UNIVERSITY OF SOUTHAMPTON

FACULTY OF ENGINEERING, SCIENCE AND MATHEMATICS

School of Mathematics

Generalized Operations on Hypermaps

by

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Thesis for the degree of Doctor of Philosophy

April 2006

UNIVERSITY OF SOUTHAMPTON

ABSTRACT

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Just as a map on a surface is an embedding of a topological realization of a graph, a hypermap is an embedding of a topological realization of a hypergraph. Well-known transformations of maps and hypermaps (such as the operations which interchange hypervertices, hyperedges and hyperfaces) arise naturally in the algebraic theory of maps and hypermaps which we summarize in Chapter 1.

There is a group of six invertible topological operations on maps which are induced by automorphisms of a certain Coxeter group that can be identified with an extended Fuchsian triangle group. In Chapter 2 we study how these operations behave with respect to the property of orientability of maps, and we determine the orbits under the group of operations on reflexible torus maps of Euclidean type.

The corresponding groups of operations on hypermaps are infinite, and they partition the sets of symmetrical hypermaps having the same automorphism group into orbits. In Chapter 3, given a group from one of several infinite families (cyclic, dihedral, affine general linear), we use the theory of T-systems to examine how the size of the orbits increase with the size of the group.

In Chapters 4 and 5 we generalize the concept of (hyper)map operations by considering functors induced by more general homomorphisms between triangle groups and extended triangle groups. In the Fuchsian and Euclidean cases we determine the automorphisms, and then we make use of the classification of two-generator Fuchsian groups to determine the remaining homomorphisms subject only to the constraint that they send (possibly infinite-order) rotations to rotations. This gives a classification of such functors, some of which correspond to well-known transformations (such as truncation, induced by a triangle group inclusion) while others are new.

The concept of a 2-dimensional algebraic map can be generalized to n dimensions, and the group of operations on n-dimensional maps has order 8 for $n > 2$. In Chapter 6 we give a combinatorial description of these operations, and examine the orbits of small 3-maps and of certain reflexible n-torus maps. We then consider a further generalization of operations as isomorphism-induced equivalences between categories of different map-like objects. In particular, we exhibit a representation of orientable 3-maps without boundary by (unrestricted) hypermaps, and another of general 3-maps by a certain family of maps.

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Acknowledgements

First and foremost, I would like to thank my supervisor Prof. Gareth Jones for his constant encouragement, patience and friendship throughout my time as a research student. Without his support, this thesis would never have reached completion. Thanks must also go to my advisor, Prof. David Singerman; my understanding of the topic of hypermap theory was much enriched by the discussions we shared.

A big thanks to my friends who have been there for me during the eventful journey. Firstly, the bheys with whom I shared bombs (well if *I* had a tequila...), segments, adhesive cats, (I could go on. I won't). Secondly, my friends from before, for always looking out for me.

A big thankyou to my family and particularly my parents for their love and support during my student years. This thesis is for you—I know you'll enjoy the read....

I am grateful to Don Hatch for his permission to use his impressive hyperbolic tessellation applet to produce some of the figures in this thesis; and to the EPSRC for funding my studies. Finally, cheers to my friends and colleagues in the department.

Chapter 1

Introduction

1.1. Orientable maps without boundary

1.1.1. Two different viewpoints

Topological maps

Initially, we define a *map* $\mathscr{M}$ to be a connected graph $\mathcal{G}$ (possibly with loops, multiple edges and free edges) embedded without crossings in a connected, orientable surface $\mathcal{S}$ without boundary, the *faces* of $\mathscr{M}$ (the connected components of $\mathcal{S} \setminus \mathcal{G}$) being homeomorphic to an open disc. This definition is widened in §1.2.

$\mathscr{M}$ has *type* (m,n) if m and n are respectively the least common multiples of the valencies of the vertices and faces. We say that $\mathscr{M}$ has *finite type* if m and n are finite, and that $\mathscr{M}$ is *finite* if the total number of vertices, edges and faces is finite (or equivalently, if the underlying surface is compact). Further, $\mathscr{M}$ has *type dividing* (a,b) if $m \mid a$ and $n \mid b$.

Algebraic maps

Each topological map $\mathscr{M}$ can be described by means of two permutations ρ_0, ρ_1 of its set Ω of *darts*, or directed edges: each non-free edge carries two darts (one for each incident vertex), and ρ_1 is the permutation transposing each such pair while fixing the single dart carried by each free edge. Around each vertex v of $\mathscr{M}$, the orientation of $\mathcal{S}$ imposes a cyclic order on the darts directed into v, and ρ_0 is the permutation with these as its disjoint cycles. The cycles of ρ_0 and ρ_1 thus correspond to the vertices and edges of $\mathcal{M}$, while the cycles of $\rho_2 = (\rho_0\rho_1)^{-1}$ correspond to the faces, again following the orientation. Since $\mathcal{G}$ is connected, ρ_0 and ρ_1 generate a transitive group $G \leqslant S^{\Omega}$, the *monodromy group* of $\mathcal{M}$. Thus $\mathcal{M}$ determines an *algebraic map*, that is, a transitive permutation representation $\Pi\colon \mathscr{C}_2^+ \to G$, $r_i \mapsto \rho_i$ of the *cartographic group*

$$\mathscr{C}_2^+ = \langle\, r_0, r_1, r_2 \mid r_1^2 = r_0 r_1 r_2 = 1 \,\rangle \cong C_\infty * C_2.$$

On the other hand, every map arises from some algebraic map, and so we have a correspondence which forms the basis of a unified theory of maps developed in [31]. We give here a brief outline of some of the fundamental aspects of this theory.

1.1.2. The unified theory of maps

Triangle groups and universal maps

If a map $\mathcal{M}$ has type (m,n) dividing (a,b) then we can equally well represent $\mathcal{M}$ as a permutation representation Π of the group

$$\mathscr{C}_2^+(a,b) = \langle\, r_0, r_1, r_2 \mid r_0^a = r_1^2 = r_2^b = r_0 r_1 r_2 = 1 \,\rangle.$$

Each such group can be represented as a group $\Delta(a,2,b)$ of conformal transformations of a simply-connected Riemann surface $\mathscr{X}$, leaving invariant a triangular tessellation of $\mathscr{X}$,[1] the internal angles of each triangle T being $\frac{\pi}{a}, \frac{\pi}{2}, \frac{\pi}{b}$. Such a group $\Delta(m_0, m_1, m_2)$ is known as a *triangle group*, and when $\frac{1}{m_0} + \frac{1}{m_1} + \frac{1}{m_2} < 1$ it is an example of a Fuchsian group—a discrete group of orientation-preserving isometries of the hyperbolic plane $\mathcal{H}$. Fuchsian triangle groups have the rigidity property of being determined, up to conjugation in $\mathrm{Aut}(\mathcal{H}) = \mathrm{PSL}(2,\mathbb{R})$, by the m_i. The theory of the geometry of Fuchsian groups (see, for example, [1]) has close links with much of the theory of maps, and results concerning Fuchsian triangle groups and their subgroups usually have analogies in the Euclidean and spherical cases. The union of one of the triangles T with its reflection in one of its sides is a fundamental region R for $\Delta(a,2,b)$. The *universal map* $\mathscr{U}_{\Delta(a,2,b)}$ of type $(a,2,b)$ is the map on $\mathscr{X}$ (or on the augmentation of $\mathscr{X}$ by ideal vertices in the case that a or b is infinite) whose edges and vertices are the images under the action of $\Delta(a,2,b)$ of the side of T subtended by the angle $\frac{\pi}{b}$ and the vertex of T

[1] If a or b is infinite then the corresponding vertices of the triangular tessellation are taken to be ideal vertices adjoined to $\mathscr{X}$.

at the angle $\frac{\pi}{a}$ respectively. We may identify the darts of $\mathcal{U}_{\Delta(a,2,b)}$ with the fundamental region R and its images.

Map subgroups, paths and coverings

We define the *map subgroup* M in $\mathcal{C}_2^+(a,b)$ of a map $\mathcal{M}$ of type (m,n) dividing (a,b) to be $M = \Pi^{-1}(\mathrm{Stab}_G(\alpha))$ for some $\alpha \in \Omega$. It is said to be *canonical* for $\mathcal{M}$ in the case $(m,n) = (a,b)$. M is uniquely determined up to conjugacy, and it is natural to identify Ω with $\mathcal{C}_2^+(a,b)/M$ via the bijection $\omega g \mapsto Mg$ so that the action of $\mathcal{C}_2^+(a,b)$ by right multiplication on the cosets Mg is isomorphic to its action on Ω. In this way, words in $\mathcal{C}_2^+$(a,b) correspond to paths through the darts of $\mathcal{M}$, with group multiplication corresponding to composition of paths. Indeed, a map is essentially equivalent to the Schreier coset diagram of the subgroup M of $\mathcal{C}_2^+$(a,b).

Map coverings

For $i = 1,2$, let M_i be the map subgroup in $\mathcal{C}_2^+(a,b)$ for a map $\mathcal{M}_i$. We say that $\mathcal{M}_1$ *covers* $\mathcal{M}_2$ if there is a branched covering of the underlying surface of $\mathcal{M}_2$ by that of $\mathcal{M}_1$ which maps the vertices and edges of $\mathcal{M}_1$ onto those of $\mathcal{M}_2$, branching being permitted only over the vertices, midpoints of edges and centres of faces. Then $\mathcal{M}_1$ covers $\mathcal{M}_2$ if and only if $M_1 \leqslant M_2$ (up to conjugacy).

$\mathcal{M}$ is covered by $\mathcal{U}_{\mathcal{C}_2^+(a,b)}$: it is isomorphic to $\mathcal{U}_{\mathcal{C}_2^+(a,b)}/M$, and its underlying surface is homeomorphic to $\mathcal{X}/M$. Conversely, any subgroup $M \leqslant \mathcal{C}_2^+(a,b)$ is the map subgroup for the quotient map $\mathcal{U}_{\mathcal{C}_2^+(a,b)}/M$.

Map automorphisms

An *orientation-preserving automorphism* of a map $\mathcal{M}$ is a permutation of Ω preserving the relations of incidence; in other words, it is a permutation which commutes with ρ_0 and ρ_1, or equivalently with G. Automorphisms are clearly determined by their effect on any one dart, and they form a group $\mathrm{Aut}^+(\mathcal{M})$—the *orientation-preserving automorphism group* of $\mathcal{M}$—which is the natural generalization of the rotation group of a polyhedron or tessellation. This group is the centralizer $\mathrm{C}_{S^\Omega}(G)$ of G in S^Ω, and it acts faithfully and freely on Ω. If $\mathrm{N}_{\mathcal{C}_2^+(a,b)}(M)$ denotes the normalizer of the map subgroup M of $\mathcal{M}$ in $\mathcal{C}_2^+(a,b)$ then $\mathrm{Aut}^+(\mathcal{M})$ can be identified with $\mathrm{N}_{\mathcal{C}_2^+(a,b)}(M)/M$;

its action on Ω can be realized as the action of $\mathrm{N}_{\mathscr{C}_2^+(a,b)}(M)/M$ by left multiplication on the set $\mathscr{C}_2^+(a,b)/M$ of M-cosets.

Actions of triangle groups

We observe from the above remarks that $\Delta(a,2,b)$ has two different actions on $\mathscr{U}_{\Delta(a,2,b)}$: it acts as the monodromy group (via a right action) and as the orientation-preserving automorphism group (via a left action). In this thesis we use the convention that automorphisms of every mathematical object considered are composed on the left, so that $h \circ k$ means "apply k then h". Identifying Ω with $\mathscr{C}_2^+(a,b)/M$ as before, the right monodromy action is $Mg \mapsto Mga$ while the left automorphism action is $Mg \mapsto Mhg$ (or equivalently, $Mg \mapsto hMg$ since $h \in \mathrm{N}_\Delta(M)$).

1.2. Extensions to the theory

1.2.1. More general surfaces

Maps on more general surfaces

Analogous to (and encompassing) the theory of orientable maps on surfaces without boundary, a theory of maps on more general surfaces has been established [4]. In this theory, the surface S may be non-orientable and may have boundary; the graph $\mathcal{G}$ is embedded in such a way that vertices and free ends of free edges may be contained within the boundary, while no interior point of a (possibly free) edge may be thus contained unless the whole of that edge is. Edges which intersect the boundary are called *boundary edges*, while other edges are called *interior edges*. Faces may intersect the boundary in (at most) one component. The set Ω now consists of the *blades*—triples consisting of a vertex, edge and face, all mutually incident—and three permutations of Ω are specified: τ_0 transposes any pair of blades at either end of an edge but on the same side of it; τ_1 transposes any pair of blades with a vertex and face in common; τ_2 transposes any pair of blades with a common vertex on either side of an edge. Further, τ_0 acts by transposing the two blades incident with any free edge whose free end lies in the interior of the surface. Each permutation then fixes any blade which does not form part of an appropriate pair. Two blades are *adjacent* if they make up a 2-cycle of some τ_i.

The *type* of a map is the triple $(|\tau_1\tau_2|, |\tau_2\tau_0|, |\tau_0\tau_1|)$, which coincides with the earlier definition in terms of the least common multiples of the valencies of the vertices, edges and faces for maps without boundary (but which is slightly more cumbersome to describe when boundary is present).

The algebraic theory

A correspondence may be established between maps and permutation representations of the *full cartographic group*

$$\mathscr{C}_2 = \langle\, t_0, t_1, t_2 \,|\, t_0^2 = t_1^2 = t_2^2 = (t_2 t_0)^2 = 1 \,\rangle,$$

and between maps of type dividing (a, b) and permutation representations of the group

$$\mathscr{C}_2(a, b) = \langle\, t_0, t_1, t_2 \,|\, t_0^2 = t_1^2 = t_2^2 = (t_1 t_2)^a = (t_2 t_0)^2 = (t_0 t_1)^b = 1 \,\rangle.$$

All of the results for orientable maps without boundary go through with only slight modification in this wider theory. We may identify the blades with the triangles T discussed in §1.1.2 for $\mathscr{U}_{\Delta(a,2,b)}$; these are the fundamental regions for $\mathscr{C}_2(a, b)$. While we lose the connection with Riemann surface theory, the maps can instead be regarded as lying on Klein surfaces.

A (possibly non-orientable) map has neither boundary nor free edges if and only if its map subgroup in $\mathscr{C}_2$ is torsion-free. Indeed, much of the theory of general maps restricts to this smaller category.

1.2.2. Hypermaps

Generalizations of maps

We can go further and remove the somewhat artificial restriction that an edge must be at most 2-valent. Informally, an *orientable hypermap without boundary* is a cellular embedding in an orientable surface without boundary of a *hypergraph*: a set Ω of elements called *hyperdarts* which has two partitions, the elements of each partition being known as *hypervertices* and *hyperedges* respectively, with *incidence* corresponding to non-empty intersection. (See [7] for a survey of combinatorial hypermap theory.) Note that there is an obvious correspondence between maps and those hypermaps whose hyperedges have valency at most 2. We often describe the hypervertices, hyperedges and hyperfaces of a hypermap as the *i-components* of the hypermap for $i = 0, 1, 2$ respectively, and i is called the *dimension* of the component.

The Cori and Walsh representations

Whereas there is a natural topological definition of a map, there are several ways of representing an orientable hypermap $\mathscr{H}$ on a surface $\mathcal{S}$. The Cori representation [6] of orientable hypermaps without boundary uses closed polygonal discs to represent the hypervertices and hyperedges. Hypervertices are mutually disjoint, as are hyperedges, and hypervertices meet hyperedges at a finite set of points—the hyperdarts (Cori called them 'brins'). The hyperfaces are the complementary regions of $\mathcal{S}$, homeomorphic to open discs. The orientation of $\mathcal{S}$ induces cyclic orderings of the hyperdarts around each hypervertex and hyperedge, and these are the cycles of two permutations ρ_0 and ρ_1; the cycles of $\rho_2 = (\rho_0\rho_1)^{-1}$ correspond to the hyperfaces.

In the Walsh representation [60], an orientable hypermap is modelled by an embedding of a bipartite graph as a map $\mathscr{M}$. The vertices in one partite set represent the hypervertices; those in the other represent the hyperedges; the faces of $\mathscr{M}$ represent the hyperfaces; and the edges of $\mathscr{M}$ represent the hyperdarts.

The James representation

Both the Cori and Walsh representations disguise the clear algebraic triality between the i-components. In the James representation [24], a hypermap is modelled as a trivalent map $\mathscr{T}$ on $\mathcal{S}$ with the faces labelled $i = 0$, 1 and 2 so that each edge of $\mathscr{T}$ separates faces with different labels. The i-components are represented by the faces labelled i. When $\mathcal{S}$ is orientable, the hyperdarts are represented by those edges of $\mathscr{T}$ which border a hypervertex and a hyperedge. The James representation also extends naturally to surfaces which may be non-orientable or with boundary. As with maps, we must replace the concept of hyperdarts (corresponding to hypervertex-hyperedge incidence) with that of *hyperblades* which correspond to hypervertex-hyperedge-hyperface incidence. The set Ω of hyperblades is easily identified with the vertex set of $\mathscr{T}$; we label each edge of $\mathscr{T}$ with the complement of the labels of its incident faces, and for $i = 0, 1, 2$ we define τ_i to be the permutation of Ω which transposes each pair of hyperblades that form the ends of an edge coloured i. (If $\mathcal{S}$ has boundary then further conditions have to be imposed on the embedding of $\mathscr{T}$, and the τ_i must be allowed fixed points.)

The algebraic theory

Every aspect of the theory of algebraic hypermaps follows analogously to the theory of maps [59, 8] but using the *hypercartographic* and *full hypercartographic* groups

$$\mathscr{H}_2^+ = \langle\, r_0, r_1, r_2 \mid r_0 r_1 r_2 = 1 \,\rangle \cong \Delta(\infty, \infty, \infty) \cong C_\infty * C_\infty$$

and

$$\mathscr{H}_2 = \langle\, t_0, t_1, t_2 \mid t_0^2 = t_1^2 = t_2^2 = 1 \,\rangle,$$

and their quotients $\mathscr{H}_2^+(m_0, m_1, m_2)$ and $\mathscr{H}_2(m_0, m_1, m_2)$.

The definition of *type* given for general maps is equally valid for hypermaps. The Euler characteristic of a hypermap $\mathscr{H}$ on a surface without boundary is $N_0+N_1+N_2-N$ where N_i denotes the number of i-components and N denotes the number of hyperdarts.

1.3. Symmetry in hypermaps

A hypermap $\mathscr{H}$ without boundary of type (m_0, m_1, m_2) is said to be *uniform* if every i-component has the same valency m_i for each i. If $\mathscr{H}$ is of finite type, then it is uniform if and only if its canonical hypermap subgroup is torsion-free; and if $\mathscr{H}$ is finite, then it is uniform if and only if its canonical hypermap subgroup is a surface group.

A hypermap $\mathscr{M}$ is *reflexible* if its *full automorphism group* $\mathrm{Aut}(\mathscr{M})$ (consisting of those permutations of the hyperblades which commute with the permutations τ_0, τ_1 and τ_2) acts transitively and hence regularly on Ω. Note that every reflexible hypermap without boundary must be uniform; in fact reflexible hypermaps have the greatest possible degree of symmetry. Equivalent ways of expressing the reflexibility condition are: $\mathrm{Aut}(\mathscr{M})$ is as large as possible; the monodromy group G acts regularly as a permutation group on Ω; the hypermap subgroup M is the kernel of the natural homomorphism $\pi\colon \mathscr{H}_2 \to G$; M is normal in $\mathscr{H}_2$. When $N = |\Omega|$ is finite, an equivalent condition is that $\mathrm{Aut}(\mathscr{M})$ has order $N = |\Omega|$.

This definition of reflexibility is equivalent to that given by Coxeter and Moser [11, §8.1] for maps without boundary; Wilson [61] calls such maps ‘regular’, a term which Coxeter and Moser use for a condition which is equivalent to reflexibility for non-orientable maps without boundary, but slightly weaker for orientable ones, merely requiring that M be normal in the *even subgroup* (which may be identified with $\mathscr{H}_2^+$) consisting of the words of even length in the generators τ_0, τ_1 and τ_2. This latter condition is equivalent to $\mathrm{Aut}^+(\mathscr{M})$ acting transitively on the hyperdarts of $\mathscr{M}$, and it is this meaning that we too attach to the term *regular* in the more general context of orientable hypermaps without boundary. (Some authors use the term *rotary* in place of regular.) Every reflexible orientable hypermap without boundary is regular. A regular, non-reflexible hypermap is called *chiral*.

On the sphere, the reflexible maps are the platonic solids, the finite maps of type $(m,2,2)$ and $(2,2,m)$ (dipoles and dihedra), and the finite star maps. The sphere admits no chiral hypermaps.

For conciseness, we may use the term *symmetrical* in place of regular (respectively, reflexible) when we are talking in the context of orientable hypermaps without boundary (respectively, completely general hypermaps), particularly if do not wish to distinguish these contexts.

1.4. Triangular groups

We have seen that the groups $\mathscr{H}_2^+(m_0,m_1,m_2)$ and $\mathscr{H}_2(m_0,m_1,m_2)$ (which include the groups $\mathscr{C}_2^+(a,b)$ and $\mathscr{C}_2(a,b)$) are central to the algebraic theory of hypermaps. These groups have the abstract structure of triangle groups and extended triangle groups, the former being generated by *rotations* (orientation-preserving isometries) of a simply-connected Riemann surface $\mathscr{X}$ about the (possibly ideal) vertices of a triangle T, and the latter being generated by reflections (orientation-reversing isometries of $\mathscr{X}$) in the sides of T. We attach to triangle groups and extended triangle groups the adjective *triangular*, and describe them as *spherical*, *Euclidean* or *Fuchsian* according to the geometry of the surface of $\mathscr{X}$, often using the term *planar* for the non-spherical ones. (Of course, an extended Fuchsian triangle group is not itself Fuchsian!)

Abusing the common notation $\Delta(m_0,m_1,m_2)$ for triangle groups, we will usually use it to denote the abstract structure as given by $\mathscr{H}_2^+(m_0,m_1,m_2)$, as well as to denote concrete isometry groups. We call the m_i the *periods* of the group, and we regard them as ordered in this notation so that there is more than one notation for the same triangle group. This removes any ambiguity as to the universal hypermap $\mathscr{U}_{\Delta(m_0,m_1,m_2)}$ that the group induces, whose quotients are the hypermaps of type dividing (m_0,m_1,m_2) where

m_i represents the valency of the i-component of $\mathcal{U}_{\Delta(m_0,m_1,m_2)}$. This hypermap lies on the Riemann sphere, the complex plane or the hyperbolic plane $\mathcal{H}$ (with adjoined ideal points if some $m_1 = \infty$) according as $\frac{1}{m_0} + \frac{1}{m_1} + \frac{1}{m_2} > 1, = 1, < 1$. We often ignore the complex structure and identify the Riemann sphere and the complex plane with the topological sphere S^2 and the Euclidean plane respectively. The universal hypermaps on the sphere are the reflexible maps, while the infinite-valent dipole $\mathcal{U}_{\Delta(\infty,2,2)}$ and the infinite-valent star map $\mathcal{U}_{\Delta(\infty,1,\infty)}$ both lie on the Euclidean plane augmented by ideal points; the former consists of parallel lines (with two ideal vertices) and the latter consists of parallel half-lines (with one ideal vertex).

We use the notation $\Delta[m_0, m_1, m_2]$ for extended triangle groups. (Note that $\mathcal{C}_2 \cong \Delta[\infty, 2, \infty]$ and that $\mathcal{H}_2 \cong \Delta[\infty, \infty, \infty]$.) Such a group has $\Delta(m_0, m_1, m_2)$ as a subgroup of index 2. There is no distinction between $\mathcal{U}_{\Delta(m_0,m_1,m_2)}$ and $\mathcal{U}_{\Delta[m_0,m_1,m_2]}$, but of course the subtessellations consisting of fundamental regions for $\Delta(m_0, m_1, m_2)$ and $\Delta[m_0, m_1, m_2]$ differ.

1.5. Signatures of Fuchsian groups

The theory described here shall be useful for §4.5. Every finitely-generated Fuchsian group G has a *signature*

$$(g; m_0, \ldots, m_{r-1}; s; t), \tag{i}$$

which means that G has a presentation with generators

$a_0, b_0, \ldots, a_{g-1}, b_{g-1},$	(hyperbolic elements)
$x_0, \ldots, x_{r-1},$	(elliptic elements)
$y_0, \ldots, y_{s-1},$	(parabolic elements)
$z_0, \ldots, z_{t-1}$	(hyperbolic boundary elements)

and defining relations

$$x_0^{m_0} = \cdots = x_{r-1}^{m_{r-1}} = [a_0, b_0] \ldots [a_{g-1}, b_{g-1}] x_0 \ldots x_{r-1} y_0 \ldots y_{s-1} z_0 \ldots z_{t-1} = 1,$$

where $[a, b] = a^{-1}b^{-1}ab$. The *measure* $\mu(G)$ of G is the measure (or hyperbolic area) $\mu(R)$ of a fundamental polygon R for G; if $\mu(G) < \infty$ then there are no hyperbolic boundary elements, and the signature is written $(g; m_0, \ldots, m_{r-1}; s)$. In this case,

$$\mu(G) = 2\pi\left(2g - 2 + \sum_{i=0}^{r-1}\left(1 - \frac{1}{m_i}\right) + s\right)$$

and g is the genus of the compactification of the quotient surface R/G. The subgroups $\langle x_i \rangle$, each of order m_i, are representatives of the conjugacy classes of maximal elliptic subgroups of G, while the subgroups $\langle y_i \rangle$, each of infinite order, are representatives of the conjugacy classes of maximal parabolic subgroups.

We are interested in these matters from the point of view of triangle groups, where R is the union of a hyperbolic triangle with its reflection in one of its sides. In this case, $g = 0$ and the s parabolic elements are often regarded as 'infinite-order elliptic rotations' (with periods $m_i = \infty$) about a vertex subtending an angle of $\frac{1}{\infty} = 0$. This gives rise to a common alternative signature: $(3, 2, \infty)$ in place of $(0; 3, 2; 1)$, for example. (We note here that much of the theory described in the section applies equally or has a close analogue in the case of Euclidean or spherical triangle groups.)

If $\mu(G) < \infty$ and H is a subgroup of finite index in G, then

$$\mu(H) = |G\!:\!H|\,\mu(G) < \infty;$$

this is the *Riemann-Hurwitz* formula. Moreover, H is finitely-generated and has a signature that is closely related to that of G, through the action of G by multiplication on the set G/H of cosets Hg of H in G. This relationship is embodied in Singerman's Theorem [55].

Theorem 1.5.1 (Singerman). *Let G have signature* (i). *Then G contains a subgroup H of index n with signature*

$$(g'; l_{01}, l_{02}, \ldots, l_{0\rho_0}, \ldots, l_{(r-1)1}, l_{(r-1)2}, \ldots, l_{(r-1)\rho_{r-1}}; s'; t')$$

if and only if

(i) *There exists a finite permutation group P transitive on n points, and an epimorphism $\theta\colon G \to P$ satisfying the following conditions:*

(a) *The permutation $\theta(x_i)$ has precisely ρ_i cycles of lengths less than m_i, the lengths of these cycles being $m_i/l_{i1}, \ldots, m_i/l_{i\rho_i}$.*

(b) *If we denote the number of cycles in the permutation* $\theta(\gamma)$ *by* $\delta(\gamma)$ *then*

$$s' = \sum_{j=0}^{s-1} \delta(y_j), \qquad t' = \sum_{k=0}^{t-1} \delta(z_k).$$

(ii) $\mu(H) = n\mu(G)$. ✠

A sketch proof of part of this result is as follows. An elliptic generator x_i of G, of order m_i, must be represented on G/H by cycles (the orbits of the induced action of $\langle x_i \rangle$ on G/H) of lengths l_{ij} dividing m_i, with $\sum_j l_{ij} = |G:H|$. If Hg lies in a *long cycle* of x_i (one of length m_i) then no conjugate of a non-trivial power of x_i lies in H and hence x_i does not give rise to an elliptic element of H.

On the other hand, if Hg lies in a *short cycle* of x_i (one of length $l_{ij} < m_i$) then $Hgx_i^{l_{ij}} = Hg$, so $gx_i^{l_{ij}}g^{-1}$ is an elliptic element x_{ij} of H of order $m_{ij} = m_i/l_{ij}$. Changing the coset representative from g to $g' = hg$ ($h \in H$) merely replaces x_{ij} with the conjugate element $hx_{ij}h^{-1}$ of H, and choosing a different coset Hgx_i^k from the same cycle has no effect on x_{ij}. Thus each short cycle of x_i gives rise to a conjugacy class of elliptic elements of H, conjugate in G to a power of x_i.

Conversely, every elliptic element of H is also an elliptic element of G and is therefore conjugate to a power of some x_i, so it arises in this way. Thus the conjugacy classes of maximal elliptic subgroups of H correspond to the short cycles of the elliptic generators of G, each cycle of x_i of length $l_{ij} < m_i$ yielding an elliptic period $m_{ij} = m_i/l_{ij}$ of H. (Note that fixed-points count as cycles of length 1, even though they are often omitted when writing a permutation as a product of disjoint cycles.)

We can show similarly that the conjugacy classes of maximal parabolic subgroups of H correspond to the cycles of the parabolic generators y_i of G on the cosets of H, each cycle of length l_{ij} yielding a parabolic element conjugate in G to $y_i^{l_{ij}}$. Hence the parabolic class number of H is equal to the total number of cycles of the parabolic generators of G on G/H.

When $\mu(G) < \infty$ the Riemann-Hurwitz formula can now be used to determine the genus of H.

Chapter 2

Map operations

2.1. Outer automorphisms of $\mathscr{C}_2$

There is a bijection between maps and transitive permutation representations of the full cartographic group

$$\mathscr{C}_2 = \langle\, l, r, t \mid l^2 = r^2 = t^2 = (tl)^2 = 1 \,\rangle,$$

or more strictly between isomorphism classes in each category. (See §1.2; instead of t_0, t_1, t_2 we use the symbols l, r, t to stand for 'longitudinal', 'rotary' and 'transverse' which describe the action of the corresponding permutations $\lambda = \tau_0$, $\rho = \tau_1$, $\tau = \tau_2$ on the set Ω of blades.) Following Jones and Thornton [35] we define an *operation* on maps to be any transformation of maps induced by a group automorphism of $\mathscr{C}_2$: let θ be such an automorphism, and let $\mathscr{M}$ be a map, with map subgroup M, associated with a permutation representation $\Pi\colon \mathscr{C}_2 \to S^{\Omega}$; then we define $\Theta(\mathscr{M})$ to be the map, with map subgroup $M^{\Theta} = \theta(M)$,[1] associated with the representation $\theta \circ \Pi\colon \mathscr{C}_2 \to S^{\Omega}$ of $\mathscr{C}_2$. Isomorphic maps have conjugate map subgroups and so the inner automorphism group $\mathrm{Inn}(\mathscr{C}_2)$ acts trivially on maps; we thus have an induced action of the outer automorphism group $\mathrm{Out}(\mathscr{C}_2) = \mathrm{Aut}(\mathscr{C}_2)/\,\mathrm{Inn}(\mathscr{C}_2)$.

It is clear that $\mathscr{C}_2$ is a free product $\mathscr{C}_2 = K * C$, where

$$K = \langle\, l, t \mid l^2 = t^2 = (tl)^2 = 1 \,\rangle$$

[1] We use the convention that operations have a left action on maps; in [35] operations act on the right and so $M^{\Theta} = \theta^{-1}(M)$ is used.

is a Klein four-group and

$$C = \langle\, r \mid r^2 = 1 \,\rangle$$

is a cyclic group of order 2. It is shown in [35] that Aut($\mathscr{C}_2$) is a split extension of Inn($\mathscr{C}_2$) $\cong$ $\mathscr{C}_2$ by a complement $S \cong S_3$ which fixes r and permutes $\{l, t, tl\}$. Thus Out($\mathscr{C}_2$) is isomorphic to S_3, and the six cosets of Inn($\mathscr{C}_2$) in Aut($\mathscr{C}_2$) are represented by the elements of S. Since a map's vertices, edges, faces and *Petrie circuits* (closed 'zig-zag' paths in the underlying graph such that at each vertex the adjacent edges enclose a single face on the right and on the left alternately) correspond to the orbits of the subgroups $\langle\, r, t \,\rangle$, $\langle\, l, t \,\rangle$, $\langle\, l, r \,\rangle$ and $\langle\, r, tl \,\rangle$, we see that there are precisely six distinct map operations (including the identity operation I which acts trivially): their effect on a map is to interchange the sets of vertices, faces and Petrie circuits while leaving the set of edges invariant.

Since Aut($\mathscr{C}_2$) preserves inclusions and normalizers of subgroups, the induced operations preserve coverings and automorphism groups of maps. Moreover, Aut($\mathscr{C}_2$) preserves normality and finiteness of index of subgroups of $\mathscr{C}_2$, and so the operations preserve reflexibility and finiteness of maps. If M is a torsion-free map subgroup in $\mathscr{C}_2$ then so is M^{Θ}, and thus map operations restrict to the category of maps without free edges on surfaces without boundary.

An earlier, non-algebraic description of these six map operations was given by Wilson [61] and Lins [41]. Moreover, the operation P described in Example 2.1.2 below is implicit in [11]. More recently, map operations have been considered by Léger [38] as part of a wider characterization of autoequivalences, up to natural equivalence, of the category of G-sets as outer automorphisms of G, where G is any group.

Example 2.1.1. The familiar duality operation D, which transposes the sets of vertices and faces of a map $\mathscr{M}$ whilst leaving edges and Petrie circuits invariant, is induced by the automorphism of $\mathscr{C}_2$ which transposes l and t. D($\mathscr{M}$) is respectively reflexible, regular, uniform if and only if $\mathscr{M}$ is. ▲

Example 2.1.2. If we 'dissolve' the faces of a map $\mathscr{M}$ to leave a skeleton of vertices and edges, and then span by a membrane each cycle of edges which forms a Petrie circuit in $\mathscr{M}$, then the resulting object (the *Petrie dual* of $\mathscr{M}$) is also a map on a surface, although in general a different surface from that of $\mathscr{M}$. This operation P is induced by the automorphism of $\mathscr{C}_2$ which transposes l and tl, and it interchanges faces and Petrie circuits whilst leaving the underlying graph of $\mathscr{M}$ invariant. P($\mathscr{M}$) is reflexible if and only if $\mathscr{M}$ is; however, if $\mathscr{M}$ is chiral, P($\mathscr{M}$) must be uniform but not necessarily regular. ▲

Example 2.1.3. Wilson [61] described a third map operation, Opp, called the *opposite* operation. This involution is obtained by composing the dual and Petrie dual operations as Opp = PDP = DPD, and it is induced by the automorphism of $\mathscr{C}_2$ which transposes t and tl. Opp($\mathscr{M}$) can be obtained from a map $\mathscr{M}$ by making a cut along each edge and rejoining corresponding sides of the cut in opposite directions. Thus Opp preserves faces, but Petrie circuits in $\mathscr{M}$ correspond to vertices in Opp($\mathscr{M}$), and vice versa, with circuit length corresponding to vertex valency. If $\mathscr{M}$ is chiral, Opp($\mathscr{M}$) must be uniform but not necessarily regular. ▲

We note that the group of map operations is generated by any two of D, P and Opp.

2.2. Orientability of maps

In this section we provide topological and combinatorial criteria for the image of a map under a map operation to be orientable.

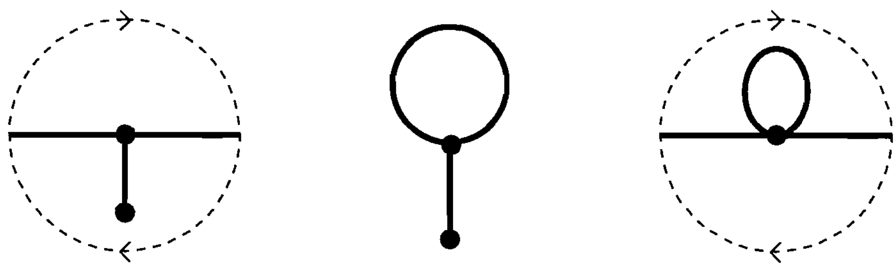
Proposition 2.2.1. *Duality is the only non-trivial operation which necessarily preserves orientability.*

Proof. It is immediate that duality preserves orientability since it preserves the underlying surface of a map. The simple example shown in Figure 2.1 demonstrates the result: $\mathscr{M}$ is a loop incident with a second edge, embedded in the sphere, while P($\mathscr{M}$) and Opp($\mathscr{M}$) are a dual pair of maps on the projective plane. ■

A map is orientable without boundary if and only if its map subgroup in $\mathscr{C}_2$ lies in the index-2 even subgroup $\langle rt, tl, lr \rangle$ which can be identified with $\mathscr{C}_2^+$. Hence, if we let Θ be a map operation and $\mathscr{M}$ be a map, then $\Theta(\mathscr{M})$ is orientable without boundary if and only if $M^\Theta \leqslant \mathscr{C}_2^+$, that is, $M \leqslant \theta^{-1}(\mathscr{C}_2^+)$. An equivalent condition is that $\mathscr{M}$ covers the 2-blade map $\mathscr{N}_\Theta$ whose map subgroup is $\theta^{-1}(\mathscr{C}_2^+)$.

Theorem 2.2.2. *Let $\mathscr{M}$ be a map on a surface without boundary.*

(i) *$\mathscr{M}$ is orientable if and only if it is possible to 2-colour the blades so that adjacent blades have different colours.*



P($\mathcal{M}$) $\mathcal{M}$ Opp($\mathcal{M}$)

Figure 2.1. A self-dual map $\mathcal{M}$ on the sphere, and its images under P and Opp on the projective plane.

(ii) Opp($\mathcal{M}$) *is orientable without boundary if and only if it is possible to direct the edges of* $\mathcal{M}$ *so that around each face, the edges follow a constant direction. The same is true for* D P($\mathcal{M}$).

(iii) P($\mathcal{M}$) *is orientable without boundary if and only if* $\mathcal{M}$ *has no free edges and it is possible to 2-colour the blades of* $\mathcal{M}$ *so that for each blade* β, β *and* $\beta\lambda$ *have the same colour, while* $\beta\tau$ *and* $\beta\rho$ *have the other colour. The same is true for* P D($\mathcal{M}$).

Proof. For each operation Θ, Table 2.1 gives the permutation of $\{l, t, tl\}$ induced by θ, together with the image under θ^{-1} of $\mathcal{C}_2^+$, and the map $\mathcal{N}_\Theta$ such that $\Theta(\mathcal{M})$ is orientable without boundary if and only if $\mathcal{M}$ covers $\mathcal{N}_\Theta$. Observe that $\mathcal{N}_\Theta = \mathcal{N}_{\mathrm{D}\Theta}$. We see that $\mathcal{N}_\mathrm{I}$ is the *trivial orientable map without boundary* consisting of a single vertex and a free edge on the sphere; $\mathcal{N}_\mathrm{P}$ consists of a single vertex in the interior of the disc, with a free edge whose free end lies on the boundary; and $\mathcal{N}_{\mathrm{Opp}}$ consists of a loop running around the boundary of the disc. The map $\mathcal{M}$ covers $\mathcal{N}_\Theta$ if and only if it is possible to 2-colour the blades of $\mathcal{M}$ so that the monodromy action of $\mathcal{C}_2$ restricts to an action on the colour sets that is isomorphic to the monodromy action of $\mathcal{C}_2$ on $\mathcal{N}_\Theta$. Each of λ, ρ and τ interchanges the two blades of $\mathcal{N}_\mathrm{I}$ and so (i) follows immediately.

Both λ and ρ interchange the two blades of $\mathcal{N}_{\mathrm{Opp}}$ while τ fixes them, and so $\mathcal{M}$ covers $\mathcal{N}_{\mathrm{Opp}}$ if and only $\mathcal{M}$ has no free edges and there is a 2-colouring of its darts[2] such that the two darts on each edge are coloured differently, and the valency of each vertex is even with the darts alternating in colour in cyclic order around the vertex. An

[2]For a map without boundary, a dart can be represented as the union of a blade and its image under the permutation τ.

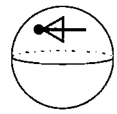
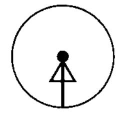
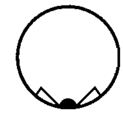
Operation Θ	Permutation	$\theta^{-1}(\mathscr{C}_2^+)$	$\mathscr{N}_\Theta$
I	1	$\langle rt, tl, lr \rangle = \mathscr{C}_2^+$	
D	(l, t)	$\langle rl, lt, tr \rangle = \mathscr{C}_2^+$	
P	(l, tl)	$\langle rt, l, tlr \rangle = \langle rt, l \rangle$	
D P	(t, l, tl)	$\langle rtl, l, tr \rangle = \langle rt, l \rangle$	
P D	(l, t, tl)	$\langle rl, t, tlr \rangle = \langle rl, t \rangle$	
D P D = Opp	(t, tl)	$\langle rtl, t, lr \rangle = \langle rl, t \rangle$	

Table 2.1. The six map operations.

equivalent condition is given in (ii).

To prove (iii), we may either argue directly as before, or note that $\mathscr{N}_{\mathrm{P}} = \mathrm{D}(\mathscr{N}_{\mathrm{Opp}})$, while $\mathscr{M}$ covers $\mathscr{N}_\Theta$ if and only if $\mathrm{D}(\mathscr{M})$ covers $\mathrm{D}(\mathscr{N}_\Theta)$, so that the condition in (iii) is the dual of that given above for Opp. ∎

Theorem 2.2.2 (i) gives the (almost classical) criterion for a map to be orientable without boundary. However, when attempting to apply a map operation Θ, it is also useful to have criteria for properties of $\Theta(\mathscr{M})$ based on properties of $\mathscr{M}$ itself. This theorem provides such criteria.

Remark 2.2.3. Recall that the map operations restrict to the category of maps without free edges and boundary. For non-trivial Θ other than D, Theorem 2.2.2 shows that if $\mathscr{M}$ is without boundary and $\Theta(\mathscr{M})$ is orientable then in fact both $\mathscr{M}$ and $\Theta(\mathscr{M})$ are without free edges and boundary. ♦

Remark 2.2.4. For an orientable map $\mathscr{M}$ without free edges and boundary, Theorem 2.2.2 tells us that $\mathrm{Opp}(\mathscr{M})$ is orientable if and only if $\mathscr{M}$ is 2-face-colourable, and that $\mathrm{P}(\mathscr{M})$ is orientable if and only if $\mathscr{M}$ is 2-vertex-colourable. The latter extends the analogous result of Wilson [61] in the case that $\mathscr{M}$ is, in addition, reflexible.

We shall soon see that when $\mathscr{M}$ is orientable and without boundary, but has free edges, the 2-vertex-colourability of $\mathscr{M}$ is still equivalent to the orientability of $\mathrm{P}(\mathscr{M})$ (but the presence of free edges in $\mathscr{M}$ ensures that $\mathrm{P}(\mathscr{M})$ has boundary); however, this criterion fails when $\mathscr{M}$ is non-orientable or with boundary, as shown in Figure 2.2.

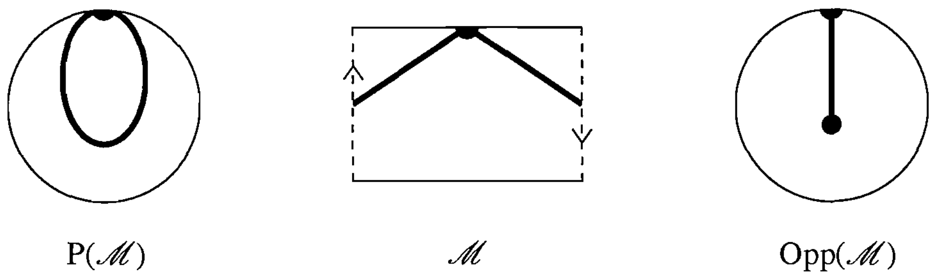


FIGURE 2.2. A self-dual map $\mathscr{M}$ on the Möbius band, and its images under P and Opp on the disc.

Likewise, when $\mathscr{M}$ is orientable and without free edges, but has boundary, the 2-face-colourability of $\mathscr{M}$ is still equivalent to the orientability of Opp($\mathscr{M}$) (but the presence of boundary ensures that Opp($\mathscr{M}$) has boundary or free edges); however, this criterion fails when $\mathscr{M}$ is non-orientable or has free edges, as Figure 2.2 also demonstrates. ♦

We now develop criteria for the orientability of the images under the six map operations of a map $\mathscr{M}$ on a surface which possibly has boundary.

Definition 2.2.5. A 2-colouring of the blades of a map is called *proper* if adjacent blades (distinct blades in each orbit of the elements l, r and t of $\mathscr{C}_2$) have different colours. •

The following criterion for the orientability of a map is given in [4] as an immediate consequence of results in [22] on the structure of subgroups of discontinuous groups containing reflections.

Theorem 2.2.6 (Bryant and Singerman). *A map $\mathscr{M}$ is orientable if and only if it admits a proper 2-blade-colouring.* ✢

Hence the result for maps without boundary extends to the category of all maps. Let us define another blade colouring property which a map $\mathscr{M}$ may possess.

Property (P1). There exists a 2-colouring of the blades of $\mathscr{M}$ such that the following hold.

(i) For each blade β of $\mathscr{M}$ not fixed by λ, β and $\beta\lambda$ have different colours if and only if their common edge is free or contained in the boundary of $\mathscr{M}$.

(ii) For each blade β of $\mathscr{M}$ not fixed by ρ, β and $\beta\rho$ have different colours.

(iii) For each blade β of $\mathscr{M}$ not fixed by τ, β and $\beta\tau$ have different colours.

Note that if a map $\mathscr{M}$ containing a loop has Property P1 then that loop is completely contained in a boundary component.

Proposition 2.2.7. *Let $\mathscr{M}$ be a map, possibly with boundary. Then* P($\mathscr{M}$) *is orientable if and only if $\mathscr{M}$ has Property P1.*

Proof. Suppose P($\mathscr{M}$) is orientable. Then we may properly 2-colour the blades so that adjacent blades have different colours. Applying the operation P to P($\mathscr{M}$) then induces a colouring of the blades of the map P(P($\mathscr{M}$)) = $\mathscr{M}$. Denote the longitudinal, rotary and transpose permutations of the blades of P($\mathscr{M}$) by $\overline{\lambda}$, $\overline{\rho}$ and $\overline{\tau}$.

Since $\rho = \overline{\rho}$ we have $\beta \neq \beta\rho = \beta\overline{\rho}$. As β and $\beta\overline{\rho}$ have different colours in P($\mathscr{M}$), so β and $\beta\rho$ have different colours in the induced colouring of $\mathscr{M}$. The same argument holds for β and $\beta\tau$.

Now suppose $\beta \neq \beta\lambda$.

If β and $\beta\lambda$ lie on a free edge of $\mathscr{M}$ then the free end of this free edge does not lie on the boundary. Hence $\beta = \beta\lambda\tau = \beta\overline{\lambda}$ and thus $\beta\overline{\tau} = \beta\overline{\lambda}\overline{\tau} = \beta\lambda$. As β and $\beta\overline{\tau}$ have different colours in P($\mathscr{M}$), so β and $\beta\lambda$ have different colours in the induced colouring of $\mathscr{M}$.

If β and $\beta\lambda$ lie on a non-free edge contained in the boundary of $\mathscr{M}$ then $\beta = \beta\tau = \beta\overline{\tau}$ and thus $\beta\overline{\lambda} = \beta\overline{\tau}\overline{\lambda} = \beta\lambda$. Thus β and $\beta\overline{\lambda}$ have different colours in the colourings of both P($\mathscr{M}$) and $\mathscr{M}$.

If β and $\beta\lambda$ lie on a non-free edge which is not fully contained in the boundary of $\mathscr{M}$ then β, $\beta\lambda$ and $\beta\lambda\tau$ are mutually distinct, and hence so are β, $\beta\overline{\lambda}\overline{\tau}$ and $\beta\overline{\lambda}$. Thus β and $\beta\overline{\lambda}\overline{\tau} = \beta\lambda$ have the same colour.

Conversely, suppose we can 2-colour the blades of $\mathscr{M}$ according to Property P1. Then essentially the same argument in reverse shows that in the induced colouring of P($\mathscr{M}$), adjacent blades have different colours. ∎

For an orientable map $\mathscr{M}$, possibly with boundary, Property P1 is equivalent to

Property (P2). The vertices of $\mathscr{M}$ may be 2-coloured such that those joined by an edge contained in the boundary have the same colour, while those joined by other edges have different colours.

Proof. If the blades of $\mathscr{M}$ are 2-coloured according to Property P1 then two blades in an orbit of τ, when regarded as part of a cyclic ordering of blades around a vertex following the orientation of the surface, induce an ordering of their distinct colours. The ordering is the same for all such pairs of blades at a given vertex, and it is clear from Property P1 that we can 2-colour the vertices accordingly.

Conversely, given such a 2-vertex-colouring of $\mathscr{M}$, we may also take a proper 2-blade-colouring since $\mathscr{M}$ is orientable. Choose one of the vertex colours and then reverse the colour of each blade incident with each vertex of that colour. The resulting colouring exhibits Property P1 for $\mathscr{M}$. ∎

Note that no orientable map with Property P1 may possess two vertices joined both by an edge contained in the boundary and another not. A *circuit* of a map $\mathscr{M}$ is a circuit of its underlying graph, that is, a finite ordered set $\{e_j\}$ of distinct non-free edges such that e_j and e_{j+1} (modulo $|\{e_j\}|$) are adjacent; we regard a loop as a circuit of length 1. It is easy to verify that Property P2 is equivalent to

Property (P3). The number of interior edges in any circuit of $\mathscr{M}$ is even.

When $\mathscr{M}$ is without boundary, Properties P2 and P3 simply reduce to the criterion of 2-vertex-colourability presented in Remark 2.2.4.

Another blade colouring property which a map $\mathscr{M}$ may possess is the following.

Property (O1). There exists a 2-colouring of the blades of $\mathscr{M}$ such that the following hold.

(i) For each blade β of $\mathscr{M}$ not fixed by λ, β and $\beta\lambda$ have different colours.

(ii) For each blade β of $\mathscr{M}$ not fixed by ρ, β and $\beta\rho$ have different colours.

(iii) For each blade β of $\mathscr{M}$ not fixed by τ, β and $\beta\tau$ have different colours if and only if their common edge is free.

Proposition 2.2.8. *Let $\mathscr{M}$ be a map, possibly with boundary. Then* Opp($\mathscr{M}$) *is orientable if and only if $\mathscr{M}$ has Property O1.*

Proof. Suppose Opp($\mathscr{M}$) is orientable. Then we may properly 2-colour the blades so that adjacent blades have different colours. Applying the operation Opp to Opp($\mathscr{M}$) then induces a colouring of the blades of the map Opp(Opp($\mathscr{M}$)) $= \mathscr{M}$. Denote the longitudinal, rotary and transpose permutations of the blades of Opp($\mathscr{M}$) by $\overline{\lambda}$, $\overline{\rho}$ and $\overline{\tau}$.

Since $\rho = \overline{\rho}$ we have $\beta \neq \beta\rho = \beta\overline{\rho}$. As β and $\beta\overline{\rho}$ have different colours in Opp($\mathscr{M}$), so β and $\beta\rho$ have different colours in the induced colouring of $\mathscr{M}$. The same argument holds for β and $\beta\lambda$.

Now suppose $\beta \neq \beta\tau$.

If β and $\beta\tau$ lie on a free edge of $\mathscr{M}$ whose free end intersects the boundary then $\beta = \beta\lambda = \beta\overline{\lambda}$ and thus $\beta\overline{\tau} = \beta\overline{\lambda}\overline{\tau} = \beta\tau$. As β and $\beta\overline{\tau}$ have different colours in Opp($\mathscr{M}$), so β and $\beta\tau$ have different colours in the induced colouring of $\mathscr{M}$.

If β and $\beta\tau$ lie on a free edge whose free end is contained in the interior of $\mathscr{M}$ then $\beta = \beta\lambda\tau = \beta\overline{\tau}$ and thus $\beta\overline{\lambda} = \beta\overline{\tau}\overline{\lambda} = \beta\tau$. Thus β and $\beta\overline{\lambda}$ have different colours in the colourings of both Opp($\mathscr{M}$) and $\mathscr{M}$.

If β and $\beta\tau$ lie on a non-free edge of $\mathscr{M}$ then β, $\beta\tau$ and $\beta\lambda\tau$ are mutually distinct, and hence so are β, $\beta\overline{\tau}\overline{\lambda}$ and $\beta\overline{\tau}$. Thus β and $\beta\overline{\lambda}\overline{\tau} = \beta\tau$ have the same colour.

Conversely, suppose we can 2-colour the blades of $\mathscr{M}$ according to Property O1. Then essentially the same argument in reverse shows that in the induced colouring of Opp($\mathscr{M}$), adjacent blades have different colours. ∎

When $\mathscr{M}$ has no free edges, Property O1 simply reduces to

Property (O2). The edges of $\mathscr{M}$ may be directed so that around each face, the edges follow a constant direction.

This is the property given in Theorem 2.2.2. In general, this property is not equivalent to 2-face-colourability. However, when we restrict our attention to less general maps, the following result shows that they are closely related.

A map $\mathscr{M}$ is *almost 2-face-colourable* if it admits a 2-colouring of its faces in which the faces on either side of a non-free edge have different colours. Let $\mathscr{M}_0$ be the map-like structure obtained from $\mathscr{M}$ by erasing any free edges. (If $\mathscr{M}$ has free edges which intersect a boundary component only at their free ends then the 'faces' of $\mathscr{M}_0$ may not be homeomorphic to an open disc or half disc.) Then $\mathscr{M}$ is almost 2-face-colourable if $\mathscr{M}_0$ is 2-face-colourable.

For an orientable map $\mathscr{M}$, Property O1 is equivalent to

Property (O3). $\mathscr{M}$ is almost 2-face-colourable.

Proof. It is easy to see that $\mathscr{M}$ has Property O1 if and only if $\mathscr{M}_0$ has, which is the same as saying that the edges of $\mathscr{M}_0$ can be directed so that the edges around each face follow a constant direction. Since $\mathscr{M}_0$ is orientable, this is equivalent to saying that the faces are 2-colourable, as we may colour them according to whether the direction of their incident edges agrees with a chosen orientation. ∎

For maps without free edges and boundary, this property simply reduces to that presented in Remark 2.2.4.

2.3. Map operations and reflexible torus maps

In this section we examine the orbits of the reflexible torus maps of type $(4,2,4)$, $(3,2,6)$ and $(6,2,3)$ under the action of the group of map operations. It is well known (e.g. [11, Ch. 8]) that all uniform torus maps are of one of these types, and that they can all be obtained as quotients of the universal maps of these types by identifying opposite sides of a parallelogram or hexagon with parallel opposite sides in the complex plane (the underlying Riemann surface of these universal maps). To simplify the notation, we identify the complex plane with the real plane $\mathbb{R}^2$ in the usual way.

2.3.1. Reflexible torus maps of type (4, 2, 4)

Consider the universal map $\mathscr{U} = \mathscr{U}_{\Delta(4,2,4)}$ on $\mathbb{R}^2$, whose vertices are the points with integer coordinates and whose edges join all pairs of vertices that are unit length apart. Note that the Petrie circuits of $\mathscr{U}$ are of infinite length (with ends 'at infinity') and fall into two infinite classes of parallel paths. We obtain a uniform map $\mathscr{M}$ of type $(4,2,4)$ on the torus by identifying opposite sides of the parallelogram whose vertices have coordinates

$$(0,0) \quad (b,c) \quad (-p,q) \quad (b-p,c+q)$$

for integers b, c and positive integers p, q. The parallelogram has area $bq + cp$ and hence $\mathscr{M}$ has this number of vertices. This map is regular if and only if $p = c$ and $q = b$ (the parallelogram is actually a square), and all regular torus maps of type $(4,2,4)$ can be constructed in this manner. Following [11] we denote such a regular map by $\{4,4\}_{b,c}$; an example is shown in Figure 2.3. The map is reflexible if and only if it is regular with $bc(b-c) = 0$.

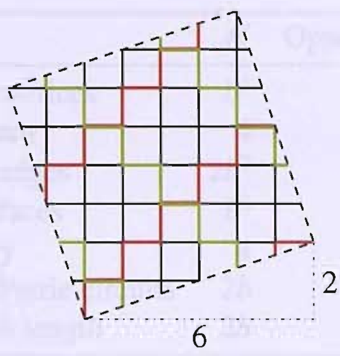


Figure 2.3. The map $\{4,4\}_{6,2}$ with two of its Petrie circuits highlighted.

Let $b = rj$ and $c = rk$ where $r = \mathrm{HCF}(b,c)$, and let $p = de$ and $q = df$ where $d = \mathrm{HCF}(p,q)$. We say that two vertices on the universal map $\mathscr{U}$ are *equivalent* if they cover the same vertex on $\mathscr{M}$ via the obvious covering. We may write the relative coordinates of one vertex v_1 from an equivalent vertex v_0 on $\mathscr{U}$ as $\lambda(b,c) + \mu(-p,q)$, where λ and μ can range over the integers. If in particular we set $\lambda = d(e+f)$ and $\mu = r(j-k)$ then these relative coordinates become $(rd(fj+ek), rd(fj+ek))$ and so v_0 and v_1 lie on the same Petrie circuit P of $\mathscr{U}$ (as do their images on $\mathscr{M}$). Now let $h = \mathrm{HCF}(\lambda,\mu)$, $s = \mathrm{HCF}(r,d)$ and $t = \mathrm{HCF}(e+f, j-k)$; simple calculation gives $h = st$. The relative coordinates from v_0 to the *nearest* equivalent vertex on P in the direction of v_1 are clearly $\frac{\lambda}{h}(b,c) + \frac{\mu}{h}(-p,q) = (\frac{rd}{h}(fj+ek), \frac{rd}{h}(fj+ek))$.

The set of Petrie circuits on $\mathscr{M}$ can be partitioned into two parallel classes (i.e. the graph-paths induced by the circuits in each class have no edges in common). Our calculations above tell us that the length (number of edges) of each path in the class containing P is $\frac{2rd}{st}(fj+ek)$, and since each vertex lies on two circuits in each class, the number of circuits in that class is

$$\frac{2(bq+cp)}{\frac{2rd}{st}(fj+ek)} = st.$$

Similar calculations tell us that the number of circuits in the second class is st' where $t' = \mathrm{HCF}(j+k, e-f)$, and each has length $2\frac{rd}{st'}(fj+ek)$. We define the *intersection multiplicity* of a circuit to be the number of distinct edge-intersections of that circuit with any given one which it intersects, which is simply the length of the first divided by the size of the class to which the second belongs. If all circuits have the same intersection multiplicity, we call this common value the *intersection multiplicity* of the

Property	$\mathcal{M}$	Opp($\mathcal{M}$)	P($\mathcal{M}$)
Number of vertices	b^2	$2b$	b^2
Vertex valency	4	$2b$	4
Number of edges	$2b^2$	$2b^2$	$2b^2$
Number of faces	b^2	b^2	$2b$
Face valency	4	4	$2b$
Number of Petrie circuits	$2b$	b^2	b^2
Petrie circuit length	$2b$	4	4

Table 2.2. Properties of the torus map $\mathcal{M} = \{4,4\}_{b,0}$ and of its images under the group of map operations.

map.

In determining the orbits of the reflexible torus maps $\{4,4\}_{b,c}$, there are essentially just two cases to consider: $c = 0$ and $b = c$. (The cases $c = 0$ and $b = 0$ give rise to isomorphic maps.) All torus maps $\mathcal{M} = \{4,4\}_{b,c}$ are self-dual and so lie in an orbit of length at most 3; we need only determine Opp($\mathcal{M}$) and its dual P($\mathcal{M}$).

Case $c = 0$

The Petrie circuits of $\mathcal{M} = \{4,4\}_{b,0}$ have length $2b$ and fall into two parallel classes, each of size b. Hence the underlying graph of Opp($\mathcal{M}$) is bipartite, each partite set having size b, and each vertex having valency $2b$. The intersection multiplicity of $\mathcal{M}$ is 2 and so this is the edge multiplicity of Opp($\mathcal{M}$).

Table 2.2 summarizes some basic combinatorial information. The underlying graph of Opp($\mathcal{M}$) is $K_{b,b}^{(2)}$, the graph obtained from the complete bipartite graph $K_{b,b}$ by 'doubling up' each of its edges. Opp($\mathcal{M}$) is a map of type $(2b,2,4)$ on a surface of characteristic $b(2-b)$. By Theorem 2.2.2, Opp($\mathcal{M}$) is orientable if and only if b is even, in which case the genus is $\frac{1}{2}(2-2b+b^2)$.

The dual P($\mathcal{M}$) of Opp($\mathcal{M}$) is an embedding of the underlying graph of $\mathcal{M}$ as a map of type $(4,2,2b)$ on the underlying surface of Opp($\mathcal{M}$).

The automorphism group of these maps is

$$(4,4 \mid 2,b) = \langle A,B,C \mid A^4 = B^4 = (AB)^2 = (A^{-1}B)^b = 1 \rangle$$

in the notation of [11, Table 7], isomorphic to $\mathcal{C}_2^+(4,4)/\langle (r_0r_1r_0)^b \rangle$ in the notation

Property	$\mathcal{M}$	Opp($\mathcal{M}$)	P($\mathcal{M}$)
Number of vertices	$2b^2$	$4b$	$2b^2$
Vertex valency	4	$2b$	4
Number of edges	$4b^2$	$4b^2$	$4b^2$
Number of faces	$2b^2$	$2b^2$	$4b$
Face valency	4	4	$2b$
Number of Petrie circuits	$4b$	$2b^2$	$2b^2$
Petrie circuit length	$2b$	4	4

TABLE 2.3. Properties of the torus map $\mathcal{M} = \{4,4\}_{b,b}$ and of its images under the group of map operations.

of §1.1 (identifying A with r_2 and B with r_0). This group is a semidirect product of a normal subgroup $C_b \times C_b$ (acting as 'translations' of $\mathcal{M}$) by a complement D_8 (stabilizing a vertex); it has order $8b^2$.

Note that $\mathcal{M}$ is invariant under the group of map operations if and only if $b = 2$.

Case $b = c$

The Petrie circuits of $\mathcal{M} = \{4,4\}_{b,b}$ have length $2b$ and fall into two parallel classes, each of size $2b$. Hence the underlying graph of Opp($\mathcal{M}$) is bipartite, each partite set having size $2b$, and each vertex having valency $2b$. The intersection multiplicity of $\mathcal{M}$ is 1 and so the underlying graph of Opp($\mathcal{M}$) is simple.

Table 2.3 summarizes some basic combinatorial information. Opp($\mathcal{M}$) has the complete bipartite graph $K_{2b,2b}$ as its underlying graph. It is a map of type $(2b, 2, 4)$ on a surface of characteristic $2b(2-b)$. By Theorem 2.2.2, Opp($\mathcal{M}$) is orientable for all b, with genus $(b-1)^2$. Hence Opp($\mathcal{M}$) is the map $\{4, 2b\}_4$ described in [11, §8.6].

P($\mathcal{M}$) is then $\{2b, 4\}_4$ listed in [11, Table 8]. The automorphism group of these maps is

$$G^{4,4,2b} = \langle A, B, C \mid A^4 = B^4 = C^{2b} = (AB)^2 = (BC)^2 = (CA)^2 = (ABC)^2 = 1 \rangle$$

in the notation of [11, Table 7], isomorphic to $\mathcal{C}_2(4,4)/\langle (trl)^{2b} \rangle$ in the notation of §1.1 and §2.1 (identifying A with lr, B with rt and C with trl). This group, of order $16b^2$, has the same structure as in the case $c = 0$ except that the 'translation' group is $C_{2b} \times C_b$.

Note that $\mathcal{M}$ is invariant under the group of map operations if and only if $b = 2$.

2.3.2. Reflexible torus maps of type (3, 2, 6) and (6, 2, 3)

Consider the universal map $\mathcal{U} = \mathcal{U}_{\Delta(6,2,3)}$ on the plane $\mathbb{R}^2$, a tessellation by equilateral triangles. The Petrie circuits of $\mathcal{U}$ are of infinite length and fall into three infinite classes of parallel paths. We construct two axes aligned at an angle $2\pi/3$, with the centre C of one of the equilateral triangles T as origin, each axis passing through the midpoint of an edge of that triangle; and we define the unit length to be the distance from C to a vertex of T. We obtain a uniform map $\mathcal{M}$ of type $(6, 2, 3)$ on the torus by identifying opposite sides of the hexagon whose vertices have coordinates

$$(0,0) \quad (b+c,c) \quad (-c,b) \quad (b-c,2b+c) \quad (2b,2b+2c) \quad (2b+c,b+2c)$$

for integers b, c. The hexagon has area $b^2 + bc + c^2$ and hence $\mathcal{M}$ has this number of vertices. Following [11] we denote this map $\mathcal{M}$ by $\{3,6\}_{b,c}$. This map is in fact regular, and all regular maps of this type on the torus are constructed in this way. The map is reflexible if and only if $bc(b-c) = 0$.

Let $b = rj$ and $c = rk$ where $r = \mathrm{HCF}(b, c)$. We may write the relative coordinates of one vertex v_1 from an equivalent vertex v_0 on $\mathcal{U}$ as $\lambda(b+2c, c-b) + \mu(b-c, 2b+c)$, where λ and μ can range over the integers. If in particular we set $\lambda = k$ and $\mu = j+k$ then these relative coordinates become $(r(j^2+jk+k^2), r(j^2+jk+k^2))$ and so v_0 and v_1 lie on the same Petrie circuit P of $\mathcal{U}$ (as do their images on $\mathcal{M}$). Now let $h = \mathrm{HCF}(\lambda, \mu)$; the relative coordinates from v_0 to the *nearest* equivalent vertex on P in the direction of v_1 are clearly $(\frac{r}{h}(j^2 + jk + k^2), \frac{r}{h}(j^2 + jk + k^2))$.

The set of Petrie circuits on $\mathcal{M}$ can be partitioned into three parallel classes. Our calculations above tell us that the length of each circuit is $\frac{2r}{h}(j^2 + jk + k^2)$, and since each vertex lies on two circuits in each class, the number of circuits in each class is

$$\frac{2(b^2 + bc + c^2)}{\frac{2r}{h}(j^2 + jk + k^2)} = rh.$$

Here, the intersection multiplicity of the map is equal to the length of the Petrie circuits divided by twice the class size.

We denote the dual of the map $\mathcal{M} = \{3,6\}_{b,c}$ by $\{6,3\}_{b,c}$; this is a uniform torus map of type $(3, 2, 6)$. The orbit of $\mathcal{M}$ under the group of map operations usually has

Property	$\mathcal{M}$	$\mathrm{D}(\mathcal{M})$	$\mathrm{P}(\mathcal{M})$	$\mathrm{D\,P}(\mathcal{M})$	$\mathrm{P\,D}(\mathcal{M})$	$\mathrm{Opp}(\mathcal{M})$
Number of vertices	b^2	$2b^2$	b^2	$3b$	$2b^2$	$3b$
Vertex valency	6	3	6	$2b$	3	$2b$
Number of edges	$3b^2$	$3b^2$	$3b^2$	$3b^2$	$3b^2$	$3b^2$
Number of faces	$2b^2$	b^2	$3b$	b^2	$3b$	$2b^2$
Face valency	3	6	$2b$	6	$2b$	3
Number of Petrie circuits	$3b$	$3b$	$2b^2$	$2b^2$	b^2	b^2
Petrie circuit length	$2b$	$2b$	3	3	6	6

Table 2.4. Properties of the torus map $\mathcal{M} = \{3,6\}_{b,0}$ and of its images under the group of map operations.

length 6. In considering the reflexible torus maps $\{3,6\}_{b,c}$ and $\{6,3\}_{b,c}$, there are two cases to consider: $c = 0$ and $b = c$.

Case $c = 0$

The Petrie circuits of $\mathcal{M} = \{3,6\}_{b,0}$ have length $2b$ and fall into three parallel classes, each of size b. Hence the underlying graph of $\mathrm{Opp}(\mathcal{M})$ is tripartite, each partite set having size b, and each vertex having valency $2b$. The intersection multiplicity of $\mathcal{M}$ is 1 and so the underlying graph of $\mathrm{Opp}(\mathcal{M})$ is simple.

Table 2.4 summarizes some basic combinatorial information. $\mathrm{Opp}(\mathcal{M})$ has the complete tripartite graph $K_{b,b,b}$ as its underlying graph. It is a map of type $(2b,2,3)$ on a surface of characteristic $b(3-b)$. By Theorem 2.2.2, $\mathrm{Opp}(\mathcal{M})$ is orientable for all b, with genus $(b-1)(b-2)/2$. Hence $\mathrm{Opp}(\mathcal{M})$ is the unique reflexible orientable triangular embedding of $K_{b,b,b}$, uniqueness following from Theorem A.1.1 in Appendix A.

$\mathrm{D\,P}(\mathcal{M}) = \mathrm{Opp}(\mathrm{D}(\mathcal{M}))$ is an embedding of $K_{b,b,b}$ as a map of type $(2b,2,6)$ on a non-orientable surface of characteristic $b(3-2b)$. Indeed, the orbit of $\mathcal{M}$ consists of the maps $\{p,q\}_r$ of [11, § 8.6] where $\{p,q,r\} = \{3,6,2b\}$. The automorphism group of these maps is

$$G^{3,6,2b} = \langle A,B,C \mid A^3 = B^6 = C^{2b} = (AB)^2 = (BC)^2 = (CA)^2 = (ABC)^2 = 1 \rangle$$

in the notation of [11, Table 7], isomorphic to $\mathscr{C}_2(3,6)/\langle (trl)^{2b} \rangle$ in the notation of §1.1 and §2.1 (identifying A with lr, B with rt and C with trl). Similarly to the previous cases, this group is a semidirect product of $C_b \times C_b$ by D_{12}; it has order $12b^2$.

Property	$\mathcal{M}$	$D(\mathcal{M})$	$P(\mathcal{M})$	$DP(\mathcal{M})$	$PD(\mathcal{M})$	$Opp(\mathcal{M})$
Number of vertices	$3b^2$	$6b^2$	$3b^2$	$3b$	$6b^2$	$3b$
Vertex valency	6	3	6	$6b$	3	$6b$
Number of edges	$9b^2$	$9b^2$	$9b^2$	$9b^2$	$9b^2$	$9b^2$
Number of faces	$6b^2$	$3b^2$	$3b$	$3b^2$	$3b$	$6b^2$
Face valency	3	6	$6b$	6	$6b$	3
Number of Petrie circuits	$3b$	$3b$	$6b^2$	$6b^2$	$3b^2$	$3b^2$
Petrie circuit length	$6b$	$6b$	3	3	6	6

Table 2.5. Properties of the torus map $\mathcal{M} = \{3,6\}_{b,b}$ and of its images under the group of map operations.

$\mathcal{M}$ is neither self-dual nor self-Petrie, but it is self-Opp when $b = 3$ in which case $P(\mathcal{M})$ and $DP(\mathcal{M}) = Opp(D(\mathcal{M}))$ are self-dual. In the case $b = 2$ we obtain the cube $PD(\mathcal{M})$ and its images, as shown in [61].

(Let X_b be the *generalized Fermat curve* given by $x^b + y^b = 1$. The projection $\pi\colon (x,y) \mapsto x$ has critical values at the b^{th} roots of 1 (and possibly ∞), so if we compose it with $\beta_b\colon x \mapsto x^b$ we obtain a Belyĭ function $\beta = \beta_b \circ \pi\colon (x,y) \mapsto x^b$ of degree b^2. The maps in the orbit of $\mathcal{M} = \{3,6\}_{b,0}$ arise from the generalized Fermat hypermap $\mathcal{F}_b$ of genus $(b-1)(b-2)/2$ associated with β [33]: the Walsh representation $W(\mathcal{F}_b)$ of $\mathcal{F}_b$ is a regular map with $2b$ vertices of valency b and b faces, all $2b$-gons. $Opp(\mathcal{M})$ carries a triangulation isomorphic to the *Fermat triangulation* obtained by stellating $W(\mathcal{F}_b)$ [32].)

Case $b = c$

The Petrie circuits of the torus map $\mathcal{M} = \{3,6\}_{b,b}$ have length $6b$ and fall into three parallel classes, each of size b. Hence the underlying graph of $Opp(\mathcal{M})$ is tripartite, each partite set having size b, and each vertex having valency $6b$. The intersection multiplicity of $\mathcal{M}$ is 3 and so this is the edge multiplicity of $Opp(\mathcal{M})$.

Table 2.5 summarizes some basic combinatorial information. $Opp(\mathcal{M})$ has $K^{(3)}_{b,b,b}$ as its underlying graph. It is a map of type $(6b,2,3)$ on a surface of characteristic $3b(1-b)$. By Theorem 2.2.2, $Opp(\mathcal{M})$ is orientable for all b, with genus $(3b^2 - 3b + 2)/2$.

$DP(\mathcal{M}) = Opp(D(\mathcal{M}))$ is an embedding of $K^{(3)}_{b,b,b}$ as a map of type $(6b,2,6)$ on a non-orientable surface of characteristic $3b(1-2b)$. The automorphism group of these maps is

$$\mathrm{Aut}(\mathscr{M}) = \mathscr{C}_2(3,6)/\langle\, (lrtrtr)^b, (lrtrt)^{2b}\, \rangle$$

[11, §8.4]. This group, of order $36b^2$, has the same structure as in the case $c = 0$ except that the 'translation' group is $C_{3b} \times C_b$.

$\mathscr{M}$ is neither self-dual nor self-Petrie, but it is self-Opp when $b = 1$ in which case $\mathrm{P}(\mathscr{M})$ and $\mathrm{D\,P}(\mathscr{M}) = \mathrm{Opp}(\mathrm{D}(\mathscr{M}))$ are self-dual.

Chapter 3

Hypermap operations

3.1. Outer automorphisms of $\mathcal{H}_2$ and $\mathcal{H}_2^+$

For each permutation π of $\{0, 1, 2\}$ there is a transformation Φ_π of hypermaps that permutes the i-components (the hypervertices, hyperedges and hyperfaces). These transformations were first described by Machí [44] for orientable hypermaps; they are analogous to the duality operation D for maps which transposes vertices and faces. The automorphism $\phi_\pi : t_i \mapsto t_{\pi(i)}$ of the full hypercartographic group

$$\mathcal{H}_2 = \langle t_0, t_1, t_2 \mid t_0^2 = t_1^2 = t_2^2 = 1 \rangle$$

induces Φ_π by its left action on the associated transitive permutation representations of $\mathcal{H}_2$. We define an *operation* on all hypermaps or on orientable hypermaps without boundary to be any transformation similarly induced by an automorphism of $\mathcal{H}_2$ or $\mathcal{H}_2^+$.

Example 3.1.1. Corresponding to the opposite operation Opp on maps (see Example 2.1.3), James [24] describes an operation on hypermaps which we shall call Hopp. It is induced by the automorphism which fixes t_1 and t_2, and conjugates t_0 by t_2. Its effect on a hypermap $\mathcal{H}$ may be described as follows: shrink each hyperface of the James representation of $\mathcal{H}$ to a point, make a cut along each edge of the resulting map, rejoin corresponding sides of the cut in opposing directions, and place a small disc (a new hyperface) around each resultant vertex to obtain the James representation of Hopp($\mathcal{H}$). ▲

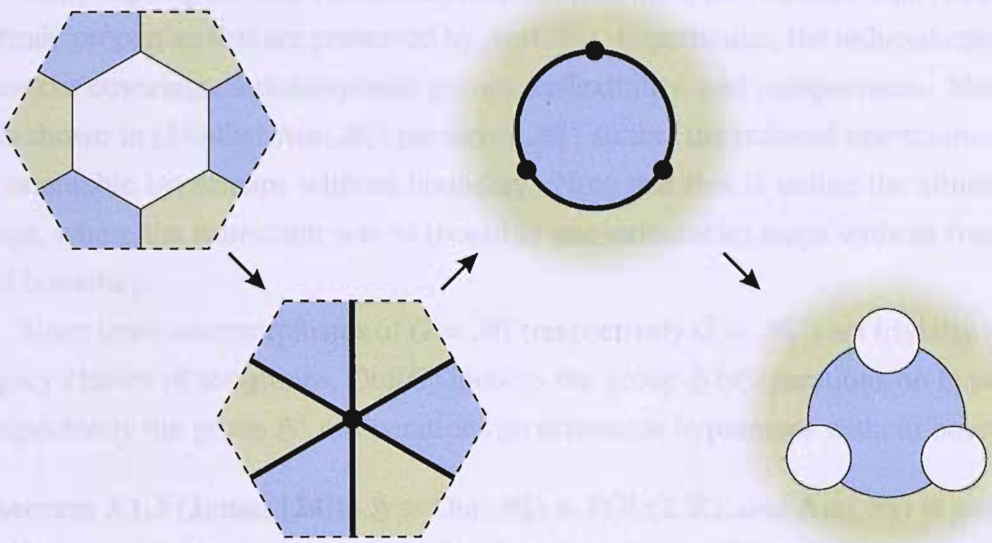


FIGURE 3.1. The procedure for applying Hopp to a simple torus hypermap.

We see that the effect of Hopp on a hypermap $\mathscr{H}$ is to shrink the hyperfaces of the James representation of $\mathscr{H}$ to obtain the (2-face-coloured) dual $\mathscr{M}_*$ of the Walsh representation of $\mathscr{H}$, then to perform the map operation Opp on $\mathscr{M}_*$, and then to expand the vertices to obtain hyperfaces again. It follows that Hopp preserves hyperedges and hyperfaces (since Opp preserves faces) but not hypervertices in general. Figure 3.1 shows this procedure being applied to a simple torus hypermap (obtained by identifying opposite sides of the hexagon); $\mathscr{M}_*$ is the torus map $\{3,6\}_{1,0}$, whose image under Opp we know from §2.3.2: it is the digon of type $(2,2,3)$ on the sphere.

Clearly, $\mathscr{H}$ is self-Hopp if and only if $\mathscr{M}_*$ is self-Opp. Despite the close relationship between Hopp and Opp, the two operations act very differently on maps:

Proposition 3.1.2. *All maps are self-*Hopp.

Proof. Let $\mathscr{H}$ be a map, and consider the James representation of $\mathscr{H}$ in which each face is coloured with the dimension of the component which it represents. This induces a 2-face-colouring of the map $\mathscr{M}_*$, in which the faces coloured 1 are all 1- or 2-valent. It is easy to see that any such map is self-Opp since applying Opp amounts to 'flipping' these faces.

Alternatively, simply note that the permutations τ_0 and τ_2 of §1.2.2 commute for hypermaps which are maps, and so Hopp acts trivially on such objects. ∎

Many topological and combinatorial properties of hypermaps are equivalent to algebraic properties that are preserved by $\mathrm{Aut}(\mathscr{H}_2)$. In particular, the induced operations preserve coverings, automorphism groups, reflexibility, and compactness. Moreover, it is shown in [24] that $\mathrm{Aut}(\mathscr{H}_2)$ preserves $\mathscr{H}_2^+$ so that the induced operations restrict to orientable hypermaps without boundary. Note that this is unlike the situation for maps, where the restriction was to (possibly non-orientable) maps without free edges and boundary.

Since inner automorphisms of $G = \mathscr{H}_2$ (respectively $G = \mathscr{H}_2^+$) act trivially on conjugacy classes of subgroups, $\mathrm{Out}(G)$ induces the group $\mathfrak{H}$ of operations on hypermaps (respectively the group $\mathfrak{H}^+$ of operations on orientable hypermaps without boundary).

Theorem 3.1.3 (James [24]). $\mathfrak{H} \cong \mathrm{Out}(\mathscr{H}_2) \cong \mathrm{PGL}(2, \mathbb{Z})$, *and* $\mathrm{Aut}(\mathscr{H}_2)$ *is generated by* Hopp *and the Machí operations. Further,* $\mathfrak{H}^+ \cong \mathrm{Out}(\mathscr{H}_2^+) \cong \mathrm{GL}(2, \mathbb{Z})$. ✣

3.2. Orbits of hypermap operations

3.2.1. Systems of transitivity

Each finite hypermap (or orientable hypermap without boundary) lies in an orbit of finite size under $\mathfrak{H}$ (respectively $\mathfrak{H}^+$) since $\mathscr{H}_2$ and $\mathscr{H}_2^+$, being finitely generated, have only finitely many subgroups of a given finite index. Regular orientable hypermaps without boundary are in one-to-one correspondence with normal subgroups of $\mathscr{H}_2^+$, which is isomorphic to the free group $F_2 = \langle x_1, x_2 \rangle$ of rank 2. Of these hypermaps $\mathscr{H}$, those for which $\mathrm{Aut}^+(\mathscr{H}) \cong G$ for some fixed group G correspond to subgroups N such that $F_2/N \cong G$ under an isomorphism denoted by ϕ_N. Note that such hypermaps can exist only if G is a one- or two-generator group. In order to count—or estimate—the number of such hypermaps, we turn to the theory of generating bases and T-systems (see, for example, [20], [48], [47], [15]).

For what follows, let F be a finitely-generated group with generators $x_1, \ldots, x_n$ and defining relations $\mathcal{R}_j(x_1, \ldots, x_n) = 1$ $(j \in J)$, and let G be a group.

Definition 3.2.1. A *G-defining subgroup of F* is a normal subgroup $N \trianglelefteq F$ such that $F/N \cong G$. If $\{w_k(x_1, \ldots, x_n)\}$ is any set of elements whose normal closure in F is N then the relations $w_k(x_1, \ldots, x_n) = 1$ and $\mathcal{R}_j(x_1, \ldots, x_n) = 1$ form a system of defining relations for G. Such a system is called an *F-definition for G belonging to N*.

An F-*base* of G is an ordered set $\mathbf{a} = (a_1, \ldots, a_n)$ of n elements (not necessarily distinct) which generate G and for which $\mathcal{R}_j(\mathbf{a}) = 1$ for all $j \in J$. Denote by $\phi_F(G)$ the cardinality of the set $\mathcal{B}_G^F$ of all F-bases of G. Two F-bases $\mathbf{a}, \mathbf{b}$ of G are *equivalent* if there is an automorphism α of G for which $\alpha(a_i) = b_i$ for each $i \in \{1, \ldots, n\}$. The equivalence classes of F-bases are called Aut(G)-*classes*.

In the case where F is the free group F_n of rank n, we talk of n-bases (rather than F_n-bases) and denote $\phi_{F_n}(G)$ by $\phi_n(G)$. •

The following theorem is given in [20].

Theorem 3.2.2 (Hall). *Let G be a group. There is a one-to-one correspondence between the* Aut(G)*-classes of F-bases of G and the G-defining subgroups of F.*

Proof. To each F-base $\mathbf{a}$ of G corresponds the G-defining subgroup $R(\mathbf{a})$ of F (called the *relation group* for $\mathbf{a}$) consisting of all the elements $w_k(x_1, \ldots, x_n)$ of F for which $w_k(\mathbf{a}) = 1$ in G. (In fact, the F-definition for G involving these elements is the maximal F-definition for G belonging to $R(\mathbf{a})$ with respect to inclusion of subsets.) Conversely, each G-defining subgroup N of F is $R(\mathbf{a})$ for all F-bases $\mathbf{a}$ which are images of $(x_1, \ldots, x_n)$ under isomorphisms $F/N \cong G$.

It is easy to see that F-bases of G in the same Aut(G)-class have the same relation group, and that two F-bases with the same relation group satisfy the same F-definitions for G and are thus equivalent. ∎

Another way to view this is to identify $\mathcal{B}_G^F$ with the set of epimorphisms $\alpha\colon F \to G$ via $\alpha \leftrightarrow \alpha(x_1, \ldots, x_n)$, observing that the kernel of each epimorphism $F \to G$ is a G-defining subgroup of F, and that two such epimorphisms have the same kernel if and only if they differ by an automorphism of G.

Clearly Aut(G) acts freely (although generally intransitively) on $\mathcal{B}_G^F$, and so each Aut(G)-class is of size $a(G) := |\text{Aut}(G)|$. Hence we may write

$$\phi_F(G) = a(G)\, d_F(G)$$

where $d_F(G)$ is the number of G-defining subgroups of F. Given that we know $a(G)$, finding $d_F(G)$ amounts to counting F-bases or epimorphisms.

For various groups G we wish to count symmetrical hypermaps which have G as their automorphism group, and so we must determine $d_F(G)$ where F is $\mathscr{H}_2^+$ or $\mathscr{H}_2$.[1] Note that in the case $F = \mathscr{H}_2^+$, the formula above gives an obvious upper bound on this number:

$$d_F(G) \leqslant \frac{|G|^2 - 1}{|\mathrm{Aut}(G)|}.$$

This bound is sharp since it is attained for cyclic groups of prime order. In general, however, it is crude. More than calculating or estimating $d_F(G)$ where F is $\mathscr{H}_2^+$ or $\mathscr{H}_2$, we are ultimately interested in how our symmetric hypermaps are partitioned into orbits under the action of the groups $\mathfrak{H}^+$ and $\mathfrak{H}$ of operations induced by Out(F). Agreeably, this too arises naturally in the theory.

For general F, Aut(F) has an induced action on the set $\mathscr{B}_G^F$ of F-bases of G via $\psi\mathbf{a} = \mathbf{b}$ where $\psi(x_i) = w_i(x_1, \ldots, x_n)$ and $b_i := w_i(a_1, \ldots, a_n)$; in general this action is neither free nor transitive. The actions of Aut(G) and Aut(F) commute, and we denote by P_G^F the group generated by all the permutations of $\mathscr{B}_G^F$ arising from the action of $\mathrm{Aut}(G) \times \mathrm{Aut}(F)$.

Definition 3.2.3. Let G be a group. The transitivity classes (orbits) of $\mathscr{B}_G^F$ under P_G^F are called the *T_F-systems* of G. When F is the free group of rank n we denote $d_F(G)$ by $d_n(G)$, $P_G^{F_n}$ by P_G^n and we talk of *T_n-systems* or simply of *T-systems*. •

Rather than regarding a T_F-system as being a transitivity class of F-bases of G, it is often more useful to regard it as a transitivity class of Aut(G)-classes of F-bases, or (equivalently) of G-defining subgroups of F.

Example 3.2.4. Consider $G = C_p$, the cyclic group of prime order p. The number of generating pairs for G is $p^2 - 1$ (since only the identity fails to generate the group), while $\mathrm{Aut}(C_p) \cong C_{p-1}$ (since it is the group of units of a finite field); thus there are $(p^2 - 1)/(p - 1) = p + 1$ Aut(C_p)-orbits of generating pairs, each of size $p - 1$. These orbits correspond to the different normal subgroups N of F_2 such that $F_2/N \cong C_p$, and so there are $p + 1$ regular orientable hypermaps without boundary with cyclic orientation-preserving automorphism group of order p.

Representatives of the Aut(C_p)-orbits are $g_i = (1, i)$ for $i = 0, 1, \ldots, p - 1$, and $g_p = (0, 1)$. Now, $\mathrm{Aut}(F_2) \cong \mathrm{Aut}(\mathscr{H}_2^+)$ is generated by the automorphisms $\rho\colon x_1 \mapsto x_1$, $x_2 \mapsto x_1x_2$ and $\mu\colon x_1 \mapsto x_2$, $x_2 \mapsto x_1$ (see §3.2.2 below). It is easily verified that the

[1] Although we shall enumerate the F-bases of various groups G directly, $d_F(G)$ can also be calculated indirectly—and elegantly—using Hall's Enumeration Principle [20]; see also §3.2.6.

induced action of ρ, via the $\mathrm{Aut}(C_p)$-orbits, on the subscripts of the representatives g_j is given by the permutation

$$(0, 1, 2, \ldots, p-1)(p)$$

while that of μ is given by the permutation

$$\prod_{\substack{0 \leqslant j \leqslant p \\ j < j^{-1}}} (j, j^{-1})$$

where j^{-1} is the multiplicative inverse of j modulo p for $1 \leqslant j \leqslant p-1$, and $0^{-1} = p$. It follows immediately that $\mathrm{Out}(F_2)$ acts transitively on the $\mathrm{Aut}(C_p)$-orbits, and our $p+1$ hypermaps lie in a single orbit under the group $\mathfrak{H}^+$ of operations. ▲

Knowledge of the F-bases theoretically provides us with complete insight as to the nature of the symmetrical hypermaps $\mathscr{H}$, since G also acts as their monodromy group G'. The right action of F on the cosets of a hypermap subgroup N is isomorphic to the right regular action of G on itself. The monodromy generators which, together with G', define the hypermap are the images of the generators of F under the corresponding permutation representation. On the other hand, the images of the generators of F under the homomorphisms $\phi_N \colon F \to G$ whose kernel is N together form an $\mathrm{Aut}(G)$-class of F-bases $\mathbf{a}$ of G; and each permutation group P generated by the images of the components of some such $\mathbf{a}$ under the permutation representation of G is isomorphic as a permutation group to G'; the P differ from each other only by a relabelling of the points on which they act.

It follows that each $\mathscr{H}$ can be reconstructed by choosing one F-base $\mathbf{a}$ from each $\mathrm{Aut}(G)$-class and considering how its components a_i act on G. In particular, basic combinatorial information about $\mathscr{H}$ such as the number of components of each dimension can be obtained simply by observing the order of each a_i and of some of their binary products. Such details regarding the cyclic groups of Example 3.2.4 above are discussed in §3.2.3.

3.2.2. Orbits of $\mathfrak{H}^+$ and $\mathfrak{H}$

Generating sets of automorphisms

To facilitate the calculation of orbits of the groups $\mathfrak{H}^+$ and $\mathfrak{H}$ of hypermap operations, it is useful to identify some sets of generators. If F_n is a free group of finite rank n with basis X, then every automorphism of F_n carries X onto another basis, and, conversely, every injective map from X onto any basis for F_n determines an automorphism. The following result is proved in [42, Prop. 4.1].

Theorem 3.2.5. *Let F_n be a free group of rank n with basis $(x_1, \dots, x_n)$. Let α_i be the endomorphism $x_i \mapsto x_i^{-1}$, $x_j \mapsto x_j$ $(j \neq i)$. Let $\beta_{i,j}$ be the endomorphism $x_i \mapsto x_i x_j$, $x_k \mapsto x_k$ $(j, k \neq i)$. Then the α_i and the $\beta_{i,j}$ are automorphisms, and together they generate* $\mathrm{Aut}(F_n)$. ✠

Of course, there are other possibilities for elementary generators of $\mathrm{Aut}(F_2) \cong \mathscr{H}_2^+$ and we will choose whichever suits our purpose. For example, a presentation given in [48] is

$$\mathrm{Aut}(F_2) = \langle \rho, \sigma \mid (\rho\sigma)^2 = (\rho^2\sigma^3)^2 = (\rho^3\sigma^2)^4 = (\rho^4\sigma^2)^3 = 1 \rangle$$

where ρ and σ are the automorphisms $\rho\colon x_1 \mapsto x_1$, $x_2 \mapsto x_1 x_2$ and $\sigma\colon x_1 \mapsto x_2$, $x_2 \mapsto x_2^{-1} x_1$; also $\mathrm{Aut}(F_2) = \langle \rho, \mu \rangle$ where $\mu = \rho\sigma\colon x_1 \mapsto x_2, x_2 \mapsto x_1$.

It is shown in [24] that the restriction $\mathrm{Aut}(\mathscr{H}_2) \to \mathrm{Aut}(\mathscr{H}_2^+)$ is surjective, and so the automorphisms inducing Hopp and the Machí operations, which generate $\mathrm{Aut}(\mathscr{H}_2)$, restrict to generating automorphisms for $\mathrm{Aut}(\mathscr{H}_2^+)$. These are given in Table 3.1. (Note that we use the presentation $\langle t_0, t_1, t_2 \mid t_0^2 = t_1^2 = t_2^2 = 1 \rangle$ for $\mathscr{H}_2$, and we set $x_3 = (x_1 x_2)^{-1}$ and identify x_1 and x_2 with $t_1 t_2$ and $t_2 t_0$ respectively, which gives the alternative presentation $\langle x_1, x_2, t_2 \mid (x_1 t_2)^2 = (t_2 x_2)^2 = t_2^2 = 1 \rangle$ for $\mathscr{H}_2$.) We shall not need to make use of any other generating sets for $\mathrm{Aut}(\mathscr{H}_2)$.

Invariants of the orbits

For a 2-base (a, b) of a group G, the union of the conjugacy classes of the commutator $[a, b] = aba^{-1}b^{-1}$ and its inverse is a well-known invariant of the $\mathrm{Aut}(F_2)$-class of (a, b),[2] called the Higman invariant [47]. (This is easily checked using any generating

[2] An $\mathrm{Aut}(F_n)$-class is often called a *Nielsen class* and the automorphisms of F_n, *Nielsen moves*.

Operation	Automorphism of $\mathscr{H}_2$	Restriction to $\mathscr{H}_2^+$
Hopp	$t_0 \mapsto t_2t_0t_2, t_1 \mapsto t_1, t_2 \mapsto t_2$	$x_1 \mapsto x_1, x_2 \mapsto x_2^{-1}, x_3 \mapsto (x_2x_1)^{-1}$
$\Phi_{(0,1)}$	$t_0 \leftrightarrow t_1, t_2 \mapsto t_2$	$x_1 \mapsto x_2^{-1}, x_2 \mapsto x_1^{-1}, x_3 \mapsto x_3^{-1}$
$\Phi_{(0,2)}$	$t_0 \leftrightarrow t_2, t_1 \mapsto t_1$	$x_1 \mapsto x_1x_2, x_2 \mapsto x_2^{-1}, x_3 \mapsto x_1^{-1}$
$\Phi_{(1,2)}$	$t_1 \leftrightarrow t_2, t_0 \mapsto t_0$	$x_1 \mapsto x_1^{-1}, x_2 \mapsto x_1x_2, x_3 \mapsto x_2^{-1}$
$\Phi_{(0,1,2)}$	$t_0 \mapsto t_1, t_1 \mapsto t_2, t_2 \mapsto t_0$	$x_1 \mapsto x_2, x_2 \mapsto x_3, x_3 \mapsto x_1$
$\Phi_{(0,2,1)}$	$t_0 \mapsto t_2, t_2 \mapsto t_1, t_1 \mapsto t_0$	$x_1 \mapsto x_3, x_2 \mapsto x_0, x_3 \mapsto x_1$

Table 3.1. The restrictions to $\mathscr{H}_2^+$ of the automorphisms of $\mathscr{H}_2$ which induce Hopp and the Machí operations.

set of automorphisms of F_2.) It follows that the order c of $[a, b]$ is an invariant of the T-system of (a, b). We may interpret this as follows. The element $[a, b]$ induces a Petrie circuit in a hypermap (see §4.4.3) under the action of G as the monodromy group of the hypermap $\mathscr{H}$ induced by (a, b). The Petrie circuits of $\mathscr{H}$ all have length $2c$ since $\mathscr{H}$ is regular; Higman's result thus implies that Petrie circuit length is an invariant of the action of the group $\mathfrak{H}^+$ of operations on orientable hypermaps without boundary.[3]

These facts motivate us to determine whether Petrie circuit length is also an invariant of the action of the group $\mathfrak{H}$ of general hypermaps, and indeed it proves to be so: let (τ_0, τ_1, τ_2) be an $\mathscr{H}_2$-base of a group G which gives rise to a hypermap $\mathscr{H}$ satisfying $\mathrm{Aut}(\mathscr{H}) \cong G$. The element $\tau_0\tau_1\tau_2$ induces a Petrie circuit (of length $2|\tau_0\tau_1\tau_2|$) in $\mathscr{H}$ (*ibid.*).

Proposition 3.2.6. *The union of the conjugacy classes of the element $\tau_0\tau_1\tau_2$ and its inverse is an invariant of the* $\mathrm{Aut}(\mathscr{H}_2)$*-class of* (τ_0, τ_1, τ_2).

Proof. The image of (τ_0, τ_1, τ_2) under the action of Machí automorphism of $\mathscr{H}_2$ induced by a permutation π of $\{0, 1, 2\}$ is $(\tau_{\pi(0)}, \tau_{\pi(1)}, \tau_{\pi(2)})$. The element $\tau_{\pi(0)}\tau_{\pi(1)}\tau_{\pi(2)}$ is a conjugate of $\tau_0\tau_1\tau_2$ or its inverse by one of τ_0, τ_1 or τ_2.

Let $\phi_{(1,2)}$ be the Machí automorphism which fixes τ_0 and transposes τ_1 and τ_2. Let ξ be the automorphism which induces Hopp. The image of (τ_0, τ_1, τ_2) under the action of $\phi_{(1,2)}\xi$ is $(\tau_2^{-1}\tau_0\tau_2, \tau_2, \tau_1)$; we observe that $\tau_2^{-1}\tau_0\tau_2\,\tau_2\,\tau_1 = \tau_2\tau_0\tau_1 = (\tau_0\tau_1\tau_2)^{\tau_2}$.

Since the Machí automorphisms and $\phi_{(1,2)}\xi$ form a generating set for $\mathrm{Aut}(\mathscr{H}_2)$, the result holds. ∎

[3]Similar arguments show that the length of a $(j, j)^{\text{th}}$-order Petrie circuit (an orbit of $a^jb^ja^{-j}b^{-j}$; *ibid.*) is also such an invariant.

Transitivity properties

Theorem 3.2.7. *An abelian group possessing n-bases has just one T_k-system for all $k \geqslant n$.* ✠

This result appears to be part of folklore. It was known to Neumann [47] in 1956 but a proof is not readily found in the literature. In fact, the best point of reference is the following recent result [12]:[4]

Theorem 3.2.8 (Diaconis and Graham). *Let G be a finite abelian group, given as*

$$G \cong \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \ldots \times \mathbb{Z}_{m_r},$$

where $m_1 \mid m_2 \mid \ldots \mid m_r$. Then G possesses r-bases but not $(r-1)$-bases and

(i) *the k-bases of G lie in a single* $\mathrm{Aut}(F_k)$*-class for all $k \geqslant r+1$,*

(ii) *the r-bases of G lie in $\phi(m_1)$* $\mathrm{Aut}(F_r)$*-classes of equal size.* ✠

The $\mathrm{Aut}(F_r)$-classes can be shown to lie in the same $\mathrm{Aut}(G)$-orbit. We provide here an alternative direct proof of the transitivity of $\mathrm{Aut}(F_2)$ on the 2-bases of a cyclic group since this at least can be done briefly and we make use of the result later.

Proposition 3.2.9. $\mathrm{Aut}(F_2)$ *acts transitively on the 2-bases of a cyclic group.*

Proof. We first show that the 2-bases in any given $\mathrm{Aut}(C_m)$-class lie in the same orbit under $\mathrm{Aut}(F_2)$. Denote the elements of C_m by $0, 1, \ldots, m-1$ with addition taken modulo m. An automorphism of C_m is determined by the image of the element 1, and so the $\mathrm{Aut}(C_m)$-class of a 2-base (u, v) consists of 2-bases (zu, zv) where z generates C_m.

Given a 2-base (u, v), let j be the product of those prime divisors of m which divide neither u nor v, taking the empty product to be 1. Then both $ju + v$ and $u + jv$ are generators of C_m since each prime divisor of m divides precisely one of the summands in each case. The generating automorphisms ρ, μ of $\mathrm{Aut}(F_2)$ given on page 35 give us

[4] The result in [12] is formulated in the slightly different language of product replacement graphs.

the following elements in the orbit of (u, v):

$(u,v) \mapsto (u, ju+v)$	under the j^{th} power of ρ
$\mapsto (ju+v, u)$	under μ
$\mapsto (ju+v, z(u+jv))$	under a power of ρ, noting that $ju+v$ is a generator
$\mapsto (z(u+jv), ju+v)$	under μ
$\mapsto (z(u+jv), zv)$	under a power of ρ, noting that $z(u+jv)$ is a generator
$\mapsto (zv, z(u+jv))$	under μ
$\mapsto (zv, zu)$	under the $(-j)^{\text{th}}$ power of ρ
$\mapsto (zu, zv)$	under μ.

Now, similarly, $(u, v) \mapsto (ju + v, u) \mapsto (ju + v, 0)$ under automorphisms of F_2, and the latter 2-base is in the same $\mathrm{Aut}(C_m)$-class (and hence $\mathrm{Aut}(F_2)$-class) as $(1, 0)$. Hence there is just one $\mathrm{Aut}(F_2)$-class. ∎

3.2.3. Cyclic groups

Consider C_{p^α}, a cyclic group of prime-power order. A 2-base for such a group must contain a generator, and so

$$\begin{aligned}\phi_2(C_{p^\alpha}) &= \phi(p^\alpha)p^\alpha + (p^\alpha - \phi(p^\alpha))\,\phi(p^\alpha)\\ &= \phi(p^\alpha)\,(2p^\alpha - \phi(p^\alpha))\\ &= p^{2\alpha}\left(1 - \frac{1}{p^2}\right).\end{aligned}$$

Let us denote the distinct primes dividing m by $p_1, \ldots, p_k$; for general m we may write $C_m \cong \bigoplus_{i=1}^{k} C_{p_i^{\alpha_i}}$. A 2-base of C_m may be identified with a generating pair $(\mathbf{u}, \mathbf{v})$, where $\mathbf{u} = [u_1, \ldots, u_k]$ and $\mathbf{v} = [v_1, \ldots, v_k]$ are elements of $C_{p_1^{\alpha_1}} \oplus \cdots \oplus C_{p_k^{\alpha_k}}$. Clearly (u_i, v_i) is then a 2-base of $C_{p_i^{\alpha_i}}$ for each i. Conversely, any choice of 2-base (u_i, v_i) of $C_{p_i^{\alpha_i}}$, one for each i, gives a 2-base $(\mathbf{u}, \mathbf{v})$ for C_m; this is because the group generated by $(\mathbf{u}, \mathbf{v})$ contains an element of the form $[1, w_2, \ldots, w_k]$ and hence contains $p_2 \ldots p_k[1, w_2, \ldots, w_k] = [p_2 \ldots p_k, 0, \ldots, 0]$ and then $[1, 0, \ldots, 0]$, and similarly it contains the other standard generators $[0, 1, 0, \ldots, 0], \ldots, [0, \ldots, 0, 1]$. We thus have

$$\begin{aligned}\phi_2(C_m) &= \prod_{i=1}^{k} \phi_2(C_{p_i^{\alpha_i}}) \\ &= \prod_{i=1}^{k} p^{2\alpha_i} \prod_{i=1}^{k} \left(1 - \frac{1}{p_i^2}\right) \\ &= m^2 \prod_{p|m} \left(1 - \frac{1}{p^2}\right).\end{aligned}$$

Now, since $|\mathrm{Aut}(C_m)| = \phi(m) = m \prod_{p|m} \left(1 - \frac{1}{p}\right)$ we have

$$d_2(C_m) = m \prod_{p|m} \left(1 + \frac{1}{p}\right),$$

and the regular orientable hypermaps without boundary whose orientation-preserving automorphism group is C_m lie in a single $\mathfrak{H}^+$-orbit of this size.

When $m = p^\alpha$ we have $d_2(C_m) = p^\alpha + p^{\alpha-1}$. As was seen in the calculation of $\phi_2(C_{p^\alpha})$, the 2-bases for C_{p^α} either consist of a generator in the first component and an arbitrary element in the second, or a non-generator in the first component and a generator in the second. Representative 2-bases from the $\mathrm{Aut}(C_{p^\alpha})$-classes may be obtained by choosing the generator in question to be 1 in each case. It is convenient now to represent each 2-base (g_0, g_1) as a $\mathscr{C}_2^+$-base (g_0, g_1, g_2) by adjoining a third component $g_2 = (g_0 g_1)^{-1}$. For $i \in \{0, 1, 2\}$ the number of i-components of the corresponding hypermap is $m/|g_i|$ (where $|g_i|$ is the order of g_i), and they each have valency $|g_i|$.

Three of the representative $\mathscr{C}_2^+$-bases contain 0: they are $(0, 1, -1)$, $(1, 0, -1)$ and $(1, -1, 0)$. These correspond to the *star hypermaps* which form the Machí orbit of the *star map* consisting of a single (hyper)vertex and (hyper)face, both of valency p^α, together with p^α 1-valent (hyper)edges. These elementary hypermaps are all spherical. The $p^\alpha - 2p^{\alpha-1}$ representatives consisting of three generators correspond to hypermaps consisting of one p^α-valent component of each dimension i; they lie on a surface of genus $(p^\alpha - 1)/2$. The remaining representative $\mathscr{C}_2^+$-bases consist of two generators and a non-trivial non-generator. The non-generator is of the form $cp^{\alpha-\beta}$ where $\mathrm{HCF}(c, p) = 1, \beta < \alpha$; it is not difficult to see that there are $p^{\beta-1}(p-1)$ such elements, corresponding to hypermaps consisting of a single i-component and a single j-component (both of

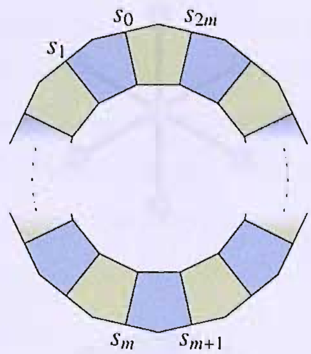


Figure 3.2. Forming orientable hypermaps $\mathscr{H}$ without boundary having a component of valency m and satisfying $\mathrm{Aut}^+(\mathscr{H}) \cong C_m$.

valency p^α) and $p^{\alpha-\beta}$ k-components (each of valency p^β), where i, j, k (distinct) vary over $\{0, 1, 2\}$. These hypermaps lie on a surface of genus $(p^\alpha - p^{\alpha-\beta})/2$.

For general m, similar calculations go through but the enumerations are more involved and we do not include them here. One point to note is that for fixed $i \in \{0, 1, 2\}$ there are m regular orientable hypermaps $\mathscr{H}$ without boundary satisfying $\mathrm{Aut}^+(\mathscr{H}) \cong C_m$ and whose i-component has valency m—the maximum possible. For $1 \leqslant t \leqslant m$ they are formed by identifying side s_0 of the polygon of Figure 3.2 with side s_{2t-1} and then making similar identifications with the remaining sides. For example, Figure 3.3 shows the orientable hypermaps $\mathscr{H}$ without boundary which satisfy $\mathrm{Aut}^+(\mathscr{H}) \cong C_6$, and of these, all nine with positive genus arise in this way, although the diagrams are not centred about the 6-valent components. (The Walsh representation is used for the three star hypermaps, and the James representation is used for the remainder. To form the genus-2 hypermaps, identify each side of the 12-gon with the side at distance 3 away to form two 12-sided polygons representing 6-valent components. To form the torus hypermaps simply identify opposite sides of the hexagon in the usual way. Note that two of the torus hypermaps are maps.)

Of course, it is not necessarily the case that an orientable hypermap without boundary satisfying $\mathrm{Aut}^+(\mathscr{H}) \cong C_m$ must have an i-component of valency m; there exist $\mathscr{C}_2^+$-bases of C_m consisting of three non-generators whenever three or more distinct primes divide m. For example six such hypermaps exist when $m = 30$: they have two i-components of valency 15, three j-components of valency 10 and six k-components of valency 5 (where i, j, k vary over $\{0, 1, 2\}$) and they lie on a surface of genus 11.

It is clear that C_m has no $\mathscr{H}_2$-bases for $m > 2$, and so there are no reflexible hyper-

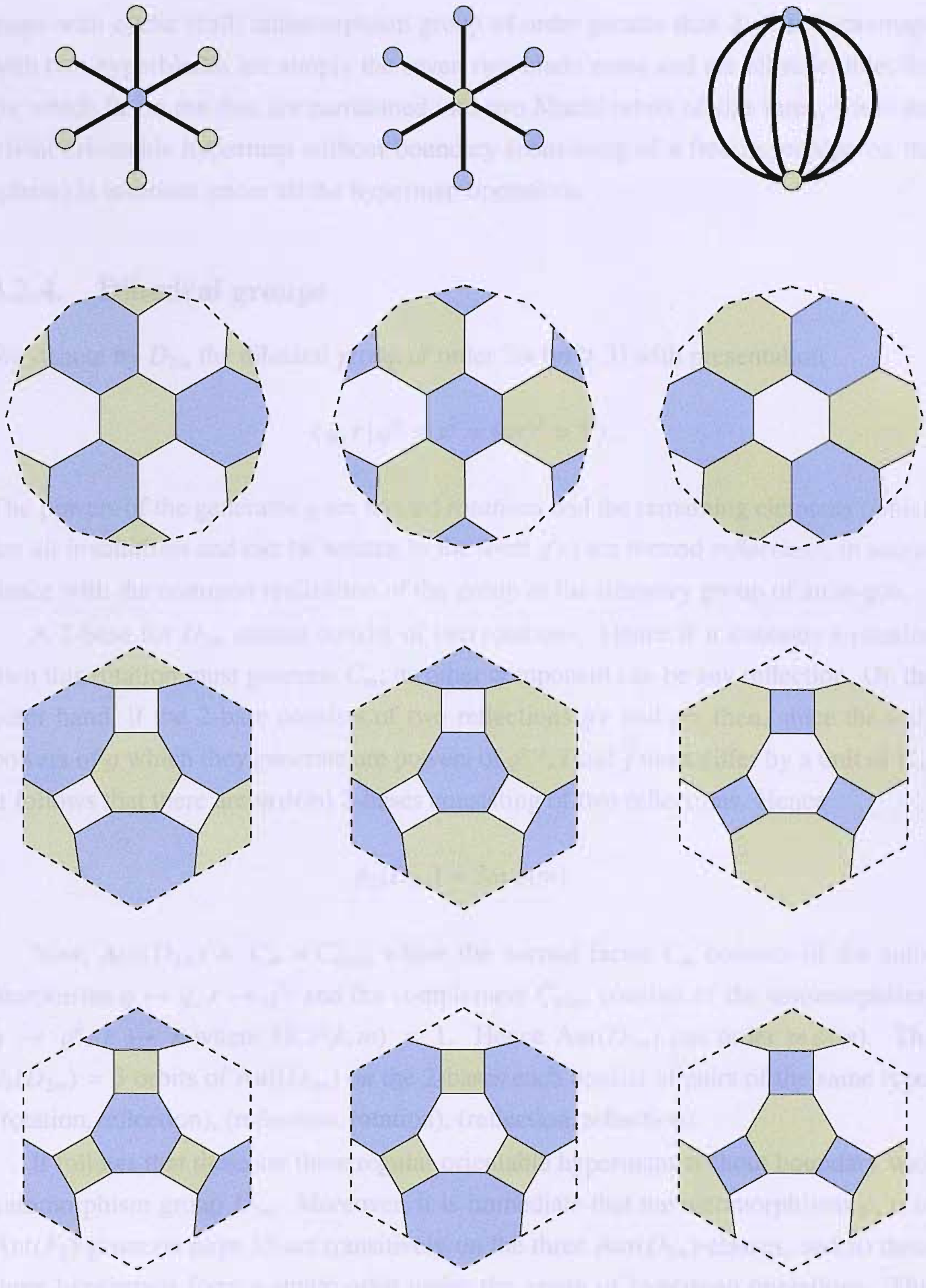


FIGURE 3.3. The regular orientable hypermaps $\mathscr{H}$ without boundary which satisfy $\mathrm{Aut}^+(\mathscr{H}) \cong C_6$.

maps with cyclic (full) automorphism group of order greater than 2. The hypermaps with two hyperblades are simply the seven two-blade maps and are all reflexible; the six which lie on the disc are partitioned into two Machí orbits of size three, while the trivial orientable hypermap without boundary (consisting of a free hyperedge on the sphere) is invariant under all the hypermap operations.

3.2.4. Dihedral groups

We denote by D_{2m} the dihedral group of order $2m$ ($m \geqslant 3$) with presentation

$$\langle\, g, r \mid g^m = r^2 = (gr)^2 = 1\,\rangle.$$

The powers of the generator g are termed *rotations* and the remaining elements (which are all involutions and can be written in the form $g^i r$) are termed *reflections*, in accordance with the common realization of the group as the isometry group of an m-gon.

A 2-base for D_{2m} cannot consist of two rotations. Hence if it contains a rotation then this rotation must generate C_m; its other component can be any reflection. On the other hand, if the 2-base consists of two reflections $g^i r$ and $g^j r$ then, since the only powers of g which they generate are powers of g^{i-j}, i and j must differ by a unit of C_m. It follows that there are $m\,\phi(m)$ 2-bases consisting of two reflections. Hence

$$\phi_2(D_{2m}) = 3m\,\phi(m).$$

Now, $\mathrm{Aut}(D_{2m}) \cong C_m \rtimes C_{\phi(m)}$ where the normal factor C_m consists of the automorphisms $g \mapsto g$, $r \mapsto g^i r$ and the complement $C_{\phi(m)}$ consists of the automorphisms $g \mapsto g^k$, $r \mapsto r$ where $\mathrm{HCF}(k, m) = 1$. Hence $\mathrm{Aut}(D_{2m})$ has order $m\,\phi(m)$. The $d_2(D_{2m}) = 3$ orbits of $\mathrm{Aut}(D_{2m})$ on the 2-bases each consist of pairs of the same type: (rotation, reflection), (reflection, rotation), (reflection, reflection).

It follows that there are three regular orientable hypermaps without boundary with automorphism group D_{2m}. Moreover, it is immediate that the automorphisms ρ, μ of $\mathrm{Aut}(F_2)$ given on page 35 act transitively on the three $\mathrm{Aut}(D_{2m})$-classes, and so these three hypermaps form a single orbit under the group of hypermap operations. This is the orbit of the dipole $\{m, 2\}$; note that Hopp and half of the Machí operations act trivially.

To determine the $\mathscr{H}_2$-bases of D_{2m} we consider the cases where m is even and odd separately. When m is odd, the only involutions are the m reflections. We have seen that there are $m\phi(m)$ 2-bases consisting of reflections, and so there are $3m\phi(m)$ $\mathscr{H}_2$-bases which contain the identity, partitioned into three $\mathrm{Aut}(D_{2m})$-classes and forming a single T-system: the automorphisms inducing the Machí operations act transitively on the three classes while that inducing Hopp acts trivially. Now, a triple (g^ir, g^jr, g^kr) of reflections generates the group

$$\langle g^ir, g^jr, g^kr \rangle$$

which we may rewrite as

$$\langle g^ir, g^{i-j}, g^{i-k} \rangle$$

using the correspondence $(a, b, c) \sim (a, ab, ac)$, and so the triple is an $\mathscr{H}_2$-base if and only if $\mathbf{a} = (i - j, i - k)$ is a 2-base of C_m. Hence there are $m\phi_2(C_m)$ such $\mathscr{H}_2$-bases. Moreover, we claim that they lie in a single T-system. Consider the $\mathscr{H}_2$-automorphisms $\tilde{\mu}$: $t_0 \mapsto t_0$, $t_1 \mapsto t_2$, $t_2 \mapsto t_1$ and $\tilde{\sigma}$: $t_0 \mapsto t_2$, $t_1 \mapsto t_2t_0t_2$, $t_2 \mapsto t_1$. ($\tilde{\mu}$ induces a Machí operation while $\tilde{\sigma}$ induces the composition of a Machí operation with Hopp.) Under the action of $\tilde{\mu}$, the $\mathscr{H}_2$-base $(g^ir, g^jr, g^kr) \sim (g^ir, g^{i-j}, g^{i-k})$ is mapped to $(g^ir, g^kr, g^jr) \sim (g^ir, g^{i-k}, g^{i-j})$; this induces the map $\mathbf{a} = (i - j, i - k) \mapsto (i - k, i - j)$ which is the action of $\mu \in \mathrm{Aut}(F_2)$ (from page 35) on $\mathbf{a}$. Under the action of $\tilde{\sigma}$, (g^ir, g^jr, g^kr) is mapped to $(g^kr, g^{2k-i}r, g^jr) \sim (g^kr, g^{i-k}, g^{k-j})$, which is then mapped to $(g^ir, g^kr, g^{i+j-k}r) \sim (g^ir, g^{i-k}, g^{k-j})$ under the action of the D_{2m}-automorphism $g \mapsto g$, $r \mapsto g^{i-k}r$; this induces the map $\mathbf{a} = (i - j, i - k) \mapsto (i - k, k - j)$ which is the action of $\sigma \in \mathrm{Aut}(F_2)$ on $\mathbf{a}$. Hence the combined action of $\langle \tilde{\mu}, \tilde{\sigma} \rangle$ and $\mathrm{Aut}(D_{2m})$ on $\mathscr{H}_2$-bases (g^ir, g^jr, g^kr) where i is fixed induces an action on 2-bases $(i - j, i - k)$ with a subaction isomorphic to that of $\langle \mu, \sigma \rangle = \mathrm{Aut}(F_2)$ on these 2-bases. The latter is transitive by Proposition 3.2.9, and since $\mathrm{Aut}(D_{2m})$ acts transitively on reflections g^ir the claim is proved.

Thus for odd m,

$$\begin{aligned}\phi_{\mathscr{H}_2}(D_{2m}) &= 3m\phi(m) + m\phi_2(C_m)\\ &= 3m\phi(m) + m\phi(m)\,d_2(C_m)\end{aligned}$$

and

$$d_{\mathscr{H}_2}(D_{2m}) = 3 + d_2(C_m).$$

The hypermaps with automorphism group D_{2m} (m odd) lie in two orbits under $\mathfrak{H}$: one of size three and one of size $d_2(C_m)$. The orbit of size three consists of the *disc star hypermaps*; it is the orbit of the *disc star map* which lies on the disc and consists of a single (hyper)vertex in the interior and m 1-valent (hyper)edges incident with the boundary. The other orbit simply consists of the orientable hypermaps $\mathscr{H}$ without boundary satisfying $\mathrm{Aut}^+(\mathscr{H}) \cong C_m$, described in §3.2.3; this shows that each of these regular hypermaps is reflexible with dihedral full automorphism group.

When m is even there is a rotation $g^{m/2}$ of order 2. The $\mathscr{H}_2$-bases consisting of reflections are enumerated as before, as are those involving the identity. Consider a D_{2m}-triple consisting of $g^{m/2}$ together with two reflections $g^i r$, $g^j r$. These elements generate the group

$$\langle g^i r, g^j r, g^{m/2} \rangle = \langle g^i r, g^{i-j}, g^{m/2} \rangle$$

and so the triple is an $\mathscr{H}_2$-base if and only if $(i-j, m/2)$ is a 2-base of C_m. This is certainly the case when $i-j$ is a generator of C_m, and since $\langle g^{m/2} \rangle$ is characteristic in $\mathrm{Aut}(D_{2m})$ it is easy to see that the $\mathscr{H}_2$-bases $(g^i r, g^j r, g^{m/2})$ with $i-j$ as such lie in a single $\mathrm{Aut}(D_{2m})$-class. Further, since the $\mathscr{H}_2$-automorphism which induces Hopp acts trivially on this class, the three such classes (indexed by the position of $g^{m/2}$) together form a T-system. Now, by representing C_m as $C_{2^{\alpha_1}} \oplus C_{p_2^{\alpha_2}} \oplus \cdots \oplus C_{p_k^{\alpha_k}}$ and representing 1 as $[1, 1, \ldots, 1]$ which identifies $m/2$ with $[2^{\alpha_1 - 1}, 0, \ldots, 0]$ we see that $(i-j, m/2)$ is a also a 2-base of C_m when $\alpha_1 = 1$ and $i-j$ is twice a generator of C_m, yielding $m\,\phi(m)$ further $\mathscr{H}_2$-bases; again they lie in a single T-system.

Thus for even m,

$$\phi_{\mathscr{H}_2}(D_{2m}) = \begin{cases} 6m\,\phi(m) + m\,\phi_2(C_m) & \text{if } m/2 \text{ is even,} \\ 9m\,\phi(m) + m\,\phi_2(C_m) & \text{if } m/2 \text{ is odd;} \end{cases}$$

$$= \begin{cases} 6m\,\phi(m) + m\,\phi(m)\,d_2(C_m) & \text{if } m/2 \text{ is even,} \\ 9m\,\phi(m) + m\,\phi(m)\,d_2(C_m) & \text{if } m/2 \text{ is odd;} \end{cases}$$

and

$$d_{\mathscr{H}_2}(D_{2m}) = \begin{cases} 6 + d_2(C_m) & \text{if } m/2 \text{ is even,} \\ 9 + d_2(C_m) & \text{if } m/2 \text{ is odd.} \end{cases}$$

There are three orbits under $\mathfrak{H}$ of hypermaps with automorphism group D_{2m} when $4 \mid m$: two of size three and one of size $d_2(C_m)$. Of these, the orbit of size three not present when m is odd is a Machí orbit of the map on the projective plane consisting of the reflexible embedding of a circuit of length $2m$ to give a single $2m$-valent (hyper)face. When $4 \nmid m$ there is another orbit in addition to these three; this is the orbit of the dipole $\mathscr{H} = \{m/2, 2\}$ and it too has size three; note that the elements of $\mathrm{Aut}(\mathscr{H}) \cong D_{2m}$ of order m are not rotations about the vertices of the dipole. However, we have already seen that $\mathrm{Aut}^+(\mathscr{H}) \cong D_m$, whence the orientation-preserving automorphisms of order $m/2$ are indeed such rotations.

3.2.5. Affine general linear groups

The group $\mathrm{AGL}(1, q)$ is the group of all affine transformations

$$f \mapsto uf + a \qquad (u, a \in \mathrm{GF}(q), u \neq 0)$$

of the field $F = \mathrm{GF}(q)$ of prime-power order $q = p^e$. It has a normal subgroup $T \cong (F, +)$ consisting of the translations

$$t_a \colon f \mapsto f + a \qquad (a \in F)$$

complemented by S, the stabilizer of 0, consisting of the scalar transformations

$$s_u \colon f \mapsto uf \qquad (u \in F \setminus \{0\}),$$

and isomorphic to the multiplicative group $F^* = F \setminus \{0\}$. The order of $\mathrm{AGL}(1, q)$ is $q(q-1)$, and each element can be written uniquely in the form $g = s_u t_a$. Moreover,

$$(s_u t_a)^{-1} = s_{u^{-1}} t_{-au^{-1}}$$

and

$$(s_u t_a)^{s_v t_b} = s_u t_{uav^{-1} + (u-1)bv^{-1}}$$

(so that u is an invariant of the conjugacy class of $s_u t_a$). Hence for $q > 2$, $\mathrm{AGL}(1, q)$ has trivial centre and so $\mathrm{Inn}(\mathrm{AGL}(1, q)) \cong \mathrm{AGL}(1, q)$.

Suppose $q > 2$. Let $s_u t_a$, $s_v t_b$ be elements of $\mathrm{AGL}(1, q)$ which do not both lie in the same cyclic subgroup. Consider the case $e = 1$: $\mathrm{AGL}(1, p)$ has order $p(p-1)$ and so it is clear that these elements form a 2-base if and only if (u, v) is a 2-

base for $F^* \cong C_{p-1}$. For $e > 1$ one cannot see from the group order alone that the latter condition is sufficient. However, the result still holds: for fixed $u, v \in F \setminus \{0\}$, $\mathrm{Inn}(\mathrm{AGL}(1,q))$ preserves and acts freely on the set $\Sigma_{u,v}$ of 2-bases $(s_u t_a, s_v t_b)$. This set contains at most $q(q-1)$ pairs since $\mathrm{AGL}(1,q)$ is the disjoint union of cyclic subgroups, namely T and the q conjugates S^g of S (which are the stabilizers of the q elements of $\mathrm{GF}(q)$), and $s_u t_a$ and $s_v t_b$ cannot lie in the same such subgroup. Yet $q(q-1)$ is the order of $\mathrm{Inn}(\mathrm{AGL}(1,q))$, and so $\Sigma_{u,v}$ has precisely this size. It follows that every pair $(s_u t_a, s_v t_b)$ for which (u, v) is a 2-base for $F^* \cong C_{q-1}$ is a 2-base for $\mathrm{AGL}(1,q)$ unless both components are powers of a common element. (We note here that $s_u t_a s_v t_b = s_{uv} t_{ub+a}$ and $s_v t_b s_u t_a = s_{uv} t_{va+b}$, and so $\langle s_u t_a, s_v t_b \rangle$ is non-abelian if and only if $ub + a \neq va + b$, that is, if and only if $a \neq 0$ and $v \neq 1$ when $u = 1$, and $b \neq a(v-1)/(u-1)$ otherwise.) Thus

$$\phi_2(\mathrm{AGL}(1,q)) = \begin{cases} q(q-1)\,\phi_2(C_{q-1}) = q(q-1)^3 \displaystyle\prod_{\substack{t \mid q-1 \\ t \text{ prime}}} \left(1 - \frac{1}{t^2}\right) & \text{if } q > 2, \\ 3 & \text{if } q = 2. \end{cases}$$

Let $\mathrm{Gal}(F)$ denote the Galois group of $F = \mathrm{GF}(q)$ (over its prime field $\mathrm{GF}(p)$). Every automorphism of $\mathrm{AGL}(1,q)$ is the composition of an inner automorphism and a 'field automorphism' [26]; in other words, $\mathrm{Aut}(\mathrm{AGL}(1,q))$ can be identified with the group of transformations $f \mapsto uf^\theta + a$ $(u \in F^*, a \in F, \theta \in \mathrm{Gal}(F))$, acting by conjugation on its normal subgroup $\mathrm{AGL}(1,q)$. Now, $\mathrm{Gal}(F)$ is cyclic of order e, being generated by the Frobenius automorphism $f \mapsto f^p$. It follows that

$$d_2(\mathrm{AGL}(1,q)) = \begin{cases} \frac{\phi_2(C_{q-1})}{e} = \frac{(q-1)^2}{e} \displaystyle\prod_{\substack{t \mid q-1 \\ t \text{ prime}}} \left(1 - \frac{1}{t^2}\right) & \text{if } q > 2, \\ 3 & \text{if } q = 2. \end{cases}$$

An $\mathrm{Aut}(\mathrm{AGL}(1,q))$-class of 2-bases may be represented by any one of e 2-bases of the form $(s_u, s_v t_1)$, corresponding to e 2-bases (u, v) of C_{q-1}. Since $\mathrm{Aut}(F_2)$ acts transitively on the 2-bases of cyclic groups by Proposition 3.2.9, the $\mathrm{Aut}(\mathrm{AGL}(1,q))$-classes lie in a single T-system. The types of the hypermaps in the corresponding $\mathfrak{H}^+$-orbit are easily determined: if we represent (u, v) as a $\mathscr{C}_2^+$-base $(u, v, (uv)^{-1})$ then the corresponding hypermap $\mathscr{H}$ has type $(|u|, |v|, |(uv)^{-1}|)$. (The e $\mathscr{C}_2^+$-bases representing

$\mathscr{H}$ are conjugate in the obvious way under Gal(F) and so the stated type is independent of the choice of representative.)

It is of interest to partition the T-system into orbits according to the action of Aut(C_{q-1}) on the 2-bases of C_{q-1}; using the results from §3.2.3 we have that

$$d_2(\mathrm{AGL}(1,q)) = \frac{|\mathrm{Aut}(C_{q-1})|\, d_2(C_{q-1})}{e} = \frac{\phi(q-1)\, d_2(C_{q-1})}{e}$$

for $q > 2$. Thus there are $d_2(C_{q-1})$ such sets, each of size $\phi(q-1)/e$, and hypermap type (and hence genus) is an invariant of the set. Note that different sets may give rise to the same type (as can easily be seen when q is prime; cf. §3.2.3). One of these sets consists of the regular orientable embeddings of the complete graph K_q classified in [26]; these were demonstrated to form an orbit under the action of Wilson's operators H_j described in §4.4.3 below. In general however, our sets are not orbits under the H_j or even under the more general operators $\mathrm{H}_{j,k}$ (*ibid.*), as can be seen by observing that these operators do not in general preserve type, sending a $\mathscr{C}_2^+$-base $(u, v, (uv)^{-1})$ to $(u^j, v^k, (u^j v^k)^{-1})$ which corresponds to a hypermap of type $(|u|, |v|, |(u^j v^k)^{-1}|)$. However, all of these operators do at least act transitively within the T-system on hypermaps of a given type, since elements of the same order in a cyclic group are necessarily powers of each other.

3.2.6. Asymptotic behaviour of orbits

Let F be $\mathscr{H}_2^+$ or $\mathscr{H}_2$, and let $\{G_i\}$ be a sequence of groups belonging to the same infinite 'family' (cyclic groups, dihedral groups, etc.), such that $|G_{i+1}| > |G_i|$. Define $f\colon \mathbb{N} \to \mathbb{N}$ such that $f(i) = |T_{\max}|$ where $T_{\max}$ is the largest T_F-system of G_i. It is interesting to examine the asymptotic behaviour of f. Can we have arbitrarily large orbits of hypermap operations, and if so how fast can they grow? To approach these questions, it is natural to first determine the asymptotic behaviour of d_F (where, as usual, $d_F(G_i)$ is the number of symmetric hypermaps with automorphism group G_i), and then to ask how these are partitioned into orbits under $\mathfrak{H}^+$ or $\mathfrak{H}$. It seems that the latter question is rather difficult.

Historically, the problem of evaluating d_F goes back to Philip Hall [20], who expressed the function as a Möbius-type summation of $d_F(H)$ over all intersections of maximal subgroups $H \subset G$. This shifted the focus of the problem from F-bases to subgroup lattices. He referred to d_F as an *Eulerian function* of G since in the special case of cyclic groups we have $d_1(C_m) = \phi(m)$. Hall used the principle to calculate d_F for a number of low-order groups and for the simple groups $\mathrm{PSL}(2, p)$ of order $\frac{1}{2}p(p^2 - 1)$ where p is a prime exceeding 3.[5] More recently, its application has led to formulae (involving elementary number-theoretic functions) for $d_F(G_i)$ in the case where $F = \Delta(3, 2, 7)$ and $\{G_i\}$ is the family of simple groups of Ree type of characteristic 3 [27], and in the case where $F = \Delta(4, 2, 5)$ and $\{G_i\}$ is the family of simple groups of Suzuki type [30].

One may also employ probabilistic methods to estimate $\phi_F(G)$ and hence $d_F(G)$. Attention has mainly been focused on the case $F = F_n$ (see, for example, surveys in [54], [50]). Dixon [13] showed that two randomly-chosen elements of the alternating group $G = A_n$ form a 2-base with probability approaching 1 as $|G| \to \infty$, and that for $G = S_n$ this probability approaches $\frac{3}{4}$. The result for A_n has been generalized to all finite simple groups G ([36], [40]). Moreover, $\mathrm{Inn}(G) \cong G$ for all finite groups G which are either simple or non-abelian symmetric. It follows that for alternating groups A_n (which satisfy $\mathrm{Aut}(A_n) \cong S_n$ for all $n > 3$ and $n \neq 6$) we have

$$d_2(G) \sim \frac{n!}{4};$$

while for symmetric groups S_n (which have no non-trivial outer automorphisms for $n \neq 2, 6$) we have

$$d_2(G) \sim \frac{3n!}{4};$$

and for the Chevalley and twisted Chevalley groups G we have

$$d_2(G) \sim \frac{|G|^2}{|\mathrm{Aut}(G)|} = \frac{|G|}{|\mathrm{Out}(G)|}.$$

The remaining non-abelian finite simple groups are sporadic. The outer automorphism group $\mathrm{Out}(G)$ of a Chevalley or twisted Chevalley group G is usually a very low multiple of the degree f of the finite field $\mathrm{GF}(p^f)$ over which G is constructed [5]. In particular, $\mathrm{Out}(G)$ is cyclic of order f for many of the families when $p = 2$ and for some of these when p is almost unrestricted. Most notably, the families $G_2(p)$, $F_4(p)$ and $E_8(p)$ have trivial outer automorphism group for $p > 3$, and for these groups G we have $d_2(G) \sim |G|$.

It is also possible—although rather difficult in general—to estimate the asymptotic

[5] Hall denotes the group $\mathrm{PSL}(2, p)$ by M^p.

behaviour of the number of T-systems of a family of groups. A recent result of Guralnick and Pak [19] which states that the number of T-systems of generating pairs of $\mathrm{SL}(2, p)$ approaches ∞ as primes $p \to \infty$ is a notable success in this regard.

Despite all this, the techniques used for estimation shed little light on individual T-systems, and asymptotic formulae for the size of T-systems seem hard to achieve. (Describing the T-systems of the alternating groups is an open problem proposed by Pak [50].)

Of course, as we have demonstrated in the previous sections, T-systems can be calculated directly for certain families of groups whose elements are 'well understood', and here we can make better progress. Consider first the cyclic groups. Recall from §3.2.3 that

$$d_2(C_m) = m\prod_{p|m}\left(1 + \frac{1}{p}\right).$$

If $m = p$ for some prime p then the size of the orbit or regular orientable hypermaps without boundary with automorphism group C_p is $p + 1$ and thus grows linearly with the order of the groups in this family. We can do better if we let $m_k = p_1 p_2 \dots p_k$ be the product of the first k primes. In this case we have

$$\begin{aligned} d_2(C_{m_k}) &= m_k \prod_{i=1}^{k}\left(1 + \frac{1}{p_i}\right) \\ &= m_k\left(1 + \sum_{i=1}^{k}\frac{1}{p_i} + \sum_{1\leqslant i<j\leqslant k}\frac{1}{p_i p_j} + \dots + \prod_{i=1}^{k}\frac{1}{p_k}\right). \end{aligned}$$

The sum of the reciprocals of the first k primes is divergent, while the other terms in the product expansion above converge. Hence the hypermap orbit grows faster than linearly for this family. The rate of divergence is known from a standard result of analytic number theory (e.g. [49]): there exists a constant C such that

$$\sum_{\substack{p\leqslant N \\ p \text{ prime}}} \frac{1}{p} = \log\log N + C + \mathrm{O}\left(\frac{1}{\log N}\right).$$

Now consider the affine general linear groups. Recall from §3.2.5 that

$$d_2(\mathrm{AGL}(1, q)) = \frac{(q-1)^2}{e}\prod_{\substack{t|q-1 \\ t \text{ prime}}}\left(1 - \frac{1}{t^2}\right)$$

if $q = p^e > 2$ is a prime power. By the Euler product formula for the Riemann zeta function $\zeta(s)$ we have

$$\prod_{t \text{ prime}} \left(1 - \frac{1}{t^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2},$$

and so

$$\frac{6}{\pi^2}\frac{(q-1)^2}{e} \leqslant d_2(\mathrm{AGL}(1,q)) \leqslant \frac{(q-1)^2}{e}.$$

We see that the orbit of regular orientable hypermaps without boundary with automorphism group $\mathrm{AGL}(1,q)$ grows linearly with the size of this group.

Chapter 4

Generalized operations I

4.1. Hypermap categories and their functors

4.1.1. Operational functors

We have seen in §1.1 how the orientable hypermaps without boundary whose type divides (m_0, m_1, m_2) correspond to transitive permutation representations of the group

$$\mathscr{H}_2^+(m_0, m_1, m_2) = \langle\, r_0, r_1, r_2 \mid r_0^{m_0} = r_1^{m_1} = r_2^{m_2} = r_0 r_1 r_2 = 1 \,\rangle \cong \Delta(m_0, m_1, m_2),$$

with coverings between such hypermaps corresponding to inclusions (up to conjugacy) of hypermap subgroups in $\mathscr{H}_2^+(m_0, m_1, m_2)$. Such hypermaps and coverings give us the objects and morphisms of a category $C_{(m_0,m_1,m_2)}$. Similarly, the transitive permutation representations of $\mathscr{H}_2$ give rise to the wider category $C_{[m_0,m_1,m_2]}$ of *all* hypermaps whose type divides (m_0, m_1, m_2).

Each map operation and hypermap operation is induced by an automorphism of the full cartographic group and full hypercartographic group, and gives a functor Φ from $C_{[\infty,2,\infty]}$ to itself and from $C_{[\infty,\infty,\infty]}$ to itself respectively. We are motivated by the following idea of James [25] to generalize these operations. For $i = 1, 2$ let C_i be a category of objects which correspond to conjugacy classes of subgroups in a group G_i. Suppose we have an epimorphism $\psi\colon M \to G_1$ from some subgroup M of G_2. For each object $\mathscr{M}_1$ in C_1 with corresponding subgroup M_1 in G_1, define $\mathscr{M}_1^{\psi}$ to be the object with corresponding subgroup $\psi^{-1}(M_1)$. The mapping Ψ taking each $\mathscr{M}_1$ to $\mathscr{M}_1^{\psi}$ provides a representation of C_1 by C_2.

In this and the following chapter we concern ourselves with the related problem in which the epimorphism is from G_1 onto a subgroup of G_2—in other words, a homomorphism from G_1 to G_2. (Of course, the two approaches to representations coincide when the epimorphism is injective, giving an inclusion of G_1 in G_2.) In particular, we investigate functors between categories of hypermaps defined by type induced by homomorphisms between triangle groups and between extended triangle groups. To elaborate, if $\phi\colon \Delta(m_0, m_1, m_2) \to \Delta(m'_0, m'_1, m'_2)$ is a homomorphism between triangle groups then it induces a functor Φ (which we shall call an *operational functor*) between $C_{(m_0,m_1,m_2)}$ and $C_{(m'_0,m'_1,m'_2)}$: for a hypermap $\mathscr{H}$ of type dividing (m_0, m_1, m_2) with hypermap subgroup H, the image $\Phi(\mathscr{H})$ is the hypermap of type dividing (m'_0, m'_1, m'_2) with hypermap subgroup $H^\Phi = \phi(H)$. This is well-defined since hypermap subgroups are defined up to conjugacy and any homomorphism sends conjugate subgroups into a single conjugacy class.

If ϕ is not surjective and yet its image has the structure of a triangle group, then Φ can be regarded as the composition of two operational functors corresponding to the composition of an epimorphism with an inclusion.

These ideas go through in exactly the same way for homomorphisms between extended triangle groups.

4.1.2. Restrictions

Each operational functor Φ can be restricted to a functor $\Phi|$ from a subcategory of its domain defined by type. On the other hand we may or may not find it possible to regard the range of $\Phi|$ as a proper such subcategory of that of Φ.

We also note that restrictions of operational functors are not necessarily themselves operational functors. For example, the topological representations of orientable hypermaps discussed in §1.2 give rise to functors $C_{(\infty,\infty,\infty)} \to C_{(\infty,2,\infty)}$ by regarding the image of a hypermap $\mathscr{H}$ to be the underlying map of its topological representation. These functors permit restrictions as follows:

Cori representation	$\mathrm{C}\colon C_{(m_0,m_1,m_2)} \to C_{(4,2,\mathrm{LCM}(m_0,m_1,2m_2))}$
Walsh representation	$\mathrm{W}\colon C_{(m_0,m_1,m_2)} \to C_{(\mathrm{LCM}(m_0,m_1),2,2m_2)}$
James representation	$\mathrm{J}\colon C_{(m_0,m_1,m_2)} \to C_{(3,2,2\,\mathrm{LCM}(m_0,m_1,m_2))}$

and the functor arising from the James representation extends naturally to one defined similarly between categories of general hypermaps. For certain types (m_0, m_1, m_2) some of these restrictions coincide with operational functors, as discussed further in §4.5.1. In the main, however, these restrictions do not arise from homomorphisms.

4.2. Inclusions as functors

In this chapter we classify the operational functors arising from inclusions between triangle groups and between extended triangle groups. In Chapter 5 we classify the remaining homomorphisms and their functors.

Inclusions fall into three classes: automorphisms of a given triangular group; isomorphisms between different triangular groups; and proper inclusions $\phi\colon \Delta \hookrightarrow \Gamma$ (for which $\phi(\Delta)$ is a proper subgroup of Γ). As discussed in §1.4, we regard the order of the periods in the notation as fixed, so that $\Delta(a,b,c)$ and $\Delta(c,b,a)$ denote different—but isomorphic—groups.

Any inclusion $\phi\colon \Delta \hookrightarrow \Gamma$ between triangle groups is an identification of Δ with a subgroup of Γ, and so we may treat the fixed points of the rotation elements of Δ as being fixed by elements of Γ. In other words, the centres of the hypermap components of the universal hypermap $\mathscr{U}_\Delta$ can be identified with a subset of those of $\mathscr{U}_\Gamma$ (not necessarily respective of the dimension i of the component), thus superimposing $\mathscr{U}_\Delta$ on $\mathscr{U}_\Gamma$. If $\Delta = \Delta(m_0, m_1, m_2)$ and $\Gamma = \Delta(m'_0, m'_1, m'_2)$ then ϕ gives rise to a functor $\Phi : C_{(m_0,m_1,m_2)} \to C_{(m'_0,m'_1,m'_2)}$. Let $\mathscr{X}$ be the simply-connected Riemann surface (augmented by ideal points if some $m_i = \infty$) on which $\mathscr{U}_\Gamma$ (and hence $\mathscr{U}_\Delta$) lies. Then Δ is the hypermap subgroup of itself corresponding to the trivial hypermap $\mathscr{T}$ on the Riemann surface $\mathscr{X}/\Delta$ homeomorphic to S^2; it is also the hypermap subgroup within Γ of a hypermap $\mathscr{D} = \mathscr{U}_\Gamma/\Delta$ on $\mathscr{X}/\Delta$. This hypermap, the image under Φ of $\mathscr{T}$, is the quotient of $\mathscr{U}_\Gamma$ obtained by identifying sides of a fundamental region for Δ. We actually have a functor from $C_{(m_0,m_1,m_2)}$ to the subcategory of $C_{(m'_0,m'_1,m'_2)}$ consisting of hypermaps which cover $\mathscr{D}$ (Figure 4.1).

The image under Φ of a regular hypermap $\mathscr{H}$ with hypermap subgroup H is not necessarily regular since normal subgroups of Δ need not be normal in Γ. On the other hand, preimages under Φ of a regular hypermap $\Phi(\mathscr{H})$ covering $\mathscr{D}$ are themselves regular. The automorphism group of $\mathscr{H}$ is a subgroup of that of $\Phi(\mathscr{H})$ and has index $|N_\Gamma(H) : N_\Delta(H)|$.

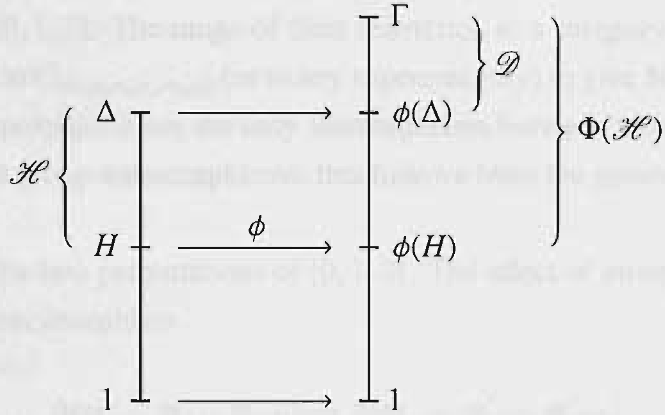


FIGURE 4.1. Homomorphism diagram for an inclusion ϕ between triangle groups.

An inclusion functor Φ sending $\mathscr{H}$ to $\Phi(\mathscr{H})$ does not in general have an inverse; rather, each hypermap $\Phi(\mathscr{H})$ has an inverse image which may consist of several hypermaps. This is because subgroups $H \leqslant \Delta$ which are conjugate in Γ need not be conjugate in Δ, and so the mapping between conjugacy classes of subgroups of Δ and Γ induced by an inclusion is not necessarily injective. As a result, a number of arbitrary choices will need to be made in the reverse construction. In particular, if Δ has index n in Γ then there are n distinct ways of superimposing a marked $\mathscr{U}_\Delta$ on $\mathscr{U}_\Gamma$ so that the marked hyperdart lies within a given fixed fundamental region for Δ. Any attempt to reverse the construction for Φ will necessarily involve (at least) making an arbitrary choice from n possibilities, giving rise to n preimages, no two of which need necessarily be isomorphic.

These reverse constructions can be obtained in a straightforward manner when $\Phi(\mathscr{H})$ is regular and $\phi(H)$ is canonical in Γ, simply by reversing the description for Φ. We provide examples of this in §4.5.1. To find the inverse image of a more general hypermap $\mathscr{M}$ covering $\mathscr{D}$, one has little choice but to find the inverse image of the minimal regular cover of $\mathscr{M}$ and form quotients of the hypermaps by the appropriate group.

4.3. Functors from isomorphisms

The Machí operations on the category $C_{(\infty,\infty,\infty)}$ of orientable hypermaps without boundary are induced by permutations of the i-components and hence by permutations π of the elements of $\{0,1,2\}$. The range of their restriction to a category $C_{(m_0,m_1,m_2)}$ can itself be restricted to $C_{(m_{\pi(0)},m_{\pi(1)},m_{\pi(2)})}$ (or to any supercategory) to give *Machí functors* Φ_π. The Machí automorphisms are the only isomorphisms between triangle groups up to composition with group automorphisms; this follows from the geometric definition of the groups.

Let π and π' be two permutations of $\{0,1,2\}$. The effect of an operational functor arising from a homomorphism

$$\Delta(m_{\pi(0)}, m_{\pi(1)}, m_{\pi(2)}) \to \Delta(m'_{\pi'(0)}, m'_{\pi'(1)}, m'_{\pi'(2)})$$

is the same as the effect of the composition $\Phi_\pi \circ \Phi \circ \Phi_{\pi'^{-1}}$ where Φ arises from a homomorphism $\phi\colon \Delta(m_0,m_1,m_2) \to \Delta(m'_0,m'_1,m'_2)$. It follows that when determining operational functors it is enough to work with just one of the possible orderings of the periods of each triangle group involved.

These ideas also hold true in the wider context of general hypermaps and extended triangle groups.

4.4. Functors from automorphisms

4.4.1. Triangle groups with finite periods

The automorphisms of a triangle group $\Delta(m_0,m_1,m_2)$ give rise to functors from the category $C_{(m_0,m_1,m_2)}$ to itself. When the periods are finite the group is cocompact; it is well known that the automorphisms of finitely-generated planar discrete cocompact groups are geometrically induced (see, for example, [62, §6.6]). For infinite triangle groups with finite periods, it follows that none, two or six Machí functors are induced by outer automorphisms according as none, two or all three of the periods are the same; and that there is precisely one other outer automorphism, represented by the automorphism υ which maps a given pair of standard elliptic generators to their inverses. This outer automorphism induces the *chiral duality functor* Υ which sends each orientable hypermap $\mathscr{H}$ without boundary to its mirror image or *chiral dual*, the hypermap obtained by reflecting $\mathscr{H}$ in the projection onto $\mathscr{H}$ of that side the principal fundamental triangle of $\mathscr{U}_\Delta$ which joins the fixed points of the given generators. (This same geometric transformation induces an inner automorphism of the corresponding extended triangle group, and so there is no notion of chirality in the wider context of hypermaps which may be non-orientable or have boundary.)

For the spherical triangle groups the situation can be different. The automorphism υ which induces chiral duality represents the only non-trivial outer automorphism of the group $\Delta(3,2,3) \cong A_4 \cong \mathrm{Aut}^+(\mathcal{T})$ where $\mathcal{T}$ is the tetrahedron; it is induced by conjugation in $\Delta[3,2,3] \cong S_4$. (Note that the maps in $C_{(3,2,3)}$ are all self-dual.) This automorphism is inner (as are all automorphisms) in the case of $\Delta(3,2,4) \cong S_4 \cong \mathrm{Aut}^+(C)$ where C is the cube. Hence the quotients of the cube are all non-chiral. Likewise, υ is inner in the case of $\Delta(3,2,5) \cong A_5 \cong \mathrm{Aut}^+(\mathcal{D})$, where $\mathcal{D}$ is the dodecahedron. There does however exist one outer automorphism in this case, represented by an automorphism ϕ, and it is not geometrically induced. (It is induced by conjugation in S_5.) Yet although there exist subgroups of $\Delta(3,2,5)$ which are not invariant under ϕ, the conjugacy classes of subgroups are invariant since representatives from different classes are of different sizes. Hence ϕ simply induces the identity functor.

The groups $\Delta(n,2,2) = \langle\, x, y \mid x^n = y^2 = (xy)^2 = 1 \,\rangle \cong D_{2n}$ are the orientation-preserving automorphism groups of the dipoles $\{n,2\}$ (using the notation of [11]). Now, $\mathrm{Aut}(D_{2n}) \cong C_n \rtimes C_{\phi(n)}$; when n is odd we have $\mathrm{Inn}(D_{2n}) \cong D_{2n}$ and $|\mathrm{Out}(D_{2n})| = \phi(n)/2$, while when n is even we have $\mathrm{Inn}(D_{2n}) \cong D_n$ and $|\mathrm{Out}(D_{2n})| = \phi(n)$. The outer automorphisms when n is odd are represented by automorphisms which—with the exception of the identity—permute the subgroups of D_{2n} non-trivially but act trivially on the conjugacy classes of subgroups. Hence there are no non-identity operational functors in this case. If n is even then for positive divisors m of n there are precisely two conjugacy classes of subgroups D_{2m} in D_{2n}. A subgroup of index 2 in $\mathrm{Out}(D_{2n})$ preserves them, and the other coset transposes them. Hence there is one non-identity operational functor Φ, represented by $x \mapsto x$, $y \mapsto x^{-1}y$. Regarding D_{2n} as acting on the n-gon in the usual way, this automorphism interchanges a reflection which fixes two vertices of the n-gon with one which preserves two edges. It follows that Φ is the Machí automorphism which fixes the vertices of a map $\mathcal{M}$ but transposes edges and faces.

As for each group $\Delta(n,1,n) \cong C_n$, which is the orientation-preserving automorphism group of the star map with n free edges, its subgroups are of course invariant under all of its automorphisms and so there are no non-identity functors induced in this case.

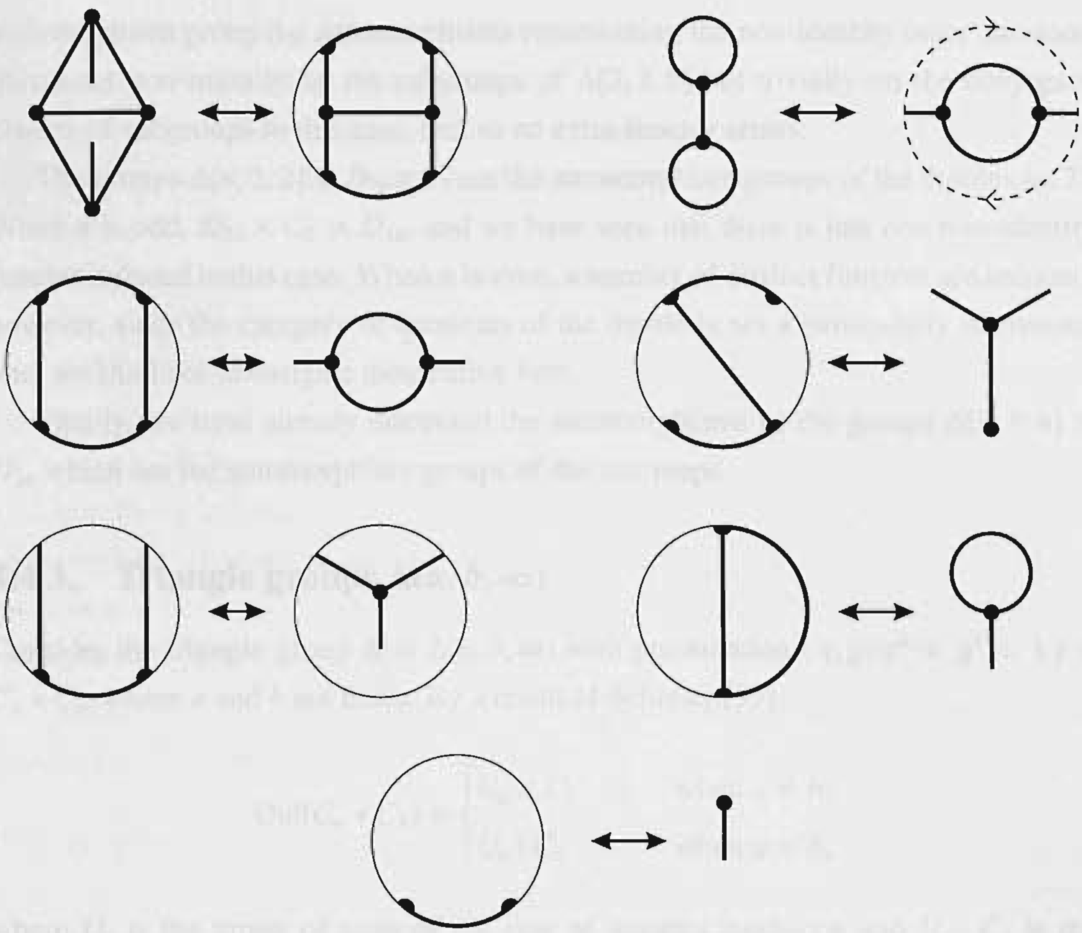


FIGURE 4.2. The non-trivial orbits of the functor induced by the non-identity outer automorphism of $\Delta[3,2,4]$.

4.4.2. Extended triangle groups with finite periods

As was the case for planar triangle groups with finite periods, the automorphisms of their extended counterparts are geometrically induced [43]. Hence the Machí functors are the only possibilities here. The automorphisms of the group $\Delta[3,2,3] \cong S_4 \cong \mathrm{Aut}(\mathcal{T})$ are all inner. The group $\Delta[3,2,4] \cong S_4 \times C_2 \cong \mathrm{Aut}(C)$ has inner automorphism group S_4; the non-identity outer automorphism is represented by an automorphism which transposes the two subgroups isomorphic to S_4, namely the rotation group of the cube and the full symmetry group of the two inscribed tetrahedra. Hence there is a single non-identity operational functor in this case; Figure 4.2 shows its effect. The automorphism group of $\Delta[3,2,5] \cong A_5 \times C_2 \cong \mathrm{Aut}(\mathcal{D})$ is isomorphic to S_5, with inner automorphism group A_5; automorphisms representing the non-identity outer automorphism act non-trivially on the subgroups of $\Delta[3,2,5]$ but trivially on the conjugacy classes of subgroups in this case, and so no extra functor arises.

The groups $\Delta[n,2,2] \cong D_{2n}\times C_2$ are the automorphism groups of the dipoles $\{n,2\}$. When n is odd, $D_{2n} \times C_2 \cong D_{4n}$, and we have seen that there is just one non-identity functor induced in this case. When n is even, a number of distinct functors are induced; however, since the category of quotients of the dipole is not a particularly interesting one, we shall not investigate their nature here.

Finally, we have already discussed the automorphisms of the groups $\Delta[n,1,n] \cong D_{2n}$ which are the automorphism groups of the star maps.

4.4.3. Triangle groups $\Delta(a,b,\infty)$

Consider the triangle group $\Delta = \Delta(a,b,\infty)$ with presentation $\langle\, x, y \mid x^a = y^b = 1 \,\rangle \cong C_a * C_b$, where a and b are finite. By a result of Schreier [53],

$$\mathrm{Out}(C_a * C_b) \cong \begin{cases} U_a \times U_b & \text{when } a \neq b; \\ U_a \wr C_2 & \text{when } a = b, \end{cases}$$

where U_n is the group of units of the ring of integers modulo n and $U_a \wr C_2$ is the wreath product of U_a and C_2, a semidirect product of $U_a \times U_a$ and C_2. (This result is a consequence of the Torsion Theorem for Free Products which implies that the images under an automorphism of x and y, being torsion elements, are conjugates of powers of x and y. Since we can express x and y as words in their images, their simple form allows us to use standard cancellation arguments together with the Normal Form Theorem for Free Products to determine the possible images and hence calculate the automorphism group.) The outer automorphisms are represented by compositions of the following automorphisms of Δ:

$$x \mapsto x^j, \quad y \mapsto y \qquad \text{where } 0 < j < a,\ \mathrm{HCF}(j,a) = 1;$$
$$x \mapsto x, \quad y \mapsto y^k \qquad \text{where } 0 < k < b,\ \mathrm{HCF}(k,b) = 1;$$
$$\text{and if } a = b, \qquad x \mapsto y, \quad y \mapsto x.$$

When $a = b$ the automorphism $\phi_{(0,1)} : x \mapsto y, y \mapsto x$ induces a Machí functor. The following definitions were first introduced by Coxeter [9].

Definition 4.4.1. Let $\mathscr{M}$ be an orientable map without boundary. A j^{th}*-order hole* in $\mathscr{M}$ is a cyclic sequence of edges, each two consecutive sharing a vertex, such that at each vertex, the surrounding two edges in the sequence subtend j faces on one side, either the right or the left but consistently throughout.

A j^{th}*-order Petrie circuit* is a similar sequence of edges, but at each vertex, j faces are subtended on the right and on the left alternately. •

First-order holes and Petrie circuits are simply the familiar notions of faces and ordinary Petrie circuits. Algebraically, j^{th}-order holes correspond to the cycles (on the set of darts) of the element $(r_0^j r_1)^{-1}$ of an appropriate triangle group (§1.1.2) while j^{th}-order Petrie circuits correspond to the cycles of $r_0^j r_1 r_0^{-j} r_1$. These concepts can be applied equally well to all maps by considering orbits on the set of blades of the subgroups $\langle (t_1(t_2t_1)^{j-1})^{-1}, t_0 \rangle = \langle t_1(t_2t_1)^{j-1}, t_0 \rangle$ and $\langle t_1(t_2t_1)^{j-1}, t_2t_0 \rangle$ of an appropriate extended triangle group (§1.2.1).

The nature of the automorphisms above motivates us to extend these concepts to hypermaps. We make the following definition.

Definition 4.4.2. Let $\mathscr{H}$ be an orientable hypermap without boundary. A $(j,k)^{th}$*-order hole* in $\mathscr{H}$ is a cyclic, alternating sequence of hypervertices and hyperedges, each pair of consecutive components adjacent, such that each hypervertex and its surrounding two hyperedges in the sequence subtend j hyperfaces consistently on one side (either the right or the left) and such that each hyperedge and its adjacent hypervertices subtend k hyperfaces consistently on that same side.

An $(j,k)^{th}$*-order Petrie circuit* is a similar sequence such that at each hypervertex and its surrounding two hyperedges, j hyperfaces are subtended on the right and on the left alternately; and such that at each hyperedge and its surrounding two hypervertices, k hyperfaces are subtended on the same side as that for the previous hypervertex in the sequence. •

So $(j,1)^{\text{th}}$-order holes and Petrie circuits in hypermaps correspond to j^{th}-order holes and Petrie circuits in maps. Algebraically, $(j,k)^{\text{th}}$-order holes and Petrie circuits in orientable hypermaps without boundary correspond to the cycles on the hyperdart-set of $(r_0^j r_1^k)^{-1}$ and $r_0^j r_1^k r_0^{-j} r_1^{-k}$. When applied to all hypermaps, these two concepts correspond to orbits on the hyperblade-set of the subgroups $\langle (t_1(t_2t_1)^{j-1})^{-1}, t_0(t_2t_0)^{k-1} \rangle$ and $\langle t_1(t_2t_1)^{j-1}, (t_2t_0)^k \rangle$.

We now see that our automorphisms $x \mapsto x^j$, $y \mapsto y^k$ of Δ induce operational functors $\mathrm{H}_{j,k}$ whose effect on a hypermap can be realized by dissolving the hyperfaces and then spanning by a membrane each cycle of hyperdarts which form a $(j,k)^{\text{th}}$-order hole. This has the effect of transforming Petrie circuits into $(j,k)^{\text{th}}$-order Petrie circuits. In general these functors come in chiral pairs: $\mathrm{H}_{-j,-k} = \mathrm{H}_{j,k} \circ \Upsilon$ where Υ is the chiral duality functor and subscripts j and k are to be taken modulo a and b.

A family H_j of map operations which fix the underlying graph but transform faces into j^{th}-order holes was introduced by Wilson [61]. When j is relatively prime to the least common multiple a of the valencies, the resulting object is a legal map. We see now that these operations arise naturally as the group of all operational functors for the category of maps of type $(a, 2, \infty)$.

4.4.4. Triangle groups $\Delta(\infty, b, \infty)$

Consider the triangle group $\Delta = \Delta(\infty, b, \infty)$ with presentation $\langle\, x, y \mid y^b = 1 \,\rangle \cong \mathbb{Z} * C_b$, where b is finite.

Proposition 4.4.3. $\mathrm{Out}(\mathbb{Z} * C_b) \cong D_{2b} \rtimes U_b$.

Proof. Representatives for an obvious family of outer automorphism classes are $x \mapsto x$, $y \mapsto y^k$ where $0 < k < b$, $\mathrm{HCF}(k,b) = 1$. Through composition with one of these representatives and an inner automorphism we may assume, by the Torsion Theorem for Free Products, that a given automorphism θ of $\mathbb{Z} * C_b$ fixes y. We write $\theta(x)$ as a word $w = w(x,y)$ and then express x as a word $w' = w'(\theta(x), y)$ in the images of the generators. The word w' is of the form $y^{i_1} w^{j_1} y^{i_2} w^{j_2} \ldots y^{i_n} w^{j_n}$. Examining the reduction that occurs in the product $ww = w^2$, we see that if half or more of the powers of x in either factor w are annihilated then they all must be; this cannot happen since w has infinite order. Hence there is a positive number of powers of x in the middle of each w which do not cancel in the reduced form of w^2. This is also true of the reduced forms of $w^{\pm} y^{i_n} w^{\pm}$ and $w^{\mp} y^{i_n} w^{\mp}$. Applying the Normal Form Theorem, we conclude that the expression of w' as a product involves precisely one w which itself involves precisely one power of x, namely x or x^{-1}. By composition with an inner automorphism we may finally assume that $\theta(x) = x^{\pm} y^i$. We deduce that the outer automorphism group is represented by the automorphisms in the group which is generated by

$$x \mapsto x, \quad y \mapsto y^k \qquad \text{where } 0 < k < b,\ \mathrm{HCF}(k,b) = 1;$$
$$x \mapsto xy, \quad y \mapsto y;$$
$$x \mapsto x^{-1}, \quad y \mapsto y.$$

The factor D_{2b} of $\mathrm{Out}(\mathbb{Z} * C_b)$ is normal: it is the kernel of the action on $\langle\, y\,\rangle$. However, an automorphism $x \mapsto x$, $y \mapsto y^k$ commutes with the automorphism $x \mapsto xy$, $y \mapsto y$ only when $k = 1$. The result follows. ∎

It follows that the group of operational functors here is generated by three of its subgroups. The first, induced by the group of automorphisms $\{x \mapsto x, y \mapsto y^k \mid 0 < k < b,\ \mathrm{HCF}(k,b) = 1\} \cong U_b$, is realized by dissolving the hyperfaces of each hypermap and then spanning by a membrane each cycle of hyperblades which form a $(1,k)^{\text{th}}$-order hole. The second, induced by the group of automorphisms $\{x \mapsto xy^i, y \mapsto y \mid 0 \leqslant i < b\} \cong C_b$, is realized by dissolving the hypervertices and hyperfaces of each hypermap to leave a skeleton of hyperedges and Petrie circuits and then recreating the hypervertices (which are $(1,0)^{\text{th}}$-order holes) as $(1,i)^{\text{th}}$-order holes. This has the effect of transforming each $(1,n)^{\text{th}}$-order hole into a $(1,n+i)^{\text{th}}$-order hole. To facilitate a simple combinatorial description, we take the third subgroup of operational functors to be the involutary group generated by the chiral duality functor Υ (induced by the automorphism $\upsilon\colon x \mapsto x^{-1}, y \mapsto y^{-1}$). Note that the Machí functor is induced directly here by $x \mapsto xy$, $y \mapsto y^{-1}$; when $b = 1$ it acts as the identity functor and υ is the identity automorphism, corresponding to the fact that the quotients of the infinite star map $\mathcal{U}_{\Delta(\infty,1,\infty)}$ are all self-dual and non-chiral. When $b > 1$ this functor has order 2.

4.4.5. Extended triangle groups $\Delta[a,b,\infty]$

Let $\Delta[a,b,\infty]$ have presentation $\langle\, l, r, t \mid l^2 = r^2 = t^2 = (rt)^a = (tl)^b = 1\,\rangle$. When $b = 2$ let π be the automorphism $r \mapsto r$, $t \mapsto t$, $l \mapsto tl$. When $a = 2$ let τ be the automorphism $r \mapsto rt$, $t \mapsto t$, $l \mapsto l$.

Theorem 4.4.4.

$$\mathrm{Out}(\Delta[a,b,\infty]) \cong \begin{cases} (U_a \times U_b)/\langle\,(-1,-1)\,\rangle & a \neq b,\ a \neq 2,\ b \neq 2 \\ ((U_a \times U_b) \rtimes C_2)/\langle\,(-1,-1)\,\rangle & a = b,\ a \neq 2. \end{cases}$$

$$\begin{aligned}
\mathrm{Out}(\overline{\Delta}[a,2,\infty]) &\cong U_a/\langle -1\rangle \times \langle \pi \rangle && \textit{when } a \neq 2.\\
\mathrm{Out}(\overline{\Delta}[2,b,\infty]) &\cong U_b/\langle -1\rangle \times \langle \tau \rangle && \textit{when } b \neq 2.\\
\mathrm{Out}(\overline{\Delta}[2,2,\infty]) &\cong (\langle \pi \rangle \times \langle \tau \rangle) \rtimes C_2 \cong D_8.
\end{aligned}$$

We begin the proof with the following lemma.

Lemma 4.4.5. *Let $\overline{\Gamma} = \Delta[m_0, m_1, m_2]$ be an extended planar triangle group with possibly infinite periods. Then an automorphism of $\overline{\Gamma}$ is fully determined by its effect on the even subgroup Γ.*

Proof. Let $\overline{\Gamma}$ have presentation $\langle\, l, r, t \mid l^2 = r^2 = t^2 = (rt)^{m_0} = (tl)^{m_1} = (lr)^{m_2} = 1 \,\rangle$. By identifying x with rt and y with tl, we have the alternative presentation $\langle\, x, y, t \mid (ty)^2 = (xt)^2 = t^2 = x^{m_0} = y^{m_1} = (xy)^{-m_2} = 1 \,\rangle$ for $\overline{\Gamma}$. Let α, β be automorphisms of $\overline{\Gamma}$ with identical effect on $\Gamma = \langle\, x, y \,\rangle$. Then $\theta = \alpha\beta^{-1}$ fixes Γ pointwise and is hence determined by its effect on t. Now, t inverts x and y, and so $\theta(t)$ also inverts $\theta(x) = x$ and $\theta(y) = y$. Thus $t\theta(t)$ is an element of the even subgroup Γ centralizing x and y; so $t\theta(t) \in \mathrm{Z}(\Gamma)$. But planar triangle groups have trivial centre, and so $t = \theta(t)$. ∎

Consider first those automorphisms of $\overline{\Delta} = \Delta[a, b, \infty]$ which leave the even subgroup Δ invariant (and hence restrict to automorphisms of Δ). Inner automorphisms of Δ extend in an obvious way to inner automorphisms of $\overline{\Delta}$, and so we examine whether representative automorphisms for $\mathrm{Out}(\Delta)$ extend. Let Δ have presentation $\langle\, x, y \mid x^a = y^b = 1 \,\rangle$ and let $\overline{\Delta}$ have alternative presentation $\langle\, x, y, t \mid (ty)^2 = (xt)^2 = t^2 = x^a = y^b = 1 \,\rangle$. It was shown in §4.4.3 that when $a \neq b$, $\mathrm{Out}(\Delta)$ is isomorphic to $U_a \times U_b$ and is represented by the automorphisms $x \mapsto x^j$, $y \mapsto y^k$ where $j \in U_a$, $k \in U_b$; while when $a = b$ it is isomorphic to $(U_a \times U_a) \rtimes C_2$ and is represented by these same automorphisms together with their compositions with the Machí automorphism $\phi_{(0,1)}: x \mapsto y,\ \ y \mapsto x$. It is clear that each of these representatives extends to an automorphism of $\overline{\Delta}$ which fixes t. We immediately recognize υ: $x \mapsto x^{-1}$, $y \mapsto y^{-1}$, $t \mapsto t$ as the inner automorphism induced by t. That this is an inner automorphism is not surprising; the restriction of this automorphism to Δ induces a chiral duality functor on orientable hypermaps without boundary, which acts trivially in the wider context of all hypermaps. (In the case $a = b = 2$ this automorphism is the identity, and $\phi_{(0,1)}$ is the only non-identity extended automorphism.)

Lemma 4.4.6. *The identity and υ are the only extended representatives which are inner automorphisms of $\overline{\Delta}$.*

Proof. The extended representatives all fix t. Now, the fixed points f_x and f_y of x and y lie on the axis of reflection R_t of t. Two geometric transformations which commute preserve each other's fixed point set, and so it is not difficult to verify that the centralizer of t in the group of isometries of the hyperbolic plane $\mathcal{H}$ (or of $\mathbb{C}$ if $a = b = 2$) consists of the hyperbolic elements whose axis is R_t (or the translations of $\mathbb{C}$ along R_t), the glide reflections along R_t, the rotations of order 2 about points on R_t, and the reflections whose axes perpendicularly bisect R_t. However, t is the only element w of the centralizer for which $w^{-1}xw$ and $w^{-1}yw$ preserve the set $\{f_x, f_y\}$ (as do the extended representatives) while lying in $\overline{\Delta}$. ∎

The effect of the functors induced by the extended automorphisms is the same as in the category of orientable hypermaps without boundary (with some extra considerations when the hypermaps have boundary).

It remains to investigate whether there exist automorphisms of $\overline{\Delta}$ which map Δ to some other index-2 subgroup. We begin by determining presentations for such subgroups. There are at most seven epimorphisms from $\overline{\Delta}$ onto $\mathbb{Z}_2$, and hence at most seven index-2 subgroups of $\overline{\Delta}$. We enumerate these as follows, giving the conditions for their existence:

$K_1 = \ker(\theta_1)$ where $\theta_1\colon l \mapsto 1, r \mapsto 0, t \mapsto 0$ (exists if and only if b is even);

$K_2 = \ker(\theta_2)$ where $\theta_2\colon l \mapsto 0, r \mapsto 1, t \mapsto 0$ (exists if and only if a is even);

$K_3 = \ker(\theta_3)$ where $\theta_3\colon l \mapsto 0, r \mapsto 0, t \mapsto 1$ (exists if and only if a and b are even);

$K_4 = \ker(\theta_4)$ where $\theta_4\colon l \mapsto 1, r \mapsto 1, t \mapsto 0$ (exists if and only if a and b are even);

$K_5 = \ker(\theta_5)$ where $\theta_5\colon l \mapsto 1, r \mapsto 0, t \mapsto 1$ (exists if and only if a is even);

$K_6 = \ker(\theta_6)$ where $\theta_6\colon l \mapsto 0, r \mapsto 1, t \mapsto 1$ (exists if and only if b is even);

$K_7 = \ker(\theta_7)$ where $\theta_7\colon l \mapsto 1, r \mapsto 1, t \mapsto 1$ (always exists: $K_7 = \Delta$).

We see that if a and b are both odd then Δ is the unique index-2 subgroup of $\overline{\Delta}$. When one of a and b is even we shall determine whether Δ is isomorphic to any other index-2 subgroup using knowledge of a presentation for each subgroup, which we obtain using the Reidemeister-Schreier method [42]. For K_1 (b even) the non-trivial generators obtained are

$$l^2, r, lrl^{-1}, t, ltl^{-1};$$

and the relators obtained are

$$l^2, r^2, (lrl^{-1})^2, t^2, (ltl^{-1})^2, (rt)^a, (lrl^{-1}.ltl^{-1})^a, (t.ltl^{-1}.l^2)^{\frac{b}{2}}, (ltl^{-1}.l^2.t)^{\frac{b}{2}},$$

giving the presentation (after dropping the redundant generator l^2)

$$\begin{aligned} K_1 &= \langle\, r, t, lrl^{-1}, ltl^{-1} \,\rangle \\ &= \langle\, \beta, \gamma, \delta, \epsilon \,|\, \beta^2 = \gamma^2 = \delta^2 = \epsilon^2 = (\beta\gamma)^a = (\delta\epsilon)^a = (\gamma\epsilon)^{\frac{b}{2}} = 1 \,\rangle, \end{aligned}$$

(respectively denoting the generators by Greek letters). Presentations for the other K_i are as follows.

$$\begin{aligned} K_2 &= \langle\, l, t, rlr^{-1}, rtr^{-1} \,\rangle \\ &= \langle\, \alpha, \gamma, \delta, \epsilon \,|\, \alpha^2 = \gamma^2 = \delta^2 = \epsilon^2 = (\epsilon\gamma)^{\frac{a}{2}} = (\gamma\alpha)^b = (\epsilon\delta)^b = 1 \,\rangle, \end{aligned}$$

$$\begin{aligned} K_3 &= \langle\, l, r, tlt^{-1}, trt^{-1} \,\rangle \\ &= \langle\, \alpha, \beta, \delta, \epsilon \,|\, \alpha^2 = \beta^2 = \delta^2 = \epsilon^2 = (\beta\epsilon)^{\frac{a}{2}} = (\delta\alpha)^{\frac{b}{2}} = 1 \,\rangle \\ &\cong D_a * D_b, \end{aligned}$$

$$\begin{aligned} K_4 &= \langle\, rl^{-1}, t, ltl^{-1} \,\rangle \\ &= \langle\, \beta, \gamma, \epsilon \,|\, \gamma^2 = \epsilon^2 = (\beta\epsilon\beta^{-1}\gamma)^{\frac{a}{2}} = (\gamma\epsilon)^{\frac{b}{2}} = 1 \,\rangle, \end{aligned}$$

$$\begin{aligned} K_5 &= \langle\, r, tl, lrl^{-1} \,\rangle \\ &= \langle\, \beta, \delta, \epsilon \,|\, \beta^2 = \epsilon^2 = (\beta\delta\epsilon\delta^{-1})^{\frac{a}{2}} = \delta^b = 1 \,\rangle, \end{aligned}$$

$$\begin{aligned} K_6 &= \langle\, l, rt, rlr^{-1} \,\rangle \\ &= \langle\, \alpha, \delta, \epsilon \,|\, \alpha^2 = \epsilon^2 = \delta^a = (\delta^{-1}\epsilon\delta\alpha)^{\frac{b}{2}} = 1 \,\rangle. \end{aligned}$$

A presentation for the even subgroup $K_7 = \Delta(a, b, \infty)$ we know already, of course.

When $a = 2$ there is an isomorphism $\Delta \to K_6$ given by $x \mapsto \delta$, $y \mapsto \alpha$; this extends to an automorphism π: $r \mapsto r$, $t \mapsto t$, $l \mapsto tl$ of $\overline{\Delta}$ of order 2. When $b = 2$ there is an isomorphism $\Delta \to K_5$ given by $x \mapsto \beta$, $y \mapsto \delta$; this extends to an automorphism $\tau : r \mapsto rt$, $t \mapsto t$, $l \mapsto l$ of $\overline{\Delta}$ of order 2. When $a = b = 2$ there is an additional isomorphism $\Delta \to K_3$ which extends to the automorphism $\pi \circ \tau$ of $\overline{\Delta}$ of order 2. We can see that there are no other cases in which Δ is isomorphic to another index-2 subgroup of $\overline{\Delta}$ by examining the abelianizations of these subgroups; they are straightforward to calculate using the presentations above.

$$K_1^{\text{ab}} \cong \begin{cases} C_2 \times C_2 \times C_2 \times C_2 & a \text{ even}, b \equiv 0 \bmod 4 \\ C_2 \times C_2 \times C_2 & a \text{ even}, b \equiv 2 \bmod 4 \\ C_2 \times C_2 & a \text{ odd}, b \equiv 0 \bmod 4 \\ C_2 & a \text{ odd}, b \equiv 2 \bmod 4, \end{cases}$$

$$K_2^{\text{ab}} \cong \begin{cases} C_2 \times C_2 \times C_2 \times C_2 & a \equiv 0 \bmod 4, b \text{ even} \\ C_2 \times C_2 \times C_2 & a \equiv 2 \bmod 4, b \text{ even} \\ C_2 \times C_2 & a \equiv 0 \bmod 4, b \text{ odd} \\ C_2 & a \equiv 2 \bmod 4, b \text{ odd}, \end{cases}$$

$$K_3^{\text{ab}} \cong \begin{cases} C_2 \times C_2 \times C_2 \times C_2 & a, b \equiv 0 \bmod 4 \\ C_2 \times C_2 \times C_2 & a \equiv 0 \bmod 4, b \equiv 2 \bmod 4 \\ C_2 \times C_2 \times C_2 & a \equiv 2 \bmod 4, b \equiv 0 \bmod 4 \\ C_2 \times C_2 & a, b \equiv 2 \bmod 4, \end{cases}$$

$$K_4^{\text{ab}} \cong \begin{cases} \mathbb{Z} \times C_2 \times C_2 & a, b \equiv 0 \bmod 4 \\ \mathbb{Z} \times C_2 & a \equiv 2 \bmod 4 \text{ or } b \equiv 2 \bmod 4, \end{cases}$$

$$K_5^{\text{ab}} \cong \begin{cases} C_b \times C_2 \times C_2 & a \equiv 0 \bmod 4 \\ C_b \times C_2 & a \equiv 2 \bmod 4, \end{cases}$$

$$K_6^{\text{ab}} \cong \begin{cases} C_a \times C_2 \times C_2 & b \equiv 0 \bmod 4 \\ C_a \times C_2 & b \equiv 2 \bmod 4. \end{cases}$$

Of course, $\Delta^{\text{ab}} \cong C_a \times C_b$, and the claim follows immediately.

Thus when $a \neq 2$ and $b \neq 2$, all automorphisms of $\overline{\Delta}$ preserve Δ and we obtain the first statement in Theorem 4.4.4. When $a \neq 2$ and $b = 2$, we must have $\text{Aut}(\overline{\Delta}) = \text{Aut}^{\Delta}(\overline{\Delta}) \cup \text{Aut}^{\Delta}(\overline{\Delta}).\pi$, the union of the two cosets of the index-2 subgroup $\text{Aut}^{\Delta}(\overline{\Delta})$ of $\text{Aut}(\overline{\Delta})$ whose elements preserve Δ. Moreover, π is easily seen to commute with our extended representatives of outer automorphisms of Δ. Similar arguments go through for τ when $a = 2$ and $b \neq 2$. Finally, when $a = 2$ and $b = 2$ the subgroups Δ, K_3, K_5 and K_6 are mutually isomorphic under restrictions of the group of automorphisms $\langle \pi, \tau \rangle$. This completes the proof of Theorem 4.4.4.

When $a = 2$ the automorphism π induces the restriction to $C_{[2,b,\infty]}$ of the Petrie dual operation P which interchanges faces and Petrie circuits. The automorphism $\tau = \pi^{\phi_{(0,1)}}$ induces the conjugation T of the Petrie dual operation by a Machí operation. In the case $a = b = 2$, the universal hypermap $\mathscr{U}_{\Delta[2,2,\infty]}$ is an infinite path in the Euclidean plane augmented by two ideal points, and so any proper quotients without boundary are on the sphere or projective plane. The functors T and P act trivially on paths and even circuits on the sphere, while they both interchange odd circuits on the sphere with those on the projective plane. However, the two functors act differently in general on maps with free edges or boundary.

4.4.6. Extended triangle groups $\Delta[\infty, b, \infty]$

Let $\overline{\Delta} = \Delta[\infty, b, \infty]$ have presentation $\langle\, l, r, t \mid l^2 = r^2 = t^2 = (tl)^b = 1 \,\rangle$.

Theorem 4.4.7.

$$\mathrm{Out}(\Delta[\infty, b, \infty]) \cong \begin{cases} ((C_b \rtimes C_2) \times U_b)/\langle\, ((0,-1),-1) \,\rangle & b > 2 \\ S_3 & b = 2, \end{cases}$$

where $C_b \rtimes C_2$ is D_{2b}, the dihedral group of order $2b$.

We proceed as in §4.4.5 and begin the proof by considering those automorphisms of $\overline{\Delta} = \Delta[\infty, b, \infty]$ which leave the even subgroup Δ invariant.

By identifying x with rt and y with tl we have the presentation $\langle\, x, y, t \mid (xt)^2 = (ty)^2 = t^2 = y^b = 1 \,\rangle$ for $\overline{\Delta}$. It was shown in the proof of Proposition 4.4.3 that $\mathrm{Out}(\Delta)$ has the automorphisms $x \mapsto x^{-1}, y \mapsto y$; $x \mapsto xy, y \mapsto y$ and $\{x \mapsto x, y \mapsto y^k \mid k \in U_b\}$ as generating representatives. The first automorphism and each from the last family extends to an automorphism of $\overline{\Delta}$ which fixes t. (Lemma 4.4.5 ensures that these extensions are unique.) By Lemma 4.4.6 the only composite of these particular automorphisms which is inner is $x \mapsto x^{-1}, y \mapsto y^{-1}, t \mapsto t$ (induced by t); so again we have a chiral duality functor Υ on orientable hypermaps without boundary which acts trivially in the wider context of all hypermaps. The second representative automorphism

$x \mapsto xy$, $y \mapsto y$ in Out(Δ) extends to one which sends t to $ty = l$; it is not inner since rotations about different vertices of a given fundamental triangle of an extended triangle group are non-conjugate. We conclude that every element of Out(Δ) extends uniquely to $\overline{\Delta}$, as was the case for $\Delta[a, b, \infty]$. Again, the functors induced by an original automorphism and its extension have the same effect (with some extra considerations when the hypermaps have boundary).

To determine whether there exist automorphisms of $\overline{\Delta}$ which map Δ to some other index-2 subgroup, we use the same enumeration of the seven epimorphisms from $\overline{\Delta}$ onto $\mathbb{Z}_2$, noting that θ_2, θ_5 and θ_7 exist for all values of b, while θ_1, θ_3, θ_4 and θ_6 exist if and only if b is even. Presentations for the kernels K_i can be read directly from the presentations derived for the case of $\Delta[a, b, \infty]$ by taking $a = \infty$ and omitting any resulting vacuous relations:

$$\begin{aligned} K_1 &= \langle\, r, t, lrl^{-1}, ltl^{-1} \,\rangle \\ &= \langle \beta, \gamma, \delta, \epsilon \mid \beta^2 = \gamma^2 = \delta^2 = \epsilon^2 = (\gamma\epsilon)^{\frac{b}{2}} = 1 \rangle \\ &\cong C_2 * D_b * C_2, \end{aligned}$$

$$\begin{aligned} K_2 &= \langle\, l, t, rlr^{-1}, rtr^{-1} \,\rangle \\ &= \langle\, \alpha, \gamma, \delta, \epsilon \mid \alpha^2 = \gamma^2 = \delta^2 = \epsilon^2 = (\gamma\alpha)^b = (\epsilon\delta)^b = 1 \,\rangle \\ &\cong D_b * D_b, \end{aligned}$$

$$\begin{aligned} K_3 &= \langle\, l, r, tlt^{-1}, trt^{-1} \,\rangle \\ &= \langle\, \alpha, \beta, \delta, \epsilon \mid \alpha^2 = \beta^2 = \delta^2 = \epsilon^2 = (\delta\alpha)^{\frac{b}{2}} = 1 \,\rangle \\ &\cong D_b * C_2 * C_2, \end{aligned}$$

$$\begin{aligned} K_4 &= \langle\, rl^{-1}, t, ltl^{-1} \,\rangle \\ &= \langle \beta, \gamma, \epsilon \mid \gamma^2 = \epsilon^2 = (\gamma\epsilon)^{\frac{b}{2}} = 1 \,\rangle \\ &\cong \mathbb{Z} * D_b, \end{aligned}$$

$$\begin{aligned} K_5 &= \langle\, r, tl, lrl^{-1} \,\rangle \\ &= \langle \beta, \delta, \epsilon \mid \beta^2 = \delta^b = \epsilon^2 = 1 \,\rangle \end{aligned}$$

$$\cong C_2 * D_{2b} * C_2,$$

$$\begin{aligned} K_6 &= \langle\, l, rt, rlr^{-1} \,\rangle \\ &= \langle\, \alpha, \delta, \epsilon \mid \alpha^2 = \epsilon^2 = (\delta^{-1}\epsilon\delta\alpha)^{\frac{b}{2}} = 1 \,\rangle, \end{aligned}$$

$$\begin{aligned} K_7 &= \langle\, rt, tl \,\rangle \\ &= \langle\, x, y \mid y^b = 1 \,\rangle \\ &\cong \mathbb{Z} * C_b. \end{aligned}$$

It is easy to show that

$$K_6^{\mathrm{ab}} \cong \begin{cases} \mathbb{Z} \times C_2 \times C_2 & b \equiv 0 \bmod 4 \\ \mathbb{Z} \times C_2 & b \equiv 2 \bmod 4, \end{cases}$$

and by comparing the abelianizations of the K_i we readily verify that isomorphisms can only exist between $\Delta = K_7$ and other index-2 subgroups of $\overline{\Delta}$ when $b = 2$. In this case we have $\Delta \cong K_4 = \langle\, lr, t \,\rangle \cong K_6 = \langle\, rt, l \,\rangle$. Here, $\overline{\Delta}$ is the full cartographic group $\mathscr{C}_2$ and we are in the familiar territory of §2.1: the automorphism $x \mapsto xy$, $y \mapsto y$, $t \mapsto ty = l$ (which fixes r) is the automorphism which induces the duality operation D, representing the sole non-trivial element of $\mathrm{Out}(\mathscr{C}_2)$ which preserves $\Delta = \mathscr{C}_2^+$. The existence of three mutually isomorphic index-2 subgroups of Δ confirm the result of Jones and Thornton [35] that $\mathrm{Out}(\mathscr{C}_2) \cong S_3$, giving the six operations on the category of all maps. (See also Table 2.1.)

4.5. Functors from non-elementary proper inclusions

Definition 4.5.1. An inclusion $\phi\colon \Delta \hookrightarrow \Gamma$ between triangle groups is said to be *elementary* if Δ is a finite cyclic or dihedral group. Similarly, an inclusion $\phi\colon \overline{\Delta} \hookrightarrow \overline{\Gamma}$ between extended triangle groups is *elementary* if ϕ restricts to an elementary inclusion $\Delta \hookrightarrow \Gamma$, where Δ and Γ are the even subgroups of $\overline{\Delta}$ and $\overline{\Gamma}$ respectively. •

In this section we examine non-elementary inclusions.

4.5.1. Inclusions between Fuchsian triangular groups

Triangle groups

A group is said to be *co-hopfian* if it is not isomorphic to any of its proper subgroups. The spherical triangle groups and Fuchsian triangle groups with finite periods are co-hopfian, while Euclidean triangle groups and the Fuchsian triangle groups with infinite periods are not. Non-co-hopfian groups have an infinite number of self-inclusions, but we shall only concern ourselves with inclusions which map parabolic elements to parabolic elements, which excludes all but a small number of cases.

In [56], Singerman determines all pairs of distinct triples (m_0, m_1, m_2) and (m'_0, m'_1, m'_2) for which there is such an inclusion between the corresponding (finite- or infinite-period) Fuchsian triangle groups Δ and Γ. We need only consider inclusions up to the action of the Machí functors, and so we may order each triple in whichever way is most convenient. In every case except one in the list, Γ has a period equal to 2; we take this to be m'_1 so that where possible, $\Phi(\mathscr{H})$ is a map for each $\mathscr{H} \in C_{(m_0,m_1,m_2)}$. In all but three cases, Γ has a period equal to 3; we take this to be m'_0 so that $\Phi(\mathscr{H})$ is trivalent.

We use the notation $\Delta \lhd_i \Gamma$ and $\Delta <_i \Gamma$ to denote that Δ is a normal or non-normal subgroup of index i in Γ. Excluding the cases of cyclic and dihedral groups, Singerman's list is as follows.

(a) $(s,s,t) \lhd_2 (s,2,2t)$	(b) $(t,t,t) \lhd_3 (3,3,t)$	(c) $(t,t,t) \lhd_6 (3,2,2t)$
(A) $(7,7,7) <_{24} (3,2,7)$	(B) $(7,2,7) <_9 (3,2,7)$	(C) $(3,3,7) <_8 (3,2,7)$
(D) $(8,8,4) <_{12} (3,2,8)$	(E) $(8,8,3) <_{10} (3,2,8)$	(F) $(9,9,9) <_{12} (3,2,9)$
(G) $(4,4,5) <_6 (4,2,5)$	(H) $(4t,4t,t) <_6 (3,2,4t)$	(I) $(2t,2t,t) <_4 (4,2,2t)$
(J) $(3t,3,t) <_4 (3,2,3t)$	(K) $(2t,2,t) <_3 (3,2,2t)$	

Singerman gives the permutations $\theta(x'_i)$ where the x'_i are the generators in the standard presentation for Γ and θ is the epimorphism of Theorem 1.5.1; these can be used to determine $\mathscr{D} = \mathscr{U}_\Gamma/\Delta$. (The hypermaps of the form $\Phi(\mathscr{H})$ in $C_{(m'_0,m'_1,m'_2)}$ are precisely those which cover $\mathscr{D}$.) It is then possible to construct a model of $\Phi(\mathscr{H})$ which is combinatorially correct (if not always conformally correct with geodesic edges).

Note how some of the inclusions give rise to 'generic' functors, ones which exist between an infinite number of pairs of categories. Most of these correspond to well-known transformations; for example (K) is the truncation operation of [57], while (a), (c) and (I) correspond to the formation of the topological hypermap representations defined in §1.2: they induce restrictions of the functors W, J and C realized by forgetting the component colours of the Walsh, James and Cori representations respectively. The inclusion (b) is shown in [28] to induce a restriction of $W^{-1} \circ J$.

In [28], Jones describes many of the functors and their inverses. Below we describe the remaining ones. Note that since this work was done, a paper by Girondo [18] has been released in which also contains a description of these functors (although not their inverses) as applied to regular hypermaps. In that paper it is recognized that some of the inclusions in Singerman's list are compositions of others, so that any inclusion can be expressed as a chain of inclusions involving just eight from the list (up to Machí automorphism of the triangle groups). These chains are:

$$
\begin{array}{ll}
\text{(c):} & (t,t,t) \overset{\text{(b)}}{\lhd_3} (3,3,t) \overset{\text{(a)}}{\lhd_2} (3,2,2t) \\
& (t,t,t) \overset{\text{(a)}}{\lhd_2} (t,2,2t) \cong (2t,2,t) \overset{\text{(K)}}{<_3} (3,2,2t) \\
\text{(A):} & (7,7,7) \overset{\text{(b)}}{\lhd_3} (3,3,7) \overset{\text{(C)}}{<_8} (3,2,7) \\
\text{(D):} & (8,8,4) \overset{\text{(I)}}{<_4} (4,2,8) \cong (8,2,4) \overset{\text{(K)}}{<_3} (3,2,8) \\
\text{(F):} & (9,9,9) \overset{\text{(b)}}{\lhd_3} (3,3,9) \cong (9,3,3) \overset{\text{(J)}}{<_4} (3,2,9) \\
\text{(H):} & (4t,4t,t) \overset{\text{(a)}}{\lhd_2} (4t,2,2t) \overset{\text{(K)}}{<_3} (3,2,4t) \\
\text{(I):} & (2t,2t,t) \overset{\text{(a)}}{\lhd_2} (2t,2,2t) \cong (2t,2t,2) \overset{\text{(a)}}{\lhd_2} (2t,2,4) \cong (4,2,2t).
\end{array}
$$

However, while compositions of the descriptions of the effect of the eight basic functors will give a description of the others, the descriptions that we derive directly in what follows are often simpler.

(B) $\Delta(7,2,7) <_9 \Delta(3,2,7)$

The action of Γ on the cosets of Δ is given by

$$x_0' \mapsto (123)(456)(789),$$
$$x_1' \mapsto (12)(34)(5)(67)(89),$$

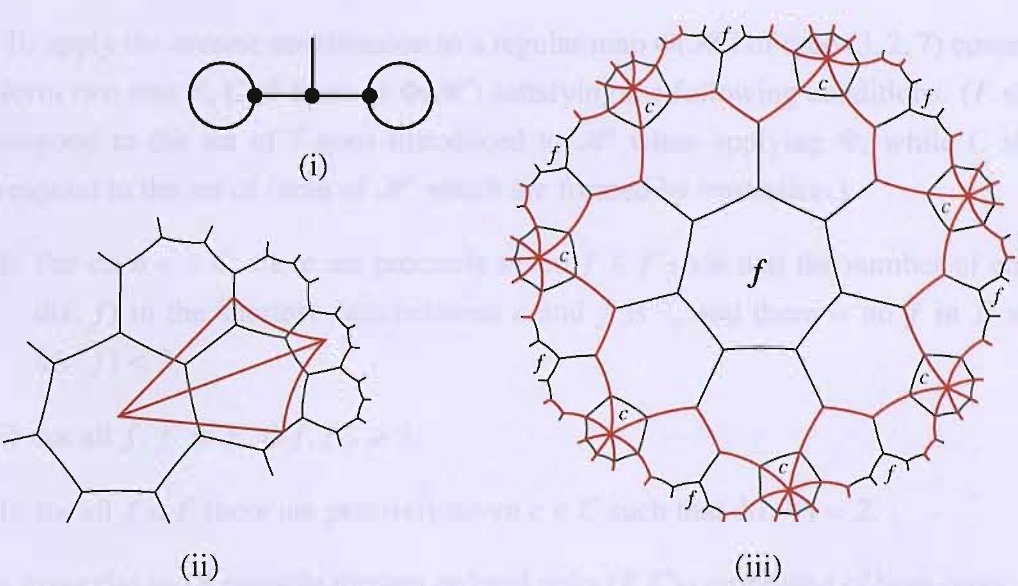


FIGURE 4.3. (i) The map $\mathscr{D}$, (ii) a fundamental region for Δ, and (iii) the universal tessellations for the inclusion (B): $\Delta = \Delta(7, 2, 7) <_9 \Delta(3, 2, 7) = \Gamma$.

$$x_2' \mapsto (1)(2369754)(8).$$

To determine $\mathscr{D}$, we observe that it must contain the following components: three vertices, each of valency 3 due to the three 3-cycles in the permutation $\theta(x_0')$; four edges and one free edge due to $\theta(x_1')$, and one 7-valent face and two 1-valent face due to $\theta(x_2')$. $\mathscr{D}$ is pictured in Figure 4.3(i). We then apply the arguments of the discussion following Theorem 1.5.1: $\theta(x_0')$ has no short cycles and so none of the centres of rotation of $\mathscr{U}_\Delta$ coincide with vertices of $\mathscr{U}_\Gamma$; $\theta(x_1')$ has one short cycle of length 1 and so the midpoints of the edges of $\mathscr{U}_\Delta$ coincide with midpoints of edges of $\mathscr{U}_\Gamma$; and $\theta(x_2')$ has two short cycles of length 1 and so the centres of the 7-valent faces and 7-valent vertices of $\mathscr{U}_\Delta$ coincide with face centres of $\mathscr{U}_\Gamma$. A fundamental region for Δ is pictured on part of the map $\mathscr{U}_\Gamma$ in Figure 4.3(ii); identify sides to obtain $\mathscr{D}$.

From this we can see, as described in [28], that to apply Φ to $\mathscr{H}$ we represent $\mathscr{H}$ as a map; truncate it; and place an n-gon inside each old face f of $\mathscr{H}$ of valency $n \in \{1, 7\}$, joining its vertices by edges to points two-thirds of the way along what remains of the old edges of f (in directions consistent with cyclic rotation around f). The resulting trivalent map is $\Phi(\mathscr{H})$. This combinatorial procedure is shown in Figure 4.3(iii), where we have distorted the edges of $\Phi(\mathscr{H})$ in order to make the 7-gons regular.

To apply the inverse construction to a regular map $\Phi(\mathscr{H})$ of type $(3, 2, 7)$ covering $\mathscr{D}$, form two sets F, C of faces of $\Phi(\mathscr{H})$ satisfying the following conditions. (F shall correspond to the set of 7-gons introduced to $\mathscr{H}$ when applying Φ, while C shall correspond to the set of faces of $\mathscr{H}$ which are formed by truncation.)

(i) For each $c \in C$, there are precisely seven $f \in F$ such that the number of edges $\mathrm{d}(c, f)$ in the shortest path between c and f is 2, and there is no f in F with $\mathrm{d}(c, f) < 2$,

(ii) for all f, f' in F, $\mathrm{d}(f, f') \geqslant 3$,

(iii) for all $f \in F$ there are precisely seven $c \in C$ such that $\mathrm{d}(c, f) = 2$.

This gives rise to 18 possible distinct ordered pairs (F, C) comprising of both orderings of each of nine unordered pairs of sets of faces. Choose any such pair and stellate each face $c \in C$ by placing a new vertex in the centre, joining it to each surrounding vertex by an edge and then deleting the old edges and vertices which bound c. Then, for each face $f \in F$, delete every edge incident with a vertex bounding f, along with all vertices incident with such edges. The resulting map is $\mathscr{H}$.

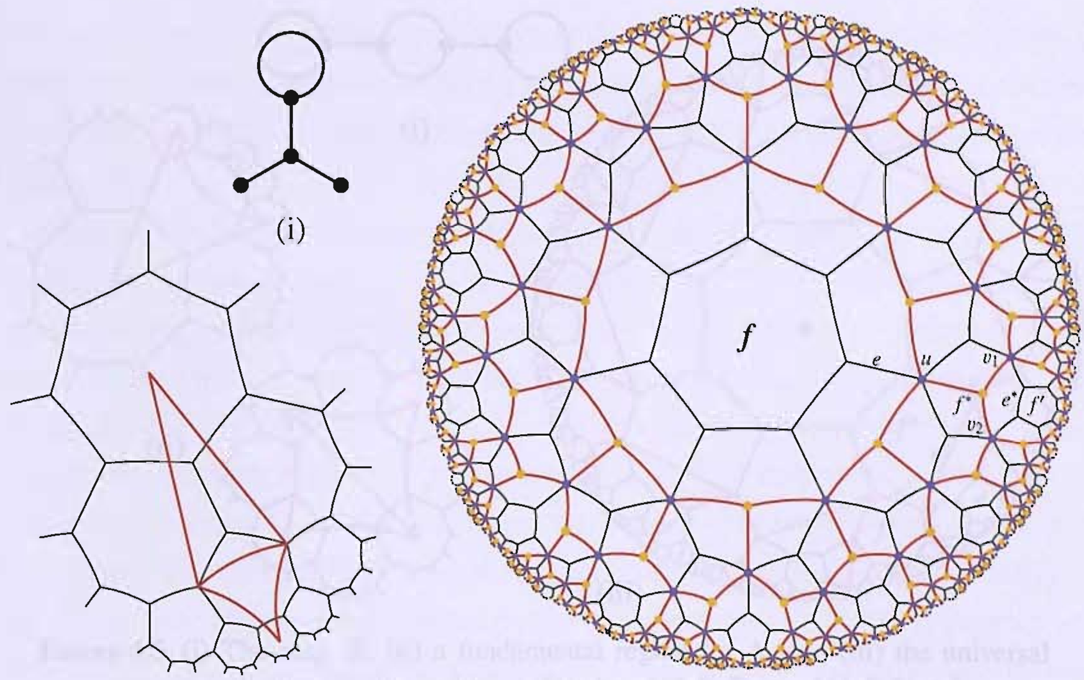
(C) $\Delta(3, 3, 7) <_8 \Delta(3, 2, 7)$

The action of Γ on the cosets of Δ is given by

$$x_0' \mapsto (123)(456)(7)(8),$$
$$x_1' \mapsto (12)(34)(57)(68),$$
$$x_2' \mapsto (1)(2368754).$$

$\mathscr{D}$ is shown in Figure 4.4(i), while a fundamental region for Δ is pictured on part of the map $\mathscr{U}_\Gamma$ in Figure 4.4(ii).

From this we can see, as described in [28], that to apply Φ to $\mathscr{H}$ we 2-colour the vertices of the James representation $\mathscr{M}$ of $\mathscr{H}$ and place an n-gon in each hyperface f of valency $n \in \{1, 7\}$ of $\mathscr{M}$, joined by new edges to the purple vertices incident with f; then we semi-stellate the hypervertices and hyperedges by placing a new vertex in the centre of each and joining it to the incident purple vertices. Finally we delete all the edges and yellow vertices of $\mathscr{M}$ (and forget the remaining vertex colours) to obtain the map $\Phi(\mathscr{H})$. This procedure is shown in Figure 4.4(iii).



(ii) (iii)

Figure 4.4. (i) The map $\mathcal{D}$, (ii) a fundamental region for Δ, and (iii) the universal tessellations for the inclusion (C): $\Delta = \Delta(3,3,7) <_8 \Delta(3,2,7) = \Gamma$.

To apply the inverse construction to a regular map $\mathcal{M} = \Phi(\mathcal{H})$ of type $(3,2,7)$ covering $\mathcal{D}$, define a relation $\to$ on the faces of $\mathcal{M}$ as follows: faces $f \neq f'$ of $\mathcal{M}$ satisfy $f \to f'$ if and only if f' shares an edge e^* with a face f^*, and there is an edge e meeting f^* at a single vertex u opposite e^*, and meeting f at a single vertex. There are eight equivalence classes under the equivalence relation generated by $\to$; choose one and denote it by F. (This class corresponds to the set of 7-gons introduced to $\mathcal{H}$ when applying Φ.) For each face $f \in F$, let f^* and u be as above and let v_1 and v_2 be the two vertices incident with f^* for which $d(v_i, u) = 2$. Then semi-stellate f^* by joining a new vertex in the centre of f^* to u, v_1 and v_2. Finally, delete $\mathcal{M}$. The resulting trivalent map is $J(\mathcal{H})$: the 14-gons represent the hyperfaces of $\mathcal{H}$, while the 6-gons can be coloured in the appropriate way as to represent the hypervertices and hyperedges (two choices).

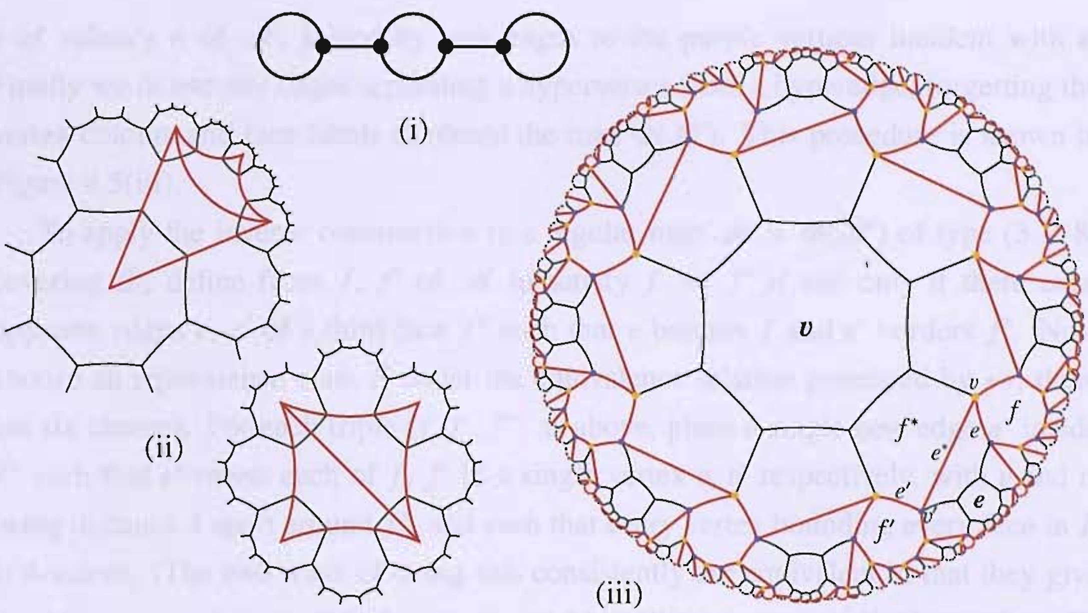


FIGURE 4.5. (i) The map $\mathscr{D}$, (ii) a fundamental region for Δ, and (iii) the universal tessellations for the inclusion (D): $\Delta = \Delta(8,8,4) <_{12} \Delta(3,2,8) = \Gamma$.

(D) $\Delta(8,8,4) <_{12} \Delta(3,2,8)$

The action of Γ on the cosets of Δ is given by

$$x_0' \mapsto (123)(456)(789)(XYZ),$$
$$x_1' \mapsto (12)(34)(57)(69)(8X)(YZ),$$
$$x_2' \mapsto (1)(2368ZX74)(59)(Y).$$

The map $\mathscr{D}$ is shown in Figure 4.5(i). A fundamental region for Δ is pictured on part of the map $\mathscr{U}_\Gamma$ in Figure 4.5(ii); the upper diagram shows the region in a form which matches Figure 4.5(iii), while the lower diagram best illustrates the symmetry of the region.

From this we can see, as described in [28], that to apply Φ to $\mathscr{H}$ we form $C(\mathscr{H})$ by omitting the face colours of the Cori representation of $\mathscr{H}$, forming the dual map, and then fully truncating it so that no original edges of the map remain. Alternatively, we 2-colour the vertices of the James representation $\mathscr{M}$ of $\mathscr{H}$ and place an n-gon in each hypervertex (0-labelled face) v of valency $n \in \{1,2,4,8\}$ of $\mathscr{M}$, joined by new edges to the yellow vertices incident with v; then we place an n-gon in each hyperedge

e of valency n of $\mathscr{M}$, joined by new edges to the purple vertices incident with e. Finally we delete any edges separating a hypervertex from a hyperedge, forgetting the vertex colours and face labels to obtain the map $\Phi(\mathscr{H})$. This procedure is shown in Figure 4.5(iii).

To apply the inverse construction to a regular map $\mathscr{M} = \Phi(\mathscr{H})$ of type $(3,2,8)$ covering $\mathscr{D}$, define faces f, f' of $\mathscr{M}$ to satisfy $f \leftrightarrow f'$ if and only if there exist opposite edges e, e' of a third face f^* such that e borders f and e' borders f'. Now choose an equivalence class F under the equivalence relation generated by $\leftrightarrow$; there are six choices. For each triple (f, f', f^*) as above, place a single new edge e^* inside f^* such that e^* meets each of f, f' in a single vertex v, v' respectively, with v and v' being distance 4 apart around f^*, and such that every vertex bounding every face in F is 4-valent. (The two ways of doing this consistently are equivalent in that they give rise to the same hypervertex, hyperedge and hyperface centres of the hypermap $\mathscr{H}$, and so they do not count as a 'choice' in the sense used elsewhere in these descriptions; this is due to the existence of a reflection in a side of a fundamental region for Δ which preserves the tessellations of both $\mathscr{U}_\Delta$ and $\mathscr{U}_\Gamma$ and leaves each orbits of elliptic fixed points invariant. This is discussed further in the treatment of extended triangle groups below.) Now delete all edges and vertices of $\mathscr{M}$ apart from those bounding the faces in F. The resulting map is $\mathrm{J}(\mathscr{H})$: the faces in F represent the hyperfaces of $\mathscr{H}$, and the 16-gons can be chosen in such a way as to represent the hypervertices and hyperedges (two choices).

(E) $\Delta(8,8,3) <_{10} \Delta(3,2,8)$

The action of Γ on the cosets of Δ is given by

$$x_0' \mapsto (123)(456)(789)(X),$$
$$x_1' \mapsto (12)(34)(9X)(57)(89),$$
$$x_2' \mapsto (1)(236X5974)(8).$$

$\mathscr{D}$ is shown in Figure 4.6(i), while a fundamental region for Δ is pictured on part of the map $\mathscr{U}_\Gamma$ in Figure 4.6(ii).

From this we can see, as described in [28] using the Cori representation, that to apply Φ to $\mathscr{H}$ we 2-colour the vertices of the James representation $\mathscr{M}$ of $\mathscr{H}$ and place an n-gon inside each hypervertex and hyperedge of valency $n \in \{1,2,4,8\}$ of

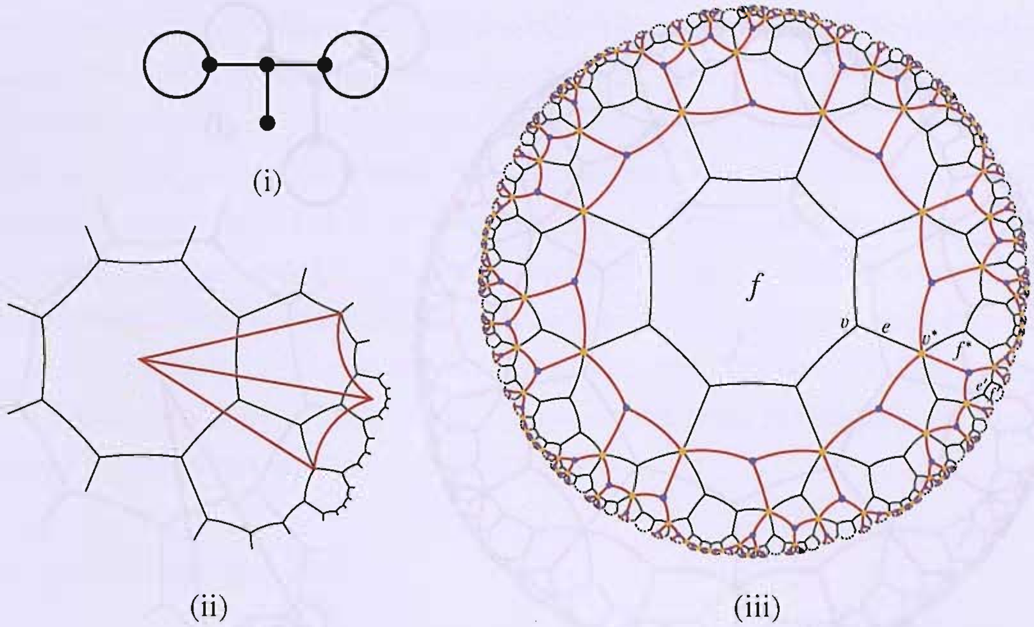


Figure 4.6. (i) The map $\mathscr{D}$, (ii) a fundamental region for Δ, and (iii) the universal tessellations for the inclusion (E): $\Delta = \Delta(8,8,3) <_{10} \Delta(3,2,8) = \Gamma$.

$\mathscr{M}$, joined by new edges to the yellow vertices incident with that component; then we semi-stellate each hyperface by placing a vertex in the centre and joining it by new edges to its incident yellow vertices. Finally we delete all the edges and yellow vertices of $\mathscr{M}$ (and forget the remaining vertex colours) to obtain the map $\Phi(\mathscr{H})$.

To apply the inverse construction to a regular map $\mathscr{M} = \Phi(\mathscr{H})$ of type $(3,2,8)$ covering $\mathscr{D}$, define faces f, f' of $\mathscr{M}$ to satisfy $f \dashrightarrow f'$ if and only if there is a face f^* and an edge e of $\mathscr{M}$ such that e meets both f and f^* at single vertices v and v^* respectively, while f^* and f' have an edge e' in common, where e' is one of the edges of f^* opposite v^*, chosen consistently according to the orientation of $\mathscr{M}$. The relation $\dashrightarrow$ generates an equivalence relation on the faces for which there are ten equivalence classes. (Choosing the edges e' against the orientation of $\mathscr{M}$ gives a different set of ten equivalence classes. This is not a problem since each of the classes in a set gives rise to a tessellation of the surface underlying $\mathscr{M}$ as we describe below, and the ten tessellations are the same for both sets. The choice for the e' simply corresponds to whether certain faces of the tessellations should represent hypervertices or hyperedges; thus we go with the orientation now—without regarding this as a 'choice'—and allow a choice later on as to the hypervertices and hyperedges.) Next, choose an equivalence

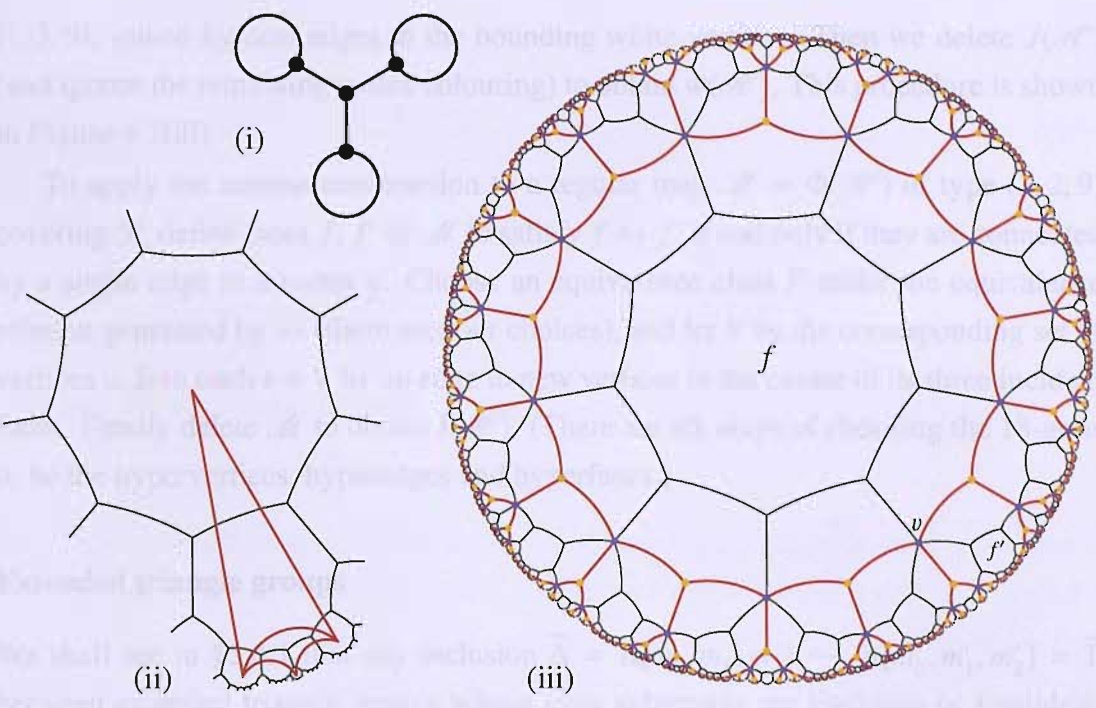


FIGURE 4.7. (i) The map $\mathscr{D}$, (ii) a fundamental region for Δ, and (iii) the universal tessellations for the inclusion (F): $\Delta = \Delta(9, 9, 9) <_{12} \Delta(3, 2, 9) = \Gamma$.

class F, and for each $f \in F$ join its corresponding vertex v^* to new vertices in the centre of each of its three incident faces. Finally, delete $\mathscr{M}$. The resulting map is $J(\mathscr{H})$: the 6-gons represent the hyperfaces, while the 16-gons can be chosen in such a way as to represent the hypervertices and hyperedges (two choices).

(F) $\Delta(9, 9, 9) <_{12} \Delta(3, 2, 9)$

The action of Γ on the cosets of Δ is given by

$$x_0' \mapsto (123)(456)(789)(XYZ),$$
$$x_1' \mapsto (12)(45)(78)(3X)(6Y)(9Z),$$
$$x_2' \mapsto (1)(4)(7)(23Z89Y56X).$$

$\mathscr{D}$ is shown in Figure 4.7(i), while a fundamental region for Δ is pictured on part of the map $\mathscr{U}_\Gamma$ in Figure 4.7(ii).

From this we can see, as described in [28], that to apply Φ to $\mathscr{H}$ we 2-colour the vertices of $J(\mathscr{H})$ and place an n-gon in the centre of each face of valency $n \in$

$\{1,3,9\}$, joined by new edges to the bounding white vertices. Then we delete $J(\mathscr{H})$ (and ignore the remaining vertex colouring) to obtain $\Phi(\mathscr{H})$. This procedure is shown in Figure 4.7(iii).

To apply the inverse construction to a regular map $\mathscr{M} = \Phi(\mathscr{H})$ of type $(3,2,9)$ covering $\mathscr{D}$, define faces f, f' of $\mathscr{M}$ to satisfy $f \leftrightarrow f'$ if and only if they are connected by a single edge to a vertex v. Choose an equivalence class F under the equivalence relation generated by $\leftrightarrow$ (there are four choices), and let V be the corresponding set of vertices v. Join each $v \in V$ by an edge to new vertices in the centre of its three incident faces. Finally delete $\mathscr{M}$ to obtain $J(\mathscr{H})$. (There are six ways of choosing the 18-gons to be the hypervertices, hyperedges and hyperfaces.)

Extended triangle groups

We shall see in §5.4.1 that any inclusion $\overline{\Delta} = \Delta[m_0, m_1, m_2] \hookrightarrow \Delta[m_0', m_1', m_2'] = \overline{\Gamma}$ between extended triangle groups whose even subgroups are Fuchsian or Euclidean restricts to one between the even subgroups. Here we determine which of the triangle group inclusions of §4.5.1 are such restrictions. By identifying $\overline{\Delta}$ with its image we regard $\overline{\Delta}$ as a subgroup of $\overline{\Gamma}$; it is generated by reflections in the sides of a fundamental triangle for $\overline{\Delta}$ and gives rise to a tessellation $\mathscr{T}_{\overline{\Delta}}$ by triangles of a simply-connected Riemann surface $\mathscr{X}$ (augmented by ideal vertices if some $m_i' = \infty$). Each such reflection, being in $\overline{\Gamma}$, is a reflection in a side of some fundamental triangle for $\overline{\Gamma}$, and hence it leaves each orbit of elliptic fixed points of $\overline{\Gamma}$ invariant. In particular, a reflection in an edge of the tessellation $\mathscr{T}_{\overline{\Delta}}$ leaves $\mathscr{T}_{\overline{\Gamma}}$ invariant.

To check whether the triangle group inclusion

$$\Delta = \Delta(m_0, m_1, m_2) \hookrightarrow \Delta(m_0', m_1', m_2') = \Gamma$$

extends to an inclusion between extended triangle groups $\overline{\Delta}$ and $\overline{\Gamma}$, it is enough to check that just one reflection R in an edge of $\mathscr{T}_{\overline{\Delta}}$ preserves $\mathscr{T}_{\overline{\Gamma}}$ and leaves the elliptic fixed point orbits invariant. This is because R is conjugate in $\overline{\Gamma}$ to a composition of any other reflection R' in an edge of $T_{\overline{\Delta}}$ with a rotation in $\overline{\Delta}$. (Rotations in $\overline{\Delta}$ are rotations in Δ and hence in Γ and so they certainly preserve $\mathscr{T}_{\overline{\Gamma}}$.)

It is straightforward to check that for each of the inclusions (B), (C), (E), (F) and (G) of Singerman's list, a reflection corresponding to Δ is not a reflection corresponding to Γ and so the inclusion does not extend to the extended triangle groups. Fig-

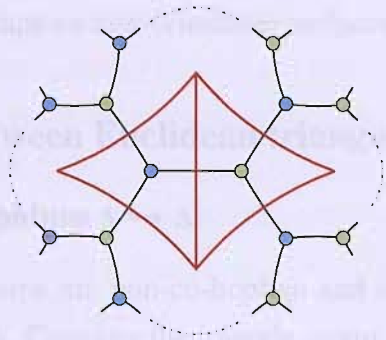


FIGURE 4.8. A fundamental region for Δ in inclusion (b): $\Delta = \Delta(n, n, n) \lhd_3 \Delta(3, 3, n) = \Gamma$.

ure 4.4(ii) shows this situation for inclusion (C); notice how one fundamental triangle T for Δ contains three whole edges of $\mathscr{U}_\Gamma$, whereas its reflection in a side of T contains none. Further, while a reflection for Δ in inclusion (b) preserves the tessellation for Γ as shown in Figure 4.8, it does not leave each orbit of elliptic fixed-points in Γ invariant; hypervertices and hyperedges are interchanged. On the other hand, the remaining inclusions do extend to the extended triangle groups. For example, Figure 4.5(ii) shows the situation for inclusion (D), where a common reflection of Δ and Γ is evident.

The inclusions between extended triangle groups are less well represented in the literature. However, the extension of inclusion (a) which induces the functor W is presented in [3], along with its compositions with map operations, one of which induces another well-known topological hypermap representation due to Vince [59].

In the situations where the inclusions do not extend, it is interesting to examine why the combinatorial description of how to apply the functors and their inverses to orientable hypermaps on surfaces without boundary fails to be applicable to the wider class of hypermaps. In the case of inclusion (B), to apply the corresponding functor Φ to a map $\mathscr{H}$ we are required to identify points two-thirds of the way along certain edges of $\mathscr{H}$, consistent with the orientation of $\mathscr{H}$. Hence the orientability of the map is fundamental to the application of the functor. In the case of inclusions (C), (E), (F) and (G) it is necessary to 2-colour the hyperblades of the hypermap $\mathscr{H}$ to apply Φ; however, such a 2-colouring is impossible when $\mathscr{H}$ is non-orientable. Note that in the case of inclusion (D) where the inclusion does extend, one description of the application of Φ given in §4.5.1 in terms of the James representation requires a 2-colouring of the hyperblades and hence fails in the extended setting; yet the other description given there in terms of the Cori representation requires no such colouring and is applicable to hypermaps on non-orientable surfaces.

4.5.2. Inclusions between Euclidean triangular groups

Triangle group monomorphisms $\Delta \hookrightarrow \Delta$

The Euclidean triangle groups are non-co-hopfian and so there exist non-surjective monomorphisms $\phi\colon \Delta \hookrightarrow \Delta$. Consider the triangle group

$$\Delta = \Delta(4,2,4) = \langle\, y_0, y_1, y_2 \,|\, y_0^4 = y_1^2 = y_2^4 = y_0y_1y_2 = 1 \,\rangle$$

and suppose that $\phi(\Delta)$ is also a triangle group $\Delta(4,2,4)$. Consider the universal map $\mathcal{U}_\Delta$ in the plane $\mathbb{R}^2$ (which we identify with the complex plane $\mathbb{C}$ in the usual way). Its vertices can be taken to be the points in $\mathbb{Z}^2 \subset \mathbb{R}^2$ so that its edges join all pairs of vertices unit length apart. Then $\phi(\Delta)$ has a presentation

$$\phi(\Delta) = \langle\, x_0, x_1, x_2 \,|\, x_0^4 = x_1^2 = x_2^4 = x_0x_1x_2 = 1 \,\rangle,$$

where x_0, x_1, x_2 are respectively rotations about points P, Q, R (say) in $\mathbb{Z}^2$. A fundamental region F for $\phi(\Delta)$ is obtained by taking the union of the triangle PQR (whose internal angles are $\pi/4$, $\pi/2$ and $\pi/4$) with its reflection in one of its sides, QR say, to form a quadrilateral $PQRP^*$ which is degenerate in the sense that P, Q and P^* are collinear. Define a point in $\mathbb{Z}^2$ to be a *y_i-point* if it is the fixed point of a conjugate in Γ of the generator y_i. It is clear that P and P^* are either both y_0-points or both y_2-points since they are fixed by conjugate elements of order 4.

Suppose first that P and P^* are both y_0-points. The universal map $\mathcal{U}_{\phi(\Delta)}$ is determined by the relative coordinates (m,n) of P from P^*, and a choice of point (a,b) corresponds to a choice of P. The square $S_{m,n}$ with vertices

$$(a,b) \quad (a+m, b+n) \quad (a+m-n, b+m+n) \quad (a-n, b+m)$$

forms a face of $\mathcal{U}_{\phi(\Delta)}$. By modifying our choice of F if necessary, we may suppose that m, $n \geqslant 0$. A subgroup Λ of Δ is conjugate to $\phi(\Delta)$ if and only if Λ gives rise to a universal tessellation $\mathcal{U}_\Lambda$ whose square faces can be obtained from $S_{m,n}$ by translation along the coordinate axes. (If $\mathcal{U}_\Lambda$ contains $S_{m,n}$ as a face then $\phi(\Delta)$ and Λ coincide and so, since $S_{m,n}$ has area m^2+n^2, there are precisely m^2+n^2 translates of $S_{m,n}$ which correspond to mutually distinct conjugates of $\phi(\Delta)$.)

Now, x_0 has order 4 and so is conjugate to either y_0 or y_0^{-1}. The inclusion is thus one of two which are possible; one is the composition of the other with the chiral duality automorphism υ of $\phi(\Delta)$ given by $x_0 \mapsto x_0^{-1}$, $x_1 \mapsto x_1^{-1}$. Denote these two inclusions by $\phi_{y_0^i,(a,b),(m,n)}$ where $i = 1, -1$. Since each is also a composition of $\phi_{y_0^i,(a,b),(n,m)}$ with υ, it follows that there is a distinct $\Phi_{y_0,(m,n)}$ for each choice of m, $n > 0$, and that $\Phi_{y_0,(0,n)} = \Phi_{y_0,(n,0)}$. The image of a uniform map of type $(4,2,4)$ under a functor $\Phi_{y_0,(m,n)}$ is obtained by subdividing each face of the map into $m^2 + n^2$ square faces.

When P is a y_2-point it is easy to see through composition of the monomorphisms $\Delta \hookrightarrow \Delta$ with the automorphism of Δ transposing y_0 and y_2 that the functors arising in this case are the functors of the previous case composed with the duality operation.

Similar arguments hold for the triangle groups $\Delta(6,2,3)$ and $\Delta(3,3,3)$; we consider $\mathscr{U}_{\Delta(6,2,3)}$ as the tessellation of $\mathbb{R}^2$ by equilateral triangles and the Walsh representation of $\mathscr{U}_{\Delta(3,3,3)}$ as the 2-vertex-coloured tessellation by regular hexagons, and we take coordinate axes to be inclined at an angle of $2\pi/3$. In the case of $\Delta(\infty,2,2) \cong D_\infty$ there are n isomorphic subgroups of each index $n \geqslant 1$, forming a single conjugacy class when n is odd (and hence inducing a single functor) but forming two classes when n is even; one of the two functors induced is the composite of the other with the Machí functor which arises for this group. Finally, in the case of the triangle group $\Delta(\infty,1,\infty) \cong C_\infty$ there is one functor for each positive integer; the duality operation acts as the identity. This gives us

Theorem 4.5.2. *Up to the action of the Machí operations there is, for each Euclidean triangle group Δ, a functor arising from monomorphisms $\Delta \hookrightarrow \Delta$ for each positive integer n. Further, for the triangle groups $\Delta = \Delta(4,2,4)$, $\Delta(3,2,6)$ and $\Delta(3,3,3)$, there is an additional such functor for each ordered pair (m,n) of positive integers.* ∎

As discussed in §4.2, the effect of each functor Φ derived from such an inclusion ϕ can be reversed (up to several arbitrary choices), and the domain of the reverse constructions is restricted to those maps $\mathscr{M}$ which cover some spherical map $\mathscr{D}$. In the case of $\Delta = \Delta(4,2,4)$, $\mathscr{D}$ has $2(m^2 + n^2)$ darts, and is the quotient of the torus map $\mathscr{M}_0$, formed by identifying opposite sides of the square $S_{m,n}$ by a reflection in one of the edges. For example, $m = 2$ and $n = 1$ gives $\mathscr{M}_0$ as one of the chiral pair of regular embeddings of K_5 on the torus. The possible values for the total number of faces for such an $\mathscr{M}_0$ is given by the theorem of Fermat which states that a positive integer k is the sum of two squares if and only if all prime factors of the form $4r-1$ have an even exponent in the prime-power factorization of k. Further, the number of non-isomorphic maps $\mathscr{D}$ with this face sum is given by Jacobi's Two Square Theorem, which states that the number of representations of a positive integer k as the sum of two squares is equal to $4(t_1 - t_3)$ where t_i is number of divisors of k congruent to i modulo 4.

Triangle group inclusions $\Delta \hookrightarrow \Gamma$

A fundamental region for $\Delta(3,3,3)$ is the union of an equilateral triangle with its reflection in one of its sides. We have already discussed functors induced by inclusions $\Delta(6,2,3) \hookrightarrow \Delta(6,2,3)$ corresponding to superimpositions of different tessellations of $\mathbb{R}^2$ by equilateral triangles, and so it is easy to see that up to composition with these and Machí functors, there is just one functor arising from an inclusion $\Delta(3,3,3) \hookrightarrow \Delta(6,2,3)$; this is the Walsh functor W induced by the Euclidean analogue of inclusion (a) between Fuchsian groups listed in §4.5.1.

Let Γ be a non-cyclic Euclidean triangle group: $\Delta(4,2,4)$, $\Delta(3,2,6)$, $\Delta(3,3,3)$ or $\Delta(\infty,2,2)$. We realize $\mathscr{U}_{\Delta(\infty,1,\infty)}$ as an infinite set of equally-spaced parallel rays in the plane $\mathbb{R}^2$ augmented by an ideal vertex, superimposed on $\mathscr{U}_\Gamma$ as follows. We have seen that the translations in Γ (the elements of infinite order) determine the functors Φ arising from inclusions $\phi\colon \Delta \hookrightarrow \Delta$ up to the action of the Machí operations. Moreover, these translations send y_i-points in $\mathscr{U}_\Gamma$ to y_i-points; given such a translation w we may take a distinguished y_i-point P, and then the images of P under the group generated by w lie on a line A and form the free ends of the free edges of $\mathscr{U}_{\Delta(\infty,1,\infty)}$. The free edges themselves can be realized as rays R in $\mathbb{R}^2$ which end at the images of P, and which lie perpendicular to and on the same side of A. It follows that, up to composition with inner automorphisms, Machí automorphisms, and the inclusions ϕ, there is just one inclusion of $\Delta(\infty,1,\infty)$ in Γ. The images of a finite star map under the corresponding operational functors lie on the doubly infinite cylinder obtained by identifying a ray R with its image under w (and identifying their corresponding reflections in A). The images themselves are the corresponding quotients of the images under the Machí functors and the inclusion functors Φ of $\mathscr{U}_\Gamma$.

Up to reordering the periods, the Euclidean triangle groups Γ containing involutions are $\Delta(4,2,4)$ and $\Delta(3,2,6)$. To realize $\mathscr{U}_{\Delta(\infty,2,2)}$ as an infinite set of equally-spaced parallel lines in $\mathbb{R}^2$ augmented by two ideal vertices, superimposed on $\mathscr{U}_\Gamma$, we do the same thing as in the previous case except that this time each edge (line) is the union of a ray R with its reflection in A. Up to composition with inner automorphisms, Machí automorphisms, and the inclusions ϕ, there are two inclusions of $\Delta(\infty, 2, 2)$ in Γ corresponding to the fact that there are three conjugacy classes of subgroups of order two in $\Delta(4, 2, 4)$ forming two orbits under Machí automorphisms, and two conjugacy classes of such subgroups in $\Delta(3, 2, 6)$. Under one of the inclusions in each case, the free ends of the rays R lie on edge centres in $\mathscr{U}_\Gamma$; under the other they lie on the centres of (even-valent) faces. The images of the quotients of $\mathscr{U}_{\Delta(\infty,2,2)}$ under the corresponding operational functors lies on a 'squashed cylinder' (topologically a plane): starting with the doubly infinite cylinder obtained by identifying a ray R with its image under w (and identifying their corresponding reflections in A), take the quotient by a rotation through π about the diameter through the free end of R of the circular cross-section at that point.

Extended triangle groups

It is geometrically clear that, up to inner automorphism, all monomorphisms $\Delta \hookrightarrow \Gamma$ between Euclidean triangle groups extend uniquely to monomorphisms between the corresponding extended triangle groups. Hence there is essentially nothing new to consider here, and all the functors described above extend to the wider setting of hypermaps which may be non-orientable or have boundary.

4.5.3. Inclusions between spherical triangular groups

Triangle groups

Up to inner automorphism and Machí automorphism the only non-elementary proper inclusions between spherical triangle groups are the obvious ones: a single inclusion of $\Delta(3, 2, 3)$ into each of $\Delta(3, 2, 4)$ and $\Delta(3, 2, 5)$. The former is the analogue on the sphere of inclusion (a) between Fuchsian groups listed in §4.5.1, and it induces a restriction of the composition of W with a Machí functor. The latter induces a functor Φ whose combinatorial effect can be deduced using the method of §4.5.1. To apply Φ to a map $\mathscr{M}$ of type dividing $(3, 2, 3)$ we edge-stellate each face f of $\mathscr{M}$ by placing a new vertex in the centre of each face and joining it by half-edges to the midpoints of the edges of f. We then place a vertex two-thirds of the way along each new half-edge, joining it by an edge to an original vertex of f consistent with a given orientation of $\mathscr{M}$ so that no edges cross. Finally we delete the edges of $\mathscr{M}$ to obtain $\Phi(\mathscr{M})$.

Extended triangle groups

The inclusion $\Delta(3,2,3) \hookrightarrow \Delta(3,2,5)$ does not extend to an inclusion between the extended triangle groups. In fact, up to inner automorphism and Machí automorphism, the only non-elementary proper inclusions between spherical extended triangle groups are two inclusions $\Delta[3,2,3] \hookrightarrow \Delta[3,2,4]$. One of these restricts to the inclusion $\Delta(3,2,3) \hookrightarrow \Delta(3,2,4)$, and the functor induced is essentially nothing new. The other inclusion does not restrict to one between the even subgroups; the functor it induces is the composite of the first with the functor induced by the unique outer automorphism of $\Delta[3,2,4]$ represented by an automorphism which transposes the two subgroups isomorphic to S_4.

4.6. Functors from elementary inclusions

As discussed in §1.5, the only finite cyclic subgroups of a triangle group

$$\Gamma = \Delta(m_0', m_1', m_2') = \langle\, y_0, y_1, y_2 \,|\, y_0^{m_0'} = y_1^{m_1'} = y_2^{m_2'} = y_0 y_1 y_2 = 1 \,\rangle$$

are the finite subgroups of the $\langle\, y_i \,\rangle$ and their conjugates. Hence we have inclusions of cyclic groups $C_n \cong \Delta(n,1,n) \hookrightarrow \Gamma$ whenever n divides some m_i'. We postpone further discussion of these inclusions until §5.3.2, where their pre-composition with epimorphisms onto cyclic groups are considered.

The following results may be obtained by inspection, and are given up to inner automorphism and Machí automorphism. There is one inclusion $D_4 \cong \Delta(2,2,2) \hookrightarrow \Delta(3,2,3)$; it is the analogue of inclusion (b) between Fuchsian groups listed in §4.5.1, and here (as it does there) it induces a restriction of the functor $\mathrm{W}^{-1} \circ \mathrm{J}$ where W and J are the Walsh and James functors. There are two inclusions $\Delta(2,2,2) \hookrightarrow \Delta(3,2,4)$ as a normal or non-normal subgroup; the first is analogous to inclusion (c) and induces a restriction of J. Next, there is one of each inclusion $D_6 \cong \Delta(3,2,2) \hookrightarrow \Delta(3,2,4)$ and $D_8 \cong \Delta(4,2,2) \hookrightarrow \Delta(3,2,4)$; the second is analogous to inclusion (K) which induces the functor realized by vertex truncation. Further, there is one of each inclusion $\Delta(2,2,2) \hookrightarrow \Delta(3,2,5)$ and $\Delta(3,2,2) \hookrightarrow \Delta(3,2,5)$. Then there are two inclusions $D_{10} \cong \Delta(5,2,2) \hookrightarrow \Delta(3,2,5)$, corresponding to the fact that each of these triangle groups has an outer automorphism of order 2, transposing the two conjugacy classes of elements of order 5. Finally, there is at least one inclusion $D_n \cong \Delta(n,2,2) \hookrightarrow \Delta(m,2,2)$

for each $n \mid m$; we do not investigate these further.

All inclusions of extended triangle groups $\Delta[n,1,n]$ and $\Delta[n,2,2]$ in spherical extended triangle groups are elementary. However, in general the inclusions between the even subgroups do not extend uniquely; again, we do not investigate these inclusions further.

Chapter 5

Generalized operations II

5.1. Homomorphisms as functors

In Chapter 4 it was shown how homomorphisms between triangular groups give rise to operational functors between categories of hypermaps defined by type, and the possible inclusions between such groups were determined. In this chapter we investigate the remaining homomorphisms.

Let $\phi\colon \Delta = \Delta(m_0, m_1, m_2) \to \Delta(m'_0, m'_1, m'_2) = \Gamma$ be a homomorphism between triangle groups with kernel K. Associated to each orientable hypermap $\mathscr{H}$ without boundary of type dividing (m_0, m_1, m_2) is a hypermap subgroup $H \leqslant \Delta$. In particular, K and HK are respectively hypermap subgroups for hypermaps $\mathscr{K}$ and $\overline{\mathscr{H}}$, say. Since HK is the smallest subgroup of Δ containing both H and K, $\overline{\mathscr{H}}$ is the largest hypermap covered by both $\mathscr{H}$ and $\mathscr{K}$ (in the sense that if $\mathscr{H}$ and $\mathscr{K}$ cover another hypermap $\mathscr{M}$ then $\overline{\mathscr{H}}$ also covers $\mathscr{M}$). Moreover, since $\phi(H) = \phi(HK)$, $\mathscr{H}$ and $\overline{\mathscr{H}}$ have the same image under the functor Φ induced by ϕ (see Figure 4.1). This image has $|\phi(H) : \Gamma|$ hyperdarts; it covers the hypermap $\mathscr{D}$ whose hypermap subgroup is $\phi(\Delta)$ with degree $|HK : \Delta|$, and it is covered by the universal map $\mathscr{U}_\Gamma = \Phi(\mathscr{K})$ for Γ with degree $|\phi(H)| = |K : HK|$. Note that since K is normal, $\mathscr{K}$ is regular (with $|K : \Delta|$ hyperdarts); hence if $\mathscr{H}$ is regular then so is $\overline{\mathscr{H}}$, although the converse is false in general. As we require, all these arguments still hold—with the same hypermaps described—when H is replaced by a conjugate subgroup.

In the case that a homomorphism $\phi\colon \Delta \twoheadrightarrow \Gamma$ between triangle groups is an epimorphism, the Correspondence Theorem for groups leads to a category equivalence

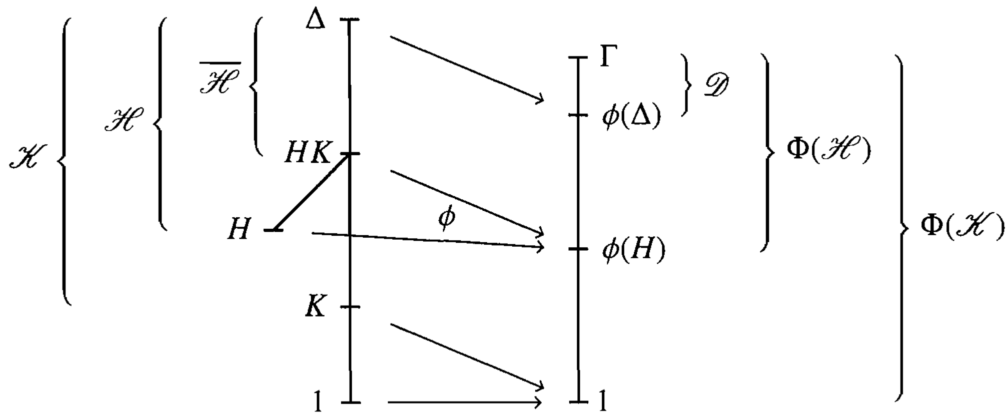


Figure 5.1. Homomorphism diagram for a homomorphism ϕ between triangle groups.

between orientable hypermaps without boundary of type (m_0, m_1, m_2) covered by $\mathscr{K}$, and orientable hypermaps without boundary of type (m'_0, m'_1, m'_2). In particular, every such hypermap $\mathscr{H}'$ of type (m'_0, m'_1, m'_2) and with hypermap subgroup H' in Γ has a canonical preimage under Φ, namely the hypermap whose hypermap subgroup is $\phi^{-1}(H')$. This is the smallest of the hypermaps in the preimage of $\mathscr{H}'$, and it is unique amongst these in that it is covered by $\mathscr{K}$. (For non-surjective homomorphisms it is not generally possible to distinguish such a hypermap, since the mapping induced by ϕ between conjugacy classes of subgroups is not necessarily injective.) Φ sends a regular hypermap $\mathscr{H}$ to a regular hypermap $\mathscr{H}'$, since if $H \trianglelefteq \Delta$ then $\phi(H) \trianglelefteq \phi(\Delta) = \Gamma$. Conversely, if $\mathscr{H}'$ is regular then so is its canonical preimage. Moreover, these two hypermaps can be regarded as sharing the same monodromy group but arising from (usually different) canonical generating sets, and so could potentially lie in the same orbit under the groups $\mathfrak{H}^+$ and $\mathfrak{H}$ of hypermap operations. (This is demonstrated in Example 5.4.10.) These ideas hold equally in the wider context of general hypermaps and extended triangle groups.

5.2. Epimorphic images of triangular groups

5.2.1. Triangle groups

Let $\phi\colon \Delta = \Delta(m_0, m_1, m_2) \to \Delta(m'_0, m'_1, m'_2) = \Gamma$ be a homomorphism between triangle groups. Since ϕ can be written as $\Delta \twoheadrightarrow \phi(\Delta) \hookrightarrow \Gamma$, we have the problem of determining the possible epimorphic images of triangle groups, and the possible inclusions of these images into triangle groups. Δ has presentation $\langle x_0, x_1, x_2 \mid x_0^{m_0} = x_1^{m_1} = x_2^{m_2} = x_0x_1x_2 = 1 \rangle$ and so $\phi(\Delta)$ is generated by elements $\phi(x_0)$, $\phi(x_1)$, $\phi(x_2)$ which satisfy the relations $\phi(x_0)^{m_0''} = \phi(x_1)^{m_1''} = \phi(x_2)^{m_2''} = \phi(x_0)\phi(x_1)\phi(x_2) = 1$, where $\phi(x_i)$ has order m_i'' dividing m_i $(0 \leqslant i \leqslant 2)$, with the usual convention that every element of $\mathbb{Z} \cup \{\infty\}$ divides ∞. Throughout this chapter, as in Chapter 4 with the exception of automorphisms, the homomorphisms we consider are assumed to map parabolic elements to parabolic or elliptic elements; this assumption shall be reiterated in the statement of theorems, but shall otherwise be used implicitly.

The group $\phi(\Delta)$ is some quotient of a (m_0'', m_1'', m_2'') triangle group, and certainly $\phi(\Delta)$ may actually be such a triangle group with presentation

$$\Lambda = \langle w_0, w_1, w_2 \mid w_0^{m_0''} = w_1^{m_1''} = w_2^{m_2''} = w_0w_1w_2 = 1 \rangle \tag{i}$$

so that ϕ maps the generating triple for Δ onto a canonical generating triple for Λ. Conversely, for any choice of positive integers m_i'' satisfying $m_i'' \mid m_i$ for $0 \leqslant i \leqslant 2$ there is, by von Dyck's Theorem [52], an epimorphism from Δ onto the group Λ with presentation (i), defined on the generators by $\phi(x_i) = w_i$ for $0 \leqslant i \leqslant 2$.

Definition 5.2.1. Let Δ and Λ be non-cyclic triangle groups. An operational functor induced by an epimorphism $\Delta \twoheadrightarrow \Lambda$ which maps a canonical generating triple for Δ onto a canonical generating triple for Λ is called a *direct derivative functor.* An operational functor induced by any other epimorphism $\Delta \twoheadrightarrow \Lambda$ is called an *indirect derivative functor.* The image of a hypermap $\mathscr{H}$ under such operational functors is respectively called a *direct derivative* or an *indirect derivative* of $\mathscr{H}$. •

It would be futile to attempt to determine all other possibilities for $\phi(\Delta)$ by examining the proper quotients of an infinite (m_0'', m_1'', m_2'') triangle group, because even the finite quotients are infinite in number. (A well-known theorem of Mal'cev [45] says that every finitely-generated linear group is residually finite, that is, has normal subgroups H_i $(i \in I)$ of finite index such that $\bigcap_{i \in I} H_i = 1$. Since subgroups of residually finite groups are also residually finite, infinite triangle groups have infinitely many finite-index normal subgroups and the claim follows.) Instead, given that we intend to embed $\phi(\Delta)$ in another triangle group, and knowing that two-generator subgroups of spherical and Euclidean triangle groups may be determined directly, our main task is to examine the possibilities for $\phi(\Delta)$ as a two-generator Fuchsian group. (Δ can clearly be generated by two elements and any subgroup of a Fuchsian (triangle) group is Fuchsian.) Such groups have been fully classified; we state the relevant parts of the classification in §5.4.1.

We now know enough to begin to determine all possible epimorphic images of triangle groups which can be embedded into other triangle groups. In §5.3.1 we investigate cyclic epimorphic images. In §5.4.1 we determine the remaining triangular epimorphic images. As a consequence, we will see there that there are no other possibilities.

5.2.2. Extended triangle groups

Proposition 5.2.2. *Let $\overline{\Delta}$ be an extended triangle group, let $\mathscr{X}$ be a simply-connected Riemann surface, and let $\overline{G}$ be a discrete subgroup of* $\mathrm{Aut}(\mathscr{X})$*. The image of a homomorphism $\phi\colon \overline{\Delta} \to \overline{G}$ which maps (possibly infinite-order) rotations to rotations or reflections is either trivial, cyclic of order 2, a triangle group $\Delta(2,2,m)$ where $2 \leqslant m \leqslant \infty$, a dihedral group, or an extended triangle group.*

Proof. It is convenient to use the presentation $\langle\, x_0, x_1, t \mid (x_0 t)^2 = (t x_1)^2 = t^2 = x_0^{m_0} = x_1^{m_1} = (x_0 x_1)^{m_2} = 1 \,\rangle$ for $\overline{\Delta}$. Under such an homomorphism $\phi\colon \overline{\Delta} \to \overline{G}$, the triangle subgroup $\Delta = \langle\, x_0, x_1 \,\rangle$ of $\overline{\Delta}$ is mapped to a discrete group G which, as introduced in §5.2.1 and proved in the sections below, has the structure of a cyclic group or a triangle group. Let x_0', x_1' and t' respectively be the images of x_0, x_1 and t. By examining the action of $x_0' t'$ and $t' x_1'$ on the fixed points of G and t', we observe the following. If t' is the identity then $G = \phi(\Delta)$ is either trivial, cyclic of order 2 or a triangle group $\Delta(2,2,m)$ with canonical generating involutions x_0' and x_1' where $2 \leqslant m \leqslant \infty$. So suppose henceforth that t' is a rotation of order 2, or a reflection.

Suppose that G is cyclic and consists of rotations. If it is trivial then $\phi(\overline{\Delta}) = \langle\, t' \,\rangle$ is cyclic of order 2; so suppose it is non-trivial. If t' is a reflection then either its axis R_t passes through the fixed point(s) f of G (in which case $\phi(\overline{\Delta})$ is dihedral), or $\mathscr{X}$ is the sphere and R_t is the equator to the polar axis of G (in which case x_0' and x_1' have orders at most 2 and $\phi(\overline{\Delta})$ is a Klein four-group). Otherwise, t' is either an order-2 rotation about f, or $\mathscr{X}$ is the sphere and t' is an order-2 rotation about the axis orthogonal to that fixed by G; either way, x_0' and x_1' have orders at most 2 and $G = \phi(\overline{\Delta})$ is cyclic of order 2. Now suppose that G is generated by a reflection v with axis R_v. If $t' = v$ then $G = \phi(\overline{\Delta})$ is cyclic of order 2. Otherwise t' is either a reflection with axis orthogonal to R_v, or an order-2 rotation. Either way, $\phi(\overline{\Delta})$ is a Klein four-group since, in the latter case, either t' fixes a point on R_v, or $\mathscr{X}$ is the sphere and R_v is the equator to the polar axis of t'.

Finally, if G is a non-cyclic triangle group then t' must be the reflection whose axis passes through the fixed points of x_0' and x_1'; it follows that $\phi(\overline{\Delta})$ is the extended triangle group whose even subgroup is G. ∎

Suppose that $\phi\colon \overline{\Delta} \to \overline{\Gamma}$ is a homomorphism between extended triangle groups. The proof of Proposition 5.2.2 shows that when the image of the restriction $\phi|$ of ϕ to Δ is not dihedral and has order greater than 2, the homomorphism ϕ is the unique extension of $\phi|$ to $\overline{\Delta}$. Moreover, $\phi(\overline{\Delta})$ is the 'expected' group in this situation: it is either a dihedral group with cyclic subgroup $\phi(\Delta)$, or it is an extended triangle group with $\phi(\Delta)$ as its even subgroup. As a result, there is little to say about most of the functors which arise from homomorphisms between extended triangle groups and which map parabolic elements to parabolic or elliptic elements, beyond what can be said about their restriction to orientable hypermaps without boundary. (Note that four of the six map operations—and the related functors T and P of §4.4.5—did not arise from extensions in this way; this is because their corresponding automorphisms map certain parabolic elements to hyperbolic ones.) We shall not consider extended triangle groups henceforth in this chapter.

5.3. Functors from epimorphisms onto cyclic groups

5.3.1. Cyclic epimorphic images

Let Δ be a triangle group $\langle x_0, x_1, x_2 \mid x_0^{m_0} = x_1^{m_1} = x_2^{m_2} = x_0x_1x_2 = 1 \rangle$, let Δ' be its commutator subgroup and let $\Delta^{ab} = \Delta/\Delta'$ be its abelianization. In order to determine the orders of the possible cyclic epimorphic images—or equivalently, the cyclic quotients—of Δ we shall make use of the following fact, which is an elementary application of the isomorphism theorems.

Lemma 5.3.1. *Let G be a group and let H be a normal subgroup of G. The quotients of G/H are identified by isomorphism with the quotients G/J of G where $H \leqslant J \trianglelefteq G$.* ∎

Remark 5.3.2. As a consequence of Lemma 5.3.1, any quotient of Δ^{ab} corresponds to an abelian quotient Δ/K where $\Delta' \leqslant K \trianglelefteq \Delta$. Conversely, if $\phi\colon \Delta \twoheadrightarrow \phi(\Delta)$ is any epimorphism with abelian image then $\phi(\Delta) \cong \Delta^{\mathrm{ab}}/(\ker(\phi)/\Delta')$. This gives a bijection between cyclic quotients of Δ and Δ^{ab}. ♦

We shall determine the orders of the cyclic quotients of Δ^{ab} by finding the structure of Δ^{ab} as the direct sum of its Sylow p-subgroups, or equivalently, of its largest p-group quotients. Suppose first that Δ has finite periods. Δ^{ab} has presentation

$$\begin{aligned}\Delta^{\mathrm{ab}} &= \langle\, x_0, x_1, x_2 \,|\, m_0x_0 = m_1x_1 = m_2x_2 = x_0 + x_1 + x_2 = 0 \,\rangle \\ &= \langle\, x_0, x_1 \,|\, m_0x_0 = m_1x_1 = m_2(x_0 + x_1) = 0 \,\rangle\end{aligned}$$

(written in additive notation as abelian groups). It is clear from the first presentation that if we let

$$\begin{aligned}G &= \langle\, x_0, x_1, x_2 \,|\, m_0x_0 = m_1x_1 = m_2x_2 = 0 \,\rangle \\ &\cong C_{m_0} \oplus C_{m_1} \oplus C_{m_2}\end{aligned}$$

and we let H be the cyclic subgroup $\langle\, x_0 + x_1 + x_2 \,\rangle$, then we have $|G| = m_0m_1m_2$, $|H| = \mathrm{LCM}(m_0, m_1, m_2)$ and $\Delta^{\mathrm{ab}} = G/H$, which gives

$$|\Delta^{\mathrm{ab}}| = m_0m_1m_2/\,\mathrm{LCM}(m_0, m_1, m_2) = \mathrm{HCF}(m_0m_1, m_0m_2, m_1m_2).$$

For any prime p, write $m_0 = p^{\alpha_p}m_0'$, $m_1 = p^{\beta_p}m_1'$ and $m_2 = p^{\gamma_p}m_2'$ where $\mathrm{HCF}(m_0', p) = \mathrm{HCF}(m_1', p) = \mathrm{HCF}(m_2', p) = 1$. Without loss of generality we may suppose temporarily that $\alpha_p \leqslant \beta_p \leqslant \gamma_p$. The quotient

$$F_p = \langle\, x_0, x_1 \,|\, m_0x_0 = m_1x_1 = m_2(x_0 + x_1) = 0, p^{\alpha_p}x_0 = p^{\beta_p}x_1 = p^{\gamma_p}(x_0 + x_1) = 0 \,\rangle$$

of Δ^{ab} has simplified presentation

$$F_p = \langle\, x_0, x_1 \,|\, p^{\alpha_p}x_0 = p^{\beta_p}x_1 = 0 \,\rangle \cong C_{p^{\alpha_p}} \oplus C_{p^{\beta_p}}$$

and so F_p is the direct sum of at most two cyclic groups; it is cyclic if and only if $\alpha_p = 0$, and it has order $|F_p| = p^{\alpha_p+\beta_p}$, which is the highest power of p dividing $\mathrm{HCF}(p^{\alpha_p+\beta_p}m_0'm_1', p^{\alpha_p+\gamma_p}m_0'm_2', p^{\beta_p+\gamma_p}m_1'm_2') = |\Delta^{\mathrm{ab}}|$. Thus F_p is the largest p-group quotient of Δ^{ab}.

For each prime p, let δ_p, ε_p, ζ_p respectively be the minimum, median and maximum of the exponents $\alpha_p, \beta_p, \gamma_p$. Then we have

$$\Delta^{\mathrm{ab}} \cong \bigoplus_{\text{primes } p} C_{p^{\delta_p}} \oplus C_{p^{\varepsilon_p}},$$

a direct sum of cyclic p-groups. It is clear that Δ^{ab} is trivial if and only if $\varepsilon_p = 0$ for all p, that is, the m_i are pairwise coprime. Similarly, Δ^{ab} is cyclic if and only if each of its Sylow p-subgroups is cyclic, which is the case precisely when $\delta_p = 0$ for all p, equivalently when no prime p divides each m_i, that is, when $\mathrm{HCF}(m_0, m_1, m_2) = 1$. Now, a largest cyclic quotient of F_p is $F_p/C_{p^{\delta_p}}$, of order p^{ε_p}, so we have

Proposition 5.3.3. *For a prime p let ε_p be the median value of the three exponents δ_p, ε_p, ζ_p of the largest powers of p dividing m_0, m_1, m_2, so that*

$$\varepsilon_p = \max\{\min\{\delta_p, \varepsilon_p\}, \min\{\varepsilon_p, \zeta_p\}, \min\{\zeta_p, \delta_p\}\}.$$

Then a largest cyclic quotient of Δ^{ab} has order $\prod_{\text{primes } p} p^{\varepsilon_p}$ and the order of any cyclic quotient of Δ^{ab} divides this. ■

We note that this result is valid when some or all of the m_i are infinite, provided that we treat the exponent of the largest power of p dividing ∞—and the prime-power itself—as infinite; indeed, it is trivial to check that the expression given above for Δ^{ab} as a direct sum of cyclic p-groups holds when $\Delta = \Delta(a, b, \infty)$. In addition, $\Delta^{\mathrm{ab}} \cong \mathbb{Z} \times C_b$ when $\Delta = \Delta(\infty, b, \infty)$, and $\Delta^{\mathrm{ab}} \cong \mathbb{Z} \times \mathbb{Z}$ when $\Delta = \Delta(\infty, \infty, \infty)$ and so the result is immediately verified in these cases too.

Example 5.3.4. Let Δ be the triangle group

$$\Delta(18, 21, 30) = \langle\, x_0, x_1, x_2 \mid x_0^{18} = x_1^{21} = x_2^{30} = x_0x_1x_2 = 1 \,\rangle.$$

Write $18 = 2.3^2$, $21 = 3.7$ and $30 = 2.3.5$. Then $\delta_2 = 0$, $\varepsilon_2 = \zeta_2 = 1$; $\delta_3 = \varepsilon_3 = 1$, $\zeta_3 = 2$; $\delta_5 = \varepsilon_5 = 0$, $\zeta_5 = 1$; $\delta_7 = \varepsilon_7 = 0$, $\zeta_7 = 1$. We have

$$\Delta^{\mathrm{ab}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3,$$

and a largest cyclic quotient of Δ has order $2^1.3^1 = 6$. Table 5.1 gives all possible

Homomorphism		Image order	Homomorphism		Image order
$x_0 \mapsto 0\ (1)$ $x_1 \mapsto 0\ (1)$ $x_2 \mapsto 0\ (1)$		1	$x_0 \mapsto 2\ (3)$ $x_1 \mapsto 4\ (3)$ $x_2 \mapsto 0\ (1)$	$x_0 \mapsto 4\ (3)$ $x_1 \mapsto 2\ (3)$ $x_2 \mapsto 0\ (1)$	3
$x_0 \mapsto 0\ (1)$ $x_1 \mapsto 2\ (3)$ $x_2 \mapsto 4\ (3)$	$x_0 \mapsto 0\ (1)$ $x_1 \mapsto 4\ (3)$ $x_2 \mapsto 2\ (3)$	3	$x_0 \mapsto 2\ (3)$ $x_1 \mapsto 2\ (3)$ $x_2 \mapsto 2\ (3)$	$x_0 \mapsto 4\ (3)$ $x_1 \mapsto 4\ (3)$ $x_2 \mapsto 4\ (3)$	3
$x_0 \mapsto 3\ (2)$ $x_1 \mapsto 0\ (1)$ $x_2 \mapsto 3\ (2)$		2	$x_0 \mapsto 1\ (6)$ $x_1 \mapsto 0\ (1)$ $x_2 \mapsto 5\ (6)$	$x_0 \mapsto 5\ (6)$ $x_1 \mapsto 0\ (1)$ $x_2 \mapsto 1\ (6)$	6
$x_0 \mapsto 3\ (2)$ $x_1 \mapsto 2\ (3)$ $x_2 \mapsto 1\ (6)$	$x_0 \mapsto 3\ (2)$ $x_1 \mapsto 4\ (3)$ $x_2 \mapsto 5\ (6)$	6	$x_0 \mapsto 1\ (6)$ $x_1 \mapsto 2\ (3)$ $x_2 \mapsto 3\ (2)$	$x_0 \mapsto 5\ (6)$ $x_1 \mapsto 4\ (3)$ $x_2 \mapsto 3\ (2)$	6
$x_0 \mapsto 2\ (3)$ $x_1 \mapsto 0\ (1)$ $x_2 \mapsto 4\ (3)$	$x_0 \mapsto 4\ (3)$ $x_1 \mapsto 0\ (1)$ $x_2 \mapsto 2\ (3)$	3	$x_0 \mapsto 1\ (6)$ $x_1 \mapsto 4\ (3)$ $x_2 \mapsto 1\ (6)$	$x_0 \mapsto 5\ (6)$ $x_1 \mapsto 2\ (3)$ $x_2 \mapsto 5\ (6)$	6

TABLE 5.1. The homomorphisms from the triangle group $\Delta = \langle\, x_0, x_1, x_2 \mid x_0^{18} = x_1^{21} = x_2^{30} = x_0x_1x_2 = 1\,\rangle$ into $\mathbb{Z}_6$, and the orders of their images.

homomorphisms from Δ into $\mathbb{Z}_6$. The homomorphisms are described by specifying the images of the generators x_0, x_1, x_2 of Δ; the numbers in parentheses are the orders of these elements in $\mathbb{Z}_6$. Some of the homomorphisms have been grouped together in pairs according to the orders of the images of the generators. It is a necessary condition for two of the homomorphisms to have the same kernel that these orders are the same under both maps. In general this is not a sufficient condition, but in this case it can be seen that one homomorphism in each pair is the composition of the other with an automorphism of $\mathbb{Z}_6$, namely that which inverts each element. ▲

5.3.2. Functors from cyclic group inclusions

Let $\Gamma = \Delta(m'_0, m'_1, m'_2) = \langle\, w_0, w_1, w_2 \mid w_0^{m'_0} = w_1^{m'_1} = w_2^{m'_2} = w_0 w_1 w_2 = 1 \,\rangle$ be a triangle group. The only cyclic subgroups of Γ are the subgroups of the $\langle\, w_i \,\rangle$ and their conjugates. Hence if $\phi\colon \Delta \to \Gamma$ is a homomorphism from $\Delta = \Delta(m_0, m_1, m_2)$ into Γ with cyclic image $\phi(\Delta)$ then the order n of $\phi(\Delta)$ must divide one of the periods m'_i of Γ (with the usual convention that every positive integer and ∞ divides a period $m'_i = \infty$).

Let $\Phi_{i,n}$ be the functor induced by ϕ, and let $\mathscr{H}$ be a hypermap of type dividing (m_0, m_1, m_2) with hypermap subgroup $H \leqslant \Delta$. The hypermaps $\Phi_{i,n}(\mathscr{H})$ are covered by the universal map $\mathscr{U}_\Gamma$ for Γ with degree $r = |\phi(H)|$; they are (usually infinite) maps on simply-connected Riemann surfaces (augmented by ideal points if some $m'_i = \infty$), invariant under a rotation of order r about the centre of the hypervertex, hyperedge or hyperface corresponding to the fixed point of w_i. Denote these hypermaps by $\mathscr{U}_{\Gamma,i,r}$, noting that $\mathscr{U}_{\Gamma,i,1} = \mathscr{U}_\Gamma$.

The functor $\Phi_{i,n}$ is completely determined by the integer i and the normal subgroup $K = \ker(\phi)$. Remark 5.3.2 tells us that normal subgroups K which give cyclic quotients Δ/K are in one-to-one correspondence with the cyclic quotients of Δ^{ab}. Hence for a positive integer n, the number of distinct functors $\Phi_{i,n}\colon C_{(m_0,m_1,m_2)} \to C_{(m'_0,m'_1,m'_2)}$ induced by homomorphisms $\phi\colon \Delta \to \Gamma$ with cyclic epimorphic images of order n is simply td, where t is the number of periods of Γ which are multiples of n, and d is the number of distinct cyclic quotients of order n of Δ^{ab}.

Example 5.3.5. Here we consider functors from spherical triangle groups. The group $\Delta = \Delta(3,2,4) \cong S_4$ gives rise to the category $C_{(3,2,4)}$ of the eleven spherical maps covered by the cube, corresponding to the conjugacy classes of subgroups of S_4. The only non-trivial cyclic epimorphic image of Δ has order 2, and so there exist non-trivial functors $\Phi_{i,2}$ from $C_{(3,2,4)}$ into each category $C_{(m'_0,m'_1,m'_2)}$ for which m'_i is even. The kernel K of all homomorphisms into $\Gamma = \Delta(m'_0, m'_1, m'_2)$ inducing such a functor is a normal subgroup of index 2 in Δ; if we identify Δ with S_4 then $K = A_4$ and $\mathscr{K}$ is the map of type $(1,2,2)$ consisting of two 1-valent vertices and single edge. The maps in $C_{(3,2,4)}$ whose image under $\Phi_{i,2}$ is $\mathscr{U}_\Gamma$ are the five which cover $\mathscr{K}$, namely the even-face-valent maps without free edges. These are precisely the bipartite maps. The image of the other six maps in $C_{(3,2,4)}$ is $\mathscr{U}_{\Gamma,i,2}$, the quotient of $\mathscr{U}_\Gamma$ by a half-turn about the centre of an i-component.

If $\Delta = \Delta(3,2,3) \cong A_4$ then $\Delta^{ab} \cong C_3$. The only non-trivial functors $\Phi\colon C_{(3,2,3)} \to C_{(m'_0,m'_1,m'_2)}$ induced by homomorphisms $\Delta(3,2,3) \to \Delta(m'_0, m'_1, m'_2) = \Gamma$ with cyclic image are the $\Phi_{i,3}$ where $3 \mid m'_i$. The map $\mathscr{K}$ whose map subgroup in $\Delta = \Delta(3,2,3)$ is the kernel of all epimorphisms $\phi\colon \Delta \twoheadrightarrow C_3 \leqslant \Gamma$ consists of one trivalent vertex and three half-edges. Of the five maps covered by the tetrahedron, those which cover $\mathscr{K}$ are precisely the three 3-vertex-valent, 3-face-valent ones; their image under $\Phi_{i,3}$ is $\mathscr{U}_\Gamma$. The image of the two other maps is $\mathscr{U}_{\Gamma,i,3}$, the quotient of $\mathscr{U}_\Gamma$ by a rotation of order 3 about the centre of an i-component.

The triangle group $\Delta = \Delta(3,2,5)$ gives rise to the icosahedral and dodecahedral maps. This group is perfect, and so its only cyclic epimorphic image is trivial. It follows that the only functors $\Phi_{i,n}$ from $C_{(3,2,5)}$ and $C_{(5,2,3)}$ are trivial in the sense that the image of every map under such a functor is $\mathscr{U}_\Gamma$. ▲

Example 5.3.6. Let $\Delta = \Delta(18,21,30)$ and let $C_{(18,21,30)}$ be the category of hypermaps which it determines. We showed in Example 5.3.4 that there are four different functors $\Phi_{i,6}$ induced by four pairs of epimorphisms from Δ onto a given cyclic subgroup of order 6 of a triangle group Γ, corresponding to the four distinct cyclic quotients of order 6 of Δ^{ab}. One of these functors, Φ, is induced by a pair of epimorphisms whose kernel is the map subgroup in Δ for a hypermap $\mathscr{K}$ whose i-components have maximum possible valency; it consists of one 6-valent hypervertex, two 3-valent hyperedges and one 6-valent hyperface on a surface whose genus is thus 2. (See Table 5.1; $\mathscr{K}$ also features in Figure 3.3.) $\mathscr{K}$ covers just the four hypermaps shown in Figure 5.2 and so there are four possible images under Φ. All spherical hypermaps in $C_{(18,21,30)}$ which are bipartite and even-hypervertex-valent have image $\mathscr{U}_{\Gamma,i,3}$; the remaining spherical hypermaps in this category have image $\mathscr{U}_\Gamma$. ▲

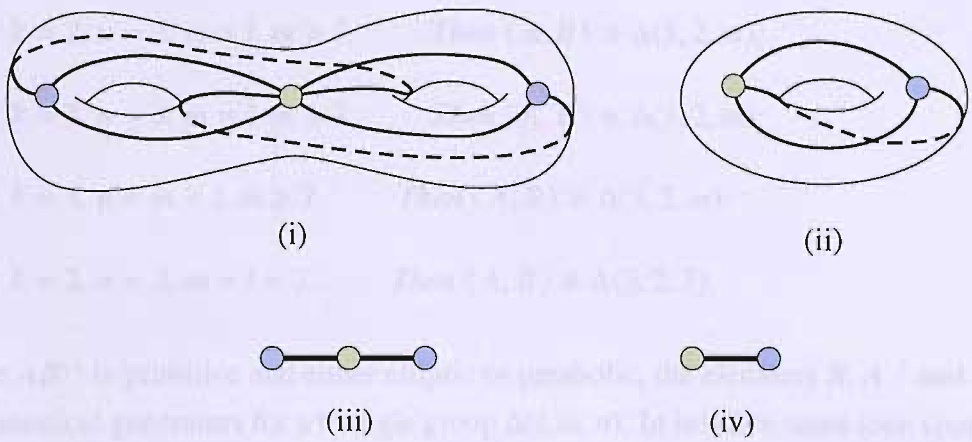


FIGURE 5.2. (i) The Walsh model of the hypermap $\mathscr{K}$ of Example 5.3.6. (ii–iv) The hypermaps covered by $\mathscr{K}$.

5.4. Functors from epimorphisms onto triangle groups

5.4.1. Triangular epimorphic images

Fuchsian triangle groups

The two-generator Fuchsian groups—discrete two-generator subgroups of the group $\mathrm{Aut}^+(\mathcal{H}) \cong \mathrm{PSL}(2,\mathbb{R})$ of conformal automorphisms of the hyperbolic plane $\mathcal{H}$—have been fully classified, and we state the results we need in the theorem below. We are interested in the cases where the two generators A and B are parabolic or elliptic elements acting with distinct fixed points. Following the geometric approach of Matelski [46] we normalize such A and B by replacing one of them with its inverse if need be so that the rotations through an angle of at most π which they determine are in opposite senses.

Definition 5.4.1. An element of $\mathrm{Aut}^+(\mathcal{H})$ is *primitive* if it is hyperbolic, parabolic, or elliptic of finite order n rotating by $2\pi/n$. •

Theorem 5.4.2 ([37], [46]). *If elements A and B of* $\mathrm{PSL}(2,\mathbb{R})$ *are non-hyperbolic, primitive and normalized as above, then they generate a discrete group if and only if AB^{-1} is primitive or one of the following five conditions holds: let A, B, AB^{-1} have orders n, m, l respectively and let AB^{-1} rotate by $k(2\pi/l)$ with k and l relatively prime.*

(i) $k = 2$, $n = m$, $1/l + 1/n < 1/2$. *Then* $\langle A, B\rangle \cong \Delta(n, 2, l)$;

(ii) $k = 2$, $n = 2$, $m = l$, $m \geqslant 7$. *Then* $\langle A, B \rangle \cong \Delta(3, 2, m)$;

(iii) $k = 3$, $n = 3$, $m = l$, $m \geqslant 7$. *Then* $\langle A, B \rangle \cong \Delta(3, 2, m)$;

(iv) $k = 4$, $n = m = l$, $m \geqslant 7$. *Then* $\langle A, B \rangle \cong \Delta(3, 2, m)$;

(v) $k = 2$, $n = 3$, $m = l = 7$. *Then* $\langle A, B \rangle \cong \Delta(3, 2, 7)$. ✠

When AB^{-1} is primitive and either elliptic or parabolic, the elements B, A^{-1} and AB^{-1} are canonical generators for a triangle group $\Delta(l, m, n)$. In just five cases (one sporadic; four concerning infinite families) is the product of the primitive generators imprimitive: in each case the group is a triangle group with a period 2. Figure 5.3 illustrates how these elements act as non-canonical generators of triangle groups.

The Fuchsian group generated by a pair of (possibly imprimitive) elliptic elements with distinct fixed points is the same as that generated by the primitive powers of these elements, and so the triangle groups just described are the only groups obtained when we insist upon the generators having an elliptic product, irrespective of their primitivity. However, since we are interested not only in the structure of the group generated but also in its specific sets of generators—which, ultimately, will determine the epimorphisms onto the group—we must examine the possibilities for imprimitive generators in full detail.

The following result is familiar to us in the case of canonical triangle group generators (which are primitive by definition). Its application to imprimitive rotations is fundamental to the investigation.

Lemma 5.4.3. *Let g_x, g_y, g_z be rotations of the Euclidean or hyperbolic plane which do not all fix the same point. Then they satisfy $g_x g_y g_z = 1$ if and only if their fixed points form the vertices of a triangle such that each rotates with the same orientation through twice the angle subtended by its fixed vertex.*

Proof. Let x, y, z be the fixed points of g_x, g_y, g_z respectively. Since $g_x g_y g_z = 1$, x is the fixed-point of $g_x^{-1} = g_y g_z$, that is, $g_y(g_z(x)) = x$, and since g_y is a rotation with respect to the usual planar metric ρ we have $\rho(y, g_z(x)) = \rho(y, x)$. Yet g_z is also a rotation and so $\rho(z, x) = \rho(z, g_z(x))$ and we have either $x = g_z(x)$ (that is, $x = z$) and thus all three fixed points are coincident, or both y and z lie on the perpendicular bisector of x and $g_z(x)$ and hence are not collinear with x.

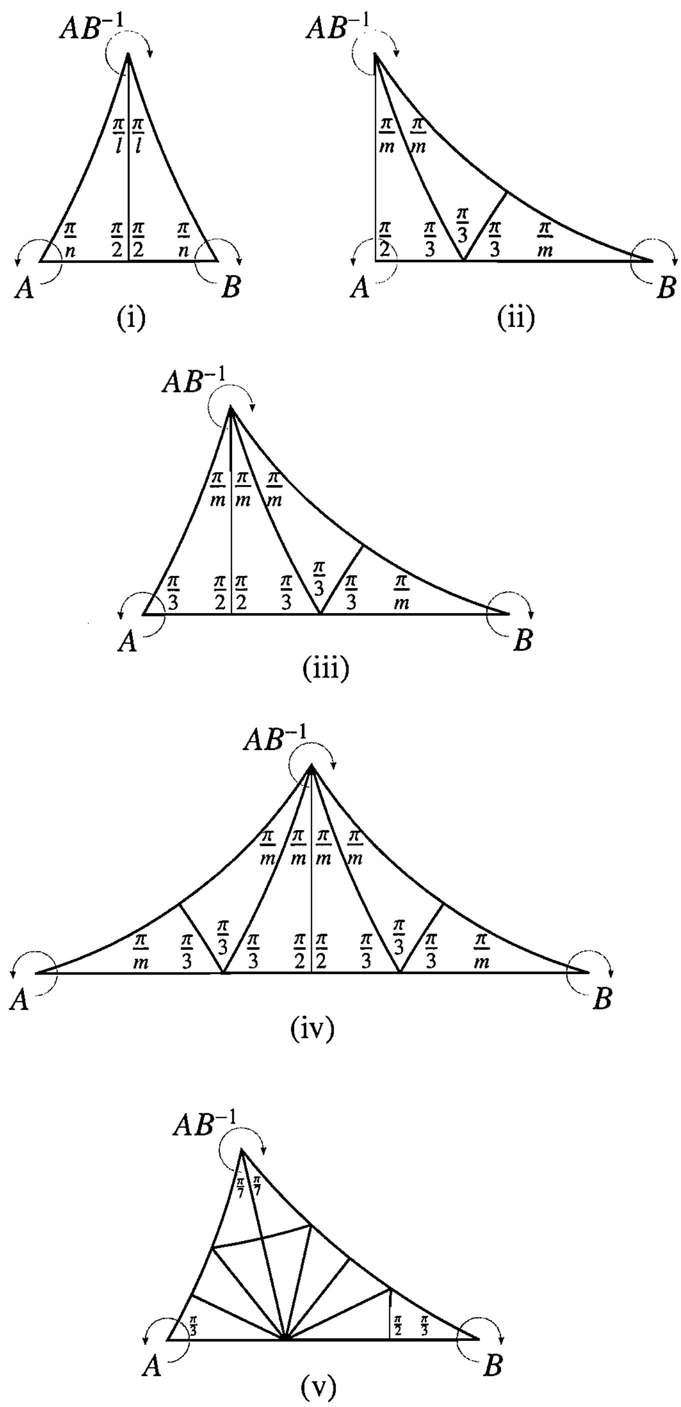
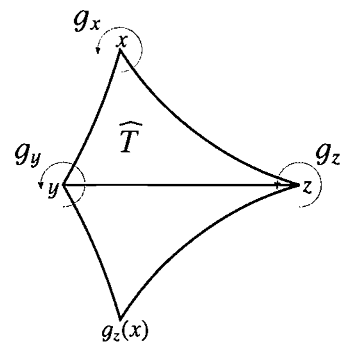


Figure 5.3. The cases (i)–(v) of Theorem 5.4.2.

In the latter case the points x, y and z form the vertices of a triangle $\widehat{T}$ whose reflection in the line segment yz is a triangle with vertices $g_z(x)$, y and z. Thus g_z is a rotation through twice the internal angle of $\widehat{T}$ at z.



Conjugations of $g_x g_y g_z$ give $g_y g_z g_x = g_z g_x g_y = 1$ and so similar arguments yield that g_x and g_y are rotations through twice the internal angle of $\widehat{T}$ at x and y respectively. ∎

This result also applies to rotations of the sphere, but we must be more careful in choosing which particular triangle to consider.

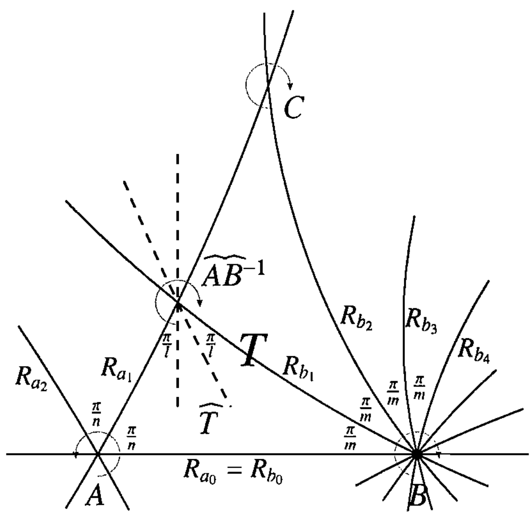
Theorem 5.4.2 describes all Fuchsian triangle groups generated by two elliptic or parabolic elements such that there is at most one imprimitive element amongst these and their product. We shall prove the following.

Theorem 5.4.4. *Let A, $B \in \mathrm{PSL}(2,\mathbb{R})$ be elliptic and normalized with orders n, m respectively. Let the product $C = AB^{-1}$ be elliptic or parabolic. Suppose that at least two of A, B and C are imprimitive. Then they generate a discrete group if and only if n is odd, $n \geqslant 7$ and they are all imprimitive with the same order n, in which case the group is a $(3,2,n)$ triangle group.*

We begin by setting out some notation. Let $\widehat{A}$ and $\widehat{B}$ be the primitive powers of A and B. Let l be the order of $\widehat{A}\widehat{B}^{-1}$ (*not the order of* $C = AB^{-1}$); let a, b and c be the fixed points of A, B and C; let T be the triangle with these fixed points as vertices; and let R_c be the reflection in the line through A and B. Let $\overline{G} = \langle \widehat{A}, \widehat{B}, R_c \rangle = \langle A, B, R_c \rangle$; then $G = \langle \widehat{A}, \widehat{B} \rangle = \langle A, B \rangle$ is the orientation preserving subgroup of $\overline{G}$ of index 2.

There are reflections in $\overline{G}$ whose axes pass through a and making angles of 0, $\pi/n, \ldots, (n-1)\pi/n$ with the line segment ab; we denote these reflections and their corresponding axes—context will make clear which we mean—by $R_{a_0}, R_{a_1}, \ldots, R_{a_{n-1}}$ so that $R_{a_0} = R_c$; similarly we denote the axes of reflection through b by $R_{b_0} = R_c$, $R_{b_1}, \ldots, R_{b_{m-1}}$.

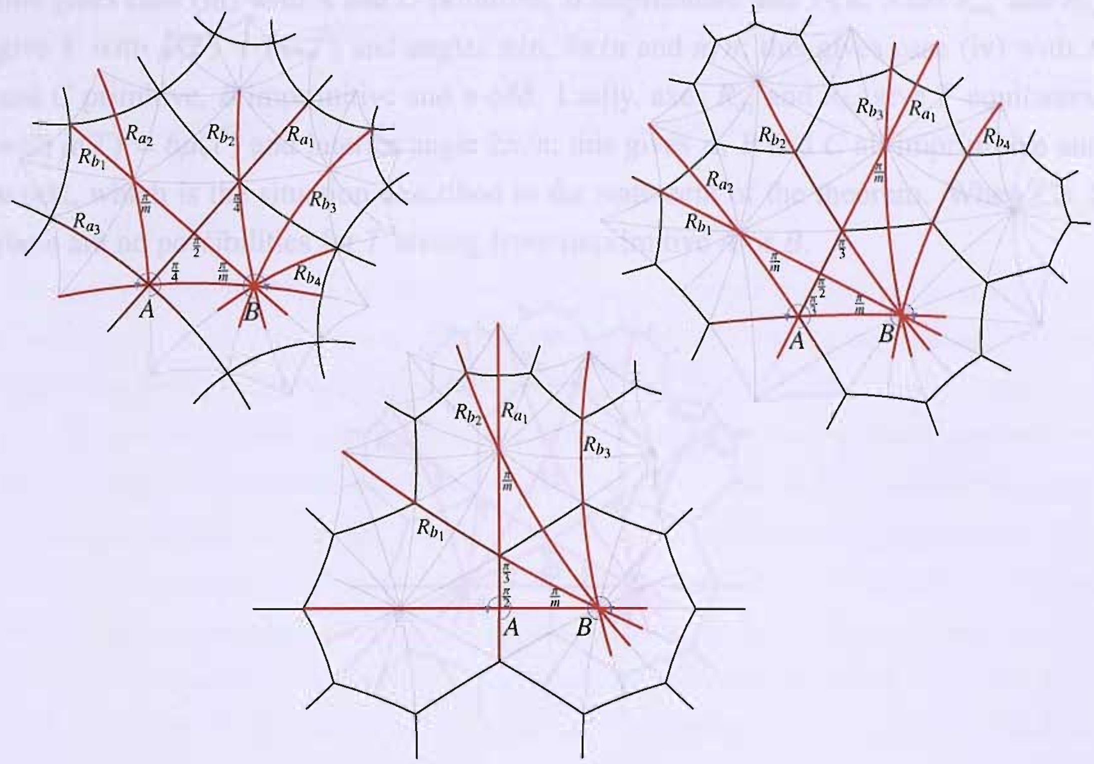
Proof. If A and B rotate through $k(2\pi/n)$ and $k'(2\pi/m)$ where k and n (respectively k' and m) are relatively prime then, by Lemma 5.4.3, the sides of T are segments of R_c, R_{a_k} and $R_{b_{k'}}$. (Hence $A = R_{b_{k'}}R_c$, $B = R_{a_k}R_c$ and $C = R_{b_{k'}}R_{a_k}$.) The lines $R_{a_1}, \ldots, R_{a_{k-1}}$ pass through the interior of T, as do $R_{b_1}, \ldots, R_{b_{k'-1}}$. Since at least one of A and B is imprimitive, either $k > 1$ or $k' > 1$ and so the fixed point d of $\widehat{AB}^{-1}$ (which, by the lemma, is the intersection point of R_{a_1} and R_{b_1}) lies on or inside T. Let $\widehat{T}$ denote the triangle whose vertices are a, b and d. We suppose that G is discrete, and we let $\overline{T}$ denote a fundamental triangle of $\overline{G}$. One of the situations of Theorem 5.4.2 now applies to $\widehat{A}$ and $\widehat{B}$.



Consider first the case where $\widehat{AB}^{-1}$ is primitive (and hence elliptic). In this case, $\widehat{T} = \overline{T}$ and $\overline{G}$ is the extended triangle group $\overline{\Delta} = \Delta[n, m, l]$. We determine which axes through a and b intersect to give a candidate for the triangle T in $\mathscr{U}_{\Delta[n,m,l]}$. We assume that $n \leqslant m$; the case $m \leqslant n$ is entirely similar and gives the same results.

If $n \geqslant 3$ then for R_{a_1} to intersect R_{b_2} or for R_{a_2} to intersect R_{b_1} we must have $l = 2$ (and hence $1/m + 1/n < 1/2$). Then R_{a_1} intersects R_{b_2} to give T with $\mu(T) = 2\mu(\overline{T})$ and angles π/n, $2\pi/m$ and π/n (see figure below); this gives two primitive generators A and C (and if B is imprimitive then m must be odd and we have simply arrived at case (i) of Theorem 5.4.2). Axes R_{a_2} and R_{b_1} intersect to give T with $\mu(T) = 2\mu(\overline{T})$

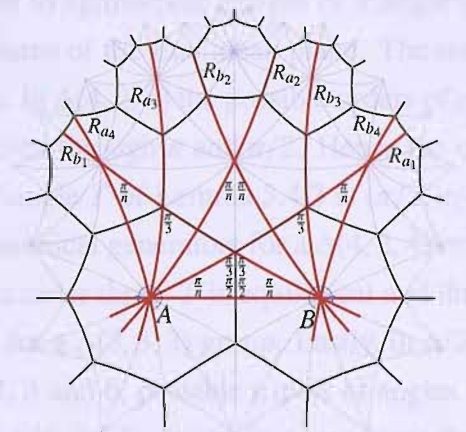
and angles π/m, $2\pi/n$ and π/m; this gives two primitive generators B and C (and if A is imprimitive then n must be odd and we have case (i) again). Axes R_{a_2} and R_{b_2} do not intersect.



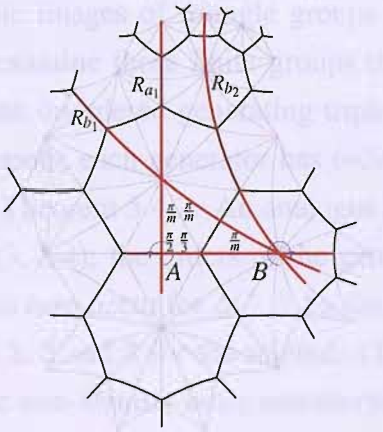
If $n = 3$ there is a further intersection: axes R_{a_1} and R_{b_3} give T with $\mu(T) = 4\mu(\overline{T})$ and angles $\pi/3$, $3\pi/m$ and π/m; this gives two primitive generators A and C (and if B is imprimitive then $3 \nmid m$ and we have case (iii) of Theorem 5.4.2). If $n = 2$ then m, $l \geqslant 3$ and for R_{a_1} and R_{b_2} to intersect we must have $l = 3$ and hence $m \geqslant 7$, giving T with $\mu(T) = 3\mu(\overline{T})$ and angles $\pi/2$, $2\pi/m$ and π/m; this gives two primitive generators A and C (and if B is imprimitive then m must be odd, giving case (ii)). The axes R_{a_1} and R_{b_3} do not intersect. To summarize, if $\widehat{AB^{-1}}$ is primitive then so is one of A and B.

Now consider the situation where $\widehat{AB^{-1}}$ is imprimitive according to case (i) of Theorem 5.4.2. We have $\mathscr{U}_{\Delta[n,2,l]}$ with fundamental region $\overline{T}$, and $\widehat{T}$ is a configuration consisting of two adjacent copies of $\overline{T}$. The vertices a and b of $\widehat{T}$ subtend angles of π/n, while the third vertex d (the fixed point of $\widehat{AB^{-1}}$) subtends an angle of $2\pi/l$. We determine which axes through a and b intersect to give a triangle T containing $\widehat{T}$, and we thus infer the possible identities of A and B.

We know that l is odd. If $l = 3$ then $n \geqslant 7$ and there are four distinct possibilities for T up to interchanging A and B. Axes R_{a_1} and R_{b_2} give T with $\mu(T) = 3\mu(\overline{T})$ and angles π/n, $2\pi/n$ and $\pi/2$; this gives case (ii) with A and C primitive, B imprimitive and n odd. Axes R_{a_1} and R_{b_3} give T with $\mu(T) = 4\mu(\overline{T})$ and angles π/n, $3\pi/n$ and $\pi/3$; this gives case (iii) with A and C primitive, B imprimitive and $3 \nmid n$. Axes R_{a_1} and R_{b_4} give T with $\mu(T) = 6\mu(\overline{T})$ and angles π/n, $4\pi/n$ and π/n; this gives case (iv) with A and C primitive, B imprimitive and n odd. Lastly, axes R_{a_2} and R_{b_2} give T equilateral with $\mu(T) = 6\mu(\overline{T})$ and interior angle $2\pi/n$; this gives A, B and C all imprimitive and n odd, which is the situation described in the statement of the theorem. When $l \geqslant 5$ there are no possibilities for T arising from imprimitive A or B.



Now consider the situation where $\widehat{AB^{-1}}$ is imprimitive according to case (ii) of Theorem 5.4.2. We have $\mathscr{U}_{\Delta[m,2,3]}$ with fundamental region $\overline{T}$, and $\widehat{A}$ and $\widehat{B}$ inducing two vertices of $\widehat{T}$ subtending angles of $\pi/2$ and π/m, with $\widehat{AB^{-1}}$ inducing the third subtending an angle of $2\pi/m$ so that $\mu(\widehat{T}) = 3\mu(\overline{T})$. We know that m is odd and $m \geqslant 7$; there are no possibilities for T arising from imprimitive A or B.



We may proceed in a similar manner to examine the situations where $\widetilde{AB^{-1}}$ is imprimitive according to cases (ii)–(v) of Theorem 5.4.2. We find that the only possibilities for T are those already given by the cases of that theorem. ∎

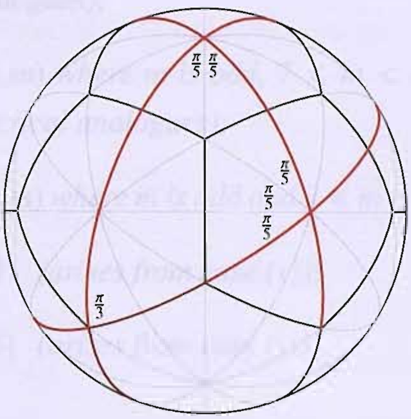
Remark 5.4.5. This result shows that there is just one infinite family of possibilities for A and B with elliptic product AB^{-1} that is not covered by the cases of Theorem 5.4.2; this situation is derived from case (i) and so we refer to it as case (i′). ♦

Euclidean triangle groups

We now turn our attention to epimorphic images of triangle groups as discrete groups of conformal automorphisms of the Euclidean plane. The results we seek may be obtained directly as follows. In $\Delta(4, 2, 4)$ the possible orders of elliptic elements are 2 and 4 corresponding to rotations through π and $\pi/2$. Hence the only possible (unordered) triple of angles for the triangle T of Lemma 5.4.3 is $(\pi/2, \pi/4, \pi/4)$, and so the corresponding rotations are canonical generators for a $\Delta(4, 2, 4)$ triangle group. In $\Delta(3, 3, 3)$ every elliptic element has order three, T is equilateral and the corresponding rotations are canonical generators for a $\Delta(3, 3, 3)$ group. Lastly, in $\Delta(3, 2, 6)$ the possible orders of elliptic elements are 2, 3 and 6; possible triples of angles for T are $(\pi/2, \pi/3, \pi/6)$, $(\pi/3, \pi/3, \pi/3)$ and $(\pi/6, \pi/6, 2\pi/3)$. In the first two cases the corresponding rotations are canonical generators for $\Delta(3, 2, 6)$ and $\Delta(3, 3, 3)$. In the third case the rotation of order 3 is imprimitive; T has angles $(\pi/6, \pi/3, \pi/6)$ and is the union of two triangles with angles $(\pi/2, \pi/3, \pi/6)$. This is the analogue in the Euclidean plane of case (i) of Theorem 5.4.2, and the group generated is a $\Delta(3, 2, 6)$ group.

Spherical triangle groups

To determine the epimorphic images of triangle groups Δ within spherical triangle groups Γ, we may simply examine these finite groups themselves. Up to automorphism action there is just one unordered generating triple $(A,\ B,\ AB^{-1})$ for $\Delta(3, 2, 3)$ containing imprimitive elements; each generator has order 3, and this is an analogue on the sphere of case (i) of Theorem 5.4.2. An analogue of this case is also the only imprimitive situation for $\Delta(3, 2, 4)$; the orders of the generators are 4, 4 and 3 here. Two further analogues of this case occur for $\Delta(3, 2, 5)$; the orders of the generators are 3, 3 and 5 for the first, and 5, 5 and 3 for the second. (The latter triple is the image under a representative of the non-identity outer automorphism of $\Delta(3, 2, 5)$ of a triple which gives rise to an analogue of case (iii) of Theorem 5.4.2.) Next, there is an analogue of case (ii); the orders of the generators are 5, 5 and 2. Then there is an analogue of case (i′), in which the generators are all of order 5. One final imprimitive situation arises for this group, and it is not the analogue of any Euclidean or Fuchsian case. The generators have orders 5, 5 and 3 as shown below.



There are no generating triples (A, B, AB^{-1}) for $\Delta(n, 2, 2)$ containing imprimitive rotations.

Remark 5.4.6. We refer to the spherical case pictured above as case (vi). ♦

Summary of results

The results of the investigation can now be summarized as follows.

Theorem 5.4.7. *The epimorphisms in the following list, together with isomorphisms between triangle groups, are sufficient to describe, through composition, any epimorphism $\Delta \twoheadrightarrow \Gamma$ between triangle groups which takes parabolic elements to parabolic or elliptic elements.*

- $\Delta(m_0, m_1, m_2) \twoheadrightarrow \Delta(1, d, 1)$ *for d dividing $\prod p^{\varepsilon_p}$ where, for each prime p, ε_p is the median value of the three exponents δ_p, ε_p, ζ_p of the largest powers of p dividing m_0, m_1, m_2 (the cyclic group case);*
- $\Delta(m_0, m_1, m_2) \twoheadrightarrow \Delta(m_0'', m_1'', m_2'')$ *where $m_i'' \mid m_i$ for each i (the direct derivative case);*

- $\Delta(n,n,l) \twoheadrightarrow \Delta(n,2,l)$ *where l is odd and $l, n \in \{2,3,\dots\}$ (arises from case (i) of Theorem 5.4.2 and its Euclidean and spherical analogues);*
- $\Delta(n,n,n) \twoheadrightarrow \Delta(n,2,3)$ *where n is odd and $7 \leqslant n < \infty$ (arises from case (i′));*
- $\Delta(2,m,m) \twoheadrightarrow \Delta(2,3,m)$ *where m is odd and $5 \leqslant m < \infty$ (arises from case (ii) and its spherical analogues);*
- $\Delta(3,m,m) \twoheadrightarrow \Delta(2,3,m)$ *where m is odd, $7 \leqslant m < \infty$ and $3 \nmid m$ (arises from case (iii) and its spherical analogues);*
- $\Delta(m,m,m) \twoheadrightarrow \Delta(2,3,m)$ *where m is odd and $7 \leqslant m < \infty$ (arises from case (iv));*
- $\Delta(3,7,7) \twoheadrightarrow \Delta(2,3,7)$ *(arises from case (v));*
- $\Delta(3,5,5) \twoheadrightarrow \Delta(3,2,5)$ *(arises from case (vi)).* ∎

5.4.2. Functors from triangle group surjections

Direct derivatives of hypermaps

Let Δ be the triangle group given by $\Delta(m_0,m_1,m_2) = \langle x_0,x_1,x_2 \mid x_0^{m_0} = x_1^{m_1} = x_2^{m_2} = x_0x_1x_2 = 1\rangle$. We have seen in §5.2 that each triangle group Γ satisfying

$$\Gamma = \Delta(m_0'',m_1'',m_2'') = \langle y_0,y_1,y_2 \mid y_0^{m_0''} = y_1^{m_1''} = y_2^{m_2''} = y_0y_1y_2 = 1\rangle$$

where $m_i'' \mid m_i$ $(0 \leqslant i \leqslant 2)$ is the image of an epimorphism ϕ which maps a canonical generator x_i of Δ onto a canonical generator y_i of Γ. This epimorphism induces a direct derivative functor $\Phi\colon C_{(m_0,m_1,m_2)} \to C_{(m_0'',m_1'',m_2'')}$ which sends i-components to i-components.

The kernel K of ϕ is the normal closure of the set $\{x_i^{m_i''} \mid 0 \leqslant i \leqslant 2\}$; this is the hypermap subgroup in Δ for a hypermap $\mathscr{K}$. As discussed in §5.1, the image under Φ of a hypermap $\mathscr{H}$ with hypermap subgroup H in Δ is the same as that of the largest hypermap $\overline{\mathscr{H}}$ (with hypermap subgroup HK in Δ) covered by both $\mathscr{K}$ and $\mathscr{H}$. Moreover, in the case of these direct derivative epimorphisms, it is clear from the point of view of hypermaps as permutation representations of triangle groups that HK and $\phi(HK) = \phi(H)$ are the hypermap subgroups for $\overline{\mathscr{H}}$ in Δ and Γ respectively. It follows that all hypermaps of type dividing (m_0'',m_1'',m_2'') are invariant under Φ. Hence $\mathscr{K}$ is the universal hypermap $\mathscr{U}_\Gamma$ corresponding to Γ, and $\Phi(\mathscr{H}) = \overline{\mathscr{H}}$ is the largest quotient of $\mathscr{H}$ of type dividing (m_0'', m_1'', m_2'').

Example 5.4.8. Let $\Delta = \Delta(3,2,10) = \langle x_0, x_1, x_2 \mid x_0^3 = x_1^2 = x_2^{10} = x_0x_1x_2 = 1 \rangle$, $\Gamma = \Delta(3,2,5) = \langle y_0, y_1, y_2 \mid y_0^3 = y_1^2 = y_2^5 = y_0y_1y_2 = 1 \rangle \cong A_5$, and let ϕ be the epimorphism induced by $x_i \mapsto y_i$ $(0 \leqslant i \leqslant 2)$ so $x_2^5 \mapsto 1$. Γ determines the category $C_{(3,2,5)}$ consisting of the nine quotients of the dodecahedron, and ϕ determines a covering of the dodecahedron by the universal map $\mathscr{U}_\Delta$, branched at the face centres. This is a covering of infinite degree, although locally (at the level of an individual face of $\mathscr{U}_\Delta$) it is a degree 2 covering resulting from a half-turn about the face centre. The relation $\leftrightarrow$ on the set of faces of $\mathscr{U}_\Delta$ defined by $f \leftrightarrow f'$ if and only if the faces f, f' are incident with opposite edges of a third face gives rise to twelve equivalence classes; these correspond to the classes of faces covering the twelve faces of the dodecahedron.

Since A_5 is a simple group, the image of any regular hypermap in $C_{(3,2,10)}$ which covers the dodecahedron is the dodecahedron, while all other regular hypermaps are sent to the trivial map and hence cover no non-trivial quotients of the dodecahedron.▲

In general, to construct the image of any hypermap $\mathscr{H}$ of type dividing (m_0, m_1, m_2) under a direct derivative functor Φ, we consider the permutation representation of Δ which determines $\mathscr{H}$, take each of the three elements $\rho_i^{m_i''}$ of the monodromy group of $\mathscr{H}$ (where ρ_i permutes the hyperdarts in cyclic order around the i-components of $\mathscr{H}$), and identify the hyperdarts which form a cycle in any of these permutations. In other words, for each hypermap component c of dimension i and valency v of $\mathscr{H}$ we identify incident hyperdarts which are consecutively m_i'' apart around it to obtain an i-component of valency $h = \mathrm{HCF}(v, m_i'')$. Unless $\mathscr{H}$ already has type dividing (m_0'', m_1'', m_2''), this process gives a covering of $\Phi(\mathscr{H})$ by Φ which has infinite degree when Δ is Euclidean or Fuchsian, but which has degree $\frac{v}{h}$ at the level of c.

While direct derivative functors send regular hypermaps to regular hypermaps, they are not so well-behaved with respect to other properties of hypermaps. The size of the automorphism group may decrease or increase: if $\mathscr{H}$ is regular and distinct from $\overline{\mathscr{H}}$ then its automorphism group is of course larger than that of $\overline{\mathscr{H}}$, whilst in the category $C_{(3,2,4)}$ the irregular 12-dart map with automorphism group of order 4 is sent to the unique regular 6-dart map of $C_{(3,2,2)}$ by the functor which reduces face valency. Next, if $\mathscr{H}_1$ and $\mathscr{H}_2$ are hypermaps within an infinite category $C_{(m_0,m_1,m_2)}$ and $\mathscr{H}_1$ covers $\mathscr{H}_2$ with degree n, then we may ask how the degree n' of the covering of $\Phi(\mathscr{H}_2)$ by $\Phi(\mathscr{H}_1)$

compares with n. Certainly $n' \leqslant n$, since if there are n cosets $H_1 g$ in H_2/H_1 then there are at most n sets $H_1 g' K = H_1 K k g'$ in $KH_2/H_1K = H_2K/H_1K$. On the other hand, the following example shows that n may be arbitrarily large compared to n', even when $n' = 1$ and $\Phi(\mathcal{H}_1) = \Phi(\mathcal{H}_2)$.

Example 5.4.9. Take the functor $\Phi\colon C_{(3,2,10)} \to C_{(3,2,5)}$ of Example 5.4.8. The dodecahedron has a reflexible double cover $\mathcal{M}_0$ of type $(3, 2, 10)$ [34]; this map has twelve 10-valent faces on a surface of genus $g = 5$. Its canonical map subgroup M_0 in $\Delta = \Delta(3, 2, 10)$ is torsion-free since $\mathcal{M}_0$ is uniform, and so M_0 is a surface group, free of rank $2g - 1 = 9$. Clearly such a group has a subgroup M_1 of index 2 (say): the kernel of any epimorphism onto a group of order 2. This is the map subgroup in Δ of a uniform map $\mathcal{M}_1$ of type $(3, 2, 10)$. M_1 has rank $2(9 - 1) + 1 = 17$ and so $\mathcal{M}_1$ lies on a surface of genus 9. This argument can be repeated arbitrarily often; the map $\mathcal{M}_n$ is uniform of type $(3, 2, 10)$ on a surface of genus $2^{n+2} + 1$. It is a degree 2^n covering of $\mathcal{M}_0$ or equivalently, $|M_0 : M_n| = 2^n$. However, both $\mathcal{M}_0$ and $\mathcal{M}_n$ cover the dodecahedron, which is thus the image of each under Φ. ▲

Indirect derivatives of hypermaps

It remains to determine the functors corresponding to the epimorphisms of Theorem 5.4.7 under which the image of a canonical generating triple for a non-cyclic triangle group $\Delta = \Delta(m_0, m_1, m_2)$ is not itself canonical. There are seven cases to consider.

(i) $\Delta(n, n, l) \twoheadrightarrow \Delta(n, 2, l)$ with l odd and $l, n \in \{2, 3, \dots\}$

Consider inclusion (a) of §4.5.1: $\Delta(n, n, \frac{l}{2}) \lhd_2 \Delta(n, 2, l)$ where l is even. By setting l even in the configuration for case (i) in Figure 5.3, we see a fundamental triangle for $\Delta[n, n, \frac{l}{2}]$ as the union of two fundamental triangles for $\Delta[n, 2, l]$. The functor induced by the inclusion is W, which is applied to a hypermap by forgetting the vertex colours of its Walsh representation. When l is odd rather than even, this construction 'folds up' to describe the functor Φ induced by our epimorphism $\phi\colon \Delta(n, n, l) = \langle\, x, y, z \mid x^n = y^n = z^l = xyz = 1 \,\rangle \twoheadrightarrow \langle\, p, q, r \mid p^n = q^2 = r^l = pqr = 1 \,\rangle = \Delta(n, 2, l)$ as follows. One realization of ϕ is given by

$$\begin{aligned} x &\mapsto p \\ y &\mapsto qpq \\ (xy)^{-1} = z &\mapsto r^2 = (pq)^{-2}. \end{aligned}$$

This is a composition of inclusion (a) given by $x \mapsto a$, $y \mapsto bab$, $z \mapsto c^2$ (where $\Delta(n,2,2l) = \langle a,b,c \mid a^n = b^2 = c^{2l} = abc = 1 \rangle$) with the direct derivative epimorphism $\Delta(n,2,2l) \to \Delta(n,2,l)$ given by $a \mapsto p$, $b \mapsto q$, $c \mapsto r$. It follows that Φ is simply the composition of W with a direct derivative functor; it is applied to a hypermap $\mathscr{H}$ of type dividing (n,n,l) where l is odd and $1/l + 1/n \geqslant 1/2$ by identifying, around each face of W($\mathscr{H}$), darts which are l edges apart.

(i′) $\Delta(n,n,n) \twoheadrightarrow \Delta(3,2,n)$ with n odd, $7 \leqslant n < \infty$

Consider inclusion (c) of §4.5.1: $\Delta(\frac{n}{2},\frac{n}{2},\frac{n}{2}) \lhd_6 \Delta(3,2,n)$ where n is even. By setting n even in the configuration for case (i′) illustrated within the proof of Theorem 5.4.4 on page 102, we see a fundamental triangle for $\Delta[\frac{n}{2},\frac{n}{2},\frac{n}{2}]$ as the union of six fundamental triangles for $\Delta[n,2,n]$. The functor induced by the inclusion is J, which is applied to a hypermap by forgetting the face colours of its James representation. When n is odd rather than even, this construction folds up, in a manner similar to the previous case, to describe the functor Φ induced by our epimorphism $\phi\colon \Delta(n,n,n) = \langle x,y,z \mid x^n = y^n = z^n = xyz = 1 \rangle \twoheadrightarrow \langle p,q,r \mid p^3 = q^2 = r^n = pqr = 1 \rangle = \Delta(3,2,n)$. One realization of ϕ is given by

$$\begin{aligned} x &\mapsto (prp^{-1})^2 \\ y &\mapsto (qrq)^2 \\ (xy)^{-1} = z &\mapsto r^2 = (pq)^{-2}. \end{aligned}$$

This is a composition of inclusion (c) given by $x \mapsto (aca^{-1})^2$, $y \mapsto (bcb)^2$, $z \mapsto c^2$ (where $\Delta(3,2,2n) = \langle a,b,c \mid a^3 = b^2 = c^{2n} = abc = 1 \rangle$) with the direct derivative epimorphism $\Delta(3,2,2n) \to \Delta(3,2,n)$ given by $a \mapsto p$, $b \mapsto q$, $c \mapsto r$. It follows that Φ is simply the composition of J with a direct derivative functor; it is applied to a hypermap $\mathscr{H}$ of type dividing (n,n,n) where n is odd and $n \geqslant 7$ by identifying, around each face of J($\mathscr{H}$), darts which are n edges apart.

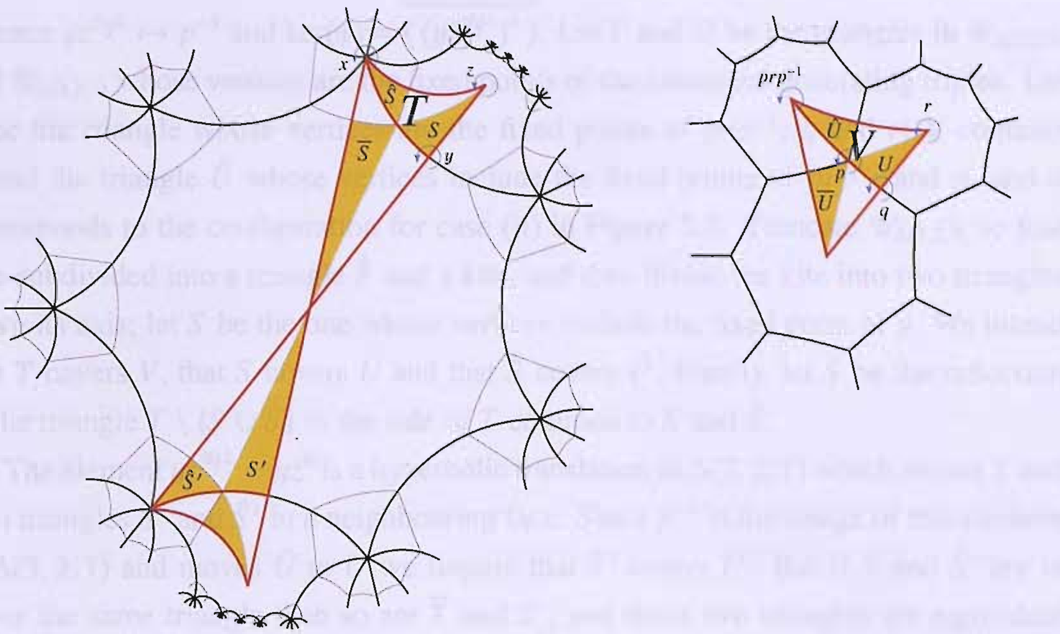


FIGURE 5.4. An application of the indirect derivative functor induced by the homomorphism of case (ii).

(ii) $\Delta(m, 2, m) \twoheadrightarrow \Delta(3, 2, m)$ with m odd, $5 \leqslant m < \infty$

Consider inclusion (K) of §4.5.1: $\Delta(m, 2, \frac{m}{2}) <_3 \Delta(3, 2, \frac{m}{2})$ where m is even. By setting m even in the configuration for case (ii) in Figure 5.3, we see a fundamental triangle for $\Delta[m, 2, \frac{m}{2}]$ as the union of three fundamental triangles for $\Delta[3, 2, m]$. The functor induced by the inclusion is applied to a map $\mathscr{M}$ by truncating its vertices. When m is odd rather than even, this construction folds up to describe the functor Φ induced by our epimorphism ϕ: $\Delta(m, 2, m) = \langle\, x, y, z \mid x^m = y^2 = z^m = xyz = 1 \,\rangle \twoheadrightarrow \langle\, p, q, r \mid p^3 = q^2 = r^m = pqr = 1 \,\rangle = \Delta(3, 2, m)$, in a sense that we now make clear.

Figure 5.4 shows parts of the universal maps for $\Delta(m, 2, m) = \langle\, x, y, z \mid x^m = y^2 = z^m = xyz = 1 \,\rangle$ and $\Delta(3, 2, m) = \langle\, p, q, r \mid p^3 = q^2 = r^m = pqr = 1 \,\rangle$ in the case $m = 7$. Φ describes a particular covering (which is branched over the face centres) of the latter by the former as follows. One realization of the epimorphism ϕ is given by

$$x \mapsto prp^{-1}$$

$$y \mapsto q$$

$$(xy)^{-1} = z \mapsto r^2 = (pq)^{-2},$$

whence $yz^{\frac{m+1}{2}} \mapsto p^{-1}$ and $\ker(\phi) = \overline{\langle (yz^{\frac{m+1}{2}})^3 \rangle}$. Let T and U be the triangles in $\mathscr{U}_{\Delta(7,2,7)}$ and $\mathscr{U}_{\Delta(3,2,7)}$ whose vertices are the fixed points of the canonical generating triples. Let V be the triangle whose vertices are the fixed points of prp^{-1}, q and r; V contains U and the triangle $\hat{U}$ whose vertices include the fixed points of prp^{-1} and p, and it corresponds to the configuration for case (ii) in Figure 5.3. Truncate $\mathscr{U}_{\Delta(7,2,7)}$ so that T is subdivided into a triangle $\hat{S}$ and a kite, and then divide the kite into two triangles down its axis; let S be the one whose vertices include the fixed point of y. We intend that T covers V, that S covers U and that $\hat{S}$ covers $\hat{U}$. Finally, let $\overline{S}$ be the reflection of the triangle $T \setminus \{S \cup \hat{S}\}$ in the side of T common to S and $\hat{S}$.

The element $yz^{\frac{m+1}{2}} = yz^4$ is a hyperbolic translation in $\Delta(7,2,7)$ which moves S and $\hat{S}$ to triangles S' and $\hat{S}'$ in a neighbouring face. Since p^{-1} is the image of this element in $\Delta(3,2,7)$ and moves $\hat{U}$ to U we require that $\hat{S}'$ covers U. But if S and $\hat{S}'$ are to cover the same triangle then so are $\overline{S}$ and S'; and these two triangles are equivalent under a rotation through 7 edges about the centre of their common 14-valent face. We may argue in the same way for the other triangles of the maps, and we see that the covering is described by identifying, around each 14-valent face of the truncation of $\mathscr{U}_{\Delta(7,2,7)}$, darts which are 7 edges apart to obtain $\mathscr{U}_{\Delta(3,2,7)}$. (We may verify that T and its image under $(yz^{\frac{m+1}{2}})^3$ are identified, as we require.)

In general, Φ is applied to a regular map $\mathscr{M}$ of type dividing $(m,2,m)$ by truncating it and then, around each face which originates from a face of $\mathscr{M}$, identifying darts which are m edges apart. Unlike in the previous two cases, Φ in this case and the ones which follow cannot be expressed simply as the composition of the inclusion functor with a direct derivative functor. Rather, once the inclusion functor has been applied, the direct derivative construction must only be applied to a subset of the hypermap components.

(iii) $\Delta(m,3,m) \twoheadrightarrow \Delta(3,2,m)$ with m odd, $7 \leqslant m < \infty$, $3 \nmid m$

Consider inclusion (J) of §4.5.1: $\Delta(m,3,\frac{m}{3}) <_4 \Delta(3,2,m)$ where $3 \mid m$. By setting m as a multiple of 3 in the configuration for case (iii) in Figure 5.3, we see a fundamental triangle for $\Delta[m,3,\frac{m}{3}]$ as the union of four fundamental triangles for $\Delta[3,2,m]$. The functor induced by the inclusion is applied to a hypermap $\mathscr{H}$ by replacing each hyperedge of the Cori representation of $\mathscr{H}$ with its stellation, that is, by placing a vertex in the centre of each hyperedge, joining it by a new edge to each bounding vertex, and then deleting the bounding edges. When $3 \nmid m$ this construction folds up to describe the functor Φ induced by our epimorphism $\phi\colon \Delta(m,3,m) = \langle x,y,z \mid x^m = y^3 = z^m = xyz = 1 \rangle \twoheadrightarrow \langle p,q,r \mid p^3 = q^2 = r^m = pqr = 1 \rangle = \Delta(3,2,m)$; one realization of the epimorphism ϕ is given by

$$\begin{aligned} x &\mapsto p \\ y &\mapsto (qr^{-1})^{-1} r (qr^{-1}) \\ (xy)^{-1} = z &\mapsto r^3 = (pq)^{-3}, \end{aligned}$$

whence $z^\alpha x \mapsto q$ and $\ker(\phi) = \overline{\langle (z^\alpha x)^2 \rangle}$ where $\alpha = \mathbb{Z} \bigcap \{\frac{m+1}{3}, \frac{2m+1}{3}\}$, and a similar argument to that used in the previous case shows that Φ is applied to a regular hypermap $\mathscr{H}$ of type dividing $(m,3,m)$ by performing the stellation procedure and then, around each face which originates from a hyperface of $\mathscr{H}$, identifying darts which are m edges apart.

(iv) $\Delta(m,m,m) \twoheadrightarrow \Delta(3,2,m)$ with m odd, $7 \leqslant m < \infty$

Consider inclusion (H) of §4.5.1: $\Delta(m,m,\frac{m}{4}) <_6 \Delta(3,2,m)$ where $4 \mid m$; it is the composition of two inclusions (a) and (K). By setting m as a multiple of 4 in the configuration for case (iv) in Figure 5.3, we see a fundamental triangle for $\Delta[m,m,\frac{m}{4}]$ as the union of six fundamental triangles for $\Delta[3,2,m]$ (and as the composition of the configuration for case (i) with that for case (ii)). Indeed, one realization of the epimorphism $\phi\colon \Delta(m,m,m) = \langle x,y,z \mid x^m = y^m = z^m = xyz = 1 \rangle \twoheadrightarrow \langle p,q,r \mid p^3 = q^2 = r^m = pqr = 1 \rangle = \Delta(3,2,m)$ induced here is given by

$$\begin{aligned} x &\mapsto prp^{-1} \\ y &\mapsto (qp) r (qp)^{-1} \\ (xy)^{-1} = z &\mapsto r^4 = (pq)^{-4}, \end{aligned}$$

which is the composition of the epimorphisms given by $x \mapsto a$, $y \mapsto bab$, $z \mapsto c^2$ and $a \mapsto prp^{-1}$, $b \mapsto q$, $c \mapsto r^2$ from cases (i) and (ii) discussed above (where $\Delta(m,2,m) = \langle a,b,c \mid a^m = b^2 = c^m = abc = 1 \rangle$). Correspondingly, the functor Φ induced by ϕ is the composition of the functors from cases (i) and (ii). To apply it to a regular hypermap $\mathscr{H}$ of type dividing (m,m,m), we may choose to leave all edge identifications until last by truncating the Walsh representation $\mathrm{W}(\mathscr{H})$ of $\mathscr{H}$ and then, around each face which originates from a hyperface of $\mathscr{H}$, identifying darts which are

m edges apart.

(v) $\Delta(3,7,7) \twoheadrightarrow \Delta(3,2,7)$

Although not quite so closely related as the previous cases to the inclusions of §4.5.1, this case does bear a strong similarity to inclusion (E), $\Delta(8,8,3) <_{10} \Delta(3,2,8)$, discussed in §4.5.1 and pictured in Figure 4.6. Here, one realization of our epimorphism $\phi\colon \Delta(3,7,7) = \langle x,y,z \mid x^3 = y^7 = z^7 = xyz = 1 \rangle \twoheadrightarrow \langle p,q,r \mid p^3 = q^2 = r^7 = pqr = 1 \rangle = \Delta(3,2,7)$ is given by

$$\begin{aligned} x &\mapsto (qr^4)p(qr^4)^{-1} \\ y &\mapsto r \\ z &\mapsto ((qpqp)^{-1}r(qpqp))^2, \end{aligned}$$

whence $z^3y^{-1}z^3 \mapsto p$, $yz^3y^{-1}z^3 \mapsto q$ and $\ker(\phi) = \overline{\langle (z^3y^{-1}z^3)^3, (yz^3y^{-1}z^3)^2 \rangle}$; a similar argument to that used in earlier cases leads to a description of Φ. However, to illustrate the similarity with the functor arising from inclusion (E), we first perform a Machí operation and then we apply our functor to a regular hypermap $\mathcal{H}$ of type $(7,7,3)$ as follows: we place an n-gon inside each hypervertex in the Cori representation of $\mathcal{H}$, joining it by a new edge to each bounding vertex; then we semi-edge-stellate each hyperface by placing a vertex in the centre and joining it by new edges to the midpoints of the bounding edges which separate the hyperface from hyperedges; then we delete the edges which bound the hypervertices; and finally, around each 14-gon arising from the hyperedges, we identify darts which are 7 edges apart.

(vi) $\Delta(3,5,5) \twoheadrightarrow \Delta(5,2,5)$

This case does not relate to an inclusion between triangle groups, yet it involves familiar constructions. One realization of our epimorphism $\phi\colon \Delta(3,5,5) = \langle x,y,z \mid x^3 = y^5 = z^5 = xyz = 1 \rangle \twoheadrightarrow \langle p,q,r \mid p^3 = q^2 = r^5 = pqr = 1 \rangle = \Delta(3,2,5)$ is given by

$$\begin{aligned} x &\mapsto (pr^{-2})p(pr^{-2})^{-1} \\ y &\mapsto (qrq)^3 \\ z &\mapsto r^2, \end{aligned}$$

whence $z^2y^2z^{-2}y^2 \mapsto p$, $y^2z^{-2}y^2 \mapsto q$, and $\ker(\phi) = \overline{\langle (z^2y^2z^{-2}y^2)^3, (y^2z^{-2}y^2)^2 \rangle}$. A similar argument to that used in previous cases shows that Φ is applied to a regular hypermap $\mathcal{H}$ of type $(3,5,5)$ as follows: first, take the James representation of $\mathcal{H}$; then semi-edge-stellate each hypervertex by placing a vertex in the centre and joining it by new edges to the midpoints of the bounding edges which separate the hypervertex from hyperedges; and finally, around each 15-gon enclosing the hyperedges and around each 10-gon arising from the hyperfaces, identify darts which are 5 edges apart.

Example 5.4.10. Let Q be Klein's quartic curve regarded as a tessellation by 24 7-gons of a genus 3 surface (see, for example, [39]); it is a regular map of type $(3,2,7)$ with 168 darts. There are indirect derivative functors $\Phi_{\mathrm{i}}, \Phi_{\mathrm{ii}}, \ldots, \Phi_{\mathrm{v}}$ and $\Phi_{\mathrm{i}'}$, arising from the cases (i)–(v) and (i′) respectively, whose range is $C_{(3,2,7)}$; and we may enquire as to the nature of the canonical preimage $\widetilde{\Phi^{-1}}(Q)$ of Q under each such functor Φ. Note that each is regular with 168 (hyper)darts.

$\widetilde{\Phi_{\mathrm{i}}^{-1}}(Q)$ has type $(3,3,7)$, with 56 hypervertices, 56 hyperedges and 24 hyperfaces; it lies on a surface of genus 17. The map $\widetilde{\Phi_{\mathrm{ii}}^{-1}}(Q)$ has type $(7,2,7)$, with 24 vertices, 84 edges and 24 faces; it lies on a surface of genus 19. The hypermaps $\widetilde{\Phi_{\mathrm{iii}}^{-1}}(Q)$ and $\widetilde{\Phi_{\mathrm{v}}^{-1}}(Q)$ both lie on a surface of genus 33; they have type $(7,3,7)$ and $(7,7,3)$ respectively. The highest genus is obtained by $\widetilde{\Phi_{\mathrm{i}'}^{-1}}(Q)$ and $\widetilde{\Phi_{\mathrm{iv}}^{-1}}(Q)$, having type $(7,7,7)$ with 24 components of each dimension, and lying on a surface of genus 49.

Using GAP [17] we can calculate the monodromy group $G \cong \mathrm{Aut}(Q) \cong \mathrm{PSL}(2,7)$ of Q and the canonical monodromy generators in G of each $\widetilde{\Phi^{-1}}(Q)$ (which are the images of x, y and z under the permutation representations defining the hypermaps $\widetilde{\Phi^{-1}}(Q)$). We can then determine whether or not the triples of canonical monodromy generators—regarded as $\mathcal{H}_2^{+}$-bases of G—lie in the same T-system of G (of which there are four: T_1, T_2, T_3, T_4 say, respectively of size 16, 16, 18, 7 with Higman invariant 4, 4, 3, 7). This tells us which of the $\widetilde{\Phi^{-1}}(Q)$ share an orbit of the group $\mathfrak{H}^{+}$ of operations on orientable hypermaps without boundary (see Chapter 3). We discover the following: $\widetilde{\Phi_{\mathrm{i}'}^{-1}}(Q)$ is the *same* genus-49 hypermap as $\widetilde{\Phi_{\mathrm{iv}}^{-1}}(Q)$, and it lies in the same orbit T_2 as $\widetilde{\Phi_{\mathrm{iii}}^{-1}}(Q)$ and hence the Petrie circuits all have length 8; $\widetilde{\Phi_{\mathrm{i}}^{-1}}(Q)$ and $\widetilde{\Phi_{\mathrm{ii}}^{-1}}(Q)$ lie in T_3 and have Petrie circuits of length 6; while $\widetilde{\Phi_{\mathrm{v}}^{-1}}(Q)$ and Q itself lie in T_1 and have Petrie circuits of length 8. In particular, we may apply Machí operations to $\widetilde{\Phi_{\mathrm{iii}}^{-1}}(Q)$ and $\widetilde{\Phi_{\mathrm{v}}^{-1}}(Q)$ to obtain two regular genus-33 hypermaps of type $(3,7,7)$ which share the same combinatorial data (i-component numbers, Petrie circuit length, automorphism group) and yet lie in different orbits of $\mathfrak{H}^{+}$. ▲

As seen in the example, it is often possible to compose two of the indirect derivative functors with Machí functors to obtain functors which share the same domain and range. However, we can see from the differing kernels of the indirect derivative epimorphisms that these functors are necessarily distinct, even if they have the same action on some non-empty subdomain.

5.5. Composition of functors

We have seen that the building blocks of operational functors between categories $C_{(m_0,m_1,m_2)}$ of hypermaps of finite type dividing (m_0, m_1, m_2) are: functors arising from automorphisms of triangle groups; functors arising from proper inclusions between triangle groups; and direct and indirect derivative direct functors arising from epimorphisms between triangle groups. We have also seen that the functors arising from the formation of topological hypermap representations restrict from the infinite period case to certain finite-period cases, as do the Machí operations. (These latter are the only map and hypermap operations which admit such restrictions.)

We may compose any number of such functors with the necessary proviso that the image of one is contained in the domain of the next. In §4.1 an operational functor was defined to be one which is induced by a homomorphism between (extended) triangle groups. Where the epimorphic image of the homomorphism is cyclic, some examples of the action of the induced functors have been given in §5.3.2. Where the epimorphic image is itself a triangle group, the homomorphism may be regarded as the composition $\phi_E \circ \phi_I$ of an epimorphism with an inclusion, which itself can be regarded as a composition $\phi_{M_1} \circ \phi_E \circ \phi_{M_2} \circ \phi_I \circ \phi_{M_1}$ where the ϕ_{M_i} are automorphisms of triangle groups, such as those giving rise to the restricted Machí functors.

Chapter 6

Higher-dimensional objects

6.1. *n*-maps

Harding [21] defines a three-dimensional topological analogue of a map (possibly with boundary) as follows. Consider a connected union of polygonal discs in a 3-manifold $\mathcal{M}$, such that their interiors are disjoint and their boundaries meet in edges and vertices. Suppose that each edge is incident with a vertex of finite valency, and that the union of vertices and edges is connected. Then the union of discs (called *faces*), edges and vertices is called a *2-graph* $\mathcal{G}$. If $\mathcal{M} \setminus \mathcal{G}$ is a disjoint union of 3-balls (called *cells*) then the structure (regarded as an embedding of $\mathcal{G}$ in $\mathcal{M}$) is called a *3-map* $\mathscr{M}$ on $\mathcal{M}$.

This definition permits 3-maps to have free edges with free ends, and free faces with free sides (which means that the boundary of a disc need not consist entirely of edges and vertices) such that each free face has just one free side which is required to be connected. The three-dimensional analogue of a blade is a *flag*: this is a tetrahedron in the barycentric subdivision of $\mathscr{M}$ (which, loosely speaking, represents mutual coincidence of a vertex, an edge, a face and a cell).

Analogous to the permutations of the blades of 2-maps described in §1.2.1, there are four involutions on the set Ω of flags of 3-maps which naturally arise: their effect is to reflect a flag β in a specified face. Informally, τ_0 'changes the vertex' by reflecting β to the flag at the other end of its edge; τ_1 'changes the edge' by reflecting β across (and on the same side of) its face; τ_2 'changes the face' by reflecting β across (and within) its cell; and τ_3 'changes the cell' by reflecting β through its face. If performing any of these reflections on β would force us to 'hit the boundary' of $\mathscr{M}$ then the corresponding involution is defined to fix β.

The involutions satisfy $(\tau_2\tau_0)^2 = (\tau_0\tau_3)^2 = (\tau_3\tau_1)^2 = 1$, corresponding to the fact that three of the dihedral angles of a tetrahedral flag are right angles. We would like to reverse the process by starting with four involutions, generating a group which is transitive on n objects and satisfying these relations, and then constructing a 3-map by considering images of a base tetrahedral flag, the action of the involutions being reflections in the appropriate faces of the flag; this is the familiar 'paste construction' that works equally well for 2-maps. Unfortunately, the resulting topological structure may not be a manifold (even one with boundary), and we find that there are a number of deficiencies with Harding's definition in the general setting. It is here that we begin to see how the agreeable unification of disparate theories of 2-maps—topological, combinatorial, algebraic, geometric—fails to hold in higher dimensions. (However, for some of the 3-maps considered below we will exhibit visual 'approximations'; these attempt to reconcile the simplicial complex approach with that of cell decompositions of manifolds by accepting structures which are more exotic than the initial topological definition allows.)

Other generalizations of the concepts and theory of maps and hypermaps to higher dimensions include the study of polytopes (such as in [14]), and the work of Tits [58] on buildings and subsequent work of Ronan [51] on coverings, carried out in the more general context of geometries and chamber systems (of which maps may be considered a "thin" case). Vince [59] approaches the subject from a combinatorial point of view in terms of edge-coloured graphs, or *combinatorial maps*. (Related concepts include the crystallizations of Gagliardi [16] and the graph-encoded maps of Lins [41].) This theory is particularly general, and is the most natural to work with for our purposes since the algebraic language used—that of Coxeter groups, Schreier representations and map subgroups—is very close to that employed in this thesis. Moreover, there is a correspondence between the set of topological 3-maps on manifolds with boundary and a certain class of combinatorial maps, while topological 3-maps of certain (very limited) types have a natural geometry arising from their representation as quotients of universal 3-maps in spherical, Euclidean and hyperbolic 3-space.

Accordingly, in this chapter we regard an n-map as being a transitive permutation representation of the following group.

Definition 6.1.1. For $n \geqslant 2$ the group $\mathscr{C}_n = \langle\, t_0, t_1, \ldots, t_n \mid t_i^2 = (t_j t_k)^2 = 1, 0 \leqslant i, j, k \leqslant n, |j-k| > 1 \,\rangle$ is the *n-cartographic group*. •

For $n = 2$, this agrees with the earlier definition of $\mathscr{C}_2$ (§1.2.1). The Coxeter diagram for $\mathscr{C}_n$ is a path on n vertices with distinct endpoints. (Such a diagram is called *linear*.) Components of dimension i correspond to orbits on the set of flags of the subgroup $\langle t_1, t_2, \ldots, \hat{t}_i, \ldots, t_n \rangle$ (where $\hat{t}_i$ denotes the omission of t_i), with incidence of components of different dimensions corresponding to non-empty intersection of these orbits.

6.2. Operations on n-maps

6.2.1. The group of operations

Generalizing the result of Jones and Thornton [35] that the outer automorphism group of $\mathscr{C}_2$ is isomorphic to S_3 (§2.1), James [23] has shown the following.

Theorem 6.2.1 (James). *Let H_n be the subgroup of automorphisms of $\mathscr{C}_n$ that is generated by θ_n: $t_i \mapsto t_{n-i}$, $0 \leqslant i \leqslant n$, and ϕ_n: $t_2 \mapsto t_0t_2$; $t_i \mapsto t_i$, $0 \leqslant i \leqslant n$, $i \neq 2$. Then* $\mathrm{Aut}(\mathscr{C}_n)$ *is a split extension of* $\mathrm{Inn}(\mathscr{C}_n)$ *by H_n. For $n \geqslant 3$, H_n is dihedral of order* 8. ✠

As discussed in §2.1, H_2 is symmetric of degree 3, permuting t_0, t_2 and t_0t_2.

We may interpret this result in terms of operations on n-maps. As was the case for 2-maps, the automorphisms for $n \geqslant 3$ induce n-map operations, and $\mathrm{Inn}(\mathscr{C}_n)$ acts trivially giving an induced action of $\mathrm{Out}(\mathscr{C}_n)$. This action is faithful: if τ_i is the permutation induced by t_i ($0 \leqslant i \leqslant n$) then the 4-flag n-map given by $\tau_0 = (1\ 4)(2\ 3)$, $\tau_j = (1\ 3)(2\ 4)$ ($1 \leqslant j \leqslant n$) lies in an orbit of size 8 for all $n \geqslant 3$. Hence the group of n-map operations is isomorphic to $\mathrm{Out}(\mathscr{C}_n)$. Table 6.1 lists representative automorphisms of $\mathscr{C}_n$; for example, the operation D induced by the automorphism θ_n is the duality operation which interchanges i-components with $(n-i)$-components.

The automorphism ϕ_n maps the subgroup $\langle t_1, t_2, t_3, \ldots, t_n \rangle$ to the subgroup $\langle t_1, t_0t_2, t_3, \ldots, t_n \rangle$. The orbits on Ω of the latter subgroup are the higher-dimensional generalizations of Petrie circuits of maps on surfaces. These *Petrie webs* are more complex than the Petrie polygons defined by Coxeter [10] for 3-maps—and generalizable to higher dimensions—as orbits of $t_0t_1t_2t_3 \ldots t_n = (t_1t_0t_2t_3 \ldots t_n)^{t_0}$ and their images under reflections. Figure 6.1 shows part of the Petrie webs for the universal cubic tessellation of Euclidean 3-space; this is discussed in more detail in §6.2.3. The Opp operation induced by ϕ_n interchanges vertices and Petrie webs but leaves the other i-components unchanged.

Operation Θ	Automorphism	Order
I	1	1
D	$t_i \mapsto t_{n-i}$	2
Opp	$t_2 \mapsto t_0 t_2,\ t_i \mapsto t_i\ (i \neq 2)$	2
D Opp	$t_2 \mapsto t_n t_{n-2},\ t_i \mapsto t_{n-i}\ (i \neq 2)$	4
D Opp D	$t_{n-2} \mapsto t_n t_{n-2},\ t_i \mapsto t_i\ (i \neq n-2)$	2
D Opp D Opp	$t_2 \mapsto t_0 t_2,\ t_{n-2} \mapsto t_n t_{n-2},\ t_i \mapsto t_i\ (i \neq 2, n-2)$	2
Opp D Opp	$t_2 \mapsto t_n t_{n-2},\ t_{n-2} \mapsto t_0 t_2,\ t_i \mapsto t_{n-i}\ (i \neq 2, n-2)$	2
Opp D	$t_{n-2} \mapsto t_0 t_2,\ t_i \mapsto t_{n-i}\ (i \neq n-2)$	4

Table 6.1. The eight n-map operations ($n \geqslant 3$).

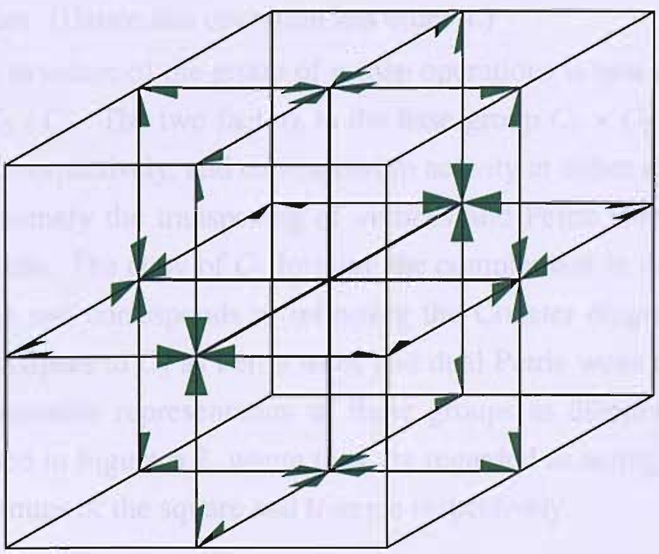


Figure 6.1. Part of a Petrie web for the universal cubic tessellation of Euclidean 3-space. (The green right-angled triangles are the intersections of the constituent flags with their incident faces, and hence each represents two flags, one in each incident cell.)

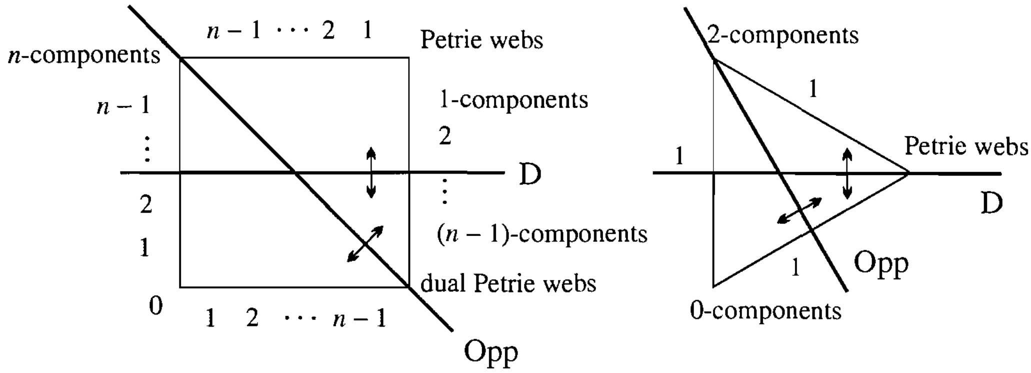


Figure 6.2. Visualizations of the groups D_8 and D_6 of 3-map and 2-map operations.

Observe that in these higher-dimensional situations, the elementary combinatorial structures which are transformed by the operations comprise not just the i-components and Petrie webs but also *dual Petrie webs*—the duals of Petrie webs—which are the orbits on the flags of the subgroup $\langle t_0, \ldots, t_{n-3}, t_n t_{n-2}, t_{n-1} \rangle$. For example, D Opp interchanges i-components with $(n-i)$-components $(1 \leqslant i \leqslant n-1)$ and sends vertices to dual Petrie webs, dual Petrie webs to Petrie webs, Petrie webs to n-components (cells), and cells to vertices. (Hence this operation has order 4.)

For $n \geqslant 3$ the structure of the group of n-map operations is best understood as the wreath product $C_2 \wr C_2$. The two factors in the base group $C_2 \times C_2$ are generated by Opp and D Opp D respectively, and correspond to activity at either end of the Coxeter diagram for $\mathscr{C}_n$, namely the transposing of vertices and Petrie webs, and of n-cells and dual Petrie webs. The copy of C_2 forming the complement in the wreath product is generated by D and corresponds to reflecting the Coxeter diagram. For 2-maps, the base group collapses to C_3 as Petrie webs and dual Petrie webs coincide as Petrie circuits. The alternative representation of these groups as dihedral groups D_8 and $D_6 = S_3$ is depicted in Figure 6.2, where they are regarded as acting in the usual way as the isometry groups of the square and triangle respectively.

6.2.2. Orbits of small 3-maps

To provide a quick visual cue as to the nature of small 3-maps, we illustrate in Figure 6.3 the five (algebraic, combinatorial) 3-maps with three flags, all of which lie on D^3 (the 3-ball with boundary S^2). In [21], Harding illustrates all fifteen 3-maps with

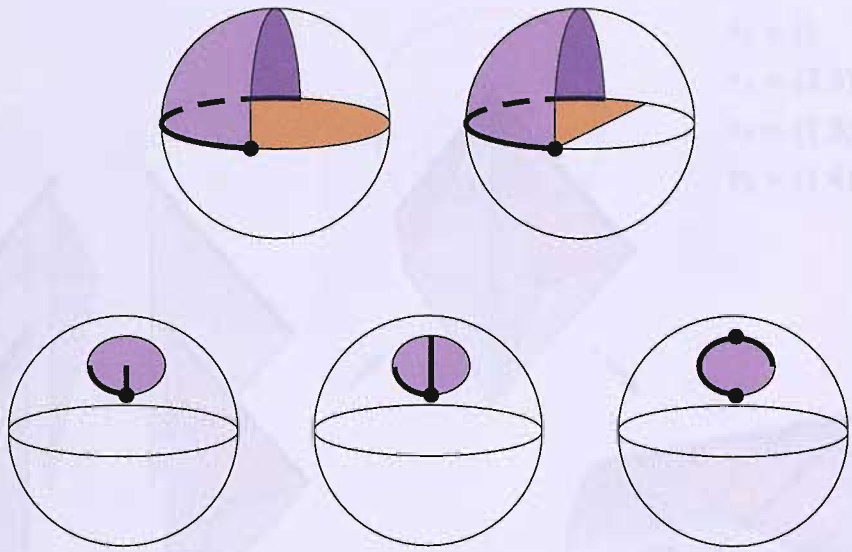


FIGURE 6.3. The five 3-flag 3-maps.

two flags[1] (all lying on orientable manifolds, of which all but one have boundary) and the 64 four-flag 3-maps which lie on manifolds (often with boundary).[2] His essentially topological definition of 3-maps does not permit other underlying structures; moreover he seems to discount non-orientable manifolds. However, in the algebraic and combinatorial context (which is the foundation for the 3-map operations above) we may still find it possible to represent more general 3-maps in a topological manner. Examining the effect of the eight operations on the 3-maps with four flags, we find that there are two which are invariant under all operations (both lying on D^3); eight orbits of size two (of which only one contains a non-orientable 3-map); fourteen orbits of size four (of which four contain non-orientable 3-maps and two others contain orientable 3-maps which do not lie on a manifold or manifold with boundary); and one orbit of size eight (containing four non-orientable manifolds with boundary). This gives a total of 82 objects, of which 18 do not lie on orientable manifolds with or without boundary.

There exist two orbits of size four which contain 3-maps that are orientable but which do not lie on a manifold with or without boundary. There are two such 3-maps in each orbit, forming a dual pair in both cases. Figure 6.4 shows the 'paste construction' of one of these 3-maps $\mathscr{M}$ from its constituent flags: the underlying

[1] In general, $\mathscr{C}_n$ has $2^{n+1} - 1$ subgroups of index 2, so this is the number of 2-flag n-maps.

[2] A small error exists in three of Harding's drawings: in numbers 32, 33 and 52 the 'free side' of the free face should be incident with some point on the equator which does not lie on the free edge.

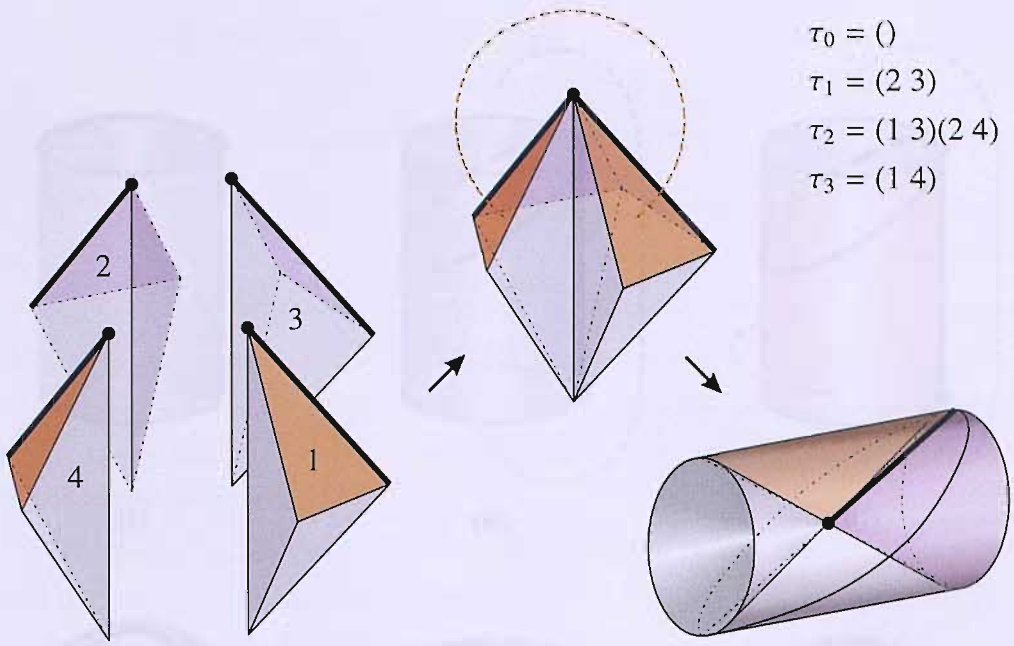


Figure 6.4. A 4-flag 3-map $\mathscr{M}$.

orbifold is the intersection of a solid cylinder with the closure of the complement of a solid double cone. There are two faces, one lying on one of the conical boundary portions and the other forming a membrane joining the conical boundary portions to the boundary of the cylinder. The single vertex lies at the cone apex, and the single edge runs along one side of the membrane. Figure 6.5(i) shows the dual 3-map $D(\mathscr{M})$; its minimal regular cover is given in Figure 6.5(ii), where the exteriors of the two conical portions should be identified as indicated so that the boundary of the 3-map (which is entirely facial) is simply the exterior of the cylinder shown; and the orientable double cover without boundary of $D(\mathscr{M})$ is given in Figure 6.5(iii) where, in addition to the same identification as before, the front and back of the (facial) exterior of the cylinder should be identified. Respectively, Figures 6.5(iv), 6.5(v) and 6.5(vi) show possible representations of these 3-maps as structures whose underlying topological spaces are manifolds (possibly with boundary). None of these representations is a valid topological 3-map as defined by Harding, but they indicate how the definition might be loosened to admit a wider range of combinatorial 3-maps: we see a 'cell' (in D^3) with disconnected non-facial boundary components; a toroidal 'cell' (in the solid 2-torus); and an annular 'face' (in S^3). Finally, Figure 6.6 shows the orbit of $\mathscr{M}$ under

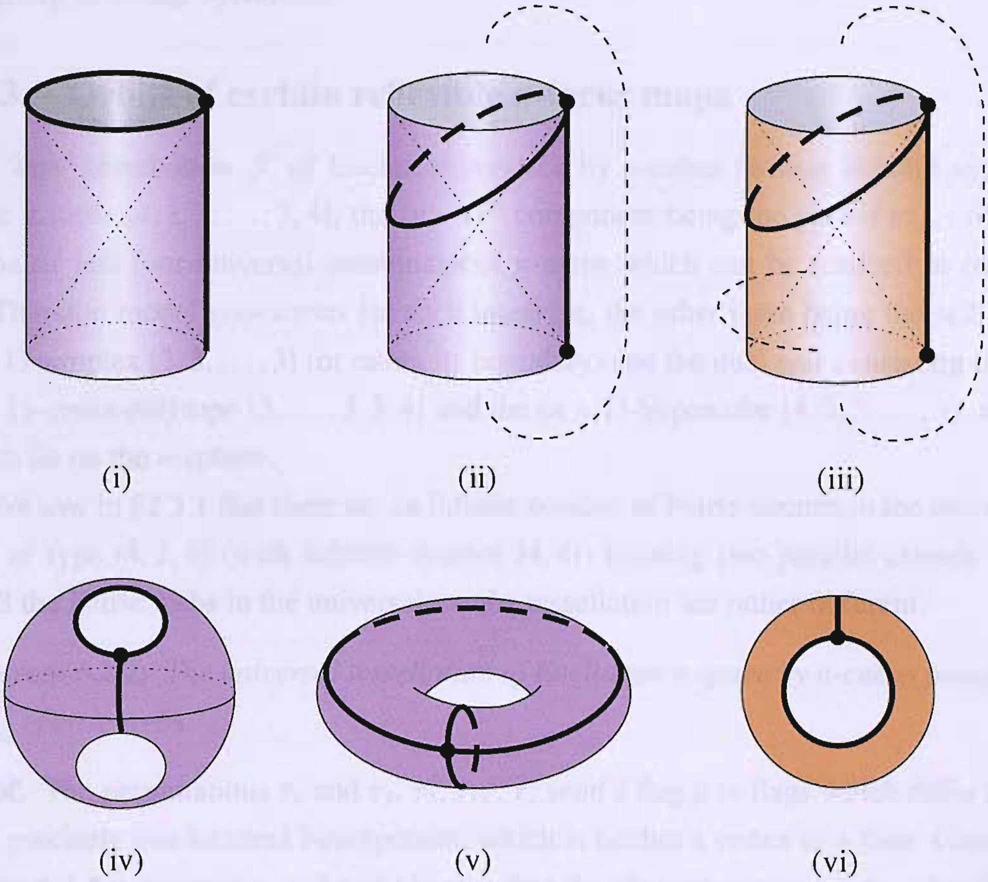


Figure 6.5. (i) The 3-map D($\mathscr{M}$); (ii) its minimal regular cover; (iii) its orientable double cover without boundary; (iv–vi) possible representations of these as objects on manifolds.

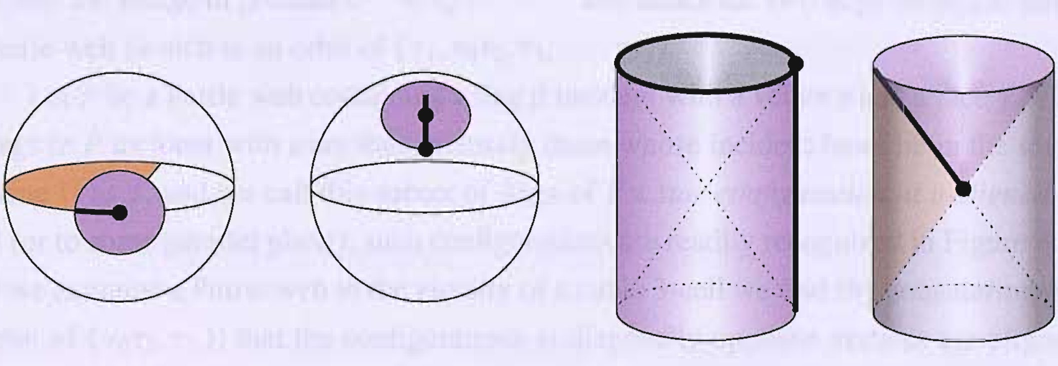


Figure 6.6. The orbit of $\mathscr{M}$ under the group of 3-map operations.

the group of 3-map operations.

6.2.3. Orbits of certain reflexible n-torus maps

The 'box' tessellation $\mathscr{T}$ of Euclidean n-space by n-cubes (whose Schläfli symbol is the n-tuple $\{4,3,3,\ldots,3,4\}$, the $(n-i)^{\text{th}}$ component being the period $m_{i,i+1}$ of $\mathscr{C}_n$) is one of just four universal combinatorial n-maps which can be realized in one of the Thurston model geometries for each integer n, the other three being the self-dual $(n+1)$-simplex $\{3,3,\ldots,3\}$ (or rather its boundary) and the dual pair consisting of the $(n+1)$–cross-polytope $\{3,\ldots,3,3,4\}$ and the $(n+1)$-hypercube $\{4,3,3,\ldots,3\}$, all of which lie on the n-sphere.

We saw in §2.3.1 that there are an infinite number of Petrie circuits in the universal map of type $(4,2,4)$ (with Schläfli symbol $\{4,4\}$) forming two parallel classes. For $n \geqslant 3$ the Petrie webs in the universal n-cube tessellation are rather different.

Theorem 6.2.2. *The universal tessellation of Euclidean n-space by n-cubes possesses* $1+\binom{n}{2}$ *Petrie webs.*

Proof. The permutations τ_1 and $\tau_3, \tau_4, \ldots, \tau_n$ send a flag β to flags which differ from β by precisely one incident i-component, which is neither a vertex or a face. Consider Figure 6.1 for the case $n = 3$ and observe that the element $\tau_2\tau_3\tau_2$ sends a flag to its reflection in its incident face. For general n, the permutation $\sigma = \tau_n^{\tau_{n-1}\tau_{n-2}\ldots\tau_2}$ sends a flag β to its reflection β' in an $(n-1)$-cell orthogonal to the plane Π determined by its incident face: β' lies in a neighbouring n-cell and shares the same vertex and edge as β while its incident face is distinct from that of β and also lies in Π. It follows that β' is also the image of β under $\sigma^{\tau_0} = \tau_n^{\tau_{n-1}\ldots\tau_3(\tau_2\tau_0)}$ and hence the two flags lie in the same Petrie web (which is an orbit of $\langle \tau_1, \tau_0\tau_2, \tau_3, \ldots, \tau_n \rangle$).

Let P be a Petrie web containing a flag β incident with a vertex v and a face f. The flags in P incident with v are thus precisely those whose incident faces lie in the same plane Π as f, and we call this subset of flags of P a *star configuration at v aligned to* Π (or to some parallel plane); such configurations are readily recognized in Figure 6.1. If we examine a Petrie web in the vicinity of a cubic 3-cell we find (by considering the orbit of $\langle \tau_0\tau_2, \tau_1 \rangle$) that the configurations at diagonally opposite vertices are aligned to the same plane; and that three of the four pairs of opposite vertices give rise to three pairs of star configurations parallel to the three pairs of planes in which the faces of the 3-cell lie, while there is no star configuration at either vertex of the fourth pair.

Finally, by looking at how adjacent 3-cells come together within the i-cells ($4 \leqslant i \leqslant n$) we see how the pattern of star configurations repeats: at any vertex v incident with a Petrie web, the set of flags in the web form a single star configuration aligned to one of the facial planes passing through v, while the vertices at distance 1 from a vertex v' which is not incident with the web are themselves all incident with it, those opposite each other across v' having parallel star configurations.

It follows that the number of Petrie webs is simply one greater than the number of distinct star configurations which can exist at a given vertex, or equivalently, one greater than the number of facial planes passing through that vertex. The result follows. ∎

Hence there are four Petrie webs in the universal tessellation $\mathscr{T}$ of Euclidean 3-space by 3-cubes. If we give $\mathscr{T}$ the usual Cartesian coordinate system so that its vertices are the points of $\mathbb{Z}^3$ then, for a, b, $c > 0$, we may identify opposite sides of a cuboid whose vertices are the points $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, $(a,b,0)$, $(0,0,c)$, $(a,0,c)$, $(0,b,c)$, (a,b,c) to obtain a 3-map $\mathscr{M}$ on a 3-torus; since the Schläfli symbol for $\mathscr{T}$ is $\{4,3,4\}$ we denote $\mathscr{M}$ by $\{4,3,4\}_{a,b,c}$. $\mathscr{M}$ is reflexible if and only if $a = b = c$.

The self-dual 3-map $\mathscr{M} = \{4,3,4\}_{1,1,1}$ consists of one vertex, three edges, three faces and one 3-cell, and it has a single Petrie web (of size 48). The 3-map $\mathrm{Opp}(\mathscr{M})$ also has these properties, but each cell corner consists of six face-corners coming together at the vertex. These 3-maps form an orbit of size two under the group of 3-map operations. In general, $\mathscr{M} = \{4,3,4\}_{a,b,c}$ consists of abc vertices, $3abc$ edges, $3abc$ faces and abc 3-cells, and is self-dual. The number of Petrie webs depends upon the parity of a, b and c: if they are all even then there are four; if just one of them is odd then there are two; while if two or all three of them are odd then there is a single Petrie web. In particular, by taking $a = b = c$ to be successive odd integers and applying Opp we have a sequence of increasingly large reflexible 3-maps with cubic cells but with just one vertex.

6.3. Even subgroups of Coxeter groups

For a given set of (possibly infinite) periods $m_{j,k}$, let $\overline{\Delta}_n$ be the Coxeter group with presentation

$$\overline{\Delta}_n = \langle t_0, t_1, \ldots, t_n \mid t_i^2 = (t_j t_k)^{m_{j,k}} = 1, 0 \leqslant i, j, k \leqslant n, j < k \rangle.$$

Let Δ_n be the even subgroup of $\overline{\Delta}_n$ consisting of all words of even length in the generators t_i. We may find a presentation P for Δ_n as follows. Let $K = K_{n+1}$ be the complete graph on $n+1$ vertices, and let us label its vertices with the generators t_i of $\overline{\Delta}_n$. We direct the edges from vertices with lower subscripts to vertices with higher, and we label each edge $\overrightarrow{t_j t_k}$ with $r_{j,k}$. The generating set of P is $\{r_{j,k}\}$ where $r_{j,k} = t_j t_k$. Now, $r_{j,k}^{m_{j,k}} = 1$, and the only other relators in P are the words of the form $t_{i_1}t_{i_2}.t_{i_2}t_{i_3}.t_{i_3}t_{i_4}\dots t_{i_m}t_{i_1}$; these correspond to circuits in K, with an edge contributing respectively $r_{j,k} = t_j t_k$ or $r_{j,k}^{-1} = t_k t_j$ to the word when it is traversed according to or against its direction. The relators given by triangles (circuits of length 3) in K are sufficient to induce relators given by longer circuits, and so we need just $\binom{n+1}{3}$ further canonical relators. For example,

$$\Delta_3 = \langle r_{0,1}, r_{0,2}, r_{0,3}, r_{1,2}, r_{1,3}, r_{2,3} \mid r_{j,k}^{m_{j,k}} = r_{0,1}r_{1,2}r_{0,2}^{-1} = r_{1,2}r_{2,3}r_{1,3}^{-1} = r_{0,2}r_{2,3}r_{0,3}^{-1} = r_{0,1}r_{1,3}r_{0,3}^{-1} = 1 \rangle.$$

When $\overline{\Delta}_n$ has a linear Coxeter diagram, we have $m_{j,k} = 2$ whenever $|j-k| \geqslant 2$. In this case, the groups $\Delta_n = \mathscr{C}_n^+$ play the same rôle for orientable n-maps without boundary as $\mathscr{C}_2^+$ plays for orientable maps without boundary. Note that the 1-dart orientable 3-map without boundary consists of an incident vertex, free edge, free face and cell in S^3.

6.4. Representing 3-maps by hypermaps

6.4.1. Motivation

In this final section we exhibit representations of 3-maps by maps and hypermaps. This is motivated by the idea in [25], discussed previously in §4, in which two categories C_1, C_2 of objects corresponding to the categories of conjugacy classes of subgroups of groups G_1, G_2 respectively, are associated by taking preimages of an epimorphism from a subgroup of G_2 onto G_1. We have examined the related idea in which the epimorphism is from G_1 onto a subgroup of G_2, in other words, a homomorphism from G_1 to G_2. An example is given in [25] of a representation of the category of all 3-maps by the category of orientable maps without boundary via preimages of an epimorphism from an index-3 subgroup of $\mathscr{C}_2^+$ onto $\mathscr{C}_3$. Whereas before we classified the operational functors arising in the case where G_1 and G_2 were both triangle groups or were both extended triangle groups, we too now look to represent three-dimensional objects by two-dimensional ones; we shall do this using inclusions, which give rise to faithful representations.

6.4.2. Representing orientable 3-maps without boundary

A correspondence with hypermaps

Consider the case in which $m_{j,j+1} = \infty$ for each $j \in \{0, 1, \ldots, n-1\}$ in the presentation for Δ_n described in §6.3. It is not difficult to see that the generators $r_{j,j+1}$ and all of the relators given by triangles involving them may be eliminated from the presentation, leaving (usually longer) relators given by circuits in the complement in the complete graph K_{n+1} of the path $r_{0,1}r_{1,2}\ldots r_{n-1,n}$. However, when $n = 3$ there are no such circuits and hence the presentation becomes

$$\Delta_3 = \langle\, r_{0,2}, r_{0,3}, r_{1,3} \,|\, r_{0,2}^{m_{0,2}} = r_{0,3}^{m_{0,3}} = r_{1,3}^{m_{1,3}} = 1 \,\rangle \cong C_{m_{0,2}} * C_{m_{0,3}} * C_{m_{1,3}},$$

the free product of three cyclic groups. In particular, the even subgroup $\mathscr{C}_3^+$ of $\mathscr{C}_3$ is the free product of three cyclic groups of order 2, and so it is isomorphic to the full hypercartographic group $\mathscr{H}_2$. This gives us

Theorem 6.4.1. *There is an isomorphism between the category of orientable 3-maps without boundary and that of (unrestricted) hypermaps.* ∎

Hence we have a 2-dimensional representation of orientable 3-maps without boundary. The correspondence preserves automorphism groups and coverings, as well as symmetry (discussed below).

It is useful to use the alternative notation l, r, t, p for the generators $t_0, \ldots, t_3$ of $\mathscr{C}_3$ so that

$$\mathscr{C}_3 = \langle\, l, r, t, p \,|\, l^2 = r^2 = t^2 = p^2 = (lp)^2 = (rp)^2 = (lt)^2 = 1 \,\rangle;$$

the corresponding monodromy permutations of 3-map darts are $\lambda = \tau_0$, $\rho = \tau_1$, $\tau = \tau_2$ and $\phi = \tau_3$. Then

$$\mathscr{C}_3^+ = \langle\, lp, rp, lt \,|\, (lp)^2 = (rp)^2 = (lt)^2 = 1 \,\rangle,$$

while $\lambda\phi$, $\rho\phi$ and $\lambda\tau$ are half-turns about edges of the fundamental tetrahedra represented by 3-map flags. Indeed, $\lambda\phi$ induces the rotation of each face f and its two incident cells about the perpendicular bisector through f of an incident edge; $\rho\phi$ induces the rotation of each face f and its two incident cells about the bisector through f of the angle subtended by two adjacent edges; and $\lambda\tau$ induces the rotation of a cell about the midpoint of an edge.

Let $\mathscr{H}_2$ have presentation

$$\mathscr{H}_2 = \langle \bar{l}, \bar{r}, \bar{t} \mid \bar{l}^2 = \bar{r}^2 = \bar{t}^2 = 1 \rangle$$

with corresponding monodromy permutations $\bar{\lambda}, \bar{\rho}$ and $\bar{\tau}$ (to complement the presentation for $\mathscr{C}_2$ used in §2.1). We choose the isomorphism between $\mathscr{H}_2$ and $\mathscr{C}_3^+$ to be given by

$$\begin{aligned} \bar{l} &\leftrightarrow lp \\ \bar{r} &\leftrightarrow rp \\ \bar{t} &\leftrightarrow lt, \end{aligned}$$

although, by Theorem 3.1.3, there are an infinite number of possible isomorphisms at our disposal. Under the resulting correspondence between hypermaps and orientable 3-maps without boundary we have $\langle \bar{l}, \bar{r} \rangle \leftrightarrow \langle lp, rp \rangle$ and so hyperfaces in a hypermap $\mathscr{H}$ represent faces in the corresponding 3-map $\mathscr{M}$. We also have $\langle \bar{l}, \bar{t} \rangle \leftrightarrow \langle lp, lt \rangle$ so that hyperedges in $\mathscr{H}$ represent edges in $\mathscr{M}$. Further, we have $\langle \bar{r}, \bar{t}\bar{l} \rangle \leftrightarrow \langle rp, tp \rangle$ and so $(1,1)^{\text{th}}$-order Petrie circuits (Definition 4.4.2) represent cell corners in $\mathscr{M}$. Lastly we have $\langle \bar{r}, \bar{t} \rangle \leftrightarrow \langle rp, lt \rangle$ and so hypervertices in $\mathscr{H}$ represent right-handed Petrie circuits as defined by Coxeter [10].[3]

It is also interesting to examine how the existence of boundary in $\mathscr{H}$ translates to the 3-map $\mathscr{M}$. Fixed points of $\bar{\lambda}$ (which result from a hyperedge meeting a hyperface on the boundary) correspond to fixed points of $\lambda\phi$, which result from the free end of free edges lying on the free side of a free face. Similarly, fixed points of $\bar{\rho}$ correspond to vertices incident with the free sides of free faces, and fixed points of $\bar{\tau}$ correspond to free edges which are incident with precisely one face. In brief, the existence of boundary in $\mathscr{H}$ ensures the existence of either free edges or free faces in $\mathscr{M}$.

[3]Strictly speaking, Coxeter's right-handed Petrie polygons are orbits of the subgroup generated by $\lambda\rho\tau\phi = (\lambda\tau\phi\rho)^{\lambda\rho}$, and thus consist of one flag from each edge rather than two. This minor difference between Coxeter's definition and ours also arises in the 2-map case discussed in earlier chapters.

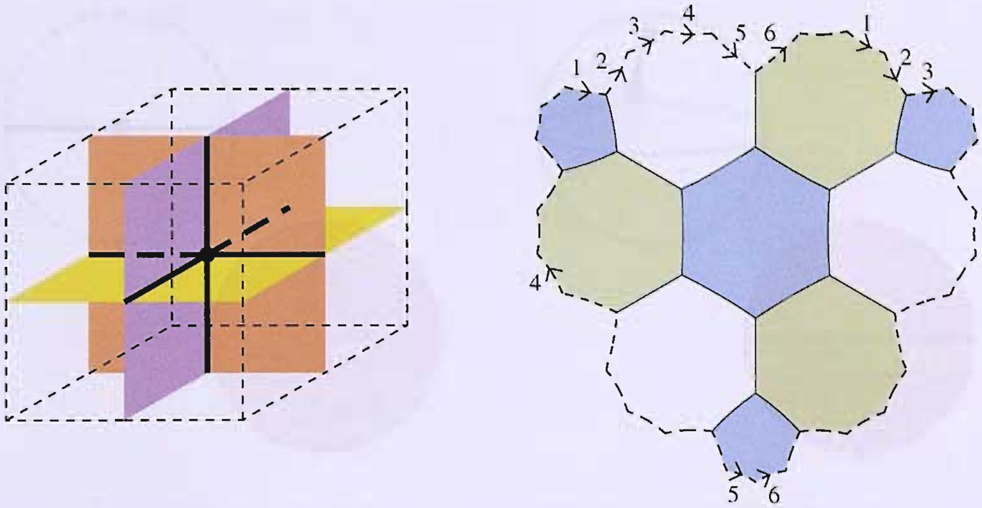


FIGURE 6.7. The 3-torus 3-map $\{4,3,4\}_{1,1,1}$ and its corresponding hypermap.

Example 6.4.2. Consider the 3-torus 3-map $\mathscr{M} = \{4,3,4\}_{1,1,1}$. In forming the corresponding hypermap $\mathscr{H}$, reflexible and non-orientable of characteristic -2, the three 4-valent faces of $\mathscr{M}$ become hyperfaces, the three edges become 4-valent hyperedges, and the four right-handed Petrie circuits become 3-valent hypervertices. Both $\mathscr{M}$ and $\mathscr{H}$ are pictured in Figure 6.7. To obtain $\mathscr{M}$, identify opposite sides of the cube; to obtain $\mathscr{H}$, identify the directed edges of the region according to their labels, and identify the remaining sides in the same way according to the symmetry of $\mathscr{H}$. ▲

Remark 6.4.3. Orientable hypermaps without boundary correspond to 3-maps which cover the particular 4-flag one shown in Figure 6.9 (discussed below). It is more difficult to see how general orientability of hypermaps translates to orientable 3-maps without boundary under the correspondence. Figure 6.8 shows two small (hyper)maps and their corresponding 3-maps. The first pair consists of a loop on the projective plane, and a 3-map comprising an edge on the 2-sphere embedded as a face in S^3. The second pair consists of an edge on the 2-sphere, and a 3-map comprising a loop on the projective plane embedded as a face in S^3. In particular, the non-orientable hypermap translates to a 3-map without any non-orientable components, while the orientable hypermap—which can in some sense be regarded as the 'orientable analogue' of the first—translates to a 3-map with a non-orientable face. (Of course, the 3-map itself is orientable.) ♦

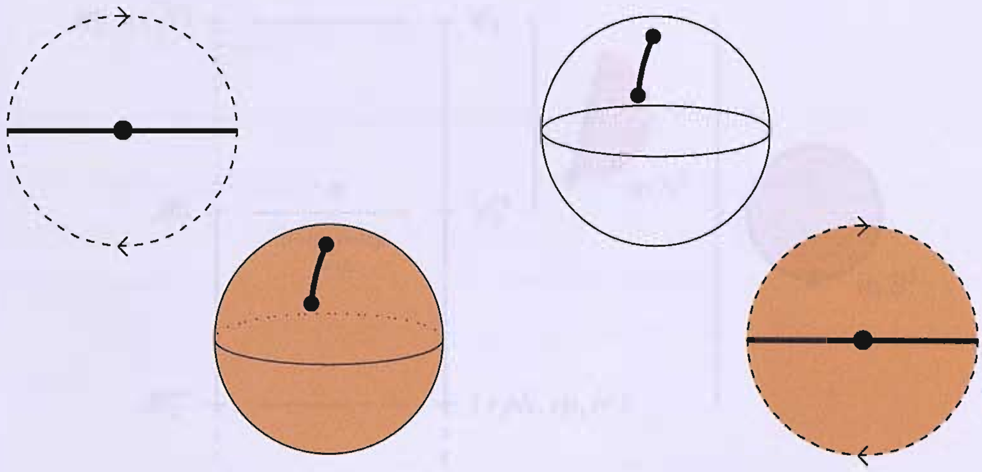


FIGURE 6.8. Two small (hyper)maps and their corresponding 3-maps.

A description of symmetry

Figure 6.9 shows a subgroup diagram for the correspondence. It is natural to ask how symmetry in hypermaps translates to 3-maps. Recall that the hypermap operation Hopp is induced by the automorphism of $\mathscr{H}_2$ which fixes t_1 and t_2, and conjugates t_0 by t_2 (in the notation of the presentation in §3.1).

Theorem 6.4.4. *Under the correspondence established between hypermaps and orientable* 3*-maps without boundary,*

(i) *regular* 3*-maps correspond to reflexible hypermaps;*

(ii) *reflexible* 3*-maps correspond to reflexible self-*Hopp *hypermaps.*

Proof. A hypermap is reflexible if and only if its hypermap subgroup in $\mathscr{H}_2$ is normal. Under the correspondence these subgroups are seen as subgroups of $\mathscr{C}_3^+$, and the first claim is immediate. Now consider the Coxeter group $\mathscr{C}_3$ which contains $\mathscr{C}_3^+$ as an index-2 subgroup; $\mathscr{C}_3$ is a split extension of $\mathscr{C}_3^+ = \langle\, lp, rp, lt \,|\, (lp)^2 = (rp)^2 = (lt)^2 = 1 \,\rangle$ by the involutary subgroup $\langle\, p \,\rangle$. This extension can be regarded as the external semidirect product of $\mathscr{C}_3^+$ and $\langle\, p \,\rangle$ specified by the conjugation action of p on $\mathscr{C}_3^+$. (The isomorphism between $\mathscr{C}_3$ and this product is given by $cq \mapsto (c, q)$ where $c \in \mathscr{C}_3^+$ and $q \in \langle\, p \,\rangle$, and the elements of this product satisfy the multiplication rule $(c_1, q_1)(c_2, q_2) = (c_1 q_1^{-1} c_2 q_1, q_1 q_2)$.)

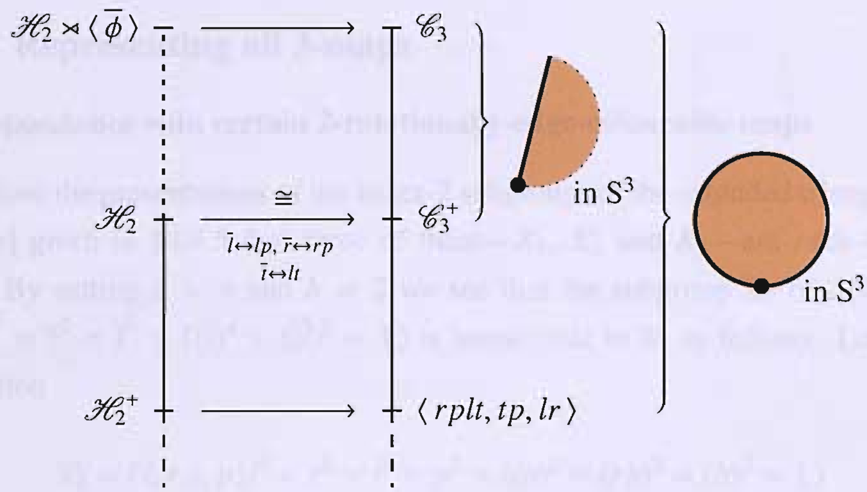


FIGURE 6.9. A subgroup diagram for the correspondence between hypermaps and orientable 3-maps without boundary.

Correspondingly, we construct the extension of $\mathscr{H}_2$ by the automorphism $\overline{\phi}$ which has the same action on $\mathscr{H}_2$ as p has on $\mathscr{C}_3^+$: $\overline{\phi}$ fixes $\overline{l}$ and $\overline{r}$, and sends $\overline{t}$ to its conjugate by $\overline{l}$ since $(lt)^p = pltp = lp.tl.lp = (lt)^{lp}$. Hence $\overline{\phi}$ is the automorphism ξ which induces the hypermap operation Hopp. A normal subgroup of $\mathscr{H}_2$ is normal in the extension if and only if it is invariant under ξ. Such a subgroup gives rise to a reflexible self-Hopp hypermap and, via the correspondence, to a reflexible 3-map. ■

Remark 6.4.5. By Proposition 3.1.2, all maps are self-Hopp and hence the reflexible ones correspond to a subset of the reflexible orientable 3-maps without boundary. Such 3-maps represented by maps are rather degenerate since each edge must be incident with at most two cells. ♦

In [2], Breda d'Azevedo defines a K-conservative hypermap to be one whose hypermap subgroup H in $\mathscr{H}_2$ lies in some normal subgroup $K \trianglelefteq \mathscr{H}_2$. If H is normal in K then the hypermap is said to be K-restricted-regular. Much of the theory of K-conservative hypermaps and K-restricted regularity described therein can be generalized to n-maps by taking $\mathscr{C}_n$ or $\mathscr{C}_n^+$ in place of $\mathscr{H}_2$.

Orientable hypermaps without boundary are regular if and only if their hypermap subgroups in $\mathscr{H}_2^+$ are normal. The corresponding 3-maps, whilst not necessarily regular, are restricted-regular with respect to the image $\langle rplt, tp, lr \rangle$ of $\mathscr{H}_2^+$; they are the regular covers of the 4-flag 3-map shown in Figure 6.9.

6.4.3. Representing all 3-maps

A correspondence with certain 2-rotationally-edge-colourable maps

We see from the presentations of the index-2 subgroups of the extended triangle groups $\Delta[a, b, \infty]$ given in §4.4.5 that three of them—K_1, K_2 and K_3—are rank-4 Coxeter groups. By setting $a = 4$ and $b = 2$ we see that the subgroup K_2 of $\Delta[4, 2, \infty] = \langle \bar{l}, \bar{r}, \bar{t} \mid \bar{l}^2 = \bar{r}^2 = \bar{t}^2 = (\bar{r}\bar{t})^4 = (\bar{t}\bar{l})^2 = 1 \rangle$ is isomorphic to $\mathscr{C}_3$ as follows. Let $\mathscr{C}_3$ have presentation

$$\mathscr{C}_3 = \langle l, r, t, p \mid l^2 = r^2 = t^2 = p^2 = (lp)^2 = (rp)^2 = (lt)^2 = 1 \rangle$$

as before. We know that $K_2 = \langle \bar{l}, \bar{t}, \bar{r}\bar{l}\bar{r}^{-1}, \bar{r}\bar{t}\bar{r}^{-1} \rangle$ has presentation

$$\langle \alpha, \gamma, \delta, \epsilon \mid \alpha^2 = \gamma^2 = \delta^2 = \epsilon^2 = (\epsilon\gamma)^2 = (\gamma\alpha)^2 = (\epsilon\delta)^2 = 1 \rangle.$$

We choose the isomorphism to be given by

$$l \leftrightarrow \epsilon = \bar{r}\bar{t}\bar{r}$$
$$r \leftrightarrow \alpha = \bar{l}$$
$$t \leftrightarrow \delta = \bar{r}\bar{l}\bar{r}$$
$$p \leftrightarrow \gamma = \bar{t}.$$

Again, much of the theory of K-conservative hypermaps described in [2] can be generalized from normal subgroups K of $\mathscr{H}_2$ to normal subgroups of any triangular group. We shall describe maps whose map subgroups lie in K_2 as *K_2-conservative* (*with respect to* $\Delta[4, 2, \infty]$). Such maps cover the 2-blade map consisting of a single vertex and two incident half-edges, both lying on the boundary of a disc.

Definition 6.4.6. A map is *n-rotationally-edge-colourable* if its blades β can be n-coloured so that β and $\beta\bar{p}$ have different colours while the set of blades incident with each edge have the same colour. •

Theorem 6.4.7. *There is an isomorphism between the category of* 3*-maps and the category of* K_2*-conservative maps, that is,* 2*-rotationally-edge-colourable maps whose interior vertices are* 2*- or* 4*-valent and whose boundary vertices have a neighbourhood containing no facial boundary.* ■

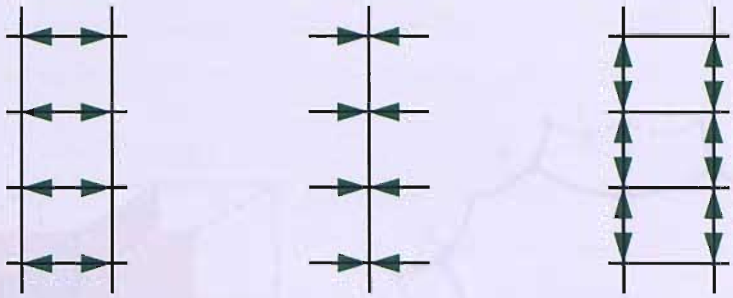


Figure 6.10. The typical appearance of 'runged ladders', 'runged poles' and 'train tracks' in a K_2-conservative map.

Under this correspondence we have $\langle r, t, p \rangle \leftrightarrow \langle \bar{l}, \bar{r}\bar{l}\bar{r}, \bar{t} \rangle$ and so vertices in a 3-map $\mathscr{M}$ represent 'runged ladders' (orbits of the latter group) in the corresponding map $\mathscr{E}$; these features are shown in Figure 6.10. Similarly, we have $\langle l, t, p \rangle \leftrightarrow \langle \bar{r}\bar{t}\bar{r}, \bar{r}\bar{l}\bar{r}, \bar{t} \rangle$ so that edges in $\mathscr{M}$ represent 'runged poles' in $\mathscr{E}$. We also have $\langle l, r, p \rangle \leftrightarrow \langle \bar{r}\bar{t}\bar{r}, \bar{l}, \bar{t} \rangle$ and so faces in $\mathscr{M}$ represent monochrome paths in $\mathscr{E}$. Finally, we have $\langle l, r, t \rangle \leftrightarrow \langle \bar{r}\bar{t}\bar{r}, \bar{l}, \bar{r}\bar{l}\bar{r} \rangle$ and so cells in $\mathscr{M}$ represent 'train tracks' in $\mathscr{E}$.

Conversely we have $\langle \bar{r}\bar{t}\bar{r}, \bar{t} \rangle \leftrightarrow \langle l, p \rangle$ and so vertices in $\mathscr{E}$ represent face-edge incidence in $\mathscr{M}$. Next, we have $\langle \bar{l}, \bar{t} \rangle \leftrightarrow \langle r, p \rangle$ so that edges in $\mathscr{E}$ represent face-vertex incidence in $\mathscr{M}$. We also have $\langle \bar{r}\bar{l}\bar{r}, \bar{l} \rangle \leftrightarrow \langle t, r \rangle$ and so faces in $\mathscr{E}$ represent cell corners in $\mathscr{M}$. Finally, we have $\langle \bar{r}\bar{t}\bar{r}, \bar{l} \rangle \leftrightarrow \langle l, r \rangle$ and so second-order holes in $\mathscr{E}$ represent faces in $\mathscr{M}$.

Fixed points of $\bar{l}$ (and hence of $\bar{r}\bar{l}\bar{r}$) correspond to fixed points of r and t, while fixed points of $\bar{t}$ (and hence of $\bar{r}\bar{t}\bar{r}$) correspond to fixed points of p and l. It follows that $\mathscr{E}$ has boundary if and only if $\mathscr{M}$ has. Moreover, if $\Delta = K_7$ is the even subgroup of $\Delta[4, 2, \infty]$, then $K_2 \bigcap \Delta = \langle (\bar{r}\bar{t})^2, \bar{l}\bar{t}, \bar{r}\bar{t}\bar{l}\bar{r} \rangle \leftrightarrow \langle lp, rp, lt \rangle = \mathscr{C}_3^+$ and so the correspondence also preserves the property of being orientable without boundary.

Example 6.4.8. Consider again the 3-torus 3-map $\mathscr{M} = \{4, 3, 4\}_{1,1,1}$. Figure 6.11 shows its corresponding K_2-conservative map $\mathscr{E}$, which is reflexible with 12 vertices, 24 edges and eight 8-valent faces, and lies on an orientable surface of genus 3. ▲

A description of symmetry

Figure 6.12 shows a subgroup diagram for the correspondence. We conclude by examining how symmetry in K_2-conservative maps translates to 3-maps.

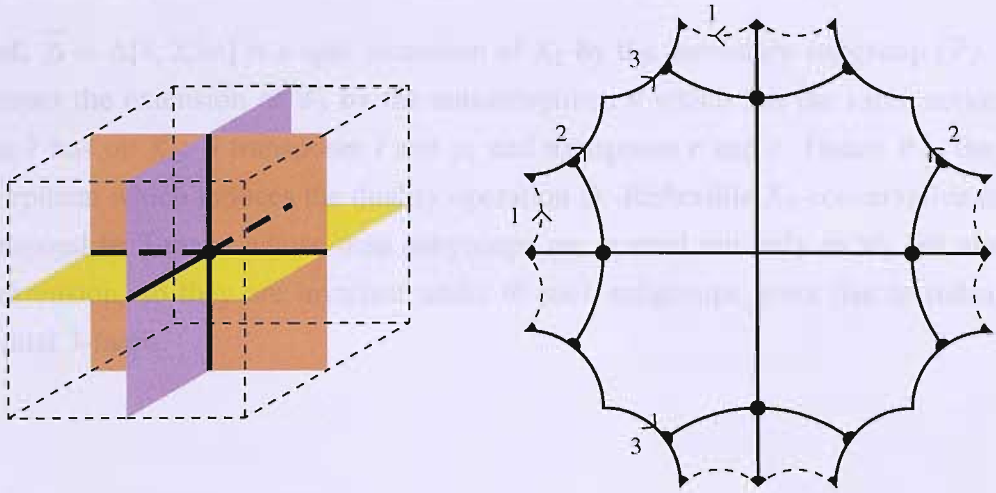


Figure 6.11. The 3-torus 3-map $\{4,3,4\}_{1,1,1}$ and its corresponding K_2-conservative map.

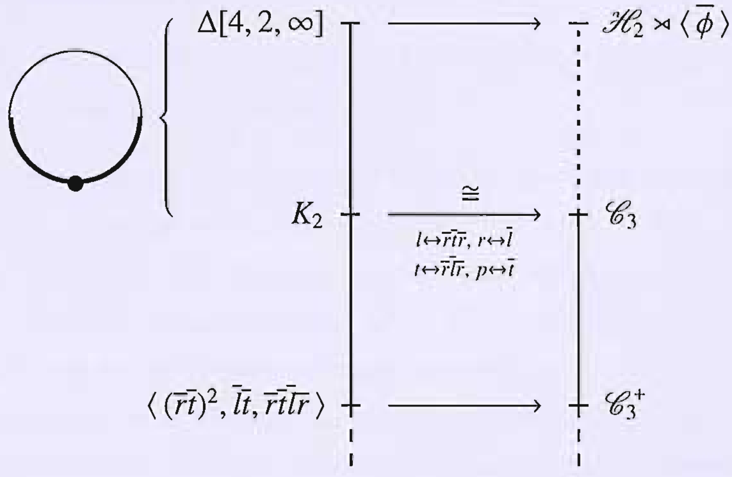


Figure 6.12. A subgroup diagram for the correspondence between K_2-conservative maps and 3-maps.

Theorem 6.4.9. *Under the correspondence established between K_2-conservative maps and 3-maps, reflexible maps correspond to reflexible self-dual 3-maps.*

Proof. $\overline{\Delta} = \Delta[4, 2, \infty]$ is a split extension of K_2 by the involutary subgroup $\langle \overline{r} \rangle$. We construct the extension of $\mathscr{C}_3$ by the automorphism θ which has the same action on $\mathscr{C}_3$ as $\overline{r}$ has on K_2: θ transposes l and p, and transposes r and t. Hence θ is the automorphism which induces the duality operation D. Reflexible K_2-conservative maps correspond to 3-maps whose map subgroups are normal not only in $\mathscr{C}_3$ but also in the extension, so they are invariant under θ; such subgroups gives rise to reflexible self-dual 3-maps. ∎

Appendix A

Complete tripartite graphs

A.1. Reflexible orientable embeddings

It is shown in [29] that the complete bipartite graph $K_{n,n}$ has a unique reflexible embedding in an orientable surface if and only if n is coprime to $\phi(n)$ where ϕ is Euler's function. Part of the approach used is to establish a correspondence between such embeddings and triples (G, x, y) where G is a group with cyclic subgroups $X = \langle x \rangle$ and $Y = \langle y \rangle$ of order n, where $G = XY$, $X \cap Y = 1$, and where there exists an automorphism of G transposing x and y.

Theorem A.1.1. *There is a unique reflexible orientable triangular embedding of the complete tripartite graph $K_{n,n,n}$ for each integer n.*

Proof. A correspondence similar to that established for complete bipartite graphs can be drawn between reflexible orientable triangular embeddings of $K_{n,n,n}$ and triples (G, x, y), where G is a group with mutually disjoint cyclic subgroups $X = \langle x \rangle$, $Y = \langle y \rangle$ and $Z = \langle y^{-1}x^{-1} \rangle$ of order n such that $G = XY = XZ = YZ$ and such that there exists a group S_3 of automorphisms of G permuting x, y and $y^{-1}x^{-1}$.

There are several steps in forming this correspondence. Firstly, if G is a group with distinct subgroups X, Y and Z, a tripartite graph T can be formed by taking the vertices to be the cosets of X, Y and Z, and taking the edges between pairs of partite sets to be the elements of G, incidence being given by containment of the edge element within the vertex cosets. G then acts by left multiplication as a group of automorphisms of T, acting regularly on each set of edges joining a pair of partite vertex sets. Conversely, if T is a tripartite graph containing a triangle Δ and possessing a group G of automorphisms acting regularly on each inter-partite edge set then T arises in the above way: label the edges of Δ with the identity, label with α those edges that are the image under the automorphism α of some edge of Δ, and take X, Y and Z to be the subgroups of G fixing one of the three vertices of Δ. Under this construction, T contains no multiple edges if and only if X, Y and Z are mutually disjoint, and is complete (as a tripartite graph) if and only if G can be written as a product $G = XY = XZ = YZ$.

Secondly, if $X = \langle x \rangle$, $Y = \langle y \rangle$ and $Z = \langle z \rangle$ are cyclic groups then a local orientation can be established around each vertex which determines an embedding $\mathcal{M}$ of T in an oriented surface; the action of G on T extends to an action on the map $\mathcal{M}$, preserving its partite sets of vertices and its orientation. Conversely, given an oriented embedding of a tripartite graph containing a triangular face (with vertices u, v, w say) such that there exists a group of orientation- and partite-set–preserving map-automorphisms acting regularly on each set of interpartite edges, then X (respectively Y, Z) can be taken as the cyclic subgroup generated by the automorphism x (y, z) fixing u (v, w) and sending the edge uv (vw, wu) to the 'next' edge around u (v, w) which joins vertices from the same two partite sets, following the orientation. Such an embedding is regular if and only if $xyz = 1$ and there exists a group S_3 of automorphisms of G permuting x, y and z. Our correspondence is thus established.

The reflexible orientable triangular tripartite maps $\mathcal{M}_{n,n,n}$ have type $\{3, n\}$ with $3n$ vertices, $3n^2$ edges, $2n^2$ faces, and lie on a surface of genus $(n-1)(n-2)/2$. The corresponding triples (G, x, y) are such that G possesses an automorphism fixing z but transposing x and y. But then G is abelian, and since $G = XY$ and $X \bigcap Y = 1$ we have $G = X \times Y \cong C_n \times C_n$. The result follows. ∎

Bibliography

[1] A. F. Beardon, *The Geometry of Discrete Groups*. Springer-Verlag, New York (1983).

[2] A. J. Breda d'Azevedo, A theory of restricted regularity of hypermaps. Submitted.

[3] A. J. Breda d'Azevedo and G. A. Jones, Double coverings and reflexive abelian hypermaps. *Beiträge zur Algebra und Geometrie*, **41** 2 (2000), 371–389.

[4] R. P. Bryant and D. Singerman, Foundations of the theory of maps on surfaces with boundary. *Quarterly Journal of Mathematics. Oxford. Second Series*, **36** 2 (1985), 17–41.

[5] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *ATLAS of Finite Groups*. Oxford University Press, New York (1985).

[6] R. Cori, *Un code pour les graphes planaires et ses applications*, Astérisque **27**. Soc. Math. France (1975).

[7] R. Cori and A. Machì, Maps, hypermaps and their automorphisms: a survey, I, II, III. *Expositiones Mathematicae*, **10** (1992), 403–427, 429–447, 449–467.

[8] D. Corn and D. Singerman, Regular hypermaps. *European Journal of Combinatorics*, **9** (1988), 337–351.

[9] H. S. M. Coxeter, Regular skew polyhedra in three and four dimensions, and their topological analogues. *Proceedings of the London Mathematical Society*, **43** 2 (1937), 33–62.

[10] H. S. M. Coxeter, *Twisted Honeycombs*, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics **4**. American Mathematical Society (1970).

[11] H. S. M. Coxeter and W. O. J. Moser, *Generators and Relations for Discrete Groups*. Springer-Verlag, Berlin/Heidelberg/New York, 4th ed. (1980).

[12] P. Diaconis and R. Graham, The graph of generating sets of an abelian group. *Colloquium Mathematicum*, **80** 1 (1999), 31–38.

[13] J. D. Dixon, The probability of generating the symmetric group. *Mathematische Zeitschrift*, **110** (1969), 199–205.

[14] A. W. M. Dress, Regular polytopes and equivariant tessellations from a combinatorial point of view. In *Algebraic Topology, Göttingen 1984*, Lecture Notes in Mathematics **1172**, Springer-Verlag, Berlin (1985), pp. 56–72.

[15] M. J. Dunwoody, On T-systems of groups. *Journal of the Australian Mathematical Society*, **3** (1963), 172–1793.

[16] C. Gagliardi, A combinatorial characterization of 3-manifold crystallizations. *Unione Matematica Italiana. Bollettino. A. Serie V*, **16** 3 (1979), 441–449.

[17] *GAP – Groups, Algorithms, and Programming, Version 4.4.* The GAP Group (2005). `http://www.gap-system.org`.

[18] E. Girondo, Multiply quasiplatonic Riemann surfaces. *Experimental Mathematics*, **12** 4 (2003), 463–475.

[19] R. Guralnick and I. Pak, On a question of B. H. Neumann. *Proceedings of the American Mathematical Society*, **131** (2003), 2021–2025.

[20] P. Hall, The Eulerian functions of a group. *Quarterly Journal of Mathematics*, **7** (1936), 134–151.

[21] S. J. Harding, *Some Arithmetic and Geometric Problems Concerning Discrete Groups*. Ph.D. thesis, University of Southampton (1985).

[22] A. H. M. Hoare and D. Singerman, The orientability of subgroups of plane groups. In *Groups—St. Andrews 1981 (St. Andrews, 1981)*, London Mathematical Society Lecture Note Series **71**, Cambridge University Press, Cambridge (1982), pp. 221–227.

[23] L. D. James, Complexes and Coxeter groups—operations and outer automorphisms. *Journal of Algebra*, **113** 2 (1988), 339–345.

[24] L. D. James, Operations on hypermaps, and outer automorphisms. *European Journal of Combinatorics*, **9** (1988), 551–560.

[25] L. D. James, *Representations of Maps*, Disertaciones del Seminario de Matemáticas Fundamentales **8**. Universidad Nacional de Educatión a Distancia, Madrid (1990).

[26] L. D. James and G. A. Jones, Regular orientable imbeddings of complete graphs. *Journal of Combinatorial Theory*, **39** 3 (1985), 353–367.

[27] G. A. Jones, Ree groups and Riemann surfaces. *Journal of Algebra*, **165** 1 (1994), 41–62.

[28] G. A. Jones, Inclusions between triangle groups, and functors between dessins. Preprint.

[29] G. Jones, R. Nedela and M. Škoviera, Complete bipartite graphs with unique regular embeddings. Submitted.

[30] G. A. Jones and S. A. Silver, Suzuki groups and surfaces. *Journal of the London Mathematical Society*, **48** 2 (1993), 117–125.

[31] G. A. Jones and D. Singerman, Theory of maps on orientable surfaces. *Proceedings of the London Mathematical Society*, **37** 3 (1978), 273–307.

[32] G. A. Jones and D. Singerman, Maps, hypermaps and triangle groups. In L. Schneps (Ed.), *The Grothendieck Theory of Dessins d'Enfants*, London Mathematical Society Lecture Note Series **200** (1994), pp. 115–145.

[33] G. Jones and D. Singerman, Belyĭ functions, hypermaps and Galois groups. *Bulletin of the London Mathematical Society*, **28** (1996), 561–590.

[34] G. A. Jones and D. B. Surowski, Regular cyclic coverings of the Platonic maps. *European Journal of Combinatorics*, **21** (2000), 333–345.

[35] G. A. Jones and J. S. Thornton, Operations on maps, and outer automorphisms. *Journal of Combinatorial Theory*, **35** 2 (1983), 93–103.

[36] W. M. Kantor and A. Lubotzky, The probability of generating a finite classical group. *Geometriae Dedicata*, **36** (1990), 67–87.

[37] A. W. Knapp, Doubly generated Fuchsian groups. *Michigan Mathematical Journal*, **15** (1968), 289–304.

[38] C. Léger, Opérations sur les cartes et métamorphoses de la catégorie des *G*-ensembles. *Revista Matemática de la Universidad Complutense de Madrid*, **4** 1 (1991), 45–51.

[39] S. Levy (Ed.), *The Eightfold Way: the Beauty of Klein's Quartic Curve*, MSRI Research Publications **35**. Cambridge University Press, Cambridge (1999).

[40] M. W. Liebeck and A. Shalev, The probability of generating a finite simple group. *Geometriae Dedicata*, **56** (1995), 103–113.

[41] S. Lins, Graph-encoded maps. *Journal of Combinatorial Theory Series B*, **32** (1982), 171–181.

[42] R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*. Springer-Verlag (1977).

[43] A. M. Macbeath, The classification of non-euclidean plane crystallographic groups. *Canadian Journal of Mathematics*, **19** (1967), 1192–1205.

[44] A. Machí, On the complexity of a hypermap. *Discrete Mathematics*, **42** (1982), 221–226.

[45] A. I. Mal'cev, On the faithful representations of infinite groups by matrices. *Matematicheskĭ Sbornik*, **8** (1940), 405–422. (English translation: *Transactions of the American Mathematical Society*, **45** (1965), 1–18.).

[46] J. P. Matelski, The classification of discrete 2-generator subgroups of PSL(2, $\mathbb{R}$). *Israel Journal of Mathematics*, **42** 4 (1982), 309–317.

[47] B. H. Neumann, On a question of Gaschütz. *Archiv der Mathematik*, **7** (1956), 87–90.

[48] B. H. Neumann and H. Neumann, Zwei klassen charakteristischer untergruppen und ihre faktorgruppen. *Mathematische Nachrichten*, **4** (1951), 106–125.

[49] D. J. Newman, *Analytic Number Theory*. Springer-Verlag, New York (1998).

[50] I. Pak, What do we know about the product replacement algorithm? In *Groups and computation, III (Columbus, OH, 1999)*, Ohio State Univ. Math. Res. Inst. Publ. **8**, de Gruyter, Berlin (2001), pp. 301–347.

[51] M. A. Ronan, Coverings and automorphisms of chamber systems. *European Journal of Combinatorics*, **1** 3 (1980), 259–269.

[52] J. J. Rotman, *An Introduction to the Theory of Groups*. Springer-Verlag, New York, 4th ed. (1995).

[53] O. Schreier, Über die gruppen $A^a B^b = 1$. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, **3** (1924), 167–169.

[54] A. Shalev, Simple groups, permutation groups, and probability. In *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998)*, Extra Vol. II (1998), pp. 129–137.

[55] D. Singerman, Subgroups of Fuchsian groups and finite permutation groups. *Bulletin of the London Mathematical Society*, **2** (1970), 319–323.

[56] D. Singerman, Finitely maximal Fuchsian groups. *Journal of the London Mathematical Society*, **6** 2 (1972), 29–38.

[57] D. Singerman and R. I. Syddall, The Riemann surface of a uniform dessin. *Beiträge zur Algebra und Geometrie*, **44** 2 (2003), 413–430.

[58] J. Tits, A local approach to buildings. In C. Davis, B. Grünbaum and F. A. Sherk (Eds.), *The Geometric Vein*, Springer-Verlag, New York (1981), pp. 519–547.

[59] A. Vince, Combinatorial maps. *Journal of Combinatorial Theory Series B*, **34** (1983), 1–21.

[60] T. R. S. Walsh, Hypermaps versus bipartite maps. *Journal of Combinatorial Theory Series B*, **18** (1975), 155–163.

[61] S. E. Wilson, Operators over regular maps. *Pacific Journal of Mathematics*, **81** (1979), 559–568.

[62] H. Zieschang, E. Vogt and H.-D. Coldewey, *Surfaces and Planar Discontinuous Groups*, Lecture Notes in Mathematics **835**. Springer-Verlag, Berlin (1980).