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Supersymmetric Effects in

Hadronic Top Production

by

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ABSTRACT

FACULTY OF SCIENCE

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Supersymmetric Effects in Hadronic Top Production

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We study the effects of supersymmetry on polarised cross-sections for $t\bar{t}$ production at the LHC within a MSSM framework. A numerical study is carried out for the ten benchmarks of the Snowmass accord. It is found that the higher order effects involving supersymmetric particles in internal loops can be as high as 6%, both for the cross-section and the parity even helicity asymmetry, for one particular benchmark. For other benchmarks smaller but nonetheless observable corrections are found.

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Introduction

For nearly 35 years, supersymmetry (SUSY) has been an attractive theory in particle physics [1, 2]. At the most theoretical level, it permits the construction of string theories which do not contain tachyonic states, and at the phenomenological level it offers a solution to the hierarchy problem through its reduced ultraviolet divergences, as well as providing resolutions of several puzzles arising in standard models of cosmology. It also gives rise to a correction to the running of the couplings, so that the strong, weak, and electromagnetic interactions can unify at some Grand Unified (GUT) scale.

However, to date there has been no reliable evidence that this theory describes Nature, so that if SUSY is indeed realised in nature, it must be broken at a scale higher than that reached in accelerator experiments conducted up to now. If the theory is to be effective in providing a solution to the hierarchy problem, then the SUSY breaking scale cannot be much more than about 1 TeV. This is also the scale of SUSY breaking which leads to unification of couplings. Hence, with the exception of some hidden corners of parameter space, SUSY can be discovered at the forthcoming LHC.

Clearly the most dramatic manifestation of SUSY would be the production and identification of supersymmetric partner particles such as the spin-$\frac{1}{2}$ charginos or neutralinos, or evidence that at sufficiently high energies hadrons display behaviour consistent with the existence of squarks or gluinos. However, the existence of SUSY will also have indirect but measurable effects on the (total and differential) cross-sections for the production of Standard Model (SM) particles. The LHC is expected to achieve sufficient integrated luminosity so that the statistical errors on these cross-sections are below the percent level. Assuming sufficient control over theoretical and experimental systematic errors, we will then be able to detect the effects of higher

order corrections coming from loops of SUSY particles. The loops can give rise to a significant correction to the production cross-sections even below the threshold for the production of the SUSY particles themselves, so that hints that some new physics is imminent can be deduced before the threshold energies are actually reached. Above these thresholds, differential cross-sections with respect to suitably chosen variables can display structures which can be used to experimentally constrain the parameters of the underlying model.

Due to its large mass the top quark decays predominantly via electroweak interactions. Since the electroweak interactions violate parity, information about the spin of the top quark is encoded in the angular distribution of its decay products [3]. To fully exploit future experimental data we therefore need accurate computations (within the SM and beyond) of *polarised* $t\bar{t}$ production amplitudes, i.e. amplitudes for the production of $t\bar{t}$ pairs with given helicities. Such amplitudes allow us to predict not only the total $t\bar{t}$ cross section but also single and double spin asymmetries like the ones discussed in [4]. Of particular interest are parity violating asymmetries, since they are free of QCD related systematic errors. Considering ratios of asymmetries and total cross sections also removes systematic errors related to uncertainties in the incident beam flux.

The SM predictions for hadronic $t\bar{t}$ production have already been calculated by several groups. Tree level amplitudes were first considered in [5–10]. The next-to-leading order (NLO) QCD corrections to unpolarised amplitudes have been calculated in [11–16] and the electroweak contributions ($\mathcal{O}(\alpha \alpha_s^2)$) have been studied in [17–20]. Soft gluon resummation and threshold effects have been considered in [21–25]. The NLO QCD and electroweak corrections to polarised amplitudes are presented in [26–29] and [20, 30–34], respectively. The estimates for the theoretical errors of these calculations lie at the percent level.

Partial studies of SUSY contributions to both unpolarised and polarised $t\bar{t}$ production amplitudes within the Minimal Supersymmetric SM (MSSM) also exist. The SUSY electroweak corrections to unpolarised amplitudes are calculated in [35–37]. The same study for polarised amplitudes is carried out in [30]. The NLO SUSY QCD (SQCD) corrections to top production via gluon fusion ($gg \to t\bar{t}$) are presented in [38, 39]. Top production via quark-antiquark annihilation is discussed in [40-42]. During the preparation of this thesis a complete study of NLO SQCD corrections [43] was also published. As pointed out there, the results of [40–42] disagree due to an

incorrect treatment of Majorana fermions in box diagrams (see *section 4.3*).

In this thesis we present a complete study of SUSY QCD and electroweak corrections to $t\bar{t}$ production within the MSSM framework. We have organised our calculation so that software is available to calculate differential cross-sections with all possible helicity configurations and any given set of the extra 105 parameters of the MSSM. This provides maximum flexibility for studying different SUSY breaking scenarios and exotic areas of the parameter space, even though a scan over all 105 parameters is clearly unrealistic. Our code can read input parameters in the SUSY Les Houches Accord format [44], which makes it easily combinable with other MSSM related software. As a first analysis we present numerical results for the 10 'Snowmass' benchmark points of the MSSM parameter space, which were compiled at the Snowmass meeting of 2002 [45]. We find considerable variation in the magnitudes of the corrections from these different parameter sets. Conversely, this means that accurate measurement of the $t\bar{t}$ production cross-section can be used as a tool to help identify the correct set of SUSY parameters.

At sufficiently high (partonic) energies, the SUSY corrections to $t\bar{t}$ production are expected to be dominated by single and double logarithms of incoming parton energy divided by the SUSY breaking scale, M_{SUSY}. The determination of these logarithms is independent of the SUSY parameter set, with the exception of M_{SUSY} and the ratio, $\tan\beta$, of the vacuum expectation values of the two Higgs doublets, and the calculation is simplified by the fact that the mixing of various SUSY particles to form mass eigenstates has no effect on these logarithms. The logarithmic contributions have been calculated by Beccaria et. al. [46]. One may have expected that it would have been possible to express the entire SUSY correction in terms of these logarithms plus a constant off-set, which depended on the SUSY parameter set. We have compared our results with those of [46] and although it is indeed the case that our results agree with these logarithms plus a constant off-set at sufficiently high partonic energies, this approximation is found to be unsuitable at typical partonic energies which will be reached at the LHC, and the entire calculation is required for a reliable prediction of the cross-sections at the LHC.

The structure of this thesis is as follows: In *chapter 1* we provide an introduction to the SM and MSSM. In *chapter 2* we introduce our strategy for calculating polarised top production amplitudes by combining computer algebra and numerical tools. In *chapter 3* we discuss the issue of renormalisation and explain the $\overline{\text{DR}}$ renormalisation

scheme, which was used in this calculation. In *chapter 4* we list the diagrams relevant for $t\bar{t}$ production at the LHC and explain some of the inner workings of the numerical library we wrote to compute them. In *chapter 5* we show how physical observables, i.e. cross sections and their ratios, can be obtained from the amplitudes we calculated earlier. In *chapter 6* we present our results and conclude.

Chapter 1

The Standard Model
and its Supersymmetric Extension

Since its formulation in the early 1970s the Standard Model of Particle Physics (SM) provides a remarkably successful description of the interactions of fundamental particles. Even with particle colliders probing energies at the scale of 100 GeV there exist, to date, no confirmed observations that contradict the predictions of the SM.[1] Still, it is clear that the SM does not hold for arbitrarily high energies. Certainly modifications to the SM will be necessary at the Planck scale of 10^{19} GeV, where quantum gravitational effects become important, and it seems very unlikely that no new physics exists in the 17 orders of magnitude in between.

A very popular class of models for physics beyond the Standard Model (BSM) are *supersymmetric* models. The main motivation for studying SUSY models is the fact that the SM has a very high "sensitivity" to any new physics at higher energy scales. The observed low-energy physics can only be explained by a judicious tuning of the parameters of the BSM model. This problem is known as the *Hierarchy* or *Fine Tuning problem*. As we will see later in this chapter, supersymmetric models can solve this problem if SUSY itself is realised at the 1 to 10 TeV scale.

Here we first review the main features of the SM. Then we discuss the Hierarchy

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[1]This statement is not entirely true. The discovery of *neutrino oszillations* has shown that neutrinos must have very small but nonzero masses, while the SM predicts them to be massless. However, the ad hoc introduction of massive neutrinos in the SM is not impossible but merely unsatisfying, since the lightness of the neutrinos remains unexplained. In any case, this issue has no impact on top quark physics, which is the subject matter of this thesis.

problem and show how it is solved in a supersymmetric theory. Finally, we provide an introduction to the formalism of supersymmetry and give an overview over the minimal supersymmetric extension of the SM (MSSM), which will be the framework of our phenomenological study of top quark production.

1.1   The Standard Model

The success of the Standard Model began in 1971, when G. 't Hooft and M. Velt-man discovered [47,48] that local gauge theories are renormalisable, even if the gauge symmetry is spontaneously broken. This encouraged serious phenomenological in-vestigation of local gauge theories, since they were now known to be predictive even beyond the leading order in perturbation theory. It quickly became clear that the weak interactions could be described by an old model by Glashow, Weinberg and Salam [49–51], which suggested a spontaneously broken $SU(2) \times U(1)$ gauge sym-metry. Likewise, many aspects of the strong interactions could be explained by an (unbroken) $SU(3)$ gauge theory, known as Quantum Chromodynamics (QCD).

Finally, the Standard Model emerged from the fusion of these two models. It is a $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge theory that undergoes a spontaneous breakdown to a $SU(3)_C \times U(1)_Q$ symmetry at the *electroweak breaking scale*

$$\Lambda_{\mathrm{EWSB}} = 246\,\mathrm{GeV} \quad . \tag{1.1}$$

The subscripts Y and Q stand for hypercharge and electric charge, respectively, and $SU(3)_C$ is the colour symmetry of QCD. The fields of the Standard Model are listed in *table 1.1*. The upper part of the table lists the fermionic fields: the quarks and leptons. They can be divided into three generations. Each generation comprises a lepton pair and a quark pair. The left-handed components of the fields in each pair form a doublet under the $SU(2)_L$ symmetry. The right-handed components transform as singlets under $SU(2)_L$, with the exception of the neutrino fields, for which there are no right-handed components. This difference in the transformation behaviour of left and right-handed fields is the origin of parity violation in the Standard Model. The quark fields d, u, s, c, b and t (called down, up, strange, charm, bottom and top quarks) each have three components in "colour space", which transform as triplets under the $SU(3)_C$ group. All the lepton fields are singlets under $SU(3)_C$.

		generation		
		1	2	3
leptons {	charged leptons	e	μ	τ
	neutrinos	ν_e	ν_μ	ν_τ
quarks {	up type	u	c	t
	down type	d	s	b
gauge bosons {	$SU(3)_C$	G^a $(a = 1\dots 8)$		
	$SU(2)_L$	W^1, W^2, W^3		
	$U(1)_Y$	B		
Higgs fields		$H = (H^+, H^0)$		

Table 1.1: Fields of the Standard Model

The middle part of *table 1.1* lists the gauge bosons associated with the $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$ subgroups. Initially, all the fields have to be *massless*. For the gauge bosons, mass terms are generally forbidden by the gauge symmetry, and for the fermions we cannot construct any mass terms because the left and right-handed components transform differently under the $SU(2)_L$ symmetry. Masses in the Standard Model are generated by the *Higgs mechanism*: First we couple the fermions and gauge bosons to scalar fields, the *Higgs doublet* $H \equiv (H^+, H^0)$, in a gauge invariant way. Then we construct the self-interactions of the Higgs field in such a way that it acquires a nonzero vacuum expectation value (VEV), which is not invariant under the full gauge group. This process is called *spontaneous symmetry breaking*. By expanding the Higgs field around its VEV we generate mass terms for all fields that couple to the Higgs field.

The Standard Model Higgs fields H^+ and H^0 transform as an anti-doublet under the $SU(2)_L$ group and have a hypercharge of 1 associated with the $U(1)_Y$ group. The Higgs potential is just the potential of a ϕ^4 theory with a negative mass term:

$$V(H) = -\mu^2 H^\dagger H + \frac{\lambda}{2}(H^\dagger H)^2 \quad , \tag{1.2}$$

with $\mu^2, \lambda > 0$. The Higgs Lagrangian is then

$$\mathcal{L}_H = (D_\mu H)^\dagger (D^\mu H) - V(H) + \text{Higgs-fermion interaction terms} \quad , \tag{1.3}$$

where D_μ is the $SU(2)_L \times U(1)_Y$ covariant derivative:

$$D_\mu = \partial_\mu - igW_\mu^i \frac{\sigma^i}{2} - ig'B_\mu \quad . \tag{1.4}$$

Here σ^i denote the Pauli matrices (A.34). The Higgs potential V is minimised if

$$H^\dagger H = \frac{\mu^2}{\lambda} \quad . \tag{1.5}$$

Clearly there is a continuum of field vectors H that satisfy this condition. Each of these minima corresponds to a different vacuum state, but none of these states are invariant under the full $SU(2)_L \times U(1)_Y$ gauge group. If Nature "chooses" to realise one of these vacuum states, this choice breaks the gauge symmetry. It is important to remember that the Lagrangian of the theory is still invariant under the full gauge group. The gauge symmetry is only broken by Nature's choice of the physical vacuum.

Assume that the vacuum field configuration of the Standard Model Higgs field is

$$H_{\text{vac}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \quad \text{with} \quad v = \frac{2\mu}{\sqrt{\lambda}} \quad . \tag{1.6}$$

This represents no loss of generality, since we can always redefine the field H in such a way that the vacuum configuration is of this form. To expand around this vacuum configuration we parametrise the field H as follows:

$$H = H_{\text{vac}} + \frac{1}{\sqrt{2}} \begin{pmatrix} -\phi^1 - i\phi^2 \\ h + i\phi^3 \end{pmatrix} \quad , \tag{1.7}$$

where ϕ^i and h are independent real-valued fields. The fields ϕ^i are called *Goldstone bosons* and the field h is the *physical* Higgs field. By substituting this parametrisation of H in the Lagrangian (1.3) (and dropping a constant term) we see that the Lagrangian for the field h is of the form

$$\mathcal{L}_h = \frac{1}{2}(\partial_\mu h)(\partial^\mu h) + 2\mu^2 h^2 + \text{interaction terms} \quad , \tag{1.8}$$

Hence the physical Higgs field has a mass

$$m_h = 2\mu = \sqrt{\lambda}v \quad . \tag{1.9}$$

gauge field	mass
$W_\mu^\pm = \dfrac{1}{\sqrt{2}} W_\mu^1 \pm i W_\mu^2$	$m_W = \dfrac{gv}{2} = \dfrac{ev}{2s_W}$
$Z_\mu^0 = \dfrac{g W_\mu^3 - g' B_\mu}{\sqrt{g^2 + g'^2}} = c_W W_\mu^3 - s_W B_\mu$	$m_Z = \dfrac{v}{2} \sqrt{g^2 + g'^2} = \dfrac{m_W}{c_W}$
$A_\mu = \dfrac{g W_\mu^3 + g' B_\mu}{\sqrt{g^2 + g'^2}} = c_W W_\mu^3 + s_W B_\mu$	0

Table 1.2: Physical gauge fields and gauge boson masses of the Standard Model

The constant term H_{vac} in (1.7) generates mass terms for the gauge bosons:

$$\mathcal{L}_{\text{mass}} = H_{\text{vac}}^\dagger \left(i g W_\mu^i \frac{\sigma^i}{2} + i g' B_\mu \right) \left(-i g W^{i\mu} \frac{\sigma^i}{2} - i g' B^\mu \right) H_{\text{vac}} \tag{1.10}$$

$$= \frac{1}{2} \left[\frac{g^2 v^2}{4} (W_\mu^1 + i W_\mu^2)(W^{1\mu} - i W^{2\mu}) + \frac{v^2}{4} (g W_\mu^3 - g' B_\mu)(g W^{3\mu} - g' B^\mu) \right] \quad .$$

This leads to the physical fields and masses shown in *table 1.2*. Usually it is more convenient to replace the coupling constants g and g' by the electromagnetic coupling e and the Weinberg angle θ_W. They are related by

$$c_W = \cos\theta_W = \frac{g}{\sqrt{g^2 + g'^2}} \quad , \quad s_W = \sin\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}} \quad ,$$

$$e = \frac{g g'}{\sqrt{g^2 + g'^2}} \quad . \tag{1.11}$$

The fermion masses in the Standard Model are generated in a similar manner if we introduce Yukawa interactions between the Higgs field and the fermions. To do this let us first rename the fields from *table 1.1* in a more convenient way. We introduce a generation index I running from 1 to 3 and define

$$u^I = (u, c, t) \quad , \quad d^I = (d, s, b) \quad , \quad \nu^I = (\nu_e, \nu_\mu, \nu_\tau) \quad , \quad e^I = (e, \mu, \tau) \quad . \tag{1.12}$$

We also define the $SU(2)_L$ doublets Q_i^I and L_i^I (with $i = 1, 2$) as

$$Q_i^I = \begin{pmatrix} u_L^I \\ d_L^I \end{pmatrix} \quad , \quad L_i^I = \begin{pmatrix} \nu_L^I \\ e_L^I \end{pmatrix} \quad , \tag{1.13}$$

where the subscript L denotes the left-handed spinor components. Now the Yukawa terms of the Standard Model Lagrangian can be written as

$$\mathcal{L}_{\text{Yuk}} = -\varepsilon_{ij} Y_u^{IJ} H_i^* \bar{u}_R^I Q_j^J - Y_d^{IJ} H_i \bar{d}_R^I Q_i^J - Y_e^{IJ} H_i \bar{e}_R^I L_i^J + \text{h.c.} \quad , \qquad (1.14)$$

where I, J are generation indices, i, j are $SU(2)_L$ doublet indices and $\varepsilon_{ij} = -i\sigma_{ij}^2$ is an anti-symmetric 2×2 matrix. The couplings Y_u^{IJ}, Y_d^{IJ} and Y_e^{IJ} are called *Yukawa couplings*. If we substitute the expansion (1.7) for H in (1.14) the constant term H_{vac} generates mass terms for the Standard Model quarks and charged leptons.

1.2 The Hierarchy Problem

As we mentioned in the last section, the issue of renormalisability is settled for the Standard Model by the work of 't Hooft, who proved that any local gauge theory is renormalisable, even if the gauge symmetry is spontaneously broken. However, the renormalisation of the Higgs field still causes problems, not in the dramatic sense that the theory becomes unpredictive, but in the sense that some unnatural *fine tuning* is required if we want to introduce new physics at high energy scales.

To understand this let us have a look at the higher order corrections to the Higgs propagator. At leading order the propagator is given by

$$-----\!\underset{h(p)}{} = \frac{i}{p^2 - m_h^2 + i\varepsilon} \quad . \qquad (1.15)$$

Here m_h denotes the *bare* Higgs mass, i.e. the coefficient multiplying the mass term in the (unrenormalised) Higgs Lagrangian. It should not be confused with the *pole mass* $m_{h,\text{pole}}$. The pole mass is related to the sum of all higher order corrections to the Higgs propagator:

$$-----\!\!\!\underset{h(p)}{} + -----\!\!\!\underset{h(p)}{} \bigcirc -----\!\!\!\underset{h(p)}{} + \ldots = \frac{i}{p^2 - m_{h,\text{pole}}^2 + i\varepsilon} \quad . \qquad (1.16)$$

It is a physical observable and can be measured directly, once we have discovered the Higgs particle. Even now there exist theoretical boundaries on the Higgs mass and indirect measurements from precision electroweak data. A summary of these

constraints can be found in [52]. The precision electroweak data currently favours a Higgs mass of

$$m_{h,\text{pole}} = 114^{+69}_{-45}\,\text{GeV} \quad , \tag{1.17}$$

while the lower bound from LEP data is

$$m_{h,\text{pole}} > 115\,\text{GeV} \quad . \tag{1.18}$$

In any case we expect the physical Higgs mass to be of the order of 100 GeV. The free parameters of the Standard Model, or any model including it, must be chosen in such a way that the experimental value of the Higgs mass is reproduced.

We can single out the loop contributions to $m_{h,\text{pole}}$ by writing

$$m^2_{h,\text{pole}} = m^2_h + \Delta m^2_h \quad . \tag{1.19}$$

In general the mass shift Δm^2_h will contain ultraviolet-divergent terms which have to be cancelled in the renormalisation procedure. The first step in this procedure consists in regularising the divergent integrals by some ad hoc prescription. This introduces a new energy scale to the theory, called the *ultraviolet cutoff scale* Λ. The mass shift Δm^2_h will now depend on Λ:

$$\Delta m^2_h \equiv \Delta m^2_h(\Lambda) \quad . \tag{1.20}$$

Today most people regard the occurrence of divergences in quantum field theories and the necessity of regularisation as an indication that these theories are only low-energy limits of a superior theory, which is ab initio divergence free. The cutoff scale Λ then represents the scale at which the field theory fails to approximate the superior theory. The "physical" value of Λ is therefore expected to lie close to the Planck scale $\Lambda_{\text{Pl}} = 10^{19}\,\text{GeV}$, where quantum-gravitational effects become important.

Let us now consider the contributions to Δm^2_h due to the following diagram:

$$(1.21)$$

A simple calculation shows that the mass shift due to this diagram is:

$$\Delta m_h^2(\Lambda) = \frac{|\lambda_f|^2}{16\pi^2} \left[-2\Lambda^2 + 6m_f^2 \log(\Lambda/m_f) + \ldots \right] \quad , \tag{1.22}$$

where λ_f is the coupling constant associated with the Higgs-fermion vertex and the dots indicate terms which are finite in the limit $\Lambda \to \infty$ and depend on the details of the regularisation method. The important point is that for large cutoffs Λ the mass shift Δm_h^2 is dominated by the Λ^2 term.[2] Thus, for cutoffs at the Planck scale, we would expect the pole mass $m_{h,\text{pole}}$ to be of order Λ_{Pl}. The only way to obtain a pole mass of order $100\,\text{GeV}$ is by some large cancellations between the bare Higgs mass m_h and the mass shift $\Delta m_h(\Lambda)$. Since m_h^2 is a free parameter of the theory and Δm_h^2 depends on all other parameters, such a cancellation would require very sensitive fine tuning of the parameters. While there is no physical argument that forbids such a fine tuned set of model parameters, to most people it would seem natural to ask for some sort of mechanism that explains this magical cancellation.

Even if we ignore the problem of interpreting the regularisation of loop integrals, the Higgs self-energy diagrams still cause problems if we want to introduce new massive particles into the theory. After cancelling ultraviolet divergent terms in accordance with the renormalisation programme, diagrams like (1.21) still lead to finite mass shifts of order m_f^2. This will drive the physical Higgs mass $m_{h,\text{pole}}$ up to the scale of the *heaviest* particle that couples to it. If we also consider two-loop diagrams this problem even arises if the particle f does not couple directly to the Higgs. Thus a Higgs mass at the $100\,\text{GeV}$ scale imposes a strong constraint on any theory for BSM physics and it seems logical to concentrate on such models in which the smallness of the Higgs mass is explained naturally, for example by some sort of symmetry principle.

1.3 Supersymmetry

In (unbroken) supersymmetric theories the Hierarchy problem is solved by postulating a symmetry between bosonic and fermionic fields. The basic idea is to introduce a fermionic *superpartner* for each bosonic field and a bosonic superpartner for each

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[2]This is a special feature of scalar self-interactions. For fermion self-energy diagrams the first term in (1.22) would be absent.

fermionic field. The couplings between a field and its superpartner are constrained by supersymmetry in such a way that the self-energy corrections cancel. As a consequence there is no mass renormalisation in supersymmetric theories and therefore no Hierarchy problem.

In the case of the loop corrections to the Higgs propagator from some fermion f supersymmetry requires the existence of a scalar particle $\tilde{f}$, whose couplings to h are constrained in such a way that

$$
\begin{array}{c}
\tilde{f} \\
\end{array}
\quad + \quad
\begin{array}{c}
\tilde{f} \\
\end{array}
\quad + \quad
\begin{array}{c}
f \\
\end{array}
\quad = 0 \quad . \tag{1.23}
$$

(The fermion loop contributes with opposite sign due to the minus sign associated with loops of fermionic particles.) As a result of this cancellation the mass shift Δm_h^2 is exactly zero and no fine tuning is required to drive the physical Higgs mass down to the scale of $100\,\text{GeV}$.

Unfortunately supersymmetry cannot be realised as an exact symmetry. As we will see later, the masses of a particle and its superpartner must be equal in a theory with exact supersymmetry. This would mean that there should be, for example, a super-electron with spin zero and the same mass as the electron. No such particle has ever been observed. Therefore, if supersymmetry is realised at all, it must be a broken symmetry. The trick is to break supersymmetry in such a way that the quadratically divergent term in (1.19) still vanishes. In this case we speak of a *softly broken* supersymmetry. The logarithmic term does not cause any fine tuning problems, since even for a bare Higgs mass m_h of order $100\,\text{GeV}$ the logarithm $\log(\Lambda_{\text{Pl}}/m_h)$ is only of order 10.

Even without knowing how to construct supersymmetric theories in the first place we can already work out how to break supersymmetry softly. Generally, the Lagrangian of a theory with softly broken supersymmetry will decompose as

$$
\mathcal{L} = \mathcal{L}_{\text{SUSY}} + \mathcal{L}_{\text{soft}} \quad , \tag{1.24}
$$

where $\mathcal{L}_{\text{SUSY}}$ is a supersymmetric Lagrangian and $\mathcal{L}_{\text{soft}}$ contains the soft SUSY break-

ing terms. Now the mass shift Δm_h^2 due to the first diagram in (1.23) is

$$\Delta m_h^2(\Lambda) = \frac{\lambda_{\tilde{f}}}{16\pi^2}\left[\Lambda^2 - 2m_{\tilde{f}}^2\log(\Lambda/m_{\tilde{f}}) + \ldots\right] \quad , \tag{1.25}$$

where $m_{\tilde{f}}$ is the mass of the "sfermion" $\tilde{f}$ and $\lambda_{\tilde{f}}$ is the coupling constant associated with the four scalar vertex. Comparison with (1.19) shows that, for the cancellation (1.23) to work, the couplings λ_f and $\lambda_{\tilde{f}}$ must satisfy

$$\lambda_{\tilde{f}} - 2|\lambda_f|^2 = 0 \quad . \tag{1.26}$$

Adding terms with dimensionless couplings to a supersymmetric Lagrangian would generally spoil this relation. Therefore the Lagrangian $\mathcal{L}_{\text{soft}}$ can only contain terms whose coupling constants have a nonzero (and positive) mass dimension.

1.4 Superalgebras

Let us now introduce the formalism of supersymmetry properly.[3] As mentioned above, supersymmetry is a symmetry between particles with different spin. The possibilities for constructing such symmetries are strongly constrained by a powerful no-go theorem proved by Coleman and Mandula [56] in 1967. A very simple and elegant proof for this theorem can be found in section 24.1 of [53]. It states that, under certain basic assumptions about the properties of the S matrix, the Poincaré group P can only be extended in a trivial way, i.e. by introducing an *internal* symmetry whose generators commute with the Poincaré generators (A.7).

However, the Coleman Mandula theorem is only concerned with extending the Poincaré group to a larger *Lie* group. Any symmetry of a local field theory that is described by a Lie group can only mix bosons with bosons and fermions with fermions. To see this assume that there exists a continuous symmetry that mixes a complex scalar field ϕ and a (Majorana) spinor field ψ. Then the generator Q of this symmetry has to satisfy a commutation relation of the form $[Q, \phi(x)] \propto \psi(x)$. Hence Q has to carry a spinor index and transform accordingly under Poincaré transformations. Now, if the symmetry group is a Lie group, the generators of a unitary representation of the group have to form a Lie algebra under the commutation bracket. This means that

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[3]Good introductions to supersymmetry can, for example, be found in [53–55].

the commutator of two symmetry generators must also be a symmetry generator. In a field theory generators can be written as space integrals over the time components of conserved *Noether currents* J^μ:

$$Q = \int d^3\mathbf{x} J^0 \quad . \tag{1.27}$$

Via Noether's theorem the currents J^μ can be expressed in terms of local products of field operators. The commutation relations of the symmetry generators then follow from the equal time commutation relations of the field operators:

$$[\phi(t, \mathbf{x}), \phi(t, \mathbf{y})] = \delta^{(3)}(\mathbf{x} - \mathbf{y}) \quad . \tag{1.28}$$

However, if Q transforms as a spinor, its corresponding Noether current must contain an odd number of spinor fields ψ. According to the spin-statistics theorem the spinor fields satisfy anti-commutation relations:

$$\{\psi(t, \mathbf{x}), \psi(t, \mathbf{y})\} = \delta^{(3)}(\mathbf{x} - \mathbf{y}) \quad . \tag{1.29}$$

Therefore the algebra of generators cannot close under the commutation bracket.

The lesson we learn from this is that, in order to circumvent the Coleman Mandula theorem, we need a wider mathematical concept of symmetry, which allows for generators that satisfy commutation as well as anti-commutation relations. The solution chosen in supersymmetry is to postulate a $\mathbb{Z}_2$ *grading* for the Lie algebra of symmetry generators. Let us first define the notion of a $\mathbb{Z}_2$ *graded algebra* or *superalgebra*. A superalgebra is a vector space $A = A_0 \oplus A_1$ with an associative bilinear multiplication $A \times A \to A$. Elements of A_0 are called *even* or *bosonic* and elements of A_1 are called *odd* or *fermionic*. We define the notion of *grade* on $A_0 \cup A_1$ via the map

$$|\cdot| : A_0 \cup A_1 \to \mathbb{Z}_2, \quad a \mapsto |a| = \begin{cases} 0, & a \in A_0 \\ 1, & a \in A_1 \end{cases} \quad . \tag{1.30}$$

Note that a grade cannot be assigned to all elements of A, but only to elements of the even and odd subspaces. However, any *gradeless* element of A can be uniquely written as the sum of an even and an odd element. The multiplication is required to

respect the grading:

$$|ab| = |a| + |b| \qquad \forall\, a, b \in A_0 \cup A_1 \quad . \tag{1.31}$$

Note that the addition on the right-hand side is modulo 2. For any superalgebra we define the *supercommutator* as

$$[a, b\} = ab - (-1)^{|a||b|} ba \qquad \forall\, a, b \in A_0 \cup A_1 \tag{1.32}$$

By linearity the definition also extends to ungraded elements of A. If the supercommutator vanishes for all $a, b \in A$ the algebra A is called *supercommutative*.

A $\mathbb{Z}_2$ graded Lie algebra or *Lie superalgebra* is a graded vector space $A = A_0 \oplus A_1$ with a bilinear *superbracket* $[\,\cdot\,,\,\cdot\,\} : A \times A \to A$. For $a, b, c \in A_0 \cup A_1$ the superbracket is required to satisfy

- compatibility with the $\mathbb{Z}_2$ grading

$$|[a, b\}| = |a| + |b| \quad , \tag{1.33a}$$

- super skew-symmetry

$$[a, b\} = -(-1)^{|a||b|} [b, a\} \quad , \tag{1.33b}$$

- super Jacobi-identity

$$(-1)^{|a||c|} [a, [b, c\}\} + (-1)^{|b||a|} [b, [c, a\}\} + (-1)^{|c||b|} [c, [a, b\}\} = 0 \quad . \tag{1.33c}$$

If a, b and c are all even, the properties (1.33) are just the defining properties of an ordinary Lie bracket. Furthermore, for any superalgebra the supercommutator (1.32) has the properties (1.33). Thus any superalgebra together with the supercommutator (1.32) is a Lie superalgebra.

Let us now discuss the possible ways of extending the Poincaré algebra (A.11) to a Lie superalgebra. Due to the Coleman Mandula no-go theorem we can only introduce additional *odd* generators or even generators that commute with the "old" Poincaré generators. The supercommutation relations of these new generators are strongly restricted by the space-time symmetries. According to the Haag-Lopuszanski-Sohnius

theorem[4] the most general extension to the Poincaré algebra consists of odd generators Q_α^r and even generators Z^{rs} satisfying the (anti-)commutation relations

$$[M^{\mu\nu}, Q_\alpha^r] = -\tfrac{i}{4}(\sigma^{\mu\nu})_{\alpha\beta} Q_\beta^r \quad , \tag{1.34a}$$

$$[P^\mu, Q_\alpha^r] = 0 \quad , \tag{1.34b}$$

$$[Q_\alpha^r, \bar{Q}_\beta^s] = 2P_\mu \gamma_{\alpha\beta}^\mu \delta^{rs} + \left(\tfrac{1+\gamma_5}{2}\right)_{\alpha\beta} Z^{sr\,*} + \left(\tfrac{1-\gamma_5}{2}\right)_{\alpha\beta} Z^{rs} \quad , \tag{1.34c}$$

$$[Z^{rs}, M^{\mu\nu}] = [Z^{rs}, P^\mu] = [Z^{rs}, Q_\alpha^t] = [Z^{rs}, \bar{Q}_\alpha^t]$$

$$= [Z^{rs}, Z^{tu}] = [Z^{rs}, Z^{tu\,*}] = 0 \tag{1.34d}$$

and the Majorana constraint

$$Q_\alpha^r = \mathcal{C}_{\alpha\beta} \bar{Q}_\beta^r \quad . \tag{1.35}$$

In other words, the indices $\alpha, \beta = 1, \ldots, 4$ are bi-spinor indices and the generators Q_α^r behave like Majorana spinors under Poincaré transformations. The indices r, s, t and u can run from 1 to some $N \geq 1$. The bosonic generators Z^{rs} are called *central charges*. In the minimal supersymmetric extension to the Standard Model (MSSM) we assume that $N = 1$ and that the central charge vanishes. In this case the Lie superalgebra (1.34) simplifies to

$$[M^{\mu\nu}, Q_\alpha] = -\tfrac{i}{4}(\sigma^{\mu\nu})_{\alpha\beta} Q_\beta \quad , \tag{1.36a}$$

$$[P^\mu, Q_\alpha] = 0 \quad , \tag{1.36b}$$

$$[Q_\alpha, \bar{Q}_\beta] = 2P_\mu \gamma_{\alpha\beta}^\mu \quad . \tag{1.36c}$$

$$\tag{1.36d}$$

1.5   Supergroups

For ordinary symmetries there is a close relationship between the Lie group representing the symmetry transformations and the Lie algebra of the symmetry generators. The group structure of an ordinary Lie group G is reflected by the fact that the algebra $\mathcal{L}(G)$ of its generators closes under the commutation bracket. Assume that the group manifold can be (locally) parametrised by n real numbers $\alpha^a \in \mathbb{R}$. Then a linear representation D of G can be regarded (locally) as a map from an open subset

---

[4]See [57] or section 25.2 of [53].

of $\mathbb{R}^n$ to the space of linear operators on some vector space. Without loss of generality we may choose the parametrisation in such a way that

$$D(0) = \mathbb{1} \quad . \tag{1.37}$$

The generators T^a in the representation D are related to derivatives with respect to the parameters α^a:

$$T^a = i \left. \frac{\partial D(\alpha)}{\partial \alpha^a} \right|_{\alpha=0} \quad . \tag{1.38}$$

By choosing an appropriate parametrisation we can then write the group elements $D(\alpha)$ as exponentials of the generators:

$$D(\alpha) = \exp(-i\alpha^a T^a) \quad . \tag{1.39}$$

The group structure of G implies that

$$\exp(-i\alpha^a T^a) \exp(-i\beta^b T^b) = \exp(-if^c(\alpha, \beta) T^c) \tag{1.40}$$

for some *structure function* $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. According to the Baker-Hausdorff formula the left-hand side can be written as

$$\begin{aligned}
\exp(-i\alpha^a T^a) \exp(-i\beta^b T^b) &= \exp(-i\alpha^a T^a - i\beta^b T^b - i\tfrac{1}{2}[\alpha^a T^a, \beta^b T^b] + \ldots) \\
&= \exp(-i\alpha^a T^a - i\beta^b T^b - i\tfrac{1}{2}\beta^b \alpha^a [T^a, T^b] + \ldots) \quad . \tag{1.41}
\end{aligned}$$

The higher order terms involve commutators of commutators, but no anti-commutators. Thus, for (1.40) to hold, the algebra of generators must close under the commutation bracket. A group manifold that is parametrisable by real numbers therefore automatically leads to generators that satisfy commutation relations.

Since the generators of ordinary Lie groups always form an ordinary Lie algebra, we may ask whether we can also define more general groups, *supergroups*, whose generators form Lie superalgebras. In fact, this is possible with only a minor relaxation. All we have to do is allow the group manifold to be parametrised by ordinary as well as *Grassmann* coordinates. This naturally introduces a $\mathbb{Z}_2$ grading to the parameter space. The even subspace consists of directions described by ordinary numbers and the odd subspace of the Grassmann directions. Likewise, a generator is even if it is a derivative with respect to an ordinary coordinate and odd if it is a derivative with re-

spect to a Grassmann coordinate. Due to (1.38) and the definition of derivatives with respect to Grassmann numbers the coordinates supercommute with the generators, i.e.

$$\alpha^a T^b = (-1)^{|\alpha^a||T^b|} T^b \alpha^a \qquad \Rightarrow \qquad [\alpha^a, T^b\} = 0 \quad . \tag{1.42}$$

Then contractions like $\alpha^a T^a$ are always even and

$$[\alpha^a T^a, \beta^b T^b] = \beta^b \alpha^a [T^a, T^b\} \quad . \tag{1.43}$$

This means that, for group manifolds which are parametrisable by a Z_2 graded space, the commutators in the last line of (1.41) must be replaced by supercommutators. Consequently, to satisfy (1.40) the algebra of generators must close under the super-commutation bracket.

It is now straightforward to extend the Poincaré group to a supergroup, the *super-Poincaré group SP* by exponentiating the generators of the algebra (1.36). For $N = 1$ the parameter space of the Poincaré group is extended by one (Grassmann-valued) Majorana spinor ξ_α and a general transformation can be parametrised by a Lorentz matrix $\Lambda = \mathbb{1} + \omega$, a four-vector a^μ and a Majorana spinor ξ_α. In some representation D a general group element can be written as

$$D(\mathbb{1} + \omega, a, \xi) = \exp(-i[a_\mu P^\mu + \tfrac{1}{2}\omega_{\mu\nu} M^{\mu\nu} + \bar{\xi}_\alpha Q_\alpha]) \quad . \tag{1.44}$$

1.6 Superfields

The superfield formalism is a powerful tool to construct field theories that are invariant under the super-Poincaré group (1.44). To motivate it, let us first review how we construct ordinary relativistic field theories. The starting point of any relativistic theory is defining the affine representation (A.4) of the Poincaré group on Minkowski space, which leads to the multiplication law (A.5). Then the fields ϕ of a relativistic field theory are defined as maps from Minkowski space to some (real or complex) vector space whose elements transform under a linear representation D of the Lorentz group. The representation D determines the *type* of the field. For example, for $D = D^{(0,0)}$ we speak of scalar fields, for $D = D^{(\frac{1}{2}, \frac{1}{2})}$ we speak of vector fields etc. On the space of fields of type D we can then define a unitary representation

$\mathcal{U}_D$ of the Poincaré group as follows:

$$\mathcal{U}_D(\Lambda, a)\phi(x) = D(\Lambda)\phi((\Lambda, a)^{-1}x) = D(\Lambda)\phi(\Lambda^{-1}x - a) \quad . \tag{1.45}$$

By considering infinitesimal transformations $(\mathbb{1} + \omega, \varepsilon)$ we see that, in this representation, the generators are differential operators acting on the fields ϕ:

$$\mathcal{U}_D(\mathbb{1} + \omega, \varepsilon)\phi(x) = \phi(x) - i\left[\varepsilon_\mu(-i\partial^\mu) + \tfrac{1}{2}\omega_{\mu\nu}(\tfrac{i}{2}x^\mu\partial^\nu - \tfrac{i}{2}x^\nu\partial^\mu + D(M^{\mu\nu}))\right]\phi(x) \tag{1.46}$$

Therefore

$$\mathcal{U}_D(M^{\mu\nu}) = \tfrac{i}{2}x^\mu\partial^\nu - \tfrac{i}{2}x^\nu\partial^\mu + D(M^{\mu\nu}) \quad , \tag{1.47a}$$

$$\mathcal{U}_D(P^\mu) = -i\partial^\mu \quad , \tag{1.47b}$$

where $D(M^{\mu\nu})$, $\mathcal{U}_D(M^{\mu\nu})$ and $\mathcal{U}_D(P^\mu)$ denote the Poincaré generators in the corresponding representation. To describe the dynamics of the fields we have to construct a Lagrangian density $\mathcal{L}(x)$ which consists of local products of the fields and transforms under Poincaré transformations as a scalar field plus a covariant derivative:

$$\mathcal{L}(x) \xrightarrow{(\Lambda, a)} \mathcal{L}(\Lambda^{-1}x - a) + \partial_\mu F^\mu(\Lambda^{-1}x - a) \quad . \tag{1.48}$$

Particularly useful for constructing Lagrangian densities are differential operators K that transform fields of one type D into fields of another type D'. These operators must satisfy

$$\mathcal{U}_{D'}(\Lambda, a)K\mathcal{U}_D^{-1}(\Lambda, a) = K \quad , \tag{1.49}$$

or equivalently

$$\mathcal{U}_{D'}(M^{\mu\nu})K = K\mathcal{U}_D(M^{\mu\nu}) \quad , \qquad \mathcal{U}_{D'}(P^\mu)K = K\mathcal{U}_D(P^\mu) \quad . \tag{1.50}$$

They can also be used to impose Poincaré invariant constraints $K\phi = 0$ on the fields ϕ. For example, the differential operator ∂^μ turns any field of type D into a field of

type $D^{(\frac{1}{2},\frac{1}{2})} \otimes D$ because

$$
\begin{aligned}
\partial^\rho \mathcal{U}_D(P^\mu) &= \partial^\rho(-i\partial^\mu) = (-i\partial^\mu)\partial^\rho = (\mathcal{U}_{D^{(\frac{1}{2},\frac{1}{2})}}(P^\mu) + \mathcal{U}_D(P^\mu))\partial^\rho \\
&= \mathcal{U}_{D^{(\frac{1}{2},\frac{1}{2})}\otimes D}(P^\mu)\partial^\rho \quad, \\
\partial^\rho D(M^{\mu\nu}) &= \partial^\rho[\tfrac{i}{2}(x^\mu\partial^\nu - x^\nu\partial^\mu) + D(M^{\mu\nu})] \\
&= \tfrac{i}{2}(g^{\mu\rho}\partial^\nu - g^{\nu\rho}\partial^\mu) + [\tfrac{i}{2}(x^\mu\partial^\nu - x^\nu\partial^\mu) + D(M^{\mu\nu})]\partial^\rho \\
&= \mathcal{U}_{D^{(\frac{1}{2},\frac{1}{2})}}(M^{\mu\nu})^\rho{}_\sigma\partial^\sigma + \mathcal{U}_D(M^{\mu\nu})\partial^\rho \\
&= \mathcal{U}_{D^{(\frac{1}{2},\frac{1}{2})}\otimes D}(M^{\mu\nu})^\rho{}_\sigma\partial^\sigma \quad.
\end{aligned}
$$

(1.51a)

(1.51b)

In other words, the derivative operator ∂_μ adds a Lorentz index to the field it acts on and makes it transform accordingly.

The superfield formalism can be developed by following an analogous procedure for representations of the super-Poincaré group SP. First we have to define an affine representation of SP on the space-time coordinates. Since the elements of SP depend on Grassmann-valued parameters, we can only construct a faithful affine representation of SP if we *extend Minkowski space by Grassmann-valued coordinates*. In the simplest case of an $N = 1$ supersymmetry we extend Minkowski space by a single Grassmann-valued Majorana spinor θ. The resulting vector space is called the *superspace* $\mathbb{R}^{4|4}$. The transformation behaviour of the *super-coordinates* (x, θ) under a super-Poincaré transformation (Λ, a, ξ) is then defined as:

$$
(x^\mu, \theta) \xrightarrow{(\Lambda,a,\xi)} (\Lambda^\mu{}_\nu x^\nu + a^\mu - i\bar{\xi}\gamma^\mu S(\Lambda)\theta, \ S(\Lambda)\theta + \xi) \quad,
$$

(1.52)

where $S = D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$ denotes the Dirac representation of the Lorentz group. This leads to the following multiplication law:

$$
(\Lambda', a', \xi')(\Lambda, a, \xi) = (\Lambda'\Lambda, \ \Lambda'a + a' + i\bar{\xi}'\gamma S(\Lambda')\xi, \ S(\Lambda')\xi + \xi') \quad.
$$

(1.53)

In analogy to ordinary relativistic fields we define *superfields* of type D as functions F of the super-coordinates x^μ and θ, which transform under a super-Poincaré transformation (Λ, a, ξ) according to a representation $\tilde{\mathcal{U}}_D$ defined by:

$$
\tilde{\mathcal{U}}_D(\Lambda, a, \xi)F(x, \theta) = D(\Lambda)\Phi((\Lambda, a, \xi)^{-1}(x, \theta)) \quad.
$$

(1.54)

The generators in this representation are

$$\tilde{\mathcal{U}}_D(M^{\mu\nu}) = \frac{i}{2}\left[x^\mu\partial^\nu - x^\nu\partial^\mu + \frac{1}{2}\bar{\theta}_\alpha(\sigma^{\mu\nu})_{\alpha\beta}\frac{\partial}{\partial\bar{\theta}_\beta}\right] + D(M^{\mu\nu}) \quad, \tag{1.55a}$$

$$\tilde{\mathcal{U}}_D(P^\mu) = -i\partial^\mu \quad, \tag{1.55b}$$

$$\tilde{\mathcal{U}}_D(Q_\alpha) = -i\frac{\partial}{\partial\bar{\theta}_\alpha} - (\slashed{\partial}\theta)_\alpha \quad. \tag{1.55c}$$

We also define

$$\tilde{\mathcal{U}}_D(\bar{Q}_\alpha) = \tilde{\mathcal{U}}_D(Q_\beta)\mathcal{C}_{\beta\alpha} = i\frac{\partial}{\partial\theta_\alpha} + (\bar{\theta}\slashed{\partial})_\alpha \tag{1.56}$$

It can easily be checked that these operators satisfy the super-commutation relations (1.36). We can also define a super-equivalent to the derivative ∂_μ. Just like ∂_μ adds a Lorentz index to the field it acts on, we can define a *super-derivative* $\mathcal{D}_\alpha$ which adds a spinor index to the superfield it acts on. Let

$$\mathcal{D}_\alpha = -\frac{\partial}{\partial\bar{\theta}_\alpha} - i(\slashed{\partial}\theta)_\alpha \quad, \qquad \bar{\mathcal{D}}_\alpha = \mathcal{D}_\beta\mathcal{C}_{\beta\alpha} = \frac{\partial}{\partial\theta_\alpha} + i(\bar{\theta}\slashed{\partial})_\alpha \quad. \tag{1.57}$$

By calculating their super-commutators with the generators (1.55) we can show that they satisfy

$$\tilde{\mathcal{U}}_{S\otimes D}(\Lambda, a, \xi)_{\alpha\beta}\mathcal{D}_\beta\tilde{\mathcal{U}}_D^{-1}(\Lambda, a, \xi) = \mathcal{D}_\alpha \quad. \tag{1.58}$$

To write down the most general *scalar* superfield F we expand in the Grassmann-coordinates θ. Since any power series of a Grassmann number aborts after first order we only get terms with up to 4 *different* θs. A convenient basis for the Grassmann algebra generated by the θs is

$$1 \;,\; \bar{\theta}_\alpha \;,\; \bar{\theta}\theta \;,\; \bar{\theta}\gamma^\mu\gamma_5\theta \;,\; \bar{\theta}\gamma_5\theta \;,\; (\bar{\theta}\theta)\bar{\theta}_\alpha \;,\; (\bar{\theta}\theta)^2 \quad. \tag{1.59}$$

Any product of θs can be expanded in this basis via the relations

$$\theta\bar{\theta} = \tfrac{1}{4}[-(\bar{\theta}\theta)\mathbb{1} + (\bar{\theta}\gamma_\mu\gamma_5\theta)\gamma^\mu\gamma_5 - (\bar{\theta}\gamma_5\theta)\gamma_5] \quad,$$

$$(\bar{\theta}\gamma_5\theta)\bar{\theta} = -(\bar{\theta}\theta)\bar{\theta}\gamma_5 \quad,\quad (\bar{\theta}\gamma^\mu\gamma_5\theta)\bar{\theta} = -(\bar{\theta}\theta)\bar{\theta}\gamma^\mu\gamma_5 \quad,\quad (\bar{\theta}\theta)\theta\bar{\theta} = -\tfrac{1}{4}(\bar{\theta}\theta)^2\mathbb{1} \quad.$$
$$\tag{1.60}$$

In this basis the expansion of F reads

$$F(x,\theta) = C(x) + \bar{\theta}\chi(x) + \tfrac{1}{2}(\bar{\theta}\theta)M(x) + \tfrac{i}{2}\bar{\theta}\gamma_5\theta N(x) - \tfrac{1}{2}\bar{\theta}\gamma^\mu\gamma_5\theta[V_\mu(x) + \partial_\mu C(x)]$$
$$- \tfrac{1}{2}(\bar{\theta}\theta)\bar{\theta}[\lambda(x) + \partial\!\!\!/\chi(x)] - \tfrac{1}{8}(\bar{\theta}\theta)^2[D(x) + \Box C(x)] \quad . \tag{1.61}$$

(This slightly obscure parametrisation will become clear later on.) The fields C, χ, M, N, λ and D are called *component fields*. Depending on the number of θs in front of them we speak of zeroth, first, second, third or fourth order components. The behaviour of the component fields under super-Poincaré transformations is dictated by the transformation law (1.58) (with $D(\Lambda) = 1$). Under ordinary Poincaré transformations the coefficients C, M, N and D behave like scalar fields, χ and λ like bi-spinor fields and V_μ like a vector field. If F is also required to be real, i.e. satisfy $F^\dagger = F$, the components C, M, N, D and V_μ must be real scalar and vector fields and χ and λ must be Majorana spinors.

To determine the behaviour of the component fields under a supersymmetry transformation $(\mathbb{1}, 0, \xi)$ we have to calculate

$$\delta_\xi F(x,\theta) = F(x + i\bar{\xi}\gamma\theta, \theta - \xi) - F(x,\theta)$$
$$= i\bar{\xi}\gamma^\mu\theta\partial_\mu F(x,\theta) - \bar{\xi}_\alpha\frac{\partial}{\partial\bar{\theta}_\alpha}F(x,\theta) \quad . \tag{1.62}$$

Substituting (1.61) in (1.62) and reading off the coefficients of the basis (1.59) yields the variations $\delta_\xi\phi$, $\delta_\xi\chi$, $\delta_\xi M$ etc. of the component fields. Without doing the full calculation we can already anticipate an important result: Note that the first term in the last line of (1.62) is a derivative with respect to the Minkowski coordinates x^μ. Contributions from this term to the variation of any component field will therefore always be a total derivative of some other (or the same) component field. The second term in (1.62) is a derivative with respect to the Grassmann coordinates θ. In the expansion (1.61) it reduces the number of θs in each term by one. Hence the variation of some n-th order component field due to this term only comes from component fields of order $n + 1$. In particular, the variation $\delta_\xi D$ gets no contributions from this term. Consequently the D-term transforms like a total derivative. This means that we can construct SUSY-invariant Lagrangian densities from D-terms of scalar superfields. The sum of all D terms in a Lagrangian is called the *Kähler potential*.

Real scalar superfields are an important building block in the MSSM. Another

important ingredient are *chiral superfields*. A left-chiral superfield is a (complex) superfield Φ that satisfies the constraint

$$\mathcal{D}_{R\alpha}\Phi(x,\theta) \equiv (\tfrac{1+\gamma_5}{2}\mathcal{D})_\alpha \Phi(x,\theta) = 0 \quad . \tag{1.63}$$

For a right-chiral superfield we replace the right-handed projection operator $\tfrac{1+\gamma_5}{2}$ by the left-handed projection operator $\tfrac{1-\gamma_5}{2}$. In particular, the complex conjugate of a left-chiral superfield is a right-chiral superfield because:

$$[(\mathcal{C}\gamma^0\tfrac{1+\gamma_5}{2}\mathcal{D})_\alpha\Phi]^\dagger = (\mathcal{D}^\dagger\tfrac{1+\gamma_5}{2}\gamma^0\mathcal{C}^{-1})_\alpha\Phi^\dagger = (\mathcal{D}^\mathsf{T}\tfrac{1-\gamma_5}{2})_\alpha\Phi^\dagger = (\tfrac{1-\gamma_5}{2}\mathcal{D})_\alpha\Phi^\dagger \quad , \tag{1.64}$$

and therefore $\mathcal{D}_{L\alpha}\Phi^\dagger$ vanishes if $\mathcal{D}_{R\alpha}\Phi$ vanishes. Another important feature of left (right) chiral superfields is that a product of only left (right) chiral superfields is again left (right) chiral. This follows immediately from the Leibniz rule of the superderivative:

$$\mathcal{D}_\alpha\Phi_1\Phi_2 = \Phi_2\mathcal{D}_\alpha\Phi_1 + \Phi_1\mathcal{D}_\alpha\Phi_2 \quad . \tag{1.65}$$

Let us concentrate on left-chiral fields for the moment. To find an expansion like (1.61) we use a little trick. For an arbitrary scalar superfield $\Phi(x,\theta)$ define

$$\tilde{\Phi}(x^\mu, \theta) = \Phi(x^\mu + \tfrac{i}{2}\bar{\theta}\gamma^\mu\gamma_5\theta, \theta) \quad , \tag{1.66}$$

so that

$$\begin{aligned}
\mathcal{D}_\alpha\Phi(x^\mu,\theta) &= \mathcal{D}_\alpha\tilde{\Phi}(x^\mu - \tfrac{i}{2}\bar{\theta}\gamma^\mu\gamma_5\theta, \theta) \\
&= \left[-\frac{\partial}{\partial\bar{\theta}_\alpha} - i(\slashed{\partial}\theta)_\alpha\right]\tilde{\Phi}(x^\mu - \tfrac{i}{2}\bar{\theta}\gamma^\mu\gamma_5\theta, \theta) \\
&= \left[i(\gamma^\mu\gamma_5\theta)_\alpha\partial_\mu\tilde{\Phi} - \frac{\partial\tilde{\Phi}}{\partial\bar{\theta}_\alpha} - i(\gamma^\mu\theta)_\alpha\partial_\mu\tilde{\Phi}\right]_{(x^\mu - \frac{i}{2}\bar{\theta}\gamma^\mu\theta, \theta)} \\
&= \left[-i((\mathbb{1} - \gamma_5)\gamma^\mu\theta)_\alpha\partial_\mu\tilde{\Phi} - \frac{\partial\tilde{\Phi}}{\partial\bar{\theta}_\alpha}\right]_{(x^\mu - \frac{i}{2}\bar{\theta}\gamma^\mu\theta, \theta)} \quad . \tag{1.67}
\end{aligned}$$

Then

$$(\tfrac{1+\gamma_5}{2}\mathcal{D})_\alpha\tilde{\Phi} = (\tfrac{1+\gamma_5}{2})_{\alpha\beta}\frac{\partial\tilde{\Phi}}{\partial\bar{\theta}_\beta} \quad , \tag{1.68}$$

so that the constraint $\mathcal{D}_{R\alpha}\Phi$ simply means that $\tilde{\Phi}$ must have no explicit dependence

on θ_R:

$$\tilde{\Phi}(x,\theta) = \tilde{\Phi}(x,\theta_L) \quad . \tag{1.69}$$

Since θ_L has only two independent components the expansion of $\tilde{\Phi}$ terminates at second order:

$$\tilde{\Phi}(x,\theta_L) = \phi(x) + \sqrt{2}\bar{\theta}_R\psi_L(x) + \bar{\theta}_R\theta_L F(x) \quad . \tag{1.70}$$

(Remember that $\bar{\theta}_R = \theta_L^\mathsf{T}\mathcal{C}$ due to the Majorana condition (1.35).) The components ϕ and F are scalar fields and ψ_L is a left-handed spinor field. Substituting (1.70) in (1.66) yields the θ expansion of Φ:

$$\Phi(x,\theta) = \phi(x) + \sqrt{2}\bar{\theta}\tfrac{1-\gamma_5}{2}\psi(x) + \bar{\theta}\tfrac{1-\gamma_5}{2}\theta F(x) - \tfrac{i}{2}\bar{\theta}\gamma^\mu\gamma_5\theta\partial_\mu\phi(x)$$
$$- \tfrac{i}{\sqrt{2}}(\bar{\theta}\theta)\bar{\theta}\gamma^\mu\tfrac{1-\gamma_5}{2}\partial_\mu\psi(x) - \tfrac{1}{8}(\bar{\theta}\theta)^2\Box\phi(x) \quad . \tag{1.71}$$

By comparing with (1.61) we see the reason for our awkward definition of the components of a general scalar superfield: In terms of component fields, a left-chiral superfield is a scalar superfield with

$$\chi_R = 0 \quad , \quad V_\mu = 0 \quad , \quad D = 0 \quad \text{and} \quad M = -iN \quad . \tag{1.72}$$

We also see that the vector component of $i(\Phi - \Phi^*)$ is the total derivative $\partial_\mu \operatorname{Re}\phi(x)$, which means that for a real vector field F the transformation

$$F \to F + i(\Phi - \Phi^*) \tag{1.73}$$

performs an (abelian) gauge transformation on the vector component of F. Inspection of (1.71) also shows that the highest order term for which the associated component field is not a total derivative is the F term. We know that under a supersymmetry transformation $(\mathbb{1}, 0, \xi)$ the variation of any given component field only contains non-derivative terms from higher order components. Consequently the F term of a chiral superfield must also transform as a total derivative and can therefore be used to construct supersymmetric Lagrangian densities. The sum of all such terms in a Lagrangian is called the *superpotential*.

1.7 Supersymmetric Gauge Theories

In the last section we introduced the two most important building blocks for constructing supersymmetric Lagrangians: the D terms of real scalar superfields and the F terms of chiral superfields. Before we start to construct supersymmetric Lagrangians let us briefly talk about mass dimensions. The Minkowski space coordinates x^μ have mass dimension -1. Thus a Lagrangian density must have mass dimension 4 so that its integral over Minkowski space is dimensionless. From the superspace transformation law (1.52) it follows that the Grassmann coordinates θ must have mass dimension $-1/2$. Therefore a scalar superfield must have mass dimension 1 if we want its lowest order component field to be a scalar field with the usual mass dimension 1. The same superfield will then also provide us with a spinor field of mass dimension 3/2. However, to obtain a *vector* component with mass dimension 1 we have to consider dimensionless scalar superfields.

Now assume we have a set of n left-chiral superfields Φ_i of mass dimension 1. Then the most general *renormalisable* supersymmetric Lagrangian density is

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{pot}} \quad , \tag{1.74}$$

with

$$\mathcal{L}_{\text{kin}} = 2[\Phi_i^* \Phi_i]_{(\bar\theta\theta)^2} \quad , \tag{1.75a}$$

$$\mathcal{L}_{\text{pot}} = 2[\tfrac{1}{2} m_{ij} \Phi_i \Phi_j + \tfrac{1}{3} y_{ijk} \Phi_i \Phi_j \Phi_k + \lambda_i \Phi_i + \text{h.c.}]_{\bar\theta\theta} \quad . \tag{1.75b}$$

Here $[\ldots]_{\bar\theta\theta}$ and $[\ldots]_{(\bar\theta\theta)^2}$ denote the coefficient of the corresponding term in the θ expansion of the superfield in the brackets, i.e. the F term and D term, respectively. The m_{ij} are mass parameters of mass dimension 1 and y_{ijk} are dimensionless coupling constants. Terms with more Φs would have to be contracted with couplings of negative mass dimension, which lead to non-renormalisable theories. Since the Φ_i commute we can assume without restriction that m_{ij} and y_{ijk} are symmetric in their indices.

In terms of component fields the Lagrangians read

$$\mathcal{L}_{\text{kin}} = i\bar{\psi}_{Li}\displaystyle{\not}\partial\psi_{Li} - \phi_i^*\Box\phi_i + F_i^*F_i \quad, \tag{1.76a}$$

$$\mathcal{L}_{\text{pot}} = \quad m_{ij}(\phi_i F_j - \tfrac{1}{2}(\bar{\psi}_{Li})^c\psi_{Lj})$$
$$+ y_{ijk}(\phi_i\phi_j F_k - (\bar{\psi}_{Li})^c\psi_{Lj}\phi_k + \text{cycl.}) + \lambda_i F_i + \text{h.c.} \quad, \tag{1.76b}$$

where 'c' denotes charge conjugation and, in accordance with (B.61), $\bar{\psi}^c = \psi^{\mathsf{T}}\mathcal{C}$. The abbreviation 'cycl.' indicates cyclic permutation of the indices i, j and k. Note that there are no kinetic terms for the fields F_i. This means that they do not represent independent dynamical degrees of freedom, but can be integrated out and expressed in terms of the other fields. In such a case one also speaks of *auxiliary fields*. For the Lagrangian above we have:

$$F_i = -\lambda_i^* - m_{ij}^*\phi_j^* - y_{ijk}^*\phi_j^*\phi_k^* \quad. \tag{1.77}$$

Substituting this in (1.76) yields the following kinetic terms and mass terms:

$$\mathcal{L} = i\bar{\psi}_{Li}\displaystyle{\not}\partial\psi_{Li} - \phi^*\Box\phi - \phi_i^* m_{ij}^* m_{jk}\phi_k - m_{ij}(\bar{\psi}_{Li})^c\psi_{Lj} + \dots \quad. \tag{1.78}$$

By diagonalising the mass matrix m_{ij} we see that this Lagrangian describes scalar particles and Majorana fermions with the same mass. Thus we have proved the statement we made earlier in this chapter: In a theory with unbroken supersymmetry each particle has the same mass as its superpartner.

Now let us try to construct a theory with gauge symmetry. Assume the superfields Φ_i transform under some (not necessarily irreducible) n dimensional representation of a compact semi-simple Lie group with generators T^a. A general transformation is of the form

$$\Phi_i \rightarrow \Phi_i' = \exp(-i\Lambda^a T^a)_{ij}\Phi_j \quad, \tag{1.79}$$

for some real parameters Λ^a. The Kähler potential of the Lagrangian (1.75) is automatically invariant under these transformations. By considering infinitesimal transformations we see that the superpotential is invariant if

$$m_{i'j}T^a_{i'i} + m_{ij'}T^a_{j'j} = 0 \quad, \quad y_{i'jk}T^a_{i'i} + y_{ij'k}T^a_{j'j} + y_{ijk'}T^a_{k'k} = 0 \quad, \quad \lambda_{i'}T^a_{i'i} = 0 \quad. \tag{1.80}$$

Let us assume for now that the model parameters m_{ij}, y_{ijk} and λ_i are chosen so that

these conditions are satisfied.

To make the gauge symmetry local we have to promote the parameters Λ^a to dimensionless left-chiral superfields. The requirement that Λ^a is left-chiral assures that $\Phi' = \exp(-i\Lambda^a T^a)\Phi$ is still left-chiral. Now the superpotential is still invariant, but $\Phi_i^* \Phi_i$ transforms as

$$\Phi^\dagger \Phi \rightarrow \Phi^\dagger \exp(i\mathbf{\Lambda}^\dagger)\exp(-i\mathbf{\Lambda})\Phi \quad . \tag{1.81}$$

Here we suppressed the indices i, j, etc. and used the shorthand

$$\mathbf{\Lambda} = \Lambda^a T^a \quad . \tag{1.82}$$

From now on, any bold symbol $\mathbf{X}$ is to be understood as a $n \times n$ matrix which can be written as $X^a T^a$. One might be tempted to require that the Λ^a are real, but in that case they would be left and right chiral at the same time. This would mean that *all* superderivatives of Λ^a vanish. Then Λ^a would be a constant and we are back at a *global* gauge transformation. To permit gauge transformations that are actually local we therefore have to allow the Λ^a to be complex.

In non-supersymmetric theories the gauge invariance of the kinetic terms is restored by the introduction of *gauge bosons*, which are real vector fields of mass dimension 1. As we already know that dimensionless real scalar superfields have vector components of mass dimension 1 we can easily guess the correct solution: We introduce new real scalar superfields V^a which transform in such a way that

$$e^{\mathbf{V}} \rightarrow e^{\mathbf{V}'} = e^{-i\mathbf{\Lambda}^\dagger} e^{\mathbf{V}} e^{i\mathbf{\Lambda}} \quad , \tag{1.83}$$

and rewrite the Kähler potential of (1.75) as $[\Phi^\dagger e^{\mathbf{V}} \Phi]_{(\bar\theta\theta)^2}$. In the abelian case ($n = 1$) the transformation law (1.83) reduces to $V \rightarrow V + i(\Lambda - \Lambda^*)$, which we already identified as the correct supersymmetric generalisation of an abelian gauge transformation on a vector field.

To construct a gauge invariant kinetic term for the fields V^a we exploit the fact that the trace of any expression $\mathbf{W} = W^a T^a$ that transforms like

$$\mathbf{W} \rightarrow e^{-i\mathbf{\Lambda}} \mathbf{W} e^{i\mathbf{\Lambda}} \tag{1.84}$$

is gauge invariant. This follows immediately from the cyclic property of the trace. From (1.83) we see that $e^{-\mathbf{V}}$ transforms as

$$e^{-\mathbf{V}} \rightarrow e^{-i\Lambda} e^{-\mathbf{V}} e^{i\Lambda^\dagger} \quad . \tag{1.85}$$

Because the Λ^a are left-chiral fields the matrices Λ satisfy

$$\mathcal{D}_R \Lambda = \bar{\mathcal{D}}_L \Lambda = 0 \quad , \qquad \mathcal{D}_L \Lambda^\dagger = \bar{\mathcal{D}}_R \Lambda^\dagger = 0 \quad . \tag{1.86}$$

Thus, using the Leibniz rule, we see that the derivative $e^{-\mathbf{V}} \mathcal{D}_L e^{\mathbf{V}}$ transforms in the desired way:

$$\begin{aligned}
e^{-\mathbf{V}} \mathcal{D}_{L\alpha} e^{\mathbf{V}} \rightarrow \quad & e^{-i\Lambda} e^{-\mathbf{V}} e^{i\Lambda^\dagger} \mathcal{D}_{L\alpha} e^{-i\Lambda^\dagger} e^{\mathbf{V}} e^{i\Lambda} \\
= & e^{-i\Lambda} e^{-\mathbf{V}} e^{i\Lambda^\dagger} (-i\mathcal{D}_{L\alpha} \Lambda^\dagger) e^{-i\Lambda^\dagger} e^{\mathbf{V}} e^{i\Lambda} + e^{-i\Lambda} e^{-\mathbf{V}} e^{i\Lambda^\dagger} e^{-i\Lambda^\dagger} \mathcal{D}_{L\alpha} e^{\mathbf{V}} e^{i\Lambda} \\
= & e^{-i\Lambda} e^{-\mathbf{V}} \mathcal{D}_{L\alpha} e^{\mathbf{V}} e^{i\Lambda} \quad .
\end{aligned} \tag{1.87}$$

Unfortunately the trace of this expression does not depend on $\mathbf{V}$ at all. To construct a matrix whose trace does actually depend on $\mathbf{V}$ consider the expression

$$\mathbf{W}_{L\alpha} = \bar{\mathcal{D}}_L \mathcal{D}_R e^{-\mathbf{V}} \mathcal{D}_{L\alpha} e^{\mathbf{V}} \quad . \tag{1.88}$$

It transforms as desired under gauge transformations because

$$\begin{aligned}
\mathbf{W}_{L\alpha} \rightarrow \quad & \bar{\mathcal{D}}_L \mathcal{D}_R e^{-i\Lambda} e^{-\mathbf{V}} \mathcal{D}_{L\alpha} e^{\mathbf{V}} e^{i\Lambda} \\
= & (-i\bar{\mathcal{D}}_L \mathcal{D}_R \Lambda) e^{-i\Lambda} e^{-\mathbf{V}} \mathcal{D}_{L\alpha} e^{\mathbf{V}} e^{i\Lambda} + e^{-i\Lambda} \bar{\mathcal{D}}_L \mathcal{D}_R e^{-\mathbf{V}} \mathcal{D}_{L\alpha} e^{\mathbf{V}} e^{i\Lambda} \\
= & e^{-i\Lambda} W_{L\alpha} e^{i\Lambda} \quad .
\end{aligned} \tag{1.89}$$

Furthermore, $\mathbf{W}_{L\alpha}$ satisfies

$$\mathcal{D}_{R\beta} W_{L\alpha} = 0 \tag{1.90}$$

because $\bar{\mathcal{D}}_L = \mathcal{D}_R^\top \mathcal{C}$ and $\{\mathcal{D}_{R\alpha}, \mathcal{D}_{R\beta}\} = 0$ so that a product of three or more right-handed derivatives always vanishes. Analogously we define

$$\mathbf{W}_{R\alpha} = \bar{\mathcal{D}}_R \mathcal{D}_L e^{-\mathbf{V}} \mathcal{D}_{R\alpha} e^{\mathbf{V}} \quad , \tag{1.91}$$

which also transforms like (1.84) and satisfies $\mathcal{D}_{L\beta}W_{R\alpha} = 0$. Consequently the traces

$$\mathrm{Tr}(\bar{\mathbf{W}}_R \mathbf{W}_L) \qquad \text{and} \qquad \mathrm{Tr}(\bar{\mathbf{W}}_L \mathbf{W}_R) \qquad (1.92)$$

are chiral superfields and we can use their F terms to construct the kinetic terms for the V^a. These terms also have the correct mass dimension. The full supersymmetric and gauge invariant Lagrangian now reads:

$$\mathcal{L} = \frac{1}{8g^2}\mathrm{Tr}[\bar{\mathbf{W}}\mathbf{W}]_{\bar{\theta}\theta} + 2[\Phi^\dagger e^{\mathbf{V}}\Phi]_{(\bar{\theta}\theta)^2}$$
$$+ 2[\tfrac{1}{2}m_{ij}\Phi_i\Phi_j + \tfrac{1}{3}y_{ijk}\Phi_i\Phi_j\Phi_k + \lambda_i\Phi_i + \text{h.c.}]_{\bar{\theta}\theta} \quad , \quad (1.93)$$

where g is the coupling constant associated with the gauge symmetry.

To determine the physical degrees of freedom we have to fix the gauge in the Lagrangian (1.93) and then integrate out the auxiliary component fields, which do not have a kinetic term. This is a rather tedious procedure with a very simple result: For the chiral fields Φ_i we already know that the F component is an auxiliary field. Therefore the "matter" degrees of freedom are the fermion components ψ_i and their scalar partners ϕ_i. In the case of the superfields V^a fixing the gauge and integrating out the auxiliary fields gets rid of most components. The only two components that remain are the vector component V_μ and the fermionic component λ. Thus we see that supersymmetry does exactly what we advertised at the beginning: It gives each field a superpartner, whose spin differs by $1/2$. The constraints that supersymmetry imposes on the interactions between these fields give rise to a series of non-renormalisation theorems. A discussion of these theorems is beyond the scope of this introduction but can be found in [58] or chapter 27.6 of [53]. As mentioned earlier, these theorems show that masses do not renormalise in supersymmetric theories. Therefore the hierarchy problem does not exist in a supersymmetric theory.

1.8   The MSSM

With the preparations from the last section it is now easy to write down the (super)field content of the minimal supersymmetric extension to the Standard Model. In our discussion and notations we will closely follow [59].

	component fields			gauge representation		
superfield	spin 0	spin 1/2	spin 1	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$
$(\Phi_L)_i^I$	$\tilde{L}_i^I = (\tilde{\nu}^I, \tilde{e}_L^I)$	$L_i^I = (\nu^I, e_L^I)$	—	**1**	**2**	-1
$(\Phi_e)^I$	$(\tilde{e}_R^I)^*$	$(e_R^I)^c$	—	**1**	**1**	$+2$
$(\Phi_Q)_i^I$	$\tilde{Q}_i^I = (\tilde{u}_L^I, \tilde{d}_L^I)$	$Q_i^I = (u_L^I, d_L^I)$	—	**3**	**2**	$+1/3$
$(\Phi_u)^I$	$(\tilde{u}_R^I)^*$	$(u_R^I)^c$	—	**3**	**1**	$-4/3$
$(\Phi_d)^I$	$(\tilde{d}_R^I)^*$	$(d_R^I)^c$	—	**3**	**1**	$+2/3$
V_G^a	—	Λ^a	G^a	**8**	**1**	0
V_W^i	—	λ_W^i	W^i	**1**	**3**	1
V_B	—	λ_B	B	**1**	**1**	0
$(\Phi_{H^u})_i$	(H_1^u, H_2^u)	(ψ_1^u, ψ_2^u)	—	**1**	**2**	$+1$
$(\Phi_{H^d})_i$	(H_1^d, H_2^d)	(ψ_1^d, ψ_2^d)	—	**1**	**2**	-1

Table 1.3: Superfields of the MSSM, their physical component fields and their transformation behaviour under the $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge group. The capital indices I are generation indices. The lowercase indices i correspond to the $SU(2)_L$ gauge group. Colour, spinor and Lorentz indices are suppressed. The bold numbers in the columns $SU(3)_C$ and $SU(2)_L$ denote the representation of the corresponding gauge group, under which the superfields in each row transform. The numbers in the $U(1)_Y$ column denote the (weak) hypercharge of these fields.

Basically, all we have to do is to promote all matter fields from *table 1.1* to left-chiral superfields and all gauge fields to real scalar superfields, whilst preserving their transformation behaviour under the $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge group. Our notations for these fields and superfields are summarised in *table 1.3*. Note that, once we have decided to use left-chiral superfields, all spinor component fields must be left-handed. For an arbitrary right-handed spinor ψ the charge conjugate $(\psi_R)^c$ of its right-handed component is left-handed, because

$$(\psi_R)^c = \mathcal{C}\left(\overline{\tfrac{1+\gamma_5}{2}\psi}\right)^{\mathsf{T}} = \mathcal{C}\left(\bar{\psi}\tfrac{1-\gamma_5}{2}\right)^{\mathsf{T}} = \mathcal{C}\tfrac{1-\gamma_5}{2}\bar{\psi}^{\mathsf{T}} = \tfrac{1-\gamma_5}{2}\mathcal{C}\bar{\psi}^{\mathsf{T}} = (\psi^c)_L \quad . \tag{1.94}$$

Hence we write the left-handed spinor components of Φ_e, Φ_u and Φ_d as charge-conjugates of right-handed spinors e_R, u_R and d_R. The only non-straightforward modification is the introduction of two Higgs (super-)doublets instead of one. We will soon understand the reason for this.

Once the superfield content of the MSSM is established we have to define the superpotential in such a way that the interaction terms of the Standard Model are reproduced. Since all masses in the Standard Model are generated by the Higgs

mechanism, this task reduces to reproducing the Yukawa interactions (1.14). The last two terms of (1.14) are easily obtained from the following F terms:

$$2[\varepsilon_{ij}Y_d^{IJ}(\Phi_{H^d})_i(\Phi_d)^I(\Phi_Q)_j^J]_{\bar\theta\theta} = -\varepsilon_{ij}Y_d^{IJ}H_i^d\bar d_R^I Q_j^J \quad , \tag{1.95a}$$

$$2[\varepsilon_{ij}Y_e^{IJ}(\Phi_{H^d})_i(\Phi_e)^I(\Phi_L)_j^J]_{\bar\theta\theta} = -\varepsilon_{ij}Y_e^{IJ}H_i^d\bar e_R^I L_j^J \quad , \tag{1.95b}$$

which agrees with the terms in (1.14) for $H_j = \varepsilon_{ij}H_i^d$. The only problem is the first term of (1.14). To generate masses for the up-type quarks we have to contract the $SU(2)_L$ index of the left-handed doublet Q with a Higgs doublet that transforms as a $\bar{\mathbf{2}}$ representation under $SU(2)_L$ and whose first component has a nonzero VEV. In the Standard Model such a doublet is give by the complex conjugate doublet H^*. Unfortunately the superpotential can only depend on the superfields and not their complex conjugates, so we cannot use the same trick in the MSSM. Instead we have to use two Higgs doublets with orthogonal VEVs:

$$\langle H^d\rangle = \frac{1}{\sqrt 2}\begin{pmatrix} v_1 \\ 0 \end{pmatrix} \qquad , \qquad \langle H^u\rangle = \frac{1}{\sqrt 2}\begin{pmatrix} 0 \\ v_2 \end{pmatrix} \tag{1.96}$$

This allows for another term in the MSSM superpotential: an interaction term between the two Higgs doublets, also known as μ *term*. The full MSSM superpotential now reads

$$\mathcal{L}_{\mathrm{pot}} = 2\big[\mu\varepsilon_{ij}(\Phi_{H^d})_i(\Phi_{H^u})_j + \varepsilon_{ij}Y_e^{IJ}(\Phi_{H^d})_i(\Phi_e)^I(\Phi_L)_j^J$$
$$+ \varepsilon_{ij}Y_d^{IJ}(\Phi_{H^d})_i(\Phi_d)^I(\Phi_Q)_j^J + \varepsilon_{ij}Y_u^{IJ}(\Phi_{H^u})_i(\Phi_u)^I(\Phi_Q)_j^J\big]_{\bar\theta\theta} + \text{h.c.} \quad . \tag{1.97}$$

There are other SUSY and gauge invariant terms, but they would break lepton or baryon number conservation. To get rid of these terms one usually assumes conservation of an additional discrete symmetry called *R-parity*. The *R*-parity of a field with spin j, lepton number L and baryon number B is defined as

$$R = (-1)^{L+3B+2j} \quad . \tag{1.98}$$

We should point out that the superpotential (1.97) does not allow the scalar components of the Higgs doublets to acquire nonzero VEVs and break the electroweak $SU(2)_L \times U(1)_Y$ symmetry. Electroweak symmetry breaking in the MSSM is closely

linked to the breaking of supersymmetry itself. As we explained in *section 1.3* we must assume a *softly* broken supersymmetry if we want to solve the hierarchy problem. We also saw that, in order to break supersymmetry softly, we can only add interaction terms to the Lagrangian which have a positive, nonzero mass dimension. In the case of the MSSM these soft breaking terms can be divided into four groups:

- Mass terms for the sfermions:

$$-m_{H^d}^2 (H_i^d)^* H_i^d - m_{H^u}^2 (H_i^u)^* H_i^u - (m_L^2)^{IJ} (\tilde{L}_i^I)^* \tilde{L}_i^J - (m_R^2)^{IJ} (\tilde{e}_R^I)^* \tilde{e}_R^J$$
$$-(m_Q^2)^{IJ} (\tilde{Q}_i^I)^* \tilde{Q}_i^J - (m_d^2)^{IJ} (\tilde{d}_R^I)^* \tilde{d}_R^J - (m_u^2)^{IJ} (\tilde{u}_R^I)^* \tilde{u}_R^J \qquad (1.99a)$$

- Mass terms for gauginos:

$$\frac{1}{2} M_1 \bar{\lambda}_B \lambda_B + \frac{1}{2} M_2 \bar{\lambda}_W^i \lambda_W^i + \frac{1}{2} M_3 \bar{\Lambda}^a \Lambda^a + \text{h.c.} \qquad (1.99b)$$

- Trilinear couplings of the scalar fields, analogous to the Yukawa terms in the superpotential:

$$m_{12}^2 \varepsilon_{ij} H_i^d H_j^u + \varepsilon_{ij} A_e^{IJ} H_i^d \tilde{L}_j^I \tilde{e}_R^J + \varepsilon_{ij} A_d^{IJ} H_i^d \tilde{Q}_j^I \tilde{d}_R^J + \varepsilon_{ij} A_u^{IJ} H_i^u \tilde{Q}_j^I \tilde{u}_R^J + \text{h.c.} \quad (1.99c)$$

- Trilinear couplings of the scalar fields, which are not of the same form as the Yukawa terms. These are also called *non-analytic terms* since they involve conjugated Higgs fields. Usually such terms are not considered because they are not generated in the most popular models for supersymmetry breaking:

$$A_e'^{IJ} (H_i^u)^* \tilde{L}_i^I \tilde{e}_R^J + A_d'^{IJ} (H_i^u)^* \tilde{Q}_i^I \tilde{d}_R^J + A_u'^{IJ} (H_i^d)^* \tilde{Q}_i^I \tilde{u}_R^J + \text{h.c.} \qquad (1.99d)$$

Determining the minimum of the Higgs potential in the softly broken Lagrangian leads to the following relations between the vacuum expectation values v_1 and v_2 and the soft breaking parameters $m_{H^d}^2$, $m_{H^u}^2$, m_{12}^2 and μ:

$$\left[\frac{e^2}{8 s_W^2 c_W^2} (v_1^2 - v_2^2) + m_{H^d}^2 + |\mu|^2 \right] v_1 = -m_{12}^2 v_2 \quad , \qquad (1.100a)$$

$$\left[-\frac{e^2}{8 s_W^2 c_W^2} (v_1^2 - v_2^2) + m_{H^u}^2 + |\mu|^2 \right] v_2 = -m_{12}^2 v_1 \quad . \qquad (1.100b)$$

Now that we know the full MSSM Lagrangian we have to bring it into a form fit for phenomenological studies. This procedure can be divided into four steps:

1. Absorb unphysical phases into a re-definition of the fields.

2. Expand the Higgs fields around the vacuum expectation values.

3. Fix the gauge.

4. Diagonalise the mass matrices generated in step 2.

Fortunately, this work has already been done and the results, including a complete list of Feynman rules, are given in [59]. With respect to step 1 note that all the Yukawa matrices and soft breaking parameters in the MSSM Lagrangian can, in general, be complex. Some of these phases and parameters can be removed by a re-definition of the fields. As usual for phenomenological studies, we work in the 't Hooft-Feynman gauge. The mass matrices are diagonalised by linear transformations of the component fields described by *mixing matrices*. After the diagonalisation these matrices appear in the interaction terms of the Lagrangian and consequently in the Feynman rules. To clarify our notations it is sufficient here to state which of the component fields from *table 1.3* mix into which *physical fields*. For explicit instructions on how to compute the mixing matrices we refer to [59].

- As in the Standard Model, the electroweak gauge bosons mix to give three massive modes ($W_\mu^\pm$, Z_μ^0) and one massless mode (the photon A_μ).

$$(W_\mu^1, W_\mu^2, W_\mu^3, B_\mu) \to (W_\mu^+, W_\mu^-, Z_\mu^0, A_\mu) \quad .$$

- Two Higgs fields, one from each doublet, mix to give us two charged Higgses:

$$(H_2^{d*}, H_1^u) \to (H_1, H_2) \quad .$$

However, only H_1 is a physical field. The field H_2 is an unphysical, massless Goldstone boson.

- The real parts of the other two Higgs fields mix into two neutral scalar Higgses:

$$(\mathrm{Re}\, H_1^d, \mathrm{Re}\, H_2^u) \to (H_1^0, H_2^0) \quad .$$

- The imaginary parts mix into two neutral pseudo-scalar Higgses:

$$(\mathrm{Im}\, H_1^d, \mathrm{Im}\, H_2^u) \to (A_1^0, A_2^0) \quad .$$

Again, only A_1^0 is a physical field and A_2^0 an unphysical, massless Goldstone boson.

- The left and right-handed components of the quark fields combine to give us massive Dirac fields

$$(u_L^1, u_R^1, u_L^2, u_R^2, u_L^3, u_R^3) \to (u^1, u^2, u^3) \quad ,$$
$$(d_L^1, d_R^1, d_L^2, d_R^2, d_L^3, d_R^3) \to (d^1, d^2, d^3) \quad .$$

By mixing different generations we diagonalise the Yukawa couplings. This procedure generates the *Cabibbo-Kobayashi-Maskawa matrix* of the Standard Model.

- Two electroweak gauginos and two Higgsinos combine into two charged, massive Dirac fermions called *charginos*:

$$(\lambda_W^1, \lambda_W^2, \psi_2^d, \psi_1^u) \to (\chi_1, \chi_2) \quad .$$

Note that the λs are Majorana fermions and the ψs are left-handed fermions, so they all have only two degrees of freedom. The charginos χ_1 and χ_2, on the other hand, have four each, so the number of degrees of freedom is preserved.

- The remaining electroweak gauginos and Higgsinos combine into four neutral Majorana fermions called *neutralinos*

$$(\lambda_W^3, \lambda_B, \psi_1^d, \psi_2^u) \to (\chi_1^0, \chi_2^0, \chi_3^0, \chi_4^0) \quad .$$

- The $SU(3)_C$ gauginos do not mix:

$$\Lambda^a \to \Lambda^a \quad .$$

- The partners of the left and right-handed up-type quarks mix, across genera-

tions, into six up-type squarks:

$$(\tilde{u}_L^1, \tilde{u}_R^1, \tilde{u}_L^2, \tilde{u}_R^2, \tilde{u}_L^3, \tilde{u}_R^3) \rightarrow (U_1, U_2, U_3, U_4, U_5, U_6) \quad .$$

- Likewise, the partners of the left and right-handed down-type quarks mix into six down-type squarks:

$$(\tilde{d}_L^1, \tilde{d}_R^1, \tilde{d}_L^2, \tilde{d}_R^2, \tilde{d}_L^3, \tilde{d}_R^3) \rightarrow (D_1, D_2, D_3, D_4, D_5, D_6) \quad .$$

1.9   SUSY Breaking Models

From a phenomenological point of view, the main problem with the MSSM is its large number of parameters. The soft SUSY breaking terms (1.99) introduce 105 new fundamental parameters in addition to the parameters of the SM. However, there are strong indications that the MSSM parameters are constrained by some organising principle, since most "random" sets of parameters are in strong disagreement with experimental bounds on flavour mixing and CP violation (see [60] and references therein). These disagreements can be avoided by imposing suitable *universality conditions* on the soft breaking parameters. Usually these conditions require the sfermion mass matrices from (1.99a) to be proportional to the identity matrix and the trilinear couplings from (1.99c) to be proportional to the corresponding Yukawa couplings. In other words:

$$(m_L^2)^{IJ} = m_L^2 \delta^{IJ} \quad , \quad (m_R^2)^{IJ} = m_R^2 \delta^{IJ} \quad ,$$
$$(m_Q^2)^{IJ} = m_Q^2 \delta^{IJ} \quad , \quad (m_d^2)^{IJ} = m_d^2 \delta^{IJ} \quad , \quad (m_u^2)^{IJ} = m_u^2 \delta^{IJ} \quad , \tag{1.101a}$$
$$A_e^{IJ} = A_l Y_l^{IJ} \quad , \quad A_d^{IJ} = A_d Y_d^{IJ} \quad , \quad A_u^{IJ} = A_u Y_u^{IJ} \quad , \tag{1.101b}$$
$$M_1, M_2, M_3, A_e, A_d, A_u \in \mathbb{R} \quad . \tag{1.101c}$$

The last condition assures that the soft breaking parameters do not introduce any new CP violating phases.

One might now be tempted to simply impose the universality conditions as an additional constraint on the MSSM parameter space. The problem with that approach is that the conditions (1.101) are not preserved under the *renormalisation group evolution* of the MSSM parameters. We will discuss the issue of renormalisation in

detail in *chapter 3*. Here it is sufficient to say that the regularisation of the theory via dimensional reduction forces us to introduce a new, arbitrary mass scale Q called the *renormalisation scale*. All parameters of the theory depend on Q. To assure the convergence of a perturbative expansion we have to choose Q to be of the same order as the typical energy scales of the process under consideration. From the interactions of the theory we can derive a set of differential equations, called *renormalisation group equations* (RGE), which determine the dependence of the parameters on Q. If all parameters are known at some renormalisation scale Q_1 the RGEs predict their values at any other scale Q_2. It now turns out that, if the conditions (1.101) are satisfied at Q_1, they will no longer be satisfied at Q_2. It can, however, be shown that the deviations are small, even if Q is varied over many orders of magnitude.

This indicates that the universality conditions (1.101), or stronger versions of them, could be satisfied at some large energy scale Q_0, called the *input scale*. At that scale flavour mixing and CP violation might be absent altogether due to some higher symmetry of the Lagrangian, involving the MSSM fields and other fields which have not been discovered yet. Running the parameters down to the electroweak scale then leads to small deviations from (1.101) and could explain the smallness of these effects. An indication that this programme could actually work is given by the celebrated *gauge coupling unification* of the MSSM. The existence of supersymmetric partners of the Standard Model particles with masses roughly at the 1 TeV scale modifies the RG evolution of the gauge couplings in such a way that they coincide at a scale of 10^{16} GeV (see *figure 1.1*). This unification can be taken as a strong hint towards so-called *Grand Unified Theories* (GUT). In any case it suggests that the Lagrangian has some higher symmetry at energy scales near the Planck scale.

To understand how patterns like (1.101) emerge at any scale, we have to consider models in which supersymmetry is broken *spontaneously*[5]. Unfortunately, it is impossible to break SUSY spontaneously using only the fields of the MSSM. The reason for this is that there exists a sum rule for the masses in a supersymmetric theory, namely:

$$\sum_{\text{all real scalars } \phi} m_\phi^2 \;=\; \sum_{\text{all chiral fermions } \chi} m_\chi^2 \;. \qquad (1.102)$$

This rule holds trivially for an unbroken supersymmetry, since the masses of the fermions and their superpartners are identical. However, it can be shown that (1.102)

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[5]For more information on spontaneous supersymmetry breaking see [53–55, 61].
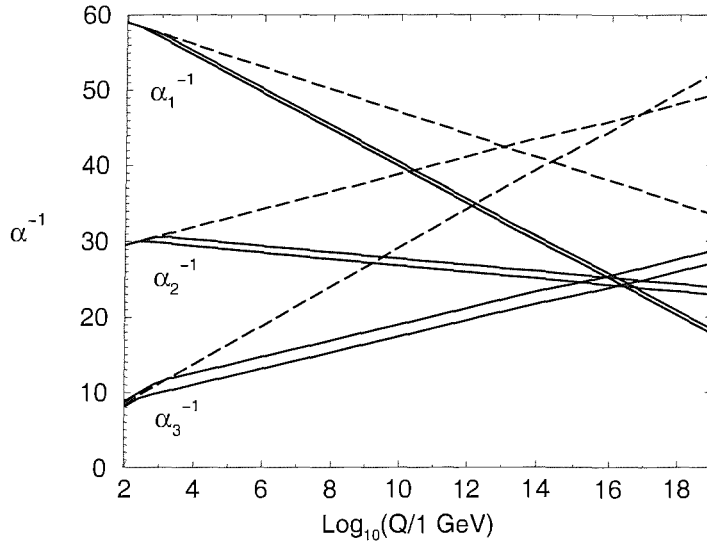
Figure 1.1: RG evolution of the inverse gauge couplings $\alpha_i^{-1}(Q)$. Here $\alpha_i = g_i^2/4\pi$, where g_3 is the strong (QCD) coupling, $g_2 = g$ and $g_1 = \sqrt{5/3}g'$. The dashed lines show the Standard Model result. In the MSSM case $\alpha_3(m_Z)$ is varied between 0.113 and 0.123 and the (universal) mass of the supersymmetric particles between 250 GeV and 1 TeV. The plot was taken from [61].

also holds if supersymmetry is spontaneously broken. In the case of the MSSM we already know that the fermion masses are all much smaller than the masses of their superpartners, so the rule (1.102) cannot hold for the MSSM spectrum alone. To construct a model with spontaneously broken supersymmetry we therefore have to extend the MSSM.

The general strategy is to postulate some *hidden sector* of fields which have no or very weak direct couplings to the "visible sector" of MSSM fields. Supersymmetry is broken spontaneously in the hidden sector and this breaking is then communicated by mediating interactions to the visible sector. If the mediating interactions are flavour blind, this would provide an explanation for universality conditions like (1.101). There are two main proposals for the nature of these mediating interactions:

Gravity-mediated SUSY breaking scenarios assume that supersymmetry breaking is communicated to the visible sector by *gravitational interactions*. This would automatically explain why the mediating interactions are flavour-blind.

Gauge-mediated SUSY breaking introduces new *messenger particles* which couple to a SUSY breaking VEV and are also charged under the $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge group.

Anomaly Mediated SUSY breaking is special type of gravity mediated SUSY breaking in which the SUSY breaking is communicated to the MSSM through the conformal anomaly.

A more detailed discussion of different SUSY breaking scenarios can be found in [61]. Here we merely wanted to convey the idea that SUSY breaking models are work in progress and that, from a practical point of view, it would be unwise to limit phenomenological studies to one specific SUSY breaking scenario with a specific set of universality conditions. Whenever possible, phenomenological calculations should be done for arbitrary values of the full set of MSSM parameters. This is the approach we take in this work. For a first analysis, however, it is sensible to study *benchmark points* of the parameter space, which assume specific SUSY breaking scenarios. At the Snowmass meeting of 2002 a collection of 10 such benchmarks was compiled [45] and we will perform our analysis of supersymmetric effects on top quark production for these 10 parameter sets.

Chapter 2

$t\bar{t}$ Production Amplitudes

As explained in the introduction, we calculate in this work the supersymmetric one-loop corrections to top-antitop production amplitudes within the MSSM framework. At the LHC top quarks will be produced from proton-proton collisions at the 1 TeV scale. At this scale protons should be thought of as a mix of light quarks and gluons, held together by strong (QCD) interactions. According to the *factorisation theorem* [62] the actual collision only takes place between two of these *partons* – one from each proton. Consequently the two production channels for top quarks at the LHC are *quark-antiquark scattering* and *gluon fusion*.

In this chapter we set up a general strategy for calculating the Feynman diagrams contributing to these two channels. The basic idea is the following: After cutting off the fermion legs from a given diagram it becomes a tensor with either two or four spinor indices. We then expand this object in a conveniently chosen basis of (tensor products of) Dirac matrices. The problem of contracting the indices of the tensor with the Dirac spinors $u^\lambda(p)$, $v^\lambda(p)$ etc. then reduces to calculating the contractions of these spinors with the basis tensors. The coefficients of the expansion can be calculated by contracting the basis tensors with the tensor representing the diagram.

The reason for this procedure is that most computer algebra systems that are currently available (we used FORM [63] for this calculation) have efficient algorithms to calculate traces of Dirac matrices, but few if none allow Dirac spinors as built-in objects. Hence the coefficients of the basis expansion can be calculated by computer algebra. The contraction of the basis tensors with the Dirac spinors has to be done on paper but turns out to be a manageable task.

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We will now explain the details of this strategy for both production channels. Most of the quantum field theory results and notations used in this chapter are summarised in the *appendices A and B*.

2.1 Quark Scattering

Consider the process

$$q(\mathbf{k}_1, \sigma_1)\bar{q}(\mathbf{k}_2, \sigma_2) \rightarrow t(\mathbf{p}_1, \lambda_1)\bar{t}(\mathbf{p}_2, \lambda_2) \quad , \tag{2.1}$$

where q denotes a massless quark and t is the top quark with mass $m_t \equiv m$. The symbols $\bar{q}$ and $\bar{t}$ denote the corresponding anti-quarks. The first argument is the spatial momentum of the particle. The second argument specifies the helicity. For the moment we ignore the colour indices. They will be accounted for when we construct the interference terms. At tree level the amplitude for the process (2.1) vanishes for $\sigma_1 = \sigma_2$. We therefore restrict ourselves to the case

$$\sigma_1 = -\sigma_2 \equiv \sigma \quad . \tag{2.2}$$

According to the Feynman rules in *table B.1* in the appendix the matrix element for the process (2.1) takes the general form

$$\mathcal{M}^{\sigma;\lambda_1\lambda_2}(k_1, k_2; p_1, p_2) = \sum_{\{\Gamma\}} \bar{v}_\alpha^\sigma(k_2)\bar{v}_\gamma^{-\lambda_1}(p_1)\Gamma_{\alpha\gamma\beta\delta}(k_1, k_2; p_1, p_2)u_\beta^\sigma(k_1)u_\delta^{\lambda_2}(p_2) \quad , \tag{2.3}$$

where α, β, γ and δ are spinor indices, the colour indices are suppressed, $k_1^0 = |\mathbf{k}_1|$, $k_2^0 = |\mathbf{k}_2|$, $p_1^0 = \sqrt{\mathbf{p}_1^2 + m^2}$ and $p_2^0 = \sqrt{\mathbf{p}_2^2 + m^2}$.

The spin tensors Γ represent the amputated Feynman diagrams. In terms of vacuum expectation values of time ordered products of field operators we have

$$\sum_{\{\Gamma\}} \Gamma_{\alpha\gamma\beta\delta}(k_1, k_2; p_1, p_2)\delta_{k_1k_2}{}^{p_1p_2}$$

$$\sim S^{-1}(k_2)_{\alpha\alpha'} S^{-1}(p_1)_{\gamma\gamma'} \langle 0| T\{q_{\alpha'k_2}\bar{q}_{\beta'k_1}t_{\gamma'}{}^{p_1}\bar{t}_{\delta'}{}^{p_2}\} |0\rangle S^{-1}(k_1)_{\beta'\beta}S^{-1}(p_2)_{\delta'\delta} \quad , \tag{2.4}$$

where q and t are the field operators corresponding to light quarks and top quarks, respectively, and S^{-1} denotes the inverse Dirac propagator (B.9). The momentum

indices represent Fourier transforms of the corresponding field operator in accordance with the notation (B.11). The tilde '$\sim$' indicates that both sides are equal in the limit where the momenta k_1, k_2, p_1 and p_2 go on shell. The sum on the left-hand side runs over all Feynman diagrams of the desired order. Each Γ can be expressed in terms of products of Dirac matrices.

Let $\{\Gamma_r^{\text{in}} | r = 1 \ldots 16\}$ and $\{\Gamma_r^{\text{out}} | r = 1 \ldots 16\}$ be two different bases of the space $\mathbb{C}^{4,4}$ of complex 4×4 matrices. Assume that they are orthogonal with respect to the trace product $\langle \cdot, \cdot \rangle$ defined in (A.48). A basis of the space $\mathbb{C}^{4,4} \otimes \mathbb{C}^{4,4}$ of rank 4 spin tensors can be constructed from tensor products of these matrices. The matrix tensor product is defined by

$$(\Gamma_r^{\text{in}} \otimes \Gamma_s^{\text{out}})_{\alpha\gamma\beta\delta} = \Gamma_{r\,\alpha\beta}^{\text{in}} \Gamma_{s\,\gamma\delta}^{\text{out}} \quad . \tag{2.5}$$

We also define an inner product on $\mathbb{C}^{4,4} \otimes \mathbb{C}^{4,4}$ by

$$\langle \Gamma, \Gamma' \rangle = \sum_{\alpha,\beta,\gamma,\delta} \Gamma_{\alpha\beta\gamma\delta} \Gamma'_{\gamma\delta\alpha\beta} \tag{2.6}$$

for arbitrary $\Gamma, \Gamma' \in \mathbb{C}^{4,4} \otimes \mathbb{C}^{4,4}$. The basis (2.5) is orthogonal with respect to this inner product. Specifically

$$\langle \Gamma_r^{\text{in}} \otimes \Gamma_s^{\text{out}}, \Gamma_{r'}^{\text{in}} \otimes \Gamma_{s'}^{\text{out}} \rangle = \langle \Gamma_r^{\text{in}}, \Gamma_{r'}^{\text{in}} \rangle \langle \Gamma_s^{\text{out}}, \Gamma_{s'}^{\text{out}} \rangle = \text{Tr}(\Gamma_r^{\text{in}} \Gamma_{r'}^{\text{in}}) \, \text{Tr}(\Gamma_s^{\text{out}} \Gamma_{s'}^{\text{out}}) \quad . \tag{2.7}$$

Expanding Γ in this basis we get

$$\Gamma = \sum_{r,s} C_{rs} \, \Gamma_r^{\text{in}} \otimes \Gamma_s^{\text{out}} \quad , \tag{2.8}$$

Now the matrix element $\mathcal{M}$ can be written as

$$\mathcal{M}^{\sigma;\lambda_1\lambda_2}(k_1, k_2; p_1, p_2) = \sum_{r,s} C_{rs} \, \bar{v}^{\sigma}(k_2)\Gamma_r^{\text{in}}u^{\sigma}(k_1) \, \bar{u}^{\lambda_1}(p_1)\Gamma_s^{\text{out}}v^{-\lambda_2}(p_2) \quad . \tag{2.9}$$

To simplify our expressions we should choose the bases $\{\Gamma_r^{\text{in}}\}$ and $\{\Gamma_r^{\text{out}}\}$ in such a way that in the sum (2.9) as many matrix elements as possible vanish.

Let us start with the basis $\{\Gamma_r^{\text{in}}\}$. A convenient basis of the space $\mathbb{C}^{4,4}$ is given by the (anti-symmetrised) products of Dirac matrices (A.46). Let Γ^{in} be an arbitrary complex 4×4 matrix. If Γ^{in} commutes with γ_5 the matrix element $\bar{v}^{\sigma}(k_2)\Gamma^{\text{in}}u^{\sigma}(k_1)$

vanishes because

$$\sigma \bar{v}^\sigma(k_2)\Gamma^{\text{in}} u^\sigma(k_1) = \bar{v}^\sigma(k_2)\Gamma^{\text{in}}\gamma_5 u^\sigma(k_1)$$
$$= \bar{v}^\sigma(k_2)\gamma_5\Gamma^{\text{in}} u^\sigma(k_1)$$
$$= -\sigma \bar{v}^\sigma(k_2)\Gamma^{\text{in}} u^\sigma(k_1) \quad . \tag{2.10}$$

In particular every product of an even number of γ matrices commutes with γ_5. Therefore, if we expand Γ^{in} in the basis (A.46) only the terms proportional to γ^μ and $\gamma^\mu\gamma_5$ contribute to the matrix element $\bar{v}^\sigma(p_b)\Gamma^{\text{in}} u^\sigma(p_a)$. Thus we can write

$$\bar{v}^\sigma(k_2)\Gamma^{\text{in}} u^\sigma(k_1) = a_\mu \bar{v}^\sigma(k_2)\gamma^\mu u^\sigma(k_1) + b_\mu \bar{v}^\sigma(k_2)\gamma^\mu\gamma_5 u^\sigma(k_1)$$
$$= c_\mu^+ \bar{v}^\sigma(k_2)\gamma^\mu \tfrac{1+\gamma_5}{2} u^\sigma(k_1) + c_\mu^- \bar{v}^\sigma(k_2)\gamma^\mu \tfrac{1-\gamma_5}{2} u^\sigma(k_1)$$
$$= c_\mu^\sigma \bar{v}^\sigma(k_2)\gamma^\mu \tfrac{1+\sigma\gamma_5}{2} u^\sigma(k_1) \quad , \tag{2.11}$$

where $c_\mu^\sigma = a_\mu + \sigma b_\mu$ and $\sigma = \pm 1$.

The incoming four-momenta k_1 and k_2 will usually be linearly independent. We can therefore choose two four-vectors e_1 and e_2 in such a way that the set $\{e_1, e_2, k_1, k_2\}$ is a basis of $\mathbb{R}^4$. We further require that e_1 and e_2 are orthogonal to k_1 and k_2 and to each other with respect to the Minkowski product and that $e_1^2 = e_2^2 = -1$. Note, however, that $\{e_1, e_2, p_a, p_b\}$ is not an orthogonal basis since, in general, $k_1 \cdot k_2 \neq 0$. Expanding the coefficients c^σ in this basis we get

$$c^\sigma = \lambda^{1\sigma} e_1 + \lambda^{2\sigma} e_2 + \mu^{1\sigma} k_1 + \mu^{2\sigma} k_2 \quad . \tag{2.12}$$

After substituting this in (2.11) we can use the fact that the Dirac spinors satisfy the momentum space Dirac equations (B.7) with $m = 0$. Therefore the terms proportional to k_1 and k_2 vanish and we can write (2.11) as

$$\bar{v}^\sigma(k_2)\Gamma^{\text{in}} u^\sigma(k_1) = \lambda^{1\sigma} \bar{v}^\sigma(k_2)\alpha_1^\sigma u^\sigma(k_1) + \lambda^{2\sigma} \bar{v}^\sigma(k_2)\alpha_2^\sigma u^\sigma(k_1) \tag{2.13}$$

with

$$\alpha_1^\sigma = \not{e}_1 \tfrac{1+\sigma\gamma_5}{2} \quad , \qquad \alpha_2^\sigma = \not{e}_2 \tfrac{1+\sigma\gamma_5}{2} \quad . \tag{2.14}$$

Using the relations (A.49) and the orthogonalities between e_1, e_2, p_a and p_b we find

$$\text{Tr}(\alpha_i^\sigma \alpha_j^{\sigma'}) = -2\delta_{ij}\delta^{\sigma,-\sigma'} \quad . \tag{2.15}$$

Thus, if we complete the set $\{\alpha_1^\sigma, \alpha_2^\sigma\}$ to a basis of $\mathbb{C}^{4,4}$ and use this basis in the expansion (2.9), only the terms with $\Gamma_r^{\text{in}} = \alpha_1^\sigma$ or α_2^σ will contribute.

The matrix elements of Γ_s^{out} have to be calculated between massive states. Therefore all basis matrices from (A.46) have to be taken into account, so we may as well use this basis in the expansion (2.9). Thus we can write the spin tensor Γ as

$$\Gamma \sim \sum_{i=1}^{2} \left[a^{i\sigma} \alpha_i^\sigma \otimes \mathbb{1} + a_\mu^{i\sigma} \alpha_i^\sigma \otimes \gamma^\mu + c_{\mu\nu}^{i\sigma} \alpha_i^\sigma \otimes \sigma^{\mu\nu} + b_\mu^{i\sigma} \alpha_i^\sigma \otimes \gamma^\mu \gamma_5 + b^{i\sigma} \alpha_i^\sigma \otimes \gamma_5 \right] \quad , \quad (2.16)$$

where the tilde '$\sim$' indicates that we have dropped terms that vanish in the expansion (2.9) for the amplitude. Note that we use a different basis expansion for each value of σ. We can project out the coefficients using equation (2.7) and the relations (A.49) and (2.15).

$$a^{i\sigma} = -\frac{1}{8} \langle \Gamma, \alpha_i^{-\sigma} \otimes \mathbb{1} \rangle \quad , \tag{2.17a}$$

$$a_\mu^{i\sigma} = -\frac{g_{\mu\nu}}{8} \langle \Gamma, \alpha_i^{-\sigma} \otimes \gamma^\nu \rangle \quad , \tag{2.17b}$$

$$c_{\mu\nu}^{i\sigma} = -\frac{g_{\mu\rho} g_{\nu\sigma}}{16} \langle \Gamma, \alpha_i^{-\sigma} \otimes \sigma^{\rho\sigma} \rangle \quad , \tag{2.17c}$$

$$b_\mu^{i\sigma} = -\frac{g_{\mu\nu}}{8} \langle \Gamma, \alpha_i^{-\sigma} \otimes \gamma^\nu \gamma_5 \rangle \quad , \tag{2.17d}$$

$$b^{i\sigma} = -\frac{1}{8} \langle \Gamma, \alpha_i^{-\sigma} \otimes \gamma_5 \rangle \quad . \tag{2.17e}$$

In the computation of the matrix element (2.3) we can exploit the fact that the Dirac spinors satisfy the on-shell conditions (B.7) and write

$$\bar{u}^{\lambda_1}(p_1) \mathbb{1} v^{-\lambda_2}(p_2) = \frac{p_{1\mu}}{m} \bar{u}^{\lambda_1}(p_1) \gamma^\mu v^{-\lambda_2}(p_2) \quad , \tag{2.18a}$$

$$\bar{u}^{\lambda_1}(p_1) \gamma_5 v^{-\lambda_2}(p_2) = \frac{p_{1\mu}}{m} \bar{u}^{\lambda_1}(p_1) \gamma^\mu \gamma_5 v^{-\lambda_2}(p_2) \quad . \tag{2.18b}$$

Therefore

$$\mathcal{M}^{\sigma;\lambda_1\lambda_2}(k_1, k_2; p_1, p_2) = \sum_{i=1}^{2} \Bigg[\left(a_\mu^{i\sigma} + a^{i\sigma} \frac{p_{1\mu}}{m} \right) \bar{u}^{\lambda_1}(p_1) \gamma^\mu v^{-\lambda_2}(p_2) \, \bar{v}^\sigma(k_2) \alpha_i^\sigma u^\sigma(k_1)$$

$$+ \left(b_\mu^{i\sigma} + b^{i\sigma} \frac{p_{1\mu}}{m} \right) \bar{u}^{\lambda_1}(p_1) \gamma^\mu \gamma_5 v^{-\lambda_2}(p_2) \, \bar{v}^\sigma(k_2) \alpha_i^\sigma u^\sigma(k_1)$$

$$+ c_{\mu\nu}^{i\sigma} \, \bar{u}^{\lambda_1}(p_1) \sigma^{\mu\nu} v^{-\lambda_2}(p_2) \, \bar{v}^\sigma(k_2) \alpha_i^\sigma u^\sigma(k_1) \Bigg] \quad . \tag{2.19}$$

To calculate the coefficients (2.17) with FORM, we still have to express them in terms of traces over spin matrices. This can be done if we assume that the spin tensor Γ factorises. There are two important cases. Assume that

$$\Gamma_{\alpha\gamma\beta\delta} = \Gamma^{\text{in}}_{\alpha\beta}\Gamma^{\text{out}}_{\gamma\delta} \quad , \tag{2.20}$$

or, in other words, $\Gamma = \Gamma^{\text{in}} \otimes \Gamma^{\text{out}}$. Then the equations (2.17) can be written as

$$a^{i\sigma} = -\frac{1}{8}\operatorname{Tr}(\Gamma^{\text{in}}\alpha_i^{-\sigma})\operatorname{Tr}(\Gamma^{\text{out}}) \quad , \tag{2.21a}$$

$$a^{i\sigma}_\mu = -\frac{g_{\mu\nu}}{8}\operatorname{Tr}(\Gamma^{\text{in}}\alpha_i^{-\sigma})\operatorname{Tr}(\Gamma^{\text{out}}\gamma^\nu) \quad , \tag{2.21b}$$

$$c^{i\sigma}_{\mu\nu} = -\frac{g_{\mu\rho}g_{\nu\sigma}}{16}\operatorname{Tr}(\Gamma^{\text{in}}\alpha_i^{-\sigma})\operatorname{Tr}(\Gamma^{\text{out}}\sigma^{\rho\sigma}) \quad , \tag{2.21c}$$

$$b^{i\sigma}_\mu = +\frac{g_{\mu\nu}}{8}\operatorname{Tr}(\Gamma^{\text{in}}\alpha_i^{-\sigma})\operatorname{Tr}(\Gamma^{\text{out}}\gamma^\nu\gamma_5) \quad , \tag{2.21d}$$

$$b^{i\sigma} = -\frac{1}{8}\operatorname{Tr}(\Gamma^{\text{in}}\alpha_i^{-\sigma})\operatorname{Tr}(\Gamma^{\text{out}}\gamma_5) \quad . \tag{2.21e}$$

Another type of factorisation, that appears in some of the box diagrams, is the following:

$$\Gamma_{\alpha\gamma\beta\delta} = \Gamma^{(1)}_{\gamma\beta}\Gamma^{(2)}_{\alpha\delta} \tag{2.22}$$

with $\Gamma^{(1)}, \Gamma^{(2)} \in \mathbb{C}^{4,4}$. In this case we can rearrange the inner products as follows

$$\langle\Gamma, \Gamma^{\text{in}}_r \otimes \Gamma^{\text{out}}_s\rangle = \Gamma_{\alpha\gamma\beta\delta}(\Gamma^{\text{in}}_r \otimes \Gamma^{\text{out}}_s)_{\beta\delta\alpha\gamma} = \Gamma^{(1)}_{\gamma\beta}\Gamma^{(2)}_{\alpha\delta}\Gamma^{\text{in}}_{r\,\beta\alpha}\Gamma^{\text{out}}_{s\,\delta\gamma}$$
$$= \operatorname{Tr}(\Gamma^{(1)}\Gamma^{\text{in}}_r\Gamma^{(2)}\Gamma^{\text{out}}_s) \tag{2.23}$$

for arbitrary basis matrices Γ^{in}_r and Γ^{out}_s. Therefore

$$a^{i\sigma} = -\frac{1}{8}\operatorname{Tr}(\Gamma^{(1)}\alpha_i^{-\sigma}\Gamma^{(2)}) \quad , \tag{2.24a}$$

$$a^{i\sigma}_\mu = -\frac{g_{\mu\nu}}{8}\operatorname{Tr}(\Gamma^{(1)}\alpha_i^{-\sigma}\Gamma^{(2)}\gamma^\nu) \quad , \tag{2.24b}$$

$$c^{i\sigma}_{\mu\nu} = -\frac{g_{\mu\rho}g_{\nu\sigma}}{16}\operatorname{Tr}(\Gamma^{(1)}\alpha_i^{-\sigma}\Gamma^{(2)}\sigma^{\rho\sigma}) \quad , \tag{2.24c}$$

$$b^{i\sigma}_\mu = +\frac{g_{\mu\nu}}{8}\operatorname{Tr}(\Gamma^{(1)}\alpha_i^{-\sigma}\Gamma^{(2)}\gamma^\nu\gamma_5) \quad , \tag{2.24d}$$

$$b^{i\sigma} = -\frac{1}{8}\operatorname{Tr}(\Gamma^{(1)}\alpha_i^{-\sigma}\Gamma^{(2)}\gamma_5) \quad . \tag{2.24e}$$

As we have seen in (2.19), the matrix element (2.3) can be written in terms

of the coefficient functions (2.17) and matrix elements of the basis matrices (A.46) between the spinors $u^\sigma(k_1)$, $\bar{v}^\sigma(k_2)$, $\bar{u}^{\lambda_1}(p_1)$ and $v^{-\lambda_2}(p_2)$. The coefficient functions can be calculated with FORM using the trace techniques described above. However, the matrix elements of the basis matrices have to be calculated manually. Momentum conservation and rotation invariance of the matrix element $\mathcal{M}$ allow us to fix the momenta k_1, k_2, p_1 and p_2 and the auxiliary vectors e_1 and e_2 in the following way:

$$k_1 = (E, -E\sin\theta, 0, E\cos\theta) \quad , \quad k_2 = (E, E\sin\theta, 0, -E\cos\theta) \quad , \tag{2.25a}$$

$$e_1 = (0, \cos\theta, 0, \sin\theta) \quad , \quad e_2 = (0, 0, 1, 0) \quad , \tag{2.25b}$$

$$p_1 = (E, 0, 0, p) \quad , \quad p_2 = (E, 0, 0, -p) \quad , \tag{2.25c}$$

with $0 \leq \theta < \pi$ and

$$p = \sqrt{E^2 - m^2} \geq 0 \quad . \tag{2.26}$$

Note that in polar coordinates we have

$$\mathbf{k}_1 = (E, \theta, \pi) \quad , \quad \mathbf{k}_2 = (E, \pi - \theta, 0) \quad . \tag{2.27}$$

For consistency we also choose the polar coordinates of $\mathbf{p}_1$ and $\mathbf{p}_2$ as

$$\mathbf{p}_1 = (p, 0, \pi) \quad , \quad \mathbf{p}_2 = (p, \pi, 0) \quad . \tag{2.28}$$

Using (B.54) and (B.32) we now obtain

$$\bar{v}^+(k_2) = \sqrt{2E} \left(\sin\tfrac{\theta}{2}\,\chi^+ + \cos\tfrac{\theta}{2}\,\chi^- \, , \, 0 \right) \quad , \tag{2.29a}$$

$$\bar{v}^-(k_2) = \sqrt{2E} \left(0 \, , \, -\cos\tfrac{\theta}{2}\,\chi^+ + \sin\tfrac{\theta}{2}\,\chi^- \right) \quad , \tag{2.29b}$$

$$\bar{u}^{\lambda_1}(p_1) = \frac{1}{\sqrt{2(E+m)}} \left(i(\lambda_1(E+m)+p)\chi^{\lambda_1} \, , \, i(\lambda_1(E+m)-p)\chi^{\lambda_1} \right) \tag{2.29c}$$

and

$$u^+(k_1) = \sqrt{2E} \begin{pmatrix} 0 \\ -i\cos\frac{\theta}{2}\chi^+ + i\sin\frac{\theta}{2}\chi^- \end{pmatrix} \quad , \tag{2.30a}$$

$$u^-(k_1) = \sqrt{2E} \begin{pmatrix} +i\sin\frac{\theta}{2}\chi^+ + i\cos\frac{\theta}{2}\chi^- \\ 0 \end{pmatrix} \quad , \tag{2.30b}$$

$$v^{-\lambda_2}(p_2) = \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} -(\lambda_2(E+m)+p)\chi^{\lambda_2} \\ (\lambda_2(E+m)-p)\chi^{\lambda_2} \end{pmatrix} \quad . \tag{2.30c}$$

Using the explicit form (A.50) of the basis matrices we obtain

$$\bar{v}^+(k_2)\gamma^1 u^+(k_1) = -2iE\cos\theta \quad , \qquad \bar{v}^-(k_2)\gamma^1 u^-(k_1) = 2iE\cos\theta \quad , \tag{2.31a}$$

$$\bar{v}^+(k_2)\gamma^2 u^+(k_1) = 2E \qquad\qquad , \qquad \bar{v}^-(k_2)\gamma^2 u^-(k_1) = 2E \quad , \tag{2.31b}$$

$$\bar{v}^+(k_2)\gamma^3 u^+(k_1) = -2iE\sin\theta \quad , \qquad \bar{v}^-(k_2)\gamma^3 u^-(k_1) = 2iE\sin\theta \quad . \tag{2.31c}$$

Thus, using the definitions (2.14) of the matrices α_i^σ, we have

$$\bar{v}^\sigma(k_2)\alpha_1^\sigma u^\sigma(k_1) = -2i\sigma E \qquad , \qquad \bar{v}^\sigma(k_2)\alpha_2^\sigma u^\sigma(k_1) = 2E \quad . \tag{2.32}$$

Furthermore

$$\bar{u}^{\lambda_1}(p_1)\gamma^0 v^{-\lambda_2}(p_2) = 0 \quad , \tag{2.33a}$$

$$\bar{u}^{\lambda_1}(p_1)\gamma^1 v^{-\lambda_2}(p_2) = -2iE\delta^{\lambda_1,-\lambda_2} \quad , \tag{2.33b}$$

$$\bar{u}^{\lambda_1}(p_1)\gamma^2 v^{-\lambda_2}(p_2) = -2\lambda_1 E\delta^{\lambda_1,-\lambda_2} \quad , \tag{2.33c}$$

$$\bar{u}^{\lambda_1}(p_1)\gamma^3 v^{-\lambda_2}(p_2) = 2i\lambda_1 m\delta^{\lambda_1,\lambda_2} \quad , \tag{2.33d}$$

$$\bar{u}^{\lambda_1}(p_1)\gamma^0\gamma_5 v^{-\lambda_2}(p_2) = 2im\delta^{\lambda_1,\lambda_2} \quad , \tag{2.33e}$$

$$\bar{u}^{\lambda_1}(p_1)\gamma^1\gamma_5 v^{-\lambda_2}(p_2) = -2ip\lambda_1\delta^{\lambda_1,-\lambda_2} \quad , \tag{2.33f}$$

$$\bar{u}^{\lambda_1}(p_1)\gamma^2\gamma_5 v^{-\lambda_2}(p_2) = -2p\delta^{\lambda_1,-\lambda_2} \quad , \tag{2.33g}$$

$$\bar{u}^{\lambda_1}(p_1)\gamma^3\gamma_5 v^{-\lambda_2}(p_2) = 0 \quad , \tag{2.33h}$$

$$\bar{u}^{\lambda_1}(p_1)\sigma^{01}v^{-\lambda_2}(p_2) = 2m\delta^{\lambda_1,-\lambda_2} \quad , \tag{2.33i}$$

$$\bar{u}^{\lambda_1}(p_1)\sigma^{02}v^{-\lambda_2}(p_2) = -2im\lambda_1\delta^{\lambda_1,-\lambda_2} \quad , \tag{2.33j}$$

$$\bar{u}^{\lambda_1}(p_1)\sigma^{03}2v^{-\lambda_2}(p_2) = -2E\lambda_1\delta^{\lambda_1,\lambda_2} \quad , \tag{2.33k}$$

$$\bar{u}^{\lambda_1}(p_1)\sigma^{12}\gamma^3 v^{-\lambda_2}(p_2) = -2ip\delta^{\lambda_1,\lambda_2} \quad , \tag{2.33l}$$

$$\bar{u}^{\lambda_1}(p_1)\sigma^{23}v^{-\lambda_2}(p_2) = 0 \quad , \tag{2.33m}$$

$$\bar{u}^{\lambda_1}(p_1)\sigma^{31}2v^{-\lambda_2}(p_2) = 0 \quad . \tag{2.33n}$$

2.2 Gluon Fusion

The second process that contributes to $t\bar{t}$ production at the LHC is *gluon fusion*. Consider the process

$$g(\mathbf{k}_1, \sigma_1)g(\mathbf{k}_2, \sigma_2) \to t(\mathbf{p}_1, \lambda_1)\bar{t}(\mathbf{p}_2, \lambda_2) \quad , \tag{2.34}$$

where g denotes a gluon and the k_i and σ_i are the gluon four-momenta and helicities, respectively. As for the quark-antiquark scattering we can use the Feynman rules in *table B.1* to write the matrix element for the process (2.1) in the general form

$$\mathcal{M}^{\sigma_1\sigma_2;\lambda_1\lambda_2}(k_1, k_2; p_1, p_2) = \sum_{\{\Gamma\}} \epsilon_\mu^{\sigma_1}(k_1)\epsilon_\nu^{\sigma_2}(k_2)\bar{v}_\alpha^{-\lambda_1}(p_1)u_\beta^{\lambda_2}(p_2)\Gamma_{\alpha\beta}^{\mu\nu} \quad , \tag{2.35}$$

where α and β are spinor indices, μ and ν Lorentz indices and the colour and gauge indices are again suppressed. Again the Γs represent the Feynman diagrams for the process, with the external legs cut off, and the sum runs over all possible diagrams of the desired order.

The contraction of the Lorentz indices can be done directly in FORM, since FORM 'understands' Lorentz vectors. Thus we define for each diagram the spin matrix

$$\Gamma_{\alpha\beta}^{\sigma_1\sigma_2} = \epsilon_\mu^{\sigma_1}(k_1)\epsilon_\nu^{\sigma_2}(k_2)\Gamma_{\alpha\beta}^{\mu\nu} \quad . \tag{2.36}$$

As in the last section we can now expand the matrix $\Gamma^{\sigma_1\sigma_2}$ in the basis matrices (A.46):

$$\Gamma^{\sigma_1\sigma_2} = a^{\sigma_1\sigma_2}\mathbb{1} + a_\mu^{\sigma_1\sigma_2}\gamma^\mu + c_{\mu\nu}^{\sigma_1\sigma_2}\sigma^{\mu\nu} + b_\mu^{\sigma_1\sigma_2}\gamma^\mu\gamma_5 + b^{\sigma_1\sigma_2}\gamma_5 \quad . \tag{2.37}$$

The coefficients a, a_μ, $c_{\mu\nu}$, b_μ and b can be projected out using the relations (A.49).

$$a^{\sigma_1\sigma_2} = +\frac{1}{4}\langle \Gamma^{\sigma_1\sigma_2}, \mathbb{1} \rangle \quad , \tag{2.38a}$$

$$a_\mu^{\sigma_1\sigma_2} = +\frac{g_{\mu\nu}}{4}\langle \Gamma^{\sigma_1\sigma_2}, \gamma^\nu \rangle \quad , \tag{2.38b}$$

$$c_{\mu\nu}^{\sigma_1\sigma_2} = +\frac{g_{\mu\rho}g_{\nu\sigma}}{8}\langle \Gamma^{\sigma_1\sigma_2}, \sigma^{\rho\sigma} \rangle \quad , \tag{2.38c}$$

$$b_\mu^{\sigma_1\sigma_2} = -\frac{g_{\mu\nu}}{4}\langle \Gamma^{\sigma_1\sigma_2}, \gamma^\nu\gamma_5 \rangle \quad , \tag{2.38d}$$

$$b^{\sigma_1\sigma_2} = +\frac{1}{4}\langle \Gamma^{\sigma_1\sigma_2}, \gamma_5 \rangle \quad . \tag{2.38e}$$

Using the relations (2.18) for matrix elements between Dirac spinors we can then write

$$\mathcal{M}^{\sigma_1\sigma_2;\lambda_1\lambda_2}(k_1, k_2; p_1, p_2) = \sum_{\{\Gamma\}}\Bigg[\ \left(a_\mu^{\sigma_1\sigma_2} + a^{\sigma_1\sigma_2}\frac{p_{1\,\mu}}{m}\right)\bar{u}^{\lambda_1}(p_1)\gamma^\mu v^{-\lambda_2}(p_2)$$

$$+ \left(b_\mu^{\sigma_1\sigma_2} + b^{\sigma_1\sigma_2}\frac{p_{1\,\mu}}{m}\right)\bar{u}^{\lambda_1}(p_1)\gamma^\mu\gamma_5 v^{-\lambda_2}(p_2)$$

$$+ c_{\mu\nu}^{\sigma_1\sigma_2}\ \bar{u}^{\lambda_1}(p_1)\sigma^{\mu\nu}v^{-\lambda_2}(p_2)\ \Bigg] \quad . \tag{2.39}$$

As in the last section we use momentum conservation and rotation invariance of the matrix element $\mathcal{M}$ to fix the momenta k_1, k_2, p_1 and p_2 according to (2.25). Consequently, the polarisation vectors of the incoming gluons are

$$\epsilon^{\sigma_1}(k_1) = (0, \cos\theta, i\sigma_1, \sin\theta) \qquad , \qquad \epsilon^{\sigma_2}(k_2) = (0, -\cos\theta, i\sigma_2, -\sin\theta) \quad . \tag{2.40}$$

As before, the coefficients $a^{\sigma_1\sigma_2}$, $a_\mu^{\sigma_1\sigma_2}$, $c_{\mu\nu}^{\sigma_1\sigma_2}$, $b_\mu^{\sigma_1\sigma_2}$ and $b^{\sigma_1\sigma_2}$ can be calculated with FORM using the projections (2.17). The contractions between the basis matrices γ^μ, $\sigma^{\mu\nu}$ and $\gamma^\mu\gamma_5$ and the Dirac spinors are given in (2.33).

For the $gg \to t\bar{t}$ several diagrams can be regarded as *crossed versions* of other diagrams. Consider two diagrams Γ and Γ' which are related to each other by

$$\Gamma'^{\mu\nu}_{\alpha\beta} = \Gamma^{\nu\mu}_{\alpha\beta} \quad . \tag{2.41}$$

Of the diagrams we will consider later on, all the t- and u-channel diagrams as well as some box diagrams are related to each other this way. Rather than calculating Γ

and Γ' as separate diagrams we can write the Γ' term in the sum (2.35) as

$$\epsilon_\mu^{\sigma_1}(k_1)\epsilon_\nu^{\sigma_2}(k_2)\bar{v}_\alpha^{-\lambda_1}(p_1)u_\beta^{\lambda_2}(p_2)\Gamma'^{\mu\nu}_{\alpha\beta} = \epsilon_\mu^{\sigma_2}(k_2)\epsilon_\nu^{\sigma_1}(k_1)\bar{v}_\alpha^{-\lambda_1}(p_1)u_\beta^{\lambda_2}(p_2)\Gamma^{\mu\nu}_{\alpha\beta} \quad , \qquad (2.42)$$

i.e. as the Γ term with the polarisation vectors for the external gluon legs swapped. For each pair of diagrams related to each other by (2.41) the BSMPRO library we introduce in *chapter 4* contains only one diagram. The the other diagram can always be obtained from the first by changing the kinematic variables in such a way that the polarisation vectors $\epsilon^{\sigma_1}(k_1)$ and $\epsilon^{\sigma_2}(k_2)$ are effectively swapped. To do this we have to replace θ with $\theta - \pi$ and swap σ_1 and σ_2.

With the formal developments discussed above we have split the task of calculating Feynman diagrams for $t\bar{t}$ production in two parts: calculating the coefficients (2.17) and (2.38) by evaluating traces of Dirac matrices and computing matrix elements of the basis matrices γ^μ, $\sigma^{\mu\nu}$ and $\gamma^\mu\gamma_5$ between between the spinors $u^\sigma(k_1)$, $\bar{v}^\sigma(k_2)$, $\bar{u}^{\lambda_1}(p_1)$ and $v^{-\lambda_2}(p_2)$ for kinematic parameters fixed according to (2.25). The first part can be done by computer algebra systems like FORM. The second part is independent of the diagram under consideration and therefore only needs to be done once. The results are given in (2.33).

Chapter 3

Renormalisation

In virtually any realistic quantum field theory we encounter divergent integrals when we attempt to make calculations beyond the leading order in perturbation theory. However, for *renormalisable* theories these divergences can be removed by the following procedure: We first introduce an ad hoc prescription to make all integrals in the perturbative expansion finite. This *regularisation method* comes at the expense of introducing a new, arbitrary parameter to the theory, which we call the *cutoff parameter*. In general, physical observables will now depend on the regularisation method and the cutoff parameter and may diverge in the limit where the cutoff is removed. However, in a renormalisable theory it is possible to cancel these divergences by introducing a suitable (divergent) cutoff dependence for the fundamental parameters of the theory.

In this chapter we review the systematics of renormalisation in the framework of *dimensional reduction*. We show how the standard integrals we encounter in perturbation theory can be regularised and calculated. Then we explain the $\overline{\text{DR}}$ renormalisation scheme and demonstrate how the divergences in self-energy and vertex corrections are removed. Troughout this chapter we will make extensive use of the notations summarised in *appendix B*.

3.1   Dimensional Reduction

From a good regularisation method we require that it preserves the crucial symmetries of the unregularised theory. For supersymmetric theories, the method of *dimensional*

reduction meets this requirement. The basic idea of dimensional reduction is to assume that the fields only depend on $d < 4$ space-time dimensions and are constant in the remaining $4 - d$ spacial dimensions. As a consequence momentum integrals can be considered as integrals in d dimensions. The final answers can be analytically continued to complex values of d and exhibit isolated poles in the physical limit $d \to 4$. Note that, unlike dimensional *regularisation*, in dimensional *reduction* the number of field components and the Clifford algebra of γ matrices remains unaltered. The reason for this is that supersymmetry mixes the components of bosonic and fermionic fields and can only be realised if the number of bosonic and fermionic degrees of freedom match. To distinguish the four-dimensional algebra from the d-dimensional algebra we introduce a d-dimensional metric tensor $\hat{g}^{\mu\nu}$ which satisfies

$$g^{\mu\nu}\hat{g}_\nu{}^\rho = \hat{g}^{\mu\rho} \qquad , \qquad g^{\mu\nu}\hat{g}_{\mu\nu} = \hat{g}^{\mu\nu}\hat{g}_{\mu\nu} = d \quad . \tag{3.1}$$

The loop integrals we encounter in one-loop perturbative calculations have the general form

$$T^{(n)}_{\mu_1\cdots\mu_m} = \int \frac{d^dq}{(2\pi)^d} \frac{q_{\mu_1}\cdots q_{\mu_m}}{D_0(q)\cdots D_{n-1}(q)} \qquad \text{with} \quad D_i(q) = (q+p_i)^2 - m_i^2 + i\varepsilon \tag{3.2}$$

and $m < n$. The term $+i\varepsilon$ indicates the way in which the poles are moved off the real axis. By shifting the integration variable we can always assure that $p_0 = 0$. Note that in (3.2) we have used the integration measure d^dq for a d-dimensional space-time. By a technique called *Veltman-Passarino reduction* we can express the *tensor integrals* $T^{(n)}_{\mu_1\cdots\mu_m}$ with $m \geq 1$ in terms of *scalar integrals*

$$T^{(n)} = \int \frac{d^dq}{(2\pi)^d} \frac{1}{D_0(q)\cdots D_{n-1}(q)} \quad . \tag{3.3}$$

We will discuss this technique later on. First, however, we have to find a way to calculate the scalar integrals (3.3).

The first step is to *Feynman parametrise* the integrand:

$$T^{(n)} = (n-1)! \int \frac{d^dq}{(2\pi)^d} \int_0^1 dx_0 \cdots \int_0^1 dx_{n-1} \frac{\delta(1 - x_0 - \ldots - x_{n-1})}{[x_0 D_0(q) + \ldots + x_{n-1}D_{n-1}(q) + i\varepsilon]^n} \quad . \tag{3.4}$$

By completing the square in the denominator and shifting the integration variable we

can write this as

$$T^{(n)} = \int_0^1 dx_0 \cdots \int_0^1 dx_{n-1}\, \delta(1 - x_0 - \ldots - x_{n-1}) I^{(n)} \tag{3.5}$$

with

$$I^{(n)} = (n-1)! \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 - \Delta + i\varepsilon)^n} \quad , \tag{3.6}$$

where Δ is a function of the Feynman parameters $x_0, \ldots, x_{n-1}$, the momenta $p_1, \ldots, p_{n-1}$ and the masses $m_0, \ldots, m_{n-1}$. To calculate $I^{(n)}$ we can rotate the contour of the q^0 integral by $90°$ in the complex plane and substitute

$$q^0 \to i\ell^0 \quad , \quad \mathbf{q} \to \boldsymbol{\ell} \quad , \quad q^2 \to -(\ell^0)^2 - \boldsymbol{\ell}^2 \equiv -\ell^2 \quad . \tag{3.7}$$

Then

$$I^{(n)} = (-1)^n i \frac{(n-1)!}{(2\pi)^d} \int d^d\ell\, \frac{1}{(\ell^2 + \Delta - i\varepsilon)} \quad . \tag{3.8}$$

Exploiting the d-dimensional rotation symmetry of the integrand we find

$$I^{(n)} = (-1)^n i \frac{\pi^{d/2}}{(2\pi)^d} (\Delta - i\varepsilon)^{d/2 - n} \Gamma(n - \tfrac{d}{2}) \quad . \tag{3.9}$$

As we will later take the physical limit $d \to 4$ we write $d = 4 - 2\delta$. Substituting this yields

$$I^{(n)} = (-1)^n \frac{i}{16\pi^2} \Delta^{2-n} (4\pi)^\delta (\Delta - i\varepsilon)^{-\delta} \Gamma(n - 2 + \delta) \quad . \tag{3.10}$$

To expand the Gamma function in δ we use

$$\Gamma(x) = (x - 1)\Gamma(x - 1) \qquad \text{and} \qquad \Gamma(\delta) = \frac{1}{\delta} + \gamma_E + \mathcal{O}(\delta) \quad , \tag{3.11}$$

where $\gamma_E \approx 0.5772\ldots$ is the Euler constant. The physical limit is now obtained by $\delta \to 0$. For the $\overline{\text{DR}}$ renormalisation it will be convenient to keep terms proportional to $(\frac{1}{\delta} + \gamma_E + \ln 4\pi)$ separate. In the following sections we will therefore write any δ-dependent quantity Q as

$$Q = (\tfrac{1}{\delta} + \gamma_E + \ln 4\pi)\hat{Q} + \bar{Q} \quad . \tag{3.12}$$

Let us now calculate the simplest scalar integrals explicitly. To keep our expressions tidy we extract a common factor of $i/16\pi^2$ from the integrals:

$$T^{(1)} = \frac{i}{16\pi^2} A_0 \quad , \quad T^{(2)} = \frac{i}{16\pi^2} B_0 \quad , \quad T^{(3)} = \frac{i}{16\pi^2} C_0 \quad \cdots \qquad (3.13)$$

For A_0 we can use the equation (3.10) directly with $\Delta = m_0^2$. This yields

$$A_0(m_0) = \frac{16\pi^2}{i} I^{(1)} \equiv (\tfrac{1}{\delta} + \gamma_E + \ln 4\pi)\hat{A}_0(m_0) + \bar{A}_0(m_0) \qquad (3.14)$$

with

$$\hat{A}_0(m_0) = m_0^2 \quad \text{and} \quad \bar{A}_0(m_0) = m_0^2(1 - \ln m_0^2) \quad . \qquad (3.15)$$

Furthermore we see that

$$T_\mu^{(1)}(m_0) = \int \frac{d^d q}{(2\pi)^d} \frac{q_\mu}{D_0} = 0 \quad \text{and} \quad \int \frac{d^d q}{(2\pi)^d} \frac{q_\mu}{D_1} = -p_\mu T^{(1)}(m_1) \quad . \qquad (3.16)$$

Using the Feynman parametrisation (3.4) and the integral formula (3.10) we obtain for the $n = 2$ scalar integral:

$$B_0(p^2, m_0, m_1) = \frac{16\pi^2}{i} T^{(2)}(p, m_0, m_1) \equiv \hat{B}_0(\tfrac{1}{\delta} + \gamma_E + \ln 4\pi) + \bar{B}_0(p^2, m_0, m_1) \quad (3.17)$$

with

$$\hat{B}_0 = 1 \quad , \quad \bar{B}_0(p^2, m_0, m_1) = -\int_0^1 dx \, \ln[\Delta(x; p^2, m_0, m_1) - i\varepsilon] \qquad (3.18)$$

and

$$\Delta(x; p^2, m_0, m_1) = p^2 x^2 - (p^2 - m_0^2 + m_1^2)x + m_1^2 \quad . \qquad (3.19)$$

To obtain a manifestly analytic expression for the integral in (3.18) we would have to distinguish several cases. We will not do this here. However, for $p^2 = 0$ and $m_0 = m_1 \equiv m$ we obtain a very simple result:

$$\bar{B}_0(0, m, m) = -\ln m^2 \equiv \bar{B}_0^{(0)}(m) \quad . \qquad (3.20)$$

By comparing (3.20) and (3.15) we find

$$\bar{A}_0(m) = m^2 \bar{B}_0^{(0)}(m) + m^2 \quad . \qquad (3.21)$$

3.2 Veltman-Passarino Reduction

The tensor integrals $T^{(n)}_{\mu_1\cdots\mu_m}$ can be expressed in terms of scalar integrals by a recursive procedure called *Veltman-Passarino Reduction*. As for the scalar integrals we first extract a factor of $i/16\pi^2$ and define

$$T^{(2)}_\mu = \frac{i}{16\pi^2}B_\mu \quad,\quad T^{(2)}_{\mu\nu} = \frac{i}{16\pi^2}B_{\mu\nu} \quad,\quad T^{(3)}_\mu = \frac{i}{16\pi^2}C_\mu \quad \ldots \tag{3.22}$$

These integrals can only depend on the momenta p_i from (3.2) and must transform as symmetric Lorentz tensors. Thus we can write them as follows:

$$B^\mu = p_1^\mu B_1 \quad, \tag{3.23a}$$

$$B^{\mu\nu} = \hat{g}^{\mu\nu}B_{00} + p_1^\mu p_1^\nu B_{11} \quad, \tag{3.23b}$$

$$C^\mu = \sum_{i=1}^{2} p_i^\mu C_i \quad, \tag{3.23c}$$

$$C^{\mu\nu} = \hat{g}^{\mu\nu}C_{00} + \sum_{i,j=1}^{2} p_i^{\{\mu}p_j^{\nu\}}C_{ij} \quad, \tag{3.23d}$$

$$C^{\mu\nu\rho} = \sum_{i=1}^{2} \hat{g}^{\{\mu\nu}p_i^{\rho\}}C_{00i} + \sum_{i,j,k=1}^{2} p_i^{\{\mu}p_j^{\nu}p_k^{\rho\}}C_{ijk} \quad \ldots \tag{3.23e}$$

The coefficients C_{ij}, C_{ijk} etc. are called *Veltman-Passarino functions*. They are symmetric in the indices i, j, k and depend only on kinematic invariants. For example, the C coefficients can be regarded as functions of the masses m_0, m_1, m_2 and the squared momenta p_1^2, p_2^2 and $(p_1+p_2)^2$. The curly brackets around the Lorentz indices indicate symmetrisation without inclusion of combinatorial factors.

To express the Veltman-Passarino functions in terms of the scalar integrals A_0, B_0 etc. the basic strategy is to substitute the integral expression (3.2) on the left-hand sides of (3.23) and then contract both sides with one of the momenta p_i or the metric tensor $\hat{g}_{\mu\nu}$. Contracting with $\hat{g}_{\mu\nu}$ gives us a factor q^2 in the numerator of the integrand, which we can write as

$$q^2 = D_0 + m_0^2 \quad. \tag{3.24}$$

Likewise, contraction with p_i gives us a factor qp_i, which we write as

$$qp_i = \tfrac{1}{2}[D_i - D_0 - p_i^2 + m_i^2 - m_0^2] \quad . \tag{3.25}$$

By cancelling the denominators and shifting the integration variable we can then write these contractions in terms of other T integrals with fewer Lorentz indices and/or fewer denominators. By applying this method recursively we can eventually express any tensor integral in terms of the scalar integrals.

To illustrate this procedure we will do it here explicitly for B_μ and $B_{\mu\nu}$. To obtain the coefficient B_1 we only have to contract both sides of (3.23a) with p^μ. Using (3.25) this yields

$$p^\mu B_\mu(p, m_0, m_1) = \tfrac{1}{2}\big[A_0(m_0) - A_0(m_1) - (p^2 + m_0^2 - m_1^2)B_0(p, m_0, m_1)\big] \tag{3.26}$$

or

$$B_1(p^2, m_0, m_1) = \frac{1}{2p^2}\big[A_0(m_0) - A_0(m_1) - (p^2 + m_0^2 - m_1^2)B_0(p^2, m_0, m_1)\big] \quad . \tag{3.27}$$

To determine B_{11} and B_{00} we need two equations. One is obtained by contracting $T^{(2)}_{\mu\nu}$ with p^μ, another one by contracting it with $g^{\mu\nu}$. Using (3.25) and (3.24) we get

$$p^\mu T^{(2)}_{\mu\nu}(p, m_0, m_1) = \tfrac{1}{2}\big[p_\nu T^{(1)}(m_1) - (p^2 + m_0^2 - m_1^2)T^{(2)}_\nu(p, m_0, m_1)\big] \quad , \tag{3.28a}$$

$$g^{\mu\nu} T^{(2)}_{\mu\nu}(p, m_0, m_1) = T^{(1)}(m_1) + m_0^2 T^{(2)}(p, m_0, m_1) \tag{3.28b}$$

or

$$p^2 B_{11} + B_{00} = \tfrac{1}{2}A_0(m_1) - \tfrac{1}{2}(p^2 + m_0^2 - m_1^2)B_1 \quad , \tag{3.29a}$$

$$p^2 B_{11} + (4 - 2\delta)B_{00} = A_0(m_1) + m_0^2 B_0 \quad , \tag{3.29b}$$

where here and for the rest of this section the functions without arguments are evaluated at (p^2, m_0, m_1). By separating the divergent and finite terms of B_{11} and B_{00} according to (3.12) we first obtain a set of equations for $\hat{B}_{11}$ and $\hat{B}_{00}$:

$$p^2 \hat{B}_{11} + \hat{B}_{00} = \tfrac{1}{2}\hat{A}_0(m_1) - \tfrac{1}{2}(p^2 + m_0^2 - m_1^2)\hat{B}_1 \quad , \tag{3.30a}$$

$$p^2 \hat{B}_{11} + 4\hat{B}_{00} = \hat{A}_0(m_1) + m_0^2 \hat{B}_0 \quad . \tag{3.30b}$$

Using (3.15) and (3.18) we find the following solution:

$$\hat{B}_{11}(p^2, m_0, m_1) = \frac{1}{2p^2}(m_1^2 - m_0^2 - \tfrac{1}{3}p^2) \quad , \tag{3.31}$$

$$\hat{B}_{00}(p^2, m_0, m_1) = \frac{1}{4}(m_1^2 + m_0^2 - \tfrac{1}{3}p^2) \quad . \tag{3.32}$$

The set of equations for the finite parts is

$$p^2 \bar{B}_{11} + \bar{B}_{00} = \tfrac{1}{2}\bar{A}_0(m_1) - \tfrac{1}{2}(p^2 + m_0^2 - m_1^2)\bar{B}_1 \quad , \tag{3.33a}$$

$$p^2 \bar{B}_{11} + 4\bar{B}_{00} - 2\hat{B}_{00} = \bar{A}_0(m_1) + m_0^2 \bar{B}_0 \tag{3.33b}$$

and has the following solution:

$$\begin{aligned}
\bar{B}_{11}(p^2, m_0, m_1) = \quad & \frac{1}{3(p^2)^2}\Big[-(p^2 + m_0^2 - m_1^2)\bar{A}_0(m_0) \\
& + (2p^2 + m_0^2 - m_1^2)\bar{A}_0(m_1) \\
& + ((p^2 + m_0^2 - m_1^2)^2 - p^2 m_0^2)\bar{B}_0\Big] \\
& - \frac{1}{6p^2}(m_0^2 + m_1^2 - \tfrac{1}{3}p^2)
\end{aligned} \tag{3.34}$$

$$\begin{aligned}
\bar{B}_{00}(p^2, m_0, m_1) = \quad & \frac{1}{12p^2}\Big[\ (p^2 + m_0^2 - m_1^2)\bar{A}_0(m_0) \\
& + (p^2 - m_0^2 + m_1^2)\bar{A}_0(m_1) \\
& + (4p^2 m_0^2 - (p^2 + m_0^2 - m_1^2)^2)\bar{B}_0\Big] \\
& + \frac{1}{6}(m_0^2 + m_1^2 - \tfrac{1}{3}p^2)
\end{aligned} \tag{3.35}$$

3.3 The $\overline{\text{DR}}$ Renormalisation Scheme

The renormalisation scheme adopted in this work is called the *modified dimensional reduction* scheme or $\overline{\text{DR}}$ scheme. To demonstrate how it is implemented let us first discuss the renormalisation procedure in a somewhat generic form. For this purpose let the fields φ_i label the entire field content of our theory, including scalar, spinor and vector fields and their *derivatives*. Furthermore, let $\mathcal{G}$ denote the full symmetry group of the theory, including Poincaré and gauge symmetry and possibly supersymmetry. The "field components" φ_i then transform under some reducible representation D of $\mathcal{G}$. To construct a Lagrangian $\mathcal{L}$ that is invariant under $\mathcal{G}$ we need tensors $\Gamma^{i_1 \cdots i_n}$

which are quasi-invariant under $\mathcal{G}$, i.e. satisfy

$$D^{i_1}_{\ i'_1} \cdots D^{i_n}_{\ i'_n} \Gamma^{i'_1 \cdots i'_n} = \Gamma^{i_1 \cdots i_n} + J^{i_1 \cdots i_n} \quad , \tag{3.36}$$

where $J^{i_1 \cdots i_n}$ must be zero if all indices are non-derivatives. For renormalisable theories the Lagrangian can only contain terms which are at most quartic in the fields. We can therefore divide $\mathcal{L}$ into three parts: a kinetic part $\mathcal{L}_{\rm kin}$, which is quadratic in the fields, a cubic part $\mathcal{L}_3$ and a quartic part $\mathcal{L}_4$:

$$\mathcal{L} = \mathcal{L}_{\rm kin} + \mathcal{L}_3 + \mathcal{L}_4 \quad . \tag{3.37}$$

Now let $\{\Gamma^{ij}_r\}$, $\{\Gamma^{ijk}_r\}$ and $\{\Gamma^{ijkl}_r\}$ denote complete sets of linearly independent tensors satisfying (3.36) and some suitable normalisation conditions. Then the three parts of the Lagrangian can be written as

$$\mathcal{L}_{\rm kin} = m^r \Gamma^{ij}_r \varphi_i \varphi_j \quad , \quad \mathcal{L}_3 = y^r \Gamma^{ijk}_r \varphi_i \varphi_j \varphi_k \quad , \quad \mathcal{L}_4 = \lambda^r \Gamma^{ijkl}_r \varphi_i \varphi_j \varphi_k \varphi_l \quad . \tag{3.38}$$

The parameters y^r and λ^r are coupling constants for the three and four-point vertices. The parameters m^r are masses. Remember that the φ_i also denote the derivatives of fundamental fields, so that the terms $\Gamma^{ij}_r \varphi_i \varphi_j$ include kinetic terms and mass terms. For those Γ^{ij}_r which couple derivatives it is understood that the coefficient m_r is just a conventional normalisation constant and not a free parameter.

All physical properties of the theory can be extracted from *Green's functions*, i.e. from vacuum expectation values of time ordered products of field operators. In momentum space a general n point Green's function can be written as

$$G_{i_1 \cdots i_n}{}^{p_1 \cdots p_n} = \langle 0 | T\{\varphi_{i_1}{}^{p_1} \cdots \varphi_{i_n}{}^{p_n}\} | 0 \rangle \quad . \tag{3.39}$$

Here we use the momentum index notation for Fourier transformed fields introduced in (B.11). Momentum conservation requires that the Green's function can be written as

$$G_{i_1 \cdots i_n}{}^{p_1 \cdots p_n} = S_{i_1 \cdots i_n}(p_1 \ldots p_{n-1}) \delta^{p_1 \cdots p_n} \quad , \tag{3.40}$$

for some tensor-valued function S of the independent momenta. To determine the

physical one-particle states we consider the two-point Green's function

$$G_{ij}{}^{pk} = S_{ij}(p)\delta^{pk} \quad .$$

(3.41)

The one particle states are associated with *poles* of $S_{ij}(p)$, i.e. with momenta p where the matrix $S^{-1}(p)$ becomes singular. By Lorentz invariance these momenta are organised in *mass shells*, i.e. in continua of momenta satisfying

$$p^2 = M_n^2$$

(3.42)

for certain masses M_n. The collection of masses M_n is the *mass spectrum* of the theory and the particles (and bound states) are associated with isolated points of the mass spectrum. In general, there is no condition on the m_n to be real. Isolated points of the mass spectrum with nonzero imaginary parts correspond to *unstable particles*. In this case we write them as

$$M_n^2 = m_n^2 + im_n\Gamma_n \quad ,$$

(3.43)

for $m_n, \Gamma_n \in \mathbb{R}$. Γ_n is called the *decay width* of the particle, since it determines the width of the resonance associated with it. For a particle with mass m and width Γ we can therefore write

$$S(p) \propto \frac{i}{p^2 - m^2 + im\Gamma} = \frac{i}{p^2 - m^2} + \frac{m\Gamma}{(p^2 - m^2)^2} + \cdots \quad .$$

(3.44)

In the last step we expanded the denominator in terms of Γ. In this form, the function $S(p)$ can have higher order poles, but they must all be located at $p^2 = m^2$.

In perturbation theory the vacuum expectation value in (3.39) is evaluated by path integral methods, using the language of Feynman diagrams. In calculations beyond leading order we encounter the divergent integrals we discussed in *section 3.1*. Once we have regularised these integrals by dimensional reduction the function G will contain terms that diverge in the limit $\delta \to 0$. Now, in a *renormalisable* theory the following thing happens: All divergent terms in G are proportional to contractions of the Γ_r^{ij}, Γ_r^{ijk} and Γ_r^{ijkl} from (3.38) and can be removed by introducing a suitable δ dependence for the parameters m^r, y^r and λ^r and by rescaling the fields φ_i by δ-dependent factors. (Since the φ_i also denote derivatives it is understood that a field and its derivative cannot be scaled independently. Furthermore, those coefficients m^r

which are not free parameters cannot renormalise, either.) The bookkeeping is easier if we introduce the δ dependence through separate *renormalisation* factors Z_i, Z_m^r, Z_y^r and Z_λ^r. Thus we replace

$$\varphi_i \to \sqrt{Z_i}\varphi_i \quad , \quad m^r \to Z_m^r m^r \quad , \quad y^r \to Z_y^r \mu^\delta y^r \quad , \quad \lambda^r \to Z_\lambda^r \mu^{2\delta} \lambda^r \quad , \quad (3.45)$$

with no implicit sum over repeated indices. We have also introduced the *renormalisation scale* μ. It is a quantity of mass dimension 1. When the space-time integration is reduced to $d = 4 - 2\delta$ dimensions the mass dimensions of all other objects in the Lagrangian must be adjusted to keep the action dimensionless. We can adjust the mass dimensions of the fields in such a way that $\int d^d x \mathcal{L}_{\text{kin}}$ remains dimensionless, but then the couplings y^r and λ^r must get mass dimensions δ and 2δ, respectively, to keep the integrals over $\mathcal{L}_3$ and $\mathcal{L}_4$ dimensionless. To extract the higher order terms in the Z factors we also define

$$\Delta Z_i = Z_i - 1 \quad , \quad \Delta Z_m^r = Z_m^r - 1 \quad , \quad \Delta Z_y^r = Z_y^r - 1 \quad , \quad \Delta Z_\lambda^r = Z_\lambda^r - 1 \quad . \quad (3.46)$$

Substituting this in the Lagrangian (3.37) (and expanding to first order in the ΔZs) leads to a number of quadratic, cubic and quartic terms proportional to the ΔZs. These terms are called *counterterms*.

To determine, for example, a ΔZ_i at a given order we have to calculate the corresponding two-point Green's function at that order, including loop diagrams and counterterms. Then ΔZ_i has to be chosen in such a way that the divergent terms cancel. However, this procedure does not determine ΔZ_i uniquely. The notion of a divergent *term* is a question of bookkeeping and might differ by a finite amount. This is where the *renormalisation scheme* comes into play. It is basically a set of rules that remove the ambiguities in the definition of the counterterms. The renormalisation scheme we adopt here is called the $\overline{\text{DR}}$ scheme and can be summarised as follows:

- Masses are renormalised in the on-shell scheme. This means that the counterterm for a mass m is chosen in such a way that the corresponding two-point Green's function has a simple pole at $p^2 = m^2$ (p being the momentum flowing trough the propagator).

- Vertices are renormalised in the *modified minimal subtraction* ($\overline{\text{MS}}$) scheme. This means that the function G from (3.40) is expressed in terms of Veltman-

Passarino functions. These functions are then decomposed into "hat" and "bar" parts in accordance with (3.12). The counterterms are chosen so that the "hat" parts cancel.

- Internal fields (i.e. those that were "pulled down" from the interaction Lagrangian) are renormalised by the same method. However, for external fields (i.e. the ones that appear in the time-ordered product (3.39)) the LSZ reduction formula requires that they are normalised like free fields. This basically means that the two-point Green's function has to look like a free field propagator.

To demonstrate how this renormalisation scheme is implemented we now discuss the renormalisation of self-energy diagrams for fermions and gluons.

3.4 Fermion Self-Energies

The one-loop self-energy corrections to the fermion propagator have the general form

$$\begin{array}{c} \xrightarrow{\;p\;}\!\!\bigcirc\!\!\xrightarrow{\;p\;} \end{array} = -i\left[a(p^2)\slashed{p} + b(p^2)\slashed{p}\gamma_5 + c(p^2)m + f(p^2)\gamma_5\right] \quad . \tag{3.47}$$

All other terms are excluded by Lorentz invariance. The Lagrangian of the free Dirac field ψ is

$$\mathcal{L}_0 = i\bar{\psi}\slashed{\partial}\psi - m\bar{\psi}\psi = i\bar{\psi}_R\slashed{\partial}\psi_R + i\bar{\psi}_L\slashed{\partial}\psi_L - m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L) \quad , \tag{3.48}$$

where ψ_L and ψ_R are the left and right-handed projections of the field ψ:

$$\psi_L = \frac{1 - \gamma_5}{2}\psi \quad , \qquad \psi_R = \frac{1 + \gamma_5}{2}\psi \quad . \tag{3.49}$$

In parity violating theories like the Standard Model or the MSSM the left and right-handed components of ψ must be treated independently under renormalisation. Thus we construct the *renormalised Lagrangian* $\mathcal{L}_{\mathrm{ren}}$ by replacing

$$\psi_L \to \sqrt{Z_L(\delta)}\,\psi_L \quad , \quad \psi_R \to \sqrt{Z_R(\delta)}\,\psi_R \quad , \quad m \to Z_m(\delta)\,m \quad . \tag{3.50}$$

To cancel the divergent terms in the coefficient f we also have to introduce a new term to the Lagrangian:

$$-iZ_5(\delta)\bar{\psi}\gamma_5\psi \equiv -iZ_5(\delta)(\bar{\psi}_L\psi_R - \bar{\psi}_R\psi_L) \quad . \tag{3.51}$$

As this term is not present in the Lagrangian of the free Dirac field we have to choose Z_5 in such a way that $Z_5(0) = 0$. Thus the renormalised Lagrangian reads

$$\begin{aligned}
\mathcal{L}_{\mathrm{ren}} = \mathcal{L}_0 &+ i\Delta Z_R\,\bar{\psi}_R\not{\partial}\psi_R + i\Delta Z_L\,\bar{\psi}_L\not{\partial}\psi_L \\
&- (\Delta Z_m + \tfrac{1}{2}\Delta Z_L + \tfrac{1}{2}\Delta Z_R)(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)m \\
&- i\Delta Z_5(\bar{\psi}_L\psi_R - \bar{\psi}_R\psi_L) \quad ,
\end{aligned} \tag{3.52}$$

where

$$Z_L = 1 + \Delta Z_L \quad , \quad Z_R = 1 + \Delta Z_R \quad , \quad Z_m = 1 + \Delta Z_m \quad , \quad Z_5 = \Delta Z_5 \quad . \tag{3.53}$$

Thus the Feynman rule for the counterterms is

$$\begin{aligned}
\overset{\displaystyle \underset{p \qquad\ p}{\longrightarrow\!\otimes\!\longrightarrow}}{} \ &= -i\Big[-\Delta Z_R\,\not{p}\tfrac{1+\gamma_5}{2} - \Delta Z_L\,\not{p}\tfrac{1-\gamma_5}{2} \\
&\qquad + (\Delta Z_m + \tfrac{1}{2}\Delta Z_L + \tfrac{1}{2}\Delta Z_R)m + i\Delta Z_5\gamma_5\Big] \\
&= -i\left[-\Delta Z_V\,\not{p} - \Delta Z_A\,\not{p}\gamma_5 + (\Delta Z_m + \Delta Z_V)m + i\Delta Z_5\gamma_5\right] \quad ,
\end{aligned} \tag{3.54}$$

with

$$\Delta Z_V = \tfrac{1}{2}(\Delta Z_R + \Delta Z_L) \quad , \qquad \Delta Z_A = \tfrac{1}{2}(\Delta Z_R - \Delta Z_L) \quad . \tag{3.55}$$

The *renormalised* one-loop correction $\Sigma(p)$ is the sum of the self-energy diagrams and the counterterms, i.e.

$$\begin{aligned}
\Sigma(p) &= i\left(\overset{\displaystyle \underset{p \qquad\ p}{\longrightarrow\!\bigcirc\!\longrightarrow}}{} + \overset{\displaystyle \underset{p \qquad\ p}{\longrightarrow\!\otimes\!\longrightarrow}}{} \right) \\
&= (a(p^2) - \Delta Z_V)\not{p} + (b(p^2) - \Delta Z_A)\not{p}\gamma_5 \\
&\quad + (c(p^2) + \Delta Z_m + \Delta Z_V)m + i(f(p^2) + \Delta Z_5)\gamma_5 \\
&= (a(p^2) - \Delta Z_V)(\not{p} - m) + (b(p^2) - \Delta Z_A)\not{p}\gamma_5 \\
&\quad + (c(p^2) + a(p^2) + \Delta Z_m)m + i(f(p^2) + \Delta Z_5)\gamma_5 \quad .
\end{aligned} \tag{3.56}$$

To calculate the corrected propagator we have to re-attach the external propagators to the self-energy diagram. The renormalisation factors have to be chosen in such a way that the correction to the propagator has a simple pole at $p^2 = m^2$, namely

$$\frac{i(\not{p} + m)}{p^2 - m^2}\left(-i\Sigma(p)\right)\frac{i(\not{p} + m)}{p^2 - m^2} = \frac{i\Pi(p)}{p^2 - m^2} \tag{3.57}$$

for some function Π which is nonzero for $p^2 = m^2$. The terms proportional to $(\not{p} - m)$ and γ_5 automatically yield simple poles when substituted on the left-hand side, because

$$\frac{(\not{p} + m)(\not{p} - m)(\not{p} + m)}{(p^2 - m^2)^2} = \frac{(\not{p} + m)(p^2 - m^2)}{(p^2 - m^2)^2} = \frac{(\not{p} + m)}{p^2 - m^2} \tag{3.58}$$

and

$$\frac{(\not{p} + m)\gamma_5(\not{p} + m)}{(p^2 - m^2)^2} = \frac{(\not{p} + m)(-\not{p} + m)\gamma_5}{(p^2 - m^2)^2} = \frac{-\gamma_5}{p^2 - m^2} \quad . \tag{3.59}$$

The third term in (3.56) would yield a double pole unless we choose ΔZ_m in such a way that it vanishes for $p^2 = m^2$. With

$$\Delta Z_m = -c(m^2) - a(m^2) \tag{3.60}$$

we have

$$c(p^2) + a(p^2) + \Delta Z_m = [c + a]_{m^2}^{p^2} = [\bar{c} + \bar{a}]_{m^2}^{p^2} \quad , \tag{3.61}$$

with $\bar{a}$ and $\bar{c}$ defined according to (3.12). We also choose the renormalisation factor Z_5 in such a way that the γ_5-term vanishes for $p^2 = m^2$:

$$\Delta Z_5 = -f(m^2) \qquad \Rightarrow \qquad f(p^2) + \Delta Z_5 = [f]_{m^2}^{p^2} \equiv [\bar{f}]_{m^2}^{p^2} \quad . \tag{3.62}$$

The renormalisation factors Z_V and Z_A have to cancel the divergent terms of a and b. In the $\overline{\text{MS}}$ scheme we only remove terms proportional to $(1/\delta + \gamma_E + \ln 4\pi)$. Note that this cancellation is only possible if the divergent terms $\hat{a}$, $\hat{b}$ and $\hat{c}$ are independent of p^2. With

$$\Delta Z_V = (\tfrac{1}{\delta} + \gamma_E + \ln 4\pi)\hat{a} \quad , \quad \Delta Z_A = (\tfrac{1}{\delta} + \gamma_E + \ln 4\pi)\hat{b} \tag{3.63}$$

we get cutoff-independent formulae for Σ and Π:

$$\Sigma(p) = \bar{a}(p^2)(\not{p} - m) + \bar{b}(p^2)\not{p}\gamma_5 + m[\bar{c} + \bar{a}]_{m^2}^{p^2} + i[\bar{f}]_{m^2}^{p^2}\gamma_5 \quad , \tag{3.64a}$$

$$\Pi(p) = \bar{a}(p^2)(\not{p} + m) + m\frac{[\bar{c} + \bar{a}]_{m^2}^{p^2}}{p^2 - m^2}(\not{p} + m)^2 - \bar{b}(p^2)\not{p}\gamma_5 - i[\bar{f}]_{m^2}^{p^2}\gamma_5 \quad . \tag{3.64b}$$

Note that the renormalisation factors Z_V, Z_A, Z_m and Z_5 must be real. The divergent parts of the coefficient functions a, b, c and f must be real, too, because otherwise the theory would not be renormalisable. However, the finite parts of the coefficient functions can have nonzero imaginary parts. In this case the cancellations discussed above are incomplete and lead to double poles in the correction to the propagator. As we saw in (3.44) these double poles can be re-summed and effectively shift the pole of the two point Green's function off the real axis, which corresponds to a nonzero decay width Γ of the fermion.

In the case of self-energy corrections on external legs we have to replace one of the propagators in (3.57) with the Dirac spinors $u(p)$, $v(-p)$, $\bar{u}(p)$ or $\bar{v}(-p)$ and take the limit $p^2 \to m^2$. Using the fact that all these spinors are eigenvectors of $\not{p}$ with eigenvalue $+\sqrt{p^2}$ we can now calculate the limits for each of the four terms in (3.64a) separately:

$$\frac{\not{p} + m}{p^2 - m^2}\bar{a}(p^2)(\not{p} - m)\,u(p) = \bar{a}(p^2)\frac{p^2 - m^2}{p^2 - m^2}u(p) \xrightarrow{p^2 \to m^2} a(m^2)u(p) \quad , \tag{3.65a}$$

$$\frac{\not{p} + m}{p^2 - m^2}\bar{b}(p^2)\not{p}\gamma_5\,u(p) = \bar{b}(p^2)\gamma_5\not{p}\frac{\not{p} - m}{p^2 - m^2}u(p)$$

$$= \bar{b}(p^2)\gamma_5\sqrt{p^2}\frac{\sqrt{p^2} - m}{p^2 - m^2}u(p) \xrightarrow{p^2 \to m^2} \frac{1}{2}b(m^2)\gamma_5 u(p) \quad , \tag{3.65b}$$

$$\frac{\not{p} + m}{p^2 - m^2}m[\bar{c} + \bar{a}]_{m^2}^{p^2}\,u(p) = m\frac{[\bar{c} + \bar{a}]_{m^2}^{p^2}}{p^2 - m^2}(\sqrt{p^2} + m)u(p)$$

$$\xrightarrow{p^2 \to m^2} 2m^2(\bar{c}'(m^2) + \bar{a}'(m^2))u(p) \quad , \tag{3.65c}$$

$$\frac{\not{p} + m}{p^2 - m^2}[\bar{f}]_{m^2}^{p^2}\gamma_5\,u(p) = [\bar{f}]_{m^2}^{p^2}\gamma_5\frac{-\sqrt{p^2} + m}{p^2 - m^2}u(p) \xrightarrow{p^2 \to m^2} 0 \quad . \tag{3.65d}$$

Analogous results are obtained for the spinors $v(-p)$, $\bar{u}(p)$ and $\bar{v}(-p)$. For $\bar{u}(p)$ and $\bar{v}(-p)$ we pick up an extra minus sign in (3.65b) from anti-commuting $\not{p}$ and γ_5.

According to the LSZ reduction formula the external fields have to be normalised in such a way that the loop corrections to the two point Green's function vanish

on-shell. Therefore one half of the loop correction to each external propagator is subtracted off. The remaining half can be accounted for by using the following rules:

$$\rangle\!\!=\!\!\xrightarrow{}\!\!\xleftarrow{p}\!\!=\!\!\xrightarrow{p}\!\!\langle = \tfrac{1}{2}\alpha + \tfrac{1}{4}\beta\gamma_5 \tag{3.66a}$$

$$\rangle\!\!=\!\!\xrightarrow{p}\!\!=\!\!\xrightarrow{p}\!\!\langle = \tfrac{1}{2}\alpha - \tfrac{1}{4}\beta\gamma_5 \tag{3.66b}$$

with

$$\alpha = \bar{a}(m^2) + 2m^2[\bar{a}'(m^2) + \bar{c}'(m^2)] \qquad , \qquad \beta = \bar{b}(m^2) \quad . \tag{3.67}$$

Equations (3.64b) and (3.66) provide us with generic expressions for the self-energy corrections of internal and external fermion propagators. To compute the corrections due to a specific self-energy diagram we only have to determine the "form factors" a, b, c and f. For the diagrams relevant for our calculations this is done explicitly in *chapter 4*.

3.5 Gauge Boson Self-Energies

The Lagrangian of a free vector field A_μ is

$$\mathcal{L}_0 = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 = \frac{1}{2}A_\mu(g^{\mu\nu}\partial^2 - (1 - \tfrac{1}{\xi})\partial^\mu\partial^\nu)A_\nu \quad , \tag{3.68}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and ξ is the gauge parameter. To obtain the renormalised Lagrangian we replace

$$A_\mu \to \sqrt{Z}A_\mu = (1 + \tfrac{1}{2}\Delta Z)A_\mu \qquad , \qquad \xi \to Z_\xi\xi = (1 + \Delta Z_\xi)\xi \quad . \tag{3.69}$$

This yields the renormalised Lagrangian

$$\mathcal{L}_{\mathrm{ren}} = \mathcal{L}_0 + \frac{1}{2}A_\mu[\Delta Z g^{\mu\nu}\partial^2 - \Delta Z(1 - \tfrac{1}{\xi})\partial^\mu\partial^\nu + \Delta Z_\xi\tfrac{1}{\xi}\partial^\mu\partial^\nu]A_\nu \quad . \tag{3.70}$$

Consequently, the Feynman rule for the counterterm is

$$\mu \xrightarrow{}\!\!\otimes\!\!\xrightarrow{} \nu \;\; = -\frac{i}{2}[\Delta Z(p^2 g^{\mu\nu} - p^\mu p^\nu) + (\Delta Z - \Delta Z_\xi)p^\mu p^\nu] \quad . \tag{3.71}$$

Due to Lorentz invariance the expressions for the one-loop self-energy diagrams have to take the general form

$$\mu \overset{\longrightarrow}{\underset{p}{\text{oooo}}} \bigcirc \overset{\longrightarrow}{\underset{p}{\text{oooo}}} \nu \;=\; i\big[f_1(p^2)p^2\hat{g}^{\mu\nu} + f_2(p^2)p^\mu p^\nu\big] \quad . \tag{3.72}$$

Note the appearance of the d-dimensional metric tensor $\hat{g}^{\mu\nu}$. Since we have regularised the diagram on the left-hand side with dimensional reduction, arguments that use Lorentz invariance can only be applied to fields on the d-dimensional subspace. The *Ward identities* require that (3.72) vanishes if both indices are contracted with p. Therefore

$$f_1(p^2) = -f_2(p^2) \equiv f(p^2) \tag{3.73}$$

and

$$\mu \overset{\longrightarrow}{\underset{p}{\text{oooo}}} \bigcirc \overset{\longrightarrow}{\underset{p}{\text{oooo}}} \nu \;=\; if(p^2)(p^2\hat{g}^{\mu\nu} - p^\mu p^\nu) \quad . \tag{3.74}$$

The calculation of gluon self-energy diagrams usually becomes much easier when we contract the two external Lorentz indices. To determine f we will therefore use

$$f(p^2) = -\frac{i}{p^2}\left(\frac{1}{3} + \frac{2}{9}\delta\right)\hat{g}_{\mu\nu}\left(\mu \overset{\longrightarrow}{\underset{p}{\text{oooo}}} \bigcirc \overset{\longrightarrow}{\underset{p}{\text{oooo}}} \nu\right) \quad , \tag{3.75}$$

which is obtained by contracting (3.74) with $\hat{g}_{\mu\nu}$, using $\hat{g}_{\mu\nu}\hat{g}^{\mu\nu} = d = 4 - 2\delta$ and expanding to first order in δ.

To cancel any divergent terms in f, the counterterm (3.71) must have the same form as (3.72), so we have to choose $\Delta Z_\xi = \Delta Z$ and consequently

$$\mu \overset{\longrightarrow}{\underset{p}{\text{oooo}}} \otimes \overset{\longrightarrow}{\underset{p}{\text{oooo}}} \nu \;=\; -\frac{i}{2}\Delta Z(p^2\hat{g}^{\mu\nu} - p^\mu p^\nu) \quad . \tag{3.76}$$

The *renormalised* self-energy correction $\Sigma^{\mu\nu}(p)$ is the sum of the self-energy diagrams and the counterterm, i.e.

$$\Sigma^{\mu\nu}(p) = -i\left(\mu \overset{\longrightarrow}{\underset{p}{\text{oooo}}} \bigcirc \overset{\longrightarrow}{\underset{p}{\text{oooo}}} \nu + \mu \overset{\longrightarrow}{\underset{p}{\text{oooo}}} \otimes \overset{\longrightarrow}{\underset{p}{\text{oooo}}} \nu\right)$$
$$= (f(p^2) - \tfrac{1}{2}\Delta Z)(p^2\hat{g}^{\mu\nu} - p^\mu p^\nu) \quad . \tag{3.77}$$

To make this expression finite we choose $\Delta Z = 2\hat{f}$ and obtain

$$\Sigma^{\mu\nu}(p) = \bar{f}(p^2)(p^2\hat{g}^{\mu\nu} - p^\mu p^\nu) \quad , \tag{3.78}$$

where $\hat{f}$ and $\bar{f}(p^2)$ are defined in analogy to (3.12). Note that the cancellation of divergent terms is only possible if $\hat{f}$ is independent of p. Thus the correction to the propagator is

$$\Pi_{\mu\nu}(p) = \frac{-ig_{\mu\rho}}{p^2}\left(i\Sigma^{\rho\sigma}\right)\frac{-ig_{\sigma\nu}}{p^2} = \frac{-i}{p^2}\bar{f}(p^2)\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) \quad . \tag{3.79}$$

For external gluons we have to replace one of the external propagators with the polarisation vector $\varepsilon_\mu^\sigma(p)$ or $\varepsilon_\mu^{\sigma*}(p)$, according to the Feynman rules for external legs shown in *table B.1*. Then we have to take the on-shell limit $p^2 \to 0$. Using $p^\mu\varepsilon_\mu^\sigma(p) = 0$ we find

$$\frac{-ig_{\mu\nu}}{p^2}(i\Sigma^{\nu\rho})\varepsilon_\rho^\sigma(p) = \bar{f}(p^2)\varepsilon_\mu^\sigma(p) \xrightarrow{p^2\to 0} \bar{f}(0)\varepsilon_\mu^\sigma(p) \quad . \tag{3.80}$$

According to the LSZ reduction formula the external fields have to be normalised in such a way that the one-loop corrections to the two-point Green's function vanish on-shell. Thus one half of the self-energy correction on an external leg is cancelled by the normalisation of the external field. To account for self-energy corrections on external gluon legs we therefore have to multiply the tree level diagram by a factor of $\frac{1}{2}\bar{f}(0)$.

Thus, as for the fermion self-energies, we have determined the generic form of self-energy corrections on internal and external gluons. Explicit expressions for the form factors f are derived in *chapter 4* for the diagrams relevant for our calculation. In principle, the renormalisation of vertex corrections can be treated in an analogous generic form: Write down the most general form for a three-point diagram, derive the Feynman rules for the counter-terms and adjust the renormalisation factors to cancel all divergences, in accordance with the renormalisation scheme. However, here the situation is simpler, because all vertices are renormalised in the modified minimal subtraction scheme. As we already know that the MSSM is renormalisable we can therefore express any vertex diagram directly in terms of Veltman-Passarino functions (3.23) and renormalise it by simply dropping all terms proportional to $(\frac{1}{\delta} + \gamma_E + \ln 4\pi)$. This is done automatically by the `LoopTools` package we used for the numerical computation of Veltman-Passarino functions.

Chapter 4

Feynman Diagrams

The objective of this work is to calculate the supersymmetric (MSSM) one-loop corrections to polarised $t\bar{t}$ production amplitudes at the TeV scale. At this scale we can consider all quark flavours except the top quark as massless. We also ignore contributions from off-diagonal elements of the CKM matrix.

Within these approximations our task is to compute all new (i.e. non Standard Model) one-loop MSSM Feynman diagrams that contribute to $t\bar{t}$ production. As we will see these diagrams involve squarks, charginos, neutralinos, gauginos and Higgs particles.

4.1 Prototype Feynman Rules

To handle the large number of diagrams we proceed in two steps: In the first step we compute a number of *prototype diagrams*. For these prototypes we only distinguish between fermions, scalar and vector particles, but not between all the different types of fermions, scalars and vector particles that appear in the MSSM. The propagators of and couplings between these prototype particles are written in a generic form, which we call the *prototype Feynman rules*:

- propagators:

$$\xrightarrow{\quad p \quad} \quad = \quad \frac{i(\not{p} + m_F)}{p^2 - m_F^2} \qquad (4.1a)$$

$$- - \!\!\blacktriangleright\!\! - - \underset{p}{} \quad = \quad - - \!\!\overset{\longrightarrow}{-}\!\! - - \underset{p}{} \quad = \frac{i}{p^2 - m_S^2} \qquad (4.1\text{b})$$

$$\mu \; \overset{\longrightarrow}{\text{\scriptsize 0000000}} \; \nu \underset{p}{} \quad = \frac{-ig_{\mu\nu}}{p^2} \qquad (4.1\text{c})$$

$$\mu \; \overset{\longrightarrow}{\sim\!\!\sim\!\!\sim} \; \nu \underset{p}{} \quad = \frac{-ig_{\mu\nu}}{p^2 - m_V^2} \qquad (4.1\text{d})$$

• vertices:

$$\mu \; \sim\!\!\sim\!\!\sim\!\!\bullet \qquad = -i\gamma^\mu (A\mathbb{1} + B\gamma_5) \qquad (4.2\text{a})$$

$$\mu \; \text{\scriptsize 0000000}\!\bullet \qquad = -i\gamma^\mu \qquad (4.2\text{b})$$

$$\mu \; \sim\!\!\sim\!\!\sim\!\!\bullet \underset{p}{\overset{k}{}} \quad = \quad \mu \; \text{\scriptsize 0000000}\!\bullet \underset{p}{\overset{k}{}} \quad = -i(p + k)^\mu \qquad (4.2\text{c})$$

$$\underset{}{\longleftarrow}\!\!\bullet \quad = \text{SM} \; \longrightarrow\!\!\bullet \quad = i(A\mathbb{1} + B\gamma_5) \qquad (4.2\text{d})$$

$$\mu \; \overset{\longrightarrow}{\underset{p_1}{\text{\scriptsize 0000000}}}\!\bullet \overset{\overset{\rho}{\underset{p_3}{}}}{\underset{\underset{\nu}{p_2}}{}} \quad = i[\quad g^{\mu\nu}(p_1 - p_2)^\rho \qquad (4.2\text{e})$$
$$+ g^{\nu\rho}(p_2 - p_3)^\mu$$
$$+ g^{\rho\mu}(p_3 - p_1)^\nu]$$

$$- - - \!\bullet \underset{}{} \quad = i \qquad (4.2\text{f})$$

$$= 2ig^{\mu\nu} \qquad (4.2\text{g})$$

As we can see from these rules, solid lines represent (Dirac) fermions and dashed lines are scalars. Complex scalar propagators are drawn with an arrow on top of the dashed line. Wavy lines stand massive vector bosons (W or Z bosons) and curly lines are massless vector bosons without axial couplings (gluons). The 'SM' label in (4.2d) indicates that the corresponding fermion is the (only) Standard Model particle of the vertex. Note that we do not provide separate prototype rules for Majorana fermions. Diagrams involving Majorana fermions can always be expressed in terms of diagrams with Dirac fermions by the techniques we discuss in *section 4.3*.

Diagrams calculated with the Feynman rules (4.1) and (4.2) depend, in addition to the kinematic parameters and the incoming and outgoing helicities, on the *generic masses* m_F, m_V, m_S and the *vertex coefficients* A and B. The actual MSSM diagrams are obtained from the prototypes by substituting the correct MSSM parameters for the generic masses and vertex coefficients.

4.2   The BSMPRO and SUSYTOP Libraries

To compute the Feynman diagrams for our calculation numerically we wrote two C++ class libraries called BSMPRO and SUSYTOP. Each of the classes in the BSMPRO library corresponds to a prototype diagram. They store information about the kinematic parameters, vertex coefficients and internal masses and provide methods to calculate the polarised amplitudes numerically. To do this we expressed, for each prototype, the coefficients (2.17) in terms of the *Passarino-Veltman functions* introduced in *section 3.2*. The algebraic manipulations were done with FORM and the results have been converted to C++ code and included in the BSMPRO library. Only the coefficient functions a, b, c and f for the self-energy diagrams are calculated by hand, following the procedure outlined in *section 3.3*. These results are given later in this chapter.

For the numerical computation of the Veltman Passarino functions BSMPRO uses the LoopTools package [64]. In accordance with the $\overline{\text{DR}}$ renormalisation scheme,

`LoopTools` regularises the ultraviolet divergent Passarino-Veltman function by removing the terms proportional to $(\frac{1}{\delta} + \gamma_E + \ln 4\pi)$. Once the coefficient functions (2.17) for a certain diagram are calculated we can obtain the corresponding amplitude by multiplying them with the helicity matrix elements (2.33). The classes of the `BSMPRO` library do this internally. In a realistic calculation there may be several diagrams which require the same Passarino-Veltman functions with identical parameters. Since the computation of Passarino-Veltman functions is numerically expensive `BSMPRO` uses the cache of the `LoopTools` package to avoid calculating the same numbers repeatedly. For this purpose, all required Passarino-Veltman functions can be stored in one or more central tables, which are then read by all instances of the `BSMPRO` classes simultaneously.

To calculate the actual *diagrams* we have to instantiate the `BSMPRO` classes, initialise the internal masses and vertex coefficients according to a given set of MSSM parameters, calculate the required Passarino-Veltman functions for a given set of kinematic parameters and call the update method for every instance of the `BSMPRO` classes to compute the polarised amplitudes. This work is done by the `SUSYTOP` library. It provides methods to read the MSSM parameters from the standard file format specified in the SUSY Les Houches Accord [44], instantiate and initialise all required `BSMPRO` classes and compute the interference terms from the amplitudes calculated by these classes. The `SUSYTOP` library also takes care of including the correct *colour factors*, which have to be calculated separately for each interference term.

We will now list all the prototype diagrams that are relevant for our calculation. The label under each diagram (set in typewriter font) is the name of the corresponding `BSMPRO` class. An asterisk behind the class name indicates that the crossed version of this diagram has to be included as well. The technique of crossing diagrams was explained in *chapter 2*. A double asterisk indicates that *only* the crossed version is needed. For each prototype we also provide a list of MSSM particles that have to be substituted for the generic propagators. To label the various MSSM particles we use the notations introduced at the end of *section 1.8*, which are essentially the same as used in [59]. However, unless stated otherwise the generation indices I, J, etc. only run over the first two generations. The third generation quarks are written explicitly as t $(= u^3)$ and b $(= d^3)$.

There are some cases in which we deviate from the Feynman rules given in *section 4.1*:

- For vertices connected to incoming fermion lines we set $A = 1$ and $B = 0$. As the incoming quarks are massless the corresponding Dirac spinors are eigenvectors of γ_5. Thus, to account for an incoming massless quark or anti-quark with helicity σ and coupling coefficients A and B we just have to multiply the diagram by $(A + \sigma B)$.

- In some diagrams we have two identical prototype vertices which can represent different MSSM vertices. In this case we have to rename the vertex coefficients for one of the vertices. A label (C, D) next to a prototype vertex indicates that the coefficients A and B in the Feynman rules (4.2) are replaced by C and D, respectively.

- In some diagrams we can have two different types of (MSSM) scalars, fermions or vectors in the same diagram. In this case we label them F_1, F_2, S_1, S_2 etc. with masses m_{F1}, m_{F2}, m_{S1}, m_{S2} etc.

4.2.1 Tree-Level Diagrams

The following prototype diagrams contribute to $t\bar{t}$ production at tree-level:

Dqqbar_sV_tree Dgg_sG_tree

$V = \gamma, Z, g$

Dgg_tF_tree*

$$F = t$$
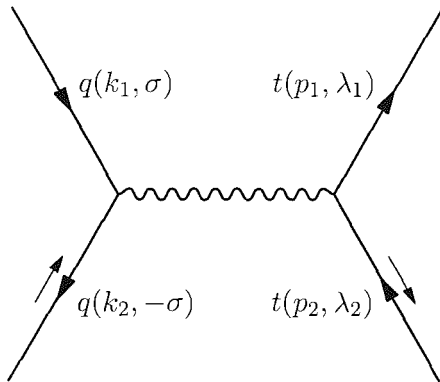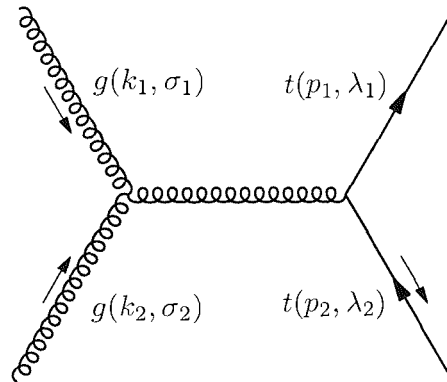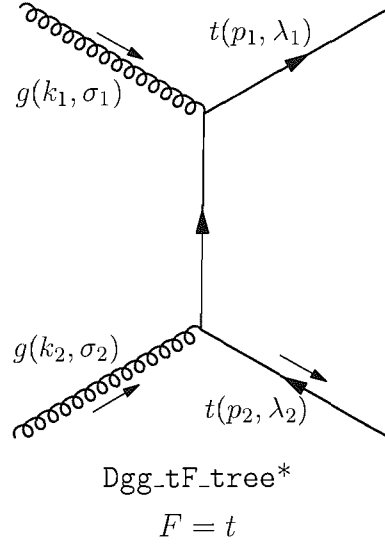
4.2.2 Self-Energy Diagrams

For our calculation we only need to consider self-energy corrections for fermions and massless vector particles (gluons). We start with the fermion self-energies:

The scalar self-energy correction to the fermion propagator is

$$= \int \frac{d^d q}{(2\pi)^d D} \left[i(A + B\gamma_5)i(\slashed{q} + m_F)i(A^* - B^*\gamma_5)i \right]$$

$$= \int \frac{d^d q}{(2\pi)^d D} \left[\quad (|A|^2 + |B|^2)\slashed{q} - (A^*B + B^*A)\slashed{q}\gamma_5 \right.$$

$$\left. + (|A|^2 - |B|^2)m_F + (A^*B - B^*A)m_F\gamma_5 \right]$$

$$= \frac{i}{16\pi^2} \left[-(|A|^2 + |B|^2)B_1\slashed{p} + 2\operatorname{Re}(A^*B)\, B_1\slashed{p}\gamma_5 \right.$$

$$\left. + (|A|^2 - |B|^2)\frac{m_F}{m}B_0\, m + 2i\operatorname{Im}(A^*B)m_F B_0\,\gamma_5 \right] \quad , \tag{4.3}$$

where $B_0 \equiv B_0(p^2, m_F, m_S)$, $B_1 \equiv B_1(p^2, m_F, m_S)$ and

$$D = (q^2 - m_F^2)((q - p)^2 - m_S^2) \quad . \tag{4.4}$$

The masses m_F and m_S correspond to the internal fermion and scalar, respectively,

and m is the renormalised mass of the external fermion. Comparing with (3.47) we define

$$a_{SF}(p^2) = \frac{|A|^2 + |B|^2}{16\pi^2} B_1 \qquad , \qquad b_{SF}(p^2) = -\frac{2\,\mathrm{Re}(A^*B)}{16\pi^2} B_1 \quad ,$$

$$c_{SF}(p^2) = -\frac{|A|^2 - |B|^2}{16\pi^2}\frac{m_F}{m} B_0 \quad , \quad f_{SF}(p^2) = -\frac{2i\,\mathrm{Im}(A^*B)}{16\pi^2} B_1 \quad . \tag{4.5}$$

Let us now move on to the gluon self-energies. The s-channel gluon propagator gets corrections from fermion and scalar loops. For the fermion loop we have

$$= -\int \frac{d^d q}{(2\pi)^d} \frac{\mathrm{Tr}\left[(-i)\gamma^\mu i(\slashed{q} - \slashed{p} + m_F)(-i)\gamma_\mu i(\slashed{q} + m_F)\right]}{(q^2 - m_F^2)((q-p)^2 - m_F^2)}$$

$$= -8\int \frac{d^d q}{(2\pi)^d} \frac{(\delta - 1)q^2 - (\delta - 1)pq + m_F^2}{(q^2 - m_F^2)((q-p)^2 - m_F^2)}$$

$$= -8\frac{i}{16\pi^2}\left[(\delta - 1)A_0 + p^2(\delta - 1)B_1 + m_F^2 B_0\right]$$

$$= -8\frac{i}{16\pi^2}\left[(\delta - 1)A_0 - \frac{p^2}{2}(\delta - 1)B_0 + m_F^2 B_0\right] \quad , \tag{4.6}$$

where $A_0 \equiv A_0(m_F)$, $B_0 \equiv B_0(p^2, m_F, m_F)$, $B_1 \equiv B_1(p^2, m_F, m_F)$ and m_F denotes the mass of the fermion in the loop. In the last step we used (3.27) for $m_0 = m_1 = m_F$. Also note that, since gluons have no axial coupling, we have set $B = 0$ in the prototype Feynman rule (4.2a). Using (3.75), (3.21) and the expressions for $\hat{A}_0$ and $\hat{B}_0$ we obtain

$$\bar{f}_F(p^2) = -\frac{1}{16\pi^2}\frac{1}{p^2}\left[-\frac{8}{3}\bar{A}_0 + \frac{4}{3}(p^2 + 2m_F^2)\bar{B}_0 + \frac{8}{3}m_F^2 - \frac{4}{9}p^2\right]$$

$$= -\frac{1}{16\pi^2}\frac{4}{3}\left[2\frac{m_F^2}{p^2}(\bar{B}_0 - \bar{B}_0^{(0)}) + \bar{B}_0 - \frac{1}{3}\right] \quad , \tag{4.7}$$

where $\bar{A}_0 \equiv \bar{A}_0(m_F)$, $\bar{B}_0^{(0)} \equiv \bar{B}_0^{(0)}(m_F)$ and $\bar{B}_0 \equiv \bar{B}_0(p^2, m_F, m_F)$.

For the scalar self-energy correction there are two diagrams which will always appear with the same coefficient: one with two three-point vertices and one with a single four-point vertex. We therefore include both diagrams in the definition of the

scalar self-energy coefficient function f_S.

$$= \int \frac{d^d q}{(2\pi)^d} \frac{(-i)(2q^\mu - p^\mu)i(-i)(2q_\mu - p_\mu)i}{(q^2 - m_S^2)((q-p)^2 - m_S^2)} + \int \frac{d^d q}{(2\pi)^d} \frac{2i(4 - 2\delta)i}{(q^2 - m_S^2)}$$

$$= \int \frac{d^d q}{(2\pi)^d} \left[\frac{4q^2 - 4pq + p^2}{(q^2 - m_S^2)((q-p)^2 - m_S^2)} - \frac{8 - 4\delta}{(q^2 - m_S^2)} \right]$$

$$= \frac{i}{16\pi^2} \left[4A_0 + 4p^2 B_1 + (p^2 + 4m_S^2)B_0 - (8 - 4\delta)A_0 \right]$$

$$= \frac{i}{16\pi^2} \left[-4(1 - \delta)A_0 + (4m_S^2 - p^2)B_0 \right] \quad , \tag{4.8}$$

where $A_0 \equiv A_0(m_S)$, $B_0 \equiv B_0(p^2, m_S, m_S)$, $B_1 \equiv B_1(p^2, m_S, m_S)$ and m_S is the mass of the scalar particle in the loop. In the last step we used (3.27) for $m_0 = m_1 = m_S$. Using (3.75), (3.21) and the expressions for $\hat{A}_0$ and $\hat{B}_0$ we obtain

$$\bar{f}_S(p^2) = \frac{1}{16\pi^2} \frac{1}{p^2} \left[-\frac{4}{3}\bar{A}_0 + \frac{1}{3}(4m_S^2 - p^2)\bar{B}_0 + \frac{4}{3}m_S^2 - \frac{2}{9}p^2 \right]$$

$$= \frac{1}{16\pi^2} \frac{4}{3} \left[\frac{m_S^2}{p^2}(\bar{B}_0 - \bar{B}_0^{(0)}) - \frac{1}{4}\bar{B}_0 - \frac{1}{6} \right] \quad , \tag{4.9}$$

where $\bar{A}_0 \equiv \bar{A}_0(m_S)$, $\bar{B}_0^{(0)} \equiv \bar{B}_0^{(0)}(m_S)$ and $\bar{B}_0 \equiv \bar{B}_0(p^2, m_S, m_S)$.

By inserting the above-mentioned self-energy corrections in individual lines of the tree level diagrams we obtain the following *self-energy diagrams*:
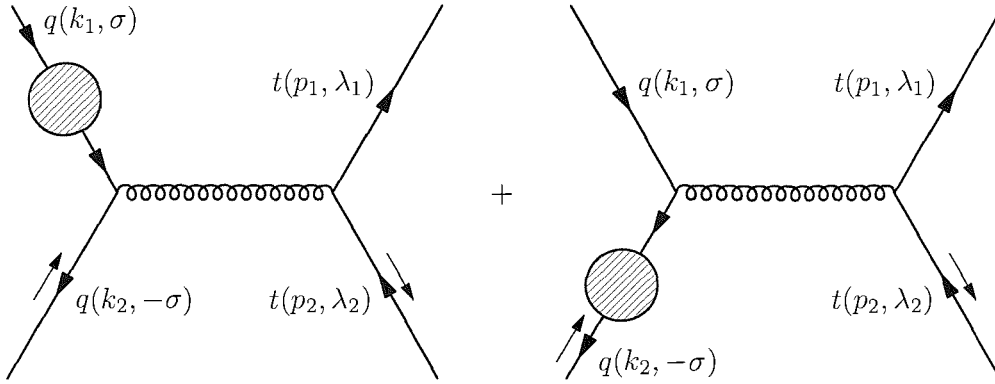
Dqqbar_sV_xseSq

$$(q, F, S) = (u^I, \chi_j^0, U_i), \quad (d^I, \chi_j^0, D_i), \quad (u^I, \chi_j, D_i), \quad (d^I, \chi_j^c, U_i), \quad (u^I, \Lambda, U_i), \quad (d^I, \Lambda, D_i)$$
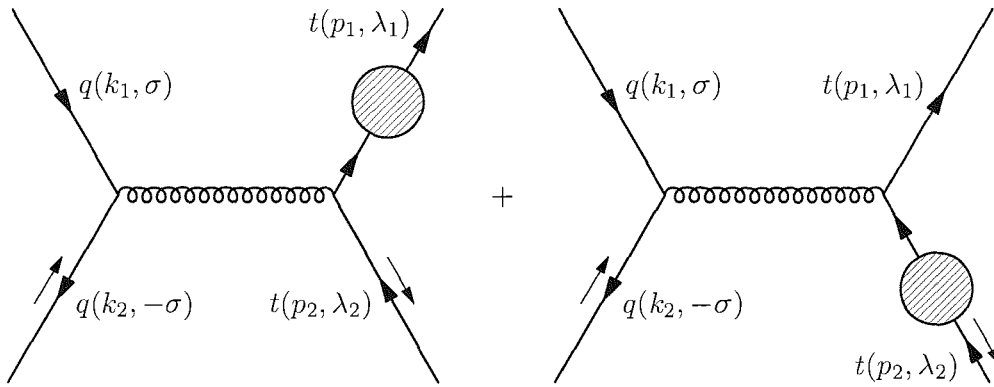
Dqqbar_sV_xseSt

$$(F, S) = (\chi_j^0, U_i), \quad (\chi_j, D_i), \quad (t, H_i^0), \quad (t, A_1^0), \quad (b, H_1), \quad (\Lambda, U_i)$$
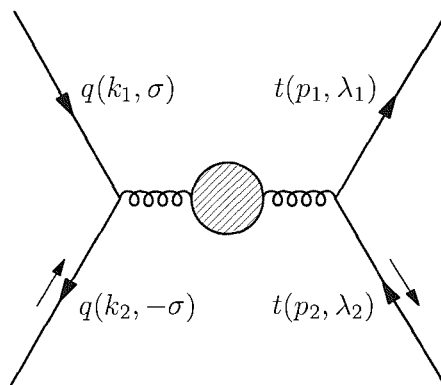
Dqqbar_sG_iseF, Dqqbar_sG_iseS

$$q = u^I, d^I \quad , \quad F = u^I, t, d^I, b, \Lambda \quad , \quad S = U_i, D_i$$

Dgg_sG_xseFg, Dgg_sG_xseSg

$$F = \Lambda \quad , \quad S = U_i, D_i$$

Dgg_sG_xseSt

$$(F, S) = (\chi_j^0, U_i), \ (\chi_j, D_i), \ (t, H_i^0), \ (t, A_1^0), \ (b, H_1), \ (\Lambda, U_i)$$

Dgg_sG_iseF, gg_sG_iseS
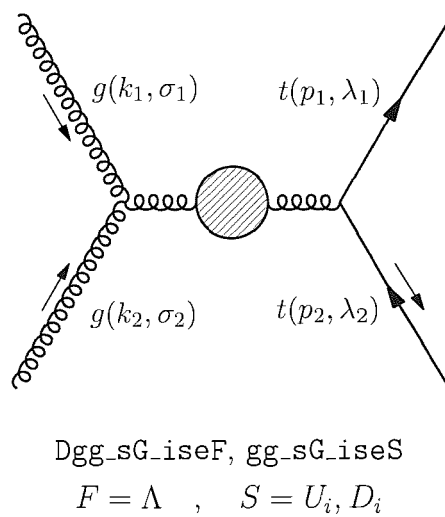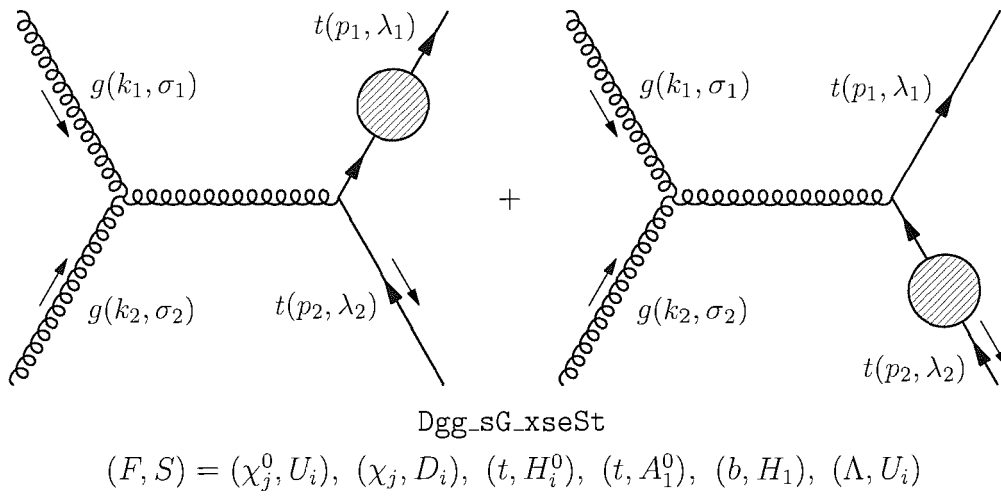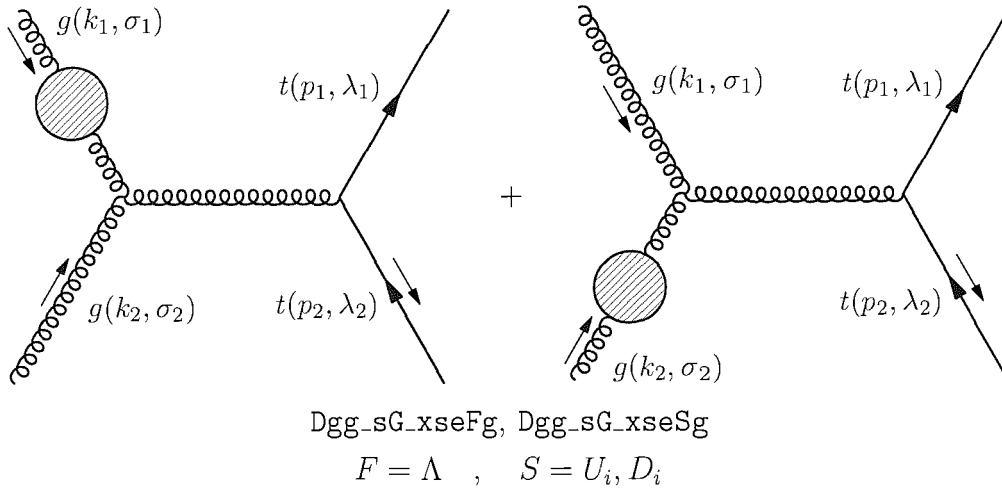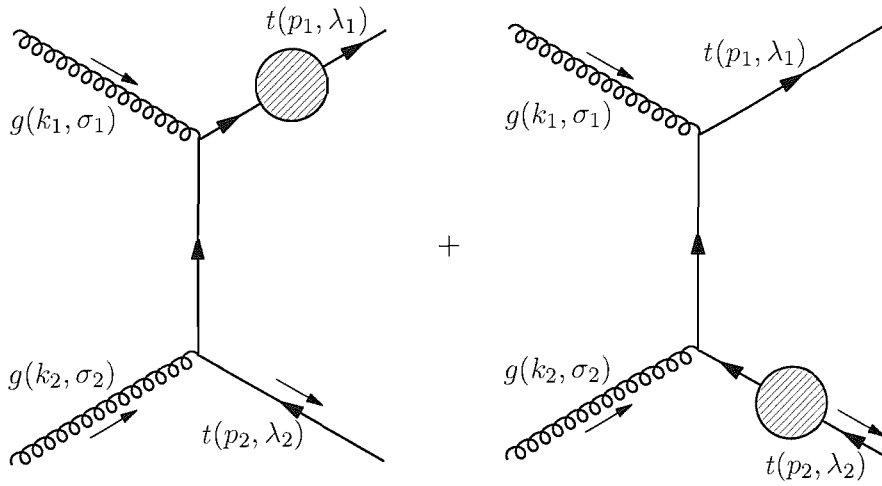
$$F = \Lambda \quad , \quad S = U_i, D_i$$

Dgg_tF_xseSt*

$$(F, S) = (\chi_j^0, U_i), \ (\chi_j, D_i), \ (t, H_i^0), \ (t, A_1^0), \ (b, H_1), \ (\Lambda, U_i)$$
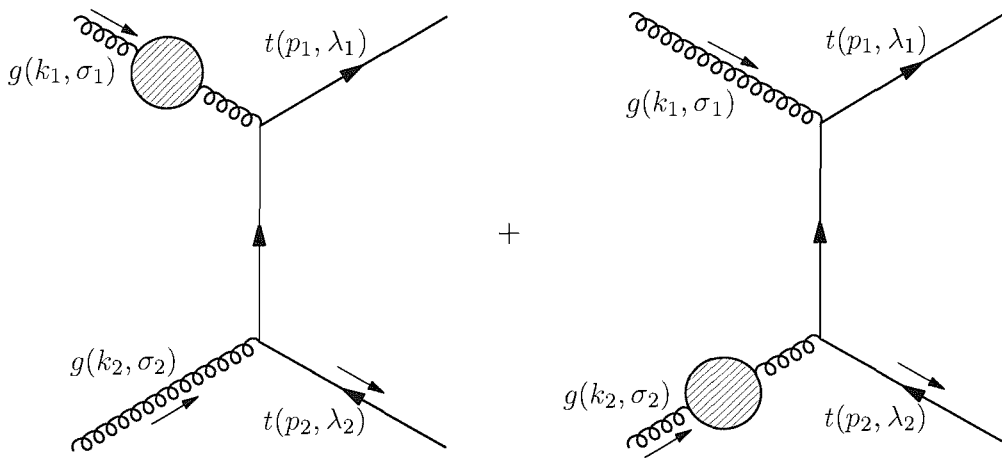
Dgg_tF_xseSt*

$$F = \Lambda \quad , \quad S = U_i, D_i$$
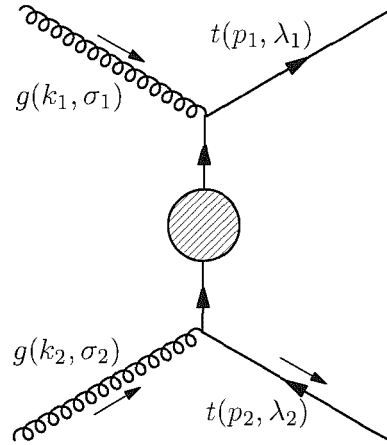
Dgg_tF_iseS*

$$(F, S) = (\chi_j^0, U_i), \quad (\chi_j, D_i), \quad (t, H_i^0), \quad (t, A_1^0), \quad (b, H_1), \quad (\Lambda, U_i)$$

In each diagram the hatched blob stands for one of the self-energy corrections discussed in the first part of this section.

4.2.3 Vertex Corrections

The prototype vertex corrections for the $q\bar{q} \to t\bar{t}$ amplitude are:
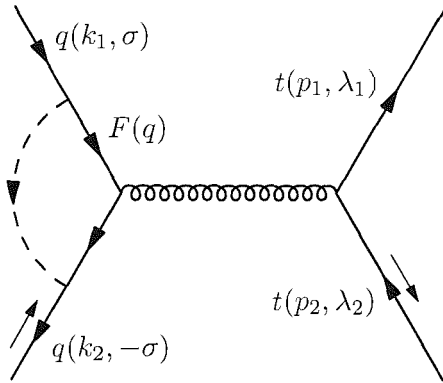
Dqqbar_sG_vertSq

$$(q, F, S) = (u^I, \Lambda, U_i), \quad (d^I, \Lambda, D_i)$$
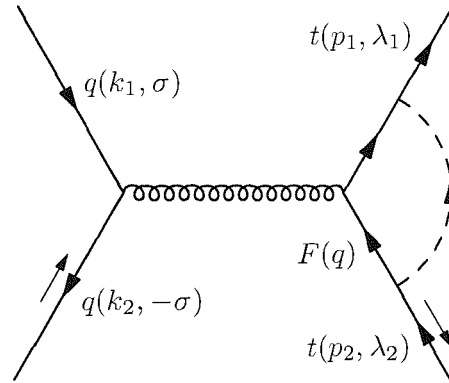
Dqqbar_sV_vertSt

$$q = u^I, d^I$$

$$(F, S) = (\Lambda, U_i), \quad (t, H_i^0), \quad (t, A_1^0), \quad (b, H_1)$$
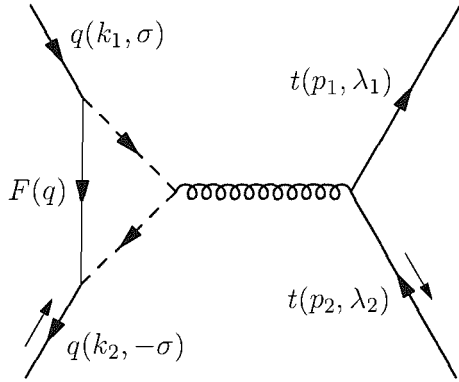
Dqqbar_sG_vertSSq

$(q, F, S) =$

$(u^I, \Lambda, U_i),\ (u^I, \chi_j^0, U_i),\ (u^I, \chi_j, D_i),$

$(d^I, \Lambda, D_i),\ (d^I, \chi_j^0, D_i),\ (d^I, \chi_j, U_i)$
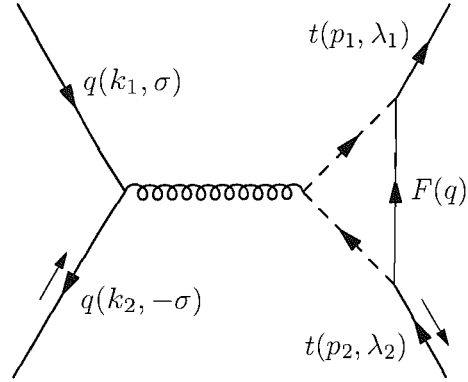
Dqqbar_sG_vertSSt

$q = u^I, d^I$

$(S, F) = (\Lambda, U_i),\ (\chi_j^0, U_i),\ (\chi_j, D_i)$

For the $gg \to t\bar{t}$ amplitude we distinguish vertex corrections for s and t-channel diagrams. The corrections to the s-channel diagrams are:
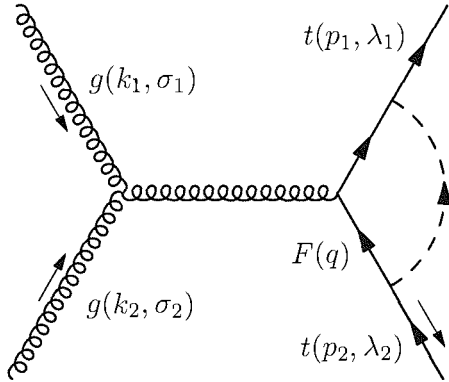
Dgg_sG_vertSt

$(F, S) = (\Lambda, U_i),\ (t, H_i^0),\ (t, A_1^0),\ (b, H_1)$
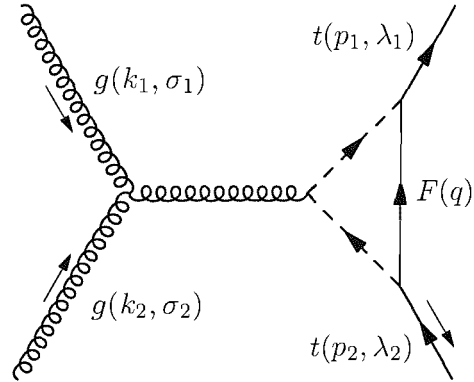
Dgg_sG_vertSSt

$(S, F) = (U_i, \Lambda),\ (U_i, \chi_j^0),\ (D_i, \chi_j)$
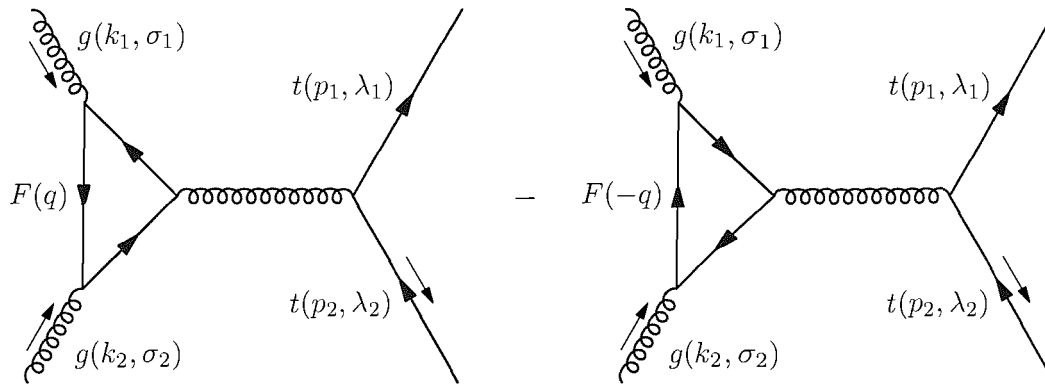
Dgg_sG_vertFg

$$F = \Lambda$$
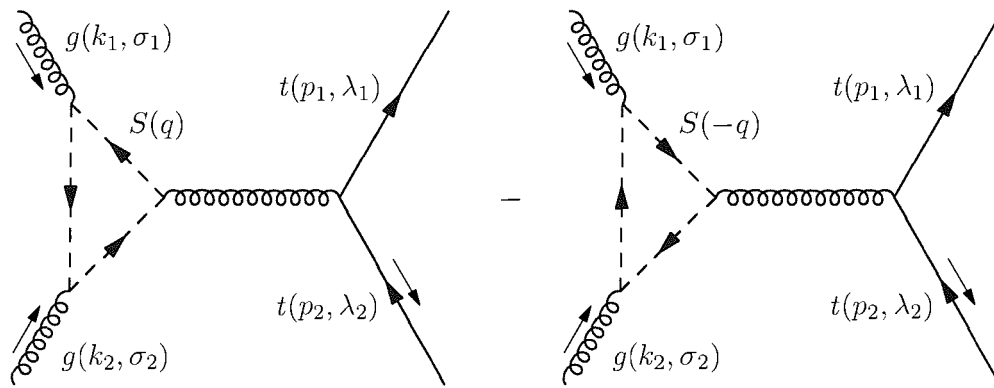
Dgg_sG_vertSg

$$S = U_i, D_i$$
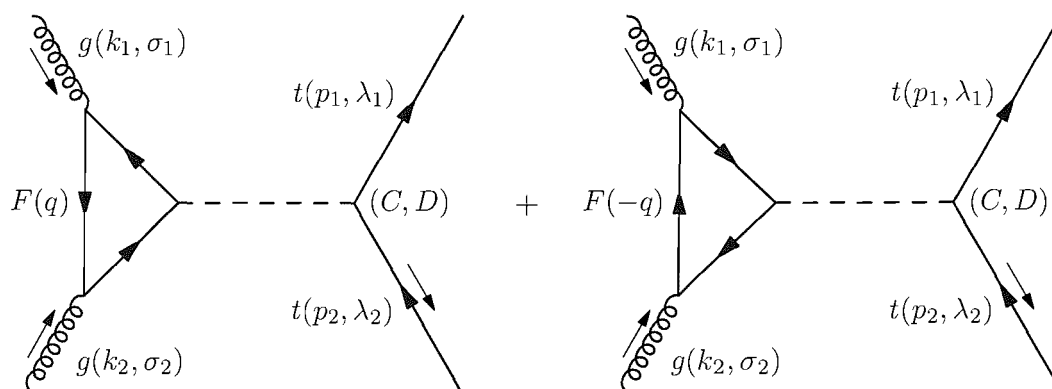
Dgg_sS_vertFg

$$F = t, b \quad , \quad S = H_i^0, A_1^0$$
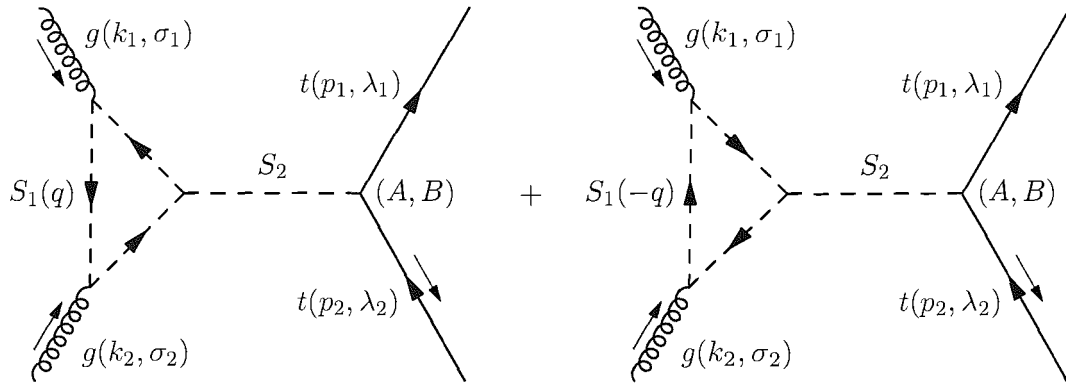
Dgg_sS_vertSg

$$S_1 = U_i, D_i \quad , \quad S_2 = H_i^0, A_1^0$$
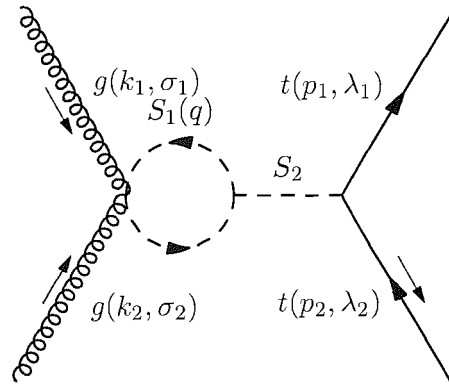
Dgg_sS_vertSSg

$$S_1 = U_i, D_i \quad , \quad S_2 = H_i^0, A_1^0$$
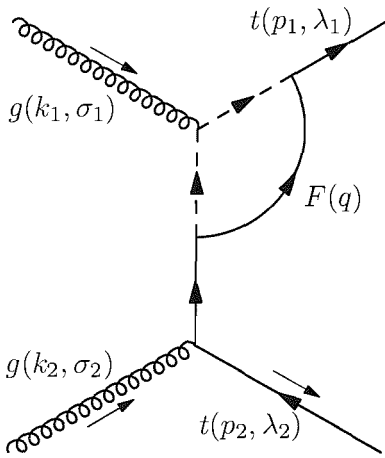
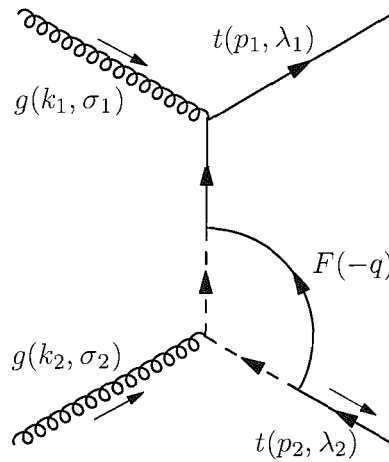The t-channel vertex corrections are:
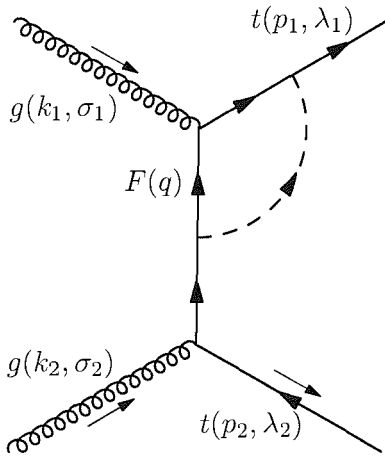
Dgg_tF_vertS1*
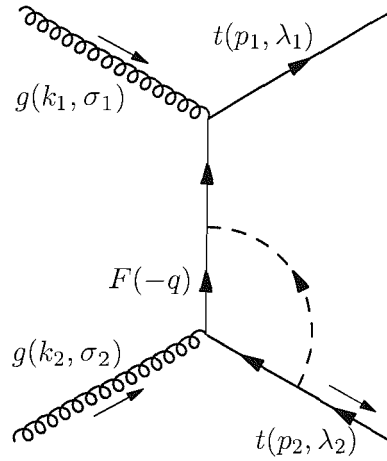
$(F, S) = (\Lambda, U_i), \ (t, H_i^0), \ (t, A_1^0), \ (b, H_1)$

Dgg_tF_vertS2*

$(F, S) = (\Lambda, U_i), \ (t, H_i^0), \ (t, A_1^0), \ (b, H_1)$

Dgg_tF_vertSS1*

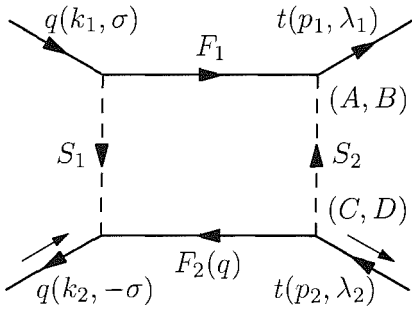$(S, F) = (U_i, \Lambda), \ (U_i, \chi_j^0), \ (D_i, \chi_j)$

Dgg_tF_vertSS2*

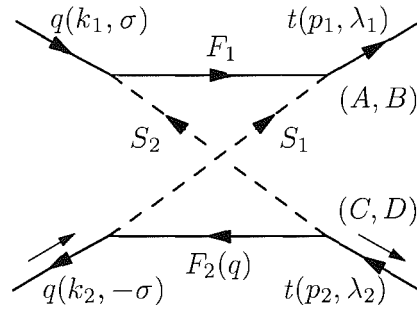$(S, F) = (U_i, \Lambda)$

4.2.4   Box Diagrams
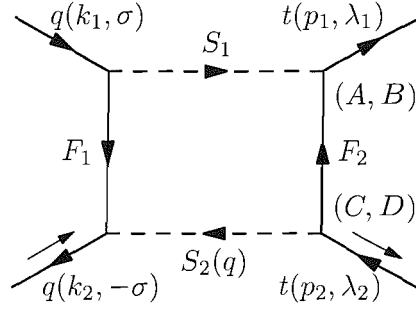
Dqqbar_boxSS

$$(q, S_1, S_2, F_1, F_2) = (u^I, U_i, U_j, \Lambda, \Lambda), \ (u^I, U_i, U_j, \Lambda, \chi_k^0), \ (u^I, U_i, U_j, \chi_k^0, \Lambda),$$
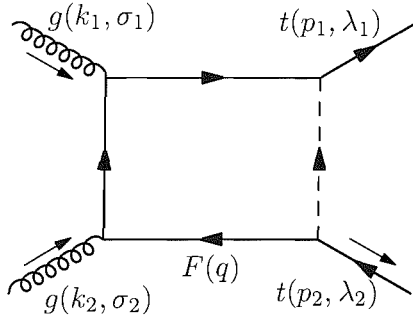$$(d^I, D_i, D_j, \Lambda, \chi_k), \ (d^I, U_i, U_j, \chi_k, \Lambda)$$

Dqqbar_fboxSS

$$(q, F_1, F_2, S_1, S_2) = (u^I, \Lambda, \Lambda, U_i, U_j),$$
$$(d^I, \Lambda, \Lambda, D_i, U_j), \ (u^I, \chi_k^0, \Lambda, U_i, U_j),$$
$$(u^I, \Lambda, \chi_k^0, U_i, U_j), \ (d^I, \chi_k^0, \Lambda, D_i, U_j),$$
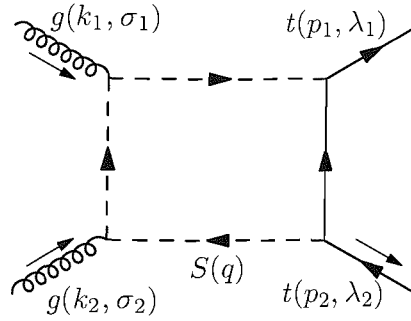$$(d^I, \Lambda, \chi_k^0, D_i, U_j)$$

Dqqbar_fboxSSx**

$$(q, F_1, F_2, S_1, S_2) = (u^I, \Lambda, \Lambda, U_i, U_j),$$
$$(d^I, \Lambda, \Lambda, D_i, U_j), \ (u^I, \chi_k^0, \Lambda, U_i, U_j),$$
$$(u^I, \Lambda, \chi_k^0, U_i, U_j), \ (d^I, \chi_k^0, \Lambda, D_i, U_j),$$
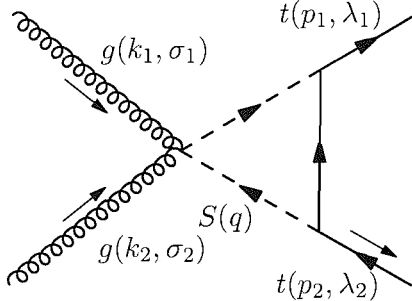$$(d^I, \Lambda, \chi_k^0, D_i, U_j)$$

Dgg_boxFS*

$(F, S) = (\Lambda, U_i), \ (t, H_1^0), \ (t, A_1^0), \ (b, H_i)$
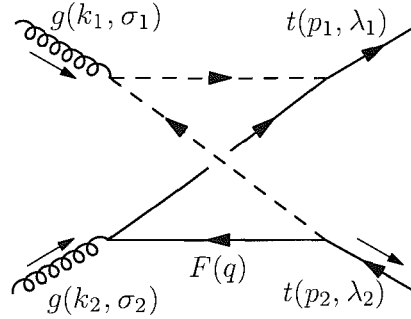

Dgg_boxSF*

$(S, F) = (U_i, \Lambda), \ (U_i, \chi_j^0), \ (D_i, \chi_j)$

Dgg_boxSF4

$(S, F) = (U_i, \Lambda), \ (U_i, \chi_j^0), \ (D_i, \chi_j)$

Dgg_boxSFx*

$(S, F) = (U_i, \Lambda)$

4.3   Majorana Fermions

Even though the gluinos and neutralinos of the MSSM are Majorana fermions, the SUSYTOP library does not provide separate prototypes for diagrams involving Majorana fermions. The reason for this is that all these diagrams can be related to diagrams where the Majorana fermions are replaced by ordinary Dirac fermions. To do this we simply use the relations (B.63) and (B.65) to eliminate the charge conjugation operator $\mathcal{C}$. In this section we show how this is done explicitly for the (sub-)diagrams relevant for our calculation.

The propagators for Majorana fermions are given in (B.71). Interaction terms involving two Majorana fields all have an extra factor of $\frac{1}{2}$ relative to their Dirac-equivalents. Thus our prototype vertex factors for vertices with two Majorana fermions are simply $\frac{1}{2}$ times those for the Dirac vertices given in (4.2). Note that here our bookkeeping is different to the one used in [59]. In [59] it is assumed that the factors

of $\frac{1}{2}$ are cancelled by the fact that there are more possibilities to contract Majorana fields than there are to contract Dirac fields (see (B.71)). In this work we represent the different ways of contracting Majorana fields by separate diagrams (namely, those containing propagators with "clashing arrows"). As a result, all our MSSM Feynman rules for vertices with two Majorana fermions will have an extra factor of $\frac{1}{2}$ compared to those given in [59].

In principle we could calculate all diagrams involving Majorana fermions separately, using the expressions (B.71) for the Majorana propagators, and then compare the results with suitable diagrams that only involve Dirac fermions. However, this would be rather tedious and error prone. Instead we will now derive a set of diagrammatic rules that allow us to "straighten out" the clashing arrows of any diagram with Majorana fermions. First consider a fermion line of the following form:

$$T = \alpha \quad \cdots \quad \beta \tag{4.10}$$

where $p_1, \ldots, p_n$ are momenta flowing into the vertices from other lines and $k_i = \sum_{j=0}^{i} p_j$. This line corresponds to the following Wick contraction:

$$T = \lambda_\alpha^{p_{n+1}} \{\bar\lambda \Gamma^{(n)} \lambda\}_{p_n} \cdots \{\bar\lambda \Gamma^{(2)} \lambda\}_{p_{n-1}} \{\bar\lambda \Gamma^{(1)} \lambda\}_{p_1} \bar\lambda_{\beta p_0} \quad, \tag{4.11}$$

where the $\Gamma^{(i)}$ denote the string of γ matrices that join the fermion fields in the corresponding vertex. The curly brackets indicate that the momentum indices of the enclosed fields are contracted with a momentum conserving delta function. The momentum indices attached to the brackets are added to that delta function. To disentangle the contractions we have to swap the fields in each vertex, so we pick up a factor of -1 for each vertex:

$$T = (-1)^n \lambda_\alpha^{p_{n+1}} \{\lambda^\mathsf{T} \Gamma^{(n)\mathsf{T}} \bar\lambda^\mathsf{T}\}_{p_n} \cdots \{\lambda^\mathsf{T} \Gamma^{(2)\mathsf{T}} \bar\lambda^\mathsf{T}\}_{p_{n-1}} \{\lambda^\mathsf{T} \Gamma^{(1)\mathsf{T}} \bar\lambda^\mathsf{T}\}_{p_1} \bar\lambda_{\beta p_0} \quad. \tag{4.12}$$

Now we can use (B.70) to write the contractions in terms of the Dirac propagator S and the charge conjugation matrix $\mathcal{C}$. Note that the fields in the contractions linking the vertices are in the wrong order. Thus we have to swap them all round, picking up a minus sign each time. As there are only $n-1$ links between the n vertices, we have one minus sign left. Furthermore, since the spinor indices get swapped along

with the fields, we have to use the transpose of the Dirac propagator for the links:

$$T = -[S(p_{n+1})\mathcal{C}^{-1}\Gamma^{(n)\mathsf{T}}S^{\mathsf{T}}(-k_{n-1})\cdots$$
$$\cdots S^{\mathsf{T}}(-k_2)\Gamma^{(2)\mathsf{T}}S^{\mathsf{T}}(-k_1)\Gamma^{(1)\mathsf{T}}\mathcal{C}^{-1}S(p_0)]_{\alpha\beta}\,\delta^{p_{n+1}}{}_{p_n\cdots p_0} \quad . \quad (4.13)$$

Now assume that the $\Gamma^{(i)}$ all satisfy

$$\mathcal{C}\Gamma^{(i)}\mathcal{C}^{-1} = \sigma_i\Gamma^{(i)\mathsf{T}} \tag{4.14}$$

with $\sigma_i = \pm 1$. Then we can use the property (B.66) to pull one factor of $\mathcal{C}^{-1}$ past all the propagators and vertices until it is next to the other $\mathcal{C}^{-1}$. Then, using $(\mathcal{C}^{-1})^2 = -\mathbb{1}$, we obtain

$$T = (\textstyle\prod_i \sigma_i)[S(p_{n+1})\Gamma^{(n)}S(k_{n-1})\cdots S(k_2)\Gamma^{(2)}S(k_1)\Gamma^{(1)}S(p_0)]_{\alpha\beta}\delta^{p_{n+1}}{}_{p_n\cdots p_0} \quad . \quad (4.15)$$

Diagrammatically this means that

Note that this logic still works if some of the fields in the chain are Dirac fields. Furthermore, if the fields have other indices like colour or weak isospin indices, the transpose applies to these indices, too. For example, the gluon-gluino coupling is proportional to

$$\Gamma^{c\mu}{}_{(\alpha a)(\beta b)} = f^a{}_{bc}\gamma^\mu_{\alpha\beta} \quad , \tag{4.17}$$

where $f^a{}_{bc}$ are the $SU(3)$ structure constants and the gluino fields are contracted with the downstairs indices. Interchanging the indices b and c gives us another minus sign, so that

$$\mathcal{C}_{\alpha\gamma}\Gamma^{c\mu}{}_{(\gamma a)(\delta b)}\mathcal{C}^{-1}_{\delta\beta} = f^c{}_{ab}(\mathcal{C}\gamma^\mu\mathcal{C}^{-1})_{\alpha\beta} = -f^c{}_{ab}\gamma^\mu_{\beta\alpha} = f^c{}_{ba}\gamma^\mu_{\beta\alpha} = \Gamma^{c\mu}{}_{(\beta b)(\alpha a)}$$
$$\Rightarrow \quad \mathcal{C}\Gamma^{c\mu}\mathcal{C}^{-1} = (\Gamma^{c\mu})^\mathsf{T} \quad . \tag{4.18}$$

Analogous derivations can be made in the cases where we replace one of the external fields with its bar-conjugate. This corresponds to the following diagrams:
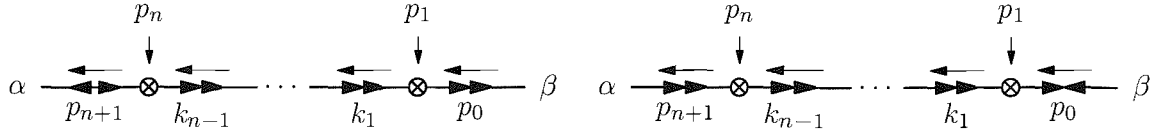
In both cases one of the contractions with the external fields ends up in the wrong order, too, so we get an extra minus sign from anti-commuting the fields in that contraction. On the other hand there is only one factor of $\mathcal{C}^{-1}$ in the game now, so we get no minus sign from using $(\mathcal{C}^{-1})^2 = -\mathbb{1}$. However, since we cannot cancel the charge conjugation matrix with anything it now appears in the final expression. To get rid of all the transposed propagators we have to pull it to the "far" end of the chain in each case. This leaves us with the following two diagrammatic rules:

$$= \left(\textstyle\prod_i \sigma_i\right) \times \alpha \cdots \gamma \ \times \mathcal{C}^{-1}_{\gamma\beta} \quad , \quad (4.19\text{a})$$

$$= \left(\textstyle\prod_i \sigma_i\right) \times \mathcal{C}^{-1}_{\alpha\gamma} \times \gamma \cdots \beta \quad . \quad (4.19\text{b})$$

Note that these rules are only applicable if we end up with a fermion line in which all arrows point towards the left index (α in this case). The order of the indices is determined by the time ordered product under consideration. However, the rules above are sufficient, since we can always rearrange the fields in a time ordered product so that the desired field appears on the left. We only have to remember that we pick up a minus sign each time we swap two fermionic fields in the time ordered product.

With these rules at hand it is now straightforward to chase through the list of prototype diagrams given in *section 4.2*, draw a version with clashing arrows for each

that involves Majorana fermions and use the rules derived above to straighten the arrows again, thus relating the new diagram to the original prototype. In most cases the rule (4.16) is sufficient. The only difficult cases are the box diagrams Dqqbar_fboxSS and Dqqbar_fboxSSx. The latter, as it is drawn in *section 4.2.4*, does not even exist in the MSSM since there are only vertices with incoming squark and outgoing quark lines or outgoing squark and incoming quark lines. However, using propagators with clashing arrows we can draw the following diagrams:

$$(4.20)$$
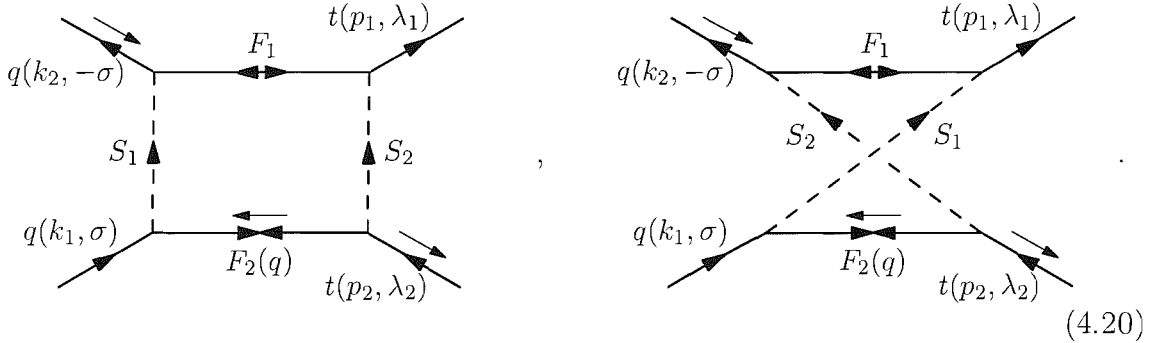
We will straighten out the arrows on the fermion lines in such a way that we end up with charge conjugation matrices acting on the incoming spinors $u^\sigma(k_1)$ and $\bar{v}^\sigma(k_2)$. Remember that our diagrams correspond to a time ordered product of the form $T\{q_{\alpha k_2}\bar{q}_{\beta k_1}t_\gamma{}^{p_1}\bar{t}_\delta{}^{p_2}\}$. On the upper fermion line we end up with arrows pointing towards the final state top quark. Thus, to apply the rules (4.20) we have to rewrite the time ordered product as

$$T\{q_{\alpha k_2}\bar{q}_{\beta k_1}t_\gamma{}^{p_1}\bar{t}_\delta{}^{p_2}\} = T\{t_\gamma{}^{p_1}q_{\alpha k_2}\bar{q}_{\beta k_1}\bar{t}_\delta{}^{p_2}\} \quad . \tag{4.21}$$

In other words, we pick up no sign from interchanging the fields in the time ordered product.

After straightening the arrows the incoming spinors get multiplied with $\mathcal{C}^{-1}$. According to the Feynman rules for external fermion legs given in *table B.1* we have to contract the incoming fermion line with $u^\sigma(k_1)$ and the incoming anti-fermion line with $\bar{v}^\sigma(k_2)$. Using (B.63) we find

$$\mathcal{C}^{-1}_{\alpha\beta}u^\sigma_\beta(k_1) = \bar{v}^{-\sigma}_\alpha(k_1) \quad , \qquad \bar{v}^\sigma_\beta(k_2)\mathcal{C}^{-1}_{\beta\alpha} = u^{-\sigma}_\alpha(k_2) \quad . \tag{4.22}$$

Thus the box diagrams (4.20) are algebraically equivalent to the "crossed" versions

of the prototypes Dqqbar_fboxSS and Dqqbar_fboxSSx:

$$, \qquad (4.23a)$$

$$. \qquad (4.23b)$$

In the first relation we also have reversed the arrow on the scalar S_1. We can do this because the direction of the arrow on scalar lines has no impact on the algebraic expression for the diagram, since the scalar propagator is symmetric in the momentum.

The upshot of all this is that for the prototypes Dqqbar_sG_iseF, Dgg_sG_iseF, Dgg_sG_vertFg the additional diagrams we get in the case of internal Majorana fermions add up to give us a factor of $1/2$. Furthermore, we have to include the crossed versions of Dqqbar_fboxSS and Dqqbar_fboxSSx. To swap the incoming particles we simply replace θ with $\theta - \pi$ and σ with $-\sigma$.

Chapter 5

Cross Sections

So far we have seen how to calculate supersymmetric one-loop corrections to the $t\bar{t}$ production amplitudes introduced in *chapter 2*. In this chapter we show how these amplitudes can be used to compute observables, specifically scattering cross sections and their ratios, which can be measured at particle colliders like the LHC.

We will first derive an expression for the $t\bar{t}$ production cross section at parton level. Then we use the *factorisation theorem* to express the proton-proton to $t\bar{t}$ cross section in terms of the parton level cross sections and *parton distribution functions* (PDF). Finally we show how information about the spins of the top quarks can be extracted from the angular distribution of their decay products.

5.1   The Cross Section at Parton Level

For the moment let us ignore the fact that, at the LHC, top quarks will be produced from proton-proton collisions and assume that we can collide individual partons (i.e. light quarks and gluons) with a given energy. In our approximation all partons are massless. Thus the flux factor is simply $1/(2M_{t\bar{t}}^2)$, where $M_{t\bar{t}}$ is the invariant mass of the top-antitop system. The differential cross section at parton level is given by

$$d\hat{\sigma}_i(M_{t\bar{t}}, \mathbf{p}_1, \mathbf{p}_2) = \frac{1}{2M_{t\bar{t}}^2} \frac{d^3\mathbf{p}_1}{(2\pi)^3 2E(\mathbf{p}_1)} \frac{d^3\mathbf{p}_2}{(2\pi)^3 2E(\mathbf{p}_2)}$$
$$\times (2\pi)^4 \delta(M_{t\bar{t}} - E(\mathbf{p}_1) - E(\mathbf{p}_2))\delta(-\mathbf{p}_1 - \mathbf{p}_2)|\mathcal{M}_i|^2 \quad . \quad (5.1)$$

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The arguments $\mathbf{p_1}$ and $\mathbf{p_2}$ are the momenta of the final state top quarks in the partonic centre of mass frame. The index i labels the type of parton in the initial state. Depending on i, the matrix element $\mathcal{M}$ is the quantity defined in (2.3) or (2.35). The spin indices are suppressed. A priori the momentum arguments of $\mathcal{M}$ can be fixed to

$$
\begin{aligned}
k_1 &= (E, 0, 0, E) & , && k_2 &= (E, 0, 0, -E) & , \\
p_1 &= (E(\mathbf{p_1}), \mathbf{p_1}) & , && p_2 &= (E(\mathbf{p_2}), \mathbf{p_2}) & ,
\end{aligned}
\tag{5.2}
$$

where

$$
E \equiv \frac{1}{2} M_{t\bar{t}}
\tag{5.3}
$$

is the energy of *one* parton or final state top quark in the centre of mass system.

Now we integrate over the momentum $\mathbf{p_2}$. The delta function yields the additional constraint $\mathbf{p_1} = -\mathbf{p_2}$ and

$$
d\hat{\sigma}_i(M_{t\bar{t}}, \mathbf{p_1}) = \frac{1}{2M_{t\bar{t}}^2} \frac{d^3\mathbf{p_1}}{(2\pi)^6 4E^2(\mathbf{p_1})} (2\pi)^4 \delta(2E - 2E(\mathbf{p_1})) |\mathcal{M}_i|^2 \quad .
\tag{5.4}
$$

To integrate out the remaining delta function we write $\mathbf{p_1}$ in polar coordinates:

$$
\mathbf{p_1} = (p, \theta, \phi) \qquad , \qquad d^3\mathbf{p_1} = dp\, p^2 d(\cos\theta) d\phi \quad .
\tag{5.5}
$$

The energy $E(\mathbf{p_1})$ depends only on the radial coordinate p:

$$
E(\mathbf{p_1}) = E(p) = \sqrt{p^2 + m^2} \quad .
\tag{5.6}
$$

Thus we may write the delta function as

$$
\delta(2E - 2E(\mathbf{p_1})) = \frac{1}{|2E'(p)|} \delta(p - p(E)) \quad ,
\tag{5.7}
$$

where $p(E)$ denotes the positive zero point of $2E - 2E(p)$. Using (5.6) we find

$$
p(E) = \sqrt{E^2 - m^2}
\tag{5.8}
$$

and

$$
E'(p) = \frac{dE(p)}{dp} = \frac{p}{\sqrt{p^2 + m^2}} = \frac{p}{E(p)} \quad .
\tag{5.9}
$$

Performing the integral over p and using $E(p(E)) = E$ we get

$$d\hat{\sigma}_i(M_{t\bar{t}}, \theta, \phi) = \frac{1}{2M_{t\bar{t}}^2} \frac{p(E)d(\cos\theta)d\phi}{(2\pi)^2 8E} |\mathcal{M}_i|^2 \quad, \tag{5.10}$$

and the momentum arguments of $\mathcal{M}$ are further constrained by $|\mathbf{p_1}| = p(E)$. By rotation invariance of $\mathcal{M}$ the integral over φ just gives us a factor of 2π. Hence

$$\frac{\partial\hat{\sigma}_i(M_{t\bar{t}}, \theta)}{\partial(\cos\theta)} = \frac{p(E)|\mathcal{M}_i|^2}{(2\pi)64E^3} \quad, \tag{5.11}$$

Furthermore, rotation invariance allows us to write the momentum arguments of $\mathcal{M}$ in terms of E and θ according to (2.25).

5.2 Proton Scattering Cross Sections

At the LHC top quarks are produced from proton collisions with centre of mass energies M_{pp} at the TeV scale. In the *parton model* the actual collision takes place between two partons, one from each proton, which carry fractions x_1 and x_2 of the proton momenta P_1 and P_2. If we neglect the proton mass the centre of mass energy $M_{t\bar{t}}$ at parton level is related to the collision energy M_{pp} of the protons by

$$M_{t\bar{t}}^2 = x_1 x_2 M_{pp}^2 \quad. \tag{5.12}$$

According to the *factorisation theorem* the cross section for producing top quarks from proton collisions is obtained by folding the cross section at parton level with the *parton distribution functions* f_i. Let the index $i = u, d, c, s, g$ denote a parton and the index $\bar{\imath} = \bar{u}, \bar{d}, \bar{c}, \bar{s}, g$ its antiparticle. Then the cross section for producing a top-antitop pair from a parton i in proton 1 and parton $\bar{\imath}$ in proton 2 is given by

$$\frac{\partial\sigma_i(M_{pp}, \theta)}{\partial(\cos\theta)} = \int_0^1 dx_1 \int_0^1 dx_2 \, f_i(x_1) f_{\bar{\imath}}(x_2) \theta(x_1 x_2 M_{pp}^2 - 4m^2) \frac{\partial\hat{\sigma}_i(\sqrt{x_1 x_2} M_{pp}, \theta)}{\partial(\cos\theta)} \quad. \tag{5.13}$$

The functions f_i and $f_{\bar{\imath}}$ are the distribution functions of the the partons i and $\bar{\imath}$. The theta function assures that the collision at parton level is above the threshold for producing the top-antitop pair.

To integrate out the theta function we make the following change of variables:

$$\tau = x_1 x_2 \qquad , \qquad \omega = (1 - x_1)x_2 \quad , \tag{5.14a}$$

$$x_1 = \frac{\tau}{\tau + \omega} \qquad , \qquad x_2 = \tau + \omega \quad , \tag{5.14b}$$

$$dx_1 \wedge dx_2 = \left(\frac{\partial x_1}{\partial \tau} \frac{\partial x_2}{\partial \omega} - \frac{\partial x_1}{\partial \omega} \frac{\partial x_2}{\partial \tau} \right) d\tau \wedge d\omega = \frac{1}{x_2} d\tau \wedge d\omega \quad . \tag{5.14c}$$

Thus equation (5.13) reads

$$\frac{\partial \sigma_i(M_{pp}, \theta)}{\partial(\cos\theta)} = \int_{4m^2/M_{pp}^2}^{1} d\tau \int_{0}^{1-\tau} d\omega \, \frac{1}{x_2} f_i(x_1) f_{\bar{\imath}}(x_2) \frac{\partial \hat{\sigma}_i(\tau M_{pp}, \theta)}{\partial(\cos\theta)} \quad , \tag{5.15}$$

Using (5.12) we can replace the integral over τ by an integral over $M_{t\bar{t}}$:

$$\frac{\partial \sigma_i(M_{pp}, \theta)}{\partial(\cos\theta)} = \int_{2m}^{M_{pp}} dM_{t\bar{t}} \frac{2M_{t\bar{t}}}{M_{pp}^2} \int_{0}^{\omega_{\max}} d\omega \, \frac{1}{x_2} f_i(x_1) f_{\bar{\imath}}(x_2) \frac{\partial \hat{\sigma}_i(M_{t\bar{t}}, \theta)}{\partial(\cos\theta)} \quad , \tag{5.16}$$

where

$$x_1 = \frac{M_{t\bar{t}}^2}{M_{t\bar{t}}^2 + \omega M_{pp}^2} \quad , \quad x_2 = \omega + \frac{M_{t\bar{t}}^2}{M_{pp}^2} \quad , \quad \omega_{\max} = 1 - \frac{M_{t\bar{t}}^2}{M_{pp}^2} \quad . \tag{5.17}$$

By omitting the integration over $M_{t\bar{t}}$ we also define the differential cross section

$$\frac{\partial^2 \sigma_i(M_{pp}, M_{t\bar{t}}, \theta)}{\partial M_{t\bar{t}} \partial(\cos\theta)} = \frac{2M_{t\bar{t}}}{M_{pp}^2} \frac{\partial \hat{\sigma}_i(M_{t\bar{t}}, \theta)}{\partial(\cos\theta)} \int_{0}^{\omega_{\max}} d\omega \, \frac{1}{x_2} f_i(x_1) f_{\bar{\imath}}(x_2) \tag{5.18}$$

for producing a top-antitop pair with a given invariant mass. If the particle i comes from proton 2 we have to interchange the labels i and $\bar{\imath}$ in (5.18) and replace θ by $\pi - \theta$. Hence we define

$$\frac{\partial^2 \bar{\sigma}_i(M_{pp}, M_{t\bar{t}}, \theta)}{\partial M_{t\bar{t}} \partial(\cos\theta)} = \frac{2M_{t\bar{t}}}{M_{pp}^2} \left. \frac{\partial \hat{\sigma}_i}{\partial(\cos\theta)} \right|_{M_{t\bar{t}}, \pi-\theta} \int_{0}^{\omega_{\max}} d\omega \, \frac{1}{x_2} f_{\bar{\imath}}(x_1) f_i(x_2) \quad . \tag{5.19}$$

Experimentally we cannot determine on an event by event basis, which constituent of which proton took part in the collision. Hence, to construct an experimentally measurable cross section, we have to sum over all possible parton types i and the two

possible ways of extracting them from the two protons. Thus we define

$$\frac{\partial^2 \sigma(M_{pp}, M_{t\bar{t}}, \theta)}{\partial M_{t\bar{t}} \partial(\cos\theta)} = \sum_{i=u,d,c,s} \left[\frac{\partial^2 \sigma_i(M_{pp}, M_{t\bar{t}}, \theta)}{\partial M_{t\bar{t}} \partial(\cos\theta)} + \frac{\partial^2 \bar{\sigma}_i(M_{pp}, M_{t\bar{t}}, \theta)}{\partial M_{t\bar{t}} \partial(\cos\theta)} \right]$$
$$+ \frac{\partial^2 \sigma_g(M_{pp}, M_{t\bar{t}}, \theta)}{\partial M_{t\bar{t}} \partial(\cos\theta)} \quad . \quad (5.20)$$

Usually it is more convenient to integrate out one more parameter. We define the *invariant mass differential cross section* as

$$\frac{\partial \sigma(M_{pp}, M_{t\bar{t}})}{\partial M_{t\bar{t}}} = \int_{-1}^{+1} d(\cos\theta) \, \frac{\partial^2 \sigma(M_{pp}, M_{t\bar{t}}, \theta)}{\partial(\cos\theta)\partial M_{t\bar{t}}} \quad . \quad (5.21)$$

If we integrate out the invariant mass argument $M_{t\bar{t}}$ it is conventional to parametrise the cross section by the transverse momentum p_T of the top quark. In terms of $M_{t\bar{t}}$ and θ the transverse momentum p_T can be written as:

$$p_T = p(\tfrac{1}{2}M_{t\bar{t}})\sin\theta \quad , \quad (5.22)$$

with p given in (5.8). Since $M_{t\bar{t}}$ lies between $2m$ and M_{pp} the transverse momentum p_T ranges from 0 to $p(\tfrac{1}{2}M_{pp})$. For a fixed value of p_T the invariant mass $M_{t\bar{t}}$ of the top-antitop system can lie between $2E(p_T)$ and M_{pp}, with E given in (5.6). Hence the *transverse momentum differential cross section* is given by the integral

$$\frac{\partial \sigma(M_{pp}, p_T)}{\partial p_T} = \int_{M_{pp}}^{2E(p_T)} dM_{t\bar{t}} \, \frac{\partial^2 \sigma(M_{pp}, M_{t\bar{t}}, \theta)}{\partial(\cos\theta)\partial M_{t\bar{t}}} \frac{d(\cos\theta)}{dp_T}$$
$$= \int_{2E(p_T)}^{M_{pp}} dM_{t\bar{t}} \, \frac{\partial^2 \sigma(M_{pp}, M_{t\bar{t}}, \theta)}{\partial(\cos\theta)\partial M_{t\bar{t}}} \frac{\tan\theta}{p} \quad , \quad (5.23)$$

where $\cos\theta$ is given in terms of p_T and $M_{t\bar{t}}$ by

$$\cos^2\theta = 1 - \frac{p_T^2}{p^2(\tfrac{1}{2}M_{t\bar{t}})} \quad . \quad (5.24)$$

5.3 Top Decays and Observables

So far we have assumed that the $t\bar{t}$ final state can be measured directly in the detectors, along with the helicities of the top quarks. This is, of course, not true. With

a half-life of the order of 10^{-25} sec the top quarks decay long before they can reach a detector. However, their lifetime is also about 20 times shorter than the natural time scale of QCD, which means that they decays predominantly via *electroweak interactions*. This gives us the unique opportunity to study spin correlations of the intermediate $t\bar{t}$ state. For other quarks the spin information is lost in the hadronisation process, but for top quarks it can be extracted from the angular distribution of the decay products, due to the parity violating nature of the weak interactions. In this section we demonstrate how to define observables on some general final state $X_1 X_2$ of a process $pp \to t\bar{t} \to X_1 X_2$, which analyse the spins of the intermediate $t\bar{t}$ state.

Let $\mathcal{M}_{\text{tot}}$ denote the total amplitude for the process $i \to t\bar{t} \to X_1 X_2$, where i can be either a gluon-gluon or quark-antiquark initial state. Let $\mathcal{M}_{\lambda,\bar{\lambda}}$ denote the amplitude for the process $i \to t\bar{t}$. The indices λ and $\bar{\lambda}$ denote the helicities of the top and anti-top, respectively, and we have suppressed the spins of the initial state and the dependence on the initial and final state momenta. Furthermore, let μ^λ and $\bar{\mu}^{\bar{\lambda}}$ denote the amplitudes for the decays $t \to X_1$ and $\bar{t} \to X_2$, respectively. Then the amplitude $\mathcal{M}_{\text{tot}}$ can be written as

$$\mathcal{M}_{\text{tot}} = \mathcal{M}_{\lambda\bar{\lambda}} \mu_\lambda \bar{\mu}_{\bar{\lambda}} \quad , \tag{5.25}$$

where a sum over repeated indices is implied. Here we neglect interactions between the decay products of the top quarks, since these contributions are of higher order than the one-loop corrections we calculated for $\mathcal{M}_{\lambda\bar{\lambda}}$. The modulus squared of $\mathcal{M}_{\text{tot}}$ can then be written as

$$|\mathcal{M}_{\text{tot}}|^2 = \mathcal{M}_{\lambda\bar{\lambda}} \mathcal{M}^*_{\lambda'\bar{\lambda}'} \mu^*_{\lambda'} \mu_\lambda \bar{\mu}^*_{\bar{\lambda}'} \bar{\mu}_{\bar{\lambda}} = R_{\lambda\lambda'\bar{\lambda}\bar{\lambda}'} \rho_{\lambda'\lambda} \bar{\rho}_{\bar{\lambda}'\bar{\lambda}} \quad , \tag{5.26}$$

with an average over the spins of the initial state implied. The *spin density matrices* R, ρ and $\bar{\rho}$ are defined as

$$R_{\lambda\lambda'\bar{\lambda}\bar{\lambda}'} = \mathcal{M}_{\lambda\bar{\lambda}} \mathcal{M}^*_{\lambda'\bar{\lambda}'} \quad , \quad \rho_{\lambda'\lambda} = \mu^*_{\lambda'} \mu_\lambda \quad , \quad \bar{\rho}_{\bar{\lambda}'\bar{\lambda}} = \bar{\mu}^*_{\bar{\lambda}'} \bar{\mu}_{\bar{\lambda}} \quad . \tag{5.27}$$

As the Pauli matrices σ^i and the identity matrix form a basis of the space $\mathbb{C}^{2,2}$ of

2×2 matrices we can expand the spin density matrices as follows:

$$\rho^{\lambda'\lambda} = a\delta_{\lambda'\lambda} + b^i\sigma^i_{\lambda'\lambda} \qquad , \qquad \bar{\rho}^{\bar{\lambda}'\bar{\lambda}} = \bar{a}\delta_{\bar{\lambda}'\bar{\lambda}} + \bar{b}^i\sigma^i_{\bar{\lambda}'\bar{\lambda}} \quad ,$$

$$R_{\lambda\lambda'\bar{\lambda}\bar{\lambda}'} = A\delta_{\lambda\lambda'}\delta_{\bar{\lambda}\bar{\lambda}'} + B^i\sigma^i_{\lambda\lambda'}\delta_{\bar{\lambda}\bar{\lambda}'} + \bar{B}^i\delta_{\lambda\lambda'}\sigma^i_{\bar{\lambda}\bar{\lambda}'} + C^{ij}\sigma^i_{\lambda\lambda'}\sigma^j_{\bar{\lambda}\bar{\lambda}'} \quad . \tag{5.28}$$

Using $\text{Tr}(\sigma^i) = 0$ and $\text{Tr}(\sigma^i\sigma^j) = 2\delta^{ij}$ we see that $|\mathcal{M}_{\text{tot}}|^2$ can be written as

$$|\mathcal{M}_{\text{tot}}|^2 = 4[aA\bar{a} + b^iB^i\bar{a} + a\bar{B}^i\bar{b}^i + b^iC^{ij}\bar{b}^j] \quad . \tag{5.29}$$

The dependence of the coefficients a, $\bar{a}$, b^i, $\bar{b}^i$, A, B^i $\bar{B}^i$ and C^{ij} on the initial and final momenta encodes the information about the spin of the top quarks and determines the angular distribution of the decay products. To define observables that can be measured with sufficient accuracy we have to integrate over most of the final state phase space. Assume that we integrate over all final state momenta except the momentum $\mathbf{q}$ of one decay product of the top quark and the momentum $\bar{\mathbf{q}}$ of one of the decay products of the anti-top. By rotation invariance the (integrated) coefficient a can only depend on $|\mathbf{q}|$ and $\mathbf{b}$ must be proportional to $\hat{\mathbf{q}}$. Analogous statements hold for $\bar{a}$, $\bar{\mathbf{b}}$ and $\bar{\mathbf{q}}$. The vectors $\hat{\mathbf{q}}$ and $\hat{\bar{\mathbf{q}}}$ can be expressed in terms of polar coordinates (θ, ϕ) and $(\bar{\theta}, \bar{\phi})$:

$$\hat{\mathbf{q}} = \sin\theta\cos\phi\,\mathbf{e_1} + \sin\theta\sin\phi\,\mathbf{e_2} + \cos\theta\,\mathbf{e_3} \quad ,$$

$$\hat{\bar{\mathbf{q}}} = \sin\bar{\theta}\cos\bar{\phi}\,\bar{\mathbf{e}}_1 + \sin\bar{\theta}\sin\bar{\phi}\,\bar{\mathbf{e}}_2 + \cos\bar{\theta}\,\bar{\mathbf{e}}_3 \quad . \tag{5.30}$$

Note that the polar axes $\mathbf{e_3}$ and $\bar{\mathbf{e}}_3$ for $\hat{\mathbf{q}}$ and $\hat{\bar{\mathbf{q}}}$ can be chosen independently. Now we also integrate over $|\mathbf{q}|$, $|\bar{\mathbf{q}}|$ and the azimuthal angles ϕ and $\bar{\phi}$. Let $d\text{LIPS}'$ denote the corresponding integration measure. Then the integral over $|\mathcal{M}_{\text{tot}}|^2$ is of the form

$$\int d\text{LIPS}'\,|\mathcal{M}_{\text{tot}}|^2 = A' + B'\cos\theta + \bar{B}'\cos\bar{\theta} + C'\cos\theta\cos\bar{\theta} \quad . \tag{5.31}$$

The coefficients A', B', $\bar{B}'$ and C' can be measured experimentally by observing the angular distribution of the final state particles. By a more careful analysis they can also be related to spin correlations of the intermediate $t\bar{t}$ state. The details of these relations depend on the decay mode under consideration. For semi-leptonic top decays, for example, they were studied in [4] within a Standard Model framework.

In the following discussion of our results for supersymmetric effects in $t\bar{t}$ pro-

duction cross sections we will only consider spin correlations at the level of the $t\bar{t}$ intermediate state. A thorough analysis of supersymmetric signals in the angular distribution of the decay products still needs to be done. For consistency such an analysis will require the computation of supersymmetric corrections to the top decay amplitudes μ_λ and $\bar{\mu}_{\bar{\lambda}}$.

Chapter 6

Results and Conclusions

In this section we present our results for the SUSY corrections to polarised $t\bar{t}$ production cross sections, which we calculated for each of the 10 Snowmass benchmarks detailed in [45]. The benchmarks 1a, 1b and 2 to 6 are derived from gravity mediated SUSY breaking scenarios. Benchmarks 7 and 8 are related to gauge mediated SUSY breaking and benchmark 9 comes from an anomaly mediated scenario. To calculate the masses of the supersymmetric particles and run the couplings to the TeV scale we used the program SOFTSUSY by B. C. Allanach [65]. The renormalisation scale μ of the scale dependent MSSM parameters was set to the geometric mean of the two stop masses, in accordance with the convention adopted in [65]. The decay widths of the MSSM Higgs particles were calculated with the program HDECAY by Djouadi, Kalinowski and Spira [66]. The Feynman rules for the MSSM vertices were taken from J. Rosiek's paper [59]. We compare our parton level cross sections with the results obtained in the leading log approximation [46]. Then we discuss our results for the total $pp \rightarrow t\bar{t}$ cross section and the asymmetries introduced in *section 5.3*.

6.1 Numerical Results

Let $\hat{\sigma}_i$ denote the total cross section for the process $i \rightarrow t\bar{t}$, where the initial state i can be a gluon pair (gg), a light up-type quark-antiquark pair $(u\bar{u})$ or a light down type quark-antiquark pair $(d\bar{d})$. We regard $\hat{\sigma}_i$ as a function of the variable $\hat{s} \equiv M_{t\bar{t}}^2$, where $M_{t\bar{t}}$ is the invariant mass of the top-antitop pair. For each of these cross sections we
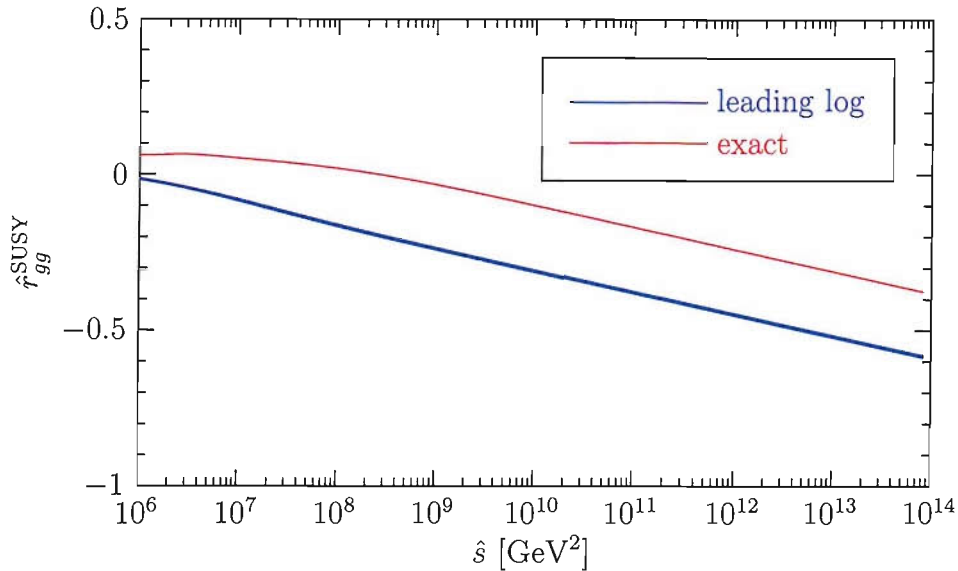
95

Figure 6.1: SUSY corrections to the $gg \rightarrow t\bar{t}$ cross section in the "exact" calculation and the leading log approximation for benchmark 5 of [45]. The thickness of the leading log graph reflects the uncertainty due to the choice of the universal SUSY scale.

have calculated the leading order contribution $\hat{\sigma}_i^{\mathrm{LO}}$ and the SUSY corrections $\hat{\sigma}_i^{\mathrm{SUSY}}$ due to the diagrams listed in *chapter 4*. The SUSY corrections can be split into super-QCD (SQCD) corrections and super-electroweak (SEW) corrections. The SQCD corrections are of order $\mathcal{O}(\alpha_s^3)$ and the SEW correction of order $\mathcal{O}(\alpha\alpha_s^2)$. Consequently the SEW corrections are one order of magnitude smaller than the SQCD corrections. We also define the ratios

$$\hat{r}_i(\hat{s}) = \frac{\hat{\sigma}_i^{\mathrm{SUSY}}(\hat{s})}{\hat{\sigma}_i^{\mathrm{LO}}(\hat{s})} \quad . \tag{6.1}$$

Figures 6.1, 6.2 and 6.3 show a comparison of our "exact" ratios with the results obtained in the leading log approximation by Beccaria, Renard and Verzegnassi [46].

We have used Snowmass benchmark 5 to compute the exact cross section, but the observations stated here are true for any of the 10 Snowmass benchmarks. The only SUSY inputs in the leading log approximation are $\tan\beta$ and a universal SUSY mass scale M_{SUSY}. Sensible values for this scale lie anywhere between the mass of the lightest and the mass of the heaviest SUSY particle. The widths of the leading log graphs in *figures 6.1, 6.2 and 6.3* reflect this uncertainty. We see that, in the leading log approximation, the ratios $\hat{r}_i^{\mathrm{SUSY}}$ are proportional to $\log(\hat{s}/M_{\mathrm{SUSY}}^2)$. For $\hat{s} \gtrsim 10^9$ GeV the exact ratio runs linear with the same slope, but with a constant offset to the leading log graph. For very large centre of mass energies this offset becomes

Figure 6.2: SUSY corrections to the $u\bar{u} \to t\bar{t}$ cross section in the "exact" calculation and the leading log approximation for benchmark 5 of [45]. The results for first and second generation up-type quarks are identical. The width of the leading log graph reflects the uncertainty due to the choice of the universal SUSY scale.
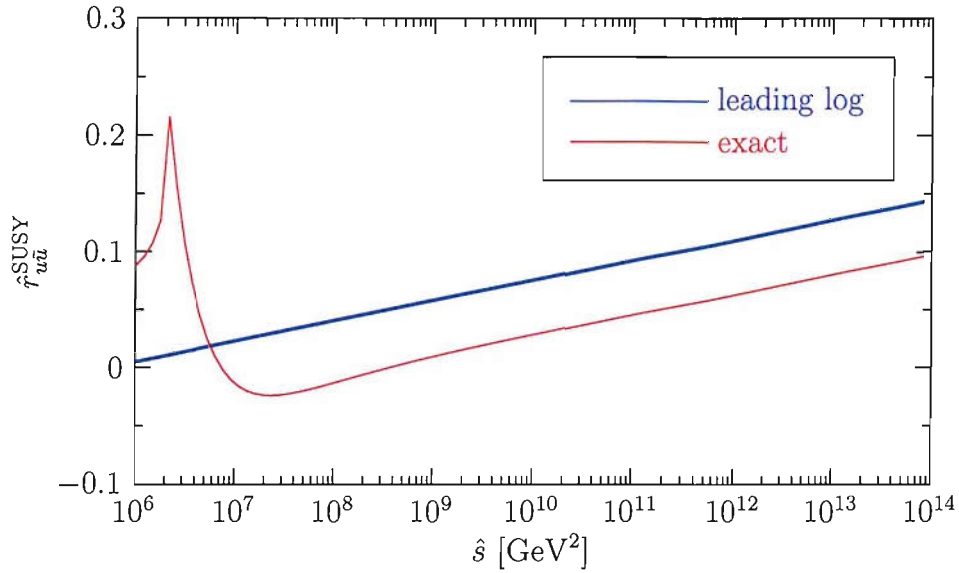

Figure 6.3: SUSY corrections to the $d\bar{d} \to t\bar{t}$ cross section in the "exact" calculation and the leading log approximation for benchmark 5 of [45]. The results for first and second generation down-type quarks are identical. The width of the leading log graph reflects the uncertainty due to the choice of the universal SUSY scale.
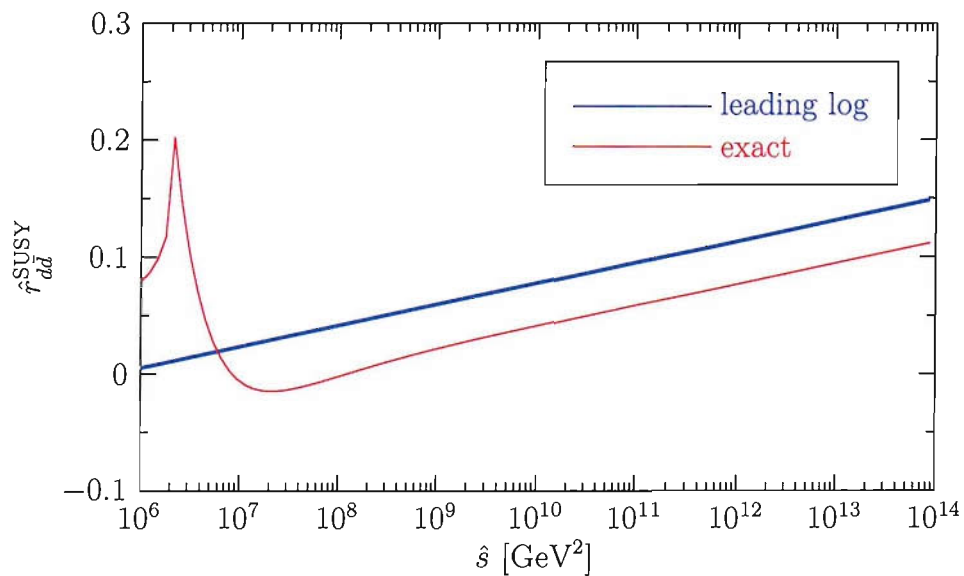
negligible. Therefore our results agree with the leading log approximation in the high energy limit. However, for collision energies that are achievable at the LHC we see that the leading log approximation fails.

To obtain the $pp \to t\bar{t}$ cross sections, the parton level cross sections $\hat{\sigma}_i$ were folded with the CTEQ6L1 set of the CTEQ v6.51 parton distribution functions [67]. The factorisation scale was set equal to the renormalisation scale. For the proton-proton collision we assumed a centre of mass energy of $14\,\mathrm{TeV}$. Let $d\sigma_{\lambda_1\lambda_2}/dM_{t\bar{t}}$ denote the invariant mass differential cross section for producing a top quark with helicity λ_1 and an anti-top quark with helicity λ_2. Then we define

$$\frac{d\sigma_{\mathrm{tot}}}{dM_{t\bar{t}}} = \frac{d\sigma_{++}}{dM_{t\bar{t}}} + \frac{d\sigma_{--}}{dM_{t\bar{t}}} + \frac{d\sigma_{+-}}{dM_{t\bar{t}}} + \frac{d\sigma_{-+}}{dM_{t\bar{t}}} \quad , \tag{6.2a}$$

$$\frac{d\sigma_{LL}}{dM_{t\bar{t}}} = \frac{d\sigma_{++}}{dM_{t\bar{t}}} + \frac{d\sigma_{--}}{dM_{t\bar{t}}} - \frac{d\sigma_{+-}}{dM_{t\bar{t}}} - \frac{d\sigma_{-+}}{dM_{t\bar{t}}} \quad , \tag{6.2b}$$

$$\frac{d\sigma_{PV}}{dM_{t\bar{t}}} = \frac{d\sigma_{+-}}{dM_{t\bar{t}}} - \frac{d\sigma_{-+}}{dM_{t\bar{t}}} \quad . \tag{6.2c}$$

For each combination we indicate the leading order and SUSY contributions by superscripts 'LO' and 'SUSY', respectively. The parity even combinations $d\sigma_{\mathrm{tot}}^{\mathrm{SUSY}}$ and $d\sigma_{LL}^{\mathrm{SUSY}}$ are dominated by the SQCD corrections. However, for the parity odd combination $d\sigma_{PV}^{\mathrm{SUSY}}$ the SQCD corrections are zero, since parity is conserved in super-QCD. For the asymmetries and the SUSY corrections we define the ratios

$$r_{LL/PV}^{\mathrm{LO}}(M_{t\bar{t}}) = \frac{d\sigma_{LL/PV}^{\mathrm{LO}}/dM_{t\bar{t}}}{d\sigma_{\mathrm{tot}}^{\mathrm{LO}}/dM_{t\bar{t}}} \quad , \qquad r_{\mathrm{tot}/LL/PV}^{\mathrm{SUSY}}(M_{t\bar{t}}) = \frac{d\sigma_{\mathrm{tot}/LL/PV}^{\mathrm{SUSY}}/dM_{t\bar{t}}}{d\sigma_{\mathrm{tot}}^{\mathrm{LO}}/dM_{t\bar{t}}} \quad . \tag{6.3}$$

Figures 6.4 and 6.5 show the results for $d\sigma_{\mathrm{tot}}^{\mathrm{LO}}/dM_{t\bar{t}}$ and $r_{LL}^{\mathrm{LO}}(M_{t\bar{t}})$, respectively, with a renormalisation scale $\mu = 464\,\mathrm{GeV}$ (corresponding to the geometric mean of the stop masses in benchmark 1a). Since there is no parity violation at leading order the ratio r_{PV}^{LO} is identically zero.

Figures 6.6 and 6.7 show the ratios $r_{\mathrm{tot}}^{\mathrm{SUSY}}(M_{t\bar{t}})$ and $r_{LL}^{\mathrm{SUSY}}(M_{t\bar{t}})$ for each of the 10 Snowmass benchmarks. The renormalisation and factorisation scales are set, for each benchmark, to the geometric mean of the stop masses. The numerical values are shown in table 6.1. We have added up the super-electroweak (SEW) and super-QCD (SQCD) corrections, but the SEW corrections are negligible compared to the SQCD corrections. We see that the SUSY corrections to the $t\bar{t}$ cross section can be as
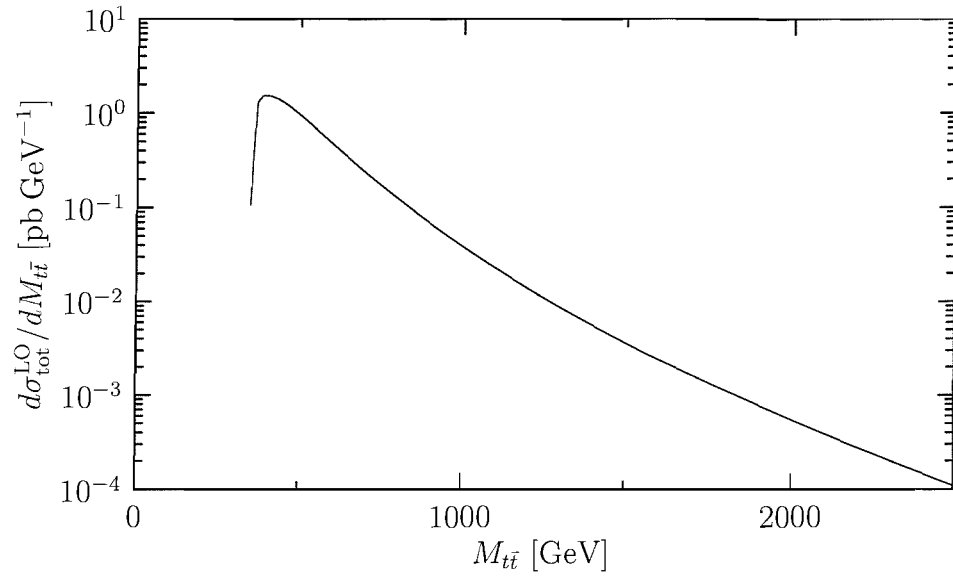
Figure 6.4: Leading order results for the invariant mass differential cross section. The renormalisation and factorisation scales are set to $464\,\text{GeV}$.
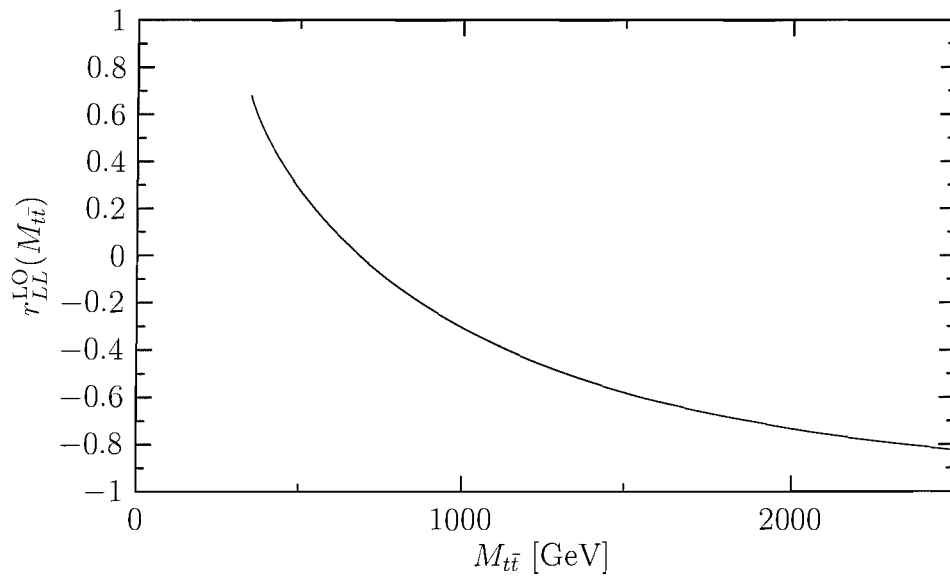
Figure 6.5: Leading order results for the invariant mass LL asymmetry. The renormalisation and factorisation scales are set to $464\,\text{GeV}$.

Figure 6.6: SUSY corrections to the invariant mass differential cross section for the Snowmass benchmarks. The numbers in the legend refer to the labelling of the benchmarks in [45].
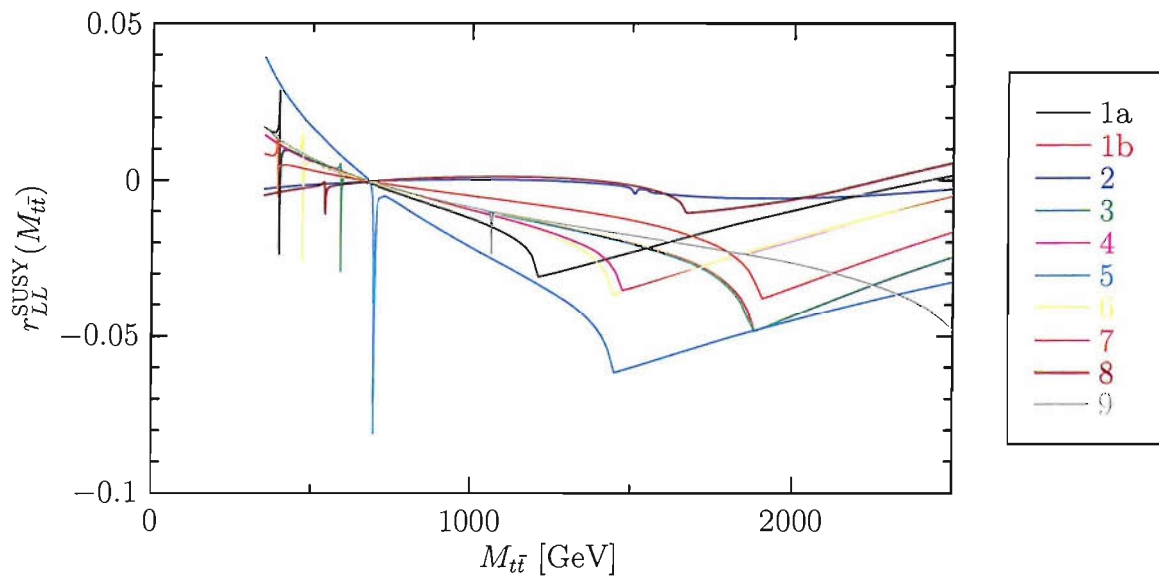
Figure 6.7: SUSY corrections to the invariant mass LL asymmetry for the Snowmass benchmarks. The numbers in the legend refer to the labelling of the benchmarks in [45].
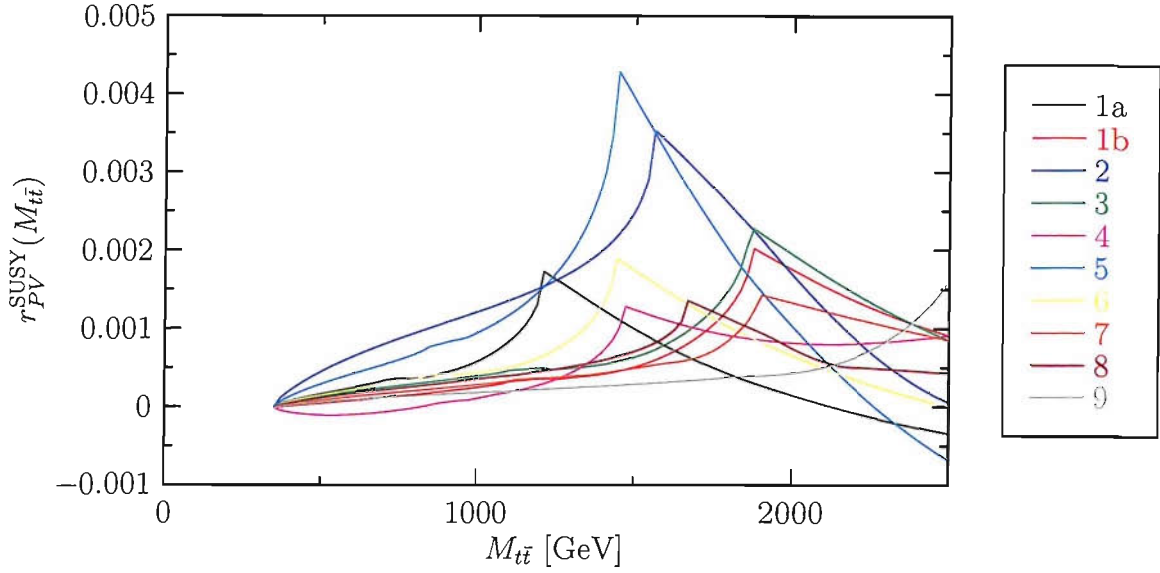
Figure 6.8: SUSY corrections to the invariant mass PV asymmetry for the Snowmass benchmarks. The numbers in the legend refer to the labelling of the benchmarks in [45].

large as 10% of the leading order cross section, but typically only reach the 5% level. In both plots we see "resonance peaks" located at the masses of the heavy and the pseudo-scalar Higgs (H_1^0 and A_1^0 in the notation of [59]). They come from the scalar s-channel propagators in the diagrams labelled Dgg_sS_vertFg, Dgg_sS_vertSg and Dgg_sS_vertSSg in *chapter 4*. Note that due to the fermion triangle the sign of the diagram Dgg_sS_vertFg is opposite that of Dgg_sS_vertSg and Dgg_sS_vertSSg. This explains why we get "troughs" instead of peaks for some of the benchmarks. Also note that, for all 10 benchmarks, the difference of the heavy and the pseudo-scalar Higgs masses is much smaller than their decay widths. Consequently we can only see two distinct peaks when these peaks have opposite signs. The kinks occurring between 1 and 2 TeV coincide, for each benchmark, with twice the gluino mass and can therefore be understood as a threshold effect due to the box diagrams Dqqbar_fboxSS and Dgg_boxFS.
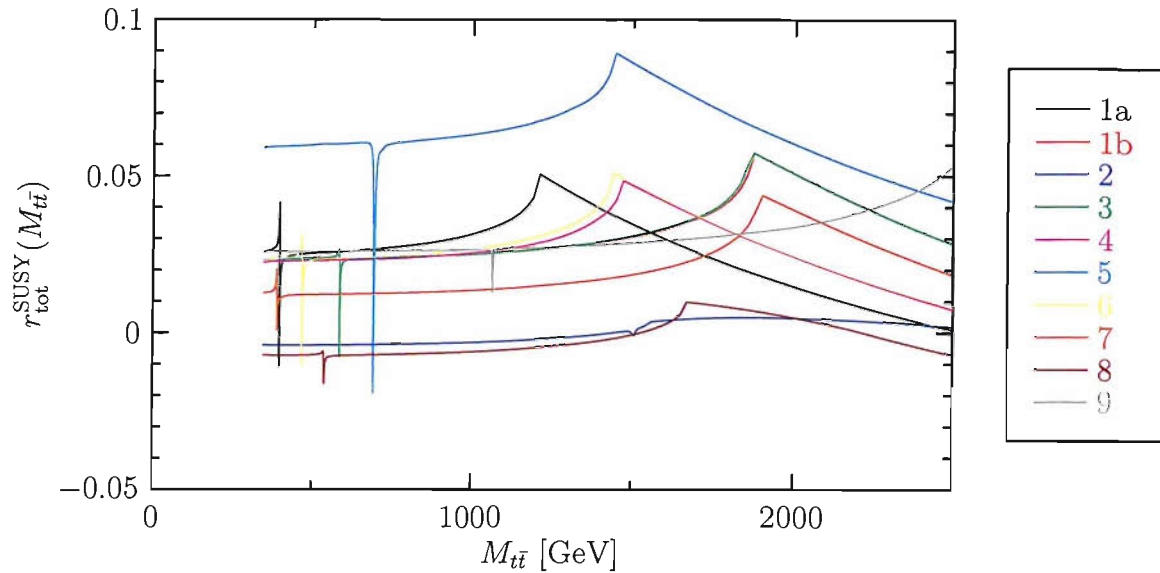
Figure 6.8 shows the SUSY corrections to the parity violating asymmetry for each of the 10 benchmarks. Here the resonance peaks are absent, because the diagrams Dgg_sS_vertFg, Dgg_sS_vertSg and Dgg_sS_vertSSg are parity-conserving. Furthermore, the SUSY corrections to the parity violating asymmetry are one order of magnitude smaller than the corrections to the parity-even observables, because it only gets super-electroweak contributions of order $\mathcal{O}(\alpha\alpha_s^2)$.

	μ	$\sigma_{\text{tot}}^{\text{LO}}$	$\sigma_{\text{tot}}^{\text{SUSY}}$	σ_{LL}^{LO}	$\sigma_{LL}^{\text{SUSY}}$	$\sigma_{PV}^{\text{SUSY}}$
1a	465	331.8	+8.5	91.99	+1.83	+0.063
1b	719	288.3	+6.9	79.50	+1.74	+0.020
2	1100	266.3	−1.0	72.84	−0.33	+0.125
3	713	289.5	+6.9	80.08	+1.75	+0.043
4	595	307.6	+7.2	85.62	+1.68	−0.018
5	402	332.1	+19.9	93.48	+5.19	+0.111
6	559	310.8	+7.4	85.74	+1.73	+0.053
7	820	284.9	+3.5	78.04	+0.80	+0.029
8	1013	271.2	−1.9	74.08	−0.62	+0.035
9	992	263.6	+6.9	72.32	+1.83	+0.019

Table 6.1: Numerical results for the integrated $t\bar{t}$ cross section and asymmetries. The numbers in the left column refer to the labelling of the Snowmass benchmarks in [45]. The renormalisation scale μ in the second column is given in GeV. The superscripts 'LO' and 'SUSY' indicate leading order results and SUSY corrections, respectively. The cross sections are given in pico-barns (pb).

By integrating the differential cross sections (6.2) over $M_{t\bar{t}}$ we obtain cross sections for producing $t\bar{t}$ pairs with arbitrary invariant mass. We define the cross sections σ_{tot}, σ_{LL} and σ_{PV} by

$$\sigma_{\text{tot}/LL/PV} = \int_{2m_t}^{M_{pp}} dM_{t\bar{t}} \frac{d\sigma_{\text{tot}/LL/PV}}{dM_{t\bar{t}}} \quad , \tag{6.4}$$

where m_t is the top mass and $M_{pp} = 14\,\text{TeV}$ is the invariant mass of the proton-proton system. Again, the leading order and SUSY contributions are indicated by superscripts 'LO' and 'SUSY', respectively. *Table 6.1* summarises our results for these cross sections. For both, σ_{tot} and σ_{LL} we see that the SUSY corrections typically make up 2% of the leading order results. However, they can be as big as 5% in the case of benchmark 5 and as small as 0.3% in the case of benchmark 2.

Experimentally it is often more convenient to parametrise the $t\bar{t}$ production cross section by the transverse momentum p_T of the top quark. For the transverse momentum differential cross section $d\sigma_{\lambda_1\lambda_2}/dp_T$ we define the total differential cross section $d\sigma_{\text{tot}}/dp_T$, the asymmetries $d\sigma_{LL}/dp_T$ and $d\sigma_{PV}/dp_T$ and the ratios $r_{LL}^{\text{LO}}(p_T)$, $r_{PV}^{\text{LO}}(p_T)$, $r_{\text{tot}}^{\text{SUSY}}(p_T)$, $r_{LL}^{\text{SUSY}}(p_T)$ and $r_{PV}^{\text{SUSY}}(p_T)$ in analogy to (6.2) and (6.3). Our results for these quantities are shown in *figures 6.9 to 6.13*. We note here that the "resonance peaks" and "troughs" from the thresholds for scalar particle exchange are smoothed out by the phase-space integration which means that $M_{t\bar{t}}$ is a far better variable in which to analyse the data in order to extract information on the SUSY parameter set, although
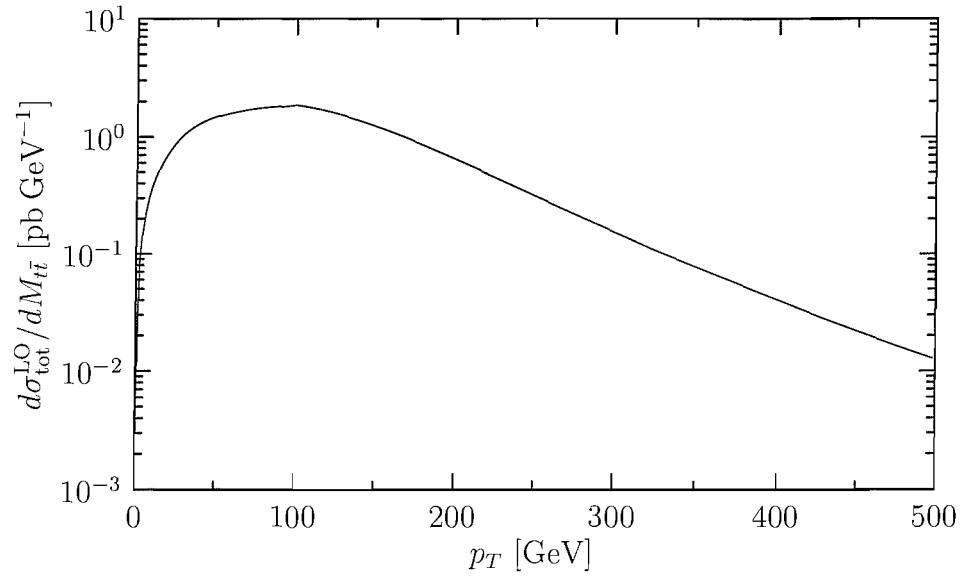
Figure 6.9: Leading order results for the transverse momentum differential cross section.

we note that some of the benchmarks give rise to an enhancement of the differential cross-section of up to 7% at large p_T.
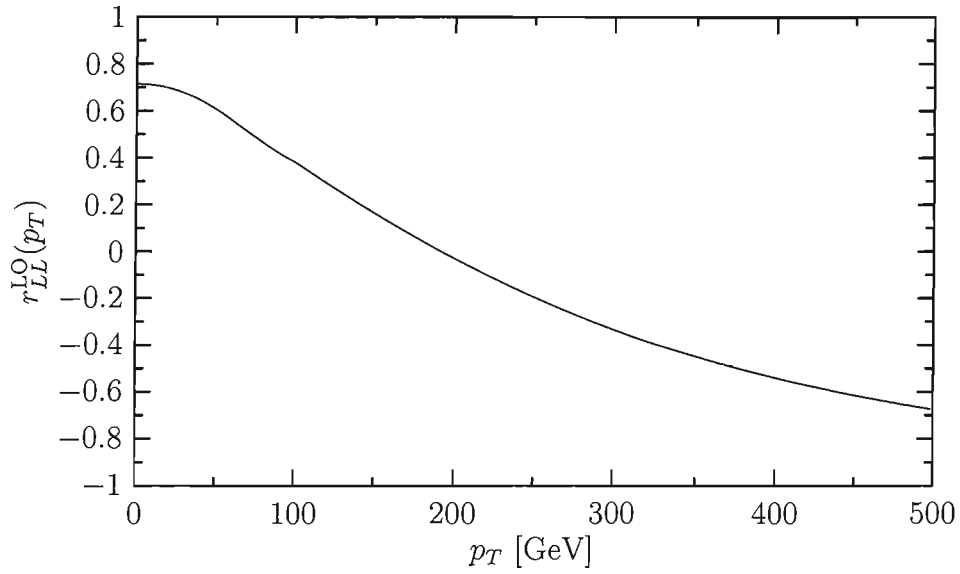
Figure 6.10: Leading order results for the transverse momentum LL asymmetry.
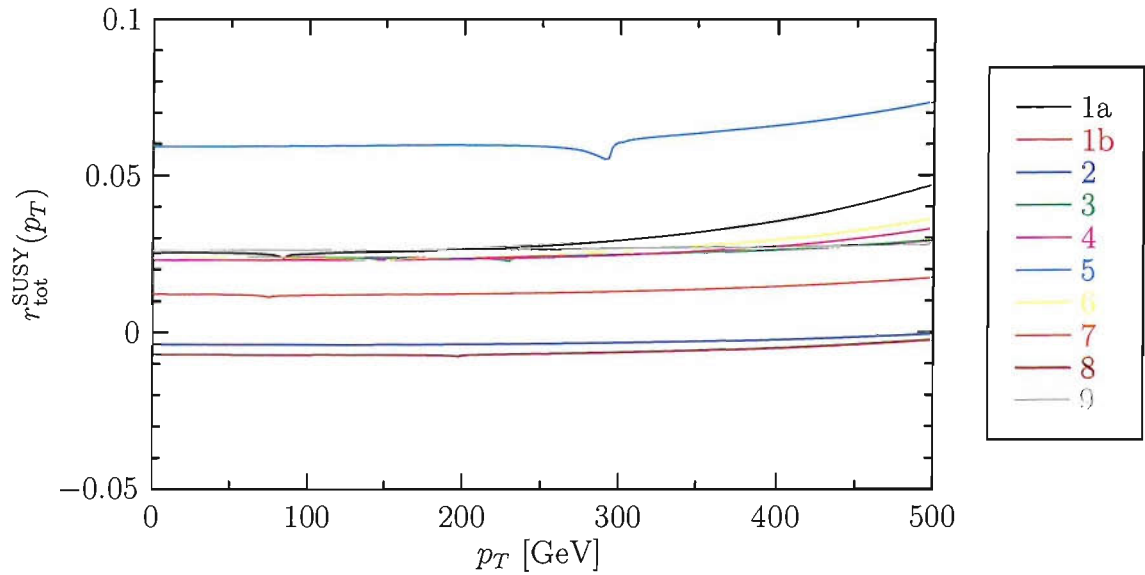
Figure 6.11: SUSY corrections to the transverse momentum differential cross section for the Snowmass benchmarks. The numbers in the legend refer to the labelling of the benchmarks in [45].
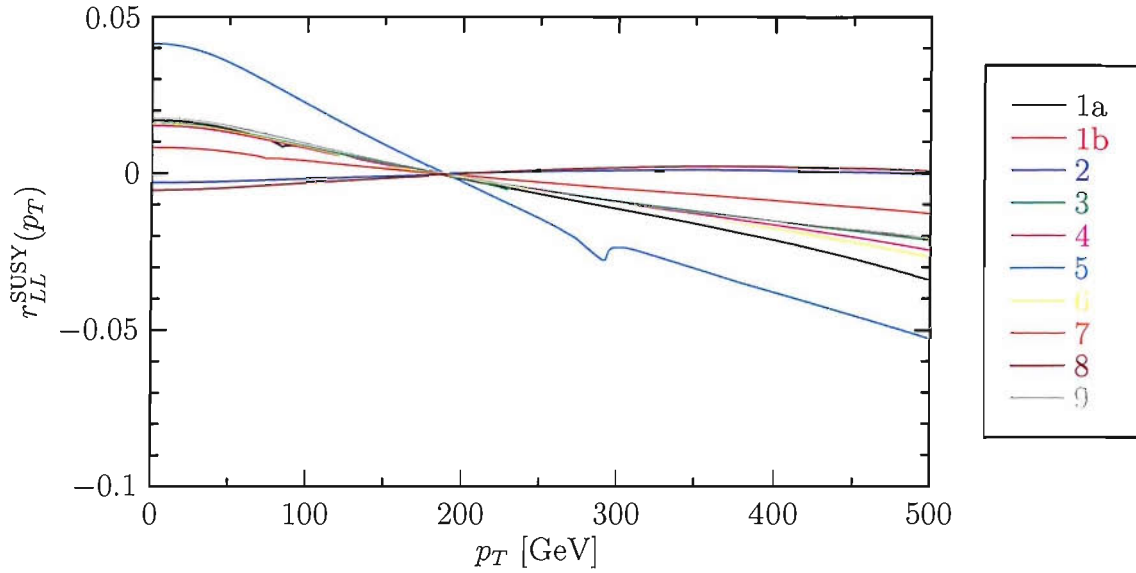
Figure 6.12: SUSY corrections to the transverse momentum LL asymmetry for the Snowmass benchmarks. The numbers in the legend refer to the labelling of the benchmarks in [45].
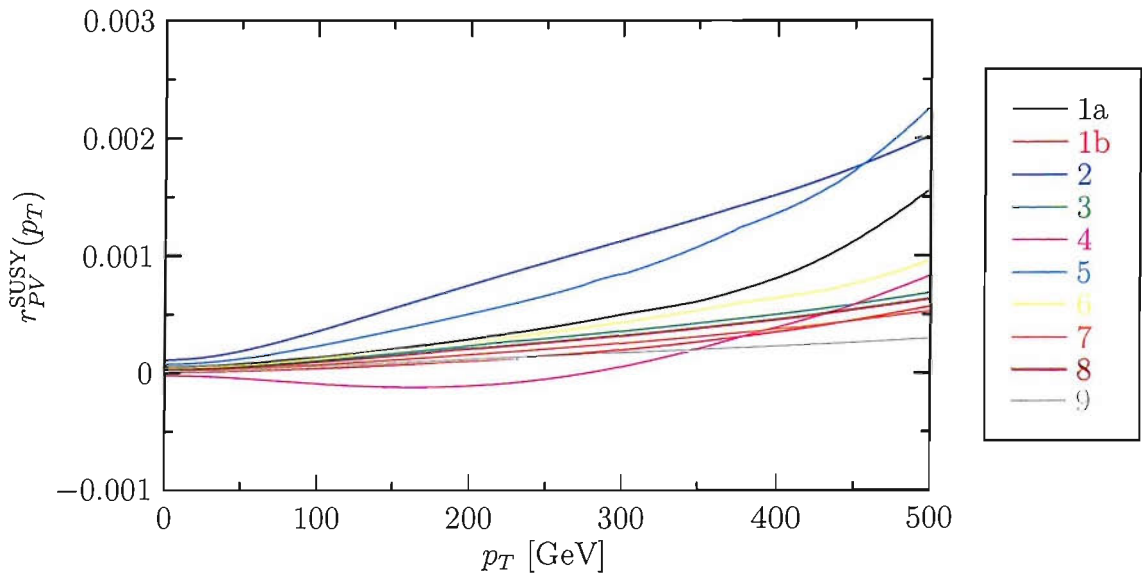
Figure 6.13: SUSY corrections to the transverse momentum PV asymmetry for the Snowmass benchmarks. The numbers in the legend refer to the labelling of the benchmarks in [45].

6.2 Conclusions

We have calculated the complete MSSM corrections to the cross-section for $t\bar{t}$ production at the LHC. The calculation has been set up in terms of prototype Feynman graphs for the polarised amplitudes at parton-level. These prototypes are independent of the underlying model and can be re-used for studying the effects of other BSM physics on top-quark production. In a second step we used these prototypes to construct a numerical library that computes the MSSM corrections for arbitrary values of the full set of MSSM parameters.

As a first analysis we have calculated numerical results for the ten Snowmass benchmark sets using the CTEQ PDFs. We find a considerable variation of the effects of the one-loop SUSY corrections between the various benchmarks. The benchmark giving the largest correction is benchmark 5, which is a super-gravity model with small $\tan\beta = 5$ and a large negative tri-linear coupling, $A_0 = -1000$ GeV. These large corrections can be understood from the fact that this large tri-linear coupling generates a light stop mass (258 GeV) thereby enhancing graphs involving a stop mass inside the loop. This gives an enhancement of 6% in the total production cross-section.

Whereas the corrections for the other benchmarks are somewhat smaller, they are usually around 3% and therefore comparable to the weak corrections calculated by Bernreuther et. al. [31] and Kühn et. al. [20]. Note that whereas the weak corrections reported in [20,31] *decrease* the prediction for the cross-sections, the SUSY corrections have a *positive* sign for most of the benchmarks considered.

Statistically, we expect these events to be easily detectable given the anticipated yield of order 10^7 events over the period of running of the LHC. We have found similar corrections in the asymmetry ratios defined in in (6.2) and (6.3). For these asymmetries we also expect cancellation of systematic errors arising from uncertainties in incoming parton fluxes and tagging efficiencies, so that these corrections of order 3% would exceed the statistical errors by a factor of ~ 100.

Given corrections of such significance, it is reasonable to assume that corrections in the differential cross-sections will also be detectable (provided sufficiently large bins are taken). We have therefore plotted the differential cross-sections with respect to the invariant mass $M_{t\bar{t}}$ of the $t\bar{t}$ system and also with respect to the transverse

momentum p_T of the t-quark. In the former case, the differential cross-sections display an interesting structure with peaks and/or troughs corresponding to thresholds for scalar particle exchanges in the gluon fusion process.

We have also determined the SUSY contribution to the parity odd helicity asymmetry. This receives only contributions from the supersymmetric partners in the weak-interaction sector, which are suppressed relative to the SQCD corrections by $\mathcal{O}(\alpha_W/\alpha_s)$. It would appear, therefore that even for benchmark 5, which produces the largest corrections, such parity violating asymmetries will be too small to observe.

Nevertheless, the rewarding result of our studies is that the existence of supersymmetric particles can have a measurable effect on the production cross sections for top-antitop pairs. Future investigations in this area should address the following issues:

For a successful comparison with experimental data it is important to estimate and, if possible, reduce the theoretical systematic errors of this calculation. The two main sources of errors are uncertainties in the parton distribution functions and higher order corrections to the parton level $t\bar{t}$ production process. A method for estimating the PDF related errors of any observable based on the CTEQ6 PDFs is suggested in [68]. It essentially consists in calculating an observable not only with the "best fit" PDFs, but also for a set of 40 *eigenvector PDFs* which span a 90% confidence range around the central fit. To estimate the error due to higher order corrections it is conventional to vary the renormalisation scale μ between one-half and twice its "ideal" value. The stability of the perturbative results can be enhanced by resummation techniques like soft gluon resummation. For unpolarised amplitudes such calculations have been done in [21–25], but, as far as I know, no such calculations exist for polarised amplitudes.

Another essential step towards a phenomenologically relevant prediction is a proper inclusion of the top decays into our analysis. Remember that the quantities (6.2) are not actually measurable, because the spin of the top quarks cannot be observed directly. In *section 5.3* we demonstrated how to define proper observables which are sensitive to the spins of the top quarks. Assuming that the supersymmetric corrections to these observables are comparable to those for the "pseudo-observables" (6.3) a proper inclusion of supersymmetric one-loop effects into the decay matrix elements might be necessary for consistency.

Finally the dependence of $t\bar{t}$ production cross sections and asymmetries on MSSM parameters has to be explored. To use experimental results for top production cross sections to constrain MSSM parameters, the relevant directions in the MSSM parameter space have to be determined. Since the electroweak SUSY corrections are too small to be measurable, the majority of the MSSM parameters become irrelevant, so that an analysis based only on low energy parameters actually seems feasible.

Appendix A

Groups and Representations

In this appendix we review the main features of the Lorentz and Poincaré group and their corresponding Lie algebras. We also discuss finite dimensional representations of the Lorentz group, in particular left and right-handed Weyl representations, the Dirac spinor representation and the Lorentz vector representation.

A.1   Lorentz and Poincaré Transformations

The Lorentz group L is the group of linear transformations on $\mathbb{R}^4$ that leave the Minkowski product

$$xy = g_{\mu\nu}x^\mu y^\nu = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3 \tag{A.1}$$

invariant. It is a Lie group of dimension 6. The transformations are described by the *Lorentz matrices* matrices $\Lambda = (\Lambda^\mu{}_\nu,\ \mu,\nu = 0\ldots 3)$, which satisfy

$$g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = g_{\rho\sigma}\quad . \tag{A.2}$$

If we write $\Lambda^\mu{}_\nu = g^\mu{}_\nu + \omega^\mu{}_\nu$ it follows from (A.2) that $\omega_{\mu\nu}\ (= g_{\mu\rho}\omega^\rho{}_\nu)$ is anti-symmetric. Therefore the Lorentz group can be locally parametrised by the six independent components of $\omega_{\mu\nu}$.

The Lorentz group L splits up into four disconnected parts $L^\uparrow_+$, $L^\uparrow_-$, $L^\downarrow_+$ and $L^\downarrow_-$. We define $L^\uparrow_+$ as the subset of Lorentz matrices Λ with $\det \Lambda = +1$ and $\Lambda^0{}_0 \geq 1$.

It is an invariant subgroup of L and we call it the group of *proper* Lorentz transformations. The other subsets can be obtained from it by parity transformations $\mathcal{P} = \text{diag}(+1, -1, -1, -1) \in L_-^{\uparrow}$ and time reflections $\mathcal{T} = \text{diag}(-1, +1, +1, +1) \in L_+^{\downarrow}$. In fact, they can be written as

$$L_-^{\uparrow} = \mathcal{P} L_+^{\uparrow} \quad , \quad L_+^{\downarrow} = \mathcal{T} L_+^{\uparrow} \quad , \quad L_-^{\downarrow} = \mathcal{P} \mathcal{T} L_+^{\uparrow} \quad . \tag{A.3}$$

The *Poincaré group* P is obtained by adding space and time translations to the Lorentz group. It can be defined as the set of *affine* Lorentz transformations of Minkowski space:

$$x^{\mu} \xrightarrow{(\Lambda, a)} [(\Lambda, a)x]^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu} \tag{A.4}$$

for $x \in \mathbb{R}^4$. Here we denote the elements of P as pairs (Λ, a), where $a \in \mathbb{R}^4$ and $\Lambda \in L$. From (A.8) we see that the product of two elements of the Poincaré group is given by:

$$(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda_2 \Lambda_1, \Lambda_2 a_1 + a_2) \quad , \tag{A.5}$$

where $(\Lambda_2 \Lambda_1)^{\mu}{}_{\nu} \equiv (\Lambda_2)^{\mu}{}_{\rho} (\Lambda_1)^{\rho}{}_{\nu}$ and $(\Lambda_2 a_1)^{\mu} \equiv (\Lambda_2)^{\mu}{}_{\rho} a^{\rho}$. We also define the disconnected subsets $P_+^{\uparrow}$, $P_-^{\uparrow}$, $P_+^{\downarrow}$ and $P_-^{\downarrow}$ of P in the obvious way.

Now consider a representation of the Poincaré group, i.e. a mapping D from P to a space $\text{Hom}\,\mathcal{V}$ of linear operators on some (finite- or infinite-dimensional) vector space $\mathcal{V}$ that satisfies

$$D\big((\Lambda_2, a_2)(\Lambda_1, a_1)\big) = D(\Lambda_2, a_2) D(\Lambda_1, a_1) \quad . \tag{A.6}$$

Representations of the elements of $P_+^{\uparrow}$ can be expressed in terms of *generators*. Let $(\Lambda, a) \equiv (\mathbb{1} + \omega, a) \in P_+^{\uparrow}$. For a given representation D we define the generators

$$D(P^{\mu}) = i \left[\frac{\partial}{\partial a_{\mu}} D(\mathbb{1} + \omega, a) \right]_{\omega = 0,\ a = 0} \quad , \tag{A.7a}$$

$$D(M^{\mu\nu}) = i \left[\frac{\partial}{\partial \omega_{\mu\nu}} D(\mathbb{1} + \omega, a) \right]_{\omega = 0,\ a = 0} \quad . \tag{A.7b}$$

In fact, the generators can be defined in a representation-independent way as elements of the tangent space of the group manifold. A linear representation D of the group on some vectors space then induces a linear representation of its generators on the same vector space. We use the symbol D to denote the group representation as well

as the induced representation of its generators. However, if it is clear or irrelevant which representation is meant we will drop the D, i.e. write $M^{\mu\nu}$ and P^{μ} instead of $D(M^{\mu\nu})$ and $D(P^{\mu})$. From the antisymmetry of $\omega_{\mu\nu}$ it follows that $M^{\mu\nu} = -M^{\nu\mu}$. As every element of $P_{+}^{\uparrow}$ can be approximated by a product of *infinitesimal* group transformations, we can write

$$D(\mathbb{1} + \omega, a) = \exp\left(-\tfrac{i}{2}\omega_{\mu\nu}M^{\mu\nu} - ia_{\mu}P^{\mu}\right) \quad . \tag{A.8}$$

Using the multiplication law (A.5) and the composition rule (A.6) we find

$$D(\Lambda, a)D(\mathbb{1}, \varepsilon)D^{-1}(\Lambda, a) = D(\mathbb{1}, \Lambda\varepsilon) \quad , \tag{A.9a}$$

$$D(\Lambda, a)D(\mathbb{1} + \omega, 0)D^{-1}(\Lambda, a) = D(\Lambda(\mathbb{1} + \omega)\Lambda^{-1}, -\Lambda(\mathbb{1} + \omega)\Lambda^{-1} + a) \quad . \tag{A.9b}$$

Expanding both sides to first order in ε and ω we obtain the following transformation laws for the generators:

$$D(\Lambda, a)P^{\mu}D^{-1}(\Lambda, a) = (\Lambda^{-1})^{\mu}_{\ \nu}P^{\nu} \quad , \tag{A.10a}$$

$$D(\Lambda, a)M^{\mu\nu}D^{-1}(\Lambda, a) = (\Lambda^{-1})^{\mu}_{\ \rho}(\Lambda^{-1})^{\nu}_{\ \sigma}(M^{\rho\sigma} + a^{\rho}P^{\sigma} - a^{\sigma}P^{\rho}) \quad . \tag{A.10b}$$

Expanding these equations for an infinitesimal transformation (Λ, a) then yields the following *commutation relations* for the generators P^{μ} and $M^{\mu\nu}$:

$$[P^{\mu}, P^{\nu}] = 0 \quad , \tag{A.11a}$$

$$[M^{\mu\nu}, P^{\rho}] = i(P^{\mu}g^{\nu\rho} - P^{\nu}g^{\mu\rho}) \quad , \tag{A.11b}$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i(g^{\mu\rho}M^{\nu\sigma} + g^{\nu\sigma}M^{\mu\rho} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma}) \quad . \tag{A.11c}$$

The linear space spanned by the generators, together with the commutation bracket, forms a representation of the *Lie algebra* $\mathcal{L}(P_{+}^{\uparrow})$ of the group of proper Poincaré transformations.

The Lie algebra of the Lorentz group L is spanned by the generators $M^{\mu\nu}$. Their commutation relations are given in (A.11c). It can be shown that these commutation relations are automatically satisfied if we construct operators γ^{μ} that satisfy the *Clifford algebra*

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}\mathbb{1} \tag{A.12}$$

and define

$$M^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \quad . \tag{A.13}$$

From the equations (A.12) and (A.13) it also follows that

$$[M^{\mu\nu}, \gamma^\rho] = i(\gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\mu\rho}) \quad . \tag{A.14}$$

This is the same relation as (A.11b). It completely determines the transformation behaviour of the γ^μ under Lorentz transformations Λ, namely

$$D(\Lambda)\gamma^\mu D^{-1}(\Lambda) = (\Lambda^{-1})^\mu{}_\nu \gamma^\nu \quad . \tag{A.15}$$

To write the commutation relations of the $M^{\mu\nu}$ in a simpler form we define

$$K^i = M^{0i} \quad , \qquad J^i = \frac{1}{2}\varepsilon^{ijk}M^{jk} \quad . \tag{A.16}$$

From (A.11c) we then obtain the following relations for $\mathbf{K}$ and $\mathbf{J}$:

$$[K^i, K^j] = -i\varepsilon^{ijk}J^k \quad , \quad [J^i, J^j] = i\varepsilon^{ijk}J^k \quad , \quad [J^i, K^j] = i\varepsilon^{ijk}K^k \quad . \tag{A.17}$$

The transformations generated by the K are called *boosts* and those generated by the J are spatial rotations. If we substitute $a^\mu = 0$ and

$$\eta^i = \omega_{0i} \quad , \qquad \theta^i = \frac{1}{2}\varepsilon^{ijk}\omega_{jk} \tag{A.18}$$

in (A.8) we obtain the following general form of a proper Lorentz transformation Λ in terms of the generators $\mathbf{K}$ and $\mathbf{J}$:

$$D(\Lambda(\boldsymbol{\eta}, \boldsymbol{\theta})) = \exp\left(-i\boldsymbol{\eta} \cdot \mathbf{K} - i\boldsymbol{\theta} \cdot \mathbf{J}\right) \quad . \tag{A.19}$$

For convenience we also define the following shorthands:

$$D(B(\boldsymbol{\eta})) = D(\Lambda(\boldsymbol{\eta}, \mathbf{0})) = \exp(-i\boldsymbol{\eta} \cdot \mathbf{K}) \quad , \tag{A.20a}$$

$$D(R(\boldsymbol{\theta})) = D(\Lambda(\mathbf{0}, \boldsymbol{\theta})) = \exp(-i\boldsymbol{\theta} \cdot \mathbf{J}) \quad . \tag{A.20b}$$

The boost parameter $\boldsymbol{\eta}$ is called *rapidity*. The transformation $R(\boldsymbol{\theta})$ describes a rotation by the angle $|\boldsymbol{\theta}|$ around the axis $\hat{\boldsymbol{\theta}}$ in counter-clockwise direction. Therefore $\boldsymbol{\theta}$ is

called the *rotation vector*. Boosts along and rotations around the i-axis ($i = 1, 2, 3$) will be denoted by $B_i(\eta)$ and $R_i(\theta)$, respectively.

We can further simplify the commutation relations (A.17) by choosing a different basis of the space of group generators. We define

$$N_\pm^i = \frac{1}{2}(J^i \pm iK^i) \quad . \tag{A.21}$$

In terms of these generators we have

$$D(\Lambda(\boldsymbol{\eta}, \boldsymbol{\theta})) = \exp\left(-i(\boldsymbol{\theta} - i\boldsymbol{\eta}) \cdot \mathbf{N}_+ - i(\boldsymbol{\theta} + i\boldsymbol{\eta}) \cdot \mathbf{N}_-\right) \quad . \tag{A.22}$$

The commutation relations for the $N_\pm^i$ read

$$[N_+^i, N_+^j] = i\varepsilon^{ijk} N_+^k \quad , \tag{A.23a}$$

$$[N_-^i, N_-^j] = i\varepsilon^{ijk} N_-^k \quad , \tag{A.23b}$$

$$[N_+^i, N_-^j] = 0 \quad . \tag{A.23c}$$

This is just the Lie algebra of an $SU(2) \times SU(2)$ group. Its irreducible representations are fully specified by the values of the two Casimir operators $\mathbf{N}_+^2 = N_+^i N_+^i$ and $\mathbf{N}_-^2 = N_-^i N_-^i$. Their possible values are $j(j + 1)$, where j is a positive half-integer number. We therefore denote a representation of L corresponding to $\mathbf{N}_+^2 = j_+(j_+ + 1)$ and $\mathbf{N}_-^2 = j_-(j_- + 1)$ by $D^{(j_+, j_-)}$.

A.2 Lorentz Vectors

At the beginning of this section we defined the elements Λ of the Lorentz group as real 4×4 matrices satisfying (A.2). This is the *defining representation* of the Lorentz group. It is equivalent to the $(\frac{1}{2}, \frac{1}{2})$ representation. That means that it is possible to find an invertible real 4×4 matrix A so that

$$AD^{(\frac{1}{2}, \frac{1}{2})}(\Lambda)A^{-1} = \Lambda \tag{A.24}$$

for every Lorentz matrix Λ. The generators in this representation are

$$(M^{\mu\nu})^\rho{}_\sigma = i(g^{\mu\rho}g^\nu{}_\sigma - g^\mu{}_\sigma g^{\nu\rho}) \quad . \tag{A.25}$$

Objects that transform according to this representation are called *Lorentz vectors*. We will now work out the explicit form of the Lorentz matrices for boosts along and rotations around the z-axis.

For boosts along the z-axis we have

$$B_3(\eta) = \exp(-i\eta K^3) = \exp(-i\eta M^{03}) \quad . \tag{A.26}$$

Using

$$-iM^{03} = \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix} \quad , \quad (-iM^{03})^2 = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \quad , \tag{A.27}$$

$$(-iM^{03})^{2n} = (-iM^{03})^2 \quad , \quad (-iM^{03})^{2n-1} = -iM^{03} \quad \forall n \geq 1$$

we find

$$\begin{aligned} B_3(\eta) &= \mathbb{1} + \sum_{n=1}^{\infty} \frac{1}{n!}(-i\eta M^{03})^n \\ &= \mathbb{1} + \sum_{n=1}^{\infty} \frac{1}{(2n)!}\eta^{2n}(-iM^{03})^2 + \sum_{n=1}^{\infty} \frac{1}{(2n-1)!}\eta^{2n-1}(-iM^{03}) \\ &= \begin{pmatrix} \cosh\eta & \cdot & \cdot & \sinh\eta \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \sinh\eta & \cdot & \cdot & \cosh\eta \end{pmatrix} \quad . \end{aligned} \tag{A.28}$$

Later on we will have to construct boosts that take the momentum $\mathring{p} = (m, 0, 0, 0)$ of a particle at rest to $p = (E, 0, 0, p^3)$, i.e. satisfy

$$\begin{pmatrix} \cosh\eta & \cdot & \cdot & \sinh\eta \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \sinh\eta & \cdot & \cdot & \cosh\eta \end{pmatrix} \begin{pmatrix} m \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} E \\ \cdot \\ \cdot \\ p^3 \end{pmatrix} \quad . \tag{A.29}$$

We see immediately that this is true if

$$\cosh \eta = \frac{E}{m} \quad , \quad \sinh \eta = \frac{p^3}{m} \quad . \tag{A.30}$$

For rotations around the z-axis we have

$$R_3(\theta) = \exp(-i\theta J^3) = \exp(-i\theta M^{12}) \quad . \tag{A.31}$$

Using

$$-iM^{12} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad , \quad (-iM^{12})^2 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad ,$$

$$(-iM^{12})^{2n} = (-1)^{n-1}(-iM^{12})^2 \quad , \quad (-iM^{12})^{2n-1} = (-1)^{n-1}(-iM^{12}) \quad \forall n \geq 1 \tag{A.32}$$

we find

$$R_3(\theta) = \mathbb{1} + \sum_{n=1}^{\infty} \frac{1}{(2n)!}(-1)^{n-1}\theta^{2n}(-iM^{12})^2$$

$$+ \sum_{n=1}^{\infty} \frac{1}{(2n-1)!}(-1)^{n-1}\theta^{2n-1}(-iM^{12})$$

$$= \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cos\theta & -\sin\theta & \cdot \\ \cdot & \sin\theta & \cos\theta & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \quad . \tag{A.33}$$

We see that this matrix rotates the spatial components of a four vector by an angle θ around the z-axis in the counter-clockwise direction.

A.3 Weyl Spinors

Objects that transform according to the $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$ representation of the Lorentz group are called *left-* or *right-handed Weyl spinors*, respectively. In other words left-handed Weyl spinors transform as singlets under the $SU(2)_-$ subgroup generated by the N_-^i and according to a $\frac{1}{2}$ representation under the $SU(2)_+$ subgroup generated by the N_+^i. Right-handed Weyl spinors transform as singlets under the $SU(2)_+$ and according to a $\frac{1}{2}$ representation under the $SU(2)_-$.

In a $\frac{1}{2}$ representation of $SU(2)$ the generators are $\sigma^i/2$, where the σ^i are the *Pauli matrices*

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} \quad , \quad \sigma^3 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \quad . \tag{A.34}$$

They satisfy the following commutation and anti-commutation relations:

$$[\sigma^i, \sigma^j] = 2i\varepsilon^{ijk}\sigma^k \quad , \quad \{\sigma^i, \sigma^j\} = 2\delta^{ij}\mathbb{1} \quad . \tag{A.35}$$

Therefore the generators of a left-handed Lorentz group representation $(\frac{1}{2}, 0)$ are

$$N_+^i = \frac{\sigma^i}{2} \quad , \quad N_-^i = 0 \quad . \tag{A.36}$$

In the right-handed representation $(0, \frac{1}{2})$ the generators are

$$N_+^i = 0 \quad , \quad N_-^i = \frac{\sigma^i}{2} \quad . \tag{A.37}$$

Consequently, using (A.22), the left and right handed representations of a Lorentz transformation $\Lambda(\boldsymbol{\eta}, \boldsymbol{\theta})$ can be written as

$$D^{(\frac{1}{2}, 0)}(\Lambda(\boldsymbol{\eta}, \boldsymbol{\theta})) = \exp\left(-i(\boldsymbol{\theta} - i\boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}\right) \tag{A.38a}$$

$$D^{(0, \frac{1}{2})}(\Lambda(\boldsymbol{\eta}, \boldsymbol{\theta})) = \exp\left(-i(\boldsymbol{\theta} + i\boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}\right) \quad . \tag{A.38b}$$

For later use we will now work out the explicit form of the representation matrices for rotations around the y-axis in counter-clockwise direction. First note that the generators of rotations are $\boldsymbol{\sigma}/2$ in both the left and the right-handed representation.

Therefore the representation matrices for pure rotations have the same form in both representations. According to (A.38) we get

$$D(R_2(\theta)) = \exp\left(-i\frac{\theta}{2}\sigma^2\right) \quad , \tag{A.39}$$

where D can be both, $D^{(\frac{1}{2},0)}$ or $D^{(0,\frac{1}{2})}$. Using

$$-i\sigma^2 = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \quad , \quad (-i\sigma^2)^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \tag{A.40}$$

$$(-i\sigma^2)^{2n} = (-1)^{n-1}(-i\sigma^2)^2 \quad , \quad (-i\sigma^2)^{2n-1} = (-1)^{n-1}(-i\sigma^2) \quad \forall n \geq 1$$

we obtain

$$\begin{aligned}
D(R_2(\theta)) &= \mathbb{1} + \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{\theta}{2}\right)^n (-i\sigma^2)^n \\
&= \mathbb{1} + \sum_{n=1}^{\infty} \frac{1}{(2n)!}\left(\frac{\theta}{2}\right)^{2n} (-1)^{n-1}(-i\sigma^2)^2 \\
&\quad + \sum_{n=1}^{\infty} \frac{1}{(2n-1)!}\left(\frac{\theta}{2}\right)^{2n-1} (-1)^{n-1}(-i\sigma^2) \\
&= \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \quad . \tag{A.41}
\end{aligned}$$

A.4 Dirac Spinors

Objects that transform according to the direct sum of a left-handed and a right-handed representation are called *Dirac spinors*. On the space of Dirac spinors a Lorentz transformation Λ is represented by the matrix

$$S(\Lambda) = D^{(\frac{1}{2},0)}(\Lambda) \oplus D^{(0,\frac{1}{2})}(\Lambda) = \begin{pmatrix} D^{(\frac{1}{2},0)}(\Lambda) & 0 \\ 0 & D^{(0,\frac{1}{2})}(\Lambda) \end{pmatrix} \tag{A.42}$$

From (A.38) and (A.22) we see immediately that the generators in this representation
are

$$N_+^i = \frac{1}{2} \begin{pmatrix} \sigma^i & \cdot \\ \cdot & 0 \end{pmatrix} \quad , \quad N_-^i = \frac{1}{2} \begin{pmatrix} 0 & \cdot \\ \cdot & \sigma^i \end{pmatrix} \tag{A.43}$$

and

$$J^i = \frac{1}{2} \varepsilon^{ijk} M^{jk} = \frac{1}{2} \begin{pmatrix} \sigma^i & \cdot \\ \cdot & \sigma^i \end{pmatrix} \quad , \quad K^i = M^{0i} = -\frac{i}{2} \begin{pmatrix} +\sigma^i & \cdot \\ \cdot & -\sigma^i \end{pmatrix} \quad . \tag{A.44}$$

We can write these generators in the form (A.13) if we define the *Dirac γ matrices*
as

$$\gamma^0 = \begin{pmatrix} \cdot & \mathbb{1} \\ \mathbb{1} & \cdot \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} \cdot & \sigma^i \\ -\sigma^i & \cdot \end{pmatrix} \quad . \tag{A.45}$$

Using (A.35) we can show that they satisfy the anti-commutation relations (A.12).

A complete basis of the space $\mathbb{C}^{4,4}$ of complex 4×4 matrices can be constructed
from anti-symmetrised products of the γ matrices. Here we choose the following basis
matrices:

$$\mathbb{1} \quad , \quad \gamma^\mu \quad , \quad \sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu] \quad , \quad \gamma^\mu \gamma_5 \quad , \quad \gamma_5 \quad , \tag{A.46}$$

with

$$\gamma_5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \quad . \tag{A.47}$$

A symmetric inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^{4,4}$ is defined by

$$\langle \Gamma_1, \Gamma_2 \rangle = \mathrm{Tr}(\Gamma_1 \Gamma_2) \qquad \text{for } \Gamma_1, \Gamma_2 \in \mathbb{C}^{4,4}. \tag{A.48}$$

The basis matrices (A.46) are orthogonal with respect to this product. They satisfy

$$\langle \mathbb{1}, \mathbb{1} \rangle = \langle \gamma_5, \gamma_5 \rangle = 4 \quad ,$$

$$\langle \gamma^\mu, \gamma^\nu \rangle = +4g^{\mu\nu} \quad , \qquad \langle \gamma^\mu \gamma_5, \gamma^\nu \gamma_5 \rangle = -4g^{\mu\nu} \quad , \tag{A.49}$$

$$\langle \sigma^{\mu\nu}, \sigma^{\rho\sigma} \rangle = 4(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \quad .$$

The explicit form of the basis matrices (A.46) is

$$
\mathbb{1} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \qquad , \qquad \gamma_5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \quad ,
$$

$$
\gamma_0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad , \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad ,
$$

$$
\sigma^{0i} = \begin{pmatrix} -i\sigma^i & 0 \\ 0 & +i\sigma^i \end{pmatrix} \qquad , \qquad \sigma^{ij} = \varepsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \quad ,
$$

$$
\gamma_0\gamma_5 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \qquad , \qquad \gamma^i\gamma_5 = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \quad .
$$

(A.50)

The Lorentz and Poincaré group representations we discussed in this appendix are of central importance for defining the transformation behaviour of relativistic fields and constructing Poincaré invariant Lagrangian densities. In the next appendix we use them to construct free Dirac fields and massless vector fields.

Appendix B

Free Fields

In this appendix we review the quantisation of free Dirac spinor and vector fields. We will derive from first principles the explicit form of the polarisation vectors $\varepsilon^{\sigma}(k)$ and the Dirac spinors $u^{\sigma}(p)$ and $v^{\sigma}(p)$ for one-particle states with given helicities σ.

B.1 The Free Dirac Field

The free Dirac field ψ is a field of Dirac spinors described by the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \quad , \tag{B.1}$$

where $\partial\!\!\!/ = \partial_{\mu}\gamma^{\mu}$. Consequently, it satisfies the *Dirac equation*

$$(i\partial\!\!\!/ - m)\psi = 0 \quad , \tag{B.2}$$

A unitary representation $\mathcal{U}$ of the Poincaré group $P_{+}^{\uparrow}$ is defined on the space solutions of (B.2) by

$$\mathcal{U}(\Lambda, a)\psi(x) = S(\Lambda)\psi(\Lambda^{-1}x - a) \quad , \tag{B.3}$$

where $S(\Lambda)$ is the matrix from (A.42). Using the property (A.15) it can easily be checked that the Dirac equation (B.2) is invariant under these transformations.

To quantise the Dirac field we promote the fields ψ to operator valued fields acting on a Fock space $\mathcal{F}$. To assure that the eigenstates of these field operators transform

according to (B.3) they have to satisfy

$$U(\Lambda, a)\psi(x)U^\dagger(\Lambda, a) = S(\Lambda^{-1})\psi(\Lambda x + a) \quad , \tag{B.4}$$

where U is a unitary representation of $P_+^\uparrow$ on $\mathcal{F}$. By solving the Dirac equation (B.2) we can write the field operators of the free Dirac field in terms of *creation* and *annihilation operators*, namely

$$\psi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E(\mathbf{p})} \sum_\lambda \left(u^\lambda(p)a_\lambda(\mathbf{p})e^{-ipx} + v^\lambda(p)b_\lambda^\dagger(\mathbf{p})e^{ipx} \right) \quad , \tag{B.5}$$

where

$$p^0 = E(\mathbf{p}) = \sqrt{|\mathbf{p}|^2 + m^2} \tag{B.6}$$

and $u^\lambda(p)$ and $v^\lambda(p)$ are complete sets of spinors satisfying

$$(\not{p} - m)u^\lambda(p) = 0 \quad , \qquad (\not{p} + m)v^\lambda(p) = 0 \quad . \tag{B.7}$$

We call them *Dirac coefficients*.

Path integral methods now allow us to calculate the *Greens functions* of the Dirac field. These are vacuum expectation values of time-ordered products of the field operators ψ and $\bar{\psi}$. Of particular importance is the two-point function

$$\overline{\psi_\alpha(x)\bar{\psi}_\beta(y)} = \langle 0| T\{\psi_\alpha(x)\bar{\psi}_\beta(y)\} |0\rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} S_{\alpha\beta}(p) \quad , \tag{B.8}$$

where $S_{\alpha\beta}(p)$ is the *Dirac propagator*

$$S_{\alpha\beta}(p) = \quad \alpha \xrightarrow{\quad p \quad} \beta \quad = (\not{p} + m)_{\alpha\beta}\Delta(p) \tag{B.9}$$

and $\Delta(p)$ the *scalar propagator*

$$\Delta(p) = \frac{i}{p^2 - m^2 + i\varepsilon} \quad . \tag{B.10}$$

For the Dirac field, the other two possible combinations, $\langle 0| T\{\psi_\alpha(x)\psi_\beta(y)\} |0\rangle$ and $\langle 0| T\{\bar{\psi}_\alpha(x)\bar{\psi}_\beta(y)\} |0\rangle$, are zero. According to Wick's theorem all Greens functions can be expressed in terms of the two-point function (B.8). First we have to find all possible ways to pair up the operators in the time-ordered product. Each pairing is

called a *Wick contraction*. Then we have to bring all the pairs next to each other, picking up a factor of -1 for each time we have to swap two fermionic fields. Finally each pair is replaced by the expression (B.8) for the two-point Greens function.

The LSZ reduction formula identifies S matrix elements as pole coefficients of Fourier transformed Greens functions. To keep our expressions readable we define the following shorthands for Fourier transformed fields:

$$\psi_{\alpha p} = \int d^4x \, e^{-ipx} \psi_\alpha(x) \quad , \quad \psi_\alpha{}^p = \int d^4x \, e^{ipx} \psi_\alpha(x) \quad . \tag{B.11}$$

Momentum conserving delta functions are written as

$$\delta^{p_1 \cdots p_n}{}_{k_1 \cdots k_m} = (2\pi)^4 \delta^{(4)}(p_1 + \ldots + p_n - k_1 - \ldots - k_m) \quad . \tag{B.12}$$

Integrals over repeated momentum indices are implied. For example:

$$\psi_{\alpha p} \delta^{pk} = \int \frac{d^4p}{(2\pi)^4} \int d^4x \, e^{-ipx} \psi_\alpha(x)(2\pi)^4 \delta^{(4)}(p+k) = \int d^4x \, e^{ikx} \psi_\alpha(x) = \psi_\alpha{}^k \quad . \tag{B.13}$$

Now the Fourier transform of the two point function (B.8) can be written as

$$\begin{aligned}
\overbrace{\psi_\alpha{}^p \bar\psi_{\beta k}} &= \int d^4x \, e^{ipx} \int d^4y \, e^{-iky} \int \frac{d^4q}{(2\pi)^4} e^{-iq(x-y)} S_{\alpha\beta}(q) \\
&= \int \frac{d^4q}{(2\pi)^4} \int d^4x \, e^{i(p-q)x} \int d^4y \, e^{i(q-k)y} S_{\alpha\beta}(q) \\
&= \int \frac{d^4q}{(2\pi)^4} (2\pi)^4 \delta(p-q)(2\pi)^4 \delta(q-k) S_{\alpha\beta}(q) \\
&= (2\pi)^4 \delta(p-k) S_{\alpha\beta}(p) \\
&= S_{\alpha\beta}(p) \delta^p{}_k \quad .
\end{aligned} \tag{B.14}$$

Note that the Dirac propagator is not symmetric in the momentum argument, so the position of the momentum indices p and k matters:

$$\overbrace{\psi_\alpha{}^p \bar\psi_{\beta k}} = S_{\alpha\beta}(p)\delta^p{}_k = S_{\alpha\beta}(k)\delta^p{}_k \quad , \quad \overbrace{\psi_{\alpha p} \bar\psi_\beta{}^k} = S_{\alpha\beta}(-p)\delta_p{}^k = S_{\alpha\beta}(-k)\delta_p{}^k \quad . \tag{B.15}$$

The transformation behaviour of the operators a and $b^\dagger$ can be determined by

substituting (B.5) in (B.4). For the right-hand side we get

$$
S(\Lambda^{-1})\psi(\Lambda x + a)
$$

$$
= \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E(\mathbf{p})} \sum_\lambda \Big(S(\Lambda^{-1})u^\lambda(p)a_\lambda(\mathbf{p})e^{-ip(\Lambda x+a)} + S(\Lambda^{-1})v^\lambda(p)b_\lambda^\dagger(\mathbf{p})e^{ip(\Lambda x+a)} \Big)
$$

$$
= \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E(\mathbf{p})} \sum_\lambda \Big(S(\Lambda^{-1})u^\lambda(p)a_\lambda(\mathbf{p})e^{-i(\Lambda^{-1}p)x}e^{-ipa}
$$

$$
+ S(\Lambda^{-1})v^\lambda(p)b_\lambda^\dagger(\mathbf{p})e^{i(\Lambda^{-1}p)x}e^{ipa} \Big)
$$

$$
= \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E(\mathbf{p})} \sum_\lambda \Big(S(\Lambda^{-1})u^\lambda(\Lambda p)a_\lambda(\Lambda\mathbf{p})e^{-ipx}e^{-i(\Lambda p)a}
$$

$$
+ S(\Lambda^{-1})v^\lambda(\Lambda p)b_\lambda^\dagger(\Lambda\mathbf{p})e^{ipx}e^{i(\Lambda p)a} \Big) \quad . \tag{B.16}
$$

Comparison with the left-hand side yields

$$
u^\lambda(p)U(\Lambda, a)a_\lambda(\mathbf{p})U^\dagger(\Lambda, a) = S(\Lambda^{-1})u^\lambda(\Lambda p)a_\lambda(\Lambda\mathbf{p})e^{-i(\Lambda p)a} \quad , \tag{B.17a}
$$

$$
v^\lambda(p)U(\Lambda, a)b_\lambda^\dagger(\mathbf{p})U^\dagger(\Lambda, a) = S(\Lambda^{-1})v^\lambda(\Lambda p)b_\lambda^\dagger(\Lambda\mathbf{p})e^{i(\Lambda p)a} \quad . \tag{B.17b}
$$

By setting $\Lambda = \mathbb{1}$ in (B.17) and expanding to first order in a, we find

$$
[P^\mu, a_\lambda^\dagger(\mathbf{p})] = p^\mu a_\lambda^\dagger(\mathbf{p}) \qquad , \qquad [P^\mu, b_\lambda^\dagger(\mathbf{p})] = p^\mu b_\lambda^\dagger(\mathbf{p}) \quad . \tag{B.18}
$$

Therefore $a_\lambda^\dagger(\mathbf{p})$ and $b_\lambda^\dagger(\mathbf{p})$ create particles with momentum $p = (E(\mathbf{p}), \mathbf{p})$.

To give a physical meaning to the index λ we consider the case where $a = 0$ and Λ is an infinitesimal counter-clockwise rotation around the axis $\hat{\mathbf{p}}$ indicated by the spatial momentum $\mathbf{p}$. Note that such a transformations leaves the four-momentum p invariant, i.e. $\Lambda p = p$. Its representations can be written as

$$
U(\Lambda) = \mathbb{1} - i\theta\,\hat{\mathbf{p}} \cdot \mathbf{J} = \mathbb{1} - \frac{i}{2}\theta\,\varepsilon^{ijk}\hat{p}^i M^{jk} \quad , \tag{B.19a}
$$

$$
S(\Lambda) = \mathbb{1} - \frac{i}{4}\theta\,\varepsilon^{ijk}\hat{p}^i\sigma^{jk} = \mathbb{1} - \frac{i}{2}\theta\,h(\hat{\mathbf{p}}) \quad , \tag{B.19b}
$$

where θ is the infinitesimal rotation angle. For the spinor representation we have defined the *helicity operator* as

$$
h(\hat{\mathbf{p}}) = \frac{1}{2}\varepsilon^{ijk}\hat{p}^i\sigma^{jk} = \frac{i}{4}\varepsilon^{ijk}\hat{p}^i[\gamma^j, \gamma^k] = \frac{i}{2}\varepsilon^{ijk}\hat{p}^i\gamma^j\gamma^k \quad . \tag{B.20}
$$

It commutes with the matrix $\not{p}$, because

$$
\begin{aligned}
[h(\hat{\mathbf{p}}), \not{p}] &= \frac{i}{2}\varepsilon^{ijk}\big(\hat{p}^i E(\mathbf{p})[\gamma^j\gamma^k, \gamma^0] - \hat{p}^i p^\ell[\gamma^j\gamma^k, \gamma^\ell]\big) \\
&= -\frac{i}{2}|\mathbf{p}|\varepsilon^{ijk}\hat{p}^i\hat{p}^\ell\big(\gamma^j\{\gamma^k, \gamma^\ell\} - \{\gamma^j, \gamma^\ell\}\gamma^k\big) \\
&= i|\mathbf{p}|\varepsilon^{ijk}\hat{p}^i\hat{p}^\ell\big(\gamma^j\delta^{k\ell} - \delta^{j\ell}\gamma^k\big) \\
&= i|\mathbf{p}|\varepsilon^{ijk}\hat{p}^i\hat{p}^\ell\big(\gamma^j\hat{p}^i\hat{p}^k - \hat{p}^i\hat{p}^j\gamma^k\big) \\
&= 0 \quad .
\end{aligned}
\tag{B.21}
$$

Therefore we can construct the Dirac coefficients $u^\lambda(p)$ and $v^\lambda(p)$ as eigenvectors of $h(\hat{\mathbf{p}})$, i.e.

$$
h(\hat{\mathbf{p}})u^\lambda(p) = \lambda u^\lambda(p) \qquad , \qquad h(\hat{\mathbf{p}})v^\lambda(p) = \lambda v^\lambda(p) \quad .
\tag{B.22}
$$

Since $h(\hat{\mathbf{p}})$ is hermitian and $h(\hat{\mathbf{p}})^2 = \mathbb{1}$ the possible values of λ are $+1$ and -1. Now substituting (B.19) in (B.17) yields

$$
[\hat{\mathbf{p}} \cdot \mathbf{J}, a_\lambda^\dagger(\mathbf{p})] = \frac{\lambda}{2}a_\lambda^\dagger(\mathbf{p}) \qquad , \qquad [\hat{\mathbf{p}} \cdot \mathbf{J}, b_\lambda^\dagger(\mathbf{p})] = -\frac{\lambda}{2}b_\lambda^\dagger(\mathbf{p}) \quad .
\tag{B.23}
$$

This tells us that $a_\lambda^\dagger$ creates particles with helicity $+\lambda$ while $b_\lambda^\dagger$ creates particles with helicity $-\lambda$.

The normalisation of the Dirac coefficients depends on the mass m of the fermions. For $m \neq 0$ we choose $u^\lambda(p)$ and $v^\lambda(p)$ in such a way that

$$
\bar{u}^\lambda(p)u^{\lambda'}(p) = \bar{v}^\lambda(p)v^{\lambda'}(p) = 2m\delta^{\lambda\lambda'} \quad , \quad \bar{u}^\lambda(p)v^{\lambda'}(p) = \bar{v}^\lambda(p)u^{\lambda'}(p) = 0 \quad , \tag{B.24}
$$

where the bar-conjugate of a Dirac spinor ψ is defined as $\bar{\psi} \equiv \psi^\dagger\gamma^0$. For massless fermions the spinors $u^\lambda(p)$ and $v^\lambda(p)$ are linearly dependent, since the equations (B.7) are equivalent for $m = 0$. However, $u^+(k)$ and $v^+(k)$ will still be orthogonal to $u^-(k)$ and $v^-(k)$. Thus, for massless particles we impose the following normalisation condition:

$$
\bar{u}^\lambda(k)u^{\lambda'}(k) = \bar{v}^\lambda(k)v^{\lambda'}(k) = 2k^0\delta^{\lambda\lambda'} \quad .
\tag{B.25}
$$

To apply the LSZ formalism, we have to construct linear combinations of field operators that create or annihilate the correct one-particle states from the vacuum. Using (B.5) and the orthogonality relations (B.24) or (B.25) we see that

- $\bar{\psi}_p u^\lambda(p)$ creates fermions with momentum p and helicity λ,

- $\bar{v}^{-\lambda}(p)\psi^p$ creates anti-fermions with momentum p and helicity λ,

- $\bar{u}^\lambda(p)\psi^p$ annihilates fermions with momentum p and helicity λ,

- $\bar{\psi}^p v^{-\lambda}(p)$ annihilates fermions with momentum p and helicity λ.

Thus, according to the LSZ reduction formula, the Feynman rules for external fermion legs are the ones shown in *table B.1* on page 135.

We will now calculate the components of the Dirac coefficients u and v. To do this we have to consider the cases $m \neq 0$ and $m = 0$ separately.

Massive Particles

For $m \neq 0$ we can normalise the Dirac coefficients according to (B.24). Then we define the *standard momentum*

$$\mathring{p} = (m, 0, 0, 0) \tag{B.26}$$

and construct the spinors $u^\lambda(\mathring{p})$ and $v^\lambda(\mathring{p})$ as eigenvectors of $\mathring{\slashed{p}}$ and $h(\hat{\mathbf{e}}_3)$, where $\hat{\mathbf{e}}_3$ is the spatial coordinate vector in z-direction. Using the explicit form (A.45) of the Dirac matrices, spinors $u^\lambda(\mathring{p})$ and $v^\lambda(\mathring{p})$ satisfying (B.7) for $p = \mathring{p}$ can be written as

$$u^\lambda(\mathring{p}) = \sqrt{m} \begin{pmatrix} \chi^\lambda \\ \chi^\lambda \end{pmatrix} \quad , \quad v^\lambda(\mathring{p}) = \sqrt{m} \begin{pmatrix} \chi^\lambda \\ -\chi^\lambda \end{pmatrix} \quad . \tag{B.27}$$

We choose the two-component vectors χ^λ as eigenvectors of σ^3, namely

$$\chi^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \chi^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \quad \sigma^3 \chi^\lambda = \lambda \chi^\lambda \quad . \tag{B.28}$$

Thus the *standard coefficients* (B.27) are also eigenvectors of $h(\hat{\mathbf{e}}_3)$, namely

$$h(\hat{\mathbf{e}}_3)u^\lambda(\mathring{p}) = \lambda u^\lambda(\mathring{p}) \quad , \quad h(\hat{\mathbf{e}}_3)v^\lambda(\mathring{p}) = \lambda v^\lambda(\mathring{p}) \quad . \tag{B.29}$$

To construct spinors $u^\lambda(p)$ and $v^\lambda(p)$ satisfying (B.22) and (B.7) for arbitrary four momenta p with $p^2 = m^2$ we can exploit the transformation behaviour (A.15) of the

gamma matrices. First of all we note that for an arbitrary Lorentz transformation Λ equation (A.15) tells us that

$$S(\Lambda)\not{p}S^{-1}(\Lambda) = p_\mu S(\Lambda)\gamma^\mu S^{-1}(\Lambda) = p_\mu(\Lambda^{-1})^\mu{}_\nu\gamma^\nu = (\Lambda p)_\nu\gamma^\nu \quad . \tag{B.30}$$

Assume that we have constructed some Lorentz transformation $\Lambda(\mathbf{p})$ that takes the standard momentum $\overset{\circ}{p}$ to p, i.e.

$$\Lambda(\mathbf{p})\overset{\circ}{p} = p \quad . \tag{B.31}$$

Then we see that the spinors

$$u^\lambda(p) \equiv \mathcal{N}(\Lambda(p))S(\Lambda(p))u^\lambda(\overset{\circ}{p}) \quad , \qquad v^\lambda(p) \equiv \mathcal{N}(\Lambda(p))S(\Lambda(p))v^\lambda(\overset{\circ}{p}) \tag{B.32}$$

satisfy (B.7). Because the matrices $S(\Lambda(p))$ may be non-unitary we have to include the normalisation factor $\mathcal{N}(\Lambda(p))$ to assure the normalisation condition (B.24). Note however that rotations are represented by unitary matrices, so $\mathcal{N}(R(\boldsymbol{\theta})) = 1$.

To assure that the spinors (B.32) are also eigenvectors of the helicity operator $h(\hat{\mathbf{p}})$ we have to choose the transformation $\Lambda(\mathbf{p})$ in a particular way. From the commutation relations (A.17) it follows that

$$S(B(\eta\hat{\mathbf{p}}))h(\hat{\mathbf{p}})S^{-1}(B(\eta\hat{\mathbf{p}})) = h(\hat{\mathbf{p}}) \quad , \tag{B.33a}$$

$$S(R(\boldsymbol{\theta}))h(\hat{\mathbf{p}})S^{-1}(R(\boldsymbol{\theta})) = h(R(\boldsymbol{\theta})\hat{\mathbf{p}}) \quad . \tag{B.33b}$$

Therefore we can construct helicity eigenvectors $u^\lambda(p)$ and $v^\lambda(p)$ for arbitrary spatial momenta $\mathbf{p}$ by first applying a boost in z-direction and then a spatial rotation to the standard spinors (B.27). According to (A.30) a Lorentz boost that transforms the standard momentum $\overset{\circ}{p}$ to a four momentum with spatial component $\mathbf{p}$ is given by

$$L(\mathbf{p}) = B(\text{Arsinh}(|\mathbf{p}/m|)\hat{\mathbf{p}}) \quad . \tag{B.34}$$

We define

$$\Lambda(\mathbf{p}) = R_3(\phi)R_2(\theta)L_3(|\mathbf{p}|) \quad , \tag{B.35}$$

where θ and ϕ are the polar angles of the vector $\mathbf{p}$. With this particular choice for $\Lambda(\mathbf{p})$ it follows from (B.33) that the spinors (B.32) are eigenvectors of $h(\hat{\mathbf{p}})$ with

eigenvalue λ.

The effect of the boosts $S(L_3(|\mathbf{p}|))$ on the Dirac coefficients $u^\lambda(\mathring{p})$ and $v^\lambda(\mathring{p})$ can be written as

$$\mathcal{N}(\Lambda(p))S(L_3(|\mathbf{p}|))u^\lambda(\mathring{p}) = \frac{E(\mathbf{p})\gamma^0 - p^3\gamma^3 + m}{\sqrt{2m(E+m)}}u^\lambda(\mathring{p}) \quad , \tag{B.36a}$$

$$\mathcal{N}(\Lambda(p))S(L_3(|\mathbf{p}|))v^\lambda(\mathring{p}) = \frac{-E(\mathbf{p})\gamma^0 + p^3\gamma^3 + m}{\sqrt{2m(E+m)}}v^\lambda(\mathring{p}) \quad . \tag{B.36b}$$

To prove this we first note that (B.30) and (B.33) imply that the spinors on the left-hand side of (B.36) satisfy (B.7) and (B.22) for $p = (E(\mathbf{p}), 0, 0, |\mathbf{p}|)$. Up to irrelevant phases, these conditions identify the spinors uniquely. In other words, the common eigenspaces of $\not{p}$ and $h(\hat{\mathbf{p}})$ are one-dimensional. If we define the spinors u' and v' as

$$u'^\lambda(p) = (\not{p} + m)u^\lambda(\mathring{p}) \quad , \quad v'^\lambda(\mathring{p}) = (-\not{p} + m)v^\lambda(\mathring{p}) \tag{B.37}$$

they satisfy

$$(\not{p} - m)u'^\lambda(p) = 0 \quad , \quad (\not{p} + m)v'^\lambda(p) = 0 \quad , \tag{B.38}$$

because

$$\begin{aligned}
(\not{p} + m)(\not{p} - m) &= p_\mu p_\nu \gamma^\mu \gamma^\nu - m^2 \\
&= \frac{1}{2}p_\mu p_\nu ([\gamma^\mu, \gamma^\nu] + \{\gamma^\mu, \gamma^\nu\}) - m^2 \\
&= p_\mu p_\nu g^{\mu\nu} - m^2 \\
&= 0 \quad .
\end{aligned} \tag{B.39}$$

If $p = (E(\mathbf{p}), 0, 0, |\mathbf{p}|)$ they are also eigenvectors of $h(\hat{\mathbf{e}_3})$ with eigenvalue λ because, as we saw in (B.21), $\not{p}$ commutes with $h(\hat{\mathbf{p}})$. Thus, because the common eigenspaces of $\not{p}$ and $h(\hat{\mathbf{p}})$ are non-degenerate, it follows that

$$u'^\lambda(p) \propto u^\lambda(p) \quad , \quad v'^\lambda(p) \propto v^\lambda(p) \quad . \tag{B.40}$$

To find the correct normalisation factor we calculate

$$
\begin{aligned}
\bar{u}'^{\lambda}(p)u'^{\lambda}(p) &= \bar{u}^{\lambda}(\mathring{p})\gamma^0(\slashed{p}^{\dagger} + m)\gamma^0(\slashed{p} + m)u^{\lambda}(\mathring{p}) \\
&= \bar{u}^{\lambda}(\mathring{p})(\slashed{p} + m)^2 u^{\lambda}(\mathring{p}) \\
&= 2m\bar{u}^{\lambda}(\mathring{p})(\slashed{p} + m)u^{\lambda}(\mathring{p}) \\
&= 2m\bar{u}^{\lambda}(\mathring{p})(E(\mathbf{p})\gamma^0 - |\mathbf{p}|\gamma^3 + m)u^{\lambda}(\mathring{p}) \\
&= 4m^2(E + m) \quad .
\end{aligned}
\tag{B.41}
$$

In an analogous calculation we obtain

$$
\bar{u}'^{\lambda}(p)u'^{\lambda}(p) = 4m^2(E + m) \quad .
\tag{B.42}
$$

After normalising the spinors (B.37) appropriately and choosing $p = (E(\mathbf{p}), 0, 0, |\mathbf{p}|)$ we see that the equations (B.40) are equivalent to (B.36).

Using (B.36) and the explicit form of the Dirac matrices (A.45) and the standard spinors (B.27) we find for $p = (E, 0, 0, p)$

$$
u^{\lambda}(p) = \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} (E - \lambda p + m)\chi^{\lambda} \\ (E + \lambda p + m)\chi^{\lambda} \end{pmatrix} \quad ,
\tag{B.43a}
$$

$$
v^{\lambda}(p) = \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} (E - \lambda p + m)\chi^{\lambda} \\ -(E + \lambda p + m)\chi^{\lambda} \end{pmatrix} \quad .
\tag{B.43b}
$$

For spatial momenta with a nonzero polar angle θ and an azimuthal angle ϕ we have to first apply a rotation by θ about the y-axis and then a rotation by ϕ about the z-axis. Using (B.28) and the results from *section A.3* we find

$$
\begin{aligned}
D^{(\frac{1}{2},0)}(R_3(\phi)R_2(\theta))\chi^{\lambda} &= D^{(\frac{1}{2},0)}(R_3(\phi)R_2(\theta))\chi^{\lambda} \\
&= \cos\frac{\theta}{2}e^{-i\lambda\phi/2}\chi^{\lambda} + \lambda\sin\frac{\theta}{2}e^{i\lambda\phi/2}\chi^{-\lambda} \quad .
\end{aligned}
\tag{B.44}
$$

Thus

$$u^\lambda(p) = \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} (E - \lambda p + m)(\cos\frac{\theta}{2} e^{-i\lambda\phi/2}\chi^\lambda + \lambda\sin\frac{\theta}{2} e^{i\lambda\phi/2}\chi^{-\lambda}) \\ (E + \lambda p + m)(\cos\frac{\theta}{2} e^{-i\lambda\phi/2}\chi^\lambda + \lambda\sin\frac{\theta}{2} e^{i\lambda\phi/2}\chi^{-\lambda}) \end{pmatrix} \quad ,$$

(B.45a)

$$v^\lambda(p) = \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} (E - \lambda p + m)(\cos\frac{\theta}{2} e^{-i\lambda\phi/2}\chi^\lambda + \lambda\sin\frac{\theta}{2} e^{i\lambda\phi/2}\chi^{-\lambda}) \\ -(E + \lambda p + m)(\cos\frac{\theta}{2} e^{-i\lambda\phi/2}\chi^\lambda + \lambda\sin\frac{\theta}{2} e^{i\lambda\phi/2}\chi^{-\lambda}) \end{pmatrix} \quad .$$

(B.45b)

Massless Particles

For $m = 0$ we use the normalisation condition (B.25) for the Dirac coefficients. Furthermore we have to use a different standard momentum, for example

$$\mathring{k} = (\mu, 0, 0, \mu) \quad , \tag{B.46}$$

where μ is an arbitrary energy scale. Using the explicit form (A.45) of the Dirac matrices, solutions of (B.7) for $p = \mathring{k}$ are

$$u^+(\mathring{k}) = v^+(\mathring{k}) = \sqrt{2\mu} \begin{pmatrix} 0 \\ \chi^+ \end{pmatrix} \quad , \qquad u^-(\mathring{k}) = v^-(\mathring{k}) = \sqrt{2\mu} \begin{pmatrix} \chi^- \\ 0 \end{pmatrix} \quad , \tag{B.47}$$

where χ^+ and χ^- are defined in (B.28). In the massless case we have $u^\lambda = v^\lambda$, because for $m = 0$ the equations (B.7) are equivalent. Also note that the standard coefficients (B.47) are eigenvectors of $h(\hat{\mathbf{e}}_3)$ and γ_5, namely

$$h(\hat{\mathbf{e}}_3)u^\lambda(\mathring{k}) = \gamma_5 u^\lambda(\mathring{k}) = \lambda u^\lambda(\mathring{k}) \quad . \tag{B.48}$$

As in the massive case we can construct spinors $u^\lambda(k)$ and $v^\lambda(k)$ satisfying (B.22) and (B.7) for arbitrary four momenta k with $k^2 = 0$ by acting on $u^\lambda(\mathring{k})$ with a Lorentz transformation

$$\Lambda(\mathbf{p}) = R_3(\phi)R_2(\theta)L_3(|\mathbf{k}|) \quad . \tag{B.49}$$

The angles θ and ϕ are the polar angles of $\mathbf{k}$ and $L_3(|\mathbf{k}|)$ is a boost in z-direction satisfying

$$L_3(|\mathbf{k}|)\mathring{k} = (|\mathbf{k}|, 0, 0, |\mathbf{k}|) \quad . \tag{B.50}$$

From the explicit form (A.28) of Lorentz matrices for boosts along the z-axis we see that

$$L_3(|\mathbf{k}|) = B_3(\ln \tfrac{|\mathbf{k}|}{\mu}) \quad . \tag{B.51}$$

According to (A.42) and (A.38) boosts along the z-axis are represented in the space of Dirac spinors by the matrix

$$S(B_3(\eta)) = \begin{pmatrix} e^{-\frac{\eta}{2}\sigma^3} & 0 \\ 0 & e^{+\frac{\eta}{2}\sigma^3} \end{pmatrix} \quad . \tag{B.52}$$

As the two-component vectors χ^λ are eigenvectors of σ^3 we obtain

$$L_3(|\mathbf{k}|)u^\lambda(\mathring{k}) = B_3(\ln \tfrac{|\mathbf{k}|}{\mu})u^\lambda(\mathring{k}) = \exp(\tfrac{1}{2}\ln \tfrac{|\mathbf{k}|}{\mu})u^\lambda(\mathring{k}) = \sqrt{\tfrac{|\mathbf{k}|}{\mu}}u^\lambda(\mathring{k}) \quad . \tag{B.53}$$

Thus, in the massless case, a boost along the z-axis leaves the standard spinors $u^\lambda(\mathring{k})$ invariant. The factor $\sqrt{|\mathbf{k}|/\mu}$ assures that the transformed spinors satisfy the normalisation condition (B.25), so we don't have to rescale them.

To calculate the spinors $u^\lambda(k)$ for spatial momenta $\mathbf{k}$ with a polar angle θ and an azimuthal angle ϕ we have to rotate the spinors (B.47) first by θ around the y-axis and then by ϕ around the z-axis. Using (B.44) we get

$$u^+(k) = \sqrt{2E} \begin{pmatrix} 0 \\ \cos\tfrac{\theta}{2}\, e^{-i\phi/2}\chi^+ + \sin\tfrac{\theta}{2}\, e^{+i\phi/2}\chi^- \end{pmatrix} \quad , \tag{B.54a}$$

$$u^-(k) = \sqrt{2E} \begin{pmatrix} -\sin\tfrac{\theta}{2}\, e^{-i\phi/2}\chi^+ + \cos\tfrac{\theta}{2}\, e^{+i\phi/2}\chi^- \\ 0 \end{pmatrix} \quad . \tag{B.54b}$$

For massless particles the helicity is a frame-independent quantity. To see this, note that for $m = 0$ the equations (B.7) read

$$\not{p}u^\lambda(p) = 0 \quad , \quad \not{p}v^\lambda(p) = 0 \quad . \tag{B.55}$$

Furthermore the on-shell condition for massless particles is $p^0 = |\mathbf{p}|$. Thus we can write (B.55) as

$$\gamma^0 u^\lambda(p) = \hat{p}^i\gamma^i u^\lambda(p) \quad , \quad \gamma^0 v^\lambda(p) = \hat{p}^i\gamma^i v^\lambda(p) \quad . \tag{B.56}$$

Multiplying with $\gamma_5\gamma^0$ from the left we get

$$
\begin{aligned}
\gamma_5 u^\lambda(p) &= \hat{p}^i \gamma_5 \gamma^0 \gamma^i u^\lambda(p) \\
&= i\hat{p}^i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^i u^\lambda(p) \\
&= -i\hat{p}^i \gamma^1 \gamma^2 \gamma^3 \gamma^i u^\lambda(p) \\
&= \frac{i}{2}\hat{p}^i \varepsilon^{ijk} \gamma^j \gamma^k u^\lambda(p) \\
&= h(\hat{\mathbf{p}}) u^\lambda(p)
\end{aligned}
\tag{B.57}
$$

and with an analogous calculation

$$
\gamma_5 v^\lambda(p) = h(\hat{\mathbf{p}}) v^\lambda(p) \quad .
\tag{B.58}
$$

So, for massless particles the helicity is simply the eigenvalue of γ_5, and this is a frame-independent quantity.

B.2 Charge Conjugation and Majorana Fermions

The free quantised Dirac field describes two types of particles: fermions, which are created by $a^\dagger$ and their corresponding *anti-fermions*, which are created by $b^\dagger$. The charge conjugation operator C swaps particles with their anti-particles. It therefore satisfies

$$
\begin{aligned}
\mathsf{C}\, a_\lambda(\mathbf{p})\mathsf{C}^{-1} = b_{-\lambda}(\mathbf{p}) \quad &, \quad \mathsf{C}\, b_{-\lambda}(\mathbf{p})\mathsf{C}^{-1} = a_\lambda(\mathbf{p}) \quad , \\
\mathsf{C}\, a_\lambda^\dagger(\mathbf{p})\mathsf{C}^{-1} = b_{-\lambda}^\dagger(\mathbf{p}) \quad &, \quad \mathsf{C}\, b_{-\lambda}^\dagger(\mathbf{p})\mathsf{C}^{-1} = a_\lambda^\dagger(\mathbf{p}) \quad .
\end{aligned}
\tag{B.59}
$$

From these relations it follows immediately that

$$
\mathsf{C}^\dagger = \mathsf{C}^{-1} = \pm\mathsf{C} \quad .
\tag{B.60}
$$

As we have seen in (B.5), the field operator $\psi(x)$ creates fermions and annihilates anti-fermions. Its charge conjugate $\mathsf{C}\,\psi(x)\mathsf{C}^{-1}$ and its bar-conjugate $\bar{\psi}(x)$ therefore create anti-fermions and annihilate fermions. To represent charge conjugation on the space of Dirac spinors we have to find a spinor matrix $\mathcal{C}$ that relates $\mathsf{C}\,\psi(x)\mathsf{C}^{-1}$ and $\bar{\psi}(x)$:

$$
\mathsf{C}\,\psi_\alpha(x)\mathsf{C}^{-1} = \mathcal{C}_{\alpha\beta}\bar{\psi}_\beta(x) \equiv \psi^c \quad .
\tag{B.61}
$$

Using (B.5) we find

$$\mathsf{C}\,\psi(x)\mathsf{C}^{-1} = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E(\mathbf{p})} \sum_\lambda \left(v^\lambda(p) a^\dagger_{-\lambda}(\mathbf{p}) e^{ipx} + u^\lambda(p) b_{-\lambda}(\mathbf{p}) e^{-ipx} \right) \quad , \quad \text{(B.62a)}$$

$$\bar{\psi}(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E(\mathbf{p})} \sum_\lambda \left(\bar{u}^\lambda(p) a^\dagger_\lambda(\mathbf{p}) e^{ipx} + \bar{v}^\lambda(p) b_\lambda(\mathbf{p}) e^{-ipx} \right) \quad . \quad \text{(B.62b)}$$

Thus $\mathcal{C}$ must satisfy

$$\mathcal{C}\bar{u}^\lambda(p)^\mathsf{T} = v^{-\lambda}(p) \quad , \quad \mathcal{C}\bar{v}^\lambda(p)^\mathsf{T} = u^{-\lambda}(p) \quad . \quad \text{(B.63)}$$

However, in the last section we have seen that the Dirac coefficients $u^\lambda(p)$ and $v^\lambda(p)$ are completely determined by the equations (B.7) and (B.22). Therefore the equations (B.63) are equivalent to

$$\mathcal{C}^{-1}\not{p}\,\mathcal{C} = -\not{p}^\mathsf{T} \quad , \quad \mathcal{C}^{-1} h(\hat{\mathbf{p}})\mathcal{C} = -h(\hat{\mathbf{p}})^\mathsf{T} \quad . \quad \text{(B.64)}$$

From the definition of $\not{p}$ and $h(\hat{\mathbf{p}})$ we see that both relations are satisfied if we choose $\mathcal{C}$ in such a way that

$$\mathcal{C}^{-1}\gamma^\mu \mathcal{C} = -\gamma^{\mu\,\mathsf{T}} \quad . \quad \text{(B.65)}$$

From this equation we immediately obtain the charge conjugation behaviour of γ_5 and the Dirac propagator $S(p)$:

$$\mathcal{C}^{-1}\gamma_5 \mathcal{C} = \gamma_5^\mathsf{T} \quad , \quad \mathcal{C}^{-1} S(p)\mathcal{C} = S^\mathsf{T}(-p) \quad . \quad \text{(B.66)}$$

In the Weyl representation the matrices γ^0 and γ^2 are symmetric, while the matrices γ^1 and γ^3 are anti-symmetric. Using the anti-commutation relations (A.12) we see that (B.65) is satisfied for

$$\mathcal{C} = i\gamma^0\gamma^2 \quad (\Rightarrow \quad \mathcal{C}^{-1} = \mathcal{C}^\dagger = \mathcal{C}^\mathsf{T} = -\mathcal{C}) \quad . \quad \text{(B.67)}$$

Majorana fermions are described by Dirac spinors λ, which satisfy the additional constraint

$$\mathcal{C}\bar{\lambda}^\mathsf{T} = \lambda \quad . \quad \text{(B.68)}$$

In other words: Majorana fermions are their own charge conjugate. This means that for Majorana fields the spinors λ and $\bar{\lambda}$ can not be treated as independent degrees of

freedom. Using (B.68) and (B.67) we can express the Lagrangian of a Majorana field entirely in terms of λ:

$$\mathcal{L} = \lambda^\mathsf{T} C (i \slashed{\partial} - m) \lambda \quad . \tag{B.69}$$

By inverting the operator $C(i\slashed{\partial} - m)$ and converting to momentum space we obtain the following two-point functions:

$$\overbrace{\lambda_\alpha{}^p \lambda_{\beta k}} = [S(p)C^{-1}]_{\alpha\beta} \delta^p{}_k \quad , \tag{B.70a}$$

$$\overbrace{\lambda_\alpha{}^p \bar\lambda_{\beta k}} = S_{\alpha\beta}(p) \delta^p{}_k \quad , \tag{B.70b}$$

$$\overbrace{\bar\lambda_\alpha{}^p \bar\lambda_{\beta k}} = [C^{-1} S(p)]_{\alpha\beta} \delta^p{}_k \quad . \tag{B.70c}$$

In Feynman diagrams we represent the Majorana propagators as follows:

$$\alpha \xrightarrow{\quad p \quad} \beta \quad = [S(p)C^{-1}]_{\alpha\beta} \quad , \tag{B.71a}$$

$$\alpha \xrightarrow{\quad p \quad} \beta \quad = S_{\alpha\beta}(p) \quad , \tag{B.71b}$$

$$\alpha \xrightarrow{\quad p \quad} \beta \quad = [(-C)S(p)]_{\alpha\beta} \quad . \tag{B.71c}$$

B.3   Gauge Fields

The gauge fields of a local $SU(N)$ gauge symmetry are massless vector fields A_μ^a, which transform under local gauge transformations according to the adjoint representation of the gauge group. In a *non-abelian* gauge theory there are interactions between gauge fields, which have to be treated perturbatively. The free Lagrangian can be written as:

$$\mathcal{L} = -\frac{1}{4} F^{a\mu\nu} F_{\mu\nu}^a - \frac{1}{2\xi} (\partial^\mu A_\mu^a)(\partial^\mu A_\mu^a) \quad , \tag{B.72}$$

with

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a \quad . \tag{B.73}$$

The second term in (B.72) is the *gauge fixing term*. In this work we are using the *Feynman gauge*, which corresponds to a value of $\xi = 1$. In this case the field equation induced by the Lagrangian (B.72) is just the homogeneous d'Alembert equation

$$\partial^\nu \partial_\nu A_\mu = 0 \quad . \tag{B.74}$$

We will suppress the 'a' index from now on, since the Lagrangian is diagonal in the gauge indices. The solutions of (B.74) can be written as

$$A_\mu(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E(\mathbf{p})} \sum_\sigma \left(\epsilon^\sigma_\mu(p) a_\sigma(\mathbf{p}) e^{-ipx} + \epsilon^{\sigma*}_\mu(p) a^\dagger_\sigma(\mathbf{p}) e^{ipx} \right) \quad, \qquad \text{(B.75)}$$

with

$$p^0 = E(\mathbf{p}) = |\mathbf{p}| \quad . \qquad \text{(B.76)}$$

A priori there is no constraint on the *polarisation vectors* ϵ so that the set $\{\epsilon^\sigma | \sigma = 0 \dots 3\}$ forms a complete basis of Minkowski space. However, two of these four degrees of freedom are unphysical and get projected out by the gauge fixing procedure.

To quantise the gluon field we promote the field components $A_\mu(x)$ to operators acting on a Fock space $\mathcal{F}$. Using path integral methods we see that the two-point function of the gauge field is given by

$$\overline{A_\mu(x)A_\nu(y)} = \langle 0 | T\{A_\mu(x)A_\nu(y)\} | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \Delta_{\mu\nu}(p) \quad, \qquad \text{(B.77)}$$

where $\Delta_{\mu\nu}(p)$ is the *gluon propagator*

$$\Delta_{\mu\nu}(p) = \quad {}_\mu \overset{\longleftarrow}{\text{\textasciitilde} \text{oooooo} \text{\textasciitilde}}_\nu \underset{p}{} = \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \Delta(p) \quad, \qquad \text{(B.78)}$$

with $\Delta(p)$ given in (B.10).

By arguments analogous to those leading to (B.23) and (B.18) we can show that $a^\dagger_\sigma(\mathbf{p})$ and $a_\sigma(p)$ create and annihilate gluons with helicity σ, if we construct the remaining two polarisation vectors as eigenvectors of the helicity operator $h(\hat{\mathbf{p}})$. In the four-vector representation we have

$$h(\hat{\mathbf{p}})^{\mu 0} = h(\hat{\mathbf{p}})^{0\mu} = 0 \qquad, \qquad h(\hat{\mathbf{p}})^{ij} = i\varepsilon^{ijk}\hat{p}^k \quad . \qquad \text{(B.79)}$$

To construct eigenvectors of $h(\hat{\mathbf{p}})$ we choose vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2 \in \mathbb{R}^3$ so that $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{p}}$ form an orthogonal, right-handed basis of $\mathbb{R}^3$. In other words:

$$\hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_s = \delta_{rs} \qquad, \qquad \hat{\mathbf{e}}_r \cdot \hat{\mathbf{p}} = 0 \qquad, \qquad \varepsilon^{ijk} \hat{e}^i_1 \hat{e}^j_2 \hat{p}^k = +1 \quad . \qquad \text{(B.80)}$$

	initial state	final state
fermions	$u^\sigma(p)$	$\bar{u}^\sigma(p)$
anti-fermions	$\bar{v}^{-\sigma}(p)$	$v^{-\sigma}(p)$
gluons	$\epsilon^\sigma(p)$	$\epsilon^{\sigma*}(p)$

Table B.1: Feynman rules for external fermions and anti-fermions with momentum p and helicity σ.

Then we see that

$$h(\hat{\mathbf{p}})^{ij}\hat{e}_1^j = -i\hat{e}_2^i \quad , \quad h(\hat{\mathbf{p}})^{ij}\hat{e}_2^j = +i\hat{e}_1^i \quad , \tag{B.81}$$

so the eigenvectors of $h(\hat{\mathbf{p}})$ are readily obtained as

$$\epsilon^\pm(p) = \frac{1}{\sqrt{2}}(0, \hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2) \quad , \tag{B.82}$$

with $\sigma = +1, -1$. They satisfy

$$h(\hat{\mathbf{p}})^\mu{}_\nu \epsilon^{\sigma\,\nu} = \sigma\epsilon^{\sigma\,\mu} \quad , \quad \epsilon_\mu^{\sigma*}\epsilon^{\sigma'\mu} = \delta^{\sigma\sigma'} \quad . \tag{B.83}$$

For the LSZ reduction formula we have to construct linear combinations of field operators that create one-particle states with the correct quantum numbers from the vacuum. Using (B.75) and the relations (B.83) we see that

- $\epsilon_\mu^\sigma(p)A^\mu{}_p$ creates gluons with momentum p and helicity σ,

- $\epsilon_\mu^{\sigma*}(p)A^{\mu p}$ annihilates gluons with momentum p and helicity σ.

Consequently the Feynman rules for external gluon legs are the ones given in *table B.1*.

Acknowledgements

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