no normal subgroups N of Γ^+ with $\Gamma^+/N \cong A_6$*

Proof: Let $\langle \alpha, \beta \rangle \hookrightarrow A_5 \subset A_6$ be an embedding of A_5 in A_6. Up to automorphisms of A_6, this embedding is unique. Let $\alpha \mapsto (1,3,5,2,4)$, $\beta \mapsto (1,2,3)$ and $\gamma \mapsto g$, so that α and β generate $A_5 < A_6$. A_6 has 45 elements g of order 2, 30 of which lie in $A_6 \backslash A_5$. Of these 30 elements, only $(1,2)(3,6)$, $(1,3)(2,6)$ and $(1,6)(2,3)$ satisfy $|(1,2,3)g| = 3$. However, $|\alpha\beta(1,2)(3,6)| = 3$, $|\alpha\beta((1,3)(2,6)| = 3$ and $|\alpha\beta(1,6)(2,3)| = 4$. So $|\alpha\beta g| \neq 2$, and the proof is complete. ■

Lemma 6.1.3

1) *There are precisely two torsion free normal subgroups K_1, K_2 whose factor group $\Gamma^+/K_i \equiv L_2(11)$.*

2) *These are the smallest index torsion free normal subgroups.*

3) *Their intersection has index 660^2 in Γ^+ and is normal in $\Omega = \langle \Gamma, \tau \rangle$, the order 2 extension of Γ in Isom($\mathbb{H}^3$).*

Proof:

1) In chapter 4 sections § 4.1.1 and § 4.2 it was shown that homomorphisms $\Gamma^+ \to L_2(11)$ can be characterised by maps of the form

$$\alpha \mapsto \frac{1}{2}\begin{pmatrix} t-f & e+1 \\ -1+e & t+f \end{pmatrix}, \beta \mapsto \frac{1}{2}\begin{pmatrix} e+1 & t+f \\ f-t & 1-e \end{pmatrix} \text{ and } \gamma \mapsto \begin{pmatrix} w & x \\ x & -w \end{pmatrix}$$

where $t = \frac{-1\pm\sqrt{5}}{2}$, $e^2 + f^2 = -2 - t$ and $ew = u - fx$, where $u = \pm 1$ and $\sqrt{5} = 4$. If $t = \frac{-1+\sqrt{5}}{2}$ then $t = 7$ mod 11, so $-2 - t = -9 \equiv 2$ mod 11. Setting $e = f = 1$ gives $e^2 + f^2 = 2$, as required. Then x satisfies the quadratic equation

$$2x^2 - 2ux + 2 \ = \ 0 \tag{6.1}$$

which has discriminant $D = 4e^2(1+t) = 4(8) = (4\sqrt{2})^2$. Now $\sqrt{2} \notin \mathbb{F}_{11}$, so equation 6.1 has no solution over $\mathbb{F}_{11}$. Letting $t = \frac{-1-\sqrt{5}}{2}$, gives $t = 3$, so $-2 - t- = -5 \equiv 6$ mod 11. Writing $e = 4$, $f = 1$, so $e^2 = 5$, $f^2 = 1$ and $e^2 + f^2 = 6$, x satisfies the polynomial

$$6x^2 - 2ux + 6 = 0 \tag{6.2}$$

which has discriminant $D = 4.5(1+3) = 5^2$ mod 11. Solving equation 6.2 gives $x = 2u \pm 5$, So for each value of u two solutions for x are obtained, giving four solutions in total.

Case I: $u = 1$: Then $x = 2 \pm 5 = 7$ or $-3 \equiv 8$ mod 11 and $4w = 1 - x$

- If $x = 7$ then $4w = 1 - 7 = -6$ so $w \equiv 4$ mod 11.

- If $x = 8$ then $4w = 1 - 8 = -7$ so $w \equiv 1 \bmod 11$.

Therefore
$$\gamma \mapsto \tilde{\gamma} \in \left\{ \begin{pmatrix} 4 & 7 \\ 7 & -4 \end{pmatrix}, \begin{pmatrix} 1 & 8 \\ 8 & -1 \end{pmatrix} \right\}$$

Case II: $u = -1$: Then $x = -2 \pm 5 = 3$ or $-7 \equiv 4 \bmod 11$ and $4w = -1 - x$

- If $x = 3$ then $4w = -1 - 3 = -4$ so we get $w \equiv -1 \bmod 11$.
- If $x = 4$ then $4w = -1 - 4 = -5$ so we get $w \equiv 7 \bmod 11$.

Therefore
$$\gamma \mapsto \tilde{\gamma} \in \left\{ \begin{pmatrix} -1 & 3 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 4 \\ 4 & -7 \end{pmatrix} \right\}$$

Observe that the solutions in the case $u = -1$ are simply those from $u = 1$ multiplied by -1 and so are equivalent in $L_2(11)$. Now $\mathrm{Aut}(L_2(11)) = PGL_2(11)$. The action of the outer automorphism on $L_2(11)$ conjugates the two conjugacy classes of A_5 in $L_2(11)$. Since A_5 is maximal in $L_2(q)$, there is no automorphism of $L_2(11)$ which fixes A_5 and conjugates the two solutions of $\overline{\gamma}$. Thus there are two distinct kernels K_1 and K_2, corresponding to the distinct epimorphisms $\theta_1, \theta_2 : \Gamma^+ \twoheadrightarrow L_2(11)$, as required.

2) Since Γ^+ is perfect, it suffices to consider only non-abelian simple quotients. The factor group of any torsion-free normal subgroup must contain faithful images of the special subgroups $A_5 \cong \langle \alpha, \beta \rangle$ and $S_4 \cong \langle \beta, \gamma \rangle$. From lemma 6.1.1 it has already been shown that there are no torsion free normal subgroups whose factor group is A_5. Neither is there a torsion free normal subgroup whose factor group is $L_2(7)$ or $L_2(8)$, since neither $L_2(7)$ nor $L_2(8)$ contain elements of order 5. Lemma 6.1.2 proves that there is no map to A_6, and the result follows.

3) Take the product map $\theta = (\theta_1, \theta_2) : \Gamma^+ \twoheadrightarrow L_2(11) \times L_2(11)$. Then letting $\ker(\theta) = K, \Gamma^+/K \cong L_2(11) \times L_2(11)$ and $K_1 \cap K_2 \leq K$. By composition with the projections onto each factor, it is clear that $K < K_i$. Thus $K \leq K_1 \cap K_2$, so $K = K_1 \cap K_2$. The K_i are not normal in Γ: If they were then there would exist some element $g \in L_2(11)$ of order 2 such that $g\alpha$, $g\alpha\beta$ and $g\alpha\beta\gamma$ all have order 2. Referring back to section § 4.2.2 of Chapter 4, this becomes the condition $e^2 = \frac{-1}{4}$ and $f^2 = \frac{-7}{4} - t$, or $e^2 = 8$ and $f^2 = 9$. Since 8 is not

a square in $\mathbb{F}_{11}$, the K_i are not normal in Γ. So for any element $g \in \Gamma\backslash\Gamma^+$ $K_i^g \neq K_i$. Since K_1 and K_2 are the unique normal subgroups of Γ^+ with factor group $L_2(11)$, $K_i^g = K_j$, for $\{i,j\} \in \{1,2\}$ and $g \in \Gamma\backslash\Gamma^+$. Then $K_1 \cap K_2$ is normal in Γ.

Write $\Omega = \langle \Gamma, \tau \rangle$, where τ is the graph automorphism of the Coxeter diagram for Γ that fixes the generators a and b and transposes the generators c and d. Since Γ^+ is normal in Ω, and K is the unique normal subgroup of Γ^+ with factor group $L_2(11) \times L_2(11)$, K is characteristic in Γ^+ and hence normal in Ω.

■

6.2 $\Gamma = T_2[2,2,3;2,5,3]$

The small manifolds arising from this group are studied in detail by Jones and Mednykh in [JM]. Their conclusions are stated in this section for completeness only. Let Γ^+ be the orientation preserving subgroup of Γ and let Ω be the normalizer of Γ in Isom $(\mathbb{H}^3)$. Then Ω is a split extension of Γ by a cyclic group of order 2. Ω has an orientation preserving subgroup Ω^+. Both results rely on the conjecture that the minimal volume hyperbolic 3-orbifold is $\mathbb{H}^3/\Omega$, where $\Omega = \langle \Gamma, \tau \rangle$ is an index two extension of T_2. Work done by Gehring and Martin [GM1], [GM2] suggests that this group is the Kleinian group with smallest co-volume.

Theorem 6.1 *There is a unique torsion-free normal subgroup K_0 of Ω of least index ($= 2640$) in Ω. This is a subgroup of Γ^+, with $\Gamma^+/K_0 \cong L_2(11)$ and $\Omega/K_0 \cong PGL_2(11) \times C_2$. The corresponding manifold $\mathcal{M}_0 = \mathcal{H}^3/K_0$ is the smallest hyperbolic 3-manifold with a large isometry group. It is orientable, with* Iso $\mathcal{M}_0 \cong PGL_2(11) \times C_2$, *and is tessellated by 11 hyperbolic icosahedra, each meeting the 10 others across two antipodal faces. The isometries preserving this tessellation form a subgroup $\Gamma/K_0 \cong L_2(11) \times C_2$ of index 2 in* Iso $\mathcal{M}_0$, *while the remaining isometries transform the tessellation to its dual. The first integer homology group $H_1(\mathcal{M}_0)$ of $\mathcal{M}_0$ is isomorphic to $\mathbf{Z}^{10}$.*

Theorem 6.2 *There are two torsion-free normal subgroups K_1 and K_2 of Ω^+ of* *least index (= 720) in Ω^+. They are subgroups of Γ^+ which are conjugate in Ω, with $\Gamma^+/K_i \cong L_2(9) \cong A_6$ and $\Omega^+/K_i \cong PGL_2(9)$ for $i = 1, 2$. The manifolds $\mathcal{M}_i = \mathcal{H}^3/K_i$ are the smallest orientable 3-manifolds with large orientation-preserving isometry groups. They form a chiral pair, with $\mathrm{Iso}\,\mathcal{M}_i = \mathrm{Iso}^+\,\mathcal{M}_i \cong PGL_2(9)$, and they are tessellated by six hyperbolic icosahedra, each meeting the other five across a set of four faces with tetrahedral symmetry. The isometries preserving this tessellation form a subgroup $\Gamma^+/K_i \cong L_2(9)$ of index 2 in $\mathrm{Iso}\,\mathcal{M}_i$, while the remaining isometries transform the tessellation to its dual. The first integer homology group $H_1(\mathcal{M}_i)$ of $\mathcal{M}_i$ is isomorphic to $\mathbf{Z}_3^6$.*

6.3 $\Gamma = T_3[2,2,4;2,3,5]$

In this section minimal index torsion free normal subgroups and minimal index torsion free subgroups of Γ^+ will be constructed. Some details on the construction of the associated manifolds will be included. The famous tessellation of $\mathbb{H}^3$ by right-angled dodecahedra is a consequence of this group. Figure 6.1, from the archives of the Geometry Center [Geom], illustrates the structure of this tessellation. This figure clearly demonstrates the octahedral vertex figure as well as the dodecahedral cells. It will be shown that the smallest manifold arising from the action of Γ^+ on $\mathbb{H}^3$ is tessellated by 2 such dodecahedral cells, while the smallest manifold with maximal symmetry is tessellated by 22 dodecahedra.

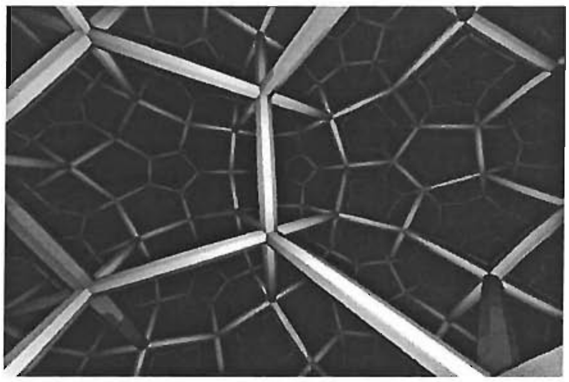
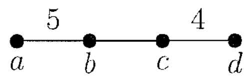


Figure 6.1: Tessellation of $\mathbb{H}^3$ by right-angled dodecahedra

Recall that $T_3[2,2,4;2,3,5]$ is the Coxeter group with Coxeter diagram



where a, b, c and d are Coxeter generators of Γ. Let $\alpha = ab$, $\beta = bc$ and $\gamma = cd$. Then the orientation preserving subgroup Γ^+ has the following presentation:

$$\Gamma^+ = \langle \alpha, \beta, \gamma \mid \alpha^5 = \beta^3 = \gamma^4 = (\alpha\beta)^2 = (\beta\gamma)^2 = (\alpha\beta\gamma)^2 = 1 \rangle$$

6.3.1 Torsion in Γ and in Γ^+

Representatives of the conjugacy classes of torsion elements in Γ are given in Table 6.3 The restriction of Table 6.3 to representatives of classes lying in Γ^+ is given in Table 6.4.

Order	Representative
Order 2	$a, d, ac, ad, (cd)^2, a(cd)^2, (bcd)^3, (abc)^5$
Order 3	bc
Order 4	$cd, b(cd)^2$
Order 5	$ab, (ab)^2$
Order 6	$c(ab)^2, bcd$
Order 10	$abc, (abc)^3, abd, (ab)^2d$

Table 6.3: Conjugacy class representatives for elements of finite order in Γ

Order	Representative
Order 2	$ac, ad, (cd)^2$
Order 3	bc
Order 4	cd
Order 5	$ab, (ab)^2$

Table 6.4: Conjugacy class representatives for elements of finite order in Γ^+

Γ^+ has 647 conjugacy classes of subgroups of index less than or equal to 60. A representative for each class was determined using the program Lowx [Lx]. For each representative, the action of Γ^+ on its cosets was calculated and the results are listed in Appendix A.3. Among the 647 induced permutation groups, 20 of the induced actions have kernels which fail to avoid torsion:

- The induced action on the representative with index 2.
- The induced action on the representative with index 5 which yields an epimorphism onto A_5.
- The induced action on the representative with index 6 which yields an epimorphism onto A_5.
- Both actions induced by the representatives with index 10 which yield an epimorphism onto A_5.
- Three of the actions induced by the nine conjugacy classes of subgroups of index 12. These correspond to the induced permutation representations A_5 and $A_5 \times C_2$ as listed in Appendix A.3
- The induced action on the representative with index 15 which yields an epimorphism onto A_5.
- Three of the actions induced by the thirteen conjugacy classes of subgroups of index 20. These correspond to the induced permutation representations A_5 and $A_5 \times C_2$ as listed in Appendix A.3.
- One of the actions induced by the twenty four conjugacy classes of subgroups of index 24. This corresponds to the induced permutation representation $A_5 \times C_2$ listed in Appendix A.3.
- Three of the actions induced by the nineteen conjugacy class of subgroups of index 30. These correspond to the induced permutation representations A_5 and $A_5 \times C_2$ as listed in Appendix A.3.
- One of the actions induced by the forty eight conjugacy class of subgroups of index 40, giving $A_5 \times C_2$
- Three of the actions induced by the three hundred and twenty three conjugacy class of subgroups of index 30.

6.3.2 Minimal index torsion free subgroups

Lemma 6.3.1 *Γ^+ has no torsion free normal subgroup N whose factor group Γ^+/N is isomorphic to $L_2(7)$ or $L_2(8)$.*

Proof: The proof is immediate since neither $L_2(7)$ nor $L_2(8)$ contain elements of order 5. ■

Lemma 6.3.2 *Γ^+ has no torsion free normal subgroup N whose factor group Γ'/N is isomorphic to A_5 or S_5.*

Proof: The case for A_5 is trivial, as A_5 contains no elements of order 4. Suppose $\Gamma^+ \twoheadrightarrow S_5$. Let $\alpha = ab$, $\beta = bc$ and $\gamma = cd$. Since $\langle \alpha, \beta \rangle \cong A_5$, we can let $\alpha \mapsto (1,3,5,2,4)$ and $\beta \mapsto (1,2,3)$ and suppose that $\gamma \mapsto g$. Now $|\{g \in S_5 | \ |g| = 4\}| = 30$. Of these elements, six of them satisfy $|\beta\gamma| = 2$: These are $(1,3,4,2)$, $(1,3,5,2)$, $(1,3,2,4)$, $(1,5,3,2)$, $(1,3,2,5)$ and $(1,4,3,2)$. Then for $g \in \{(1,3,4,2),(1,3,5,2),(1,3,2,4),(1,5,3,2)\}$, $|\alpha\beta g| = 4$ while if $g \in \{(1,3,2,5),(1,4,3,2)\}$ then $|\alpha\beta g| = 6$. ■

Lemma 6.3.3 *Let Γ^+ be the orientation preserving subgroup of $\Gamma = T_3[2,2,4;2,3,5]$. Then*

1) Γ^+ has a unique normal subgroup K with factor group $\Gamma^+/K \cong A_5$ which has 2 classes of torsion free subgroups H_i of index 120 in Γ. These are the minimal index torsion free subgroups of Γ^+ and they have abelianisation $H_i/[H_i,H_i] \cong \mathbb{Z} \oplus \mathbb{Z}_2^5$.

2) For each minimal index torsion free subgroup H_i of Γ^+, the associated manifold $\mathcal{M}_i(K,1) = \mathbb{H}^3/H_i$ is a 2-fold branched cover of $\mathbb{H}^3/K$. Further, there exist manifolds $\mathcal{M}_i(K,n)$ tessellated by $2n$ right-angled dodecahedra for any $n \in \mathbb{N}$.

Proof: 1): Write $\alpha \mapsto (1,3,5,2,4)$ and $\beta \mapsto (1,2,3)$. Let $\gamma \mapsto g$ where g is some element of A_5 such that the order of g divides 4. Since A_5 contains no elements of order 4, then $|g| \in \{1,2\}$. If $\gamma \mapsto ()$, the identity element of A_5, then $|\beta\gamma| = 2$ forces $\beta = ()$, a contradiction. So $|g| = 2$. There is a unique element in $(2,4)(3,5) \in A_5$ satisfying $|\beta g| = 2$ and $|\alpha\beta g| = 2$. Let $\theta : \Gamma^+ \twoheadrightarrow A_5$ be the induced map constructed and write $K = \mathrm{Ker}(\theta)$. Then by construction K contains the conjugacy class $\mathcal{C}(\gamma^2)$ of 2-torsion in Γ^+. GAP was used to investigate index 2 subgroups of K. There are 7 classes of subgroups in K, $\{H_i | 1 \leq i \leq 7\}$ of index 2, 5 of which avoid the conjugacy class $\mathcal{C}(\gamma^2)$ of torsion. Taking the abelianisation of each of these 5 subgroups yields 2 subgroups, H_1 and H_2, with positive first Betti number. Their abelianization, $H_i/[H_i, H_i]$, was calculated using GAP and in each case this was found to be $\mathbb{Z} \oplus \mathbb{Z}_2^5$. The fact that Γ^+ is not perfect (it maps onto C_2) means that it remains to consider the possibility that Γ^+ maps onto some small solvable group with torsion free kernel. Now, in any homomorphism from Γ or Γ^+ to a solvable group, α and β must go to the identity since they generate a perfect A_5. Setting $\alpha = \beta = 1$ in the presentation of Γ^+ gives C_2. Hence the derived series terminates with a subgroup of index 2 in Γ^+. So, any quotient of Γ^+ with torsion free kernel has the form $H : C_2$ or H, where H is perfect, Hence, in looking for small index normal subgroups is it sufficient to consider simple groups H or groups $H : C_2$, Since it has already been established that Γ^+ has no torsion free kernel with quotient A_5, the result follows.

2): Let $\mathcal{M}_i(K,1) = \mathbb{H}^3/H_i$ be the manifold associated to each H_i. Then, by construction, each $\mathcal{M}_i(K,1)$ is a 2-fold branched cover of the orbifold $\mathbb{H}^3/K$. Since $\mathbb{H}^3$ carries a tessellation by right angled dodecahedra, it follows that $\mathcal{M}_i(K,1)$ is tessellated by 2 right angled dodecahedra. By composing with the map $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, characteristic subgroups K_n of index n in each H_i can be obtained. The associated manifold $\mathcal{M}_i(K,n)$is a regular n-fold cover of the manifold $\mathcal{M}_i = \mathbb{H}^3/H_i$. Hence the manifolds $\mathcal{M}_i(K,n)$ are tesselated by $2n$ right angled dodecahedra. ∎

6.3.3 Minimal index torsion free normal subgroups

Lemma 6.3.4 Γ^+ *has no torsion free normal subgroups N with $\Gamma^+/N \cong A_6$ or S_6.*

Proof: Suppose there exists a torsion free normal subgroup. $A_6 \cong L_2(9)$ has 2 conjugacy classes of subgroups isomorphic to A_5. Representatives for each class are $\langle(1,3,5,2,4),(1,2,3)\rangle$ and $\langle(1,2,3,4,6),(1,5,6)(2,4,3)\rangle$. In the first case (when $\langle\alpha,\beta\rangle \to \langle(1,3,5,2,4),(1,2,3)\rangle$) there are nine elements g of order 4 in A_6 satisfying $|\beta g| = 2$, listed below:

$$\{(1,3,4,2)(5,6),(1,3,5,2)(4,6),(1,3,6,2)(4,5),$$
$$(1,3,2,4)(5,6),(1,3,2,5)(4,6),(1,3,2,6)(4,5),$$
$$(1,4,3,2)(5,6),(1,5,3,2)(4,6),(1,6,3,2)(4,5)\}$$

For each g in the above list, the word $\alpha\beta g$ has order $5,5,3,5,4,4,4,5,4$, respectively. In the second case, when $\langle\alpha,\beta\rangle \to \langle(1,2,3,4,6),(1,5,6)(2,4,3)\rangle$, again there are nine elements g of order 4 in A_6 satisfying $|\beta g| = 2$. They are:

$$g \in \{(1,2)(3,4,6,5),(1,2,3,5)(4,6),(1,3)(2,6,5,4),\\ (1,3,4,5)(2,6),(1,4)(2,3,6,5),(1,4,2,5)(3,6),\\ (1,6,4,2)(3,5),(1,6,2,3)(4,5),(1,6,3,4)(2,5)\}$$

For each g in the above list, the word $\alpha\beta g$ has order $5,4,5,4,3,5,4,4,5$, respectively. Therefore there are no torsion free normal subgroups N whose factor group $\Gamma^+/N \cong A_6$. The case for $\Gamma^+ \not\twoheadrightarrow S_6$ is similar. ■

Lemma 6.3.5 Γ^+ *has precisely two distinct torsion free normal subgroups* N_1 *and* N_2 *with* $\Gamma^+/N_i \cong PGL_2(11)$.

Proof: From Chapter 4, § 4.4.1, $t = \frac{-1\pm\sqrt{5}}{2}$ and $\sqrt{5} = 4$ in $\mathbb{F}_{11}$. If $t = \frac{-1+\sqrt{5}}{2}$ then $t = 7$, and if $t = \frac{-1-\sqrt{5}}{2}$ then $t = 3$.

Case I: $t = 7$: Then $e^2 + f^2 = -2 - 7 = -9 \equiv 2 \bmod 11$, giving

$$\alpha \mapsto \begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix} \text{ and } \beta \mapsto \begin{pmatrix} 1 & 4 \\ -3 & 0 \end{pmatrix}. \text{ Write } \gamma \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Next, $|\alpha\beta\gamma| = 2$ gives the identity

$$(\alpha\beta\gamma)^2 \mapsto \begin{pmatrix} c^2 - ad & cd - bd \\ ab - ac & b^2 - ad \end{pmatrix} = Id \in PGL_2(11)$$

and this forces $b = c$. Then

$$(\beta\gamma)^2 \mapsto \begin{pmatrix} (a+4b)^2 + 8a(b+4d) & (b+4d)(a+b) \\ 8a(a+b) & 9b^2 + 8a(b+5d) \end{pmatrix}$$

gives the equations $8a(a+b) = 0$ and $(b+4d)(a+b) = 0$. If $a = 0$ then either $b = 0$, in which case γ is the zero matrix, or $b = -4d$, giving

$$(\beta\gamma)^2 \mapsto d^2 \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

as the identity matrix in $PGL_2(11)$, a contradiction. If $a \neq 0$, then $b = -a$ and so

$$\gamma \mapsto \begin{pmatrix} -b & b \\ b & d \end{pmatrix}$$

Then

$$\gamma^4 \mapsto \begin{pmatrix} 8b^4+4b^2+1 & -8b^4-8b^2-4-\frac{1}{b^2} \\ -8b^4-8b^2-4-\frac{1}{b^2} & 8b^4+12b^2+9+\frac{1}{b^4}+\frac{4}{b^2} \end{pmatrix}$$

Writing $8b^4 = -8b^2-4-\frac{1}{b^2}$ gives

$$\gamma^4 \mapsto \begin{pmatrix} -4b^2-3-\frac{1}{b^2} & 0 \\ 0 & 4b^2+5+\frac{3}{b^2}+\frac{1}{b^4} \end{pmatrix}$$

Since these must be equal for this to be an element of $PGL_2(11)$, b must satisfy the equation

$$8b^6+8b^4+4b^2+1=0$$

Solving for $8x^3+8x^2+4x+1=0$ give x a cube root of unity, so b is a sixth root of unity. Since 6 does not divide $10 = 11-1$, there are no maps $\Gamma^+ \twoheadrightarrow PGL_2(11)$ with $\mathrm{trace}(\alpha) = 7$.

Case II: $t=3$: Then $e^2+f^2 = -5 \equiv 6 \bmod 11$, and set

$$\alpha \mapsto \begin{pmatrix} -2 & 1 \\ 0 & 5 \end{pmatrix} \text{ and } \beta \mapsto \begin{pmatrix} 1 & 5 \\ 2 & 0 \end{pmatrix}. \text{ Write } \gamma \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$(\alpha\beta\gamma)^2 \mapsto \begin{pmatrix} c^2-ad & cd-bc \\ ab-ac & b^2-ad \end{pmatrix} = Id \in PGL_2(11)$$

forces $b=c$. Then

$$(\beta\gamma)^2 \mapsto \begin{pmatrix} (a+5b)^2+2a(b+5d) & (b+5d)(a+7b) \\ 2a(a+7b) & 4b^2+2a(b+5d) \end{pmatrix}$$

This gives $2a(a+7b)=0$ and $(b+5d)(a+7b)=0$. If $a=0$ then either $b=0$, forcing γ to be the zero matrix, or $b=-5d$, and this gives

$$(\beta\gamma)^2 \mapsto b^2 \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$$

which cannot be the identity matrix in $PGL_2(11)$. So $a \neq 0$. Thus $a=-7b$, giving

$$(\beta\gamma)^2 \mapsto 4b(3b-d) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus

$$\gamma \mapsto \begin{pmatrix} -7b & b \\ b & d \end{pmatrix}$$

subject to $\gamma^4 = Id \in PGL_2(11)$. Now

$$\gamma^4 \mapsto g = \begin{pmatrix} 8b^4 + 8db^3 + d^2b^2 & 6b^4 + 7db^3 + 4d^2b^2 + d^3b \\ 6b^4 + 7db^3 + 4d^2b^2 + d^3b & 6b^4 + 8db^3 + 3d^2b^2 + d^4 \end{pmatrix}$$

so $b(6b^3 + 7db^2 + 4d^2b + d^3) = 0$. Since b cannot be 0, then $6b^3 + 7db^2 + 4d^2b + d^3 = 0$. This cubic equation has three solutions, $b \in \{5d, 6d, 8d\}$. If $b = 8d$ then $\det(g) = d^2$, so writing $d = 1$ gives $g \in PSL_2(11)$. For $b = 5d$, $\det(g) = 6d^2$ and for $b = 6d$, $\det(g) = 10d^2$. Because neither 6 nor 10 are squares in $\mathbb{F}_{11}$, neither matrix lies in $SL_2(11)$. They must then lie in $PGL_2(11)$. The images of α and β generate an A_5 subgroup of $PGL_2(11)$. Now A_5 is not maximal in $PGL_2(11)$ - it is contained in a maximal $L_2(11)$ subgroup. However, A_5 is maximal in $L_2(11)$, and the images of γ are, by construction, not contained in $L_2(11)$. The group elements in $PGL_2(11) \backslash L_2(11)$ acts by interchanging the two conjugacy classes of A_5 in $L_2(11)$, so they do not fix any given A_5. Since $PGL_2(11)$ has no outer automorphisms, the two maps $\Gamma^+ \twoheadrightarrow PGL_2(11)$ constructed are inequivalent, and their kernels are two distinct normal subgroups K_1 and K_2 in Γ^+. ∎

Since Γ^+ is not perfect, it is still possible that Γ^+ maps onto a small solvable group. However, in any map from Γ^+ to a solvable group, α and β must map to the identity, since they generate a perfect group. Then $\Gamma^+/\{\alpha, \beta\} \cong C_2$, so the derived series terminates with a subgroup of index 2 in Γ^+. Hence any quotient of Γ^+ has the form $G \lhd C_2$, where G is a perfect group. Hence for small index subgroups, it suffices to consider either G simple or extensions of G by a cyclic group of order 2. If the extension has the form $G \times C_2$, then it follows that Γ^+ also maps onto G. Hence the following result is obtained:

Theorem 6.3 *The smallest index torsion free normal subgroups of Γ^+ arise from epimorphisms $\Gamma^+ \twoheadrightarrow PGL_2(11)$. There are two such maps $\theta_1, \theta_2 : \Gamma^+ \twoheadrightarrow PGL_2(11)$ corresponding to two distinct normal subgroups K_1 and K_2 in Γ^+.*

Using Lowx and GAP, generating sets for K_1 and K_2 were found. Both subgroups arise from the action of Γ^+ on conjugacy classes of subgroups of index 12 and they are conjugate in Γ. The abelianisation of each K_i, $K_i/[K_i, K_i] \cong \mathbb{Z}_2^{10} \oplus \mathbb{Z}_{11}^3$ and their intersection is a normal subgroup of Γ^+ of index $660 * 1320$. The elements generating K_1 and K_2 as subgroups of Γ^+ are listed in Table 6.5

Elements of the generating set for K_1	Elements of the generating set for K_2
$bcbdcbabcdcbdabcdcba$	$bcdcbabcdcdbabcdbcba$
$cbabcdcbabdcdcbabcdb$	$cbdcbabcdcdbabcdcbab$
$cdcbabcdcbdabcdbcbab$	$dcbabcdcbadbcdcbabcb$
$acbdcbabcdcdbabcdcbaba$	$acbabcdcbabdcdcbabcdba$
$adcbabcdcbadbcdcbabcba$	$acdcbabcdcbdabcdbcbaba$
$babacbabadcbdabcdababcda$	$babacbdcbabcdcdbabcdcaba$
$babacdcbabcdcdbabcdbcaba$	$babadcbabadcdbabcdababca$
$babdcbabadcbdabcdababcba$	$babcbabadcbadbcdababcdba$
$bacbabadcbabdcdababcdaba$	$bacbabdcbabadcdbabcdabca$
$bacbabcdcbabdcdcbabcdbab$	$badcbabadcbadbcdababcaba$
$bacbadcbabadcdbabcdbabca$	$badcbabcdcbadbcdcbabcbab$
$bacdcbabcdcbdabcdbcababa$	$bcbabadcbabadcdbabcdbaba$
$bdcbabadcbabdcdababcbaba$	

Table 6.5: The generating sets for K_1 and K_2 in Theorem 6.3

6.3.4 Manifold structure

From the previous section, α, β and γ have torsion-free representations in $PGL_2(11)$ given by $\alpha \mapsto \tilde{\alpha} = \begin{pmatrix} -2 & 1 \\ 0 & 5 \end{pmatrix}$ $\beta \mapsto \tilde{\beta} = \begin{pmatrix} 1 & 5 \\ 2 & 0 \end{pmatrix}$ and $\gamma \mapsto \tilde{\gamma} \in \left\{ \begin{pmatrix} -2 & 5 \\ 5 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 6 \\ 6 & 1 \end{pmatrix} \right\}$. Corresponding to the two subgroups K_1 and K_2 are two hyperbolic manifolds $\mathcal{M}_i \cong \mathbb{H}^3/K_i$. The structure of these manifolds can be determined using the group structure of $PGL_2(11)$. In $PGL_2(11)$, $\langle \tilde{\alpha}, \tilde{\beta} \rangle \cong A_5$, and there exists a complementary dihedral subgroup D_{11} of order 22 which can be used to index the cosets of A_5 in $PGL_2(11)$. Write

$$D_{11} \cong \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 5 \\ 0 & 1 \end{pmatrix} \right\rangle$$

The following method can be used to determine the structure of each of the two tessellations $\mathcal{K}_i$ associated with $L = PGL_2(11)$, by identifying which dodecahedron contains the flag labeled by any given element of L. The 22 dodecahedra may be identified with the cosets gG in L of the subgroup $G \cong A_5$, which is the stabiliser of one dodecahedron D_0 in the monodromy representation of L on flags. Let B be the Borel subgroup of L, that is, a subgroup of the normaliser $C_{11} : C_{10}$ of a Sylow 11-subgroup of L. Let $\tilde{B} \subset B$ denote the unique subgroup D_{11} of B. Then $|\tilde{B} \cap G| = 1$ so $\tilde{B}$ has an orbit of length 22 on the dodecahedra and therefore acts transitively on them. As a result $L = \tilde{B}G$ and each element $h \in L$ has a unique form $h = bg$, where $b \in \tilde{B}$ and $g \in G$. The flag corresponding to h lies in the dodecahedron corresponding to the coset bG, and position of the flag in the dodecahedron is determined by g.

6.3.5 Other manifolds and minimal index torsion free subgroups

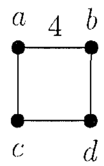
The next smallest index torsion free normal subgroup arises from the action of Γ^+ on a subgroup of index 20. This group was found by studying the induced actions of Γ^+ on low index subgroups. Table A.3 of Appendix A summarises the induced actions. The induced permutation representation is $G = A_5 \ltimes H$, where H is elementary abelian of order 32. The resulting manifold is tessellated by 32 dodecahedra and the fundamental group in this case has abelianisation $\mathbb{Z}^{15} \oplus \mathbb{Z}_2^2$. As a consequence a tower of manifolds with maximal isometry group is recovered. Each manifold in this tower consists of $32n^6$ dodecahedra and is a regular n^6-fold cover of the manifold $\mathbb{H}^3/G$. Additionally, GAP was used to verify that G has subgroups H of index 120 in G whose preimages correspond to subgroups of index 120 in Γ^+ that avoid all conjugacy classes of torsion.

6.4 $\Gamma = T_4[2,2,5;2,3,5]$

The manifolds arising from torsion free subgroups and torsion free normal subgroups of the group T_5^+ have already been studied in detail in chapter 5.

6.5 $\Gamma = T_5[2,3,3;2,4,3]$

Recall that $T_5[2,3,3;2,4,3]$ is the Coxeter group with Coxeter diagram



where a, b, c and d are Coxeter generators of Γ. Let $\alpha = ab$, $\beta = bc$ and $\gamma = cd$. Then the orientation preserving subgroup Γ^+ has the following presentation:

$$\Gamma^+ = \langle \alpha, \beta, \gamma \mid \alpha^4 = \beta^3 = \gamma^3 = (\alpha\beta)^2 = (\beta\gamma)^2 = (\alpha\beta\gamma)^3 = 1 \rangle$$

6.5.1 Torsion in Γ and in Γ^+

Representations of the conjugacy classes of torsion elements in Γ are given in Table 6.6 The restriction of this list to representatives of classes lying in Γ^+ is given in Table 6.7

Order	Representative
Order 2	a, ac, bd, $(ab)^2$, $(abc)^3$, $(bad)^3$
Order 3	bc
Order 4	ab, $c(ab)^2$, $d(ab)^2$, adc, bcd
Order 6	abc

Table 6.6: Conjugacy class representatives for elements of finite order in Γ

Order	Representative
Order 2	ac, bd, $(ab)^2$
Order 3	bc
Order 4	ab

Table 6.7: Conjugacy class representatives for elements of finite order in Γ^+

Lemma 6.5.1 *There are no torsion free normal subgroups of Γ^+ whose factor group is S_4 or S_5.*

Proof: Any torsion free normal subgroup of Γ^+ must contain $\langle\alpha,\beta\rangle \equiv S_4$. First, suppose that there is such a subgroup N with $\Gamma^+/N \simeq S_4$. Then it can be assumed that $\alpha \mapsto (1,2,3,4)$. Therefore $\beta \in \{(1,3,2),(1,4,2),(1,4,3),(2,4,3)\}$ and $\alpha\beta \in \{(3,4),(2,3),(1,2),(1,4)\}$. Now $|\gamma| = 3$ and $|\alpha\beta\gamma| = 3$ can therefore both be written as a product of an even number of transpositions. But $\alpha\beta$ has an odd number of transpositions, a contradiction! Since S_5 has a unique conjugacy class of elements of order 4, the result extends to S_5. ■

In particular, since A_5 has no element of order 4, the following result is immediate:

Proposition 6.5.1 *Γ^+ has no torsion free normal subgroup whose factor group is isomorphic to A_5.*

In Theorem 4.8 in chapter 4 it was shown that there is a unique normal subgroup K in Γ^+ whose factor group is $L_2(7)$. Since it is unique, it is characteristic. Γ^+ is normal in the normalizer Ω of Γ. Consequently K is normal in Ω

Note: In Table A.5 of Appendix A it can be seen that there are two conjugacy classes of subgroups of index 8 in Γ^+. Calculating their cores gives the same normal subgroup K of Γ^+ in each case, verifying Theorem 4.8.

The group Γ^+ is perfect: Abelianizing the presentation for Γ^+, the identities $\alpha^4 = \beta^3 = \gamma^3 = 1$, $(\alpha\beta)^2 = \alpha^2\beta^2 = 1$, $(\beta\gamma)^2 = \beta^2\gamma^2 = 1$ and $(\alpha\beta\gamma)^3 = \alpha^3 = 1$ are obtained. Hence $\alpha = 1$, so $\beta^2 = \beta^3 = 1$. Therefore $\beta = 1$, so $\gamma^2 = \gamma^3 = 1$, and so Γ^+ is a perfect group. Combining the above results, the following theorem can be stated:

Theorem 6.4 *Let K be a smallest index torsion free normal subgroup of Γ^+. Then K is unique, K has index 168 in Γ^+ and $\Gamma^+/K \cong L_2(7)$. Furthermore, K is normal in Ω, the normalizer of Γ in Isom($\mathbb{H}^3$).*

Using GAP, a presentation for K was obtained. K is the fundamental group of the associated hyperbolic manifold $\mathcal{M} = \mathbb{H}^3/K$. A presentation for $\pi_1(\mathcal{M})$ is given here:

$$\begin{aligned}\pi_1(M) = \langle F_1, F_2, F_3, F_4, F_5, F_6 \mid\ & \\ & F_6F_1^{-1}F_6^{-1}F_3F_5^{-1}F_2F_5F_2^{-1}F_3^{-1}F_1 = 1, \\ & F_4^{-1}F_1^{-1}F_5^{-1}F_1^{-1}F_3F_2F_1F_5F_3^{-1}F_2^{-1}F_4F_1 = 1, \\ & F_3F_6^{-1}F_2^{-1}F_4F_3F_2F_3^{-1}F_4^{-1}F_1^{-1}F_2^{-1}F_3^{-1}F_1F_6F_2 = 1, \\ & F_1F_6^{-1}F_1^{-1}F_4^{-1}F_2F_6F_5^{-1}F_1^{-1}F_2^{-1}F_6^{-1}F_5F_1F_4F_6 = 1, \\ & F_4^{-1}F_1^{-1}F_6^{-1}F_3F_5^{-1}F_4F_3F_2F_3^{-1}F_4^{-1}F_5F_3^{-1}F_2^{-1}F_4F_1F_6 = 1, \\ & F_1^{-1}F_3F_2F_1F_4F_3F_5^{-1}F_2^{-1}F_3^{-1}F_4^{-1}F_5F_3^{-1}F_6F_1F_4F_1^{-1}F_4^{-1}F_6^{-1} = 1\rangle\end{aligned}$$

The manifold constructed differs from the manifolds constructed using the groups T_1, T_2, T_3 and T_4. Those manifolds had as fundamental regions hyperbolic dodecahedra, and the manifolds had a straightforward decomposition into dodecahedra identified across faces. The manifolds arising from the groups T_5 to T_9 have a more complicated structure. The associated fundamental regions are pleated (or creased or pinched) across their faces. Figures 6.2 and 6.3 illustrate the difference between a standard cube and a pleated cube:

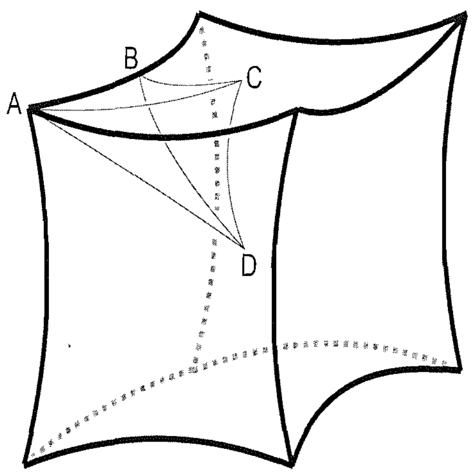


Figure 6.2: Standard hyperbolic cube

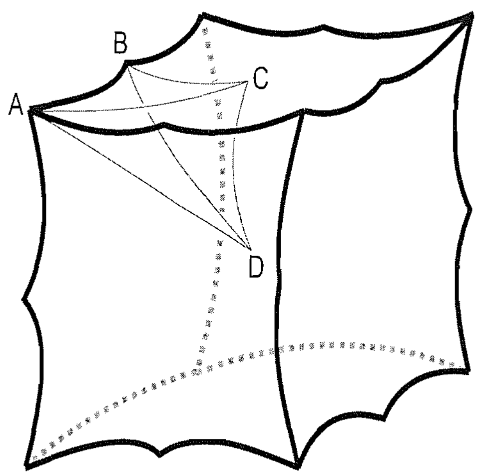


Figure 6.3: Pinched hyperbolic cube

The added creases across faces of a cube give a "pinched point" at the midpoint of each edge. Visualization of the resulting manifolds is correspondingly harder. For example, the group stabilizing the vertex C in the center of a face is, in the case of T_5, an octahedral group! The abelianisation of $K = \pi_1(\mathcal{M})$ is free abelian on six generators: $\pi_1(\mathcal{M})/[\pi_\infty(\mathcal{M})\,,\,\pi_\infty(\mathcal{M})] = \mathbb{Z}^6$, and as a result an infinite tower of covers of $\mathcal{M}$ by manifolds $\mathcal{M}_n$ consisting of $7n$ cubical cells can be constructed.

Note: The group $\pi_1(\mathcal{M})$ is not the torsion free subgroup of smallest index in Γ^+. There are 5 conjugacy classes of such groups of index 24 in Γ^+. There are four distinct normal subgroups whose factor group is $L_2(23)$. In each case the preimage under the induced epimorphism $\Gamma^+ \twoheadrightarrow L_2(11)$ of $C_{23} \ltimes C_{11} < L_2(11)$ is a torsion free subgroup of index 24. Since $C_{23} \ltimes C_{11}$ is maximal in $L_2(23)$, each preimage has trivial normalizer in Γ^+, so the associated hyperbolic manifolds have very little symmetry. The Sylow 7-subgroup C_7 of $L_2(7)$ has index 24 and its preimage is also a torsion free subgroup. The normalizer of C_7 in $L_2(7)$ has index 8 in $L_2(7)$ and pulls back to an index 8 normalizer of H in Γ^+. Since there are no futher conjugacy classes of subgroups of index 24 that are torsion free, this completes the list.

Let H be a Sylow 7-subgroup of $L_2(7)$. The elements of H can be used to index the 7 cosets of S_4 in $L_2(7)$, where S_4 stabilizes a cubical cell in the tessellation of $\mathcal{M}$. Letting the cosets correspond to cubical cells in the decomposition of $\mathcal{M}$, the manifold can then be reconstructed combinatorially as in the previous sections.

6.5.2 The action of $L_2(7)$ on $H_1(\mathcal{M})$

The subgroup $\Gamma^+/N_1 \cong L_2(7)$ of isometries of the manifold $\mathcal{M} = \mathbb{H}^3/K$ is generated by a rotation through $\pi/2$ about the center of a square face f_1, induced by α, a rotation through $2\pi/3$ about a vertex v_1 adjacent to f_i, induced by β and a rotation through $2\pi/3$ about an edge adjacent to v_1. These rotations act on $H_1(M_1)$, regarded as a 6-dimensional module over $\mathbb{Z}$, as the matrices

$$\alpha \mapsto \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 & -1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \qquad \beta \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{pmatrix}$$

$$\gamma \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -2 & -2 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 \end{pmatrix}$$

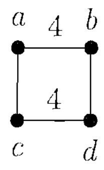
According to the Atlas of Finite Groups [Atlas], there are 6 irreducible characters for $L_2(7)$, with degrees, 1, 3, 3, 6, 7, and 8. The values of the characters of an element of order 2 are 1, -1, -1, 2, -1 and 0, respectively. Since the element $\alpha\beta$ is an element of order 2 and has trace -2, the representation cannot be primitive, so the character splits as a sum of two irreducible characters: either as $\chi_2 + \chi_5$ or as $2\chi_2$. Since the trace of α is 2 and the trace of β is 0, the action splits as $2\chi_2$. So there exists a covering manifold $\mathcal{M}_n$ of $\mathcal{M}$ whose covering transformations (deck transformations) form an abelian group, and over which the isometry group extends to the group $L_2(7) \rtimes \mathbb{Z}_n^6$.

6.5.3 Computational results

Γ^+ has 62 conjugacy classes of subgroups of index less than or equal to twenty. Letting Γ^+ act on the cosets of a representative of each conjugacy class induces a representation $\Gamma^+ \twoheadrightarrow G$, where G is some finite permutation group. Unless $G = A_5$, the induced map faithfully carries all conjugacy classes of torsion elements of Γ^+. The smallest torsion-free normal subgroup is, as previously observed, the kernel K of a map $K = \ker\theta : \Gamma^+ \twoheadrightarrow L_2(7)$. It is interesting to note that, after A_5, the next two smallest alternating groups that arise as the quotient of Γ^+ by some torsion free normal subgroup are A_{11} and A_{19}.

6.6 $\Gamma = T_6[2,3,4;2,3,4]$

Recall that $T_6[2,3,4;2,4,3]$ is the Coxeter group with Coxeter diagram



where a, b, c and d are Coxeter generators of Γ. Let $\alpha = ab$, $\beta = bc$ and $\gamma = cd$. Then the orientation preserving subgroup Γ^+ has the following presentation:

$$\Gamma^+ = \langle \alpha, \beta, \gamma \mid \alpha^4 = \beta^3 = \gamma^4 = (\alpha\beta)^2 = (\beta\gamma)^2 = (\alpha\beta\gamma)^3 = 1 \rangle$$

6.6.1 Torsion in Γ and in Γ^+

Representations of the conjugacy classes of torsion elements in Γ are given in Table 6.8 The restriction of Table 6.8 to representatives of classes lying in Γ^+ is given in Table 6.9

Order	Representative
Order 2	a, b, ac, bd, $(ab)^2$, $(cd)^2$, $(abc)^3$, $(dab)^3$, $(cda)^3$, $(bcd)^3$.
Order 3	bc, ad.
Order 4	ab, cd, $c(ab)^2$, $d(ab)^2$, $a(cd)^2$, $b(cd)^2$.
Order 6	abc, dab, acd, bcd.

Table 6.8: Conjugacy class representatives for elements of finite order in Γ

Order	Representative
Order 2	ac, bd, $(ab)^2$, $(cd)^2$.
Order 3	bc, ad.
Order 4	ab, cd.

Table 6.9: Conjugacy class representatives for elements of finite order in Γ^+

6.6.2 Maps to S_4

Lemma 6.6.1 *There are precisely two distinct torsion free normal subgroups of Γ^+ whose factor group is isomorphic to S_4.*

Proof: The proof amounts to constructing two non-conjugate epimorphisms $\Gamma^+ \twoheadrightarrow S_4$. Let $\alpha \mapsto (1,3,2,4)$ and $\beta \mapsto (1,2,3)$. Then for $\gamma \in \{(1,3,2,4),(1,3,4,2)\}$ we get $|\gamma| = 4$, $|\beta\gamma| = 2$ and $|\alpha\beta\gamma| = 3$. This proves existence of the two maps. If $\gamma \mapsto (1,3,2,4)$ then the common image of $\alpha\gamma$ and of α^2 has order 2, while if $\gamma \mapsto (1,3,4,2)$ then $\alpha\gamma \mapsto (1,4,3)$ has order 3. Hence the maps are not conjugate. ∎

As can be seen in Table A.6 in Appendix A, there are six conjugacy classes of subgroups of index 4 in Γ^+. Letting Γ^+ act on the cosets of a representative in each case gives the following results:

- 2 kernels K_1, K_2 such that $\Gamma^+/K_i \cong S_4$ with $K_i/[K_i, K_i] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_8^2$
- 4 kernels K_3, K_4, K_5, K_6 with $\Gamma^+/K_i \cong S_4$ and $K_i/[K_i, K_i] \cong \mathbb{Z}_2^5 \oplus \mathbb{Z}_3$.

The four kernels with abelianisation $\mathbb{Z}_2^5 \oplus \mathbb{Z}_3$ contain either the conjugacy class $\mathcal{C}(ab)^2$ or the conjugacy class $\mathcal{C}(cd)^2$, while the two kernels with abelianisation $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_8^2$ avoid the conjugacy classes of torsion elements and are distinct. Therefore they are the kernels of the maps constructed in Lemma 6.6.1. Their intersection is a normal subgroup of Γ^+ of index 288. Both of these kernels, K_1 and K_2, have three conjugacy classes of subgroups of index 4 whose abelianisation has positive first Betti number. Consequently an infinite family of manifolds $M(K_i, 4n)$ can be constructed as $4n$–fold covers of $M(K_i) = \mathbb{H}^3/K_i$. These manifolds are tessellated by $4n$ cubical cells. If the associated fundamental groups, $\pi_1(M(K_i, 4n)) < \Gamma^+$ are normal in Γ^+, then the manifolds $M(K_i, 4n)$ will exhibit a high degree of symmetry. Presentations for the two kernels are given in Table 6.10 and Table 6.11:

$$
\begin{aligned}
K_1 := \langle F_1, F_2, F_3 | \\
& F_1^{-1}F_2^{-1}F_1^{-1}F_3F_1^{-2}F_2^{-1}F_1^{-1}F_2^{-2}F_1F_3F_2^{-2} = 1, \\
& F_3F_1^{-1}F_3F_1^{-2}F_2^{-1}F_1^{-1}F_3F_1^{-1}F_3^{2}F_2^{-1}F_1F_3 = 1, \\
& F_1^{-1}F_3F_1^{-2}F_2^{-1}F_1^{-1}F_3^{-1}F_1^{-1}F_2F_3^{-2}F_1^{-1}F_2F_3^{-1}F_1^{-1}F_2^{2}F_3^{-1}F_1^{-1}F_2 = 1\rangle
\end{aligned}
$$

Table 6.10: Presentation for K_1

As a subgroup of Γ^+, K_1 has generators $\{bdca,\ dcba,\ bcdbab\}$. In the presentation given in Table 6.10, $F_1 = bdca$, $F_2 = dcba$ and $F_3 = bcdbab$.

$$
\begin{aligned}
K_2 := \langle F_1, F_2, F_3 |\ & \\
& F_2F_1^{-1}F_2F_1^{-1}F_3F_1^{-1}F_2F_1^{-1}F_2^2F_1F_3F_1F_2 = 1, \\
& F_1F_2^{-1}F_1F_3^{-2}F_1^{-1}F_3^{-1}F_1^{-1}F_2^{-1}F_1^{-1}F_3^{-1}F_1^{-1}F_3^{-2} = 1, \\
& F_2^{-1}F_3^2F_1^{-1}F_2F_1^{-1}F_3F_1^{-1}F_2^{-1}F_1^{-1}F_3^{-1}F_1^{-2}F_2F_1^{-1}F_3F_1^{-1}F_2^{-1}F_1^{-1}F_3^{-1}F_1^{-1}F_2^{-1} = 1\rangle
\end{aligned}
$$

Table 6.11: Presentation for K_2

As a subgroup of Γ^+, K_2 has generators $\{cbda,\ dacb,\ abcbdaba\}$. In the presentation given in Table 6.11, $F_1 = cbda$, $F_2 = dacb$ and $F_3 = abcbdaba$.

6.6.3 Geometrical description of the two manifolds

From Lemma 6.6.1 there are two distinct epimorphisms $\Gamma^+ \twoheadrightarrow S_4$ with torsion-free kernel. Letting S_4 act on the right as a monodromy action on flags, manifolds $\mathcal{M} = \mathbb{H}^3/K_i$ can be constructed by identifications on faces of a pinched cube. Letting $\alpha, \gamma \mapsto (1,3,2,4)$ gives identifications as shown in Figure 6.4. Letting $\gamma \mapsto (1,3,4,2)$ (the image of $\beta^{-1}\alpha\beta$ under the maps constructed in Lemma 6.6.1) gives identifications as shown in Figure 6.5. Two of the identifications, red square to red square and blue square to blue square, are illustrated in these diagrams. The vertex shared by two blue squares and a red square in Figure 6.4 is the "pinched point" introduced in § 6.5.1. Similarly for the vertex shared by two red squares and a blue square in Figure 6.4.

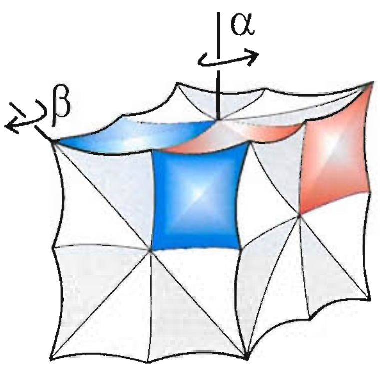


Figure 6.4: Identifications for $\gamma = (1,3,2,4)$

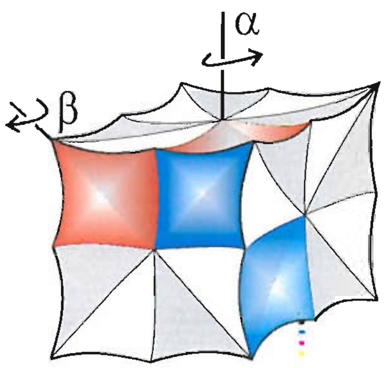


Figure 6.5: Identifications for $\gamma = (1,3,4,2)$

6.6.4 Maps to $L_2(7)$

Further small index torsion free normal subgroups can be constructed by pulling back images of normal subgroups in $\Gamma^+/K_1 \cap K_2 \cong S_4 \times S_4$. It is also interesting to study the small non-abelian simple groups which arise as quotients of Γ^+ by some torsion-free normal subgroup. The smallest such group is $L_2(7)$. There are two conjugacy classes of subgroups of index 7 in Γ^+. Their cores have index 168 in Γ^+ and they are contained in each other, so there is a unique normal subgroup K_3 satisfying $\Gamma^+/K_3 \cong L_2(7)$. The resulting manifold has $\frac{168}{24} = 7$ pinched cubical cells. Since K_3 is the unique normal subgroup of Γ^+ with quotient $L_2(7)$, it is characteristic in Γ^+ and hence it is normal in Ω, the normalizer of Γ in $\mathrm{Isom}(\mathbb{H}^3)$. A presentation for K_3 is given in Table 6.12. The generators of K_3 as a subgroup of Γ^+ are given in Table 6.13. The abelianization of K_3, $K_3/[K_3,K_3]$, is $\mathbb{Z}^{13}$.

$$
\begin{aligned}
K = \langle & F_1, F_2, F_3, F_4, F_5, F6, F7, F_8, F_9, F_{10}, F_{11}, F_{12}, F_{13}, w \,| \\
& F_4F_{12}^{-1}F_{13}F_5F_{13}^{-1}F_4^{-1}F_{12}F_5^{-1} = 1,\ F_{12}^{-1}F_{10}F_{11}^{-1}F_{10}^{-1}F_2F_{12}F_{11}F_2^{-1} = 1 \\
& F_{10}^{-1}F_{13}^{-1}F_6^{-1}F_7^{-1}F_6F_{10}F_{13}F_7 = 1,\ F_{12}F_1F_9^{-1}F_{12}^{-1}F_6^{-1}F_1^{-1}F_6F_9 = 1 \\
& F_4F_{10}^{-1}F_9F_8F_9^{-1}F_4^{-1}F_{10}F_8^{-1} = 1,\ F_{13}^{-1}F_2^{-1}F_9F_3^{-1}F_2F_9^{-1}F_{13}F_3 = 1, \\
& F_6F_{13}F_5F_3F_1^{-1}F_5^{-1}F_3^{-1}F_{13}^{-1}F_6^{-1}F_1 = 1,\ w = F_{11}F_{10}^{-1}F_4F_{13}F_3, \\
& F_3^{-1}F_{10}F_7^{-1}F_{13}^{-1}F_{10}^{-1}F_9F_8F_9^{-1}F_{13}F_3F_7F_8^{-1} = 1, \\
& F_1^{-1}F_{11}F_8^{-1}F_{12}F_1F_9^{-1}F_{12}^{-1}F_{10}F_{11}^{-1}F_{10}^{-1}F_9F_8 = 1, \\
& F_6F_{13}F_5F_2F_8^{-1}F_9^{-1}F_6^{-1}F_5^{-1}F_{13}^{-1}F_2^{-1}F_9F_8 = 1, \\
& F_7F_{11}F_{10}^{-1}F_{12}F_5^{-1}F_{10}F_7^{-1}F_{13}^{-1}F_{11}^{-1}F_{12}^{-1}F_{13}F_5 = 1, \\
& F_{10}F_{11}^{-1}F_6F_{13}F_3F_4F_{11}F_{10}^{-1}F_6^{-1}F_3^{-1}F_{13}^{-1}F_4^{-1} = 1, \\
& F_1F_9^{-1}F_2F_{13}F_7F_{10}^{-1}F_4F_9F_1^{-1}F_{12}^{-1}F_2^{-1}F_{10}F_7^{-1}F_{13}^{-1}F_4^{-1}F_{12} = 1, \\
& F_7F_8^{-1}F_9^{-1}F_2F_{13}F_5F_1(F_7^{-1}F_5^{-1}F_{13}^{-1}F_2^{-1}F_9F_8F_1^{-1})^w = 1\rangle
\end{aligned}
$$

Table 6.12: Presentation for K_3

$F_1 = badcbabcda$	$F_2 = bcdabadcba$	$F_3 = bcdcbadcda$
$F_4 = cdabadcbab$	$F_5 = cdcbadcdab$	$F_6 = dabadcbabc$
$F_7 = dcbabcdaba$	$F_8 = acdcbadcdaba$	$F_9 = bacbadcdbcda$
$F_{10} = cbadcdbcdaba$	$F_{11} = babcdacdcadcbdcdbcdaba$	$F_{12} = acbadcdbcdab$
$F_{13} = abcdabadcacbadcdbdabcabacbadcd$		

Table 6.13: Generators for K_3 in Γ^+

Thinking of $L_2(7)$ as the factor group $SL_2(7)/\{\pm I\}$, the induced map $\Gamma^+ \twoheadrightarrow L_2(7)$ obtained by the composition $\Gamma^+ \twoheadrightarrow \Gamma^+/K \cong L_2(7)$ can be expressed as an epimorphism $\theta : \Gamma^+ \to L_2(7)$ defined by the following identifications:

$$\alpha \mapsto \left[\begin{pmatrix} 0 & -3 \\ -2 & -3 \end{pmatrix}\right], \beta \mapsto \left[\begin{pmatrix} 3 & 0 \\ 2 & -2 \end{pmatrix}\right] \text{ and } \gamma \mapsto \left[\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}\right]$$

6.6.5 Maps to $A_6 \cong L_2(9)$

There are 18 conjugacy classes of subgroups of index 6 in Γ^+. Letting Γ^+ act on them induces epimorphisms onto D_6, S_4 and A_6. There are two conjugacy classes of subgroups of index 6 whose cores have index 360 in Γ^+, and these cores coincide. This unique core K_4 is torsion free. The map $\Gamma^+ \twoheadrightarrow \Gamma^+/K_4 \cong A_6$ can be characterized by $\alpha \mapsto (1,3,4,2)(5,6)$, $\beta \mapsto (1,4,3)$ and $\gamma \mapsto (1,6,3,4)(2,5)$. The group A_6 can be written as the quotient group $SL_2(9)/\{\pm I\}$. As equivalence classes of matrices, the map $\Gamma^+ \twoheadrightarrow L_2(9)$ can be expressed as

$$\alpha \mapsto \left[\begin{pmatrix} 0 & 1 \\ 2 & \sqrt{2} \end{pmatrix}\right] \quad , \quad \beta \mapsto \left[\begin{pmatrix} 0 & 2+2\sqrt{2} \\ 2+2\sqrt{2} & 1 \end{pmatrix}\right]$$

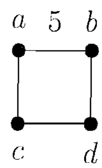
$$\text{and } \gamma \mapsto \left[\begin{pmatrix} 2+2\sqrt{2} & 1 \\ 0 & 1+2\sqrt{2} \end{pmatrix}\right]$$

Since K_4 is the unique normal subgroup of Γ^+ with quotient A_6, it is characteristic in Γ^+ and hence it is normal in Ω, the normalizer of Γ in $\mathrm{Isom}(\mathbb{H}^3)$. The following lemma summarizes these results:

Lemma 6.6.2 *There is a unique torsion free normal subgroup K_4 of Γ^+ whose factor group is $A_6 \cong L_2(9)$. Furthermore, K is normal in Ω.*

6.7 $\Gamma = T_7[2,3,3;2,5,3]$

Recall that $T_7[2,3,3;2,5,3]$ is the Coxeter group with Coxeter diagram



where a, b, c and d are Coxeter generators of Γ. Let $\alpha = ab$, $\beta = bc$ and $\gamma = cd$. Then the orientation preserving subgroup Γ^+ has the following presentation:

$$\Gamma^+ = \langle \alpha, \beta, \gamma \mid \alpha^5 = \beta^3 = \gamma^3 = (\alpha\beta)^2 = (\beta\gamma)^2 = (\alpha\beta\gamma)^3 = 1 \rangle$$

6.7.1 Torsion in Γ and in Γ^+

Representatives of the conjugacy classes of torsion elements in Γ are given in Table 6.14. The restriction of this list to representatives of classes lying in Γ^+ is given in Table 6.15.

Order	Representative
Order 2	a, ac, bd, $(abc)^5$ and $(dab)^5$
Order 3	bc
Order 4	adc and bcd
Order 5	ab and $(ab)^5$
Order 6	$c(ab)^2$ and $d(ab)^2$
Order 10	abc, $(abc)^3$, dab and $(dab)^3$

Table 6.14: Conjugacy class representatives for elements of finite order in Γ

Order	Representative
Order 2	ac and bd
Order 3	bc
Order 5	ab and $(ab)^5$

Table 6.15: Conjugacy class representatives for elements of finite order in Γ^+

Lemma 6.7.1 *Proper normal subgroups of Γ^+ are torsion free*

Proof: The proof is essentially the same as Corollary 3.1.1 for the group $T_4 = [5,3,5]$ in chapter 3. ■

6.7.2 Maps to A_5

Lemma 6.7.2

i) Γ^+ has two distinct normals subgroups K_1, K_2 whose factor groups are A_5.

ii) K_1 and K_2 are conjugate in Γ.

iii) The intersection $K_1 \cap K_2$ is normal in Ω, the full isometry group of Γ, and $\Gamma^+/K_1 \cap K_2 \cong A_5 \times A_5$.

Proof:

i) Let $\alpha \mapsto (1,3,5,2,4)$, $\beta \mapsto (1,2,3)$ be defined as in Chapter 3, Lemma 3.2.1. Then for $\gamma \in \{(2,3,4),(2,3,5)\}$, the remaining conditions $|\gamma| = 3$, $|\beta\gamma| = 2$ and $|\alpha\beta\gamma| = 3$ are satisfied. Since $(1,3,5,2,4)(2,3,4) = (1,4)(3,5)$ and $(1,3,5,2,4)(2,3,5) = (1,5,3,2,4)$, it is clear that the two normal subgroups K_1 and K_2 induce inequivalent epimorphisms and hence are distinct.

ii) Let $d \in \Gamma\backslash\Gamma^+$ act by conjugation on $\{\alpha,\beta,\gamma\}$. Then $\alpha^d = (\alpha^{-1})^{\beta\gamma}$, $\beta^d = (\beta^{-1})^{\gamma}$ and $\gamma^d = \gamma^{-1}$. Therefore, conjugating by d induce the outer automorphism $\{(1,3,5,2,4),(1,2,3),(2,3,4)\} \mapsto \{(1,3,2,4,5),(1,4,3),(2,4,3)\}$. Conjugating $\{(1,3,2,4,5),(1,4,3),(2,4,3)\}$ by $g = (2,4,5) = \beta^2\alpha^2\beta^2$ gives $\{(1,3,5,2,4),(1,2,3),(2,3,5)\}$, hence the K_i are conjugate in Γ.

iii) The intersection of K_1 and K_2 is clearly normal in Γ^+: Since the effect of $d \in \Gamma\backslash\Gamma^+$ is to transpose K_1 with K_2 then it fixes the intersection, and as a result $K_1 \cap K_2$ is normal in Γ. Finally, the effect of the outer automorphism $\tau \in \Omega\backslash\Gamma$ is to transpose the generators a and d with b and c. The equivalent action on the generators of Γ^+ is to send α to α^{-1}, β to $\alpha\beta\gamma$ and γ to γ^{-1}. For $\alpha = (1,3,5,2,4)$, $\beta = (1,2,3)$ and $\gamma = (2,3,4)$ the element $g = (1,5)(2,4) \in A_5$ has the desired action, while for $\alpha = (1,3,5,2,4)$, $\beta = (1,2,3)$ and $\gamma = (2,3,5)$ the element $g = (1,4)(2,3) \in A_5$ has the required action.

Consequently, the index 2 orientation preserving subgroup $\Omega^+ = \langle \Gamma^+,\ \tau \rangle$ of Ω has two normal subgroups $\widetilde{K}_1$ and $\widetilde{K}_2$ of index 60, each of whose factor group is A_5. The groups K_i arise as index two subgroups of the $\widetilde{K}_i$, and the factor groups $\Omega^+/K_i \cong A_5 \times C_2$. Since both K_i are normal in Ω^+, $K_1 \cap K_2$ is also normal in Ω^+. Hence $K_1 \cap K_2$ is normal in Ω. It remains to show that $\Gamma^+/K_1 \cap K_2 \cong A_5 \times A_5$. Let $\phi : \Gamma^+ \twoheadrightarrow A_5 \times A_5$ be defined by the maps

$$
\begin{aligned}
\alpha &\mapsto (\ (1,3,5,2,4),\ (1,3,5,2,4)\ ) \\
\beta &\mapsto (\ (1,2,3),\ (1,2,3)\ ) \\
\gamma &\mapsto (\ (2,3,4),\ (2,3,5)\ )
\end{aligned}
$$

Then, by composition with the restriction to each direct product, $\mathrm{Ker}(\phi)$ is a normal subgroup of both K_1 and K_2, and $\mathrm{Ker}(\phi) \subseteq K_1 \cap K_2$. Since $\mathrm{Ker}(\phi)$ has index 60 in both K_1 and K_2, $\mathrm{Ker}(\phi) = K_1 \cap K_2$.

■

6.7.3 Presentations for K_i

A generating set for K_1 as a subgroup of Γ^+ is $\{babdabca, (bcda)^2, d(abc)^2a, (dabc)^2, a(cdab)^2a\}$. More formally, K_1 can be given the abstract presentation listed in Table 6.16.

$$
\begin{aligned}
K_1 = \langle F_1, F_2, F_3, F_4, F_5 | \\
& F_1^{-1}F_4^{-1}F_5F_2^{-1}F_3^{-1}F_4^{-1}F_2^{-1}F_1F_2^{-1}F_5F_1F_3^{-1}F_1^{-1}F_2 = 1, \\
& F_2F_5^{-1}F_4F_5^{-1}F_3^{-1}F_4^{-1}F_5^{-1}F_2F_1^{-1}F_4^{-1}F_5F_4^{-1}F_1F_3^{-1} = 1, \\
& F_3F_5F_4^{-1}F_1F_3^{-1}F_1F_2^{-1}F_5F_1F_3^{-2}F_4^{-1}F_2^{-1}F_1 = 1, \\
& F_3F_1^{-1}F_4F_5^{-1}F_2^{-1}F_1F_3F_1^{-1}F_5^{-1}F_2F_5^{-1}F_4F_1F_2^{-1}F_5F_2^{-1} = 1, \\
& F_3^{-1}F_1^{-1}F_2F_4F_3F_1^{-1}F_4F_5^{-1}F_3^{-1}F_4^{-1}F_1F_3^{-1}F_2F_5^{-1}F_4F_1 = 1, \\
& F_3F_1^{-1}F_5^{-1}F_2F_1^{-1}F_4^{-1}F_2^{-1}F_1F_3F_1^{-1}F_5^{-1}F_3^{-1}F_4^{-1}F_2^{-1}F_3F_1^{-1}F_4F_5^{-1}F_1^{-1}F_4^{-1}F_5F_2^{-1} = 1 \rangle
\end{aligned}
$$

Table 6.16: Presentation for the group K_1 of Lemma 6.7.2

Table 6.17 describes each generator of K_1 as given in Table 6.16 in terms of the generators of Γ^+.

Generator of K_1	Representative in Γ^+
F_1	$babdabca$
F_2	$bdabcaba$
F_3	$(cba)^2dbca$
F_4	$abadbc(babc)^3$
F_5	$ad(acb)^2ab$

Table 6.17: Representating the generators of K_1 as elements of Γ^+

A generating set for K_2 as a subgroup of Γ^+ is $\{bacbadba, cbabdaba, (cdab)^2, dabacbab, (dcba)^2, a(dabc)^2a\}$. More formally, K_2 can be given the abstract presentation in Table 6.18.

$$
\begin{aligned}
K_1 = \langle F_1, F_2, F_3, F_4, F_5 | \\
& F_2F_5F_2^{-1}F_3F_4F_1F_3F_1F_2^{-1}F_1F_4F_5^{-1}F_2F_1^{-1} = 1, \\
& F_2F_5^{-1}F_2F_1^{-1}F_3^{-1}F_5F_4^{-1}F_1^{-1}F_2F_5F_3^{-1}F_2F_5^{-1}F_4^{-1} = 1, \\
& F_2^{-1}F_3F_1F_2^{-1}F_5^2F_2^{-1}F_3F_4F_1F_5F_3^{-1}F_4F_5F_2^{-1}F_3 = 1, \\
& F_1^{-1}F_4^{-1}F_3^{-1}F_2F_5^{-1}F_2^{-1}F_1F_4F_1F_5F_3^{-1}F_2F_5^{-1}F_4^{-1}F_1F_2^{-1}F_5F_4^{-1} = 1, \\
& F_1F_4F_3^{-1}F_4F_5F_2^{-1}F_3F_4F_1F_5F_3^{-1}F_2F_5^{-1}F_4^{-1}F_3^{-1}F_2F_5^{-1}F_2^{-1} = 1, \\
& F_5F_4^{-1}F_1^{-1}F_2F_5F_2^{-1}F_3F_1F_2^{-1}F_5F_4^{-1}F_3F_1F_5F_3^{-1}F_2F_5^{-1}F_4^{-1}F_2F_5F_2^{-1}F_3F_4F_1 = 1 \rangle
\end{aligned}
$$

Table 6.18: Presentation for the group K_2 of Lemma 6.7.2

Table 6.19 describes each generator of K_2 as given in Table 6.18 in terms of the generators of Γ^+

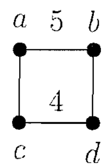
Generator of K_1	Representative in Γ^+
F_1	$bacbadba$
F_2	$cbabdaba$
F_3	$(cdab)^2$
F_4	$dabacbab$
F_5	$acbacdabab$

Table 6.19: Representing the generators of K_2 as elements of Γ^+

Both K_1 and K_2 have abelianization $\mathbb{Z}_2 \oplus \mathbb{Z}_4^4$. Both groups have 15 conjugacy classes of subgroups of index 2 whose abelianization has positive first Betti number, none of which is normal in Γ^+. Their intersection $K_1 \cap K_2$ has abelianisation $\mathbb{Z}^{59} \oplus \mathbb{Z}_2^{30}$.

6.8 $\Gamma = T_8[2,4,3;2,5,3]$

Recall that $T_8[2,4,3;2,5,3]$ is the Coxeter group with Coxeter diagram



where a, b, c and d are Coxeter generators of Γ. Let $\alpha = ab$, $\beta = bc$ and $\gamma = cd$. Then the orientation preserving subgroup Γ^+ has the following presentation:

$$\Gamma^+ = \langle \alpha, \beta, \gamma \mid \alpha^5 = \beta^3 = \gamma^4 = (\alpha\beta)^2 = (\beta\gamma)^2 = (\alpha\beta\gamma)^3 = 1 \rangle$$

6.8.1 Torsion in Γ and in Γ^+

Representatives of the conjugacy classes of torsion elements in Γ are given in Table 6.20. The restriction of this list to representatives of classes lying in Γ^+ is given in Table 6.21.

Order	Representative
Order 2	a, ac, bd, $(cd)^2$, $(adc)^3$, $(bcd)^3$, $(abc)^5$, $(dab)^5$
Order 3	bc, ad
Order 4	cd, $a(cd)^2$, $b(cd)^2$
Order 5	ab, $(ab)^2$
Order 6	adc, bcd, $c(ab)^2$, $d(ab)^2$
Order 10	abc, $(abc)^3$

Table 6.20: Conjugacy class representatives for elements of finite order in Γ

Order	Representative
Order 2	ac, bd, $(cd)^2$
Order 3	bc, ad
Order 4	cd
Order 5	ab, $(ab)^2$

Table 6.21: Conjugacy class representatives for elements of finite order in Γ^+

6.8.2 Maps to finite groups

Lemma 6.8.1 Γ^+ *contains no normal subgroups whose factor group is* A_5.

Proof: Suppose that there exists some $K \lhd \Gamma^+$ whose factor group $\Gamma^+/K \cong A_5$. Let $\theta : \Gamma^+ \twoheadrightarrow A_5$ be the induced permutation representation. Adding $\gamma = 1$ to the presentation for Γ^+ gives the trivial group, so $\gamma \not\mapsto 1$. Write $\alpha \mapsto (1,3,5,2,4)$ and $\beta \mapsto (1,2,3)$. Since A_5 contains no elements of order 4, let γ map to some element of order 2. A quick check reveals that no element of order 2 in A_5 satisfies the required conditions. ∎

Lemma 6.8.2 Γ^+ *contains no normal subgroups whose factor group is* $L_2(7)$.

Proof: Suppose that there exists some $K \lhd \Gamma^+$ whose factor group $\Gamma^+/K \cong L_2(7)$. Let $\theta : \Gamma^+ \twoheadrightarrow L_2(7)$ be the induced permutation representation. Since $7 \equiv \pm 2 \mod 5$, $L_2(7)$ contains no element of order 5, so the conjugacy class $\mathcal{C}(\alpha)$ lies in K. Adding the relation $\alpha = 1$ to the presentation for Γ^+ gives the trivial group. So K cannot exist. ∎

Lemma 6.8.3 *There exists a unique torsion free normal subgroup K of Γ^+ whose factor group $\Gamma^+/K \cong A_6 \cong L_2(9)$*

Proof: Write

$$\alpha \mapsto \frac{1}{2}\begin{pmatrix} t-f & e+1 \\ -1+e & t+f \end{pmatrix} \text{ and } \beta \mapsto \frac{1}{2}\begin{pmatrix} e+1 & t+f \\ f-t & 1-e \end{pmatrix}$$

where $e^2+f^2 = -2-t$, $t = \frac{-1\pm\sqrt{5}}{2}$. Then

$$\gamma \mapsto \begin{pmatrix} w & x \\ x+k & u-w \end{pmatrix}$$

where $u = \pm\sqrt{2}$, $k = \pm 1$, and x satisfies the quadratic equation

$$\begin{aligned}(8+4t)x^2 + (-4fu - 4ftk + 4tk + 8k)x+ & \\ (1+3t+f^2-2utk-2ft-2fuk) &= 0.\end{aligned} \tag{6.3}$$

Taking everything mod 3, $t = 2(2\pm\sqrt{2})$, since $\sqrt{5} \equiv \sqrt{2}$ mod 3.

Case I

Suppose first that $t = -(2+\sqrt{2}) = 1+2\sqrt{2}$. Then letting $e = 1+2\sqrt{2}$ and $f = 0$, $e^2+f^2 = \sqrt{2} = -2-t$ and equation 6.3 becomes

$$2\sqrt{2}x^2 + 2\sqrt{2}kx + (1+uk+2\sqrt{2}uk) = 0 \tag{6.4}$$

which has discriminant $D = \sqrt{2}+2+uk+\sqrt{2}uk$. Now $u = \pm\sqrt{2}$ and $k = \pm 1$. Then $D \in \mathbb{F}_3$ if and only if $\sqrt{2}+\sqrt{2}uk = 0$ if and only if sign(u) =sign(k) and this forces $D = 0$. Otherwise $D = \sqrt{2}+2+2+\sqrt{2} = 1+2\sqrt{2}$. For this to be a square in $\mathbb{F}_3(\sqrt{2})$ there must exist $a, b \in \mathbb{F}_3$ with $(a+b\sqrt{2})^2 = a^2+2b^2+2ab\sqrt{2} = 1+2\sqrt{2}$, so $a^2+2b^2 = 1$ (1) and $2ab = 2$ (2). (2) forces $a, b \in \mathbb{F}_3^* = \{1, 2\}$. Then $a^2 = b^2 = 1$, and hence $a^2+2b^2 \equiv 0$ mod 3. Thus D is not a square.

1) Letting $u = \sqrt{2}$ and $k = -1$ gives x as a solution to the quadratic $2x^2+x+2 = 0$. By the previous paragraph $D = 0$ so there exists a unique solution, $x = 2$. Since $2ew = eu - u - tk$ (chapter 4 § 4.9), substituting in for e, u, t and k gives $2(1+2\sqrt{2})w = 2+2\sqrt{2}$. Multiplying both sides by $(2+\sqrt{2})^{-1} = 1+\sqrt{2}$ gives $w = \sqrt{2}$. Consequently

$$\gamma \mapsto \begin{pmatrix} \sqrt{2} & 2 \\ 1 & 0 \end{pmatrix}$$

2) Letting $u = -\sqrt{2}$ and $k = 1$ gives x as a solution to the quadratic $x^2+x+1 = 0$. Again it is required that $D = 0$ so there is a unique solution for x, $x = 1$. In this case $2ew = eu - u - tk$ becomes $2(1+2\sqrt{2})w = 1+\sqrt{2}$. Multiplying both sides by $(2+\sqrt{2})^{-1} = 1+\sqrt{2}$ gives $w = 2\sqrt{2}$. Consequently

$$\gamma \mapsto \begin{pmatrix} 2\sqrt{2} & 1 \\ 2 & 0 \end{pmatrix}$$

Since the value of γ in both solutions differs only by ± 1, both solutions are equivalent.

Case II

Now let $t = -(2-\sqrt{2}) = 1+\sqrt{2}$. Then letting $e = 1+\sqrt{2}$ and $f = 0$, $e^2+f^2+3 \equiv t^2 \mod 3$ and equation 6.3 becomes

$$\sqrt{2}x^2 + \sqrt{2}kx + (1 + uk + \sqrt{2}uk) = 0 \tag{6.5}$$

which has discriminant $D = 2 + 2\sqrt{2} + uk + 2\sqrt{2}uk$. Then $D \in \mathbb{F}_3$ if and only if $2\sqrt{2}+\sqrt{2}uk = 0$, forcing sign(u) =sign(k). If sign(u) $\neq$sign(k) then $D = 1+\sqrt{2}$. If there is $a + b\sqrt{2} = \omega \in \mathbb{F}_3(\sqrt{2})$ with $a, b \in \mathbb{F}_3$ such that $\omega^2 = D$ then $a^2 + 2b^2 = 1$ and $2ab = 1$. $2ab = 1$ forces $a, b \neq 0$ so $a^2 = b^2 = 1$ and $0 = 1$, a contradiction. Hence sign(u) =sign(k) and $D = 0$.

1) Letting $u = \sqrt{2}$ and $k = 1$ gives x as a solution to the quadratic $x^2+2x+ = 0$. By the previous paragraph $D = 0$ so there exists a unique solution, $x = 2$. Since $2ew = eu - u - tk$ (chapter 4 § 4.9), substituting in for e, u, t and k gives $2(1+\sqrt{2})w = 1+2\sqrt{2}$. Multiplying both sides by $(2+2\sqrt{2})^{-1} = 1+2\sqrt{2}$ gives $w = \sqrt{2}$. Consequently

$$\gamma \mapsto \begin{pmatrix} \sqrt{2} & 1 \\ 2 & 0 \end{pmatrix}$$

2) Letting $u = -\sqrt{2}$ and $k = -1$ gives x as a solution to the quadratic $x^2+x+ = 0$. $D = 0$ gives a unique solution $x = 1$. Again, using $2ew = eu - u - tk$, $2(1+\sqrt{2})w = 2+\sqrt{2}$. Multiplying both sides by $(2+2\sqrt{2})^{-1} = 1+2\sqrt{2}$ gives $w = 2\sqrt{2}$ and as a result

$$\gamma \mapsto \begin{pmatrix} \sqrt{2} & 2 \\ 1 & 0 \end{pmatrix}$$

Since the value of γ in both solutions differs only by ± 1, both solutions are equivalent. Finally, there is an automorphism of A_6 that transposes the two epimorphisms, so there is a unique such kernel. ■

Note: The above proof is included as a demonstration of theory provided in Chapter 4. A more elementary proof would go as follows: Since both conjugacy classes of A_5 in A_6 are conjugate under the outer automorphism of S_6, it suffices to consider only $A_5 \cong \langle (1,3,5,2,4),(1,2,3)\rangle$. Then there exists a unique element g of order 5 in A_6 satisfying $|(1,2,3)g| = 2$ and $|(2,4,3,5)g| = 3$.

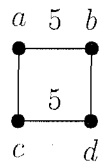
The group Γ^+ is perfect: Abelianizing the presentation for Γ^+, the identities $\alpha^4 = \beta^3 = \gamma^4 = 1$, $(\alpha\beta)^2 = \alpha^2\beta^2 = 1$, $(\beta\gamma)^2 = \beta^2\gamma^2 = 1$ and $(\alpha\beta\gamma)^3 = \alpha^3\gamma^3 = 1$ are obtained. Squaring $\beta^2\gamma^2$ gives $\beta = 1$. Hence $\alpha = 1$, so $\gamma^4 = \gamma^3 = 1$, and so Γ^+ is a perfect group. Combining the above results, the following theorem can be stated:

Theorem 6.5 *There exists a unique torsion free normal subgroup K of Γ^+ with factor group A_6. Furthermore, this subgroup K is the smallest index torsion free normal subgroup of Γ^+.*

Theorem 6.5 describes the smallest index torsion free normal subgroup, but this is not the smallest index torsion free subgroup of Γ^+. There is a unique normal subgroup N whose factor group is $L_2(11)$. The conjugacy class $\mathcal{C}(\gamma^2)$ of elements of order 2 lies in K. Let θ be the induced map $\Gamma^+ \twoheadrightarrow L_2(11)$. Then the preimage $K.C_{11}$ of the subgroup C_{11} of $L_2(11)$ of order 11 under the map θ is an index 60 subgroup which avoids all conjugacy classes of torsion, excepting the class $\mathcal{C}(\gamma^2)$ of elements of order 2. Using GAP, torsion free subgroups of $K.C_{11}$ of of index 2 were found.

6.9 $\Gamma = T_9[2,3,5;2,3,5]$

Recall that $T_9[2,5,3;2,5,3]$ is the Coxeter group with Coxeter diagram



where a, b, c and d are Coxeter generators of Γ. Let $\alpha = ab$, $\beta = bc$ and $\gamma = cd$. Then the orientation preserving subgroup Γ^+ has the following presentation:

$$\Gamma^+ = \langle \alpha, \beta, \gamma \mid \alpha^5 = \beta^3 = \gamma^5 = (\alpha\beta)^2 = (\beta\gamma)^2 = (\alpha\beta\gamma)^3 = 1 \rangle$$

6.9.1 Torsion in Γ and in Γ^+

Representatives of the conjugacy classes of torsion elements in Γ are given in Table 6.22. The restriction of this list to representatives of classes lying in Γ^+ is given in Table 6.23.

Order	Representative
Order 2	a, ac, bd
Order 3	bc, ad
Order 5	ab, $(ab)^2$, cd, $(cd)^2$
Order 6	$c(ab)^2$, $d(ab)^2$, $a(dc)^2$, $b(cd)^2$
Order 10	abc, bad, cda, bcd, $(abc)^3$, $(bad)^3$, $(cda)^3$, $(bcd)^3$

Table 6.22: Conjugacy class representatives for elements of finite order in Γ

Order	Representative
Order 2	ac, bd
Order 3	bc, ad
Order 5	ab, cd, $(ab)^2$, $(cd)^2$

Table 6.23: Conjugacy class representatives for elements of finite order in Γ^+

6.9.2 Maps to $L_2(5) \cong A_5$

Lemma 6.9.1 *There are three distinct torsion free normal subgroups K_1, K_2 and K_3 of Γ^+ with factor group A_5.*

Proof: Let $\alpha \mapsto \overline{\alpha} = (1,3,5,2,4)$ and $\beta \mapsto \overline{\beta} = (1,2,3)$. Then, for $\overline{\gamma} \in \{(1,3,5,2,4),(1,5,3,4,2),(1,5,3,2,4)\}$, the conditions $|\overline{\gamma}| = 5$, $|\overline{\beta}\overline{\gamma}| = 2$ and $|\overline{\alpha}\overline{\beta}\overline{\gamma}| = 3$ are satisfied. Hence there are three torsion-free normal subgroups whose factor group is A_5. The elements $(1,3,5,2,4)$ and $(1,5,3,4,2)$ lie in the same conjugacy class, while $(1,5,3,4,2)$ lies in the other conjugacy class of elements of order 5 in A_5. To prove that these groups are distinct, it suffices to consider the image of the word $\alpha\gamma$ under the three maps.

Setting $\overline{\gamma} = (1,5,3,4,2)$ gives $|\overline{\alpha\gamma}| = 5$, letting $\overline{\gamma} = (1,5,3,4,2)$ gives $|\overline{\alpha\gamma}| = 3$ and letting $\overline{\gamma} = (1,5,3,2,4)$ gives $|\overline{\alpha\gamma}| = 2$. Consequently, the epimorphisms are inequivalent and so there are precisely three distinct normal subgroups, as desired. ■

The above lemma gives three values for γ satisfying the presentation of Γ^+. Some facts are known about the manifolds in each case:

Case I: $\alpha \mapsto (1,3,5,2,4)$, $\beta \mapsto (1,2,3)$ so $\alpha\beta = (2,4)(3,5)$. Then let $\gamma \mapsto (1,3,5,2,4)$ (ie $\gamma = \alpha$), so $\alpha\gamma \mapsto (1,5,2,3,4)$. The kernel of this map, K_1, contains the subgroup $\langle\langle(\alpha\gamma)^5\rangle\rangle_{\Gamma^+}$ (this subgroup has infinite index in Γ^+) and K_1 has abelianisation $\mathbb{Z}_2 \oplus \mathbb{Z}_3^4 \oplus \mathbb{Z}_4^4 \oplus \mathbb{Z}_5^3$.

Case II: $\alpha \mapsto (1,3,5,2,4)$, $\beta \mapsto (1,2,3)$ so $\alpha\beta = (2,4)(3,5)$. If $\gamma \mapsto (1,5,3,4,2)$ is conjugate to α and $\alpha\gamma \mapsto (1,4,5)$. The kernel of this map, K_2, contains the subgroup $\langle\langle(\alpha\gamma)^3\rangle\rangle_{\Gamma^+}$ with finite index (this subgroup has index 622,080,000 in Γ^+) and K_2 has abelianisation $\mathbb{Z}_2 \oplus \mathbb{Z}_3^4 \oplus \mathbb{Z}_4^4 \oplus \mathbb{Z}_5^3$.

Case III: $\alpha \mapsto (1,3,5,2,4)$, $\beta \mapsto (1,2,3)$ so $\alpha\beta = (2,4)(3,5)$. Then let $\gamma \mapsto (1,5,3,2,4)$, so $\gamma \not\sim \alpha$ in A_5, but is in S_5, and $\alpha\gamma \mapsto (1,2)(4,5)$. The kernel of this map, K_3, is the subgroup $K = \langle\langle(\alpha\gamma)^2\rangle\rangle_{\Gamma^+}$ and K_3 has abelianisation $\mathbb{Z}^{11}$. Consequently there exists a k-celled manifold arising from this Coxeter group for every $k \in \mathbb{N}$. The associated manifold $\mathcal{M}_z = \mathbb{H}^3/K_3$ is Zimmermann's manifold [Z].

The group K_3 is generated by the words shown in Table 6.24

$$(bcda)^2, (cdab)^2, (dabc)^2, (dcba)^2, acdabcdaba, ((dabc)^2)^a, babacdcdca,$$
$$babcbabada, babcdcdcba, badababcba, bcbabadaba, cbdacdcadb$$

Table 6.24: A generating set for K_3

As an abstract group, K_3 has a presentation as given in Table 6.25. The relationship between the generators of K_3 as a subgroup of Γ^+ and the generators of K_3 as an abstract group is summarised in Table 6.26

$$\begin{aligned}
K_3 = \langle F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9, F_{10}, F_{11} \mid \\
& F_7^{-1}F_9F_4^{-1}F_2F_7F_2^{-1}F_9^{-1}F_4 = 1, F_{10}^{-1}F_3^{-1}F_5^{-1}F_4F_{10}F_1F_4^{-1}F_3F_5F_1^{-1} = 1, \\
& F_6F_8F_2^{-1}F_6^{-1}F_4F_{11}F_8^{-1}F_4^{-1}F_2F_{11}^{-1} = 1, \\
& F_{11}F_2^{-1}F_5F_3F_{10}F_{11}^{-1}F_5^{-1}F_{10}^{-1}F_3^{-1}F_2 = 1, \\
& F_{10}^{-1}F_3^{-1}F_9^{-1}F_4F_7^{-1}F_9F_1^{-1}F_7F_{10}F_1F_4^{-1}F_3 = 1, \\
& F_9F_6^{-1}F_3F_7^{-1}F_1F_9^{-1}F_7F_4^{-1}F_6F_3^{-1}F_4F_1^{-1} = 1, \\
& F_5^{-1}F_{10}^{-1}F_8F_2^{-1}F_5F_3F_{10}F_3^{-1}F_6F_2F_8^{-1}F_6^{-1} = 1, \\
& F_6F_{10}F_5F_{11}F_7F_8^{-1}F_6^{-1}F_5^{-1}F_{10}^{-1}F_8F_{11}^{-1}F_7^{-1} = 1, \\
& F_8F_7^{-1}F_1F_9^{-1}F_7F_4^{-1}F_6F_2F_8^{-1}F_6^{-1}F_2^{-1}F_4F_1^{-1}F_9 = 1, \\
& F_{10}^{-1}F_8F_{11}^{-1}F_7^{-1}F_9F_5^{-1}F_2F_8^{-1}F_{10}F_5F_{11}F_7F_2^{-1}F_9^{-1} = 1, \\
& F_{11}F_8^{-1}F_9^{-1}F_7F_4^{-1}F_{10}^{-1}F_8F_{11}^{-1}F_7^{-1}F_9F_3F_{10}F_3^{-1}F_4 = 1, \\
& F_5^{-1}F_2F_8^{-1}F_6^{-1}F_2^{-1}F_4F_1^{-1}F_7F_4^{-1}F_5F_6F_8F_7^{-1}F_1 = 1, \\
& F_6F_3^{-1}F_2F_{11}F_2^{-1}F_4F_{10}F_9F_6^{-1}F_3F_{11}^{-1}F_4^{-1}F_3F_{10}^{-1}F_3^{-1}F_9^{-1} = 1\rangle
\end{aligned}$$

Table 6.25: A presentation for K_3

Generator in Γ	Image in K_3
$(dcba)^2$	F_1
$babacdcdca$	F_2
$babcbabada$	F_3
$babcdcdcba$	F_4
$badababcba$	F_5
$bcbabadaba$	F_6
$(cdab)^2$	$F_{10}^{-1}F3^{-1}F_5^{-1}F_2F_8^{-1}F_6^{-1}$
$acdabcdaba$	$F_7F_2^{-1}F_9^{-1}F_4F_7^{-1}F_9F_1^{-1}F_9$
$(bcda)^2$	$F_4F_7^{-1}F_9F_1^{-1}F_7$
$(cdab)^2$	$F_7F_2^{-1}F_9^{-1}F_4F_7^{-1}F_9F_1^{-1}F_7F_4^{-1}$
$((dabc)^2)^a$	$F_{10}^{-1}F_3^{-1}F_5^{-1}F_2F_8^{-1}F_{10}F_5$
$cbdacdcadb$	$F_7F_{11}F_2^{-1}F_5F_3F_{10}F_3^{-1}F_6F_9^{-1}$

Table 6.26: The images of the generators of K_3 as a subgroup of Γ^+ in H

As has been previously observed, $H_1/[H_1, H_1] \cong \mathbb{Z}^{11}$. The induced action of $A_5 \cong \Gamma^+/H_1$ on $H_1/[H_1, H_1] \cong \mathbb{Z}^{11}$ equips $\mathbb{Z}^{11}$ with a module structure $\theta : A_5 \hookrightarrow GL_{11}(\mathbb{Z})$. The injection θ is defined by the two maps:

$$\alpha \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\beta \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ -1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

According to the Atlas of Finite Groups [Atlas], there are 5 irreducible characters for A_5, with degrees, 1, 3, 3, 4, and 5, so this representation cannot be primitive. Hence the character splits as a sum of irreducible characters. The traces of the conjugacy class representatives of A_5 in this representation are given as follows: $\mathrm{tr}(\alpha) = 1$, $\mathrm{tr}(\alpha^2) = 1$, $\mathrm{tr}(\beta) = -1$ and $\mathrm{tr}(\alpha\beta) = -1$. A calculation involving characters of representatives of the conjugacy classes of elements of A_5 gives this module as a sum of the 3-, 3- and 5-dimensional irreducible modules of A_5. So, for any positive integer n, there exists a covering manifold $\mathcal{M}_z(n)$ of $\mathcal{M}_z$ whose covering transformations (deck transformations) form an abelian group, and over which the isometry group extends to the group $A_5 \rtimes \mathbb{Z}_n^{11}$.

6.9.3 Further interesting subgroups

Lemma 6.9.2 *There are two distinct normal subgroups K_4 and K_5 of Γ^+ whose factor group is $A_6 \cong L_2(9)$.*

Proof: There are two conjugacy classes of A_5 in A_6. Letting $\alpha \mapsto \alpha_1 = (13524)$ and $\beta \mapsto \beta_1 = (123)$, we can have $\gamma_1 \in \{(14326), (16325)\}$ satisfying $|\gamma_1| = 5$, $|\beta_1\gamma_1| = 2$ and $|\alpha_1\beta_1\gamma_1| = 3$. Note that α_1^2 is conjugate to γ_1 in each case. Letting $\alpha \mapsto \alpha_2 = (13542)$ and $\beta \mapsto \beta_2 = (143)$ we get $\gamma_2 \in \{(14326), (16325)\}$. Notice that conjugating by $(24) \in S_6 \backslash A_6$ will transpose α_1, β_1 and γ_1 with α_2, β_2 and γ_2 ∎

Lemma 6.9.3 *There is a unique normal subgroup K_6 of Γ^+ whose factor group is $L_2(11)$.*

Proof: Let $\phi : \Gamma^+ \longrightarrow L_2(11)$ be a map from Γ^+ to $L_2(11)$ as constructed in § 4.10, and $K = \mathrm{Ker}(\theta)$ be the kernel of the map. There are two cases to consider: the trace($\overline{\alpha}$) =trace($\overline{\gamma}$) and trace($\overline{\alpha}$) ≠trace($\overline{\gamma}$).

Case 1 - trace($\overline{\alpha}$) = trace($\overline{\gamma}$):

Suppose first that trace($\overline{\alpha}$) = trace($\overline{\gamma}$) = t. Then $t = (-1 \pm \sqrt{5})/2 = 7$ or 3. If trace($\overline{\alpha}$) = 7 then

$$\alpha \mapsto \overline{\alpha} = \begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix} \text{ and } \beta \mapsto \overline{\beta} = \begin{pmatrix} 1 & 4 \\ 8 & 0 \end{pmatrix}$$

Suppose first that trace($\overline{\alpha}\overline{\beta}\overline{\gamma}$) = 1: then there is no solution for $\gamma \mapsto g \in L_2(11)$. If trace($\overline{\alpha}\overline{\beta}\overline{\gamma}$) = −1 then two distinct solutions for γ:

$$\gamma \mapsto g_1 = \begin{pmatrix} 4 & 0 \\ -1 & 3 \end{pmatrix} = (\overline{\alpha})^{\overline{\beta}} \text{ and } \gamma \mapsto g_2 = \begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix} = \overline{\alpha}$$

Hence two epimorphisms θ_1, $\theta_2 : \Gamma^+ \longrightarrow A_5$ are constructed. The kernels K_{θ_1}, K_{θ_2} of these maps are distinct torsion free normal subgroups of Γ^+ whose factor group Γ^+/K_{θ_i} is an A_5 subgroup of $L_2(11)$. If trace($\overline{\alpha}$) = 3 then

$$\alpha \mapsto \overline{\alpha} = \begin{pmatrix} 10 & 3 \\ 2 & 4 \end{pmatrix} \text{ and } \beta \mapsto \overline{\beta} = \begin{pmatrix} 3 & 4 \\ 1 & 9 \end{pmatrix}$$

Assume first that trace($\overline{\alpha}\overline{\beta}\overline{\gamma}$) = 1. Then, solving the equations in § 4.10, a unique solution

$$\gamma \mapsto g_3 = \begin{pmatrix} 10 & 8 \\ 9 & 4 \end{pmatrix}$$

is recovered. Since the order of $\overline{\alpha}g_3$ is six, $\overline{\alpha}g_3 \notin A_5 < L_2(11)$, so $g_3 \notin A_5$. As A_5 is maximal in $L_2(11)$, it follows that the images $\overline{\alpha}$, $\overline{\beta}$ and g_3 generate $L_2(11)$. Hence the kernel of the map, K_{θ_3}, is a distinct torsion free normal subgroup of Γ^+ whose factor group Γ^+/K_{θ_3} isomorphic to $L_2(11)$. Next, assuming that trace$(\overline{\alpha}\overline{\beta}\overline{\gamma}) = -1$ then, solving the equations in § 4.10, two distinct solutions

$$\gamma \mapsto g_4 = \begin{pmatrix} 10 & 3 \\ 2 & 4 \end{pmatrix} = \overline{\alpha} \text{ and } \gamma \mapsto g_5 = \begin{pmatrix} 4 & 9 \\ 8 & 10 \end{pmatrix} = (\overline{\alpha})^{\overline{\beta}}$$

are recovered. The kernels K_{θ_4}, K_{θ_5} of these maps are distinct torsion free normal subgroups of Γ^+ whose factor group Γ^+/K_{θ_i} is an A_5 subgroup of $L_2(11)$.

Case 2 - trace(α) $\neq$ trace(γ):

Now suppose that trace$(\overline{\alpha}) \neq$ trace$(\overline{\gamma})$. As in the previous case, $t = (-1 \pm \sqrt{5})/2 =$ 7 or 3. If trace$(\overline{\alpha}) = 7$ then

$$\alpha \mapsto \overline{\alpha} = \begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix} \text{ and } \beta \mapsto \overline{\beta} = \begin{pmatrix} 1 & 4 \\ 8 & 0 \end{pmatrix}$$

Suppose first that trace$(\overline{\alpha}\overline{\beta}\overline{\gamma}) = 1$: then a unique solution

$$\gamma \mapsto g_6 = \begin{pmatrix} 10 & 8 \\ 9 & 4 \end{pmatrix} = (\overline{\alpha}^2)^{\overline{\beta}\overline{\alpha}^{-1}}$$

is recovered, and the image of α, β and γ generate an A_5 subgroup of $L_2(11)$. If trace$(\overline{\alpha}\overline{\beta}\overline{\gamma}) = 1$: then there is no solution for $\gamma \mapsto g \in L_2(11)$. If trace$(\overline{\alpha}) = 3$ then

$$\alpha \mapsto \overline{\alpha} = \begin{pmatrix} 10 & 3 \\ 2 & 4 \end{pmatrix} \text{ and } \beta \mapsto \overline{\beta} = \begin{pmatrix} 3 & 4 \\ 1 & 9 \end{pmatrix}$$

Assume first that trace$(\overline{\alpha}\overline{\beta}\overline{\gamma}) = 1$. Then, solving the equations in § 4.10, a unique solution

$$\gamma \mapsto g_7 = \begin{pmatrix} 3 & 10 \\ 0 & 4 \end{pmatrix} = (\overline{\alpha}^2)^{\overline{\beta}\overline{\alpha}^{-1}}$$

for γ is recovered, and the image of α, β and γ generate an A_5 subgroup of $L_2(11)$. If trace$(\overline{\alpha}\overline{\beta}\overline{\gamma}) = 1$: then there is no solution for $\gamma \mapsto g \in L_2(11)$.

Hence there is a unique map $\theta_3 : \Gamma^+ \twoheadrightarrow L_2(11)$. The kernel of the map K_{θ_3}, is a unique torsion free subgroup of Γ^+ whose factor group is $L_2(11)$. The epimorphism is characterised by the matrices

$$\alpha \mapsto \begin{pmatrix} 10 & 3 \\ 2 & 4 \end{pmatrix} \quad \beta \mapsto \begin{pmatrix} 3 & 4 \\ 1 & 9 \end{pmatrix} \quad \gamma \mapsto \begin{pmatrix} 10 & 8 \\ 9 & 4 \end{pmatrix}$$

Computer calculations with GAP yielded $K/K' \cong \mathbb{Z}^{41}$. ∎

Lemma 6.9.4 *There are two distinct normal subgroups K_7 and K_8 of Γ^+ whose factor group is $L_2(19)$.*

Proof: Proceeding as in the previous lemma, two non-conjugate maps $\theta_1, \theta_2 : \Gamma^+ \twoheadrightarrow L_2(19)$ were constructed, one for each case $t = t_1$, $u = t_2$ and $t = t_2$, $u = u_1$:

$$\alpha \mapsto \begin{pmatrix} 0 & 8 \\ 7 & 14 \end{pmatrix} \qquad \beta \mapsto \begin{pmatrix} 8 & 14 \\ 0 & 12 \end{pmatrix} \qquad \gamma \mapsto \begin{pmatrix} 17 & 7 \\ 6 & 7 \end{pmatrix}$$

$$\alpha \mapsto \begin{pmatrix} 0 & 13 \\ 16 & 5 \end{pmatrix} \qquad \beta \mapsto \begin{pmatrix} 8 & 1 \\ 0 & 12 \end{pmatrix} \qquad \gamma \mapsto \begin{pmatrix} 5 & 11 \\ 4 & 9 \end{pmatrix}$$

The kernels of the maps are distinct non-conjugate torsion free normal subgroups of Γ^+. In both cases the abelianization has the form $\mathbb{Z}_2 \oplus \mathbb{Z}_3^{37} \oplus \mathbb{Z}_5^{18} \oplus \mathbb{Z}_8^{18} \oplus \mathbb{Z}_{11}^{18} \oplus \mathbb{Z}_{17}^{20} \oplus \mathbb{Z}_{19}^{3}$ ∎

6.9.4 Maps to M_{12}

Γ^+ has a natural description as the quotient of a free product with amalgamation. Write

$$\begin{aligned} A_5 &= \langle \alpha, \beta \mid \alpha^5 = \beta^3 = (\alpha\beta)^2 = 1 \rangle \\ A_5' &= \langle \beta', \gamma \mid \gamma^5 = \beta'^3 = (\beta'\gamma)^2 = 1 \rangle \end{aligned}$$

and consider the free product with amalgamation, $A_5 *_{\{\beta=\beta'\}} A_5'$. This is a group with presentation

$$\langle \alpha, \beta, \gamma \mid \alpha^5 = \beta^3 = \gamma^5 = (\alpha\beta)^2 = (\beta\gamma)^2 = 1 \rangle$$

Then Γ^+ can be written as the quotient group $A_5 *_{\{\beta=\beta'\}} A_5'/\{(\alpha\beta\gamma)^3\}$.

If Γ^+ has a subgroup H of index n, then, using the action of Γ^+ on the cosets of H, a permutation group G with permutation representation on n points can be constructed. The kernel of the induced map $\Gamma^+ \to G$ is the core of H in Γ^+. If G has a minimal value n_0 of n for which it has a faithful permutation representation on n points, then no action of Γ^+ on subgroups of index less than n_0 can generate an induced map $\Gamma^+ \to G$.

The group A_5 has transitive permutation representations on 5, 6, 10 and 12 points, corresponding to right coset actions on subgroups A_4, D_{10}, D_6 and C_5. In each case the action is unique. Choose generators x, y of A_5 so that $x^2 = y^3 = (xy^2)^5 = 1$. Diagrams representing the actions of A_5 on n points (*Higman diagrams*), for $n = 5, 6, 10, 12$, can be constructed as shown in Figure 6.6 and Figure 6.7. In these diagrams, heavy dots represent points fixed by the action of y; triangles represent vertices permuted in an anti-clockwise direction by y and any two vertices interchanged by x are joined by a dotted line. As well as these transitive actions, A_5 has a trivial action on one vertex.

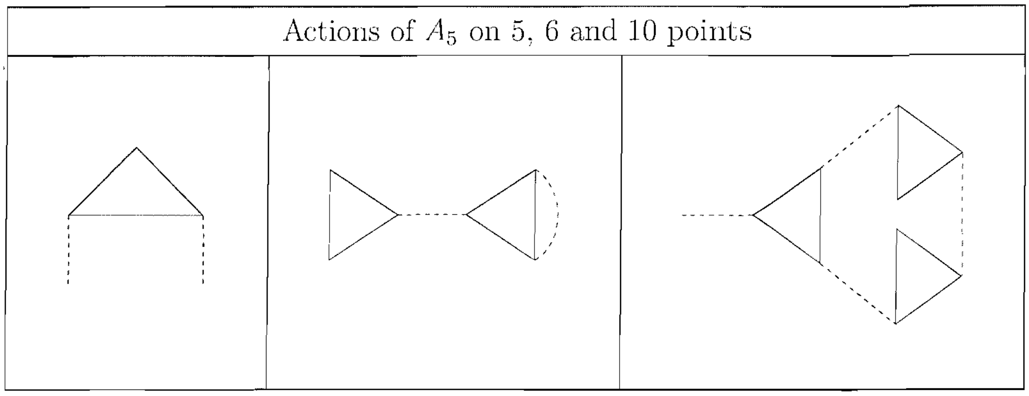


Figure 6.6: Higman diagrams for A_5 acting on 5, 6 and 10 points.

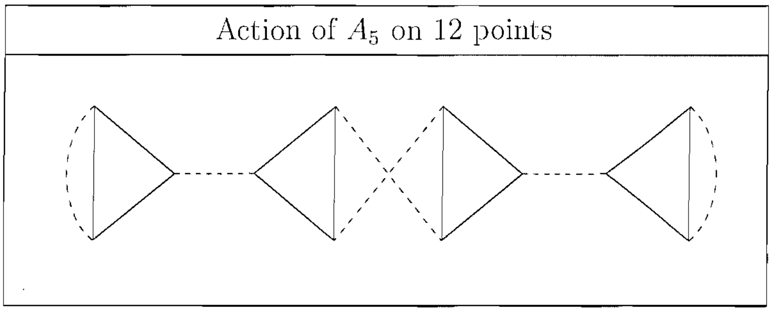


Figure 6.7: Higman diagram for A_5 acting on 12 points.

Recall that Γ^+ has presentation

$$\Gamma^+ = \langle \alpha, \beta, \gamma \mid \alpha^5 = \beta^3 = \gamma^5 = (\alpha\beta)^2 = (\beta\gamma)^2 = 1 \rangle$$

By writing $x = \alpha\beta$, $y = \beta$ and $z = \beta\gamma$, the presentation for Γ^+ can be rewritten as

$$\begin{aligned} \Gamma^+ &= \langle x, y, z \mid x^2 = y^3 = z^2 = 1, (xy^2)^5 = (y^2z)^5 = (xy^2z)^3 = 1 \rangle \\ &= \langle x, y, z \mid x^2 = y^3 = z^2 = 1, (xy^2)^5 = (y^2z)^5 = 1 \rangle / \{(xy^2z)^3\} \\ &= A_5 *_{y'=y} A_5' / \{(xy^2z)^3\} \end{aligned}$$

where $A_5 = \langle x, y \mid x^2 = y^3 = (xy^2)^5 = 1 \rangle$ and $A_5' = \langle y', z \mid y'^3 = z^2 = (y'^2 z)^5 = 1 \rangle$ Permutation representations for Γ^+ on n points can then be constructed by amalgamating Higman diagrams for A_5 on n points and adding the additional condition that $(xy^2z)^3 = 1$. The process goes as follows: suppose that a permutation representation of Γ^+ on n points, for some integer n, is required. Take two Higman diagrams D_1 and D_2, each describing the action of A_5 on n points. Let A_5 act on D_1 and A_5' act on D_2. These two diagrams are now amalgamated by identifying the triangles in D_1 with those in D_2. The resulting amalgamated diagram satisfies the relations of the free product with amalgamation, $A_5 *_{\{y'=y\}} A_5'$. The remaining vertices of D_1 not already identified with those of D_2 are then paired so that the final diagram satisfies the additional condition $(xy^2z)^3 = 1$.

It is instructive at this point to consider an example. Suppose that all possible permutation representations of Γ^+ on 5 points are desired. Take two copies, D_1 and D_2, of the diagram representing the action of A_5 on 5 points. Let D_1 describe the action of A_5 and D_2 describe the action of A_5'. Since Γ^+ can be expressed as the quotient of the free product $A_5 *_{\{y'=y\}} A_5'$, the two diagrams D_1 and D_2 can be amalgamated by identifying triangles. There are three possible ways of doing this, giving diagrams $\widetilde{D}_1$, $\widetilde{D}_2$ and $\widetilde{D}_3$. For each diagram $\widetilde{D}_i$, the condition $(xy^2z)^3 = 1$ in the presentation of Γ^+ is then used to figure out all admissible actions of z on the 5 points. Figure 6.8 illustrates all these admissible diagrams, which represent three distinct non-conjugate epimorphisms from Γ^+ to A_5, corresponding to three distinct normal subgroups of Γ^+ whose factor groups are all isomorphic to A_5. This verifies the results of Lemma 6.9.1, where it was shown that there are three distinct normal subgroups with factor group A_5.

The Mathieu group M_{12} is the automorphism group of the $S(5,6,12)$ Steiner system. The smallest index subgroup of M_{12} is the Mathieu group M_{11}, the automorphism group of the $S(4,5,11)$ Steiner system. There are two conjugacy classes of maximal subgroups M_{11} in M_{12}: one is the natural subgroup M_{11} acting on 11 points. A representative of the second class has a transitive representation on 12 points. The action by right multiplication of M_{12} on the 12 cosets of an M_{11} subgroup induces a permutation representation for M_{12} on 12 points. This is the smallest permutation representation for M_{12}. The rest of this section is devoted to proving that Γ^+ has two distinct normal subgroups N_1 and N_2 whose factor

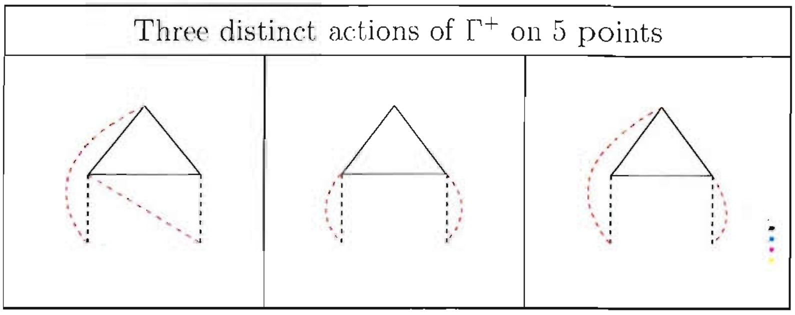


Figure 6.8: Representation of the maps $\Gamma^+ \to A_5$ constructed in Lemma 6.9.1. Black dotted lines correspond to the action of x on points, and red dotted lines correspond to the action of z on points.

group is M_{12}. Higman diagrams for the action of A_5 on 12 points will be combined as described above and GAP will be used to classify the resulting permutation groups.

Theorem 6.6 *There are two distinct normal subgroups K_{10} and K_{11} of Γ^+ whose factor group is M_{12}.*

Proof: Assume that $\Gamma^+ = A_5 *_{y'=y} A_5'/\{(xy^2z)^3\}$, where $A_5 = \langle x, y \mid x^2 = y^3 = (xy^2)^5 = 1\rangle$ and $A_5' = \langle y', z \mid y'^3 = z^2 = (y'^2z)^5 = 1\rangle$. Permutation representations of A_5 (and A_5') on 12 points are in 1-1 correspondence with partitions of 12 into orbits of A_5 of size 1, 5, 6, 10 and 12. Admissible partitions are

$$\{1,1,1,1,1,1,1,1,1,1,1,1\}, \{5,1,1,1,1,1,1,1\},$$

$$\{6,1,1,1,1,1,1\}, \{5,5,1,1\}, \{5,6,1\}, \{6,6\}, \{10,1,1\}, \{12\}$$

The proof now proceeds by listing all combinations of pairs of these partitions which yield transitive representations of $A_5 *_{y=y'} A_5'$ on 12 points and imposing the condition that $(xy^2z)^3 = 1$. The resulting permutation representation will describe a transitive permutation representation of Γ^+ on 12 points. In general, if a partition involves n_i orbits of length i, for $i = 1, 5, 6, 10, 12$, then y has $n_1 + 2n_5 + n_{10}$ fixed points, since it fixes 1, 2, 0, 1 or 0 points in each orbit of length 1, 5, 6, 10 or 12 points, respectively. Then the partition can only be combined with one where y' has the same number of fixed points.

There are a total of 49 possible pairs of partitions. The proof is divided into seven cases: in each case it is assumed that the partition induced by the action of A_5 is fixed. The only restriction on the second partition (the partition induced by A_5') is that the action of y' fixes the same number of points as y does.

1) **One of the partitions is $\{5, 1, 1, 1, 1, 1, 1, 1\}$:** In this case there are 9 points fixed by the action of y. Hence the second partition must also be of type $\{5, 1, 1, 1, 1, 1, 1, 1\}$ and there is no way of amalgamating these two diagrams to obtain a transitive representation on 12 points. So no transitive permutation diagram of an action of Γ^+ on 12 points can be constructed by amalgamating two A_5 actions with one of the actions being the $\{5, 1, 1, 1, 1, 1, 1, 1\}$ permutation representation of A_5 on 12 points.

2) **One of the partitions is $\{6, 1, 1, 1, 1, 1, 1\}$:** Since this contains 6 points stabilized by y, this partition can only be combined with partitions of type $\{5, 5, 1, 1\}$ or type $\{6, 1, 1, 1, 1, 1, 1\}$. Clearly any amalgamation of two copies of $\{6, 1, 1, 1, 1, 1, 1\}$ will result in an intransitive permutation representation with 6 fixed points, while any amalgamation of $\{6, 1, 1, 1, 1, 1, 1\}$ with $\{5, 5, 1, 1\}$ will have 2 fixed points. So no transitive permutation diagram of an action of Γ^+ on 12 points can be constructed by amalgamating two A_5 actions with one of the actions being the $\{6, 1, 1, 1, 1, 1, 1\}$ permutation representation of A_5 on 12 points.

3) **One of the partitions is $\{5, 5, 1, 1\}$:** Since this contains 6 points stabilized by y, then by the result of 2) this action can only be combined with $\{5, 5, 1, 1\}$. Let s_1 and s_2 be the two points fixed by the action of A_5. Transitivity forces the points s_1 and s_2 to be transposed by the action of $z \in A_5'$ with points not fixed by y. Similarly, x must transpose the points v_1 and v_2 fixed by y' and z. Then the only to construct a transitive diagram must be to construct a diagram similar to that of Figure 6.9

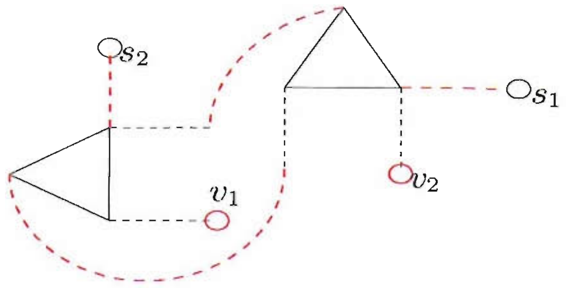


Figure 6.9: A possible transitive action of Γ^+ on 12 points

It only remains to find all such diagrams: Starting with a permutation representation of A_5 on 5 points, let S be a point permuted by the action of x and action of y. Let y' and z fix a vertex s_f. Since the final diagram must be transitive on 12 points, this vertex must be transposed by the action of x. Diagrams a) and c) in Figure 6.10 illustrate the two possible positions of s_f. The choice of S is dependent on the choice of s_f. For a fixed choice of the position of s_f (and consequently a fixed choice of the position of S), the image of S under the action of $(xy^2z)^2$ is investigated. Write $\mathcal{V} = (xy^2z)^2(S)$. There are 4 possible configurations and they are illustrated in Figure 6.10:

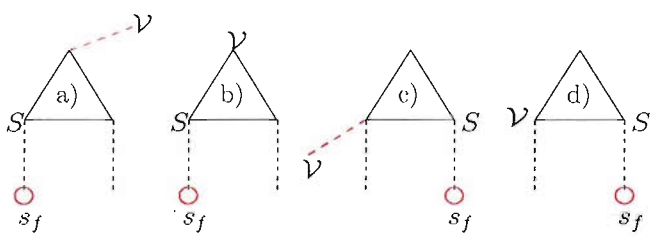


Figure 6.10: Images of S under the action of $(xy^2z)^2$

Connectivity excludes the cases a) and c), since in those cases it is impossible to adjoin an action of xy^2 such that $xy^2z(\mathcal{V}) = S$. In cases b) and d), the action of z fixes $\mathcal{V}$, so $xy^2z(\mathcal{V}) = S$ if and only if the action of z transposes two points permuted by y, which is impossible. So no transitive permutation diagram of an action of Γ^+ on 12 points can be constructed by amalgamating two A_5 actions with one of the actions being the $\{5,5,1,1\}$ permutation representation of A_5 on 12 points.

4) **One of the partitions is $\{5,6,1\}$:** Since this contains 3 points stabilized by y, this partition can only be combined with partitions of type $\{5,6,1\}$ or $\{10,1,1\}$. Suppose first that two diagrams of the form $\{5,6,1\}$ are used. Then transitivity forces a diagram containing the subscheme illustrated in Figure 6.11, with triangle 4 amalgamated either with triangle 2 or with triangle 3. However, no amalgamation of triangles 2 and 3 or triangles 2 and

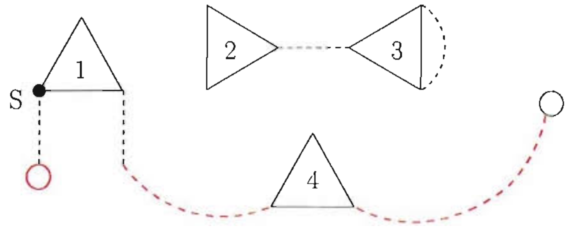


Figure 6.11: Partial subscheme containing the $\{5,6,1\}$ Higman diagram

4 satisfies the condition $(xy^2z)^3(s) = s$. Consequently all resulting permutation diagrams will fail to satisfy $(xy^2z)^3 = 1$. Suppose now that diagrams of the form $\{5,6,1\}$ and $\{10,1,1\}$ are used. Then the two vertices representing trivial actions of y' and z must be swapped by an action of x, while the vertex representing the trivial action of x and y must be swapped by an action of z. So a diagram as illustrated in Figure 6.12 is recovered, where triangle 4 is superimposed on one of triangles 1, 2 or 3.

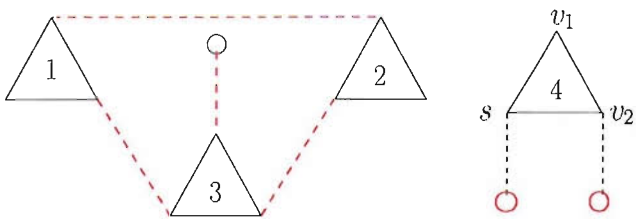


Figure 6.12: Partial subscheme containing the $\{5,6,1\}$ Higman diagram

Now, the action of $(xy^2z)^2$ sends s to $z(v_1)$. If z fixes v_1, then $(xy^2z)^3(s) = z(v_2) \neq s$, while if z swaps v_1, then a transitive permutation representation on 12 points must contain a subdiagram as shown in Figure 6.13, which is impossible.

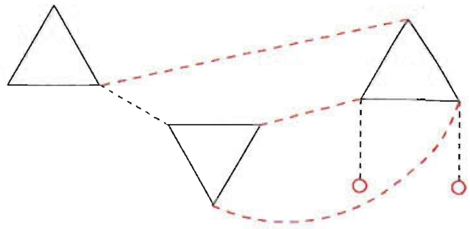


Figure 6.13: Diagram 3

So no transitive permutation diagram of an action of Γ^+ on 12 points can be constructed by amalgamating two A_5 actions with one of the actions being the $\{5,6,1\}$ permutation representation of A_5 on 12 points.

5) **One of the partitions is** $\{6,6\}$**:** Since this contains no points stabilized by y, this diagram can only be combined with a diagram of the form $\{6,6\}$ or a diagram of the form $\{12\}$. Suppose first that two diagrams of the form $\{6,6\}$ are used. Let Figure 6.14 represent the action of A_5 on 12 points. Then the action of A'_5 on 12 points is similar. To find all amalgamated diagrams $\widetilde{D}$ satisfying the relations $A_5 *_{\{y=y'\}} A'_5$ and the condition $(xy^2z)^3 = 1$, it suffices to consider all ways of adding the action of z to the diagram $\mathcal{D}$ in Figure 6.14 so that, for any vertex $\mathcal{V}$ in $\mathcal{D}$, $(xy^2z)^3(\mathcal{V}) = \mathcal{V}$.

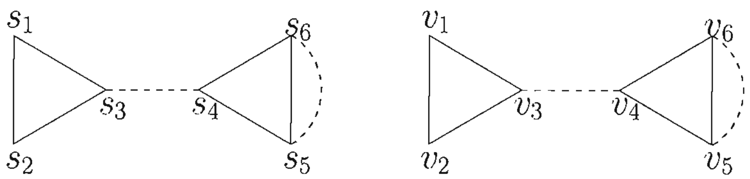


Figure 6.14: Diagram $\mathcal{D}$: A_5 action of type $\{6,6\}$ on 12 points

Suppose first that the action of z is to swap s_1 with some vertex v_i. If it assumed that z also swaps s_2 and s_3, then $(xy^2z)^2$ sends s_4 to $z(s_6)$. If z fixes s_6, then $(xy^2z)^3$ sends s_4 to $z(s_5)$. Hence $(xy^2z)^3(s_4) = z(s_5)$ and this is s_4 if and only if z swaps s_4 with s_5. Since the diagram for the action of y and z is of type $\{6,6\}$, this forces z to swap s_6 with some v_j, contradicting the assumption that it fixes s_6. So z swaps s_6 with some v_j. But then $(xy^2z)(s_4) = z(s_6) = v_j$, for some j and it is impossible for $xy^2z(v_j) = s_4$. So z cannot swap s_2 and s_3.

Let z fix s_2 and s_3. Then $(xy^2z)^2(s_4) = s_3$, so $(xy^2z)^3(s_4) = s_4$ if and only if $xy^2z(s_3) = z(s_6) = s_4$. Hence z swaps s_4 and s_6. This forces $z(s_5) = v_i$, for some i. But then $(zy^2z)^2(s_5) = v_i$ and $xy^2z(v_i) = s_5$ forces $z(s_5) = v_1$. Then $(xy^2z)^3(v_2) = v_2$ if and only if z swaps v_4 with v_6. Consequently z swaps s_1 and v_5. The resulting group action is illustrated in Figure 6.15 and describes the group $A_5 \rtimes (C_2 \times C_2 \times C_2 \times C_2)$

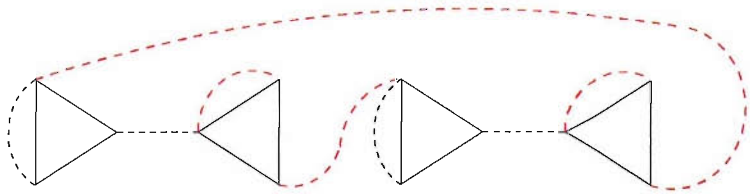


Figure 6.15: Action of Γ^+ on 12 points

By symmetry arguments, there is also only one solution if it is assumed that z swaps s_2 with some v_i. The last case is if z swaps s_3 with some v_i. Suppose then that z swaps s_1 and s_2. Then $(xy^2z)^2(s_1) = z(s_3)$, a contradiction, since there is no way of satisfying $xy^2z(v_i) = s_2$, for any i. So z fixes s_1 and s_2. Then $xy^2z(s_2) = z(s_3)$, and since $xy^2z(s_4) = s_3$, z must swap s_4 with some v_j. Now suppose that z swaps s_5 and s_6. Then $(xy^2z)^2(s_3) = z(s_4)$, or $xy^2z(z(s_4)) = s_3$. Similarly, $(xy^2z)^2(s_4) = z(s_3)$, or $xy^2z(z(s_3)) = s_4$. Combining these results, we get $s_4zxy^2z = s_3$ and $s_3zxy^2z = s_4$, so $s_4zxy^2z^2xy^2z = s_4$. Since $x = \alpha\beta = ac$, $y = \beta = bc$ and $z = \beta\gamma = bd$, the identity $zxy^2z^2xy^2z = 1$ becomes $zxy^2xy^2z = bdacbcbcacbcbcbd = bdababbd = (\alpha^2)^{\beta\gamma} = 1$, a contradiction. So z does not swap s_5 and s_6. Then $(xy^2z)^2(s_6) = z(s_4)$, again a contradiction.

Instead let the second partition be of type $\{12\}$. Let Figure 6.16 represent the action of A_5 on 12 points and Figure 6.14 represent the action of A_5'. Suppose first that z fixes s_1 and s_2, as shown in Figure 6.16:

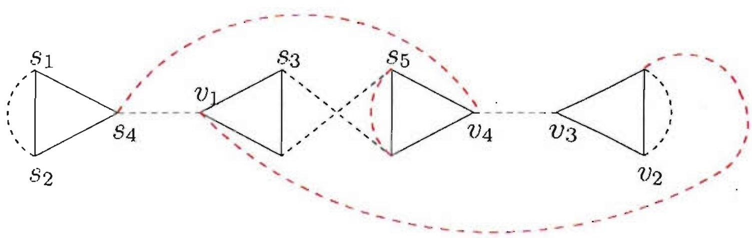


Figure 6.16: Diagram 5

If z fixes v_1, then $(xy^2z)(v_1) = s_2$, so z must transpose s_4 and s_5. Consequently, $(xy^2z)^2(s_3) = z(s_3)$. If z fixes s_3 then $(xy^2z)^3(s_4) = v_1 \neq s_2$, so z

cannot fix s_3. Since $(xy^2z)(z(s_3)) = s_3$, z must transpose v_2 and s_3. But then $(xy^2z)^3(v_3) = v_4 \neq v_3$. So v_1 cannot be fixed by z. Then there are only 3 possible partial diagrams with v_1 transposed by z, shown in Figure 6.17:

I)

II)

III)

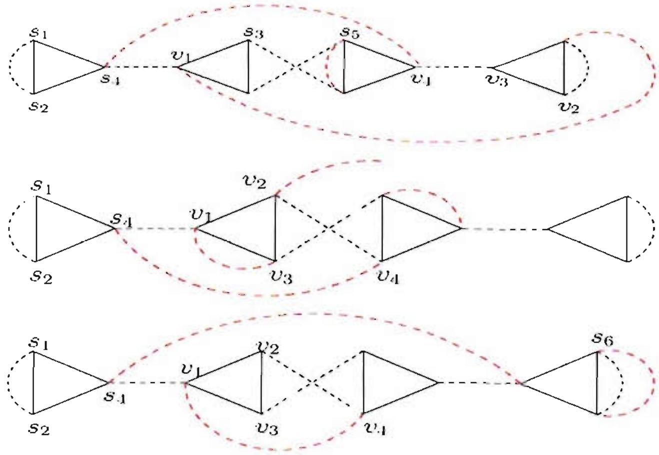


Figure 6.17: Possible diagrams

In I), $(xy^2z)^2(v_2) = z(v_3)$. If $z(v_3) = v_3$, then $(xy^2z)^3(v_2) = xy^2z(v_3) = s_3 \neq v_2$, so $z(v_3) = v_2$. But then $(xy^2z)^3(v_2) = v_4 \neq v_2$. In a similar manner, diagram II) fails (since $(xy^2z)^2(v_3) = v_3$) and diagram III) fails (there is no way to complete the diagram so that $(xy^2z)^3(s_6) = s_6$). Hence z cannot fix s_1 and s_2.

Now suppose that z transposes s_1 and s_2. Then there are two diagrams, illustrated in Figure 6.18, in which all conditions are met:

IV)

V)

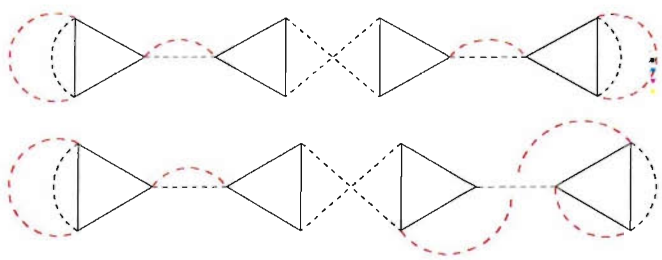


Figure 6.18: Admissible diagrams

Diagram IV) describes an action of $A_5 \ltimes (C_2 \times C_2 \times C_2 \times C_2 \times C_2)$ on 12 points, and diagram V) describes an action of A_{12} on 12 points.

The remaining case is when z fixes one of s_1 or s_2. By symmetry arguments, it suffices to consider when z fixes s_2 and transposes s_1 with some other point. Calculations similar to those above show that there are two admissible diagrams, both yielding an action of Γ^+ on 12 points. In each case the kernel in Γ^+ of this action is a normal subgroup whose factor group is $A_5 \ltimes ((C_2 \times C_2 \times C_2 \times C_2 \times C_2)$.

6) **One of the partitions is $\{10, 1, 1\}$:** Since this contains 3 points stabilized by y, this partition can only be combined with partitions of type $\{5, 6, 2\}$ or of type $\{10, 1, 1\}$. By 4) above, it suffices to consider only the case $\{10, 1, 1\}$. Since there are two points with trivial action and only one point fixed by the action of y but permuted by x or z, it is clear that there is no way of combining these diagrams to get a transitive representation on 12 points.

7) **One of the partitions is $\{12\}$:** The only remaining case is to combine this with $\{12\}$. Calculations similar to those done in the previous sections give 6 possible diagrams:

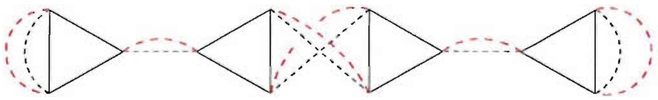


Figure 6.19: Transitive action of A_5 on 12 points with generators x, y and z

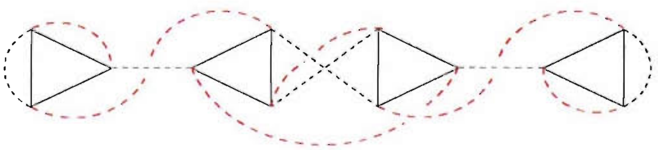


Figure 6.20: Transitive action of A_5 on 12 points with generators x, y and z

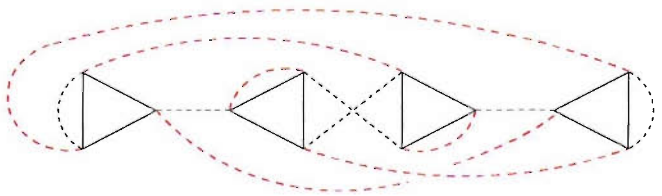


Figure 6.21: Transitive action of A_5 on 12 points with generators x, y and z

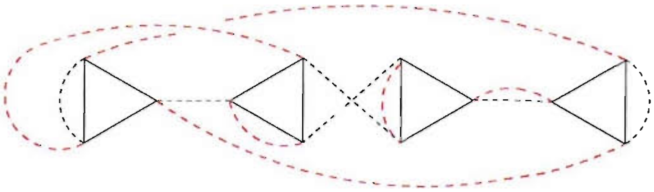


Figure 6.22: Transitive action of $L_2(11)$ on 12 points with generators x, y and z

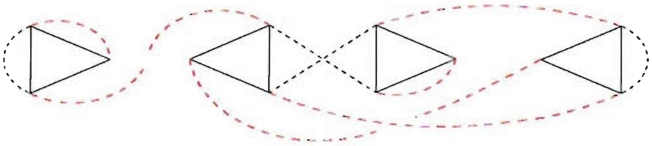


Figure 6.23: Transitive action of M_{12} on 12 points with generators x, y and z

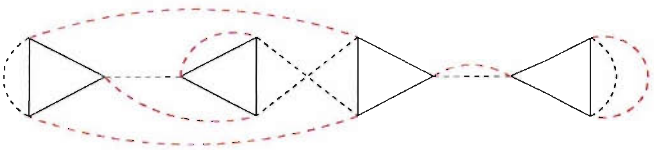


Figure 6.24: Transitive action of M_{12} on 12 points with generators x, y and z

The first three diagrams, Figures 6.19, 6.20 and 6.21, correspond to transitive actions of A_5 on 12 points and arise from the three maps $\Gamma^+ \to A_5$ constructed in Lemma 6.9.1. The fourth diagram, Figure 6.22, corresponds to a transitive action of $L_2(11)$ on 12 points. The associated epimorphism, $\Gamma^+ \to L_2(11)$, was constructed in Lemma 6.9.3. The final two diagrams, Figures 6.23 and 6.24, correspond to transitive actions of M_{12} on 12 points. the last two diagrams describe different epimorphisms, since the element $\alpha^2\gamma = (xy^2)^2y^2z$ has order 4 for the representation described in Figure 6.23, and order 6 in the representation described in Figure 6.24.

■

As a corollary, from the representation described in diagram V), the following result is immediate:

Corollary 6.9.1 *The group Γ^+ has a unique normal subgroup with factor group A_{12}*

6.10 Concluding remarks

In this chapter, examples of manifolds arising from torsion-free subgroups of the 9 Lannér groups were constructed. Using the isometry groups of these manifolds, combinatorial descriptions of several of the manifolds were given, taking advantage of the size of the isometry groups. In some interesting cases, the induced action of these isometries on the first homology was computed. This additional information was then used to investigate the construction of arbitrarily large manifolds whose isometry group is also large.

For each group, a minimal index torsion free subgroup was determined. The orientation preserving subgroups of the groups T_1, T_2, T_3, T_5 and T_8 have non-normal minimal index torsion-free subgroups, while orientation preserving subgroups of the groups T_4, T_6, T_7 and T_9 all have minimal index torsion-free normal subgroups.

Chapter 7

The 4-dimensional compact simplicial Coxeter groups

7.1 Motivation

In 1985, Mike Davis constructed his celebrated "Davis manifold" [Da], providing the first explicit construction of a four-dimensional hyperbolic manifold. This manifold arose from the existence of a torsion free subgroup of index 14400 in the group $\Gamma_4 = [5, 3, 3, 5]$, shown in Table 7.1. The manifold has Euler characteristic 26. By the Gauss-Bonnet theorem [KZ], the volume of a complete finite volume hyperbolic 4-manifold $\mathcal{M}$ is given by $\mathrm{Vol}(\mathcal{M}) = \frac{4\pi^2}{3}\chi(\mathcal{M})$: thus the volume of the Davis Manifold is $26 \times \frac{4\pi^2}{3}$.

Since then, an outstanding question has been whether or not Γ_1 has a minimal index torsion free subgroup K. Since the least common multiple of the orders of conjugacy classes of maximal finite subgroups of Γ_1 is 14400, K must have index 14400. If such a subgroup does exist, then the associated manifold will have Euler characteristic 1 and therefore it would be an example of a smallest volume compact hyperbolic 4-manifold. This chapter will study the subgroup structure of Γ_1 and seek to outline a possible approach to the construction of a smallest volume compact hyperbolic 4-manifold.

The five co-compact four-dimensional hyperbolic simplicial Coxeter groups are listed in Table 7.1.

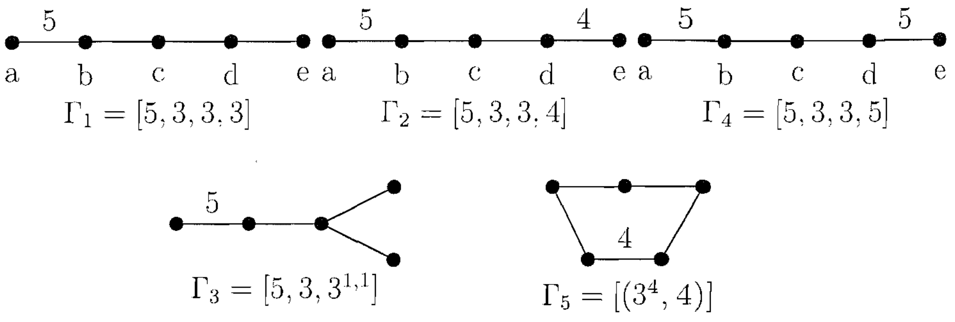


Table 7.1: The 5 co-compact simplicial Coxeter groups with a hyperbolic 4-simplex as a fundamental region

The groups Γ_1 and Γ_2 deserve particular attention. In his paper, Davis [Da] constructed an explicit epimorphism from Γ_4 onto the $[5,3,3]$ Coxeter group. This group is the full symmetry group of the 120-cell, first introduced in § 5.8. The Davis manifold has Euler characteristic 26. In his concluding remarks, Davis notes that Γ_1 and Γ_2 have rational Euler characteristic $\frac{1}{14400}$ and $\frac{17}{28800}$, respectively, and speculates as to the existence of torsion free subgroups of indices 14400 and 28000 in these groups. Such subgroups would yield compact hyperbolic manifolds of Euler characteristic 1 and 17, respectively. The analogous problem for non-compact manifolds has been solved by John Ratcliffe and Steven Tschantz in [RT1]. The manifolds they constructed arise from torsion-free subgroups of index 16×120 of the Coxeter group shown in Table 7.2. More recently, using computational

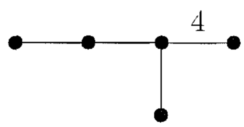


Table 7.2: Simplicial Coxeter group with Non-co-compact simplex Δ^4 in $\mathbb{H}^4$

methods, Marston Conder and Colin Maclachlan [CM] constructed a torsion free subgroup of index 8×14400 in Γ_1. This corresponds to the smallest volume co-compact hyperbolic 4-manifold currently known.

In this chapter a different approach is used to construct subgroups of Γ_1 of index 8×14400. The subgroups constructed are different to those of Conder and

Maclachlan. Additionally, some obstructions to finding a torsion free subgroup of index 14400 are presented. The method applied combines computational techniques with classical group theory, and in particular makes use of the classification of the finite simple groups. Furthermore, it provides a method to answer the question whether Γ_1 has a torsion free subgroup of index 14400.

7.2 Conjugacy class representatives

Any attempt to construct a manifold from the action of a subgroup of Γ_i on $\mathbb{H}^4$ requires information on the conjugacy classes of torsion elements: namely what they are and how they are avoided. Representatives of the conjugacy classes of torsion elements in Γ_1 are characterised using Theorem 2.2 in Chapter 2

This identifies the torsion elements, up to conjugacy. Therefore, for Γ_1, any conjugacy class of elements of finite order lies in one of the following five special subgroups listed in Table 7.3

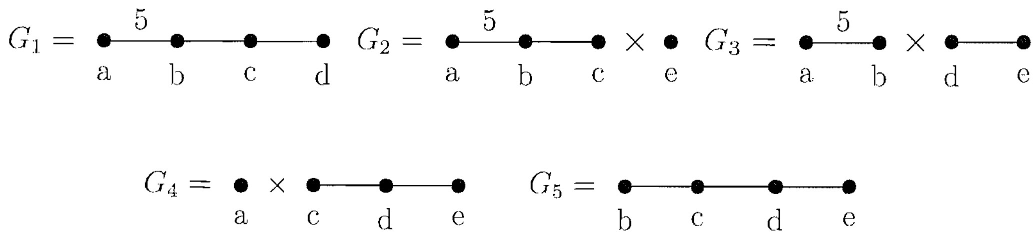


Table 7.3: Maximal finite subgroups of Γ_1

It suffices to consider only the conjugacy classes of elements of prime order. The conjugacy classes for each maximal subgroup G_1, G_2, G_3, G_4 and G_5 are summarised in Table 7.4, Table 7.6, Table 7.5, Table 7.7 and Table 7.8, respectively.

Order of representative	Class representative
2	a
2	ac
2	$(abc)^5$
2	$(abcd)^{15}$
3	bc
3	$(abcd)^{10}$
5	ab
5	$(ab)^2$
5	$(abcd)^6$
5	$(abcd)^{12}$
5	$(abcd)^{10}(abc)^{-2}$

Table 7.4: Conjugacy classes of elements of prime order in $G_1 = [5,3,3]$

Order of representative	Class representative
2	a
2	d
2	ad
3	de
5	ab
5	$(ab)^2$

Table 7.5: Conjugacy classes of elements of prime order in $G_3 \cong D_{10} \times D_6$

Order of representative	Class representative
2	e
2	a
2	ae
2	ac
2	ace
2	$(abc)^5$
2	$e(abc)^5$
3	bc
5	ab
5	$(ab)^2$

Table 7.6: Conjugacy classes of elements of prime order in $G_2 = [5,3] \times C_2 \cong A_5 \times C_2 \times C_2$

Order of representative	Class representative
2	a
2	c
2	ce
2	ac
2	ace
3	cd

Table 7.7: Conjugacy classes of elements of prime order in $G_4 \cong C_2 \times S_4$

Order of representative	Class representative
2	b
2	bd
3	bc
5	$bcde$

Table 7.8: Conjugacy classes of elements of prime order in $G_5 \cong S_5$

The elements listed in the previous tables account for all conjugacy classes of torsion of prime order in Γ. However, in many cases the same class has been counted more than once. For example, a and e represent different conjugacy classes of elements of order 2 in G_3. However, a is conjugate to b in G_1 while b is conjugate to e in G_5, so a and e represent the same conjugacy class in Γ. After removing multiple references for the same conjugacy class, representatives of the conjugacy classes of torsion of prime order in Γ are given by the following elements:

Order 2: a, ac, ace, $(abc)^5$, $(abc)^5e$ and $(abcd)^{15}$.

Order 3: bc and $(abcd)^{10}$.

Order 5: ab, $(ab)^2$, $bcde$, $(abcd)^6$, $(abcd)^{12}$ and $(abcd)^{10}(abc)^{-2}$.

7.3 Subgroups of the $[5, 3, 3, 3]$ Coxeter group

As previously observed, torsion free subgroups of the $[5, 3, 3, 3]$ Coxeter group Γ correspond to hyperbolic 4-manifolds. If Γ were to contain a torsion free subgroup of index 14400, then the associated hyperbolic 4-manifold would have Euler characteristic 1, and would thus be an example of a smallest volume compact hyperbolic 4-manifold. In this section, examples of the smallest currently known manifolds are constructed. It will be shown that these examples differ from those constructed in [CM].

7.3.1 Low index subgroups of Γ

Using Lowx [Lx], a complete list of conjugacy class representatives of subgroups of [5, 3, 3, 3] of index at most 720 was obtained. The number of such subgroups and their index in each case is summarised in Table 7.9

Index	# classes	Index	# classes	Index	# classes
2	1	385	1	595	6
85	2	420	1	600	17
120	2	425	2	624	4
136	2	436	1	625	1
156	2	445	1	640	2
170	4	480	18	641	1
240	10	505	1	650	7
255	3	510	21	660	1
272	6	521	1	675	2
300	1	540	1	676	1
312	4	544	8	680	15
325	1	555	2	685	4
340	4	556	1	691	2
360	2	565	1	720	13

Table 7.9: Number of classes of subgroups of index ≤ 720 in Γ

The data in Table 7.9 will be used to prove that if K is a torsion free normal subgroup of Γ, then Γ/K contains no sporadic simple group, nor any extension of one, among its epimorphic images. This first result proves that the construction used in Davis paper [Da] cannot be used for this group.

Proposition 7.3.1 *There exists no epimorphism $\varphi : \Gamma \mapsto \Gamma_0 := [5,3,3]$ with torsion free kernel.*

Proof: Suppose there exists such a map. Let a, b, c, d, e be the standard Coxeter generators for $\Gamma = [5,3,3,3]$, and define $\varphi(a) := \underline{a}$, $\varphi(b) := \underline{b}$, $\varphi(c) := \underline{c}$, $\varphi(d) := \underline{d}$. Then $\underline{a}, \underline{b}, \underline{c}, \underline{d}$ form a generating set for Γ_0 which satisfies the Coxeter relations for Γ_0, and so form a set of Coxeter generators for Γ_0. It remains to find an image for e under φ such that $\underline{a}\varphi(e)$, $\underline{b}\varphi(e)$ and $\underline{c}\varphi(e)$ all have order 2 and that $\underline{d}\varphi(e)$

has order 3. Recalling the decomposition in § 5.9 of the 120-cell into two linked tori consisting of 60 dodecahedra, one of these tori can be cut along a meridian curve to recover a tower of dodecahedra, which contains a core tower of 10 stacked dodecahedra. Adjacent dodecahedra share a horizontal pentagonal face, and at each edge a further dodecahedron is attached. The outer face of the tower consists of the 200 free faces of the 50 dodecahedra wrapped around the central tower. The core tower is stabilised by the dihedral group D_{10} generated by $\underline{a}$ and $\underline{b}$. Hence e must map to some element of order 2 with $|\underline{a}g| = |\underline{b}g| = 2$. In § 5.9 it has been proved that any element g of order 2 in Γ_0 which satisfies $|\underline{a}g| = |\underline{b}g| = 2$ lies in a dihedral group D_{20} which stabilizes the meridian curve as illustrated in Figure 5.12. There are 11 possible involutions w_i lying in this subgroup. Three of them, $(\underline{ababc})^3$, d^w, where $w = (\underline{cbabcd})^2(\underline{abc})^3\underline{abdc}$ and $(\underline{abcd})^{15}$, satisfy $|\underline{c}g| = 2$. However, their product with $\underline{d}$ has orders 10, 5 and 2, respectively. Since it was required that de have order 3, this proves that no such map exists. ■

Proposition 7.3.2 *Let $w = abcde$ in Γ. Then for $i < 25$, $i \neq 17$, the following results hold:*

1) $\langle\langle w^i \rangle\rangle_\Gamma = \Gamma$ if i is odd.

2) $\langle\langle w^i \rangle\rangle_\Gamma = \Gamma^+$ if i is even.

3) For $i = 17$, $\Gamma / \langle\langle w^i \rangle\rangle_\Gamma \cong S_4(4)$, the 4-dimensional simple symplectic group over the field of 4 elements. $\langle\langle w^i \rangle\rangle_\Gamma$ is the unique such subgroup with this quotient. The kernel of the map $\Gamma \mapsto \Gamma / \langle\langle w^i \rangle\rangle_\Gamma$ contains the conjugacy class $C((abcd)^{15})$ of torsion elements.

Proof: The first two results have been verified computationally. The third result was obtained as follows: Γ has two conjugacy classes of subgroups of index 85. These were obtained by using the program Lowx. The induced action of Γ on the cosets of a representative of one of these classes gives a permutation representation $\theta : \Gamma \to S_{85}$. The resulting group was found, using GAP, to be the simple group $S_4(4)$. The Atlas of simple groups [Atlas] lists $S_4(4)$ as having two conjugacy classes of maximal subgroups of index 85. These lift, under the inverse of θ, to representatives of the two conjugacy classes of subgroups of index 85 in Γ. Hence the kernel of θ is unique. ■

7.3.2 Maps to $S_4(4)$

Let Γ act on the cosets of a representative of either of the two conjugacy classes of subgroups of index 85. This action induces a permutation representation $\theta : \Gamma \twoheadrightarrow S_4(4)$. From the Atlas of Finite Groups [Atlas], it can be seen that $S_4(4)$ has two conjugacy classes of subgroups of index 85, so these subgroups lift to two conjugacy classes of subgroups of index 85 in Γ. Since Γ has only two conjugacy classes of subgroups of index 85, they must coincide with the lifted subgroups. Since $S_4(4)$ is simple, the core must be the same in each case, giving the epimorphism $\theta : \Gamma \twoheadrightarrow S_4(4)$. Write $K = \ker(\theta)$. GAP [GAP] was used to study the actions of each conjugacy class of torsion elements of Γ on the cosets of a subgroup of index 85 in Γ. It was discovered that the conjugacy class $\mathcal{C}((abcd)^{15})$ of elements of order 2 in Γ acts with fixed points on the 85 cosets, and hence lies in K. The group $S_4(4)$ contains conjugacy class of Sylow-17 groups Syl_{17} which have index 4×14400 in $S_4(4)$. The normalizer $N_{S_4(4)}(Syl_{17}) = N(Syl_{17})$ has index 14400 in $S_4(4)$ and is an extension of Syl_{17} by a cyclic group of order 4. Any element of order 2 lying in $N(S_{17})$ acts faithfully by conjugation, since otherwise it would commute with Syl_{17} and hence $S_4(4)$ would have an element of order 34, a contradiction. GAP was used to study the images of representatives of conjugacy classes of torsion elements of Γ. It was found that elements of the conjugacy class represented by ac act by inversions on Syl_{17}, therefore the preimage $N(S_{17})K$ contains elements of the conjugacy class $\mathcal{C}(ac)$.

The preimage of the group Syl_{17} under θ clearly cannot contain any of the torsion elements and is an extension of K by Syl_{17}. This group was constructed using GAP and the conjugacy classes of index 2 subgroups were determined. There are fifteen such conjugacy classes of subgroups, 8 of which avoid the conjugacy class $\mathcal{C}((abcd)^{15})$. Their abelianisation is one of the three following types:

$$\mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{17} \tag{7.1}$$

$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{17} \tag{7.2}$$

$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_{17} \tag{7.3}$$

These were found by constructing an explicit presentation for H, and hence for the index 2 subgroups.

(a) A subgroup with abelianisation $\mathbb{Z}_4^3 \oplus \mathbb{Z}_{17}$

Let $H_1 < \Gamma$ be the subgroup generated by the following elements (these generators were found using GAP):

$$cdcbabacbdedcbabcdabcababcdebabc,$$
$$abacbabadedcbabacbabcdecdbcababcdbc,$$
$$bacbedcbabacbadcbabedcbabcdecababcdbca,$$
$$abacbabacbabcdcbaedcbabcdabcababcdedbca,$$
$$ababadcbabacbdcedcbabcdeabcdabcababcdebcd,$$
$$babacdcbabacbadedcbabacdebcababcdcbabcababc$$

Then H_1 has index 8×14400 in Γ. Letting Γ act by right multiplication on the cosets of H_1 gives an induced permutation representation on 8×14400 points. Now H_1 contains an element of a conjugacy class $\mathcal{C}(g)$ of Γ if and only if the induced permutation action of g fixes some points. Using GAP, the induced actions of representatives of each of the conjugacy classes of Γ were computed. Since no representative of a conjugacy class in Γ fixed a point, H_1 avoids all conjugacy classes of torsion elements. The resultant manifold has Euler characteristic 8 and therefore has volume $\frac{32\pi^2}{3}$. The abelianisation of H, $H_1/[H_1, H_1]$, is a finite group of type $\mathbb{Z}_4^3 \oplus \mathbb{Z}_{17}$. Since the second generator has odd length, so H_1 is not a subgroup of Γ^+.

(b) A subgroup with abelianisation $\mathbb{Z}_2^3 \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}_{17}$

Let $H_2 < \Gamma$ be the subgroup generated by the following elements (these generators were found using GAP):

$$ababdcbabecdcabab,$$
$$bcbabacbdcbabedcbabcababcdecdbabc,$$
$$adcbaedcbabacbabdcbabacdedbcababcdabcababcba,$$
$$bcbabacdcbabacbabcdcbabecdcbabcababcdebabcdbca,$$
$$acbabcdcbabacbadcedcbabacbdcababcdebabcdbabcababcde$$

Then H_2 has index 8×14400 in Γ. Letting Γ act by right multiplication on the cosets of H_2 gives an induced permutation representation on 8×14400 points. Now H_2 contains an element of a conjugacy class $\mathcal{C}(g)$ of Γ if and only if the induced permutation action of g fixes some points. Using GAP, the induced actions of representatives of each of the conjugacy classes of Γ were computed. Since no representative of a conjugacy class in Γ fixed a point, H_2 avoids all conjugacy classes of torsion elements. The resultant manifold has Euler characteristic 8 and therefore has volume $\frac{32\pi^2}{3}$. The abelianisation of H_2, $H_2/[H_2, H_2]$, is a finite group of type $\mathbb{Z}_2^3 \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}_{17}$. Since the first generator has odd length, so H_2 is not a subgroup of Γ^+.

(c) A subgroup with abelianisation $\mathbb{Z}_2^3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_{17}$

Let $H_3 < \Gamma$ be the subgroup generated by the following elements (these generators were found using GAP):

$$
\begin{gathered}
deababcbabcdacbab, \\
bcbaacbdcbabedcbabcababcdecdbabc, \\
acbadcbedcbabacbabcdcbaedcbabcdbabcababcdecda, \\
bcbabdcbabacedcbabacbabdecdbcababcdebabcababc, \\
abcbdcbabacbaedcbabacbabdecdbcababcdbabcababcaba, \\
cbadcbabacbadcbaedcbabacbdecababcdcbabcababcdedcab
\end{gathered}
$$

Then H_3 has index 8×14400 in Γ. Letting Γ act by right multiplication on the cosets of H_3 gives an induced permutation representation on 8×14400 points. Now H_3 contains an element of a conjugacy class $\mathcal{C}(g)$ of Γ if and only if the induced permutation action of g fixes some points. Using GAP, the induced actions of representatives of each of the conjugacy classes of Γ were computed. Since no representative of a conjugacy class in Γ fixed a point, H_3 avoids all conjugacy classes of torsion elements. The resultant manifold has Euler characteristic 8 and therefore has volume $\frac{32\pi^2}{3}$. The abelianisation of H_3, $H_3/[H_3, H_3]$, is a finite group of type $\mathbb{Z}_2^3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_{17}$. Since the first generator has odd length, so H_3 is not a subgroup of Γ^+.

7.3.3 Maps to $S_4(5)$

There are two conjugacy classes of subgroups of index 156 in Γ. Letting Γ act on the cosets of a representative of either class produces a permutation representation $\Gamma \to H_4 < S_{156}$ into the symmetric group on 156 points. GAP was used to construct the map, and the group H_4 was discovered to be the group $S_4(5).2$, a group containing $S_4(5)$ as a normal subgroup of index 2. $S_4(5)$ is a simple symplectic group of order $2^6.3^2.5^4.13$. The group $S_4(5).2$ is its automorphism group and is isomorphic to $SO_5(5)$

Proposition 7.3.3 $S_4(5)$ *contains no subgroup of index 7200.*

Proof: A subgroup S of index 7200 would have order $650 = 2.5^2.13$, so Sylow's theorems imply that S has a unique and hence normal Sylow 13-subgroup C_{13}. Since C_{13} has no automorphism of order 5, elements of order 5 and 13 in S must commute, giving elements of order 65. But $S_4(5)$ has no elements of this order. ∎

7.4 Characterising maps to simple groups

Theorem 7.1 *Let $H < \Gamma$ have index 14400 and suppose that H is torsion free. Let $K \lhd \Gamma$ be the core of H in Γ. Then Γ/K does not admit a map onto any of the sporadic simple groups.*

Proof: Suppose it did. Let $\overline{H}$ be the image of H in $\overline{\Gamma} = \Gamma/K$ and suppose that $\varphi : \Gamma/K \to G$ is a map from the factor group $\overline{\Gamma}$ to some sporadic simple group G. Then the image of $\overline{H}$ under φ is a subgroup of index dividing 14400 in a sporadic group and avoids the images of all the conjugacy classes of torsion. Using the list of all conjugacy classes of subgroups of Γ with index ≤ 720 (Table 7.9), first look at all sporadic simple groups S with maximal subgroups of index less than 720. These are $M_i, i \in \{11, 12, 22, 23, 24\}$, J_2, J_3, HS, McL and Co_3. GAP was used to compute the actions of Γ on representatives of the conjugacy classes in each case. None of the above sporadic groups appeared in the resulting list of permutation groups. Among the remaining groups, Co_1, Fi_{23}, Fi'_{24}, HN, Th, B, M $O'N$, Ly

and J_4 contain no subgroup of index ≤ 14400 (see chapter 5 of [W]), and are therefore eliminated. The remaining groups are Suz, Co_2, He, Fi_{22}, J_3 and Ru.

Suz contains one class of maximal subgroups of index ≤ 14400. A representative of this class has index 1782 in Suz. Therefore Suz contains no subgroup of index 14400 since 1782 does not divide 14400. Co_2 contains one class of maximal subgroups of index ≤ 14400. A representative of this class has index 2300 in Co_2, and as a result it also cannot contain a subgroup of index 14400. He contains two classs of maximal subgroups of index ≤ 14400: one class has index 2058 and has representative $S_4(4) : 2$, the other has index 8330 with representative $2^2.L_3(4).S_3$. Neither 2058 nor 8330 divide 14400. Therefore the respective subgroups cannot contain A subgroup of index 14400. Fi_{22} contains three classes of maximal subgroups of index ≤ 14400: two classes of type $O_7(3)$ with index 14080 and one class of type $2.U_6(2)$ with index 3510. Because 14080 and 3510 do not divide 14400, Fi_{22} contains no subgroups of index 14400. Similarly, J_3 has one class of maximal subgroups of index less than 14400, and because this class has index 6156 in J_3, J_3 has no subgroup of index 14400. Finally, Ru also has one class of maximal subgroups of index less than 14400. This class has index 4060 in Ru, therefore Ru also contains no subgroup of index 14400. ∎

Chapter 8

Conclusion and further work

8.1 Conclusion

In this work all torsion free normal subgroups of the index two orientation preserving subgroups of the Lannér groups whose factor groups have the form $PSL_2(q)$, where $q = p^n$ and p is a prime, are classified. In the case of each group, some examples of manifolds were constructed and their homology computed. Minimal index torsion free subgroups of each Lannér group are also constructed. The computational techniques developed in this work were applied to the 4-dimensional Coxeter group $[5, 3, 3, 3]$ to study minimal volume compact hyperbolic manifolds.

In chapter 3, subgroups of the orientation preserving subgroup Γ^+ of the Lannér group $\Gamma = T_4 = [5, 3, 5]$ are studied. All such subgroups whose factor group is $L_2(q)$ are classified. The conditions under which these subgroups are normal in any extension of Γ^+ in $\mathrm{Isom}(\mathbb{H}^3)$ is also investigated. The special case $p = 5$ yields two normal subgroups N_1 and N_2, whose associated manifolds $\mathcal{M}_i = \mathbb{H}^3/N_i$ are a chiral pair of Seifert-Weber manifolds.

Chapter 4 generalises the results of chapter 3 to the other 8 Lannér groups. The classification of such subgroups, in the case of $\Gamma = T_2$, was previously described by Anna Torstensson in her PhD thesis [To]. A statement of her results was included for completeness. A classification of all torsion-free normal subgroups $N \lhd \Gamma^+ \lhd \Gamma =$

T_i of the remaining 7 Lannér groups T_i, whose factor groups are of the form $L_2(q)$, is given. Some conditions are also provided under which such subgroups N remain normal in an extension of Γ^+ in $\mathrm{Isom}(\mathbb{H}^3)$.

As well as the theoretical results of chapters 3 and 4, explicit examples of some of the manifolds were given. In chapter 5, several of the manifolds arising from torsion-free subgroups $N \lhd \Gamma^+ = [5, 3, 5]^+$ were constructed. A combinatorial description of these manifolds, using the group theoretic structure of their isometry groups, was given. Interesting chiral pairs of manifolds were discovered, and their chirality exhibited using the subgroup structure of Γ.

In chapter 6, manifolds exhibiting a large degree of symmetry which arise from the other 8 Lannér groups were investigated. The smallest manifolds with a large degree of symmetry which were associated to the Lannér group $\Gamma = T_2$ were described in a paper of Jones and Mednykh [JM]. A statement of their main results was included for completeness. For each of the remaining 7 Lannér groups, a brief description of the associated manifolds $\mathcal{M}$ is given. In the case of both the Lannér groups T_5 and T_9, the smallest index torsion-free normal subgroup $N \cong \pi_1(\mathcal{M})$ has a free abelianisation $N/[N, N] \cong H_1(\mathcal{M})$. The action of the isometry group $\mathrm{Isom}(\mathcal{M})$ on this free $\mathbb{Z}$-ring is computed. As a result a construction for an arbitrarily large hyperbolic 3-manifold with a large isometry group is given.

Computational techniques were developed to construct complete lists of conjugacy classes of subgroups of low index in these groups. A summary of these results is provided in the Appendices. These lists were then used to test the theoretical results proved in this thesis and also to search for specific normal subgroups whose factor groups were of interest.

The computational techniques developed in this work were applied to the 4-dimensional Coxeter group $[5, 3, 3, 3]$ and a study was done of the low index subgroups of this group. The existence of torsion free subgroups of index 115200 was established and a possible approach towards determining the minimal index torsion-free subgroup of this group is outlined.

8.2 Further work

Based on the work of this thesis, there are several interesting avenues of research that can be pursued in the future. Chapter 6 includes only a selection of the results obtained from the Lannér groups. It would be interesting to study more of these manifolds, providing researchers in hyperbolic geometry and abstract polytopes with a library of "ready-made" manifolds with which to test hypotheses. Using the tables in Appendix A and the methods described in chapters 5 and 6, many more interesting manifolds can be identified and constructed combinatorically.

The methods used in chapters 3 and 4 can also be applied to the 23 non-compact simplicial Coxeter groups acting on $\mathbb{H}^3$. They are often referred to as "Quasi-Lannér Groups". Vinberg has shown in [V1] that 6 of these are non-arithmetic, so torsion-free subgroups of these groups can be used to construct non-arithmetic manifolds. If, further, the subgroups are normal, then the manifolds will exhibit a high degree of symmetry.

Generalising to higher dimensions, the combination of group theory and computational techniques used in chapter 7 can be extended to characterise examples of cusped hyperbolic manifolds in dimensions 6 and 8. By combining computational techniques and classical group theory, it becomes possible to construct explicit subgroups avoiding torsion.

Additionally, it would be interesting to determine the full isometry group in $\mathrm{Isom}(\mathbb{H}^n)$ of each of these groups, and to determine subgroups of Γ that are torsion free whose associated manifolds $\mathcal{M}$ have maximal symmetry. Note that, in $\mathbb{H}^3$ at least, the isometry group of $\mathcal{M}$ is still finite.

Appendix A

Tables of low index subgroups for the simplicial groups

A.1 $\Gamma = T_1[2, 2, 3; 3, 5, 2]$

Γ^+ has 3359 conjugacy classes of low index subgroups of index ≤ 60. Their index, the induced permutation group and, where possible, the abelianisation of the kernel of the map are listed here.

Index	*# classes*	*Induced action*	*Abelianization*
5	1	A_5	$\mathbb{Z}_2^{11}$
6	1	A_5	$\mathbb{Z}_2^{11}$
10	1	A_5	$\mathbb{Z}_2^{11}$
11	4	$PSL_2(11)$	$\mathbb{Z}_2^{10} \oplus \mathbb{Z}_{11}^3$
12	9	$1 \Rightarrow A_5$	$\mathbb{Z}_2^{11}$
		$2 \Rightarrow L_2(11)$	$\mathbb{Z}_2^{10} \oplus \mathbb{Z}_{11}^3$
		$2 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2^{14}$
		$2 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{27}$
		$2 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{31}$
15	1	A_5	$\mathbb{Z}_2^{11}$
16	4	$A_5 \ltimes H_2$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2^2$

Index	*# classes*	*Induced action*	*Abelianization*
18	3	$A_5 \ltimes H_3$	not computed
20	29	$1 \Rightarrow A_5$	$\mathbb{Z}_2^{11}$
		$4 \Rightarrow A_5 \ltimes H_2$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2^2$
		$2 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2^{14}$
		$2 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{27}$
		$2 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{31}$
		$2 \Rightarrow PSL_2(19)$	$\mathbb{Z}_2^{18} \oplus \mathbb{Z}_{19}^3$
		$16 \Rightarrow A_5 \ltimes H_4$	not computed
22	1	A_{22}	not computed
24	22	$1 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2^{14}$
		$1 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{27}$
		$1 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{31}$
		$4 \Rightarrow A_5 \ltimes H_5$	$\mathbb{Z}^{43}$
		$8 \Rightarrow A_5 \ltimes H_6$	not computed
		$4 \Rightarrow A_5 \ltimes H_7$	not computed
		$3 \Rightarrow A_5 \ltimes H_{24}$	not computed
26	1	$L_2(25)$	$\mathbb{Z}^{78}$
28	1	$S_6(2)$	not computed
30	41	$1 \Rightarrow A_5$	$\mathbb{Z}_2^{11}$
		$4 \Rightarrow A_5 \ltimes H_2$	$\mathbb{Z}^{15}\mathbb{Z}_2^2$
		$4 \Rightarrow L_2(29)$	not computed
		$16 \Rightarrow A_5 \ltimes H_4$	not computed
		$3 \Rightarrow A_5 \ltimes H_8$	not computed
		$1 \Rightarrow A_5 \ltimes H_9$	
32	22	$4 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2 14$
		$4 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{27}$
		$4 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{31}$
		$2 \Rightarrow L_2(31)$	not computed
		$8 \Rightarrow A_5 \ltimes H_6$	not computed

Index	*# classes*	*Induced action*	*Abelianization*
34	3	A_{34}	not computed
		$9 \Rightarrow A_5 \ltimes H_{21}$	not computed
		$1 \Rightarrow S_6(2)$	not computed
		$12 \Rightarrow A_5 \ltimes H_{20}$	not computed
		$1 \Rightarrow A_5 \ltimes H_{19}$	not computed
		$2 \Rightarrow A_5 \ltimes H_{18}$	not computed
		$12 \Rightarrow A_5 \ltimes H_{24}$	not computed
36	73	$12 \Rightarrow A_5 \ltimes H_{25}$	not computed
		$2 \Rightarrow A_5 \ltimes H_{17}$	not computed
		$4 \Rightarrow A_5 \ltimes H_{26}$	not computed
		$5 \Rightarrow A_5 \ltimes (\times_{i=1}^{6} A_6)$	not computed
		$7 \Rightarrow A_5 \ltimes (H_1 \ltimes (\times_{i=1}^{6} A_5))$	not computed
		$2 \Rightarrow A_5 \ltimes (\times_{i=1}^{6} A_6)$	not computed
		$4 \Rightarrow A_5 \ltimes (H_1 \ltimes (\times_{i=1}^{6} A_6))$	not computed
38	2	A_{38}	not computed
		$3 \Rightarrow A_5 \ltimes H_2$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2^2$
		$2 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2^{14}$
		$3 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{27}$
		$3 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{31}$
		$4 \Rightarrow A_5 \ltimes H_5$	$\mathbb{Z}^{43}$
40	64	$12 \Rightarrow A_5 \ltimes H_4$	not computed
		$12 \Rightarrow A_5 \ltimes H_7$	not computed
		$12 \Rightarrow A_5 \ltimes H_{22}$	not computed
		$6 \Rightarrow A_5 \ltimes H_{23}$	not computed
		$2 \Rightarrow A_5 \ltimes (\times_{i=1}^{5} A_8)$	not computed
		$4 \Rightarrow A_{40}$	not computed
		$3 \Rightarrow A_5 \ltimes H_{16}$	not computed
		$4 \Rightarrow A_5 \ltimes (\times_{i=1}^{5} A_8)$	not computed
42	70	$60 \Rightarrow A_5 \ltimes (H_1 \ltimes (\times_{i=1}^{6} A_7)))$	not computed
		$3 \Rightarrow A_{42}$	not computed

Index	*# classes*	*Induced action*	*Abelianization*
44	9	$7 \Rightarrow A_{22} \ltimes H_{14}$	not computed
		$2 \Rightarrow A_{44}$	not computed
45	1	$A_5 \ltimes H_{10}$	not computed
46	18	$18 \Rightarrow A_{46}$	not computed
48	332	$1 \Rightarrow 3840$	not computed
		$18 \Rightarrow 122880$	not computed
		$6 \Rightarrow 1399680$	not computed
		$8 \Rightarrow 3932160$	not computed
		$18 \Rightarrow 7864320$	not computed
		$2 \Rightarrow 113374080$	not computed
		$20 \Rightarrow 251658240$	not computed
		$9 \Rightarrow 5733089280$	not computed
		$12 \Rightarrow 11466178560$	not computed
		$24 \Rightarrow 16106127360$	not computed
		$4 \Rightarrow 82649704320$	not computed
		$4 \Rightarrow 5289581076480$	not computed
		$2 \Rightarrow 23482733690880$	not computed
		$4 \Rightarrow 46965467381760$	not computed
		$3 \Rightarrow 1348984441405440$	not computed
		$8 \Rightarrow 1502894956216320$	not computed
		$16 \Rightarrow 3005789912432640$	not computed
		$1 \Rightarrow 17118912860651520$	not computed
		$2 \Rightarrow 34237825721303040$	not computed
		$25 \Rightarrow 43167502124974080$	not computed
		$18 \Rightarrow 1095610423081697280$	not computed
		$8 \Rightarrow A_5 \ltimes (\times_{i=1}^{6} A_8)$	not computed
		$118 \Rightarrow A_5 \ltimes (H_1 \ltimes (\times_{i=1}^{6} A_8))$	not computed
		$1 \Rightarrow A_{48}$	not computed

Index	*# classes*	*Induced action*	*Abelianization*
50	27	$1 \Rightarrow A_5 \ltimes H_{12}$	not computed
		$3 \Rightarrow A_5 \ltimes (\times_{i=1}^{10} A_5)$	not computed
		$1 \Rightarrow A_5 \ltimes H_{14}$	not computed
		$8 \Rightarrow A_5 \ltimes H_{15}$	not computed
		$14 \Rightarrow A_{50}$	not computed
52	24	$1 \Rightarrow L_2(25)$	$\mathbb{Z}^{78}$
		$6 \Rightarrow L_2(25) \ltimes H_{11}$	not computed
		$17 \Rightarrow A_{52}$	not computed
53	1	A_{53}	not computed
54	403	{	
55	14	$4 \Rightarrow L_2(11)$	$\mathbb{Z}_2^{10} \oplus \mathbb{Z}_{11}^3$
		$4 \Rightarrow A_5 \ltimes L_2(11)$	not computed
		$4 \Rightarrow L_2(11) \ltimes (\times_{i=1}^{11} A_5)$	not computed
		$2 \Rightarrow A_{55}$	not computed
56	31	$1 \Rightarrow S_6(2)$	not computed
		$14 \Rightarrow S_6(2) \ltimes H_{10}$	not computed
		$16 \Rightarrow A_{56}$	not computed
57	4	$4 \Rightarrow L_2(19)$	$\mathbb{Z}_2^{18} \oplus \mathbb{Z}_{19}^3$
58	78	$78 \Rightarrow A_{58}$	not computed
59	1	$1 \Rightarrow A_{59}$	not computed

There are also 2063 conjugacy classes of subgroups of index 60 in Γ^+. No information on the induced actions was computed.

1) H_1 is elementary abelian of order 32.

2) H_2 is elementary abelian of order 16.

3) $H_3/[H_3, H_3]$ is elementary abelian of order 32, and $[H_3, H_3]$ is elementary abelian of order 729.

4) H_4 is elementary abelian of order 1024.

5) H_5 is elementary abelian of order 64.

6) $H_6/[H_6, H_6]$ is elementary abelian of order 64, and $[H_6, H_6]$ is elementary abelian of order 32.

7) H_7 is elementary abelian of order 2048.

8) $H_8/[H_8, H_8]$ is elementary abelian of order 32, and $[H_8, H_8]$ is elementary abelian of order 5^6.

9) $H_9/[H_9, H_9]$ is elementary abelian of order 32, and $[H_9, H_9]$ is elementary abelian of order 3^{10}.

10) H_{10} is elementary abelian of order 2^{21}.

11) H_{11} is elementary abelian of order 2^{13}.

12) $H_{12}/[H_{12}, H_{12}]$ is elementary abelian of order 32, and $[H_{12}, H_{12}]$ is elementary abelian of order 5^{10}.

13) $H_{13}/[H_{13}, H_{13}]$ is elementary abelian of order 16, and $[H_{13}, H_{13}]$ is elementary abelian of order 5^{15}.

14) $H_{14}/[H_{14}, H_{14}]$ is elementary abelian of order 32, and $[H_{14}, H_{14}]$ is $\times_{i=1}^{10} A_5$.

15) $H_{15} = \times_{i=1}^{5} A_{10}$.

16) $H_{16}/[H_{16}, H_{16}]$ is elementary abelian of order 32, and $[H_{16}, H_{16}]$ is elementary abelian of order 7^6.

17) $H_{17}/[H_{17}, H_{17}]$ is elementary abelian of order 2^{11}, and $[H_{17}, H_{17}]$ is elementary abelian of order 3^{12}.

18) $H_{18}/[H_{18}, H_{18}]$ is elementary abelian of order 2^6, and $[H_{18}, H_{18}]$ is elementary abelian of order 3^{12}.

19) $H_{19}/[H_{19}, H_{19}]$ is elementary abelian of order 2^5, and $[H_{19}, H_{19}]$ is elementary abelian of order 3^{12}.

20) $H_{20}/[H_{20}, H_{20}]$ is elementary abelian of order 2^6, and $[H_{20}, H_{20}]$ is elementary abelian of order 3^6.

21) $H_{21}/[H_{21}, H_{21}]$ is elementary abelian of order 2^5, and $[H_{21}, H_{21}]$ is elementary abelian of order 3^6.

22) $H_{22}/[H_{22}, H_{22}]$ is elementary abelian of order 2^{10}, and $[H_{22}, H_{22}]$ is elementary abelian of order 16.

23) $H_{23}/[H_{23}, H_{23}]$ is elementary abelian of order 2^{11}, and $[H_{23}, H_{23}]$ is elementary abelian of order 16.

24) $H_{24}/[H_{24}, H_{24}]$ is elementary abelian of order 2^5. Write $H'_{24} = [H_{24}, H_{24}]$. Then $H'_{24}/[H'_{24}, H'_{24}]$ is elementary abelian of order 3^6 and $[H_{24}, H_{24}]$ is elementary abelian of order 2^{12}.

25) $H_{25}/[H_{25}, H_{25}]$ is elementary abelian of order 2^6. Write $H'_{25} = [H_{25}, H_{25}]$. Then $H'_{25}/[H'_{25}, H'_{25}]$ is elementary abelian of order 3^6 and $[H_{25}, H_{25}]$ is elementary abelian of order 2^{12}.

26) $H_{26}/[H_{26}, H_{26}]$ is elementary abelian of order 2^6. Write $H'_{26} = [H_{26}, H_{26}]$. Then $H'_{26}/[H'_{26}, H'_{26}]$ is elementary abelian of order 3^6 and $[H_{26}, H_{26}]$ is elementary abelian of order 2^{12}.

A.2 $\Gamma = T_2[2,2,3;2,5,3]$

Γ^+ has 143 conjugacy classes of low index subgroups of index ≤ 60. Their index, the induced permutation group and, where possible, the abelianisation of the kernel of the map are listed here.

Index	*# classes*	*Induced action*	*Abelianization*
6	4	A_6	$\mathbb{Z}_3^6$
10	2	A_6	$\mathbb{Z}_3^6$
11	2	$L_2(11)$	$\mathbb{Z}^{10}$
12	1	$L_2(11)$	$\mathbb{Z}^{10}$
15	4	A_6	$\mathbb{Z}_3^6$
17	1	$L_2(16)$	$\mathbb{Z}^{12} \oplus \mathbb{Z}_2^6$
20	4	$2 \Rightarrow A_6$	$\mathbb{Z}_3^6$
		$2 \Rightarrow L_2(19)$	$\mathbb{Z}_3^{19} \oplus \mathbb{Z}_{19}^3$
22	2	$L_2(11) \ltimes H_1$	not computed
26	1	$L_2(25)$	$\mathbb{Z}^{51}$
30	10	$4 \Rightarrow A_6$	$\mathbb{Z}_3^6$
		$2 \Rightarrow L_2(29)$	$\mathbb{Z}_3^{28} \oplus \mathbb{Z}_5^{15} \oplus \mathbb{Z}_{29}^3$
		$4 \Rightarrow A_6 \ltimes H_2$	not computed
33	2	$2 \Rightarrow L_2(11) \ltimes H_3$	not computed
36	10	$2 \Rightarrow A_6$	$\mathbb{Z}_3^6$
		$4 \Rightarrow A_6 \times A_6$	not computed
		$4 \Rightarrow A_6 \ltimes (\times_{i=1}^6 A_5)$	not computed
40	6	$2 \Rightarrow A_6$	not computed
		$4 \Rightarrow A_{40}$	not computed
42	2	$2 \Rightarrow L_2(41)$	$\mathbb{Z}^{40} \oplus \mathbb{Z}_{41}^{18}$
45	2	$2 \Rightarrow A_6$	$\mathbb{Z}_3^6$
50	4	$\Rightarrow A_6 \rtimes H_4$	not computed

Index	*# classes*	*Induced action*	*Abelianization*
51	4	$1 \Rightarrow L_2(16)$	$\mathbb{Z}^{12} \oplus \mathbb{Z}_2^6$
		$3 \Rightarrow L_2(16) \rtimes H_5$	not computed
52	3	$1 \Rightarrow L_2(25)$	$\mathbb{Z}^{51}$
		$2 \Rightarrow L_(25) \rtimes H_6$	not computed
53	2	$2 \Rightarrow A_{53}$	not computed
55	2	$2 \Rightarrow L_2(11)$	$\mathbb{Z}^{10}$
57	4	$4 \Rightarrow L_2(19)$	$\mathbb{Z}_3^{19} \oplus \mathbb{Z}_{19}^3$
60	71	$4 \Rightarrow A_6$	$\mathbb{Z}_3^6$
		$1 \Rightarrow L_2(11)$	$\mathbb{Z}^{10}$
		$2 \Rightarrow L_2(19)$	$\mathbb{Z}_3^{19} \oplus \mathbb{Z}_{19}^3$
		$2 \Rightarrow L_2(29)$	$\mathbb{Z}_3^{28} \oplus \mathbb{Z}_5^{15} \oplus \mathbb{Z}_{29}^3$
		$2 \Rightarrow L_2(59)$	$\mathbb{Z}_3^{116} \oplus \mathbb{Z}_5^{58} \oplus$ $\mathbb{Z}_{11}^{58} \oplus \mathbb{Z}_{59}^3$
		$2 \Rightarrow L_2(59)$	$\mathbb{Z}_3^{58} \oplus \mathbb{Z}_5^{58} \oplus$ $\mathbb{Z}_7^{58} \oplus \mathbb{Z}_{19}^{58} \oplus \mathbb{Z}_{59}^3$
		$4 \Rightarrow A_6 \times A_6$	not computed
		$36 \Rightarrow A_6 \ltimes H_7$	not computed
		$6 \Rightarrow L_2(19) \ltimes H_8$	not computed
		$4 \Rightarrow A_6 \ltimes (\times_{i=1}^6 A_5)$	not computed
		$4 \Rightarrow A_6 \ltimes (\times_{i=1}^{10} A_5)$	not computed
		$4 \Rightarrow A_6 \ltimes H_9$	not computed

The groups H_i

1) H_1 is an elementary abelian group of order 1024.

2) H_2 is $\times_{i=1}^6 A_5$.

3) H_3 has commutator subgroup H' elementary abelian of order 3^{11}. The factor group H/H' is also elementary abelian and has order 2^{10}.

4) $H_4 = \times_{i=1}^{10} A_5$.

5) H_5 is elementary abelian of order 3^{16}.

6) H_6 is elementary abelian of order 2^{13}.

7) H_7 is elementary abelian of order 3^6.

8) H_8 is elementary abelian of order 3^19.

9) H_9 is a group of order 782757789696000000. Its structure is not known but is possibly the semidirect product of 6 copies of A_5 with an elementary abelian group of order $2^{24} = 16777216$.

A.3 $\Gamma = T_3[2,2,4;2,3,5]$

There are 647 conjugacy classes of subgroups of index ≤ 60 in Γ^+. Their index, the induced permutation group and, where possible, the abelianisation of the kernel of the map are listed here. Note that the subgroups arising from the representatives of subgroups of index ≤ 10 all contain the torsion representative γ^2. The kernel of the map $\Gamma^+ \twoheadrightarrow PGL_2(11)$ is torsion free.

Index	*# classes*	*Induced action*	*Abelianization*
2	1	$\mathbb{Z}_2$	0
5	1	A_5	$\mathbb{Z}_2^5 \oplus \mathbb{Z}_4$
6	1	A_5	$\mathbb{Z}_2^5 \oplus \mathbb{Z}_4$
10	2	$1 \Rightarrow A_5$	$\mathbb{Z}_2^5 \oplus \mathbb{Z}_4$
		$1 \Rightarrow A_5 \times C_2$	$\mathbb{Z}_2^{11}$
12	9	$1 \Rightarrow A_5$	$\mathbb{Z}_2^5 \oplus \mathbb{Z}_4$
		$2 \Rightarrow A_5 \times C_2$	$\mathbb{Z}_2^{11}$
		$2 \Rightarrow PGL_2(11)$	$\mathbb{Z}_2^{10} \oplus \mathbb{Z}_{11}^3$
		$4 \Rightarrow H_1 \times (A_5 \ltimes H_2)$	$\mathbb{Z}^{31}$
15	1	A_5	$\mathbb{Z}_2^5 \oplus \mathbb{Z}_4$
18	1	$(A_5 \times C_2) \ltimes H_3$	not computed
20	13	$1 \Rightarrow A_5$	$\mathbb{Z}_2^5 \oplus \mathbb{Z}_4$
		$2 \Rightarrow A_5 \times C_2$	$\mathbb{Z}_2^{11}$
		$4 \Rightarrow A_5 \ltimes H_2$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2^2$
		$4 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{31}$
		$2 \Rightarrow PGL_2(19)$	$\mathbb{Z}_2^{18} \oplus \mathbb{Z}_{19}^3$
22	3	$2 \Rightarrow PGL_2(11)$	$\mathbb{Z}_2^{10} \oplus \mathbb{Z}_{11}^3$
		$1 \Rightarrow S_{22}$	not computed

Index	*# classes*	*Induced action*	*Abelianization*
24	22	$1 \Rightarrow A_5 \times C_2$	$\mathbb{Z}_2^{11}$
		$2 \Rightarrow PGL_2(11)$	$\mathbb{Z}_2^{10} \oplus \mathbb{Z}_{11}^3$
		$2 \Rightarrow A_5 \ltimes H_4$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2^{14}$
		$2 \Rightarrow A_5 \ltimes H_4$	$\mathbb{Z}^{27}$
		$6 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{31}$
		$8 \Rightarrow (A_5 \times C_2) \ltimes H_2$	not computed
		$1 \Rightarrow (A_5 \times C_2) \ltimes H_5$	not computed
26	1	$L_2(25)$	$\mathbb{Z}^{26} \oplus \mathbb{Z}_2^{27}$
28	1	$S_6(2)$	not computed
30	19	$1 \Rightarrow A_5$	$\mathbb{Z}_2^5 \oplus \mathbb{Z}_4$
		$2 \Rightarrow A_5 \times C_2$	$\mathbb{Z}_2^{11}$
		$8 \Rightarrow A_5 \ltimes H_2$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2^2$
		$4 \Rightarrow PGL_2(29)$	not computed
		$1 \Rightarrow (A_5 \times C_2) \ltimes H_6$	not computed
		$1 \Rightarrow (A_5 \times C_2) \ltimes H_7$	not computed
		$1 \Rightarrow (A_5 \times C_2) \ltimes H_8$	not computed
		$1 \Rightarrow (A_5 \times C_2) \ltimes H_9$	not computed
32	4	$2 \Rightarrow (A_5 \times C_2) \ltimes H_2$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2^2$
		$2 \Rightarrow L_2(31)$	not computed
34	1	A_{34}	not computed
36	22	$1 \Rightarrow S_6(2)$	not computed
		$7 \Rightarrow (A_5 \times C_2) \ltimes H_3$	not computed
		$2 \Rightarrow (A_5 \times C_2) \ltimes H_{10}$	not computed
		$8 \Rightarrow (A_5 \times C_2) \ltimes H_{11}$	not computed
		$2 \Rightarrow (A_5 \times C_2) \ltimes H_{12}$	not computed
		$1 \Rightarrow (A_5 \times C_2) \ltimes H_8$	not computed
		$1 \Rightarrow (A_5 \times C_2) \ltimes H_9$	not computed

Index	*# classes*	*Induced action*	*Abelianization*
40	48	$2 \Rightarrow A_5 \times C_2$	$\mathbb{Z}_2^{11}$
		$7 \Rightarrow A_5 \ltimes H_2$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2^2$
		$2 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2^{14}$
		$2 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{27}$
		$10 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{32}$
		$2 \Rightarrow PGL_2(19)$	$\mathbb{Z}_2^{18} \oplus \mathbb{Z}_{19}^3$
		$8 \Rightarrow A_5 \ltimes H_{13}$	not computed
		$8 \Rightarrow A_5 \ltimes H_{14}$	not computed
		$4 \Rightarrow A_5 \ltimes H_{15}$	not computed
		$2 \Rightarrow (A_5 \times C_2) \ltimes H_{16}$	not computed
		$2 \Rightarrow A_{40}$	not computed
42	8	$1 \Rightarrow A_5 \ltimes H_{17}$	not computed
		$2 \Rightarrow A_5 \ltimes H_{18}$	not computed
		$4 \Rightarrow (A_5 \times C_2) \ltimes H_{19}$	not computed
		$1 \Rightarrow A_{42}$	not computed
44	7	$1 \Rightarrow S_{22}$	not computed
		$6 \Rightarrow S_{22} \ltimes H_{20}$	not computed
45	1	$A_5 \ltimes H_{21}$	not computed
46	4	$2 \Rightarrow A_{46}$	not computed
		$2 \Rightarrow S_{46}$	not computed

Index	*# classes*	*Induced action*	*Abelianization*
48	89	$3 \Rightarrow (A_5 \times C_2) \ltimes H_2$	not computed
		$4 \Rightarrow (A_5 \times C_2) \ltimes H_1$	not computed
		$20 \Rightarrow (A_5 \times C_2) \ltimes H_{24}$	not computed
		$8 \Rightarrow (A_5 \times C_2) \ltimes H_{22}$	not computed
		$10 \Rightarrow (A_5 \times C_2) \ltimes H_{23}$	not computed
		$16 \Rightarrow (A_5 \times C_2) \ltimes H_{33}$	not computed
		$7 \Rightarrow (A_5 \times C_2) \ltimes H_{34}$	not computed
		$2 \Rightarrow (A_5 \times C_2) \ltimes H_{37}$	not computed
		$1 \Rightarrow A_5 \ltimes (\times_{i=1}^{6} L_2(7))$	not computed
		$2 \Rightarrow (A_5 \times C_2) \ltimes H_{36}$	not computed
		$3 \Rightarrow (A_5 \times C_2) \ltimes H_{35}$	not computed
		$2 \Rightarrow (A_5 \times C_2) \ltimes H_{31}$	not computed
		$2 \Rightarrow A_5 \ltimes (\times_{i=1}^{6} A_8)$	not computed
		$8 \Rightarrow (A_5 \times C_2) \ltimes (H_2 \ltimes (\times_{i=1}^{6} A_8))$	not computed
		$1 \Rightarrow A_{48}$	not computed
50	5	$1 \Rightarrow A_5 \ltimes H_{30}$	not computed
		$1 \Rightarrow A_5 \ltimes (\times_{i=1}^{10} A_5)$	not computed
		$1 \Rightarrow A_5 \ltimes (H_2 \ltimes (\times_{i=1}^{10} A_5))$	not computed
		$1 \Rightarrow A_{50}$	not computed
		$1 \Rightarrow S_{50}$	not computed
52	10	$1 \Rightarrow L_2(25)$	not computed
		$2 \Rightarrow L_2(25) \ltimes C_2$	not computed
		$4 \Rightarrow L_2(25) \ltimes H_{25}$	not computed
		$2 \Rightarrow A_{52}$	not computed
		$1 \Rightarrow S_{52}$	not computed
53	1	$1 \Rightarrow A_{53}$	not computed

Index	*# classes*	*Induced action*	*Abelianization*
54	27	$1 \Rightarrow (A_5 \times C_2) \ltimes H_{27}$	not computed
		$1 \Rightarrow (A_5 \times C_2) \ltimes H_{28}$	not computed
		$1 \Rightarrow (A_5 \times C_2) \ltimes H_{29}$	not computed
		$4 \Rightarrow (A_5 \times C_2) \ltimes H_{32}$	not computed
		$7 \Rightarrow A_5 \ltimes (\times_{i=1}^{6} A_9)$	not computed
		$4 \Rightarrow (A_5 \times C_2) \ltimes (\times_{i=1}^{6} A_9)$	not computed
		$7 \Rightarrow (A_5 \times C_2) \ltimes (H_2 \ltimes (\times_{i=1}^{6} A_9))$	not computed
		$1 \Rightarrow A_{54}$	not computed
		$1 \Rightarrow S_{54}$	not computed
55	4	$4 \Rightarrow PGL_2(11)$	$\mathbb{Z}_2^{10} \oplus \mathbb{Z}_{11}^3$
56	9	$1 \Rightarrow S_6(2)$	not computed
		$2 \Rightarrow S_6(2) \ltimes C_2$	not computed
		$4 \Rightarrow S_6(2) \ltimes H_{26}$	not computed
		$2 \Rightarrow S_{56}$	not computed
58	8	$6 \Rightarrow A_{58}$	not computed
		$2 \Rightarrow S_{58}$	not computed
59	1	$1 \Rightarrow S_{59}$	not computed

There are also 323 conjugacy classes of subgroups of index 60 in Γ^+. No information on the induced actions was computed.

1) H_1 is elementary abelian of order 64.

2) H_2 is elementary abelian of order 32.

3) H_3' is elementary abelian of order 729, and H_3/H_3' is elementary abelian of order 32.

4) $H_4 = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$.

5) H_5/H_5' is elementary abelian of order 32. Now $K = H_5'$ has an elementary derived subgroup of order 2^{12} and the factor group K/K' is elementary abelian of order 3^5.

6) H_6' is elementary abelian of order 5^6 and H_6/H_6' is elementary abelian of order 32.

7) H_7' is elementary abelian of order 3^{10} and H_7/H_7' is elementary abelian of order 32.

8) H_8 is the direct product of six copies of A_5.

9) H_9' is the direct product of six copies of A_5 and H_9/H_9' is elementary abelian of order 32.

10) H_{10}' is elementary abelian of order 3^{12} and H_{10}/H_{10}' is elementary abelian of order 32.

11) $H_{11} = H_2 \ltimes (H_2 \ltimes (H/H' \ltimes H'))$ where H' is elementary abelian of order 2^{12} and H/H' is elementary abelian of order 729.

12) H_{12}' is elementary abelian of order 3^{12} and H_{12}/H_{12}' is elementary abelian of order 2^{11}.

13) H_{13}' is elementary abelian of order 16 and $H_{13}/H13' = \mathbb{Z}_2^5 \oplus \mathbb{Z}_4$

14) H_{14}' is elementary abelian of order 32 and $H_{14}/H14' = \mathbb{Z}_2^5 \oplus \mathbb{Z}_4$

15) H_{15}' is elementary abelian of order 512 and $H_{14}/H14' = \mathbb{Z}_2^5 \oplus \mathbb{Z}_4$

16) $H_{16} = A_8 \times A_8 \times A_8 \times A_8 \times A_8$

17) H_{17}' is elementary abelian of order 7^6 and $H_{17}/H17'$ is elementary abelian of order 64.

18) $H_{18} = A_7 \times A_7 \times A_7 \times A_7 \times A_7 \times A_7$

19) H_{19}

20) H_{20} is elementary abelian of order 2^21

21) H_{21} has an elementary abelian derived subgroup of order 3^{15}. The factor group $H_{21}/H_{21}'$is elementary abelian of order 2^5.

22) $H_{22}/[H_{22}, H_{22}]$ is elementary abelian of order 2^5. $[H_{22}, H_{22}]$ is elementary abelian of order 2^{11}.

23) $H_{23}/[H_{23}, H_{23}]$ is elementary abelian of order 2^6. $[H_{23}, H_{23}]$ is elementary abelian of order 2^{11}.

24) H_{24} is elementary abelian of order 2^{11}.

25) H_{25} is elementary abelian of order 2^{13}.

26) H_{26} is elementary abelian of order 2^{21}.

27) $H_{27}/[H_{27}, H_{27}]$ is elementary abelian of order 2^5. $[H_{27}, H_{27}] \cong \times_{i=1}^{6} C_9$.

28) $H_{28}/[H_{28}, H_{28}]$ is elementary abelian of order 2^5. $[H_{28}, H_{28}]$ is elementary abelian of order 3^{12}.

29) $H_{29}/[H_{29}, H_{29}]$ is elementary abelian of order 2^{11}. $[H_{28}, H_{28}]$ is elementary abelian of order 3^{12}.

29) $H_{29}/[H_{29}, H_{29}]$ is elementary abelian of order 2^{11}. $[H_{28}, H_{28}]$ is elementary abelian of order 3^{12}.

30) $H_{30}/[H_{30}, H_{30}]$ is elementary abelian of order 2^5. $[H_{30}, H_{30}]$ is elementary abelian of order 5^{10}.

31) $H_{31}/[H_{31}, H_{31}]$ is elementary abelian of order 2^{10}. Write $H'_{31} = [H_{31}, H_{31}]$. Then $H'_{31}/[H'_{31}, H'_{31}]$ is elementary abelian of order 3^{12} and $[H_{31}, H_{31}]$ is elementary abelian of order 2^{24}

32) Let $H_{32}^{(i)} = [H_{32}^{(i-1)}, H_{32}^{(i-1)}]$ be the i^{th} derived subgroup of H_{32}, and $K^{(i)} = H_{32}^{(i-1)}/H_{32}^{(i)}$. Then the $K^{(i)}$ are elementary abelian, $|H^{(1)}| = 2^5$, $|H^{(2)}| = 3^6$, $|H^{(3)}| = 2^{12}$ and $H_{32}^{(3)}$ is also elementary abelian of order 3^{18}.

33) $H_{33}/[H_{33}, H_{33}]$ is elementary abelian of order 2^{11}. $[H_{33}, H_{33}]$ is elementary abelian of order 2^{11}.

34) $H_{34}/[H_{34}, H_{34}]$ is elementary abelian of order 2^5. Write $H'_{34} = [H_{34}, H_{34}]$. Then $H'_{34}/[H'_{34}, H'_{34}]$ is elementary abelian of order 3^6 and $[H_{34}, H_{34}]$ is elementary abelian of order 2^{12}.

35) $H_{35}/[H_{35}, H_{35}]$ is elementary abelian of order 2^5 and $[H_{35}, H_{35}] \cong \times_{i=1}^{6} L_2(7)$.

36) $H_{36}/[H_{36}, H_{36}]$ is elementary abelian of order 2^5. Write $H'_{36} = [H_{36}, H_{36}]$. Then $H'_{36}/[H'_{36}, H'_{36}]$ is elementary abelian of order 3^{12} and $[H_{36}, H_{36}]$ is elementary abelian of order 2^{24}.

37) $H_{37}/[H_{37}, H_{37}]$ is elementary abelian of order 2^5. Write $H'_{37} = [H_{37}, H_{37}]$. Then $H'_{37}/[H'_{37}, H'_{37}]$ is elementary abelian of order 3^6 and $[H_{37}, H_{37}]$ is elementary abelian of order 2^{24}.

A.4 $\Gamma = T_4[2,2,5;2,3,5]$

There are 147 conjugacy classes of subgroups of index ≤ 60 in Γ^+. Their index, the induced permutation group and, where possible, the abelianisation of the kernel of the map are listed here.

Index	*# classes*	*Induced action*	*Abelianization*
5	2	$2 \Rightarrow A_5$	$\mathbb{Z}_5^3$
6	2	$2 \Rightarrow A_5$	$\mathbb{Z}_5^3$
10	2	$2 \Rightarrow A_5$	$\mathbb{Z}_5^3$
12	2	$2 \Rightarrow A_5$	$\mathbb{Z}_5^3$
15		$2 \Rightarrow A_5$	$\mathbb{Z}_5^3$
17	1	$L_2(16)$	$\mathbb{Z}^{33} \oplus \mathbb{Z}_2^{12}$
20	5	$2 \Rightarrow A_5$	$\mathbb{Z}_5^3$
		$1 \Rightarrow L_2(19)$	$\mathbb{Z}^{56}$
		$2 \Rightarrow L_2(19)$	$\mathbb{Z}_5^{18} \oplus \mathbb{Z}_{19}^3$
25	5	$1 \Rightarrow A_5 \times A_5$	$\mathbb{Z}^{41} \oplus \mathbb{Z}_2^{12}$
		$4 \Rightarrow A_{25}$	not computed
30	25	$2 \Rightarrow A_5$	$\mathbb{Z}_5^3$
		$2 \Rightarrow A_5 \times A_5$	$\mathbb{Z}^{41} \oplus \mathbb{Z}_2^{12}$
		$10 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}_5^{15} \oplus \mathbb{Z}_{25}^{16} \oplus \mathbb{Z}_{125}^{11}$
		$2 \Rightarrow L_2(29)$	not computed
		$2 \Rightarrow A_5 \ltimes H_2$	not computed
		$7 \Rightarrow A_{30}$	not computed
32	2	$2 \Rightarrow L_2(31)$	not computed
34	1	$L_2(16) \ltimes H_3$	not computed
36	3	$1 \Rightarrow A_5 \times A_5$	$\mathbb{Z}^{41} \oplus \mathbb{Z}_2^{12}$
		$2 \Rightarrow A_5 \ltimes H_2$	not computed
42	2	$2 \Rightarrow L_2(41)$	not computed
46	2	$2 \Rightarrow A_{46}$	not computed

Index	*# classes*	*Induced action*	*Abelianization*
50	11	$2 \Rightarrow A_5 \times A_5$	$\mathbb{Z}^{41} \oplus \mathbb{Z}_2^{12}$
		$2 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}_5^{15} \oplus \mathbb{Z}_{25}^{16} \oplus \mathbb{Z}_{125}^{11}$
		$2 \Rightarrow L_2(49)$	not computed
		$1 \Rightarrow (A_5 \times A_5) \ltimes H_3$	not computed
		$2 \Rightarrow A_{25} \ltimes H_4$	not computed
		$2 \Rightarrow A_5 \ltimes H_5$	not computed
51	5	$1 \Rightarrow L_2(16)$	$\mathbb{Z}^{33} \oplus \mathbb{Z}_2^{12}$
		$3 \Rightarrow L_2(16) \ltimes H_6$	not computed
		$1 \Rightarrow L_2(16) \ltimes H_7$	not computed
55	4	$2 \Rightarrow A_5 \ltimes H_8$	not computed
		$2 \Rightarrow A_{55}$	not computed
57	6	$2 \Rightarrow L_2(19)$	$\mathbb{Z}^{56}$
		$4 \Rightarrow L_2(19)$	$\mathbb{Z}_5^{18} \oplus \mathbb{Z}_{19}^{3}$
59	1	A_{59}	not computed
60	64	$2 \Rightarrow A_5$	$\mathbb{Z}_5^{3}$
		$2 \Rightarrow L_2(19)$	$\mathbb{Z}_5^{18} \oplus \mathbb{Z}_{19}^{3}$
		$1 \Rightarrow L_2(19)$	$\mathbb{Z}^{56}$
		$6 \Rightarrow A_5 \times A_5$	$\mathbb{Z}^{41} \oplus \mathbb{Z}_2^{12}$
		$10 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}_5^{15} \oplus \mathbb{Z}_{25}^{16} \oplus \mathbb{Z}_{125}^{11}$
		$2 \Rightarrow L_2(29)$	not computed
		$2 \Rightarrow L_2(59)$	not computed
		$4 \Rightarrow A_5 \ltimes (\times_{i=1}^{6} A_5)$	not computed
		$3 \Rightarrow L_2(19) \ltimes (\times_{i=1}^{19} C_3)$	not computed
		$4 \Rightarrow A_5 \ltimes (\times_{i=1}^{6} A_6)$	not computed
		$2 \Rightarrow A_5 \ltimes (\times_{i=1}^{5} M_{11})$	not computed
		$8 \Rightarrow A_5 \ltimes (\times_{i=1}^{5} M_{12})$	not computed
		$6 \Rightarrow A_{30} \ltimes H_9$	not computed
		$12 \Rightarrow A_{60}$	not computed

1) H_1 is elementary abelian of order 125.

2) H_2 is the direct product of six copies of A_5.

3) H_3 is elementary abelian of size 2^{16}.

4) H_4 is elementary abelian of size 2^{24}.

5) H_5 is the direct product of six copies of A_{10}.

6) H_6 is elementary abelian of size 3^{16}.

7) H_7' is elementary abelian of size 3^{17} and H/H' is elementary abelian of size 2^{16}.

8) H_8 is the direct product of six copies of M_11.

9) H_9 is elementary abelian of order 2^{29}.

A.5 $\Gamma = T_5[2,3,3;2,4,3]$

There are 62 conjugacy classes of subgroups of index ≤ 20 in Γ^+. Their index, the induced permutation group and, where possible, the abelianisation of the kernel of the map are listed here.

Index	*# classes*	*Induced action*	*Abelianization*
5	1	A_5	$\mathbb{Z}_2^{11}$
6	1	A_5	$\mathbb{Z}_2^{11}$
7	2	$L_2(7)$	$\mathbb{Z}^6$
8	5	$1 \Rightarrow L_2(7)$	$\mathbb{Z}^6$
		$4 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{27}$
10	1	A_5	$\mathbb{Z}_2^{11}$
11	1	A_{11}	not computed
12	7	$1 \Rightarrow A_5$	$\mathbb{Z}_2^{11}$
		$4 \Rightarrow A_5 \ltimes H_2$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2^{10} \oplus \mathbb{Z}_4^3$
		$2 \Rightarrow A_5 \ltimes H_2$	$\mathbb{Z}^{25} \oplus \mathbb{Z}_2^8$
14	6	$2 \Rightarrow L_2(7)$	$\mathbb{Z}^6$
		$4 \Rightarrow L_2(7) \ltimes H_3$	$\mathbb{Z}^{27}$
15	1	A_5	$\mathbb{Z}_2^{11}$
16	6	$4 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2^2$
		$2 \Rightarrow L_2(7) \ltimes H_4$	$\mathbb{Z}^{193}$
18	2	$L_2(17)$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4^{16} \oplus \mathbb{Z}_{17}^3$
19	2	A_{19}	not computed
20	27	$1 \Rightarrow A_5$	$\oplus_{i=1}^{11} \mathbb{Z}_2$
		$4 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2^2$
		$4 \Rightarrow A_5 \ltimes H_2$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2^{10} \oplus \mathbb{Z}_4^3$
		$2 \Rightarrow A_5 \ltimes H_2$	$\mathbb{Z}^{25} \oplus \mathbb{Z}_2^8$
		$16 \Rightarrow A_5 \ltimes H_5$	$\mathbb{Z}^{325} \oplus \mathbb{Z}_2^{50} \oplus \mathbb{Z}_4^{24} \oplus \mathbb{Z}_8^{52}$

1) H_1 is elementary abelian of order 16.

2) H_2 is elementary abelian of order 32.

3) H_3 is elementary abelian of order 8.

4) H_4 where $|H/H'| = 64$ is elementary abelian and H' is cyclic of order 2.

5) H_3 is elementary abelian of order 1024.

A.6 $\Gamma = T_6[2, 3, 4; 2, 3, 4]$

There are 388 conjugacy classes of subgroups of index $\leq$ 20 in Γ^+. Their index, the induced permutation group and, where possible, the abelianisation of the kernel of the map are listed here.

Index	*# classes*	*Induced action*	*Abelianization*
2	1	C_2	$\mathbb{Z}_3 \oplus \mathbb{Z}_3$
3	4	$2 \Rightarrow D_6$	$\mathbb{Z}_3$
		$2 \Rightarrow D_6$	$\mathbb{Z}_2^4 \oplus \mathbb{Z}_3$
4	6	$4 \Rightarrow S_4$	$\mathbb{Z}_2^5 \oplus \mathbb{Z}_3$
		$2 \Rightarrow S_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_8^2$
6	18	$2 \Rightarrow D_6$	$\mathbb{Z}_3$
		$2 \Rightarrow D_6$	$\mathbb{Z}_2^4 \oplus \mathbb{Z}_3$
		$4 \Rightarrow S_4$	$\mathbb{Z}_2^5 \oplus \mathbb{Z}_3$
		$2 \Rightarrow S_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_8^2$
		$2 \Rightarrow A_6$	$\mathbb{Z}^9 \oplus \mathbb{Z}_2^2 \oplus \mathbb{Z}_3^6$
7	2	$2 \Rightarrow L_2(7)$	$\mathbb{Z}^{13}$
8	25	$4 \Rightarrow S_4$	$\mathbb{Z}_2^5 \oplus \mathbb{Z}_3$
		$2 \Rightarrow S_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_8^2$
		$2 \Rightarrow C_2 \ltimes (H_1)$	$\mathbb{Z}_3 \oplus \mathbb{Z}_8^3$
		$1 \Rightarrow L_2(7)$	$\mathbb{Z}^{13}$
		$12 \Rightarrow C_2 \ltimes H_2$	$\mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{16}^2$
		$4 \Rightarrow L_2(7) \ltimes H_3$	$\mathbb{Z}^{55}$
9	1	D_{18}	$\mathbb{Z}_2^8$
10	1	A_6	$\mathbb{Z}^9 \oplus \mathbb{Z}_2^2 \oplus \mathbb{Z}_3^6$
11	4	S_{11}	not computed

Index	*# classes*	*Induced action*	*Abelianization*
12	98	$12 \Rightarrow S_4$	$\mathbb{Z}_2^{11} \oplus \mathbb{Z}_4^2$
		$12 \Rightarrow C_2 \ltimes (H_4)$	$\mathbb{Z}_2^5 \oplus \mathbb{Z}_8^2$
		$6 \Rightarrow C_2 \ltimes H_4$	$\mathbb{Z}_3 \oplus \mathbb{Z}_8^3$
		$16 \Rightarrow C_2 \ltimes H_1$	$\mathbb{Z}_2^{48} \oplus \mathbb{Z}_4$
		$4 \Rightarrow C_2 \ltimes H_5$	$\mathbb{Z}_2^{11} \oplus \mathbb{Z}_4^2$
		$12 \Rightarrow C_2 \ltimes H_5$	$\mathbb{Z}^9 \oplus \mathbb{Z}_2^6$
		$2 \Rightarrow C_2 \ltimes H_5$	$\mathbb{Z}^9 \oplus \mathbb{Z}_2^8$
		$2 \Rightarrow C_2 \times A_6$	$\mathbb{Z}^{19} \oplus \mathbb{Z}_3^6$
		$24 \Rightarrow C_2 \times H_6$	$\mathbb{Z}_3^{25} \oplus \mathbb{Z}_{16}^3$
		$2 \Rightarrow PGL_2(11)$	$\mathbb{Z}^{11} \oplus \mathbb{Z}_3^{11} \oplus \mathbb{Z}_4^{10} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{11}^3$
		$4 \Rightarrow A_6 \ltimes H_7$	$\mathbb{Z}^{273} \oplus \mathbb{Z}_2^{32} \oplus \mathbb{Z}_3^6$
		$2 \Rightarrow M_{12}$	not computed
13	2	$2 \Rightarrow S_{13}$	not computed
14	34	$2 \Rightarrow L_2(7)$	$\mathbb{Z}^{13}$
		$2 \Rightarrow C_2 \times L_2(7)$	$\mathbb{Z}^{13} \oplus \mathbb{Z}_3^2$
		$4 \Rightarrow L_2(7) \ltimes H_3$	not computed
		$4 \Rightarrow C_2 \times (L_2(7) \ltimes H_3)$	$\mathbb{Z}^{55} \oplus \mathbb{Z}_2^{28} \oplus \mathbb{Z}_3^2$
		$8 \Rightarrow L_2(7) \ltimes H_8$	$\mathbb{Z}^{251} \oplus \mathbb{Z}_2^8$
		$8 \Rightarrow C_2 \times (L_2(7) \ltimes H_8)$	not computed
		$4 \Rightarrow A_{14}$	not computed
15	8	$2 \Rightarrow A_6$	$\mathbb{Z}^9 \oplus \mathbb{Z}_2^2 \oplus \mathbb{Z}_3^6$
		$2 \Rightarrow C_2 \ltimes H_9$	not computed
		$4 \Rightarrow C_2 \ltimes H_{10}$	not computed

Index	*# classes*	*Induced action*	*Abelianization*
16	46	$2 \Rightarrow C_2 \ltimes H_1$	as above
		$6 \Rightarrow C_2 \ltimes (C_3 \ltimes H_{12})$	$\mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{16}^2$
		$2 \Rightarrow C_2 \ltimes H_5$	$\mathbb{Z}_2^{48} \oplus \mathbb{Z}_4$
		$7 \Rightarrow C_2 \ltimes H_5$	$\mathbb{Z}^9 \oplus \mathbb{Z}_2^8$
		$1 \Rightarrow L_2(7) \times C_2$	$\mathbb{Z}^{13} \oplus \mathbb{Z}_3^2$
		$8 \Rightarrow C_2 \ltimes (C_3 \ltimes H_{13})$	$\mathbb{Z}_3^{13} \oplus \mathbb{Z}_8^2 \oplus \mathbb{Z}_{32}$
		$8 \Rightarrow C_2 \ltimes (C_3 \ltimes (C_3 \ltimes H_{13}))$	$\mathbb{Z}^{33} \oplus \mathbb{Z}_2^6$
		$4 \Rightarrow C_2 \ltimes (L_2(7) \ltimes H_3)$	$\mathbb{Z}^{55} \oplus \mathbb{Z}_2^{28} \oplus \mathbb{Z}_3^2$
		$4 \Rightarrow A_6 \ltimes H_{11}$	$\mathbb{Z}^{129} \oplus \mathbb{Z}_2^{32} \oplus \mathbb{Z}_3^6$
		$4 \Rightarrow L_2(7) \ltimes (C_2 \ltimes H_8)$	not computed
17	4	$4 \Rightarrow A_{17}$	not computed
18	93	$28 \Rightarrow A_{18}$	not computed
		$12 \Rightarrow S_{18}$	not computed
19	6	$6 \Rightarrow A_{19}$	not computed
20	35	$1 \Rightarrow A_6$	$\mathbb{Z}^9 \oplus \mathbb{Z}_2^2 \oplus \mathbb{Z}_3^6$
		$2 \Rightarrow A_6 \times C_2$	$\mathbb{Z}^{19} \oplus \mathbb{Z}_3^6$
		$8 \Rightarrow A_6 \ltimes H_7$	not computed
		$4 \Rightarrow A_6 \ltimes H_{14}$	not computed
		$4 \Rightarrow S_4 \wr A_5$	not computed
		$4 \Rightarrow H_{15} \wr A_5$	not computed
		$12 \Rightarrow S_{20}$	not computed

1) H_1' is elementary abelian of order 16 and $H/H' = C_3$ is cyclic.

2) $H_2 = C_3 \ltimes (C2 \times C2 \times C4 \times C4)$.

3) H_3 is elementary abelian of order 8.

4) H_4' is elementary abelian of order 4 and H/H' is elementary abelian of order 9.

5) H_5' is elementary abelian of order 16 and H/H' is elementary abelian of order 9.

6) $H_6 = C_2 \ltimes (C_3 \ltimes H)$, where H' is elementary abelian of order 8 and H/H' is elementary abelian of order 16.

7) H_7 is elementary abelian of order 32.

8) H_8 is elementary abelian of order 64.

9) $H_9 = C_3 \ltimes (A_5 \ltimes (A_5 \times A_5))$.

10) $H_{10} = C_3 \ltimes (C_2 \ltimes (C_2 \ltimes (A_5 \ltimes (A_5 \times A_5))))$.

11) H_{11} is elementary abelian of order 16.

12) $H_{12}' = C_2$ and H_{12}/H_{12}' is elementary abelian of order 16.

13) H_{13}' is elementary abelian of order 4 and H_{13}/H_{13}' is elementary abelian of order 16.

14) H_{14} is elementary abelian of order 512.

15) $H_{15} = S_4 \ltimes H_3$.

A.7 $\Gamma = T_7[2,3,3;2,5,3]$

Index	*# classes*	*Induced action*	*Abelianization*
5	2	$2 \Rightarrow A_5$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4^4$
6	2	$2 \Rightarrow A_5$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4^4$
7	1	A_7	$\mathbb{Z}^{69}$
10	4	$2 \Rightarrow A_5$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4^4$
		$2 \Rightarrow A_{10}$	not computed
11	4	$4 \Rightarrow L_2(11)$	$\mathbb{Z}_2 \oplus \mathbb{Z}_8^{10} \oplus \mathbb{Z}_{11}^3$
12	8	$2 \Rightarrow A_5$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4^4$
		$2 \Rightarrow L_2(11)$	$\mathbb{Z}_2 \oplus \mathbb{Z}_8^{10} \oplus \mathbb{Z}_{11}^3$
		$4 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2^8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8^{10} \oplus \mathbb{Z}_{32}$
14	3	$3 \Rightarrow A_7 \ltimes H_2$	not computed
15	4	$2 \Rightarrow A_5$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4^4$
		$2 \Rightarrow A_7$	$\mathbb{Z}^{69}$
16	8	$8 \Rightarrow A_5 \ltimes H_3$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4^{10} \oplus \mathbb{Z}_{16}$
20	20	$2 \Rightarrow A_5$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4^4$
		$8 \Rightarrow A_5 \ltimes H_3$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4^{10} \oplus \mathbb{Z}_{16}$
		$4 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2^8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8^{10} \oplus \mathbb{Z}_{32}$
		$6 \Rightarrow A_{10} \ltimes H_4$	not computed

1) H_1 is elementary abelian of order 32.

2) H_2 is elementary abelian of order 64.

3) H_3 is elementary abelian of order 16.

3) H_3 is elementary abelian of order 512.

A.8 $\Gamma = T_8[2,4,3;2,5,3]$

There are 103 conjugacy classes of subgroups of index ≤ 20 in Γ^+. Their index, the induced permutation group and, where possible, the abelianisation of the kernel of the map are listed here.

Index	*# classes*	*Induced action*	*Abelianization*
6	6	$4 \Rightarrow A_6$	$\mathbb{Z}_2^{61} \oplus \mathbb{Z}_3^6$
		$2 \Rightarrow A_6$	$\mathbb{Z}^{20} \oplus \mathbb{Z}_3^2$
10	7	$3 \Rightarrow A_6$	as above
		$4 \Rightarrow A_{10}$	not computed
11	2	$L_2(11)$	$\mathbb{Z}_{10} \oplus \mathbb{Z}_2^{101}$
12	11	$1 \Rightarrow L_2(11)$	as above
		$2 \Rightarrow M_{12}$	not computed
		$6 \Rightarrow A_6 \ltimes H_1$	not computed
		$2 \Rightarrow A_{12}$	not computed
15	12	$6 \Rightarrow A_6$	as above
		$6 \Rightarrow A_{12}$	not computed
16	17	$4 \Rightarrow A_6 \ltimes H_2$	$\mathbb{Z}^{215} \oplus \mathbb{Z}_2^3$
		$8 \Rightarrow A_6 \ltimes H_2$	$\mathbb{Z}_2^{42} \oplus \mathbb{Z}_3^{21} \oplus \mathbb{Z}_8^{15}$ $\oplus \mathbb{Z}_{16}^8 \oplus \mathbb{Z}_{32}^{52}$
		$5 \Rightarrow A_{16}$	not computed
17	10	$1 \Rightarrow L_2(16)$	$\mathbb{Z}^{17} \oplus \mathbb{Z}_2^{652} \oplus \mathbb{Z}_4^{12}$
		$9 \Rightarrow A_{17}$	not computed
18	8	$2 \Rightarrow A_6 \ltimes C_3$	$\mathbb{Z}^{62}$
		$2 \Rightarrow A_6 \ltimes H_3$	not computed
		$4 \Rightarrow A_{18}$	not computed

Index	*# classes*	*Induced action*	*Abelianization*
20	29	$1 \Rightarrow A_6$	$\mathbb{Z}^{20} \oplus \mathbb{Z}_3^2$
		$2 \Rightarrow A_6$	$\mathbb{Z}_2^{61} \oplus \mathbb{Z}_3^6$
		$2 \Rightarrow L_2(19)$	$\mathbb{Z}_2^{571} \oplus \mathbb{Z}_3^{19} \oplus \mathbb{Z}_{19}^3$
		$12 \Rightarrow A_6 \ltimes H_1$	not computed
		$6 \Rightarrow A_6 \ltimes H_4$	not computed
		$4 \Rightarrow A_{10} \ltimes H_4$	not computed
		$2 \Rightarrow A_{20}$	not computed

1) H_1 is elementary abelian of order 32.

2) H_2 is elementary abelian of order 16.

3) H_3' is elementary abelian of order 729 and H_3/H_3'.

4) H_4 is elementary abelian of order 512. is elementary abelian of order 32.

A.9 $\Gamma = T_9[2,3,5;2,3,5]$

Γ^+ has 457 conjugacy classes of low index subgroups of index ≤ 24. Their index, the induced permutation group and, where possible, the abelianisation of the kernel of the map are listed here.

Index	*# classes*	*Induced action*	*Abelianization*
5	3	$2 \Rightarrow A_5$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3^4 \oplus \mathbb{Z}_4^4 \oplus \mathbb{Z}_5^3$
		$1 \Rightarrow A_5$	$\mathbb{Z}^{11}$
6	7	$2 \Rightarrow A_5$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3^4 \oplus \mathbb{Z}_4^4 \oplus \mathbb{Z}_5^3$
		$1 \Rightarrow A_5$	$\mathbb{Z}^{11}$
		$4 \Rightarrow A_6$	$\mathbb{Z}^9 \oplus \mathbb{Z}_2^{16} \oplus \mathbb{Z}_3^6$
10	7	$2 \Rightarrow A_5$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3^4 \oplus \mathbb{Z}_4^4 \oplus \mathbb{Z}_5^3$
		$1 \Rightarrow A_5$	$\mathbb{Z}^{11}$
		$2 \Rightarrow A_6$	$\mathbb{Z}^9 \oplus \mathbb{Z}_2^{16} \oplus \mathbb{Z}_3^6$
		$2 \Rightarrow A_{10}$	not computed
11	10	$2 \Rightarrow L_2(11)$	$\mathbb{Z}^{41}$
		$8 \Rightarrow A_{10}$	not computed
12	20	$2 \Rightarrow A_5$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3^4 \oplus \mathbb{Z}_4^4 \oplus \mathbb{Z}_5^3$
		$1 \Rightarrow A_5$	$\mathbb{Z}^{11}$
		$1 \Rightarrow L_2(11)$	$\mathbb{Z}^{41}$
		$4 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2^8 \oplus \mathbb{Z}_3^4 \oplus \mathbb{Z}_4 \oplus$ $\mathbb{Z}_5^3 \oplus \mathbb{Z}_8^{10} \oplus \mathbb{Z}_{19}^{12} \oplus \mathbb{Z}_{32}$
		$6 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{103} \oplus \mathbb{Z}_2^2$
		$4 \Rightarrow M_{12}$	not computed
		$2 \Rightarrow A_{12}$	not computed

Index	*# classes*	*Induced action*	*Abelianization*
15	26	$3 \Rightarrow A_5$	computed above
		$4 \Rightarrow A_6$	computed above
		$6 \Rightarrow A_5 \ltimes H_2$	$\mathbb{Z}^{40} \oplus \mathbb{Z}_2^{81} \oplus \mathbb{Z}_3^{32} \oplus$ $\mathbb{Z}_4^4 \oplus \mathbb{Z}_5^{33} \oplus \mathbb{Z}_7^{20} \oplus \mathbb{Z}_9^{12}$
		$3 \Rightarrow A_5 \ltimes H_2$	$\mathbb{Z}^{191} \oplus \mathbb{Z}_2^{40} \oplus \mathbb{Z}_3^2$
		$10 \Rightarrow A_{15}$	not computed
16	40	$4 \Rightarrow A_5 \ltimes H_3$	$\mathbb{Z}^{71}$
		$8 \Rightarrow A_5 \ltimes H_3$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3^4 \oplus$ $\mathbb{Z}_4^{10} \oplus \mathbb{Z}_5^3 \oplus \mathbb{Z}_{16}$
		$8 \Rightarrow A_6 \ltimes H_3$	$\mathbb{Z}^{129} \oplus \mathbb{Z}_2^{56} \oplus \mathbb{Z}_3^6 \oplus \mathbb{Z}_4^{16}$
		$20 \Rightarrow A_{16}$	not computed
18	17	$1 \Rightarrow A_5 \ltimes H_4$	not computed
		$4 \Rightarrow A_5 \ltimes H_5$	not computed
		$8 \Rightarrow A_5 \ltimes H_6$	not computed
		$4 \Rightarrow A_{18}$	not computed
20	108	$3 \Rightarrow A_5$	computed above
		$2 \Rightarrow A_6$	computed above
		$12 \Rightarrow A_5 \ltimes H_3$	computed above
		$10 \Rightarrow A_5 \ltimes H_1$	computed above
		$2 \Rightarrow L_2(19)$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3^{37} \oplus \mathbb{Z}_5^{18} \oplus$ $\mathbb{Z}_8^{18} \oplus \mathbb{Z}_{11}^{18} \oplus \mathbb{Z}_{17}^{20} \oplus \mathbb{Z}_{19}^3$
		$16 \Rightarrow A_5 \ltimes H_7$	not computed
		$4 \Rightarrow A_6 \ltimes H_8$	not computed
		$15 \Rightarrow A_5 \ltimes H_9$	not computed
		$38 \Rightarrow A_{20}$	not computed
21	34	$10 \Rightarrow L_3(4)$	not computed
		$24 \Rightarrow A_{21}$	not computed

Index	*# classes*	*Induced action*	*Abelianization*
22	30	$6 \Rightarrow L_3(4) \ltimes H_{10}$	not computed
		$8 \Rightarrow A_{11} \ltimes H_{10}$	not computed
		$16 \Rightarrow A_{22}$	not computed
24	155	$2 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2^8 \oplus \mathbb{Z}_3^4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5^3 \oplus \mathbb{Z}_8^{10} \oplus \mathbb{Z}_{19}^{12} \oplus \mathbb{Z}_{32}$
		$3 \Rightarrow A_5 \ltimes H_1$	$\mathbb{Z}^{103} \oplus \mathbb{Z}_2^2$
		$4 \Rightarrow A_5 \ltimes H_{11}$	$\mathbb{Z}^{167} \oplus \mathbb{Z}_2^{32}$
		$4 \Rightarrow A_5 \ltimes H_{12}$	not computed
		$4 \Rightarrow A_5 \ltimes H_{13}$	not computed
		$36 \Rightarrow A_5 \ltimes H_{14}$	not computed
		$1 \Rightarrow L_2(11) \ltimes H_{13}$	not computed
		$5 \Rightarrow A_5 \ltimes H_{15}$	not computed
		$12 \Rightarrow A_5 \ltimes H_{16}$	not computed
		$4 \Rightarrow M_{12} \ltimes H_{13}$	not computed
		$40 \Rightarrow A_5 \ltimes (H_1 \ltimes H_{16})$	not computed
		$2 \Rightarrow A_{12} \ltimes H_7$	not computed
		$38 \Rightarrow A_{24}$	not computed

1) H_1 is an elementary abelian group of order 32.

2) H_2 is an elementary abelian group of order 81.

3) H_3 is an elementary abelian group of order 16.

4) H_4 is an elementary abelian group of order 3^5.

5) H_5 is an elementary abelian group of order 3^6.

6) H_6 is the semidirect product of two elementary abelian groups. H_6' is elementary abelian of order 3^6 and the factor group is elementary abelian order 32.

7) H_7 is an elementary abelian group of order 1024.

8) H_8 is an elementary abelian group of order 512.

9) H_9 is the semidirect product of two elementary abelian groups. H_9' is elementary abelian of order 2^{10} and the factor group is elementary abelian order 3^4.

10) H_{10} is an elementary abelian group of order 2^{10}.

11) H_{11} is an elementary abelian group of order 64.

12) $H_{12} = C_4 \times C_4 \times C_4 \times C_4 \times C_4$.

13) H_{13} is an elementary abelian group of order $2^{11} = 2048$.

14) H_{14} has order 2048 and has elementary abelian derived subgroup of order 32. The factor group is also elementary abelian

15) H_{15} is the semidirect product of two elementary abelian groups. H_{15}' is elementary abelian of order 2^{12} and the factor group is elementary abelian order 3^5.

16) H_{16} is the semidirect product of two elementary abelian groups. H_{16}' is elementary abelian of order 2^{12} and the factor group is elementary abelian order 3^6.

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