UNIVERSITY OF SOUTHAMPTON

FACULTY OF ENGINEERING, SCIENCE AND MATHEMATICS

School of Mathematics

The Geometry of Weak Gravitational Singularities

by

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ABSTRACT

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We introduce a new class of *weak* curvature singularity with the critical feature that, although the curvature may be unbounded on approach to the singularity, the curvature remains square Lebesgue integrable.

Previous work looking at analogous singularities in the context of Yang-Mills gauge theory, suggests that a two-dimensional singularity, with square Lebesgue integrable curvature, will have a connection approaching that of a flat connection on approach to the singularity. By considering a 2-dimensional, timelike and static, weak singularity and by treating General Relativity as a gauge theory, we are able to apply these methods to gravitational singularities. We show that

1. A limit holonomy exists and is independent of position on the singularity
2. The connection tends to the conical connection in an L_1^2 Sobolev norm
3. The metric tends to the conical metric in an L_2^2 Sobolev norm

In the final chapter we review previous work on the use of Colombeau's theory of generalised functions in describing the curvature of conical spacetimes as distributions. Using the results stated above, we are able to extend this work to a class of weak curvature singularities. We show that the distributional part of the curvature of a weak, two-dimensional, timelike and static, curvature singularity is associated (in the sense of Colombeau algebras) to the distributional curvature of a 4-dimensional cone, which may be described in terms of 2-dimensional delta functions.

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Conventions

Throughout this thesis we shall adopt the following conventions.

The letters a through to h will be used for space-time indices, letters i to n will be used for Lie algebra and Lie group indices and Greek letters α, β, etc... will be used for indices of the Lie algebra $so(3)$.

The different types of derivative are given in the following way.

Ordinary derivative	d
Partial derivative	∂ or $,$
Exterior derivative	d
Covariant exterior derivative	D
Space-time covariant derivative	∇ or $;$

We shall use c_i for $i \in \mathbb{N}$ to denote generic constants. Unless otherwise stated it can be assumed that any c_i in a theorem or proof will not necessarily be the same when written again in a different theorem or proof.

As in [53] we shall sometimes refer to ω as a differential p-form and sometimes we refer to $\omega_{a_1 \ldots a_p}$ as the p-form. The distinction is not important since the index structure of differential forms is trivial. It will be clearly indicated where necessary when an index-free letter represents a function and when that letter represents a differential form written without coordinate indices.

If we have an object with Lie algebra or Lie group indices we distinguish between the full object and just one particular Lie algebra or Lie group component of the object by the use of the normal notation and the normal notation with a check mark above it respectively.

For example, A^i_{ja} is the Lie algebra valued connection 1-form and $\check{A}^i_{ja}$ is the (i,j)th scalar valued 1-form component of A^i_{ja}.

In Chapter 4 when we write e^u where u is a matrix (in the Lie algebra $so(3)$) we mean to interpret the exponential as an expansion. i.e.

$$e^u = \mathrm{Id} + u + \frac{u^2}{2} + \frac{u^3}{3!} + \frac{u^4}{4!} + \dots$$

In Chapter 3 and Chapter 4, if a Sobolev norm is written without reference to a domain it is to be taken over, then it is assumed that the Sobolev space is to be taken locally. For example, we may shorten $A \in L^2_{1,\mathrm{loc}}$ to just $A \in L^2_1$.

Nomenclature

We include a list of some of the notation used in this thesis which has not been explained in the 'Conventions' summary. We omit notation for the standard Lie groups and Lie algebras which are listed in Appendix A.1. Notation given for one chapter will have the same meaning in following chapters unless stated otherwise.

Chapter 1

G_{ab}	Einstein tensor	T_{ab}	Energy-momentum tensor
$\{t, r, \theta, z\}$	Cylindrical polar coordinates	$2\pi(1-A)$	Angular deficit of a cone

Chapter 2

M	Manifold	g_{ab}	Metric
m	Mass of black hole	$\bar{M}$	Cauchy completion of M
∂M	Complement of M in $\bar{M}$	Γ^a_{bc}	Metric connection
T_pM	Tangent space of M at p	C_{abcd}	Weyl tensor
		LM	Frame bundle
R_{abcd}	Riemann tensor	$[X]$	Equivalence class of X
L^i_j	Lorentz transformation	$\underline{\omega}_p$	Connection 1-form on bundle at p
φ	Soldering 1-form		
e^a_i	Basis of vectors	ϑ^i_a	Dual basis of 1-forms
G_{ab}	Positive definite metric on LM	Π	Projection from bundle to base space
C^r	Space of r times differentiable functions	β_λ	Loop space of a path λ
		κ	Closed path
$\mathrm{Hol}(E(0), \kappa)$	Holonomy of κ	σ	Volume element
K	Gaussian curvature	Ω_{ab}	Curvature 2-form

U	Region homeomorphic to a disk	$P\exp$	Path ordered exponential operator
κ_g	Geodesic curvature	L^p_q	Sobolev space
Λ	Subspace of $\mathbb{R}^n$	$\Vert\ \Vert_{L^p_q}$	L^p_q Sobolev norm

Chapter 3

P	Bundle space	G	A Lie group
F_{ab}	Yang-Mills curvature 2-form	$\underline{A}_q$	Yang-Mills connection 1-form at q
s	A gauge	s_*	Pushforward of s
s^*	Pullback of s	A^i_{ja}	Gauge dependent connection 1-form in Yang-Mills
I	Action		
J_r	Holonomy	$[J_r]$	Conjugacy class of J_r
Σ	2-dimensional singularity	Σ_0	Point on the singularity
N	Neighbourhood of Σ	N_0	Neighbourhood of Σ_0
$(g_r)^i_j$	A gauge transformation	λ_r, μ_r	Paths in the bundle
E^a_i	Basis element	γ_r	Path in the manifold
m	The holonomy number	${A^\flat}^i_{ja}$	Prototype of a flat connection 1-form
$\{B_\alpha\}, \{U_\alpha\}$	Sets of open balls		
s^i_j	A specific gauge transformation	λ	A cut-off function
		s_0	The limit of s as $r \to 0$

Chapter 4

$SO(3)$	A specific Lie group	$so(3)$	The Lie algebra of $SO(3)$
Q	The bundle space with gauge group $SO(3)$	L	Lorentz group
		γ^i_{jk}	Ricci rotation coefficients
$L^\uparrow_+$	Proper orthocronous Lorentz subgroup	ω	Gauge dependent connection 1-form
e^a_i	Frame of basis vectors	ϑ^i_a	Dual basis of covectors
$G, \hat{G}, \tilde{G}, h, e^u, e^\psi$	Gauge transformations	${\omega^\flat}^i_{ja}$	(Prototype) connection 1-form of a flat 4-d cone
χ_r	A path in the manifold		
$\lambda_i, i \in \mathbb{N}$	Eigenvalues	$\mathbf{\Gamma^a_{bc}}$	Levi Civita connection
g_{ab}	Metric	$m(N)$	Measure of a set N

Chapter 5

G	Gravitational constant	$\mathcal{D}, \mathcal{A}_0, \mathcal{A}_q$	Spaces of C^∞ functions with compact support
β	Rotation produced by holonomy		
Φ	A test function	$\mathcal{E}(\mathbb{R}^n)$	An algebra
$\mathcal{E}_M(\mathbb{R}^n)$	The subalgebra of $\mathcal{E}(\mathbb{R}^n)$	$\mathcal{N}(\mathbb{R}^n)$	The ideal of $\mathcal{E}_M(\mathbb{R}^n)$
$\mathcal{G}(\mathbb{R}^n)$	The Colombeau algebra	$\mathcal{E}, \mathcal{E}_M, \mathcal{I}$	Pointwise values of $\mathcal{E}(\mathbb{R}^n)$, $\mathcal{E}_M(\mathbb{R}^n)$ and $\mathcal{N}(\mathbb{R}^n)$
$\bar{\mathbb{C}}$	Algebra of generalised numbers		
$\vdash$	Associativity relation between elements of $\bar{\mathbb{C}}$ and real complex numbers	$\approx$	Associativity relation between elements of the Colombeau algebra
$R\sqrt{g}$	Curvature density	h_{ab}	Singular part of the metric
$\acute{R}$	Non-singular part of the curvature away from origin	μ	Mass per unit length

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Chapter 1

Introduction

A singularity in General Relativity is a point at the edge of a space-time where causal geodesics come to an end. An object traveling along such a geodesic has a point beyond which it has no past and/or a point beyond which it has no future. Singularities have attracted much interest over the last fifty years, but the task of understanding such phenomena has not been without obstacles since the nature of singularities prevents their direct observation, as proposed by the cosmic censorship hypothesis [33, 38]. However, properties can be deduced by the effect they have on the surrounding space-time.

Einstein's field equations, given by $G_{ab} = 8\pi G T_{ab}$ are the basis from which we understand General Relativity. The meaning of these equations and the effect of gravitation have been summarised by the theoretical physicist John Wheeler:

"Space tells bodies how to move and bodies tell space how to curve."

Where there exists a massive body in space-time, the curvature of the space-time around that body will be increased.

Curvature singularities can be thought of as an extreme of this situation. As we approach a curvature singularity the curvature becomes unbounded. For example we consider the case of the Schwarzschild solution, with line element

$$ds^2 = \left(\frac{r-2m}{r}\right) dt^2 - \left(\frac{r}{r-2m}\right) dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{1.1}$$

The effect is that the gravitational attraction, resulting from the distortion of the geometry of the space-time around the singularity, becomes so strong that, at a certain distance from the singularity, not even light can escape. At this distance we have what is known as an 'event horizon'. The region which the event horizon bounds is defined to be a 'black hole'. According to the cosmic censorship hypothesis all (generic) curvature singularities are hidden behind an event horizon in this way.

However, not all gravitational singularities are black holes. An important class of such singularities is given by cosmological or big-bang singularities such as that given by the Friedmann solution with line element

$$ds^2 = dt^2 - R^2(t)\left(\frac{1}{1-\kappa r^2}dr^2 + r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2\right) \tag{1.2}$$

where $R(t)$ is a scale-factor found by solving Friedmann's equation. This space-time is singular at $t = 0$ where R vanishes.

Another class of singularities which do not have event horizons is given by quasi-regular singularities. A quasi-regular singularity does not have this rapidly increasing curvature [12] and in fact the curvature is bounded as measured in a parallely propagated frame. The curvature is well-defined in the neighborhood but not at the singularity itself, which is not a point of the space-time manifold. The standard model for a quasi-regular singularity is the 4-dimensional cone. Our intuitive understanding of the familiar two-dimensional cone shows that at any point away from the singularity, although the cone may be curved, locally the cone is flat (i.e. has zero curvature). Likewise we see that at the point of the cone, we cannot describe the curvature and the point is degenerate.

For the four-dimensional cone the singularity is not a point but a 2-dimensional surface. If coordinates on the singularity are given by (t, z) then for each point (t_1, z_1) the coordinates in the surrounding cone are (r, θ). The line element of the four-dimensional cone is

$$ds^2 = dt^2 - dr^2 - r^2 A^2 d\theta^2 - dz^2 \tag{1.3}$$

Although all quasi-regular singularities have an undefined curvature, the integral of the curvature over a neighborhood of the singularity is defined. We find the curvature can be described using a distribution [49]. The *holonomy* of a closed path is given by the integral of the connection around the path, which may also be given in terms of an integral of the curvature over the spanning surface. Using $G_{ab} = 8\pi K T_{ab}$ and the holonomy, we can measure the integral of the corresponding energy-momentum tensor at the singularity. The idea of a distributional curvature at a quasi-regular singularity is covered extensively in [49] and in Chapter 2.

In the past singularities have often been considered to be exceptions to many of the laws and theorems governing space-times. By providing a distributional solution to Einstein's equations for quasi-regular singularities we can work with them in much the same way that we do for point and line charges in electromagnetism. This enables one to regard them as interior points of the space-time (see [5] for details).

In this thesis we will focus on a class of singularities for which the singularity is stronger than for quasi-regular singularities but for which the curvature is locally square integrable. We will refer to this class of singularities as weak curvature singularities.

The goal of this thesis is to extend the holonomy analysis to include weak curvature singularities. This is used to find a description of the connection and metric in the neighbourhood of the singularity. The method of Colombeau algebras is then used to calculate the distributional curvature of the singularity.

In Chapter 2 we provide a brief review of singular space-times and key concepts regarding completeness, distributional curvature and holonomy. We also introduce some important concepts from analysis such as Sobolev spaces.

We often think of General Relativity by considering a metric; and an explicit form for the metric is obtained by making a coordinate choice. Instead we will think of GR as a gauge theory by looking at connections on the orthonormal frame bundle. This allows us to make comparisons with Yang-Mills theory where similar holonomy dependent theorems have been developed, also with the restriction that the curvature is square Lebesgue integrable [39, 40]. In Chapter 3 we briefly describe the gauge theory formalism for Yang-Mills fields and then review two theorems of particular interest in [39] which tell us how, as we radially tend towards the singularity, the geometry of the Yang-Mills field in a neighborhood of the singularity increasingly resembles the geometry of a space with a flat connection.

Our aim is to take these theorems and adapt the method of proof so they apply in General Relativity. This shows that we can describe the curvature at a weak curvature singularity in much the same way as a conical singularity. Chapter 4 details the main results of this thesis, with a complete proof, at the expense of some repetition of material from the previous chapter. In the GR case we have a metric as well as a connection and this enables us to obtain a description of the limiting geometry in terms of the metric.

In Chapter 5 we review work from the last two decades on generalised functions and Colombeau algebras and look at succesful efforts to find the distributional curvature of a four-dimensional conical singularity. We then use a similar method to establish an expression for the distributional curvature of a weak curvature singularity.

Chapter 2

Singularities in General Relativity

2.1 What is a singularity?

A singularity is an exceptional point in space-time with which we must take especial care when applying the laws of physics. Singularities are best described by the effect they have on the surrounding geometry of a space-time and so perhaps a better question would be to ask, "where is a singularity?". If we consider a space-time to be described by the pair (M, g), a manifold and a metric, then we realise that singularities cannot exist within a space-time. The differential geometry of the manifold and the metric cannot describe such an unnatural point as a singularity and so we say that singularities exist at the boundary of space-time.

A familiar singularity is that of the Schwarzschild solution. This singularity exists within an event horizon and so is causally separated from any external observer. We understand that the curvature increases to infinity as we approach the singularity. However, we also note that at a distance of $2m$ away from the singularity (where m is the mass of the black hole in geometrised units) the metric in Schwarzschild coordinates is undefined. This may compel us to believe there exists a singularity at $r = 2m$. However, the singularity here is due to the particular choice of coordinates we have used to describe the Schwarzschild solution, and in fact $r = 2m$ is not a true singularity. We call such phenomena, 'coordinate singularities'.

Another problem which may arise in trying to identify singularities is with our notion of a limit. Since singularities do not reside in any space-time, one way to measure their properties is to take limits as we approach the singular point. If our space-time has a fixed background metric and/or the metric is positive definite then this is not a problem, as we have a clear understanding of distance and hence the shortest distance between points. Hence, when dealing with electro-magnetism which is usually described by modeling it in Minkowski space-time (which has a fixed background metric), we do not have problems in taking limits on approach to singularities. In General Relativity the metric also has Lorentzian signature and hence (in this thesis) has the sign convention

$$\begin{pmatrix} + & & & \\ & - & & \\ & & - & \\ & & & - \end{pmatrix}$$

However, without a fixed positive definite background metric we have no obvious way to measure distance. Instead we regard singular space-times as ones which are in some sense incomplete and also cannot be extended to make them complete. We first look at various definitions of completeness, and show how to attach a boundary to incomplete space-times.

Cauchy completeness

We start by looking at the concept of Cauchy completeness. We say that a metric space (M, d) is Cauchy complete (m-complete) if all Cauchy convergent sequences of points converge to points also within the space. A sequence of points (x_n) is Cauchy convergent if

$$\forall\ \epsilon > 0\ \exists\ N_\epsilon \in \mathbb{N} \text{ such that } \forall\ n, m \geq N_\epsilon,\ \ d(x_n, x_m) < \epsilon \tag{2.1}$$

If one has a Cauchy sequence that does not converge to a point in M then this indicates the space is not complete. By working with equivalence classes of such Cauchy sequences one can attach some additional ideal points to the space M to obtain the Cauchy completion $\bar{M}$ (see for example [1]). The boundary points $\partial M := \bar{M} \backslash M$ form the boundary of the space. However, since we have no natural way to measure the distance between x_n and x_m we find this to be an inadequate way of identifying boundary points in GR.

Geodesic completeness

An alternative notion of completeness more suitable for the GR case is provided by geodesic completeness. A manifold is geodesically complete (g-complete) if all geodesics are of infinite length and never leave the manifold.

Geodesic completeness and Cauchy completeness are interchangeable when dealing with a space with a positive definite metric (by the Hopf-Rinow theorem, see for example [10]). However, for a metric of Lorentzian signature, g-completeness still makes sense where m-completeness does not. An affinely parameterised geodesic is a C^2 curve, $\mu : t \mapsto \mu(t) \in M$ through a space-time satisfying the geodesic equation given by

$$\frac{d^2\mu^a}{dt^2} + \Gamma^a_{bc}\frac{d\mu^b}{dt}\frac{d\mu^c}{dt} = 0 \tag{2.2}$$

If there is a value beyond which the affine parameter t cannot be extended the geodesic is incomplete. A smooth curve is a geodesic if the tangent to the geodesic is parallely transported along itself. Parallel transport preserves norms and it is the norm of the tangent vector to the geodesic that determines if the geodesic is timelike, spacelike or null. Hence geodesics always remain either timelike, spacelike or null. For the case of timelike geodesics we have time as our parameter. Massless particles (e.g. photons) travel along null geodesics and particles with mass can travel along timelike geodesics. Hence, it seems satisfactory to say that should a manifold of an inextendible space-time feature a non-spacelike geodesic which is incomplete, then that space-time has a boundary point at the point of incompleteness. However, Geroch [16] has constructed an example of a rocket-ship with bounded acceleration which could leave a g-complete space-time at a point which we must consider to be a boundary point. Hence geodesic completeness does not necessarily imply that the space-time is without boundary.

Instead of just g-completeness it seems natural to demand that a space-time without a boundary is complete for all non-spacelike curves; that is to say, containing no non-spacelike curves which are either future or past incomplete. This leads us onto our next definition of completeness.

Bundle completeness

In Riemannian geometry (where we have a positive definite metric g) we define $d(p, q)$ as the length of the shortest of the curves $\gamma : t \mapsto \gamma(t)$ joining $p \in M$ to $q \in M$ where γ is continuous and twice piecewise differentiable. We write this explicitly as

$$d(p,q) = \inf_{\gamma}\left(\int_{\gamma}\sqrt{g(\dot{\gamma},\dot{\gamma})}dt\right) \tag{2.3}$$

This gives us the arc-length of the geodesic. In GR affine parameters give the arc-length of (timelike) geodesics. Generalised affine parameters (GAPs) measure the length of any path. We again consider a parameterised curve $\mu : t \mapsto \mu(t)$ (although not necessarily geodesic) through a point $p \in M$. We choose a basis (e_i^a) for T_pM the tangent space at p and then look at the tangent space for the entire curve, $T_{\mu(t)}M$. By parallely propagating (e_i^a) along μ we obtain a basis $(e_i^a(t))$ for $T_{\mu(t)}M$ for each t. The tangent vector can be expressed using this basis by

$$V^a(t) = V^i(t)e_i^a(t) \tag{2.4}$$

We define the GAP u to be

$$u = \int \left(V^i(t)V^j(t)\delta_{ij}\right)^{\frac{1}{2}} dt \tag{2.5}$$

Note that u is not uniquely defined by μ, but also depends on the initial choice of basis e_i^a for T_pM. However, if there is a value for a GAP for a curve beyond which it cannot be extended then it can be shown that this will be true for all GAPs of that curve. Hence whether or not a curve can or cannot be extended does not depend on the initial choice of basis but only on μ. If there exists a value beyond which the GAP cannot be extended then such a curve is b-incomplete. We describe a space-time as being b-complete if it permits no such incomplete curves and b-incomplete otherwise. We note that since a b-complete space-time contains only b-complete curves, all geodesics must also be complete and hence b-completeness implies g-completeness.

The b-boundary

Since the singularity lies outside the space-time, we would like to define its location in another space so it is more than a conceptual position. To do this we can consider various ideas such as the a-boundary [37] (abstract boundary), b-boundary [35] (bundle boundary), c-boundary [17] (causal boundary) or the g-boundary [15] (geodesic boundary). In this thesis we consider the b-boundary, the construction of which involves Cauchy sequences and bundles.

Cauchy sequences

In general, if we have a metric space (M, d), we say M is *complete* if all Cauchy convergent sequences are convergent to a point in M. If M is not complete then we look at $\bar{M}$, the space of equivalence classes of Cauchy sequences on M, where two sequences are equivalent if they converge to the same point. We can identify elements of $\bar{M}$ with points, not all of which lie in M. We call $\bar{M}$ the Cauchy completion of M. We define the boundary of M as

$$\partial M = \bar{M} \backslash M \tag{2.6}$$

Now we wish to apply this method to model singularities. We choose a space-time (M, g) where g is C^2 and look at a completion of M by adding ideal points to get $\bar{M}$. The singularities are located at the boundary $\partial M = \bar{M} \backslash M$. However, since we are not using a positive definite metric, it is not possible simply to use the Cauchy completion of M. To continue we shall need to construct a positive definite metric on the bundle space above the manifold.

Frame Bundles

Any point x in a manifold M, which we shall take to be four dimensional, can be mapped to $L_x M$ (a set of 4 linearly independent vectors at the point x) which lies in LM, the *frame bundle.*

$$\begin{aligned} M &\longrightarrow LM \\ x &\mapsto L_x M \end{aligned} \tag{2.7}$$

The *fibre* above a point x on a manifold is the set of all points in the bundle which project down onto x. In LM the fibre is a space of sets of four linearly independent vectors. A point in this space could be $(X_0^a, X_1^a, X_2^a, X_3^a)$. We can transform from this point to any other point in the space $(Y_0^a, Y_1^a, Y_2^a, Y_3^a)$ by means of a matrix $L \in GL(4, \mathbb{R})$ where $GL(4, \mathbb{R})$ is the group of 4×4 invertible matrices with elements in $\mathbb{R}$.

$$Y_i^a = L_i^j X_j^a \tag{2.8}$$

We say that $X \sim Y \iff Y = LX$ and we find that X is equivalent to all points in $L_x M$. Hence we can project any point X in the space to $[X]$, the equivalence class of X. We identify $[X]$ with x.

We now have a way to take LM to M

$$\begin{aligned} \Pi : LM &\longrightarrow M \\ X &\mapsto [X] = x \end{aligned} \tag{2.9}$$

We now use the soldering 1-form and the connection 1-form to put a positive definite metric on LM and hence give it the structure of a metric space.

The soldering 1-form (or canonical 1-form)

We have a projection from the frame bundle LM to the manifold M

$$\Pi : LM \rightarrow M \tag{2.10}$$

The derivative of the projection, $D\Pi$ maps the tangent space of LM to the tangent space of M

$$D\Pi : T(LM) \rightarrow TM \tag{2.11}$$

We now define the soldering 1-form φ as a map from the tangent space of LM, at the point $p \in LM$, to $\mathbb{R}^4$

$$\varphi : T_p(LM) \rightarrow \mathbb{R}^4 \tag{2.12}$$

Suppose $v \in T_p(LM)$ is a vector in the tangent space to the point $p \in LM$ such that $\Pi(p) = x$. We let

$$\begin{aligned} D\Pi_p : T_p(LM) &\rightarrow T_xM \\ v &\mapsto D\Pi_p(v) \end{aligned} \tag{2.13}$$

We can think of elements of LM as being given by a frame and a point in M. So $p = (\{e_i^a\}_{i=0}^3, x)$ and we can look at the components of $D\Pi_p(v)$ using the frame $\{e_i^a\}_{i=0}^3$ at the point p, i.e.

$$(D\Pi_p(v))^a = W^i e_i^a \tag{2.14}$$

and we use this W^i to define the soldering 1-form, so that $\varphi(v) = W^i$.

Definition 2.1 *The soldering 1-form φ is defined by its action on the tangent space to the bundle LM at a point $p = (\{e_i^a\}_{i=0}^3, x) \in LM$*

$$\begin{aligned} \varphi : T_p(LM) &\rightarrow \mathbb{R}^4 \\ v &\mapsto (D\Pi_p(v))^a (e^{-1})_a^i = W^i = \left(W^0, W^1, W^2, W^3\right) \end{aligned} \tag{2.15}$$

where $(e^{-1})_a^i$, also written ϑ_a^i, is the dual basis to e_i^a and so $\vartheta_a^i e_i^b = \delta_a^b$.

See Appendix A.3 for more on duals of differential forms.

If $\varphi(v) = 0$ then $W = 0$ and so $D\Pi_p(v) = 0$. This implies $v \in T_pLM$ provides a zero tangent vector at the point x. The converse of this argument is also true and so we have $\varphi(v) = 0$ if and only if v is 'vertical' in LM. That is to say, a vertical v represents a change in the frame direction on the manifold but not a change in position. We define $V_p \subset T_pLM$ to be the space of all vectors in T_pLM that are vertical in LM.

The connection 1-form on the bundle

Let $gl(4, \mathbb{R})$ be the real vector space equal to the set of all left-invariant vector fields on the Lie group $GL(4, \mathbb{R})$, where $GL(4, \mathbb{R})$ is the Lie group of 4×4 matrix transformations acting on fibres of LM. We call $gl(4, \mathbb{R})$ the Lie algebra of the Lie group $GL(4, \mathbb{R})$ (Lie groups, Lie algebras and left-invariance are discussed in Appendix A.1). The Lie group action moves points in the bundle along the fibre and so the vector field A^* which is tangent to the orbit under this action is also tangent to the fibre. We call A^* the fundamental vector field corresponding to $A \in gl(4, \mathbb{R})$. The map from A to A^* is denoted by ς where

$$\varsigma : gl(4, \mathbb{R}) \rightarrow \mathfrak{X}(LM) \tag{2.16}$$

and $\mathfrak{X}(LM)$ is the space of vector fields on LM. We note that ς is a Lie algebra homomorphism given by

$$\varsigma([A, B]) = [\varsigma(A), \varsigma(B)] \tag{2.17}$$

where $[,]$ denotes the bracket in the Lie algebra $gl(4, \mathbb{R})$ on the left-hand side and the Lie bracket between vector fields on the right hand side (see for example [27] for more details).

We now define the connection 1-form $\underline{\omega}$ on the bundle LM. We write $\underline{\omega}$ in order to distinguish this gauge independent connection 1-form from the gauge dependent connection 1-form ω on M which is used later in this thesis.

Definition 2.2 *The connection 1-form $\underline{\omega}_p$ at a point $p \in LM$ is defined to be a $gl(4,\mathbb{R})$-valued one-form on the bundle LM that satisfies the following conditions*

1. $\underline{\omega}_p(A^*) = A \quad \forall p \in LM,\ A \in gl(4,\mathbb{R})$
2. $(R_{g^*}\underline{\omega})_p(v) = \mathrm{Ad}_{g^{-1}}(\underline{\omega}_p(v)) \quad , \forall v \in T_pLM$

Where $(R_{g^*}\underline{\omega})_p = \underline{\omega}_p g^*$, the right action of g^* on $\underline{\omega}_p$ and the adjoint map $\mathrm{Ad}_g : LM \to LM$ is defined for each $f \in LM$ by

$$\mathrm{Ad}_g(f) := gfg^{-1} \tag{2.18}$$

We say that $v \in T_pLM$ is horizontal if and only if $\underline{\omega}_p(v) = 0$. A horizontal vector v represents a change in the frame position on the manifold corresponding to parallel propagation with respect to the connection $\underline{\omega}$. We define $H_p \subset T_pLM$ to be the space of all vectors in T_pLM that are horizontal in LM.

From our definitions of V_p and H_p we have [35]

$$T_pLM = V_p \oplus H_p \tag{2.19}$$

The metric on LM

We may now use the soldering 1-form φ and the connection 1-form $\underline{\omega}$ to define a symmetric covariant tensor of rank 2 on LM.

Definition 2.3 *G is a covariant tensor of rank 2 defined by*

$$\begin{aligned} G : T_p(LM) \times T_p(LM) &\to \mathbb{R} \\ (m,n) &\mapsto \varphi(m)\cdot\varphi(n) + \underline{\omega}_p(m)\cdot\underline{\omega}_p(n) \end{aligned} \tag{2.20}$$

where

$$\varphi(m)\cdot\varphi(n) = \varphi(m)^i\varphi(n)^j\delta_{ij}, \tag{2.21}$$

$$\underline{\omega}_p(m) \cdot \underline{\omega}_p(n) = \underline{\omega}_p(m)^i_j \underline{\omega}_p(n)^k_l \delta_{ik} \delta^{jl} \tag{2.22}$$

and $\varphi(m)^i$ are the components of $\varphi(m) \in \mathbb{R}^4$ and $\underline{\omega}_p(m)^i_j$ are the components of $\underline{\omega}_p(m) \in gl(4, \mathbb{R})$.

Proposition 2.4 *G is a positive definite metric on LM.*

Proof

We observe that

$$G(m, m) = \varphi(m) \cdot \varphi(m) + \underline{\omega}_p(m) \cdot \underline{\omega}_p(m) \geq 0 \tag{2.23}$$

and

$$\begin{aligned} G(m, m) &= 0 \\ \iff \varphi(m) \cdot \varphi(m) + \underline{\omega}_p(m) \cdot \underline{\omega}_p(m) &= 0 \\ \iff \varphi(m) = 0 \quad \text{and} \quad \underline{\omega}_p(m) &= 0 \end{aligned}$$

and so m is both vertical and horizontal. Since $H_p \cap V_p = \{0\}$ [35], we therefore know that $m = 0$. So

$$G(m, m) = 0 \iff m = 0 \tag{2.24}$$

Hence the metric G satisfies the requirements of a positive definite metric on LM. $\square$

The positive definite metric G endows LM with the structure of a metric space. This gives us a way to measure the b-length (bundle length) of a path in LM and hence we can now look at $\overline{LM}$, the Cauchy completion of LM. The transformation

$$\begin{aligned} GL(4, \mathbb{R}) \times LM &\to LM \\ (L^i_j, X^a_i) &\mapsto L^i_j X^a_i = Y^a_i \end{aligned} \tag{2.25}$$

is uniformly continuous and so the transformation extends to $GL(4, \mathbb{R}) \times \overline{LM} \to \overline{LM}$ (see [48]). The projection $\bar{\Pi}$ takes elements of $\overline{LM}$ to $\bar{M}$ by identifying $[X]$ with x as before, and this is how we define the completion of M. In the same way that $M \simeq LM/GL(4, \mathbb{R})$, we define $\bar{M} = \overline{LM}/GL(4,\mathbb{R})$.

With the projection

$$\bar{\Pi} : \overline{LM} \longrightarrow \bar{M} \tag{2.26}$$

we can now define the b-boundary, $\partial M = \bar{M}\backslash M$. Note that the b-boundary should be used with caution as it is not without problems. For example, a topological manifold must, by definition, be Hausdorff and yet it has been shown that the b-boundary is non-Hausdorff ([4] and more recently [41]).

Extensions

Let us now consider a space-time that is half Minkowski space; that is to say, identical to Minkowski space but only defined for $0 < x < \infty$. Any point at $x = 0$ is a boundary point but these boundary points are not singular because we have not considered a full inextendible space-time. In some sense, our choice of space-time was simply not large enough. We can extend the space time to include $x \leq 0$ and in particular the points at $x = 0$.

We now provide the following definition and proposition regarding extendibility which are stated and proved in [4].

Definition 2.5 *An extension of a space-time (M, g) is an isometric imbedding $\theta : M \to M'$, where (M', g') is a space-time and θ is onto a proper subset of M'.*

If a space-time has an extension then it is termed extendible. The following result demonstrates that extendible space-times are timelike g-incomplete (and hence also timelike b-incomplete).

Proposition 2.6 *If M has an extension $\theta : M \to M'$ then there is an incomplete timelike geodesic γ in M such that $\theta \circ \gamma$ is extendible.*

We will use extendibility to determine if the point of incompleteness is a *singular* or *regular* boundary point. Let us consider a g-incomplete timelike geodesic γ. Suppose there is an isometry θ of the space-time (M, g) into a larger space-time (M', g') where γ now has an endpoint $p \in M' \backslash \theta(M)$. If the Riemann tensor of (M', g') is C^r then (M', g') is a C^r extension of (M, g). If (M', g') is C^r at p then γ is said to terminate at a C^r regular boundary point. If there is no isometry θ for which the space-time (M', g') admits a point p which is the endpoint of γ, then the boundary point is singular.

In the example of half Minkowski space above we now realise that the points on the line $x = 0$ are regular boundary points.

2.2 Classification of singularities

Ellis and Schmidt [12] gave a flow-chart describing the general way in which one classifies singularities, a modified version of which is reproduced here (Figure 2.1).

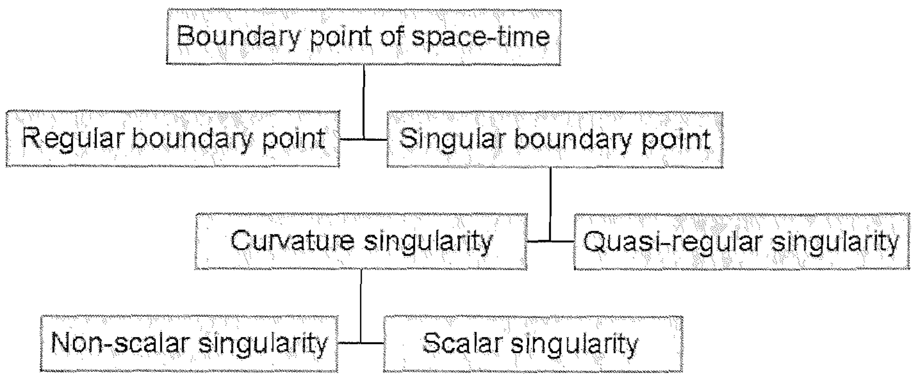

Figure 2.1: A simple flow-chart to classify singularities.

The first step towards classifying a singularity is to ensure that a candidate boundary point is indeed a singularity. We recall that if a boundary point is singular then there is no extension to the space-time which includes the boundary point. If all inextendible paths are b-complete then the space-time has no singularities. However, if a boundary point is accessible only by spacelike curves it is not clear that this represents a physical singularity since it cannot be reached by a material particle (which travels along timelike or null curves). Hence we regard non-spacelike b-incompleteness of an inextendible space-time as a sufficient condition for the presence of a physical singularity.

Once we have established that our boundary point is singular we then observe how the curvature behaves as we increase the parameter t of a path $\gamma(t)$, which terminates at the singularity.

If any of the components of the curvature tensor are unbounded as we parallely propagate a basis along a b-incomplete curve that terminates at a singularity, then we say that the singular point is a parallely propagated curvature singularity.

If any of the scalar polynomials in g_{ab}, C_{abcd} (the Weyl tensor) and R_{abcd} (and their derivatives) are unbounded on the incomplete curve, then the curve terminates at a scalar polynomial curvature singularity, an example of which is given by the Schwarzschild solution in spherical polar coordinates where we find that the quantity

$$C_{abcd}C^{abcd} = \frac{48m^2}{r^6} \tag{2.27}$$

is divergent on approach to the singularity ($r \to 0$). We call such quantities *scalar invariants* since they remain the same in all coordinate systems.

We note that parallely propagated curvature singularities and scalar polynomial curvature singularities are sometimes respectively shortened to curvature singularities and scalar curvature singularities.

If the Riemann tensor tends to finite limits in some orthonormal frame along $\gamma(t)$, but not in any such frame parallely propagated along $\gamma(t)$, we have a particular class of non-scalar singularity called a *whimper* or *intermediate* singularity [26].

If the curvature tends to a limit in a parallely propagated frame along all paths on approach to the singularity then we have a *quasi-regular singularity.* If the curvature is unbounded on approach to the singularity but remains in L^2 then we have a *weak singularity.* These last two singularities are of particular interest to us and will be discussed further below.

Quasi-regular singularities

A quasi-regular singularity arises when the curvature tensor components (when measured in a parallely propagated frame) tend to a well-defined limit along all curves terminating at the singularity. It is not so much the singularity which is imposing itself on the space-time, more that the singularity is a natural result of the topology of the space-time. As discussed in the introduction to this thesis, the prototype for a quasi-regular singularity is the four-dimensional conical singularity given by the line element

$$d\sigma^2 = dt^2 - dr^2 - A^2 r^2 d\theta^2 - dz^2 \qquad 0 \leq \theta < 2\pi \qquad A \neq 1 \tag{2.28}$$

the metric of which is a direct product of the metric for the two-dimensional cone and the metric for two-dimensional flat space-time $(ds^2 = -dt^2 + dz^2)$. If we set $\tilde{\theta} = A\theta$ we obtain

$$d\sigma^2 = dt^2 - dr^2 - r^2 d\tilde{\theta}^2 - dz^2 \qquad 0 \leq \tilde{\theta} < 2\pi A \tag{2.29}$$

We now show why the two-surface given by $r = 0$ is a quasi-regular singularity in the case of a four-dimensional cone. Since the metric given in (2.29) is flat then the components of the curvature tensor in a parallely propagated frame are necessarily well behaved. Points on the surface $r = 0$ therefore cannot be points of a curvature singularity. If we have a circle around any regular (in the sense of being non-singular) point, then the ratio of the circumference of the circle to its distance has a limiting value of 2π as the radius tends to zero. This is not the case for a four-dimensional cone at $r = 0$ where the ratio is in fact $2\pi A$ and so, while not a curvature singularity, points on the surface $r = 0$ can also not be thought of as regular. Instead such a surface is a quasi-regular singularity.

To understand the geometry of the four-dimensional conical space-time it is best to consider the space-time of a two dimensional cone with line element

$$d\sigma^2 = dr^2 + A^2 r^2 d\theta^2 \qquad 0 \leq \theta < 2\pi \tag{2.30}$$

where we will assume $0 < A < 1$. We can take this cone and cut a slit from its point in a straight line away from the point. We then 'unravel' the cone to have a manifold with a section missing from it (see Figure 2.2). The angle of the section is $2\pi(1 - A)$. As we would expect, we see that this space is analogous to Minkowski space-time. At all non-singular points on a cone the metric is locally flat.

We now let $\tilde{\theta} = A\theta$ and so (2.30) becomes

$$d\sigma^2 = dr^2 + r^2 d\tilde{\theta}^2 \qquad 0 \leq \tilde{\theta} < 2\pi A \tag{2.31}$$

Taking $\tilde{x} = r\cos\tilde{\theta}$ and $\tilde{y} = r\sin\tilde{\theta}$ for $0 < \tilde{\theta} < 2\pi A$ we have

$$d\sigma^2 = d\tilde{x}^2 + d\tilde{y}^2 \tag{2.32}$$

This is the same as Minkowski space-time but with a wedge taken out since $\tilde{\theta}$ only varies between 0 and $2\pi A$. We identify the two lines $\tilde{\theta} = 2\pi A$ and $\tilde{\theta} = 0$. But this now implies

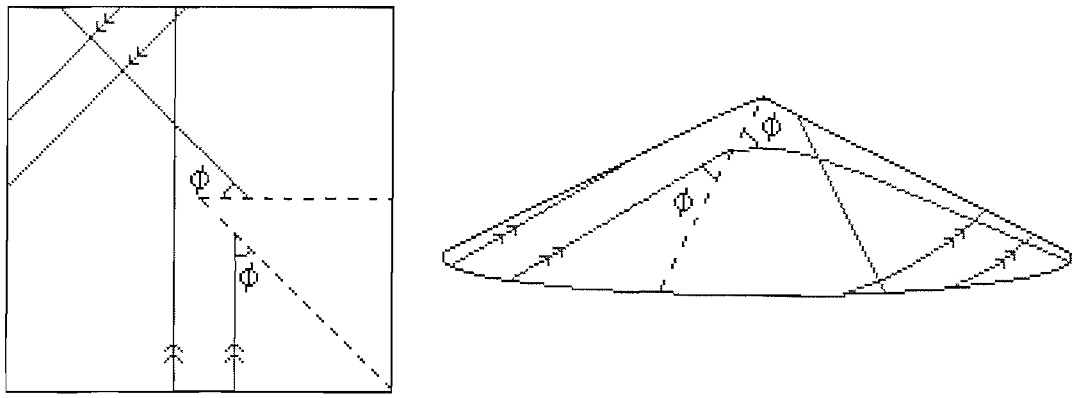

Figure 2.2: The first diagram illustrates the construction of a cone from Minkowski space. Both diagrams show how parallel geodesics on a cone do or don't meet depending on whether the apex of the cone is or is not between the two lines.

that a geodesic will have a kink as it passes over this line of identification. It will be focused more towards the singularity at $r = 0$. If we fold this space into a cone, it will be a smooth regular cone except along the line of identification where any grid drawn on the original Minkowski space will have a kink in it. Along this line on the cone the coordinates are singular but the space-time is not.

If we draw two parallel lines which both pass the singularity on the same side, then the lines will never meet. However, if we take parallel lines either side of the singularity, then we see that the lines do meet and hence there must exist a non-zero curvature at the singularity. This is also illustrated in Figure 2.2.

2.3 Distributional curvature

In General Relativity the naive expectation of the requirement to find the curvature of a space-time is for the metric to be C^2. This is because the equation for the curvature in terms of the metric has first and second order derivatives of the metric. On closer examination we find that the metric need only be C^{2-}, meaning that the metric is C^1 and its first derivatives satisfy the Lipschitz condition. However, to satisfy the contracted Bianchi identities and hence the conservation of the energy-momentum tensor (from Einstein's equation) we require higher differentiability.

Findings in [18] indicate that we can circumvent this problem by interpreting the curvature as a distribution (see Appendix A.5 for more on classical distribution theory). However, the curvature tensor is a non-linear function of the metric and its first two derivatives which means we cannot simply take weak derivatives and then multiply terms. This in turn implies that we cannot just lower the required differentiability of the metric to include a broader class of space-times as in general we cannot interpret the curvature as a distribution in this case. However, Geroch and Traschen have set out conditions [18] on a metric for it to be considered 'GT-regular' and for which the required products make sense as a distribution and hence for which the components of the curvature are well defined as a distribution. They later deduced that GT-regular metrics can only have their singular support on a submanifold of codimension one. This final condition clearly rules out conical singularities having a GT-regular metric. However, a non-linear theory of generalised functions has been proposed by Colombeau [8, 9] to circumvent this problem and is discussed in detail in Chapter 5.

More recently Garfinkle [14] has studied a wider class of semi-regular metrics, which contains the class of GT-regular metrics, whose curvature makes sense as a distribution. Garfinkle defines a metric g_{ab} to be a semi-regular metric provided that

1. g_{ab} and g^{ab} exist almost everywhere and are locally integrable
2. The weak first derivative of g_{ab} exists
3. The Christoffel tensor Γ^c_{ab} is locally integrable
4. $\Gamma^d_{e[b}\Gamma^e_{a]c}$ is locally integrable

Hence weak curvature singularities are not necessarily semi-regular since no assumption is made about the behaviour of the metric and so there is no guarantee that Garfinkle's first condition is satisfied.

In the next section we discuss a heuristic method to determine distributional curvature at a conical singularity by considering its holonomy. In Chapter 5 we shall use this holonomy method in combination with Colombeau's theory to find the distributional curvature of conical and weak singularities.

2.4 Holonomy

What is holonomy?

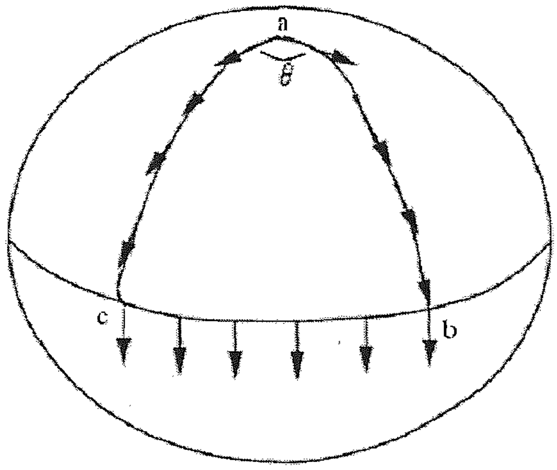

Figure 2.3: A diagram to illustrate holonomy on the 2-dimensional surface of a sphere.

An easy way to gain an initial understanding of holonomy is to consider the 2-dimensional surface of a sphere. We take a vector at a point a and parallely transport it along one of the great circles passing through a until we reach a point b. We then transport along a different great circle passing through b until we reach a point c. We then parallely transport along the great circle which connects c and a until we are back where we started at a. We compare the directions of the initial vector and the final vector and the rotation required to take one to the other tells us the holonomy around a closed path on the surface of the sphere.

The way we will calculate holonomy in more complex geometries is by comparing initial and final frames. Let us consider a closed path κ_t. Any frame propagated completely around the path will be related to the initial frame by a $GL(n, \mathbb{R})$ transformation. For GR we use a metric connection and orthonormal frames, so that the holonomy is an element of the Lorentz group. The holonomy is the rotation which takes the initial frame to the frame after it has been completely parallely propagated around a closed path. We include a more formal definition similar to that given in [55].

Definition 2.7 (Holonomy of a closed curve) *Let $\kappa : [0, 1] \to M$ be a closed curve in the manifold M. Suppose that the frame $E(s)$ at point $\kappa(s)$ is defined by parallely propagating $E(0)$ around κ, then the holonomy of κ,* $\mathrm{Hol}(E(0), \kappa)$ *is defined to be the $GL(4, \mathbb{R})$ transformation such that*

$$E_i(1) = \mathrm{Hol}(E(0), \kappa)_i^j E_j(0) \tag{2.33}$$

The holonomy of a singularity in General Relativity

Let us take a manifold M which has a singular boundary which is a surface of codimension two. We consider a b-incomplete path in this space $\lambda : [0,1) \to M$ which terminates at a point on the singularity. We define the loop space β_λ of λ to be the set of C^1 maps $\mu : [0,1] \times [0,1) \to M$ such that $\mu(0,t) = \mu(1,t) = \lambda(t)$ and the b-length of the s-parameterised (t fixed) closed path $\mu(s,t) = \mu_t(s)$ tends to zero as t tends to 1. The b-length of a path, as shown earlier, is the length of the lifted path measured in the frame bundle LM, so to find the b-length of μ we first perform a 'horizontal lift' taking $\mu_t(s)$ into LM to $\bar{\mu}_t : [0,1] \to LM$ by parallely propagating a frame $E_t(0)$ at $\mu_t(0)$ along μ_t. The loop space β_λ is composed of closed paths encircling the singular point and the act of increasing t is to 'tighten the noose' around the point.

If we lift the loop μ_t into the frame bundle then the path $\bar{\mu}_t$ is no longer closed. The path starts and finishes at the same fibre but not the same point, since position in LM indicates both the position and direction of the parallely propagated frame in M. The holonomy is the rotation which the frame has undergone to get from its initial orientation to its final orientation and so looking at this in LM, it is the group element $\mathrm{Hol}(E_t(0), \mu_t) \in GL(4, \mathbb{R})$ taking $\bar{\mu}(0,t)$ to $\bar{\mu}(1,t)$. The *limit* holonomy is found by taking the limit of $\mathrm{Hol}(E_t(0), \mu_t)$ as t tends to 1. We provide a formal definition

Definition 2.8 (Limit holonomy for a loop space) *Let $\lambda : [0,1) \to M$ be a b-incomplete curve and β_λ be the loop space consisting of the C^1 maps $\mu : [0,1] \times [0,1) \to M$ such that*

1. $\mu(0,t) = \mu(1,t) = \lambda(t)$
2. *The b-length of $\mu(s,t) = \mu_t(s)$ tends to zero as t tends to* 1.

Then the limit holonomy of the loop space β_λ, where it exists, is defined to be

$$\mathrm{Hol}(0) = \lim_{t \to 0} \mathrm{Hol}(E_t(0), \mu_t) \tag{2.34}$$

We shall now use the holonomy method to calculate the integrated curvature at conical singularities as shown in [49]. We first consider the Gauss-Bonnet theorem (for a region with boundary), that states that for a regular 2-dimensional Riemannian manifold,

$$\int_U K d\sigma = 2\pi - \int_{\partial U} \kappa_g ds \tag{2.35}$$

where σ is the volume element, K is the Gaussian curvature, U is a region homeomorphic to a disk and ∂U is the boundary of U. The geodesic curvature κ_g is the norm of the geodesic curvature vector $\nabla_T T$, so that

$$g(\nabla_T T, \nabla_T T) = \kappa_g^2 \tag{2.36}$$

where T is the vector tangent to ∂U. Since U in our case contains the conical singularity we can regard the right hand side of (2.35) as defining the integrated curvature of the singularity. Vickers has shown [47] that we can rewrite (2.35) in the more relevant form

$$\exp \int_U \Omega = L \tag{2.37}$$

where Ω is the curvature 2-form taking values in $so(2)$ and L is the rotation matrix which describes parallel propagation around ∂U and represents a rotation through $2\pi(1 - A)$ no matter how small U is, as long as it contains the origin. As we make U smaller we can see that in the limiting case the conical singularity will have delta function curvature.

We can generalise (2.37) for four dimensions to the equation

$$L = P\exp\left(\int_U \Omega\right) \tag{2.38}$$

where $P\exp$ is a path ordered exponential operator (as given in [47]), L is the Lorentz transformation relating an initial orthonormal frame to one obtained by parallel propagation around ∂U and Ω is the curvature 2-form of U.

We now have an equation (2.38) which relates curvature to holonomy and vice-versa. We may extend this to the distributional setting and use it heuristically to relate the distributional curvature to the limit holonomy. However it remains to show that it makes mathematical sense to describe curvature as a distribution in this way. In Chapter 5 we demonstrate how Clarke, Vickers and Wilson [7] show that curvature can be thought of as a distribution by using the Colombeau algebra (as described in [8, 9]). They then apply the results of the above holonomy method to find the distributional curvature and energy-momentum tensor of a conical singularity.

The aim of this thesis is to apply the above ideas to a wider class of singularities for which the holonomy is defined. This will require the curvature to (at least) be integrable. The main condition we will use is that the curvature is square integrable in a neighbourhood U of the singularity in the sense that

$$\int_U |\Omega|^2 < C < \infty \tag{2.39}$$

We will later include an additional condition on the gauge dependent connection 1-form. What it means to 'square the modulus' of a Lie algebra valued 2-form will be discussed in detail in Section 2.6. We will call curvature singularities with curvature that satisfies (2.39) 'weak curvature singularities'.

We now have the motivation for the next chapters to find the holonomy of different geometries featuring weak singularities in both Yang-Mills theory and General Relativity.

2.5 Results from the holonomy method for quasi-regular singularities

We briefly discuss the main results from Vickers [49] for quasi-regular singularities. The theorem is restricted to singularities with bounded curvature on approach to the singularity and is obtained in a similar fashion to that used in the next chapter for singularities in Yang-Mills. We wish to extend the results obtained in [49] to weak curvature singularities. We include here a simplified version of the theorem and the physical and geometric significance of its results in order to show the sort of results we hope to obtain in Chapter 4.

Theorem 2.9 *Let Σ be a (timelike) two dimensional quasi-regular singularity and let L_r be the holonomy obtained by parallel propagation around a loop of radius r round a point $p \in \Sigma$, then*

1. *The limit holonomy $\lim_{r \to 0} L_r = L_0$ exists*
2. *The tangent space to Σ is a fixed point of L_0*
3. *L_0 is independent of p*

The first result implies that the quasi-regular singularity locally looks like a conical singularity. The second tells us that the axis of rotation about which we establish the holonomy is parallel to the singularity and the second result combined with the first result informs us that the singularity has a similar geometric structure to a cosmic string. The third result implies that the angle of the cone is the same no matter which point we take on the singularity and that the integral of the curvature and hence the distributional energy momentum tensor of the 'string' are conserved. The second and third results also tell us that the singularity is totally geodesic (i.e. geodesics in the submanifold Σ are also geodesics in the manifold). The notion of totally geodesic makes sense in this context since for such a quasi-regular singularity we may use the b-completion of tensor fields on M to obtain tensor fields on $\bar{M}$ (see [49] for details).

Unruh et al. [46] have obtained similar results modelling cosmic strings on conical singularities, with particular focus on those models with angular defecit smaller than π. They show that an idealized cosmic string with conical angular defecit $\triangle\phi$ must be straight in the sense that the one-currvature κ of a parallel curve at a small, constant distance d from the string tends to zero with d according to

$$\kappa \sim a^{-1}\left(\frac{d}{a}\right)^{\eta-1} \tag{2.40}$$

where

$$\eta \equiv \left(1 - \frac{\triangle\phi}{2\pi}\right)^{-1} \tag{2.41}$$

and a is a characteristic global length associated with the loop. They also show that the divergence of the Riemann tensor is weak enough to preserve integrability of the Riemann curvature for positive angular defecits. They use this to show that the cosmic string's shape and orbit are geodesics of the spacetime.

More precisely, they consider the history of the idealized string to be a timelike two-space S_{zt}, whose points are conical singularities of the space-time. They show that S_{zt} is totally geodesic if the Ricci tensor is bounded near S_{zt}. Space-time geodesics orthogonal to S_{zt} at a point Q sweep out a spacelike two-space S_{xy} having a conical structure at the point Q where it intersects S_{zt}. One can then select any intrinsic geodesic L_t of S_{zt}, parametrised by proper distance t and with tangent

$$u^a = \frac{dx^a}{dt} \tag{2.42}$$

Unruh et al. then prove that L_t, as a locus of conical singularities of the four-metric, must be a geodesic.

2.6 The Sobolev spaces $L^p_q(X)$

We include a discussion on Sobolev norms and spaces, since they will be used extensively in the following chapters.

The standard context for consideration of Sobolev spaces is for functions $f : \mathbb{R}^n \to \mathbb{R}$ in Euclidean space. We define the L^p_q norm of f over an arbitrary domain $\Lambda \subset \mathbb{R}^n$ in the following way.

Definition 2.10 *Let $f : \mathbb{R}^n \to \mathbb{R}$ and let $\Lambda \subset \mathbb{R}^n$. The L^p_q norm of f over Λ is $\|f\|_{L^p_q(\Lambda)}$ where p is a positive real number, q is a non-negative integer and*

$$\|f\|_{L^p_q(\Lambda)} = \left(\int_\Lambda \sum_{0 \le |\alpha| \le q} |\partial^\alpha f|^p d^n x \right)^{\frac{1}{p}} \tag{2.43}$$

Here α is a multi-index with dimension n and ∂ is the usual partial derivative but taken in the weak sense (as defined in Appendix A.5).

We now define the Sobolev space $L^p_q(\Lambda)$.

Definition 2.11 *$L^p_q(\Lambda)$ is a space of functions such that $\|f\|_{L^p_q(\Lambda)} < \infty$ for all $f \in L^p_q(\Lambda)$ and $\partial^\alpha f$ is a measurable function for all $0 \le |\alpha| \le q$.*

Note that in the case of compact Λ, $f \in C^q(\Lambda)$ implies that $f \in L^p_q(\Lambda)$, although the conditions in Definition 2.11 are weaker since we allow weak derivatives of f provided $\partial^\alpha f$ is a measurable function, and $\|f\|_{L^p_q(\Lambda)}$ is finite.

The 'L' used to denote a Sobolev space comes from the French mathematician Henri Lebesgue and it is in reference to him that we say that $L^2(\Lambda)$ is the set of all functions on Λ which are square Lebesgue integrable. Note that unless it is unclear in what domain the Sobolev space lies we shall adopt the convention of shortening $\|\ \|_{L^p_q(\Lambda)}$ to $\|\ \|_{L^p_q}$.

In a similar way to that found in [20] we define Sobolev norms for space-times with metrics that are not necessarily flat. We first take a positive definite background metric ξ_{ab} on the manifold. The physical space-time as given in the following chapters will be Lorentzian and contain a singularity and so while the introduced background metric is unphysical, it nevertheless provides a method to measure magnitudes and volumes of physical objects.

Definition 2.12 *Let h be a scalar function in an n-dimensional space-time (M, g) and let $\Lambda \subset M$. The L^p_q norm of h over Λ is $\|h\|_{L^p_q}$ where p is a positive real number, q is a non-negative integer and*

$$\|h\|_{L^p_q} = \left(\int_\Lambda \sum_{0 \le |\alpha| \le q} |\nabla^\alpha h|^p d\sigma \right)^{\frac{1}{p}} \tag{2.44}$$

Here $d\sigma$ is the volume element induced by the positive definite metric ξ_{ab} on M. The operator ∇ is the covariant derivative with respect to ξ_{ab} and α is a multi-index with dimension n.

Note that this thesis follows the notation in [39] in that $\sum_{|\alpha|=n} |\nabla^\alpha h|^p$ will sometimes be shortened to $|\nabla^{(n)} h|^p$.

We now look at Sobolev norms of non-scalar functions with space-time components.

Definition 2.13 *For $i, j \in \mathbb{N}$ let $K^{a_1 \ldots a_i}{}_{b_1 \ldots b_j}$ be a function in an n-dimensional space-time (M, g) and let $\Lambda \subset M$. The L^p_q norm of K over Λ is $\|K\|_{L^p_q}$ where p is a positive real number, q is a non-negative integer and*

$$\|K\|_{L^p_q} = \left(\int_\Lambda \sum_{0 \le |\alpha| \le q} |\nabla^\alpha K|^p d\sigma \right)^{\frac{1}{p}} \tag{2.45}$$

We define the magnitude norm $|\ |$ by

$$|K|^2 = K^{a_1 \ldots a_i}{}_{b_1 \ldots b_j} K^{c_1 \ldots c_i}{}_{d_1 \ldots d_j} \xi_{a_1 c_1} \ldots \xi_{a_i c_i} \xi^{b_1 d_1} \ldots \xi^{b_j d_j} \tag{2.46}$$

We shall choose our background metric to be locally flat so, when using Cartesian coordinates, we can take all the covariant derivatives to be partial derivatives. That is to say, we let $\xi_{ab} \stackrel{*}{=} \delta_{ab}$, where $\stackrel{*}{=}$ indicates a coordinate dependent equality (the flat metric is different depending on what coordinates we are in).

For an example of a Sobolev norm on a coordinate indexed object we can consider l_a, a differential 1-form. $l \in L^2_1$ requires that

$$||l_a||_{L_1^2} = \left(\int \sum_{0 \le |\alpha| \le 1} \nabla^\alpha l_a \nabla^\alpha l_b \xi^{ab} d\sigma \right)^{\frac{1}{2}} < \infty \tag{2.47}$$

In gauge theory one also considers scalar functions taking values in the Lie group or its Lie algebra, elements of which we can think of as square matrices. Here we define the Sobolev norm in the same way as for Definition 2.10 but we now extend our definition of the modulus $|\ |$ to accommodate Lie indices, as given in [34]. Below, the raised $\dagger$ indicates the complex conjugate of the transpose (also known as the Hermitian conjugate).

Definition 2.14 *Let H be an object taking values in a Lie algebra or Lie group. We think of H as being matrix valued. The modulus of H is*

$$|H|^2 = Tr(HH^\dagger) \tag{2.48}$$

or in index notation

$$|H^i_j|^2 = \delta^k_i H^i_j H^{\dagger j}_{\ k} = H^i_j \bar{H}^i_j = \sum_{i,j} |\breve{H}^i_j|^2 \tag{2.49}$$

where $\breve{H}^i_j$ denotes the particular ${}^i_j$ component of the matrix H^i_j and so the operator $|\ |$ in the instance on the right hand side of (2.49) is just the standard scalar modulus defined in the usual way for real and complex numbers.

As an example we consider a Lie group valued object g^i_j (which shall be described in the next chapter as a gauge transformation) where

$$g : M \to SU(2) \tag{2.50}$$

$g \in L^2_2$ requires that

$$||g^i_j||_{L^2_2} = \left(\int \sum_{0 \le |\alpha| \le 2} \sum_{i,j} |\nabla^\alpha \breve{g}^i_j|^2 d\sigma \right)^{\frac{1}{2}} < \infty \tag{2.51}$$

The above definitions naturally lead on to a consideration of Sobolev norms of additional objects which are used in gauge theory, functions with both space-time and Lie indices. In this case the definition of a Sobolev norm is analogous to Definition 2.13, but we now define the modulus of an object with both space-time and Lie indices. To define the modulus we will make use of both (2.46) and (2.49).

Definition 2.15 *The modulus of an object J taking values in a Lie algebra or Lie group and with m raised and p lowered space-time indices is defined by*

$$\begin{aligned} & |J_j^{ia_1\ldots a_m}{}_{b_1\ldots b_p}|^2 \\ = \; & J_j^{ia_1\ldots a_m}{}_{b_1\ldots b_p} \bar{J}_j^{ic_1\ldots c_m}{}_{d_1\ldots d_p} \xi_{a_1c_1}\ldots\xi_{a_mc_m}\xi^{b_1d_1}\ldots\xi^{b_pd_p} \end{aligned} \tag{2.52}$$

In the next chapter we discuss the Yang-Mills connection 1-form A^i_{ja} but we show here a Sobolev norm of this connection as a useful example of a mixed indexed object. $A \in L^2_1$ only if

$$\begin{aligned} ||A^i_{ja}||_{L^2_1} &= \left(\int \sum_{0\le|\alpha|\le 1} |\nabla^\alpha A^i_{ja}|^2 d\sigma\right)^{\frac{1}{2}} \\ &= \left(\int |A^i_{ja}|^2 + |\delta^b_t\nabla_b A^i_{ja}|^2 + |\delta^b_r\nabla_b A^i_{ja}|^2 + |\delta^b_\theta\nabla_b A^i_{ja}|^2 + |\delta^b_z\nabla_b A^i_{ja}|^2 d\sigma\right)^{\frac{1}{2}} \\ &< \infty \end{aligned} \tag{2.53}$$

where we have also illustrated how we take the covariant derivative with the multi-index α.

Finally we define the local Sobolev space $L^p_{q,\mathrm{loc}}(\Lambda)$ for non-compact regions by saying that $f \in L^p_{q,\mathrm{loc}}(\Lambda)$ if $\nabla^\alpha f$ is a measurable function for all $0 \le |\alpha| \le q$ and for all compact subsets C of Λ we have

$$\begin{aligned} & ||J_j^{ia_1\ldots a_m}{}_{b_1\ldots b_n}||_{L^p_{q,\mathrm{loc}}(\Lambda)} \\ = \; & \left(\int_C \sum_{0\le|\alpha|\le q} |\nabla^\alpha J_j^{ia_1\ldots a_m}{}_{b_1\ldots b_n}|^p d\sigma\right)^{\frac{1}{p}} < \infty \end{aligned}$$

Chapter summary

In this chapter we reviewed some of the important background material needed for this thesis, such as different methods of defining singularities. We also introduced more complex ideas such as quasi-regular and weak curvature singularities, distributional curvature, a definition of a positive definite metric on the frame bundle and its use in the construction of the b-boundary.

We also showed how the holonomy around a singularity could be used to find the distributional curvature of the singularity and discussed how previous work by Vickers [49] applied this method for conical singularities.

Finally we introduced Sobolev spaces, which play a crucial role in the following chapters.

Chapter 3

Singularities in Yang-Mills theory

In this chapter we show a method for analysing weak curvature singularities in Yang-Mills theory given by [39] whose GR analogy in Chapter 4 is used to understand the properties of certain curvature singularities.

Although in general the curvature diverges on approach to a singularity, it is possible to assign a distributional curvature in some cases. This method has been used by Vickers [49] to gain an insight into quasi-regular singularities but the restrictions needed (i.e. curvature tending to a limit in a parallely propagated frame) are too strong for curvature singularities. It is our intention to find a method to examine the properties of certain *weak* curvature singularities, those singularities whose curvature is *square Lebesgue integrable.* Sibner and Sibner [39, 40] and Råde [34] show two ways to classify singular Sobolev connections in Yang-Mills gauge theory by examining the holonomy around the singularity. One of the requirements of the Sibners' method is square Lebesgue integrability of the curvature. This has led us to believe that the method is transferable to General Relativity with a similar requirement. We discuss how this might be done in the next chapter.

Råde tactfully comments [34] that the main proof in Sibner and Sibner [39] "is hard to follow". In actual fact, while the theorems are useful for mathematicians and physicists working with Yang-Mills theory, as well as having an analogous application in GR (given in this thesis), the presentation of the proofs is not only lacking in detail but is also incorrect in many places. In most cases the errors seem likely to have been made in the final stages of writing the paper and are typographical and hence do not result in a need to modify the proofs or theorems. Instead it has been necessary to locate and correct the errors in individual places. Because of the lack of detail given in [39], this has been a quite difficult and tedious process and is probably the cause for Råde to have presented his alternate approach.

We start by looking at Yang-Mills fields on Minkowski space for which there exist Minkowski coordinates (t, x, y, z) such that the metric is

$$g_{ab} \stackrel{*}{=} \eta_{ab} \tag{3.1}$$

For the purpose of calculating norms we will also use a positive definite background metric which is given in these coordinates by

$$\xi_{ab} \stackrel{*}{=} \delta_{ab} \tag{3.2}$$

Note that the choice of Minkowski coordinates is not unique so that this prescription does not define the background metric ξ_{ab} uniquely. However, none of the results depend on the particular choice of background metric (up to a change in constants).

The goal of this chapter is to look at fields which are singular on a smooth 2-dimensional submanifold Σ and show how we find the holonomy around Σ. We then demonstrate two important theorems. The first will show that for smooth local curvature and connection in the neighbourhood of a singularity a limit holonomy exists and is independent of position on the singularity. The second theorem shows that by looking at smaller and smaller closed paths around the singularity, we see that the L_1^2 norm of the difference between the connection and a flat connection goes to zero. Hence as one approaches the singularity, the field looks more and more like a singular Yang-Mills field with a locally flat connection.

Much of the process for establishing the two main theorems of this chapter is directly analogous to the process we shall be using in the next chapter and so the technical details of the proofs shall be reserved for the General Relativity counterpart. Introductions, methods, proof outlines and most theorems, corollaries and lemmas will be presented in both chapters, at the expense of some repetition.

Yang-Mills gauge theory

Current understanding of physics is that there exist four natural forces; gravitational, electromagnetic, weak nuclear and strong nuclear. The basic foundation of all but the first of these forces lies in gauge theory. Gauge theories are a class of physical theories based on the concept that symmetry transformations can be performed locally as well as globally. Most physical theories are characterised by Lagrangians which are invariant under certain transformations - those that are identically performed at every point in the space-time; we say that the Lagrangians have global symmetries. Gauge theory extends this idea by requiring that the Lagrangians must have local symmetries as well. Local symmetry transformations can be performed in a particular region of space-time without affecting what happens in another region.

An example of a local gauge symmetry is that in electromagnetic theory, a gauge transformation of the vector potential A leaves the electromagnetic field tensor F unaffected. The local gauge transformation is given by $A \to A + \mathrm{d}f$. However, F and A are related by the equation

$$F = \mathrm{d}A \tag{3.3}$$

so that $F \to \mathrm{d}(A + \mathrm{d}f) = \mathrm{d}A = F$ since $\mathrm{d}^2 = 0$. Hence both A and $A + \mathrm{d}f$ lead to the same field strengths (see for example [30]).

An example of a global gauge symmetry is the Lorentz invariance of Maxwell's equations. Elements L^a_b of the Lorentz group are global gauge transformations and gauge transform the electromagnetic field tensor in the following way

$$\tilde{F}_{ab} = F_{cd} L^c_a L^d_b \tag{3.4}$$

Hence if F_{ab} is a solution of Maxwell's equations then $\tilde{F}_{cd}$ is also a solution (since it is the same solution but in different coordinates).

In general gauge theory is the study of connections on vector bundles with the action of Lie groups. A choice of gauge in a space-time provides an element of the bundle at each point in a manifold. A gauge can thus be seen as a section though the bundle which associates a unique point in the bundle with a unique point on the base manifold. A gauge transformation is a transformation between two sections. Gauge theory provides an abstract mathematical description of various different Yang-Mills fields such as the strong and weak nuclear forces.

We now introduce some of the key concepts for Yang-Mills theory.

The bundle

Definition 3.1 *A bundle is defined to be a triple (P, Π, M) where P and M are topological spaces and $\Pi : P \to M$ is a continuous map. The space P is called the bundle space of the bundle. M is the base space of the bundle. The map Π is the projection and the inverse image $\Pi^{-1}(\{x\})$ is the fibre over $x \in M$.*

We note that in all existing applications in physics the bundles that arise have the special property that the fibres $\Pi^{-1}(\{x\})$, $x \in M$, are all homeomorphic to a common space F known as the fibre of the bundle. These bundles are said to be fibre bundles.

Definition 3.2 *If G is a Lie group then a right G-space is a manifold P equipped with an action by G in such a way that any $g \in G$ acts on P on the right.*

Definition 3.3 *A bundle (P, Π, M) is defined to be a G-bundle if P is a right G-space and if (P, Π, M) is isomorphic to the bundle $(P, \rho, P/G)$ where P/G is the orbit of the space of the G-action on P and ρ is the projection $\rho : P \to P/G$.*

Definition 3.4 *If G acts freely on P then (P, Π, M) is defined to be a principal G-bundle, and G is then called the structure group of the bundle.*

Let (P, Π, M) be a principal fibre bundle over the manifold M with structure group G and projection $\Pi : P \to M$. Given $x \in M$ then $\Pi^{-1}(x)$ is a closed submanifold of P which is diffeomorphic to G and is called the fibre at x. The group G acts freely on P on the right

$$\begin{aligned} R : P \times G &\to P \\ (u, g) &\mapsto ug = R_g u \end{aligned} \tag{3.5}$$

For every open ball $U \in M$ we can look at $\Pi^{-1}(U) \in P$. There exists a homeomorphism ϕ which takes $\Pi^{-1}(U)$ to $U \times G$ which is part of the product bundle space $(M \times G)$ which we call the trivial bundle.

We define a Yang-Mills theory to follow from a functional of the form

$$I = \int_M F^i_{jab} F^{jab}_i d^4x \tag{3.6}$$

where F^i_{jab} is the curvature 2-form of a principal fibre bundle with arbitrary Lie fibre group, G.

For our description of Yang-Mills theory we will take P to be a principal, $SU(2)$-bundle over Minkowski space M. The use of the Lie group $SU(2)$ provides a special case for Yang-Mills theory but it has all the features of more general gauge groups (e.g. it is non-Abelian) so we use it as an illustration and a vehicle for showing concrete calculations.

The connection

From Chapter 2 we already understand the concept of a gauge independent connection 1-form on the bundle LM

$$\underline{\omega}_p : T_p(LM) \to gl(4, \mathbb{R}) \tag{3.7}$$

where $p \in LM$.

In Yang-Mills we have the connection 1-form $\underline{A}_q$ on the bundle P

$$\underline{A}_q : T_q P \to su(2) \tag{3.8}$$

where $q \in P$.

We can now choose a gauge σ (also known as a section) through the bundle

$$\sigma : M \to P \tag{3.9}$$

noting that $\Pi \cdot \sigma : M \to M$ is the identity map on the manifold but $\sigma \cdot \Pi : P \to P$ is not the identity map on the bundle. This arises because σ is injective and Π is surjective (neither are bijective).

The pushforward of σ is given by the derivative of σ

$$\sigma_* : T_x M \to T_{\sigma(x)} P \tag{3.10}$$

for any $x \in M$. If $X \in T_x M$ then we can define $\sigma^* \underline{A}_q$, the pullback of σ on $\underline{A}_q$, in terms of the pushforward

$$\left(\sigma^* \underline{A}_q\right)(X) := \underline{A}_q \left(\sigma_*(X)\right) \tag{3.11}$$

Using the gauge σ we can now define the gauge dependent connection 1-form A on M.

Definition 3.5 *The pullback of σ acting on the gauge independent connection 1-form $\underline{A}_q$ is defined to be the gauge dependent connection 1-form A, with elements taking values in the Lie algebra $su(2)$, such that*

$$\begin{aligned} A : T_xM &\rightarrow su(2) \\ X^a &\mapsto (A(X))^i_j = X^a A^i_{ja} \end{aligned} \tag{3.12}$$

From here onwards reference to the 'connection 1-form' always pertains to the gauge dependent connection 1-form, unless otherwise stated.

We now consider a Lie group valued function s such that

$$s : M \rightarrow SU(2) \tag{3.13}$$

We use s to make a gauge transformation on σ to a new gauge $\tilde{\sigma}$ in the following way

$$R_s\sigma = \tilde{\sigma} : M \rightarrow P \tag{3.14}$$

In this new gauge we now define a different connection 1-form on M

$$\tilde{\sigma}^*\underline{A} = \tilde{A} \tag{3.15}$$

As a consequence of (3.11), (3.14) and (3.15) we now have a relationship between A and $\tilde{A}$, given by

$$\tilde{A} = \tilde{\sigma}^*\underline{A} = (R_s\sigma)^*\,\underline{A} = s^{-1}As + s^{-1}ds \tag{3.16}$$

Two connection 1-forms A and $\tilde{A}$ are gauge equivalent if there is a gauge transformation $s : M \rightarrow SU(2)$ such that for all $x \in M$ (3.16) is satisfied. More explicitly

$$\tilde{A}^i_{ja} = ((s(x))^{-1})^i_k A^k_{la}(s(x))^l_j + ((s(x))^{-1})^i_k(\mathrm{d}(s(x)))^k_{ja} \tag{3.17}$$

In the future we will shorten this and similar expressions to $\tilde{A}^i_{ja} = (s^{-1})^i_k A^k_{la} s^l_j + (s^{-1})^i_k(\mathrm{d}s)^k_{ja}$ or just $\tilde{A} = s^{-1}As + s^{-1}\mathrm{d}s$ and remember, when omitting x, that gauge transformations vary with position on the manifold.

We now consider the exterior covariant derivative $D = \mathrm{d} + A$, as given in [23] (note, in accordance with the custom in Pure Mathematics, D is referred to as the connection in [23]). Here d is the exterior derivative of the space-time part and A is the Lie algebra valued connection 1-form. The exterior covariant derivative acts on some Lie algebra valued scalar α^i_j in the following way.

$$\begin{aligned} D\alpha^i_j &= \alpha^i_{j,a}dx^a + A^i_{ka}\alpha^k_j dx^a \\ \text{or } (D\alpha)^i_{ja} &= \alpha^i_{j,a} + A^i_{ka}\alpha^k_j \end{aligned} \tag{3.18}$$

and acts on a Lie algebra valued p-form $\beta^i_{ja_1\ldots a_p}$ like

$$(D\beta)^i_{ja_1\ldots a_p b} = (-1)^p \beta^i_{j[a_1\ldots a_p,b]} + A^i_{k[b}\beta^k_{|j|a_1\ldots a_p]} \tag{3.19}$$

The curvature

We can now define the curvature 2-form in Yang-Mills.

Definition 3.6 *The curvature 2-form F on M has elements which take values in $su(2)$ and is antisymmetric on its spacetime indices*

$$\begin{aligned} F : T_xM \times T_xM &\to su(2) \\ (X^a, Y^b) &\mapsto X^a Y^b F^i_{jab} \end{aligned} \tag{3.20}$$

and is defined by its relation to the connection 1-form as given by

$$\begin{aligned} F = DA = &= dA + [A, A] \\ &\quad or \\ F^i_{jab} &= 2A^i_{j[b,a]} + A^i_{ka}A^k_{jb} - A^i_{kb}A^k_{ja} \end{aligned} \tag{3.21}$$

Note that this has the analogue $R^a{}_{bcd} = 2\Gamma^a_{b[d,c]} + \Gamma^a_{ec}\Gamma^e_{bd} - \Gamma^a_{ed}\Gamma^e_{bc}$ in General Relativity.

An important feature of the curvature F is that it transforms homogeneously under a gauge transformation (unlike the connection A).

Lemma 3.7 *If $F = dA + [A, A]$ and $\hat{F} = d\hat{A} + [\hat{A}, \hat{A}]$ and A and $\hat{A}$ are gauge equivalent, then $\hat{F} = s^{-1}Fs$.*

Proof

$$
\begin{aligned}
\hat{F} &= \mathrm{d}\hat{A} + [\hat{A}, \hat{A}] \\
&= \hat{A}^i_{jb,a} - \hat{A}^i_{ja,b} + \hat{A}^i_{ka}\hat{A}^k_{jb} - \hat{A}^i_{kb}\hat{A}^k_{ja} \\
&= \left((s^{-1})^i_k A^k_{lb} s^l_j + (s^{-1})^i_k s^k_{j,b}\right)_{,a} - \left((s^{-1})^i_k A^k_{la} s^l_j + (s^{-1})^i_k s^k_{j,a}\right)_{,b} \\
&\quad + \left((s^{-1})^i_m A^m_{na} s^n_k + (s^{-1})^i_m s^m_{k,a}\right)\left((s^{-1})^k_p A^p_{qb} s^q_j + (s^{-1})^k_p s^p_{j,b}\right) \\
&\quad - \left((s^{-1})^i_m A^m_{nb} s^n_k + (s^{-1})^i_m s^m_{k,b}\right)\left((s^{-1})^k_p A^p_{qa} s^q_j + (s^{-1})^k_p s^p_{j,a}\right) \\
&= (s^{-1})^i_k A^k_{lb,a} s^l_j - (s^{-1})^i_k A^k_{la,b} s^l_j + (s^{-1})^i_k A^k_{na} A^n_{lb} s^l_j - (s^{-1})^i_k A^k_{nb} A^n_{la} s^l_j \\
&= s^{-1} F s \qquad \square
\end{aligned}
\tag{3.22}
$$

Hence F is homogeneous under a gauge transformation.

Corollary 3.8 *Connections that have zero curvature in one gauge transform to connections that have zero curvature in another gauge. We call these connections, flat connections.*

The action in Yang-Mills theory is given by

$$
I = \int_M \mathrm{Tr}(F \wedge *F) \tag{3.23}
$$

where $*$ is the Hodge star operator as described in [31]. Since F transforms homogeneously we know that the Yang-Mills action is gauge-invariant. In index notation we can rewrite (3.23) as

$$
I = \frac{1}{4}\int_M F^i_{jab} F_i^{jab} d^4x \tag{3.24}
$$

Minimising the action gives the source free Yang-Mills equations $D * F = 0$.
The electromagnetic action

$$
I = \int_M F_{ab} F^{ab} d^4x \tag{3.25}
$$

is one specific example of the Yang-Mills action of (3.24) for which the gauge group is $U(1)$. The Yang-Mills equations are then simply $\nabla \cdot F = 0$ which is equivalent to the Maxwell's equations

$$
\nabla \cdot E = 0 \qquad \text{and} \qquad \nabla \times B = \dot{E} \tag{3.26}
$$

where B is the magnetic field, E is the electric field and $\dot{E}$ is the derivative of E with respect to time.

3.1 The holonomy method in Yang-Mills

We now set out the general method by which we establish the existence of limit holonomy around a weak singularity in Yang-Mills theory. Note that our terminology differs from that in [39] where holonomy is treated as being the conjugacy class $[J_r]$ of phase factors J_r. This is the *gauge invariant* approach. We will simply consider J_r to be the holonomy but bear in mind that this holonomy will change depending on what gauge we are in. As we shall show later, if a connection A has holonomy J_r (forming the conjugacy class $[J_r]$), then if we gauge transform A, the new holonomy will also be conjugate to J_r and thus will be in the same conjugacy class $[J_r]$ although the holonomy itself will not necessarily be J_r.

Conditions on the connection and curvature

We first discuss the geometry of the space in which we carry out the method and then give some preliminary conditions on the curvature and connection forms.

The manifold M we are to work with is 4-dimensional with a 2-dimensional smoothly imbedded, timelike, orientable, connected singular set $\Sigma \in M$. We let coordinates on the singular 2-surface be (u, v). For each point on the singularity we then look at flat space-like 2-planes which are normal to the singularity at that point.

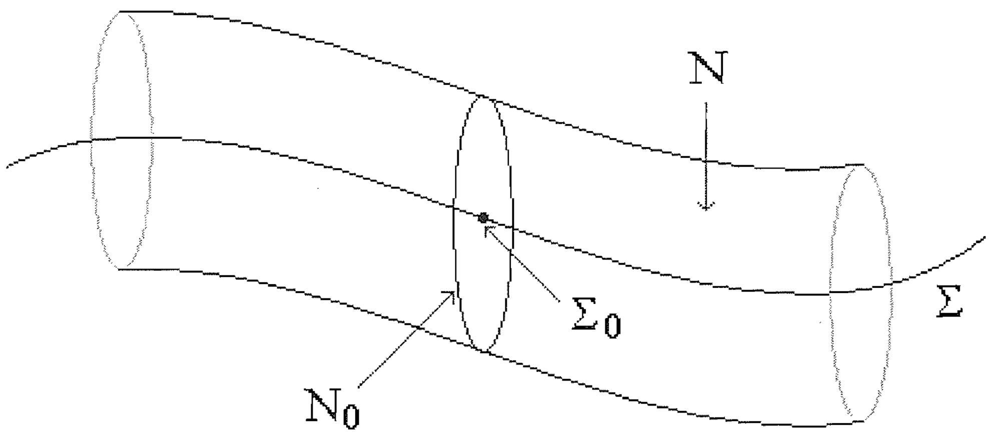

Figure 3.1: A diagram showing the singularity Σ, the ϵ-neighbourhood N, a point on the singularity Σ_0 and the disk N_0 which is normal to Σ at Σ_0.

Since the singularity is not necessarily a plane we must ensure that our normal planes, for each point on the singularity, do not intersect. For this purpose when we look at normal planes to the singularity we do so only within a ball of radius ϵ, where ϵ is sufficiently small that the normal plane does not intersect other normal planes (normal at different points on the singularity) within the ball. We label this ϵ-neighbourhood of the singularity, N. For ϵ sufficiently small, N is locally a product of Σ with a normal disk (see Figure 3.1). We then look at a given point on the singularity Σ_0 and its corresponding small disk N_0 which lies in the plane normal to Σ at Σ_0. Since the metric on this 2-plane is flat and positive definite, there exist coordinates such that the metric is given by

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{3.27}$$

and so points on the normal plane to each point on the singularity are given by the polar coordinates (r, θ).

We let $X = N \backslash \Sigma$ and $X_0 = N_0 \backslash \Sigma_0$.

We require $A \in L^2_{1,\text{loc}}(X)$ as one of our conditions and call such a connection a 'Sobolev connection'. As shown in Chapter 2, $A \in L^2_{1,\text{loc}}(X)$ means that $\nabla^\alpha A$ is a measurable function for all $0 \leq |\alpha| \leq 1$ and for all compact subsets C of X we have from (2.53)

$$\|A^i_{ja}\|_{L^2_{1,\text{loc}}(X)} = \left(\int_C \sum_{0 \leq |\alpha| \leq 1} |\nabla^\alpha A^i_{ja}|^2 d\sigma \right)^{\frac{1}{2}} < \infty \tag{3.28}$$

We also demand that the curvature 2-form F belongs to $L^2(N)$, which is equivalent to the requirement that the Yang-Mills action I is finite. Note that $F \in L^2_{\text{loc}}(X)$ is an immediate consequence of the formula $F = \mathrm{d}A + [A, A]$ and using the Sobolev imbedding $L^2_1 \to L^4$ on $A \in L^2_{1,\text{loc}}(X)$ (see Appendix B.1 and [1]). However we also require the stonger global condition $F \in L^2(X)$ (not simply $L^2_{\text{loc}}(X)$) in some of the proofs and so we impose this as an additional requirement.

Holonomy around the singularity

We now find the holonomy around the singularity.

We take a particular point on the 2-dimensional singularity and consider the normal (r, θ) plane. We look at the singularity and a closed path given by $\gamma_r(\theta)$ in the manifold, starting at a point $x \in \mathbb{R}^4$. The path encircles the singularity at distance r. As θ increases we move along the path. γ_r (and later also λ_r, μ_r and g_r) is an object that depends on r, but r is fixed unless otherwise stated.

We consider the bundle above the manifold which is locally $SU(2) \times X_0$ where each point in the bundle represents an element of the Lie group $SU(2)$ at a point in the manifold X_0. The global structure of the bundle may be more complicated but since we are only looking at an open ball around a singularity, we need only consider local structure. A projection, Π, takes points in $SU(2) \times X_0$ to X_0. The projection discards information about the element leaving only its position. The fibre above $y \in X_0$ is the set of all points $\alpha \in SU(2) \times X_0$ such that $\Pi(\alpha) = y$. The fibre represents all possible $SU(2)$ group elements at a point in X_0.

We let $\gamma_r(\theta)$ be the projection of a particular path $\lambda_r(\theta)$ in $SU(2) \times X_0$ which describes the parallel propagation of a basis element E_i^a. The path $\lambda_r(\theta)$ contains not only the information of the position of the path $\gamma_r(\theta)$ in the manifold but also the information of how the element is propagated around this path.

Now consider a reference section (or gauge choice) through $SU(2) \times X_0$, passing through the point $\lambda_r(0)$ which projects down to $x \in X_0$. On this plane we draw, starting at $\lambda_r(0)$, the closed path which projects down to γ_r and we shall call this path $\mu_r(\theta)$. Since $\mu_r(\theta)$ is a closed path, we know that $\mu_r(0) = \mu_r(2\pi)$. Note also that $\mu_r(\theta)$ and $\lambda_r(\theta)$ both lie on the same fibre above $\gamma_r(\theta)$ for each θ.

There is a gauge transformation $g : M \to SU(2)$ such that for each θ we have $g_r(\theta) \in SU(2) : SU(2) \to SU(2)$, that takes $\mu_r(\theta)$ to $\lambda_r(\theta)$. i.e.

$$\lambda_r(\theta) = R_{g_r(\theta)}\mu_r(\theta) \tag{3.29}$$

Since $\mu_r(0) = \lambda_r(0)$ we have $g_r(0) = I$.

We now wish to establish a first order differential equation for the gauge transformation $g_r(\theta)$ in terms of the connection 1-form A. The solution of this equation at $\theta = 2\pi$ will give us the holonomy around the singularity.

Let $\gamma^a(s) = (r, s, u_0, v_0)$ be a closed path in the manifold around a point on the singularity (u_0, v_0) at a fixed distance r. We consider a point in the fibre, E. The gauge transformation

g_r takes E to the point $\tilde{E}$, where $\tilde{E}$ is the result of parallel propagation of E along the path γ. We have

$$\tilde{E}_i^a = g_i^k E_k^a \tag{3.30}$$

Since we are applying parallel propagation we have

$$D_T \tilde{E}_i^a = 0 \tag{3.31}$$

where

$$T^a = \frac{d\gamma^a(s)}{ds} = \delta_\theta^a \tag{3.32}$$

We now evaluate (3.31).

$$\begin{aligned}
D_T \tilde{E}_i^a &= 0 \\
D_T(g_i^k E_k^a) &= 0 \\
(D_T g_i^k) E_k^a + g_i^k D_T E_k^a &= 0 \\
T^c E_k^a D_c g_i^k + g_i^k T^c D_c E_k^a &= 0 \\
T^c E_k^a g_{i,c}^k + g_i^k T^c A_{kc}^j E_j^a &= 0 \\
\delta_\theta^c E_k^a g_{i,c}^k + \delta_\theta^c g_i^k A_{kc}^j E_j^a &= 0 \\
E_k^a g_{i,\theta}^k + g_i^k A_{k\theta}^j E_j^a &= 0 \\
{E^{-1}}^l_{\ a} E_k^a g_{i,\theta}^k + g_i^k A_{k\theta}^j E_j^a {E^{-1}}^l_{\ a} &= 0 \\
\delta_k^l g_{i,\theta}^k + g_i^k A_{k\theta}^j \delta_j^l &= 0 \\
g_{i,\theta}^l + g_i^k A_{k\theta}^l &= 0 \quad \square
\end{aligned} \tag{3.33}$$

Which we can write as

$$\frac{dg_r}{d\theta} + A_\theta g_r = 0 \tag{3.34}$$

since g_r for fixed r depends only on θ so $\frac{dg}{d\theta} = \frac{\partial g}{\partial \theta}$.

Since $\mu_r(0) = \lambda_r(0)$, our initial point is the identity and we have the initial value problem

$$\frac{dg_r}{d\theta} + A_\theta g_r = 0, \qquad g_r(0) = I \tag{3.35}$$

the solution of which would be obtained by finding the integral of a path ordered exponential as given in [55].

The transformation taking the initial point at $\theta = 0$ to the final point at $\theta = 2\pi$ is the holonomy which we now define as $g_r(2\pi) = J_r$. If it exists, the limit holonomy $J^\flat$ of the singularity is the limit as r tends to zero of J_r.

Later in this chapter we shall prove that this limit holonomy exists for the given conditions on the connection and curvature.

Properties of holonomy

The proofs of the two critical theorems in this chapter rely on various properties of holonomy, connections and, in particular, flat connections which we include below as lemmas.

We first define conjugacy for a group G.

Definition 3.9 *Two elements a and b of G are conjugate if there exists some $c \in G$ such that $a = c^{-1}bc$.*

We would like to show that $[J_r]$, the conjugacy class of J_r, is gauge invariant and we express this in the following lemma.

Lemma 3.10 *If g_r and $\hat{g}_r$ are two solutions of (3.35), for gauge equivalent connections $D =\mathrm{d}+A$ and $\hat{D} =\mathrm{d}+\hat{A}$, then $g_r(2\pi)$ and $\hat{g}_r(2\pi)$ are conjugate in $SU(2)$.*

Proof

We let $s : X_0 \to SU(2)$ be the continuous gauge transformation between A and $\hat{A}$. The connection $\hat{A}$ in the new gauge is given by $\hat{A} = s^{-1}As + s^{-1}ds$. We are always taking gauge transformations at different points on the same path γ and so we shall simplify $s(\gamma(\theta))$ to just $s(\theta)$. Since $\gamma_r(0) = \gamma_r(2\pi)$ we have $s(0) = s(2\pi)$. (3.35) now becomes

$$\frac{d\hat{g}_r}{d\theta} + \left(s^{-1}A_\theta s + s^{-1}\frac{ds}{d\theta}\right)\hat{g}_r = 0 \qquad \hat{g}_r(0) = I \tag{3.36}$$

Now, substitute the function $g_r(\theta) = s(\theta)\hat{g}_r(\theta)s^{-1}(0)$ into (3.35)

$$
\begin{aligned}
\frac{d}{d\theta}\left(s\hat{g}_r s^{-1}(0)\right) + A_\theta s\hat{g}_r s^{-1}(0) &= 0 \\
\frac{ds}{d\theta}\hat{g}_r s^{-1}(0) + s\frac{d\hat{g}_r}{d\theta}s^{-1}(0) + A_\theta s\hat{g}_r s^{-1}(0) &= 0 \\
s^{-1}\frac{ds}{d\theta}\hat{g}_r + \frac{d\hat{g}_r}{d\theta} + s^{-1}A_\theta s\hat{g}_r &= 0 \\
\frac{d\hat{g}_r}{d\theta} + \left(s^{-1}A_\theta s + s^{-1}\frac{ds}{d\theta}\right)\hat{g}_r &= 0
\end{aligned}
\tag{3.37}
$$

and so $s(\theta)\hat{g}_r(\theta)s^{-1}(0)$ solves (3.35) in the original gauge. Hence, by uniqueness

$$
\begin{aligned}
g_r(\theta) &= s(\theta)\hat{g}_r(\theta)s^{-1}(0) \\
\Longrightarrow g_r(2\pi) &= s(2\pi)\hat{g}_r(2\pi)s^{-1}(2\pi) \qquad \square
\end{aligned}
\tag{3.38}
$$

We note that Lemma 3.10 holds if A is related to $\hat{A}$ by a 'weak' gauge transformation $s \in L^2_{2,\mathrm{loc}}$, which need neither be smooth nor continuous [39]. Hence if A is weakly gauge equivalent to a connection $\hat{A}$ for which $[\hat{J}]$ has a limit, then the limit holonomy for A also exists and belongs to the conjugacy class $[\hat{J}]$.

We now wish to show that if the connection were flat, then the conjugacy class of the holonomy, $[J_r]$, would be a homotopy invariant and hence independent of r.

Lemma 3.11 *If $D = \mathrm{d}+A$ describes a flat connection then the associated conjugacy class of the holonomy is homotopy invariant.*

Proof

The proof shall consider three cases.

Case 1

We look at two homotopic closed paths around the singularity, γ_1 and γ_2, which start (and hence finish) at the same point. We wish to show that for a flat connection, the holonomy generated by these loops are the same. The holonomy of a closed path is related to the exponential of the integral of the curvature over the region enclosed. Since each path encircles the singularity, where the curvature is undefined, we instead look at the region enclosed by the path $\gamma_2^{-1}\gamma_1$. This region does not encircle the singularity and so the integral of the curvature over this path is zero. Therefore the exponential is the identity and hence we have, where $\mathrm{Hol}(E(0,\gamma_1))$ is the rotation generated by parallely propagating a frame around γ_1

$$(\mathrm{Hol}(E(0,\gamma_2)))^{-1}\mathrm{Hol}(E(0,\gamma_1)) = I \quad \Longrightarrow \quad \mathrm{Hol}(E(0,\gamma_1)) = \mathrm{Hol}(E(0,\gamma_2)) \tag{3.39}$$

and hence the holonomy of a loop does not change when the loop is continuously deformed as long as the two loops share a point.

Case 2

Let us consider two loops A and B around a singularity which do not intersect each other. We take a loop C around the singularity with the particular feature that it intersects A and B in exactly one place each. From Case 1 we know that loop A has the same holonomy up to conjugacy as loop C and likewise that loop C has the same holonomy up to conjugacy as loop B. Hence A and B have the same holonomy up to conjugacy.

Case 3

We must also consider the case where two loops A and B around a singularity intersect each other in multiple places. Once again we choose a third loop C around the singularity which does not intersect either loop. Now from Case 2 we know that A and C have the same holonomy up to conjugacy and also from Case 2 we know that B and C have the same holonomy up to conjugacy. Hence A and B have the same holonomy up to conjugacy.

These three cases combined show that if we have a flat connection then the conjugacy class of the holonomy is homotopy invariant. □

Lemma 3.12 *There is a unique correspondence between conjugacy classes of holonomy and flat connections.*

As we have shown before, conjugacy classes of holonomy are conjugacy classes in $SU(2)$. For positive values of θ we can uniquely describe conjugacy classes in $SU(2)$ by their trace (since the other matrix invariant is the determinant, which is always one) which will be between -2 and 2. To uniquely describe any one conjugacy class in $SU(2)$ we use its diagonal representative of the form

$$\begin{pmatrix} e^{-2\pi i m} & 0 \\ 0 & e^{2\pi i m} \end{pmatrix} \qquad m \in \mathbb{R} \tag{3.40}$$

The prototype of the flat connection is given by

$$A^{\flat} = \begin{pmatrix} im & 0 \\ 0 & -im \end{pmatrix} d\theta = m\hat{i}d\theta \qquad m \in \mathbb{R} \tag{3.41}$$

where

$$\hat{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{3.42}$$

We now solve (3.35) with $A_\theta = A^{\flat}_\theta = m\hat{i}$ to get

$$g_r = \begin{pmatrix} e^{-im\theta} & 0 \\ 0 & e^{im\theta} \end{pmatrix} \tag{3.43}$$

We know from Lemma 3.11 that the solution of (3.35) is homotopy invariant for flat bundles and so in this case $g_r(\theta) = g(\theta)$. Hence the holonomy of the prototype of the flat connection is

$$g(2\pi) = \begin{pmatrix} e^{-2\pi i m} & 0 \\ 0 & e^{2\pi i m} \end{pmatrix} \tag{3.44}$$

which is equal to the diagonal representative of the conjugacy class given by (3.40). Note that two different values of m which are separated by integer values, will yield the same diagonal representative. Hence there is a unique correspondence between flat connections (modulo 1) and conjugacy classes of holonomy. $\square$

Lemma 3.13 *If $m_1 = m_2 + n$, $n \in \mathbf{Z}$, then there exists a gauge transformation g such that $g^{-1}A^{\flat}_2 g + g^{-1}\mathrm{d}g = A^{\flat}_1$ where*

$$A^{\flat}_1 = \begin{pmatrix} im_1 & 0 \\ 0 & -im_1 \end{pmatrix} d\theta \qquad A^{\flat}_2 = \begin{pmatrix} im_2 & 0 \\ 0 & -im_2 \end{pmatrix} d\theta \tag{3.45}$$

Proof

Take

$$g(\theta) = \begin{pmatrix} e^{in\theta} & 0 \\ 0 & e^{-in\theta} \end{pmatrix} \tag{3.46}$$

Since g and $A_2^\flat$ are diagonal, $g^{-1}A_2^\flat g = A_2^\flat$. So gauge transforming $A_2^\flat$ with this $g(\theta)$ it follows that

$$\begin{aligned} A_2^\flat + g^{-1}\mathrm{d}g &= \begin{pmatrix} im_2 & 0 \\ 0 & -im_2 \end{pmatrix} d\theta + \begin{pmatrix} e^{-in\theta} & 0 \\ 0 & e^{in\theta} \end{pmatrix} \begin{pmatrix} ine^{in\theta} & 0 \\ 0 & -ine^{-in\theta} \end{pmatrix} d\theta \\ &= \left[\begin{pmatrix} im_2 & 0 \\ 0 & -im_2 \end{pmatrix} + \begin{pmatrix} in & 0 \\ 0 & -in \end{pmatrix} \right] d\theta \\ &= \begin{pmatrix} i(m_2+n) & 0 \\ 0 & -i(m_2+n) \end{pmatrix} d\theta \\ &= \begin{pmatrix} im_1 & 0 \\ 0 & -im_1 \end{pmatrix} d\theta \\ &= A_1^\flat \end{aligned} \tag{3.47}$$

and hence $A_1^\flat$ and $A_2^\flat$ are gauge equivalent. $\square$

We note that since n is an integer, we have

$$\begin{aligned} g(2\pi) &= \begin{pmatrix} e^{2i\pi n} & 0 \\ 0 & e^{-2i\pi n} \end{pmatrix} \\ &= I \end{aligned} \tag{3.48}$$

Lemma 3.14 *If m is an integer then D is gauge equivalent to* d.

Proof

From Lemma 3.13 we know that $A_1^\flat$ and $A_2^\flat$ are gauge equivalent and hence so are $D_1 = \mathrm{d} + A_1^\flat$ and $D_2 = \mathrm{d} + A_2^\flat$. Since m_1 is an integer we can choose n to be m_1 also and hence $m_2 = 0$.

$$\Longrightarrow A_2^\flat = \begin{pmatrix} i0 & 0 \\ 0 & -i0 \end{pmatrix} d\theta = \underline{0} \tag{3.49}$$

Hence $D_2 = \mathrm{d}$. Therefore $D_1 = \mathrm{d} + A_1^\flat$ is gauge equivalent to d. $\square$

From this it can be seen that, locally, flat bundles can be indexed by m, which we shall call the holonomy number, belonging to the finite interval

$$0 \leq m < 1 \tag{3.50}$$

Lemma 3.15 *If $f(2\pi)$ and $h(2\pi)$ are two conjugate elements of a group and $f(0) = h(0) = \mathrm{Id}$ then there exists a continuous periodic $k(\theta)$ in the group such that $f(\theta) = k^{-1}(\theta)h(\theta)k(2\pi)$ which provides the conjugacy at $\theta = 2\pi$.*

Proof

Since $f(2\pi)$ and $h(2\pi)$ are conjugate, we know that there is an element of the group a such that $f(2\pi) = a^{-1}h(2\pi)a$. Now let a be the specific value for the group element $k(\theta)$ when $\theta = 2\pi$, so $a = k(2\pi)$. Since we have $k(2\pi) = h(2\pi)k(2\pi)f^{-1}(2\pi)$, we now define $k(\theta)$ for each value of θ as $k(\theta) = h(\theta)k(2\pi)f^{-1}(\theta)$. Since $h(0) = f(0) = \mathrm{Id}$ we know that $k(0) = \mathrm{Id}k(2\pi)\mathrm{Id}^{-1} = k(2\pi)$ and so k is continuous and periodic.

Lemma 3.16 *Let $D = \mathrm{d} + A$ be a flat connection with holonomy m. There is a gauge in which $D = \mathrm{d} + A^\flat$ where*

$$A^\flat = \begin{pmatrix} im & 0 \\ 0 & -im \end{pmatrix} d\theta \tag{3.51}$$

Proof

From Lemma 3.12 we know that the holonomy $g(2\pi)$ of the flat connection A will be conjugate to the holonomy $g^\flat(2\pi) = \begin{pmatrix} e^{-2\pi im} & 0 \\ 0 & e^{2\pi im} \end{pmatrix}$ of the prototype of the flat connection $A^\flat$. From Lemma 3.15 we know that the conjugacy is provided by a continuous periodic $k(\theta)$ such that $g^\flat(\theta) = k^{-1}(\theta)g(\theta)k(2\pi)$ which we rewrite as

$$g = kg^\flat k^{-1}(2\pi) \tag{3.52}$$

remembering that the non-constants are functions of θ.

For our original flat connection A we have $\frac{dg}{d\theta} + A_\theta g = 0$. Using (3.52) we now get

$$\begin{aligned}
\frac{d}{d\theta}(kg^\flat k^{-1}(2\pi)) + A_\theta(kg^\flat k^{-1}(2\pi)) &= 0 \\
\frac{dk}{d\theta}g^\flat k^{-1}(2\pi) + k\frac{dg^\flat}{d\theta}k^{-1}(2\pi) + A_\theta kg^\flat k^{-1}(2\pi) &= 0 \\
\frac{dk}{d\theta}g^\flat + k\frac{dg^\flat}{d\theta} + A_\theta kg^\flat &= 0 \\
\frac{dg^\flat}{d\theta} + (k^{-1}A_\theta k + k^{-1}\frac{dk}{d\theta})g^\flat &= 0 \qquad (3.53)
\end{aligned}$$

Since $g^\flat$ is the solution for the prototype of the flat connection

$$\frac{dg^\flat}{d\theta} + A^\flat_\theta g^\flat = 0 \qquad (3.54)$$

we have by uniqueness

$$k^{-1}A_\theta k + k^{-1}\frac{dk}{d\theta} = A^\flat_\theta \qquad (3.55)$$

Hence there exists a gauge transformation k, which takes a flat connection A to the prototype of the flat connection $A^\flat$. □

We have now established important lemmas concerning flat connections and holonomy but we have yet to show that the limit holonomy given by $\lim_{r\to 0} J_r = J^\flat$ exists. The aim of the next section is to be able to state that

"$\lim_{r\to 0}[J_r] = [J]$ exists for almost all $P \in \Sigma$, and is independent of P."

3.2 Existence of limit holonomy

The theorem below gives sufficient analytic conditions for the limit holonomy condition to be satisfied.

Theorem 3.17 (Sibner and Sibner) *Let N be a normal ϵ-neighbourhood of the 2-dimensional submanifold Σ of a 4-manifold M. If D =d+A with $A \in L^2_{1,loc}(N\backslash\Sigma)$ and $F \in L^2(N)$ then locally, there is a gauge in which the components of A have a limit in Σ with, in particular, $A_\theta \to m\hat{i}$ almost everywhere. The holonomy limit of D at Σ exists and is independent of the normal plane to the singular 2-surface.*

Overview of proof

There are three key statements we wish to prove.

A) There exists a gauge such that $A_\theta \to \begin{pmatrix} im & 0 \\ 0 & -im \end{pmatrix}$ as $r \to 0$.

B) The limit holonomy exists at Σ.

C) The limit holonomy is independent of the point at which we take the normal plane to the singularity.

We first gauge transform the connection into the radial gauge where $A_r = 0$. We then look at the θ component of A and show that $\lim_{r\to 0} A_\theta = C_\theta$ where $C = C_\theta d\theta$ is a flat connection. We know from Lemma 3.16 that any flat connection is gauge equivalent to another flat connection

$$A^\flat = \begin{pmatrix} im & 0 \\ 0 & -im \end{pmatrix} d\theta \qquad \text{for some } 0 \leq m < 1 \tag{3.56}$$

and m classifies the gauge equivalence class. So there is an s such that $s^{-1}Cs + s^{-1}\mathrm{d}s = A^\flat$ and since C and $A^\flat$ depend only on θ, so too does s. We then gauge transform A with s to get $\hat{A} = s^{-1}As + s^{-1}\mathrm{d}s$. We know that since s only depends on θ

$$\lim_{r\to 0} \hat{A}_\theta = \lim_{r\to 0}(s^{-1}A_\theta s + s^{-1}\frac{ds}{d\theta}) = s^{-1}C_\theta s + s^{-1}\frac{ds}{d\theta} \tag{3.57}$$

and

$$\lim_{r\to 0} \hat{A}_\theta d\theta = \lim_{r\to 0}(s^{-1}As + s^{-1}\mathrm{d}s) = A^\flat \tag{3.58}$$

Hence, by first gauge transforming to the radial gauge and then gauge transforming using s we find that there exists a gauge such that $A_\theta \to \begin{pmatrix} im & 0 \\ 0 & -im \end{pmatrix}$ as $r \to 0$.

$\square$A

g_r solves (3.35) for $D = \mathrm{d} + A$ and $\hat{g}_r$ is the solution for $\hat{D} = \mathrm{d} + \hat{A}$. Since D is gauge equivalent to $\hat{D}$ we can apply Lemma 3.10 to show that $g_r(2\pi) = J_r$ and $\hat{g}_r(2\pi) = \hat{J}_r$ are conjugate in $SU(2)$. We know that $\hat{g}_r(2\pi) = \hat{J}_r$ depends only on $\hat{A}_\theta$ and having shown that $\lim_{r\to 0} \hat{A}_\theta = A^\flat$ we can then apply Lemma 3.12 to deduce that $\lim_{r\to 0} \hat{J}_r = \hat{J}$ exists. Since

J_r and $\hat{J}_r$ are conjugate, we know that $[J_r] = [\hat{J}_r]$ and hence that $\lim_{r \to 0}[J_r] = [J]$ exists. □B

Having established a limit holonomy for one point P on the singularity, we then show that this limit does not change if we choose a different point on the singularity, say P'. We first change the coordinates on the singularity so that P and P' both lie on the line $v = v_0$. We then show that the difference between the holonomy at P and the holonomy at P' is bounded by an expression which we know tends to zero as r tends to zero. Hence the limit holonomy is independent of the point at which we take the normal plane to the singularity. □C

The detailed proof of Theorem 3.17 has a direct analogy to the proof in the GR case and so is included in Chapter 4 but not here.

3.3 Similarities to a flat connection as $r \to 0$

In this section we bring together ideas needed for the second main result of [39]. We first consider the set of connections $\mathcal{A}^p$ where $p \geq 2$

$$\mathcal{A}^p = \left\{ D = \mathrm{d} + A | A \in L^p_{1,\mathrm{loc}}(X_0) \text{ and } F \in L^p(N_0) \right\} \tag{3.59}$$

In the previous section we have demonstrated the existence of limit holonomy for $D \in \mathcal{A}^2$. It follows from Theorem 3.17 that if $D \in \mathcal{A}^p$ then

"A limit holonomy exists and is given by the real number m as shown in (3.50)". (H_m)

We call the above m-dependent holonomy condition (H_m). If we know that (H_m) is satisfied for a connection in $\mathcal{A}^p$ then we can find a prototype of the flat connection $A^\flat = m\hat{i}d\theta$ which shares the same holonomy.

We introduce a new space $L^p_{1,\Upsilon}$ where Υ is a flat connection. A 1-form ζ is in $L^p_{1,\zeta}$ if

$$\|\zeta\|_{L^p_{1,\Upsilon}} = \|\zeta\|_{L^p} + \|\nabla_\Upsilon \zeta\|_{L^p} = \|\zeta\|_{L^p} + \|\mathrm{d}\zeta + [\zeta, \Upsilon]\| < \infty \tag{3.60}$$

We now give the main result for this section.

Theorem 3.18 (Sibner and Sibner) *There exists a constant $k > 0$ such that, for $D \in \mathcal{A}^2$ with $\|F\|_{L^2(N_0)} < k$ there is a real number m (with corresponding flat connection $A^\flat =$*

$m\hat{i}d\theta$) and a gauge in which $D = \mathrm{d}+A$ with $A - A^{\flat} \in L^2_{1,A^{\flat}}(N_0)$. Moreover, for some constant C,

$$\|A - A^{\flat}\|_{L^2_{1,A^{\flat}}} \leq C\|F\|_{L^2} \tag{3.61}$$

We already know from Theorem 3.17 that locally there is a gauge in which the components of A have a limit at the singularity with $A_\theta \to m\hat{i}$ almost everywhere. Theorem 3.18 states that the $L^2_{1,A^{\flat}}(N_0)$ norm of the difference between the connection A and the flat connection $A^{\flat}$ is less than or equal to a constant multiplied by the L^2 norm of the curvature F. By taking the curvature over smaller and smaller regions around the singularity we can make this term go to zero. Hence on approach to the singularity we find that A tends to $A^{\flat}$. In other words, for any given connection, there is a gauge in which this connection is, near the singular 2-manifold, asymptotic to a flat connection $A^{\flat} = m\hat{i}d\theta$. This is stronger than the statement from Theorem 3.17 since it involves all the components of A (not just the θ component).

All connections in $\mathcal{A}^p$ have a holonomy number m, telling us which conjugacy class the holonomy of the connection belongs to. Since all these connections with corresponding m are asymptotic to the same $A^{\flat}$, the holonomy number $m \in \mathbb{R}$ provides a useful way to classify connections in $\mathcal{A}^p$.

We now define $\mathcal{A}^p_{m,k}$, a subset of $\mathcal{A}^p$, to be used in the proposition below.

$$\mathcal{A}^p_{m,k} = \{D \in \mathcal{A}^p | \, \|F\|_{L^2} \leq k \, \text{and} \, (H_m) \, \text{holds}\} \tag{3.62}$$

The main technical result of this section is

Proposition 3.19 *Let $p > 2$. Then there exists k and $\hat{c}$, and an explicit flat connection Γ with holonomy m, such that if $D \in \mathcal{A}^p_{m,k}$, then there exists a gauge in which $D = \mathrm{d}+\hat{A}$ and*

$$\|\hat{A} - \Gamma\|_{L^q_{1,\Gamma}(N_0)} \leq \hat{c}\|F\|_{L^q(N_0)}, \quad 2 \leq q \leq p \tag{3.63}$$

In Chapter 4 we will then show how Proposition 3.19 leads on to Corollary 3.20:

Corollary 3.20 *Let $p > 2$. There exists a gauge in which $D = \mathrm{d}+A \in \mathcal{A}^p_{m,k}$ with $A - A^{\flat} \in L^q_{1,A^{\flat}}(N_0)$ and*

$$\|A - A^{\flat}\|_{L^q_{1,A^{\flat}}(N_0)} \leq c\|F\|_{L^q(N_0)}, \quad 2 \leq q \leq p \tag{3.64}$$

As shall be demonstrated in Chapter 4, from Corollary 3.20 we can then show Theorem 3.18.

Overview of Proof of Proposition 3.19

To prove Proposition 3.19 we will need to show that there does exist a flat connection such that in a particular gauge the inequality in (3.63) is satisfied. We now provide a brief overview of the proof. Once again, the proof of Proposition 3.19 has a direct analogy to the proof in the GR case and so the details of the proof are included in Chapter 4 but not here.

For open balls B_α we can perform a gauge transformation such that in these balls the following properties hold for the gauge transformed connection 1-form A^α.

$$\mathrm{d} * A^\alpha = 0 \tag{3.65}$$

$$||A^\alpha||_{L_1^q(B_\alpha)} \leq K||F||_{L^q(B_\alpha)}, \qquad 2 \leq q \leq p \tag{3.66}$$

$$\frac{1}{\mathrm{Vol}(B_\alpha)} \int_{B_\alpha} A^\alpha dV = 0 \tag{3.67}$$

We call this gauge the Coulomb gauge for short since (3.65) is the Coulomb property for differential forms.

We construct a *composite* gauge transformation g using cutoff functions such that in two overlapping balls which are adjacent in the θ direction the transformation g will continuously change from the gauge transformation which makes A Coulomb in one ball to the gauge transformation which makes A Coulomb in the other. The composite gauge transformation is the result of a careful composition of many gauge transformations, the first acting on all balls except the first and then each successive one acting on smaller and smaller regions going around the singularity.

Note that this gauge transformation does not patch together the gauge transformation needed to make A Coulomb in the last ball and that needed to make A Coulomb in the first ball and so is not continuous at $\theta = 0 = 2\pi$. Hence the composite gauge transformation is non-global. We look at the gauge transformation which bridges the discontinuity. We take the Coulomb connection at $\theta = 0$, undo the gauge transformation that made the connection Coulomb in the first ball and then make the gauge transformation that makes the connection Coulomb in the final ball. We call this gauge transformation s.

The complete non-global composite gauge transformation will also patch together overlapping balls in the radial direction and in both directions spanning the singular 2-surface. In this new gauge A will either be Coulomb or 'close' to Coulomb everywhere, the 'closeness' being bounded by a cutoff function and its derivatives.

We know from Theorem 3.17 that the limit holonomy of $D = d + A$ at the singularity exists. If we have a global gauge and the origin is a regular point then the limit holonomy is trivial. In our case we start with a global gauge and a singular origin. We use Taubes' theorem (and property 1. above) to establish that $\lim_{r\to 0} s = s_0$ exists.

We have shown earlier in this chapter how we can use parallel propagation to establish a non-global composite gauge in which the limit 'jump' is the limit holonomy. We will call this the holonomy gauge. We will force our new non-global composite gauge to look like the holonomy gauge in the limit as r tends to zero. Hence they will have the same limit 'jump'. Since the limit as r tends to zero of the jump in the holonomy gauge is the limit holonomy, we know that the limit as r tends to zero of the jump in the composite gauge is also the (same) limit holonomy. Hence the limit holonomy for the connection A is s_0. This is proved by first showing that s_0 is the limit holonomy of a constant flat connection A_∞ and then showing that A_∞ and A have the same limit holonomy.

Also from before we know that any flat connection is gauge equivalent to $A^\flat = m\hat{i}d\theta$ for some $m \in \mathbb{R}$. We say that our flat connection has holonomy number m.

To get the final requirement to show (3.63) we must first, before applying the gauge transformation g, have changed to a radial gauge (so $A_r = 0$) and we can then find that we have a bound on $||A - A_\infty||_{L^q}$.

One final obstacle to proving Proposition 3.19 is that we require that our composite gauge transformation be global. We use another cutoff function to make the composite gauge transformation continuous and periodic and hence global. Since we are now using a different global gauge transformation, some more calculations are required before we establish the final result that $||\hat{A} - \Gamma||_{L^q_{1,\Gamma}(N_0)} \leq \hat{c}||F||_{L^q}$. Then since $D \in \mathcal{A}^p_{m,k}$ we have proved

Proposition 3.19.

Chapter summary

In this chapter we have introduced Yang-Mills gauge theory and explained the concepts of limit holonomy in this context. We gave the important definitions and lemmas which are crucial for the proofs of the two main theorems of Sibner and Sibner [39]. The proofs are summarised but we delay giving all the details of the proofs until the next chapter, where the analogous results are established in the context of General Relativity.

The purpose of this chapter has been to show that for certain 2-dimensional singularities, with connection in $L^2_{1,\mathrm{loc}}(X)$ and curvature in $L^2(N)$, a limit holonomy exists which is the same at all points on the singularity. We then went on to show that the connection tends in $L^2_{1,A^\flat}(X)$ to that of the prototype flat connection $A^\flat$. In the next chapter, the latter result will be the key to showing that the metric of a weak singularity tends to that of a conical metric.

Chapter 4

Weak Curvature Singularities in General Relativity

In Chapter 2 we have discussed quasi-regular singularities. These are singularities that have the particular feature of curvature tending to a limit in a parallely propagated frame. It has been shown that the holonomy around a 2-dimensional timelike quasi-regular singularity can be calculated and then used to establish a distributional value for the curvature at the singularity [49].

In this chapter we wish to find a holonomy method for weak curvature singularities for which curvature can be unbounded on approach to the singularity, but has the particular constraint that the curvature is in L^2, i.e. the integral of the square of the modulus (as given in Chapter 2) of the curvature is bounded. In the previous chapter we demonstrated a holonomy method used in Yang-Mills theory [39] where we had the corresponding constraint that the curvature is in L^2. We now wish to apply this holonomy method from Yang-Mills to a method for General Relativity to be applied to weak singularities.

We will look at 2-dimensional timelike weak singularities given by Σ. As in Chapter 3 we will first show that there exists a limit holonomy. We will also show that this limit holonomy acts in a surface transverse to the singularity. The second main theorem of the chapter will show how, in the limiting case, the connection tends to that of the flat 4-dimensional cone, as used to model such quasi-regular singularities. In addition we find that the metric also tends to that of the 4-dimensional cone.

In Chapter 5 we will go on to show that these findings enable us to compare weak curvature singularities with quasi-regular singularities. Methods used by Wilson and Vickers [49, 55] for quasi-regular singularities (based on Colombeau algebras) can then be applied, enabling us to find distributional curvature of weak curvature singularities.

The usual way of describing a space-time is to look at how a metric describes distance between points, located by a choice of coordinates. Instead of thinking of tensors in a coordinate basis we can look at tensors with respect to an orthonormal frame. Instead of using the metric to derive the Christoffel connection, we find the Ricci rotation coefficients (see Appendix A.2), again using a frame. This gives us a local description of the connection on the frame bundle and hence a description of GR as a gauge theory on the frame bundle, with the Lorentz group as fibres. This way of thinking allows comparisons between GR and Yang-Mills theory and is our first step to using [39] to understand weak curvature singularities in GR.

Let t be the time coordinate corresponding to the static Killing vector T so that the hypersurfaces which are orthogonal to T are given by t equal to a constant. Let Σ_{t_0} be the intersection of Σ with the hypersurface $t = t_0$. Then Σ_{t_0} is a curve in a 3-dimensional space. We now introduce cylindrical polar type coordinates (r, θ, z) in a neighbourhood of Σ_{t_0} on the 3-dimensional space such that z is a coordinate along the singularity, r gives the radial distance from the singularity and θ is an angular coordinate. The precise definition is given below and is similar to that of Unruh et al. [46], who also looked at a class of singularities for which the curvature was unbounded but only diverged slowly.

We let z be some coordinate such that the z =constant surfaces are transverse to Σ_{t_0}, so that z parameterises points on the singularity (see Figure 4.1). We now look at a fixed 2-surface $S_{(t_0,z_0)}$, given by $t = t_0$ =constant and $z = z_0$ =constant. We let r give the geodesic distance of points in z_0 from the singular point in $S_{(t_0,z_0)}$ and we let θ be a coordinate which is constant along such geodesics and is 2π-periodic in θ. Note however, this does not fix θ uniquely. Furthermore, for θ to be well defined we need to be in a sufficiently small neighbourhood to ensure that geodesics are unique. This will determine the size of our ϵ-neighbourhood.

We may do this on any such 2-surface for different values of z and all we require is that the θ coordinates fit together smoothly (that is to say if $\gamma(f)$ is a smooth curve, $\theta(f)$ is a

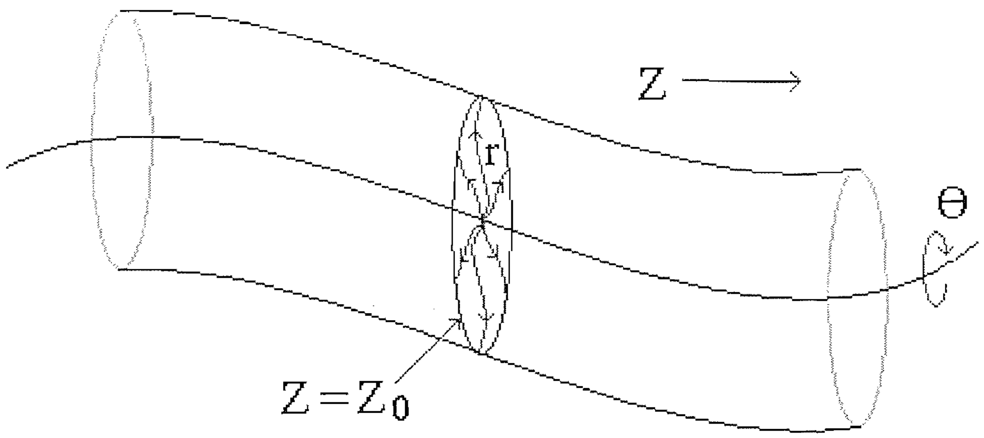

Figure 4.1: A diagram showing the cylindrical polar type coordinates around the singualrity Σ and the surface $z = z_0$ transverse to Σ.

smooth function).

Thus (t, r, θ, z) give coordinates in the neighbourhood of the singularity so that t and z give coordinates that parameterise points on the singularity (which is given by $r = 0$) and (r, θ) give polar type coordinates on 2-surfaces transverse to the singularity.

At this stage it is important that we also highlight the differences between gauge theory for Yang-Mills and for General Relativity.

The bundle and connection in GR

In Yang-Mills theory we considered a connection on an $SU(2)$ bundle. In formulating General Relativity as a gauge theory we take the bundle to be OM (the bundle of orthonormal frames) with gauge group the Lorentz group $L = O(1,3)$. The analogue of the Yang-Mills connection 1-form $A^i_j = A^i_{ja}dx^a$ (taking values in $su(2)$) is provided by the connection 1-form

$$\omega^i_j = \gamma^i_{jk}\vartheta^k_a dx^a \tag{4.1}$$

which takes values in the Lie algebra $\mathfrak{l}$ of the Lorentz group. Here $\vartheta^k_a(x)$ are dual to the frames $e^a_k(x)$ and γ^i_{jk} are the Ricci rotation coefficients (see Appendix A.2 for details of this and Cartan's description of the connection).

Cartan's description provides us with many of the properties and identities involving the connection 1-form and the Ricci rotation coefficients, which will be used later in this thesis. We recall from Chapter 2 an alternative description of the connection in terms of a 1-form $\underline{\omega}$ on the bundle OM taking values in $\mathfrak{l}$.

We can now choose a section σ (also known as a gauge) through the bundle

$$\sigma : M \to OM \tag{4.2}$$

Let $\omega = \sigma^*\underline{\omega}$ be the pullback of $\underline{\omega}$ under the section σ. Then ω is a 1-form on M taking values in $\mathfrak{l}$ which give the (gauge dependent) connection 1-form on M. Hence ω is a map

$$\begin{aligned} \omega : T_xM &\rightarrow \mathfrak{l} \\ X^a &\mapsto (\omega(X))^i_j = X^a\omega^i_{ja} \end{aligned} \tag{4.3}$$

Note this is related to the definition (4.1) given by Cartan's description through

$$\omega^i_{ja} = \gamma^i_{jk}\vartheta^k_a \tag{4.4}$$

From here onwards reference to the 'connection 1-form' always pertains to the gauge dependent connection 1-form on M, unless otherwise stated.

The curvature

Instead of the curvature 2-form $F^i_j = F^i_{jab}dx^a \wedge dx^b \in su(2)$ in Yang-Mills, we now use the curvature 2-form $\Omega^i_j = \Omega^i_{jab}dx^a \wedge dx^b = \frac{1}{2}R^a{}_{bcd}\vartheta^i_a e^b_j dx^c \wedge dx^d \in \mathfrak{l}$ where $R^a{}_{bcd}$ is the standard Riemann curvature tensor.

Definition 4.1 *The curvature 2-form Ω on M has elements which take values in $\mathfrak{l}$ and is antisymmetric on its spacetime indices*

$$\begin{aligned} \Omega : T_xM \times T_xM &\rightarrow \mathfrak{l} \\ (X^a, Y^b) &\mapsto X^aY^b\Omega^i_{jab} \end{aligned} \tag{4.5}$$

and is defined by its relation to the connection 1-form as given by

$$\Omega = d\omega + [\omega, \omega] \tag{4.6}$$

or

$$\Omega^i_{jab} = 2\omega^i_{j[b,a]} + \omega^i_{ka}\omega^k_{jb} - \omega^i_{kb}\omega^k_{ja} \tag{4.7}$$

In the proofs of Theorem 3.17 and Theorem 3.18 we made use of the fact that $SU(2)$ is compact. It is important to note that the Lorentz group is not compact and it is unclear whether or not compactness is a requisite for the proof of these theorems. This will be the subject of discussion at the end of the chapter. Instead we shall, for the purposes of this thesis, restrict our application of the holonomy theorems to static space-times.

The bundle and connection for static space-times

For static space-times we know that there is a timelike Killing vector T which is orthogonal to a spacelike surface. We will choose an orthonormal basis adapted to this description in order to reduce a bundle OM to the bundle Q with gauge group $SO(3)$ (which is compact). This construction is described below.

We choose Q to consist of frames (e^a_i) such that the zeroth vector points in the $\hat{T}$ direction. If e^a_i and $\tilde{e}^a_i$ are two orthonormal frames such that both $e^a_0 = \hat{T}^a$ and $\tilde{e}^a_0 = \hat{T}^a$ then

$$\tilde{e}^a_i = L^j_i e^a_j \tag{4.8}$$

where

$$L^i_j = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & c \\ 0 & d & e & f \\ 0 & g & h & i \end{pmatrix} \tag{4.9}$$

and

$$G^\alpha_\beta = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in SO(3) \tag{4.10}$$

Hence we have an action

$$\begin{aligned} SO(3) \times Q &\to Q \\ (m, e) &\mapsto \tilde{e} \end{aligned} \tag{4.11}$$

We therefore consider the bundle Q with gauge group $SO(3)$. We now show that the connection on OM induces a connection on Q.

Let e_i^a be a basis for our space-time with ϑ_a^i the corresponding dual basis. Then we can choose e_0^a to be the basis vector pointing in the $\hat{T}$ direction and e_α^a, with $\alpha = \{1, 2, 3\}$, to be the three other vectors of the basis, all orthogonal to the $\hat{T}$ direction. Since the space-time is static, all the hypersurfaces orthogonal to the timelike Killing vector are identical and we may choose ϑ_a^α to be the same for all t. Hence parallel propagation in the e_0^a direction leaves the ϑ_a^α covectors fixed. Thus

$$\nabla_{e_0} \vartheta^\alpha = 0 \tag{4.12}$$

From here we can proceed to show that $\omega_\alpha^0 = \omega_0^\alpha = 0$. We start with (4.12)

$$\begin{aligned} \nabla_{e_0} \vartheta^\alpha &= 0 \\ \Longrightarrow \gamma_{i0}^\alpha \vartheta^i &= 0 \\ \Longrightarrow \gamma_{i0}^\alpha &= 0 \end{aligned} \tag{4.13}$$

where γ_{jk}^i are the Ricci rotation coefficients as given in Appendix A.2. We know that, because the metric connection is torsion free,

$$\nabla_{e_i} e_j - \nabla_{e_j} e_i = [e_i, e_j] \tag{4.14}$$

Hence $\gamma_{jk}^i - \gamma_{kj}^i = C_{kj}^i$, where C_{kj}^i are the structure constants of the Lie algebra [22] defined in terms of the basis vectors,

$$C_{jk}^i e_i = [e_j, e_k] \tag{4.15}$$

We are interested in the term

$$C_{0\beta}^\alpha = \langle [e_0, e_\beta], \vartheta^\alpha \rangle \qquad (\beta = \{1, 2, 3\}) \tag{4.16}$$

Since we are in a static space-time we can apply the hypersurface orthogonality condition which shows that $[e_0, e_\beta] \propto \frac{\partial}{\partial t}$ and hence we can show that

$$
\begin{aligned}
C^{\alpha}_{0\beta} &= 0 \\
\Longrightarrow \gamma^{\alpha}_{j0} &= \gamma^{\alpha}_{0j} \\
\Longrightarrow \gamma^{\alpha}_{0i} &= 0 \qquad \text{(from (4.13))} \\
\Longrightarrow \gamma^{\alpha}_{0i}\vartheta^{i} &= 0 \\
\Longrightarrow \omega^{\alpha}_{0} &= 0
\end{aligned} \tag{4.17}
$$

From the symmetry of (A.21) we can further deduce that $\omega^{0}_{\alpha} = 0$.

Since $\omega^{0}_{\alpha} = \omega^{\alpha}_{0} = 0$ the only non-zero terms in the connection are ω^{α}_{β}. Furthermore

$$
\begin{aligned}
\delta_{\alpha\gamma}\omega^{\gamma}_{\beta} &= -\eta_{\alpha i}\omega^{i}_{\beta} \\
&= -\omega_{\alpha\beta} \\
&= \omega_{\beta\alpha} \\
&= \eta_{\beta j}\omega^{i}_{\alpha} \\
&= -\delta_{\beta\gamma}\omega^{\gamma}_{\alpha}
\end{aligned} \tag{4.18}
$$

Thus

$$
\omega_{\alpha\beta} = -\omega_{\beta\alpha} \tag{4.19}
$$

We now want to look at how ω^{α}_{β} transforms under a gauge transformation. In General Relativity the standard symbol for the metric is g and so in this chapter the GR analogue of the gauge transformation previously denoted as g in Chapter 3 will be G. Hence a gauge transformation is a map

$$
\begin{aligned}
G : M &\rightarrow SO(3) \\
\mathbf{x} &\mapsto G^{\alpha}_{\beta}
\end{aligned} \tag{4.20}
$$

This induces a map on the frame $\{e^{a}_{j}\}$ given by the corresponding L^{i}_{j} through (4.8) and (4.9). The connection in the new frame is

$$
\tilde{\omega}^{i}_{j} = (L^{-1})^{i}_{k}\omega^{k}_{l}L^{l}_{j} + (L^{-1})^{i}_{k}(\mathrm{d}L)^{k}_{j} \tag{4.21}
$$

However, $L^{0}_{0} = 1$, $L^{0}_{\alpha} = 0$, $L^{\alpha}_{0} = 0$ and $L^{i}_{j,0} = 0$ (since the space-time is static) so that (4.21) reduces to

$$\tilde{\omega}^{\alpha}_{\beta} = (G^{-1})^{\alpha}_{\gamma}\omega^{\gamma}_{\delta}L^{\delta}_{\beta} + (G^{-1})^{\alpha}_{\gamma}(\mathrm{d}G)^{\gamma}_{\beta} \tag{4.22}$$

Hence we may regard ω as an $SO(3)$ connection on the bundle Q. From now onwards ω will represent this induced connection on Q rather than the connection on OM.

The curvature

We now define the curvature 2-form for our restricted bundle.

Definition 4.2 *The curvature 2-form Ω on M has elements which take values in $so(3)$ and is antisymmetric on its spacetime indices*

$$\begin{aligned} \Omega : T_xM \times T_xM &\rightarrow so(3) \\ (X^a, Y^b) &\mapsto X^aY^b\Omega^{\alpha}_{\beta ab} \end{aligned} \tag{4.23}$$

and is defined by its relation to the connection 1-form as given by

$$\Omega = d\omega + [\omega, \omega] \tag{4.24}$$

or

$$\Omega^{i}_{jab} = 2\omega^{\alpha}_{\beta[b,a]} + \omega^{\alpha}_{\gamma a}\omega^{\gamma}_{\beta b} - \omega^{\alpha}_{\gamma b}\omega^{\gamma}_{\beta a} \tag{4.25}$$

Comparing to (4.7) we see that for a static space-time $\Omega^{0}_{\alpha} = \Omega^{\alpha}_{0} = 0$ and Ω^{α}_{β} are just the (α, β) components of Ω^{i}_{jab} since $\omega^{0}_{\alpha} = \omega^{\alpha}_{0} = 0$. Hence

$$\Omega^{\alpha}_{\beta cd} = \frac{1}{2}R^{a}_{bcd}\vartheta^{\alpha}_{a}e^{b}_{\beta} \tag{4.26}$$

where R^{a}_{bcd} is the usual Riemann curvature of the static space-time.

4.1 The holonomy method in General Relativity

We now set out the general method by which we establish the existence of a limit holonomy around a weak singularity in General Relativity. By partially following the convention in the previous chapter we shall label the neighbourhood of the singularity N and the complement of the singularity in N we shall label X (i.e. $X = N\backslash\Sigma$). We also relabel our transverse 2-surface $S_{(t_0,z_0)}$ as N_0. We define $X_0 = N_0\backslash\Sigma_{(t_0,z_0)}$ the complement of the singular point

$\Sigma_{(t_0,z_0)}$ in the transverse surface $N_0 = S_{(t_0,z_0)}$.

We require $\omega \in L^2_{1,\text{loc}}(X)$ as one of our conditions. As shown in Chapter 2, $\omega \in L^2_{1,\text{loc}}(X)$ means that $\nabla^\alpha \omega$ is a measurable function for all $0 \leq |\alpha| \leq 1$ and for all compact subsets C of X we have from (2.53)

$$||\omega^\gamma_{\beta a}||_{L^2_{1,\text{loc}}(X)} = \left(\int_C \sum_{0 \leq |\alpha| \leq 1} |\nabla^\alpha \omega^\gamma_{\beta a}|^2 d\sigma \right)^{\frac{1}{2}} < \infty \tag{4.27}$$

Since we are working locally we will for convenience take the positive definite background metric ξ_{ab} to have line element

$$\xi_{ab} dx^a dx^b = dt^2 + dr^2 + r^2 d\theta^2 + dz^2 \tag{4.28}$$

where (t, r, θ, z) are the coordinates in the neighbourhood of the singularity introduced at the beginning of this chapter. However, we find that the results of this thesis are not sensitive to this particular choice of positive definite metric.

We also require $\Omega \in L^2(N)$. Note that $\Omega \in L^2_{\text{loc}}(X)$ is an immediate consequence of the formula $\Omega = \mathrm{d}\omega + [\omega, \omega]$ and using the Sobolev imbedding $L^2_1 \to L^4$ on $\omega \in L^2_{1,\text{loc}}(X)$ (see Appendix B.1 and [1]). However, as in the previous chapter, the proof requires the stronger condition $\Omega \in L^2(X)$ and so we impose this additional requirement.

Holonomy around the singularity

Below, with the aid of Figure 4.2 we first set out the general method by which we establish holonomy around a singularity in General Relativity and then show the algebra involved in the process.

We take a particular point on the 2-dimensional singularity and consider the transverse (r, θ) surface (for the diagram we suppress dimensions tangent to the singularity). We take a closed path $\chi_r(\theta)$ in X_0, around the singularity at distance r, beginning and ending at the same point $x \in X_0$. The path χ_r (and later also λ_r, μ_r and G_r) depends on r, but r is fixed unless otherwise stated. We shall take a frame on χ_r and parallely propagate round the path until the location of the frame is the same, but the direction is related to the original by a gauge transformation.

We consider the bundle above the manifold (see Figure 4.2) which locally is given by $SO(3) \times X_0$. The global structure of the bundle is that of Q but, since we are only looking at an open ball around a singularity, we need only consider local structure. A projection, Π, takes points in $SO(3) \times X_0$ to X_0. The projection discards information about the element leaving only its position. The fibre above $y \in X_0$ is the set of all points $\alpha \in SO(3) \times X_0$ such that $\Pi(\alpha) = y$. The fibre represents all possible $SO(3)$ group elements (and hence all frames in Q) at a point in X_0.

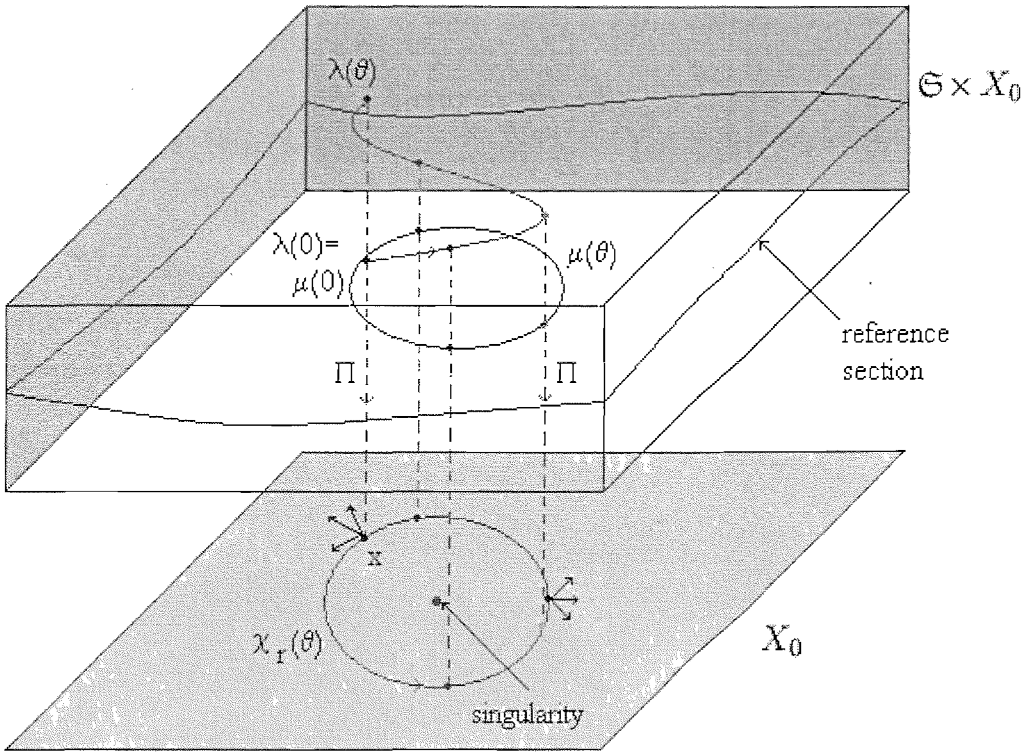

Figure 4.2: A diagram showing how we establish holonomy of a singularity, using the (local structure of the) bundle Q.

$\chi_r(\theta) \in M$ is the projection of a particular path $\lambda_r(\theta) \in SO(3) \times X_0$. This path in the frame bundle details the transformation of the parallely transported frame e^a_α, indicating position on the path $\chi_r(\theta)$ in the manifold and how the frame is propagated around this path. We now consider a reference section through $SO(3) \times X_0$, intersecting the point $\lambda_r(0)$ which projects down to $x \in X_0$. A section through the frame bundle can be seen as a choice of gauge, with $SO(3)$ transformations between gauges taking one section to another. On this plane we draw, starting at $\lambda_r(0)$, the closed path which projects down to χ_r and we shall call this path $\mu_r(\theta)$.

Since $\mu_r(\theta)$ is a closed path, we know that $\mu_r(0) = \mu_r(2\pi)$. This path corresponds to a frame rotating around χ_r which, on returning to x has the same direction as when it started. We note that $\mu_r(\theta)$ and $\lambda_r(\theta)$ both lie on the same fibre above $\chi_r(\theta)$ for each θ. The fibre above $y \in M$ is the set of all points $\alpha \in Q$ such that $\Pi(\alpha) = y$ and represents all possible frame configurations at a point in M.

There is a gauge transformation $G : M \to SO(3)$ such that for each θ we have $G_r(\theta) \in SO(3) : SO(3) \to SO(3)$, that takes $\mu_r(\theta)$ to $\lambda_r(\theta)$. i.e.

$$\lambda_r(\theta) = R_{G_r(\theta)}\mu_r(\theta) \tag{4.29}$$

Since $\mu_r(0) = \lambda_r(0)$ we have $G_r(0) = I$. We let $G_r(2\pi) = J_r$, and the limit as r tends to zero of $G_r(2\pi)$ is $J^\flat$, the holonomy of the singularity. We would like to show that the holonomy limit

$$\lim_{r\to 0} J_r = J^\flat \tag{4.30}$$

is well defined and this is given by Theorem 4.8.

As in the previous chapter we now wish to set up a differential equation containing the gauge transformation G and the connection 1-form ω.

We take an initial frame to be the basis e^a_α, which lies on a path $\chi^a(\theta) = (t_0, r_0, \theta, z_0)$ around a point on the singularity (t_0, z_0) at a constant distance (r_0). We look at the gauge transformation G which takes e to the frame $\tilde{e}$ which is e parallely propagated around the path χ. So we have

$$\tilde{e}^a_\beta = G^\alpha_\beta e^a_\alpha \tag{4.31}$$

Since we are looking at parallel propagation we have

$$\nabla_T \tilde{e}^a_\beta = 0 \tag{4.32}$$

where

$$T^a = \frac{d\chi^a(s)}{ds} = \delta^a_\theta \tag{4.33}$$

We now evaluate (4.32).

$$\begin{aligned}
\nabla_T(G^\alpha_\beta e^a_\alpha) &= 0 \\
(\nabla_T G^\alpha_\beta)e^a_\alpha + G^\alpha_\beta \nabla_T e^a_\alpha &= 0 \\
T^c G^\alpha_{\beta,c} e^a_\alpha + G^\alpha_\beta T^c \nabla_c e^a_\alpha &= 0 \\
\delta^c_\theta G^\alpha_{\beta,c} e^a_\alpha + G^\alpha_\beta T^\epsilon e^c_\epsilon \nabla_c e^a_\alpha &= 0 \\
&\text{see Appendix A.2} \\
\frac{dG^\alpha_\beta}{d\theta} e^a_\alpha + G^\alpha_\beta T^\epsilon \gamma^\zeta_{\alpha\epsilon} e^a_\zeta &= 0 \\
\left(\frac{dG^\alpha_\beta}{d\theta} + G^\zeta_\beta T^\epsilon \gamma^\alpha_{\zeta\epsilon}\right) e^a_\alpha &= 0 \\
&\text{since } e^a_\alpha \text{ is invertible} \\
\frac{dG^\alpha_\beta}{d\theta} + G^\zeta_\beta T^\alpha \vartheta^\epsilon_\alpha \gamma^\alpha_{\zeta\epsilon} &= 0 \\
\frac{dG^\alpha_\beta}{d\theta} + G^\zeta_\beta \delta^a_\theta \omega^\alpha_{\zeta a} &= 0 \\
\frac{dG^\alpha_\beta}{d\theta} + \omega^\alpha_{\zeta\theta} G^\zeta_\beta &= 0 \quad \square
\end{aligned} \tag{4.34}$$

In index free notation we write

$$\frac{dG}{d\theta} + \omega_\theta G = 0 \tag{4.35}$$

Since $\mu_r(0) = \lambda_r(0)$ we know that the initial frame is at $\theta = 0$ and so we obtain the initial condition $G(0) = I$. We now have an initial value problem which has a solution given by a path ordered integral.

$$\frac{dG}{d\theta} + \omega_\theta G = 0 \quad G(0) = I \tag{4.36}$$

Properties of holonomy

The proofs of the two critical theorems in this chapter rely on various properties of holonomy, connections and, in particular, flat connections which we include below as lemmas. We first note that the definitions of gauge equivalence and conjugacy for our connection ω and curvature Ω are analogous to Lemma 3.7, Corollary 3.8 and Definition 3.9 in Chapter 3.

We wish to show that the conjugacy class $[J_r]$ of the holonomy $J_r = G_r(2\pi)$ is gauge independent, i.e. that the transformation from the initial element to the '2π-evolved' element will always have the same 'magnitude' of rotation, although the actual transformation will change depending on the initial frame.

If G_r is the solution of (4.35) for connection ω, then $\hat{G}_r$ is the solution for connection $\hat{\omega}$. Let ω and $\hat{\omega}$ be gauge equivalent. That is to say there exists some gauge transformation $s : X \to Q$ such that $\hat{\omega} = s^{-1}\omega s + s^{-1}ds$. Since χ_r is a closed path we have $s(0) = s(2\pi)$. In the new gauge we have

$$\frac{d\hat{G}}{d\theta} + \hat{\omega}_\theta \hat{G} = 0 \tag{4.37}$$

which we can rewrite using the connection of the original gauge as

$$\frac{d\hat{G}}{d\theta} + \left(s^{-1}\omega_\theta s + s^{-1}\frac{ds}{d\theta} \right) \hat{G} = 0 \tag{4.38}$$

(4.35) and (4.38) are two equations with three unknowns (for a given s) and so any solution relating G with $\hat{G}$ without the ω terms must satisfy both (4.35) and (4.38). If we let $\hat{G} = s^{-1}Gs(0)$ then (4.38) becomes (4.35)

$$\begin{aligned}
\frac{d(s^{-1}Gs(0))}{d\theta} + (s^{-1}\omega_\theta s + s^{-1}\frac{ds}{d\theta})s^{-1}Gs(0) &= 0 \\
\frac{ds^{-1}}{d\theta}Gs(0) + s^{-1}\frac{dG}{d\theta}s(0) + & \\
+ s^{-1}\omega_\theta s s^{-1}Gs(0) + s^{-1}\frac{ds}{d\theta}s^{-1}Gs(0) &= 0 \\
s\frac{ds^{-1}}{d\theta}G + \frac{dG}{d\theta} + \omega_\theta G + \frac{ds}{d\theta}s^{-1}G &= 0 \\
\frac{d(ss^{-1})}{d\theta}G + \frac{dG}{d\theta} + \omega_\theta G &= 0 \\
\frac{dG}{d\theta} + \omega_\theta G &= 0
\end{aligned} \tag{4.39}$$

and so by uniqueness we have $G = s\hat{G}s^{-1}(0)$. Since we know that $s(0) = s(2\pi)$ we have

$$G(2\pi) = s(2\pi)\hat{G}(2\pi)s^{-1}(2\pi) \tag{4.40}$$

and so

$$J_r = G_r(2\pi) = s(2\pi)\hat{G}_r(2\pi)s^{-1}(2\pi) = s(2\pi)\hat{J}_r s^{-1}(2\pi) \tag{4.41}$$

and hence we have

Lemma 4.3 *If G and $\hat{G}$ are solutions of (4.35) for gauge equivalent connections ω and $\hat{\omega}$ then J_r and $\hat{J}_r$ are conjugate in $SO(3)$; and so $[J_r]$ is gauge invariant.*

Lemma 3.11 in Chapter 3 has a direct analogy to the GR case and so is not reproduced here, but we recall that if the connection were flat, then the conjugacy class of the holonomy, $[J_r]$, would be a homotopy invariant and hence independent of r. Hence for conical singularities the holonomy does not change with distance from the singularity, because of the flat (local) topology of the 4-dimensional cone. However, for weak curvature singularities, the holonomy is dependent on position away from the singularity.

In the next two sections we shall provide two theorems. We first want to show that if we let r tend to zero then the limit holonomy exists; it follows immediately from Lemma 4.3 that if the limit holonomy exists in one gauge, then it will exist in all gauges obtained by smooth gauge transformations. We note (from [39]) that Lemma 4.3 still holds if ω and $\hat{\omega}$ are related by a weak gauge transformation $s \in L^2_{2,\mathrm{loc}}$. This means that if ω is weakly gauge equivalent to a connection $\hat{\omega}$ for which $[\hat{J}]$ has a limit, then $[J]$ also has a limit. The second result we wish to show is that, as we let r tend to zero, we find that the connection tends to that of a conical singularity as measured in an L^2_1 norm.

Flat connections and flat bundles

Using a similar method to that given in [21] and making use of Appendix A.2, we shall now find the prototype for a flat conical connection 1-form by starting with the line element for a four dimensional cone

$$ds^2 = dt^2 - dr^2 - r^2 A^2 d\theta^2 - dz^2 \tag{4.42}$$

where $0 < A < 1$ provides the angular deficit of the cone $2\pi(1 - A)$. We shall temporarily suppress use of space-time coordinates in our calculations and let

$$ds^2 = \mu_{ij}\vartheta^i\vartheta^j \tag{4.43}$$

where we shall choose μ_{ij} to be the Minkowski metric $\mathrm{Diag}(1, -1, -1, -1)$ and ϑ^i is the basis of one-forms

$$\vartheta^0 = dt \qquad \vartheta^1 = dr \qquad \vartheta^2 = rAd\theta \qquad \vartheta^3 = dz \tag{4.44}$$

We wish to find the associated flat connection 1-form $\omega^\flat$ and so we first establish some symmetries from the equation

$$\omega^{(i}{}_k\mu^{m)k} = 0 \tag{4.45}$$

We find that ${\omega^\flat}^i_j = 0$ for all $i = j$ and that ${\omega^\flat}^i_j = -{\omega^\flat}^j_i$ for all $i \neq j \neq 0 \neq i$. Hence the prototype will have the form

$$\omega^\flat = \begin{pmatrix} 0 & A & B & C \\ A & 0 & D & E \\ B & -D & 0 & F \\ C & -E & -F & 0 \end{pmatrix}$$

where each $A \dots F$ is a 1-form. It follows that $d\vartheta^0 = d\vartheta^1 = d\vartheta^3 = 0$. We also have

$$\begin{aligned} d\vartheta^2 &= \frac{\partial \vartheta^2_a}{\partial x^b} dx^b \wedge dx^a \\ &= \frac{\partial \vartheta^2_2}{\partial x^b} dx^b \wedge d\theta \\ &= \frac{\partial (rA)}{\partial r} dr \wedge d\theta \\ &= A\vartheta^1 \wedge (\frac{1}{rA}\vartheta^2) \\ &= \frac{1}{r}\vartheta^1 \wedge \vartheta^2 \end{aligned} \tag{4.46}$$

We now use the equation $d\vartheta^i = -{\omega^\flat}^i_k \wedge \vartheta^k$ (again from Appendix A.2) and find that

$$\begin{aligned} A \wedge \vartheta^1 + B \wedge \vartheta^2 + C \wedge \vartheta^3 &= 0 \\ A \wedge \vartheta^0 + D \wedge \vartheta^2 + E \wedge \vartheta^3 &= 0 \\ B \wedge \vartheta^0 - D \wedge \vartheta^1 + F \wedge \vartheta^3 &= -\frac{1}{r}\vartheta^1 \wedge \vartheta^2 \\ C \wedge \vartheta^0 - E \wedge \vartheta^1 - F \wedge \vartheta^2 &= 0 \end{aligned} \tag{4.47}$$

Solving these simultaneous equations gives us, for some constant c

$${\omega^\flat}^i_j = \begin{pmatrix} 0 & 0 & c\vartheta^3 & c\vartheta^2 \\ 0 & 0 & -\frac{1}{r}\vartheta^2 & 0 \\ c\vartheta^3 & \frac{1}{r}\vartheta^2 & 0 & c\vartheta^0 \\ c\vartheta^2 & 0 & -c\vartheta^0 & 0 \end{pmatrix} \tag{4.48}$$

From before we know that connection 1-forms in static space-times are such that $\omega^\alpha_0 = \omega^0_\alpha = 0$ and so we choose $c = 0$. Finally we recall that $\vartheta^2 = rAd\theta$ and so we have our prototype for the flat connection

$$\omega^{b\,i}_{\;\;j} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -A & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} d\theta = \Psi d\theta \tag{4.49}$$

Lemma 4.4 *There is a unique correspondence between conjugacy classes of holonomy and flat connections.*

As we have shown before, conjugacy classes of holonomy are conjugacy classes in $SO(3)$. Elements of the same conjugacy class for the group $GL(3,\mathbb{R})$ have three invariants under conjugacy. For a matrix $B \in GL(3,\mathbb{R})$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3$, these invariants are

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &= \mathrm{Inv}_1 (= \mathrm{Tr} B) \\ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 &= \mathrm{Inv}_2 \\ \lambda_1\lambda_2\lambda_3 &= \mathrm{Inv}_3 (= \det B) \end{aligned} \tag{4.50}$$

However, we are effectively using $SO(3)$ (since the t component is unity) which has the two restrictions that $\det B = 1$ and BB^T =Id. To find the eigenvalues of a 3×3 matrix one must solve a cubic equation, a solution of which must be real. Since the transformation represented by an $SO(3)$ element is length preserving we have that the real eigenvalue is ± 1.

If the other eigenvalues are real too, they must also be ± 1. Since $\det B = \mathrm{Inv}_3 = 1$, we know that either $\lambda_1 = \lambda_2 = \lambda_3 = 1$ or, without loss of generality (invariants are symmetric), $\lambda_1 = 1, \lambda_2 = \lambda_3 = -1$. In both cases $\mathrm{Inv}_1 = \mathrm{Inv}_2$.

If the other eigenvalues are non-real then they are complex conjugate to one another, so the real eigenvalue is $\lambda_1 = 1$ and the complex eigenvalues are (since $\lambda_1\lambda_2\lambda_3 = 1$) $\lambda_2 = e^{i\phi}$ and $\lambda_3 = e^{-i\phi}$. We find that once again $\mathrm{Inv}_1 = \mathrm{Inv}_2$.

Since $\mathrm{Inv}_3 = 1$ and $\mathrm{Inv}_1 = \mathrm{Inv}_2$ for all elements of $SO(3)$, we can uniquely describe a pair of conjugacy classes in $SO(3)$ by their trace (which will be between 0 and 4), where one conjugacy class is identical to the other except has the opposite orientation (i.e. represents rotations in the opposite direction). To uniquely describe any one conjugacy class in $SO(3)$ we use a representative of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\pi A & \sin 2\pi A & 0 \\ 0 & -\sin 2\pi A & \cos 2\pi A & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.51}$$

The prototype of the flat connection is given by

$$\omega^{\flat}{}^{i}{}_{j} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -A & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} d\theta \tag{4.52}$$

The solution of (4.35) where $\omega_\theta = \omega^{\flat}{}_\theta$ can be found.

$$\begin{aligned} \frac{\partial G^i_j}{\partial \theta} + \omega^{\flat}{}^{i}{}_{k\theta} G^k_j = 0 \quad \Longrightarrow \quad & \frac{\partial G^0_j}{\partial \theta} = 0 \;\; , \quad \frac{\partial G^1_j}{\partial \theta} - A G^2_j = 0 \\ & \frac{\partial G^2_j}{\partial \theta} + A G^1_j = 0 \quad \text{and} \quad \frac{\partial G^3_j}{\partial \theta} = 0 \quad \forall j \end{aligned} \tag{4.53}$$

Solving these simultaneous equations shows us that G^i_j is of the form

$$G^i_j = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ H_1 & H_2 & H_3 & H_4 \\ J_1 & J_2 & J_3 & J_4 \\ c_5 & c_6 & c_7 & c_8 \end{pmatrix} \tag{4.54}$$

where $c_1 \ldots c_8$ are real constants, $H_i = m_i \sin A\theta - n_i \cos A\theta$, $J_i = m_i \cos A\theta + n_i \sin A\theta$ and m_i and n_i are real constants.

We can determine all of the above constants by noting that at $\theta = 0$ we have G^i_j =Id. The solution of (4.35) is

$$G^i_j = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos A\theta & \sin A\theta & 0 \\ 0 & -\sin A\theta & \cos A\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.55}$$

It is easy to see from (4.55) that these gauge transformations represent rotations in the surface transverse to the (t, z) singularity.

We know from Lemma 3.11 in Chapter 3 that the solution of (4.35) is homotopy invariant for flat bundles and so we can confirm that $G^i_j(\theta)$ is also independent from r. Hence the holonomy of the prototype of the flat connection is

$$G(2\pi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\pi A & \sin 2\pi A & 0 \\ 0 & -\sin 2\pi A & \cos 2\pi A & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.56}$$

which is equal to the representative of the conjugacy class as given by (4.51). Note that two different values of A which are separated by integer values, will yield the same representative. Hence there is a unique correspondence between flat connections (modulo 1) and conjugacy classes of holonomy. $\square$

Lemma 4.5 *If $A_1 = A_2 + n$, $n \in \mathbf{Z}$, then there exists a gauge transformation G such that $G^{-1}\omega_2^\flat G + G^{-1}\mathrm{d}G = \omega_1^\flat$ where*

$$\omega_1^\flat = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -A_1 & 0 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} d\theta \qquad \omega_2^\flat = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -A_2 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} d\theta \tag{4.57}$$

Proof

Take

$$G(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos n\theta & -\sin n\theta & 0 \\ 0 & \sin n\theta & \cos n\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.58}$$

We note that $G^{-1}\omega_2^\flat G = \omega_2^\flat$. So gauge transforming $\omega_2^\flat$ with this $G(\theta)$ it follows that

$$
\begin{aligned}
\omega_2^\flat + G^{-1}\mathrm{d}G &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -A_2 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} d\theta + \\
& \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos n\theta & \sin n\theta & 0 \\ 0 & -\sin n\theta & \cos n\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -n\sin n\theta & -n\cos n\theta & 0 \\ 0 & n\cos n\theta & -n\sin n\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} d\theta \\
&= \left[\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -A_2 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -n & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] d\theta \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -A_1 & 0 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} d\theta \\
&= \omega_1^\flat
\end{aligned} \tag{4.59}
$$

and hence $\omega_1^\flat$ and $\omega_2^\flat$ are gauge equivalent. □

We note that since n is an integer, we have

$$
G(2\pi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} d\theta = I \tag{4.60}
$$

Lemma 4.6 *If A is an integer then D is gauge equivalent to* d.

Proof

From Lemma 4.5 we know that $\omega_1^\flat$ and $\omega_2^\flat$ are gauge equivalent and hence so are $D_1 = \mathrm{d} + \omega_1^\flat$ and $D_2 = \mathrm{d} + \omega_2^\flat$. Since A_1 is an integer we can choose n to be A_1 also and hence $A_2 = 0$. Therefore

$$
\omega_2^\flat = \underline{0} \tag{4.61}
$$

Hence $D_2 = \mathrm{d}$. Therefore $D_1 = \mathrm{d} + \omega_1^\flat$ is gauge equivalent to d. □

Lemma 4.7 *Let $D = \mathrm{d}+\omega$ be a flat connection with holonomy A. There is a gauge in which $D = \mathrm{d}+\omega^\flat$ where*

$$\omega^\flat = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -A & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} d\theta \tag{4.62}$$

Proof

From Lemma 4.4 we know that the holonomy $G(2\pi)$ of the flat connection ω will be conjugate to the holonomy

$$G^\flat(2\pi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\pi A & \sin 2\pi A & 0 \\ 0 & -\sin 2\pi A & \cos 2\pi A & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.63}$$

of the prototype of the flat connection $\omega^\flat$. From Lemma 3.15 we know that the conjugacy is provided by a continuous periodic $k(\theta)$ such that $G^\flat(\theta) = k^{-1}(\theta)G(\theta)k(2\pi)$ which we rewrite as

$$G = kG^\flat k^{-1}(2\pi) \tag{4.64}$$

remembering that the non-constants are functions of θ.

For our original flat connection ω we have $\frac{dG}{d\theta} + \omega_\theta G = 0$. Using (4.64) we now get

$$\begin{aligned} \frac{d}{d\theta}(kG^\flat k^{-1}(2\pi)) + \omega_\theta(kG^\flat k^{-1}(2\pi)) &= 0 \\ \frac{dk}{d\theta}G^\flat k^{-1}(2\pi) + k\frac{dG^\flat}{d\theta}k^{-1}(2\pi) + \omega_\theta kG^\flat k^{-1}(2\pi) &= 0 \\ \frac{dk}{d\theta}G^\flat + k\frac{dG^\flat}{d\theta} + \omega_\theta kG^\flat &= 0 \\ \frac{dG^\flat}{d\theta} + (k^{-1}\omega_\theta k + k^{-1}\frac{dk}{d\theta})G^\flat &= 0 \end{aligned} \tag{4.65}$$

Since $G^\flat$ is the solution for the prototype of the flat connection

$$\frac{dG^\flat}{d\theta} + \omega_\theta^\flat G^\flat = 0 \tag{4.66}$$

we have by uniqueness

$$k^{-1}\omega_\theta k + k^{-1}\frac{dk}{d\theta} = \omega_\theta^\flat \tag{4.67}$$

Hence there exists a gauge transformation k, which takes a flat connection ω to the prototype of the flat connection $\omega^\flat$. $\square$

We have now established important lemmas concerning flat connections and holonomy but we have yet to show that the limit holonomy given by (4.30) exists. The aim of the next section is to be able to state that

"$\lim_{r\to 0}[J_r] = [J]$ exists for almost all $P \in \Sigma$, and is independent of P."

4.2 Existence of limit holonomy

We now state and prove the theorem governing existence of limit holonomy.

Theorem 4.8 *Let N be a transverse ϵ-neighborhood of the 2-dimensional submanifold Σ of a 4-manifold M. If $\omega \in L^2_{1,loc}(N\backslash\Sigma)$ and $\Omega \in L^2(N)$ then the holonomy limit exists at Σ. Locally there is a gauge in which the components of ω have a limit at Σ with, in particular $\omega_\theta \to \omega_\theta^\flat$ almost everywhere, where the flat connection of the 4-d cone is given by*

$$\omega^\flat = \omega_\theta^\flat d\theta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & -A & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} d\theta \tag{4.68}$$

Overview of proof

There are three key statements we wish to prove.

A) There exists a gauge such that $\omega_\theta \to \omega_\theta^\flat$ as $r \to 0$.

B) The limit holonomy exists at Σ.

C) The limit holonomy is independent of the point at which we take the surface transverse to the singularity.

We first gauge transform the connection into the radial gauge where $\omega_r = 0$. We then look at the θ component of ω and show that $\lim_{r\to 0}\omega_\theta = C_\theta$ where $C = C_\theta d\theta$ is a flat connection. We know from Lemma 4.7 that any flat connection is gauge equivalent to another flat connection, specifically

$$\omega^\flat = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & -A & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} d\theta \tag{4.69}$$

and A classifies the gauge equivalence class. So there is an s such that $s^{-1}Cs + s^{-1}\mathrm{d}s = \omega^\flat$ and since C and $\omega^\flat$ depend only on θ, so too does s. We then gauge transform ω with s to get $\hat{\omega} = s^{-1}\omega s + s^{-1}\mathrm{d}s$. We know that since s only depends on θ

$$\lim_{r\to 0}\hat{\omega}_\theta = \lim_{r\to 0}(s^{-1}\omega_\theta s + s^{-1}\frac{ds}{d\theta}) = s^{-1}C_\theta s + s^{-1}\frac{ds}{d\theta} \tag{4.70}$$

and

$$\lim_{r\to 0}\hat{\omega}_\theta d\theta = \lim_{r\to 0}(s^{-1}\omega s + s^{-1}\mathrm{d}s) = \omega^\flat \tag{4.71}$$

Hence, by first gauge transforming to the radial gauge and then gauge transforming using s we find that there exists a gauge such that $\omega_\theta \to \omega^\flat_\theta$ as $r \to 0$.
□A

G_r solves (4.36) for $D = \mathrm{d} + \omega$ and $\hat{G}_r$ is the solution for $\hat{D} = \mathrm{d} + \hat{\omega}$. Since D is gauge equivalent to $\hat{D}$ we can apply Lemma 4.3 to show that $G_r(2\pi) = J_r$ and $\hat{G}_r(2\pi) = \hat{J}_r$ are conjugate in $SO(3)$. We know that $\hat{G}_r(2\pi) = \hat{J}_r$ depends only on $\hat{\omega}_\theta$ and having shown that $\lim_{r\to 0}\hat{\omega}_\theta = \omega^\flat$ we can then apply Lemma 4.4 to deduce that $\lim_{r\to 0}\hat{J}_r = \hat{J}$ exists. Since J_r and $\hat{J}_r$ are conjugate, we know that $[J_r] = [\hat{J}_r]$ and hence that $\lim_{r\to 0}[J_r] = [J]$ exists.
□B

Having established a limit holonomy for one point P on the singularity, we then show that this limit does not change if we choose a different point on the singularity, say P'. We first change the coordinates on the singularity so that P and P' both lie on the line $t = t_0$.

We then show that the difference between the holonomy at P and the holonomy at P' is bounded by an expression which we know tends to zero as r tends to zero. Hence the limit holonomy is independent of the point at which we take the surface transverse to the singularity.

□C

Proof

The proof is divided into many steps which we shall discuss individually in detail.

We wish to show that there exists a gauge transformation such that the radial component of the connection term, ω_r, disappears. Let G be the gauge transformation which is the solution of

$$\frac{dG}{dr} + \omega_r G = 0 \tag{4.72}$$

We would like to show that $\omega \in L^2_{1,\mathrm{loc}}$ implies that $G \in L^2_{2,\mathrm{loc}}$.

We have in (4.72) a first order ODE for the gauge transformation, but only in the r-direction. All the large changes around a singularity happen on approach to the singularity (i.e. by decreasing r). In the case of a flat or near flat conical singularity changes induced by derivatives of the gauge transformation in the θ, t and z direction are small and smooth. If we vary parameters smoothly then the solution will also change smoothly. Therefore we can assume that any gauge transformation will be bounded in the non-r directions allowing us to focus solely on proving the boundedness of G in the r direction.

We are only considering the Lie algebra and Lie group valued r-components of (4.72). We perform the following calculation treating the matrices as scalars. The results found for scalars also work for matrices. Let us simplify the notation in (4.72) to

$$\begin{aligned} G' &= -\omega_r G \\ \Longrightarrow G'' &= (-\omega_r G)' \\ &= -\omega_r' G - \omega_r G' \\ &= -\omega_r' G - \omega_r(-\omega_r G) \\ &= \omega_r^2 G - \omega_r' G \end{aligned} \tag{4.73}$$

where $'$ denotes taking the derivative with respect to r. We wish to show that $\omega_r \in L^2_{1,\mathrm{loc}}$ implies that $G \in L^2_{2,\mathrm{loc}}$. We start by looking at the square of the $L^2_{2,\mathrm{loc}}$ norm of G.

$$\begin{aligned} \|G^\epsilon_\beta\|^2_{L^2_2} &= \int \sum_{0\leq|\alpha|\leq 2} |\nabla^\alpha G^\epsilon_\beta|^2 d\sigma \\ &= \int |G^\epsilon_\beta|^2 + |\delta^a_r \nabla_a G^\epsilon_\beta|^2 + |\delta^a_r \delta^b_r \nabla_b(\nabla_a G^\epsilon_\beta)|^2 d\sigma \end{aligned} \tag{4.74}$$

We know that $\nabla_a G^\epsilon_\beta = \partial_a G^\epsilon_\beta = G^\epsilon_{\beta,a}$. However, we must take care with the second covariant derivative $\nabla_b \nabla_a G^\epsilon_\beta = \nabla_b(G^\epsilon_{\beta,a}) = G^\epsilon_{\beta,ab} - \mathbf{\Gamma}^c_{ab} G^\epsilon_{\beta,c}$ where $\mathbf{\Gamma}^c_{ab}$ is the background metric connection. We find that

$$\delta^a_r \delta^b_r \nabla_b(\nabla_a G^\epsilon_\beta) = G^\epsilon_{\beta,rr} - \mathbf{\Gamma}^c_{rr} G^\epsilon_{\beta,c} = G'' \tag{4.75}$$

This is because the $\mathbf{\Gamma}^c_{rr}$ terms for the 4-d cylindrical polar flat metric are zero for each c. Hence

$$\|G^\epsilon_\beta\|^2_{L^2_2} = \int |G''|^2 + |G'|^2 + |G|^2 d\sigma \tag{4.76}$$

Since GG^T =Id it follows that

$$\begin{aligned} |\check{G}^\epsilon_\beta| &\leq 1 \qquad \forall \;\; \epsilon, \beta \\ \Longrightarrow \sum_{\epsilon,\beta} |\check{G}^\epsilon_\beta|^2 &< \infty \\ \Longrightarrow |G^\epsilon_\beta|^2 &< \infty \end{aligned} \tag{4.77}$$

Hence we know that G is bounded (so $|G| \leq c$ for some constant c) and so

$$\begin{aligned} \int |G''|^2 + |G'|^2 + |G|^2 d\sigma &= \int |\omega_r^4 G^2 - 2\omega_r^2 \omega_r' G^2 + {\omega_r'}^2 G^2| + |\omega_r^2 G^2| + |G^2| d\sigma \\ &\leq c^2 \int |\omega_r^4| - 2|\omega_r^2 \omega_r'| + |{\omega_r'}^2| + |\omega_r^2| + 1 d\sigma \end{aligned} \tag{4.78}$$

From $\omega_r \in L^2_{1,\text{loc}}$ we know that $\int |\omega_r^2| + |\omega_r'^2| d\sigma$ is finite.

We now apply the Sobolev Imbedding Theorem to our problem following Appendix B.1 very carefully. We let the domain Ξ (called Ω in the appendix) be a 4-D subset of $\mathbb{R}^4$ in the neighbourhood of the singular set $N\backslash\Sigma$. Using the notation of Appendix B.1, so far we have $k = n = 4$. Now since $\omega_r \in L^2_{1,\text{loc}}$ we want to take $p = 2$ and $j + m = 1$. To satisfy the initial restrictions of *Case A*, we must have $2m < 4$ and $4 - 2m < 4 \leq 4$. Simplifying this it follows that $0 < m < 2$ or $m = 1$ and hence $j = 0$. We can now extract results from the theorem and in particular from (B.3)

$$L^2_{1,\text{loc}}(N\backslash\Sigma) \to L^q_{\text{loc}}(N\backslash\Sigma), \qquad 2 \leq q \leq 4 \tag{4.79}$$

So by taking an imbedding into a different space and taking q to be 4, we can see that $\int |\omega_r^4| d\sigma$ is also finite. Going back to (4.78) we see that it only remains to show that $\int |\omega_r^2 \omega_r'| d\sigma < \infty$.

$$\int |\omega_r^2| + |\omega_r'^2| d\sigma < \infty \Longrightarrow \int |\omega_r'^2| d\sigma < \infty \tag{4.80}$$

Let us say that $\omega_r^2\omega_r' > \omega_r^4$ and $\omega_r^2\omega_r' > \omega_r'^2$. This implies that $\omega_r' > \omega_r^2$ and $\omega_r^2 > \omega_r'$ which is a contradiction. Hence $\omega_r^2\omega_r' \leq \omega_r^4$ and/or $\omega_r^2\omega_r' \leq \omega_r'^2$. This implies $\int |\omega_r^2\omega_r'| d\sigma \leq \int |\omega_r^4| d\sigma < \infty$ and/or $\int |\omega_r^2\omega_r'| d\sigma \leq \int |\omega_r'^2| d\sigma < \infty$.

Therefore

$$\int |\omega_r^2 \omega_r'| d\sigma < \infty \tag{4.81}$$

Hence

$$\begin{aligned} c^2 \int |\omega_r^4| - 2|\omega_r^2\omega_r'| + |\omega_r'^2| + |\omega_r^2| + 1 d\sigma &< \infty \\ \Longrightarrow \int |G''|^2 + |G'|^2 + |G|^2 d\sigma &< \infty \\ \Longrightarrow G &\in L^2_{2,\text{loc}} \qquad \square \end{aligned} \tag{4.82}$$

and so we know that

$$\omega \in L^2_{1,\text{loc}} \Longrightarrow G \in L^2_{2,\text{loc}} \tag{4.83}$$

Since the neighborhood N of Σ is locally a product we can apply a consequence of Fubini's theorem (see Appendix B.2)

$$\begin{aligned} G &\in L_2^2(N) \\ \Longrightarrow \int_N |G|^2 + |\partial G|^2 + |\partial(\partial G)|^2 d^4x &< \infty \\ \Longrightarrow \int_{N_0} |G|^2 + |\partial G|^2 + |\partial(\partial G)|^2 d^2x &< \infty \text{ almost everywhere} \\ \Longrightarrow G &\in L_2^2(N_0) \text{ almost everywhere} \end{aligned} \tag{4.84}$$

where N_0 is the surface transverse to the singularity. We now return to the Sobolev Imbedding Theorem and look at *Case C* from [1] (once again, refer to Appendix B.1).

Our domain is N_0 so $n = 2$. Let us choose $m = p = 2$ which satisfies the requirement for *Case C*, that $mp > n$. And so we have

$$L_{j+2}^2 \to C^j \tag{4.85}$$

and hence

$$L_2^2 \to C^0 \tag{4.86}$$

Therefore, since we have from Fubini that $G \in L_2^2(N_0)$, we can imbed G into a space of continuous functions and therefore G is continuous in almost all transverse surfaces. The obvious periodicity from using polar coordinates tells us also that G is periodic in θ with period 2π.

We now let ω and $\tilde{\omega}$ be related by the gauge transformation G. Hence

$$\begin{aligned} \tilde{\omega} &= G^{-1}\omega G + G^{-1}\mathrm{d}G \\ \tilde{\omega}_r &= G^{-1}\omega_r G + G^{-1}\frac{dG}{dr} \\ &= G^{-1}\left(\frac{dG}{dr} + \omega_r G\right) \\ &= 0 \qquad \text{from (4.72)} \end{aligned} \tag{4.87}$$

We can see that in this gauge $\tilde{\omega}_r = 0$ and so we have put ω into a radial gauge. Since $\omega \in L_1^2$ and $G \in L_2^2$, we know that $\tilde{\omega} \in L_1^2$. We now re-label $\tilde{\omega}$ as ω.

We have by definition,

$$\Omega^{\alpha}_{\beta ab} = 2\omega^{\alpha}_{\beta[b,a]} + \omega^{\alpha}_{\gamma a}\omega^{\gamma}_{\beta b} - \omega^{\alpha}_{\gamma b}\omega^{\gamma}_{\beta a} \tag{4.88}$$

and since $\omega_r = 0$ in the new radial gauge, we have

$$\begin{aligned}
\Omega^{\alpha}_{\beta r\theta} &= 2\omega^{\alpha}_{\beta[\theta,r]} + \omega^{\alpha}_{\gamma r}\omega^{\gamma}_{\beta\theta} - \omega^{\alpha}_{\gamma\theta}\omega^{\gamma}_{\beta r} \\
&= \omega^{\alpha}_{\beta\theta,r} - \omega^{\alpha}_{\beta r,\theta} + 0 - 0 \\
&= \omega^{\alpha}_{\beta\theta,r} \\
\text{Hence} \qquad \Omega_{r\theta} &= \frac{\partial\omega_\theta}{\partial r}
\end{aligned} \tag{4.89}$$

We take ω_θ to be a Fourier series with complex coefficients a_n and so in a Fourier series expansion

$$\omega_\theta = \sum_{n=-\infty}^{\infty} a_n(r)e^{in\theta} \tag{4.90}$$

We now note, again from Fubini's theorem, that Ω is in L^2 on almost all transverse surfaces. We note that Ω is piecewise continuous with periodicity 2π and so we can apply Parseval's equality (see Appendix B.3) for complex coefficients. We take a transverse surface at (u_0, v_0) and use Parseval's equality, (4.89) and the knowledge that when integrating between r_1 and r_2 we have $r_1 \leq r$

$$\begin{aligned}
\frac{\partial\omega_\theta}{\partial r} &= \Omega_{r\theta} \\
\sum_{n=-\infty}^{\infty} \frac{\partial a_n}{\partial r}e^{in\theta} &= \Omega \\
\sum_{n=-\infty}^{\infty} \left|\frac{\partial a_n}{\partial r}\right|^2 &= \int |\Omega|^2 d\theta \\
\int_{r_1}^{r_2} \sum_{n=-\infty}^{\infty} \left|\frac{\partial a_n}{\partial r}\right|^2 dr &= \int\int_{r_1}^{r_2} |\Omega|^2 dr\, d\theta \\
&\leq \int\int_{r_1}^{r_2} |\Omega|^2 \frac{r}{r_1} dr\, d\theta \\
\Longrightarrow \int_{r_1}^{r_2} \sum_{n=-\infty}^{\infty} \left|\frac{\partial a_n}{\partial r}\right|^2 dr &\leq c\int\int_{r_1}^{r_2} |\Omega|^2 r\, dr\, d\theta \qquad \square
\end{aligned} \tag{4.91}$$

We let r_m be a sequence of radii tending to zero. We let $\omega_\theta(r_m) = \omega_m$ with $m = \{0, 1, 2, \ldots\}$. We would like to show that for all $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that $|\omega_m - \omega_p| < \epsilon$ for all $m, p > N$ and hence that ω_m is a Cauchy sequence. Let us take

$$\begin{aligned}
|\omega_k - \omega_l| &= \left| \sum_{n=-\infty}^{\infty} a_n(r_k)e^{in\theta} - \sum_{n=-\infty}^{\infty} a_n(r_l)e^{in\theta} \right| \\
&= \left| \sum_{n=-\infty}^{\infty} e^{in\theta}\left(a_n(r_k) - a_n(r_l)\right) \right| \\
&\leq \sum_{n=-\infty}^{\infty} |e^{in\theta}(a_n(r_k) - a_n(r_l))| \\
&= \sum_{n=-\infty}^{\infty} |a_n(r_k) - a_n(r_l)| \qquad (4.92)
\end{aligned}$$

and applying Parseval's theorem again it follows that

$$\int_0^{2\pi} |\omega_k - \omega_l|^2 d\theta \leq \sum_{n=-\infty}^{\infty} |a_n(r_k) - a_n(r_l)|^2 \qquad (4.93)$$

Now let $r_k = r_l + \delta r_{kl}$. Using the Fundamental Theorem of Calculus it follows that

$$a_n(r_k) = a_n(r_l) + \int_{r_l}^{r_l+\delta r_{kl}} \frac{\partial a_n(r)}{\partial r} dr \qquad (4.94)$$

Hence

$$\begin{aligned}
|a_n(r_k) - a_n(r_l)| &= \left| \int_{r_l}^{r_k} \frac{\partial a_n(r)}{\partial r} dr \right| \\
&\leq \int_{r_l}^{r_k} \left| \frac{\partial a_n(r)}{\partial r} \right| dr \qquad (4.95)
\end{aligned}$$

Now we use Hölder's inequality with $p = 2$ (see Appendix B.4) to get

$$\begin{aligned}
|a_n(r_k) - a_n(r_l)| &\leq \int_{r_l}^{r_k} |1 \cdot \frac{\partial a_n(r)}{\partial r}| dr \\
&\leq \|1\|_{L^2} \|\frac{\partial a_n(r)}{\partial r}\|_{L^2} \\
&= \left(\int_{r_l}^{r_k} 1^2 dr \right)^{\frac{1}{2}} \left(\int_{r_l}^{r_k} \left| \frac{\partial a_n(r)}{\partial r} \right|^2 dr \right)^{\frac{1}{2}} \\
&\leq c_1 \left(\int_{r_l}^{r_k} \left| \frac{\partial a_n(r)}{\partial r} \right|^2 dr \right)^{\frac{1}{2}} \qquad (4.96)
\end{aligned}$$

Hence

$$|a_n(r_k) - a_n(r_l)|^2 \leq c_2 \int_{r_l}^{r_k} \left| \frac{\partial a_n(r)}{\partial r} \right|^2 dr \qquad (4.97)$$

and using (4.91) it follows that

$$\sum_{n=-\infty}^{\infty} |a_n(r_k) - a_n(r_l)|^2 \leq c_2 \int_{r_l}^{r_k} \sum_{n=-\infty}^{\infty} \left|\frac{\partial a_n}{\partial r}\right|^2 dr \leq c_3 \int\int_{r_l}^{r_k} |\Omega|^2 r dr d\theta \tag{4.98}$$

From (4.93)

$$\int_0^{2\pi} |\omega_k - \omega_l|^2 d\theta \leq c_3 \int\int_{r_l}^{r_k} |\Omega|^2 r dr d\theta \tag{4.99}$$

Since r_k is a sequence of radii tending to zero we see that as r_l approaches r_k, the right hand side of (4.99) tends to zero. Hence

$$\begin{aligned}
\lim_{k,l\to\infty} \int_0^{2\pi} |\omega_\theta(r_k) - \omega_\theta(r_l)|^2 d\theta &= 0 \\
\Longrightarrow \int_0^{2\pi} \left(\lim_{k,l\to\infty} |\omega_\theta(r_k) - \omega_\theta(r_l)|\right)^2 d\theta &= 0 \\
\Longrightarrow \lim_{k,l\to\infty} |\omega_\theta(r_k) - \omega_\theta(r_l)| &= 0 \qquad \text{almost everywhere}
\end{aligned}$$

since if the integral of the square of a function is zero then the function can only be non-zero on a set of measure zero. Hence $\omega_\theta(r_m) = \sum_{n=-\infty}^{\infty} a_n(r_m)e^{in\theta}$ is a Cauchy sequence almost everywhere. □

The Cauchy Convergence Criterion states that a sequence in $\mathbb{R}^p$ is convergent if and only if it is a Cauchy sequence. We have shown that the Fourier series above forms a Cauchy sequence, so the sequence converges to a limit. The limit of (4.90) as r tends to zero is

$$C_\theta = \sum_{n=-\infty}^{\infty} a_n(0)e^{in\theta} \tag{4.100}$$

We have a connection 1-form which only has non-zero coefficients for $d\theta$ and that coefficient depends only on θ. Hence the curvature 2-form given by the connection $C = C_\theta d\theta$ is

$$\begin{aligned}
\Omega^\alpha_{\beta ab} &= C^\alpha_{\beta a,b} - C^\alpha_{\beta b,a} + C^\alpha_{\gamma a}C^\gamma_{\beta b} - C^\alpha_{\gamma b}C^\gamma_{\beta a} \\
&= C^\alpha_{\beta\theta,\theta} - C^\alpha_{\beta\theta,\theta} + C^\alpha_{\gamma\theta}C^\gamma_{\beta\theta} - C^\alpha_{\gamma\theta}C^\gamma_{\beta\theta} \\
&= 0
\end{aligned} \tag{4.101}$$

Hence the 1-form $C = C_\theta d\theta$ defines a flat connection on the transverse surface through (t_0, z_0)

Lemma 4.7 states that any flat connection is gauge equivalent to another flat connection, which is given by the prototype for a flat connection $\omega^\flat$. Therefore we have

$$s^{-1}Cs + s^{-1}\mathrm{d}s = \omega^{\flat} = \omega^{\flat}_{\theta}d\theta \tag{4.102}$$

for some s. Since C and $\omega^{\flat}$ only depend on θ we know that s only depends on θ. By varying t and z we find that the extended s is still independent of r for all of X_0.

Now we return to the connection ω for which we have gauged away the radial component ($\omega_r = 0$) and gauge transform by s to obtain

$$\hat{\omega} = s^{-1}\omega s + s^{-1}\mathrm{d}s \tag{4.103}$$

Since s is independent of r, we still have $\hat{\omega}_r = 0$. So now, just as before for ω_θ, we have the inequality (4.91) for the Fourier coefficients of $\hat{\omega}_\theta$ and so likewise $\hat{\omega}_\theta$ converges to a limit as $r \to 0$. However

$$\begin{aligned}
\lim_{r\to 0} \hat{\omega}_\theta &= \lim_{r\to 0}\left(s^{-1}\omega_\theta s + s^{-1}\frac{ds}{d\theta}\right) \\
&= s^{-1}(\lim_{r\to 0}\omega_\theta)s + s^{-1}\frac{ds}{d\theta} \\
&= s^{-1}C_\theta s + s^{-1}\frac{ds}{d\theta}
\end{aligned} \tag{4.104}$$

and

$$\begin{aligned}
\lim_{r\to 0} \hat{\omega}_\theta d\theta &= \lim_{r\to 0}\hat{\omega} \\
&= \lim_{r\to 0}\left(s^{-1}\omega s + s^{-1}\mathrm{d}s\right) \\
&= s^{-1}(\lim_{r\to 0}\omega)s + s^{-1}\mathrm{d}s \\
&= s^{-1}Cs + s^{-1}\mathrm{d}s \\
&= \omega^{\flat}
\end{aligned} \tag{4.105}$$

From Lemma 4.3, since g_r and $\hat{g}_r$ are solutions of (4.36) for gauge equivalent connections $D = \mathrm{d} + \omega$ and $\hat{D} = \mathrm{d} + \hat{\omega}$, we know that $g_r(2\pi) = J_r$ and $\hat{g}_r(2\pi) = \hat{J}_r$ are conjugate in $SO(3)$. We know $\hat{g}_r(2\pi) = \hat{J}_r$ depends only on $\hat{\omega}_\theta(r)$. As $r \to 0$ we have $\hat{\omega} \to \omega^{\flat}$, hence $\lim_{r\to 0}\hat{J}_r = \hat{J}$ exists. The conjugacy class $[\hat{J}_r]$ of $\hat{J}_r$ is gauge invariant and so $[\hat{J}_r] = [J_r]$. Hence the solution of (4.36) at $\theta = 2\pi$ converges and the limit holonomy condition exists. $\square$

The existence of limit holonomy can be shown for any (t, z) but it remains to show the limit is independent of the actual values t and z take. We let two points, P_1 and P_2 be points on the singularity. We change the coordinates t and z to $\tilde{t}$ and $\tilde{z}$ in such a way that one of the two coordinates denoting position on the singularity is the same for both points, i.e. $P_1 = (\tilde{t}_0, \tilde{z}_1)$ and $P_2 = (\tilde{t}_0, \tilde{z}_2)$. We can now consider a sequence of cylinders $C_{r_i} = \{(\tilde{t}_0, \tilde{z}, r_i, \theta) | \tilde{z}_1 \leq \tilde{z} \leq \tilde{z}_2, 0 \leq \theta \leq 2\pi\}$ as $r_i \to 0$. We recall that $\Omega \in L^2(N)$ and so, again as a consequence of Fubini's theorem, we can separate the $d\tilde{t}$ and dr components and be left with a finite integral over the θ and $\tilde{z}$ directions

$$\int_0^{2\pi} \int_{\tilde{z}_1}^{\tilde{z}_2} |\Omega|^2 r_i d\tilde{z} d\theta = \int\int_{C_{r_i}} |\Omega|^2 dS < \infty \qquad \text{almost everywhere} \tag{4.106}$$

where $dS = d\tilde{z} r d\theta$. Hence

$$\lim_{r_i \to 0} r_i \int\int_{C_{r_i}} |\Omega|^2 dS = 0 \tag{4.107}$$

We now choose a particular gauge so that $\omega_{\tilde{z}} = 0$ and, referring back to (4.88) and (4.89), we obtain $\frac{\partial \omega_\theta}{\partial \tilde{z}} = \Omega_{\tilde{z}\theta}$ in an analogous way. Integrating $\Omega_{\tilde{z}\theta}$ with respect to $\tilde{z}$ between $\tilde{z}_1$ and $\tilde{z}_2$ we obtain

$$\begin{aligned} |\omega_\theta(\tilde{z}_2, \tilde{t}_0, r_i, \theta) - \omega_\theta(\tilde{z}_1, \tilde{t}_0, r_i, \theta)| &= \left| \int_{\tilde{z}_1}^{\tilde{z}_2} \Omega_{\tilde{z}\theta} d\tilde{z} \right| \\ &\leq \int_{\tilde{z}_1}^{\tilde{z}_2} |\Omega_{\tilde{z}\theta}| d\tilde{z} \end{aligned} \tag{4.108}$$

We know that $|\Omega|^2 = |\Omega^i_{jab}|^2 = \Omega^i_{jab}\bar{\Omega}^i_{jcd}\xi^{ac}\xi^{bd}$ where ξ_{ab} is the positive definite background metric in 4-D cylindrical polar coordinates. Hence $|\Omega|^2$ is equal to the sum of positive definite terms one of which is $\frac{1}{r_i^2}|\Omega_{\tilde{z}\theta}|^2$. Therefore $\frac{1}{r_i^2}|\Omega_{\tilde{z}\theta}|^2$ must be less than or equal to $|\Omega|^2$ and so $|\Omega_{\tilde{z}\theta}| \leq r_i|\Omega|$. Now continuing from (4.108) we have

$$\begin{aligned} |\omega_\theta(\tilde{z}_2, \tilde{t}_0, r_i, \theta) - \omega_\theta(\tilde{z}_1, \tilde{t}_0, r_i, \theta)| &\leq r_i \int_{\tilde{z}_1}^{\tilde{z}_2} |\Omega| d\tilde{z} \\ |\omega_\theta(\tilde{z}_2, \tilde{t}_0, r_i, \theta) - \omega_\theta(\tilde{z}_1, \tilde{t}_0, r_i, \theta)|^2 &\leq r_i^2 \left(\int_{\tilde{z}_1}^{\tilde{z}_2} |\Omega| d\tilde{z} \right)^2 \\ & \qquad \text{using Hölder's inequality} \\ &\leq k r_i^2 \int_{\tilde{z}_1}^{\tilde{z}_2} |\Omega|^2 d\tilde{z} \\ \int_0^{2\pi} |\omega_\theta(\tilde{z}_2, \tilde{t}_0, r_i, \theta) - \omega_\theta(\tilde{z}_1, \tilde{t}_0, r_i, \theta)|^2 d\theta &\leq k r_i^2 \int_0^{2\pi} \int_{\tilde{z}_1}^{\tilde{z}_2} |\Omega|^2 d\tilde{z} d\theta \\ &= k r_i \int\int_{C_{r_i}} |\Omega|^2 dS \\ & \qquad \text{using Hölder's inequality} \\ \Longrightarrow \left(\int_0^{2\pi} |\omega_\theta(\tilde{z}_2, \tilde{t}_0, r_i, \theta) - \omega_\theta(\tilde{z}_1, \tilde{t}_0, r_i, \theta)| d\theta \right)^2 &\leq K r_i \int\int_{C_{r_i}} |\Omega|^2 dS \end{aligned} \tag{4.109}$$

We let G_1 and G_2 be solutions of (4.36) at $r = r_i$ in the surfaces transverse to the singularity at $(\tilde{z}_1, \tilde{t}_0)$ and $(\tilde{z}_2, \tilde{t}_0)$ respectively. So

$$\begin{aligned}
\frac{dG_1}{d\theta} + \omega_\theta(\tilde{z}_1, \tilde{t}_0, r_i, \theta)G_1 &= 0, \qquad G_1(0) = I \\
\Longrightarrow \frac{dG_1}{d\theta} + \omega_\theta(\tilde{z}_2, \tilde{t}_0, r_i, \theta)G_1 &= (\omega_\theta(\tilde{z}_2, \tilde{t}_0, r_i, \theta) - \omega_\theta(\tilde{z}_1, \tilde{t}_0, r_i, \theta))G_1 \\
\Longrightarrow \frac{dG_1}{d\theta}G_1^{-1} + \omega_\theta(\tilde{z}_2, \tilde{t}_0, r_i, \theta) &= \omega_\theta(\tilde{z}_2, \tilde{t}_0, r_i, \theta) - \omega_\theta(\tilde{z}_1, \tilde{t}_0, r_i, \theta) \\
\Longrightarrow \int_0^{2\pi} |\frac{dG_1}{d\theta}G_1^{-1} + \omega_\theta(\tilde{z}_2, \tilde{t}_0, r_i, \theta)|d\theta &= \int_0^{2\pi} |\omega_\theta(\tilde{z}_2, \tilde{t}_0, r_i, \theta) - \omega_\theta(\tilde{z}_1, \tilde{t}_0, r_i, \theta)|d\theta \\
&\leq \left(Kr_i \int\int_{C_{r_i}} |\Omega|^2 dS\right)^{\frac{1}{2}} \qquad \text{from (4.109)}
\end{aligned}$$

We know that

$$\frac{dG_2}{d\theta} + \omega_\theta(\tilde{z}_2, \tilde{t}_0, r_i, \theta)G_2 = 0 \Longrightarrow \omega_\theta(\tilde{z}_2, \tilde{t}_0, r_i, \theta) = -\frac{dG_2}{d\theta}G_2^{-1} \tag{4.110}$$

So

$$\begin{aligned}
\left(Kr_i \int\int_{C_{r_i}} |\Omega|^2 dS\right)^{\frac{1}{2}} &\geq \int_0^{2\pi} \left|\frac{dG_1}{d\theta}G_1^{-1} - \frac{dG_2}{d\theta}G_2^{-1}\right| d\theta \\
&= \int_0^{2\pi} \left|\frac{dG_1}{d\theta}G_1^{-1} + G_2\frac{dG_2^{-1}}{d\theta}\right| d\theta \qquad \text{(Leibniz's rule)} \\
&= \int_0^{2\pi} \left|G_2(G_2^{-1}\frac{dG_1}{d\theta} + \frac{dG_2^{-1}}{d\theta}G_1)G_1^{-1}\right| d\theta \\
&= \int_0^{2\pi} |G_2| \left|(G_2^{-1}\frac{dG_1}{d\theta} + \frac{dG_2^{-1}}{d\theta}G_1)\right| |G_1^{-1}| d\theta \\
&\geq C\left|\int_0^{2\pi} G_2^{-1}\frac{dG_1}{d\theta} + \frac{dG_2^{-1}}{d\theta}G_1 d\theta\right| \\
&= C\left|\left[G_2^{-1}G_1\right]_0^{2\pi}\right| \\
&= C\left|G_2^{-1}(2\pi)G_1(2\pi) - G_2^{-1}(0)G_1(0)\right| \\
&= C\left|G_2^{-1}(2\pi)G_1(2\pi) - I\right| \qquad \text{since } G_1(0) = G_2(0) = I
\end{aligned}$$

Hence

$$C|G_2^{-1}(2\pi)G_1(2\pi) - I| \leq \left(Kr_i \int\int_{C_{r_i}} |\Omega|^2 dS\right)^{\frac{1}{2}} \to 0 \tag{4.111}$$

This proves that as $r_i \to 0$ we have that $G_1(2\pi) = G_2(2\pi)$ and hence the holonomy limit is the same at P_1 and P_2. $\square$

We now show that the directions tangent to the singularity are left invariant by the holonomy. Since the space-time is static we know that the holonomy is a rotation keeping the $\hat{T}$ direction fixed. Hence the t-direction of the singularity remains fixed. We thus need to show that the z-direction is also invariant.

We therefore work at a fixed time t_0 and look at directions tangent to Σ_{t_0}. To show that these directions remain invariant we adapt the method of [6]. However, rather than considering fixed loops surrounding a thick cosmic string, we will consider a family of loops surrounding the singularity. We may use the method in [6] since the curvature diverges more slowly than the lengths of the loops tends to zero. We start by considering a map $\lambda(r, \theta, z)$ which parameterises a solid cylinder in a neighbourhood of some point on the singularity. Let

$$\begin{aligned} \lambda : (0,1] \times [0, 2\pi] \times [0,1] &\rightarrow N \cap \Sigma_{t_0} \\ (r, \theta, z) &\mapsto \lambda(r, \theta, z) \end{aligned} \tag{4.112}$$

be a map with the following properties (see Figure 4.3)

1. $\lambda_{r,z}(\theta) := \lambda(r\theta, z)$ is a family of loops with $\lambda_{r,z}(0) = \lambda_{r,z}(2\pi)$
2. The length of $\lambda_{r,z} \rightarrow 0$ as $r \rightarrow 0$
3. The curves $r \mapsto \lambda(r, \theta, z)$ for fixed θ and z are curves which tend to points on the singularity as $r \rightarrow 0$
4. The curves $z \mapsto \lambda(r, \theta, z)$ for fixed r and θ are geodesics

Let e_i^a be a frame at the point $\lambda(1, 0, 0)$. We first parallelly propagate this in along the curve $\lambda(r, 0, 0)$ and then around each of the loops $\lambda_{r,0}(\theta)$. Finally we propagate it along the curve $z \mapsto \lambda(r, \theta, z)$ to obtain a frame at each of the points on the cylinder.

We now consider tangent vectors to the cylinder $r = const.$ with coordinate components

$$X^a = \frac{\partial \lambda^a}{\partial z} \qquad \text{and} \qquad Y^a = \frac{\partial \lambda^a}{\partial \theta} \tag{4.113}$$

We can rewrite X^a with frame indices by contracting with the dual frame ϑ_a^i.

$$X^i = X^a \vartheta_a^i \tag{4.114}$$

By parallely propagating the frame around $\lambda_{r,0}$ we then have

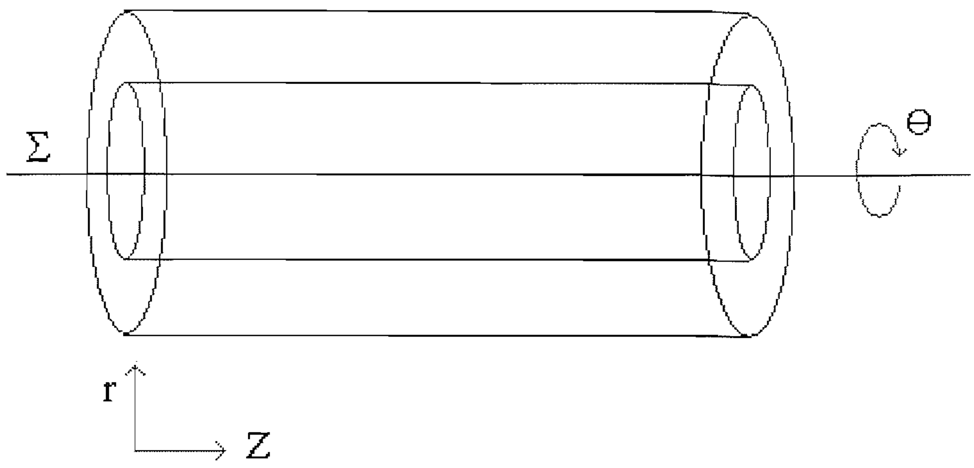

Figure 4.3: A diagram showing the map λ.

$$X^i(r, 2\pi, 0) = \vartheta^i_a(r, 2\pi, 0)X^a(r, 0, 0) \tag{4.115}$$

Let $R_i^j(r)$ be the holonomy around Σ such that

$$e_i^a(r, 0, 1) = R_i^j(r)e_j^a(r, 0, 0) \tag{4.116}$$

Hence

$$\vartheta^i_a(r, 2\pi, 0) = \bar{R}^i_j(r)\vartheta^j_a(r, 0, 0) \tag{4.117}$$

where $\bar{R}^i_j(r)$ is the inverse of $R^i_j(r)$. Therefore

$$\begin{aligned} X^i(r, 0, 1) &= \vartheta^i_a(r, 0, 1)X^a(r, 0, 0) \\ &= \bar{R}^i_j(r)\vartheta^j_a(r, 0, 0)X^a(r, 0, 0) \\ &= \bar{R}^i_j(r)X^i(r, 0, 0) \end{aligned} \tag{4.118}$$

Hence

$$\begin{aligned} |\bar{R}(r)X(r, 0, 0) - X(r, 0, 0)| &= |X(r, 0, 1) - X(r, 0, 0)| \\ &= \left| \int_0^{2\pi} \nabla_Y X d\theta \right| \\ &\leq \int_0^{2\pi} |\nabla_Y X| \, d\theta \end{aligned} \tag{4.119}$$

Since the connection is torsion free and X and Y are surface forming

$$\nabla_X Y - \nabla_Y X = [X, Y] = 0 \tag{4.120}$$

Hence

$$|\bar{R}(r)X(r,0,0) - X(r,0,0)| \leq \int_0^{2\pi} |\nabla_X Y| \, d\theta \tag{4.121}$$

We now estimate $|\nabla_X Y|$. Since the curves $z \mapsto \lambda(r,\theta,z)$ are geodesics the equation of geodesic deviation gives

$$\frac{d^2Y^i}{dz^2} = R^i_{jkl}X^jX^kY^l \tag{4.122}$$

Solving this equation using the Green's functions for $\frac{d^2f}{dz^2} = 0$ gives

$$\begin{aligned} Y^i(r,\theta,z) &= zY^i(r,\theta,1) + (1-z)Y^i(r,\theta,0) + (z-1)\int_0^z z' R^i_{jkl}(r,\theta,z')X^jX^kY^l dz' \\ &+z\int_z^1 (z'-1)R^i_{jkl}(r,\theta,z')X^jX^kY^l dz' \end{aligned} \tag{4.123}$$

Differentiating and setting $z = 0$ gives

$$\frac{dY^i}{dx}(r,\theta,0) = Y^i(r,\theta,1) - Y^i(r,\theta,0) + \int_0^1 (z'-1)R^i_{jkl}(r,\theta,z')X^jX^kY^l dx' \tag{4.124}$$

Since $\nabla_X e^a_i = 0$, $\frac{dY^i}{dx}(r,\theta,0)$ gives the frame components of $\nabla_X Y^i(r,\theta,0)$ and hence

$$|\nabla_X Y^i(r,\theta,0)| \leq |Y^i(r,\theta,1)| + |Y^i(r,\theta,0)| + \int_0^1 (z'-1)|R^i_{jkl}(r,\theta,z')X^jX^kY^l| dx' \tag{4.125}$$

substituting into (4.121) and using the fact that $0 < z' - 1 < 1$, then gives

$$\begin{aligned} |\bar{R}(r)X(r,0,0) - X(r,0,0)| &\leq \int_0^{2\pi} |Y^i(r,\theta',1)| d\theta' + \int_0^{2\pi} |Y^i(r,\theta',0)| d\theta \\ &+ \int_0^{2\pi}\int_0^1 |R^i_{jkl}(r,\theta,z')X^jX^kY^l| dx' d\theta' \end{aligned} \tag{4.126}$$

To estimate (4.126) we use the fact that the length of the loops given by $r = const.$ are $O(r)$ as $r \to 0$. Hence we have

$$|Y^i(r,\theta,z)| < kr \tag{4.127}$$

Hence the first two terms on the right hand side of (4.126) tend to zero as $r \to 0$. To estimate the final term we write

$$X^i = f\hat{X}^i, \qquad Y^i = g\hat{Y}^i \tag{4.128}$$

where $\hat{X}^i$ and $\hat{Y}^i$ are unit vectors as measured by the background metric. Then by the construction of λ we have

$$|f| < c_1 \qquad \text{and} \qquad |g| < c_2 r \tag{4.129}$$

Then

$$|R^i_{jkl}X^jX^kY^l| < c_3|R^i_{jkl}\hat{X}^j\hat{X}^k\hat{Y}^l|r \tag{4.130}$$

However the curvature is in L^2 so that

$$\int_0^{2\pi}\int_0^r |R^i_{jkl}\hat{X}^j\hat{X}^k\hat{Y}^l|^2 r\,dr\,d\theta < \infty \tag{4.131}$$

Hence $|R^i_{jkl}\hat{X}^j\hat{X}^k\hat{Y}^l|^2 r$ diverges more slowly than $\frac{1}{r}$. Thus $|R^i_{jkl}\hat{X}^j\hat{X}^k\hat{Y}^l|$ diverges more slowly than $\frac{1}{r}$ so that $|R^i_{jkl}\hat{X}^j\hat{X}^k\hat{Y}^l|r$ tends to zero as r tends to zero. Hence by (4.130) we have

$$|R^i_{jkl}X^jX^kY^l| \to 0 \qquad \text{as} \qquad r \to 0 \tag{4.132}$$

Hence

$$\lim_{r\to 0}\int_0^{2\pi}\int_0^1 |R^i_{jkl}(r,\theta',z')X^jX^kY^l|dx'd\theta' = 0 \tag{4.133}$$

Thus taking the limit as $r \to 0$ in (4.121) we have

$$\lim_{r\to 0}|\bar{R}(r)X(r,0,0) - X(r,0,0)| = 0 \tag{4.134}$$

But as r tends to zero, $X(r,0,0)$ is tangent to the singularity and $\bar{R}(r) \to \bar{R}$, the holonomy of the singularity at $z = 0$. Hence we must have that the tangent directions to the singularity are invariant under the limit holonomy as claimed.

4.3 Similarities to a flat connection as $r \to 0$

In this section we bring together ideas needed for the second main result of [39]. We first consider the set of connections $\mathcal{W}^p$ where $p \geq 2$

$$\mathcal{W}^p = \left\{D = \mathrm{d} + \omega | \omega \in L^p_{1,\mathrm{loc}}(X_0) \,\text{and}\, \Omega \in L^p(N_0)\right\} \tag{4.135}$$

In the previous section we have demonstrated the existence of limit holonomy for $D \in \mathcal{W}^2$. It follows from Theorem 4.8 that if $D \in \mathcal{W}^p$ then

"A limit holonomy exists and is given by the real number A as shown in (4.51)". (H_A)

We call the above A-dependent holonomy condition (H_A). If we know that (H_A) is satisfied for a connection in $\mathcal{W}^p$ then we can find a prototype of the flat connection $\omega^\flat$ which shares the same holonomy.

We introduce a new space $L^p_{1,\Upsilon}$ where Υ is a flat connection. A 1-form ζ is in $L^p_{1,\zeta}$ if

$$\|\zeta\|_{L^p_{1,\Upsilon}} = \|\zeta\|_{L^p} + \|\nabla_\Upsilon \zeta\|_{L^p} = \|\zeta\|_{L^p} + \|\mathrm{d}\zeta + [\zeta, \Upsilon]\| < \infty \tag{4.136}$$

We now give the main result for this section.

Theorem 4.9 *There exists a constant $k > 0$ such that, for $D \in \mathcal{W}^2$ with $\|\Omega\|_{L^2(N_0)} < k$ there is a real number A (with corresponding flat connection $\omega^\flat$) and a gauge in which $D = \mathrm{d} + \omega$ with $\omega - \omega^\flat \in L^2_{1,\omega^\flat}(N_0)$. Moreover, for some constant C,*

$$\|\omega - \omega^\flat\|_{L^2_{1,\omega^\flat}} \leq C\|\Omega\|_{L^2} \tag{4.137}$$

We already know from Theorem 4.8 that locally there is a gauge in which the components of ω have a limit at the singularity with $\omega_\theta \to \omega^\flat$ almost everywhere. Theorem 4.9 states that the $L^2_{1,\omega^\flat}(N_0)$ norm of the difference between the connection ω and the flat connection $\omega^\flat$ is less than or equal to a constant multiplied by the L^2 norm of the curvature Ω. By taking the curvature over smaller and smaller regions around the singularity we can make this term go to zero. Hence on approach to the singularity we find that ω tends to $\omega^\flat$. In other words, for any given connection, there is a gauge in which this connection is, near the singular 2-manifold, asymptotic to a flat connection $\omega^\flat$. This is stronger than the statement from Theorem 4.8 since it involves all the components of ω (not just the θ component).

All connections in $\mathcal{W}^p$ have a holonomy number A, telling us which conjugacy class the holonomy of the connection belongs to. Since all these connections with corresponding A are asymptotic to the same $\omega^\flat$, the holonomy number $A \in \mathbb{R}$ provides a useful way to classify connections in $\mathcal{W}^p$.

We now define $\mathcal{W}^p_{A,k}$, a subset of $\mathcal{W}^p$, to be used in the proposition below.

$$\mathcal{W}^p_{A,k} = \{D \in \mathcal{W}^p | \, ||\Omega||_{L^2} \leq k \,\text{and}\, (H_A) \text{ holds}\} \tag{4.138}$$

The main technical result of this section is Proposition 4.10 which leads on to Corollary 4.11 which then implies Theorem 4.9.

Proposition 4.10 *Let $p > 2$. Then there exists k and $\hat{c}$, and an explicit flat connection Γ with holonomy A, such that if $D \in \mathcal{W}^p_{m,k}$, then there exists a gauge in which $D = \mathrm{d} + \hat{\omega}$ and*

$$||\hat{\omega} - \Gamma||_{L^q_{1,\Gamma}(N_0)} \leq \hat{c}||\Omega||_{L^q(N_0)}, \quad 2 \leq q \leq p \tag{4.139}$$

We now wish to show a corollary of the above proposition featuring a similar inequality to (4.139) but for the prototype of the flat connection $\omega^\flat$.

Corollary 4.11 *Let $p > 2$. There exists a gauge in which $D = \mathrm{d} + \omega$ with $\omega - \omega^\flat \in L^p_{1,\omega^\flat}(N_0)$ and*

$$||\omega - \omega^\flat||_{L^q_{1,\omega^\flat}(N_0)} \leq c||\Omega||_{L^q(N_0)}, \quad 2 \leq q \leq p \tag{4.140}$$

Proof

Since Γ is a flat connection, we know from Lemma 4.7 that there is a gauge in which Γ is given by $\omega^\flat$. In the limit as $r \to 0$ both these connections are given by $a_\infty d\theta$ for some constant a_∞ (see (4.250)). Hence Γ and $\omega^\flat$ are related by a gauge transformation $\grave{G}$ with the property that $\lim_{r\to 0} \grave{G} = \mathrm{Id}$ and $\lim_{r \to 0} \mathrm{d}\grave{G} = 0$. Now

$$\begin{aligned}
||\omega - \omega^\flat||_{L^q_{1,\omega^\flat}(N_0)} &= ||\omega - \omega^\flat||_{L^q(N_0)} + ||\nabla_{\omega^\flat}(\omega - \omega^\flat)||_{L^q(N_0)} \\
&= ||\grave{G}^{-1}\hat{\omega}\grave{G} - \grave{G}^{-1}\Gamma\grave{G}||_{L^q(N_0)} + ||\mathrm{d}\omega + [\omega, \omega^\flat]||_{L^q(N_0)} \\
&= c_1||\hat{\omega} - \Gamma||_{L^q(N_0)} + ||\mathrm{d}(\grave{G}^{-1}\hat{\omega}\grave{G}) + \grave{G}^{-1}\hat{\omega}\Gamma\grave{G} - \grave{G}^{-1}\Gamma\hat{\omega}\grave{G}||_{L^q(N_0)} \\
&= c_1||\hat{\omega} - \Gamma||_{L^q(N_0)} + ||\grave{G}^{-1}\left(\mathrm{d}\hat{\omega} + \hat{\omega}\Gamma - \Gamma\hat{\omega}\right)\grave{G}||_{L^q(N_0)} \\
&\quad + ||\mathrm{d}\grave{G}^{-1}\hat{\omega}\grave{G} + \grave{G}^{-1}\omega\mathrm{d}\grave{G}||_{L^q(N_0)} \\
&\leq c_1\left(||\hat{\omega} - \Gamma||_{L^q(N_0)} + ||\nabla_\Gamma(\hat{\omega} - \Gamma)||_{L^q(N_0)}\right) \\
&\quad + ||\mathrm{d}\grave{G}^{-1}\hat{\omega}\grave{G} + \grave{G}^{-1}\omega\mathrm{d}\grave{G}||_{L^q(N_0)} \\
&= c_1||\hat{\omega} - \Gamma||_{L^1_{1,\Gamma}(N_0)} + ||-\grave{G}^{-1}\mathrm{d}\grave{G}\grave{G}^{-1}\hat{\omega}\grave{G} + \grave{G}^{-1}\omega\mathrm{d}\grave{G}||_{L^q(N_0)} \\
&\leq C||\Omega||_{L^q(N_0)} + ||-\grave{G}^{-1}\mathrm{d}\grave{G}\grave{G}^{-1}\hat{\omega}\grave{G} + \grave{G}^{-1}\omega\mathrm{d}\grave{G}||_{L^q(N_0)}
\end{aligned} \tag{4.141}$$

Since $\mathrm{d}\grave{G} \to 0$ as $r \to 0$ for sufficiently small N_0, we may ignore the second term at the expense of an increase in the constant C. Hence

$$\|\omega - \omega^{\flat}\|_{L^q_{1,\omega^{\flat}}(N_0)} \leq C\|\Omega\|_{L^q(N_0)} \qquad \square \tag{4.142}$$

We now wish to show that Corollary 4.11 implies Theorem 4.9.

Let us take a connection 1-form $\tilde{\omega} \in L^2_{1,\mathrm{loc}}$. We can approximate $\tilde{\omega}$ by a smooth sequence $\tilde{\omega}_j$ such that $\|\Omega_{\tilde{\omega}_j}\|_{L^2(N_0)} < k$ (uniformly bounded). Since the $\tilde{\omega}_j$ are smooth we have that $\tilde{\omega}_j \in L^p_{1,\mathrm{loc}}$ for $p > 2$. For each j we can apply Corollary 4.11. Hence for each $\tilde{D}_j = \mathrm{d} + \tilde{\omega}_j$ we know there is a gauge such that $D_j = \mathrm{d} + \omega_j$ in which

$$\|\omega_j - \omega_j^{\flat}\|_{L^q_{1,\omega_j^{\flat}}(N_0)} \leq c\|\Omega_j\|_{L^2(N_0)} \qquad p > 2 \qquad 2 \leq q \leq p \tag{4.143}$$

where $\omega_j^{\flat}$ approximates $\omega^{\flat}$ and ω_j approximates ω. We take limits as $j \to \infty$ and see that a subsequence of $\omega_j - \omega_j^{\flat}$ converges weakly in L^2_1 to some $\omega - \omega^{\flat}_{\infty}$. This means that the integral given by $\|\omega_j - \omega_j^{\flat}\|_{L^2_{1,\omega_j^{\flat}}(N_0)}$ converges to the integral given by $\|\omega - \omega^{\flat}_{\infty}\|_{L^2_{1,\omega_{\infty}}(N_0)}$. Since $D_j = \mathrm{d} + \tilde{\omega}_j$ can be gauge transformed to $D_j = \mathrm{d} + \omega_j$ for each j, there exists a sequence $s_j \in L^2_{2,\mathrm{loc}}(X_0)$ such that

$$s_j^{-1}\tilde{\omega}_j s_j + s_j^{-1}\mathrm{d}s_j = \omega_j \tag{4.144}$$

It follows (see [39]) that a diagonal subsequence of $\{s_j\}$ converges weakly to $s \in L^2_{2,\mathrm{loc}}$. Hence as $j \to \infty$ we have

$$s_j^{-1}\tilde{\omega}_j s_j + s_j^{-1}\mathrm{d}s_j = \omega_j \quad \to \quad s^{-1}\tilde{\omega}s + s^{-1}\mathrm{d}s = \omega \tag{4.145}$$

A remark earlier in this chapter states that Lemma 4.3 holds if ω and $\tilde{\omega}$ are related by such a weak gauge transformation. Hence the holonomy J_r of ω is conjugate to the holonomy $\tilde{J}_r$ of $\tilde{\omega}$ ($[J_r] = [\tilde{J}_r]$).

From before, we know there is a constant $k > 0$ such that $\tilde{D} = \mathrm{d} + \tilde{\omega}$ with $\tilde{D} \in \mathcal{W}^2$ and $\|\Omega\|_{L^2(N_0)} < k$. Using s we know there is a gauge such that for some flat connection $\omega^{\flat}$ (given by an $A \in \mathbb{R}$) we have $\omega - \omega^{\flat} \in L^2_{1,\omega^{\flat}}(N_0)$ and

$$\|\omega - \omega^b\|_{L^2_{1,\omega^b}} \leq C\|\Omega\|_{L^2} \tag{4.146}$$

Hence we have proved Theorem 4.9.

The remainder of this section is concerned with the proof of Proposition 4.10. We first include an overview of the proof and then the full proof. Concepts and theorems by Taubes that appear here without reference are those that originally appeared in [39].

Overview of Proof

To prove Proposition 4.10 we will need to show that there does exist a flat connection such that in a particular gauge the inequality in (4.139) is satisfied. We now provide a brief overview of the proof.

For open balls B_α we can perform a gauge transformation such that in these balls the following properties hold for the gauge transformed connection 1-form ω^α.

$$\mathrm{d} * \omega^\alpha = 0 \tag{4.147}$$

$$\|\omega^\alpha\|_{L^q_1(B_\alpha)} \leq K\|\Omega\|_{L^q(B_\alpha)}, \qquad 2 \leq q \leq p \tag{4.148}$$

$$\frac{1}{\mathrm{Vol}(B_\alpha)} \int_{B_\alpha} \omega^\alpha dV = 0 \tag{4.149}$$

We call this gauge the Coulomb gauge for short since (4.147) is the Coulomb property for differential forms.

We construct a *composite* gauge transformation G using cutoff functions such that in two overlapping balls which are adjacent in the θ direction the transformation G will continuously change from the gauge transformation which makes ω Coulomb in one ball to the gauge transformation which makes ω Coulomb in the other. The composite gauge transformation is the result of a careful composition of many gauge transformations, the first acting on all balls except the first and then each successive one acting on smaller and smaller regions going around the singularity.

Note that this gauge transformation does not patch together the gauge transformation needed to make ω Coulomb in the last ball and that needed to make ω Coulomb in the first ball and so is not continuous at $\theta = 0 = 2\pi$. Hence the composite gauge transformation is non-global. We look at the gauge transformation which bridges the discontinuity. We take the Coulomb connection at $\theta = 0$, undo the gauge transformation that made the connection Coulomb in the first ball and then make the gauge transformation that makes the connection Coulomb in the final ball. We call this gauge transformation s.

The complete non-global composite gauge transformation will also patch together overlapping balls in the radial direction and in both directions spanning the singular 2-surface. In this new gauge ω will either be Coulomb or 'close' to Coulomb everywhere, the 'closeness' being bounded by a cutoff function and its derivatives.

We know from Theorem 4.8 that the limit holonomy of $D = d + \omega$ at the singularity exists. If we have a global gauge and the origin is a regular point then the limit holonomy is trivial. In our case we start with a global gauge and a singular origin. We use Taubes' theorem (and property 1. above) to establish that $\lim_{r\to 0} s = s_0$ exists.

We have shown earlier in this chapter how we can use parallel propagation to establish a non-global composite gauge in which the limit 'jump' is the limit holonomy. We will call this the holonomy gauge. We will force our new non-global composite gauge to look like the holonomy gauge in the limit as r tends to zero. Hence they will have the same limit 'jump'. Since the limit as r tends to zero of the jump in the holonomy gauge is the limit holonomy, we know that the limit as r tends to zero of the jump in the composite gauge is also the (same) limit holonomy. Hence the limit holonomy for the connection ω is s_0. This is proved by first showing that s_0 is the limit holonomy of a constant flat connection ω_∞ and then showing that ω_∞ and ω have the same limit holonomy.

Also from before we know that any flat connection is gauge equivalent to $\omega^\flat = m\hat{i}d\theta$ for some $m \in \mathbb{R}$. We say that our flat connection has holonomy number m.

To get the final requirement to show (4.139) we must first, before applying the gauge transformation G, have changed to a radial gauge (so $\omega_r = 0$) and we can then find that we have a bound on $\|\omega - \omega_\infty\|_{L^q}$.

One final obstacle to proving Proposition 4.10 is that we require that our composite gauge transformation be global. We use another cutoff function to make the composite gauge transformation continuous and periodic and hence global. Since we are now using a different global gauge transformation, some more calculations are required before we establish the final result that $\|\hat{\omega} - \Gamma\|_{L^q_{1,\Gamma}(N_0)} \leq \hat{c}\|\Omega\|_{L^q}$. Then since $D \in \omega^p_{m,k}$ we have proved Proposition 4.10.

Proof of Proposition 4.10

We cover the space $X_0 = N_0 \backslash \Sigma_0$ by a countable collection of balls. We note that on each ball we have a Coulomb gauge, as given by Uhlenbeck [45]. In this gauge the L^2_1 norm of the connection form can be estimated by the L^2 norm of the curvature. We cover X_0 by balls B_α in which the Coulomb gauge is given by the following strengthening of the theorem from [45] (see [39])

Theorem 4.12 *For K sufficiently small, there is a gauge in which $D = d + \omega^\alpha \in \mathcal{W}^p$ with*

$$\begin{aligned} d * \omega^\alpha &= 0 & (4.150) \\ \|\omega^\alpha\|_{L^q_1(B_\alpha)} &\leq K\|\Omega\|_{L^q(B_\alpha)}, \qquad 2 \leq q \leq p & (4.151) \\ \frac{1}{\mathrm{Vol} B_\alpha}\int_{B_\alpha} \omega^\alpha dV &= 0 & (4.152) \end{aligned}$$

Proof

(4.150) and (4.151) are proved in [45] and so we know there exists some $\omega^\diamond$ such that $d{*}\omega^\diamond = 0$ and $\|\omega^\diamond\|_{L^q_1(B_\alpha)} \leq K\|\Omega\|_{L^q(B_\alpha)}$ with $2 \leq q \leq p$. We now look at the sequence $\omega_n = e^{-u_n}\omega^\diamond e^{u_n} + du_n$ where each ω_i is gauge equivalent to $\omega^\diamond$ under the the gauge transformation using $s_i = e^{u_i}$. We let u_n have the property that

$$\Delta u_n = e^{-u_{n-1}}[\omega^\diamond, du_{n-1}]e^{u_{n-1}} \tag{4.153}$$

We now solve (4.153) to get the particular solution such that $\int_{B_\alpha} du_n dV = -\int_{B_\alpha} e^{-u_{n-1}}\omega^\alpha e^{u_{n-1}} dV$, where the integration here is on each space-time component of the 1-forms.

$$
\begin{aligned}
\Delta u_n = *d*du_n &= e^{-u_{n-1}}[\omega^\alpha, du_{n-1}]e^{u_{n-1}} \\
d*du_n &= e^{-u_{n-1}}(du_{n-1}*\omega^\alpha - *\omega^\alpha du_{n-1})e^{u_{n-1}} \\
du_n &= -e^{-u_{n-1}}\omega^\alpha e^{u_{n-1}} \quad \text{is one solution} \\
du_n &= -e^{-u_{n-1}}\omega^\alpha e^{u_{n-1}} \\
\int_{B_\alpha} du_n dV &= -\int_{B_\alpha} e^{-u_{n-1}}\omega^\diamond e^{u_{n-1}} dV
\end{aligned} \tag{4.154}
$$

From [39] we have that u_n converges to u and so ω_n converges to $\omega = e^{-u}\omega^\diamond e^u + du$. We see that (4.153) and (4.154) become

$$
\Delta u = e^{-u}[\omega^\diamond, du]e^u \tag{4.155}
$$

$$
\int_{B_\alpha} dudV = -\int_{B_\alpha} e^{-u}\omega^\diamond e^u dV \tag{4.156}
$$

We should now look at (4.155) in more detail

$$
\begin{aligned}
\Delta u &= e^{-u}[\omega^\diamond, du]e^u \\
(\Delta u)^\alpha_\beta &= (e^{-u})^\alpha_\mu \hat{g}^{af}[(\omega^\diamond)^\mu_{\nu a}(du)^\nu_{\gamma f} - (du)^\mu_{\nu f}(\omega^\diamond)^\nu_{\gamma a}](e^u)^\gamma_\beta
\end{aligned}
$$

where $\hat{g}$ is the background metric. Since $\hat{g}^{af} = \delta^f_h \hat{g}^{ah} = -\epsilon^{bcdf}\epsilon^a{}_{bcd}$ we have

$$
\begin{aligned}
(\Delta u)^\alpha_\beta &= -(e^{-u})^\alpha_\mu \epsilon^{bcdf}\epsilon^a{}_{bcd}[(\omega^\diamond)^\mu_{\nu a}(du)^\nu_{\gamma f} - (du)^\mu_{\nu f}(\omega^\diamond)^\nu_{\gamma a}](e^u)^\gamma_\beta \\
(*d*du)^\alpha_\beta &= \epsilon^{bcdf}\left(-(e^{-u})^\alpha_\mu \epsilon^a{}_{bcd}[(\omega^\diamond)^\mu_{\nu a}(du)^\nu_{\gamma f} - (du)^\mu_{\nu f}(\omega^\diamond)^\nu_{\gamma a}](e^u)^\gamma_\beta\right) \\
\Longrightarrow (d*du)^\alpha_{\beta bcdf} &= -(e^{-u})^\alpha_\mu \epsilon^a{}_{bcd}[(\omega^\diamond)^\mu_{\nu a}(du)^\nu_{\gamma f} - (du)^\mu_{\nu f}(\omega^\diamond)^\nu_{\gamma a}](e^u)^\gamma_\beta \\
&= (e^{-u})^\alpha_\mu \left((du)^\mu_{\nu f}(\omega^\diamond)^\nu_{\gamma a}\epsilon^a{}_{bcd} - (\omega^\diamond)^\mu_{\nu a}\epsilon^a{}_{bcd}(du)^\nu_{\gamma f}\right)(e^u)^\gamma_\beta \\
&= (e^{-u})^\alpha_\mu \left((du)^\mu_{\nu f}(*\omega^\diamond)^\nu_{\gamma bcd} - (*\omega^\diamond)^\mu_{\nu bcd}(du)^\nu_{\gamma f}\right)(e^u)^\gamma_\beta \\
\therefore d*du &= e^{-u}(du*\omega^\diamond - *\omega^\diamond du)e^u
\end{aligned} \tag{4.157}
$$

We now use (4.156) and (4.157) to check that ω satisfies each of (4.150), (4.151) and (4.152).

$$
\begin{aligned}
d*\omega &= d*(e^{-u}\omega^\diamond e^u + du) \\
&= d(e^{-u}*\omega^\diamond e^u) + d*du \\
&= -e^{-u}du*\omega^\diamond e^u + e^{-u}d*\omega^\diamond e^u + e^{-u}*\omega^\diamond due^u + d*du \\
&= -e^{-u}du*\omega^\diamond e^u + e^{-u}*\omega^\diamond due^u + e^{-u}(du*\omega^\diamond - *\omega^\diamond du)e^u \\
&= 0 \qquad \square(4.150)
\end{aligned} \tag{4.158}
$$

It can be seen from an earlier result (4.83), that $\|u_n\|_{L_2^2(B_\alpha)} \leq \|\omega^\alpha\|_{L_1^2(B_\alpha)}$ and hence also that $\|u_n\|_{L_2^q(B_\alpha)} \leq \|\omega^\alpha\|_{L_1^q(B_\alpha)}$. Since u_n converges to u we have

$$\begin{aligned}
\|\omega\|_{L_1^2(B_\alpha)} &= \|e^{-u}\omega^\diamond e^u + du\|_{L_1^2(B_\alpha)} \\
&\leq \|e^{-u}\omega^\diamond e^u\|_{L_1^2(B_\alpha)} + \|du\|_{L_1^2(B_\alpha)} \qquad \text{(triangle inequality)} \\
&\leq c_1\|\omega^\diamond\|_{L_1^2(B_\alpha)} + \|u\|_{L_2^2(B_\alpha)} \qquad (\text{since } u \in L_2^2) \\
&\leq c_2\|\omega^\diamond\|_{L_1^2(B_\alpha)} \\
&\leq c\|\Omega\|_{L^2(B_\alpha)} \\
\Longrightarrow \|\omega\|_{L_1^q(B_\alpha)} &\leq C\|\Omega\|_{L^q(B_\alpha)} \qquad \square(4.151)
\end{aligned} \tag{4.159}$$

$$\begin{aligned}
\frac{1}{\mathrm{Vol}B_\alpha}\int_{B_\alpha}\omega dV &= \frac{1}{\mathrm{Vol}B_\alpha}\left(\int_{B_\alpha} e^{-u}\omega^\diamond e^u dV + \int_{B_\alpha} du dV\right) \\
&= \frac{1}{\mathrm{Vol}B_\alpha}\left(\int_{B_\alpha} e^{-u}\omega^\diamond e^u dV - \int e^{-u}\omega^\diamond e^u dV\right) \\
&= 0 \qquad \square(4.152)
\end{aligned} \tag{4.160}$$

Hence there is a gauge in which $D = d + \omega^\alpha$ satisfying all three Coulomb conditions as required.

Proposition 4.13 *Under a constant gauge transformation e^a, all three Coulomb results continue to hold.*

Proof

Let ω^α be Coulomb so we already have $d * \omega^\alpha = 0$, $\|\omega^\alpha\|_{L_1^q(B_\alpha)} \leq k\|\Omega\|_{L_1^q(B_\alpha)}$ for $2 \leq q \leq p$ and $\frac{1}{\mathrm{Vol}B_\alpha}\int_{B_\alpha}\omega^\alpha dV = 0$. Then

$$\begin{aligned}
\omega &= e^{-a}\omega^\alpha e^a \\
*\omega &= e^{-a} * \omega^\alpha e^a \\
d * \omega &= d(e^{-a} * \omega^\alpha e^a) \\
&= e^{-a} d * \omega^\alpha e^a \\
\therefore d * \omega &= 0
\end{aligned} \tag{4.161}$$

and

$$\begin{aligned}
\omega &= e^{-a}\omega^{\alpha}e^{a} \\
\|\omega\|_{L_1^q(B_\alpha)} &= \|e^{-a}\omega^{\alpha}e^{a}\|_{L_1^q(B_\alpha)} \\
&\leq c\|\omega^{\alpha}\|_{L_1^q(B_\alpha)} \qquad \text{(since } e^a \text{ is constant)} \\
\therefore \|\omega\|_{L_1^q(B_\alpha)} &\leq c\|\Omega\|_{L^q(B_\alpha)} \quad \text{for } 2 \leq q \leq p
\end{aligned} \tag{4.162}$$

and

$$\begin{aligned}
\omega &= e^{-a}\omega^{\alpha}e^{a} \\
\frac{1}{\mathrm{Vol}B_\alpha}\int_{B_\alpha}\omega dV &= \frac{1}{\mathrm{Vol}B_\alpha}\int_{B_\alpha} e^{-a}\omega^{\alpha}e^{a}dV \\
&= e^{-a}\frac{1}{\mathrm{Vol}B_\alpha}\int_{B_\alpha}\omega^{\alpha}dV\,e^{a} \\
\therefore \frac{1}{\mathrm{Vol}B_\alpha}\int_{B_\alpha}\omega dV &= 0 \qquad \square
\end{aligned} \tag{4.163}$$

Hence we have some gauge freedom since the connection ω^α is not uniquely specified.

We now look at the covering as constructed in [39] and then look at the composite gauge transformation over the balls surrounding the singularity.

The global covering

Assume that $\{B_\alpha\}$ is an open covering of X_0, which contains a subcovering $\{U_\alpha\}$ which we now define. We allow our two dimensional (local) singularity Σ_0, to be diffeomorphic to a square in $\mathbb{R}^2$. The points in the balls U_α are determined by three parameters contained in $\alpha = (n, l, x)$ where $n \in \mathbb{N}$, $l \in \mathbb{N}$ $(l \leq 7)$ and x is a point of Σ_0 lying on the standard square lattice Λ_n of side length 2^{-n-1}. The open ball U_α consists of all points restricted as follows:

1. $2^{-n-2} < r < 2^{-n}$
2. $\frac{\pi l}{4} < \theta < \frac{\pi}{4}(l+2)$
3. y belongs to the open square centered at x of the length 2^{-n}

For each point on the square, the ball U_α is a section of an annulus. For any connection ω, there exists a gauge transformation h_α such that $\omega^\alpha = h_\alpha^{-1}\omega h_\alpha + h_\alpha^{-1}dh_\alpha$ is a Coulomb connection in U_α.

If $U_\alpha \cap U_\beta \neq \emptyset$ then we can gauge transform ω^α to ω^β by first undoing h_α and then applying h_β, i.e. gauge transform by $h_\alpha^{-1}h_\beta$ (since both h_α and h_β are defined in $U_\alpha \cap U_\beta$) as we now show.

$$\begin{aligned}
& h_\beta^{-1}h_\alpha\omega^\alpha h_\alpha^{-1}h_\beta + h_\beta^{-1}h_\alpha d(h_\alpha^{-1}h_\beta) \\
= \; & h_\beta^{-1}h_\alpha(h_\alpha^{-1}\omega h_\alpha + h_\alpha^{-1}dh_\alpha)h_\alpha^{-1}h_\beta + h_\beta^{-1}h_\alpha(dh_\alpha^{-1})h_\beta + h_\beta^{-1}h_\alpha h_\alpha^{-1}(dh_\beta) \\
= \; & h_\beta^{-1}\omega h_\beta + h_\beta^{-1}dh_\alpha h_\alpha^{-1}h_\beta + h_\beta^{-1}h_\alpha dh_\alpha^{-1}h_\beta + h_\beta^{-1}dh_\beta \\
= \; & h_\beta^{-1}\omega h_\beta + h_\beta^{-1}dh_\alpha h_\alpha^{-1}h_\beta + h_\beta^{-1}h_\alpha dh_\alpha^{-1}h_\beta + h_\beta^{-1}dh_\beta \\
= \; & h_\beta^{-1}\omega h_\beta + h_\beta^{-1}d(h_\alpha h_\alpha^{-1})h_\beta + h_\beta^{-1}dh_\beta \\
= \; & h_\beta^{-1}\omega h_\beta + h_\beta^{-1}dh_\beta \\
= \; & \omega^\beta
\end{aligned} \tag{4.164}$$

We let $h_\alpha^{-1}h_\beta = e^u$ and for points $x \in U_\alpha \cap U_\beta$ we have a gauge transformation (leaving x unchanged)

$$\begin{aligned}
e^u = h_\alpha^{-1}\cdot h_\beta : (U_\alpha \cap U_\beta)\times SO(3) &\rightarrow (U_\alpha \cap U_\beta)\times SO(3) \\
(x, G) &\mapsto \big(x, (h_\alpha^{-1}\cdot h_\beta)G\big) = (x, e^u G)
\end{aligned} \tag{4.165}$$

where, $u \in so(3)$.

From before we have

$$\|u\|_{L_2^q(B_\alpha)} \leq c\|\Omega\|_{L^q(B_\alpha)} \tag{4.166}$$

Since $\{B_\alpha\}$ is an open covering of X_0, which contains a subcovering U_α, we know that B_α is the smallest ball containing U_α and so $\|u\|_{L_2^q(U_\alpha)} \leq c\|\Omega\|_{L^q(U_\alpha)}$ also. Hence on $U_\alpha \cup U_\beta$, for a point P lying in $U_\alpha \cap U_\beta$ we have

$$
\begin{aligned}
& \frac{1}{d_\alpha^2}\|u-u(P)\|_{L^q(U_\alpha\cup U_\beta)}+\frac{1}{d_\alpha}\|\nabla u\|_{L^q(U_\alpha\cup U_\beta)}+\|\nabla(\nabla u)\|_{L^q(U_\alpha\cup U_\beta)} \\
= & \frac{1}{d_\alpha^2}\|u-u(P)\|_{L^q(U_\alpha\cup U_\beta)}+\frac{1}{d_\alpha}\|\nabla(u-u(P))\|_{L^q(U_\alpha\cup U_\beta)}+\|\nabla(\nabla(u-u(P))\|_{L^q(U_\alpha\cup U_\beta)} \\
\leq & C\|u-u(P)\|_{L^q_2(U_\alpha\cup U_\beta)} \qquad \text{(where C is a constant dependent on d_α)} \\
\leq & C\left(\|u\|_{L^q_2(U_\alpha\cup U_\beta)}+\|u(P)\|_{L^q_2(U_\alpha\cup U_\beta)}\right) \\
= & C\left(\|u\|_{L^q_2(U_\alpha)}+\|u\|_{L^q_2(U_\beta)}-\|u\|_{L^q_2(U_\alpha\cap U_\beta)}+\|u(P)\|_{L^q_2(U_\alpha\cap U_\beta)}\right) \\
= & C\left(\|u\|_{L^q_2(U_\alpha)}+\|u\|_{L^q_2(U_\beta)}\right) \qquad \text{(since $P\in U_\alpha\cap U_\beta$)} \\
\leq & C\left(\|\Omega\|_{L^q(U_\alpha)}+\|\Omega\|_{L^q(U_\beta)}\right) \qquad (4.167)
\end{aligned}
$$

Hence we have

Corollary 4.14

$$\frac{1}{d^2}\|u-u(P)\|_{L^q}+\frac{1}{d}\|\nabla u\|_{L^q}+\|\nabla(\nabla u)\|_{L^q}\leq C\left(\|\Omega\|_{L^q(U_\alpha)}+\|\Omega\|_{L^q(U_\beta)}\right) \qquad (4.168)$$

on $U_\alpha\cup U_\beta$, where P lies in $U_\alpha\cap U_\beta$, and d is the larger of the diameters of the two balls B_α and B_β.

We note that d is the maximum distance between the fixed point $P\in U_\alpha\cap U_\beta$ and any other point in $U_\alpha\cup U_\beta$.

The composite gauge transformation

The non-Abelian nature of $SO(3)$ means that we cannot glue more than two local gauges by a cutoff function. We glue together regions of X_0 which are related to one another by transformations as detailed above. We first glue together the eight regions as related by increasing l by one from 0 to 7 (keeping x and n constant). This will give us a complete annulus around the singularity. Now we glue together those annuli which are related to each other by a shift of one on the square lattice Λ_n, creating what we can think of as a thick walled cylinder.[1] We finally glue together regions related to each other by increasing n by one, which effectively fills out the 'cylinder' radially so that it is a solid object and we have covered all of X_0. This will produce a multi-valued gauge in the sense that $G(r,2\pi,y)\neq G(r,0,y)$.

[1]The analogy to a cylinder would be correct if the singularity was one-dimensional. However, Σ is two-dimensional.

We use a number of cutoff functions to glue together local gauge functions. Each cutoff function yields a value of 0, 1 or some real number between 0 and 1 for all locations in X_0. What value a cutoff function takes between 0 and 1 is not required in the proof, except for determining derivatives (which we address in Appendix A.4). It is sufficient to know that the cutoff function is smooth between $\lambda = 0$ and $\lambda = 1$. We begin by looking at overlapping regions around the annulus. We start with the function

$$\lambda(t) = \begin{cases} 1, & 0 \leq t \leq \frac{\pi}{16} \\ 0, & \frac{3\pi}{16} \leq t \leq 2\pi \end{cases} \tag{4.169}$$

and we let $\lambda_l(\theta) = \lambda(\theta - \frac{\pi l}{4})$ be periodic in θ of period 2π.

$$\lambda_l(\theta) = \begin{cases} 1, & \frac{\pi l}{4} \leq \theta \leq \frac{\pi}{16} + \frac{\pi l}{4} \\ 0, & 0 \leq \theta < \frac{\pi l}{4}, \quad \frac{3\pi}{16} + \frac{\pi l}{4} \leq \theta < 2\pi \end{cases} \tag{4.170}$$

With n and x fixed we now define a gauge transformation on the entire annulus $\bigcup_{l=0}^{7} U_{(n,l,x)}$ by

$$\begin{aligned} h_{n,x}|U_{(n,l,x)} &= h_{(n,l,x)} \exp(\lambda_l u_l) \\ &= \begin{cases} h_{(n,l,x)} \exp(u_l), & \frac{\pi l}{4} \leq \theta \leq \frac{\pi}{16} + \frac{\pi l}{4} \\ h_{(n,l,x)}, & 0 \leq \theta < \frac{\pi l}{4}, \quad \frac{3\pi}{16} + \frac{\pi l}{4} \leq \theta < 2\pi \end{cases} \end{aligned} \tag{4.171}$$

where $\exp u_l = h_{(n,l,x)}^{-1} h_{(n,l-1,x)}$. So we can rewrite (4.171) as

$$h_{n,x}|U_{(n,l,x)} = \begin{cases} h_{(n,l-1,x)}, & \frac{\pi l}{4} \leq \theta \leq \frac{\pi}{16} + \frac{\pi l}{4} \\ h_{(n,l,x)}, & 0 \leq \theta < \frac{\pi l}{4}, \quad \frac{3\pi}{16} + \frac{\pi l}{4} \leq \theta < 2\pi \end{cases} \tag{4.172}$$

We will use (4.172) to tell us how, for every value of l, we can glue the gauge to its adjacent gauge on the annulus.

We now define the cutoff function used to glue together a pair of overlapping annuli on the singularity. We move along the singularity from a point (x_1, x_2) on the lattice Λ_n to a neighboring point $x' = (x_1, x_2 - 2^{-(n+1)})$. Our cutoff function is

$$\lambda(y) = \begin{cases} 1, & y \leq x_2 - 2^{-(n+1)} \\ 0, & y \geq x_2 \end{cases} \tag{4.173}$$

We let $u(n,x,x')$ be defined by $\exp u(n,x,x') = h_{n,x}^{-1} h_{n,x'}$. Then on $U_{(n,l,x)} \cap U_{(n,l,x')}$ our global gauge transformation is given by

$$\begin{aligned}
h_n &= h_{n,x} \exp(\lambda(y) u(n,x,x')) \\
&= \begin{cases} h_{n,x} \exp u(n,x,x'), & y \leq x_2 - 2^{-(n+1)} \\ h_{n,x}, & y \geq x_2 \end{cases} \\
&= \begin{cases} h_{n,x'}, & y \leq x_2 - 2^{-(n+1)} \\ h_{n,x}, & y \geq x_2 \end{cases}
\end{aligned} \tag{4.174}$$

(4.174) tells us how, for every value of x, we can glue the local gauge of the annulus to all annuli up our 'cylinder' (note that in contrast to the first cutoff function, this one is dependent on another variable n). Since the singularity is two dimensional, we must define an analogous transformation addressing overlapping regions on the singularity with squares centred at (x_1, x_2) and $(x_1 - 2^{-(n+1)}, x_2)$.

Finally we establish the cutoff function used to glue together the 'cylinders' of different inner and outer radii by looking at overlapping regions $U_{(n,l,x)}$ and $U_{(n+1,l,x)}$. Our final cutoff function is

$$\lambda(r) = \begin{cases} 0, & r \leq 2^{-(n+1)} \\ 1, & r \geq \frac{3}{2} 2^{-(n+1)} \end{cases} \tag{4.175}$$

we now define $G = h_n \exp(\lambda u)$ on the appropriate intersection where $\exp u = h_n^{-1} h_{n+1}$. So

$$\begin{aligned}
G &= \begin{cases} h_n, & r \leq 2^{-(n+1)} \\ h_n \exp u, & r \geq \frac{3}{2} 2^{-(n+1)} \end{cases} \\
&= \begin{cases} h_n, & r \leq 2^{-(n+1)} \\ h_{n+1}, & r \geq \frac{3}{2} 2^{-(n+1)} \end{cases}
\end{aligned} \tag{4.176}$$

and this tells us how, for each pair of radii, we can glue the local gauge of the 'cylinder' to that of the next larger or smaller radius cylinder.

We now show in detail how to construct the composite gauge transformation by using the above cutoff functions. It is sufficient to show how this is done just in the θ direction since this is where we will address the complication arising from G being discontinuous at $\theta = 2\pi$. Analogous algorithms are carried out for the the radial and parallel to singularity directions.

In a ball $U_{(n,l,x)}$ the gauge transformation $h_{(n,l,x)}$ takes some connection ω to $\omega^{(n,l,x)}$ where $\omega^{(n,l,x)}$ is Coulomb.

We refer back to (4.170), (4.171) and (4.172) and let

$$e^{u_{(l-1,l)}} = \left(h_{n,x}|U_{(n,l-1,x)}\right)^{-1}\left(h_{n,x}|U_{(n,l,x)}\right) \tag{4.177}$$

and so on $U_{(n,l-1,x)} \cap U_{(n,l,x)}$

$$e^{u_{(l-1,l)}} = \left(h_{n,x}|U_{(n,l-1,x)}\right)^{-1}\left(h_{n,x}|U_{(n,l,x)}\right) = \begin{cases} \mathrm{Id} & \frac{\pi l}{4} < \theta \leq \frac{\pi l}{4} + \frac{\pi}{16} \\ h^{-1}_{(n,l-1,x)} h_{(n,l,x)} & \frac{\pi l}{4} + \frac{3\pi}{16} \leq \theta < \frac{\pi l}{4} + \frac{\pi}{4} \end{cases} \tag{4.178}$$

For reasons that shall become clear later we extend $e^{u_{(l-1,l)}}$ over all $X = \Sigma \backslash N$

$$e^{u_{(l-1,l)}} = \left(h_{n,x}|U_{(n,l-1,x)}\right)^{-1}\left(h_{n,x}|U_{(n,l,x)}\right) = \begin{cases} h^{-1}_{(n,l-1,x)} h_{(n,l,x)} & 0 \leq \theta \leq \frac{\pi l}{4} - \frac{\pi}{4} \\ h^{-1}_{(n,l-2,x)} h_{(n,l,x)} & \frac{\pi l}{4} - \frac{\pi}{4} \leq \theta \leq \frac{\pi l}{4} - \frac{3\pi}{16} \\ h^{-1}_{(n,l-1,x)} h_{(n,l,x)} & \frac{\pi l}{4} - \frac{\pi}{16} \leq \theta \leq \frac{\pi l}{4} \\ \mathrm{Id} & \frac{\pi l}{4} \leq \theta \leq \frac{\pi l}{4} + \frac{\pi}{16} \\ h^{-1}_{(n,l-1,x)} h_{(n,l,x)} & \frac{\pi l}{4} + \frac{3\pi}{16} \leq \theta < 2\pi \end{cases} \tag{4.179}$$

We see above that the function over all θ is multi-valued at $\frac{\pi l}{4}$ and $\frac{\pi l}{4} - 4$. From (4.179) we have on $\bigcup_{i=l}^{6} U_{(n,i,x)}$

$$e^{u_{(l-1,l)}} = \left(h_{n,x}|U_{(n,l-1,x)}\right)^{-1}\left(h_{n,x}|U_{(n,l,x)}\right) = \begin{cases} \mathrm{Id} & \frac{\pi l}{4} < \theta \leq \frac{\pi l}{4} + \frac{\pi}{16} \\ h^{-1}_{(n,l-1,x)} h_{(n,l,x)} & \frac{\pi l}{4} + \frac{3\pi}{16} \leq \theta \leq 2\pi \end{cases} \tag{4.180}$$

Since neither $\frac{\pi l}{4}$ nor $\frac{\pi l}{4} - 4$ are in $\bigcup_{i=l}^{6} U_{(n,i,x)}$, we are not affected by the fact that the cutoff function is multi-valued.

On $U_{(n,l-1,x)} \cap U_{(n,l,x)}$ we can get from $\omega^{(n,l-1,x)}$ to $\omega^{(n,l,x)}$ via the gauge transformation given by $e^{u_{(l-1,l)}}$

$$e^{-u_{(l-1,l)}}\omega^{(n,l-1,x)}e^{u_{(l-1,l)}} + du_{(l-1,l)} = \begin{cases} \omega^{(n,l-1,x)} & \frac{\pi l}{4} < \theta \leq \frac{\pi l}{4} + \frac{\pi}{16} \\ \omega^{(n,l,x)} & \frac{\pi l}{4} + \frac{3\pi}{16} \leq \theta < \frac{\pi l}{4} + \frac{\pi}{4} \end{cases} \tag{4.181}$$

Let us, for example, look at this gauge change for $l = 1$. On $U_{(n,0,x)} \cap U_{(n,1,x)}$ we can get from $\omega^{(n,0,x)}$ to $\omega^{(n,1,x)}$ via the gauge transformation given by $e^{u_{(0,1)}}$

$$e^{-u_{(0,1)}}\omega^{(n,0,x)}e^{u_{(0,1)}} + du_{(0,1)} = \begin{cases} \omega^{(n,0,x)} & \frac{\pi}{4} < \theta \leq \frac{5\pi}{16} \\ \omega^{(n,1,x)} & \frac{7\pi}{16} \leq \theta < \frac{\pi}{2} \end{cases} \tag{4.182}$$

Constructing the composite gauge

(Since we are only currently looking at gluing the balls in the angular direction, we shall abbreviate $U_{(n,l,x)}$ to U_l, $\omega^{(n,l,x)}$ to ω^l and $h_{(n,l,x)}$ to h_l)

We start by gauge transforming some connection ω to ω^0 by h_0 everywhere. So in U_0, ω^0 is Coulomb. Now we apply the following algorithm:

1. On $U_0 \cap U_1$ we use $e^{u_{(0,1)}}$ to take ω^0 to ω^1
2. Extend $e^{u_{(0,1)}}$ to act on all $\bigcup_{i=0}^{6} U_i \backslash U_0$
3. (a) In $U_0 \backslash U_1$ the connection is ω^0
   (b) In $U_0 \cap U_1$ the connection is ω^0 changing to ω^1
   (c) In $\left(\bigcup_{i=0}^{6} U_i\right) \backslash U_0$ and at $\theta = 2\pi$ the connection is ω^1

We then repeat the algorithm for successively larger values of l

1. On $U_1 \cap U_2$ we use $e^{u_{(1,2)}}$ to take ω^1 to ω^2
2. Extend $e^{u_{(1,2)}}$ to act on all $\bigcup_{i=0}^{6} U_i \backslash (U_0 \cup U_1)$
3. (a) In $U_0 \backslash U_1$ the connection is ω^0
   (b) In $U_0 \cap U_1$ the connection is ω^0 changing to ω^1
   (c) At $\theta = \frac{\pi}{2}$ the connection is ω^1
   (d) In $U_1 \cap U_2$ the connection is ω^1 changing to ω^2
   (e) In $\left(\bigcup_{i=0}^{6} U_i\right) \backslash (U_0 \cup U_1)$ and at $\theta = 2\pi$ the connection is ω^2

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1. On $U_{l-1} \cap U_l$ we use $e^{u_{(l-1,l)}}$ to take ω^{l-1} to ω^l
2. Extend $e^{u_{(l-1,l)}}$ to act on all $\bigcup_{i=0}^{6} U_i \backslash (\bigcup_{i=0}^{l-1} U_i)$

3. (a) In $U_0\backslash U_1$ the connection is ω^0

   (b) In $U_{i-1}\cap U_i$ the connection is ω^{i-1} changing to ω^i for all $1\le i\le l$

   (c) At $\theta=\frac{\pi(j+1)}{4}$ the connection is ω^j for all $1\le j<l$

   (d) In $\left(\bigcup_{i=0}^6 U_i\right)\backslash(\bigcup_{i=0}^{l-1}U_i)$ and at $\theta=2\pi$ the connection is ω^l

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Finally, for $l=7$

1. On $U_6\cap U_7$ we use $e^{u_{(6,7)}}$ to take ω^6 to ω^7

2. Extend $e^{u_{(6,7)}}$ to act on all $\bigcup_{i=0}^6 U_i\backslash(\bigcup_{i=0}^6 U_i)$. **Note that** $\bigcup_{i=0}^6 U_i\backslash(\bigcup_{i=0}^6 U_i)=\emptyset$

3. (a) In $U_0\backslash U_1$ the connection is ω^0

   (b) In $U_{i-1}\cap U_i$ the connection is ω^{i-1} changing to ω^i for all $1\le i\le 7$

   (c) At $\theta=\frac{\pi(j+1)}{4}$ the connection is ω^j for all $1\le j<7$

   (d) In $\left(\bigcup_{i=0}^6 U_i\right)\backslash(\bigcup_{i=0}^6 U_i)$ and at $\theta=2\pi$ the connection is ω^7. **Again, note that** $\bigcup_{i=0}^6 U_i\backslash(\bigcup_{i=0}^6 U_i)=\emptyset$

Looking at point 2. above for $l=7$ we see that we are no longer extending the gauge transformation and so we stop here. Importantly, we do not continue this algorithm for the intersection $U_7\cap U_0$.

To summarise: We start in one gauge and look at overlapping balls U_l around the singularity. As we cross into a new section we make a continuous gauge transformation into a different gauge. We do this as we enter a new ball all the way up to $U_6\cap U_7$. Since these gauge transformations are all continuous, we can compose them all into one continuous gauge transformation G (note that $\omega|_{\theta=2\pi}\neq\lim_{x\to0^+}\omega|_{\theta=x}$ and so we do not have continuity as we go across $\theta=2\pi=0$). When we gauge transform in an intersection, as we are changing from ω^{i-1} to ω^i, the connection is *not* Coulomb in the in-between stages. However, as shall be discussed later, the deviation of the connection is controlled by the first and second derivatives of λ for which we have estimates on upper bounds (see Appendix A.4).

We have established that $e^{u_{(l-1,l)}}=(h_{n,x}|U_{(n,l-1,x)})^{-1}(h_{n,x}|U_{(n,l,x)})$ takes $\omega^{(n,l-1,x)}$ to $\omega^{(n,l,x)}$ in the region $U_{(n,l-1,x)}\cap U_{(n,l,x)}$. The equivalent result for the entire space is $e^{u_{[(n,l,x),(m,p,y)]}}=$

$(g|U_{(n,l,x)})^{-1}(g|U_{(m,p,y)})$ taking $\omega^{(n,l,x)}$ to $\omega^{(m,p,y)}$ in the region $U_{(n,l,x)} \cap U_{(m,p,y)}$ where either $m = n$ or $n+1$, $p = l$ or $l+1$ and $y = (x_1, x_2)$, $(x_1, x_2 - 2^{-(n+1)})$ or $(x_1 - 2^{-(n+1)}, x_2)$.

We include a summary of the effect of each component gauge transformation in the region $U_{(n,l,x)}$:

$h_{(n,l,x)}$ takes any connection ω to $\omega^{(n,l,x)}$ (which is Coulomb in $U_{(n,l,x)}$).

$h_{n,x}$ takes ω to $\omega^{(n,l-1,x)}$ in some regions and $\omega^{(n,l,x)}$ in others. In all remaining transitionary regions the connection is not Coulomb.

h_n takes ω to $\omega^{(n,l-1,x)}$, $\omega^{(n,l,x)}$, $\omega^{(n,l-1,x')}$, $\omega^{(n,l,x')}$ or some smoothly changing non-Coulomb connection (dependent on region).

G takes ω to $\omega^{(n,l-1,x)}$, $\omega^{(n,l,x)}$, $\omega^{(n,l-1,x')}$, $\omega^{(n,l,x')}$, $\omega^{(n+1,l-1,x)}$, $\omega^{(n+1,l,x)}$, $\omega^{(n+1,l-1,x')}$, $\omega^{(n+1,l,x')}$ or some smoothly changing non-Coulomb connection (dependent on region).

$e^{u_{[(n,l,x),(m,p,y)]}} = (G|U_{(n,l,x)})^{-1}(G|U_{(m,p,y)})$ smoothly takes $\omega^{(n,l,x)}$ to $\omega^{(m,p,y)}$, where $U_{(n,l,x)} \cap U_{(m,p,y)} \neq \emptyset$.

By composing the above functions and their inverses we construct the composite gauge transformation. In this gauge we rewrite ω as $\dot{\omega}$.

As noted earlier, constant gauge transformations are not uniquely specified and we can use this gauge freedom to choose a particularly simple form for u in the regions of overlapping balls in the θ direction. We extend this notion to include radial directions and directions parallel to the singularity and have the following lemma.

Lemma 4.15 *In the region $U_\alpha \cap U_\beta$, where $\beta = (n', l', x')$, we can choose the function u such that $u(P) = 0$ if any of the below conditions apply*

1. $n = n'$, $x = x'$ *and* $l = l' + 1$ *with* $1 \leq l' < 7$
2. $n = n'$, $l = l'$ *and* x *and* x' *are adjacent on the square lattice* Λ_n
3. $n = n' + 1$, $l = l'$ *and* $x = x'$

That is to say, if U_α and U_β differ by one or more of either a rotation of $\frac{\pi}{4}$, a displacement of 1 on the grid Λ_n or a radial skew by a factor of two, then the function u at any point P in $U_\alpha \cap U_\beta$ is zero. Of importance is that $l' < 7$ and so $u(P)$ cannot be zero in an intersection where $l' = 7$. This is because no matter what gauge we take, we can never make the gauge 'match up' in the region $U_0 \cap U_7$.

The effect of Lemma 4.15 is to choose our non-global composite gauge to look like the holonomy gauge (achieved by parallel propagation) in the limit as r tends to zero. This will later be used to establish the limit holonomy.

We recall that each of the u's occurring on an intersection of two balls belongs to L_2^q for $q > 2$. We now note that each ball U_α has a boundary which does not consist of slits or cusps and in fact is Lipschitz continuous (see Appendix B.5). This means we can apply Sobolev's Lemma (as given in Appendix B.6) where $N = 4$, $m = 2$ and $q > 2$. Hence $N < mq$ and for all $u \in L_2^q$ we have

$$\|u\|_{L^\infty} \leq C\|u\|_{L_2^q} \leq c\|\Omega\|_{L^q(U_\alpha \cup U_\beta)} \tag{4.183}$$

Hence all u's occurring on an intersection belong to L^∞.

From Theorem 4.14 and Lemma 4.15, for all intersections not involving $l = 0$ and $l = 7$ we have the following equation which will be important later in this chapter.

$$\frac{1}{d^2}\|u\|_{L^q} + \frac{1}{d}\|\nabla u\|_{L^q} + \|\nabla(\nabla u)\|_{L^q} \leq \text{const}(\|F\|_{L^q(U_\alpha \cup U_\beta)}) \tag{4.184}$$

We refer back to our previous definition of the composite gauge transformation $G = h_n \exp(\lambda u)$ and rewrite it in full in terms of its cutoff functions.

$$G = h_{(n,l,x)} \exp(\lambda_l u_l) \exp(\lambda(y) u(n, x, x')) \exp(\lambda u) \tag{4.185}$$

In Appendix A.4 we show that we have upper bounds on the cutoff functions and their first two derivatives and we know that each u is bounded in the manner described in (4.184). We also know that the L_2^2 norm of the gauge transformation $h_{(n,l,x)}$, used to make a connection Coulomb in $U_{(n,l,x)}$, is itself less than or equal to a constant times the L^2 norm of Ω (as we saw in the proof of Theorem 4.12). From this we can deduce that on an intersection not involving $\theta = 0$ and $\theta = 2\pi$, the 1-form $\kappa = G^{-1}\mathrm{d}G \in L_1^q$ behaves like

$$\|\kappa\|_{L^q_1} \leq \text{const}\|\Omega\|_{L^q(U_\alpha \cup U_\beta)} \tag{4.186}$$

We have used the non-global composite gauge to glue together Coulomb gauges using cutoff functions. In this gauge we still require the second Coulomb condition on the connection, (4.151), to hold. If the condition holds without cutoff functions (i.e. in the Coulomb case) then it will hold with cutoff functions up to a larger constant coefficient of $\|\Omega\|_{L^2}$. We can make the effect of the λs arbitrarily small by using the previously discussed gauge freedom. Hence

$$\|\dot{\omega}\|_{L^q_1} \leq k\|\Omega\|_{L^q} \tag{4.187}$$

At $\theta = 2\pi$ we use G to gauge transform from $\omega^{(n,7,x)}$ to $\omega^{(n,0,x)}$ by $s = G(r, 2\pi, y)^{-1}G(r, 0, y)$.

We now look at the two values of G given by $G_0 = G|U_{(n,0,x)}$ and $G_7 = G|U_{(n,7,x)}$. The transformation used to gauge transform from ω^0 to ω^7 is defined to be $s = G_0^{-1}G_7$ on $\bigcup_{(n,x)} \{U_{(n,0,x)} \cap U_{(n,7,x)}\}$. It remains to identify three important features of this transformation and we shall go through these in turn. We first show s has a pointwise limit as r tends to zero and we identify this limit with the holonomy. We then use a carefully normalised local radial gauge to show that the connection satisfies good curvature related estimates. From this we can modify s to obtain a global gauge in which the connection form satisfies the inequality of Proposition 4.10.

We define a ball parameterised only by x and n which we shall call $V = V[n, x] = \{(r, \theta, y)\}$ where

$$\tfrac{3}{8}2^{-n} < r < \tfrac{7}{8}2^{-n}$$

$$-\tfrac{\pi}{8} < \theta < \tfrac{\pi}{8}$$

$$y \text{ belongs to the open square centered at } x \text{ of side length } \tfrac{7}{8}2^{-n}$$

We let $\Omega[n, x]$ be the collection of all indices $\alpha = (n', l', x')$ such that the intersection of U_α and V is non-empty. We let the union of all of these non-zero V-intersecting U_αs be $W[n, x]$. We let $X[n, x]$ be the union of all $W[n', x]$s where $n' \geq n$. Finally we let the point $P[n, x] = (2^{-(n+1)}, 0, x)$. So in equation form

$$
\begin{aligned}
W[n,x] &= \bigcup_{\alpha\in\Omega} U_\alpha \\
X[n,x] &= \bigcup_{n'\geq n} W[n',x] \\
P[n,x] &= \left(2^{-(n+1)},0,x\right)
\end{aligned}
\tag{4.188}
$$

Using this we can apply the following theorem:

Theorem 4.16 *The gauge transformation s obeys the inequalities*

$$(a) \qquad \sup_{(r,\theta,y)\in V} \|s(r,\theta,y)-s(P[n,x])\| \leq c\|\Omega_\omega\|_{L^2(W)} \tag{4.189}$$

$$(b) \quad |s(P[n,x])-s(P[n',x'])| \leq c\|\Omega_\omega\|_{L^2(X[n,x]\cup X[n',x'])} \tag{4.190}$$

Part (a) states that the least upper bound in V of the norm of the difference between s and $s(P)$ is bounded above by a constant times the L^2 norm of Ω_ω (the curvature for a connection ω) in the region W, the unity of all the V-intersecting balls U_α.

Part (b) states that the norm of the difference between s at two different points, $P[n,x]$ and $P[n',x']$ is bounded above by a constant times the L^2 norm of Ω_ω in the region given by the union of $X[n,x]$ and $X[n',x']$.

Proof (a)

The way to prove (4.189) is to show first that the gauge transformation $s=G_0^{-1}G_7$ has a limit as one tends to the singular set or, in other words, that s is continuous as n tends to infinity. The standard way to show continuity would be to use Morrey's lemma [32] (see Appendix B.7) to show that s is Hölder continuous (see Appendix B.8).

If we take $q=2$ for Theorem 4.14 then we find that $\frac{1}{d}\|\nabla u\|_{L^2} \leq K(\|\Omega\|_{L^2(U_\alpha)}+\|\Omega\|_{L^2(U_\beta)}) < \infty$ and hence $\int|\partial u|^2 d\sigma$ is bounded and we can use Morrey's lemma (for $p=2$). If we had taken any higher values of q then we could not bound $\int|\partial u|^p d\sigma$ since we only know that $\Omega\in L^2$. Note that $u\in L_2^q$ and hence, when $q=2$ we have $u\in L_2^2$; this in turn implies that $u\in L_1^2$ satisfying another condition of Morrey's lemma. However we have the requirement that $q>2$ in order to apply Sobolev's lemma to show that the u's in the intersection

$U_\alpha \cap U_\beta$ belong to L^∞. Hence the problem must be solved in a different fashion.

If we let Ω be in L^r for $r > 2$ then we can now use Theorem 4.14 for higher values of q (up to and including r) in order to get the initial conditions required for Morrey's lemma, in the same way as before. With $q > 2$ we can now still use Sobolev's Lemma. However, this has the drawback that our condition on the curvature is now even stronger and hence Theorem 4.9 is less versatile.

Instead we shall show that s is continuous by using a proof given by Taubes in the appendix of [44]. The method is quite complex but we include here a summary of the procedure.

Our aim is to show that a gauge transformation from a Coulomb gauge to another Coulomb gauge is continuous. The way we do this is to use the fact that the Laplacian of a gauge transformation s is L where L depends only on s and the two connection 1-forms (no derivatives are involved). We mollify this expression, divide by a potential, integrate by parts, and then show that the remaining potentials are continuous functions.

For two connection 1-forms a and b related by the gauge transformation s in an open ball U we have the equation

$$a = sbs^{-1} + sds^{-1} \tag{4.191}$$

(analogous to our gauge transformation equation) and the first Coulomb condition that $d * a = d * b = 0$. We note that even though we are considering connections in the non-global composite gauge, which is not in general Coulomb, the Coulomb properties hold since in the region we are currently working, $U_0 \cap U_7$, the Coulomb gauge and the non-global composite gauge are identical (the cutoff functions are not used in this intersection). From this Coulomb condition and (4.191) we have

$$^*d^*ds = sb \cdot b - 2a \cdot sb + a \cdot as \tag{4.192}$$

We now use a standard mollifier j_ϵ on both sides of (4.192). A mollifier is a smooth function with special properties, used to create sequences of smooth functions approximating non-smooth functions.

$$d^*d(j_\epsilon * s)(y) = (j_\epsilon * L)(y) \quad \text{at } y \in U_\epsilon \tag{4.193}$$

where L is the RHS of (4.192) and $*$ indicates a convolution. We consider the cut-off function

$$\beta^x(y) = \begin{cases} 1 & |x-y| < \frac{1}{4}\text{dist}(x, \partial U) \\ 0 & |x-y| > \frac{1}{2}\text{dist}(x, \partial U) \end{cases} \tag{4.194}$$

and multiply both sides of (4.193) by $|x-y|^{-2}\beta^x(y)$ and then integrate the resulting equation over U using the Green's function for $\Delta = *d * d$ to get

$$\begin{aligned}(j_\epsilon * s)(x) &= 2\pi^2 \int_U (j_\epsilon * L)(y)|x-y|^{-2}\beta^x(y)d^4y \\ &\quad +2\pi^2 \int_U (j_\epsilon * s)(y)\left(2d\beta^x \frac{d|x-y|}{|x-y|^3} - \frac{d^* d\beta^x}{|x-y|^2}\right)d^4y \end{aligned} \tag{4.195}$$

We note that the first part of the RHS is what we would have if the convolution $(j_\epsilon * L)$ was not a mollification and was simply L.

Taubes proves that for $\epsilon = 0$ the RHS of (4.195) defines a continuous function from U to M_n and hence the LHS, $(j_\epsilon * s)(x)$, does the same for $\epsilon = 0$. He does this by splitting the RHS into two terms $s_1^\epsilon(y)$ and $s_2^\epsilon(y)$. We find that $s_2^\epsilon(y)$ is continuous from a lemma stating that for any function v we have

Lemma 4.17 (Taubes) *The map $h_2 : U \times L^1(U) \to \mathbb{R}$ defined to be*

$$h_2(x, v) = 2\pi^2 \int_U v(y)\left(2d\beta^x \frac{d|x-y|}{|x-y|^3} - \frac{d^* d\beta^x}{|x-y|^2}\right)d^4y \tag{4.196}$$

is jointly continuous.

The proof that $s_1^\epsilon(y)$ is continuous is more complicated than the proof for the continuity of $s_2^\epsilon(y)$. We establish that the $\epsilon = 0$ limit of s_1^ϵ exists as a map $s_1 : U \to M_n$ and then that the map $h_1 : U \times (L^1 \cap L^\infty) \times_2 L_1^2(U) \to \mathbb{R}$, defined as

$$h_1(x, u, v, w) = \int_U \frac{uvx}{|x-y|^2} d^4y \tag{4.197}$$

is well defined and has the properties given by the lemma below.

Lemma 4.18 (Taubes) *The map h_1, above, is continuous on its domain*

$$K \equiv U \times (L^1 \cap L^\infty) \times_2 L_1^2(U) \tag{4.198}$$

Hence $s_1^\epsilon(x)$ and thus the entire RHS of (4.195) is continuous for $\epsilon = 0$.

Since $j_\epsilon * s$ converges strongly to s in $L^p_{\text{loc}}(U)$ for any $1 \leq p \leq \infty$ (which we are in) we know that the gauge transformation s is continuous and hence on the intersection of the first and seventh coordinate patches s has a limit as we tend towards the singular set.

Having shown that s is continuous we now wish to finish the proof of part a) of the theorem, in order to prove part b) which will tell us *how* s tends to s_0. As we have shown in (4.183), $||u||_{L^\infty} \leq c||\Omega||_{L^q(U_\alpha \cup U_\beta)}$. We recall that s is dependent on $h \in L^2_2$ and exponentials of u. Since s is continuous we know that this dependence is also continuous and from this we can show that $\sup_V ||s - s(p)||$ is bounded above as in Theorem 4.16. $\square$

We now look at the proof of the second part of Theorem 4.16

Proof (b)

The proof of (b) comes by using (a),

$$
\begin{aligned}
|s(P[n,x]) - s(P[n',x'])| &= \left|s(P[n,x]) - s(P) - \big(s(P[n',x'] - s(P)\big)\right| \\
&\leq |s(P[n,x]) - s(P)| + |s(P[n',x']) - s(P)| \\
&\leq \sup_V |s(P[n,x]) - s(P)| + \sup_V |s(P[n',x']) - s(P)| \\
&\leq c_1||\Omega_\omega||_{L^2(W[n,x])} + c_2||\Omega_\omega||_{L^2(W[n',x'])} \\
&\leq c||\Omega_\omega||_{L^2(X[n,x]\cup X[n',x'])} \qquad (4.199)
\end{aligned}
$$

We can go from the fourth to the fifth line because $W[n,x] \cup W[n',x']$ is smaller than $X[n,x] \cup X[n',x']$.

Part (b) shows us that as we increase n and n' to infinity the region $X[n,x] \cup X[n',x']$ will shrink to only the U_α balls intersecting the shrinking region V around the singularity. V will certainly be shrinking not just radially but also on the singularity since, as n increases, the side length of the square on the singularity equal to $\frac{7}{8}2^{-n}$ is also shrinking. So as $n \to \infty$ the sequence $\{s(P[n,x])\}$ converges to some pointwise limit $s_0 \in SO(3)$. This s_0 in $SO(3)$ is independent of the value of x we choose, since s is not affected by position on the singularity, but only the size of the open square around that position. The size of the open square is not dependent on x but on n.

The limit as r tends to zero of the jump in the holonomy gauge is the limit holonomy and so, from Lemma 4.15, we have an indication that the limit as r tends to zero of the jump in the composite gauge is also the (same) limit holonomy. The proof that the limit s_0 can be identified with the limit holonomy of ω is as follows.

First we wish to show that s_0 is equal to the holonomy J_∞ of a constant flat connection ω_∞. Since s_0 is a constant it can be rewritten as $\exp(-2\pi a_\infty)$ where a_∞ is a constant matrix. It follows from the discussion at the end of Section 4.2 that a_∞ takes values in $so(3)$. Now let us look at some constant flat connection $\omega_\infty = a_\infty d\theta$. We can work out the holonomy, $G(2\pi)$, of ω_∞ by solving

$$\frac{dG}{d\theta} + a_\infty G = 0 \tag{4.200}$$

We find the holonomy J_∞ of this constant flat connection ω_∞ is $G(2\pi) = \exp(-2\pi a_\infty)$. Hence $s_0 = J_\infty$. Up to a choice of gauge, ω_∞ is the flat connection to which we will show our connection ω will asymptotically tend. More precisely, we will show ω tends to a prototype flat connection which is gauge equivalent to ω_∞.

We now wish to show that ω_∞ and ω have the same limit holonomy. To do this we need the following lemma which will also be crucial to the proof of Proposition 4.10.

Lemma 4.19 *In any ball B_α, a point P may be chosen and a radial gauge ($\omega_r = 0$) found, in which $\omega(P) = \omega_\infty = a_\infty d\theta$ and the inequality,*

$$r^{-2}||\omega_\theta - a_\infty||_{L^q} + r^{-1}||\omega_z||_{L^q} + r^{-1}||\omega_t||_{L^q} \leq c||\Omega||_{L^q} \quad \textit{for } q \geq 2 \tag{4.201}$$

is satisfied.

We use the above lemma to show that an initial gauge can be chosen such that the connection in the composite gauge tends to the constant flat connection ω_∞ as r tends to zero. Hence ω_∞ and ω have the same limit holonomy. From before we know that s_0 is equal to J_∞, the holonomy of ω_∞. This concludes the proof that s_0 can be identified with the limit holonomy of ω. $\square$

Proof of Lemma 4.19

From before we know that there exists a radial gauge such that $\lim_{r\to 0}\omega = \omega_\infty$ almost everywhere. If ω' is related to ω by a gauge transformation G' then

$$\omega' = G'^{-1}\omega G' + G'^{-1}\mathrm{d}G' \tag{4.202}$$

and

$$\omega'_z(P) = G'^{-1}\omega_z(P)G' + G'^{-1}\frac{\partial G'}{\partial z} \tag{4.203}$$

If we let G' be the solution to the equation

$$\frac{dG'}{dz} + \omega_z(P)G' = 0 \tag{4.204}$$

which is $G' = e^{-\omega_z(P)z}$, then it follows that $\omega'_z(P) = 0$. If we then transform ω' further using the gauge transformation $G'' = e^{-\omega'_t(P)t}$ then we have a new radial gauge

$$\omega'' = e^{\omega'_t(P)t}\left[e^{\omega_z(P)z}\omega e^{-\omega_z(P)z} - \omega_z(P)e^{-\omega_z(P)z}dz\right]e^{-\omega'_t(P)} - e^{\omega'_t(P)t}\omega'_t(P)e^{-\omega'_t(P)t}dt \tag{4.205}$$

which still satisfies $\omega''_z(P) = 0$ but also $\omega''_t(P) = 0$. We now work in this gauge and so restrict the notation ω'' to ω. We now make a constant gauge transformation to fix $\omega_\theta(P) = a_\infty$. Since $\omega_r = 0$ it follows that $\frac{\partial\omega_\theta}{\partial r} = \Omega_{r\theta}$. Now we integrate both sides of this equation with respect to r (using a dummy variable ρ) between $r(P)$ and r to get

$$|\omega_\theta - a_\infty| \le \int_{\rho=r(P)}^{r} |\Omega_{\rho\theta}|d\rho \tag{4.206}$$

and since here $r(P) \le \rho \le r$ we have

$$|\omega_\theta - a_\infty| \le \int_{\rho=r(P)}^{r} \rho|\Omega|d\rho \tag{4.207}$$

Now we use (4.207) and Hölder's inequality.

$$
\begin{aligned}
|\omega_\theta - a_\infty| &\leq \int_{\rho=r(P)}^{r} \rho|\Omega|d\rho \\
&\leq \|1\|_{L^{\frac{q}{q-1}}} \|\rho|\Omega|\|_{L^q} \\
&= \left(\int_{\rho=r(P)}^{r} 1 d\rho\right)^{\frac{q-1}{q}} \left(\int_{\rho=r(P)}^{r} |\rho\Omega|^q d\rho\right)^{\frac{1}{q}} \\
&= (r - r(P))^{\frac{q-1}{q}} \left(\int_{\rho=r(P)}^{r} |\rho\Omega|^q d\rho\right)^{\frac{1}{q}} \\
&\leq r^{\frac{q-1}{q}} \left(\int_{\rho=r(P)}^{r} |\rho\Omega|^q d\rho\right)^{\frac{1}{q}} \\
|\omega_\theta - a_\infty|^q &\leq r^{q-1} \int_{\rho=r(P)}^{r} |\rho\Omega|^q d\rho
\end{aligned}
\tag{4.208}
$$

Now we integrate over the volume of a ball B_α with radius r. Nothing significant is happening in the z and t directions and any integral over θ is bounded so (introducing a multiplicative constant) we only need to look at the integral in the radial direction, remembering to introduce the additional r when integrating over the area. So we have

$$
\begin{aligned}
\int |\omega_\theta - a_\infty|^q dr &\leq \int r^{q-1} \int_{\rho=r(P)}^{r} |\rho\Omega|^q d\rho r dr \\
&= \int r^q \int_{\rho=r(P)}^{r} |\rho\Omega|^q d\rho dr
\end{aligned}
\tag{4.209}
$$

Now using integration by parts we obtain

$$
\begin{aligned}
\int |\omega_\theta - a_\infty|^q dr &\leq \frac{r^{q+1}}{q+1} \int_{\rho=r(P)}^{r} |\rho\Omega|^q d\rho - \int \frac{r^{q+1}}{q+1} |r\Omega|^q dr \\
&\leq \frac{r^{q+1}}{q+1} \int_{\rho=r(P)}^{r} |\rho\Omega|^q d\rho \\
\Longrightarrow \left(\int |\omega_\theta - a_\infty|^q dr\right)^{\frac{1}{q}} &\leq c r^{\frac{q+1}{q}} \left(\int_{\rho=r(P)}^{r} |\rho\Omega|^q d\rho\right)^{\frac{1}{q}}
\end{aligned}
\tag{4.210}
$$

Now we use integration by parts again

$$
\begin{aligned}
\int_{\rho=r(P)}^{r} \rho^q |\Omega^q| d\rho &= \left[\rho^q \int |\Omega|^q d\rho\right]_{\rho=r(P)}^{r} - q \int_{\rho=r(P)}^{r} \rho^{q-1} d\rho \int |\Omega|^q d\rho \\
&\leq (r^q - r^q(P)) \int_{\rho=r(P)}^{r} |\Omega|^q d\rho \\
&\leq r^q \int_{\rho=r(P)}^{r} |\Omega|^q d\rho
\end{aligned}
\tag{4.211}
$$

So (4.210) becomes

$$\begin{aligned}\left(\int |\omega_\theta - a_\infty|^q dr\right)^{\frac{1}{q}} &\leq cr^{\frac{q+1}{q}}\left(r^q \int_{\rho=r(P)}^{r} |\Omega|^q d\rho\right)^{\frac{1}{q}} \\ &= cr^{\frac{2q+1}{q}}\left(\int_{\rho=r(P)}^{r} |\Omega|^q d\rho\right)^{\frac{1}{q}} \end{aligned} \tag{4.212}$$

However, $q \geq 2$ implies that $\frac{2q+1}{q} \geq 2$ and since r is small, we know that

$$r^{\frac{2q+1}{q}} \leq r^2 \tag{4.213}$$

and so (4.212) becomes

$$\begin{aligned}\left(\int |\omega_\theta - a_\infty|^q dr\right)^{\frac{1}{q}} &\leq cr^2\left(\int_{\rho=r(P)}^{r} |\Omega|^q d\rho\right)^{\frac{1}{q}} \\ \Longrightarrow r^{-2}||\omega_\theta - a_\infty||_{L^q} &\leq c||\Omega||_{L^q}\end{aligned} \tag{4.214}$$

With a similar method we can obtain estimates on ω_z and ω_t thus concluding our proof for Lemma 4.19. $\square$

When we refer to performing the composite gauge transformation on the connection, we mean that we apply the composite gauge transformation on the connection in the radial gauge.

We let ω represent the connection 1-form in the radial gauge of Lemma 4.19 in the ball containing $U_0 = U_{(n,0,x)}$, i.e. the ball over the sector $0 < \theta < \frac{\pi}{2}$. We let ω_0 be the connection 1-form in the *Coulomb gauge* on this same ball. The gauge transformation h takes ω to ω_0 using the equation $\mathrm{d}h + \omega h = h\omega_0$. We can rewrite this as (remembering that $\omega_r = 0$)

$$\begin{aligned}&\frac{\partial h}{\partial \theta}d\theta + \frac{\partial h}{\partial z}dz + \frac{\partial h}{\partial t}dt + \frac{\partial h}{\partial r}dr + \omega_\theta h d\theta + \omega_z h dz + \omega_t h dt = \\ &\qquad h\omega_{0\theta}d\theta + h\omega_{0z}dz + h\omega_{0t}dt + h\omega_{0r}dr\end{aligned} \tag{4.215}$$

Taking coefficients of $d\theta$

$$\frac{\partial h}{\partial \theta} + \omega_\theta h = h\omega_{0\theta} \tag{4.216}$$

Let $h = e^{a_\infty \theta}\tilde{h}$. This implies

$$
\begin{aligned}
a_\infty e^{a_\infty\theta}\tilde{h} + e^{a_\infty\theta}\frac{\partial\tilde{h}}{\partial\theta} + \omega_\theta e^{a_\infty\theta}\tilde{h} &= e^{a_\infty\theta}\tilde{h}\omega_{0\theta} \\
\Longrightarrow a_\infty\tilde{h} + \frac{\partial\tilde{h}}{\partial\theta} + \omega_\theta\tilde{h} &= \tilde{h}\omega_{0\theta} \\
\Longrightarrow \frac{\partial\tilde{h}}{\partial\theta} &= \tilde{h}\omega_{0\theta} - (a_\infty + \omega_\theta)\tilde{h} \\
\Longrightarrow \tilde{h} &= \int_{\theta_Q}^{\theta} \tilde{h}(s)\omega_{0\theta}(s) - (a_\infty + \omega_\theta(s))\tilde{h}(s)ds \\
&\quad + I + f(z,t,r)
\end{aligned}
$$

and so

$$
h = e^{a_\infty\theta}\tilde{h} = e^{a_\infty\theta}\left(I + \int_{\theta_Q}^{\theta} e^{-a_\infty\theta}[h\omega_{0\theta} - (a_\infty + \omega_\theta)h]d\theta + f\right) \tag{4.217}
$$

We now use a result from measure theory (see [39]) which states that there is a point $Q \in U_0$ with $0 < \theta_Q < \frac{\pi}{2}$ such that

$$
\int_{U_0\cap(\theta=\theta_Q)} |\omega - \omega_\infty|^q r dr dz dt \leq \frac{2\|\omega - \omega_\infty\|^q_{L^q(U_0)}}{\frac{\pi}{2}} \tag{4.218}
$$

With Q defined this way and $f(Q) = 0$ it follows that

$$
|\nabla f| \leq \text{const}\left\{(|\omega_z| + |\omega_t|)_{(r,\theta_Q,z,t)} + \int_{\theta_Q}^{\theta}(|\omega_0| + |\omega_z| + |\omega_t| + r|\Omega|d\theta)\right\} \tag{4.219}
$$

We then take L^q norms and use the measure theoretic fact above, Theorem 4.14 and the Poincaré inequality (see Appendix B.9) to get

Lemma 4.20 *Let h be the gauge transformation on U_0 from the radial to the Coulomb gauge. Then, $h = e^{a_\infty\theta}(I + R_0)$ where, for $q \geq 2$*

$$
\|R_0\|_{L^q(U_0)} \leq cr^2\|\Omega\|_{L^q(U_0)} \tag{4.220}
$$

for some remainder term R_0.

An analogous computation for the ball $U_7 = U(n, 7, x)$ gives us an expression for the gauge transformation k from the radial gauge to the Coulomb gauge

$$
k = e^{a_\infty(\theta - 2\pi)}(I + R_7) \tag{4.221}
$$

with

$$
\|R_7\|_{L^q(U_0)} \leq cr^2\|\Omega\|_{L^q(U_0)} \tag{4.222}
$$

Returning to the transformation $s = G_0^{-1}G_7 = h^{-1}k$ we can rewrite it in terms of $e^{-2\pi a_\infty}$ and the remainder functions

$$\begin{aligned} s &= \left(e^{a_\infty \theta}(I+R_0)\right)^{-1} e^{a_\infty(\theta-2\pi)}(I+R_7) \\ &= (I+R_0)^{-1}e^{-a_\infty\theta}e^{a_\infty\theta}e^{-2\pi a_\infty}(I+R_7) \\ &= (I+R_0)^{-1}e^{-2\pi a_\infty}(I+R_7) \end{aligned} \tag{4.223}$$

Now we define the set $N = \{x \in U_0 \cap U_7 | \, |R_7(x)| > 1\}$ and let $m(N)$ be its measure. Then

$$\begin{aligned} m(N) &= \int_N dx \\ &< \int_N |R_7(x)|dx \qquad \text{since } |R_7(x)| > 1 \\ &< \int_N |R_7(x)|^q dx \\ &= \left(\|R_7(x)\|_{L^q(N)}\right)^q \\ &\leq \left(\|R_7(x)\|_{L^q(U_7)}\right)^q \qquad \text{since } U_7 \supseteq U_7 \cap U_0 \\ &\leq \left(cr^2\|\Omega\|_{L^q(U_7)}\right)^q \\ &\leq cr^{2q}\|\Omega\|^q_{L^q(U_7)} \\ &\leq Cr^{2q}o(1) \qquad \text{since } \|\Omega\|_{L^q(U_7)} \text{ tends to } 0 \end{aligned} \tag{4.224}$$

Since r is small and $q \geq 2$ we have $m(N) \leq Cr^4 o(1)$. One can approximate the measure of the intersection of the two balls U_0 and U_7 to be r^4 and from this we know $m(N) < \frac{1}{2}m(U_0 \cap U_7) \sim r^4$. If we let $V = (U_0 \cap U_7)\backslash N$ then we see that $V = \{x \in U_0 \cap U_7 | \, |R_7(x)| \leq 1\}$ so $|R_7| \leq 1$ on V. Since $m(V) + m(N) = m(U_0 \cap U_7)$, if $m(N) < \frac{1}{2}m(U_0 \cap U_7)$ then $m(V) \geq \frac{1}{2}m(U_0 \cap U_7)$. The average value over V of $|R_0(P)|^q$ is given by

$$\frac{\int_V |R_0|^q dx}{m(V)} \tag{4.225}$$

and so we can find a point $P \in V$ such that

$$|R_0(P)|^q \leq \frac{\int_V |R_0|^q dx}{m(V)} \tag{4.226}$$

and hence

$$|R_0(P)|^q \leq \frac{2\int_V |R_0|^q dx}{m(U_0 \cap U_7)} \tag{4.227}$$

Since $P \in V$ we know that $R_7(P)| \leq 1$. By taking the qth root of (4.227) and then applying Lemma 4.20, we have

Lemma 4.21 *There exists a point P in $U_0 \cap U_7$ and a constant C such that*

$$|R_0(P)| < \frac{2Cr^2\|\Omega\|_{L^q(U_0)}}{m(U_0 \cap U_7)^{\frac{1}{q}}} \qquad \textit{and} \qquad |R_7(P)| \leq 1 \tag{4.228}$$

We now look at $P = P[n, x]$, the point of $U_0 \cap U_7$ given by Lemma 4.21 and the gauge transformation $s = s_0 e^{\psi} = G_0^{-1} G_7$ on $U_{(n,0,x)} \cap U_{n,7,x)}$ where $\psi \in L_2^q$ and $2 \leq q \leq p$. Since s is on $U_{(n,0,x)} \cap U_{(n,7,x)}$ we can use Theorem 4.14 to get

$$\begin{aligned} \|\nabla(\nabla\psi)\|_{L^q} + r^{-1}\|\nabla\psi\|_{L^q} + r^{-2}\|\psi\|_{L^q} &\leq \|\Omega\|_{L^q(U_0 \cap U_7)} \\ \Longrightarrow \|\nabla(\nabla\psi)\|_{L^q} + r^{-1}\|\nabla\psi\|_{L^q} &\leq \|\Omega\|_{L^q(U_0 \cap U_7)} \end{aligned} \tag{4.229}$$

We shall use this result later for Proposition 4.22.

We now observe that for small r we can use a Taylor series expansion on $(I + R_0(P))^{-1}$ to get

$$(I + R_0(P))^{-1} = I - R_0(P) + \text{higher order terms} \tag{4.230}$$

Let us call the higher order terms, H. We know that by taking a sufficiently small space near to $R_0(P)$ we have that $|H| \leq R_0(P)$. So now let

$$(I + R_0(P))^{-1} = I - \tilde{R}_0(P) \tag{4.231}$$

where

$$\begin{aligned} \tilde{R}_0(P) &= R_0(P) - H \\ &\leq R_0(P) + R_0(P) \\ \Longrightarrow |\tilde{R}_0(P)| &\leq 2|R_0(P)| \end{aligned} \tag{4.232}$$

Now we use (4.223) and (4.231) to get

$$\begin{aligned} s(P) &= (I + R_0(P))^{-1} e^{-2\pi a_\infty}(I + R_7(P)) \\ &= e^{-2\pi a_\infty}(I - \tilde{R}_0(P))(I + R_7(P)) \end{aligned} \tag{4.233}$$

We let $s = s_0 e^{\psi}$ so we have

$$s(P) = s_0 e^{\psi(P)} = e^{-2\pi a_\infty}(I - \tilde{R}_0(P))(I + R_7(P)) \tag{4.234}$$

The remainder terms vanish from the point we expand about, so $\tilde{R}_0(0) = R_7(0) = 0$ and we know that

$$s(0) = e^{-2\pi a_\infty} = s_0 \tag{4.235}$$

so, using (4.232) and the inequality $|R_7(P)| \leq 1$ from Lemma 4.21, it follows that

$$\begin{aligned}
e^{\psi(P)} &= I - \tilde{R}_0(P)R_7(P) - \tilde{R}_0(P) + R_7(P) \\
|I - e^{\psi(P)}| &= |\tilde{R}_0(P)R_7(P) + \tilde{R}_0(P) - R_7(P)| \\
&\leq |\tilde{R}_0(P)R_7(P) + \tilde{R}_0(P)| \\
&\leq |2\tilde{R}_0(P)| \\
&\leq 4|R_0(P)|
\end{aligned} \tag{4.236}$$

Now let $e^{\psi(P)} = I + W$. Then

$$\begin{aligned}
|I - I - W| &\leq 4|R_0(P)| \\
\Longrightarrow |W| &\leq 4|R_0(P)|
\end{aligned} \tag{4.237}$$

$|\ln(I+W)| \leq 2|W|$ for all values of W except when $-1 \leq W \leq p \approx -0.797$ (considering just the one dimensional case for illustrative purposes). However, since we know that $|W| \leq 4|R_0|$ we can see from Lemma 4.21 that if we are considering $U_0 \cap U_7$ close enough to the singularity we can ensure that $|W| < 0.797$. Hence,

$$\begin{aligned}
|\psi(P)| &= |\ln(I + W)| \\
&\leq 2|W| \\
&\leq 8|R_0|
\end{aligned} \tag{4.238}$$

So now using Lemma 4.21 again we have the result

$$\begin{aligned}
|\psi(P)| &\leq 8|R_0| \\
&\leq \frac{Cr^2\|\Omega\|_{L^q(U_0)}}{(m(U_0 \cap U_7))^{\frac{1}{q}}} \\
&\leq \frac{Cr^2\|\Omega\|_{L^q(U_0 \cup U_7)}}{(m(U_0 \cap U_7))^{\frac{1}{q}}}
\end{aligned} \tag{4.239}$$

We now combine (4.229) and (4.239) to get

Proposition 4.22 *The gauge transformation $s = s_0 e^{\psi} = G_0^{-1}G_7$ on $U_{(n,0,x)} \cap U_{(n,7,x)}$, where $\psi \in L_2^q$, $2 \leq q \leq p$, and obeys the inequalities,*

$$\begin{aligned} \|\nabla(\nabla\psi)\|_{L^q} + r^{-1}\|\nabla\psi\|_{L^q} &\leq \|\Omega\|_{L^q(U_0\cup U_7)} \\ |\psi(P)| &\leq \frac{Cr^2\|\Omega\|_{L^q(U_0\cup U_7)}}{(m(U_0\cap U_7))^{\frac{1}{q}}} \end{aligned}$$

where $P = P[n,x]$ is the point of $U_0 \cap U_7$ given by Lemma 4.21.

From Proposition 4.22 we can see that since $|\psi|$ approaches zero in the limit we have e^{ψ} =Id in the limit. Hence $s = s_0 e^{\psi}$ tends to s_0 as we would expect. We now have all the requirements to calculate the inequality in (4.139) except that the gauge transformation G is not global as a result of using the composite gauge transformation which is discontinuous at $\theta = 0 = 2\pi$. We use the following cut-off function to modify the non-global composite gauge to be global in such a way that the required inequalities for ω and κ are preserved up to constants.

$$\lambda(\theta) = \begin{cases} 1 & \theta = 0 \\ 0 & \theta \geq \frac{\pi}{8} \end{cases} \tag{4.240}$$

Now, instead of G we shall have our new *global* gauge transformation $\hat{G} = Gs^{\lambda}$ where $\hat{G}$ is identical to G except in the region $U_{(n,0,x)} \cap U_{(n,7,x)}$. In this region we use $s = G_0^{-1}G_7$ and the cutoff function and observe that

$$\hat{G}(\theta) = \begin{cases} G_7 & \theta = 0 \\ G_0 & \theta \geq \frac{\pi}{8} \end{cases} \tag{4.241}$$

Now let $\hat{\omega} = \hat{G}^{-1}\omega\hat{G} + \hat{G}^{-1}\mathrm{d}\hat{G}$ and $\hat{\kappa} = \hat{G}^{-1}\mathrm{d}\hat{G}$. We wish to show that the L_1^q norms of both of these objects in this global gauge are bounded above by $c\|\Omega\|_{L^q}$ (this is the only Coulomb property we still require at this stage).

Recall that $\dot{\omega}$ is the original connection gauge transformed once into the radial gauge and then into the non-global composite gauge. We know that $\|\dot{\omega}\|_{L_1^q} \leq c\|\Omega\|_{L^q}$ and hence

$$\begin{aligned}
||\hat{\omega}||_{L_1^q} &= ||s^{-\lambda}\dot{\omega}s^{\lambda} + s^{-\lambda}\mathrm{d}(s^{\lambda})||_{L_1^q} \\
&\leq c_1||\dot{\omega}||_{L_1^q} + ||s^{-\lambda}\mathrm{d}(s^{\lambda})||_{L_1^q} \\
&\leq c_2||\Omega||_{L^q} + ||(\ln s_0)\nabla\lambda||_{L_1^q} + c_3||\psi\nabla\lambda||_{L_1^q} + ||\lambda\nabla\psi||_{L_1^q} \\
&\leq C||\Omega||_{L^q} \qquad (4.242)
\end{aligned}$$

In the above calculation we make use of Proposition 4.22 and bounds on the cutoff function and its first two derivatives, obtained in a similar manner to Appendix A.4. Note that it is only necessary to apply Proposition 4.22 in the region $U_0 \cap U_7$ since elsewhere $G = \hat{G}$ and so $||\hat{\omega}||_{L_1^q} \leq C||\Omega||_{L^q}$ anyway.

We had before that $||\kappa||_{L_1^q} = ||G^{-1}\mathrm{d}G||_{L_1^q} \leq c||\Omega||_{L^q}$. Hence

$$\begin{aligned}
||\hat{\kappa}||_{L_1^q} &= ||\hat{G}^{-1}\mathrm{d}\hat{G}||_{L_1^q} \\
&= ||(\ln s)\nabla\lambda + \lambda\nabla\psi + s^{-\lambda}\kappa s^{\lambda}||_{L_1^q} \\
&\leq ||(\ln s)\nabla\lambda||_{L_1^q} + ||\lambda\nabla\psi||_{L_1^q} + c||\kappa||_{L_1^q} \\
&\leq c||\Omega||_{L^q} \qquad (4.243)
\end{aligned}$$

We now deduce some important results to assist in our final calculation

1. ω_∞ is a flat connection which implies that $\Gamma = \hat{G}^{-1}\omega_\infty\hat{G} + \hat{G}^{-1}\mathrm{d}\hat{G}$ is a flat connection.

2. Since Γ is a flat connection we know that its curvature $\Omega_\Gamma = \mathrm{d}\Gamma + [\Gamma, \Gamma] = 0$

3.

$$\begin{aligned}
||\mathrm{d}\omega||_{L^q} &= ||(\mathrm{d}\omega)^{\alpha}_{\beta ab}||_{L^q} \\
&= ||\omega^{\alpha}_{\beta b,a} - \omega^{\alpha}_{\beta a,b}||_{L^q} \\
&\leq ||\omega^{\alpha}_{\beta b,a}||_{L^q} + ||\omega^{\alpha}_{\beta a,b}||_{L^q} \\
&= 2||\omega^{\alpha}_{\beta b,a}||_{L^q} \\
&= 2||\partial\omega||_{L^q}
\end{aligned}$$

4. We wish to show that

$$||\hat{\omega}\Gamma||_{L^q} \leq c||\hat{\omega}||_{L_1^q}||\Gamma||_{L_1^q} \qquad (4.244)$$

To show this we will first look at the case where ω and Γ are scalars.

$$\begin{aligned}
\|\hat{\omega}\Gamma\|_{L^q} &= \left(\int |\hat{\omega}\Gamma|^q d\sigma\right)^{\frac{1}{q}} \\
&= \left(\int |\hat{\omega}^q||\Gamma^q| d\sigma\right)^{\frac{1}{q}} \\
&\leq (\|\hat{\omega}^q\|_{L^2}\|\Gamma^q\|_{L^2} d\sigma)^{\frac{1}{q}} \quad \text{(Hölder's inequality)} \\
&= \left(\int |\hat{\omega}|^{2q} d\sigma\right)^{\frac{1}{2q}} \left(\int |\Gamma|^{2q} d\sigma\right)^{\frac{1}{2q}} \qquad (4.245)
\end{aligned}$$

By the Sobolev imbedding theorem (see proof of Theorem 5.23 in [1]) we know that if $mp \leq n$ and $p \leq r \leq \frac{np}{n-mp}$ then

$$\int |w|^r dx \leq k\|w\|^r_{L^p_m} \qquad (4.246)$$

for some scalar w, constant k. Hence

$$\|\hat{\omega}\Gamma\|_{L^q} \leq c\|\hat{\omega}\|_{L^q_1}\|\Gamma\|_{L^q_1} \qquad (4.247)$$

As a result of the flat background metric and the construction of the Sobolev norm as shown in Chapter 2, it can be shown that (4.247) applies in the non-scalar case also. Hence we know that

$$\begin{aligned}
\|[\hat{\omega},\Gamma]\|_{L^q} &\leq 2\|\hat{\omega}\Gamma\|_{L^q} \\
&\leq c_2\|\hat{\omega}\|_{L^q_1}\|\Gamma\|_{L^q_1} \\
&\leq c_2\|\hat{\omega}\|_{L^q_1}\|\hat{G}^{-1}\omega_\infty\hat{G}+\hat{\kappa}\|_{L^q_1} \\
&\leq c_2\|\hat{\omega}\|_{L^q_1}(\|\hat{G}^{-1}\omega_\infty\hat{G}\|_{L^q_1}+\|\hat{\kappa}\|_{L^q_1}) \\
&\leq c_3\|\hat{\omega}\|_{L^q_1}+c_2\|\hat{\omega}\|_{L^q_1}\|\hat{\kappa}\|_{L^q_1} \\
&\leq c_4\|\Omega\|_{L^q}+c_5\|\Omega\|^2_{L^q} \\
&\leq c_4\|\Omega\|_{L^q}+c_6\|\Omega\|_{L^q} \\
&\leq c\|\Omega\|_{L^q} \qquad (4.248)
\end{aligned}$$

We now finish the proof of this section by applying the above results and Lemma 4.19. We introduce $\mathcal{C}$ as a finite set of αs such that $\bigcup_{\alpha\in\mathcal{C}} B_\alpha$ forms a covering space for N_0.

$$
\begin{aligned}
\|\hat{\omega}-\Gamma\|_{L^q_{1,\Gamma}} &= \|\hat{\omega}-\Gamma\|_{L^q}+\|\nabla_\Gamma(\hat{\omega}-\Gamma)\|_{L^q}\\
&= \|\hat{\omega}-\Gamma\|_{L^q}+\|\mathrm{d}(\hat{\omega}-\Gamma)+[\hat{\omega}-\Gamma,\Gamma]\|_{L^q}\\
&= \|\hat{\omega}-\Gamma\|_{L^q}+\|\mathrm{d}\hat{\omega}+[\hat{\omega},\Gamma]-(\mathrm{d}\Gamma+[\Gamma,\Gamma])\|_{L^q}\\
&= \|\hat{G}^{-1}\omega\hat{G}-\hat{G}^{-1}\omega_\infty\hat{G}\|_{L^q}+\|\mathrm{d}\hat{\omega}+[\hat{\omega},\Gamma]\|_{L^q}\\
&\le \|\hat{G}^{-1}(\omega-\omega_\infty)\hat{G}\|_{L^q}+\|\mathrm{d}\hat{\omega}\|_{L^q}+\|[\hat{\omega},\Gamma]\|_{L^q}\\
&\le c_2\|\omega-\omega_\infty\|_{L^q}+2\|\partial\hat{\omega}\|_{L^q}+c_1\|\Omega\|_{L^q}\\
&\le c_2\left(\sum_{\alpha\in\mathcal{C}}\|\omega-\omega_\infty\|_{L^q(B_\alpha)}\right)+c_3\|\hat{\omega}\|_{L^q_1}+c_1\|\Omega\|_{L^q}\\
&\le c_4\|\Omega\|_{L^q}+c_5\|\Omega\|_{L^q}\\
&\le c\|\Omega\|_{L^q} &(4.249)
\end{aligned}
$$

Since $D\in\mathcal{W}^p_{m,k}=\{D=\mathrm{d}+\omega|\omega\in L^p_{1,\mathrm{loc}}(X_0),\Omega\in L^p(N_0),\|\Omega\|_{L^2}\le k$ and (H_A) holds$\}$ we have thus shown Proposition 4.10. Γ is given by A_∞ in the composite global gauge and has a limit given by

$$\lim_{r\to\infty}\Gamma=a_\infty d\theta \tag{4.250}$$

Hence, as previously shown, we obtain Corollary 4.11 by noting that in the limit $\Gamma=a_\infty d\theta$ is a constant flat connection and so is gauge equivalent to $\omega^\flat$. Corollary 4.11 then leads on to Theorem 4.9. □

4.4 Limiting behaviour of the metric

We have established Theorem 4.9 demonstrating that the connection is asymptotically the same as the connection of a 4-dimensional cone. We would now like to establish the following theorem relating the metric with the metric of a four dimensional cone.

Theorem 4.23 *There exist coordinates such that the metric g satisfies*

$$\|g-g^\flat\|_{L^2_{2,\nabla^\flat}(N_0)}\le C\|\Omega\|_{L^2(N_0)} \tag{4.251}$$

where $g^\flat$ is the metric for the four-dimensional cone and the $L^2_{2,\nabla^\flat}$ norm is where the covariant derivatives are taken with respect to the conical metric.

Proof

Let us first consider the 4-dimensional conical space-time, with angular deficit $2\pi(1-A)$ and connection 1-form $\omega^\flat$. We look at a point x_0 away from the singularity which we can say is $x_0 = (0, r_0, 0, 0)$, without loss of generality. We choose a dual basis at x_0 to be

$$\vartheta^{\flat\alpha}_{\ \ a}(x_0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.252}$$

We Fermi-Walker transport [20] $\vartheta^{\flat\alpha}_{\ \ a}(x_0)$ using the metric connection $\mathbf{\Gamma}^\flat$, first in along the radial line $\theta = 0$, then around a loop (for each value of r), then in both the t and z directions. Note that we employ Fermi-Walker transportation since the lines along which we propagate are not geodesics and hence parallel propagation will not preserve orthogonality of the dual basis. Since we have used the metric connection we know that the coframe remains orthonormal. This will give $\vartheta^{\flat\alpha}_{\ \ a}(t, r, \theta, z) = \vartheta^{\flat\alpha}_{\ \ a}$. We note that $\vartheta^{\flat\alpha}_{\ \ a}(t, r, 2\pi, z) \neq \vartheta^{\flat\alpha}_{\ \ a}(t, r, 0, z)$. However, there exists an element of $SO(3)$, $L^{\flat\alpha}_{\ \ \beta}(t, r, z) = L^{\flat\alpha}_{\ \ \beta}$ such that

$$\vartheta^{\flat\alpha}_{\ \ a}(t, r, 2\pi, z) = L^{\flat\alpha}_{\ \ \beta}\vartheta^{\flat\beta}_{\ \ a}(t, r, 0, z) \tag{4.253}$$

Since $\vartheta^{\flat\alpha}_{\ \ a}$ is orthonormal, the flat metric $g^\flat_{\ ab}(t, r, \theta, z)$ is given by

$$g^\flat_{\ ab} = \eta_{\alpha\beta}\vartheta^{\flat\alpha}_{\ \ a}\vartheta^{\flat\beta}_{\ \ b} \tag{4.254}$$

where $\eta_{\alpha\beta}$ is the Minkowski metric. Hence $g^\flat_{\ ab}(t, r, 2\pi, z) = g^\flat_{\ ab}(t, r, 0, z)$.

Now we consider the space-time with connection 1-form ω. We choose a dual basis $\vartheta^\alpha_a(x_0)$ at the point x_0 and Fermi-Walker transport in an analogous way to the above, using the metric connection $\mathbf{\Gamma}$, to get the coframe $\vartheta^\alpha_a(t, r, \theta, z) = \vartheta^\alpha_a$. However we choose the initial coframe in such a way that

$$\lim_{r\to 0} \vartheta^\alpha_a(0, r, 0, 0) = \lim_{r\to 0} \vartheta^{\flat\alpha}_{\ \ a}(0, r, 0, 0) \tag{4.255}$$

Then in the same fashion as before we find the $SO(3)$ transformation L^α_β such that

$$\vartheta^\alpha_a(t, r, 2\pi, z) = L^\alpha_\beta\vartheta^\beta_a(t, r, 0, z) \tag{4.256}$$

and we also have

$$g_{ab} = \eta_{\alpha\beta}\vartheta^\alpha_a\vartheta^\beta_b \tag{4.257}$$

We now want to show that $\|g_{ab} - g^\flat_{ab}\|_{L^2} \leq K\|\Omega\|_{L^2}$. We first define $h^\alpha_\beta(t,r,\theta,z) = h^\alpha_\beta$ by

$$\vartheta^\alpha_a = h^\alpha_\beta {\vartheta^\flat}^\beta_a \tag{4.258}$$

$$\Longrightarrow \vartheta^\alpha_a - {\vartheta^\flat}^\alpha_a = h^\alpha_\beta {\vartheta^\flat}^\beta_a - {\vartheta^\flat}^\alpha_a = (h^\alpha_\beta - \delta^\alpha_\beta){\vartheta^\flat}^\beta_a \tag{4.259}$$

Hence

$$\|\vartheta^\alpha_a - {\vartheta^\flat}^\alpha_a\|_{L^2} \leq c\|h^\alpha_\beta - \delta^\alpha_\beta\|_{L^2} \tag{4.260}$$

We now let $\tilde{\vartheta}^\alpha_a(t,r,\theta,z)$ be a global section in the global composite gauge $\hat{g}$ from before, such that

$$\vartheta^\alpha_a = p^\alpha_\beta\tilde{\vartheta}^\beta_a \tag{4.261}$$

From [20] we know the definition of the Fermi-Walker derivative of a vector and from this it is not hard to deduce the Fermi-Walker derivative of a 1-form

$$\nabla_{\text{F-W}}\vartheta_a = \nabla_T\vartheta_a + \vartheta_d g_{ac}\nabla_T T^c T^d - \vartheta_d g_{ac} T^c \nabla_T T^d \tag{4.262}$$

Since we are applying Fermi-Walker transport it follows that

$$\nabla_{\text{F-W}}\vartheta^\alpha_a = 0 \tag{4.263}$$

Using previous methods to find (4.35)we can derive a similar expression

$$p^\alpha_{\gamma,\theta} - p^\alpha_\mu\omega^\mu_{\gamma\theta} = \tilde{e}^a_\gamma\vartheta^\alpha_d g_{ac}(T^c\nabla_T T^d - T^d\nabla_T T^c) \tag{4.264}$$

In the same way we let

$${\vartheta^\flat}^\alpha_a = q^\alpha_\beta\tilde{\vartheta}^\beta_a \tag{4.265}$$

We find that

$$q^\alpha_{\beta,\theta} - q^\alpha_\gamma{\omega^\flat}^\gamma_{\beta\theta} = \tilde{e}^a_\beta{\vartheta^\flat}^\alpha_d g^\flat_{ac}(T^c\nabla^\flat_T T^d - T^d\nabla^\flat_T T^c) \tag{4.266}$$

Note that using Gronwall's inequality (see Appendix B.10) we see that even if p and q were elements of the full Lorentz group (as opposed to the subgroup $SO(3)$), they would still be bounded. From (4.258) we have

$$\begin{aligned}
\vartheta^{\alpha}_{a} &= h^{\alpha}_{\beta}{\vartheta^{\flat}}^{\beta}_{a} \\
\Longrightarrow p^{\alpha}_{\beta}\tilde{\vartheta}^{\beta}_{a} &= h^{\alpha}_{\beta}q^{\beta}_{\gamma}\tilde{\vartheta}^{\gamma}_{a} \\
\Longrightarrow p^{\alpha}_{\gamma} &= h^{\alpha}_{\beta}q^{\beta}_{\gamma} \qquad (4.267)
\end{aligned}$$

In coordinate free notation we write $h = pq^{-1}$. This implies

$$\begin{aligned}
\frac{dh}{d\theta} &= \frac{dp}{d\theta}q^{-1} + p\frac{dq^{-1}}{d\theta} \\
&= \frac{dp}{d\theta}q^{-1} - pq^{-1}\frac{dq}{d\theta}q^{-1} \\
&= p(\omega_\theta - \omega^{\flat}_{\theta})q^{-1} + g(\tilde{e},T)\omega_\theta\vartheta_\theta q^{-1} - g^{\flat}(\tilde{e},T)\omega^{\flat}_{\theta}\vartheta_\theta q^{-1} - g(\tilde{e},e\omega_\theta\vartheta_\theta)\vartheta_\theta q^{-1} \\
&\quad + g^{\flat}(\tilde{e},e\omega^{\flat}_{\theta}\vartheta_\theta)\vartheta_\theta q^{-1} \quad \text{(from (4.264) and (4.266))} \qquad (4.268)
\end{aligned}$$

Reverting back to using indices we have

$$\begin{aligned}
\frac{dh^{\alpha}_{\nu}}{d\theta} &= p^{\alpha}_{\gamma}(\omega^{\gamma}_{\mu\theta} - {\omega^{\flat}}^{\gamma}_{\mu\theta}){q^{-1}}^{\mu}_{\nu} + (g_{ac} - g^{\flat}_{ac})\tilde{e}^{a}_{\mu}{q^{-1}}^{\mu}_{\nu}\vartheta^{\gamma}_{\theta}{\omega^{\flat}}^{\beta}_{\gamma\theta}(T^{c}\delta^{\alpha}_{\beta} - e^{c}_{\beta}\vartheta^{\alpha}_{\theta}) \\
&\quad + g_{ac}\tilde{e}^{a}_{\mu}\vartheta^{\gamma}_{\theta}(\omega^{\beta}_{\gamma\theta} - {\omega^{\flat}}^{\beta}_{\gamma\theta})(T^{c}\delta^{\alpha}_{\beta} - e^{c}_{\beta}\vartheta^{\alpha}_{\theta}){q^{-1}}^{\mu}_{\nu} \qquad (4.269)
\end{aligned}$$

It can be shown that

$$g_{ab} - g^{\flat}_{ab} = \eta_{\alpha\beta}h^{\beta}_{\lambda}(h^{\alpha}_{\gamma} - \delta^{\alpha}_{\gamma}){\vartheta^{\flat}}^{\gamma}_{a}{\vartheta^{\flat}}^{\lambda}_{b} + \eta_{\alpha\beta}\delta^{\beta}_{\lambda}(h^{\alpha}_{\gamma} - \delta^{\alpha}_{\gamma}){\vartheta^{\flat}}^{\gamma}_{a}{\vartheta^{\flat}}^{\lambda}_{b} \qquad (4.270)$$

Hence, since $\tilde{e}$, ϑ, $\vartheta^{\flat}$, η, $\omega^{\flat}$, T, e, q^{-1} and δ are bounded we can rewrite (4.269) as

$$\frac{dh^{\alpha}_{\nu}}{d\theta} = p^{\alpha}_{\gamma}(\omega^{\gamma}_{\mu\theta} - {\omega^{\flat}}^{\gamma}_{\mu\theta}){q^{-1}}^{\mu}_{\nu} + C^{\gamma\alpha}_{\beta\nu}(\omega^{\beta}_{\gamma\theta} - {\omega^{\flat}}^{\beta}_{\gamma\theta}) + D^{\gamma\alpha}_{\beta\nu}(h^{\beta}_{\gamma} - \delta^{\beta}_{\gamma}) \qquad (4.271)$$

where $C^{\gamma\alpha}_{\beta\nu}$ and $D^{\gamma\alpha}_{\beta\nu}$ are bounded tensors. Integrating both sides and choosing the constant of integration to be the identity matrix I we have, on suppressing indices,

$$\begin{aligned}
h &= I + \int_{\phi=0}^{\theta} p(\omega^b - \omega)q^{-1} + C(\omega^b - \omega) + D(h - I)d\phi \\
\Longrightarrow h - I &= \int_{\phi=0}^{\theta} p(\omega^b - \omega)q^{-1}d\phi + \int_{\phi=0}^{\theta} C(\omega^b - \omega)d\phi \\
&\quad + \int_{\phi=0}^{\theta} D(h - I)d\phi \\
\Longrightarrow |h - I| &\leq \int_{\phi=0}^{\theta} |p(\omega^b - \omega)q^{-1}|d\phi + \int_{\phi=0}^{\theta} |C(\omega^b - \omega)|d\phi \\
&\quad + \int_{\phi=0}^{\theta} D|(h - I)|d\phi \\
&\leq c_1 \int_{\phi=0}^{\theta} |\omega^b - \omega|d\phi + c_2 \int_{\phi=0}^{\theta} |h - I|d\phi \\
&\quad \text{(Using Gronwall's Lemma)} \\
|h - I| &\leq c_1 e^{c_2\theta} \int_{\phi=0}^{\theta} |\omega^b - \omega|d\phi \\
&\leq c_3 \int_{\phi=0}^{\theta} |\omega^b - \omega|d\phi
\end{aligned}$$

Hence

$$\begin{aligned}
|h - I|^2 &\leq c_3^2 \left(\int_{\phi=0}^{\theta} |\omega^b - \omega|d\phi \right)^2 \\
&\leq 2\pi c_4 \int_0^{2\pi} |\omega^b - \omega|^2 d\theta \qquad \text{(Hölder's inequality)} \\
\Longrightarrow \int\int\int_{r=0}^{r_1} |h - I|^2 r dr dz dt &\leq 2\pi c_4 \|\omega^b - \omega\|_{L^2}^2 \\
\Longrightarrow \int\int\int_0^{2\pi}\int_{r=0}^{r_1} |h - I|^2 r dr d\theta dz dt &\leq 4\pi^2 c_4 \|\omega^b - \omega\|_{L^2}^2 \qquad \text{(Integrating WRT } \theta\text{)} \\
\Longrightarrow \|h - I\|_{L^2}^2 &\leq 4\pi^2 c_4 \|\omega^b - \omega\|_{L^2}^2 \\
\Longrightarrow \|h - I\|_{L^2} &\leq 2\pi c_4 \|\omega^b - \omega\|_{L^2} \\
\Longrightarrow \|\vartheta - \vartheta^b\|_{L^2} &\leq c_4 \|\omega^b - \omega\|_{L^2} \qquad \text{(from (4.260)} \\
\Longrightarrow \|\vartheta - \vartheta^b\|_{L^2} &\leq c_5 \|\Omega\|_{L^2} \qquad \text{(from Theorem 4.9)} \qquad (4.272)
\end{aligned}$$

We now return to the metrics

$$\begin{aligned}
\|g_{ab} - g^b_{ab}\|_{L^2} &= \|\eta_{\alpha\beta}\vartheta^\alpha_a\vartheta^\beta_b - \eta_{\alpha\beta}\vartheta^{b\,\alpha}_{\ a}\vartheta^{b\,\beta}_{\ b}\|_{L^2} \\
&= \|\eta_{\alpha\beta}\vartheta^\alpha_a\vartheta^\beta_b - \eta_{\alpha\beta}\vartheta^\alpha_a\vartheta^{b\,\beta}_{\ b} + \eta_{\alpha\beta}\vartheta^\alpha_a\vartheta^{b\,\beta}_{\ b} - \eta_{\alpha\beta}\vartheta^{b\,\alpha}_{\ a}\vartheta^{b\,\beta}_{\ b}\|_{L^2} \\
&\leq \|\eta_{\alpha\beta}\vartheta^\alpha_a\vartheta^\beta_b - \eta_{\alpha\beta}\vartheta^\alpha_a\vartheta^{b\,\beta}_{\ b}\|_{L^2} + \|\eta_{\alpha\beta}\vartheta^\alpha_a\vartheta^{b\,\beta}_{\ b} - \eta_{\alpha\beta}\vartheta^{b\,\alpha}_{\ a}\vartheta^{b\,\beta}_{\ b}\|_{L^2} \\
&\leq c_1\|\vartheta^\alpha_b - \vartheta^{b\,\alpha}_{\ b}\|_{L^2} + c_2\|\vartheta^\alpha_a - \vartheta^{b\,\alpha}_{\ a}\|_{L^2} \\
&\leq c_3\|\Omega\|_{L^2} \qquad (4.273)
\end{aligned}$$

Since we wish to estimate $||g - g^\flat||_{L^2_{2,\nabla^\flat}}$, the covariant derivatives are taken with respect to the connection of the conical metric. In order to simplify the calculations we make a change of coordinates to quasi-Cartesian coordinates so that $g^\flat_{ab} \stackrel{*}{=} \eta_{ab}$, the connection vanishes and the covariant derivative is simply the partial derivative. The effect of this coordinate change is to treat the conical space-time as being Minkowski space but with a sector of angle $2\pi(1 - A)$ missing (see Section 2.2).

We now determine inequalities for the first and second derivative terms making up the L^2_2 norm of $g - g^\flat$.

In quasi-Cartesian coordinates we have

$$||\omega||_{L^2} + ||\mathrm{d}\omega||_{L^2} \leq C||\Omega||_{L^2} \tag{4.274}$$

If an object is in L^2 then its components will also be in L^2 and so

$$||\mathbf{\Gamma}||_{L^2} + ||\mathrm{d}\mathbf{\Gamma}||_{L^2} \leq C||\Omega||_{L^2} \tag{4.275}$$

The metric is related to the connection $\mathbf{\Gamma}$ by the equation

$$g_{ab,c} = 2g_{d(a}\mathbf{\Gamma}^d_{\ b)c} \tag{4.276}$$

Hence taking line integrals we have

$$g_{ab} = g^0_{ab} + \int g_{da}\mathbf{\Gamma}^d_{bc}dx^c + \int g_{db}\mathbf{\Gamma}^d_{ac}dx^c \tag{4.277}$$

where g^0_{ab} is a constant of integration. Taking the sup norm of both sides shows us that

$$|g_{ab}| \leq |g^0_{ab}| + 2\int |g_{da}||\mathbf{\Gamma}^d_{bc}|dx^c \tag{4.278}$$

Using the equivalence of norms, we can now let the norm be the magnitude norm defined in Section 2.6. We can then apply Gronwall's inequality (see Appendix B.10) and hence

$$|g_{ab}| \leq |g^0_{ab}| \exp\left(2\int |\mathbf{\Gamma}^d_{bc}|dx^c\right) \tag{4.279}$$

Since we are taking norms locally we know that

$$
\begin{aligned}
\mathbf{\Gamma} &\in L^2_{1,\mathrm{loc}} \\
\Longrightarrow \mathbf{\Gamma} &\in L^1_{,\mathrm{loc}} \\
\Longrightarrow \int |\mathbf{\Gamma}^d_{bc}| dx^c &< \infty \\
\Longrightarrow \exp\left(2\int |\mathbf{\Gamma}^d_{bc}| dx^c\right) &< \infty \\
\Longrightarrow |g_{ab}| &< \infty
\end{aligned}
\tag{4.280}
$$

From (4.276) we now have

$$
\begin{aligned}
|g_{ab,c}| &= 2|g_{d(a}\mathbf{\Gamma}^d_{b)c}| \\
&\leq c|\mathbf{\Gamma}^d_{bc}| \\
\Longrightarrow \|\mathrm{d}g\|_{L^2} &\leq c\|\mathbf{\Gamma}\|_{L^2} \\
\Longrightarrow \|\mathrm{d}g\|_{L^2} &\leq C\|\Omega\|_{L^2}
\end{aligned}
\tag{4.281}
$$

Now we establish an inequality for the second derivative of g.

$$
\begin{aligned}
g_{ab,c} &= g_{da}\mathbf{\Gamma}^d_{bc} + g_{db}\mathbf{\Gamma}^d_{ac} \\
|g_{ab,ce}| &= \left|g_{da,e}\mathbf{\Gamma}^d_{bc} + g_{da}\mathbf{\Gamma}^d_{bc,e} + g_{db,e}\mathbf{\Gamma}^d_{ac} + g_{db}\mathbf{\Gamma}^d_{ac,e}\right| \\
&= |(g_{fd}\mathbf{\Gamma}^f_{ae} + g_{fa}\mathbf{\Gamma}^f_{de})\mathbf{\Gamma}^d_{bc} + g_{da}\mathbf{\Gamma}^d_{bc,e} \\
&\quad +(g_{fd}\mathbf{\Gamma}^f_{be} + g_{fb}\mathbf{\Gamma}^f_{de})\mathbf{\Gamma}^d_{ac} + g_{db}\mathbf{\Gamma}^d_{ac,e}|
\end{aligned}
\tag{4.282}
$$

Since $|g|$ is bounded we know from (4.275) that

$$
\|g\partial\mathbf{\Gamma}\|_{L^2} \leq c_1\|\partial\mathbf{\Gamma}\|_{L^2} \leq C_1\|\Omega\|_{L^2} \tag{4.283}
$$

Again since $|g|$ is bounded and applying the sub-multiplicativity property of (4.247) we have

$$
\|g\mathbf{\Gamma}^2\|_{L^2} \leq c_2\|\mathbf{\Gamma}^2\|_{L^2} \leq c_3\|\mathbf{\Gamma}\|_{L^2_1}\|\mathbf{\Gamma}\|_{L^2_1} \leq C_2\|\Omega\|^2_{L^2} \leq C_3\|\Omega\|_{L^2} \tag{4.284}
$$

for a small enough region around the singularity. Hence

$$
\|\mathrm{d}^2 g\|_{L^2} \leq C_4\|\omega\|_{L^2} \tag{4.285}
$$

and finally

$$\|g - g^\flat\|_{L^2_{2,\nabla^\flat}(N_0)} \leq C\|\Omega\|_{L^2(N_0)} \tag{4.286}$$

as required.

Chapter summary

In this chapter we discussed the details of the conditions for a weak curvature singularity in order that we might find analogous theorems in General Relativity to those found in Chapter 3. As well as the connection 1-form being in $L^2_{1,\mathrm{loc}}(X)$ and the curvature being in $L^2(N)$, we imposed the additional condition that the space-time is static. This has the effect that the Lorentz transformations are restricted to rotations (see the discussion below) and hence the gauge group is $SO(3)$. We discussed properties of holonomy and then constructed the locally flat connections which in the GR case are given by the four-dimensional cones.

We stated and proved the GR analogue of the two main theorems from Sibner and Sibner [39]. These theorems state that

1. Limiting holonomy exists for weak curvature singularities and is identical at all locations on the singularity.
2. As r tends to 0, the $L^2_{1,\omega^\flat}(N_0)$ norm of the difference between the connection ω and the conical connection $\omega^\flat$ is bounded by the $L^2(N_0)$ norm of the curvature.

The proof of Theorem 3.18, in particular, is complex and required considerable patience and time to complete due to the brevity and inaccuracy of the proof offered in [39].

In the GR case there were two new issues which required additional attention. Firstly, at the end of the proof of Theorem 4.8 we needed to show that the existence and invariance of the limit holonomy meant that the axis of the rotation of the holonomy agreed with the direction of the singularity. Secondly, in GR the connection is given by the metric. We were able to use the results of Theorem 4.9 to show that the metric of the weak curvature singularity is close to that of a conical singularity. More precisely we showed that

3. As r tends to 0, the $L^2_{2,\nabla^\flat}(N_0)$ norm of the difference between the metric g and the conical metric $g^\flat$ is bounded by the $L^2(N_0)$ norm of the curvature.

This property is required for our work in the next chapter.

We now offer two discussions on issues relating to this chapter.

Using the full Lorentz group

Earlier in this chapter we explained that it is necessary to take a Lie subgroup $SO(3)$ of the Lorentz group as the gauge group for our bundle space. The reason behind this is that it is not certain whether or not boundedness of the Lie group elements is required in order for the main results of this chapter to hold. For example, in our proof of (4.78) we certainly require G to be bounded.

In the case of a 2-dimensional and timelike quasi-regular singularity, where the full Lorentz group is used as the gauge group, the holonomy is a rotation and not a boost. So one might expect our theorems to apply to 2-dimensional and timelike square integrable singularities with the Lorentz gauge group, since we would also expect holonomy to be a rotation and hence bounded. In which case the G from (4.78) is bounded anyway.

Use of the Lorentz group could allow for the consideration of non-static space-times and hence a considerably larger scope for application of Theorem 4.8 and Theorem 4.9 to General Relativity.

However, even if it was not possible to show that the elements of the group remained bounded, it might be possible to prove Theorem 4.8 and Theorem 4.9 with a weaker condition.

An alternative proof?

In [39] the Sibners offer an alternative proof for (the Yang-Mills analogue of) Theorem 4.9 using a separate theorem of Taubes. The idea they propose is to go directly to the critical Sobolev case of $p = 2$ in Theorem 4.9 and hence bypass some of the later lemmas in their paper as well as Proposition 4.10 and Corollary 4.11 (since these are concerned with lowering $p > 2$ to $p = 2$). However, the method relies on the application of Theorem 4.16 which we have deduced can only be applied in relation to L^p norms of Ω where $p > 2$. This ultimately comes from the fact that $u \in L^\infty$ only if $u \in L_2^q$ for $q > 2$ (Sobolev's lemma).

For the alternate proof to work we would perhaps have to approximate $u \in L_2^2$ by a sequence of smooth $u \in L_2^q$ in a similar fashion to the method shown to derive Theorem 4.9 from Corollary 4.11.

Chapter 5

Distributional Curvature

In this chapter we shall discuss how the findings of the previous chapter, regarding the metric of a weak curvature singularity, allow us to calculate the distributional curvature of the singularity. We have shown that the L_2^2 norm of the difference between the metric of a conical singularity in four dimensions (4-cone) and the metric of a weak singularity tends to zero on approach to the singularity. We therefore expect that the distributional curvature and hence also the energy momentum tensor of these two classes of singularity will show a similar relationship.

As previously noted, the 4-cone is a relatively simple example of a quasi-regular singularity. We showed in Chapter 2 a heuristic method used in [49] to find the distributional curvature of a 4-cone, by calculating the holonomy around the two dimensional singularity. Clarke, Vickers and Wilson have shown a rigorous method [7, 55] to verify the value of the distributional curvature. The process used (and summarised in this chapter) is to regard the singularity as a distributional solution to Einstein's equations by using Colombeau's non-linear theory of generalised functions [8, 9] to describe the space-time geometry.

From the distributional curvature of the 4-cone found using the holonomy method, it follows that the related energy-momentum tensor is of the form

$$T_t^t = T_z^z = \pi\mu\delta^{(2)} \tag{5.1}$$

where $\delta^{(2)}$ is a two-dimensional delta function with compact support, $\mu = \beta/\left(4G(2\pi - \beta)\right)$, G is the gravitational constant and β is the rotation provided by the holonomy. The energy-momentum of the form in (5.1) is precisely the form of the energy momentum tensor of a cosmic string in the thin string limit. We shall explain what this means below.

Cosmic strings are considered to be topological defects in a space-time, formed when different regions of space-time undergo thermodynamic phase transitions, resulting in domain boundaries between the two regions when they meet. They have immense density and so represent significant gravitational sources.

Cosmic strings, for which the thickness of the string tends to zero, can be modeled by 2-dimensional, timelike quasi-regular singularities (for example, the 4-cone, as shown above). This 'thin string' model is not an exact physical model of a cosmic string since they do have a small but non-zero thickness. However, it has been shown that the thin string limit is a good estimate since almost all of the matter is confined to a region with the thickness of the Higgs Compton wavelength [24, 52]. Further validation that the thin string model is an accurate estimate can be found in [49].

By looking at a suitable class of weak curvature singularities instead of quasi-regular singularities we find that we get the same energy-momentum tensor from a distributional point of view. Hence weak singularities also display properties of cosmic strings.

In Section 5.1 we review the methods used to construct the generalised Colombeau algebra [7]. In Section 5.2 we review previous successes in finding the distributional curvature of a standard conical singularity [7, 42] and also for variations on the four-dimensional cone [55]. In Section 5.3 we extend this work to find the distributional curvature of a weak curvature singularity.

5.1 Construction of the Colombeau algebra

In this section we include the full technical details of the construction of the Colombeau algebra and then briefly summarise the process and its application.

We start by defining the following sets of functions.

$$\begin{aligned}
\Phi \in \mathcal{D}(\mathbb{R}^n) &\iff \Phi \in \mathcal{C}^\infty \text{ and } \Phi \text{ has compact support} \\
\Phi \in \mathcal{A}_0(\mathbb{R}^n) &\iff \Phi \in \mathcal{D}(\mathbb{R}^n) \text{ and } \int_{\mathbb{R}^n} \Phi(\mathbf{x})d\mathbf{x} = 1 \\
\Phi \in \mathcal{A}_q(\mathbb{R}^n) &\iff \Phi \in \mathcal{A}_0(\mathbb{R}^n) \text{ and } \int_{\mathbb{R}^n} \Phi(\mathbf{x})\mathbf{x}^{\mathbf{i}}d\mathbf{x} = 0 \quad (1 \leq \mathbf{i} \leq q)
\end{aligned} \tag{5.2}$$

where $\mathbf{i}$ is a multi-index such that $\mathbf{x}^{\mathbf{i}} = (x^1)^{i_1}(x^2)^{i_2}\ldots(x^n)^{i_n}$ where $i_k \in \mathbb{N}$ and $|\mathbf{i}| = i_1 + i_2 + \ldots + i_k$. We define the n-dimensional δ-function as having the property

$$\int_{\mathbb{R}^n} \delta^{(n)}(\mathbf{x})d\mathbf{x} = 1 \tag{5.3}$$

Let $\Phi \in \mathcal{A}_0(\mathbb{R}^n)$, then the ϵ-parametrised function $\Phi_\epsilon(\mathbf{x}) = \frac{1}{\epsilon^n}\Phi(\frac{\mathbf{x}}{\epsilon}) \quad (0 < \epsilon < 1)$ is known as a model δ-net and is also in $\mathcal{A}_0(\mathbb{R}^n)$, although with a smaller support and larger amplitude than Φ. For a locally integrable function $f : \mathbb{R}^n \to \mathbb{C}$ we can look at its smoothed function $\tilde{f}$ such that

$$\tilde{f}(\Phi_\epsilon, \mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{x}+\mathbf{u})\Phi_\epsilon(\mathbf{u})d\mathbf{u} = \int_{\mathbb{R}^n} f(\mathbf{u})\Phi_\epsilon(\mathbf{u}-\mathbf{x})d\mathbf{u} \tag{5.4}$$

Hence

$$\tilde{f}(\Phi_\epsilon, \mathbf{x}) = \frac{1}{\epsilon^n}\int f(\mathbf{x}+\mathbf{u})\Phi\left(\frac{\mathbf{u}}{\epsilon}\right)d\mathbf{u} \tag{5.5}$$

Letting $\frac{\mathbf{u}}{\epsilon} = \mathbf{v}$ we have

$$\tilde{f}(\Phi_\epsilon, \mathbf{x}) = \int f(\mathbf{x}+\epsilon\mathbf{v})\Phi(\mathbf{v})d\mathbf{v} \tag{5.6}$$

In the case where f is continuous we therefore have

$$\lim_{\epsilon\to 0}\tilde{f}(\Phi_\epsilon, \mathbf{x}) = \int f(\mathbf{x})\Phi(\mathbf{v})d\mathbf{v} = f(\mathbf{x}) \tag{5.7}$$

which justifies the use of the term delta-net to describe Φ_ϵ.

Definition 5.1 *The function R is an element of the algebra $\mathcal{E}(\mathbb{R}^n)$ iff*

$$\begin{aligned} R : \mathcal{A}_0(\mathbb{R}^n)\times\mathbb{R}^n &\to \mathbb{C} \\ (\Phi, \mathbf{x}) &\mapsto R(\Phi, \mathbf{x}) \end{aligned}$$

and for fixed Φ, $B_\Phi \in C^\infty$ where $\Phi \in \mathcal{A}_0(\mathbb{R}^n)$ and

$$\begin{aligned} B_\Phi : \mathbb{R}^n &\to \mathbb{C} \\ \mathbf{x} &\mapsto B_\Phi(\mathbf{x}) = R(\Phi, \mathbf{x}) \end{aligned}$$

Definition 5.2 *The function R is an element of the moderate subalgebra $\mathcal{E}_M(\mathbb{R}^n)$ of $\mathcal{E}(\mathbb{R}^n)$ iff for all $K \subset\subset \mathbb{R}^n$ and for all $\mathbf{i} \in \mathbb{N}^n$, there is some $N \in \mathbb{N}$ such that: If $\Phi \in \mathcal{A}_N(\mathbb{R}^n)$ there exists $c, \eta > 0$ such that*

$$\sup_{\mathbf{x}\in K} |\partial_{\mathbf{x}}^{\mathbf{i}} R(\Phi_\epsilon, \mathbf{x})| \leq c\epsilon^{-N} \qquad (0 < \epsilon < \eta) \tag{5.8}$$

Note that ϵ^{-N} increases as a polynomial of $\frac{1}{\epsilon}$, not an exponential. A feature of this subalgebra is that if $R_1, R_2 \in \mathcal{E}_M(\mathbb{R}^n)$ then the product R_1R_2 does not coincide with usual multiplication of C^∞ functions. If we have a smooth function f, we can map this into an element of $\mathcal{E}_M(\mathbb{R}^n)$ in two ways. We can leave it unchanged by using

$$\hat{f}(\Phi_\epsilon, \mathbf{x}) = f(\mathbf{x}) \tag{5.9}$$

or smooth the function by using

$$\tilde{f}(\Phi_\epsilon, \mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{x}+\mathbf{u})\Phi_\epsilon(\mathbf{u})d\mathbf{u} \qquad \text{(a generalised function)} \tag{5.10}$$

where $\tilde{f}$ and $\hat{f}$ are both in $\mathcal{E}_M(\mathbb{R}^n)$ and $\Phi_\epsilon \in \mathcal{A}_q(\mathbb{R}^n)$. However, on one hand we have

$$\hat{f}\hat{g}(\Phi, \mathbf{x}) := \hat{f}(\Phi, \mathbf{x}) \cdot \hat{g}(\Phi, \mathbf{x}) = \widehat{(fg)}(\Phi, \mathbf{x}) \tag{5.11}$$

and on the other

$$\tilde{f}\tilde{g}(\Phi, \mathbf{x}) := \tilde{f}(\Phi, \mathbf{x}) \cdot \tilde{g}(\Phi, \mathbf{x}) \neq \widetilde{(fg)}(\Phi, \mathbf{x}) \tag{5.12}$$

We would like to be using $\tilde{f}$ as our approximation but we would also like the multiplicativity property of (5.11). We will bunch together the two representations by letting $\tilde{f} \sim \hat{f}$ for all Φ_ϵ so there is just one regularisation of f given by the equivalence class $[\hat{f}]$, which we will show belongs to a differential algebra, namely Colombeau's generalised function algebra [8, 9].

Before constructing our equivalence relation we must first look at the ideal $\mathcal{N}(\mathbb{R}^n)$ of the subalgebra $\mathcal{E}_M(\mathbb{R}^n)$.

Definition 5.3 *The function R is an element of the ideal $\mathcal{N}(\mathbb{R}^n)$ iff for all $K \subset\subset \mathbb{R}^n$ and for all $\mathbf{i} \in \mathbb{N}^n$, there is some $N \in \mathbb{N}$ and some increasing and unbounded sequence $\{\gamma_q\}_{q\in\mathbb{N}}$ such that: If $\Phi \in \mathcal{A}_q(\mathbb{R}^n)$, for $q \geq N$, there exists $c, \eta > 0$ such that*

$$\sup_{\mathbf{x}\in K} |\partial_{\mathbf{x}}^{\mathbf{i}} R(\Phi_\epsilon, \mathbf{x})| \leq c\epsilon^{\gamma_q - N} \qquad (0 < \epsilon < \eta) \tag{5.13}$$

The important point here is that for sufficiently large q, $\epsilon^{\gamma_q - N}$ tends to zero.

Proof that $\mathcal{N}(\mathbb{R}^n)$ is an ideal

If $g \in \mathcal{N}(\mathbb{R}^n)$ then g and all its derivatives are less than or equal to $c_1\epsilon^{\gamma_q - N}$. If $f \in \mathcal{E}_M$ then f and all its derivatives are less than or equal to $c_2\epsilon^{-N}$. Hence

$$\begin{aligned} fg \leq c_1 c_2 \epsilon^{\gamma_q - N}\epsilon^{-N} &= c_3\epsilon^{\gamma_q - 2N} \\ (fg)' = f'g + fg' \leq c_1c_2\epsilon^{\gamma_q - 2N} + c_1c_2\epsilon^{\gamma_q-2N} &= 2c_3\epsilon^{\gamma_q - 2N} \\ (fg)'' = (f'g + fg')' = f''g + 2f'g' + fg'' &\leq 4c_3\epsilon^{\gamma_q-2N} \quad \text{etc.} \end{aligned} \tag{5.14}$$

So letting $2N = N'$ we see that if $g \in \mathcal{N}(\mathbb{R}^n)$ and $f \in \mathcal{E}_M(\mathbb{R}^n)$ then $fg \in \mathcal{N}(\mathbb{R}^n)$ which is the property of $\mathcal{N}(\mathbb{R}^n)$ being an ideal. □

We now wish to show that $\tilde{f} - \hat{f} \in \mathcal{N}(\mathbb{R}^n)$. Let us look at the 1-D case, which has a result analogous to the higher dimensional result.

$$\begin{aligned} \tilde{f} - \hat{f} &= \int_{\mathbb{R}} f(x+u)\Phi_\epsilon(u)du - f(x) \\ &= \int_{\mathbb{R}} \Bigg(f(x) + uf'(x) + \frac{u^2}{2}f''(x) + \dots \\ &\qquad \dots + \frac{u^{q+1}}{(q+1)!}f^{(q+1)}(x+\theta u)\Bigg) \Phi_\epsilon(u)du - f(x) \qquad (0<\theta<1) \\ &= \int_{\mathbb{R}} f(x)\Phi_\epsilon(u)du + \int_{\mathbb{R}} \frac{u^{q+1}}{(q+1)!}f^{(q+1)}(x+\theta u)\Phi_\epsilon(u)du - f(x) \qquad \text{since } \phi \in A_q(\mathbb{R}^n) \\ &= f(x) + \int_{\mathbb{R}} \frac{u^{q+1}}{(q+1)!}f^{(q+1)}(x+\theta u)\Phi_\epsilon(u)du - f(x) \\ &= \int_{\mathbb{R}} \frac{u^{q+1}}{(q+1)!}f^{(q+1)}(x+\theta u)\frac{1}{\epsilon}\Phi\left(\frac{u}{\epsilon}\right)du \\ &= \epsilon^{q+1}\int_{\mathbb{R}} \frac{v^{q+1}}{(q+1)!}f^{(q+1)}(x+\epsilon\theta v)\Phi(v)dv \qquad \text{letting } v = \frac{u}{\epsilon} \\ &= \epsilon^{q+1+N-N}\int_{\mathbb{R}} \frac{v^{q+1}}{(q+1)!}f^{(q+1)}(x+\epsilon\theta v)\Phi(v)dv \\ &\leq c\epsilon^{\gamma_q - N} \qquad \text{where } \gamma_q = q+1+N \end{aligned} \tag{5.15}$$

Since $\Phi(v)$ is of compact support and the function is finite everywhere, the integrand on the penultimate line is bounded by a constant. From (5.15) we see that $\tilde{f} - \hat{f} \in \mathcal{N}(\mathbb{R}^n)$.

We now define our equivalence relation.

Definition 5.4 *Let R_1 and R_2 be in $\mathcal{E}_M(\mathbb{R}^n)$, then $R_1 \sim R_2$ iff $R_1 - R_2 \in \mathcal{N}(\mathbb{R}^n)$.*

From the previous result we get that $\tilde{f} \sim \hat{f}$ as required. The equivalence classes formed under this relation are elements of the quotient set $\frac{\mathcal{E}_M(\mathbb{R}^n)}{\mathcal{N}(\mathbb{R}^n)}$. Let us look at two elements of $\frac{\mathcal{E}_M(\mathbb{R}^n)}{\mathcal{N}(\mathbb{R}^n)}$ which we shall call $[R_1]$ and $[R_2]$. Let $a, b \in \mathcal{E}_M(\mathbb{R}^n)$ such that $a \in [R_1]$ and $b \in [R_2]$ and let $n_i \in \mathcal{N}(\mathbb{R}^n)$ $\forall i \in \mathbb{N}$. Then $a = R_1 - n_1$ and $b = R_2 - n_2$. So

$$ab = (R_1 - n_1)(R_2 - n_2) = R_1R_2 - R_1n_2 - n_1R_2 + n_1n_2 \tag{5.16}$$

Since $n_1, n_2 \in \mathcal{N}(\mathbb{R}^n)$ and $R_1, R_2 \in \mathcal{E}_M(\mathbb{R}^n)$ we have $R_1n_2, n_1R_2 \in \mathcal{N}(\mathbb{R}^n)$ and also $n_1n_2 \in \mathcal{N}(\mathbb{R}^n)$ since $\mathcal{N}(\mathbb{R}^n) \subset \mathcal{E}_M(\mathbb{R}^n)$. Hence

$$ab = R_1R_2 - n_3 \iff R_1R_2 - ab = n_3 \iff R_1R_2 \sim ab \tag{5.17}$$

Therefore if $a \in [R_1]$ and $b \in [R_2]$ then $ab \in [R_1R_2]$ and so we still have the required closure group property. We let $\frac{\mathcal{E}_M(\mathbb{R}^n)}{\mathcal{N}(\mathbb{R}^n)} = \mathcal{G}$, Colombeau's generalised function algebra. $\mathcal{G}$ is a differential algebra which means we can differentiate and take products of elements in a well defined manner. To find the generalised function corresponding to the product of two distributions or a distribution and a continuous function, we take the equivalence class of the product of the two smoothed distributions or the product of the smoothed distribution and the smoothed continuous function respectively. Hence, if we can express the metric as a generalised function (an element of $\mathcal{G}$) then we can express the curvature as a generalised function.

Later in this section we shall be interested in generalised functions which in some way correspond to distributions. This correspondence shall be defined using a notion of *weak equivalence* which we shall now derive, following the program outlined in [7].

We would like to understand what it means to take the integral of a generalised function. In an analogous way to that done before we will first define a new algebra, sub-algebra and ideal space which are in effect pointwise values of the spaces $\mathcal{E}(\mathbb{R}^n)$, $\mathcal{E}_M(\mathbb{R}^n)$ and $\mathcal{N}(\mathbb{R}^n)$ respectively.

Definition 5.5 *The algebra $\mathcal{E}$ is defined to be the set of functions*

$$\beta : \mathcal{A}_1(\mathbb{R}^n) \to \mathbb{C} \tag{5.18}$$

Definition 5.6 *The subalgebra $\mathcal{E}_M$ is the set of all $\beta \in \mathcal{E}$ such that there exists an $N \in \mathbb{N}$ such that if $\Phi \in \mathcal{A}_N$ then there exists $c, \eta > 0$ such that*

$$|\beta(\Phi^\epsilon)| \leq c\epsilon^{-N} \qquad (0 < \epsilon < \eta) \tag{5.19}$$

Each element $R \in \mathcal{E}_M(\mathbb{R}^n)$ is, for a test function Φ, a representative for a corresponding generalised function $G = [R(\Phi, \mathbf{x})] \in \mathcal{G}(\mathbb{R}^n)$. The function

$$\begin{aligned} \rho_R : \mathcal{A}_1 &\to \mathbb{C} \\ \Phi &\mapsto \int_{\mathbb{R}^n} R(\Phi, \mathbf{x}) d\mathbf{x} \end{aligned} \tag{5.20}$$

is an element of $\mathcal{E}_M$. If R_1 and R_2 are representatives for the same G, then ρ_{R_1} and ρ_{R_2} differ by a function belonging to the ideal $\mathcal{I}$ of the algebra $\mathcal{E}_M$.

Definition 5.7 *The ideal $\mathcal{I}$ of the algebra $\mathcal{E}_M$ is the set of functions ρ such that there is some $N \in \mathbb{N}$ and some increasing and unbounded sequence $\{\gamma_q\}_{q \in \mathbb{N}}$ such that if $\Phi \in \mathcal{A}_q(\mathbb{R}^n)$ for $q \geq N$ then there exists $c, \eta > 0$ such that*

$$|\rho(\Phi^\epsilon)| \leq c\epsilon^{\gamma_q - N} \qquad (0 < \epsilon < \eta) \tag{5.21}$$

The equivalence class obtained from the quotient of $\mathcal{E}_M$ and $\mathcal{I}$ is independent of which representative we take of the generalised function G and we think of this quotient as being the value for the integral of G. We define elements of this quotient to be generalised complex numbers.

Definition 5.8 *The algebra of generalised numbers is defined to be*

$$\bar{\mathbb{C}} = \frac{\mathcal{E}_M}{\mathcal{I}}$$

For any $z \in \mathbb{C}$ we can look at the constant function $\rho_z(\Phi) = z$. The equivalence class $[\rho_z]$ of this function is a generalised complex number. Hence for any classical complex number we can associate a generalised complex number.

Definition 5.9 *$\bar{z} \in \bar{\mathbb{C}}$ is associated to $z \in \mathbb{C}$ if there is a representative $\rho \in \mathcal{E}_M$ of $\bar{z}$ such that for $\Phi \in A_q(\mathbb{R}^n)$ and with a large enough $q \in \mathbb{N}$,*

$$\lim_{\epsilon \to 0} \rho(\Phi_\epsilon) = z \tag{5.22}$$

If $\bar{z} \in \bar{\mathbb{C}}$ is associated to $z \in \mathbb{C}$ then we write $\bar{z} \vdash z$. Note that if $\bar{z} \vdash z$ then it is not necessary that $\bar{z}$ is the equivalence class $[\rho_z]$. We now define association between two generalised complex numbers.

Definition 5.10 *$\bar{z}_1, \bar{z}_2 \in \bar{\mathbb{C}}$ are associated to each other if and only if $\bar{z}_1 - \bar{z}_2 \vdash 0 \in \mathbb{C}$.*

In a similar way we can now define weak equivalence (or association) between elements of the Colombeau algebra.

Definition 5.11 *If $G_1, G_2 \in \mathcal{G}(\mathbb{R}^n)$ then G_1 is weakly equivalent to G_2 if and only if for each $\Psi \in \mathcal{D}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} (G_1(\mathbf{x}) - G_2(\mathbf{x}))\, \Psi(\mathbf{x}) d\mathbf{x} \vdash 0 \in \mathbb{C} \tag{5.23}$$

Weak equivalence between G_1 and G_2 is written $G_1 \approx G_2$.

Finally we say that a generalised function G corresponds to a distribution T if $G \approx \tilde{T}$.

The key point about a Colombeau algebra is that it takes a non-smooth function $f(\mathbf{x})$ and embeds it *uniquely* into the algebra as a smoothed function $\tilde{f}(\mathbf{x}, \Phi_\epsilon)$. However, in doing so, the smoothed function is no longer simply a function of $\mathbf{x}$, but also of the ϵ-parameterised kernel, Φ_ϵ. For fixed Φ, $\tilde{f}$ is a 1-parameter family of smoothed functions. Since the embedded objects are smooth we can now perform calculations involving differentiation, multiplication and addition with other elements of the Colombeau algebra and still have a resulting answer in the algebra.

To return the function from the algebra to the set of distributions we use association. We consider how the smoothed function acts on a test function $\Psi \in \mathcal{D}$. If the limit as $\epsilon \to 0$ of this integral is independent of Φ, but agrees with the action of some distribution T acting on Ψ, then we say that the algebra element is associated to that distribution.

5.2 Distributional curvature of a conical singularity

Many alternative methods have been used to determine information about cosmic strings. Vilenkin [50] deduced, using a weak field theory, that the exterior metric of a cosmic string is the 4-dimensional conical metric. Gott [19] and Linet [28] came to the same conclusion by starting with the conditions on the energy-momentum tensor "$T^t_t = T^z_z$ and $T^a_b = 0$ for other components" and deducing the exterior metric which gave rise to such conditions. Vilenkin [51] and Vickers [49] used the Nambu action and then the four dimensional Lagrangian action to calculate the energy-momentum density of the string, but from this it is quite hard to obtain the metric.

Using the Colombeau algebra, Clarke, Vickers and Wilson [7, 55] found the distributional curvature for a conical singularity, determined a precise energy-momentum tensor and hence calculated the mass per unit length of a cosmic string. Wilson [55] has also applied these methods to other quasi-regular singularities including conical non-flat singularities (for specifically behaving curvature) and also conical singularities with variable angular deficit.

In previous chapters we have shown that a suitable class of weak curvature singularities asymptotically agree with conical singularities as $r \to 0$. Hence we expect the singular part of the energy-momentum tensor provided by a weak curvature singularity to be associated to the distributional energy-momentum tensor for a cosmic string.

We start by reviewing the calculation of the distributional curvature of a cone. Following the method in [55], we determine distributional curvature (density), first for a 2-D cone and then for a 4-D cone. We first notice that in both cases, since the Geroch-Traschen conditions [18] are not satisfied we cannot immediately recognise that R_{abcd} can be interpreted as a distribution.

Since the cone does not satisfy the GT-regularity conditions, we cannot calculate the distributional curvature directly. One approach is to replace $g^\flat_{ab}$ by a family of smooth metrics $g^\flat_{ab\epsilon}$. We could now try to calculate the curvature R_{abcd_ϵ} of $g^\flat_{ab_\epsilon}$ for fixed ϵ and then take the limit of R_{abcd_ϵ} as $\epsilon \to 0$, as has been attempted in [2, 18, 29]. By making a suitable choice of smoothing, the limiting curvature can be used to recover, for example, the Ricci scalar

$$R = \lim_{\epsilon \to 0} \tilde{R}_\epsilon \tag{5.24}$$

However, it has been demonstrated that this type of method is only applicable in certain situations [18] and in the case of the 4-cone ultimately yields different answers for the mass per unit length of the cosmic string.

The advantage of using the Colombeau approach is that one has a canonical embedding of non-smooth objects into the algebra. We embed the metric into the Colombeau algebra by taking a representative of the family of smoothed metrics (given by an equivalence class). In the algebra we can now perform the necessary calculations to find a representative of the curvature density. To get back to the level of distributions we are then required to use the relationship of weak equivalence (or association). We now outline the calculation for the case of a two dimensional cone (see [55] for details). We start by looking at the metric for the 2-cone in Cartesian coordinates. This avoids any confusion which may arise when distinguishing between coordinate and actual singularities. We then split the metric into singular and non singular parts

$$g^{\flat}_{ab} = \frac{1}{2}(1-A^2)h_{ab} + \frac{1}{2}(1+A^2)\delta_{ab} \tag{5.25}$$

where

$$h_{ab} = \begin{pmatrix} \frac{x^2-y^2}{x^2+y^2} & \frac{2xy}{x^2+y^2} \\ \frac{2xy}{x^2+y^2} & \frac{y^2-x^2}{x^2+y^2} \end{pmatrix} \tag{5.26}$$

we then smooth $g^{\flat}_{ab}$ using a model δ-net $\Phi \in \mathcal{A}_1(\mathbb{R}^2)$ to find the family of smooth metrics

$$\tilde{g}^{\flat}_{ab_\epsilon}(x,y) = \int_{\mathbb{R}^2} g^{\flat}_{ab}(\lambda,\mu)\Phi_\epsilon(\lambda - x, \mu - y)d\lambda d\mu \tag{5.27}$$

where Φ has a radius of

$$R_0 = \sup\left\{ (x^2+y^2)^{\frac{1}{2}} \,\middle|\, |\Phi(x,y)| > 0 \right\} \tag{5.28}$$

Due to the nature of Φ, the smoothed metric will belong to the algebra $\mathcal{E}_M(\mathbb{R}^2)$. We note that

$$\tilde{g}^{\flat}_{ab_\epsilon} = \frac{1}{2}(1+A^2)\delta_{ab} + \frac{1}{2}(1-A^2)h_{ab_\epsilon} \tag{5.29}$$

and so we need only evaluate $\tilde{h}_{ab_\epsilon}$. Wilson shows that

$$\begin{aligned} \tilde{h}_{ab_\epsilon} &= \begin{pmatrix} C_4 + C_6\frac{x}{\epsilon} + C_7\frac{y}{\epsilon} & C_5 + C_8\frac{x}{\epsilon} + C_9\frac{y}{\epsilon} \\ C_5 + C_8\frac{x}{\epsilon} + C_9\frac{y}{\epsilon} & -C_4 - C_6\frac{x}{\epsilon} - C_7\frac{y}{\epsilon} \end{pmatrix} + O\left(\frac{x^2+y^2}{\epsilon^2}\right) \quad (r < \epsilon R_0) \\ \tilde{h}_{ab_\epsilon} &= \begin{pmatrix} \frac{x^2-y^2}{x^2+y^2} & \frac{2xy}{x^2+y^2} \\ \frac{2xy}{x^2+y^2} & \frac{y^2-x^2}{x^2+y^2} \end{pmatrix} + O\left(\frac{\epsilon^{q+1}}{(x^2+y^2)^{\frac{q+1}{2}}}\right) \quad (r > \epsilon R_0) \end{aligned} \tag{5.30}$$

where the C_k are constants.

Since $\tilde{g}^{\flat}_{ab_\epsilon}$ is as an element of $\mathcal{E}_M(\mathbb{R}^2)$, we can embed into the Colombeau algebra $\mathcal{G}$ by taking the representative $[\tilde{g}^{\flat}_{ab}] \in \mathcal{G}$ as described in Section 5.1. Working in the Colombeau algebra, we can now calculate the representative Ricci scalar $\tilde{R}_\epsilon$ from $\tilde{g}^{\flat}_{ab_\epsilon}$ in the usual manner. Note that we wish to regard the dirac delta $\delta^{(2)}$ as a scalar density (so that it is dual to a function) and we therefore choose to calculate the Ricci scalar density $\tilde{R}_\epsilon\sqrt{\tilde{g}^{\flat}_\epsilon}$ rather than the Ricci scalar. The Ricci scalar density $\tilde{R}_\epsilon\sqrt{\tilde{g}^{\flat}_\epsilon}$ gives a representative of the equivalence class $[\tilde{R}\sqrt{\tilde{g}^{\flat}}]$ in $\mathcal{G}$. To return from the Colombeau algebra to the space of distributions, Wilson demonstrates that for each $\Phi \in \mathcal{C}^\infty$ with compact support

$$\lim_{\epsilon\to 0}\int_K \tilde{R}_\epsilon\sqrt{\tilde{g}^{\flat}_\epsilon}(x,y)\Psi(x,y)dxdy = \lim_{\epsilon\to 0}\int_K 4\pi(1-A)\delta^{(2)}\Psi(x,y)dxdy \tag{5.31}$$

This shows that

$$[\tilde{R}\sqrt{\tilde{g}^{\flat}}] \approx 4\pi(1-A)\delta^{(2)} \tag{5.32}$$

Hence we have our associated curvature distribution $4\pi(1-A)\delta^{(2)}$. The calculation (5.31) can be done either by evaluating the integrals directly or by using a holonomy (Gauss-Bonnet) argument. We note that since we are taking limits as $\epsilon \to 0$ we make good use of approximations involving ϵ.

The important point of this method is that, by using the Colombeau algebra, we can take the curvature of the representative of the metric to find the representative of the curvature of the metric. Note also that since weak equivalence is independent of the representative in $\mathcal{E}_M$, the right hand side of (5.32) is also independent of both Φ and the representative we take for $\mathcal{G}$. This shows that at the level of weak equivalence the curvature of the 2-dimensional cone is uniquely given by $4\pi(1-A)\delta^{(2)}$.

For the 4-D case the calculations are very similar and we find that for the full smoothed Riemann density tensor, the only non-zero component is $[\tilde{R}^{xy}{}_{xy}\sqrt{-\tilde{g}^{\flat}}] \approx 2\pi(1-A)\delta^{(2)}$, as we expected from our heuristic holonomy method in Chapter 2. By a simple contraction on the components of the Riemann tensor we obtain the distributional mixed energy-momentum tensor density, with non-zero components

$$[\tilde{T}^t_t\sqrt{-\tilde{g}^{\flat}}] = [\tilde{T}^z_z\sqrt{-\tilde{g}^{\flat}}] \approx -2\pi(1-A)\delta^{(2)}(x,y) \tag{5.33}$$

Wilson then calculates the mass density per unit length of the cosmic string to be $[\tilde{\mu}(t,z)] \approx 2\pi(1-A)$.

Variations on the 4-dimensional cone

Physical models of a cosmic string are likely to require a model less simple than that of a 4-dimensional cone. Clarke, Wilson and Vickers [7, 55] have proposed a number of alternatives to the conical metric

$$ds^2 = dt^2 - dr^2 - A^2r^2d\phi^2 - dz^2 \tag{5.34}$$

two of which we briefly discuss. Note that the convention in [7, 55] is to adopt the Lorentzian sign convention $\mathrm{Diag}(-1,1,1,1)$, but this has no significant effect on any results.

Firstly, the constant A can be considered to be a function $A(t,z)$. The metric

$$ds^2 = dt^2 - dr^2 - A^2(t,z)r^2d\phi^2 - dz^2 \tag{5.35}$$

describes a conical metric but with a variable deficit angle dependent on what point of the singularity we are looking at. Note that the above metric does not describe a quasi-regular singularity but, with significance to this thesis, it does describe a spacetime with square Lebesgue integrable curvature. Wilson [55] shows that the distributional part of the energy-momentum tensor behaves like a cosmic string with a mass per unit length that changes with position on the string.

If we introduce a perturbation of a conical metric that vanishes sufficiently fast as we approach the axis we would still expect to be able to calculate the distributional curvature using a method similar to that of a cone. We first embed the components of the metric into the Colombeau algebra and then calculate the generalised function curvature density $\tilde{R}^{ab}{}_{cd}\sqrt{-\tilde{g}}$. This curvature density will have both a distributional contribution from the axis of the perturbed cone and a regular contribution from the metric. By subtracting the regular part from $\tilde{R}^{ab}{}_{cd}\sqrt{-\tilde{g}}$ we obtain the contribution from the axis.

Wilson demonstrates the procedure for the special case of a metric given by

$$ds^2 = dt^2 - dr^2 - A^2r^2(1+k(r))d\phi^2 - dz^2 \tag{5.36}$$

where the term $k(r)$ is a perturbation function of r that is $O(r^2)$ and can be thought of as a C^∞ function of the Cartesian coordinates. He shows that the resultant Riemann curvature density has both a distributional and a regular component. The non-zero part is given by

$$[\tilde{R}^{xy}{}_{xy}\sqrt{-\tilde{g}}] \approx R^{xy}{}_{xy}\sqrt{-g} + 2\pi(1-A)\delta^{(2)}(x,y) \tag{5.37}$$

The significance to our work of this last model is in the derivation of the curvature. We use a similar method and in fact find that the curvature of a weak singuaraity has the same distributional and regular contributions as given by (5.37). We discuss this in the following section.

5.3 Distributional curvature of a weak singularity

We now apply the previous results from this chapter and Chapter 4 to calculate the curvature of a weak singularity. We show that the distributional part of the curvature is identical to that of the conical singularity and that at the singularity, weak and conical singularities have the same curvature. This follows from the theorem for this section which we now state.

Theorem 5.12 *In Cartesian coordinates for $g^\flat_{ab}$, a weak curvature singularity has a distributional curvature*

$$[\tilde{R}^{xy}{}_{xy}\sqrt{-\tilde{g}}] \approx \acute{R}^{xy}{}_{xy}\sqrt{-g} + 2\pi(1-A)\delta^{(2)}(x,y) \tag{5.38}$$

and for the other components

$$[\tilde{R}^{ab}{}_{cd}\sqrt{-\tilde{g}}] \approx \acute{R}^{ab}{}_{cd}\sqrt{-g} \tag{5.39}$$

where $\acute{R}$ is the non-singular curvature away from the origin.

Note that in the conical case, $\acute{R}$ is zero.

Proof of Theorem 5.12

We first consider comparisons between the conical and weak singularities in the two dimensional case. The metric for both singularities will have coordinate singularities at $r = 0$ in polar coordinates so we use Cartesian coordinates to avoid confusion.

As before we write the metric for the conical singularity by splitting it into regular and singular parts

$$g^\flat_{ab} = \frac{1}{2}(1+A^2)\delta_{ab} + \frac{1}{2}(1-A^2)h_{ab} \tag{5.40}$$

We now do the same for a weak singularity

$$g_{ab} = \frac{1}{2}(1+A^2)\delta_{ab} + \frac{1}{2}(1-A^2)\tilde{h}_{ab} \tag{5.41}$$

for some $\tilde{h}_{ab}$. We smooth both metrics with a function $\Phi \in \mathcal{A}_1(\mathbb{R}^2)$ such that

$$\begin{aligned}
\tilde{g}^\flat_{ab,\epsilon}(x,y) &= \int_{\mathbb{R}^2} g^\flat_{ab}(u,v)\Phi_\epsilon(u-x,v-y)dudv & (5.42)\\
\tilde{g}_{ab,\epsilon}(x,y) &= \int_{\mathbb{R}^2} g_{ab}(u,v)\Phi_\epsilon(u-x,v-y)dudv & (5.43)
\end{aligned}$$

We recall that

$$\Phi_\epsilon(u-x,v-y) = \frac{1}{\epsilon^2}\Phi\left(\frac{1}{\epsilon}(u-x,v-y)\right) \tag{5.44}$$

Letting $(s,t) = (1/\epsilon)(u-x,v-y)$ we have

$$\tilde{g}^\flat_{ab,\epsilon}(x,y) = \int_{\mathbb{R}^2} g^\flat_{ab}(x+\epsilon s, y+\epsilon t)\Phi(s,t)dsdt \tag{5.45}$$

Also, since

$$\int_{\mathbb{R}^2} \Phi(s,t)dsdt = I \tag{5.46}$$

it follows that

$$g^\flat_{ab}(x,y) = \int_{\mathbb{R}^2} g^\flat_{ab}(x,y)\Phi(s,t)dsdt \tag{5.47}$$

Hence,

$$\begin{aligned}
\left|\tilde{g}^\flat_{ab,\epsilon}(x,y) - g^\flat_{ab}(x,y)\right| &= \left|\int_{\mathbb{R}^4}\left(g^\flat_{ab}(x+\epsilon s,y+\epsilon t) - g^\flat_{ab}(x,y)\right)\Phi(s,t)dsdt\right|\\
\Longrightarrow |\tilde{g}^\flat_{ab,\epsilon} - g^\flat_{ab}| &= \mathcal{O}(\epsilon)
\end{aligned}$$

where $\mathcal{O}(\epsilon)$ is a function of ϵ that tends to zero as $\epsilon \to 0$. Similar estimates apply for the first and second derivatives provided $\Phi \in \mathcal{A}_q$ for $q \geq 2$. We also have

$$|\tilde{g}_{ab,\epsilon} - g_{ab}| = \mathcal{O}(\epsilon) \tag{5.48}$$

and again similar estimates for the first and second derivatives apply. It follows that

$$\|\tilde{g}^{\flat}_{ab,\epsilon} - g^{\flat}_{ab}\|_{L^2_2} = \mathcal{O}(\epsilon) \qquad \|\tilde{g}_{ab,\epsilon} - g_{ab}\|_{L^2_2} = \mathcal{O}(\epsilon) \tag{5.49}$$

From Theorem 4.23 we know that

$$\|g - g^{\flat}\|_{L^2_2(B(a))} \leq C\|\acute{R}\|_{L^2(B(a))} \tag{5.50}$$

Hence

$$\begin{aligned} \|\tilde{g}_\epsilon - \tilde{g}^{\flat}_\epsilon\|_{L^2_2(B(a))} &= \|\tilde{g}_\epsilon - g + g - g^{\flat} + g^{\flat} - \tilde{g}^{\flat}_\epsilon\|_{L^2_2(B(a))} \\ &\leq \|\tilde{g}_\epsilon - g\|_{L^2_2(B(a))} + \|g - g^{\flat}\|_{L^2_2(B(a))} + \|g^{\flat} - \tilde{g}^{\flat}_\epsilon\|_{L^2_2(B(a))} \\ &\leq C_1\|\acute{R}\|_{L^2(B(a))} + \mathcal{O}(\epsilon) \\ &\leq O(a) + \mathcal{O}(\epsilon) \end{aligned} \tag{5.51}$$

Hence considering the formula for the curvature in terms of the connection and the metric, and employing an argument similar to that used to prove Theorem 4.9, we find that for any fixed function $\Phi \in \mathcal{A}_1(\mathbb{R}^2)$

$$\begin{aligned} \left|\int_{B(a)} (\tilde{R}_\epsilon\sqrt{\tilde{g}_\epsilon} - \tilde{R}^{\flat}_\epsilon\sqrt{g^{\flat}_\epsilon})\Phi dxdy\right| &\leq O(a) + \mathcal{O}(\epsilon) \\ \Longrightarrow \lim_{\epsilon\to 0}\left|\int_{B(a)} (\tilde{R}_\epsilon\sqrt{\tilde{g}_\epsilon} - \tilde{R}^{\flat}_\epsilon\sqrt{g^{\flat}_\epsilon})\Phi dxdy\right| &\leq O(a) \end{aligned} \tag{5.52}$$

From the results of [55] given earlier we know that

$$\lim_{\epsilon\to 0}\int_{B(a)} \tilde{R}^{\flat}_\epsilon\sqrt{\tilde{g}^{\flat}_\epsilon}\Phi dxdy = 4\pi(1-A)\Phi(0,0) = \int_{B(a)} 4\pi(1-A)\delta^{(2)}\Phi dxdy \tag{5.53}$$

Hence from (5.52) we have

$$\lim_{\epsilon\to 0}\left|\int_{B(a)} (\tilde{R}_\epsilon\sqrt{\tilde{g}_\epsilon} - 4\pi(1-A)\delta^{(2)})\Phi dxdy\right| \leq O(a) \tag{5.54}$$

We now let U be the support of Φ where we have chosen a such that $B(a) \subset U$. We recall that $\tilde{R}_\epsilon\sqrt{\tilde{g}_\epsilon} \approx \acute{R}\sqrt{g}$ and so, from the definition of weak equivalence we know that

$$\lim_{\epsilon\to 0}\left|\int_{U\setminus(0,0)} (\tilde{R}_\epsilon\sqrt{\tilde{g}_\epsilon} - \acute{R}\sqrt{g})\Phi dxdy\right| = 0 \tag{5.55}$$

Finally, we show that the following limit is zero

$$\begin{aligned}
&\lim_{\epsilon\to 0}\left|\int_{\mathbb{R}^2}\left(\tilde{R}_\epsilon\sqrt{\tilde{g}_\epsilon}-\acute{R}\sqrt{g}-4\pi(1-A)\delta^{(2)}\right)\Phi dxdy\right| \qquad (5.56)\\
=\ &\lim_{\epsilon\to 0}\left|\int_{U}\left(\tilde{R}_\epsilon\sqrt{\tilde{g}_\epsilon}-\acute{R}\sqrt{g}-4\pi(1-A)\delta^{(2)}\right)\Phi dxdy\right|\\
=\ &\lim_{\epsilon\to 0}\left|\int_{U\backslash B(a)}\left(\tilde{R}_\epsilon\sqrt{\tilde{g}_\epsilon}-\acute{R}\sqrt{g}\right)\Phi dxdy\right.\\
&\left.+\int_{B(a)}\left(\tilde{R}_\epsilon\sqrt{\tilde{g}_\epsilon}-\acute{R}\sqrt{g}-4\pi(1-A)\delta^{(2)}\right)\Phi dxdy\right|
\end{aligned}$$

Since (5.56) is independent of a we can write

$$\begin{aligned}
&\lim_{\epsilon\to 0}\left|\int_{\mathbb{R}^2}\left(\tilde{R}_\epsilon\sqrt{\tilde{g}_\epsilon}-\acute{R}\sqrt{g}-4\pi(1-A)\delta^{(2)}\right)\Phi dxdy\right|\\
=\ &\lim_{a\to 0}\lim_{\epsilon\to 0}\left|\int_{U\backslash B(a)}\left(\tilde{R}_\epsilon\sqrt{\tilde{g}_\epsilon}-\acute{R}\sqrt{g}\right)\Phi dxdy\right.\\
&\left.+\int_{B(a)}\left(\tilde{R}_\epsilon\sqrt{\tilde{g}_\epsilon}-\acute{R}\sqrt{g}-4\pi(1-A)\delta^{(2)}\right)\Phi dxdy\right|\\
\leq\ &\lim_{\epsilon\to 0}\left|\int_{U\backslash(0,0)}\left(\tilde{R}_\epsilon\sqrt{\tilde{g}_\epsilon}-\acute{R}\sqrt{g}\right)\Phi dxdy\right| \qquad (5.57)\\
&+\lim_{a\to 0}\lim_{\epsilon\to 0}\left|\int_{B(a)}\left(\tilde{R}_\epsilon\sqrt{\tilde{g}_\epsilon}-4\pi(1-A)\delta^{(2)}\right)\Phi dxdy\right|+\lim_{a\to 0}\left|\int_{B(a)}\acute{R}\sqrt{g}\Phi dxdy\right|
\end{aligned}$$

From (5.55) we know that the first integral vanishes. Likewise, from (5.54) we know the second integral vanishes. Finally, since $\acute{R}$ is the regular part of the curvature, we know that the third integral vanishes. Therefore

$$\lim_{\epsilon\to 0}\left|\int_{\mathbb{R}^2}\left(\tilde{R}_\epsilon\sqrt{\tilde{g}_\epsilon}-\acute{R}\sqrt{g}-4\pi(1-A)\delta^{(2)}\right)\Phi dxdy\right|=0 \qquad (5.58)$$

Again from the definition of weak equivalence we find that

$$[\tilde{R}\sqrt{g}]\approx\acute{R}\sqrt{g}+4\pi(1-A)\delta^{(2)}(x,y) \qquad (5.59)$$

Using similar methods to those found in Chapter 5 of [55] we can then extend this result to the four dimensional case.

$$[\tilde{R}^{xy}{}_{xy}\sqrt{-\tilde{g}}]\approx\acute{R}^{xy}{}_{xy}\sqrt{-g}+2\pi(1-A)\delta^{(2)}(x,y) \qquad (5.60)$$

Chapter summary

Heuristic arguments suggest that the curvature of a conical singularity is given by a delta function. However, the metric does not lie in the class of GT-regular metrics and therefore one cannot simply use classical distribution theory to calculate the curvature. An alternative approach is to regularise the metric through some smoothing procedure. However, as shown by Geroch and Traschen, this method also has a number of difficulties and in particular, different smoothings provide different answers. Because of this we followed [55] and applied a method involving the use of Colombeau's algebras (see Section 5.1 for a full description of these).

We embed the metric into the Colombeau algebra by taking a representative of the family of smoothed metrics. In the algebra we can now perform the necessary calculations to find a representative of the curvature density. To get back to the level of distributions we are then required to use the relationship of association. For the cone we find that the Colombeau distributional curvature density is associated to a delta function which is exactly the result expected from the holonomy method given in [49].

We then briefly discussed using similar methods for finding the distributional curvature of metrics which represent a perturbation of a conical metric. We then recalled the results from Chapter 4 which show that a weak curvature singularity looks like a cone close to the singularity. We combined the previous results of this chapter for calculating the distributional curvature of the cone, with the results from Chapter 4, to calculate the distributional curvature density of a weak singularity. In doing so we confirmed the expected result, that weak curvature singularities have a regular part given by the regular part of the metric and a singular part with a distributional curvature equal to that of a 4-dimensional cone.

Chapter 6

Conclusions

The objective of this thesis has been to find the distributional curvature of weak curvature singularities, those singularities with the property (amongst others) of having square Lebesgue integrable curvature. We examined theorems from Yang-Mills gauge theory and derived analogous theorems for General Relativity. These theorems state that weak singularities have a limit holonomy and also have connection and metric tending to that of a conical singularity as $r \to 0$. We then considered known results for the distributional curvature of a conical singularity and demonstrated that this is the same as the distributional part of the curvature of a weak singularity.

Summary

Since this thesis concerns singularities, in Chapter 2 we included a brief review of modern understanding of some crucial concepts such as "what a singularity is" and "where a singularity is". In defining a singularity we introduced the ideas of Cauchy, geodesic and bundle completeness. We found that the metric and manifold alone are inadequate tools to describe the location of a singularity and so in this thesis we chose to adopt the method of locating singularities on a b-boundary. We showed how constructing the b-boundary requires the establishment of a positive definite metric on the frame bundle using both the canonical and connection 1-forms in order to find the Cauchy completion of the frame bundle.

We then went on to classify different types of singularities and in particular looked at quasi-regular singularities. We then introduced the idea of distributional curvature as a way of describing curvature at a singularity. The concept of holonomy was explained and we showed how holonomy can be used to measure distributional curvature of a singularity.

As an illustration of some of the results that have been found for quasi-regular singularities using the holonomy method we summarised the main findings in [49].

Next we introduced the particular type of singularity with which we are concerned, the weak curvature singularity. The curvature of the space-time may approach infinity towards a weak singularity but is, however, square Lebesgue integrable everywhere. Hence a distributional curvature can be assigned at the singularity.

This thesis relies on the use of Sobolev spaces and norms and so a detailed explanation was given, particularly on how to take Sobolev norms of objects with space-time and/or Lie group or Lie algebra indices.

In Chapter 3 we considered the work of Sibner and Sibner [39] in Yang-Mills gauge theory. We first introduced Yang-Mills theory and then showed the method by which the Sibners demonstrated the existence of limit holonomy at all points on a singular 2-surface. It was demonstrated that this limit holonomy is independent of the limiting point on the singularity. The remainder of Chapter 3 was concerned with the second main result of Sibner and Sibner which showed that, in a neighbourhood of a singularity, the L^2_1 norm of the difference between a connection and some constant flat connection is bounded by the L^2 norm of the curvature. Any connection has a corresponding flat connection to which it tends in the limit. This connection and flat connection share the same limit holonomy. Many of the details of the proofs were omitted particularly where a directly analogous proof was applied in the case of General Relativity in Chapter 4.

In Chapter 4 we adapted the theorems shown in Chapter 3 to be applicable to certain static space-times in General Relativity which have 2-dimensional and timelike, weak curvature singularities. We showed the important differences between the Sibners problem in Yang-Mills and our problem in GR. Most notably we discussed the gauge group $SO(3)$, a subgroup of the Lorentz group, assigned to our bundle space and demonstrated that the connection 1-form ω^i_j and the curvature 2-form Ω^i_j take values in the Lie algebra $so(3)$ of $SO(3)$. We also provided the construction of the prototype flat connection, which for GR is the flat 4-dimensional cone.

We included the full details of the proofs needed to establish Theorem 4.8 and Theorem 4.9.

We also showed two more results integral for our problem in GR. The first used work done in [6] to show that the axis of rotation of the holonomy agrees with the singularity. The second showed that it is not just the connection which tends to a flat conical connection on approach to the singularity, but also the metric which tends to the metric of the flat cone. This last result ultimately allows us to find the distributional curvature of a weak singularity.

In Chapter 5 we gave a detailed explanation of Colombeau's theory of generalised functions and then summarised a method given in [55] to find the distributional curvature of a conical singularity. We briefly discussed further work in [55] to find the distributional curvature of a perturbed conical singularity. We then applied a similar method to calculate the distributional part of the curvature of a weak singularity to be the same as the distributional curvature of a conical singularity.

By extending previous results for conical singularities, to include weak curvature singularities, we have increased the size of the class of singularities which possess curvature that can be described using a distribution. By treating these singularities as a distributional solution of Einstein's equations we can consider them, at least in some sense, to be part of the space-time. This will enable physicists to work with singularities in a considerably more rigorous fashion.

Appendix A

Relevant methods and properties

A.1 Lie groups and Lie algebras

We define a Lie group using the definition in [22].

Definition A.1 *A Lie group G is a group in the usual sense but is also a differentiable manifold with the properties that taking the product of two group elements, and taking the inverse of a group element, are smooth operations. Specifically the maps*

$$\begin{aligned} \mu : G \times G &\to G \\ (g_1, g_2) &\mapsto g_1 g_2 \end{aligned} \tag{A.1}$$

and

$$\begin{aligned} \iota : G &\to G \\ g &\mapsto g^{-1} \end{aligned} \tag{A.2}$$

are both C^∞.

Some examples of Lie groups are:

The general linear group

$$GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) | detA \neq 0\} \tag{A.3}$$

The special linear group

$$SL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) | detA = 1\} \tag{A.4}$$

The orthogonal group

$$O(n) = O_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) | A^T A = I\} \tag{A.5}$$

The special orthogonal group

$$SO(n) = SO_n(\mathbb{R}) = \{A \in O(n) | detA = 1\} \tag{A.6}$$

The unitary group (where $A^* = \overline{A^T}$, the transpose of the complex conjugate of A)

$$U(n) = U_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) | A^* A = I\} \tag{A.7}$$

The special unitary group

$$SU(n) = SU_n(\mathbb{C}) = \{A \in U(n) | detA = 1\} \tag{A.8}$$

The right and left translations of a Lie group G are diffeomorphisms of G labelled by the elements $g \in G$ and defined by

$$\begin{aligned} R_g : G \to G \qquad & L_g : G \to G \\ g' \mapsto g'g \qquad & g' \mapsto gg' \end{aligned} \tag{A.9}$$

Isham [22] defines a vector field X on a Lie group G as left-invariant if it is L_g-related to itself for all $g \in G$. That it is to say

$$L_{g_*} X = X \quad \text{for all } g \in G \tag{A.10}$$

To every Lie group G, we can associate a Lie algebra $\mathfrak{g}$, the set of all left-invariant vector fields on G, whose underlying vector space is the tangent space of G at the identity element. $\mathfrak{g}$ completely captures the local structure of the group since there is a one-to-one association between one-parameter subgroups of the Lie group and its Lie algebra (see page 166 in [22] for details). Lie algebra indices are given by lower case letters.

As an example of a construction of a Lie algebra let us consider g in the group $SO(3)$ (rotations in 3 dimensions), with $g(0) = I$. Here δ is the Euclidean metric and $\dot{g}$ is the derivative of g with respect to t.

$$
\begin{aligned}
g^i_k(t)\delta_{ij}g^j_l(t) &= \delta_{kl} \\
\dot{g}^i_k(t)\delta_{ij}g^j_l(t) + g^i_k(t)\delta_{ij}\dot{g}^j_l(t) &= 0 \\
\dot{g}^i_k(0)\delta_{ij}g^j_l(0) + g^i_k(0)\delta_{ij}\dot{g}^j_l(0) &= 0 \\
\dot{g}^i_k(0)\delta_{il} + \delta_{ki}\dot{g}^i_l(0) &= 0
\end{aligned} \tag{A.11}
$$

Now let $\dot{g}(0) = B \in so(3)$. Lowering the indices we get

$$B_{lk} + B_{kl} = 0 \tag{A.12}$$

so B is anti-symmetric. Note that this lowering of indices will be different when we work with the Lorentz group and Minkowski metric. The basis for $so(3)$ is e_i where

$$\underset{\sim}{e}_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \underset{\sim}{e}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad \underset{\sim}{e}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tag{A.13}$$

We say that a matrix C takes values in the Lie algebra $so(3)$ iff C can be composed from a linear combination of $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$ and $\underset{\sim}{e}_3$.

A.2 Relating basis vectors to connection 1-forms

Once we have chosen the metric on a manifold, then we can choose the basis vectors from which the metric is derived such that their inner products are constant. If we let $\{\vartheta^i, i = 0,1,2,3\}$ be the basis of one-forms (the dual of the basis of vectors $\{e_i, i = 0,1,2,3\}$) then the metric is

$$ds^2 = \mu_{ij}\vartheta^i\vartheta^j \tag{A.14}$$

where μ_{ij} is a matrix of constants. In GR we often take μ_{ij} to be the Minkowski metric η_{ij}. We note that ϑ^i is a 1-form and so could just as well be written $\vartheta^i_a dx^a$.The covariant derivative of ϑ^i in the e_j direction is a 1-form and hence is a linear combination of the ϑ^k. We may therefore write [21]

$$e^c_j\nabla_c\vartheta^i_a = -\gamma^i_{kj}\vartheta^k_a \tag{A.15}$$

The scalars γ^i_{kj} (since i, k, j are numerical labels) are the Ricci rotation coefficients and determine the connection. Also, since $g^{ab}\vartheta^i_a\vartheta^j_b = \mu^{ij}$ — that is to say, the ϑ^i have constant inner product μ^{ij} — and since the ϑ^j form a basis dual to the e_i, we have $\vartheta^j_b e^c_j = \delta^c_b$ (and also $\vartheta^j_a e^a_i = \delta^j_i$). From this we have the following

$$
\begin{aligned}
\nabla_c\delta^b_a &= 0 \\
\nabla_c\vartheta^i_a e^b_i &= 0 \\
\vartheta^i_a\nabla_c e^b_i &= -e^b_i\nabla_c\vartheta^i_a \\
e^c_j\vartheta^i_a\nabla_c e^b_i &= -e^b_i e^c_j\nabla_c\vartheta^i_a \\
e^c_j\vartheta^i_a\nabla_c e^b_i &= e^b_i\gamma^i_{kj}\vartheta^k_a \\
\vartheta^i_b e^c_k\nabla_c e^a_i &= e^a_j\gamma^j_{ik}\vartheta^i_b \\
e^c_k\nabla_c e^a_i &= \gamma^j_{ik}e^a_j
\end{aligned}
\tag{A.16}
$$

We can contract both sides of (A.15) with $g^{ab}\vartheta^m_b$ to find

$$
\begin{aligned}
g^{ab}\vartheta^m_b e^c_j\nabla_c\vartheta^i_a &= -g^{ab}\vartheta^m_b\gamma^i_{kj}\vartheta^k_a \\
&= -\gamma^i_{kj}\mu^{mk} \\
\implies g^{ab}\vartheta^m_b e^c_j\nabla_c\vartheta^i_a + g^{ab}\vartheta^i_b e^c_j\nabla_c\vartheta^m_a &= -\gamma^i_{kj}\mu^{mk} - \gamma^m_{kj}\mu^{ik} \\
\implies g^{ab}\vartheta^m_b e^c_j\nabla_c\vartheta^i_a + g^{ab}\vartheta^i_a e^c_j\nabla_c\vartheta^m_b &= -\left(\gamma^i_{kj}\mu^{mk} + \gamma^m_{kj}\mu^{ik}\right) \\
&\quad \text{reverse Leibniz} \\
\implies g^{ab}e^c_j\nabla_c(\vartheta^i_a\vartheta^m_b) &= -\left(\gamma^i_{kj}\mu^{mk} + \gamma^m_{kj}\mu^{ik}\right) \\
&\quad \text{since } \nabla_c g^{ab} = 0 \\
\implies e^c_j\nabla_c(g^{ab}\vartheta^i_a\vartheta^m_b) &= -\left(\gamma^i_{kj}\mu^{mk} + \gamma^m_{kj}\mu^{ik}\right) \\
\implies e^c_j\nabla_c\mu^{im} &= -\left(\gamma^i_{kj}\mu^{mk} + \gamma^m_{kj}\mu^{ik}\right) \\
&\quad \text{since } \mu^{ij} \text{ are constants} \\
\implies 0 &= -\left(\gamma^i_{kj}\mu^{mk} + \gamma^m_{kj}\mu^{ik}\right) \\
\implies \gamma^{(i}{}_{kj}\mu^{m)k} &= 0
\end{aligned}
\tag{A.17}
$$

We now define (suppressing space-time indices) the connection 1-forms ω^i_k by

$$\omega^i_k = \gamma^i_{kj}\vartheta^j \tag{A.18}$$

For the GR case we could now write (A.17) as

$$\eta_{ik}\omega^k_j + \eta_{jk}\omega^k_i = 0 \tag{A.19}$$

which tells us that ω^i_j takes values in the Lie algebra of the Lorentz group.

The Ricci rotation coefficients may be used to obtain an expression for the coordinate components of the connection

$$\omega^i_{ja}\vartheta^j_b e^c_i = \gamma^i_{jk}\vartheta^k_a\vartheta^j_b e^c_i = \Gamma^c_{ab} \tag{A.20}$$

where Γ denotes the standard Levi-Civita connection.

The connection 1-forms ω^i_k also contain the information about the connection and by (A.17) they are skew-symmetric if an index is raised by μ^{ij}, as is shown here

$$\begin{aligned}
\omega^i_k\mu^{mk} &= \gamma^i_{kj}\mu^{mk}\vartheta^j \\
\Longrightarrow \frac{1}{2}\left(\omega^i_k\mu^{mk} + \omega^m_k\mu^{ik}\right) &= \frac{1}{2}\vartheta^j\left(\gamma^i_{kj}\mu^{mk} + \gamma^m_{kj}\mu^{ik}\right) \\
\Longrightarrow \omega^{(i}{}_k\mu^{m)k} &= \vartheta^j\gamma^{(i}{}_{kj}\mu^{m)k} \\
\Longrightarrow \omega^{(i}{}_k\mu^{m)k} &= 0 \quad \square
\end{aligned} \tag{A.21}$$

Now from (A.15), if we contract both sides with ϑ^j_b we find

$$\begin{aligned}
\vartheta^j_b e^c_j\nabla_c\vartheta^i_a &= -\vartheta^j_b\gamma^i_{kj}\vartheta^k_a \\
\Longrightarrow \delta^c_b\nabla_c\vartheta^i_a &= -\omega^i_{kb}\vartheta^k_a \\
\Longrightarrow \nabla_b\vartheta^i_a &= -\omega^i_{kb}\vartheta^k_a
\end{aligned} \tag{A.22}$$

We now skew this on $[a, b]$, and then revert to index free notation

$$\begin{aligned}
\nabla_b\vartheta^i_a &= -\omega^i_{kb}\vartheta^k_a \\
\Longrightarrow \nabla_b\vartheta^i_a - \nabla_a\vartheta^i_b &= -\omega^i_{kb}\vartheta^k_a + \omega^i_{ka}\vartheta^k_b \\
\Longrightarrow \partial_b\vartheta^i_a - \partial_a\vartheta^i_b &= -\omega^i_{kb}\vartheta^k_a + \omega^i_{ka}\vartheta^k_b \\
\Longrightarrow \mathrm{d}\vartheta^i &= -\omega^i_k \wedge \vartheta^k
\end{aligned} \tag{A.23}$$

(A.23) is called the first structure equation.

A.3 Duals of differential forms

It is important to understand that a dual of a differential form (or Hodge dual) is different from the dual of a vector space.

Let ω be a p-form on an n-dimensional oriented manifold with metric g_{ab}. We define the dual $*\omega$, of ω by

$$*\omega_{\nu_1\nu_2\ldots\nu_{n-p}} = \frac{1}{p!}\omega^{\mu_1\mu_2\ldots\mu_p}\epsilon_{\mu_1\ldots\mu_p\nu_1\ldots\nu_{n-p}} \tag{A.24}$$

where $\epsilon_{\mu_1\ldots\mu_n}$ is the natural volume element on M, the totally antisymmetric tensor field which satisfies

$$\epsilon^{\mu_1\ldots\mu_n}\epsilon_{\mu_1\ldots\mu_n} = (-1)^s n! \tag{A.25}$$

and

$$s = \begin{cases} 0 & \text{signature of } g_{ab} \text{ is non-negative} \\ 1 & \text{signature of } g_{ab} \text{ is negative} \end{cases}$$

It can be seen that the dual of a k-form in n dimensions is an $n-k$-form. For example, in Chapter 3 the curvature $F = \frac{1}{2}F_{ab}dx^a \wedge dx^b$ is a 2-form and $*F$ is the dual of F written as

$$(*F)_{ab} = \epsilon_{ab}^{\ \ cd}F_{cd} \tag{A.26}$$

where

$$\epsilon_{abcd} = \begin{cases} 1 & \text{even permutation of } 0123 \\ 0 & \text{repeated entry} \\ -1 & \text{odd permutation of } 0123 \end{cases} \tag{A.27}$$

A.4 Estimating derivatives of the cut-off function λ

This appendix relates to the cutoff functions used in Section 4.3 and should be read with reference to that section.

We show that for our cutoff functions λ we have

$$|\nabla\lambda| \leq cd^{-1} \tag{A.28}$$

recalling that d is the larger of the diameters of the two balls B_α and B_β. We will show how to estimate the cutoff function in the radial direction $\lambda(r)$. Analogous estimates may be found for the cutoff functions $\lambda(y)$ and $\lambda_l(\theta)$. We begin with

$$\lambda(r) = \begin{cases} 0 & r \leq a \\ 1 & r \geq \frac{3a}{2} \end{cases} \tag{A.29}$$

where $a = 2^{-(n+1)}$.

We wish to define the function in the region $a < r < \frac{3a}{2}$ in such a way that $\lambda(r)$ is smooth. We first note that

$$e^{\frac{-1}{(r-a)^2}} e^{\frac{-1}{(r-\frac{3a}{2})^2}} \tag{A.30}$$

is zero at $r = a$ and $r = \frac{3a}{2}$ and positive in between, so this is a good start for a function for the gradient of λ. A little thought shows us that we can now write λ as

$$\lambda(r) = \begin{cases} 0 & r \leq a \\ C\int_a^r e^{\frac{-1}{(x-a)^2}} e^{\frac{-1}{(x-\frac{3a}{2})^2}}\, dx & a < r < \frac{3a}{2} \\ 1 & r \geq \frac{3a}{2} \end{cases}$$

where the constant C is defined as

$$C = \left(\int_a^{\frac{3a}{2}} e^{\frac{-1}{(r-a)^2}} e^{\frac{-1}{(r-\frac{3a}{2})^2}}\, dr \right)^{-1} \tag{A.31}$$

With this definition

$$|\nabla\lambda(r)| = \begin{cases} 0 & r \leq a \\ Ce^{\frac{-1}{(r-a)^2}} e^{\frac{-1}{(r-\frac{3a}{2})^2}} & a < r < \frac{3a}{2} \\ 0 & r \geq \frac{3a}{2} \end{cases}$$

We now wish to find the maximum of $|\nabla\lambda|$ by looking at its derivatives. After some calculations we obtain

$$\frac{\partial|\nabla\lambda|}{\partial r} = \begin{cases} 0 & r \le a \\ 2Ce^{\frac{-1}{(r-a)^2}}e^{\frac{-1}{(r-\frac{3a}{2})^2}}\left(\frac{1}{(r-a)^3}+\frac{1}{(r-\frac{3a}{2})^3}\right) & a < r < \frac{3a}{2} \\ 0 & r \ge \frac{3a}{2} \end{cases}$$

So the maximum of $|\nabla\lambda|$ is when

$$2Ce^{\frac{-1}{(r-a)^2}}e^{\frac{-1}{(r-\frac{3a}{2})^2}}\left(\frac{1}{(r-a)^3}+\frac{1}{(r-\frac{3a}{2})^3}\right) = 0 \quad \text{for} \quad a < r < \frac{3a}{2} \tag{A.32}$$

From (A.32) we have

$$\begin{aligned} 0 &= \frac{1}{(r-a)^3}+\frac{1}{(r-\frac{3a}{2})^3} \\ &= (r-\frac{3a}{2})^3+(r-a)^3 \\ &= 2r^3-\frac{15ar^2}{2}+\frac{39a^2r}{4}-\frac{35a^3}{8} \\ \Longrightarrow r &= \frac{5a}{4} \end{aligned} \tag{A.33}$$

So

$$\max|\nabla\lambda(r)| = |\nabla\lambda(\frac{5a}{4})| = Ce^{\frac{-32}{a^2}} = Ce^{-(2^{2n+7})} \tag{A.34}$$

The maximum possible diameter of the ball will be less than the outside arc-length added to the difference between the radial distance of the outside edge and the radial distance of the inside edge from the singularity. So

$$\begin{aligned} d &< \frac{\pi}{2}2^{-n}+2^{-n}-2^{-n-2} \\ &= (2\pi+3)2^{-n-2} \\ \Longrightarrow d^{-1} &> \frac{2^{n+2}}{2\pi+3} \end{aligned} \tag{A.35}$$

To show that $|\nabla\lambda| \le cd^{-1}$ we will first show that $|\nabla\lambda| \le \frac{2^{n+2}}{2\pi+3}$. For this we shall use a proof by contradiction. Suppose $|\nabla\lambda| > \frac{2^{n+2}}{2\pi+3}$. Then

$$\begin{aligned} Ce^{-(2^{2n+7})} &> \frac{2^{n+2}}{2\pi+3} \\ \Longrightarrow -(2^{2n+7}) &> \ln 2^{n+2}-\ln C(2\pi+3) \\ \Longrightarrow \ln C(2\pi+3) &> \ln 2^{n+2}+2^{2n+7} \end{aligned} \tag{A.36}$$

Let

$$e^{\frac{-1}{(r-a)^2}} e^{\frac{-1}{(r-\frac{3a}{2})^2}} = f(r) \tag{A.37}$$

Then

$$C = \left(\int_a^{\frac{3a}{2}} f(r)dr \right)^{-1} \tag{A.38}$$

We calculate that

$$f\left(\frac{9a}{8}\right) = f\left(\frac{11a}{8}\right) = e^{\frac{-640}{9a^2}} \tag{A.39}$$

Since f is monotonic increasing for $r \leq \frac{5}{4}$ and monotonic decreasing for $r \geq \frac{5}{4}$ (in the region $a < r < \frac{3a}{2}$) we know that

$$f(r) \geq e^{\frac{-640}{9a^2}} \quad \forall \quad \frac{9a}{8} < r < \frac{11a}{8} \tag{A.40}$$

Therefore the integral $\int_a^{\frac{3a}{2}} f(r)dr$ is greater than the area of the rectangle with sides $\frac{9a}{8} \leq r \leq \frac{11a}{8}$ and $0 \leq f(r) \leq \exp(\frac{-640}{9a^2})$

$$\begin{aligned} \int_a^{\frac{3a}{2}} f(r)dr &\geq e^{\frac{-640}{9a^2}} \left(\frac{a}{4}\right) \\ &= e^{\frac{5}{9}(2^{2n+9})} 2^{-n-3} \\ \Longrightarrow C &\leq e^{\frac{-5}{9}(2^{2n+9})} 2^{n+3} \end{aligned} \tag{A.41}$$

The right hand side decreases as n increases so we find that the maximum of the right hand side is when $n = 0$, the smallest value of n we can take. Therefore we know that

$$\begin{aligned} C &\leq e^{\frac{-5}{9}(512)} 2^3 \\ &= 8e^{\frac{-2560}{9}} \\ \Longrightarrow \ln C(2\pi + 3) &\leq \ln 8e^{\frac{-2560}{9}} (2\pi + 3) \\ &= \ln 8(2\pi + 3) - \frac{2560}{9} \\ &< -280 \end{aligned} \tag{A.42}$$

Going back to the right hand side of (A.36) we observe that

$$\ln C(2\pi + 3) > \ln 2^{n+2} + 2^{2n+7} \geq \ln 4 + 2^7 > 128 \tag{A.43}$$

and so we have a contradiction, which proves that $|\nabla(\lambda)| \leq \frac{2^{n+2}}{2\pi+3} \leq cd^{-1}$ for any constant $c > 1$.

The calculation to show that $|\nabla(\nabla\lambda)| \leq cd^{-2}$ is more complex but is done in a similar way to that used above.

A.5 Distributions

We now provide a brief review of distributions. For the sake of simiplicity we look at functions in one dimension. Analogous results exist in higher dimensions.

Let us consider the following function f, mapping $\mathbb{R}$ to $\mathbb{R}$, where $\alpha \in \mathbb{R}^+$, the set of positive real numbers.

$$f(x) = \begin{cases} 2\alpha(1-\alpha x) & x \in [0, 1/\alpha] \\ 0 & x \notin [0, 1/\alpha] \end{cases} \tag{A.44}$$

We see that for all $\alpha \in \mathbb{R}^+$ the integral is 1. We can increase α to any real number we wish and still we have a bounded function. However, if we take the limit as $\alpha \to \infty$ then we have a function which yields 0 at all points except one (the origin) where it is undefined. However, although the function is undefined at zero, the integral is still 1. In the limit as $\alpha \to \infty$, we call this type of function a distribution. An underlying idea for distributions is that if a function has a certain property then so too might the limit of the function to the distribution.

Distributions are objects like δ-functions. The way they are defined classically is as dual spaces to some function space. If we let α be a distribution, then α is a map from functions f to real numbers. We call f a test-function and the set of test-functions D. Likewise the set of distributions is D'.

$$\begin{aligned} \alpha \in D' \qquad & f \in D \\ < \alpha, f > \quad &= \quad c \in \mathbb{R} \end{aligned} \tag{A.45}$$

Usually we take distributions to be dual to the space of functions f where the f are smooth and of compact support. The support of a function is those elements which are not mapped to zero. Compact means closed and bounded for subsets of $\mathbb{R}$, so f vanishes outside some finite interval. We can imbed an ordinary function in the space of distributions in the following way (for one dimension). Given an ordinary function g in $\mathbb{R}$ we define the corresponding distribution $\tilde{g} \in D'$ by

$$< \tilde{g}, f >= \int_{-\infty}^{\infty} f(x)g(x)dx = c \in \mathbb{R} \tag{A.46}$$

We label the imbedding operation $i : L^1_{\text{loc}} \longrightarrow D'$, where L^1_{loc} is the space of locally integrable functions.

Now we define $\tilde{g}'$, the *weak derivative* of $\tilde{g}$, as follows. First we suppose that g is differentiable. Then we set $(\tilde{g})' = (\tilde{g'})$. We bear in mind that since f has compact support, it is zero at $\pm\infty$.

$$\begin{aligned}
\langle (\tilde{g})', f \rangle &= \left\langle \tilde{g'}, f \right\rangle \\
&= \int_{-\infty}^{\infty} f g' dx \qquad \text{(Integration by parts)} \\
&= [fg]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f' g dx \\
&= -\int_{-\infty}^{\infty} f' g dx \\
&= -\langle \tilde{g}, f' \rangle
\end{aligned} \tag{A.47}$$

Even if g is not differentiable, we can still define the weak derivative

$$\langle (\tilde{g})', f \rangle = -\langle \tilde{g}, f' \rangle \tag{A.48}$$

and in general, for some $\alpha \in D'$ we have

$$\langle \alpha', f \rangle = -\langle \alpha, f' \rangle \tag{A.49}$$

For more on distributions see [13].

Appendix B

Definitions, lemmas and theorems

B.1 The Sobolev imbedding theorem

We now state, without proof, the parts of the Sobolev Imbedding Theorem, as given in [1], which are relevant to this thesis (notation is changed from [1] to remain consistent with this thesis). An imbedding is a homeomorphism of one topological space to a subspace of another topological space. The whole of the theorem applies to many different cases. We state only those cases relevant to the work in this thesis.

Let Ω be a domain in $\mathbb{R}^n$ and let Ω^k be the k-dimensional domain obtained by intersecting Ω with a k-dimensional plane in $\mathbb{R}^n$, $1 \le k \le n$. (Thus $\Omega^n \equiv \Omega$.) Let j and m be non-negative integers and let p satisfy $1 \le p < \infty$.

There exists the following imbeddings:

Case A: Suppose $mp < n$ and $n - mp < k \le n$. Then

$$L^p_{j+m}(\Omega) \to L^q_j(\Omega^k), \qquad p \le q \le \frac{kp}{n-mp}, \tag{B.1}$$

and in particular,

$$L^p_{j+m}(\Omega) \to L^q_j(\Omega), \qquad p \le q \le \frac{np}{n-mp}, \tag{B.2}$$

and taking $j = 0$ we get

$$L^p_m(\Omega) \to L^q(\Omega), \qquad p \le q \le \frac{np}{n-mp} \tag{B.3}$$

Case C: Suppose $mp > n$. Then

$$L^p_{j+m}(\Omega) \to C^j(\Omega) \tag{B.4}$$

B.2 Fubini's theorem

Following the description in [54] we state Fubini's theorem for $\mathbb{R}^2$.

Suppose that $f \in L^1(\mathbb{R}^2)$. Then

$$\int_{\mathbb{R}^2} f(x,y)d(x,y) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)dy \right) dx \tag{B.5}$$

We interpret (B.5) as meaning that

$$F(x) = \int_{\mathbb{R}} f(x,y)dy \tag{B.6}$$

exists for almost all $x \in \mathbb{R}$ and that

$$\int_{\mathbb{R}} F(x)dx \tag{B.7}$$

exists and equals

$$\int_{\mathbb{R}^2} f(x,y)d(x,y) \tag{B.8}$$

As a consequence of (B.5) we can see that if $f(x,y) = g(x)h(y)$ then

$$\int_{\mathbb{R}^2} f(x,y)d(x,y) = \int_{\mathbb{R}} g(x)dx \int_{\mathbb{R}} h(y)dy < \infty \tag{B.9}$$

B.3 Parseval's equality

Parseval's equality states that for a continuous function f, of period 2π, with Fourier coefficients a_k and b_k we have

$$\int |f|^2 d\theta = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_k^2 + b_k^2) \tag{B.10}$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta \tag{B.11}$$

and

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta \tag{B.12}$$

Parseval's theorem also applies for Fourier series with complex coefficients. i.e. for

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \tag{B.13}$$

where

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}d\theta \tag{B.14}$$

we have

$$\int |f|^2 d\theta = \sum_{n=-\infty}^{\infty} |c_n^2| \tag{B.15}$$

B.4 Hölder's inequality

Hölder's inequality states that if $1 < p < \infty$ and $\frac{1}{p}+\frac{1}{p'} = 1$, then for two functions $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ we have

$$\int_\Omega |f(x)g(x)|dx \leq \|f\|_{L^p}\|g\|_{L^{p'}} \tag{B.16}$$

B.5 Lipschitz-continuous subsets

A function f such that

$$|f(x) - f(y)| \leq C|x - y| \tag{B.17}$$

for all x and y, where C is a constant independent of x and y , is called a Lipschitz function. For example, any function with a bounded first derivative must be Lipschitz.

A subset is Lipschitz continuous if the boundary of the subset is Lipschitz continuous. In simpler geometric terms this means that the boundary can have corners but not cusps or slits.

B.6 Sobolev's lemma

Let Ω be an open Lipschitz-continuous subset of $\mathbb{R}^N$ and let $q \in \mathbb{R}$ with $1 \leq q < \infty$ and $m \in \mathbb{N}$. Then there is a constant C such that for all $u \in L_m^q$

$$\|u\|_{L^\infty} \leq C\|u\|_{L_m^q} \tag{B.18}$$

for $N < mq$.

B.7 Morrey's lemma

Let $u \in L_1^p(B_R(x_0))$, $1 \leq p \leq n$, and suppose that there are constants $\mu > 0$ and $\nu > 0$ such that

$$\int_{B_r(x)} |\partial u|^p dx \leq \nu^p (\frac{r}{\delta})^{n-p+\mu p}, \quad 0 < r < \delta = R - |x - x_0| \tag{B.19}$$

for every ball $B_r(x)$, $x \in B_R(x_0)$. Then $u \in C[B_r(x_0)]$ for $r < R$ and

$$|u(\xi) - u(x)| \leq \frac{4}{\mu} \nu \delta^{1-\frac{n}{p}-\mu} V_n^{-\frac{1}{p}} |\xi - x|^\mu, \quad |\xi - x| \leq \frac{\delta}{2} \tag{B.20}$$

where V_n is the volume of the unit-ball.

B.8 Hölder continuity

Let (M, d_1) and (N, d_2) be two metric spaces. If a function $f : M \to N$, satisfies

$$d_2(f(x), f(y)) \leq k d_1(x, y)^\alpha \tag{B.21}$$

for some constants $k \geq 0$, $\alpha > 0$ and all x and y, it is said to be Hölder continuous. The number α is called the exponent of the Hölder condition. If $\alpha = 1$, then the function satisfies a Lipschitz condition.

B.9 The Poincaré inequality

For all $u(x) \in L_1^p(\Omega)$ and $1 \leq p < n$, $0 < q \leq np/(n-p)$, if Ω is a bounded Lipschitz domain then the function u satisfies the (q, p)-Poincaré inequality

$$\inf_{a\in\mathbb{R}^n}\left(\int_\Omega |u-a|^q dx\right)^{\frac{1}{q}} \leq C\left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{q}} \tag{B.22}$$

where C is some constant.

B.10 Gronwall's lemma

If, for $t_0 \leq t \leq t_1$, $\phi(t) \geq 0$ and $\psi(t) \geq 0$ are continuous functions such that the inequality

$$\phi(t) \leq k + L\int_{t_0}^{t} \psi(s)\phi(s)ds \tag{B.23}$$

holds on $t_0 \leq t \leq t_1$, with k and L positive constants, then

$$\phi(t) \leq k\exp\left(L\int_{t_0}^{t} \psi(s)ds\right) \tag{B.24}$$

on $t_0 \leq t \leq t_1$.

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